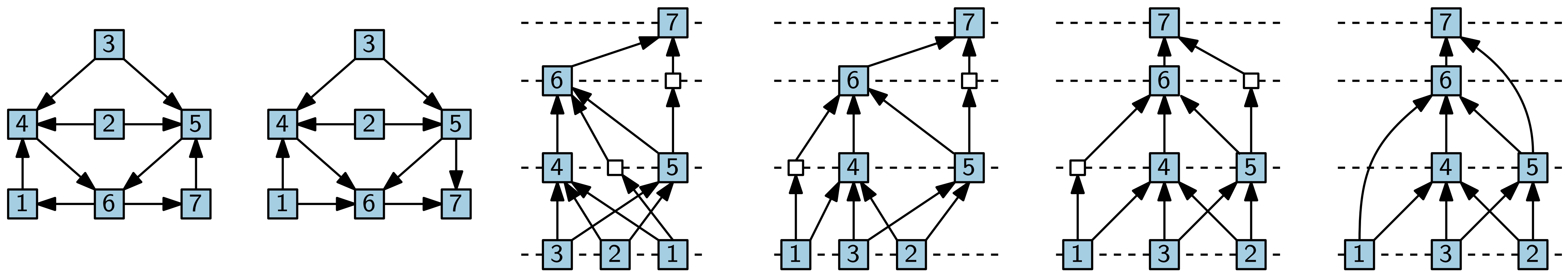
Visualisation of graphs

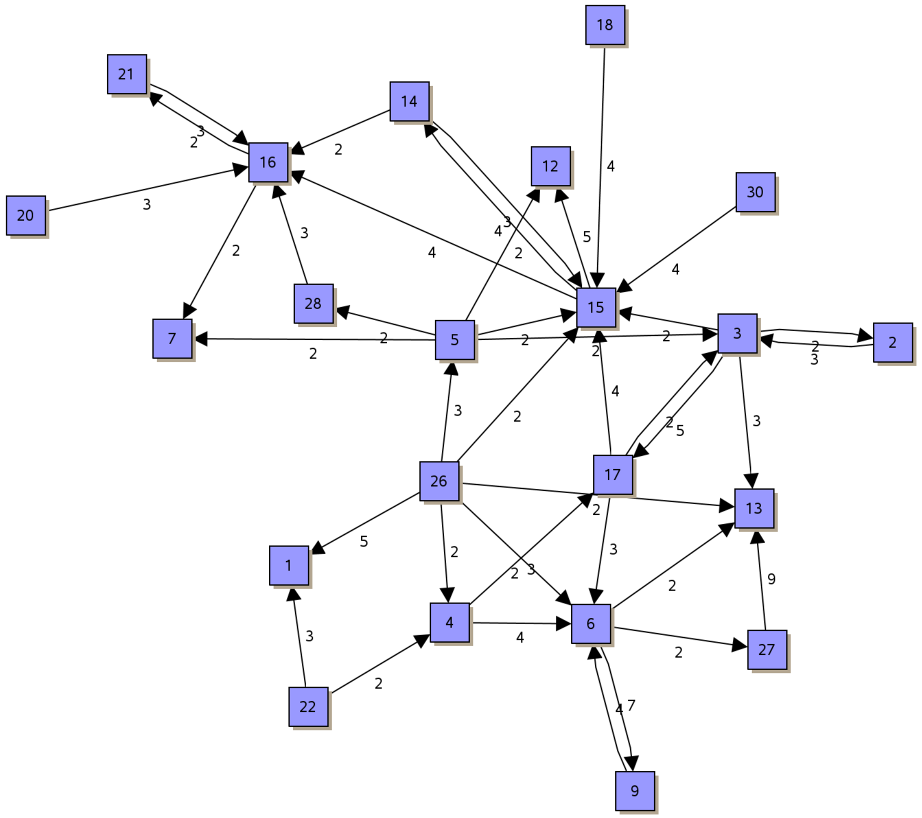
Hierarchical layouts

Sugiyama framework

Jonathan Klawitter · Summer semester 2020



Hierarchical drawings – motivation



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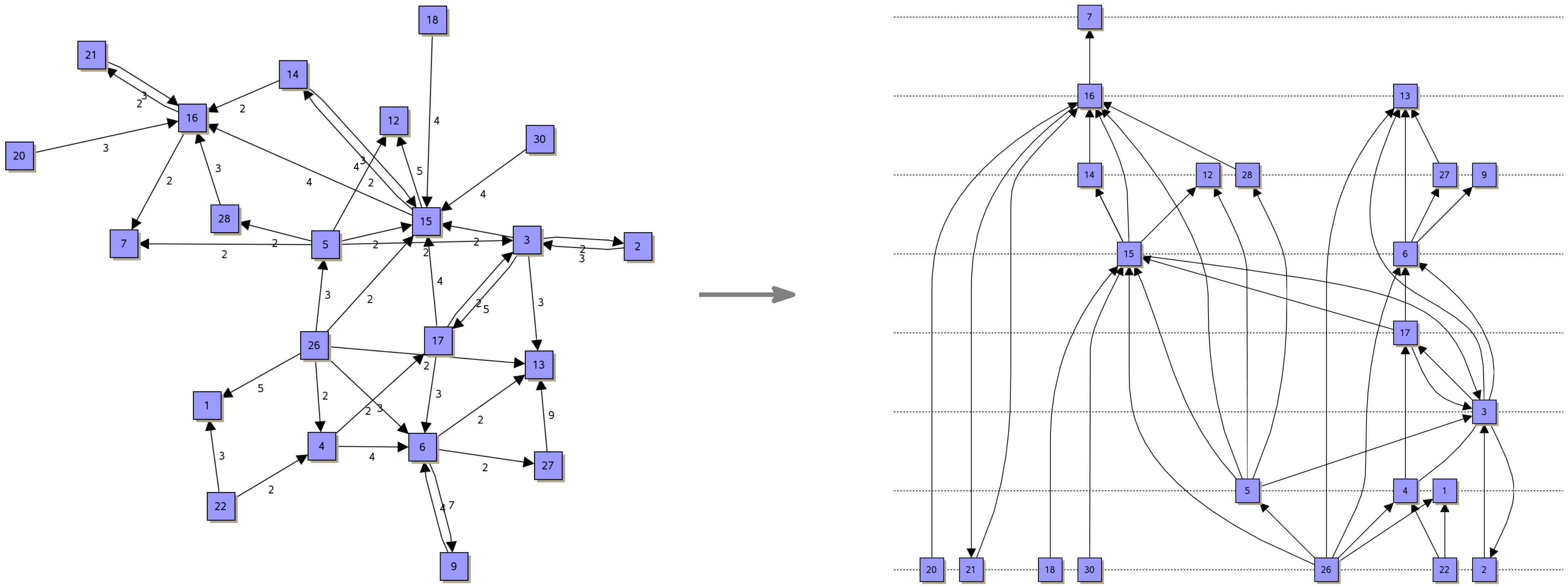
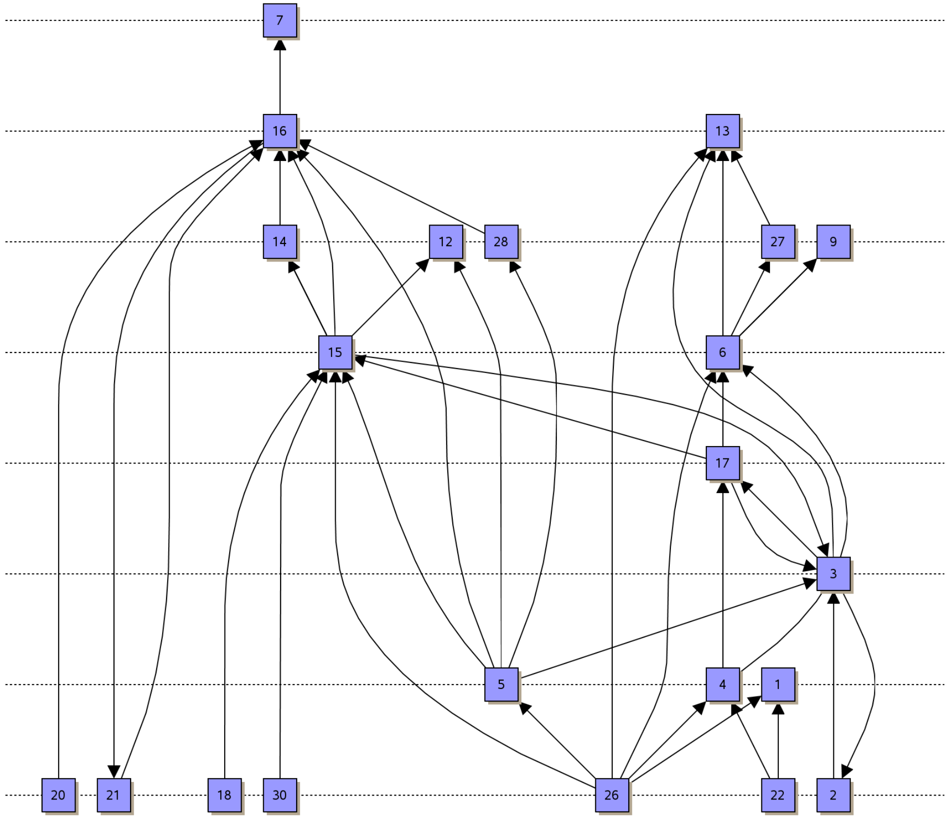


Hierarchical drawing

Problem statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



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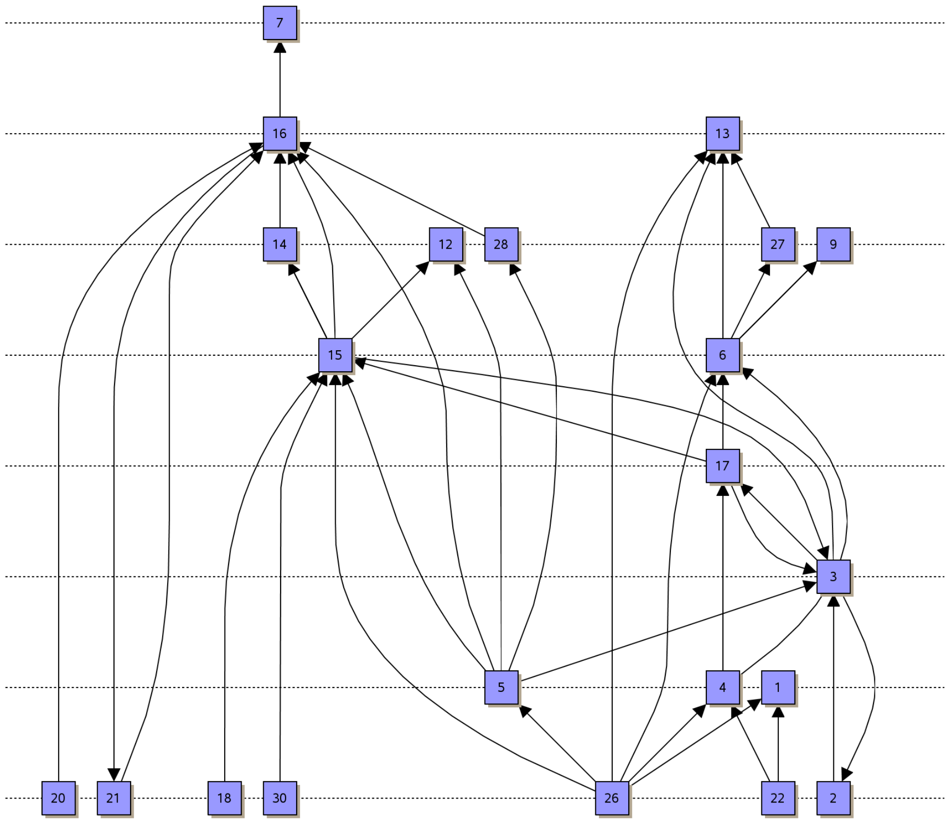
Desireable properties.



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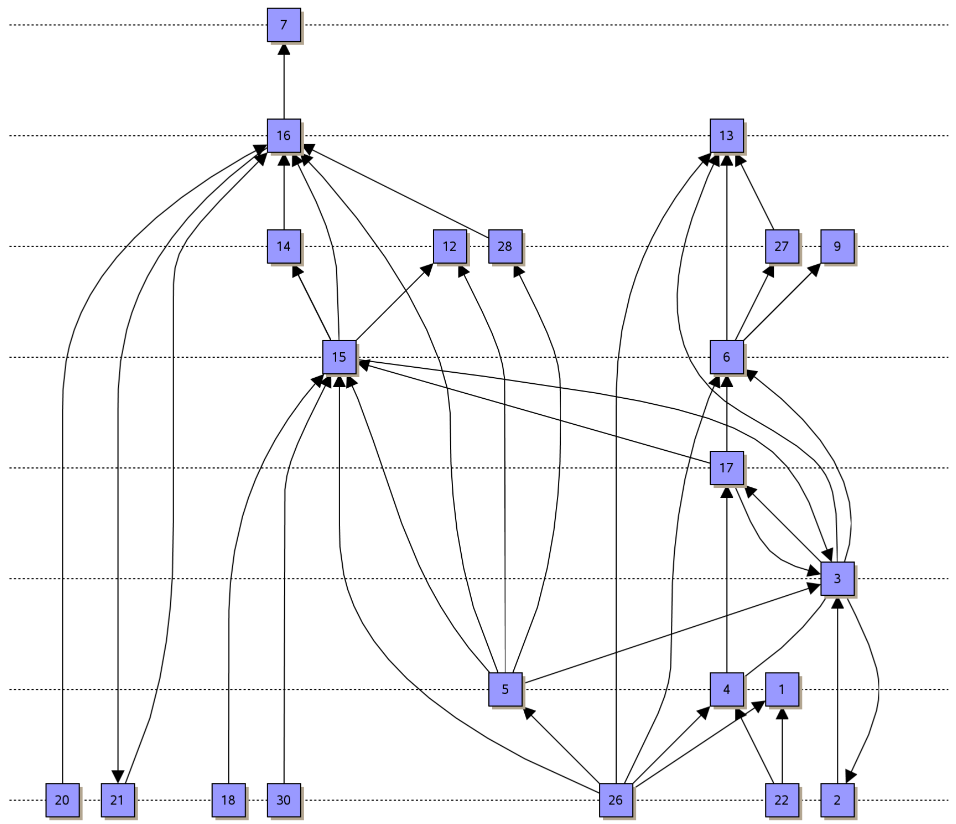
- vertices occur on (few) horizontal lines



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Desireable properties.

- vertices occur on (few) horizontal lines
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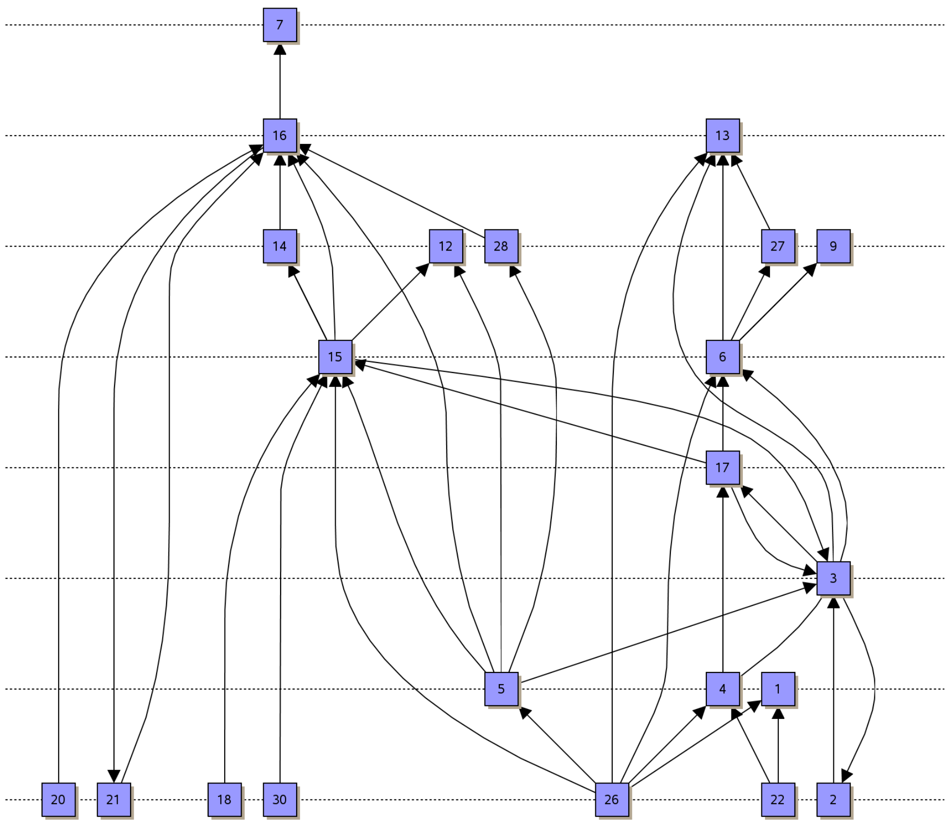

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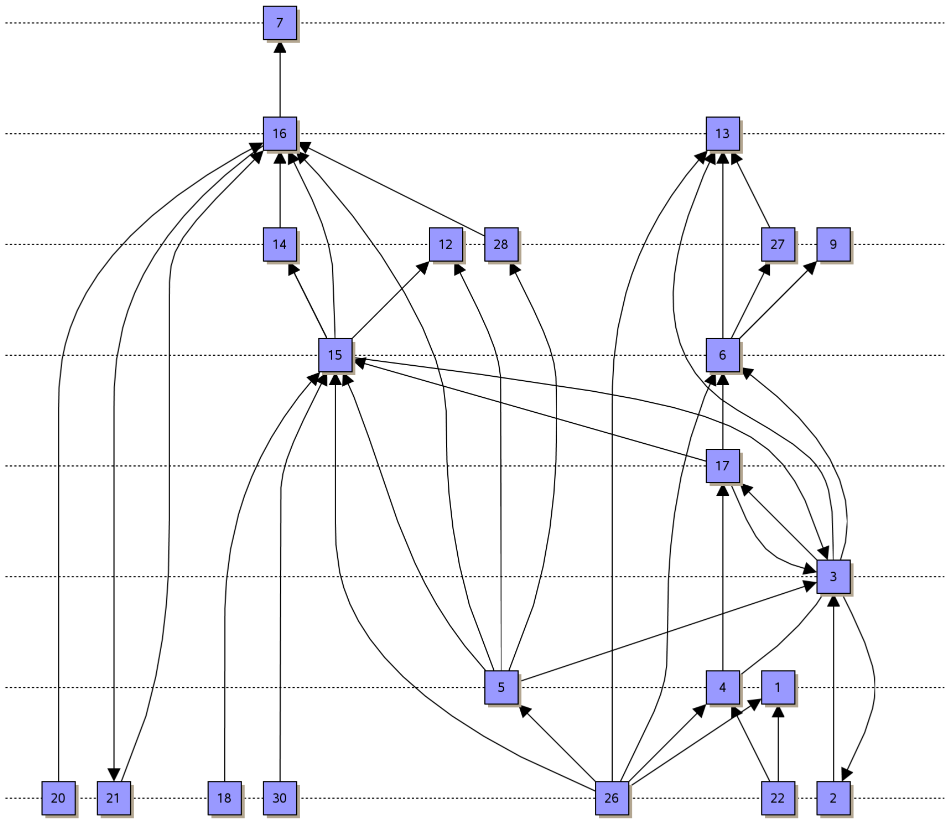

Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized

Hierarchical drawing

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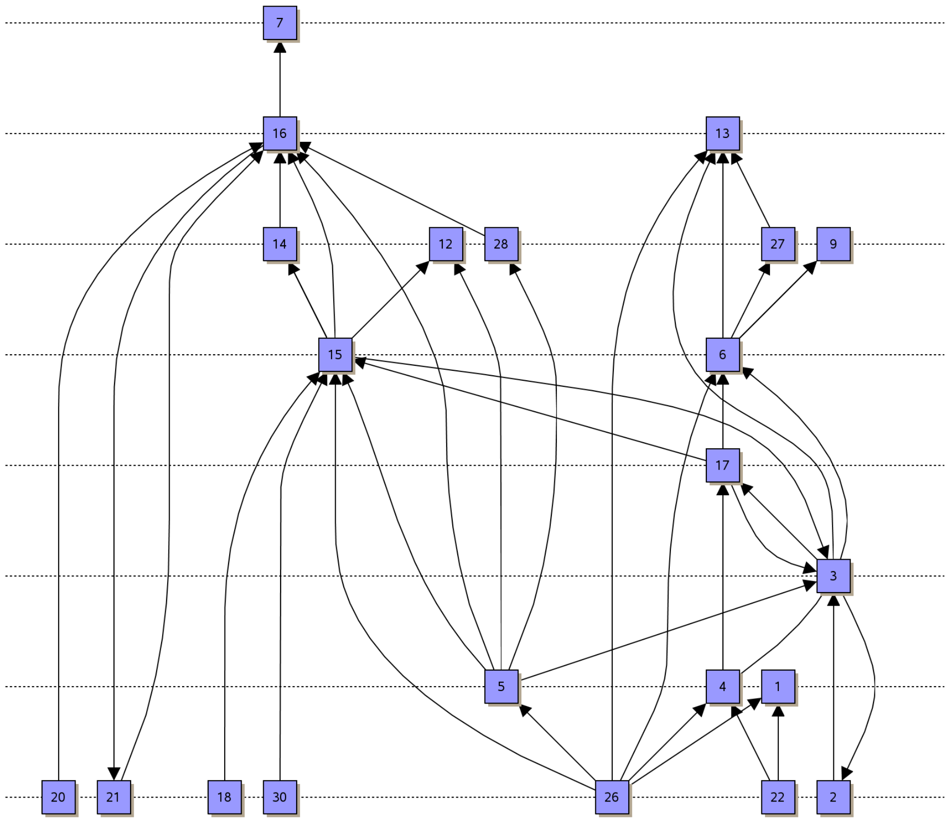
Desireable properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges upward, straight, and short as possible



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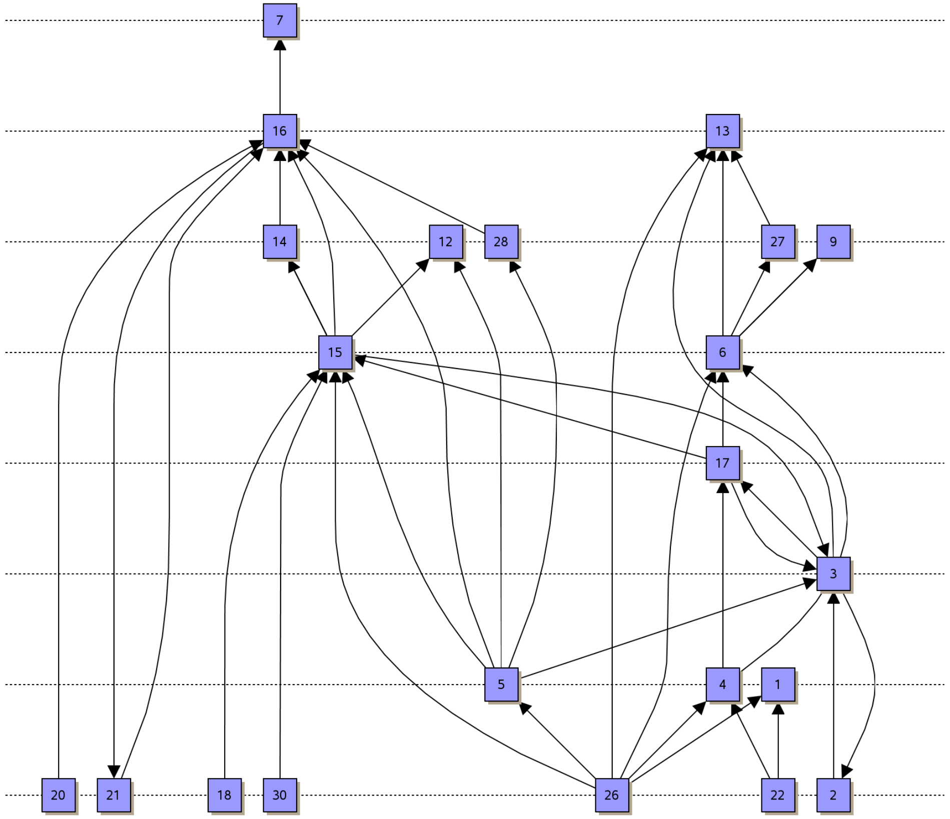
- edges upward, straight, and short as possible
- vertices evenly spaced



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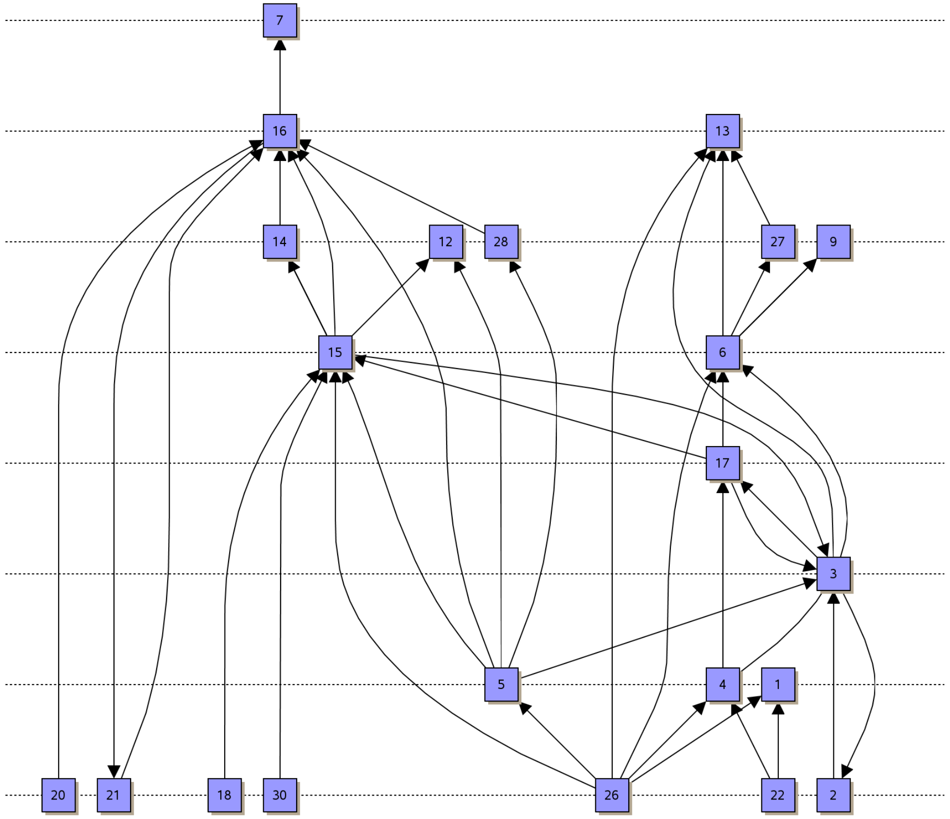
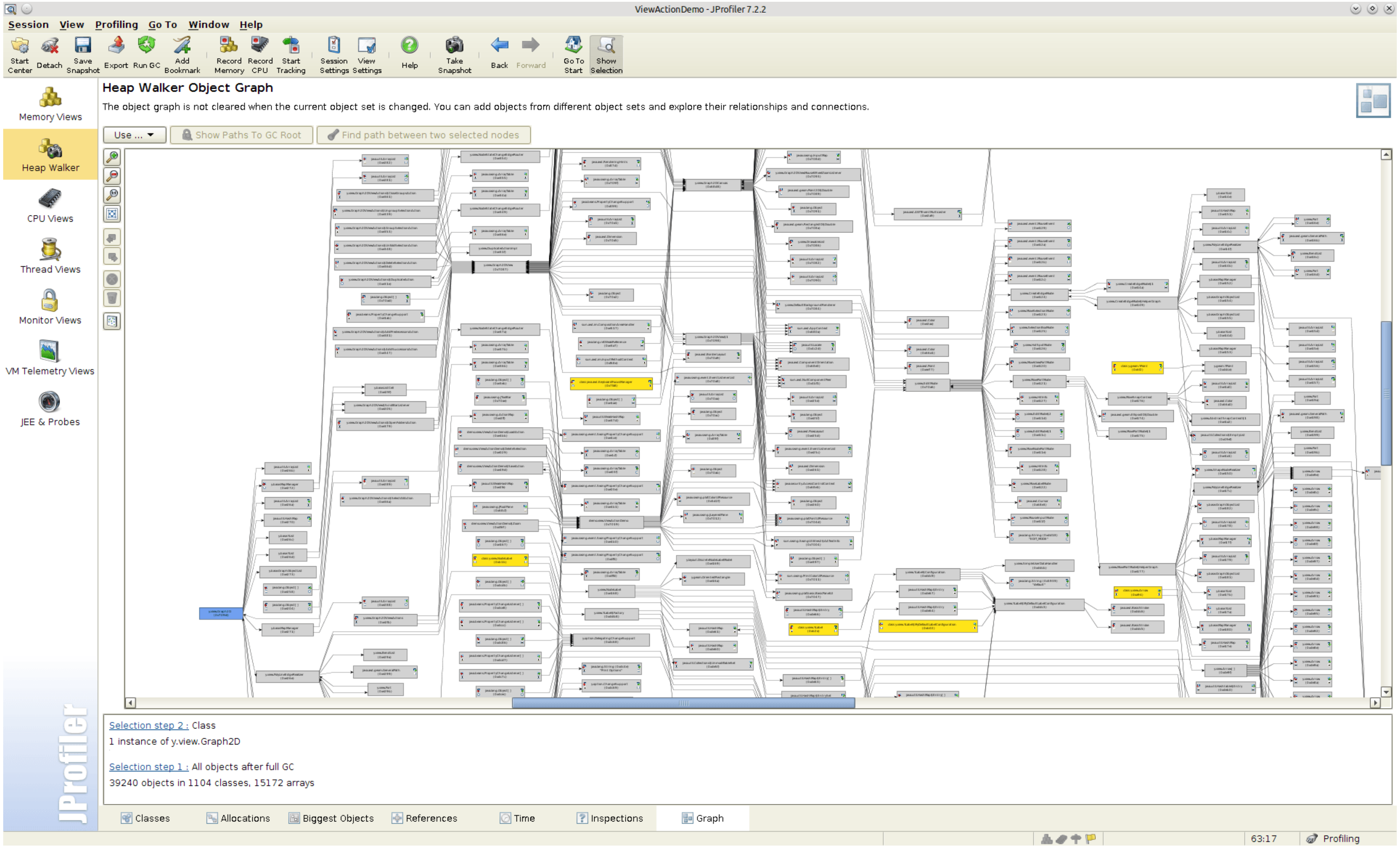
Criteria can be contradictory!



