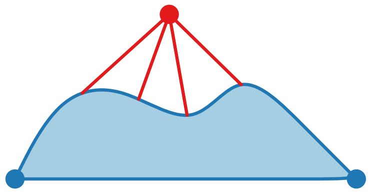
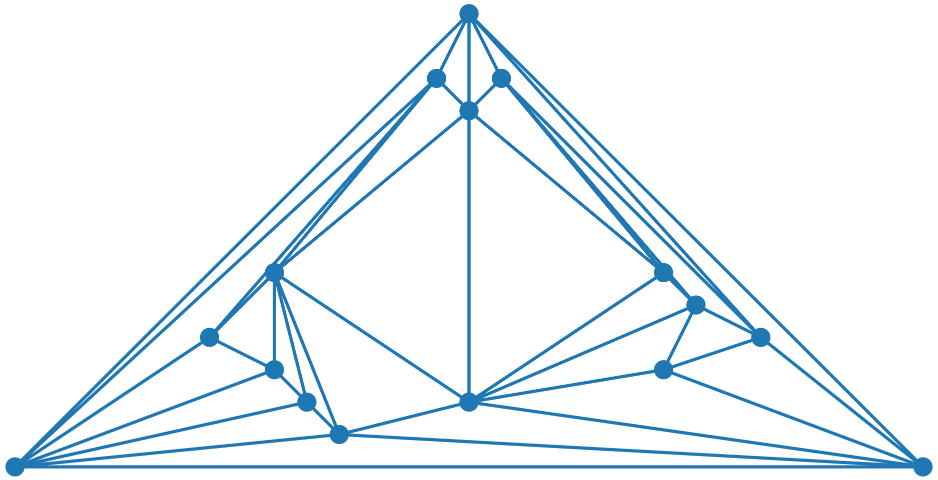
Visualisation of graphs

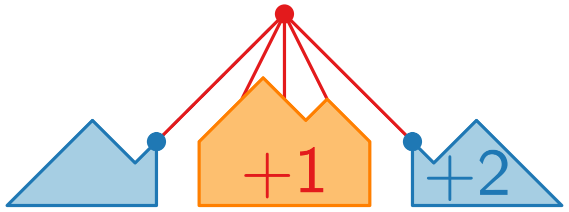
Planar straight-line drawings

Canonical order & shift method

Jonathan Klawitter · Summer semester 2020







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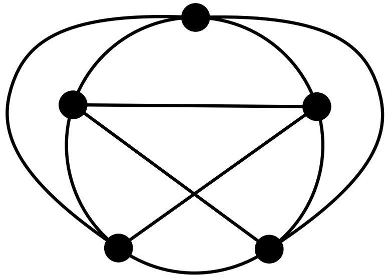
- Why straight-line? Why planar?
- Bennett, Ryall, Spaltzeholz and Gooch, 2007
  "The Aesthetics of Graph Visualization"

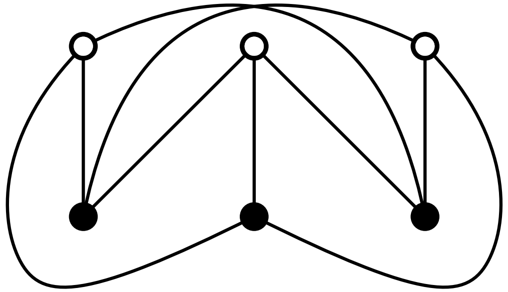
3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

Planar graphs

- **Characterisation:** A graph is planar iff it contains neither a K_5 nor a $K_{3,3}$ minor.
  [Kuratowski 1930]





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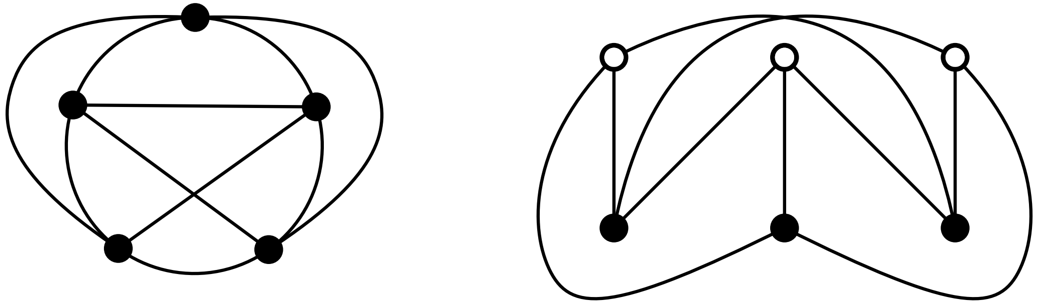

- **Recognition:** For a graph G with n vertices, there is an $\mathcal{O}(n)$ time algorithm to test if G is planar. [Hopcroft & Tarjan 1974]
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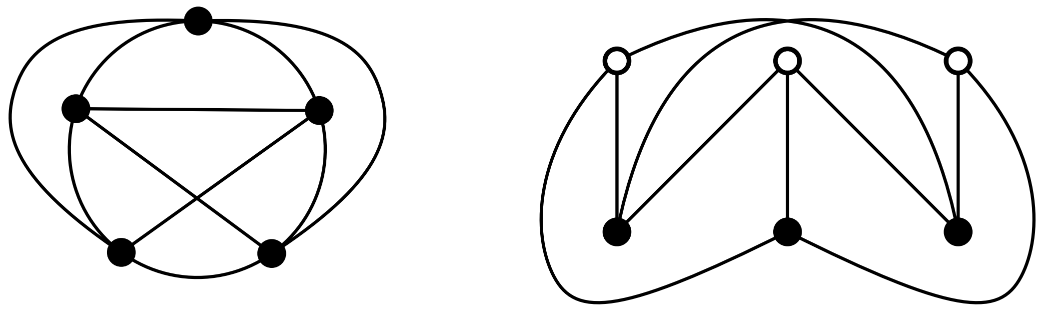

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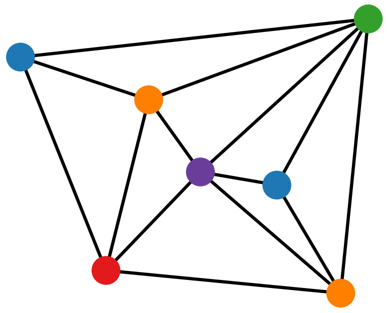
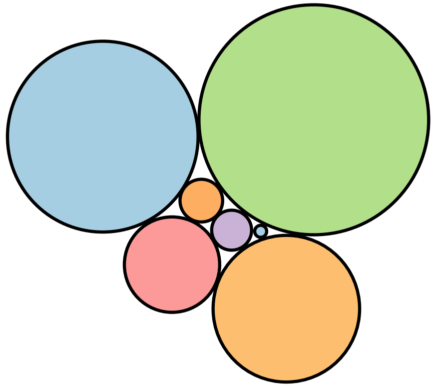
- **Straight-line drawing:** Every planar graph has an embedding where the edges are straight-line segments. [Wagner 1936, Fáry 1948, Stein 1951]
  - The algorithms implied by this theory produce drawings with area not bounded by any polynomial on n.

Planar graphs

- **Coin graph:** Every planar graph is a circle contact graph (implies straight-line drawing). [Koebe 1936]





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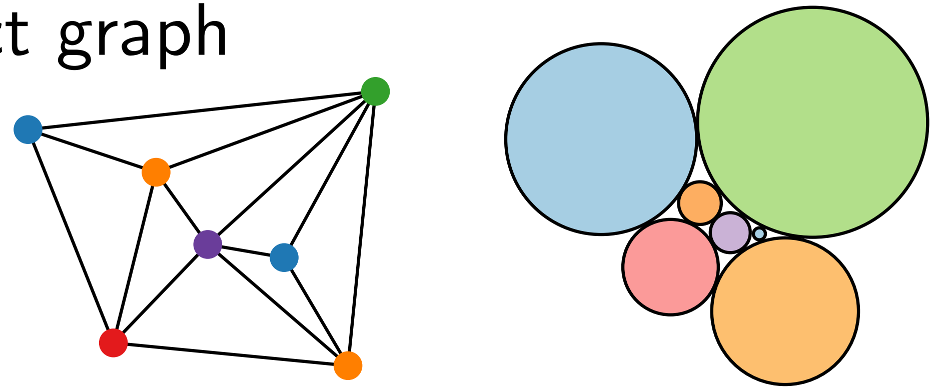

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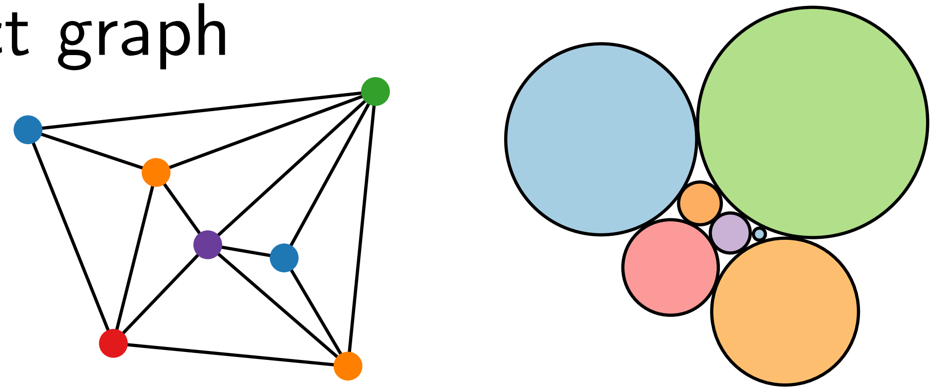

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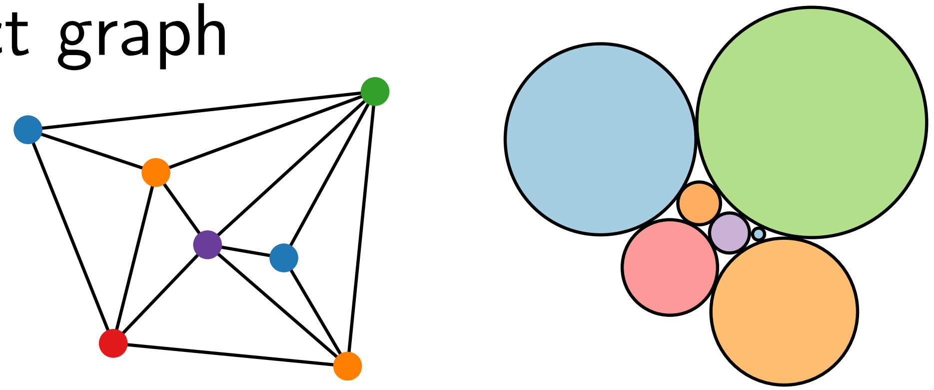

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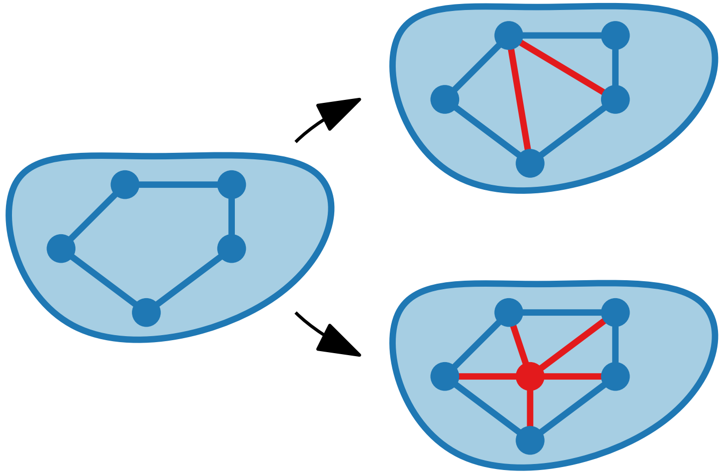

Planar straight-line drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

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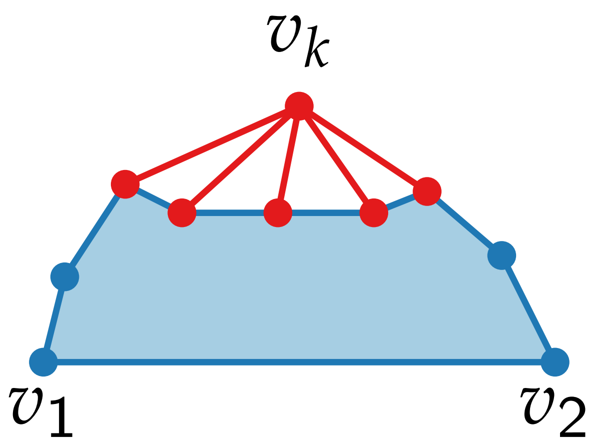
Idea.

- Start with singe edge (v_1, v_2). Let this be G_2.
- To obtain G_{i+1}, add v_{i+1} to G_i so that neighbours of v_{i+1} are on the outer face of G_i.
- Neighbours of v_{i+1} in G_i have to form path of length at least two.



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Canonical order – definition

Definition.
Let $G = (V, E)$ be a triangulated plane graph on $n \geq 3$ vertices.
An order $\pi = (v_1, v_2, \ldots, v_n)$ is called a **canonical order**, if the following conditions hold for each k, $3 \leq k \leq n$:

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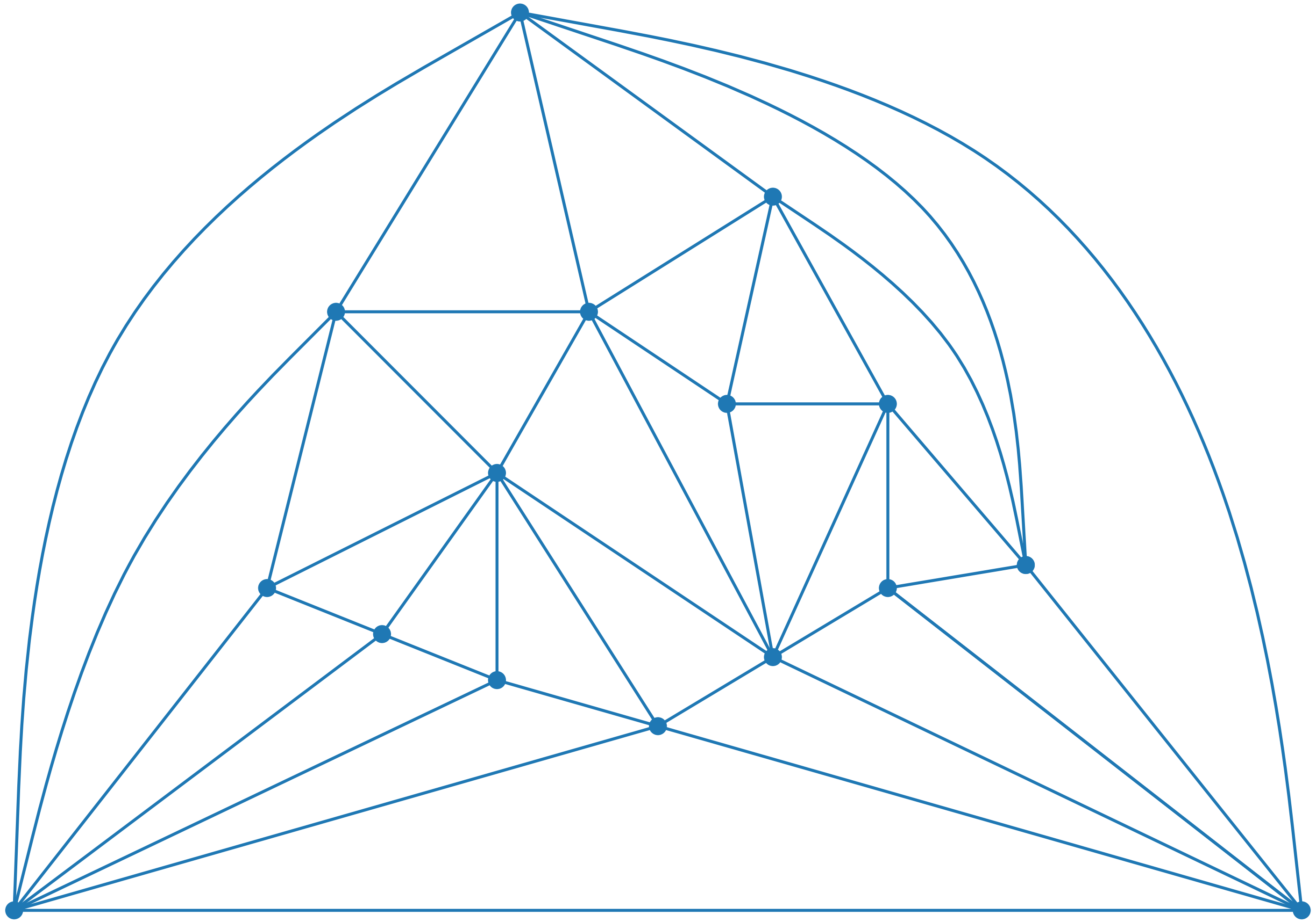
- (C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and all neighbors of v_{k+1} in G_k appear on the boundary of G_k consecutively.

Canonical order – example



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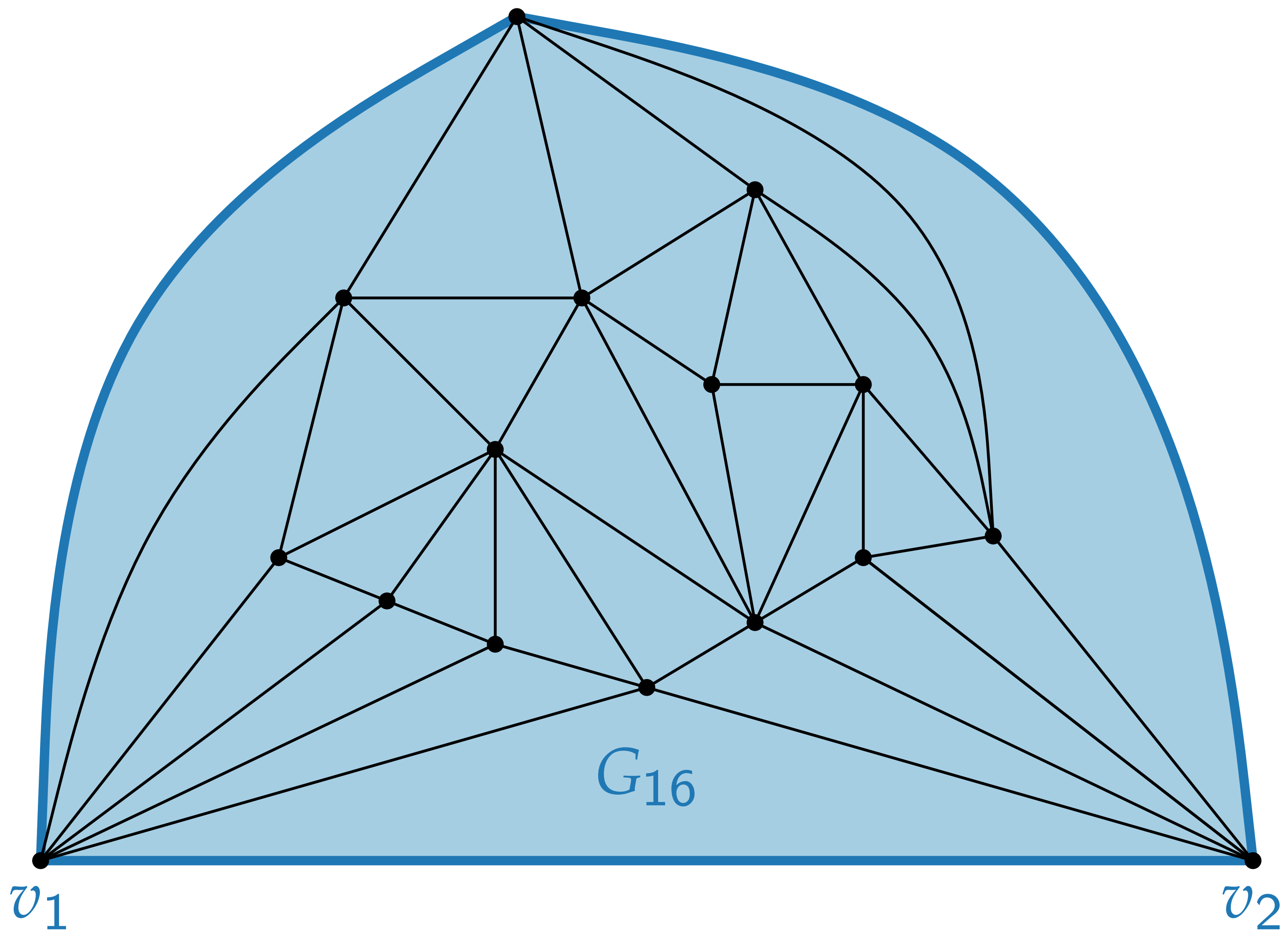

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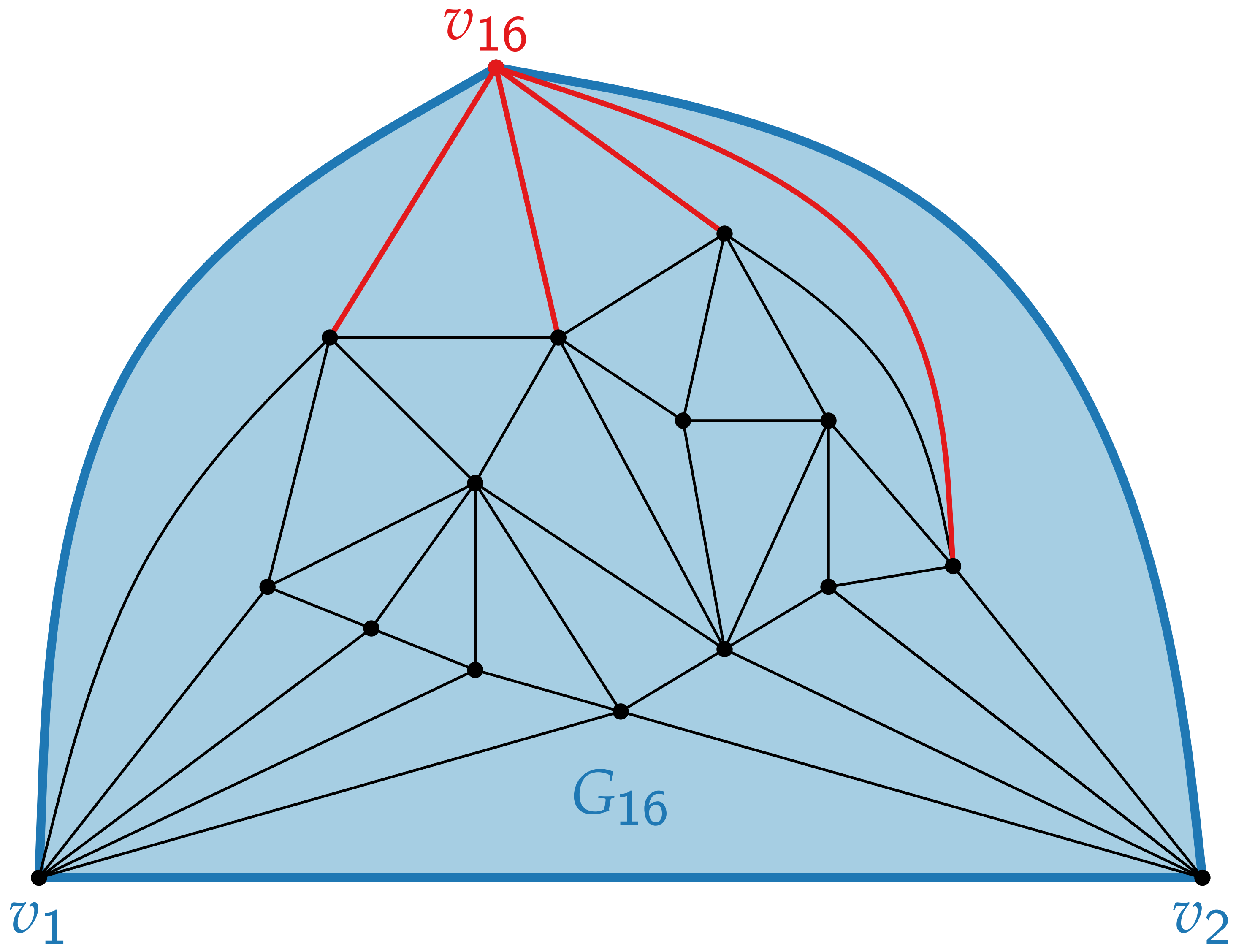

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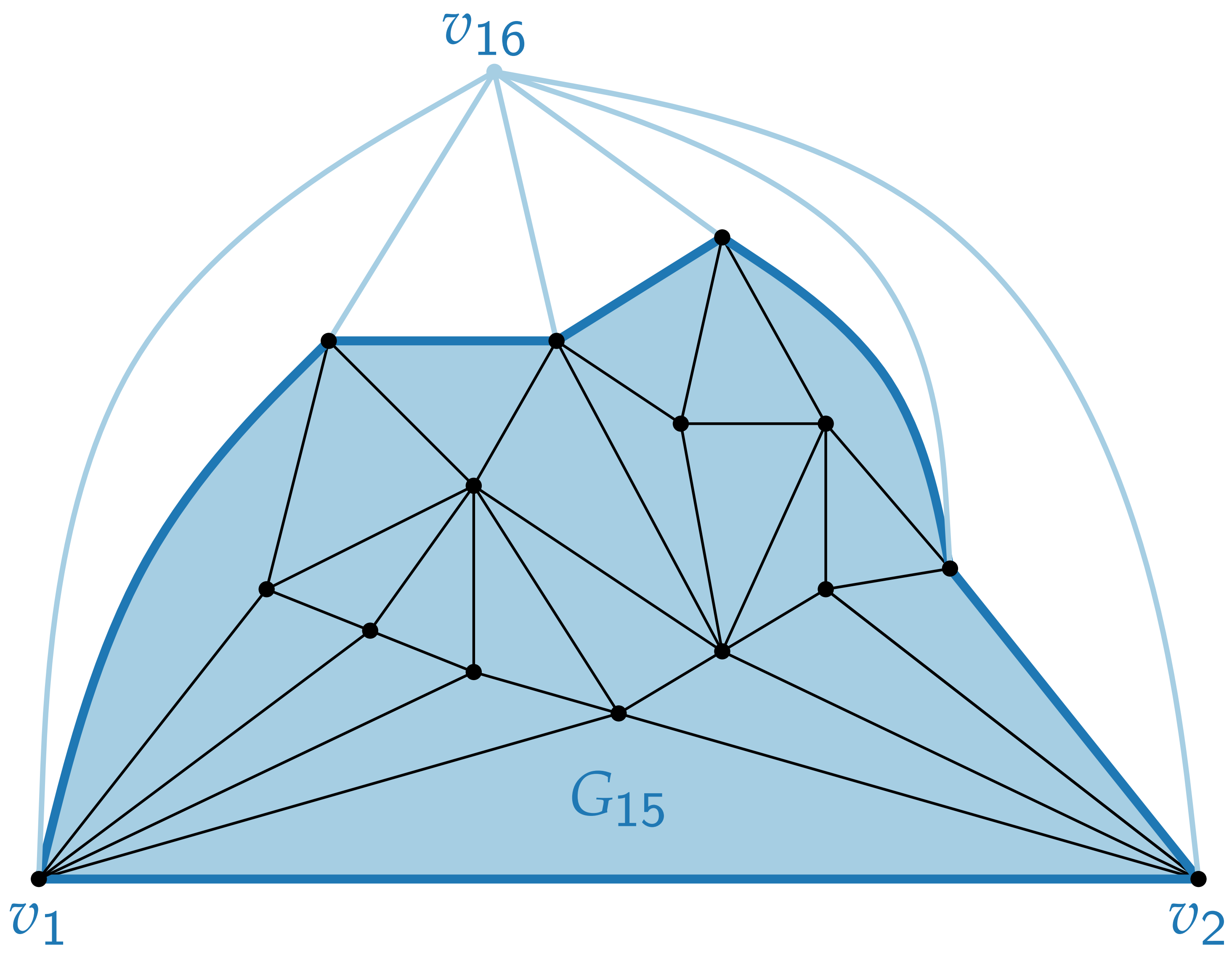

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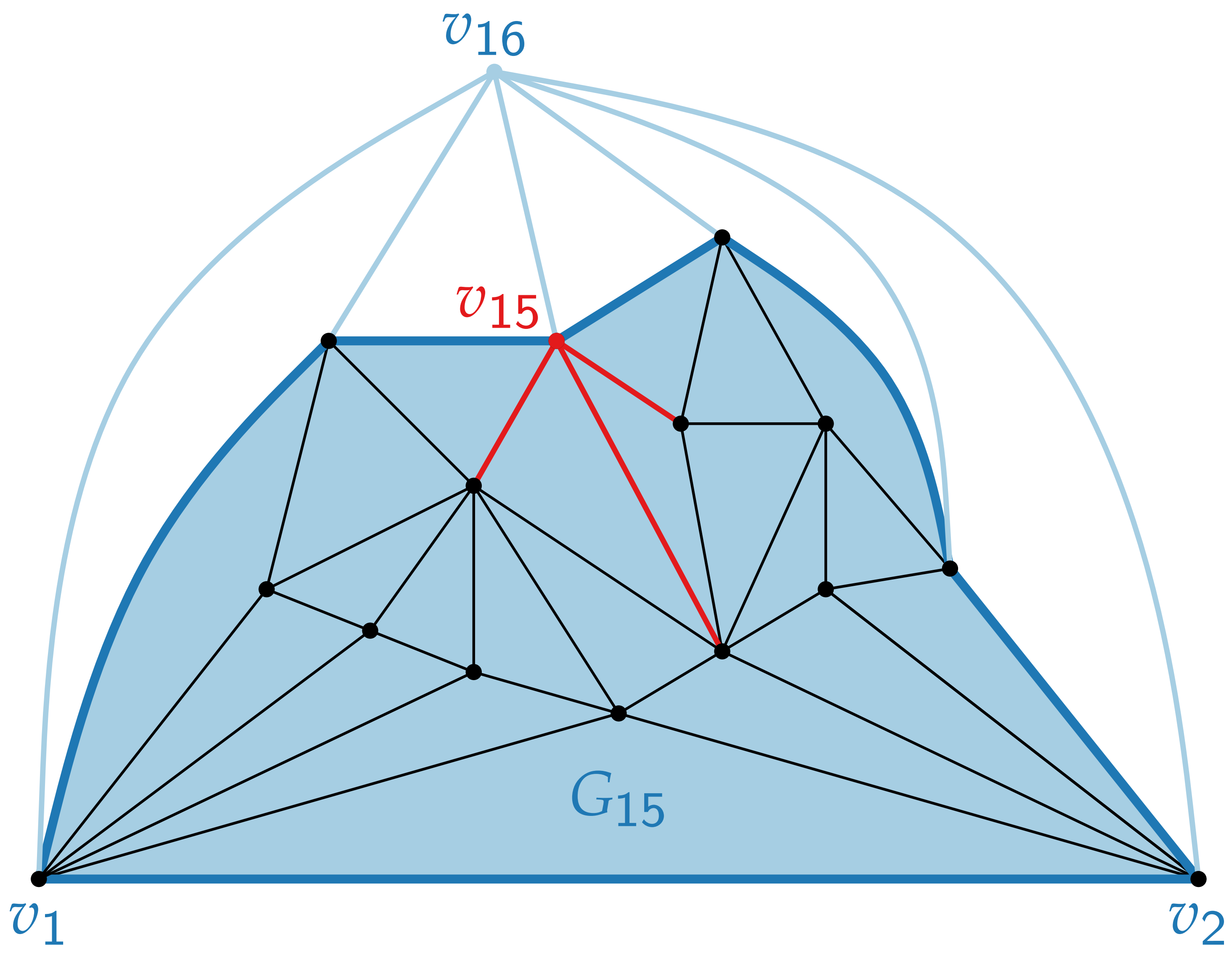

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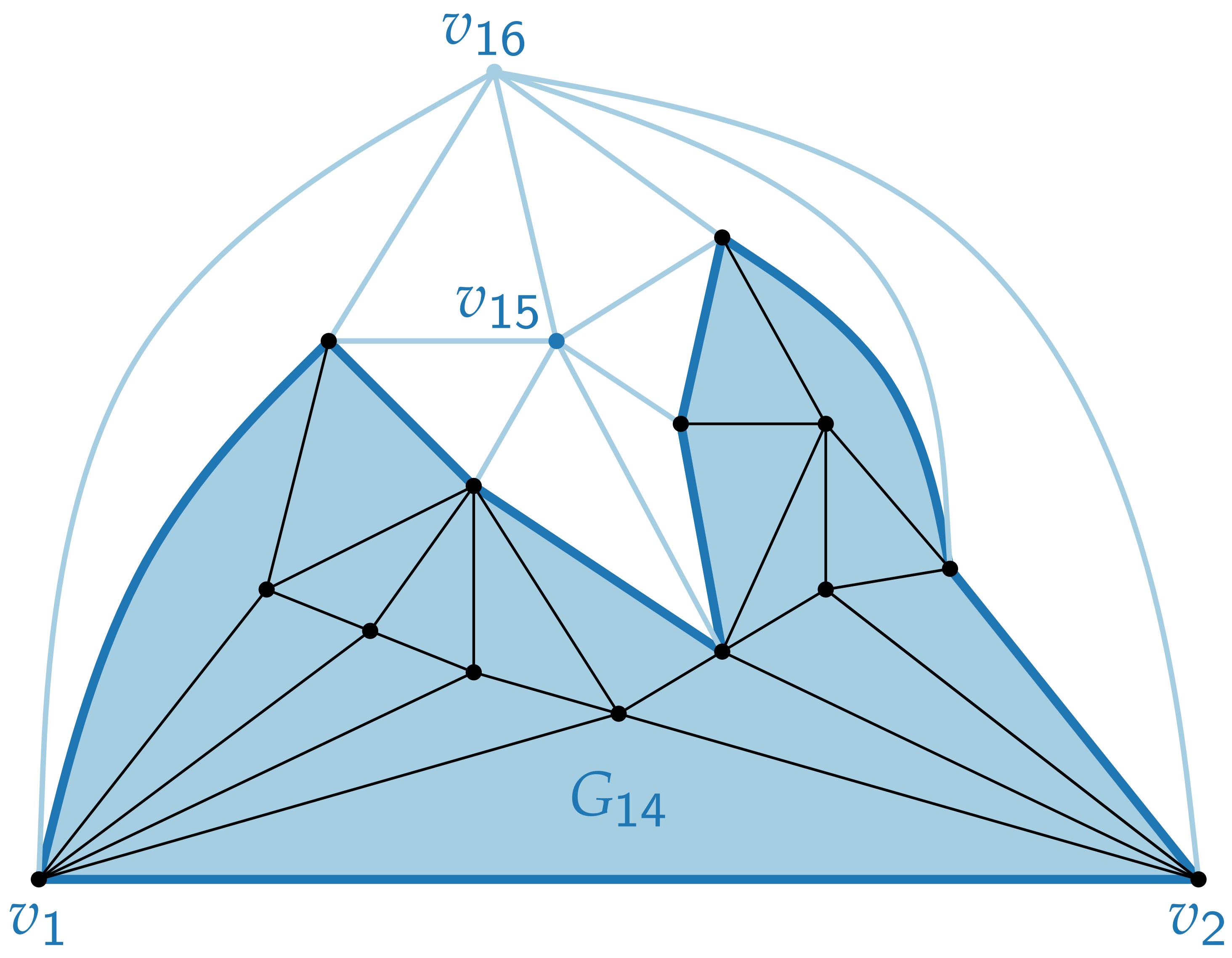

