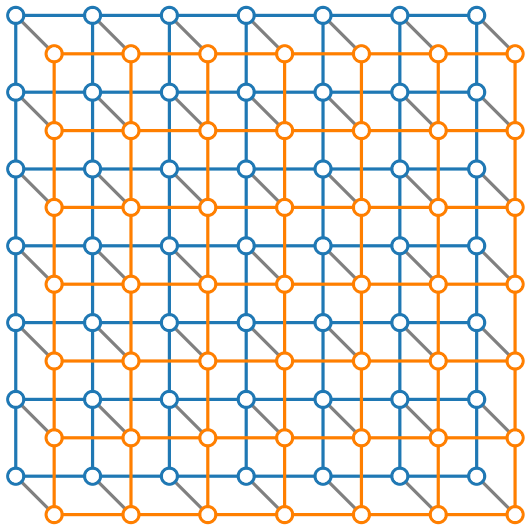
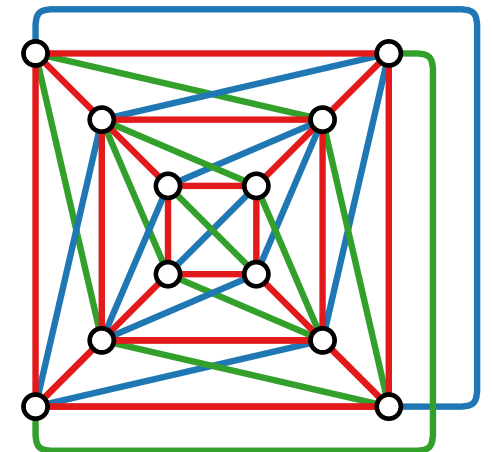
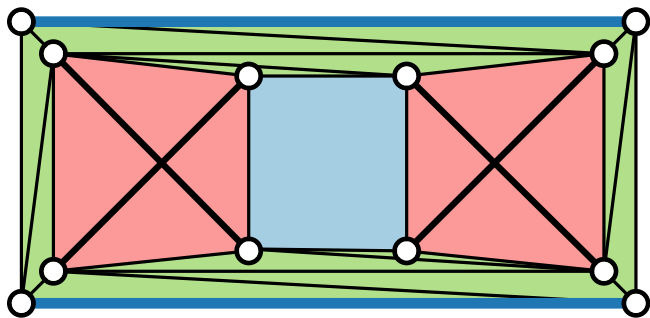
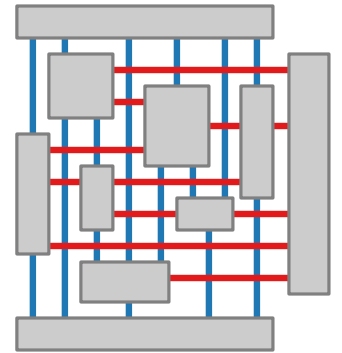


Visualization of Graphs



Lecture 11: Beyond Planarity Drawing Graphs with Crossings

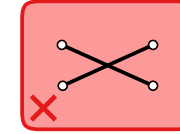


Alexander Wolff

Summer semester 2026

Planar Graphs

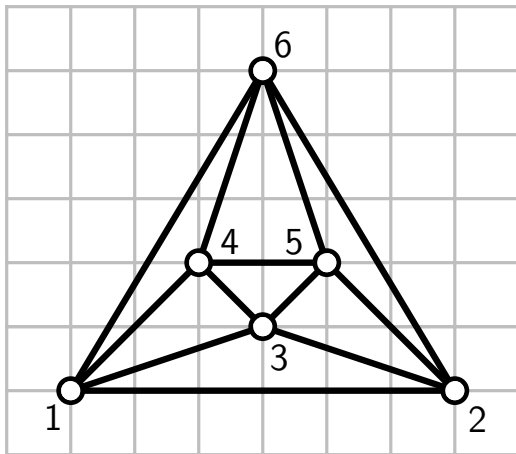
Planar graphs admit drawings in the plane without crossings.



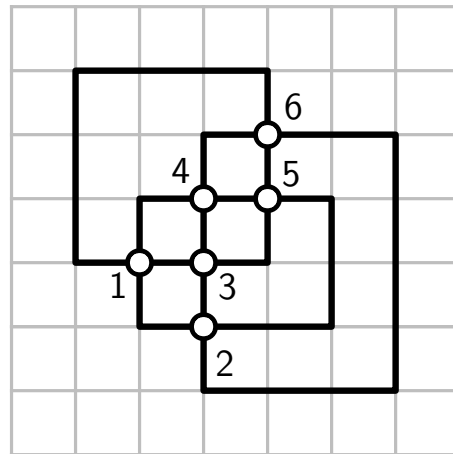
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

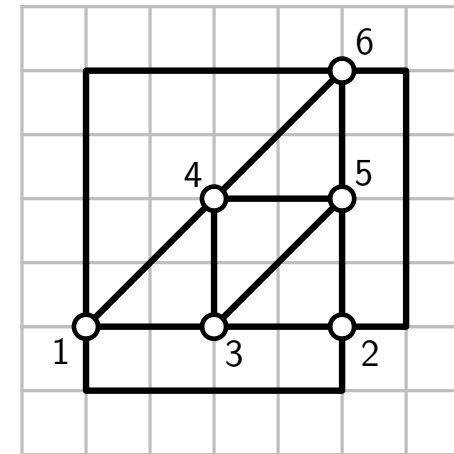
Different drawing styles ...