Hierarchical drawing – applications

yEd Gallery: Java profiler JProfiler using yFiles



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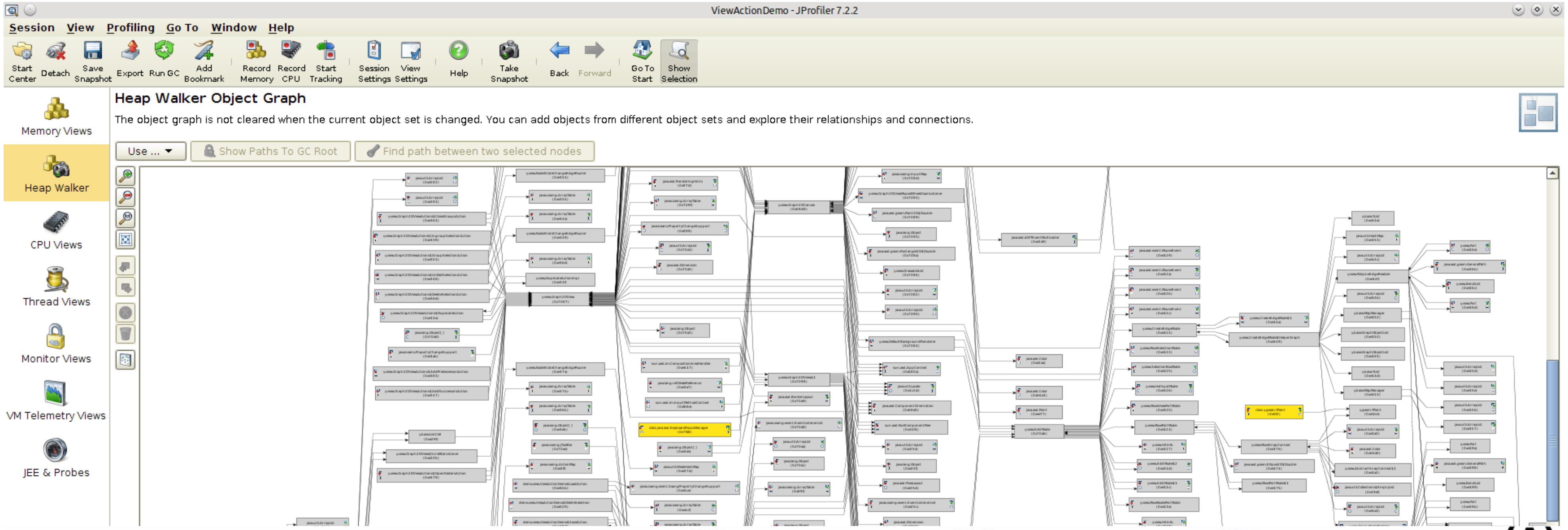

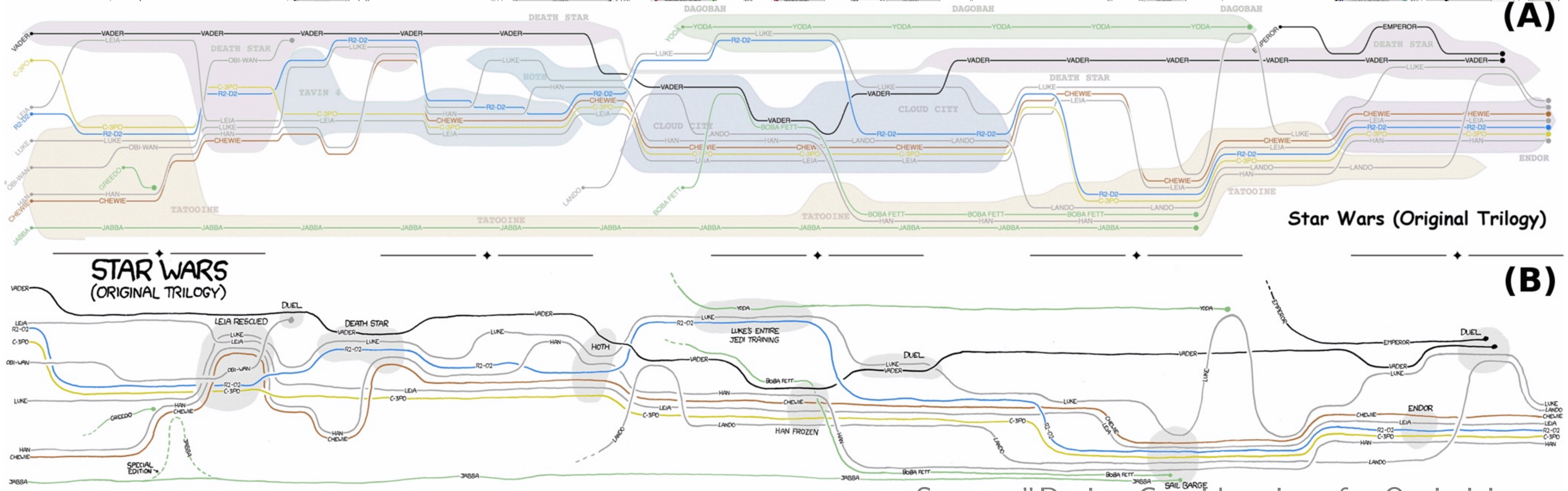

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

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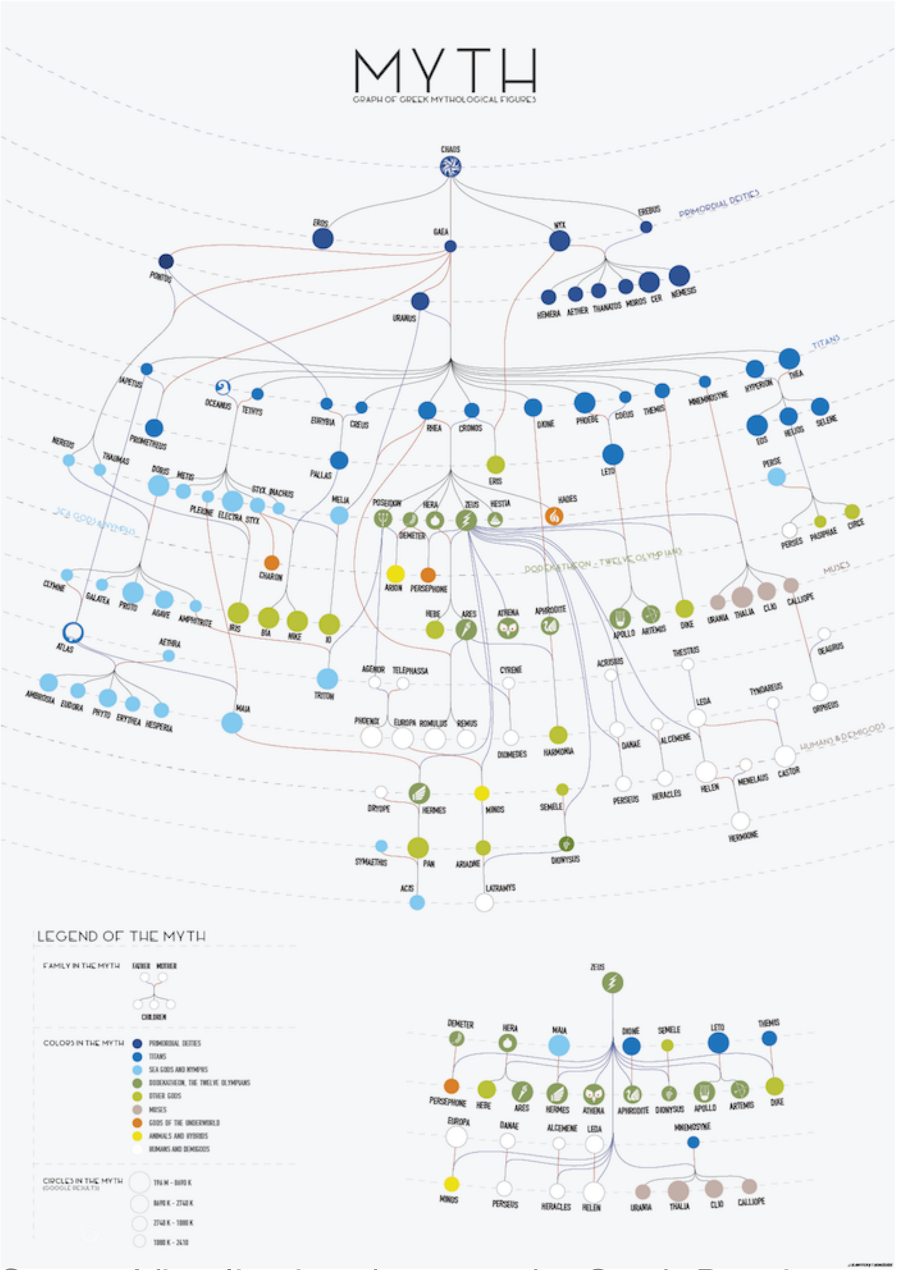

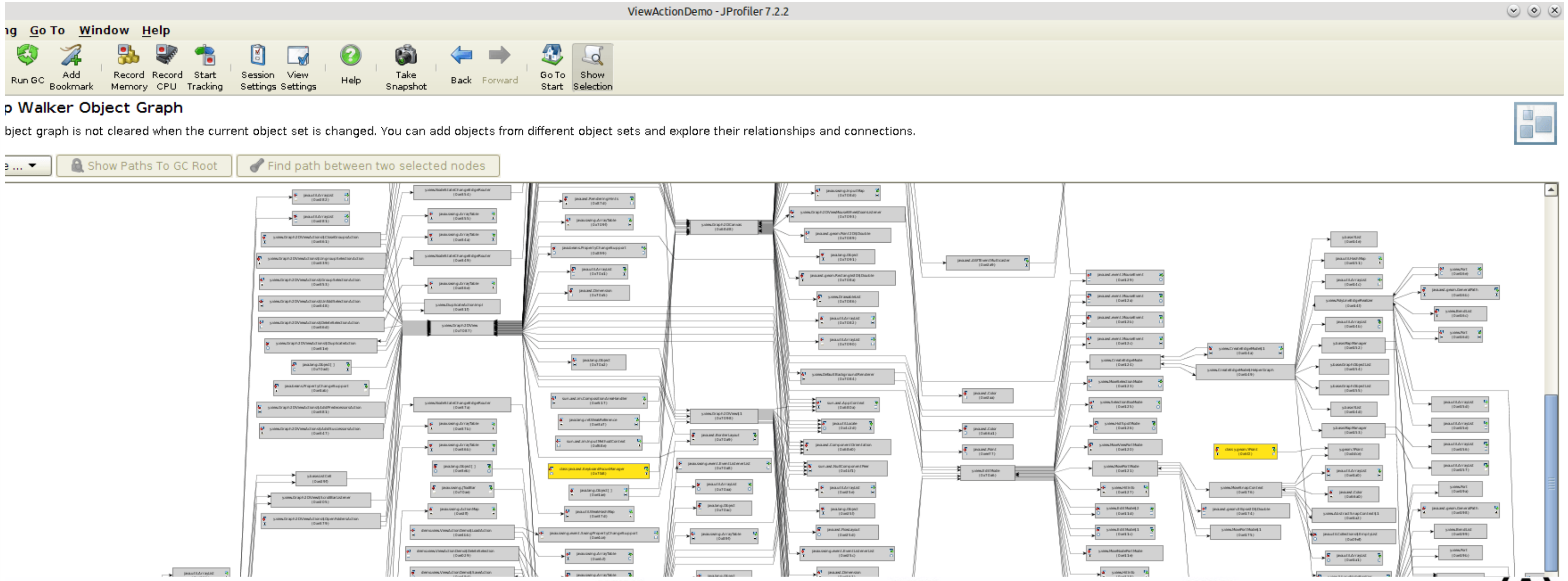

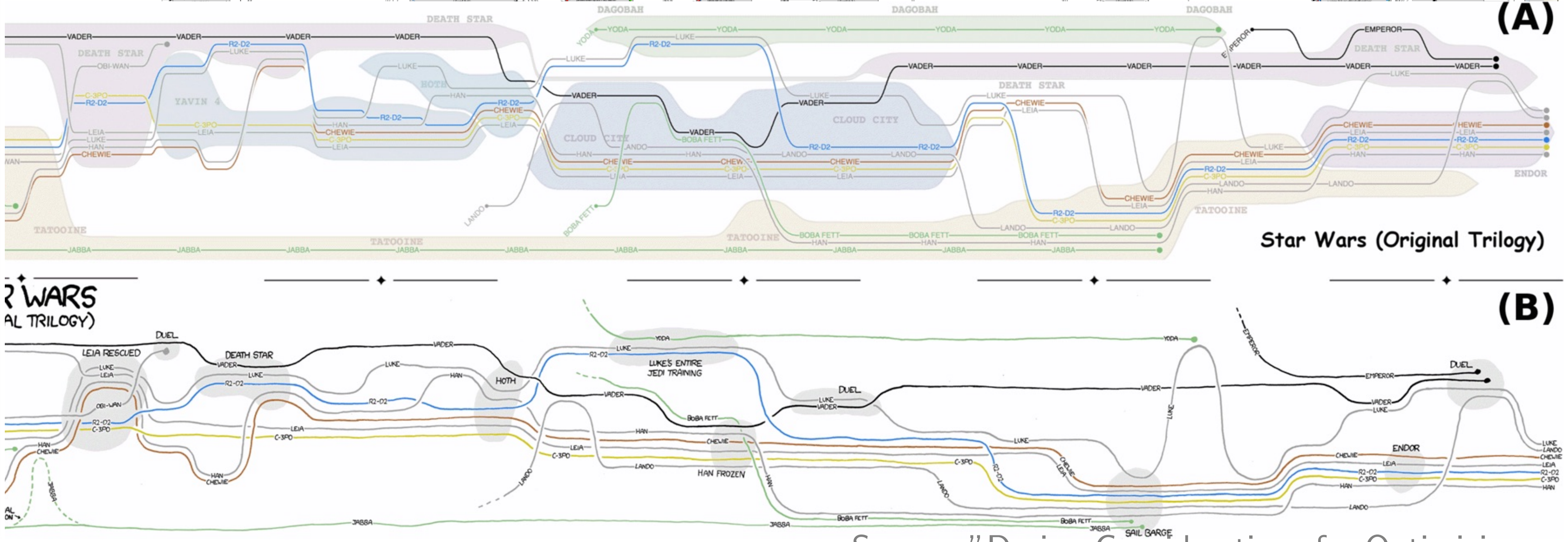

Source: Visualization that won the Graph Drawing contest 2016. Klawitter & Mchedlidze

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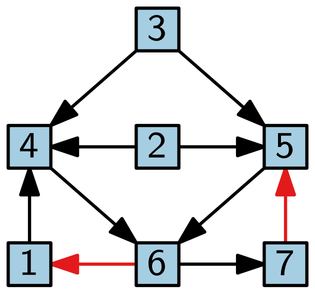
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]

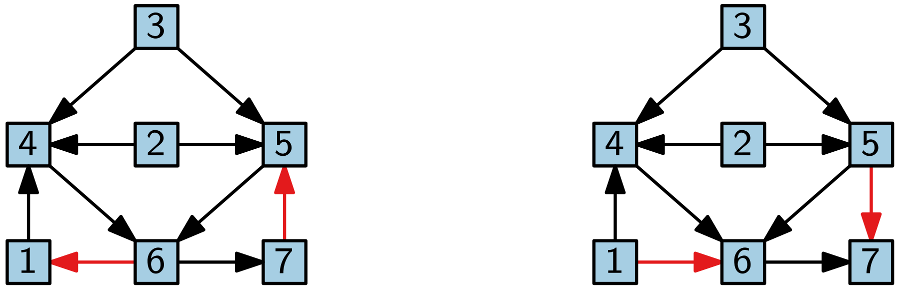
Input



Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]

Input → Cycle breaking



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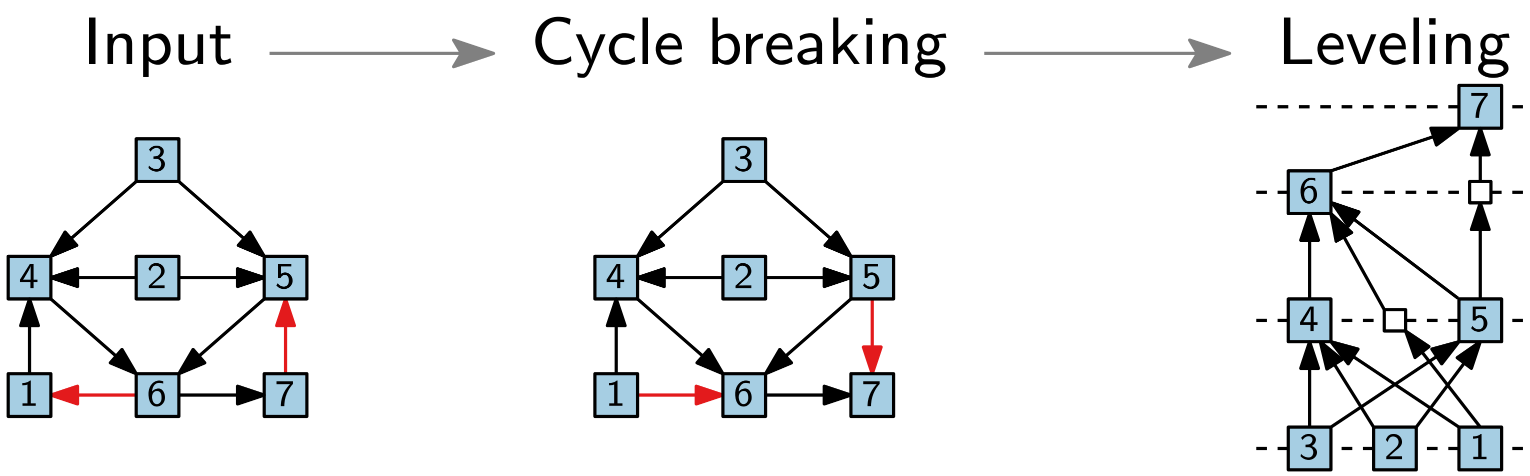

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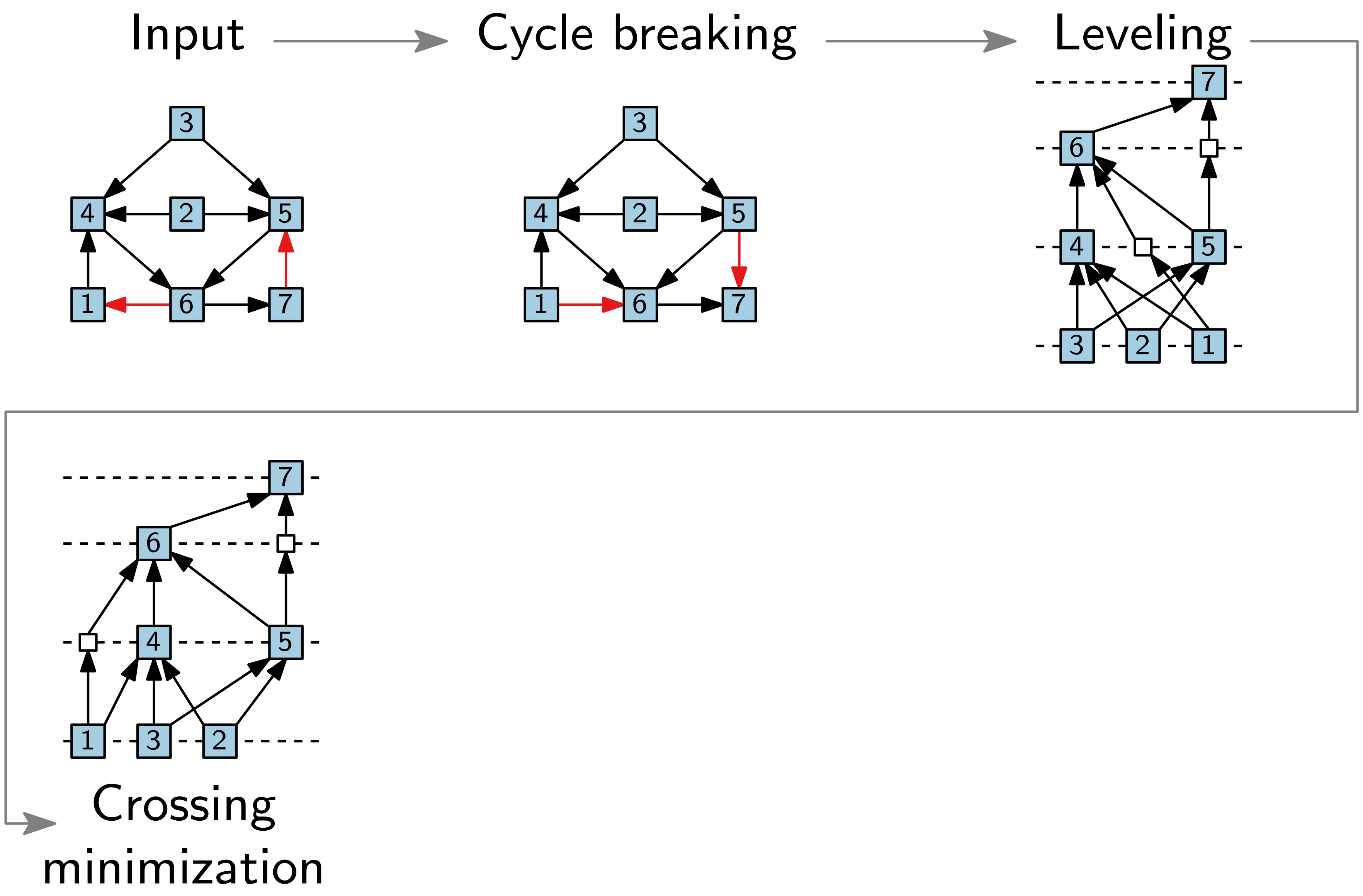

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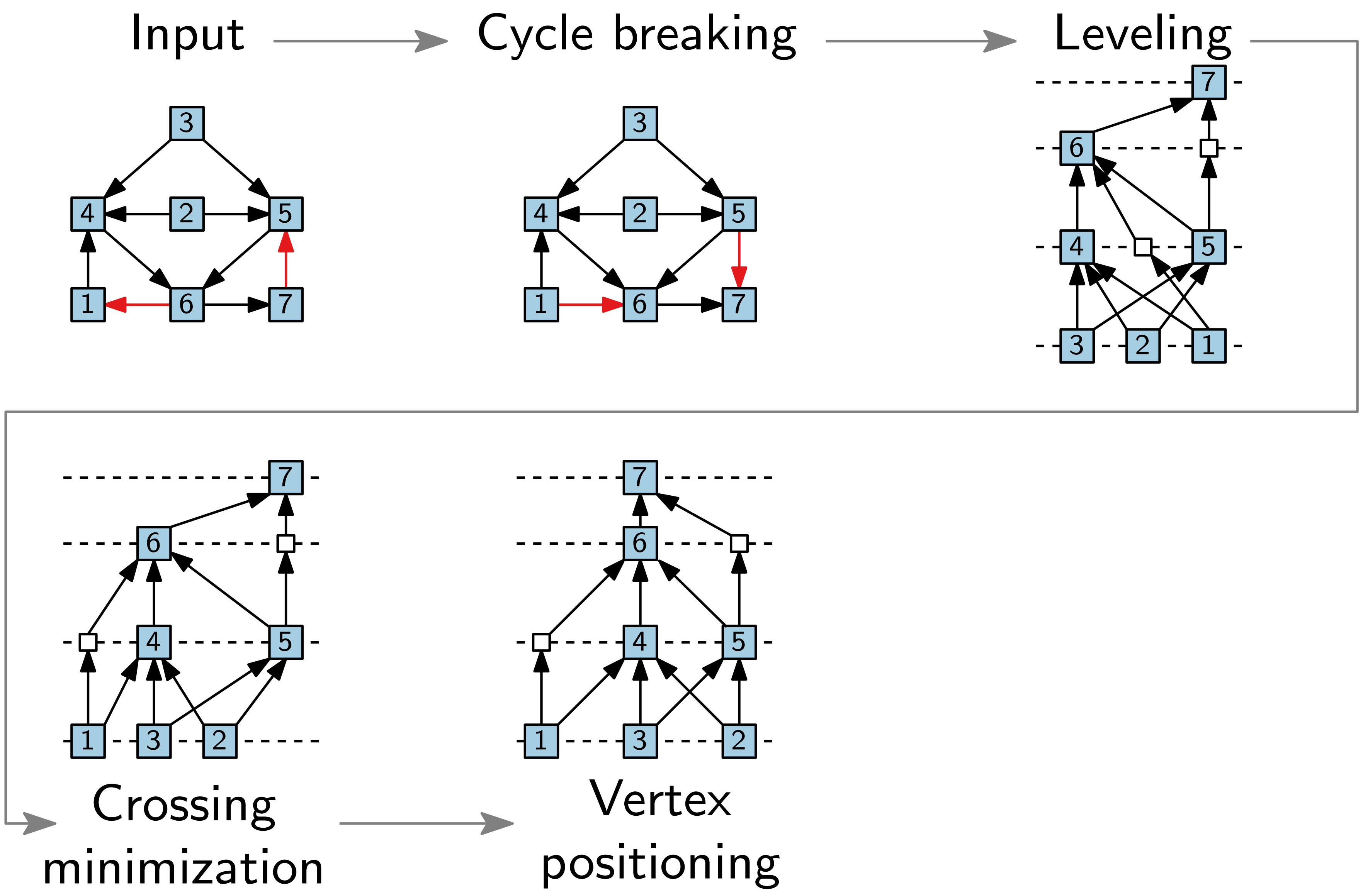

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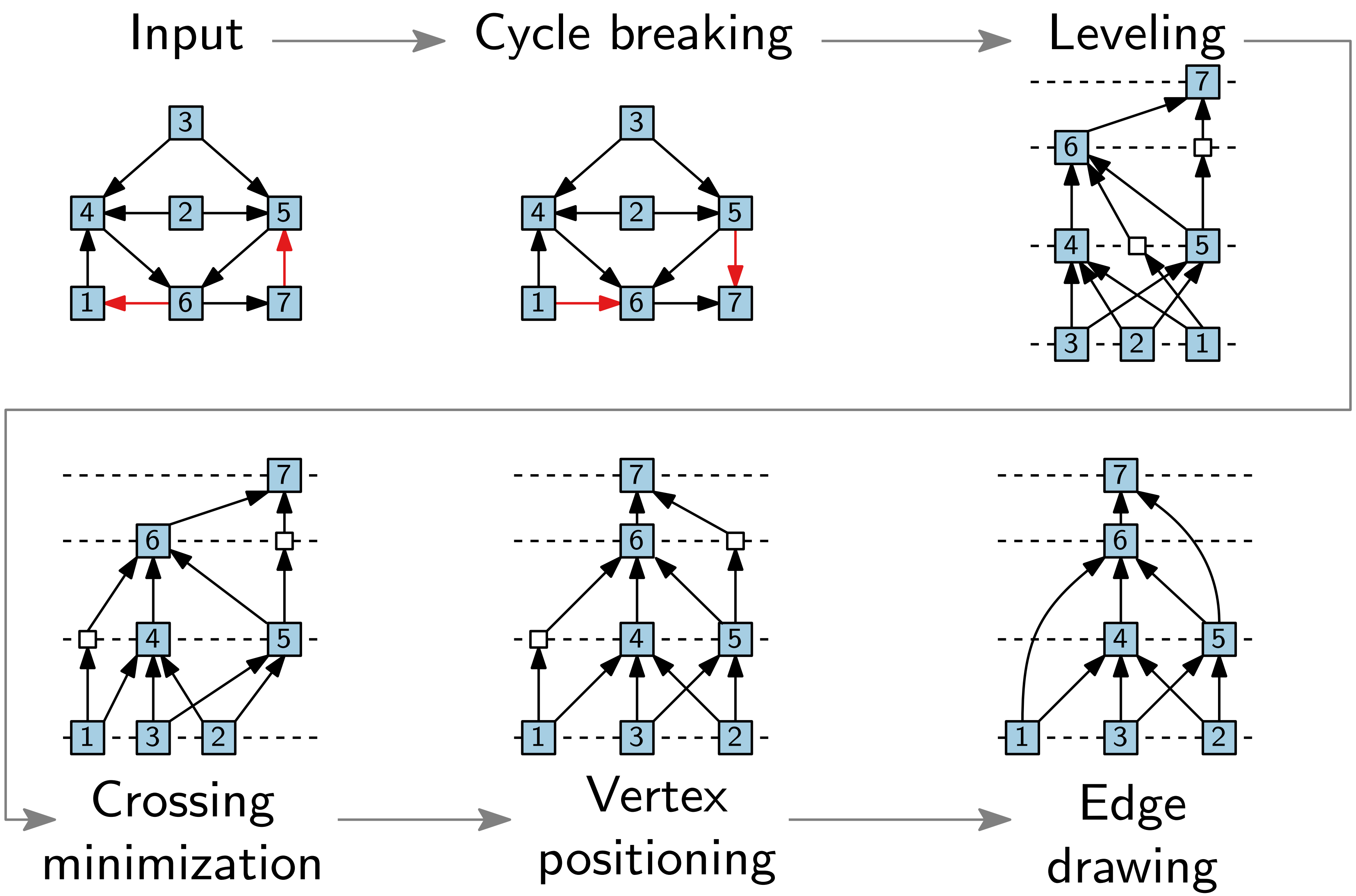
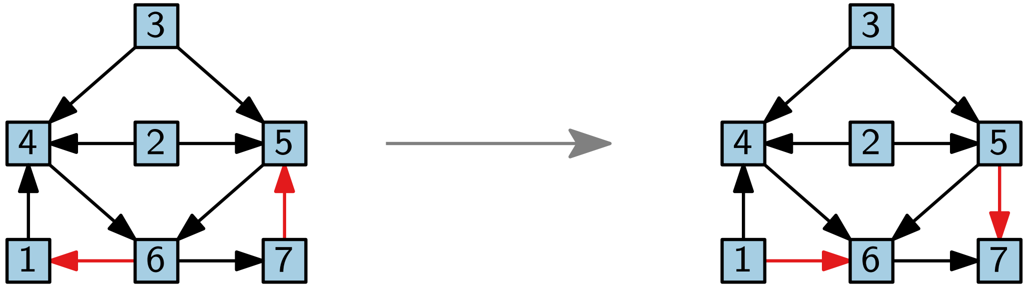


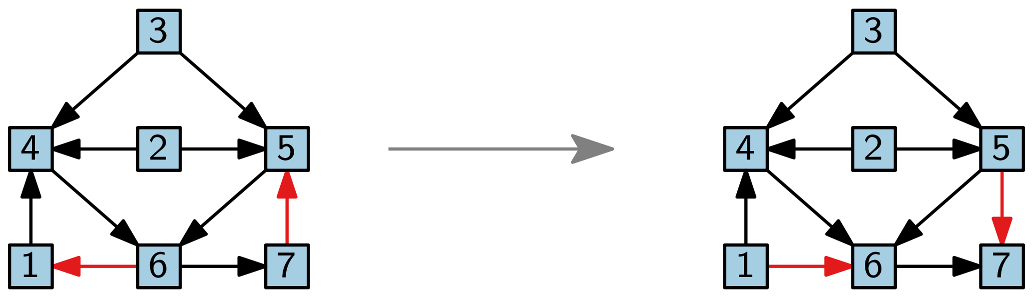
Step 1: Cycle breaking



Approach.