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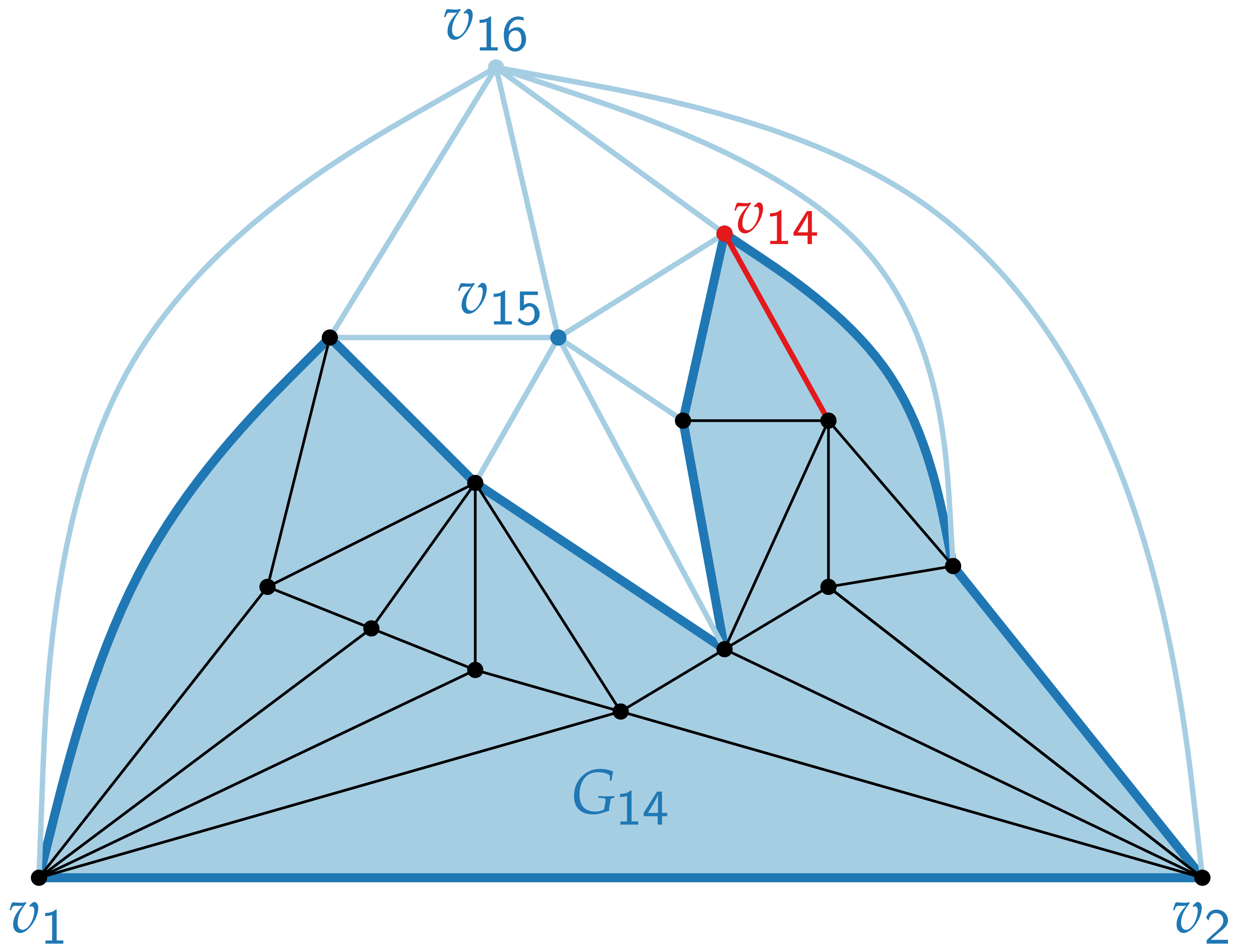

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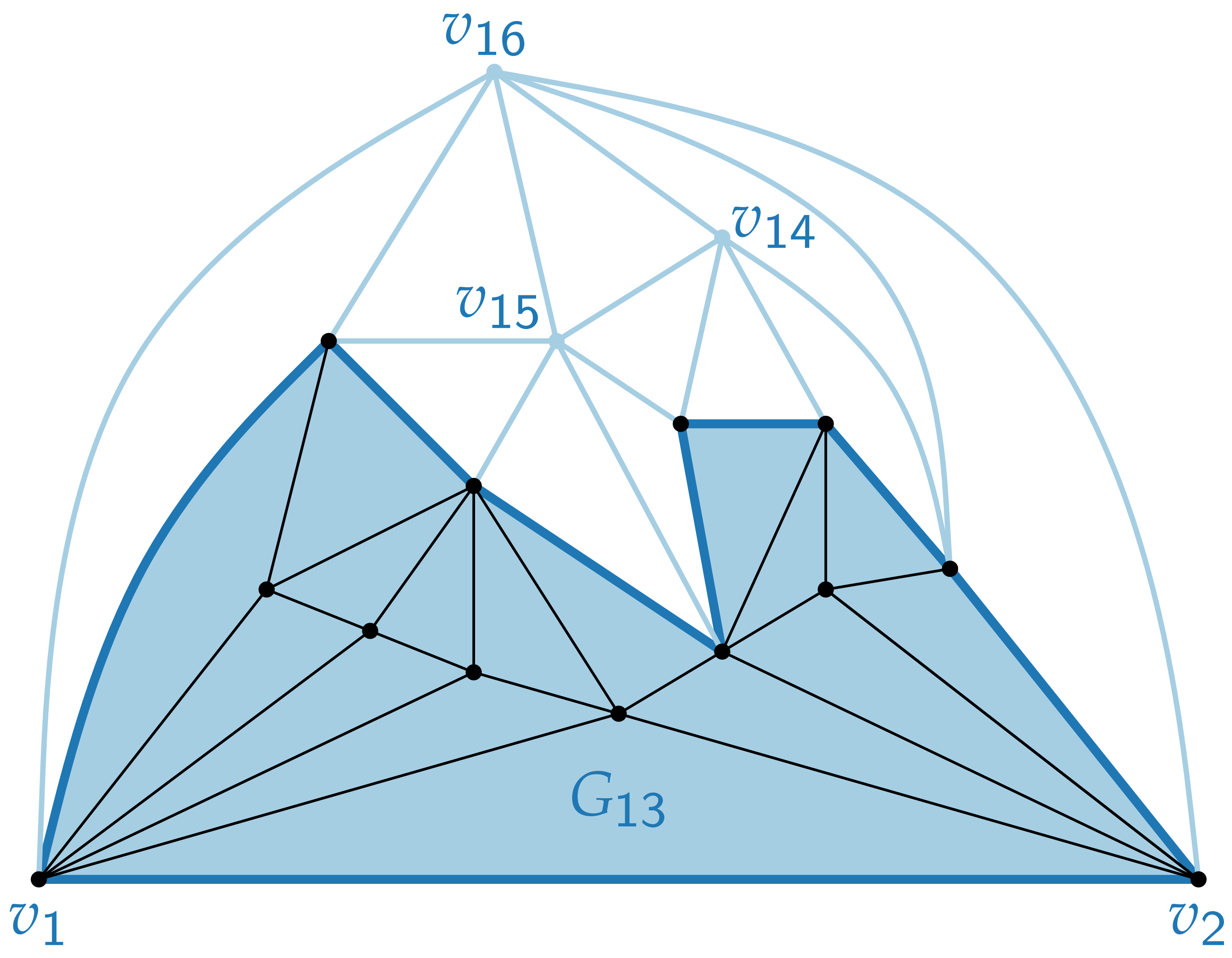

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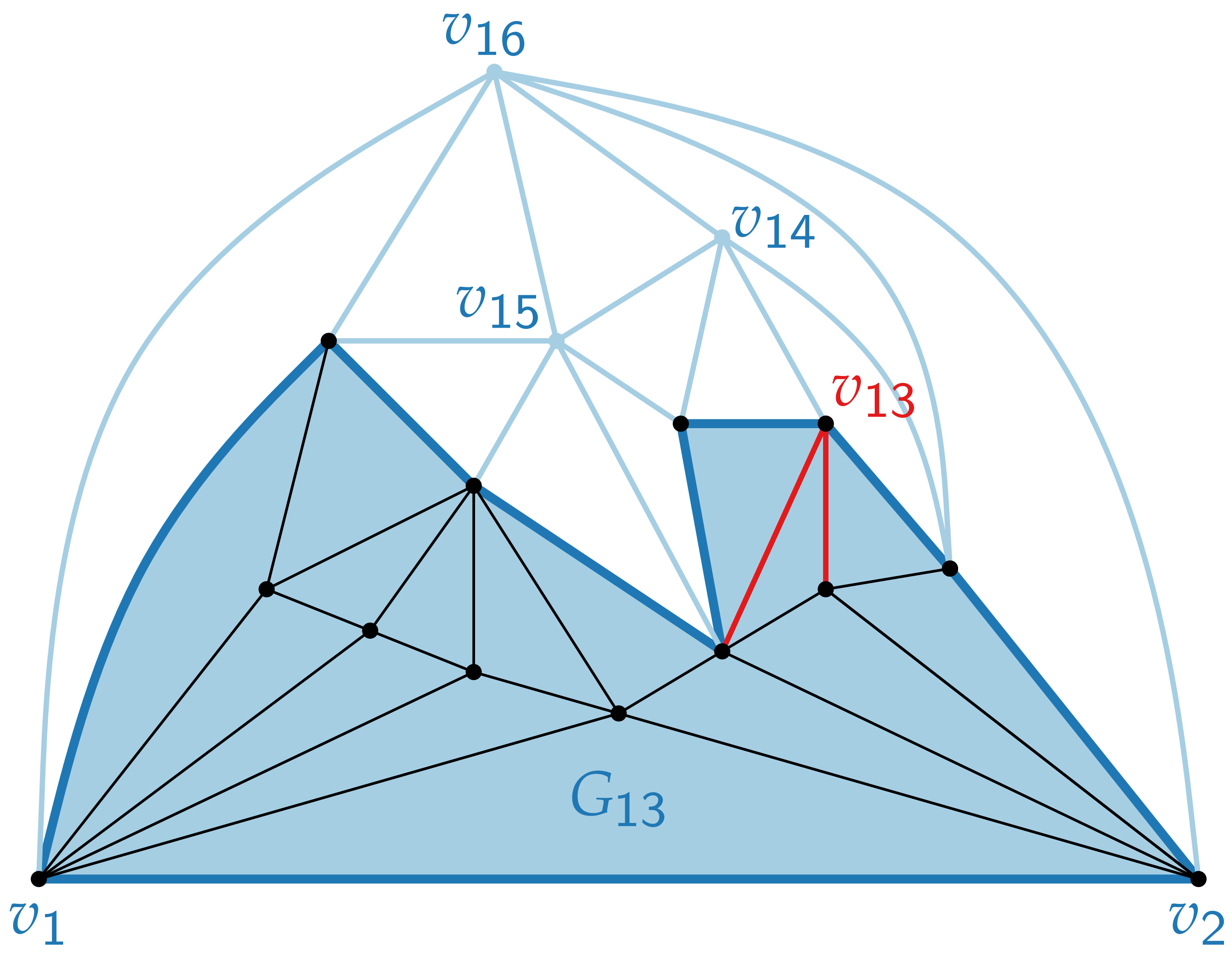

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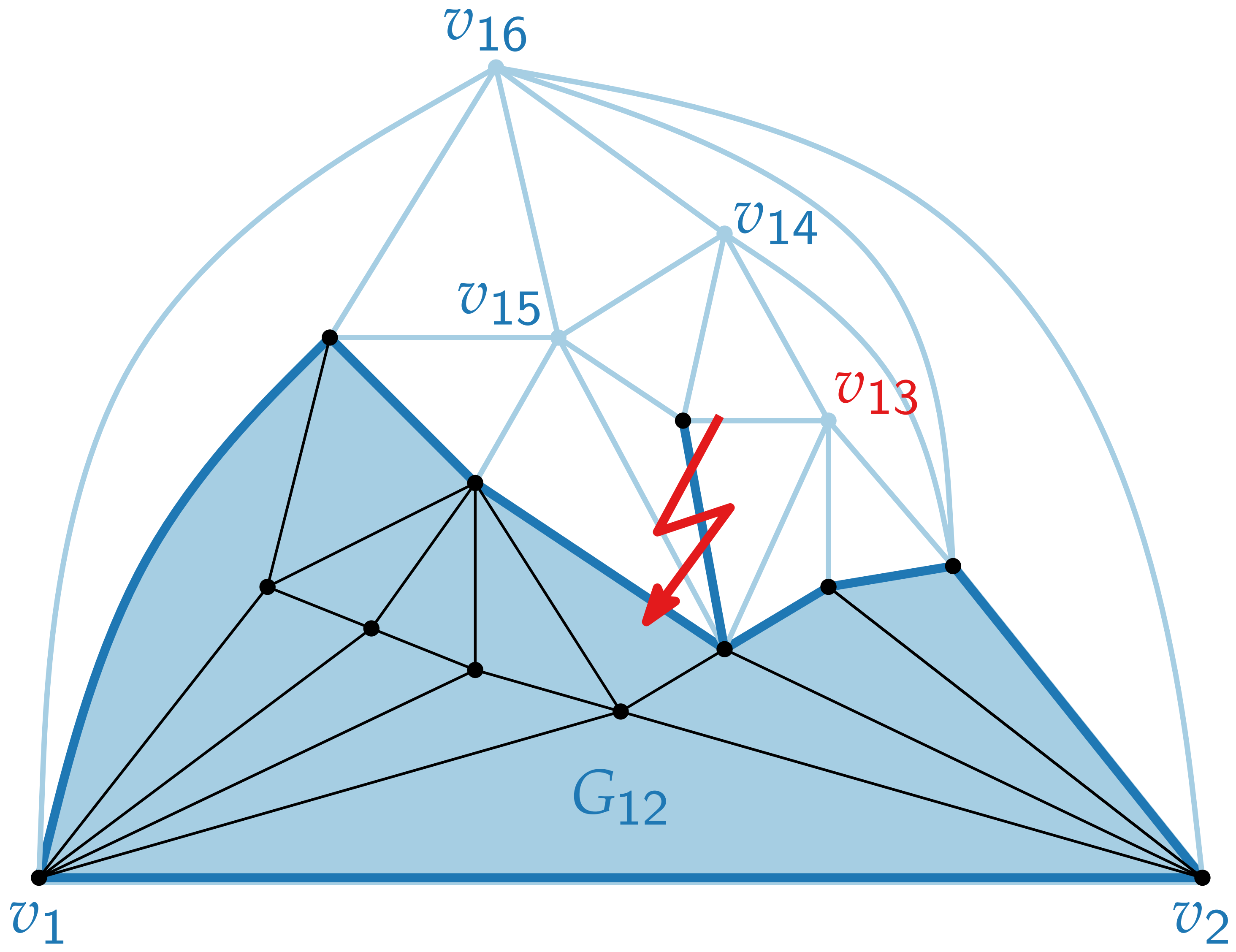

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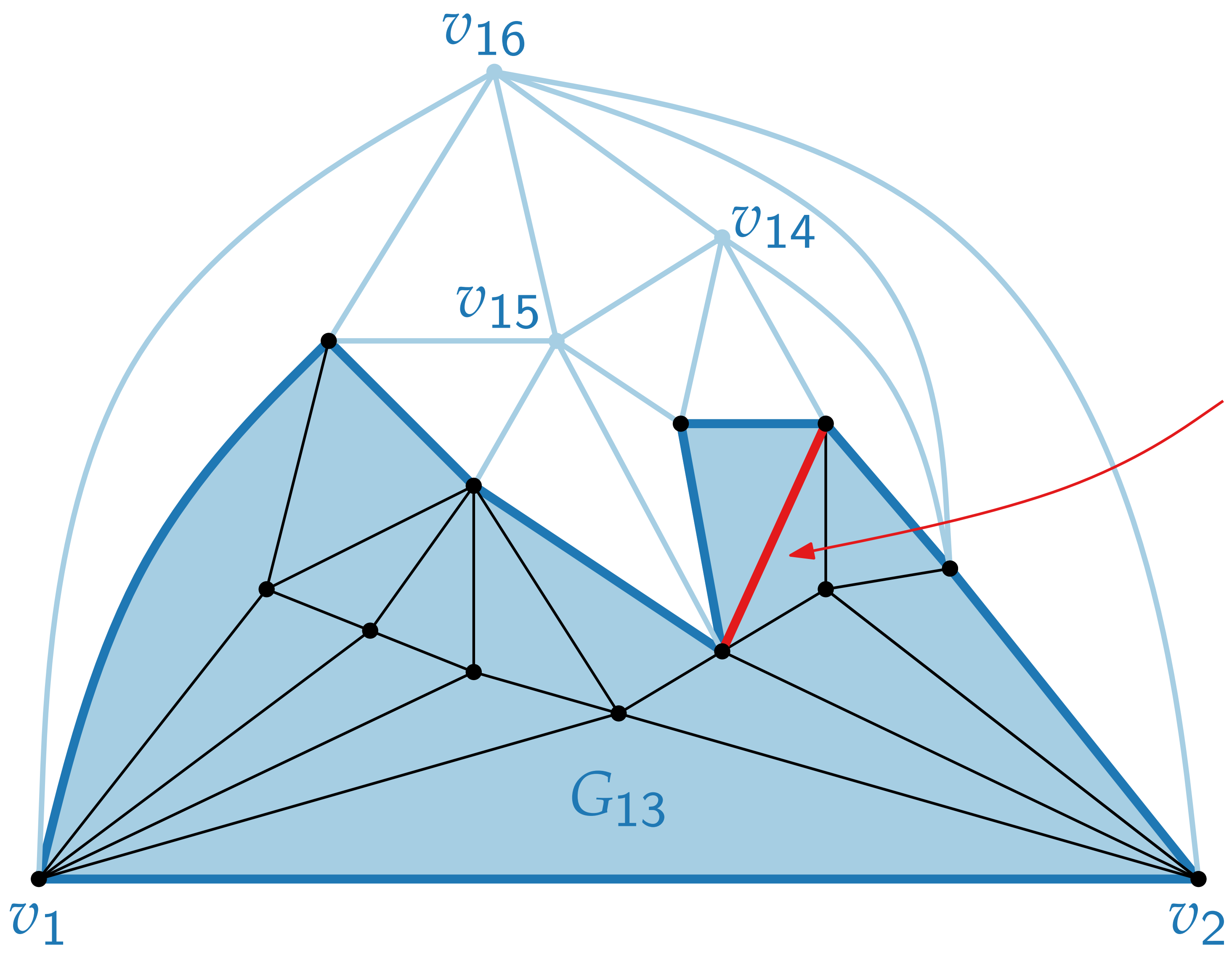

chord
edge joining two nonadjacent vertices in a cycle

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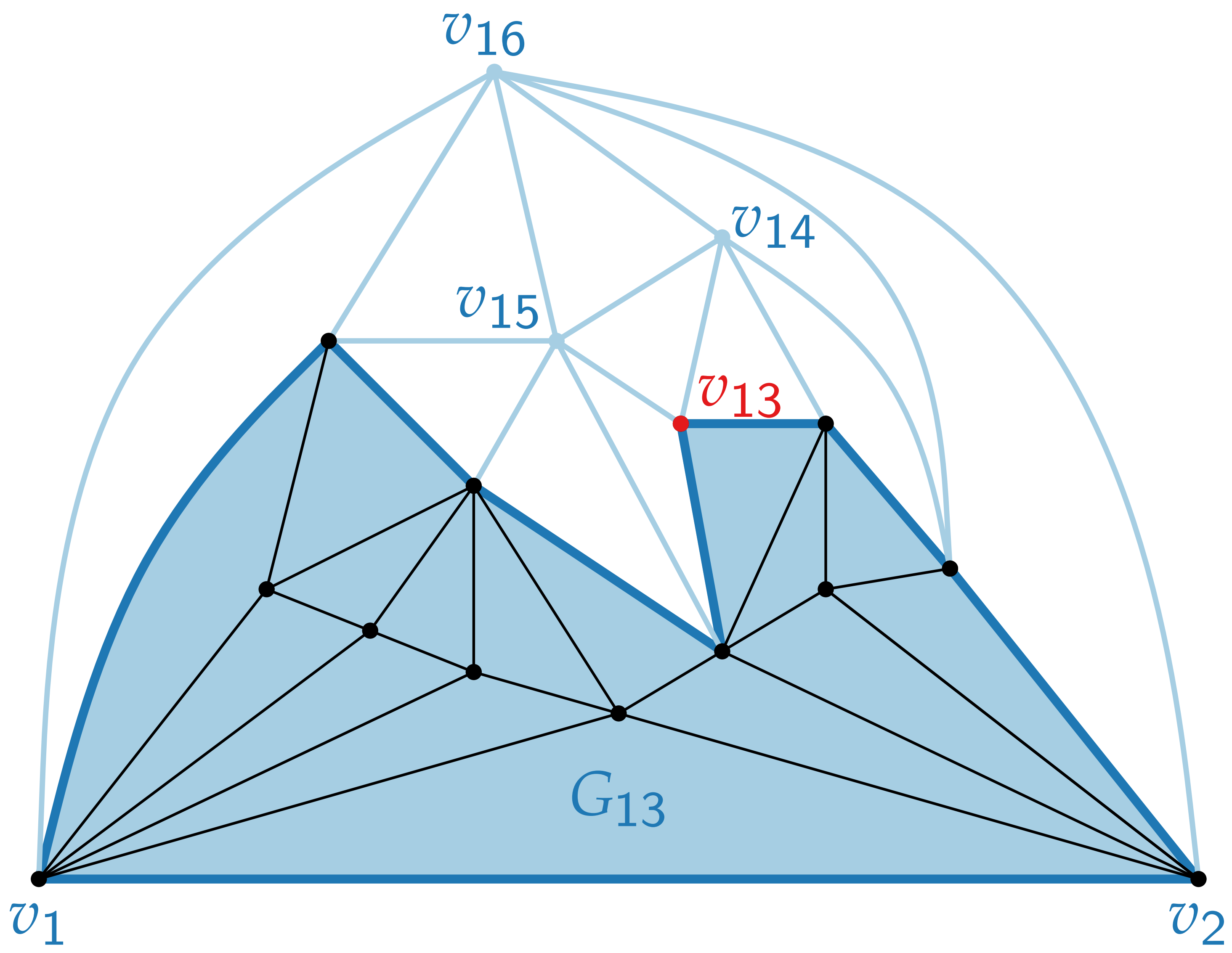

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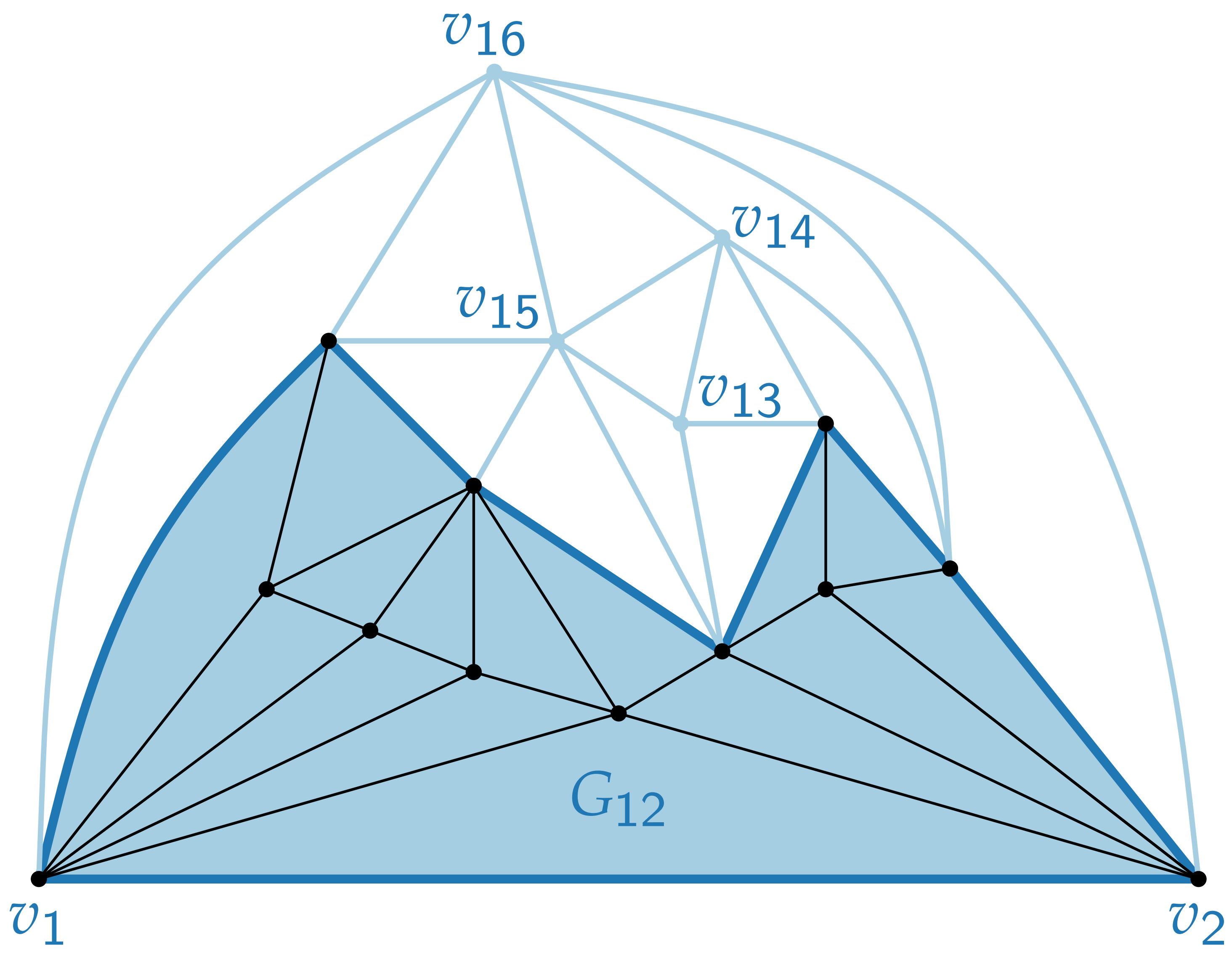

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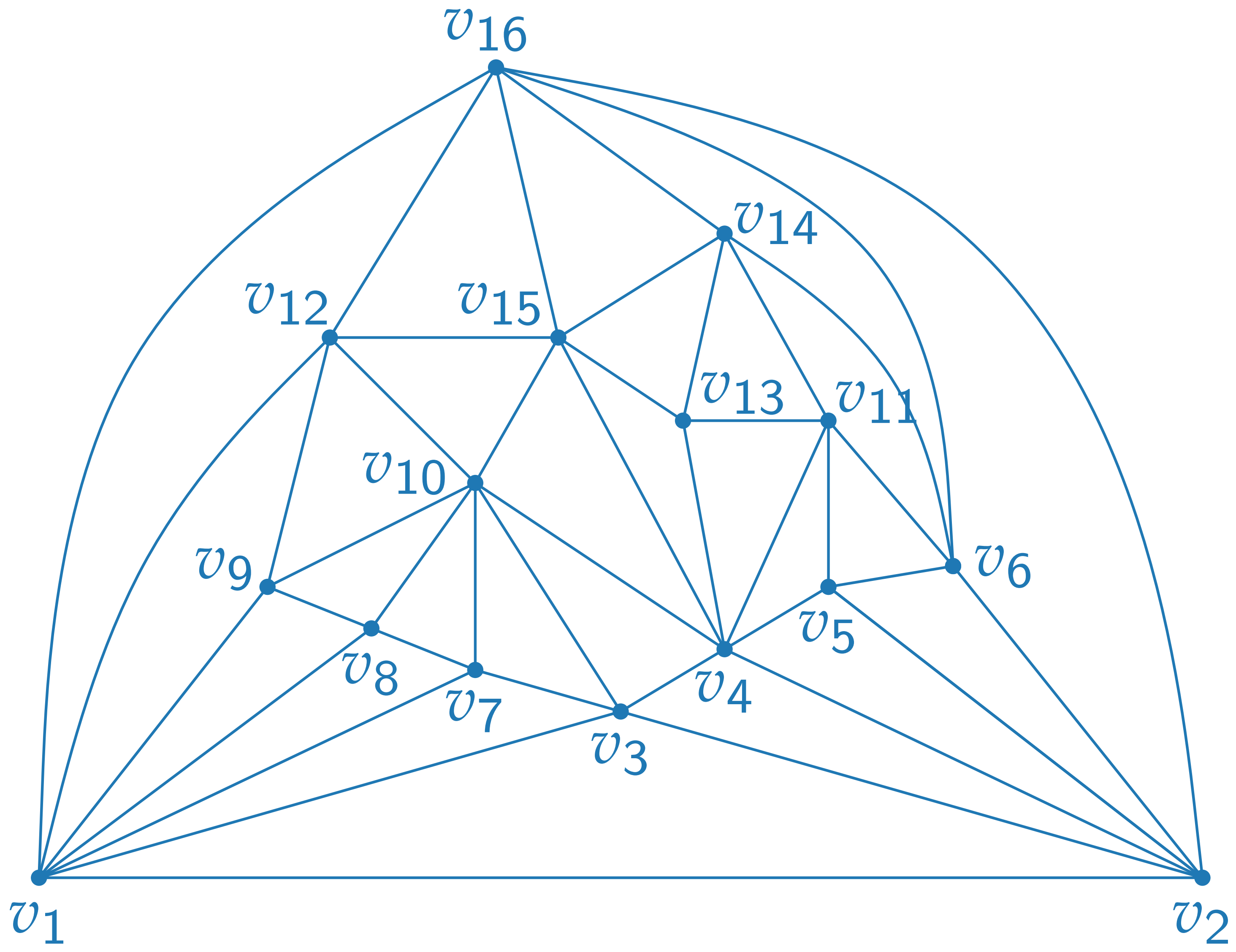

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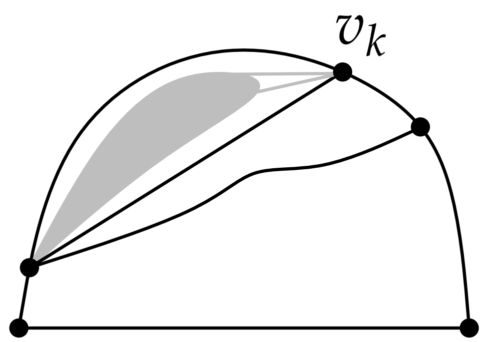

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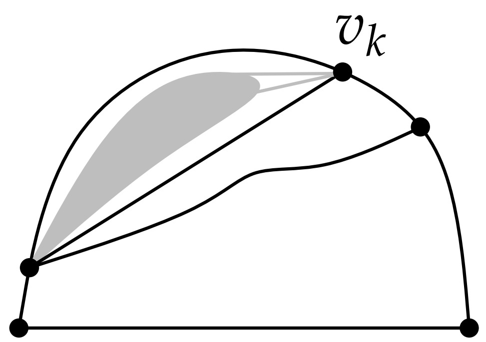

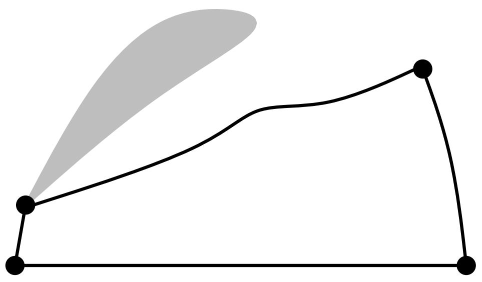

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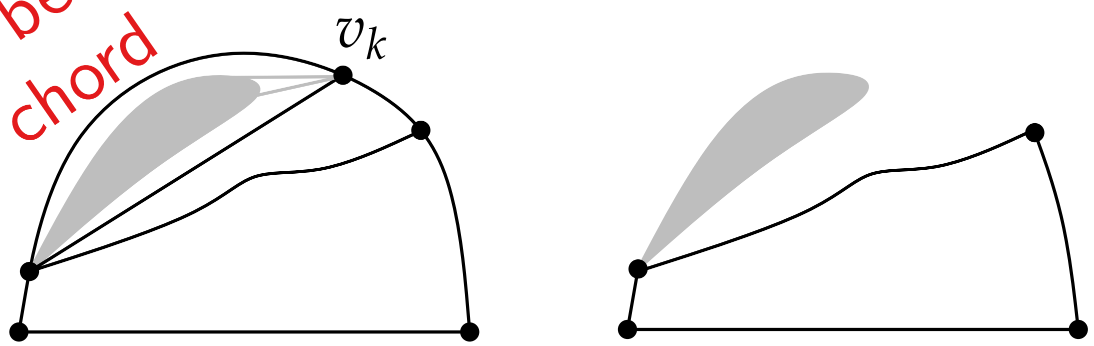
v_k should not be adjacent to a chord



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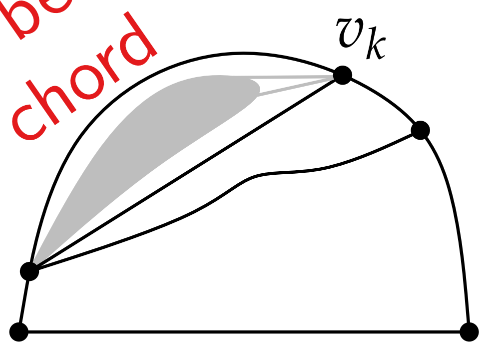

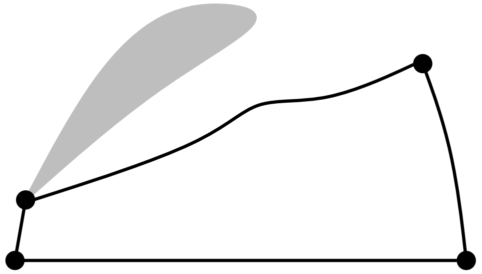

Have to show:

1. v_k not adjacent to chord is sufficient
2. Such v_k exists

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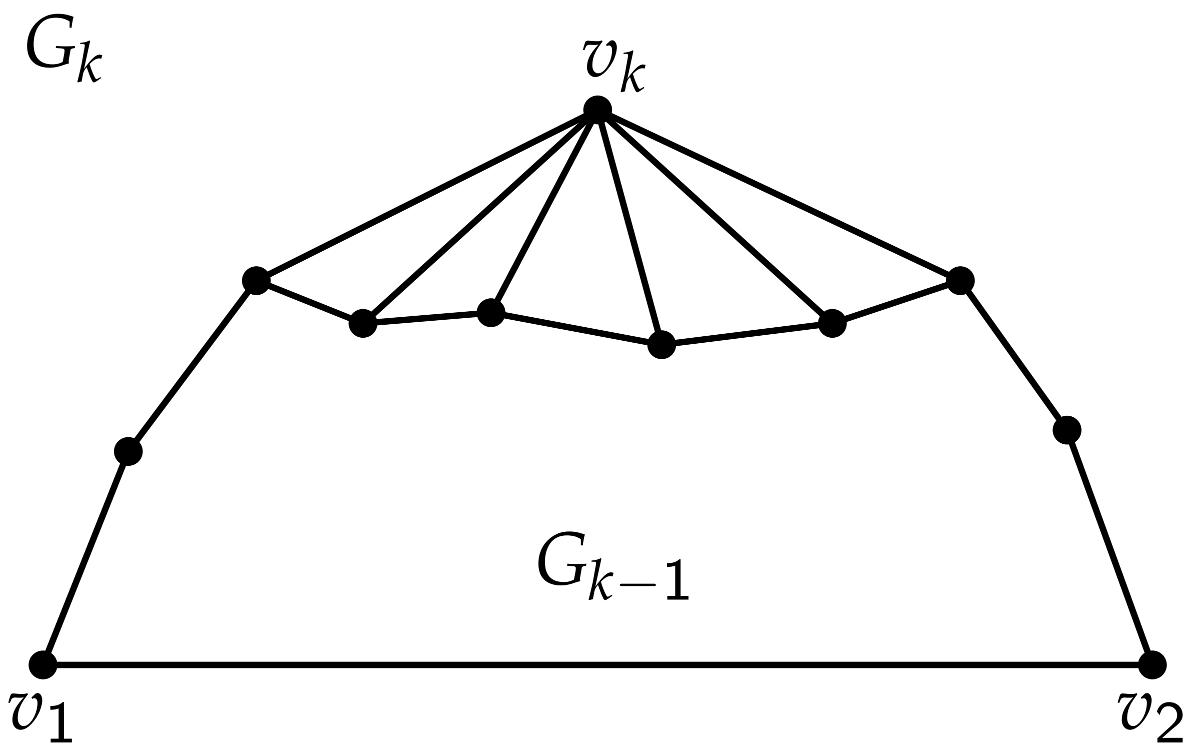

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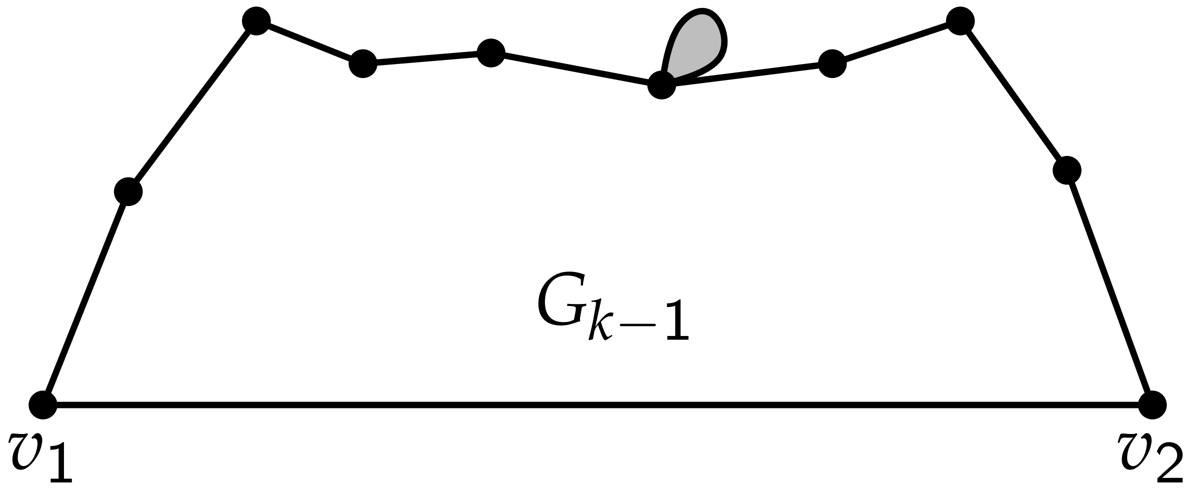