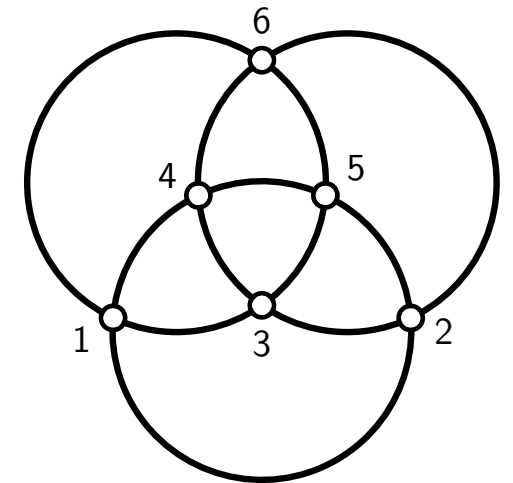
straight-line drawing



orthogonal drawing



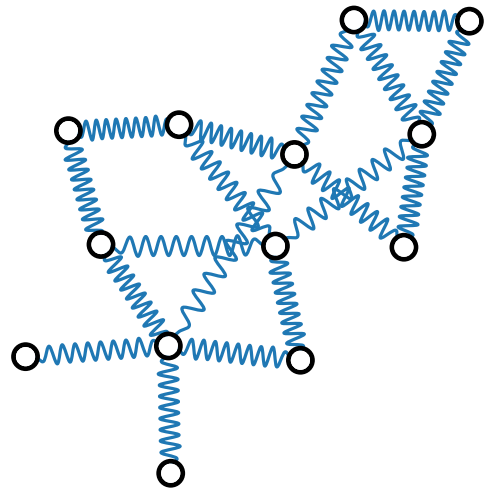
grid drawing with
bends & 3 slopes



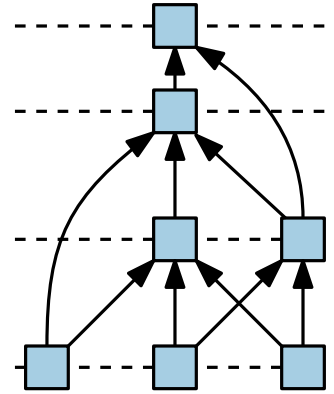
circular-arc drawing

And Non-Planar Graphs?

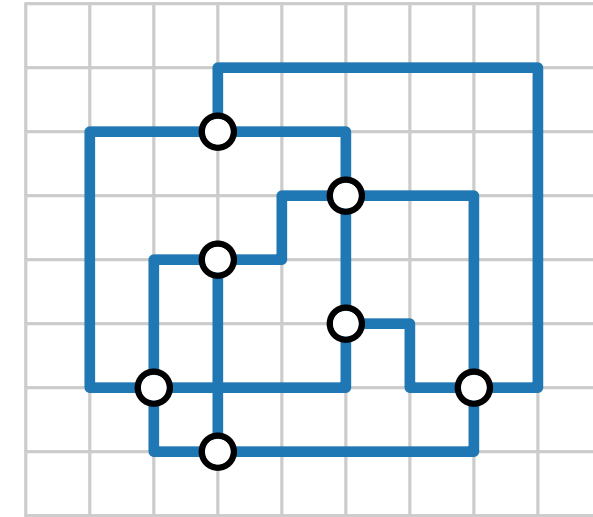
We have seen a few drawing styles:



force-directed drawing

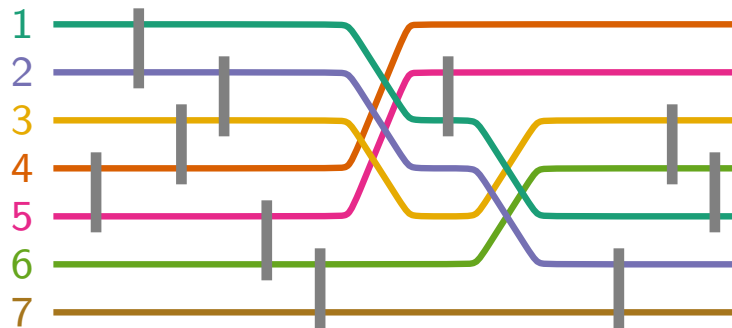


hierarchical drawing

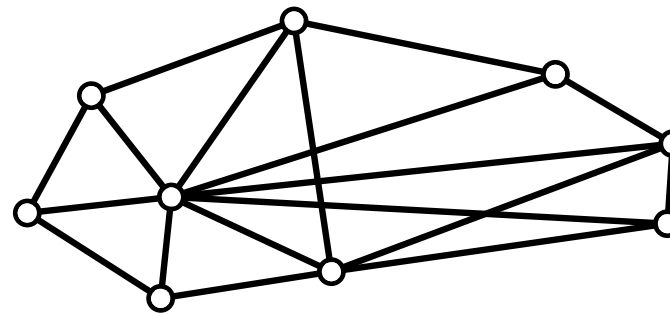


orthogonal layouts
(via planarization)

Maybe not all crossings are equally bad?



block crossings



Which crossings feel worse?

Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results: no crossings

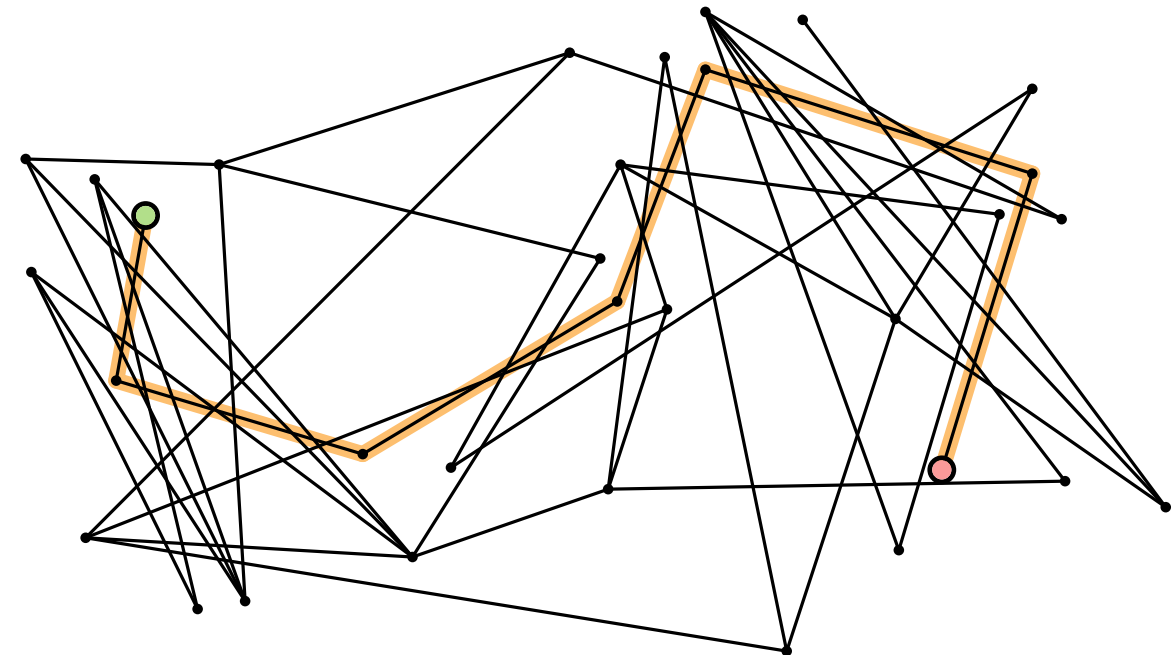
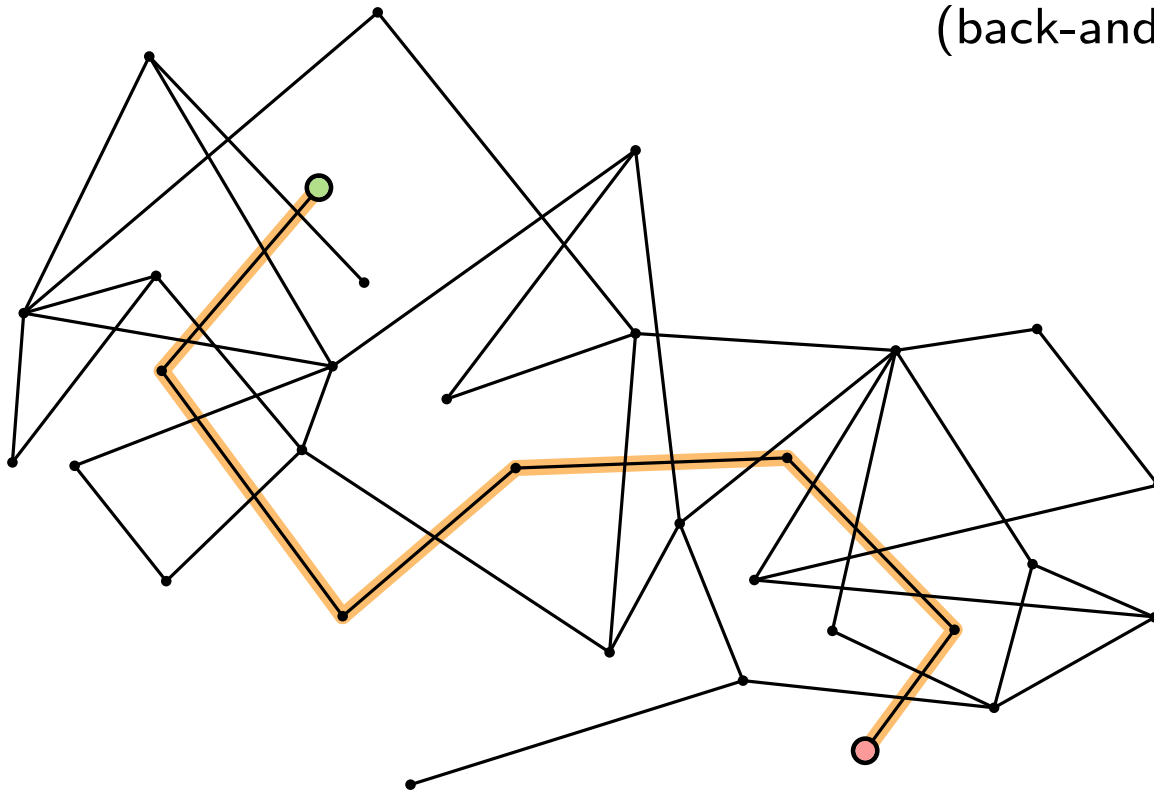
eye movements smooth and fast