- Find minimum set $E^{\star}$ of edges which are not upwards.
- Remove $E^{\star}$ and insert reversed edges.

Step 1: Cycle breaking



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- Find minimum set $E^\star$ of edges which are not upwards.
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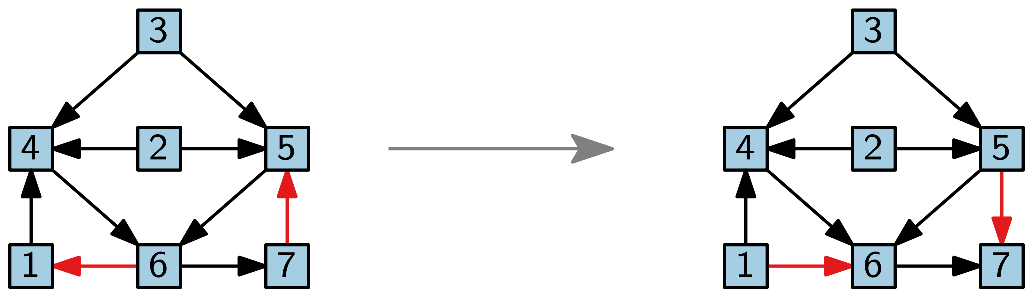
Problem Minimum Feedback Arc Set **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

Step 1: Cycle breaking



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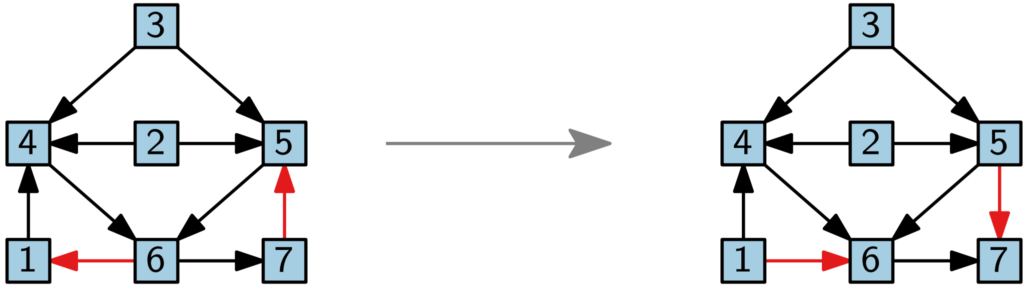
Problem Minimum Feedback ~~Arc~~ Set (F~~A~~S).

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that ~~$G - E^\star$~~ $G - E^\star + E_r^\star$ acyclic

Step 1: Cycle breaking



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Problem MINIMUM FEEDBACK ~~ARC~~ SET **(F~~A~~S).**

- Input: directed graph $G = (V, E)$
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. . . NP-hard :-(

Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$
foreach $v \in V$ **do**
  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  else
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  remove v and $N(v)$ from G.
return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
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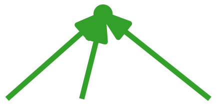

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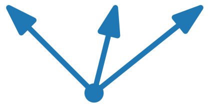

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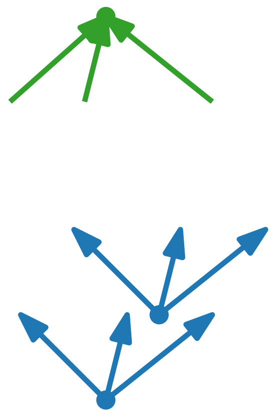

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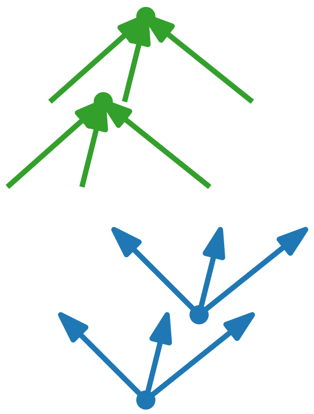

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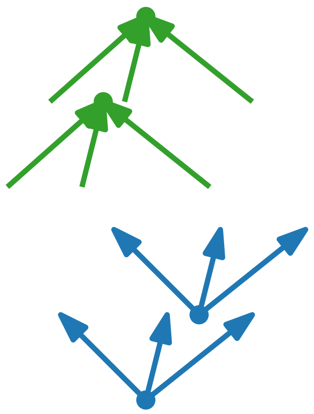
  - we create an order on V



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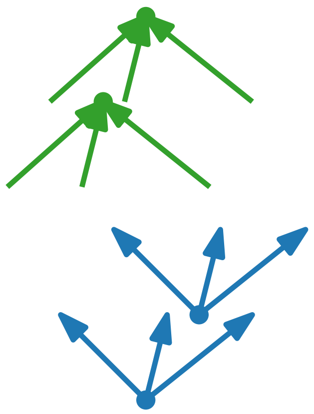
- $E \setminus E'$ is a feedback arc set



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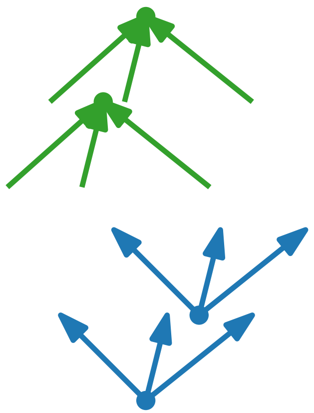

- $G' = (V, E')$ is a DAG
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- Time:

Heuristic 1

[Berger, Shor '90]

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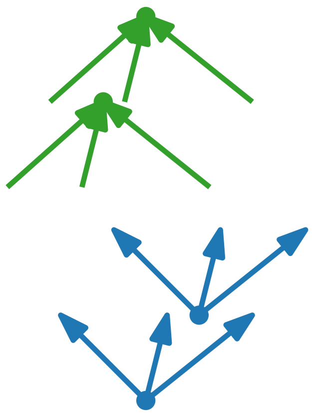

- Time: $\mathcal{O}(|V| + |E|)$

Heuristic 1

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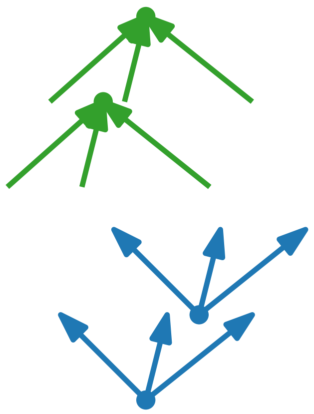

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- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq$

Heuristic 1

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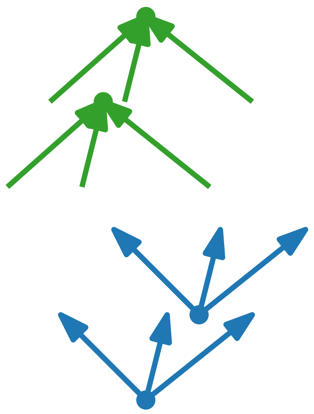

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- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq |E|/2$

Heuristic 2

[Eades, Lin, Smyth ’93]

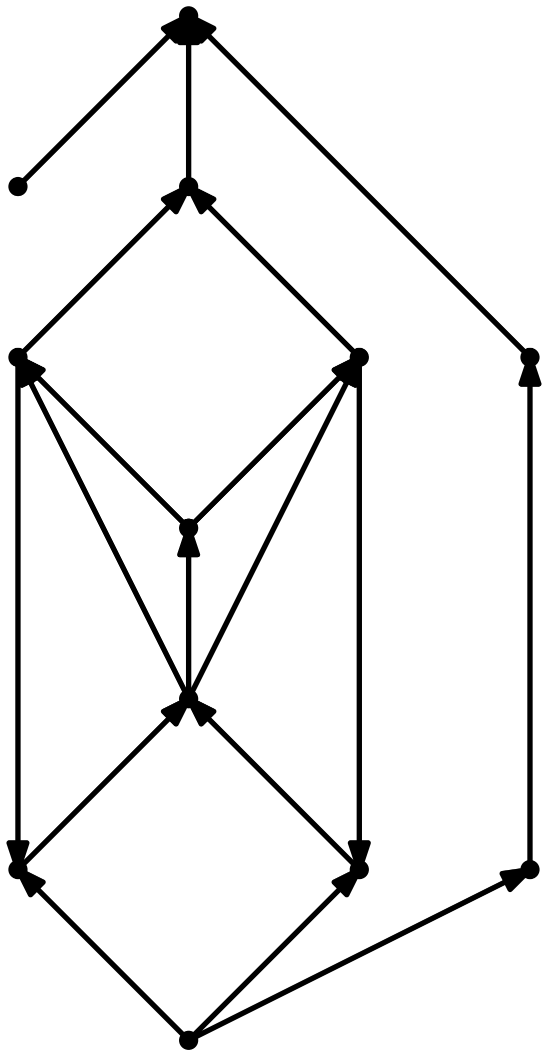
$E' \leftarrow \emptyset$

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

    remove v and $N^{\leftarrow}(v)$



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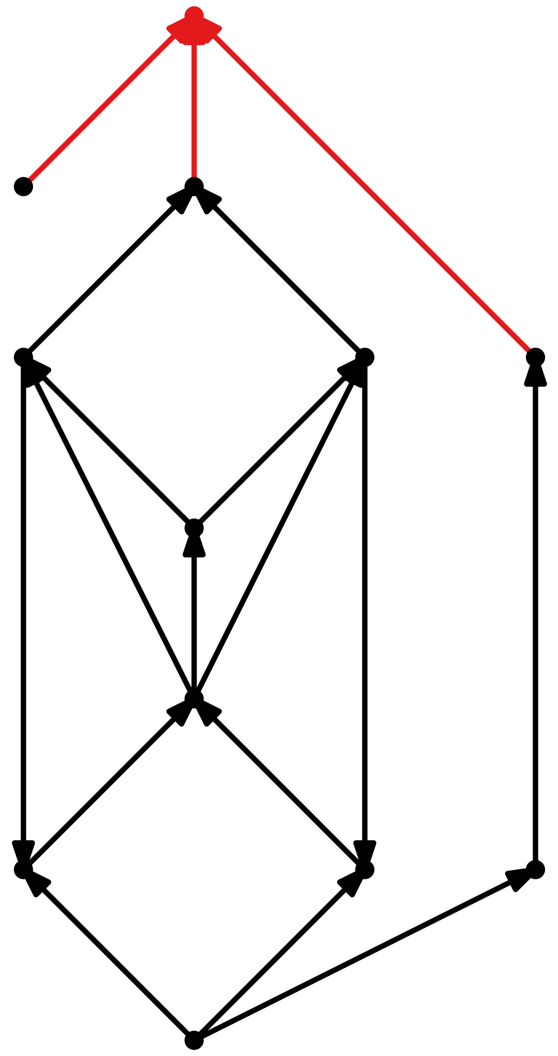

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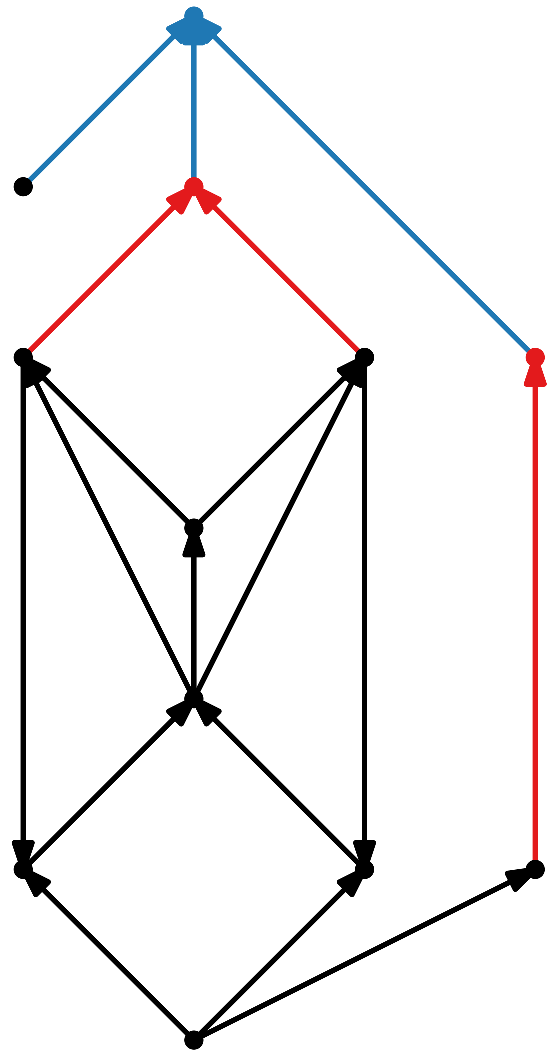

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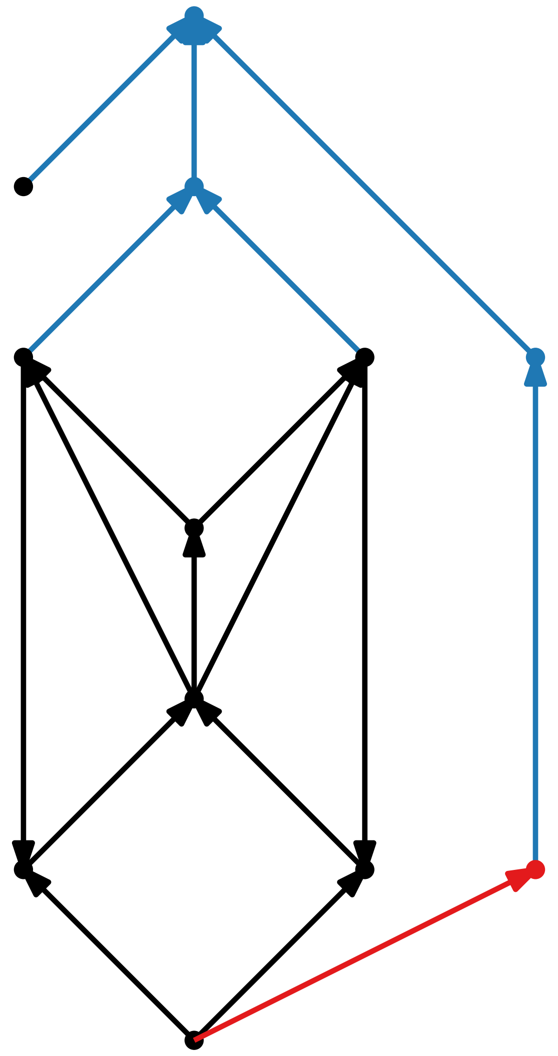

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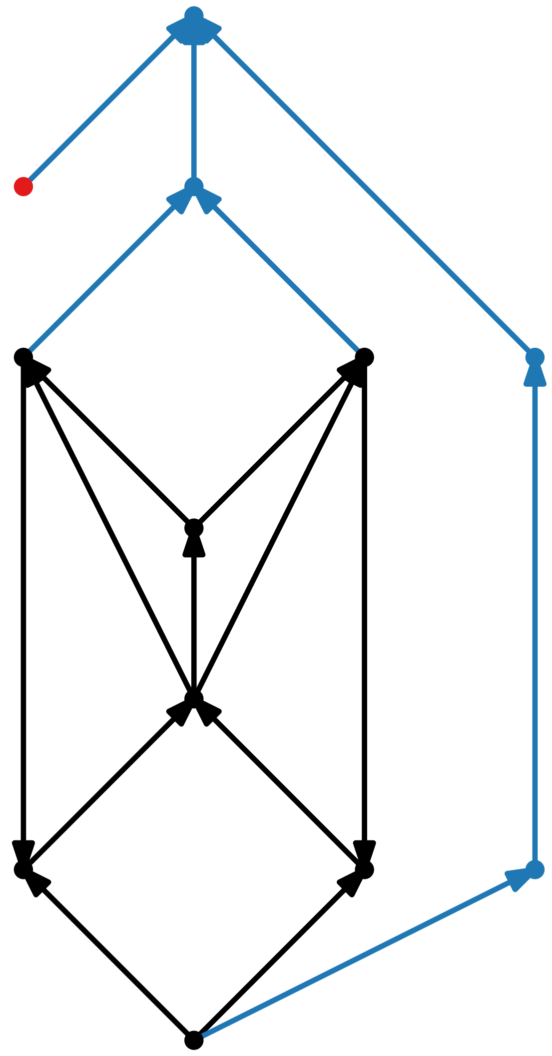

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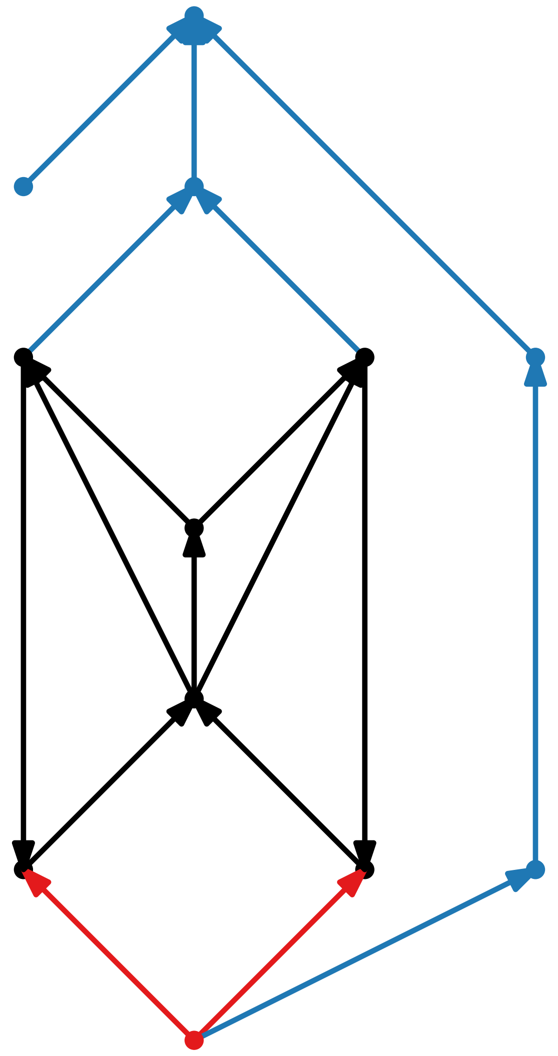

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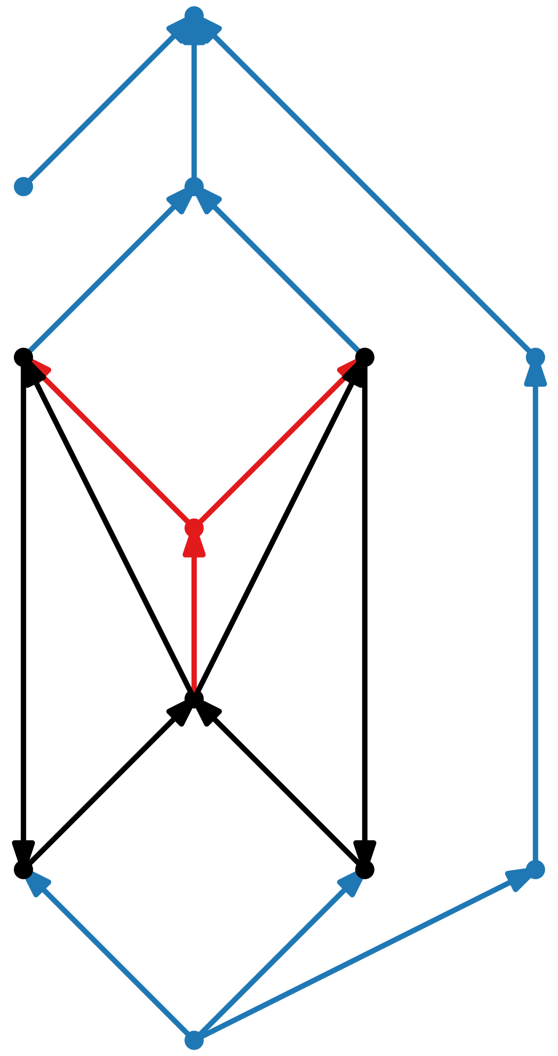

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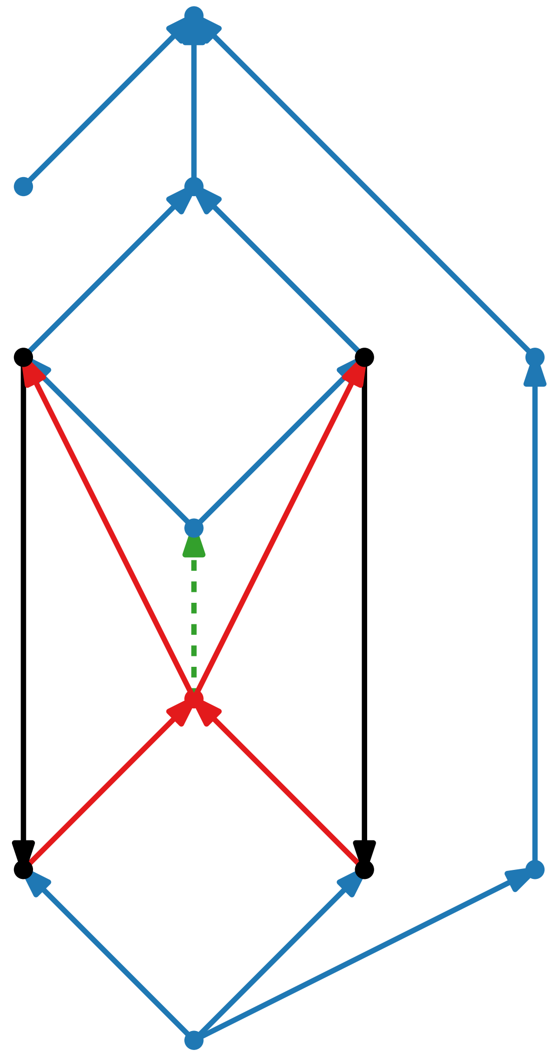

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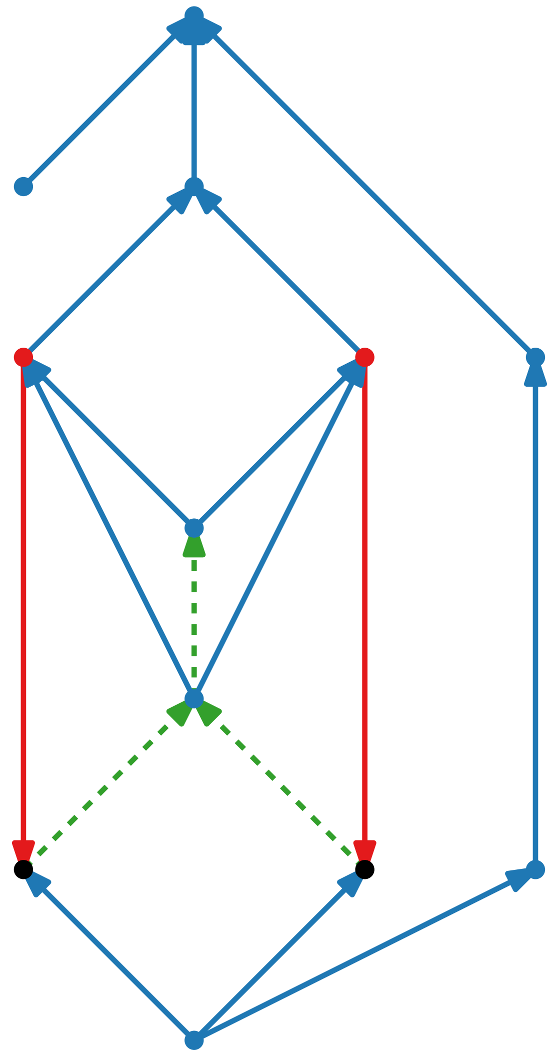

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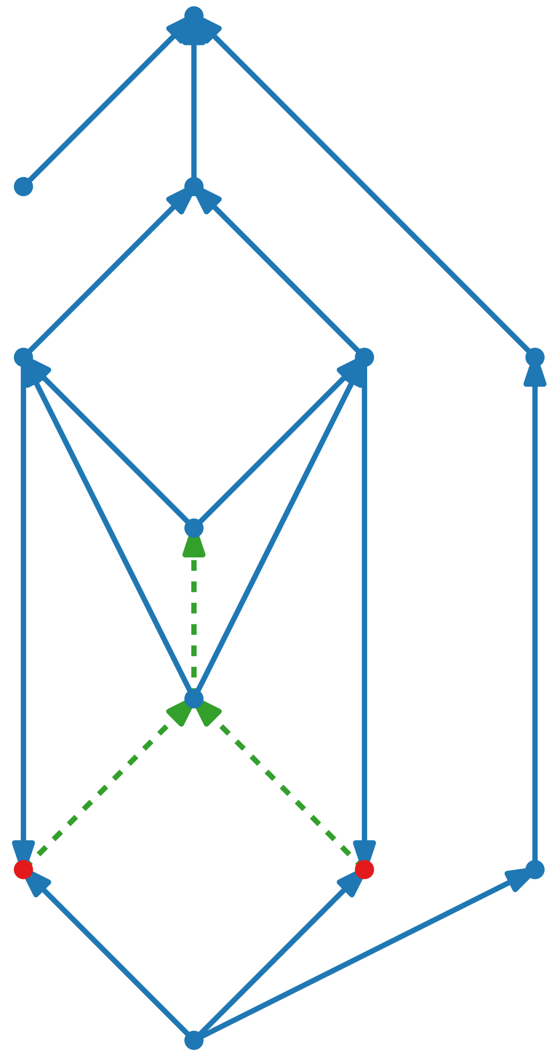

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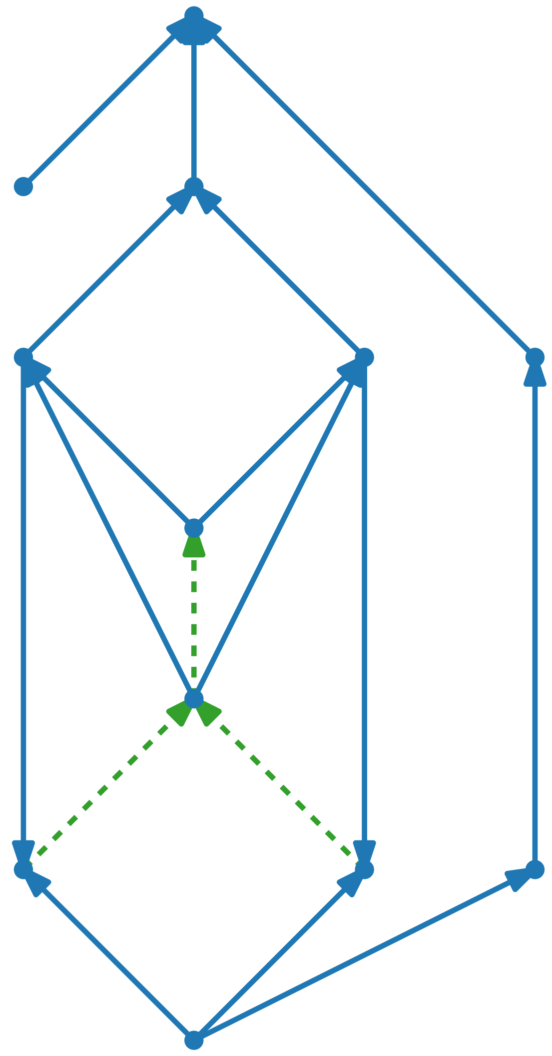

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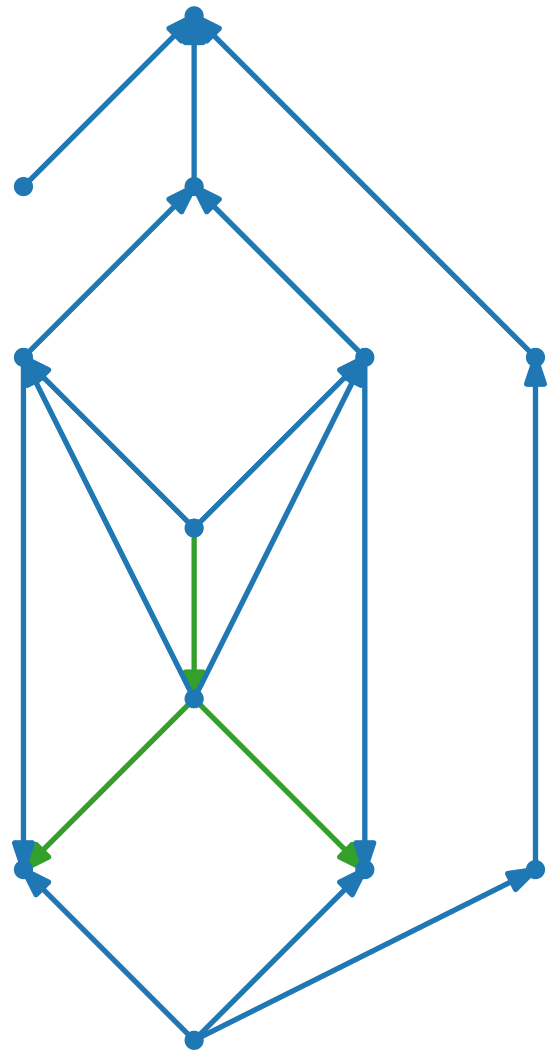

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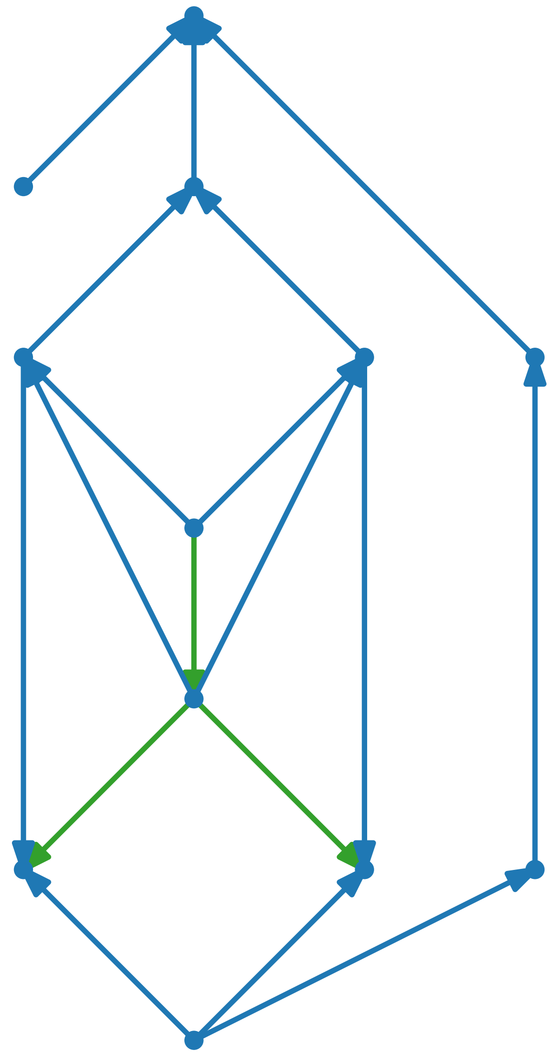
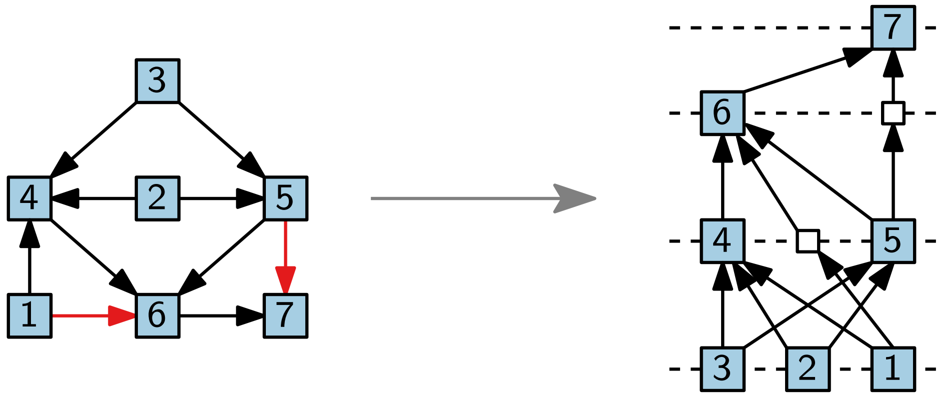


- Time: $\mathcal{O}(|V| + |E|)$
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$

Step 2: Leveling



Problem.