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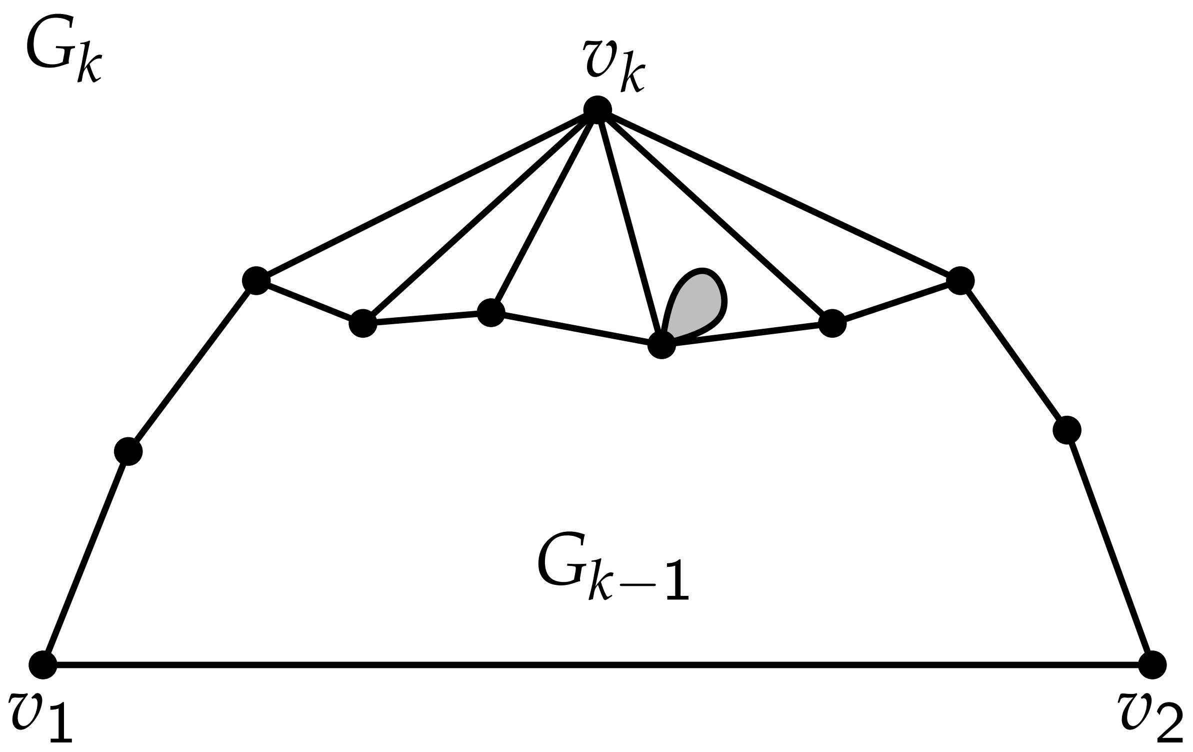

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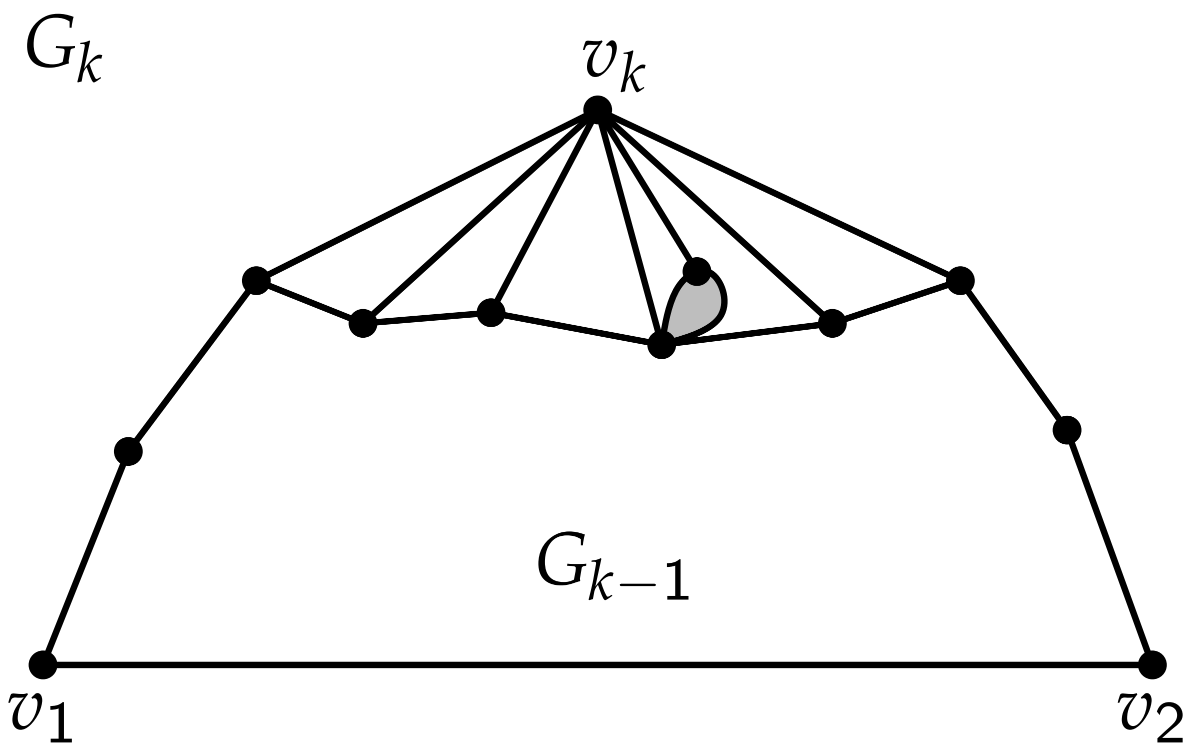

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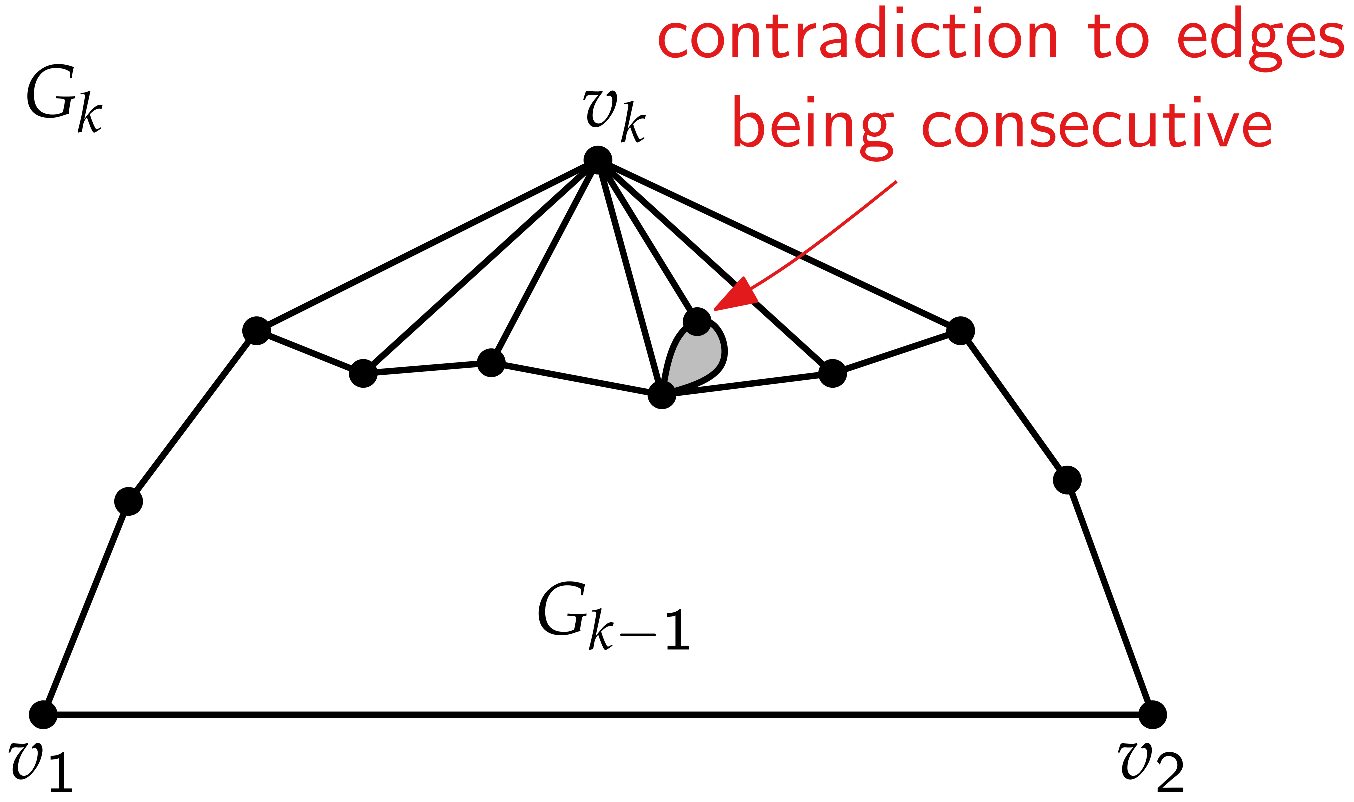

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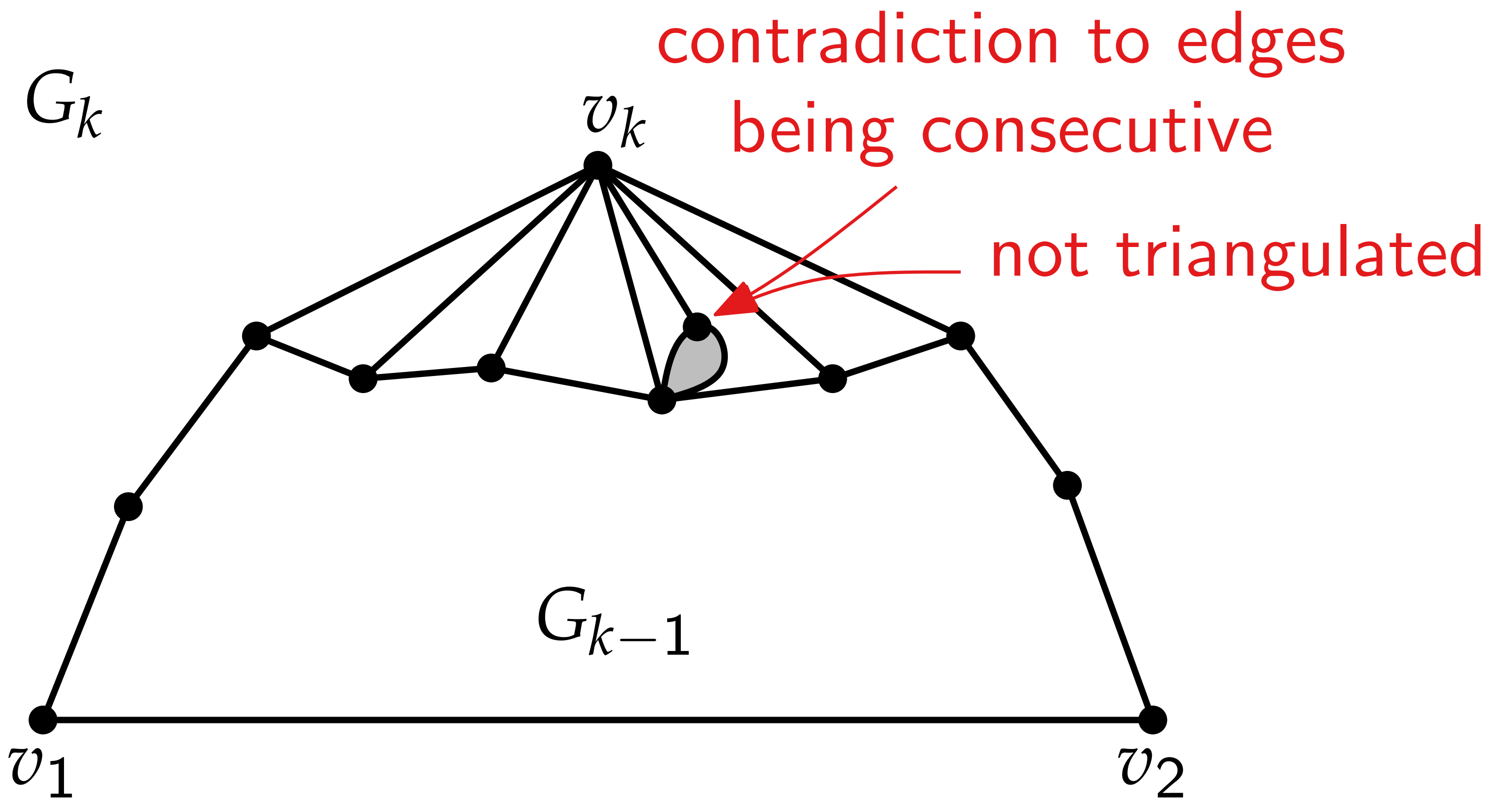

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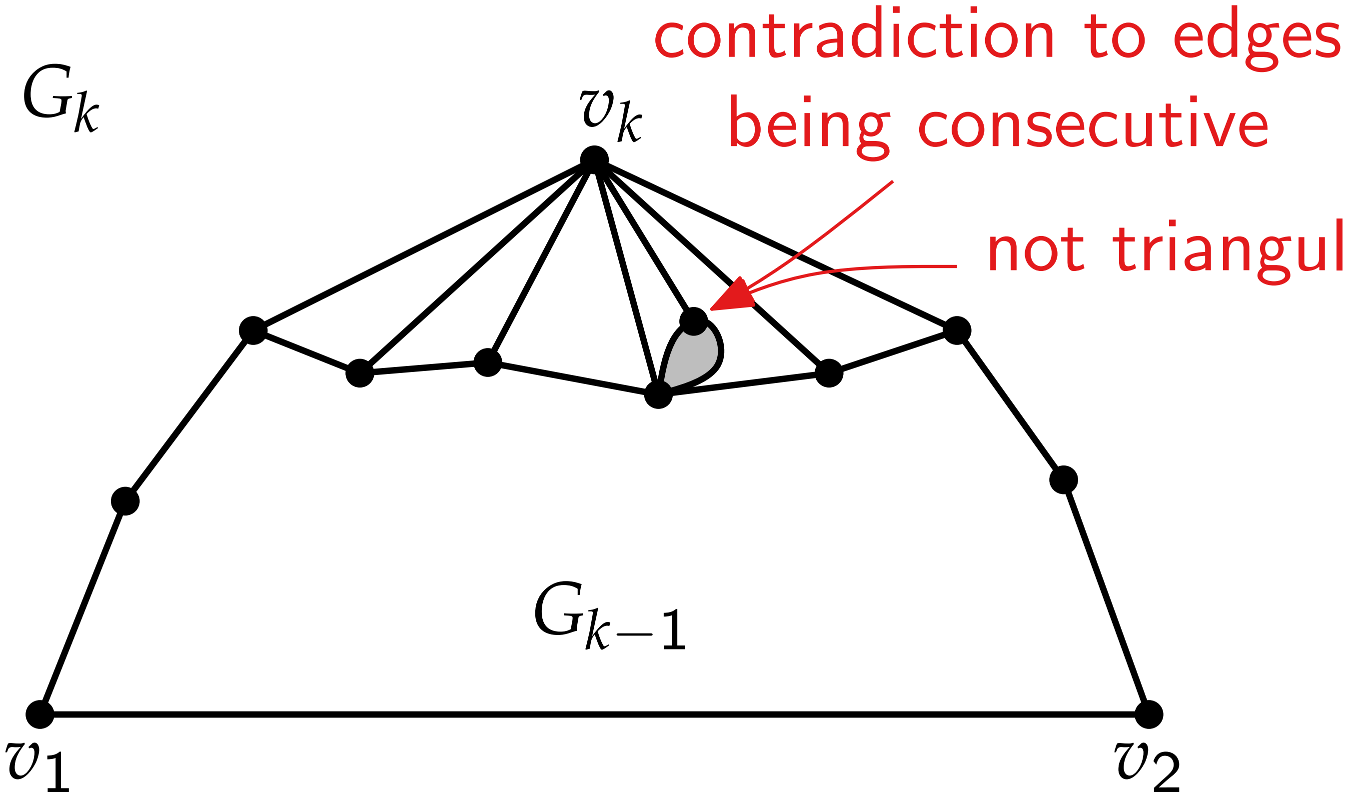

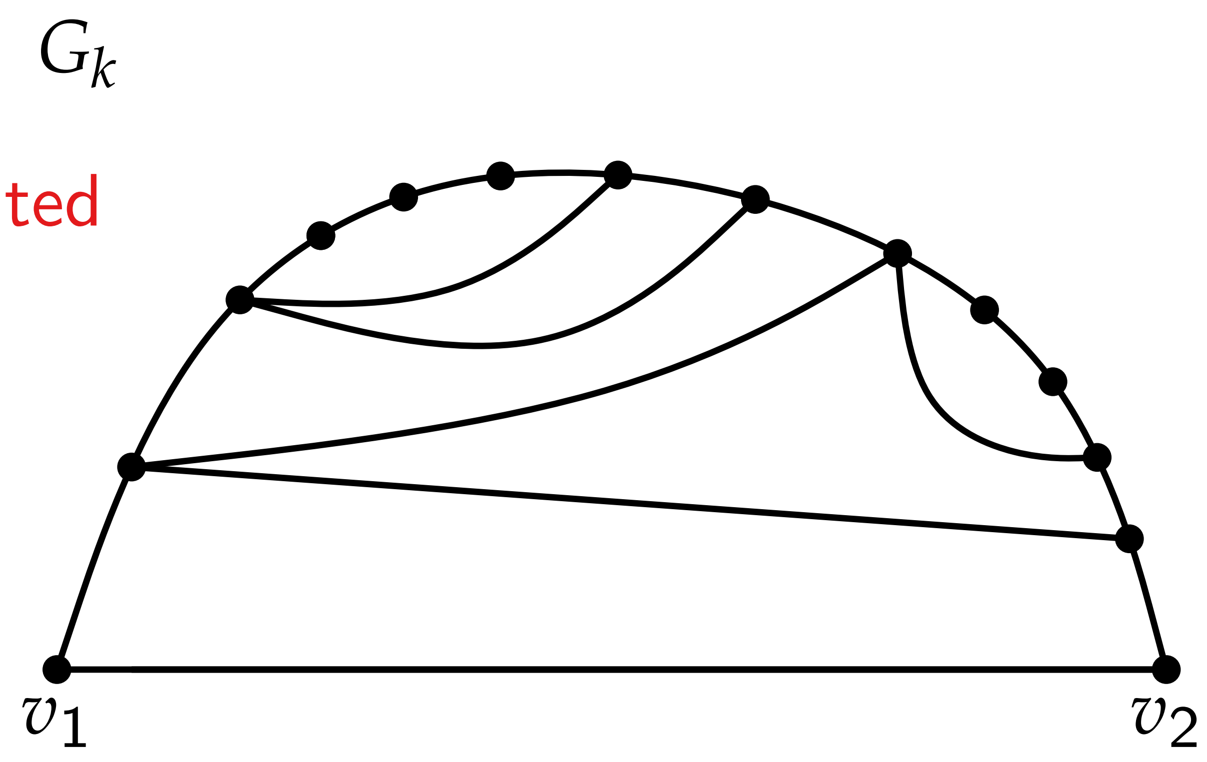

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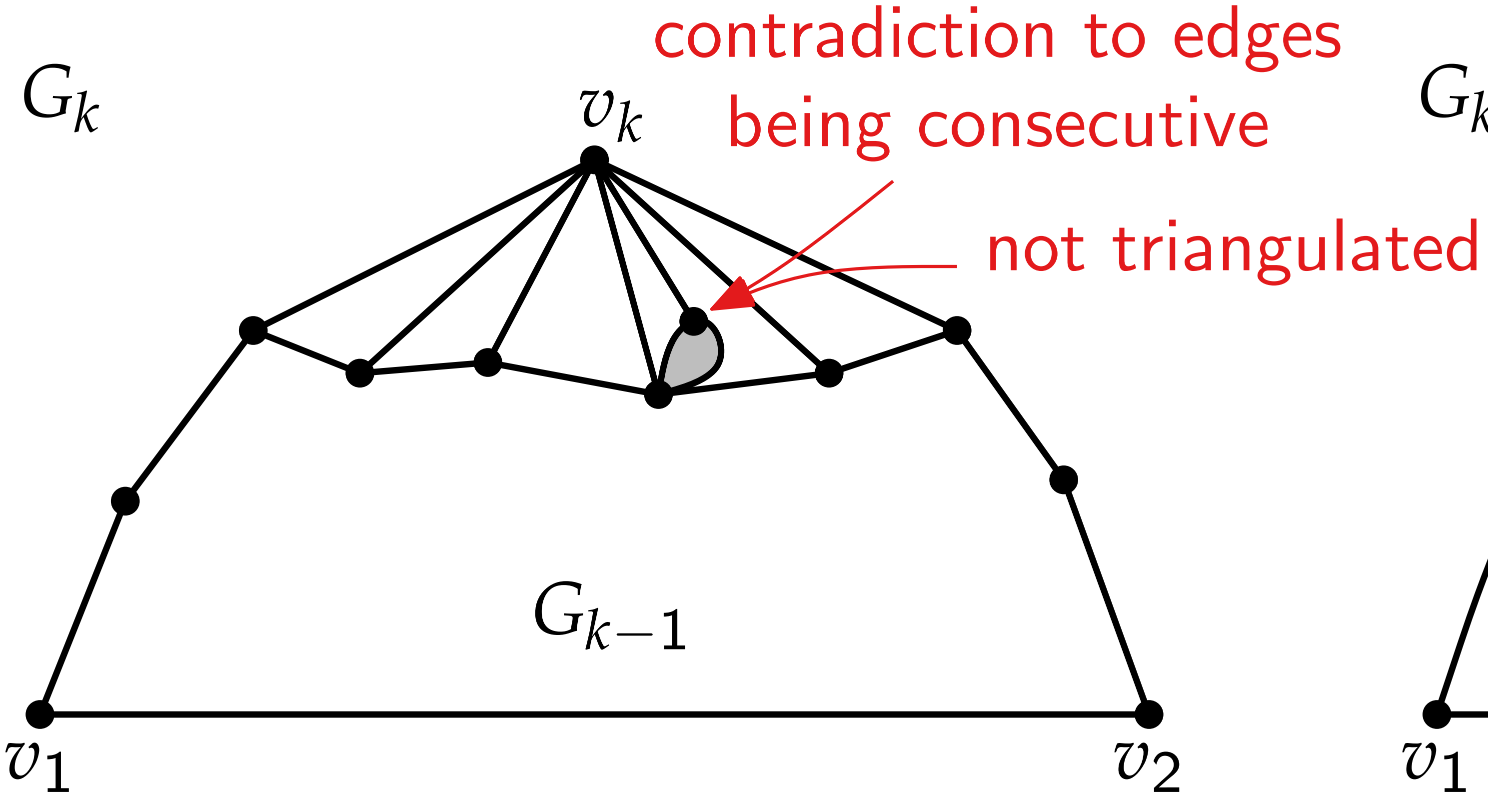

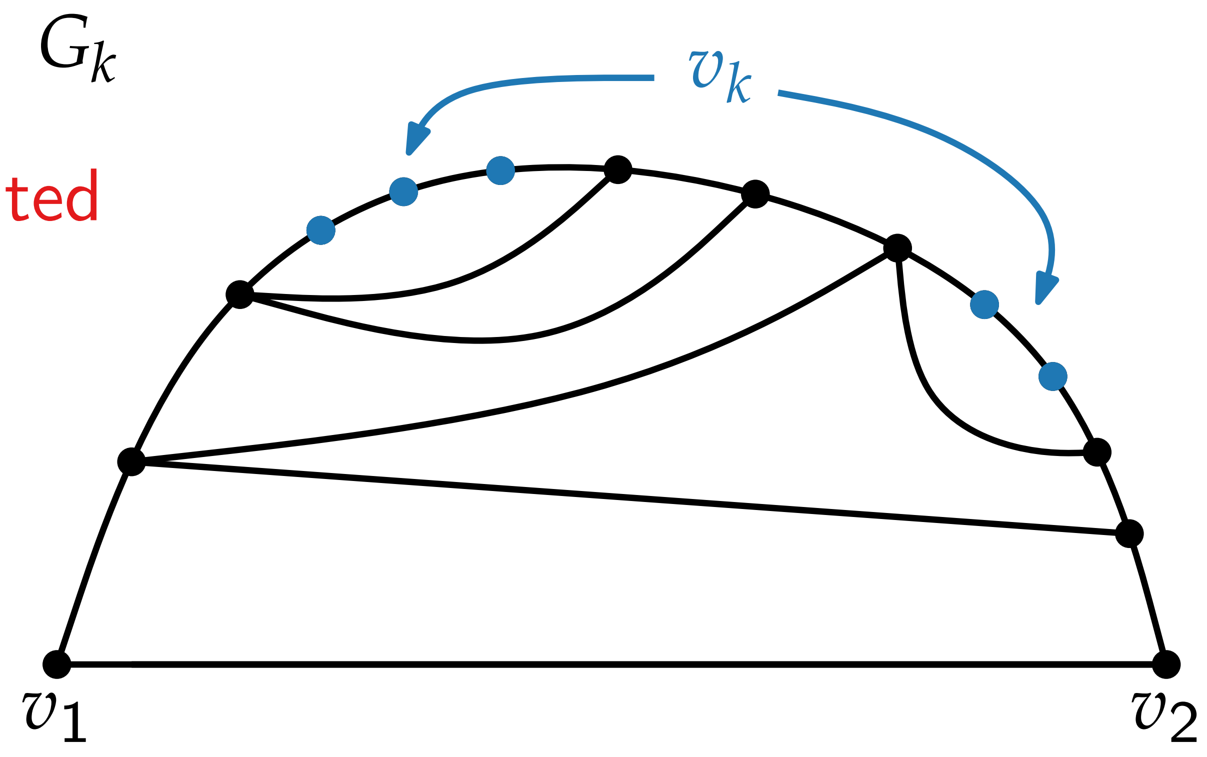

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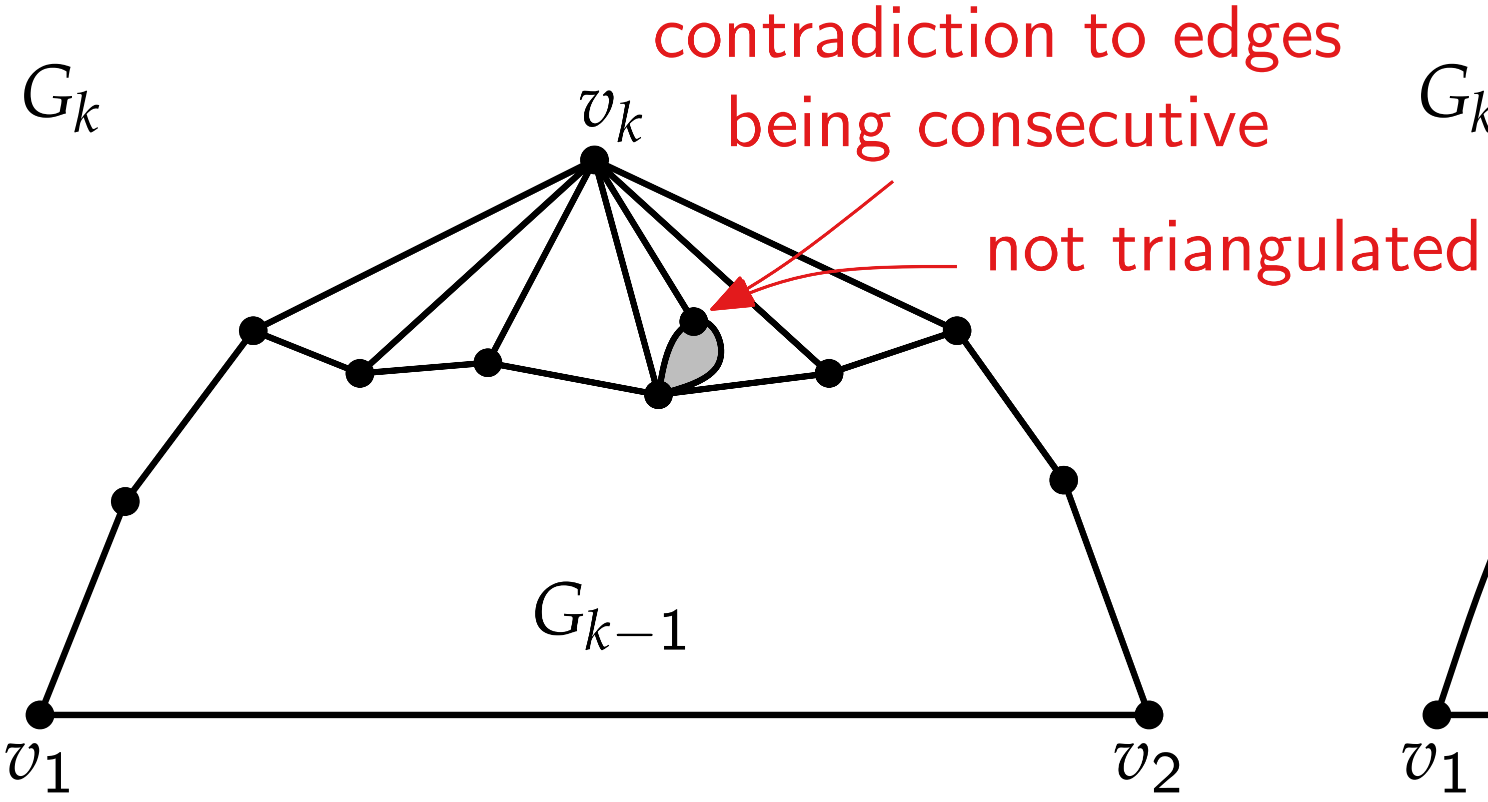

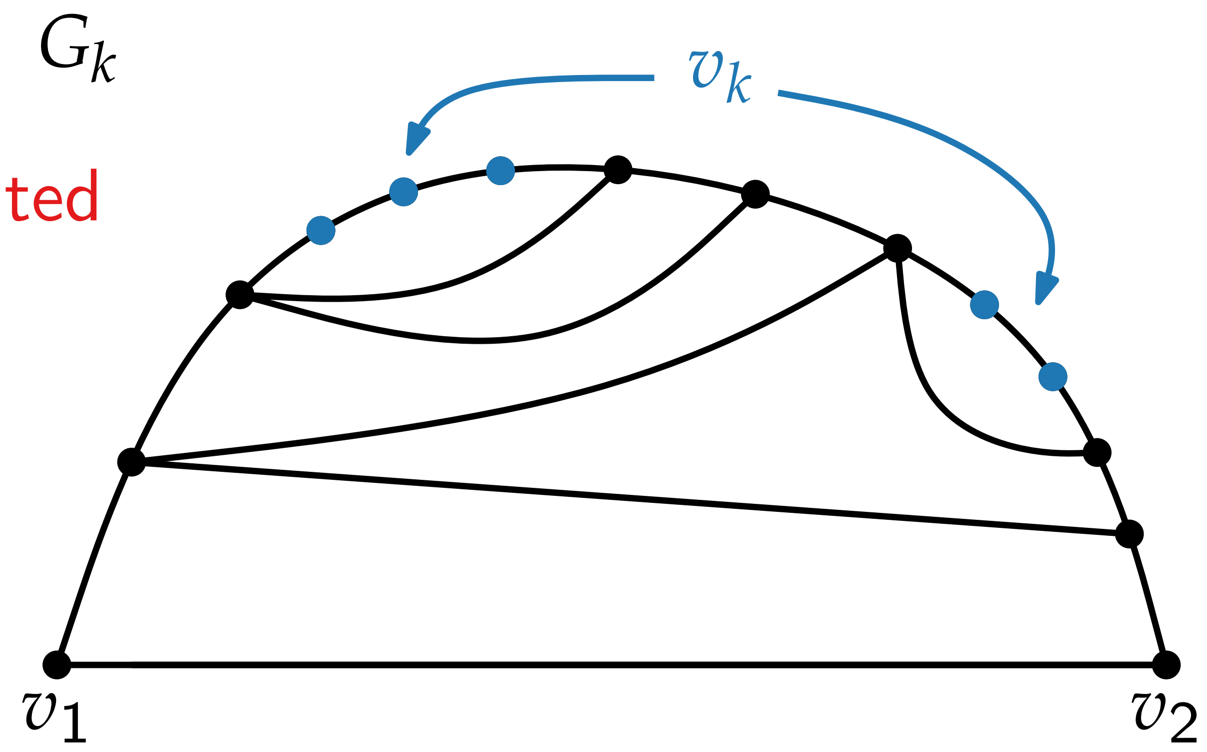

This completes proof of Lemma. □

Canonical order – implementation

Algorithm CanonicalOrder

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false;

out(v_1), out(v_2), out(v_n) $\leftarrow$ true

for $k = n$ **to** 3 **do**
  choose $v \neq v_1, v_2$ such that mark(v) $=$ false,
   out(v) $=$ true, and chords(v) $= 0$
  $v_k \leftarrow v$; mark(v) $\leftarrow$ true
  // Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{k-1} and let $w_p, \ldots, w_q$ be the unmarked neighbors of v_k
  out(w_i) $\leftarrow$ true for all $p < i < q$
  update number of chords for w_i and its neighbours

- chord(v) – # chords adjacent to v
- mark(v) $=$ true iff vertex v was numbered
- out(v) $=$ true iff v is currently outer vertex

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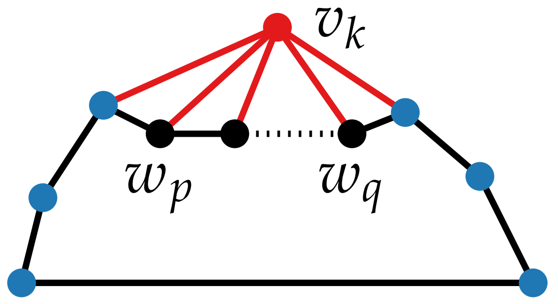

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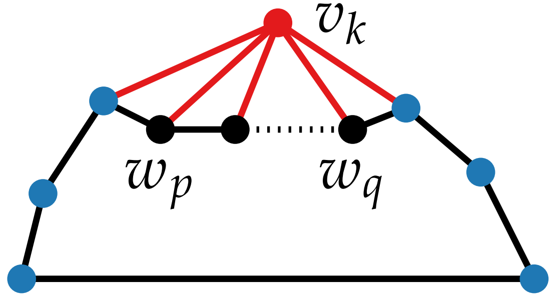

Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

Shift method

Algorithm invariants/constraints:

G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-4,0)$,
- boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone,
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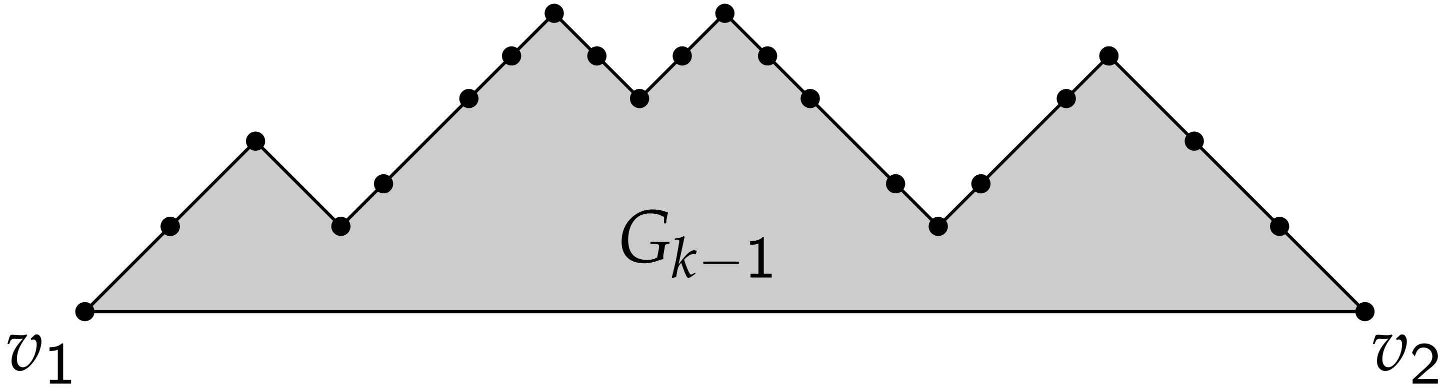