large crossing angles

eye movements smooth but slightly slower

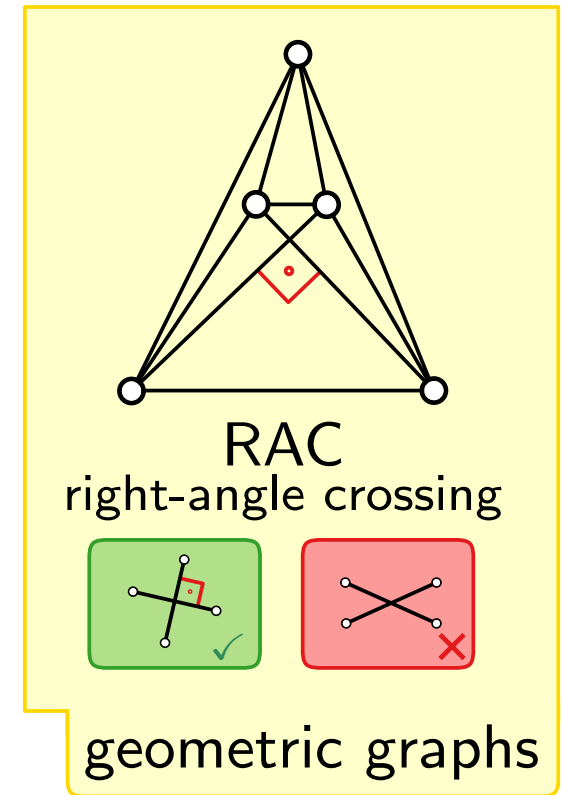
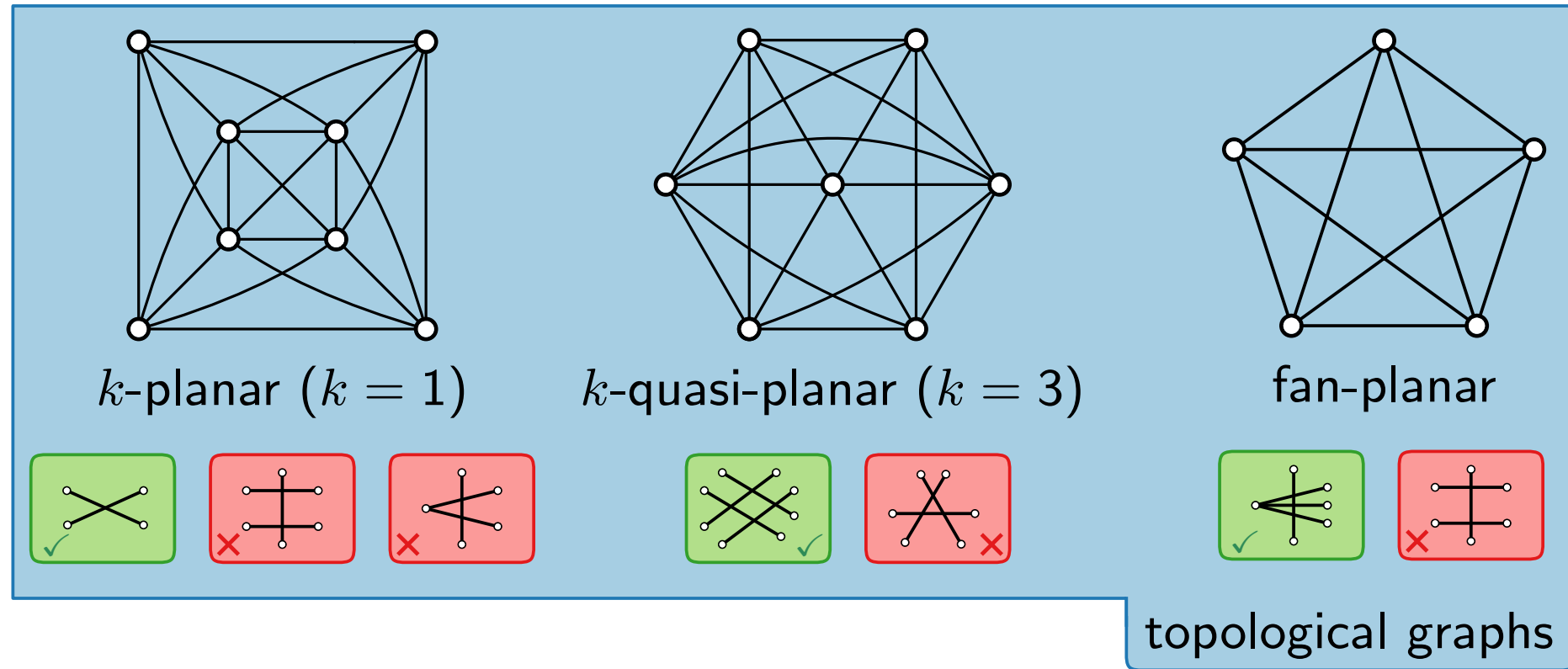
small crossing angles

eye movements no longer smooth and very slow
(back-and-forth movements at crossing points)



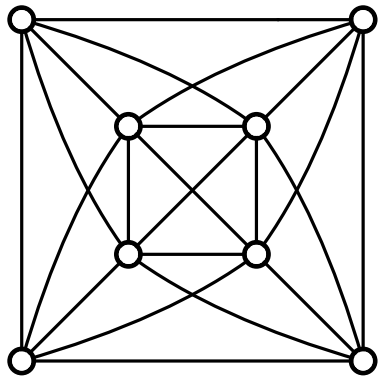
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.

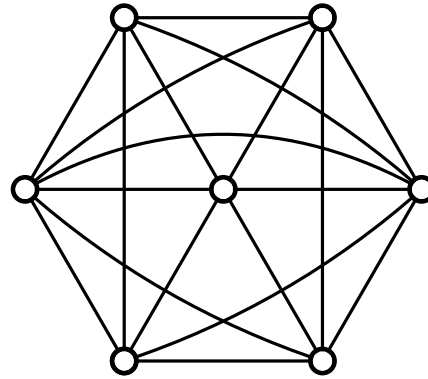
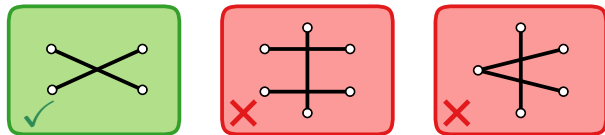


Some Beyond-Planar Graph Classes

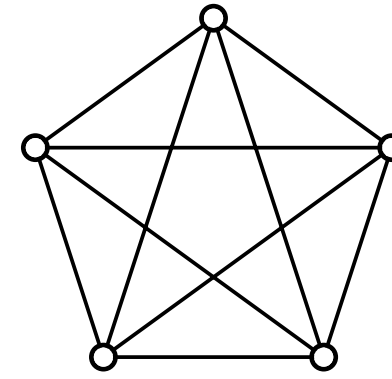
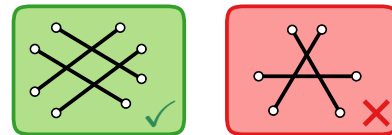
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



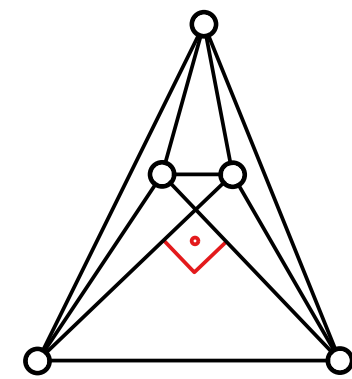
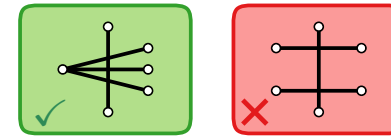
k -planar ($k = 1$)