- Input: acyclic, digraph $G = (V, E)$
- Output: Mapping $y \colon V \to \{1, \ldots, |V|\}$,
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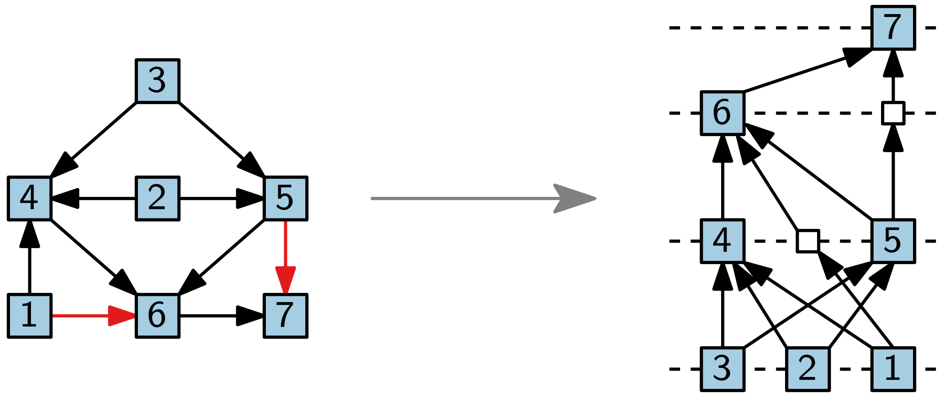

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Objective is to *minimize . . .*

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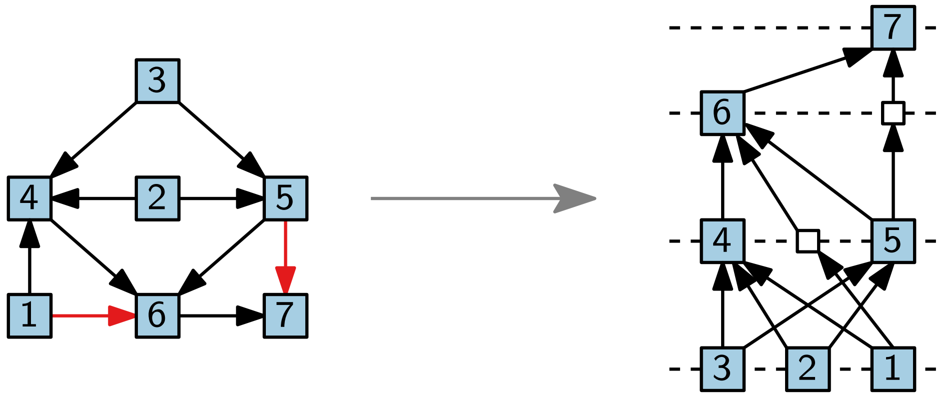

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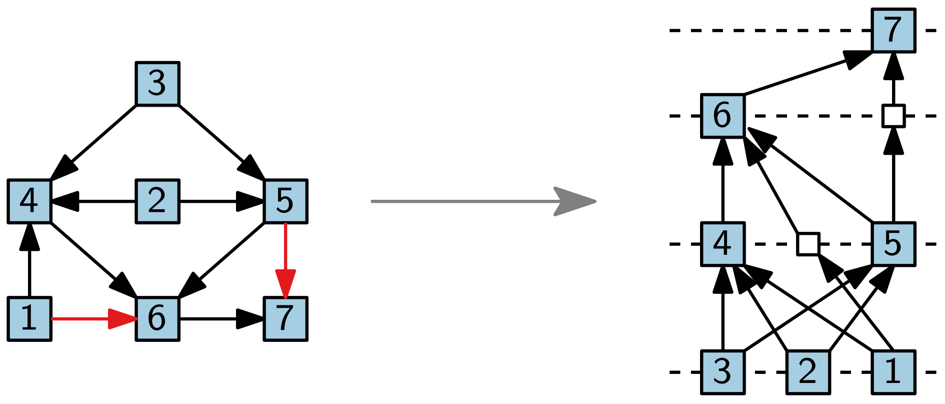
Objective is to *minimize* ...

- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in A} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices

Min number of layers

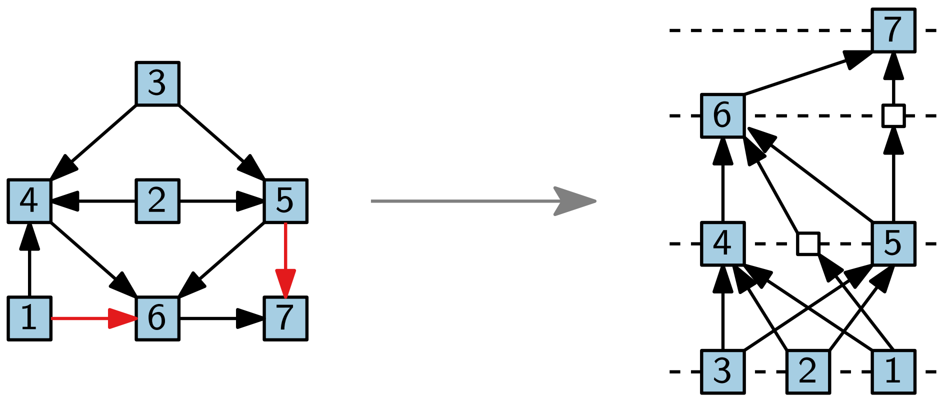
Algorithm.



Min number of layers

Algorithm.

- for each source q
  set $y(q) := 1$



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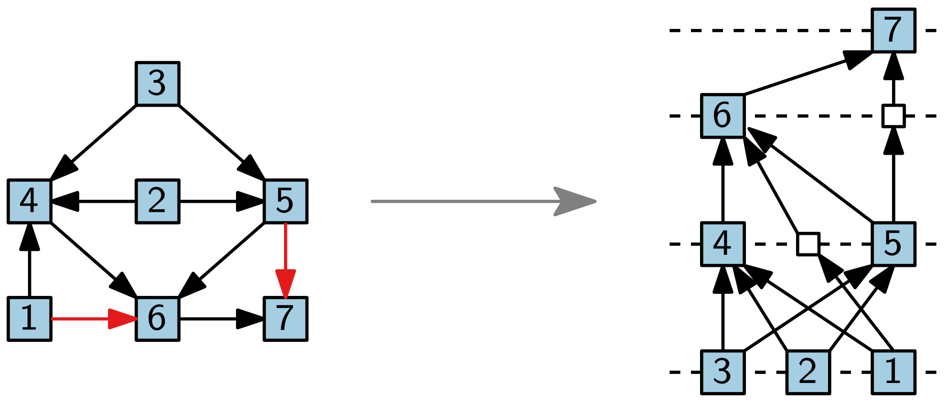

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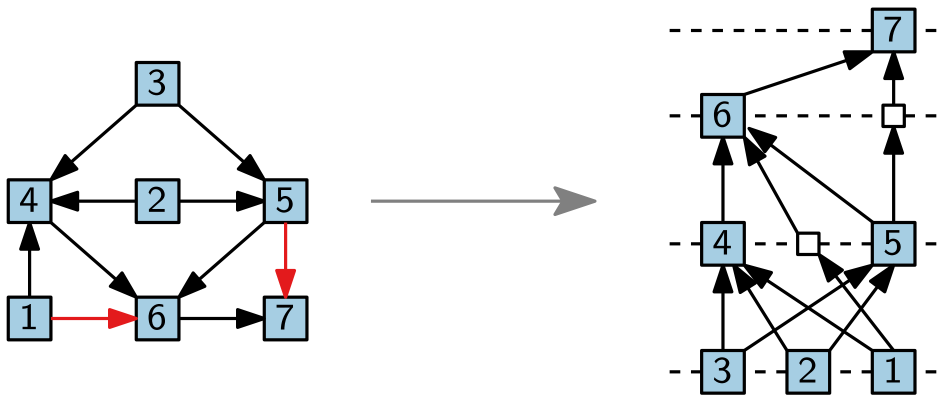

Observation.

- $y(v)$

Min number of layers

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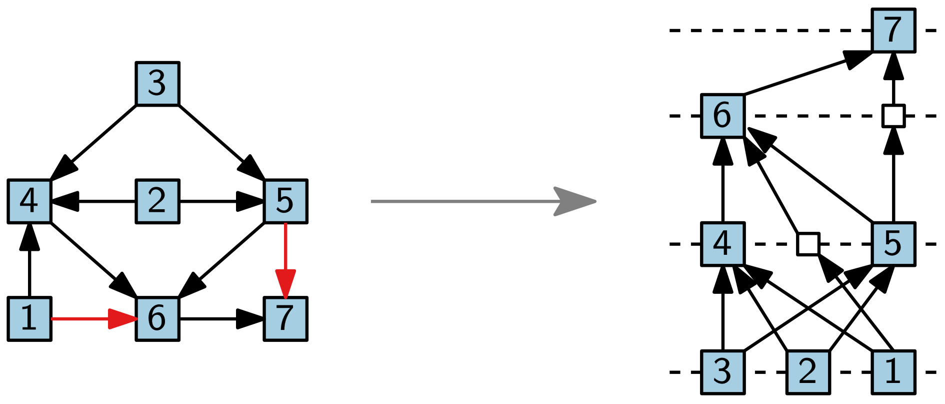

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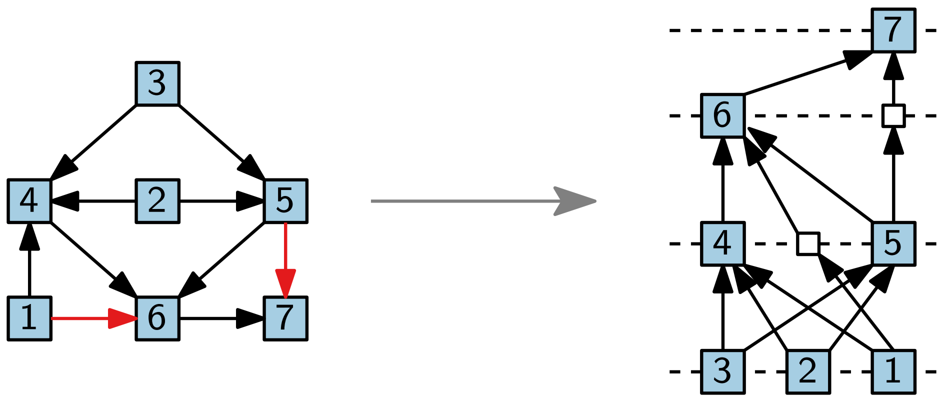

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  ... which is optimal!
- Can be implemented in linear time with recursive algorithm.

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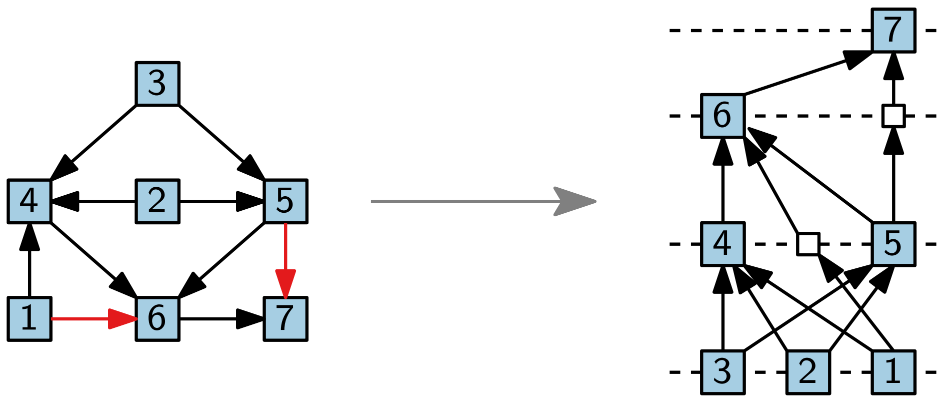

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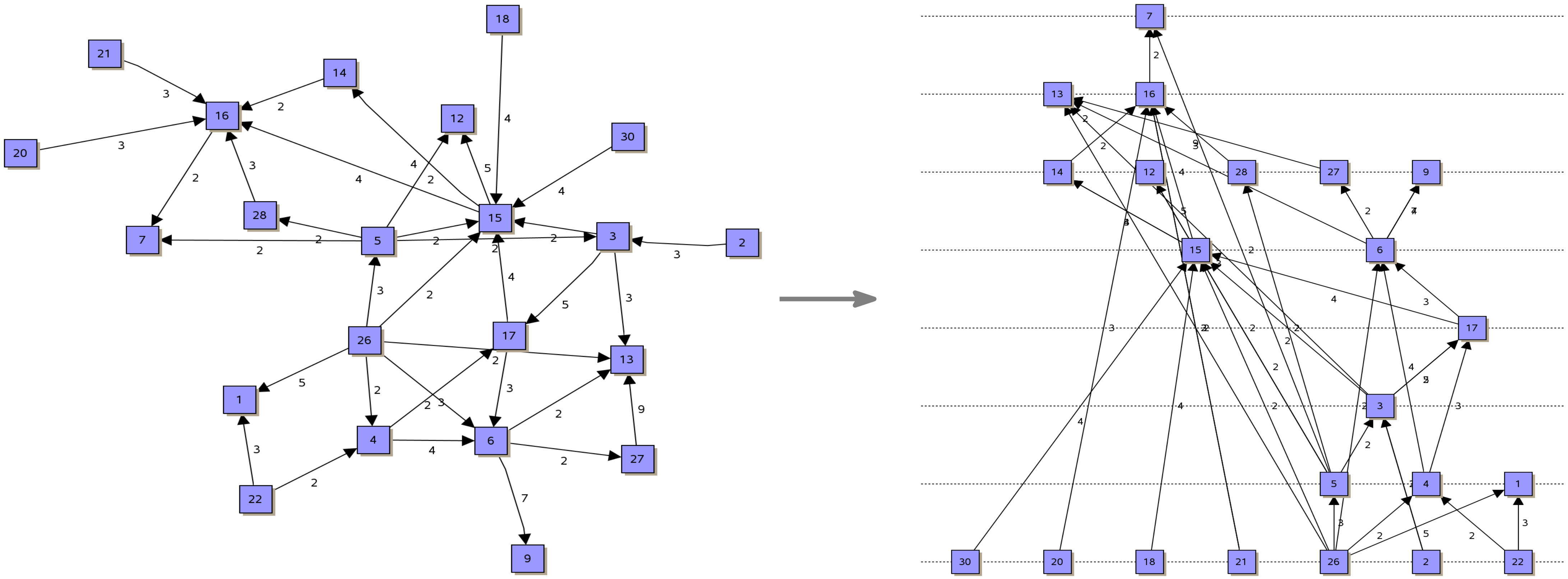
Example


21 18 14 16 12 30 20 28 15 7 5 3 2 26 17 13 1 4 6 27 22 9
7 13 16 14 12 28 27 9 15 6 17 3 5 4 1 30 20 18 21 26 2 22

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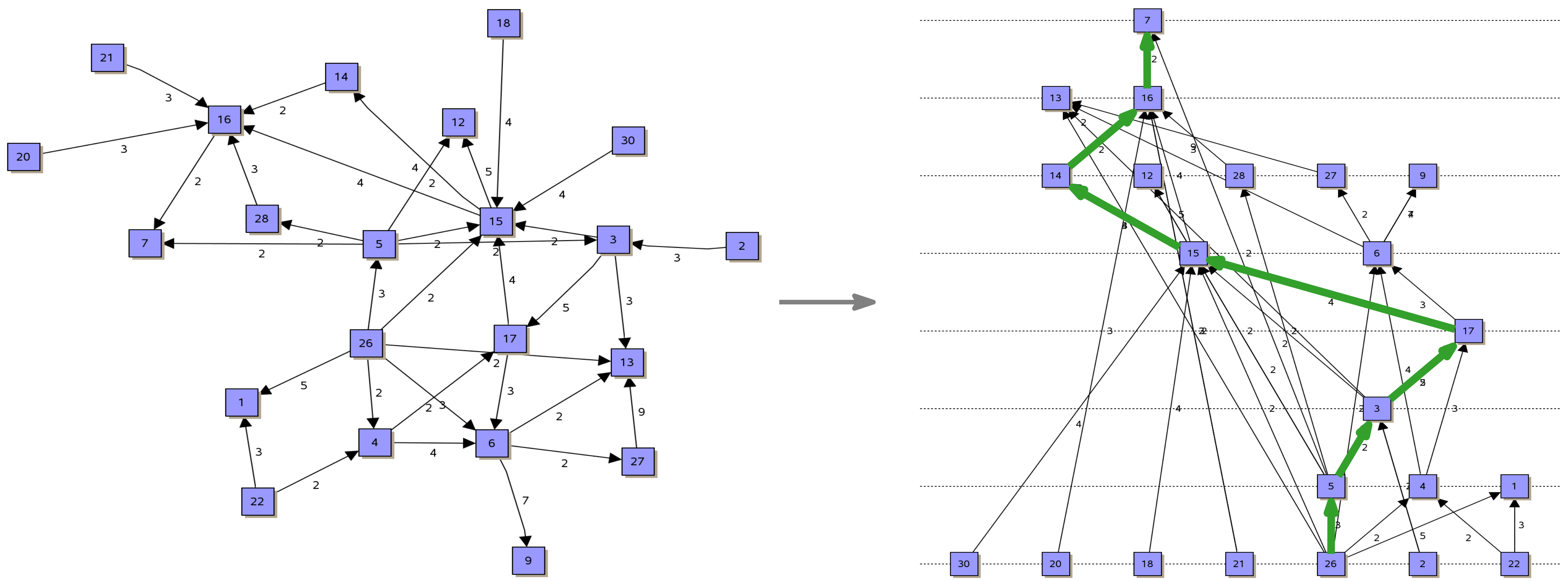

21
18
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27
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15
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Total edge length – ILP

Can be formulated as an integer linear program:

$$
\begin{aligned}
\text{min} \quad & \textstyle\sum_{(u,v)\in E} (y(v) - y(u)) & \\
\text{subject to} \quad & y(v) - y(u) \geq 1 & \forall (u,v) \in E \\
& y(v) \geq 1 & \forall v \in V \\
& y(v) \in \mathbb{Z} & \forall v \in V
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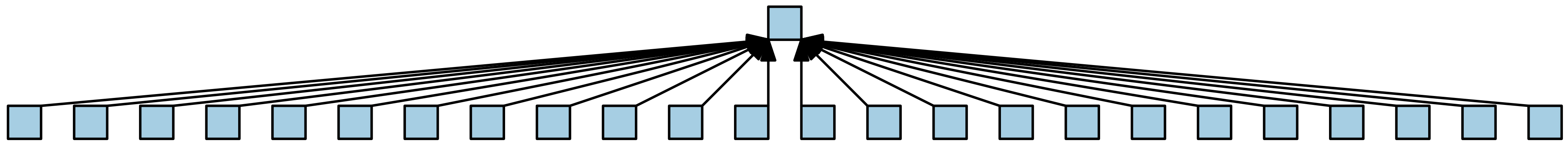
\end{array}
$$

One can show that:

- Constraint-matrix is **totally unimodular**
  $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time

Width



Drawings can be very wide.

Narrower layer assignment

Problem: Leveling with a given width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

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Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
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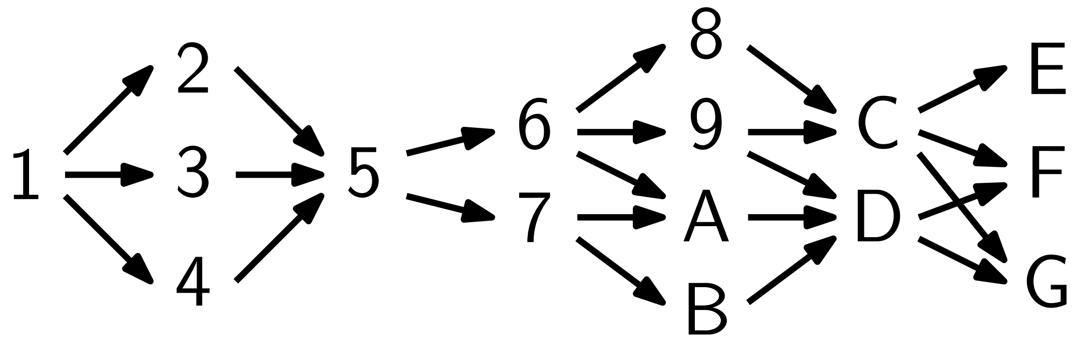
- NP-hard, $(2 - \frac{2}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. $(W \geq 3)$.

Approximating PCMPS

- jobs stored in a list L (in any order, e.g., topologically sorted)
- for each time $t = 1, 2, \ldots$ schedule $\leq W$ available jobs
- a job in L is *available* when all its predecessors have been scheduled
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



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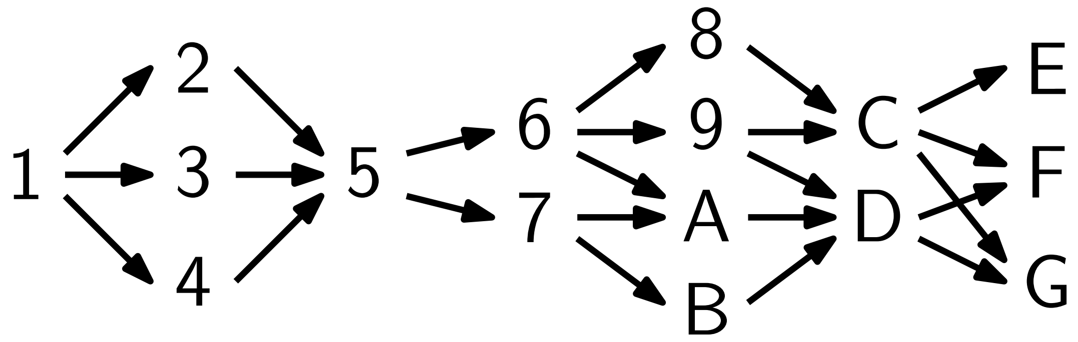

Number of Machines is $W = 2$.

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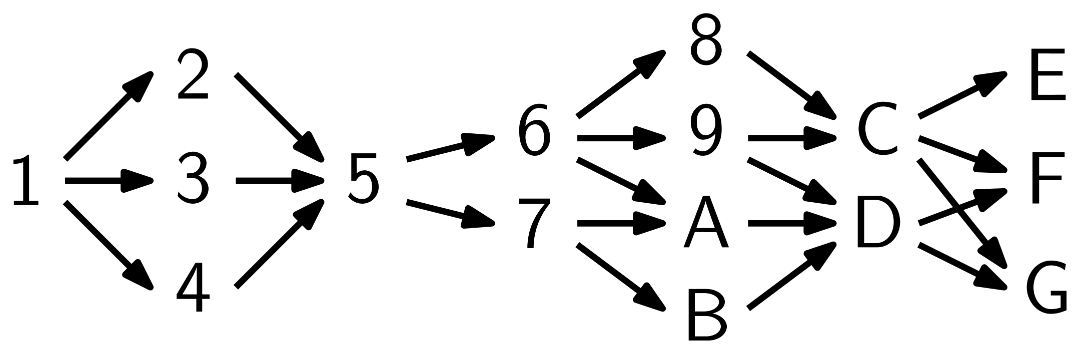

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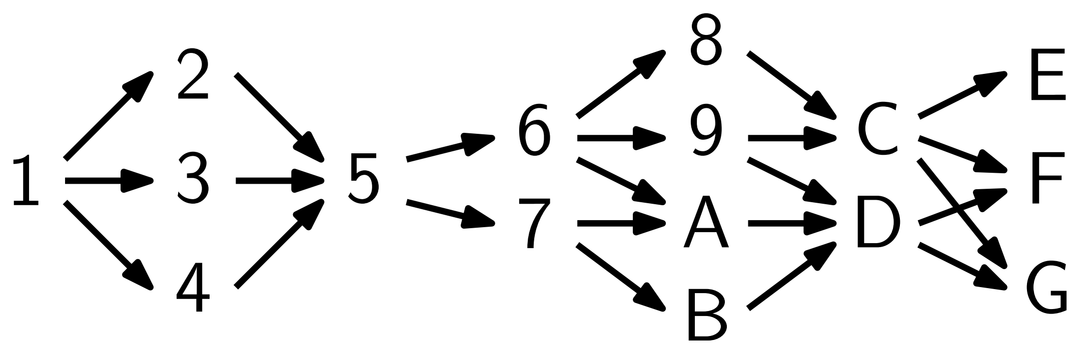

Number of Machines is $W = 2$.

Output: Schedule

M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

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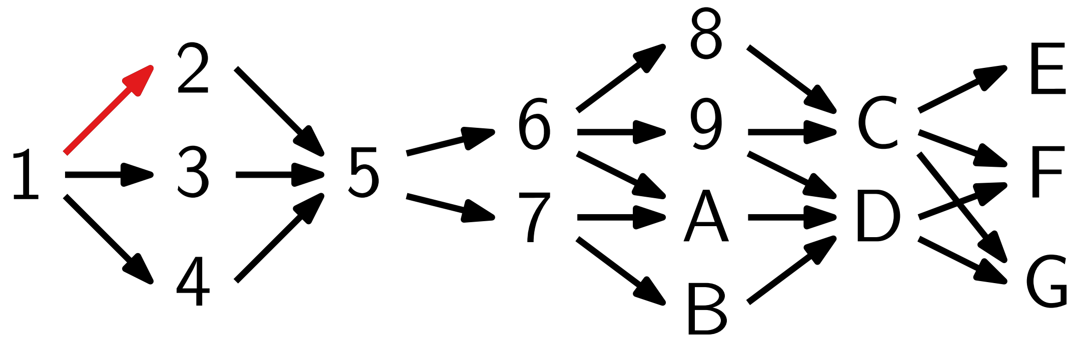

Number of Machines is $W = 2$.

Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

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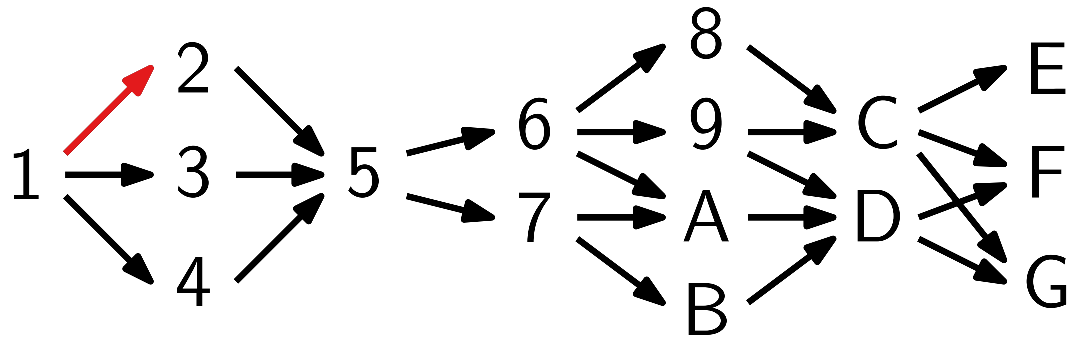

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Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

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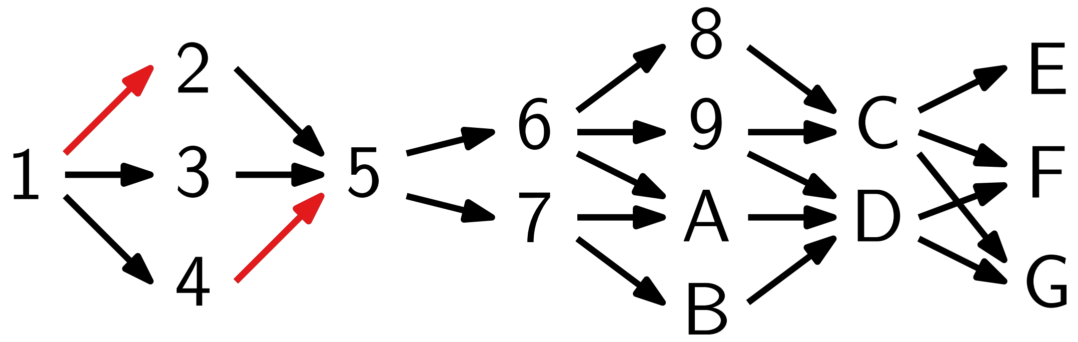

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

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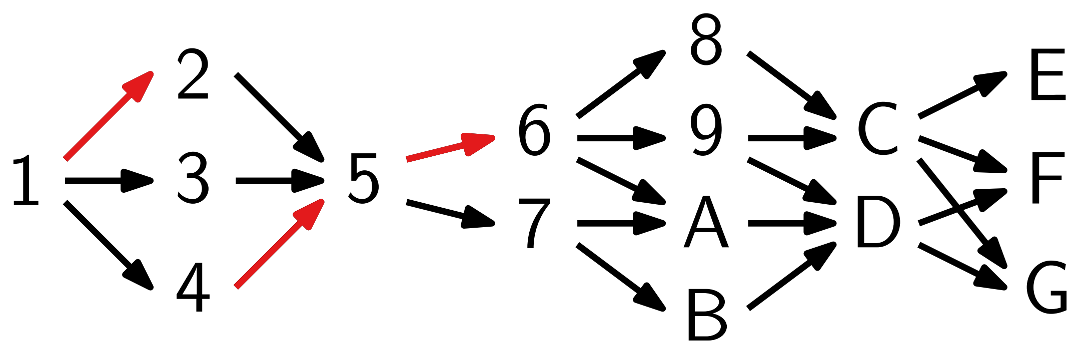

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

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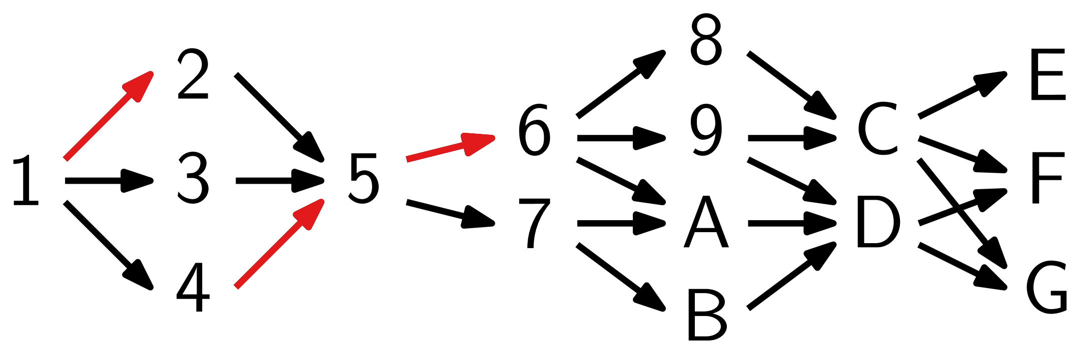

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

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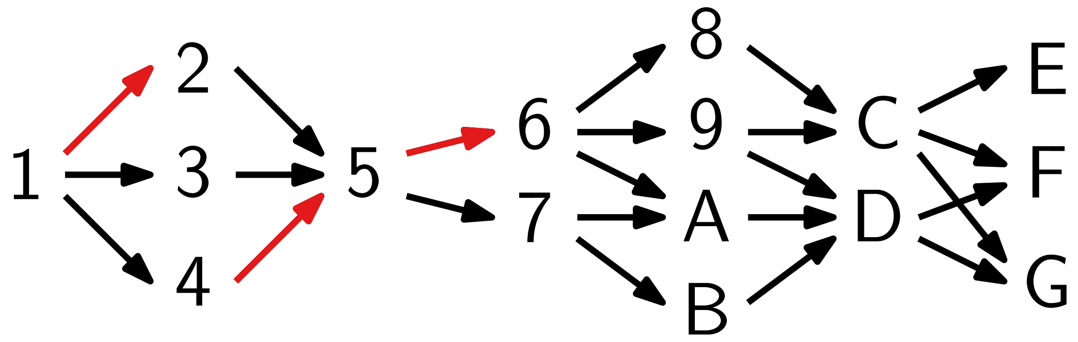


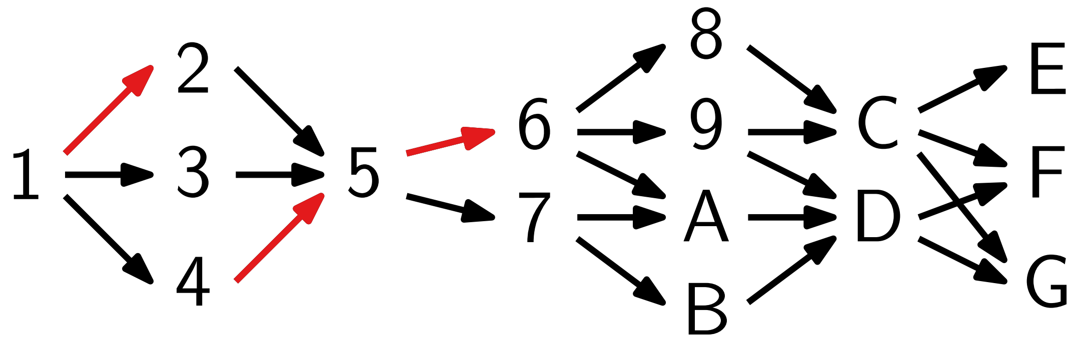
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



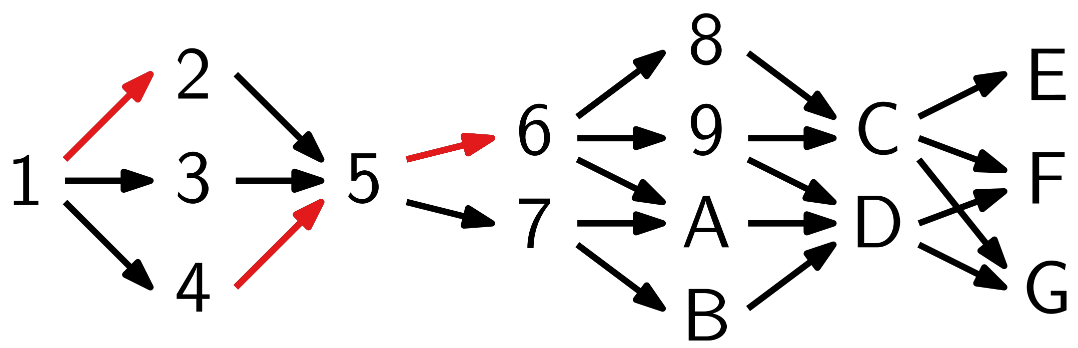
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



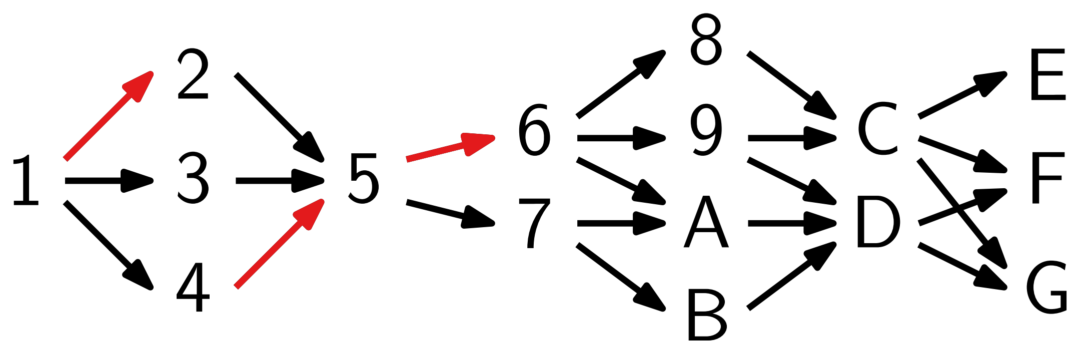
Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	
M_2	–	3	–	–	7	9	B	D	F	
t	1	2	3	4	5	6	7	8	9	10

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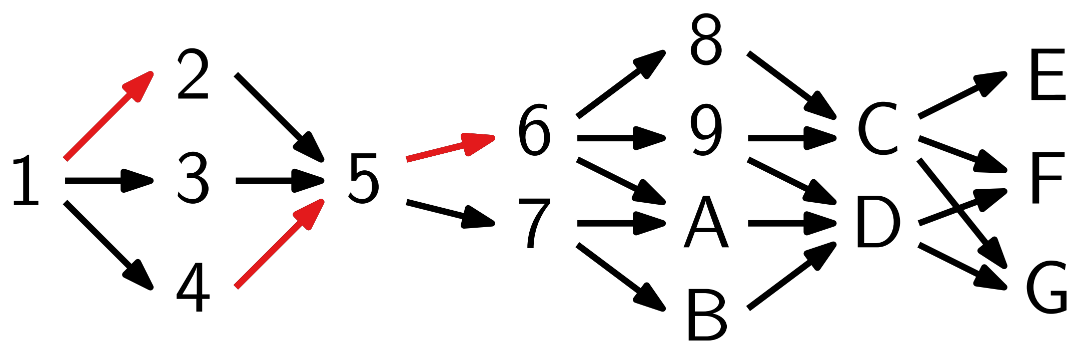

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
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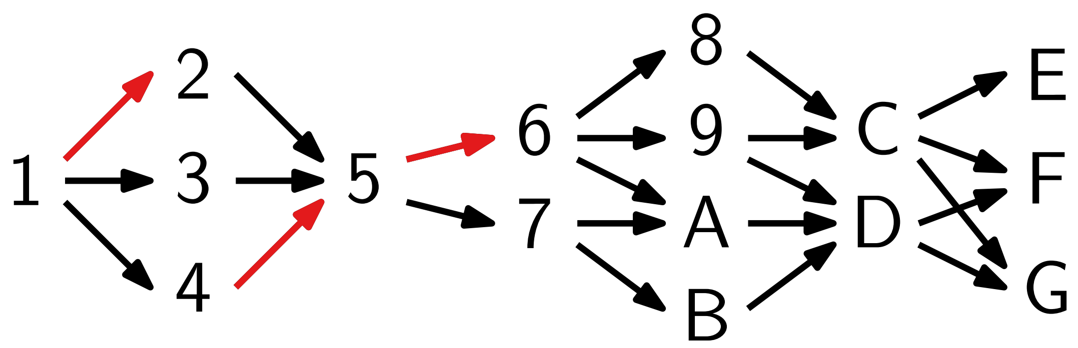

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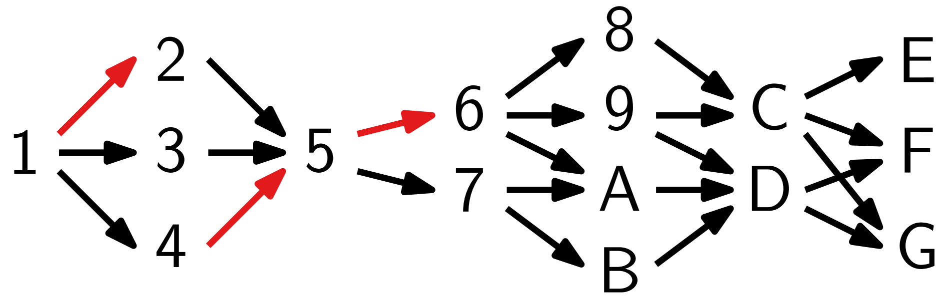
Question: Good approximation factor?

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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M_1	1	2	4	5	6	8	A	C	E	G
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„The art of the lower bound“

Approximating PCMPS - analysis for $W = 2$

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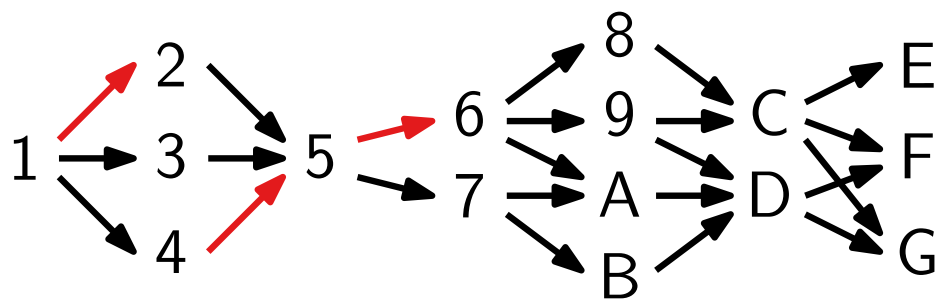

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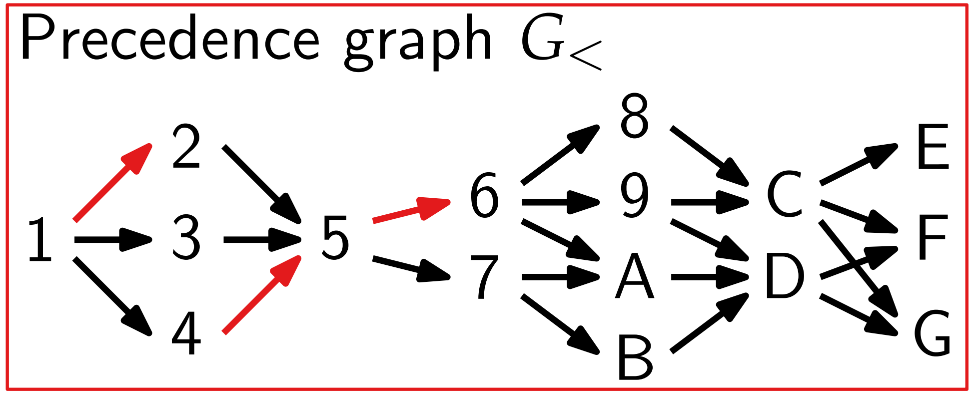
„The art of the lower bound“

OPT $\geq$

Approximating PCMPS - analysis for $W = 2$



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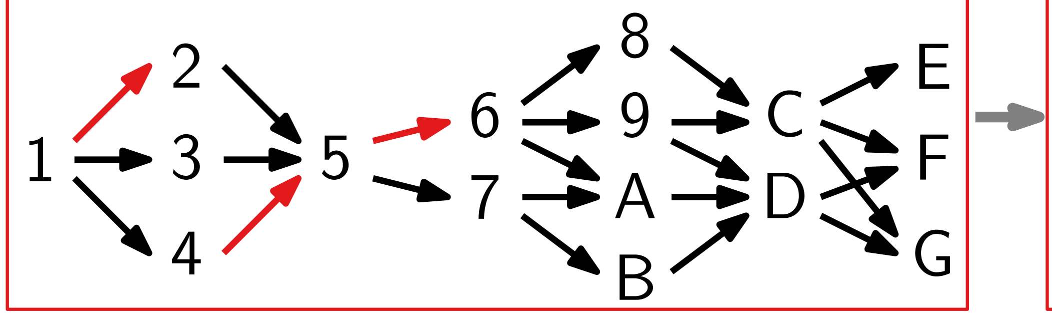
$\text{OPT} \geq \lceil n/2 \rceil$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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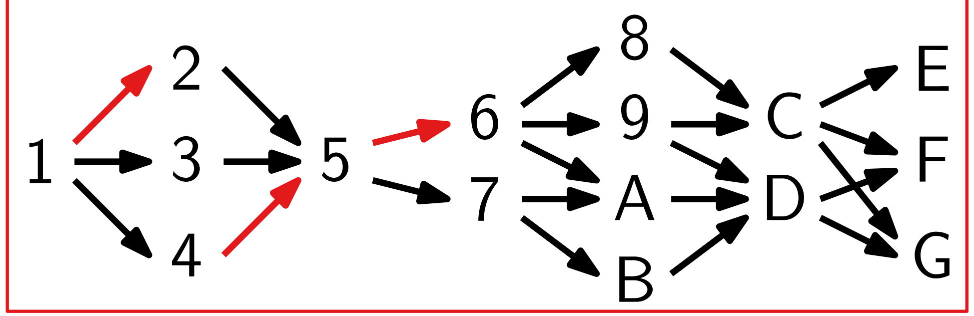
„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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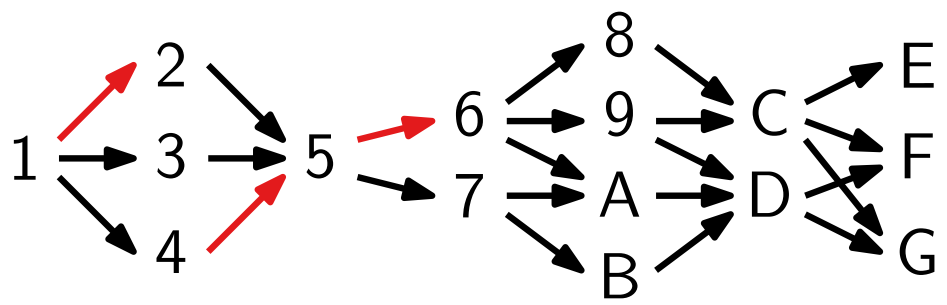
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ \ell :=$ Number of layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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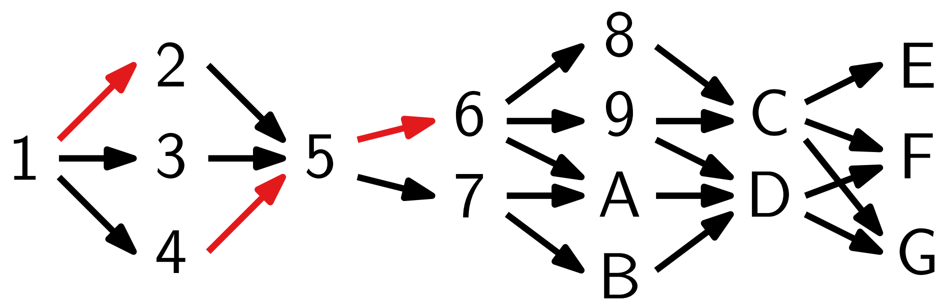
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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Bound. ALG $\leq$

Approximating PCMPS - analysis for $W = 2$

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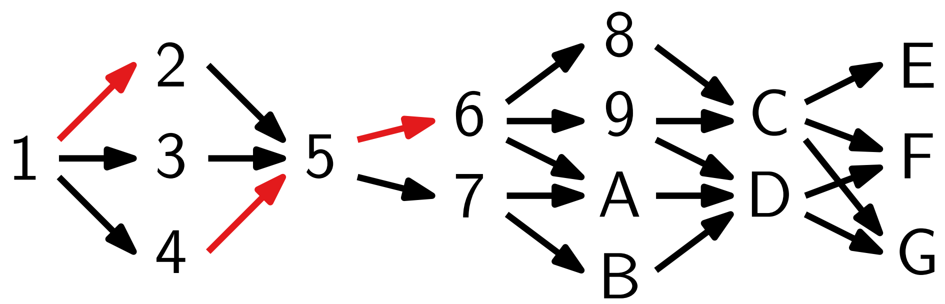

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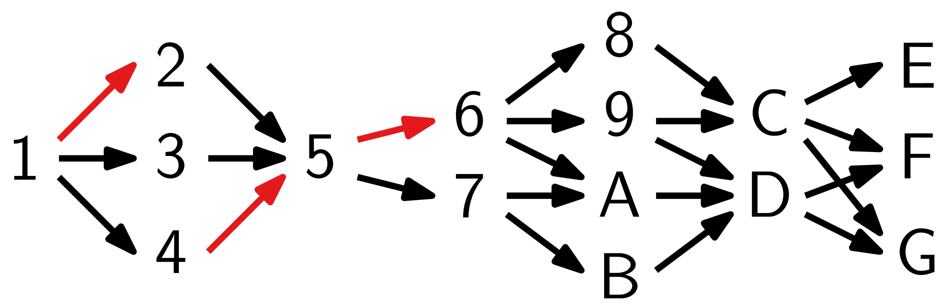

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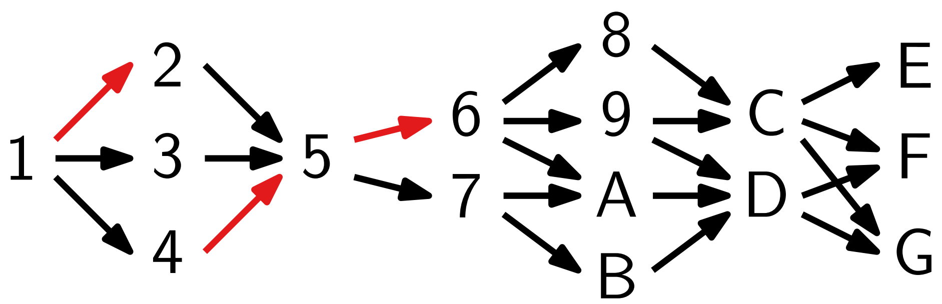
insertion of pauses (–) in the schedule
(except the last) maps to layers of $G_<$

Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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Goal: measure the quality of our algorithm using the lower bounds

Bound. $\text{ALG} \leq \left\lceil \frac{n+\ell}{2} \right\rceil$

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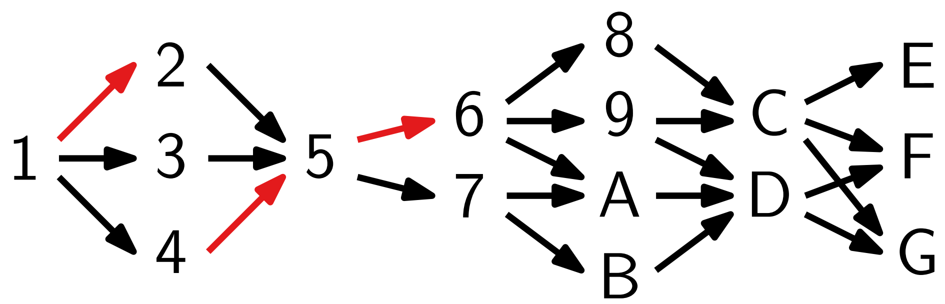
Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx$

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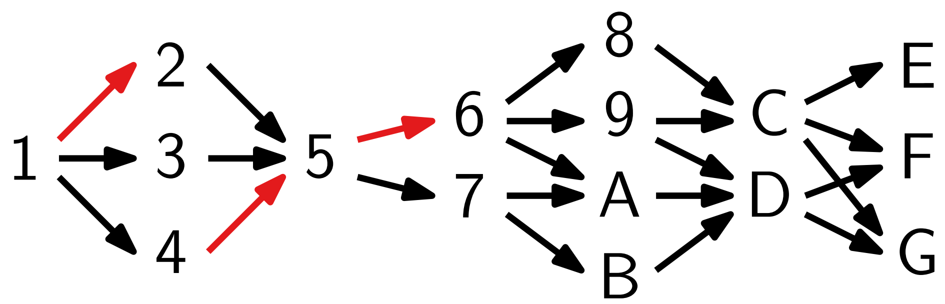
Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2$

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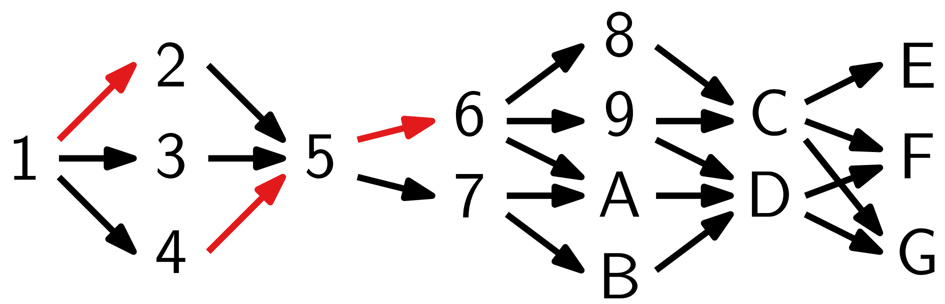
Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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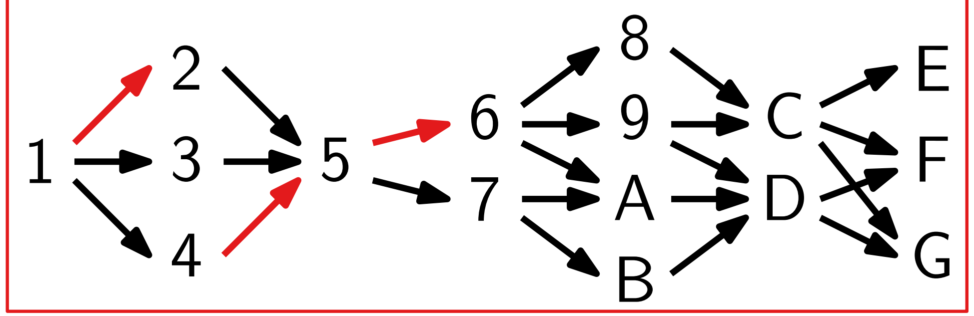
Approximating PCMPS - analysis for $W = 2$

Precedence graph $G_<$



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Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

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Approximating PCMPS - analysis for $W = 2$

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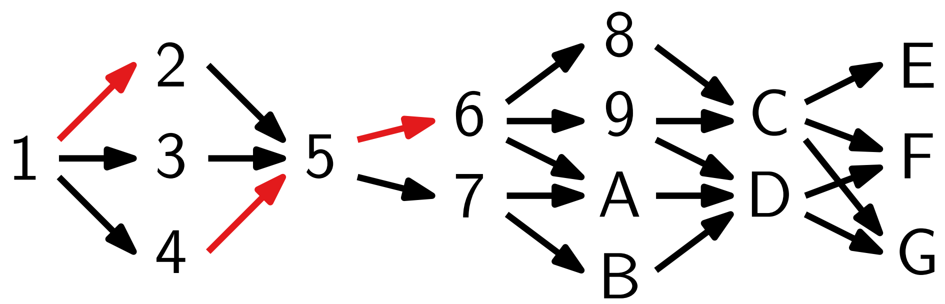

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M_1	1	2	4	5	6	8	A	C	E	G
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OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

$\leq (2 - 1/W) \cdot$ OPT in general case

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

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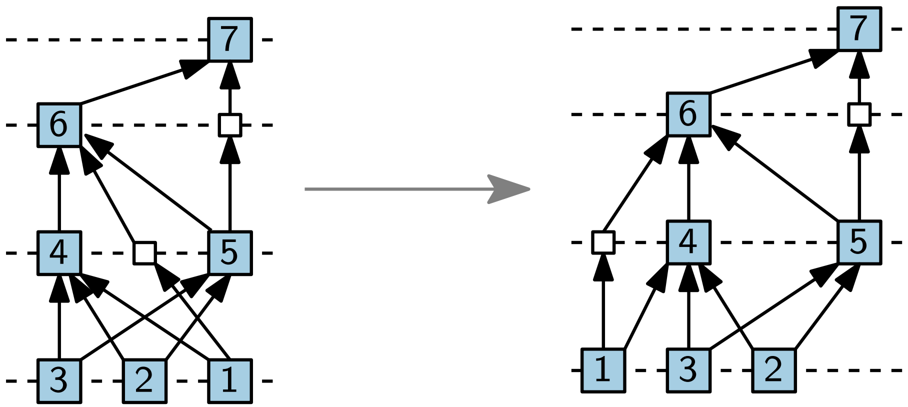
Step 3: Crossing minimization



Problem.

- Input: Graph G, layering $y\colon V \to \{1, \ldots, |V|\}$
- Output: (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

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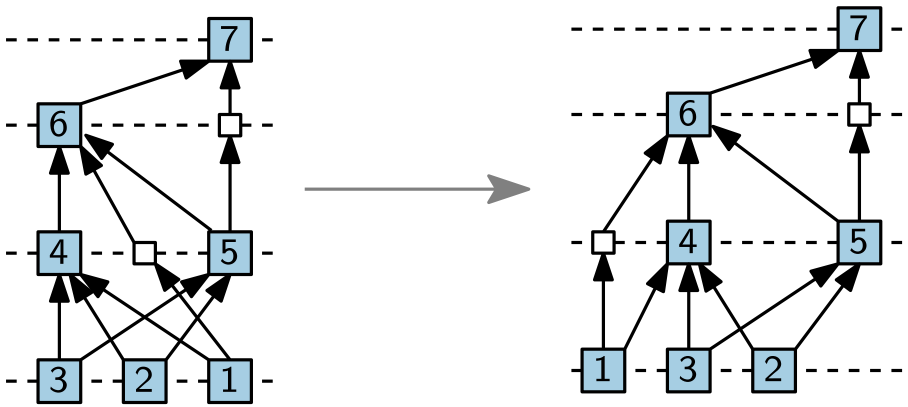

Problem.

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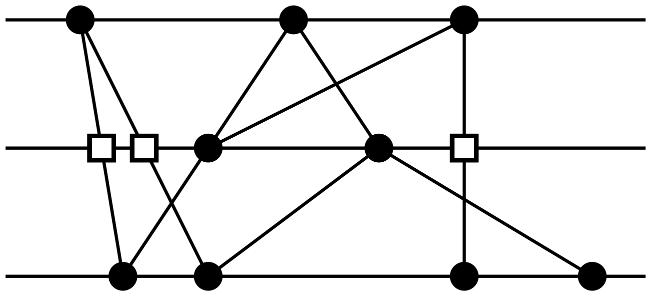
- NP-hard, even for 2 layers [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers :(

Iterative crossing reduction – idea

Observation.

The number of crossings only depends on permutations of adjacent layers.



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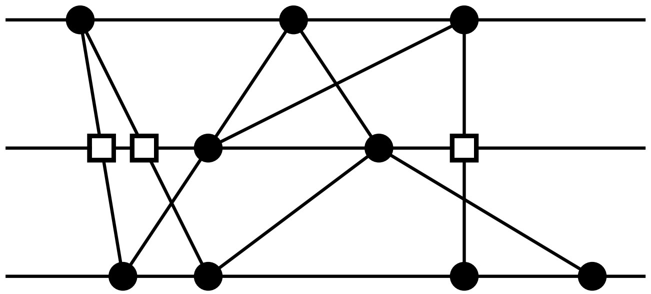

- Add dummy-vertices for edges connecting “far” layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.
- Minimize crossings by permuting L_{i+1} while keeping L_i fixed.