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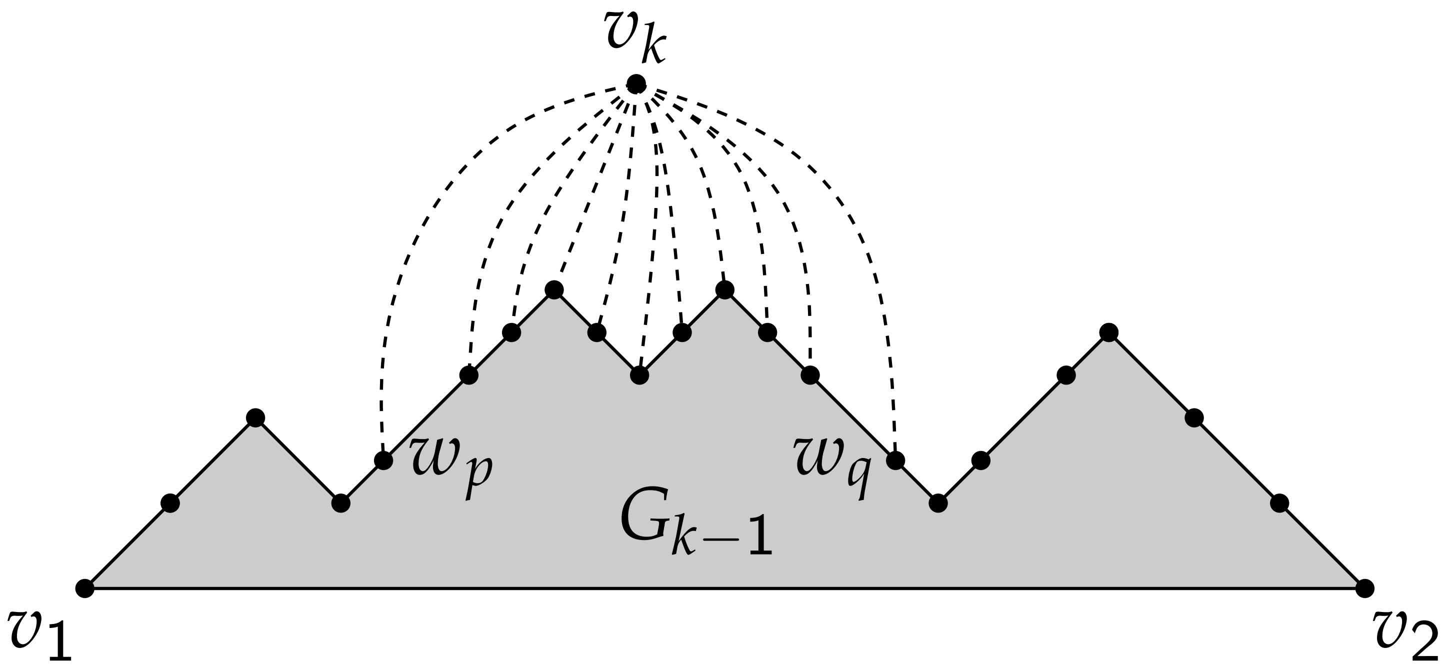

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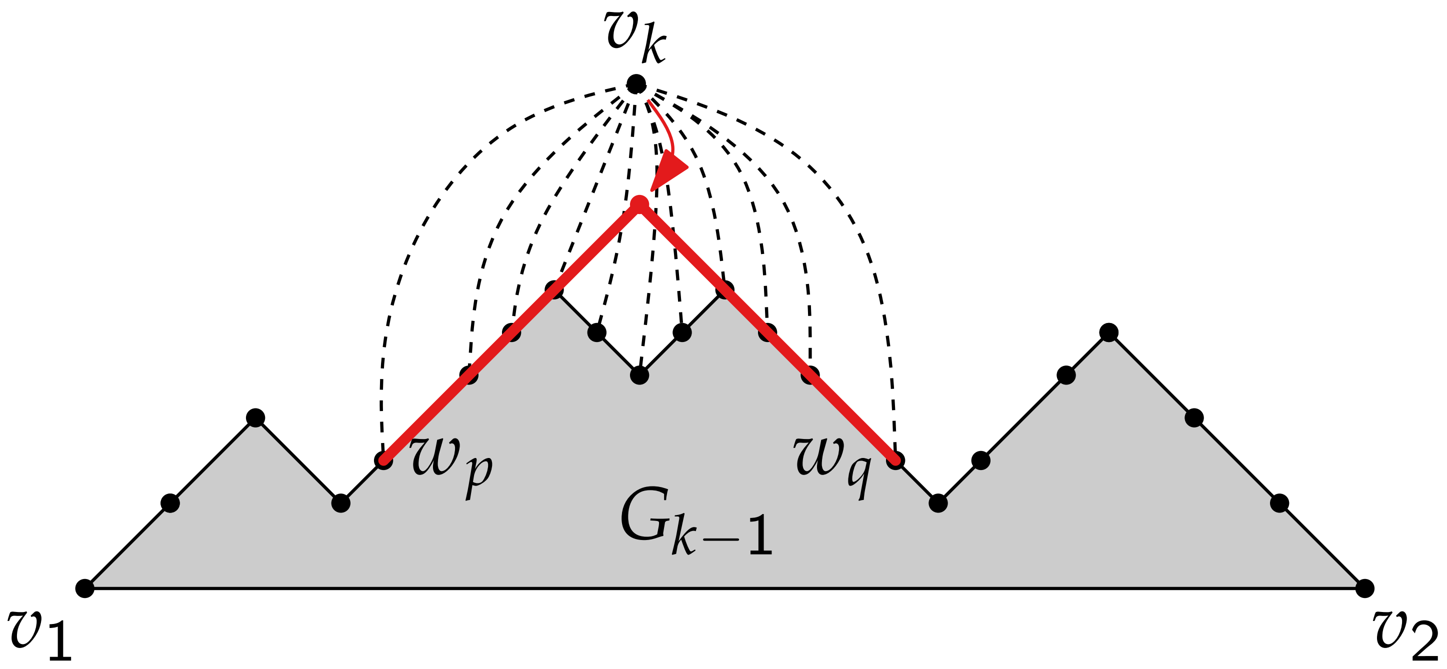

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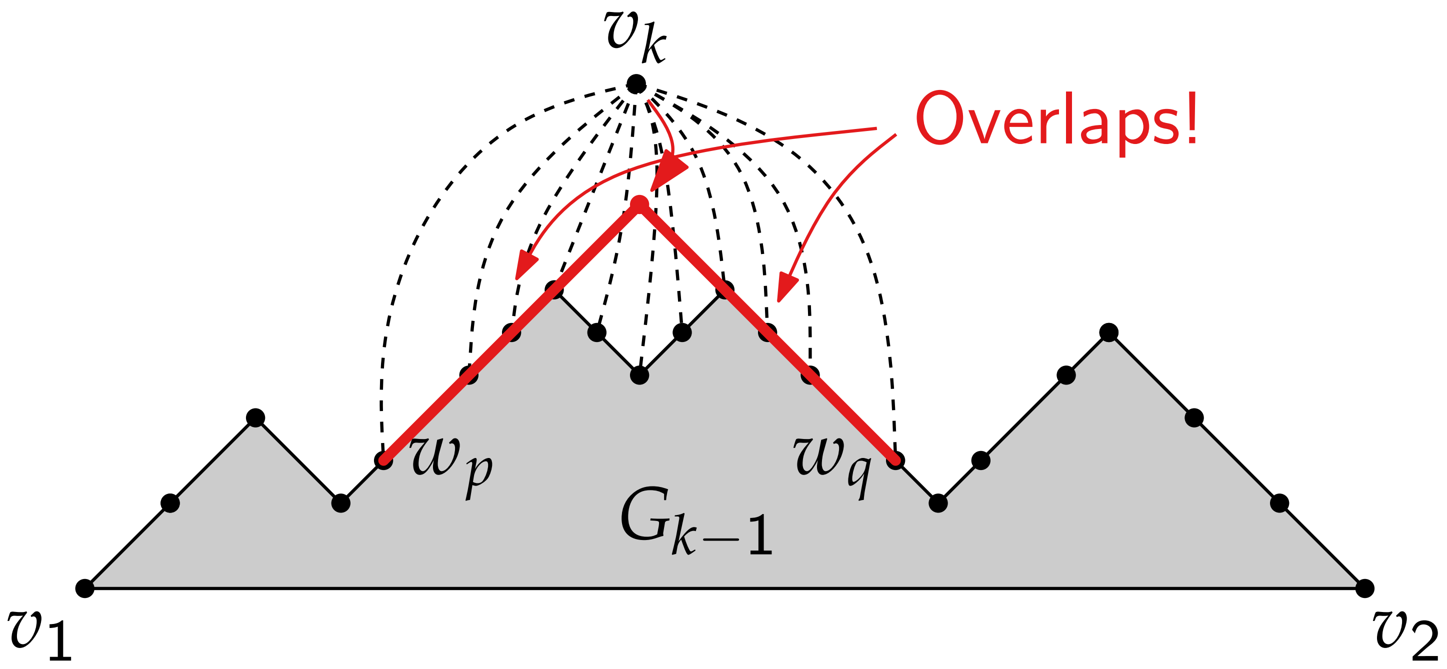

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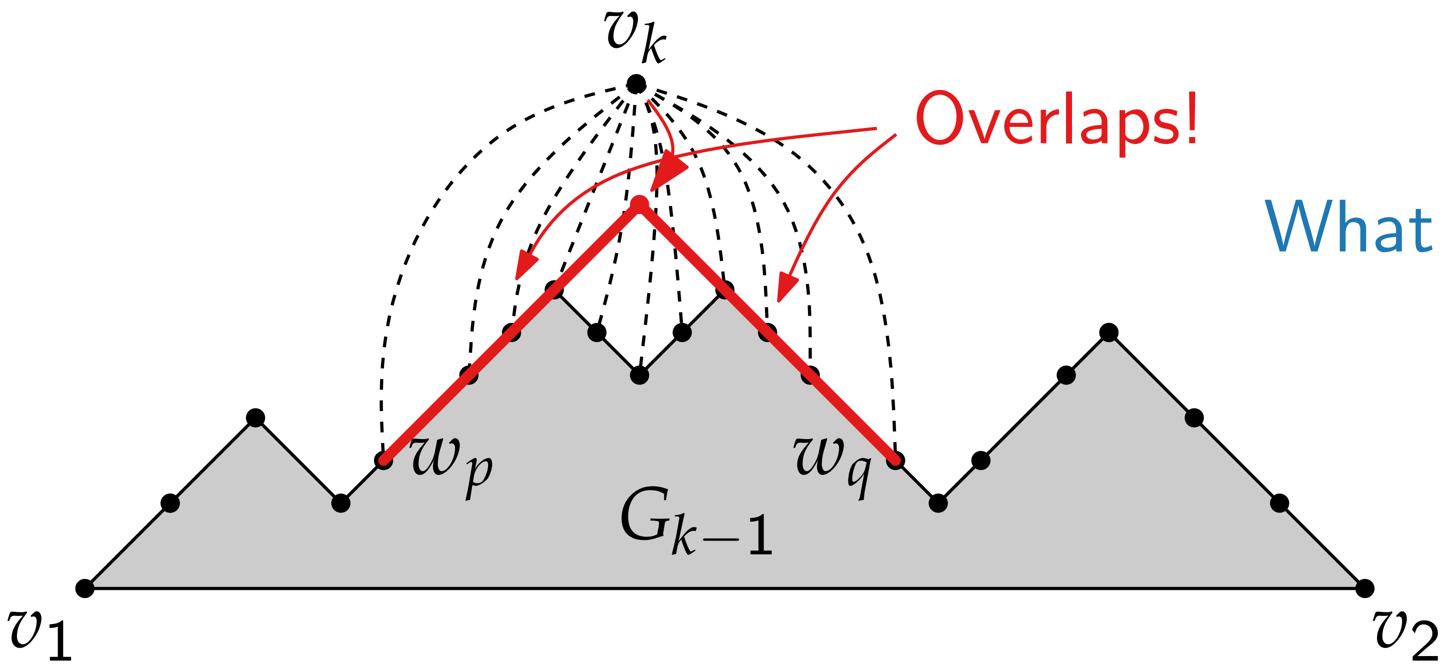

What could be the solution?

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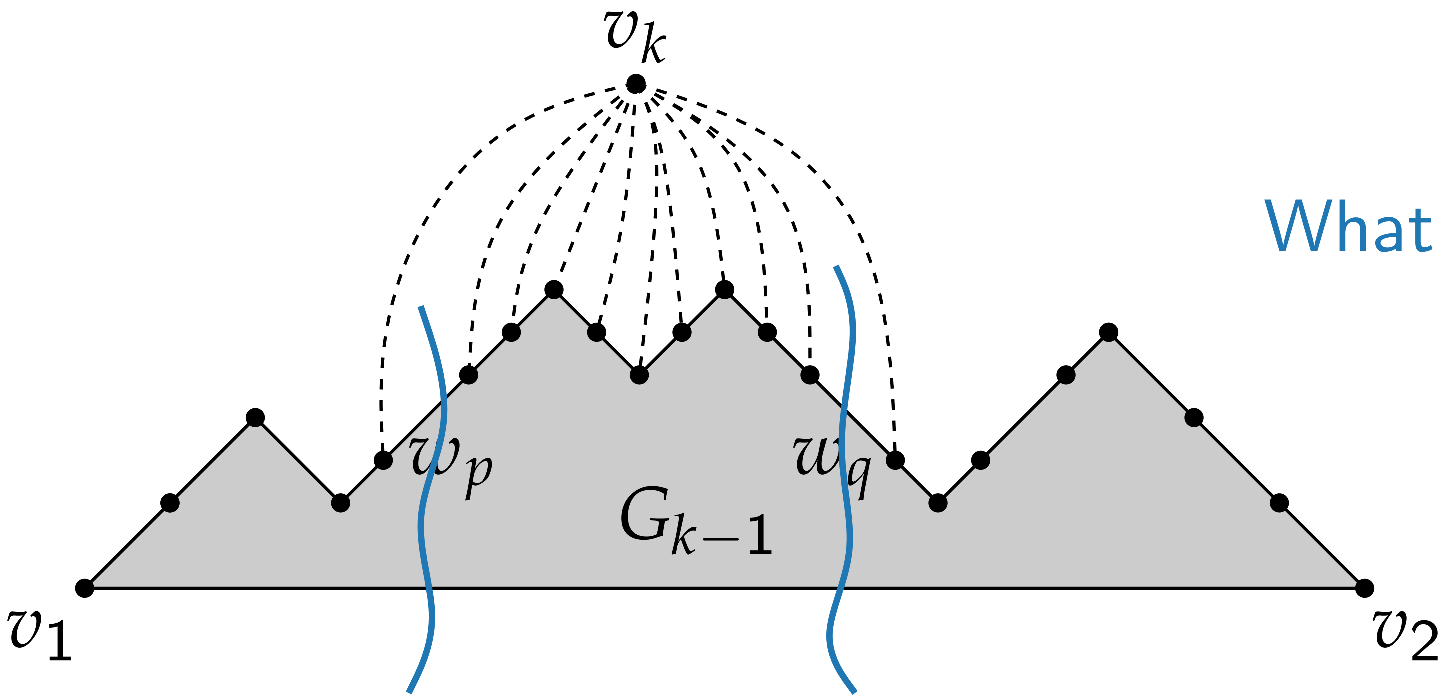

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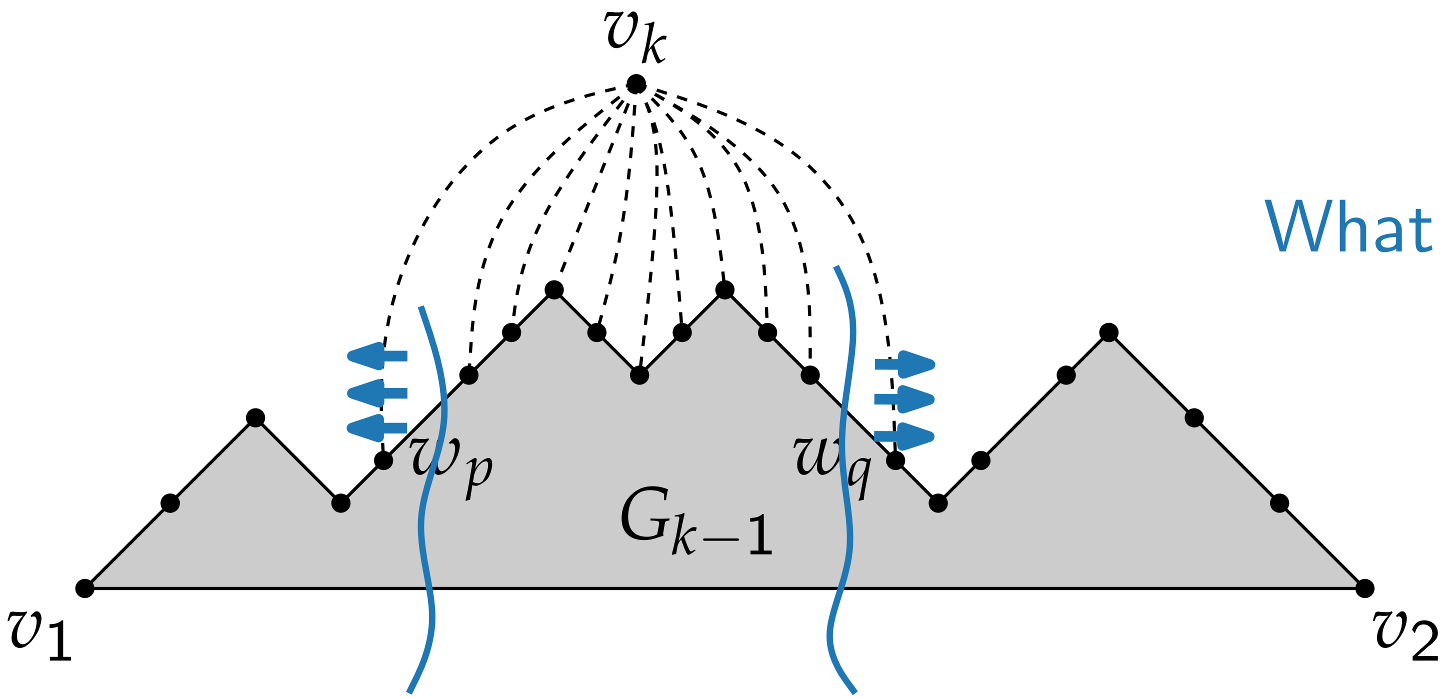

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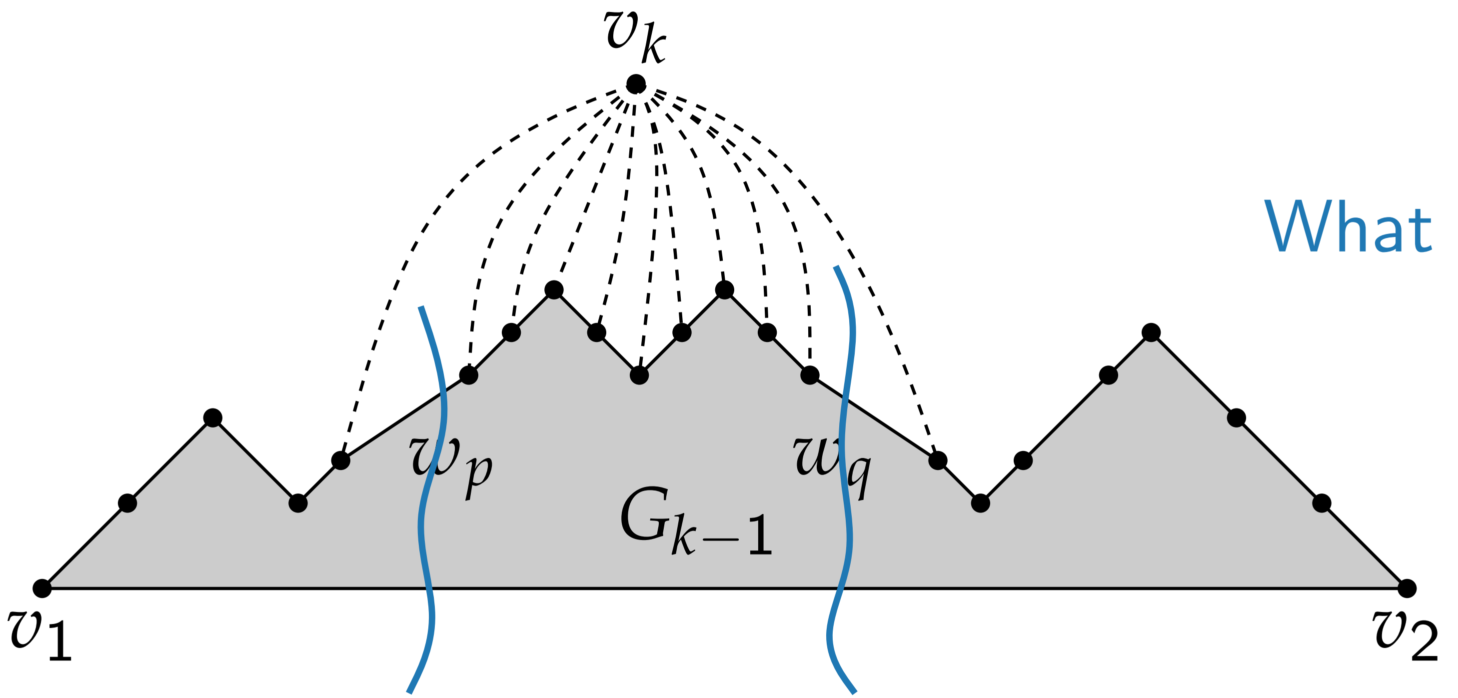

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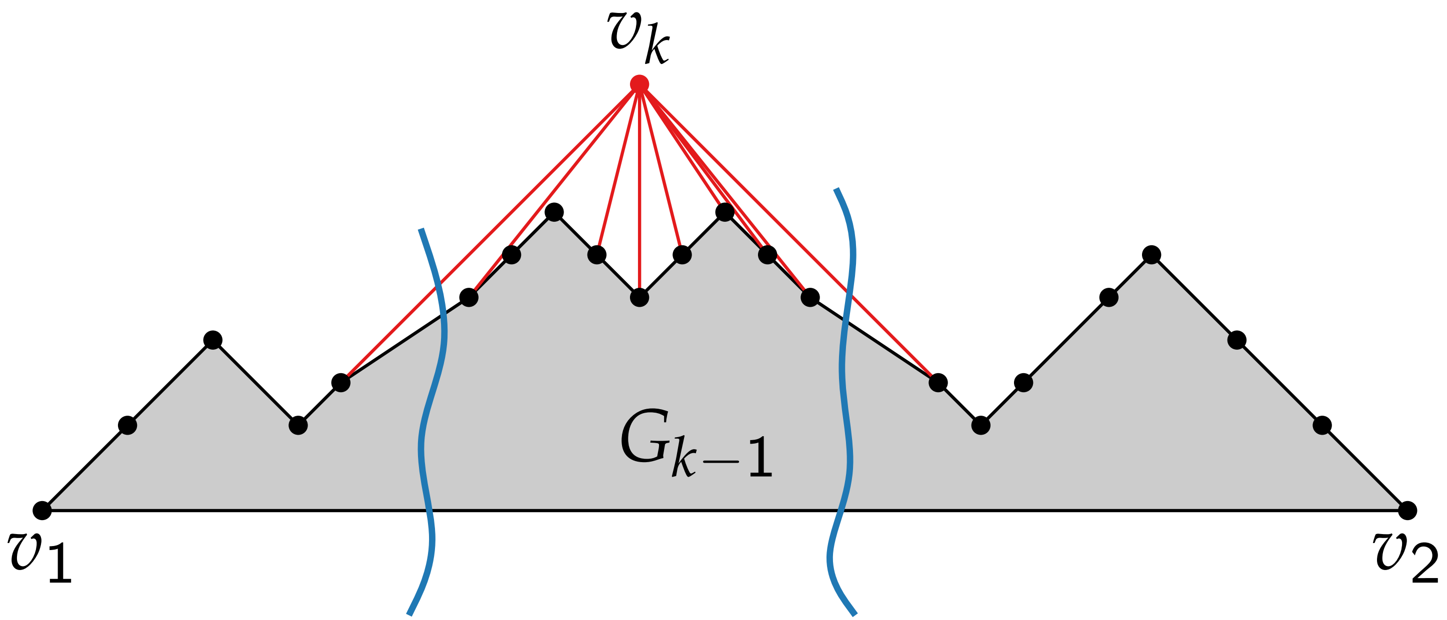

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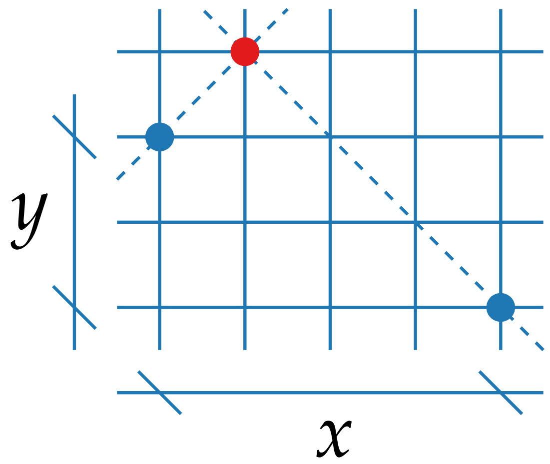

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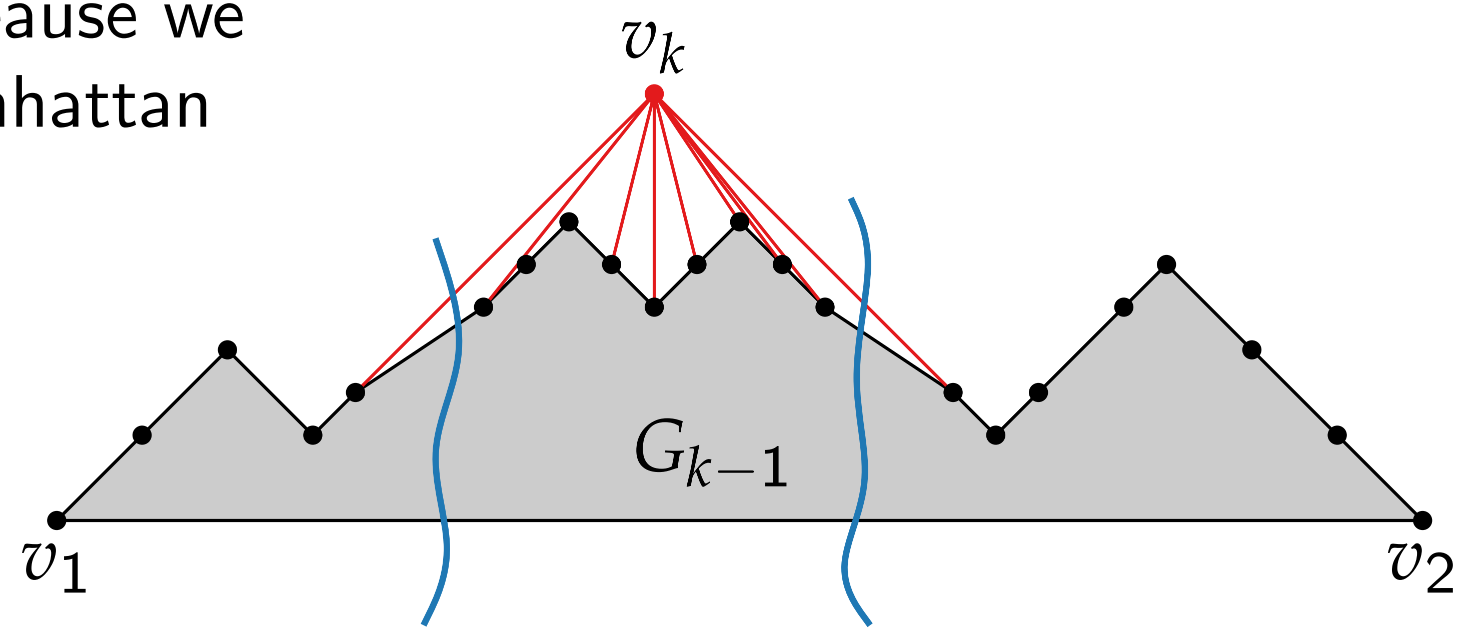
- v_k on grid, beause we had even Manhattan distance



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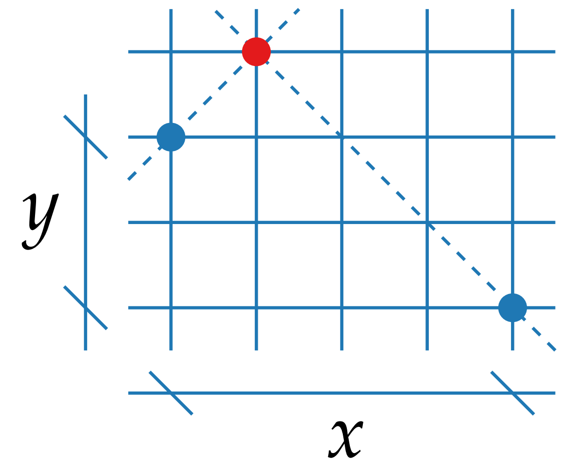

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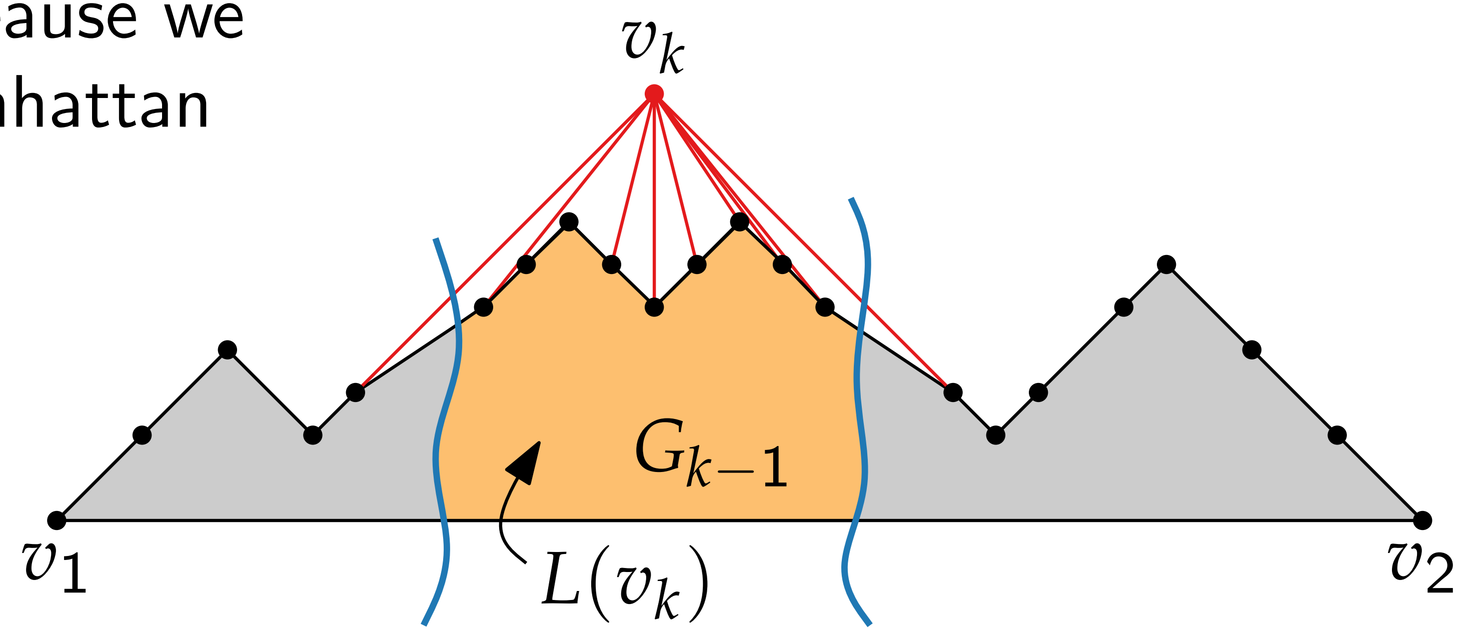
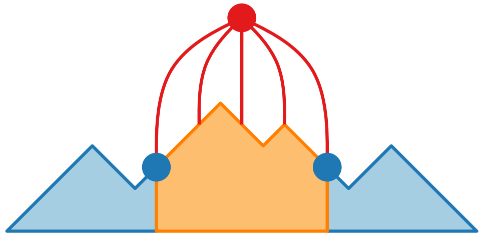
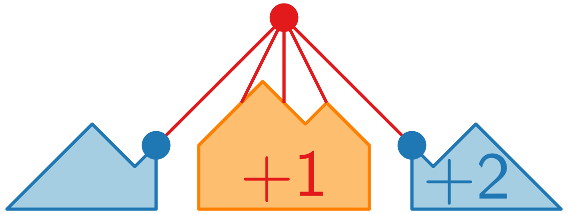
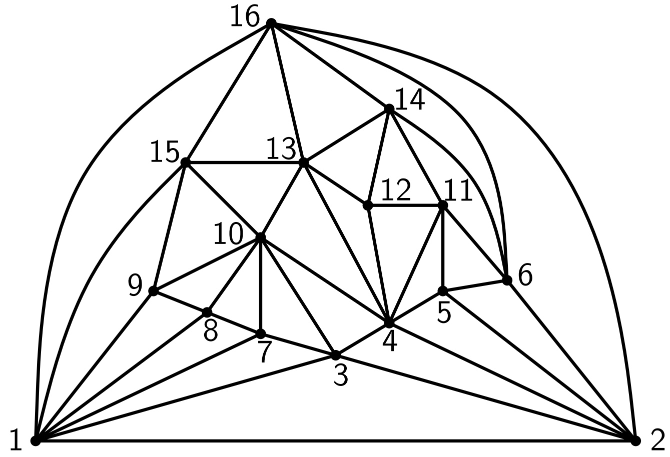


Shift method – example







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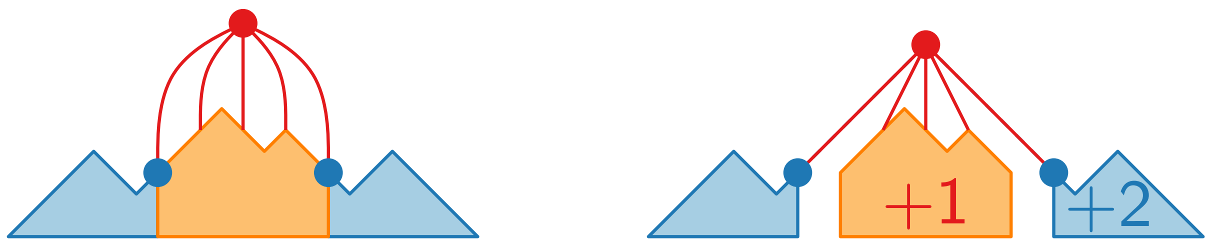

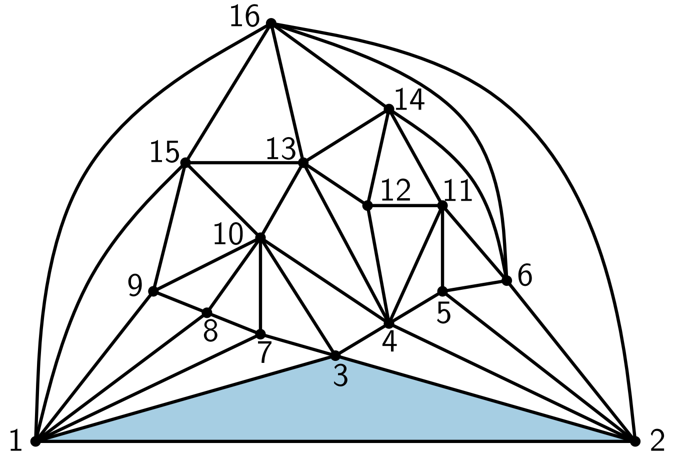

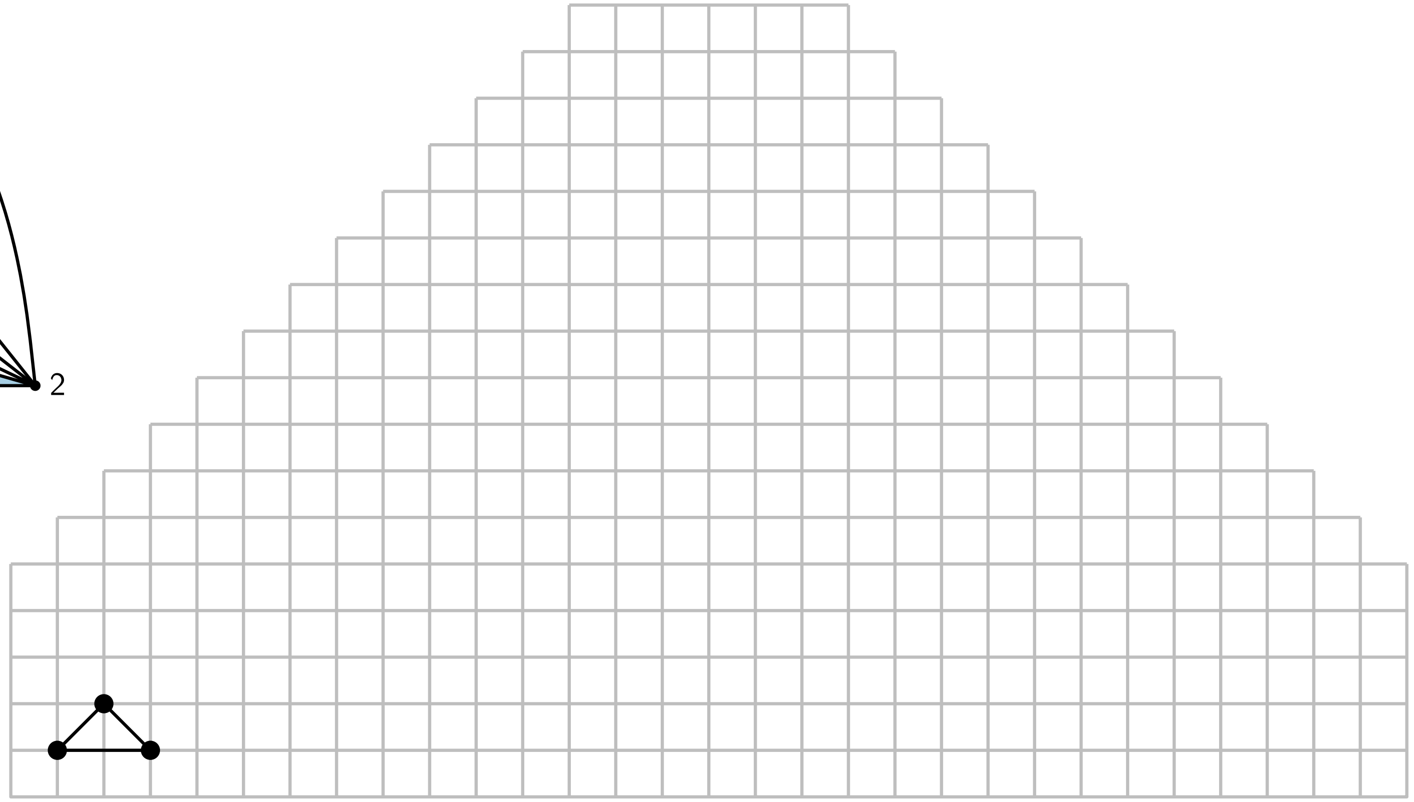

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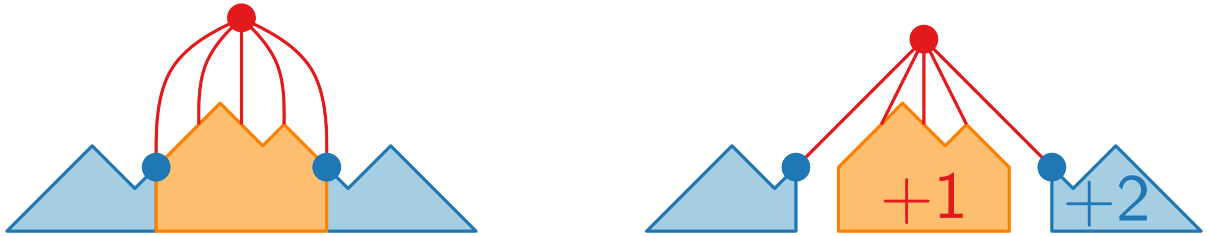