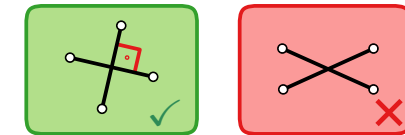
k -quasi-planar ($k = 3$)



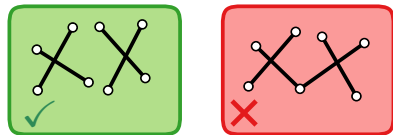
fan-planar



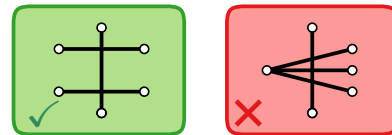
RAC
right-angle crossing



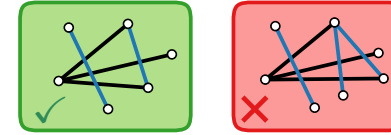
There are many more beyond-planar graph classes...



IC (independent crossing)



fan-crossing-free

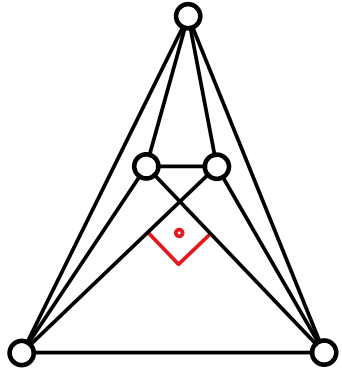


skewness- k ($k = 2$)

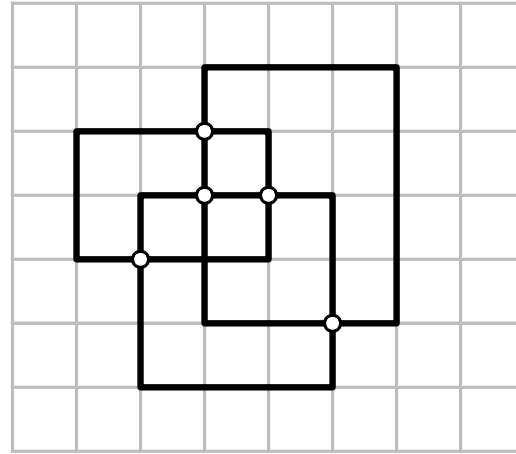
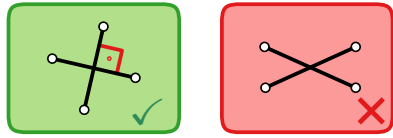
remove $\leq k$ edges to make it planar

combinations, ...

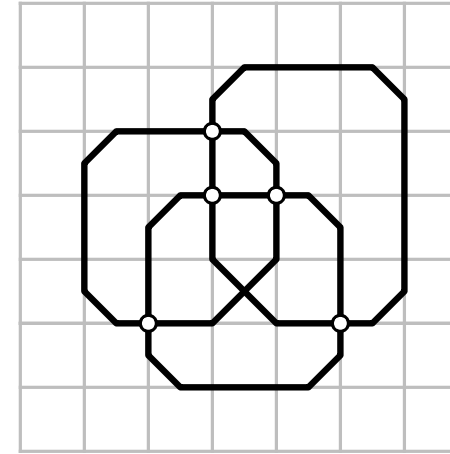
Drawing Styles for Crossings



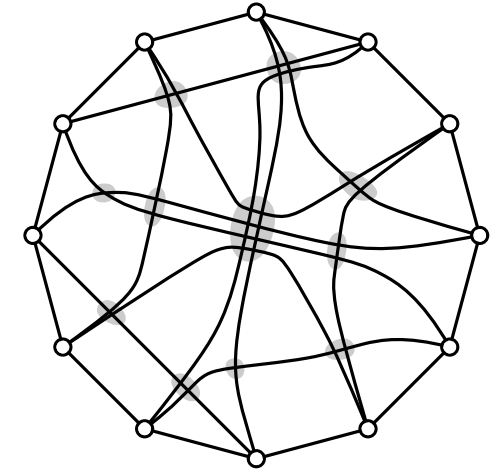
RAC
right-angle crossing



orthogonal

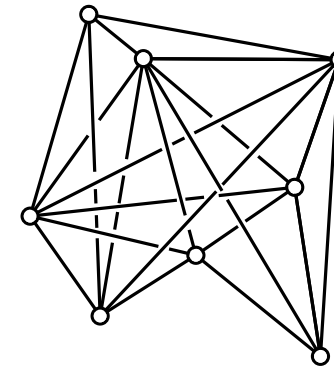


slanted orthogonal

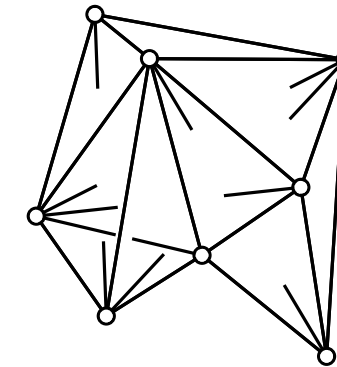


block / bundled crossings

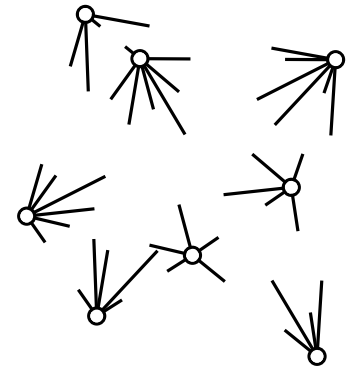
circular layout: 28 individual
vs. 12 bundle crossings