Iterative crossing reduction – algorithm

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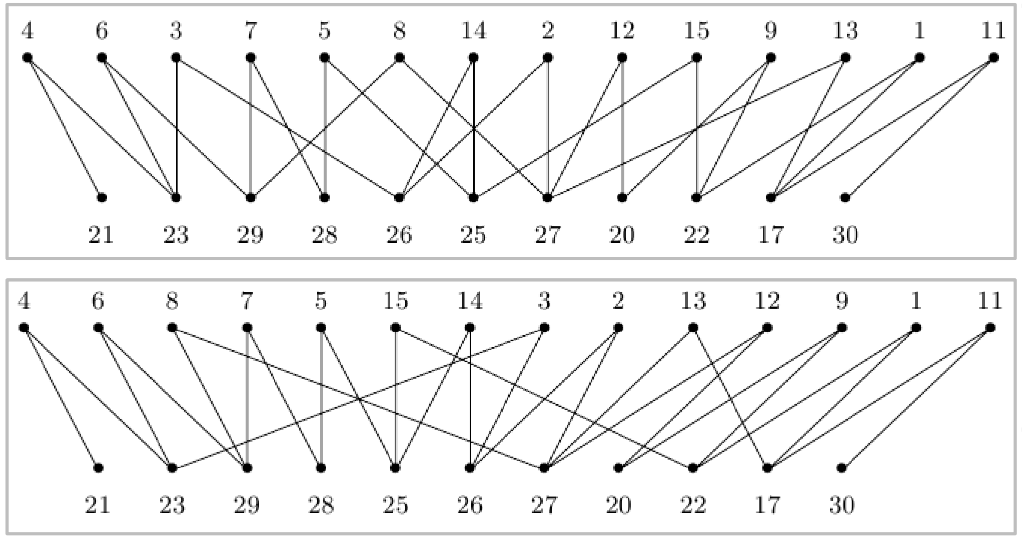
One-sided crossing minimization

Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.



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One-sided crossing minimization is NP-hard.
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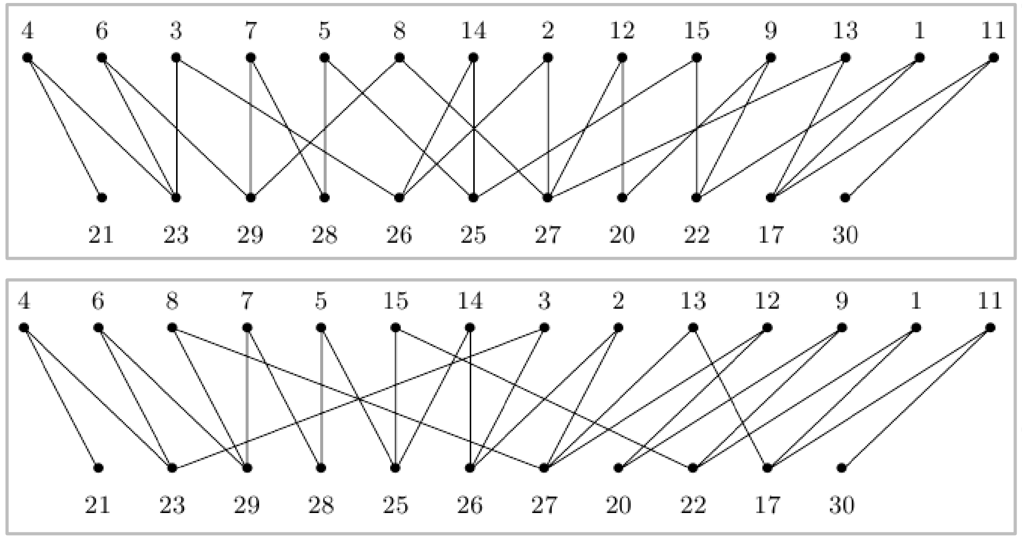

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Algorithms.

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP
- ...

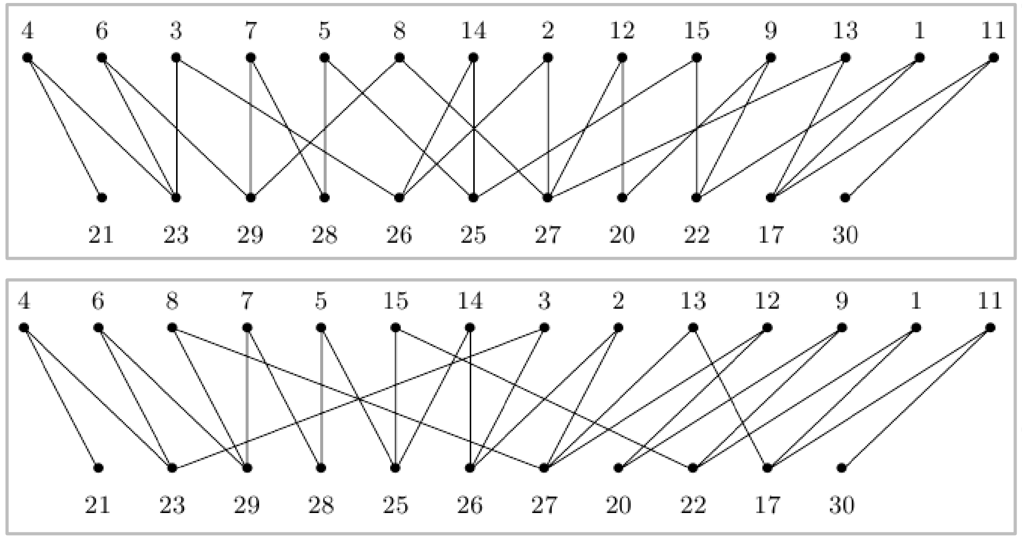

Abb. aus [Kaufmann und Wagner: Drawing Graphs]
(c) Springer-Verlag

Barycentre heuristic

[Sugiyama et al. '81]

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$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

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- relatively good results
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- $O(\sqrt{n})$-approximation factor

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- linear runtime
- relatively good results
- optimal if no crossings are required ← *exercise!*
- $O(\sqrt{n})$-approximation factor

Worst case?

Barycentre heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbours
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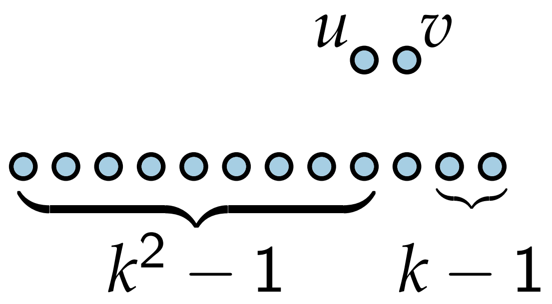

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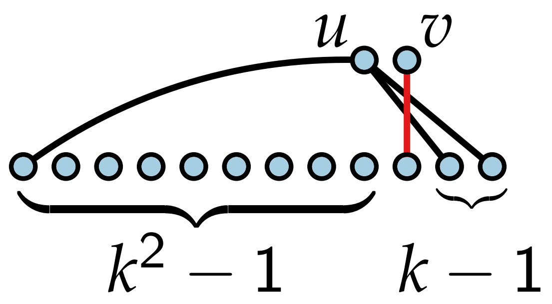

Median heuristic

[Eades & Wormald '94]

- $\{v_1, \ldots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \cdots < \pi_1(v_k)$
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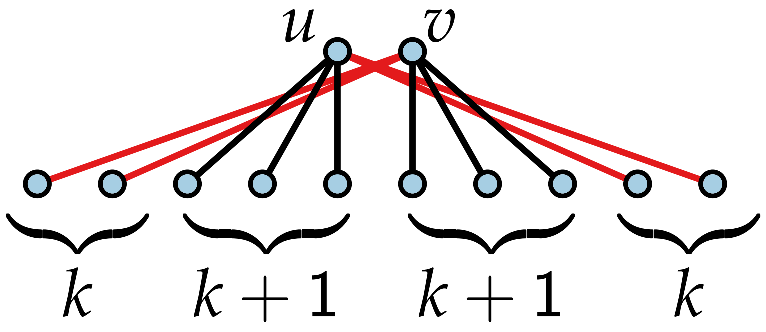

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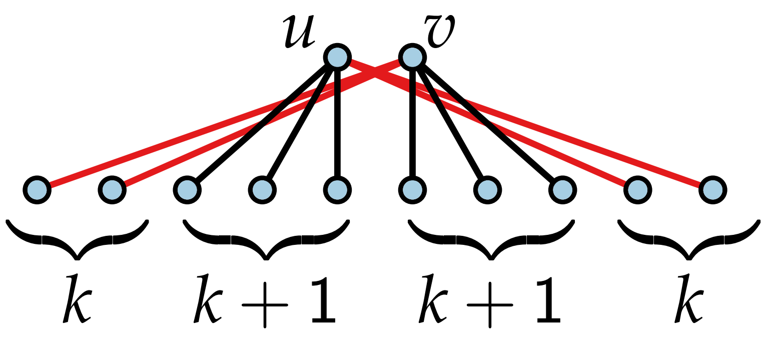

$2k(k+1) + k^2$ vs. $(k+1)^2$

Greedy-switch heuristic

- iterately swap each adjacent node as long as crossings decrease
- runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- suitable as post-processing for other heuristics

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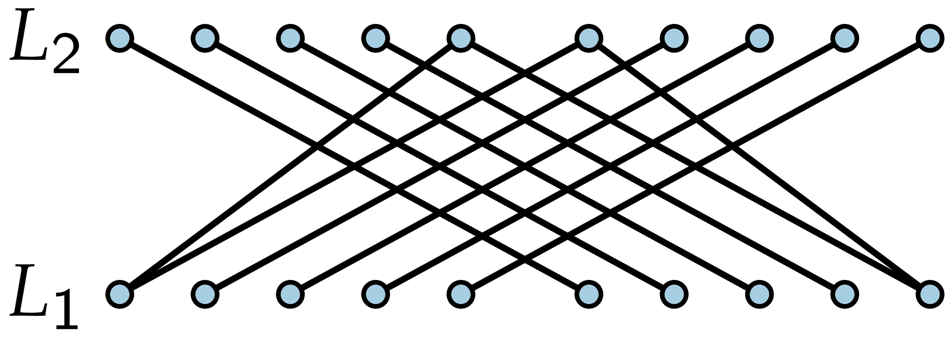

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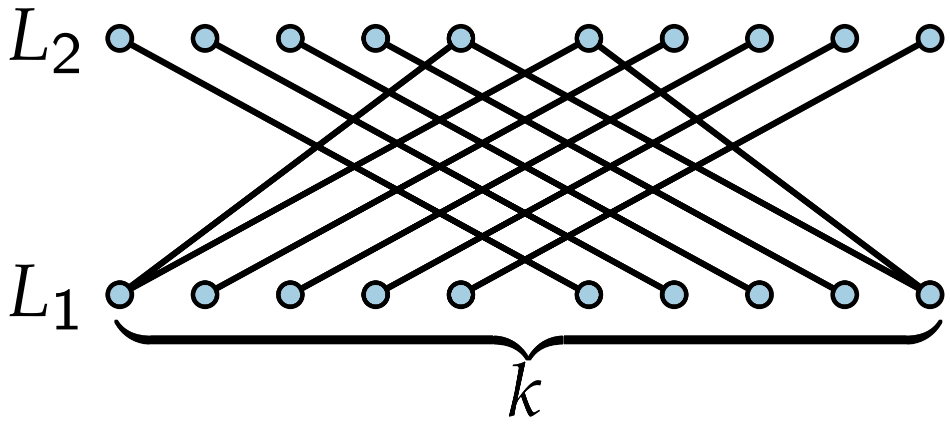

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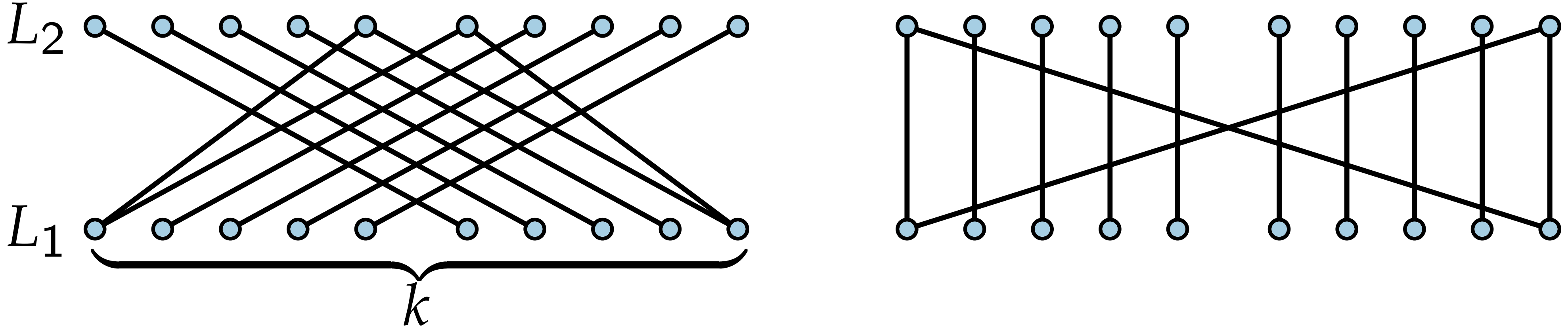

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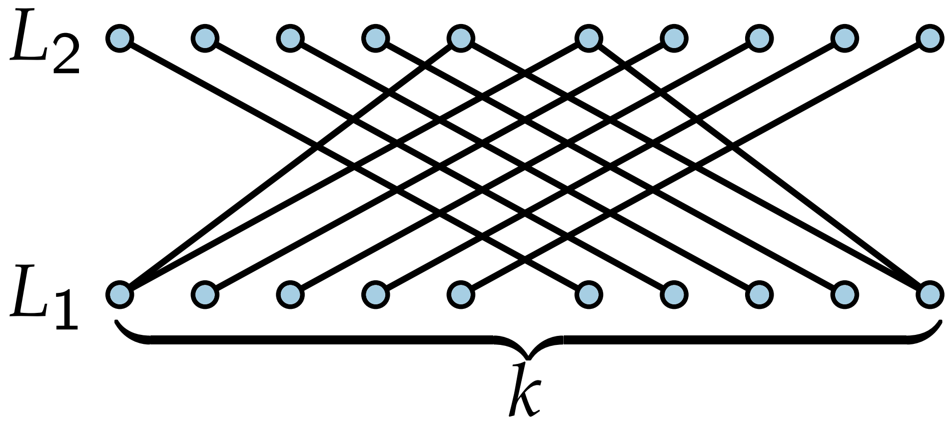
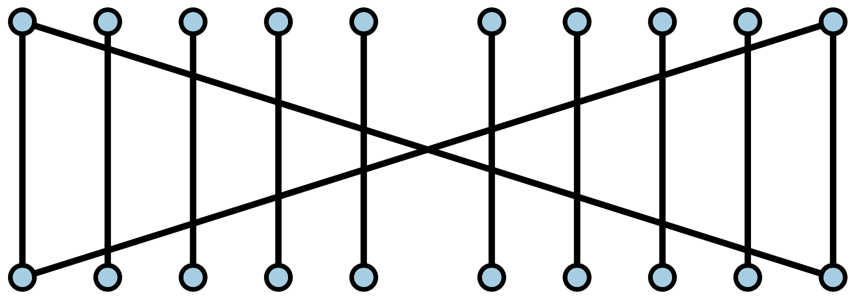


$\approx k^2/4$

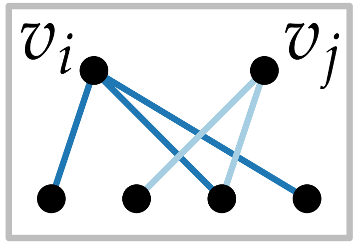


$\approx 2k$

Integer linear program

[Jünger & Mutzel, '97]

- Constant $c_{ij} := \#$ crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$



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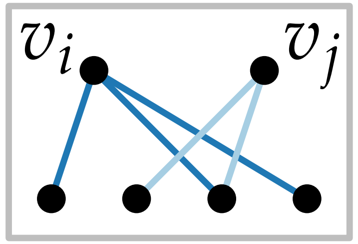

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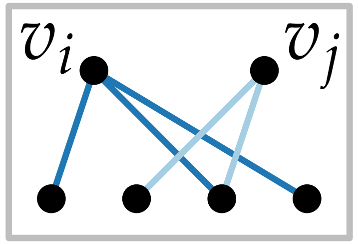

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- The number of crossings of a permutations π_2

$$\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij} + \underbrace{\sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}}_{\text{constant}}$$

Integer linear program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

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$$0 \le x_{ij} + x_{jk} - x_{ik} \le 1 \qquad \text{for } 1 \le i < j < k \le n_2$$

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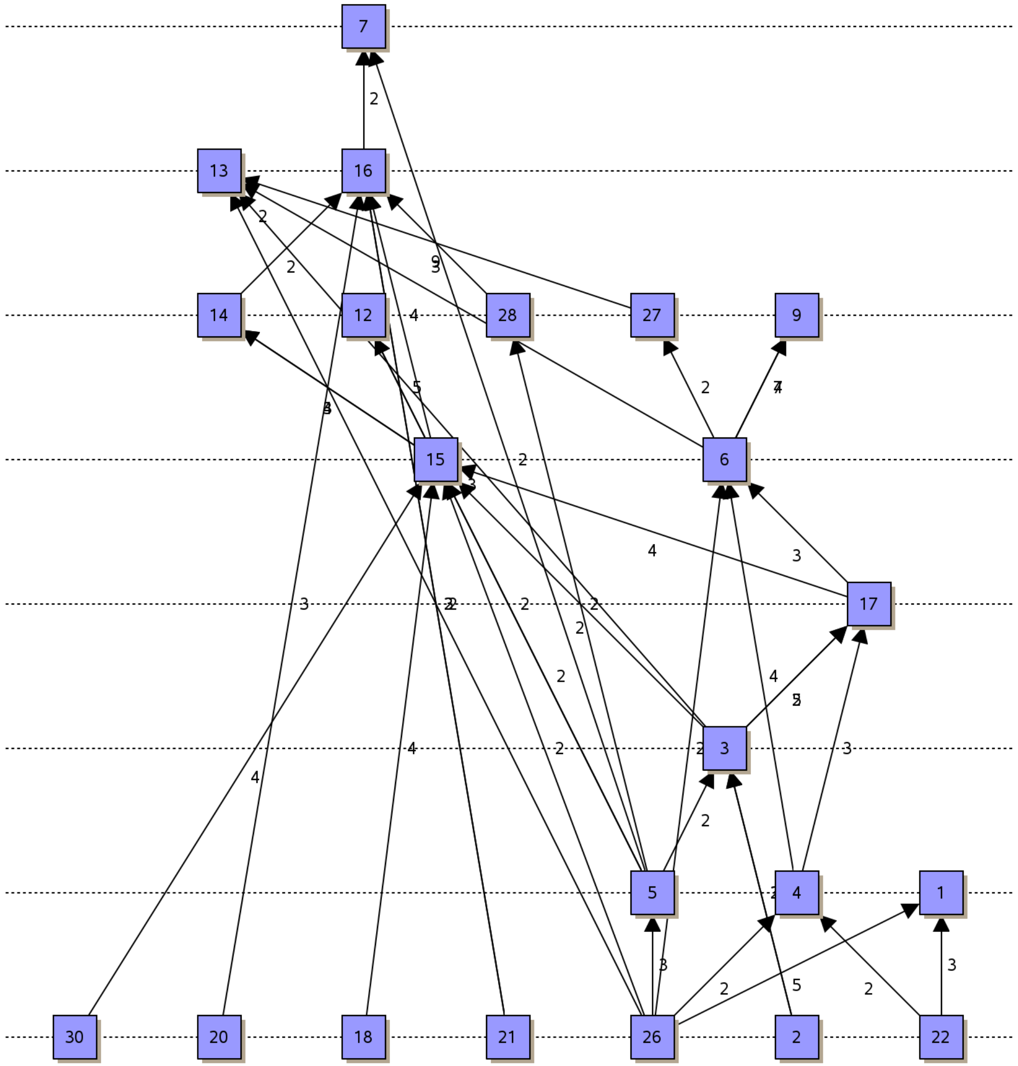
Properties.

- branch-and-cut technique for DAGs of limited size
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

Iterations on example



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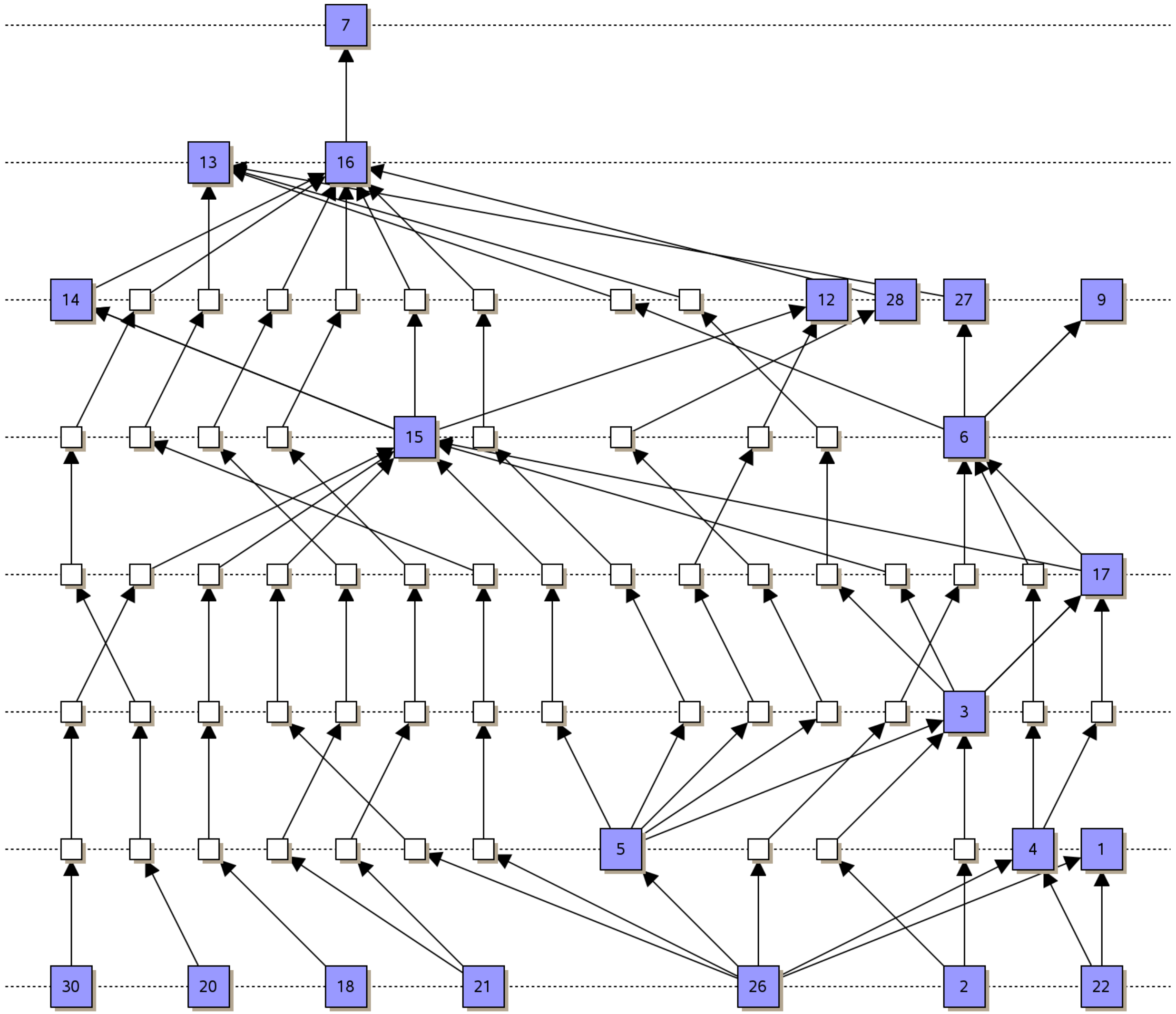

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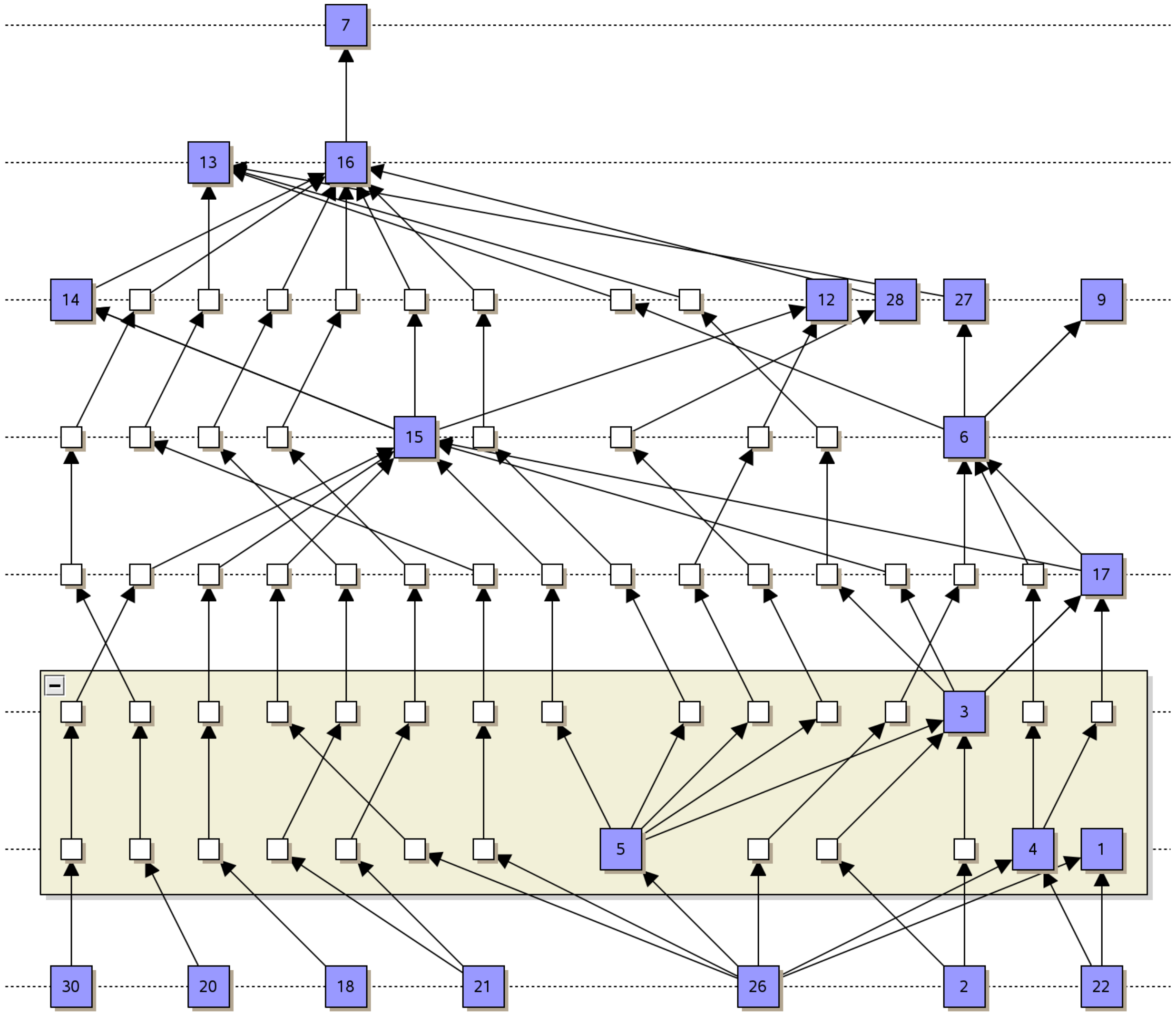

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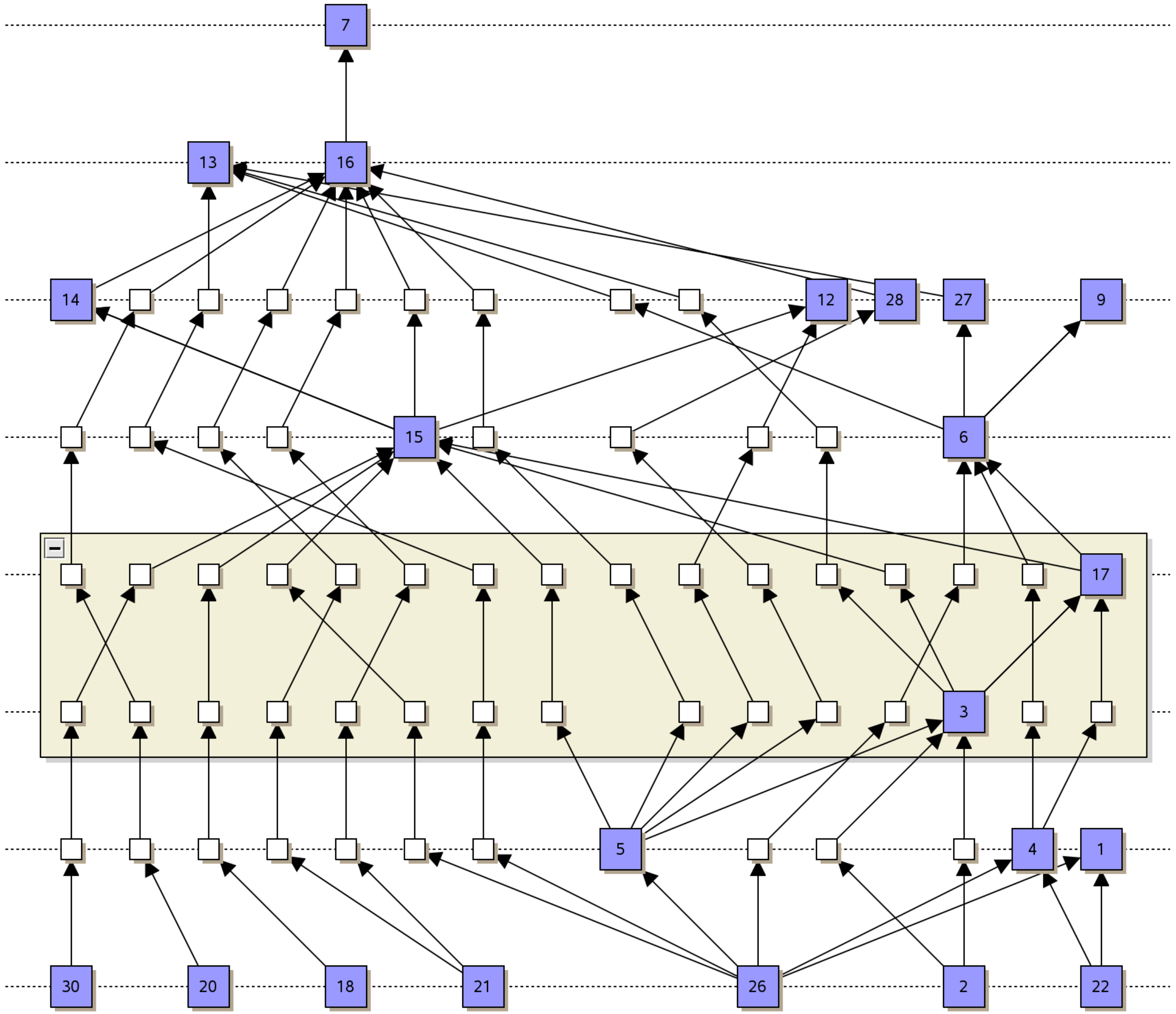