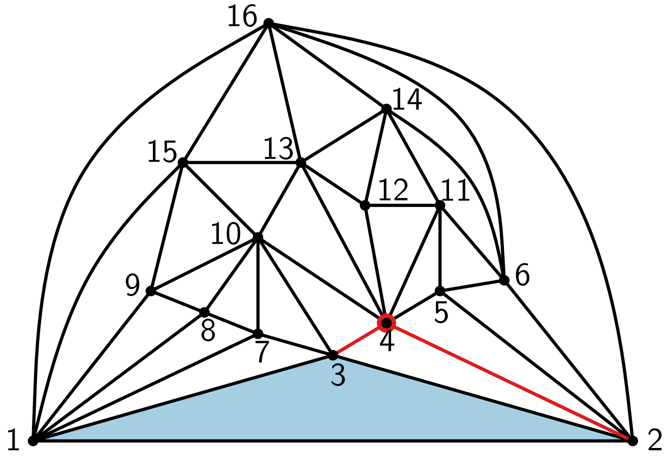

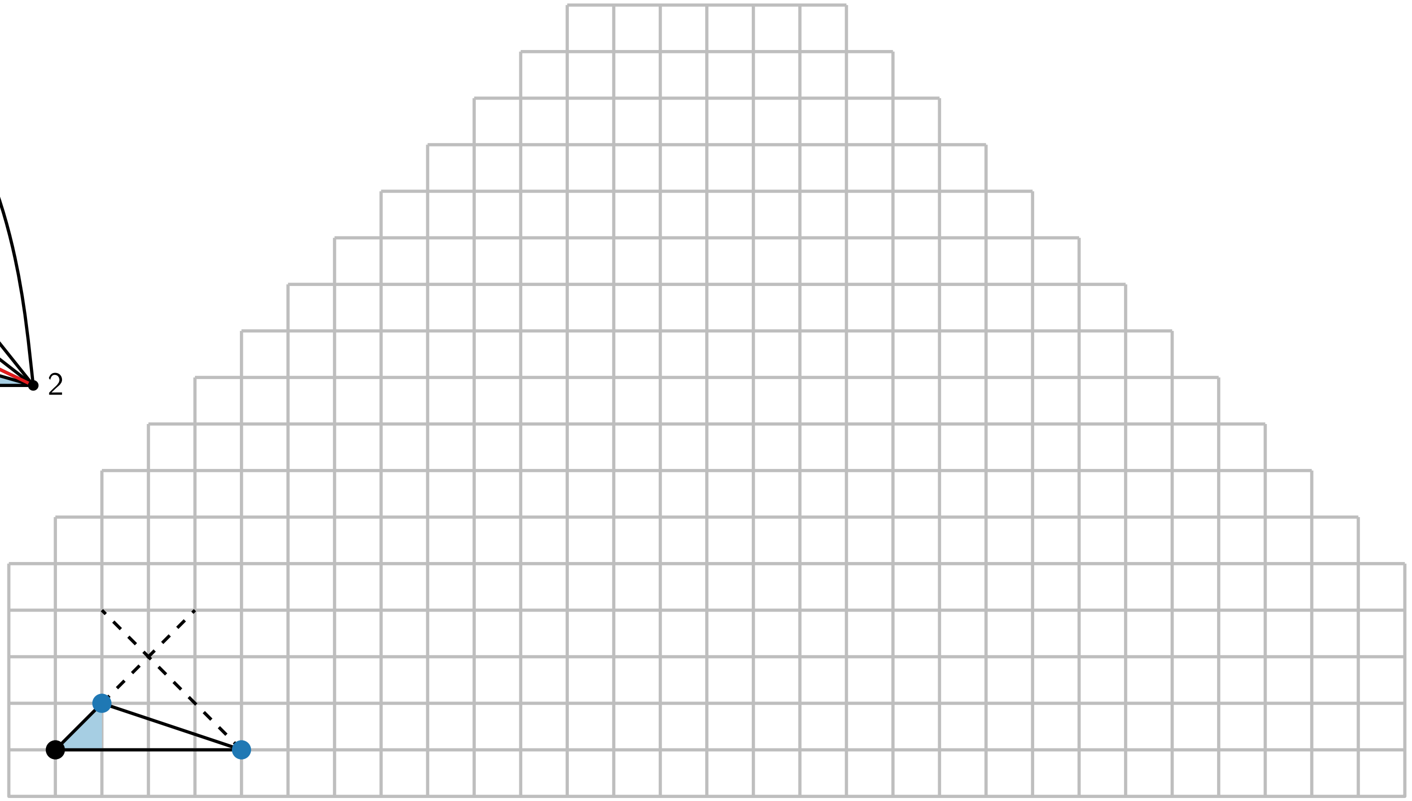

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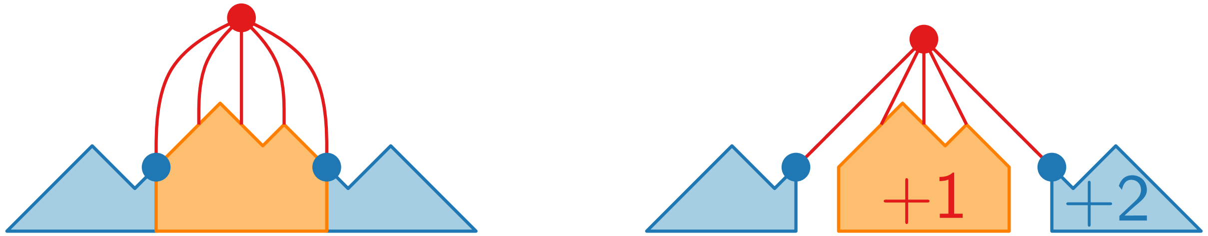

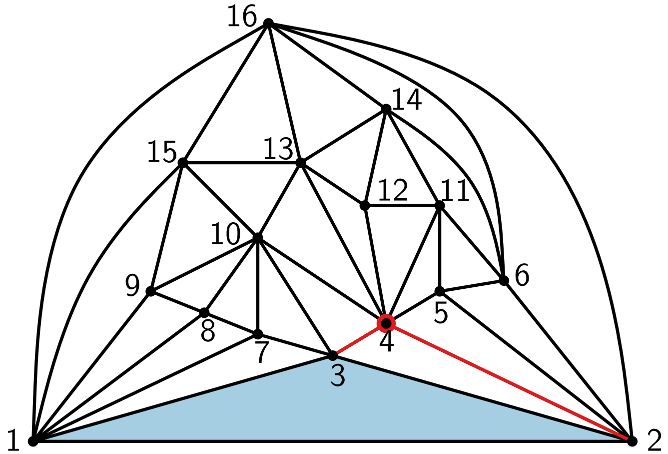

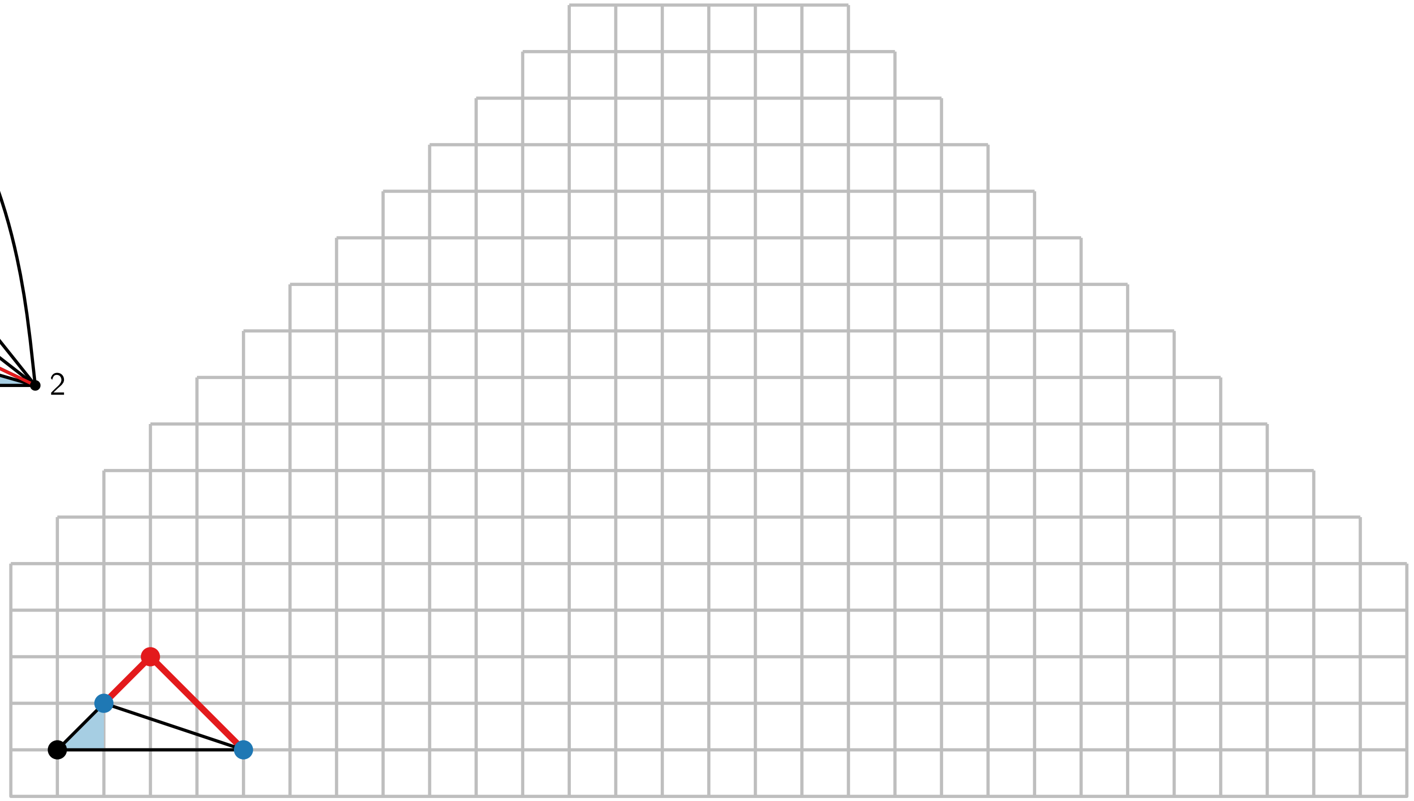

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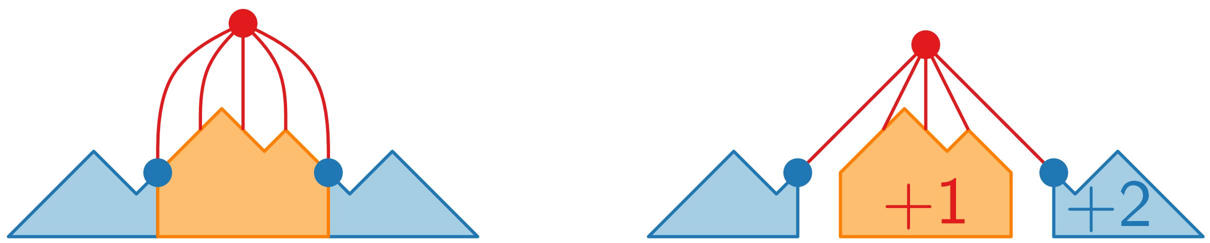

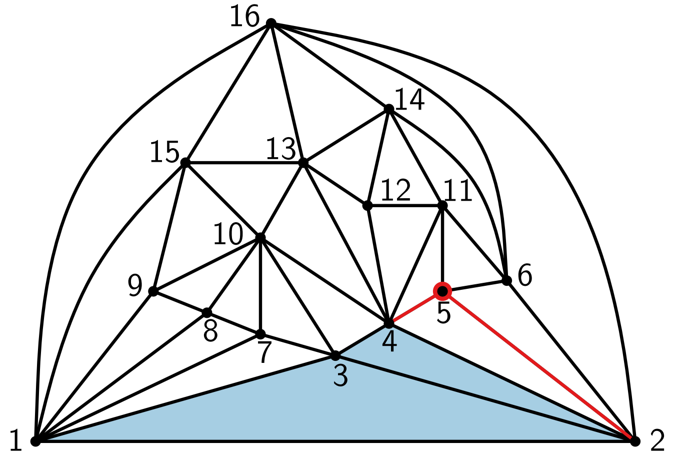

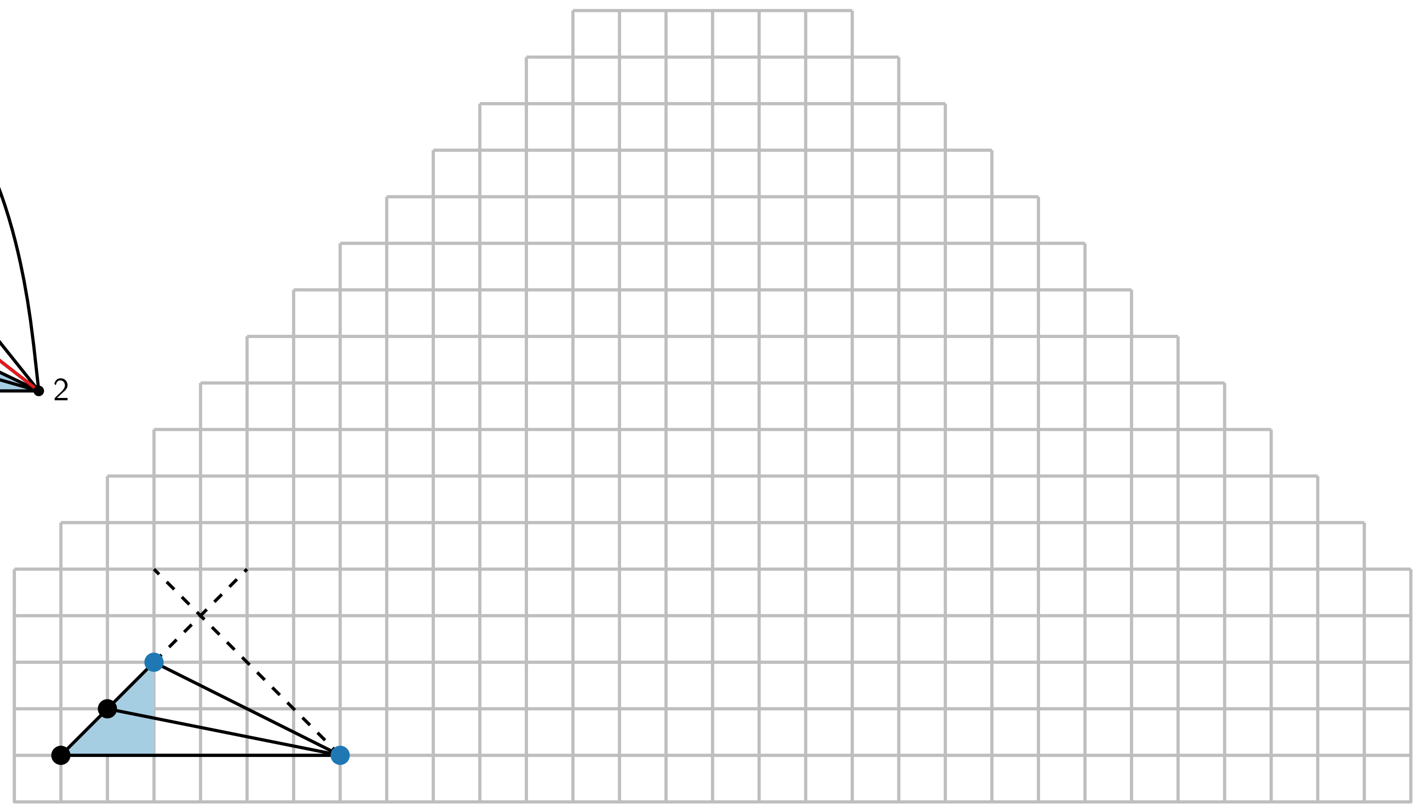

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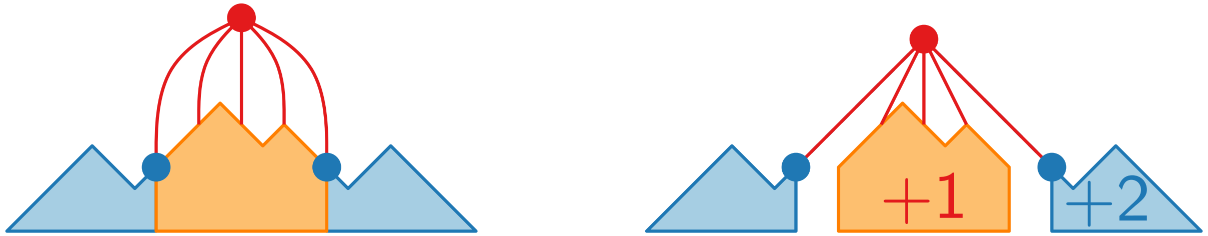

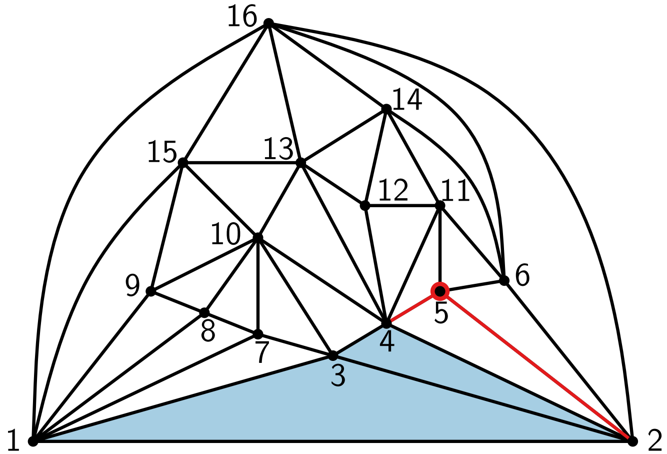

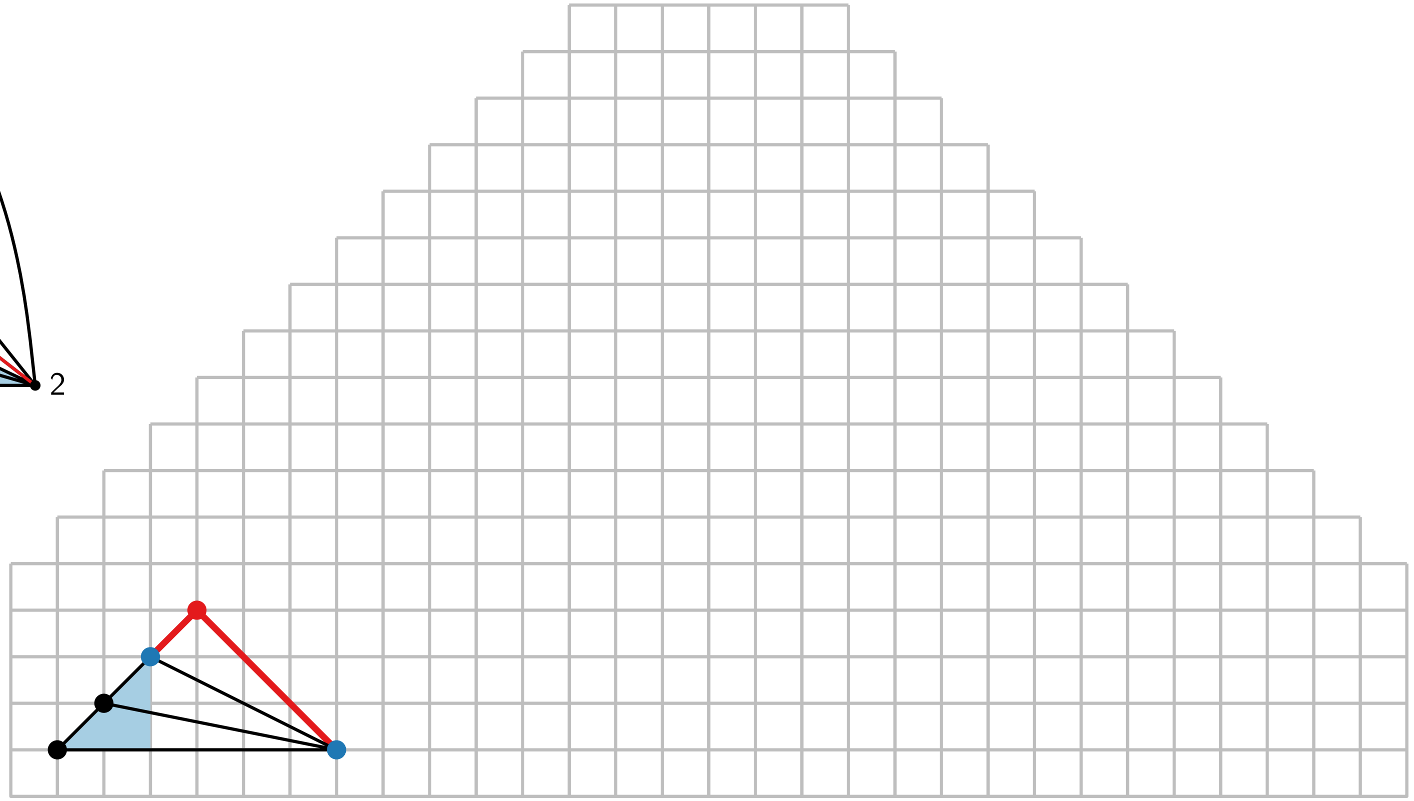

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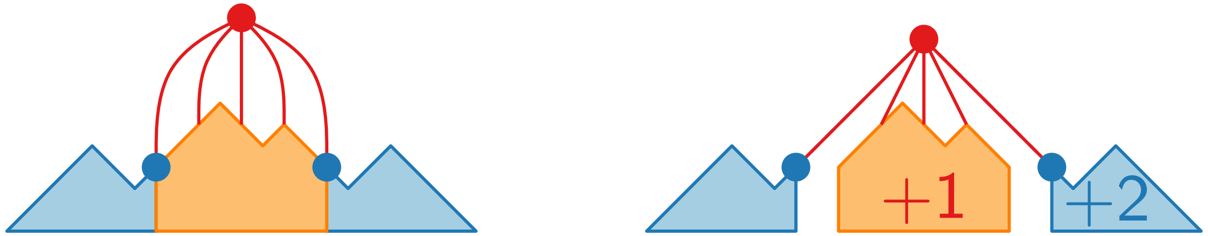

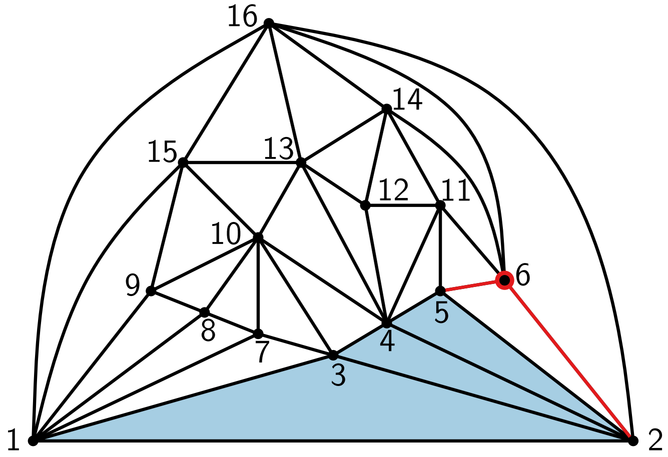

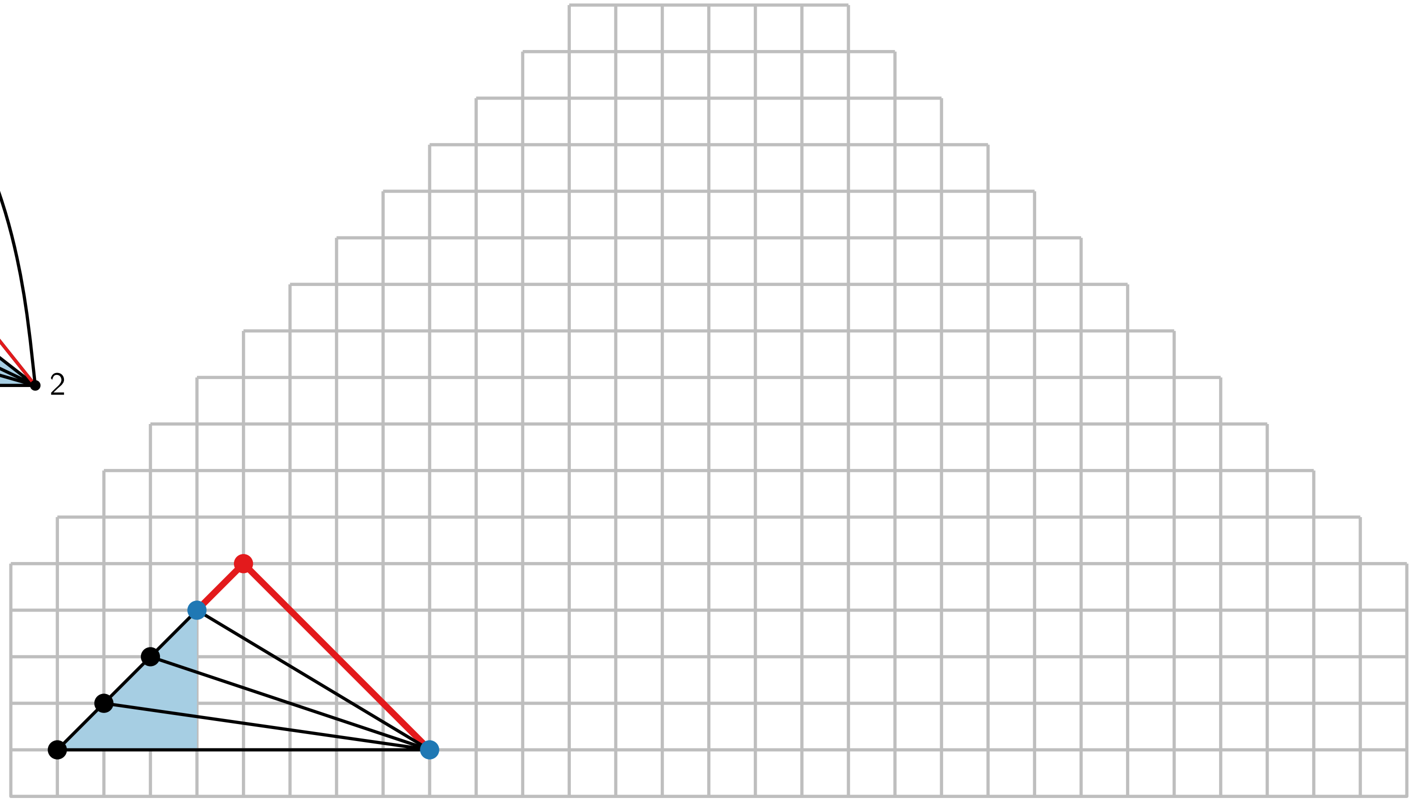

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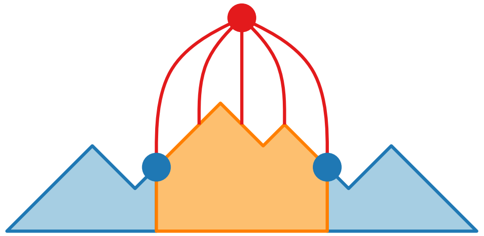

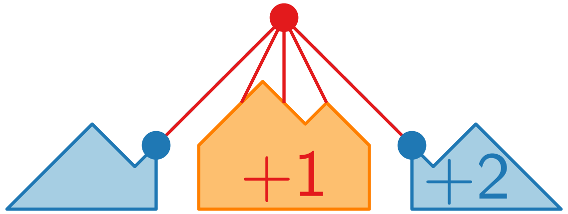

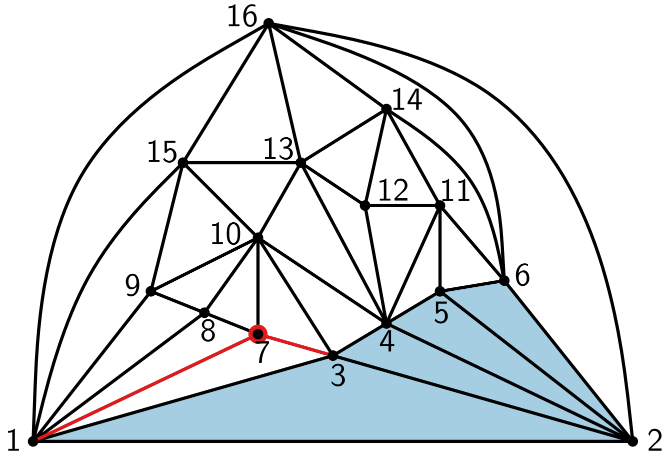

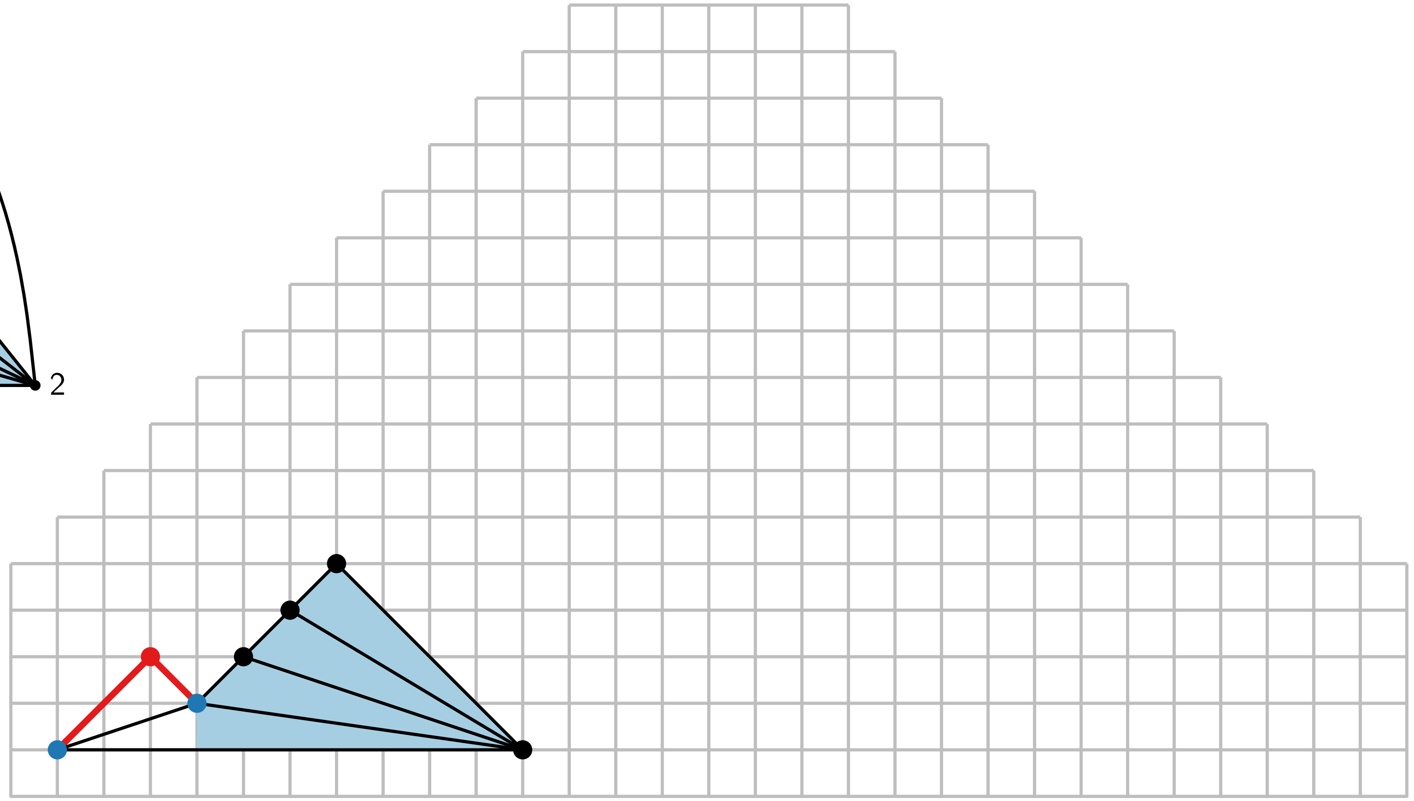

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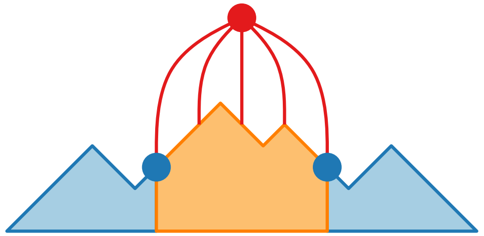