cased crossings

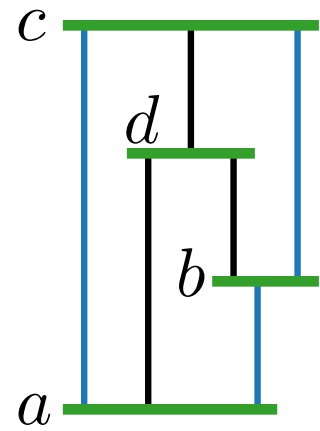
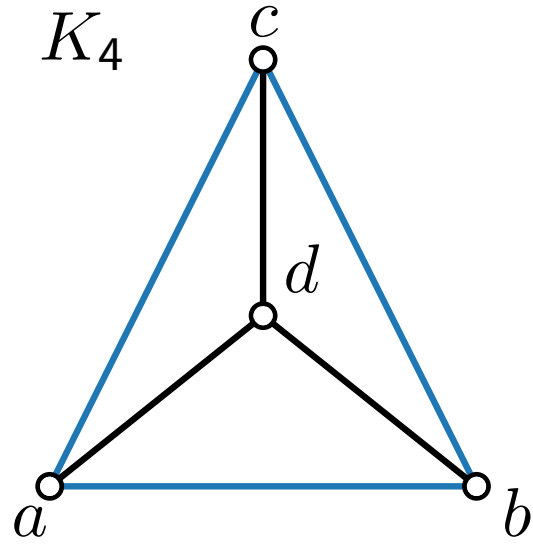


symmetric partial
edge drawing



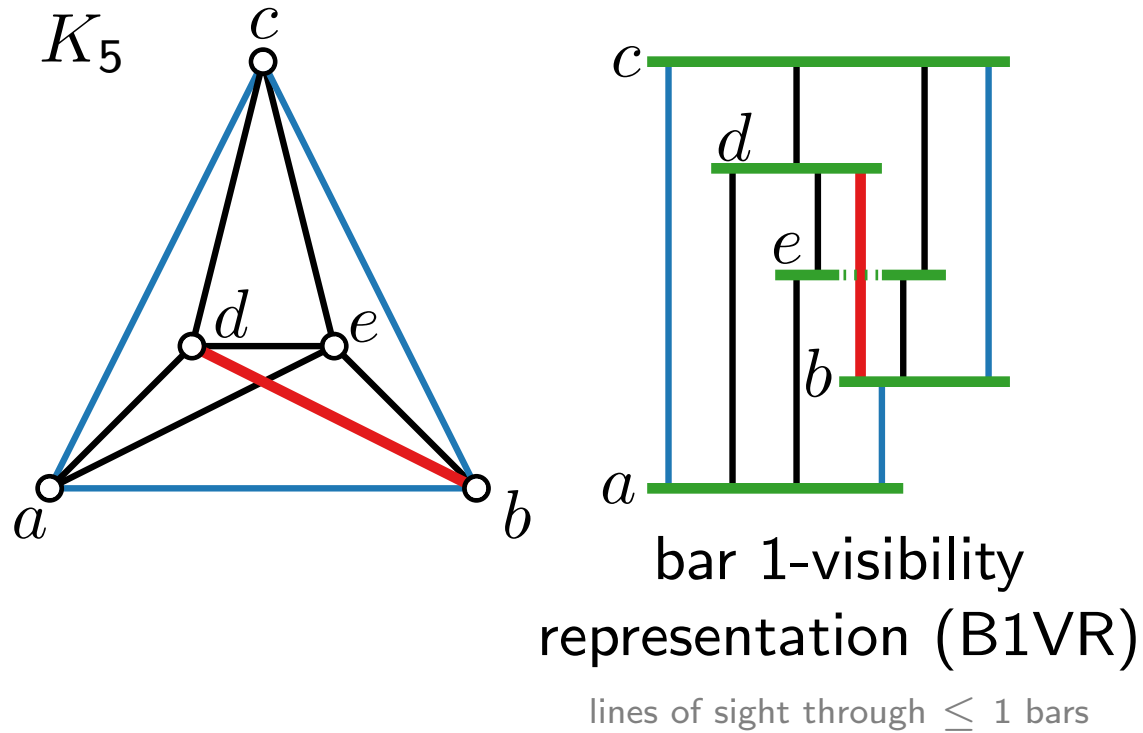
1/4-SHPED
symmetric homogenous
partial edge drawing

Geometric Representations

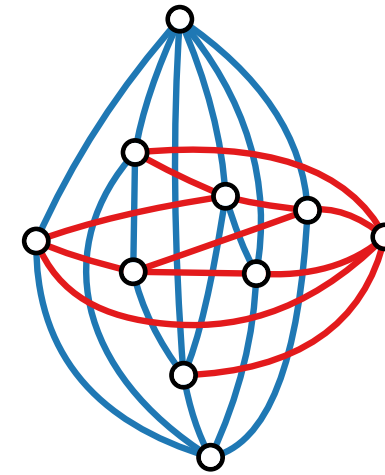


bar visibility
representation

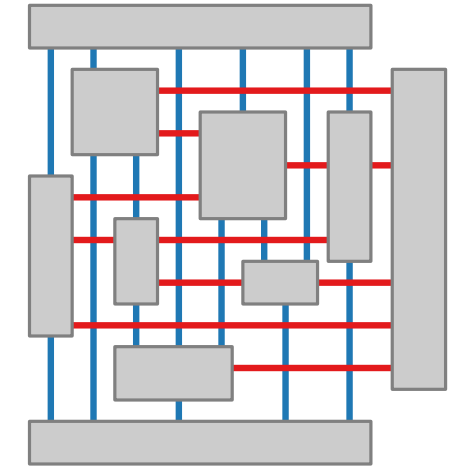
Geometric Representations



- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

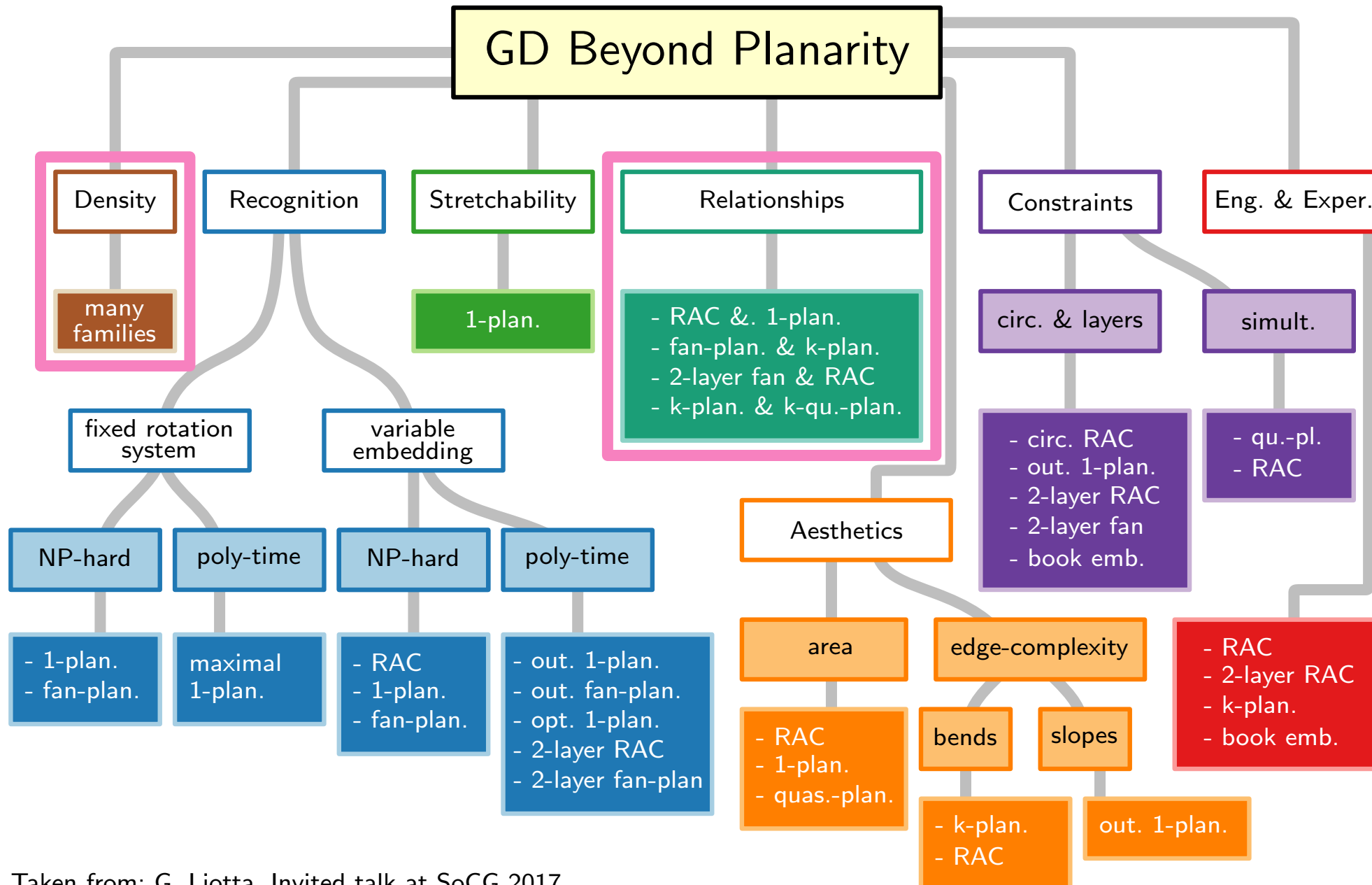


decompose into 2 planar graphs



- Rectangle visibility graphs (RVGs) have $\leq 6n - 20$ edges. [Hutchinson, Shermer, Vince 1996]
- Recognizing thickness-2 graphs and RVGs is NP-hard. [Mansfields 1983] [Shermer 1996]
- RVGs can be recognized efficiently if embedding is fixed. [Biedl, Liotta, Montecchiani 2018]

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]

A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.

Proof sketch.

- Let the **red** edges be those that do not cross.
- Each **blue** edge crosses a **green** edge.
- This yields a **red-blue** plane graph G_{rb} with

$$m_{rb} \leq 3n - 6$$

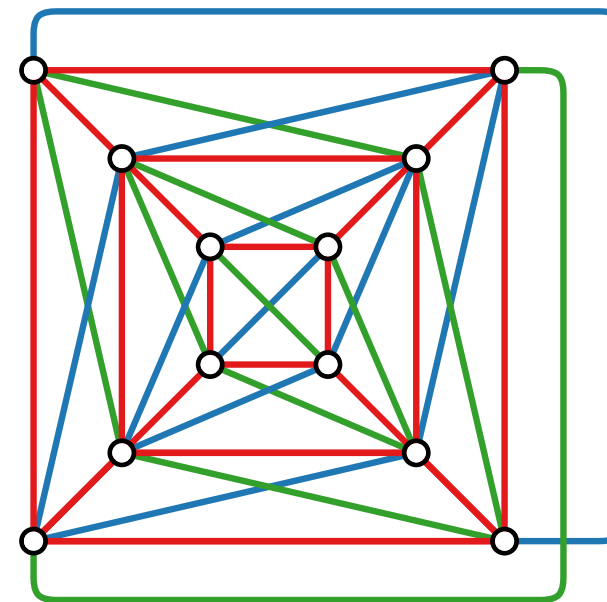
- and a **green** plane graph G_g with

$$m_g \leq 3n - 6 \quad \Rightarrow \quad m \leq m_{rb} + m_g \leq 6n - 12$$

- Observe that each **green** edge joins two faces in G_{rb} .

$$m_g \leq f_{rb}/2 \leq (2n - 4)/2 = n - 2$$

$$\Rightarrow m = m_{rb} + m_g \leq 3n - 6 + n - 2 = 4n - 8$$



Lower-bound construction:

$2n - 4$ edges

$n - 2$ faces

Edges per face: 2 edges

Total: $4n - 8$ edges

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]

A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.