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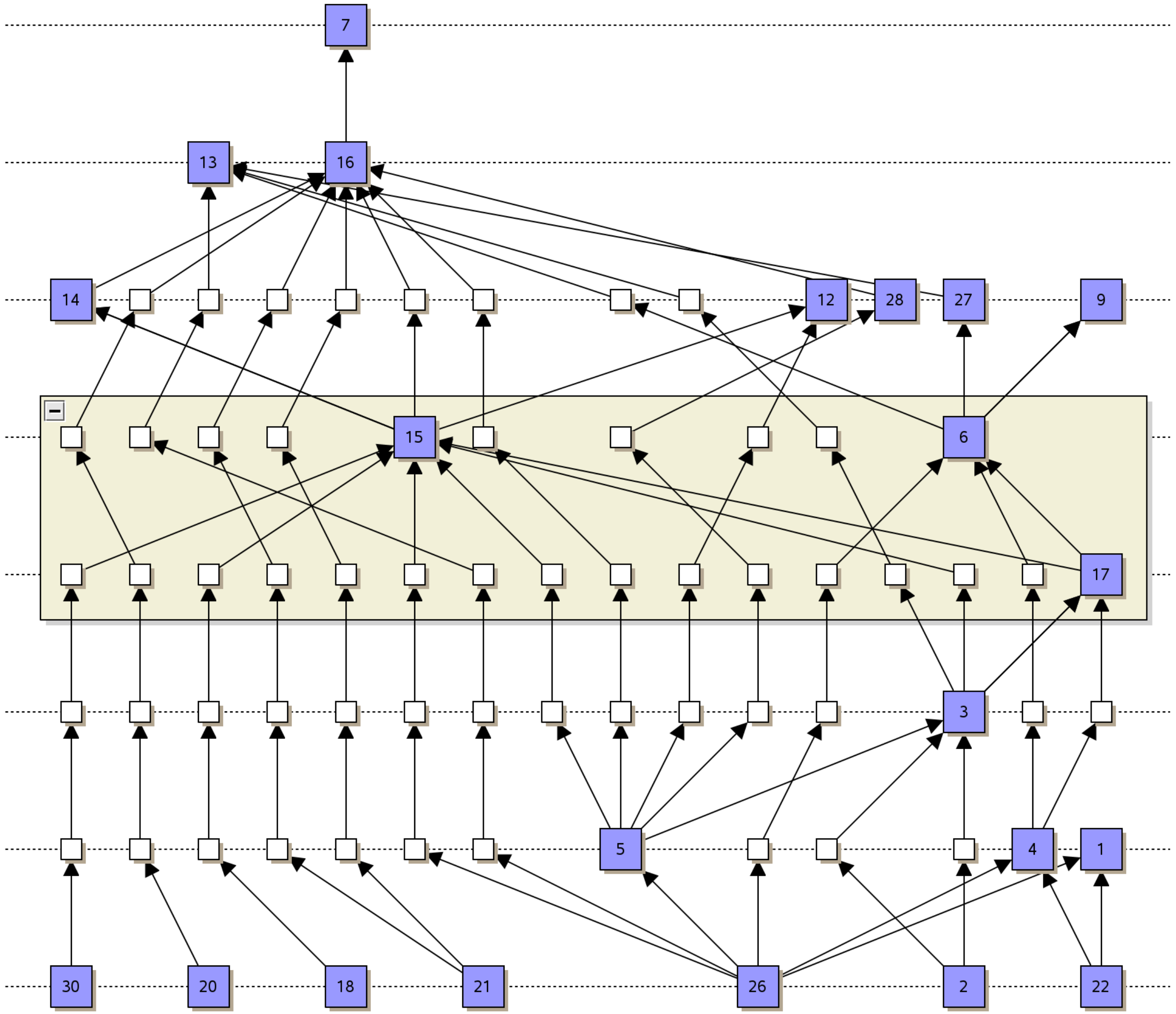

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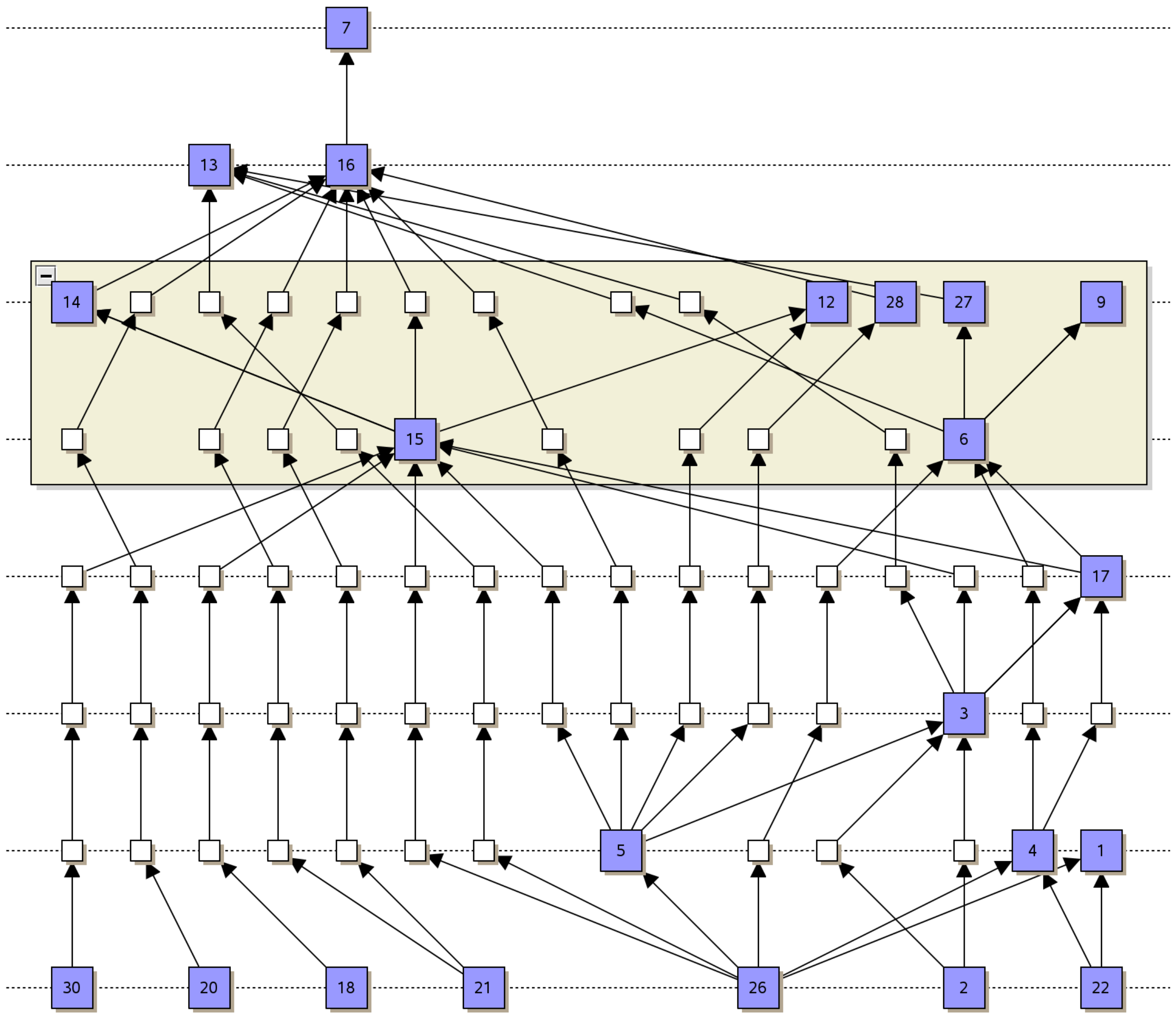

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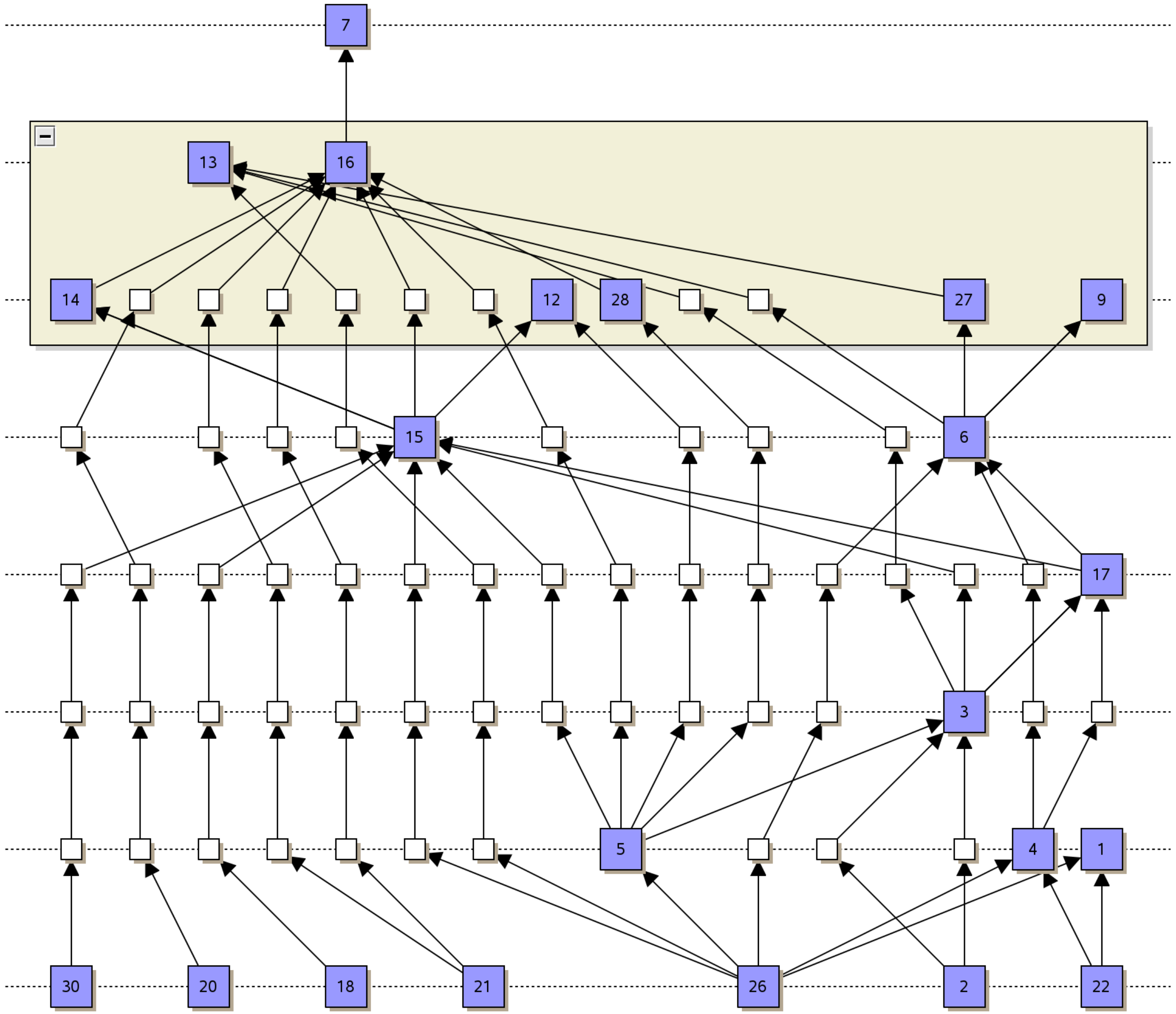

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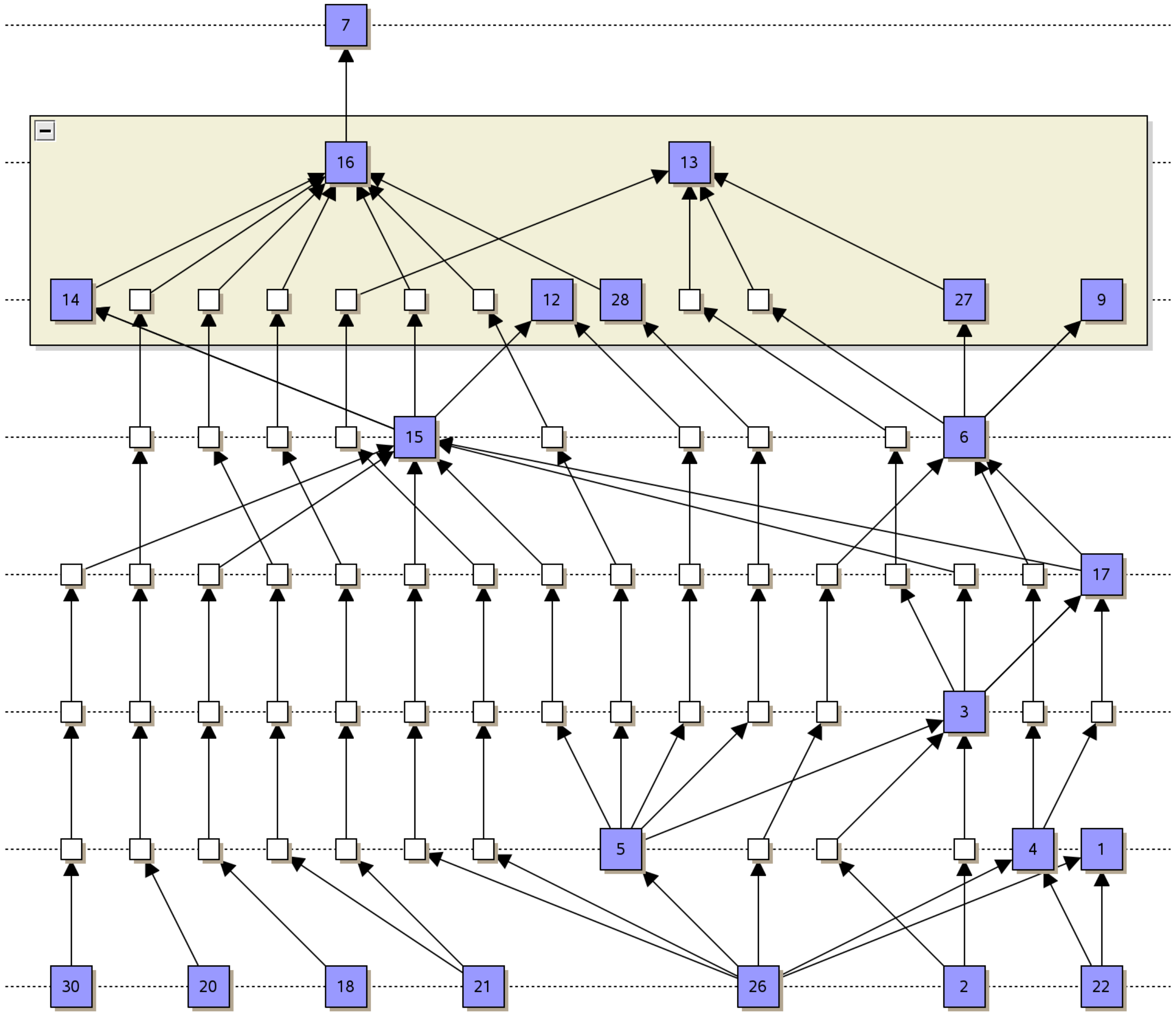

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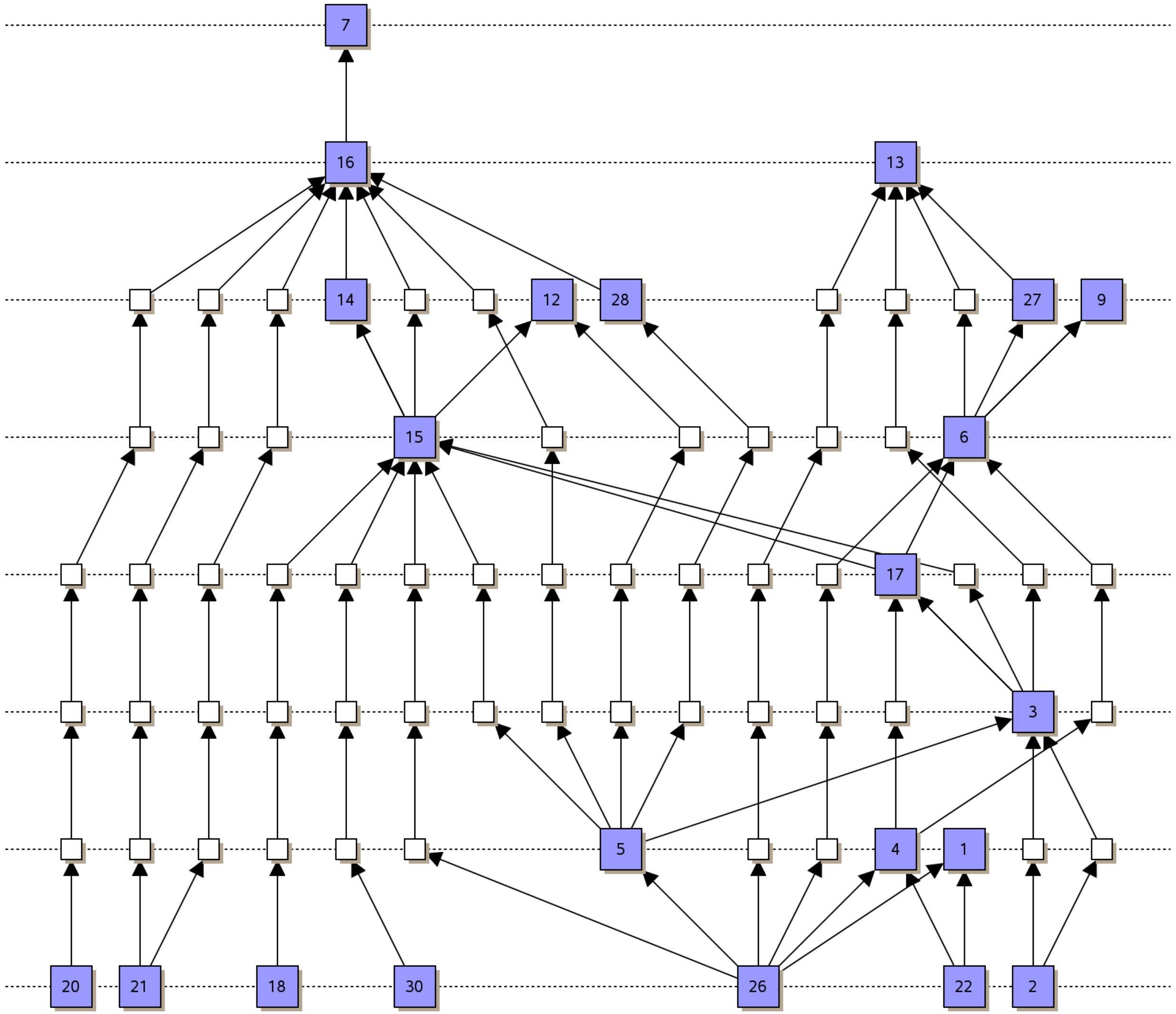
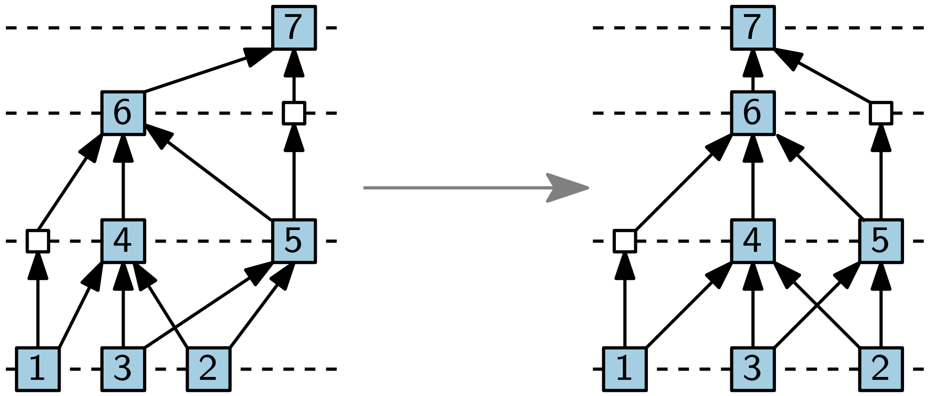


Step 4: Vertex positioning



Goal.

paths should be close to straight, vertices evenly spaced

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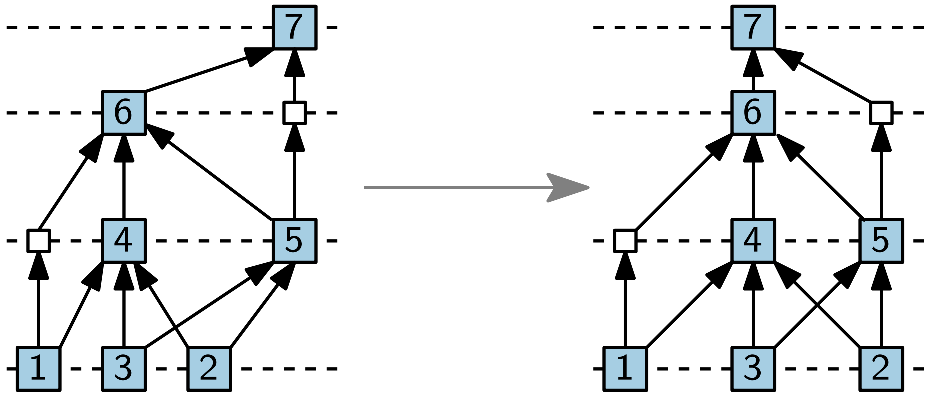

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- **Exact:** Quadratic Program (QP)
- **Heuristic:** iterative approach

Quadratic Program

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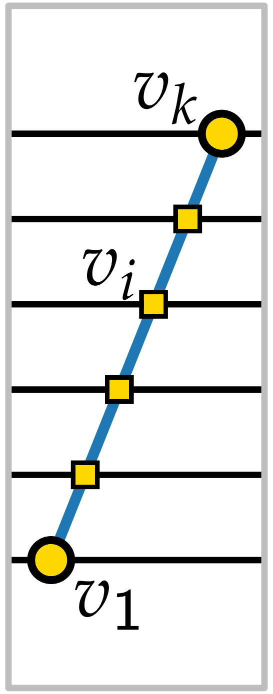

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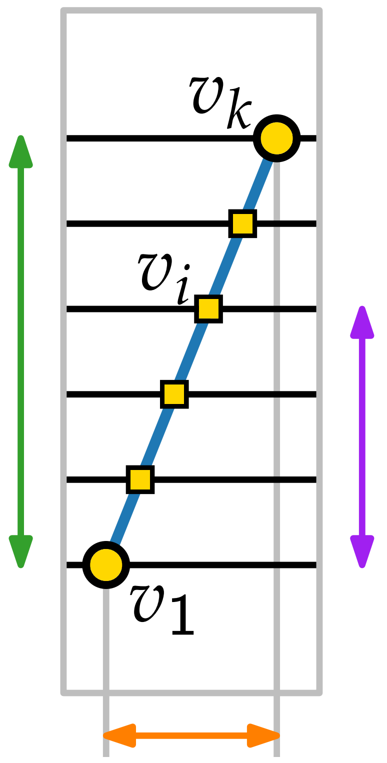

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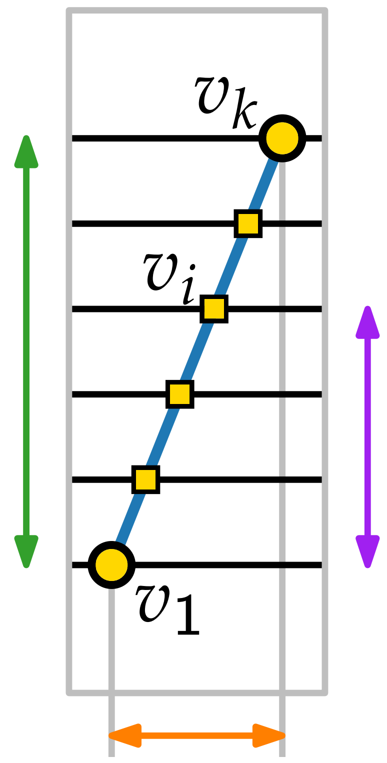

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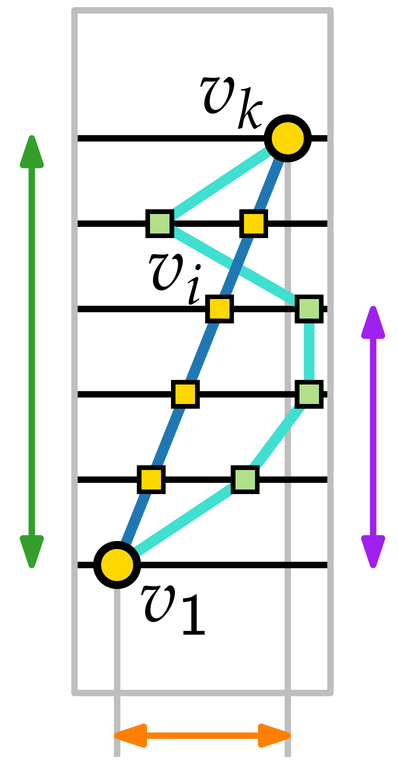

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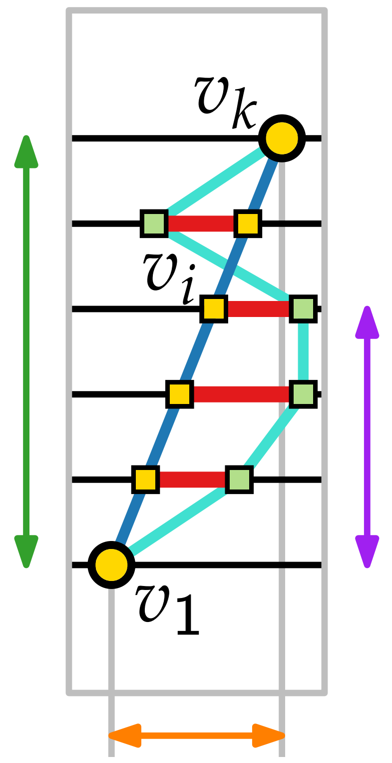

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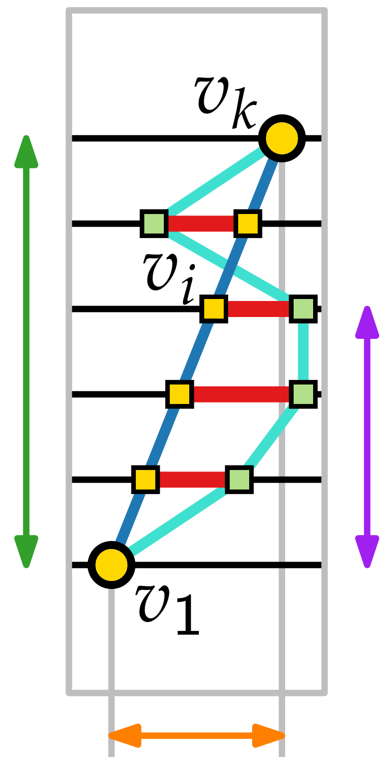

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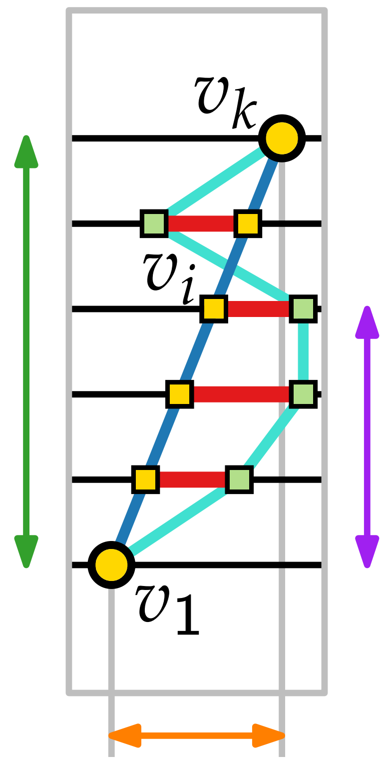

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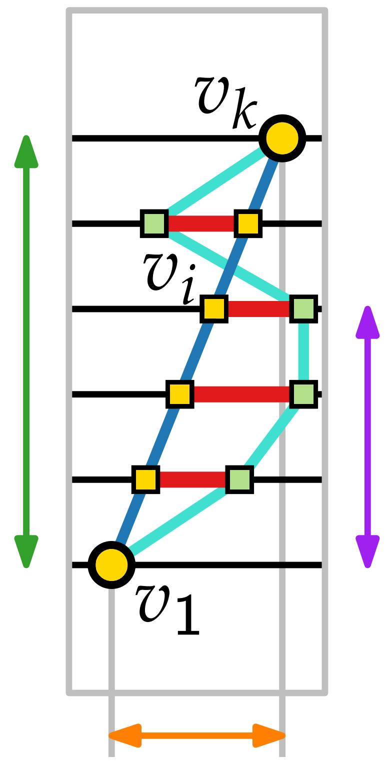

