

## Homework Assignment #1

### Approximation Algorithms (Winter Semester 2026/27)

#### Exercise 1 – Maximum acyclic subgraph

Devise a factor-1/2 approximation algorithm for the following problem:

Let  $G$  be a given directed graph. Find a set of edges  $E' \subseteq E(G)$  with maximum cardinality such that the resulting subgraph  $(V(G), E')$  is acyclic.

[5 points]

#### Exercise 2 – Smallest maximal matching

Give a factor-2 approximation algorithm for the following problem:

Let  $G$  be a given undirected graph. Find a maximal matching with smallest cardinality in  $G$ .

[5 points]

#### Exercise 3 – Greedy for MAXIMUM CUT

Let  $G$  be a given graph. The MAXIMUM CUT problem asks for a partition of the vertex set  $V(G)$  of  $G$  into two sets  $S$  and  $\bar{S}$  such that the number of edges connecting these sets is as large as possible.

Consider the following greedy algorithm for MAXIMUM CUT. For a set  $U \subseteq V(G)$  and a vertex  $v \in V(G) \setminus U$ , let  $\deg(v, U)$  denote the number of edges between  $v$  and  $U$ .

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**Algorithm 1: Greedy-Max-Cut( $G$ )**

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choose two arbitrary vertices  $v_1, v_2 \in V(G)$

$S \leftarrow \{v_1\}$

$\bar{S} \leftarrow \{v_2\}$

**foreach**  $v \in V(G) \setminus \{v_1, v_2\}$  **do**

**if**  $\deg(v, S) \geq \deg(v, \bar{S})$  **then**

$\bar{S} \leftarrow \bar{S} \cup \{v\}$

**else**

$S \leftarrow S \cup \{v\}$

**return** Cut  $(S, \bar{S})$

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- a) Let  $G^n$  be the undirected graph with vertex set  $\{v_1, \dots, v_n\}$  and edge set  $\{v_1v_i, v_2v_i : 3 \leq i \leq n\}$ . Which result does Greedy-Max-Cut yield, assuming the vertices are handled in the order of their indices? What is the optimal solution? What is the approximation ratio of the algorithm for the graph class  $\{G^n : n \in \mathbb{N}, n \geq 3\}$ ?

[3 points]

- b) Show that the greedy algorithm is a factor- $\frac{1}{2}$  approximation algorithm for arbitrary graphs (with at least two vertices).

*Suggestion:* The following definition could be helpful in your analysis. Let  $G$  be a graph, let  $S$  be the vertex set defined by the algorithm above, and let  $v \in S$ . Let  $\text{inter}(v)$  and  $\text{intra}(v)$  denote the values  $\deg(v, \bar{S})$  and  $\deg(v, S)$  *at the point of time* when the algorithm considers  $v$ , respectively. Note that these values change during the execution of the algorithm. For  $v \in \bar{S}$ ,  $\text{inter}(v)$  and  $\text{intra}(v)$  are defined symmetrically.

**[7 points]**