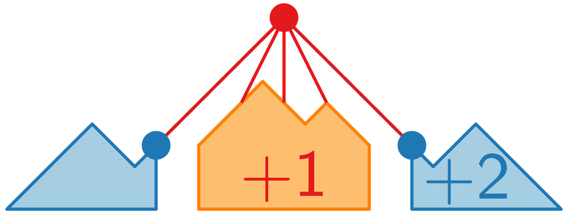

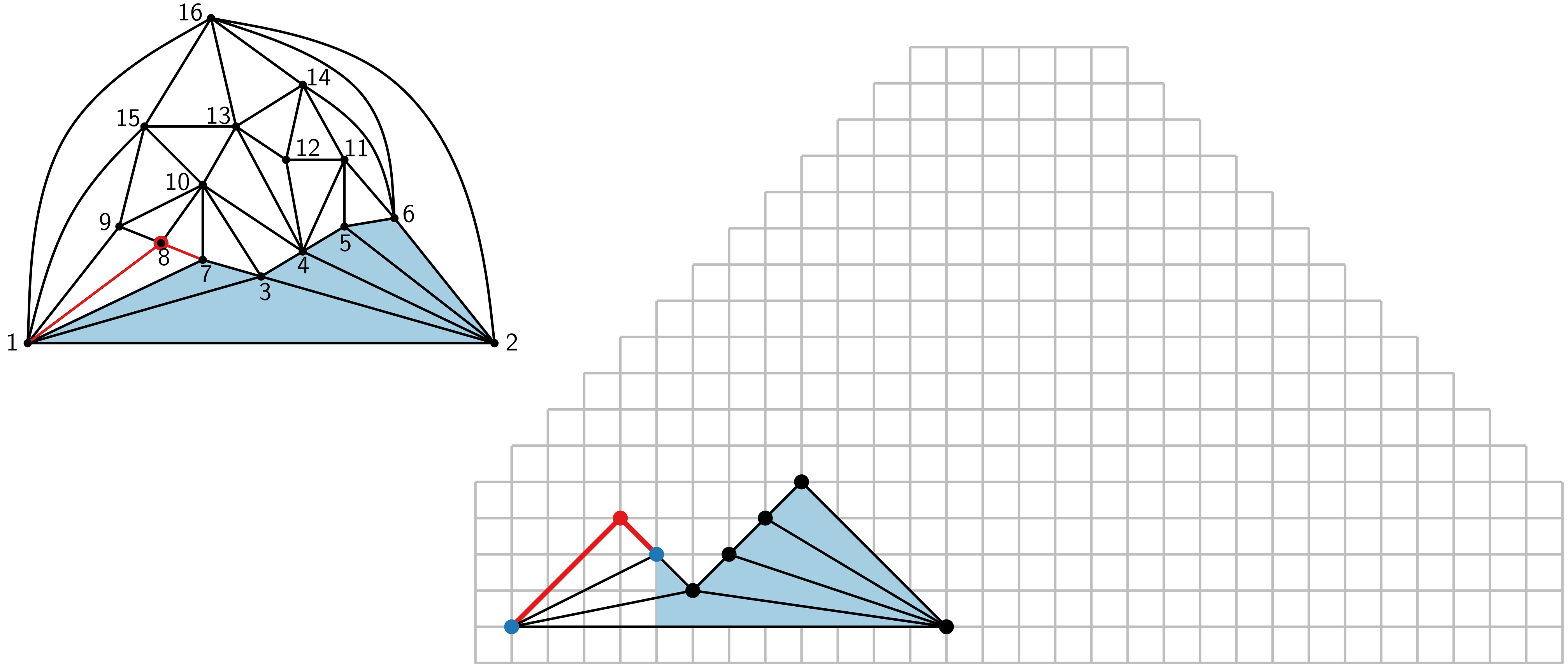

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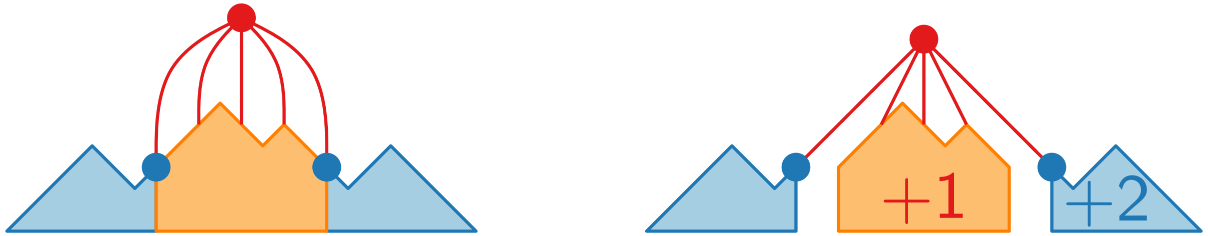

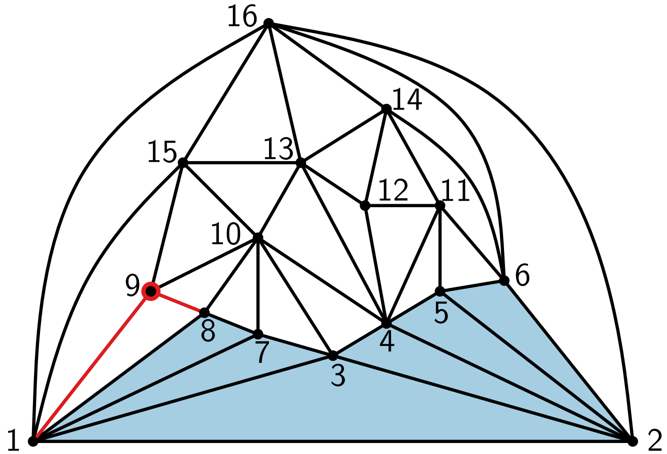

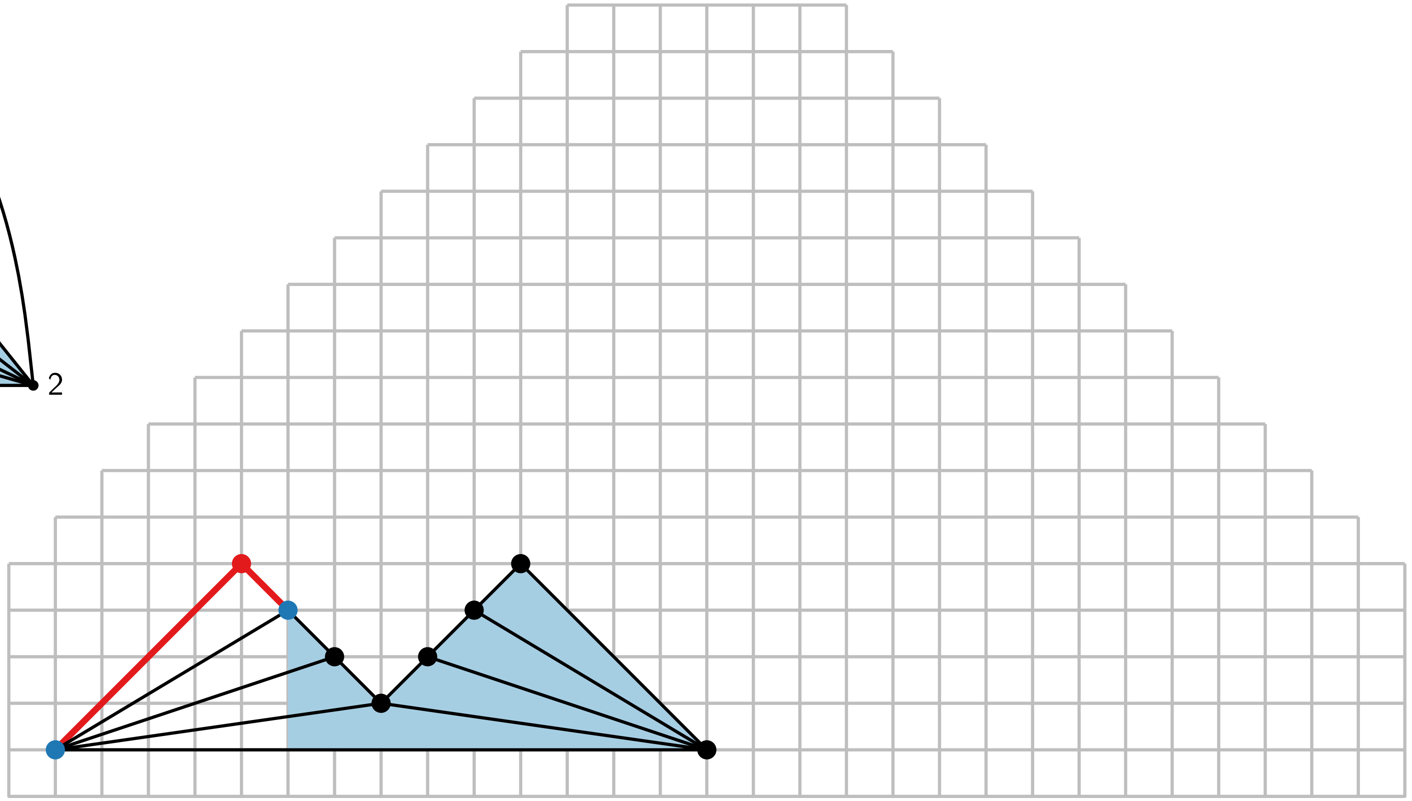

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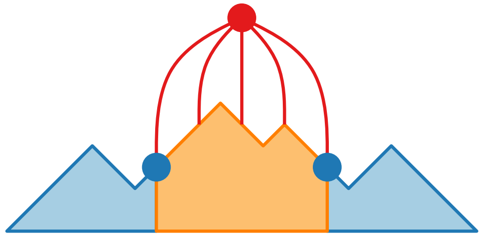

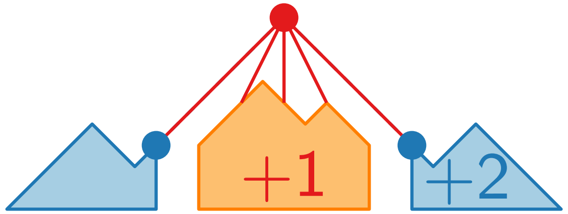

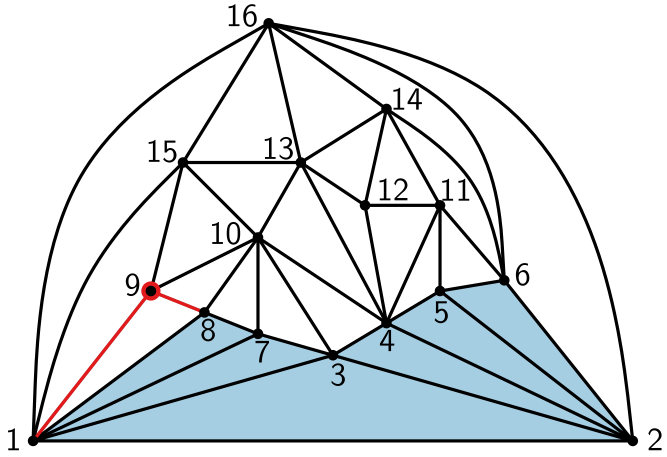

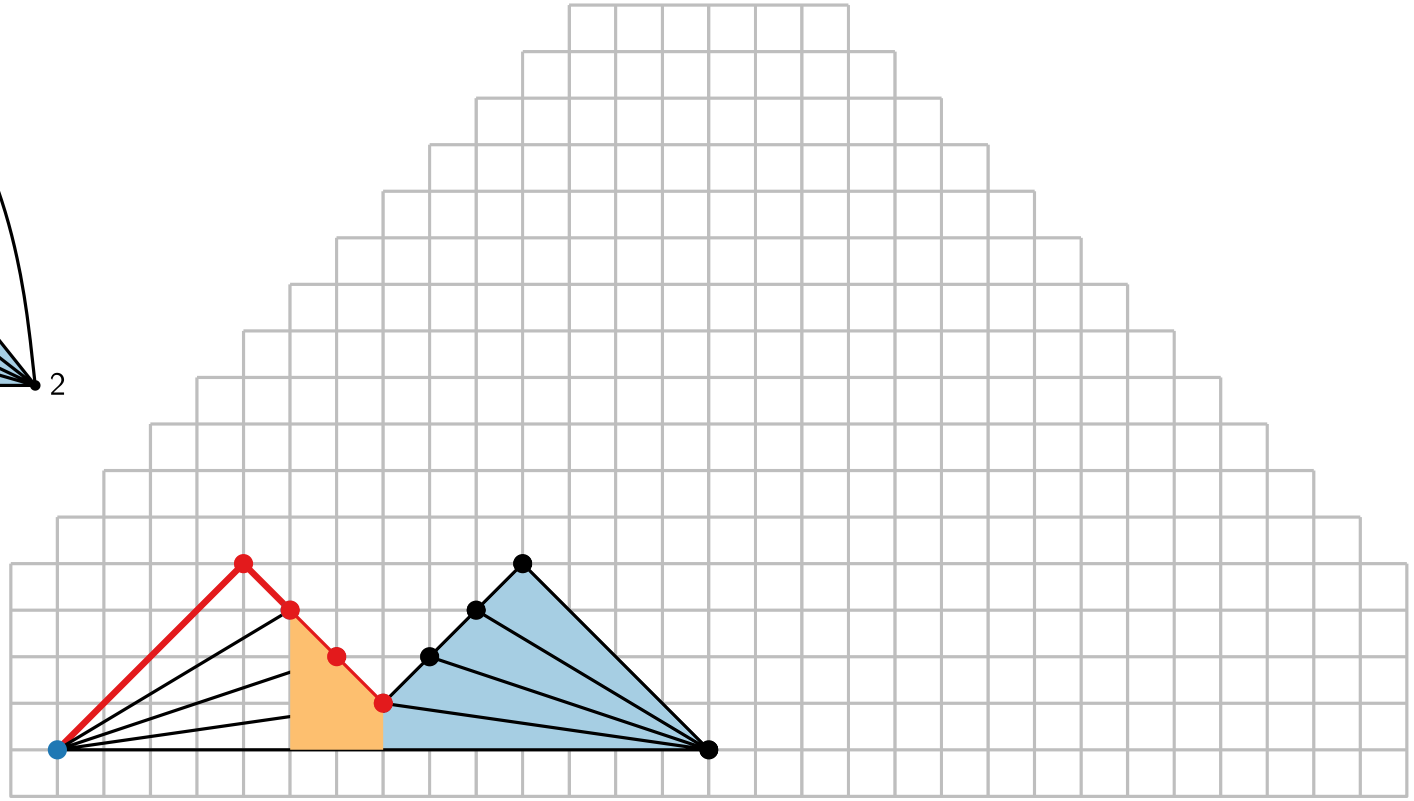

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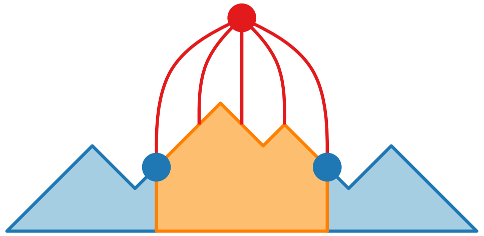

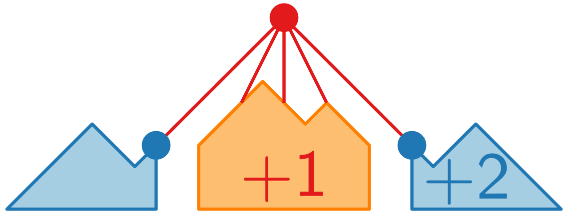

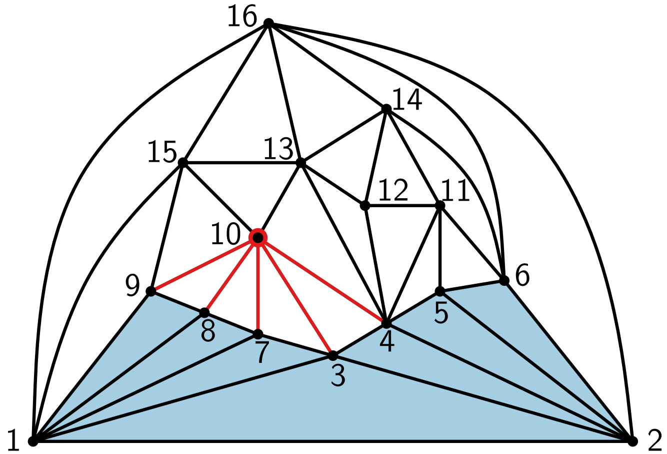

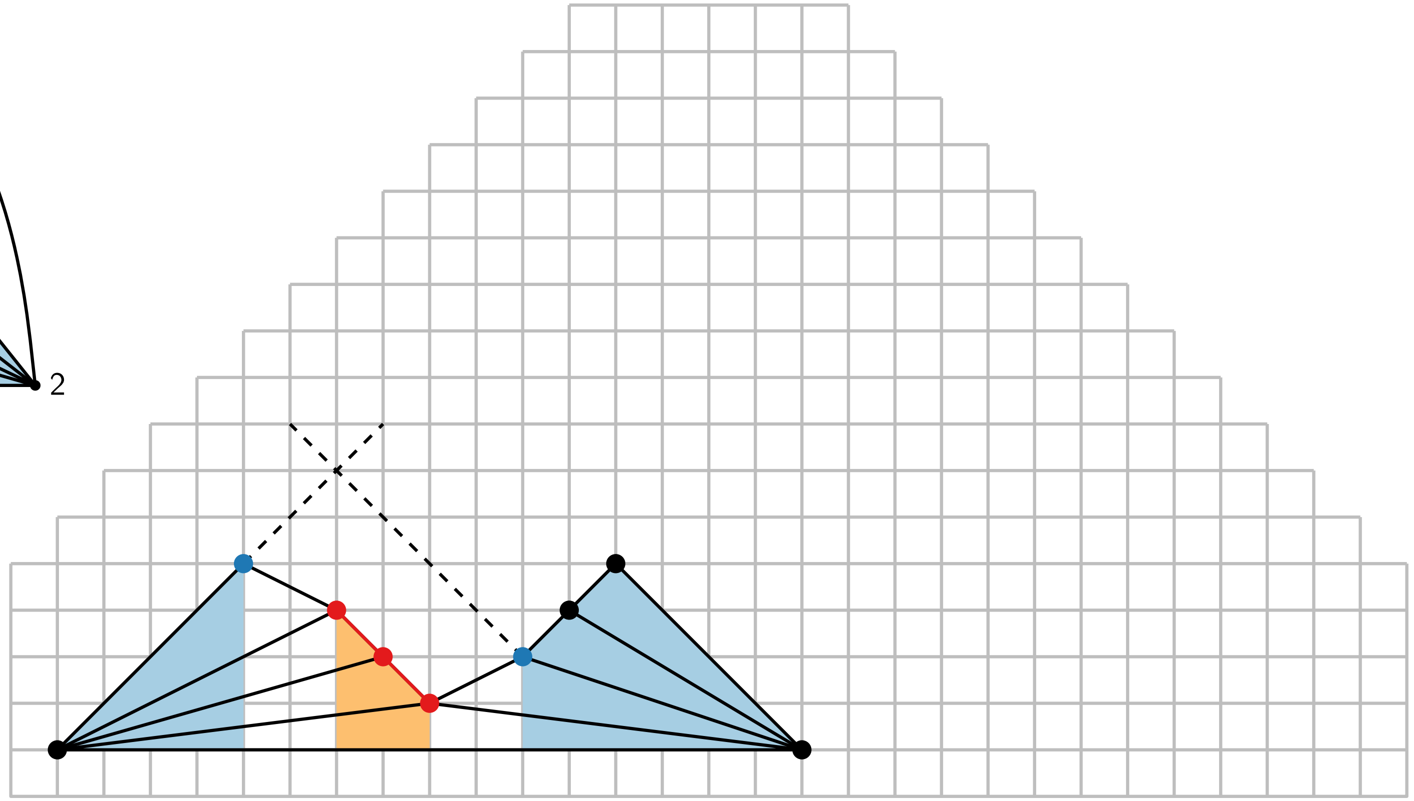

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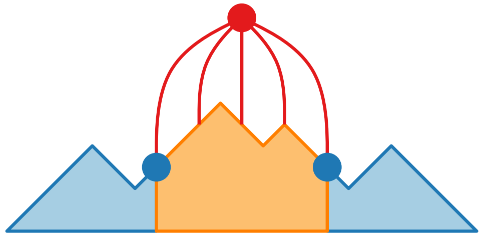
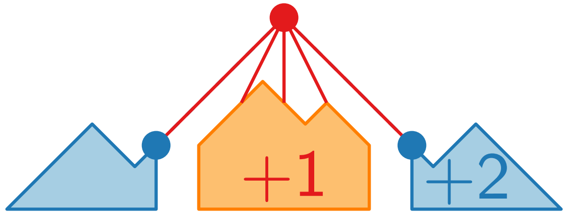
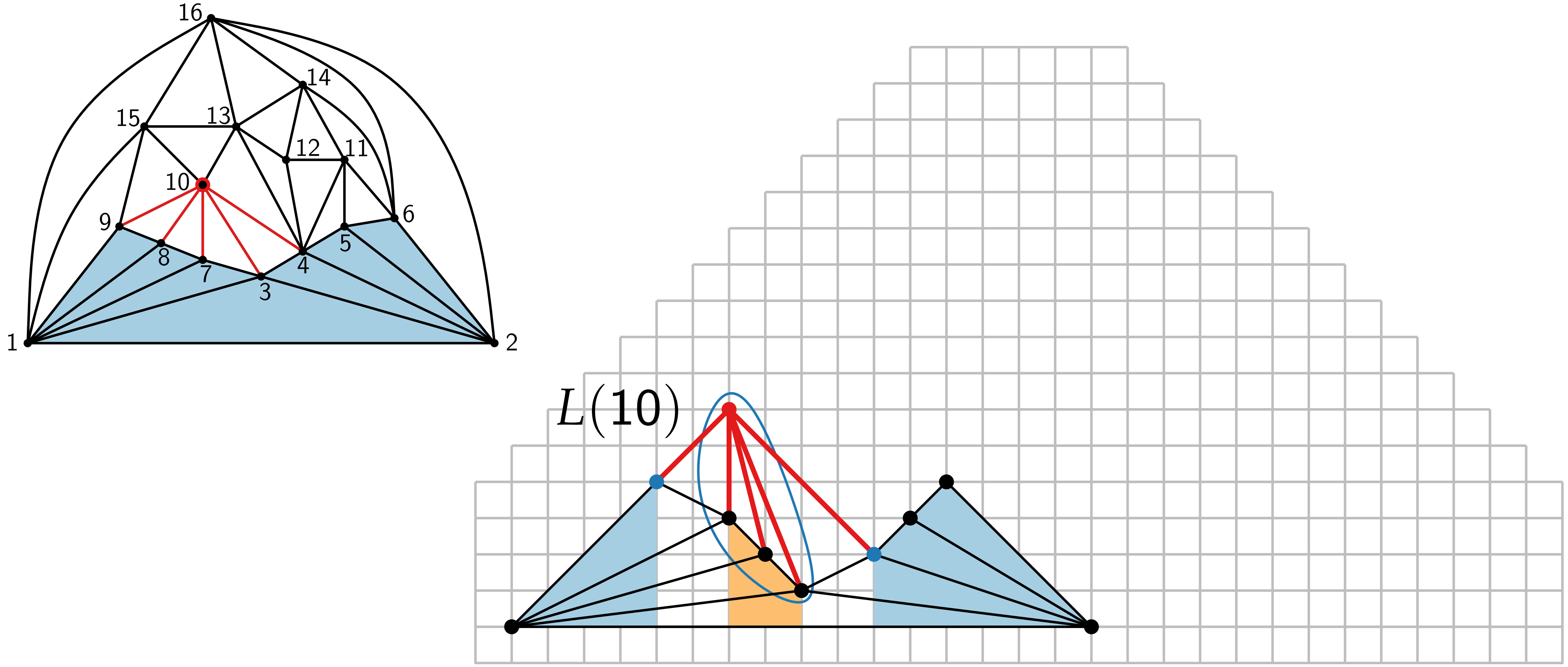
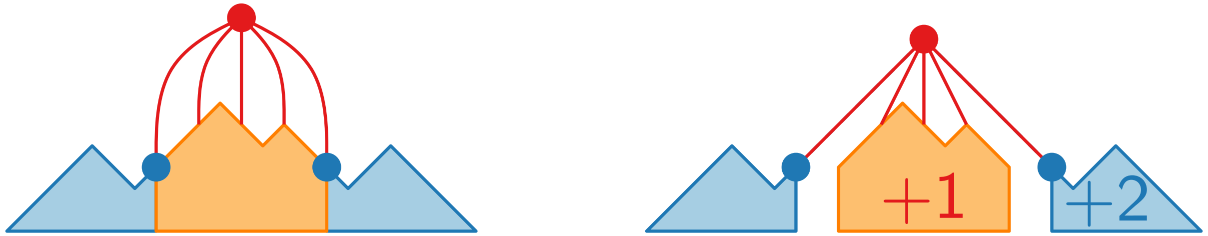
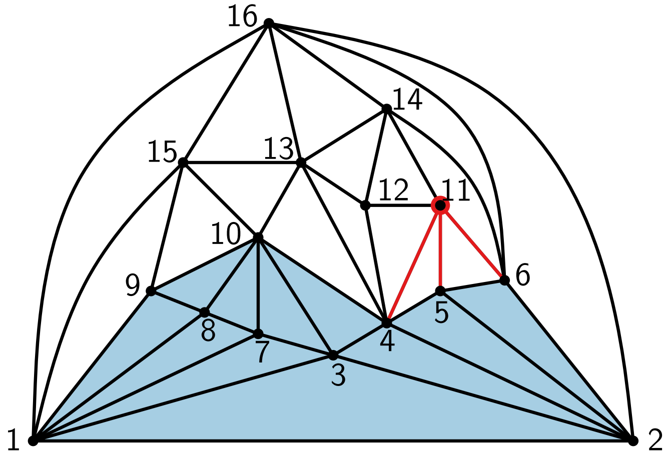


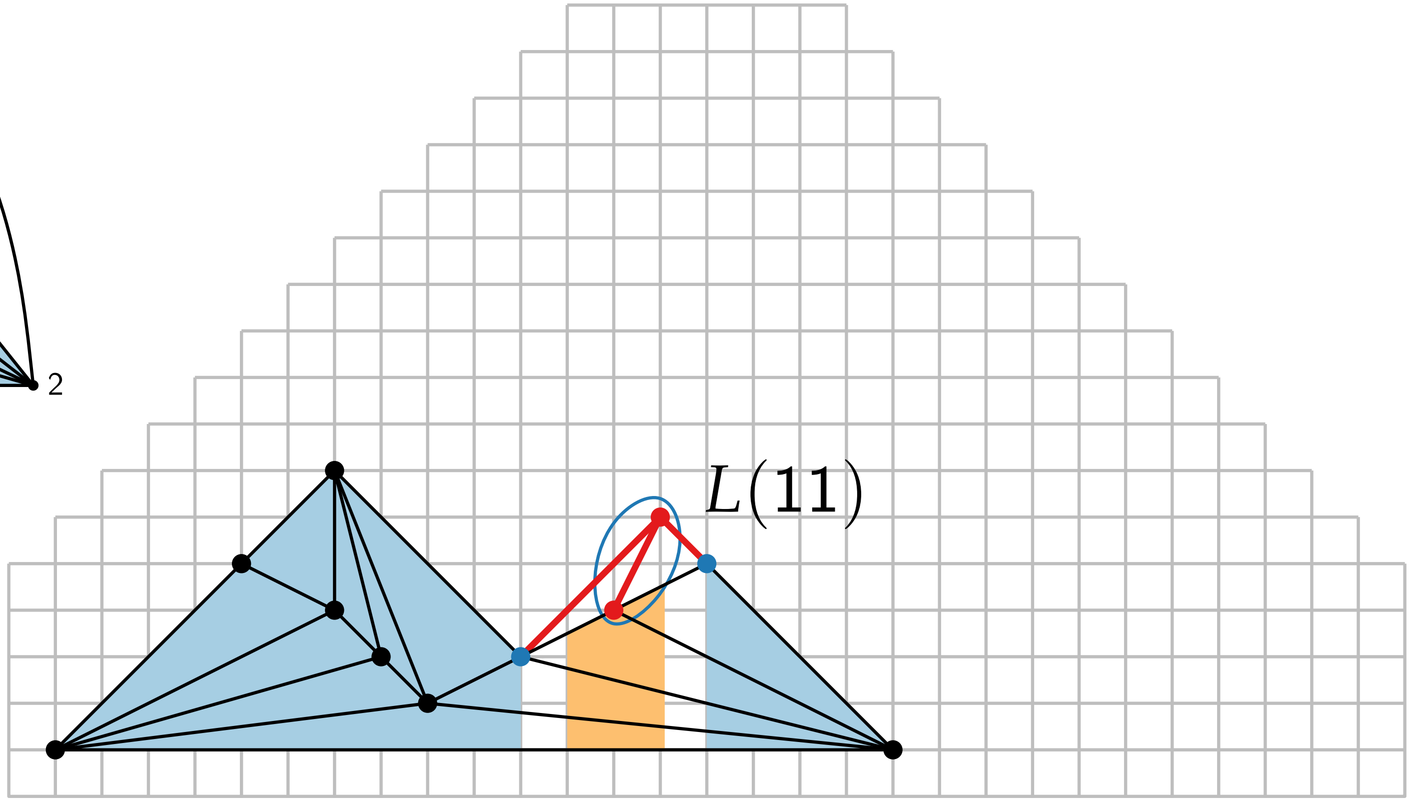




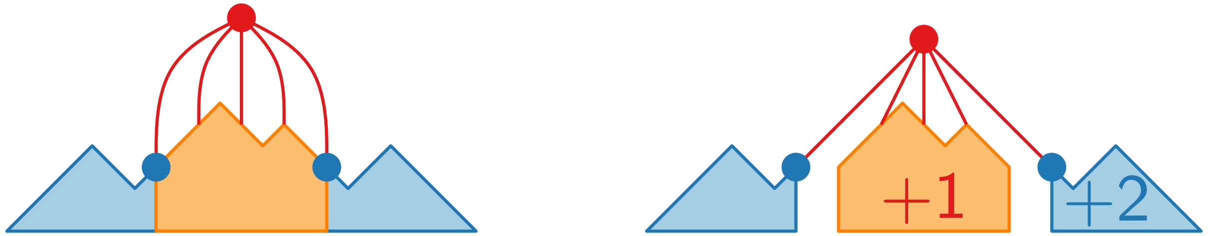
Shift method – example

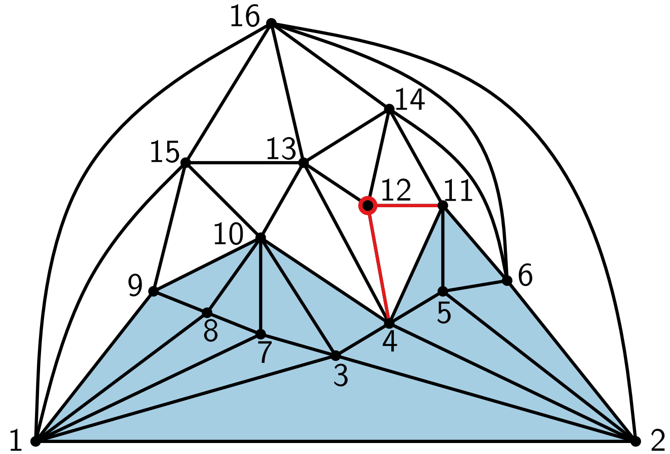


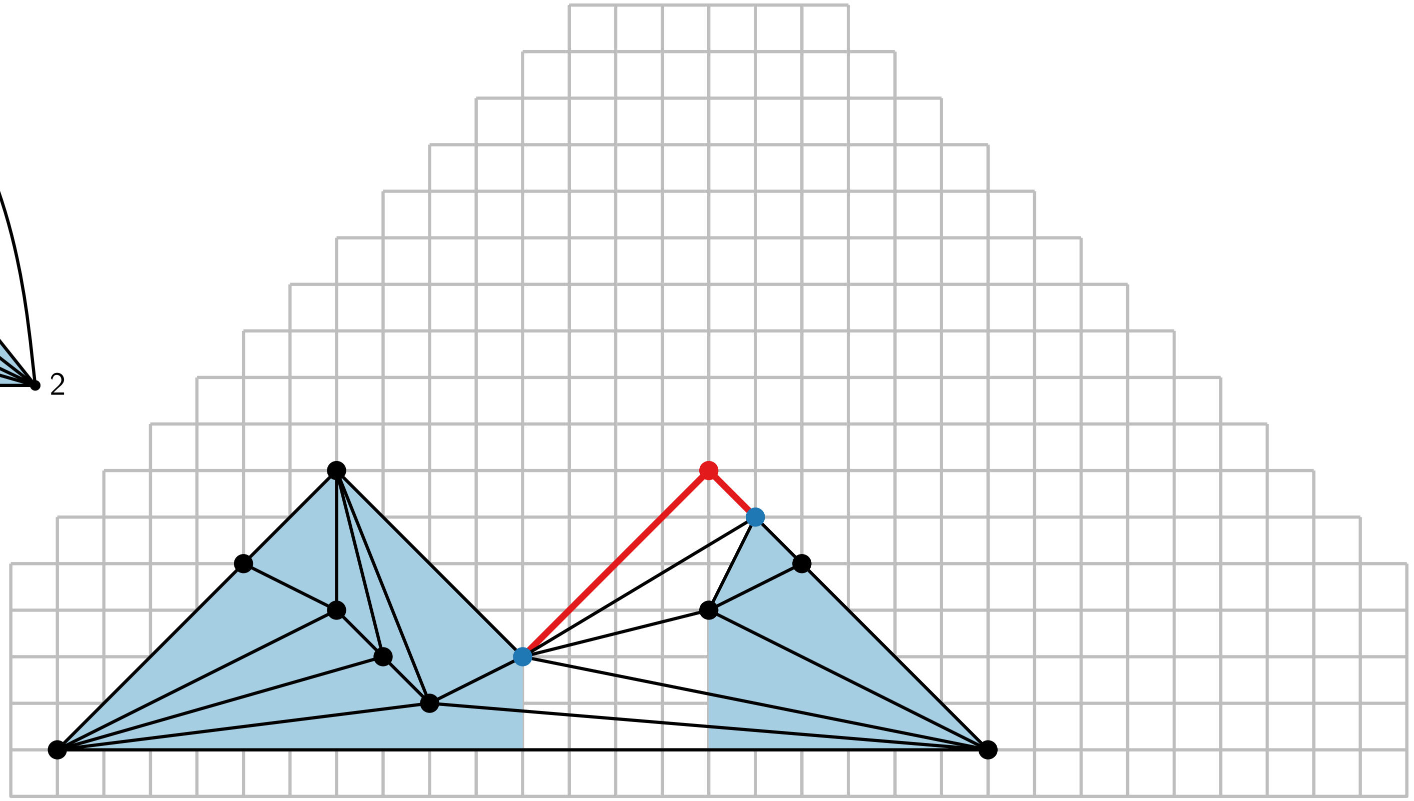




Shift method – example







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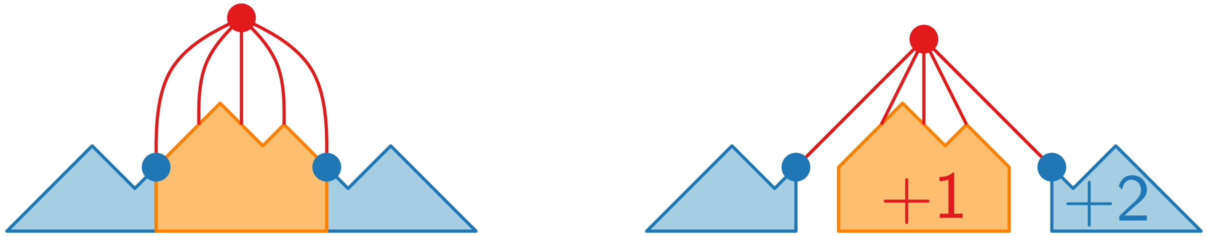

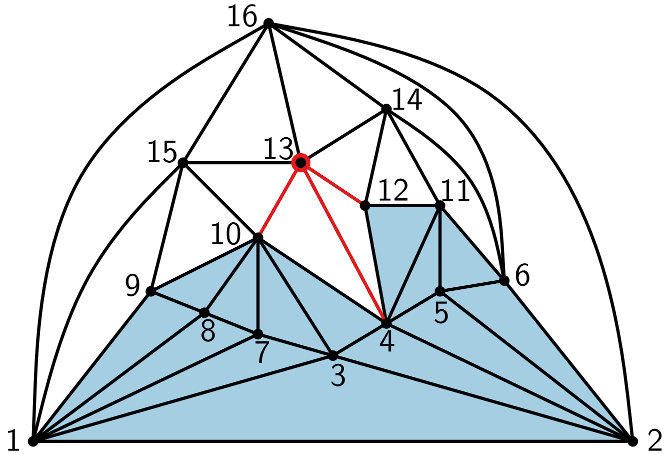

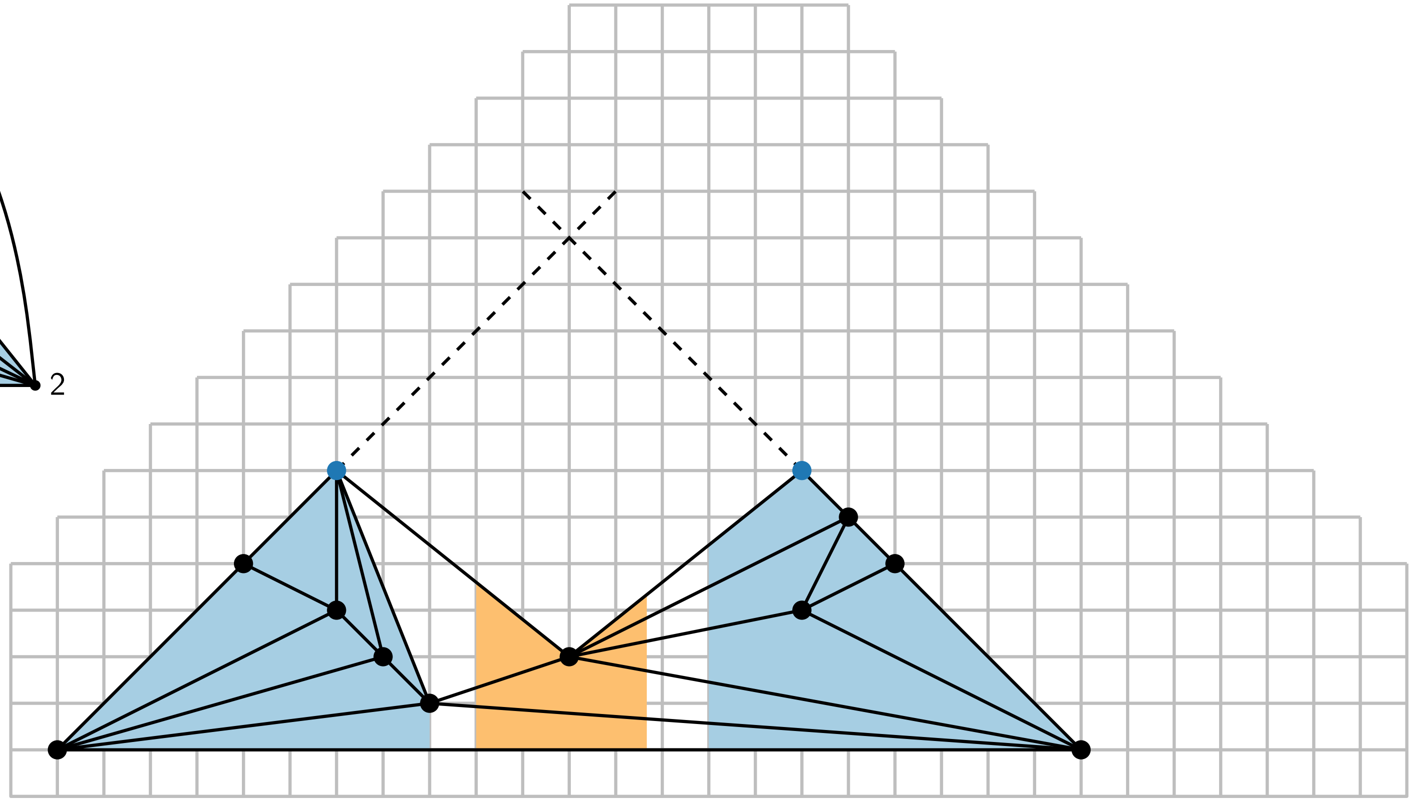

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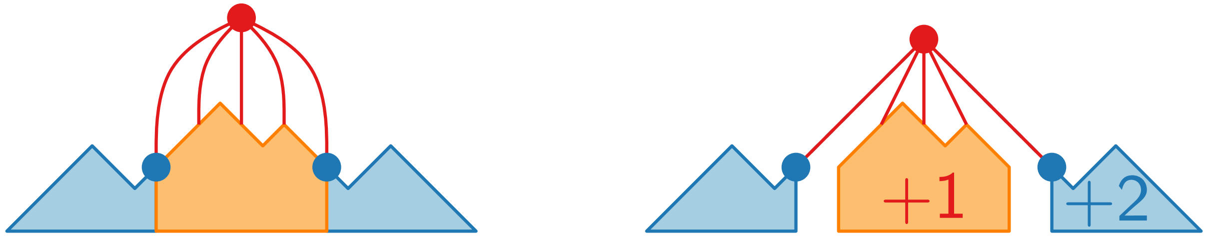