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min. horizontal distance

- QP is time-expensive
- width can be exponential

Iterative heuristic

- compute an initial layout

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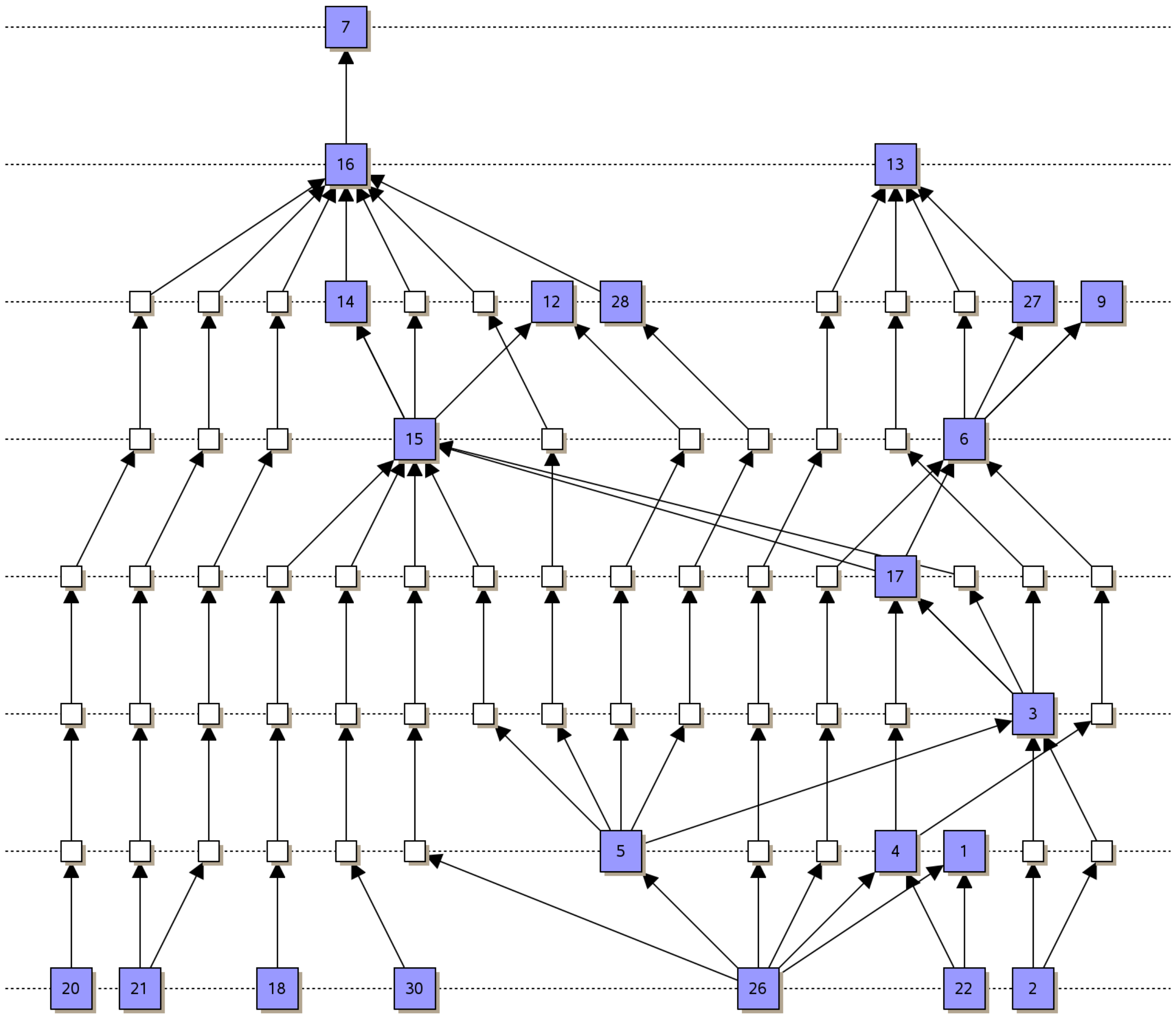
- apply the following steps as long as improvements can be made:

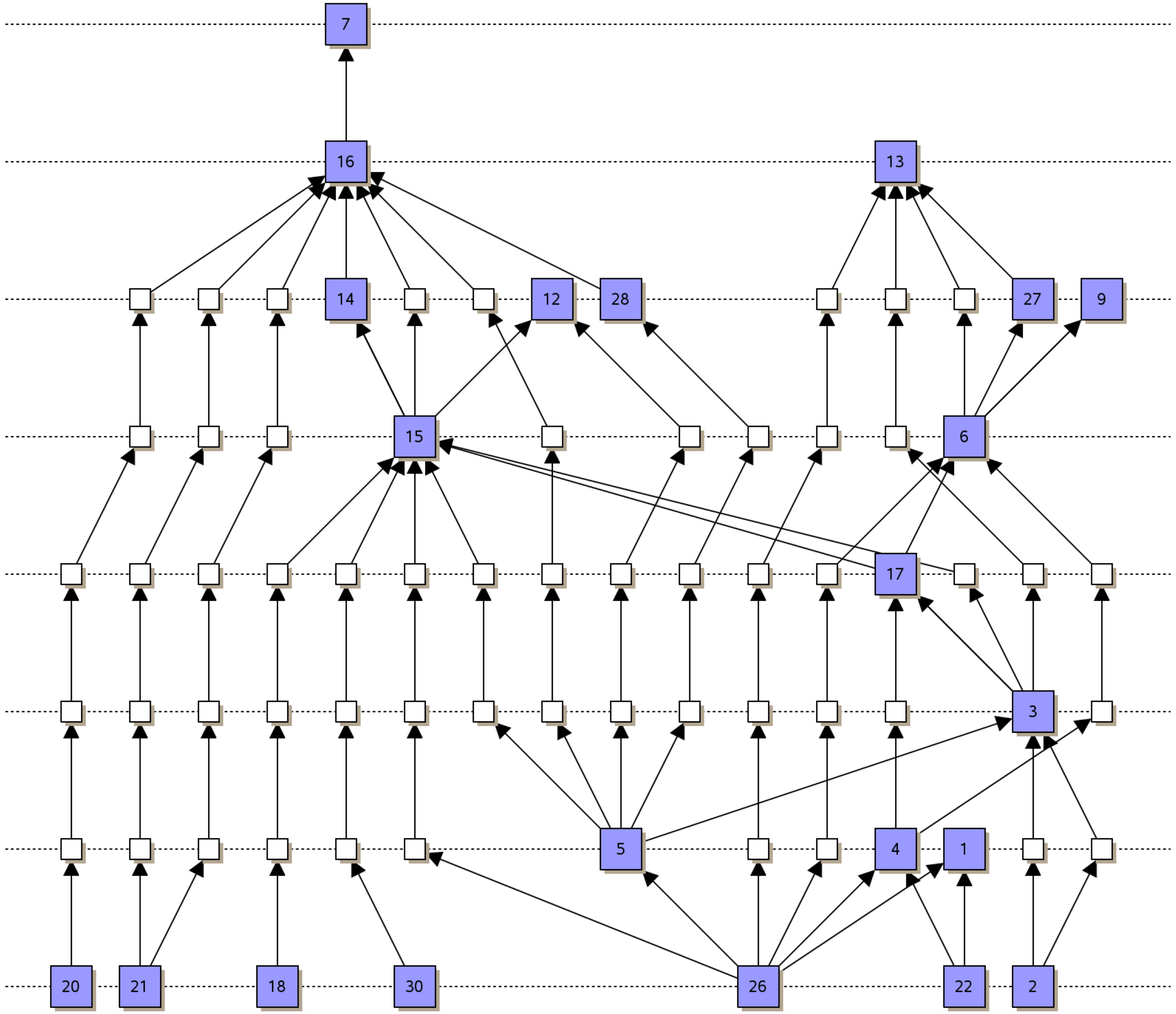
Iterative heuristic

- compute an initial layout
- apply the following steps as long as improvements can be made:
  1. vertex positioning,
  2. edge straightening,
  3. compactifying the layout width.

Example



Example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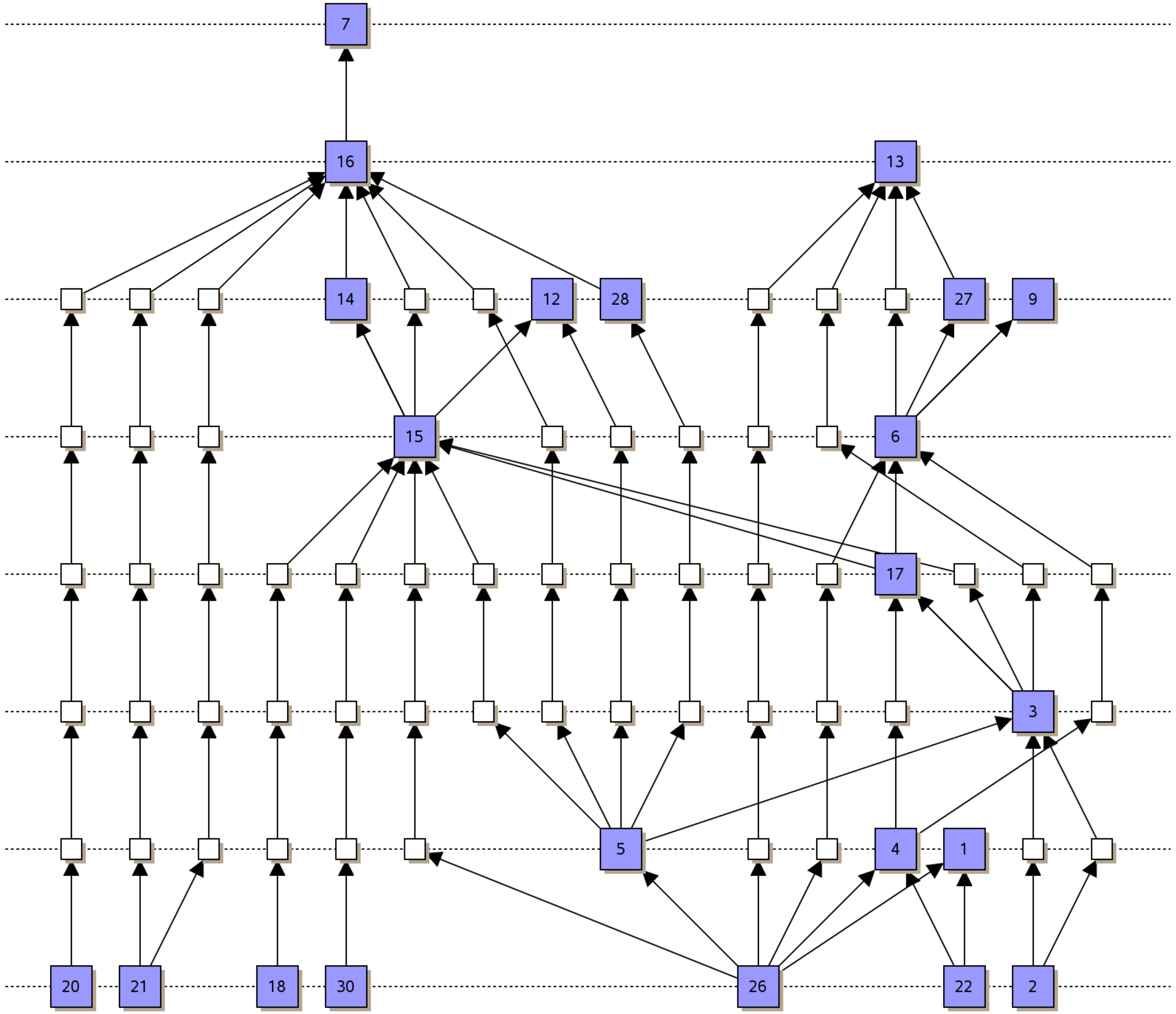
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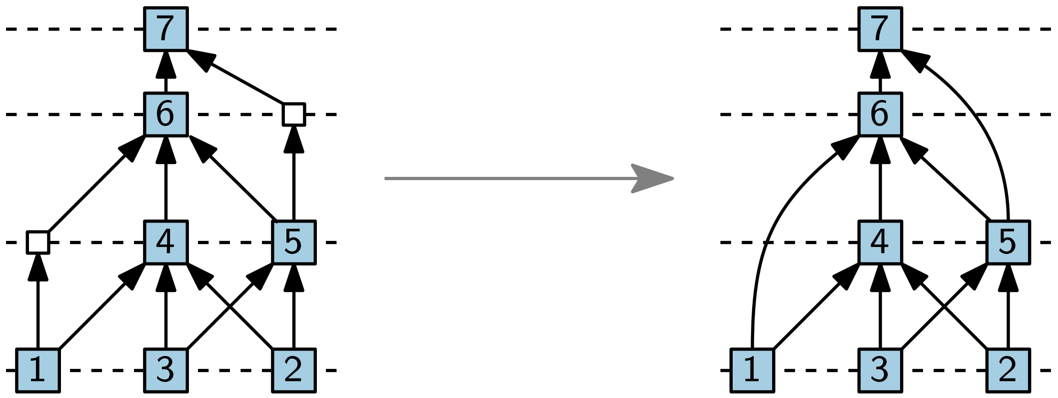
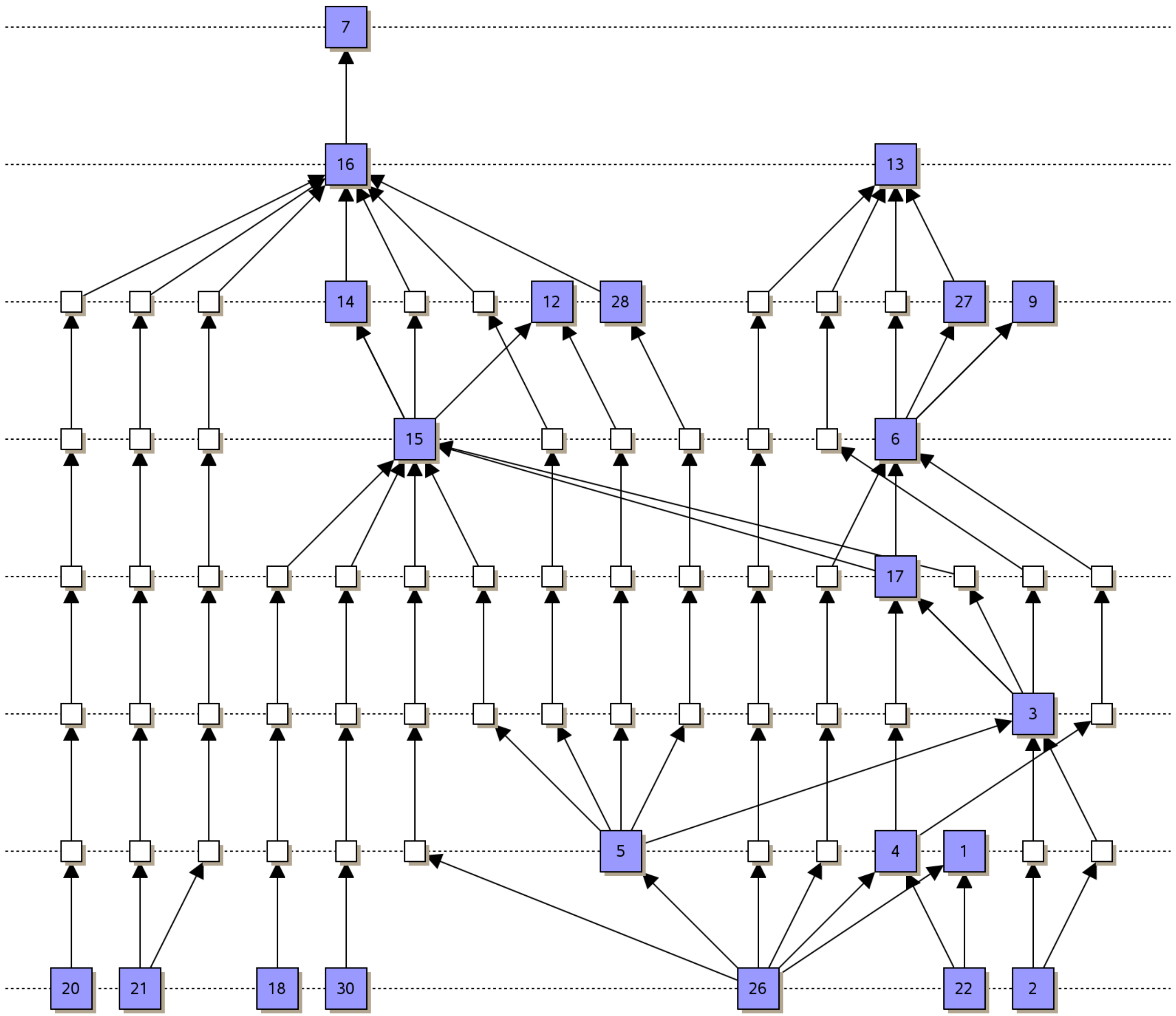
Step 5: Drawing edges



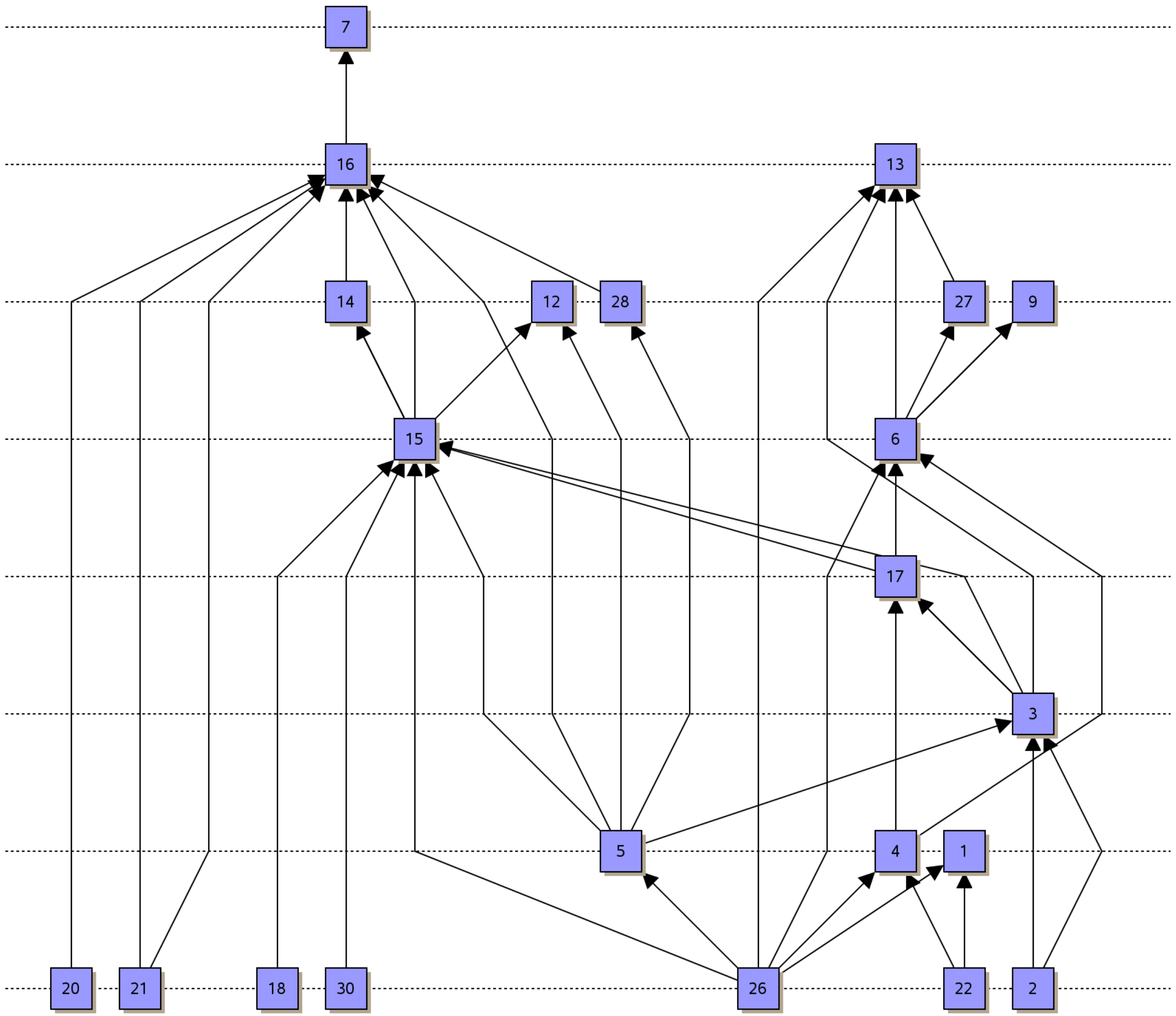
Possibility.

Substitute polylines by Bézier curves

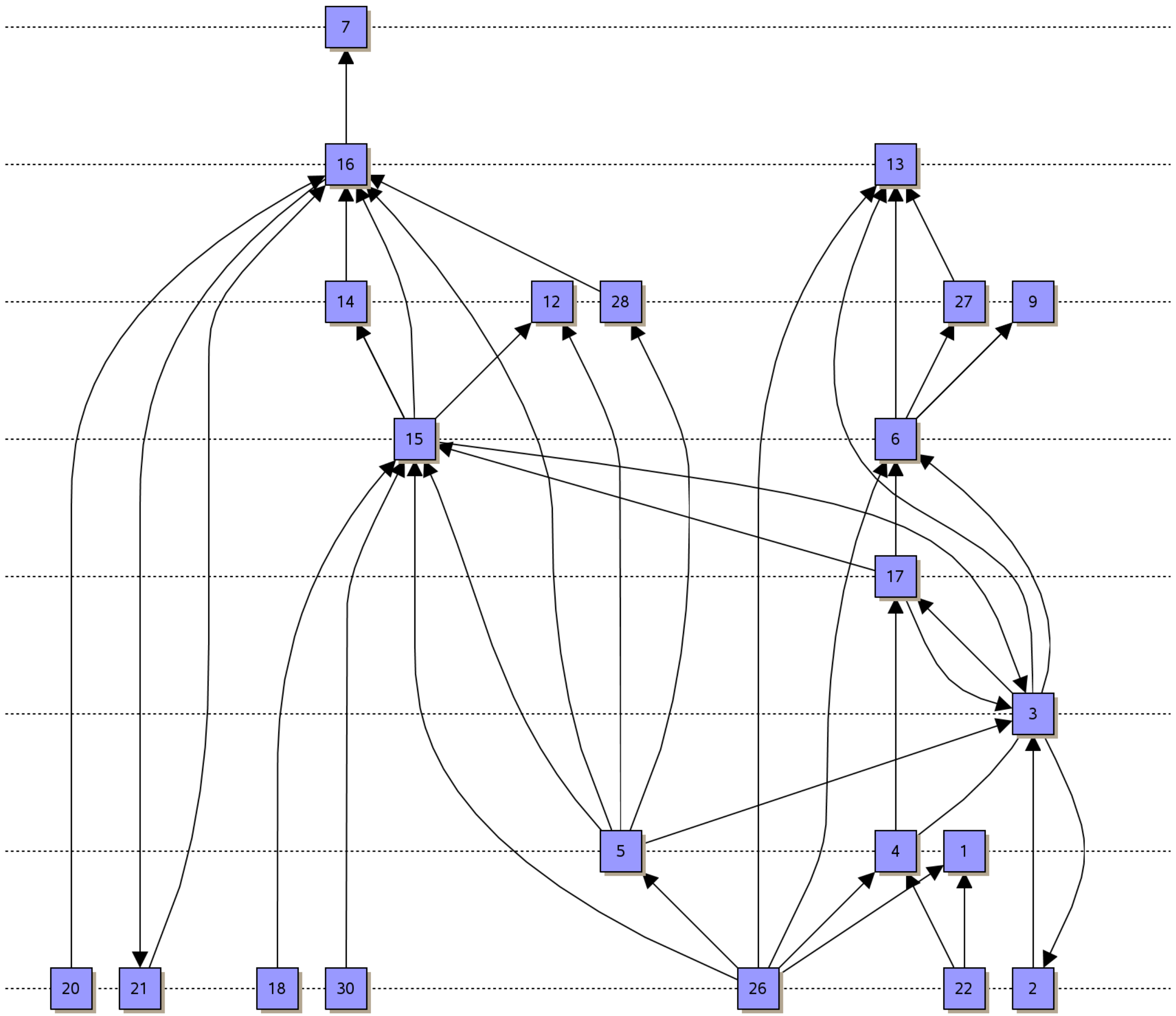
Example



Example

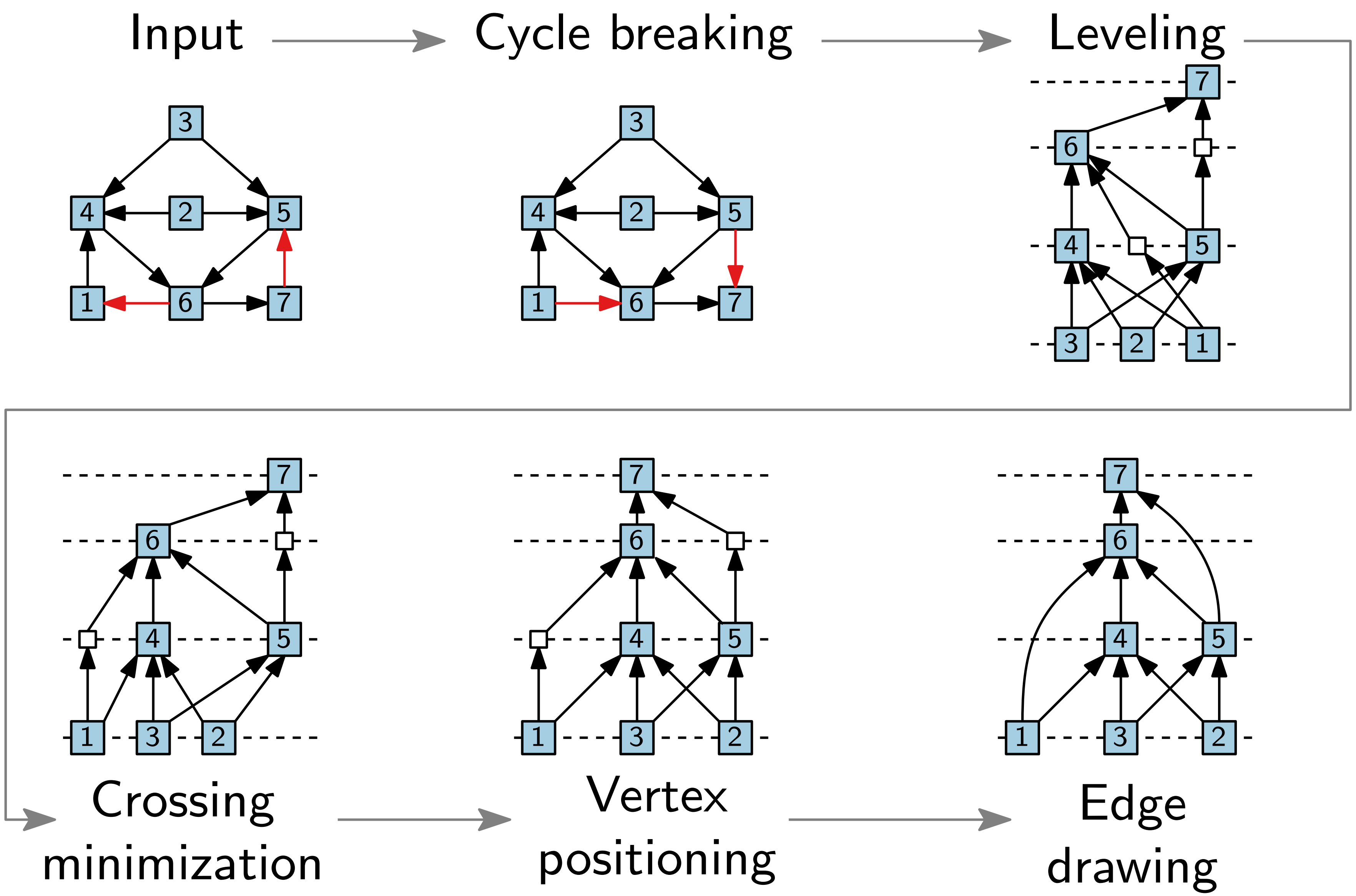


Example



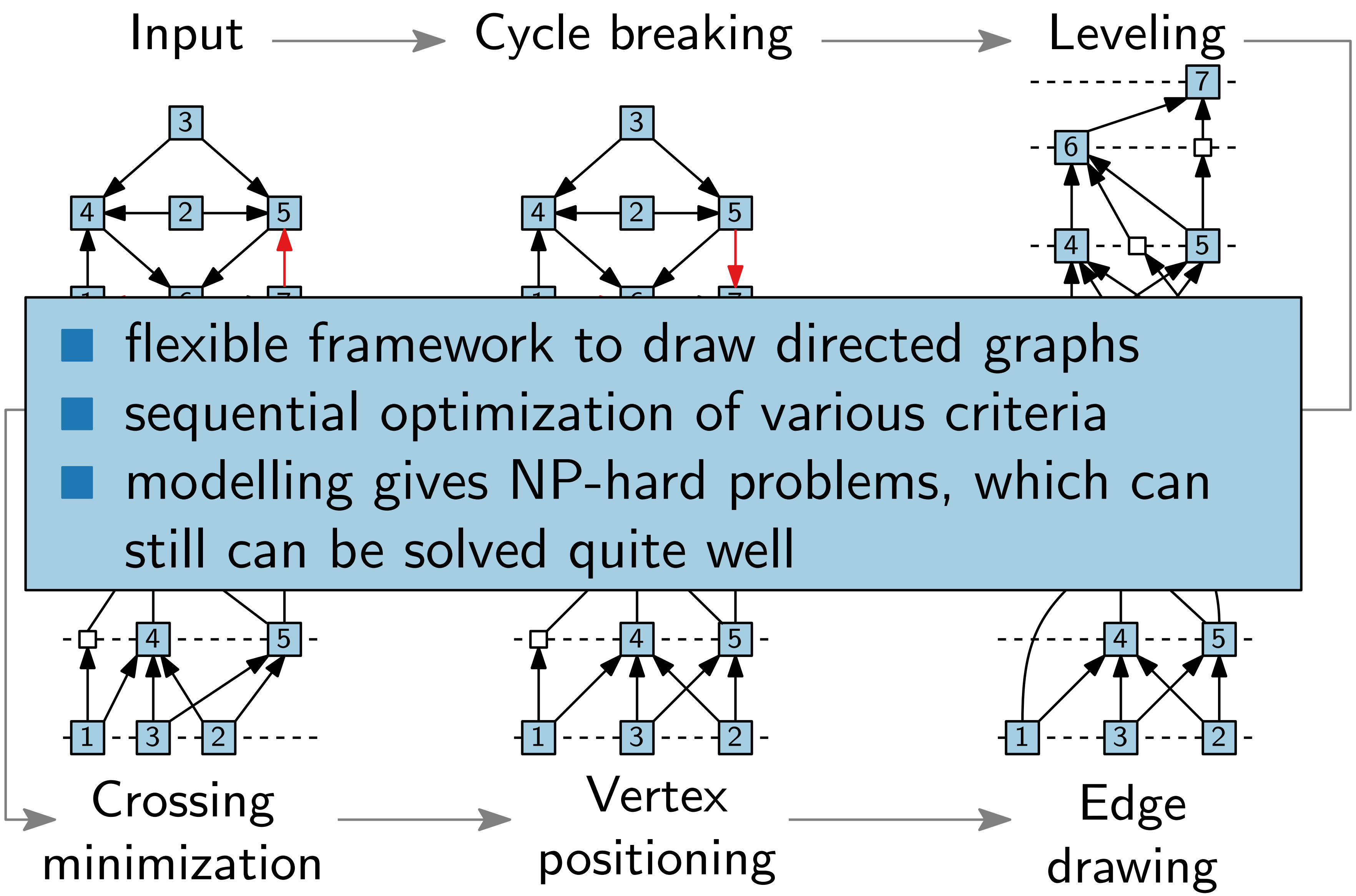
Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



Classical approach – Sugiyama framework

[Sugiyama, Tagawa, Toda ’81]



- flexible framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation alogorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms