

Homework Assignment #2

Approximation Algorithms (Winter Semester 2026/27)

Exercise 1 – MAXIMUM k-COVERAGE

In this exercise, we consider the MAXIMUM k-COVERAGE problem, which is a variant of the SET COVER problem introduced in the lecture. Let U be a set, let $\mathcal{S}$ be a set of subsets of U such that $U = \bigcup \mathcal{S}$, and let $k \leq |\mathcal{S}|$ be a natural number. The problem MAXIMUM k-COVERAGE asks for a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| = k$ such that the number $|\bigcup \mathcal{S}'|$ of elements covered by $\mathcal{S}'$ is maximized.

For this problem, we consider a greedy algorithm that iteratively performs the following step: In each iteration, select a set from $\mathcal{S}$ that contains the largest number of uncovered elements. The algorithm terminates after at most k iterations or when U is covered. Let $\ell \leq k$ be the number of iterations that the algorithm performs.

In the following, we analyze the quality of this algorithm. For $i \in \{1, \dots, \ell\}$, let ALG_i be the number of elements that are covered by the sets that have been selected in the steps up to and including step i . We set $\text{ALG}_0 = 0$.

- a) Show that $\text{ALG}_1 \geq \frac{1}{k} \cdot \text{OPT}$. In other words, show that the greedy algorithm covers at least OPT / k elements with the set selected in the first step. [1 point]
- b) Show by induction that for each $i \in \{0, \dots, \ell\}$, the following holds: $\text{OPT} - \text{ALG}_i \leq \left(1 - \frac{1}{k}\right)^i \cdot \text{OPT}$.
Hint: Use the fact that, at the beginning of iteration i , there exist at least $\text{OPT} - \text{ALG}_{i-1}$ uncovered elements in the optimal solution. [7 points]
- c) Show that the greedy algorithm has approximation ratio $\left(1 - \frac{1}{e}\right) \approx 0.63$.
Hint: Use that, for any positive integer j , it holds that $\left(1 - \frac{1}{j}\right)^j \leq \frac{1}{e}$. [2 points]

Exercise 2 – Integer Linear Programs (ILPs)

- a) Formulate SET COVER as an ILP. [2 points]
- b) Formulate MAXIMUM k-COVERAGE as an ILP. [3 points]

Don't forget to mention what the variables represent.

Exercise 3 – Factor- h approximation for SET COVER

In this exercise, we consider algorithms that approximate SET COVER with a frequency based factor. A SET COVER instance $(U, \mathcal{S})$ has *frequency* h if each element from U is contained in at most h sets from $\mathcal{S}$. (VERTEX COVER is a special case of SET COVER with frequency 2.)

- a) Assume that we are given an unweighted SET COVER instance, i.e., all sets in $\mathcal{S}$ have the same cost. Consider the following greedy algorithm:

Algorithm 1: FrequencyGreedySetCover($\mathcal{U}, \mathcal{S}$)

```
 $\mathcal{S}' \leftarrow \emptyset$ 
while  $\mathcal{U} \neq \emptyset$  do
    mark an arbitrary element  $u \in \mathcal{U}$ 
     $\mathcal{R} \leftarrow \{S \in \mathcal{S} \mid u \in S\}$ 
     $\mathcal{S}' \leftarrow \mathcal{S}' \cup \mathcal{R}$ 
     $\mathcal{U} \leftarrow \mathcal{U} \setminus \bigcup \mathcal{R}$  // remove all elements from  $\mathcal{U}$  that are covered by  $\mathcal{R}$ 
return  $\mathcal{S}'$ 
```

Show that this algorithm yields a factor- h approximation and that the algorithm generalizes the factor-2 approximation algorithm for VERTEX COVER from the lecture.

Hint: Consider the set $\mathcal{U}_m$ of elements marked by the algorithm and find a relation of $|\mathcal{U}_m|$ and OPT. **[4 points]**

- b) Find an instance of weighted SET COVER where FrequencyGreedySetCover does not yield a factor- h approximation. **[1 point]**
- c) Design a factor- h approximation algorithm for weighted SET COVER. **[5 bonuspoints]**