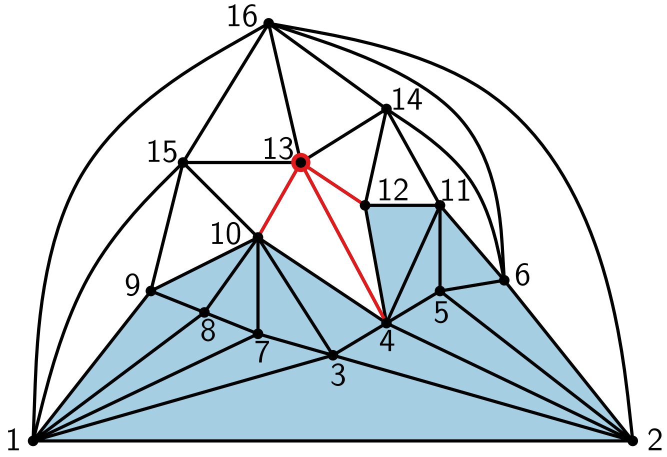

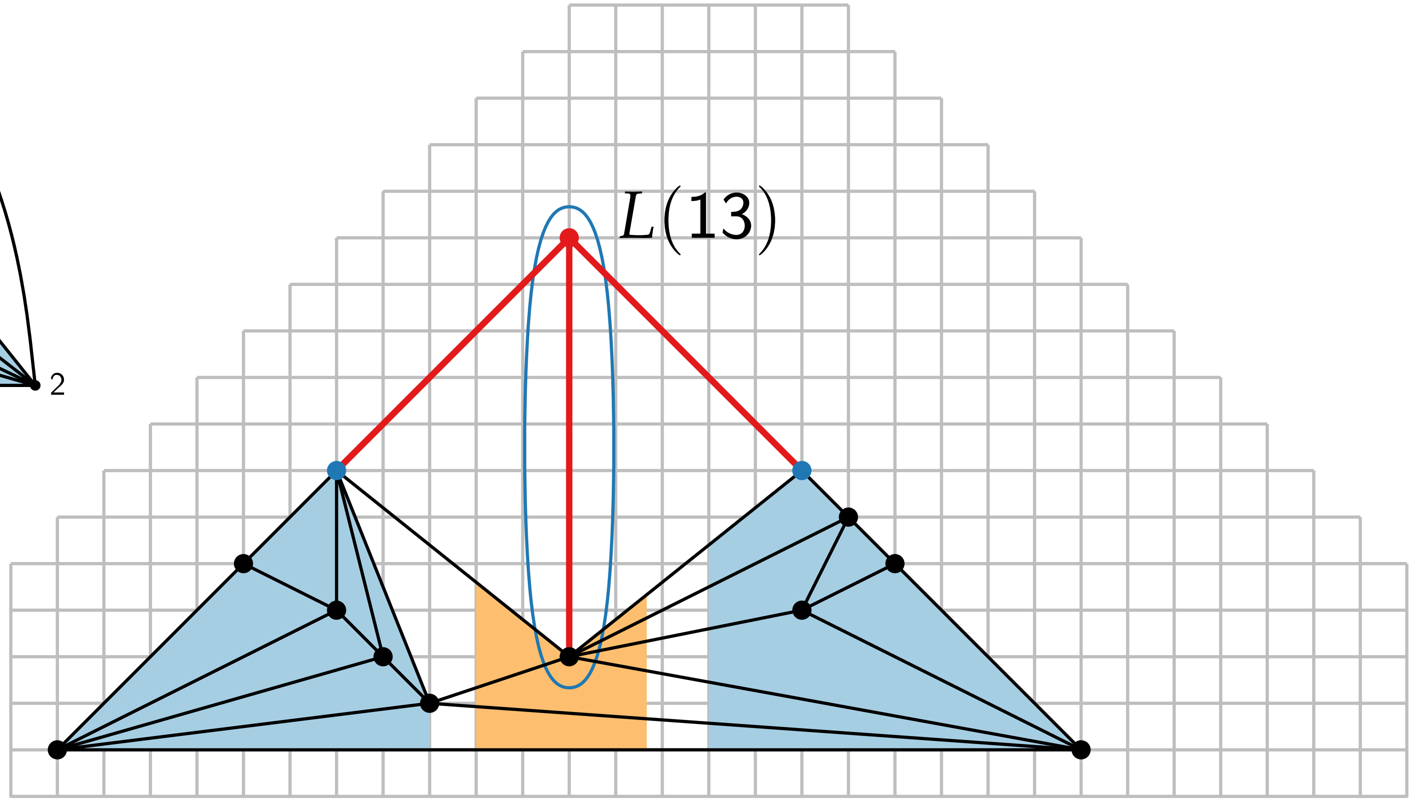

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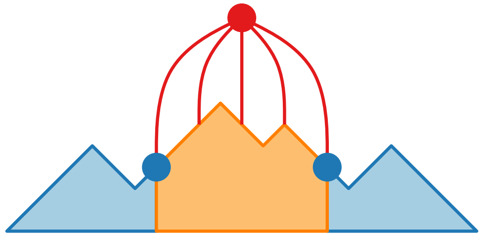

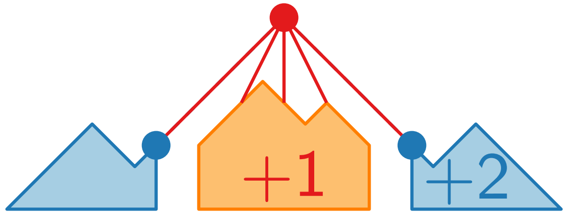

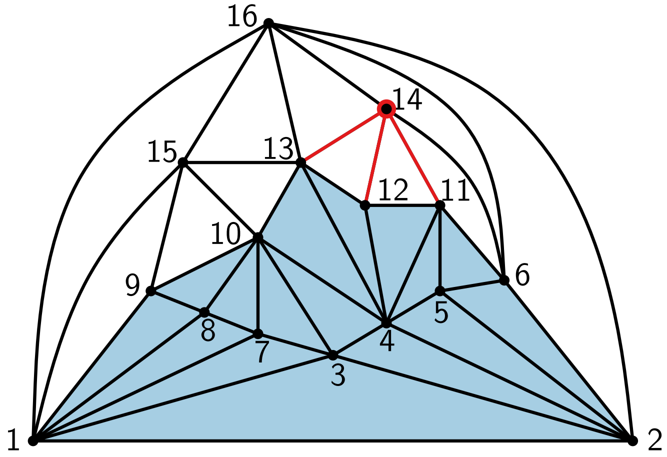

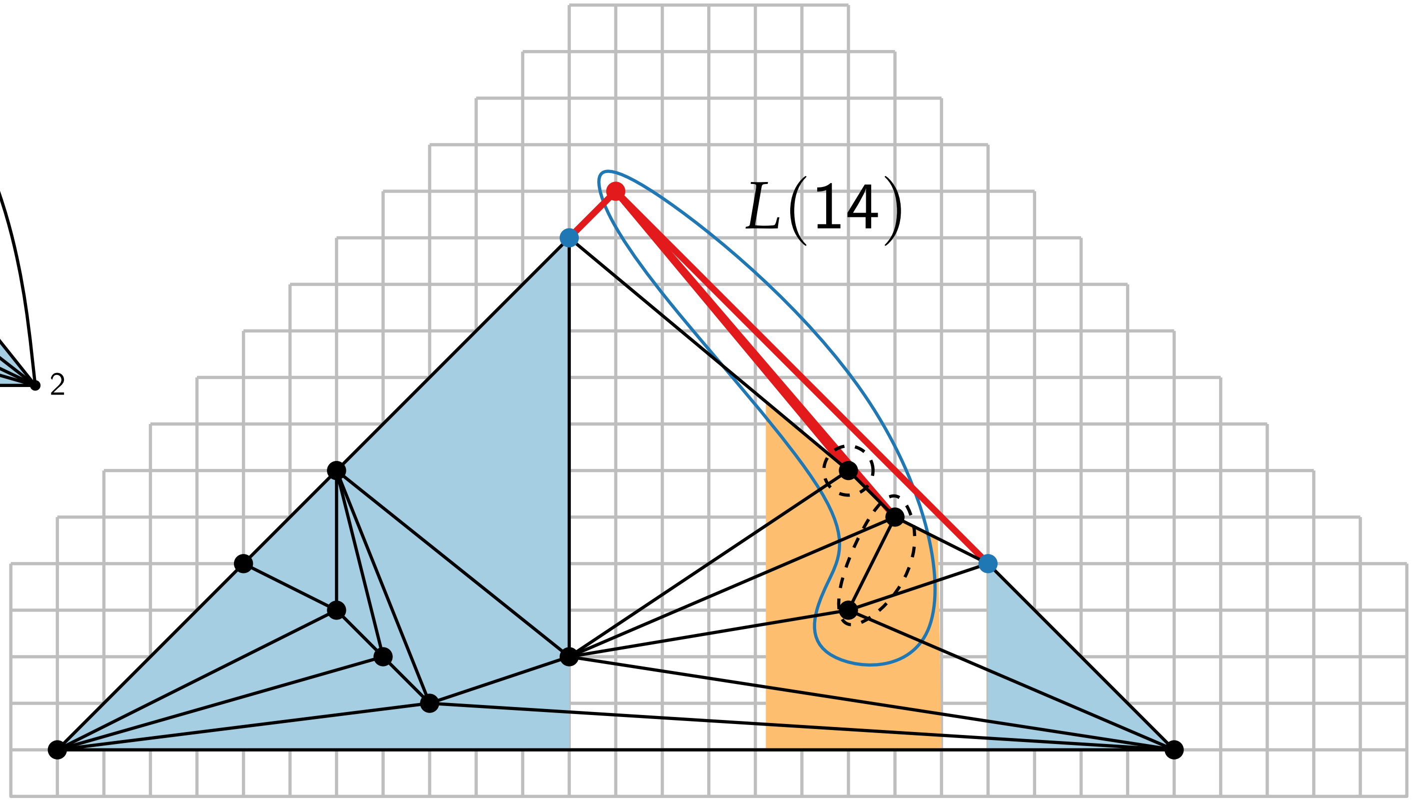

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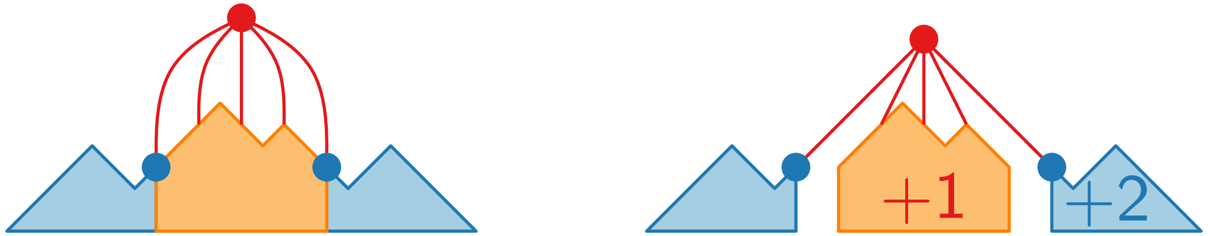

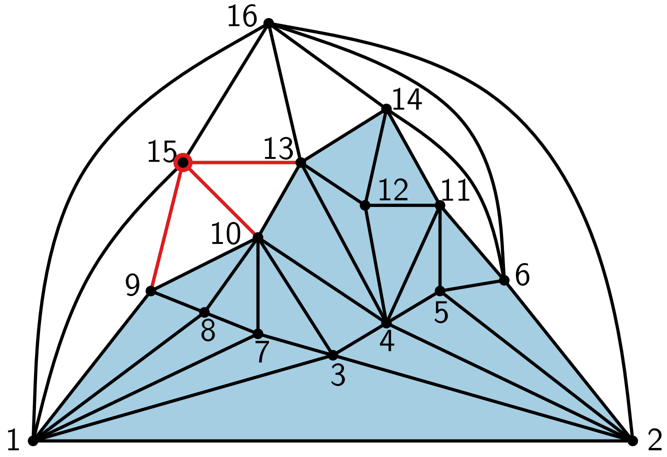

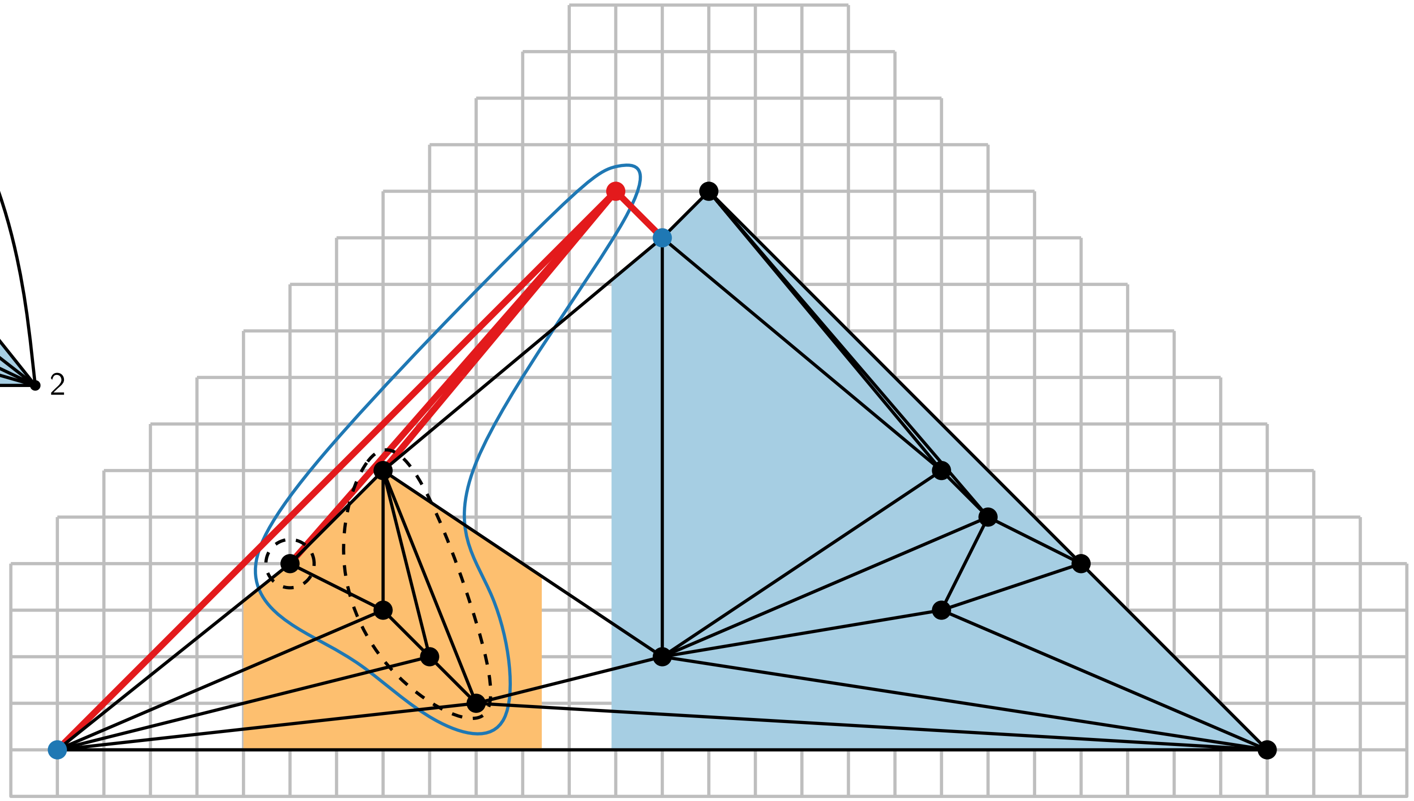

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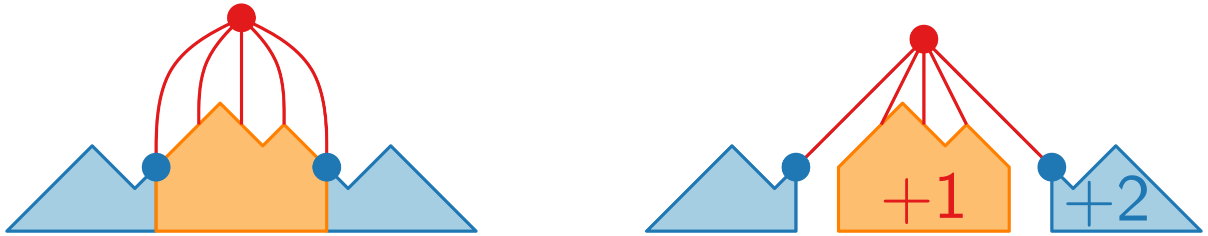

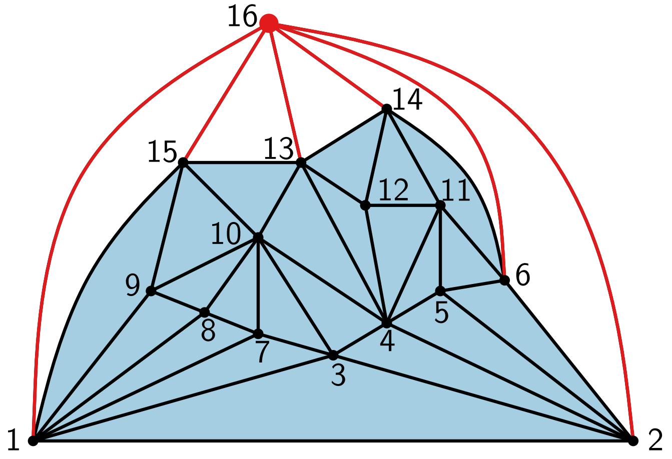

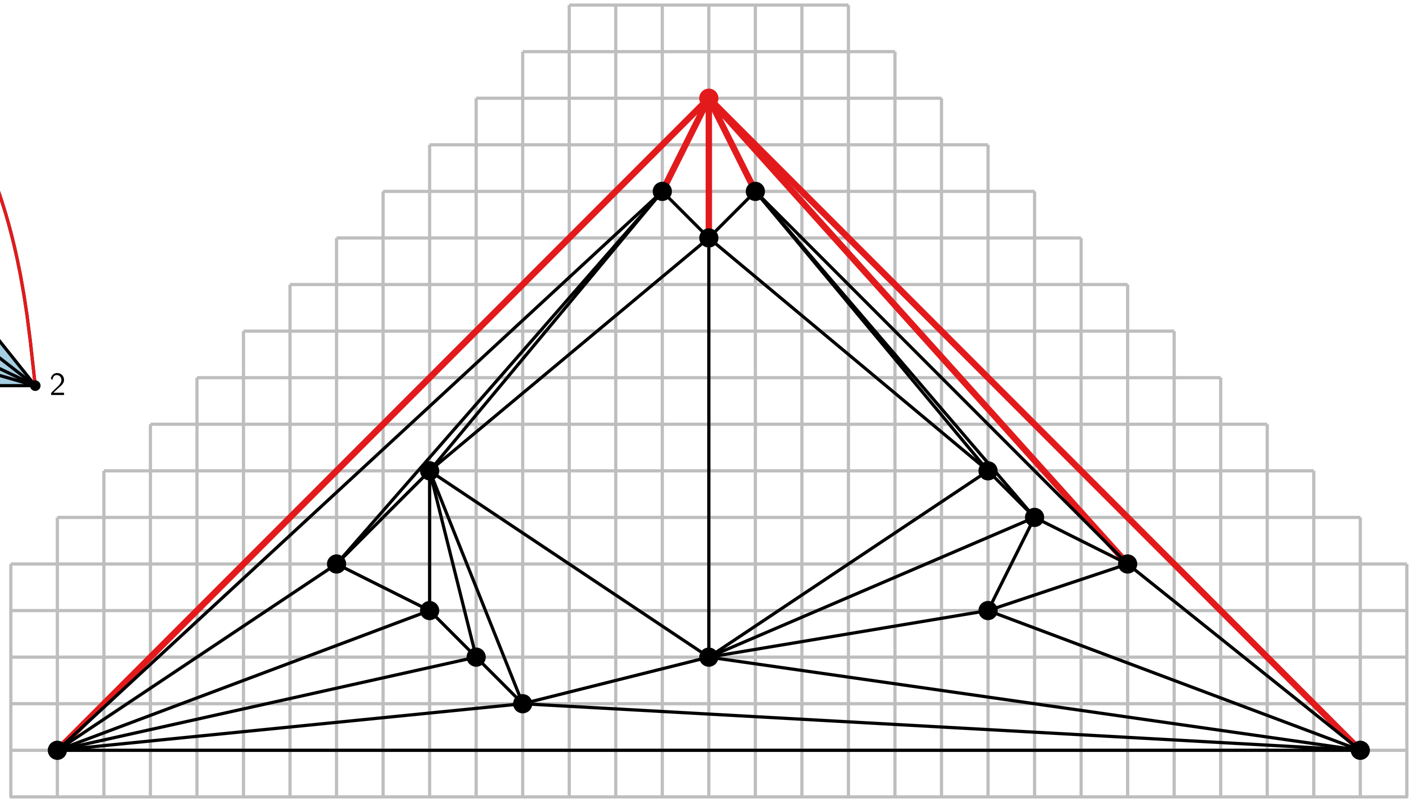

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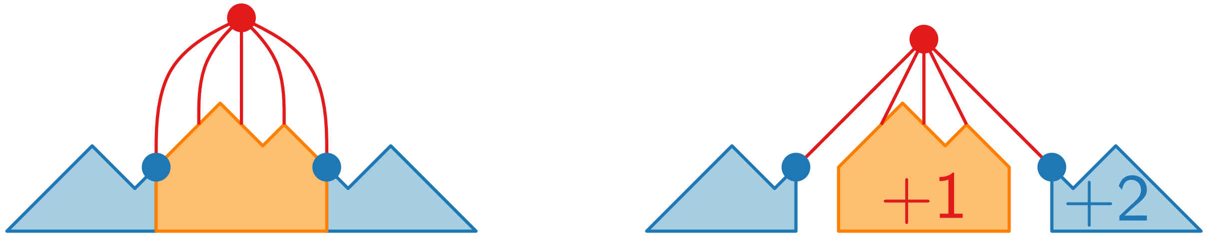
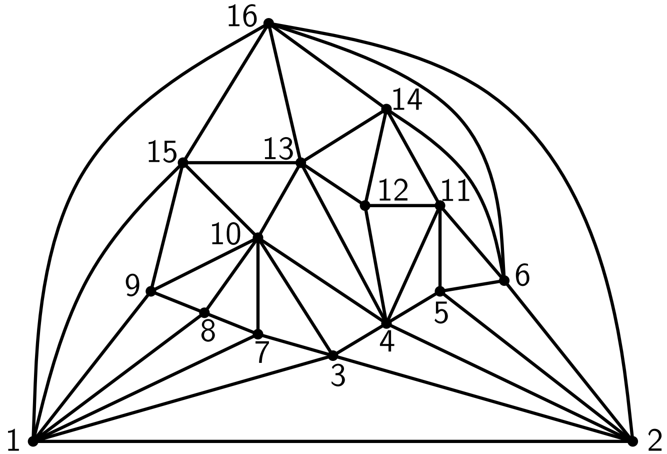
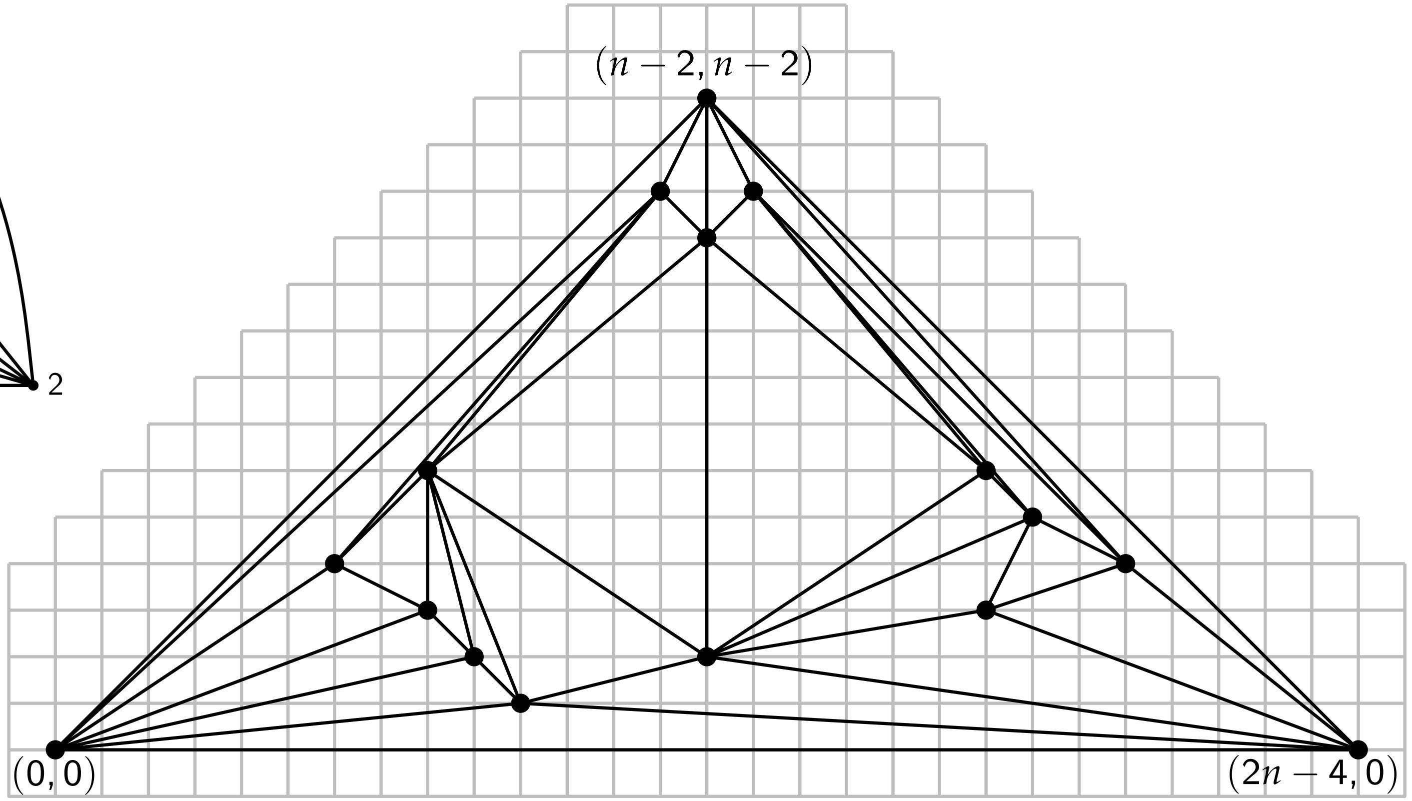
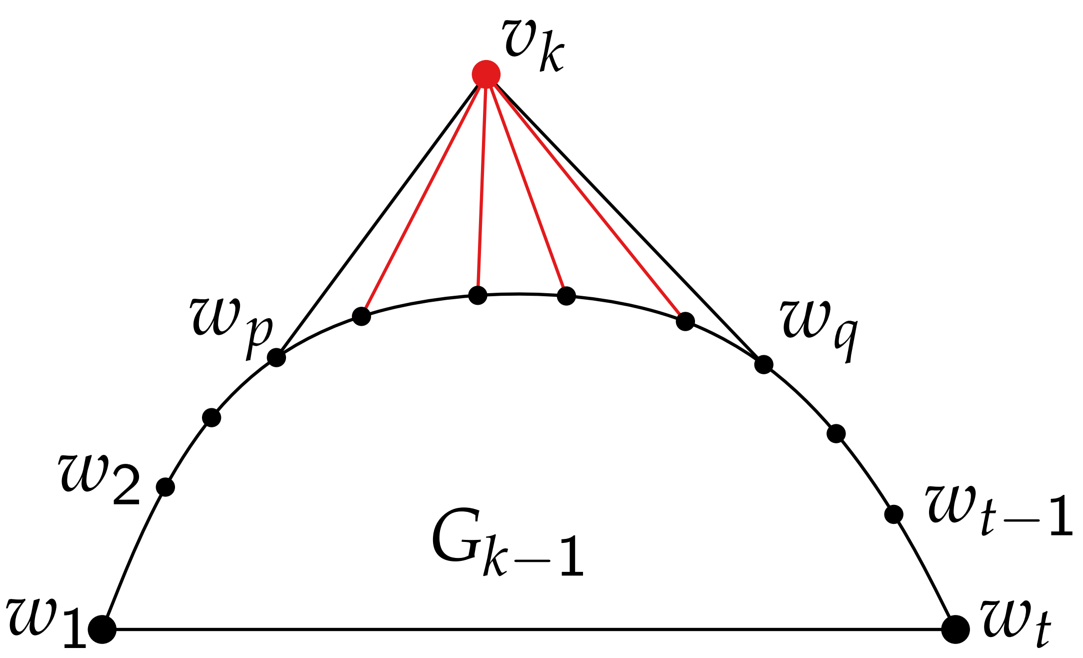






Shift method – planarity



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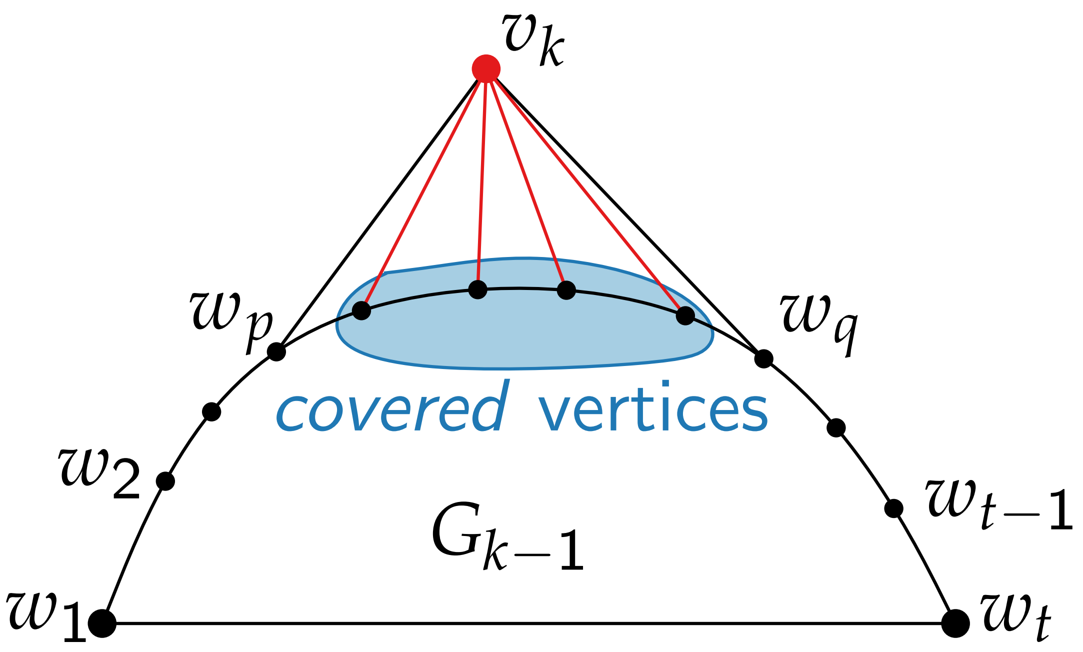

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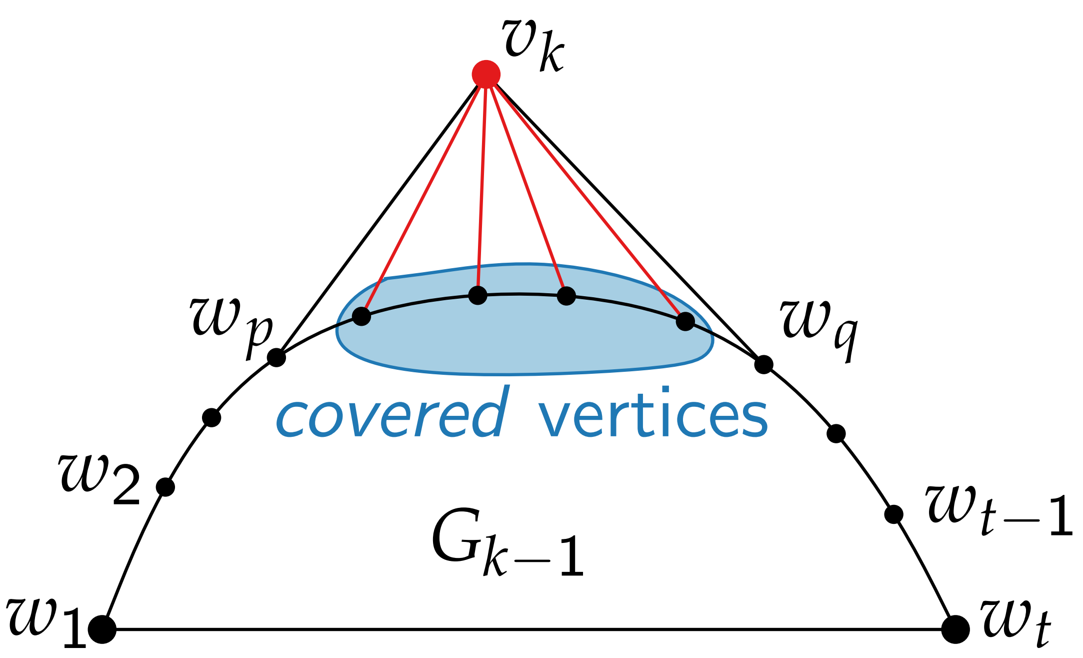
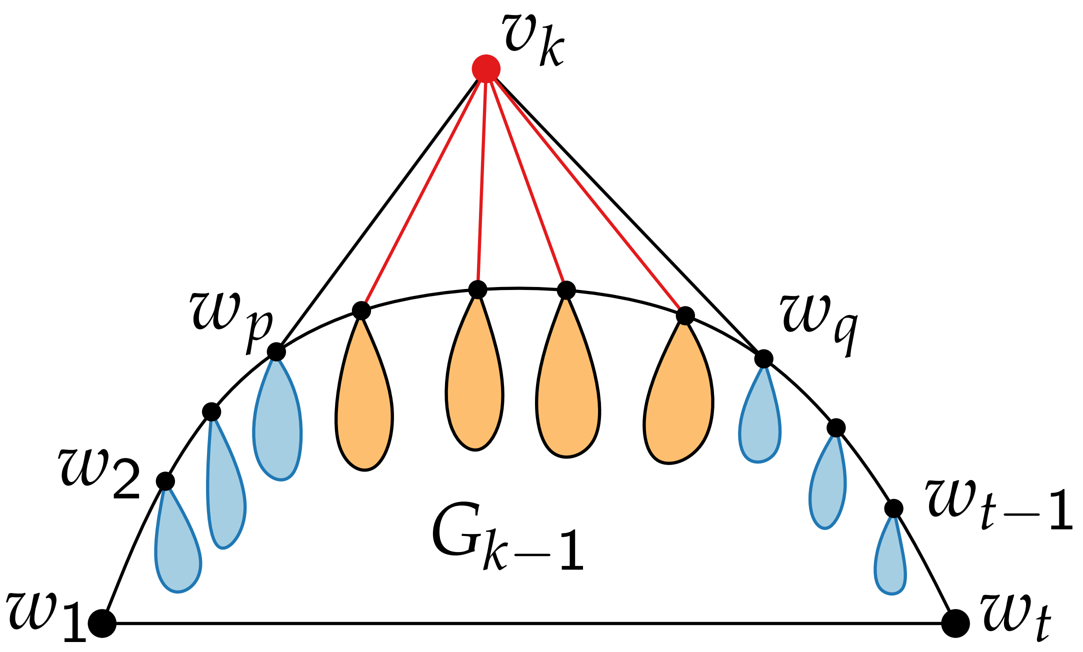


Observations.

- Each internal vertex is covered exactly once.
- Covering relation defines a tree in G
- and a forest in G_i, $1 \leq i \leq n-1$.

Shift method – planarity



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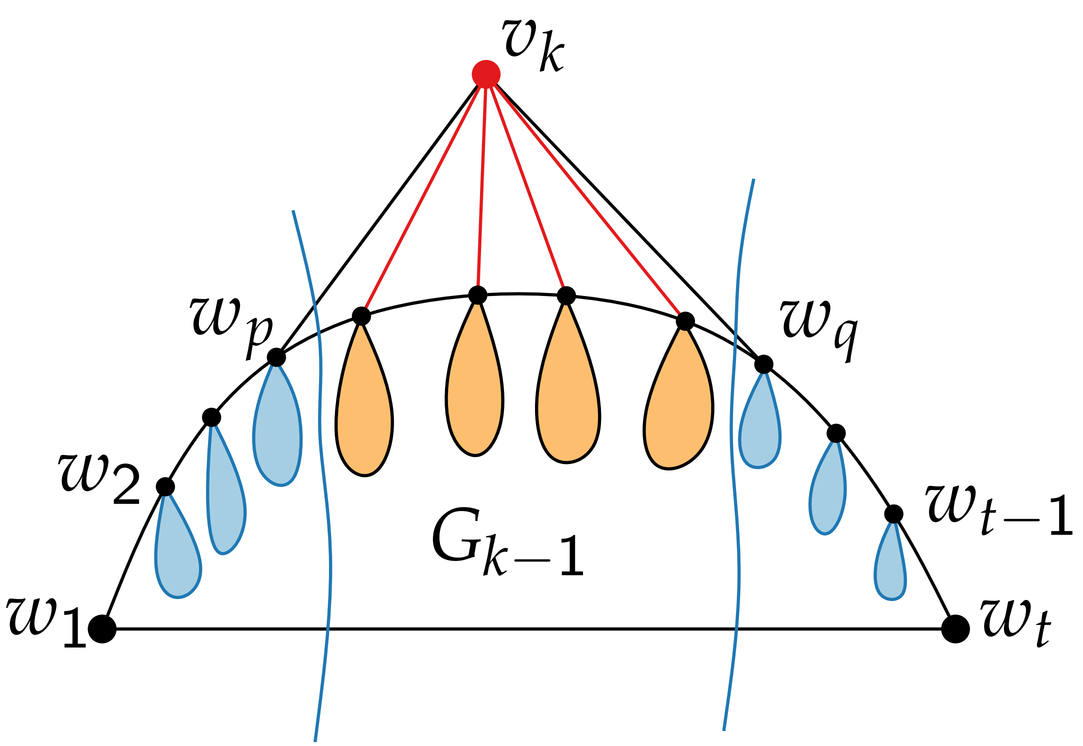

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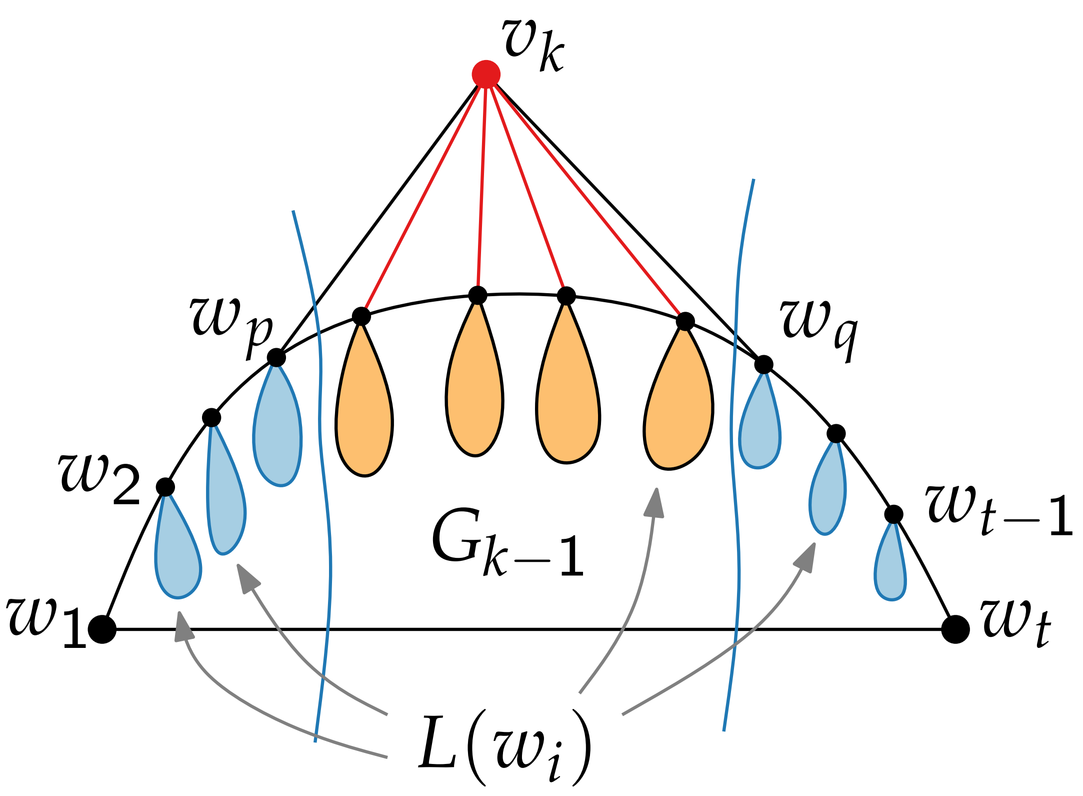

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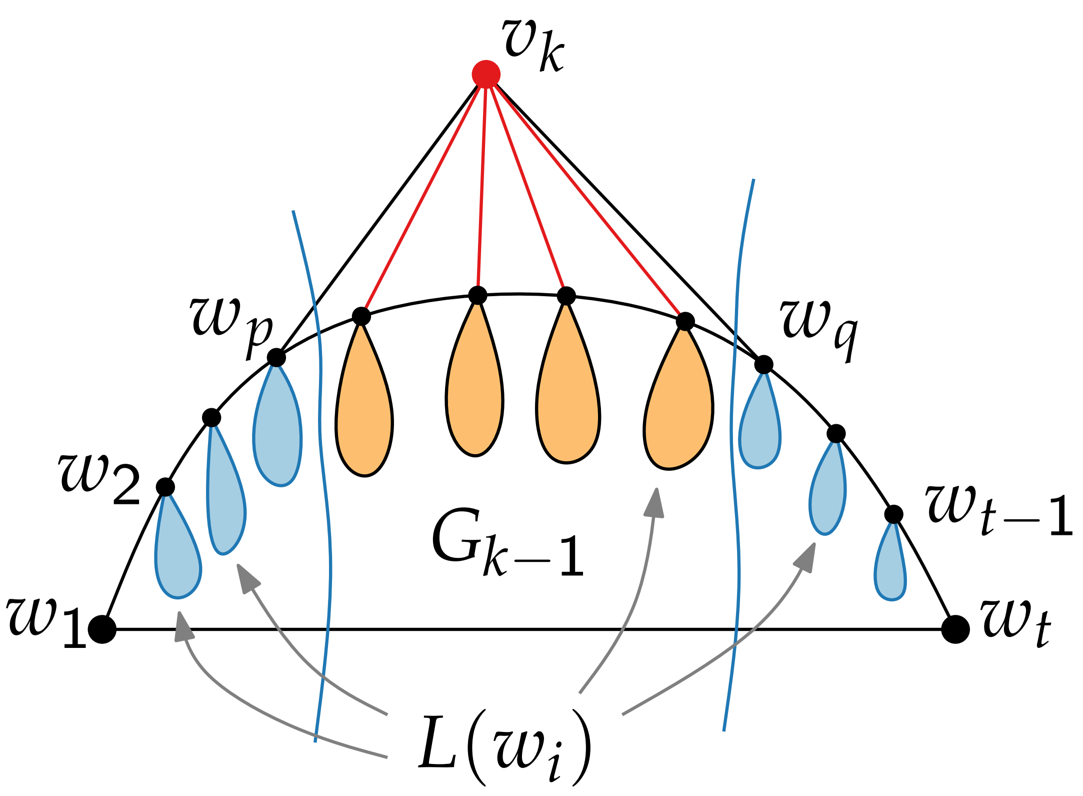
Shift method – planarity

Lemma. Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.
If we shift $L(w_i)$ by δ_i to the right, we get a planar straight-line drawing.



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Shift method – planarity

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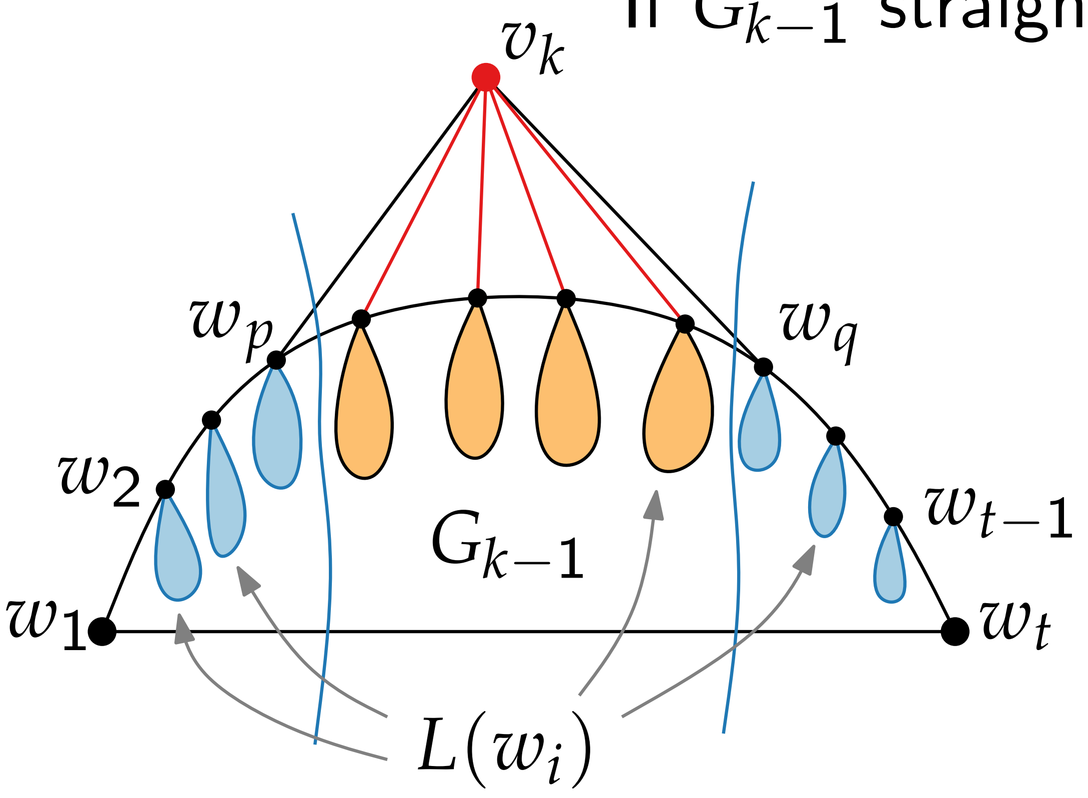
If we shift $L(w_i)$ by δ_i to the right, we get a planar straight-line drawing.

Proof by induction:
If G_{k-1} straight-line planar, then also G_k.



Observations.

- Each internal vertex is covered exactly once.
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Shift method – pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

  $L(v_i) \leftarrow \{v_i\}$

$P(v_1) \leftarrow (0,0); P(v_2) \leftarrow (2,0), P(v_3) \leftarrow (1,1)$

for $i = 4$ to n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1} and let $w_p, \ldots, w_q$ be the neighbours of v_k

  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 1$

  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 2$

  $P(v_i) \leftarrow$ intersection of $+1/-1$ edges from $P(w_p)$ and $P(w_q)$

  $L(v_i) \leftarrow \cup_{j=p+1}^{q-i} L(w_j) \cup \{v_i\}$

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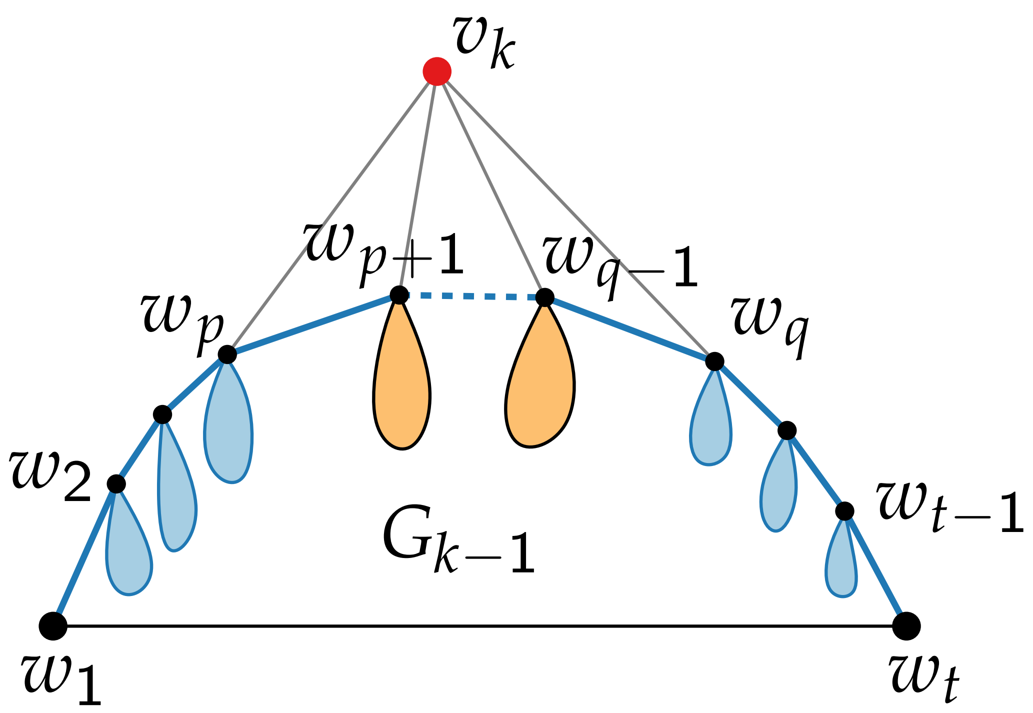
- Runtime $\mathcal{O}(n^2)$
- Can we do better?

Shift method – linear time implementation

- **Idea 1.** To compute $x(v_k)$ & $y(v_k)$, we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$



Shift method – linear time implementation

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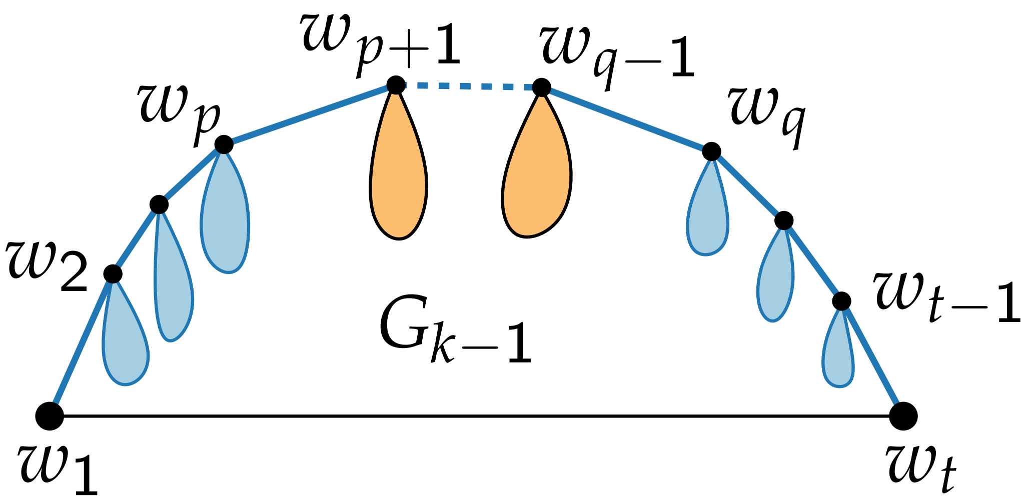

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

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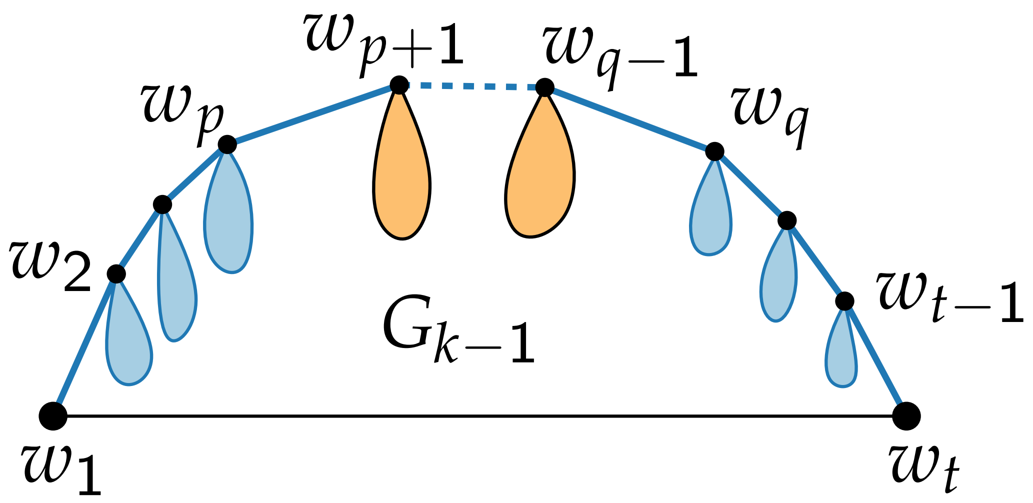
Shift method – linear time implementation

- **Idea 1.** To compute $x(v_k)$ & $y(v_k)$, we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$
- **Idea 2.** Instead of storing explicit x-coordinates, we store certain x differences.



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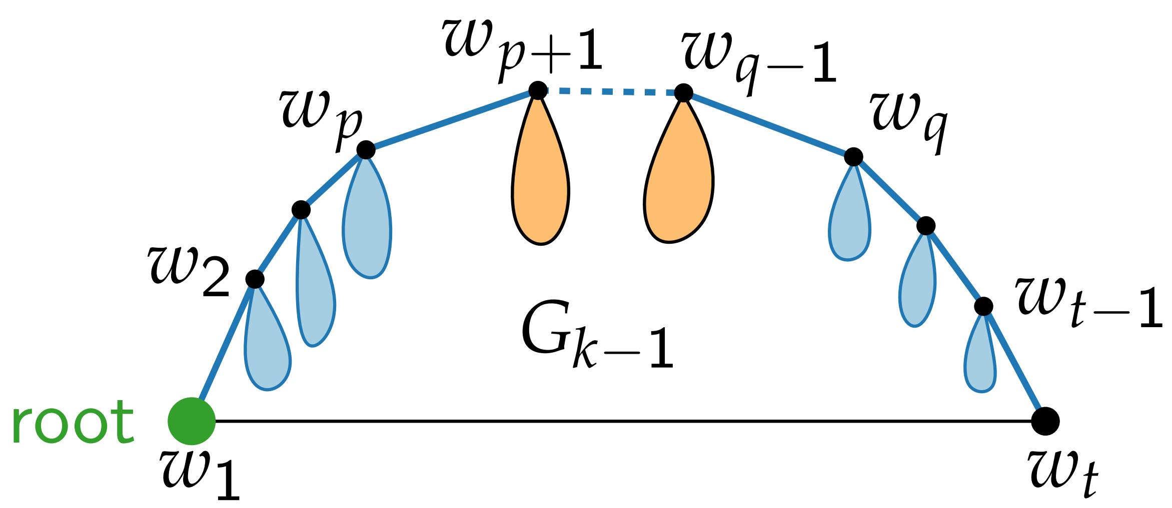
Shift method – linear time implementation

Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



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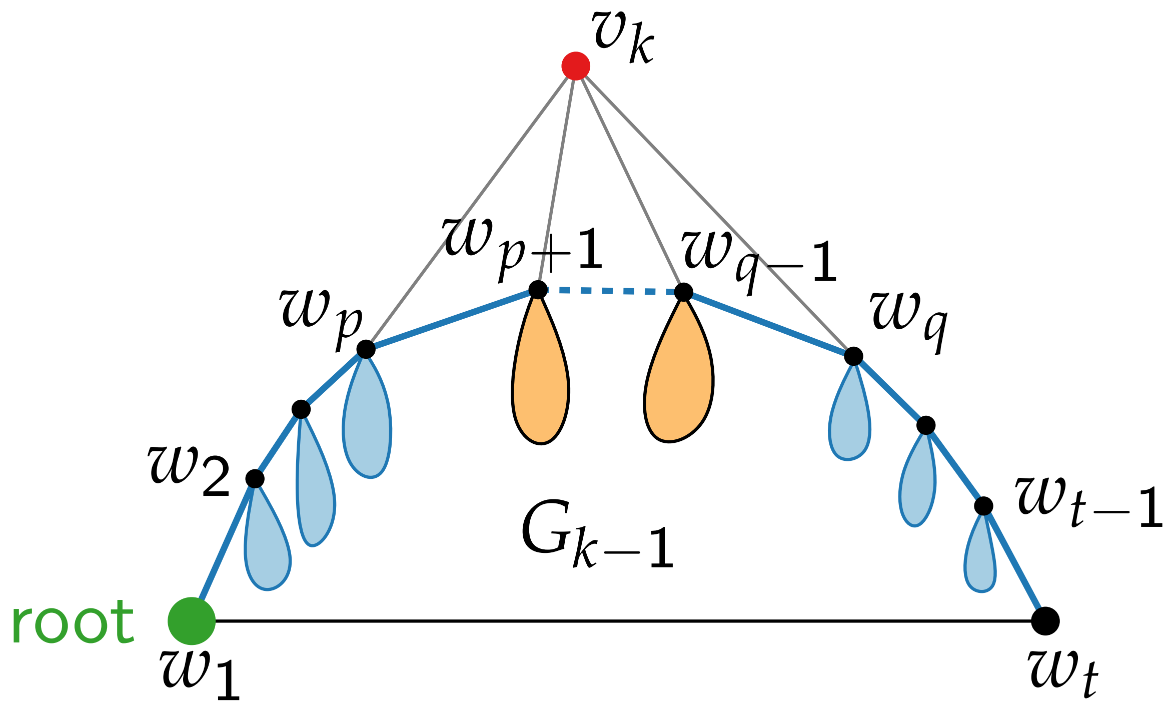

Shift method – linear time implementation

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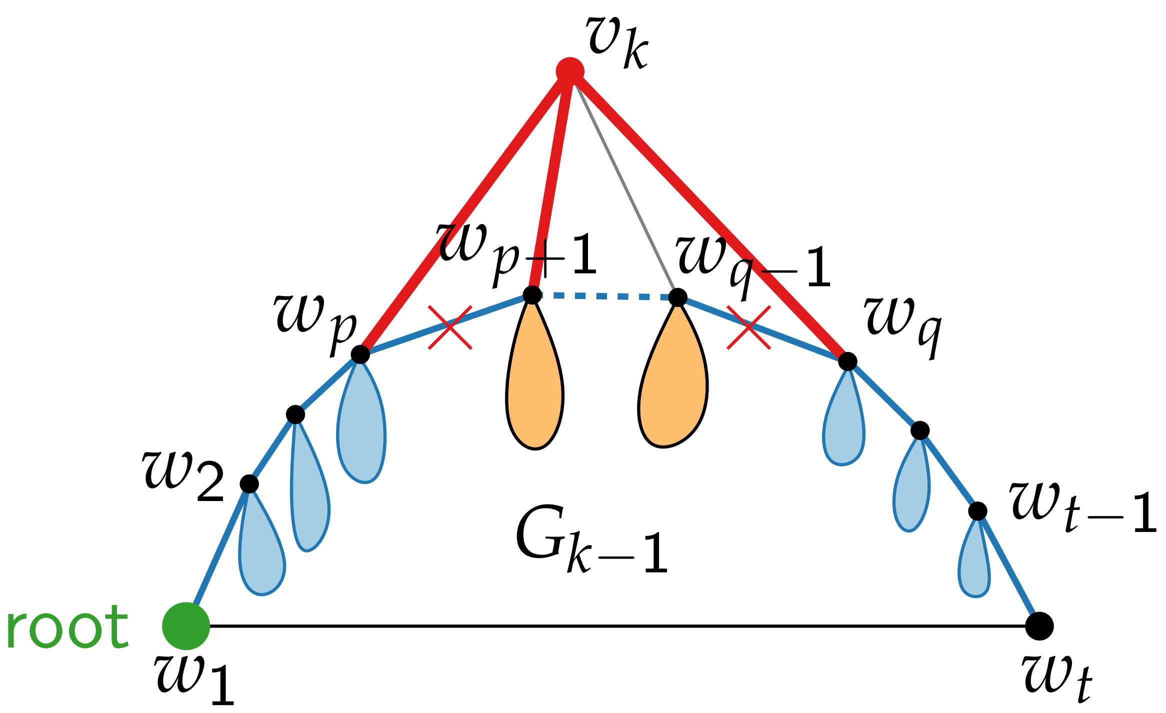

Shift method – linear time implementation

Relative x distance tree.

For each vertex v store
- x-offset $\Delta_x(v)$ from parent
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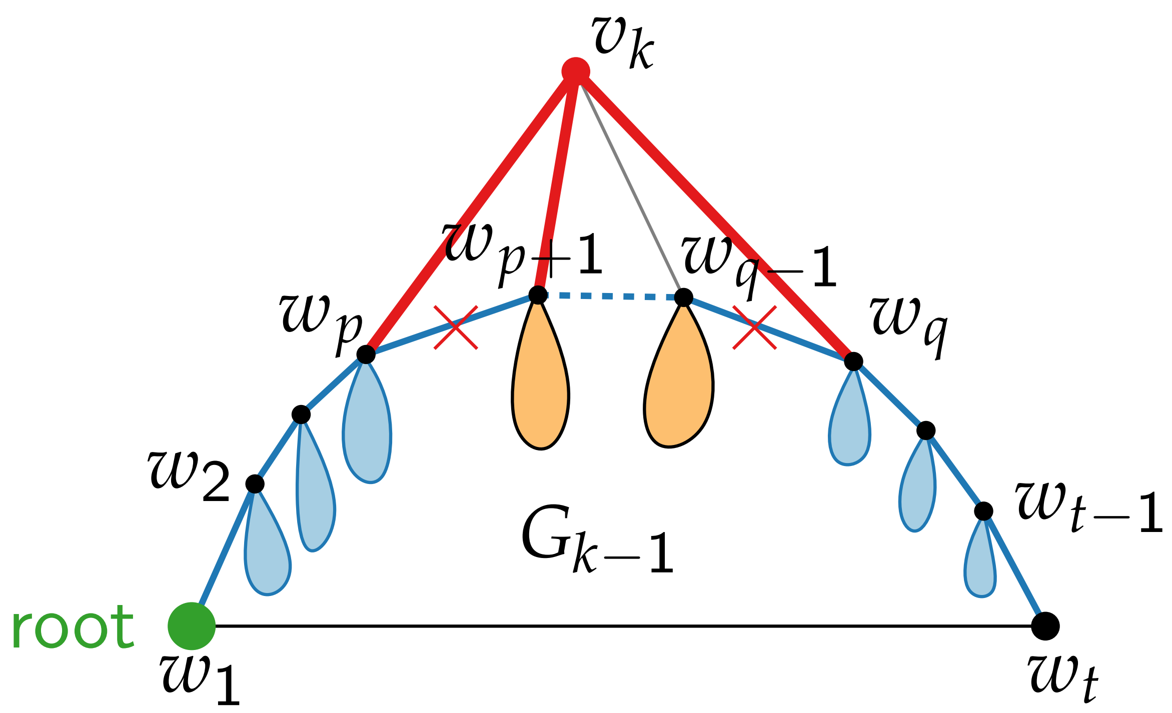
Calculations.
- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++



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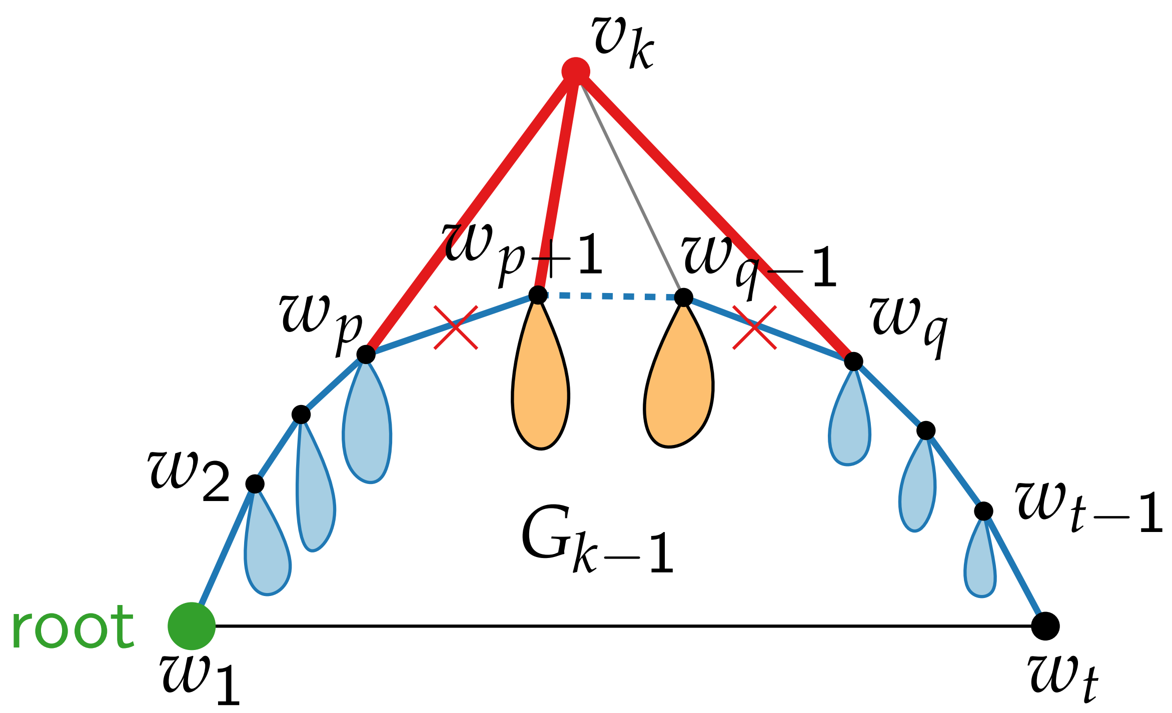
Shift method – linear time implementation



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- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)

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Shift method – linear time implementation

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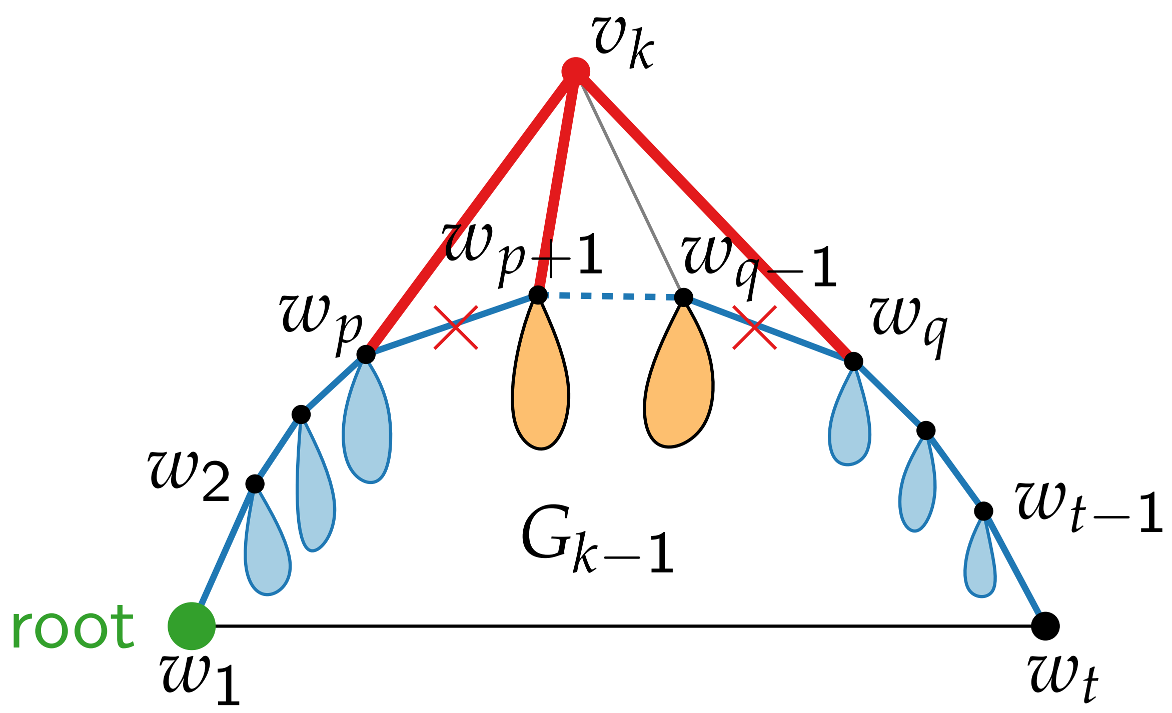

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Shift method – linear time implementation

Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

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- $\Delta_x(v_k)$ by (3)
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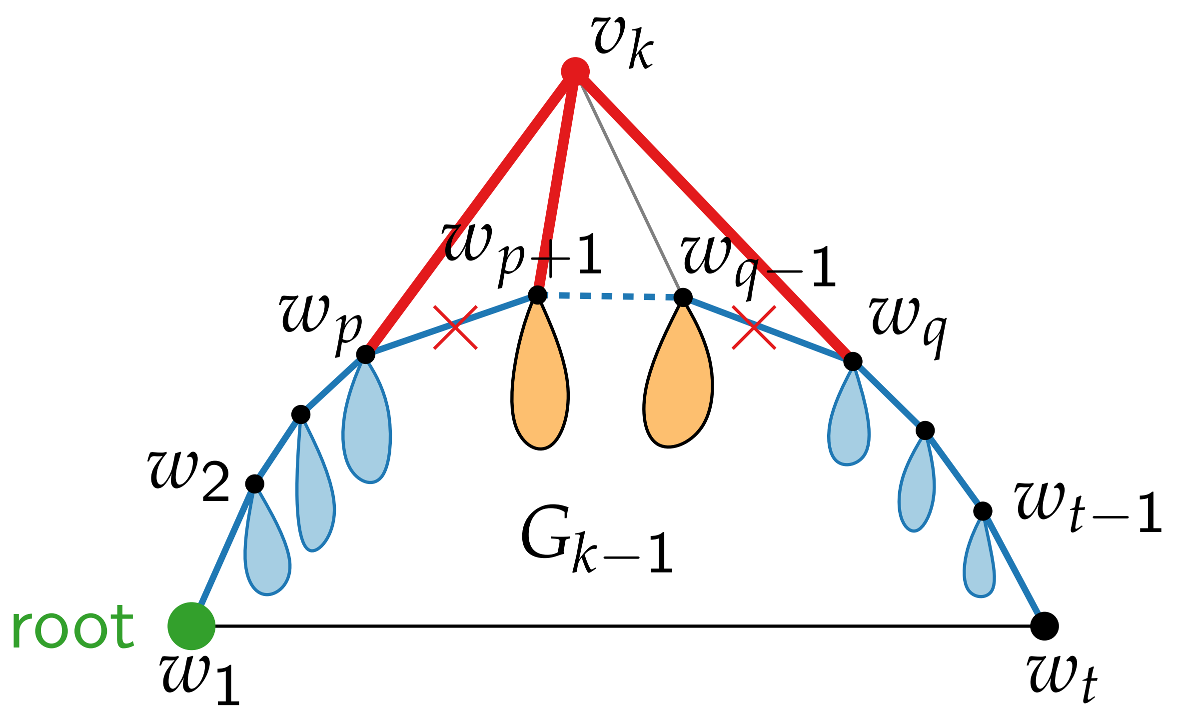

- After v_n, use preorder traversal to compute x-coordinates

Literature

- [PGD Ch. 4.2] for detailed explanation of shift method
- [dFPP90] de Fraysseix, Pach, Pollack "How to draw a planar graph on a grid" 1990 – original paper on shift method