A 1-planar graph with n vertices is called **optimal** if it has exactly $4n - 8$ edges.

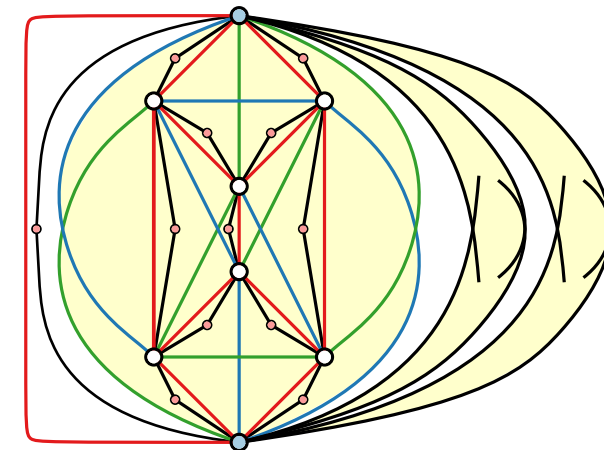
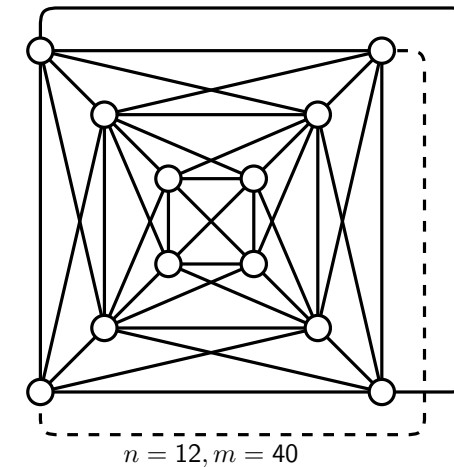
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.

Theorem. [Brandenburg et al. 2013]

There are **maximal** 1-planar graphs with n vertices and $45/17n - O(1) \approx 2.65n - O(1)$ edges.

Theorem. [Didimo 2013]

A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n - 9$ edges.



Idea: in a drawing of an optimal 1-planar graph, we cannot realize the crossing on the outer face with two straight-line edges.

Density of k -Planar Graphs

Theorem.

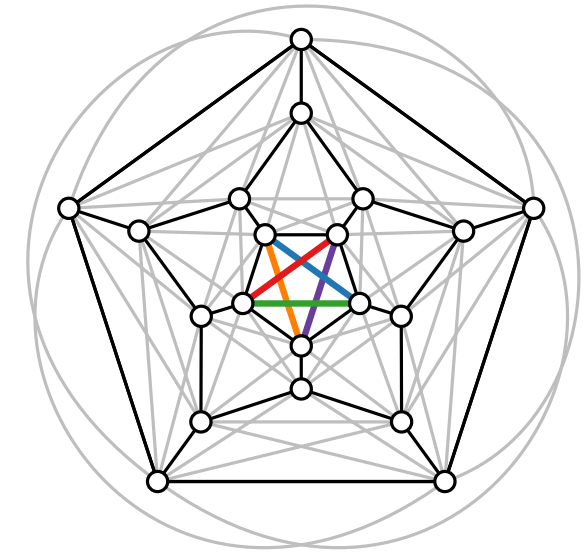
A k -planar graph with n vertices has at most:

k	number of edges	
0	$3(n - 2)$	Euler's formula
1	$4(n - 2)$	[Ringel 1965]
2	$5(n - 2)$	[Pach and Tóth 1997]

$$n - m + f = 2$$

$$m = c \cdot f ?$$

$$m = \frac{5}{2} f$$



optimal 2-planar

Planar structure:

$$\frac{5}{3}(n - 2) \text{ edges}$$

$$\frac{2}{3}(n - 2) \text{ faces}$$

Edges per face: 5 edges

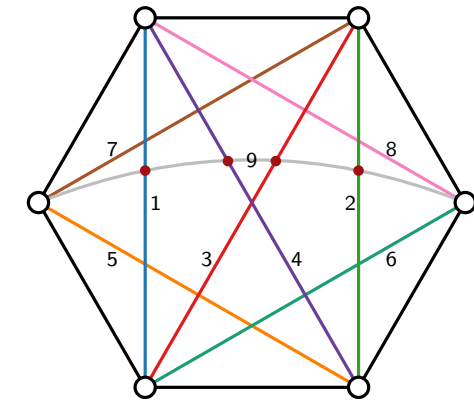
Total: $5(n - 2)$ edges

Density of k -Planar Graphs

Theorem.

A k -planar graph with n vertices has at most:

k	number of edges	
0	$3(n - 2)$	Euler's formula
1	$4(n - 2)$	[Ringel 1965]
2	$5(n - 2)$	[Pach and Tóth 1997]
3	$5.5(n - 2)$	[Pach et al. 2006]



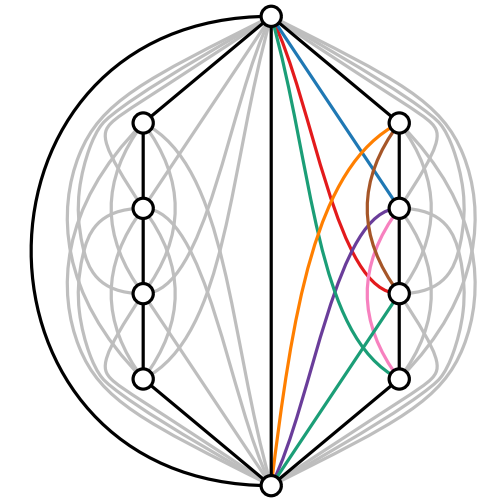
optimal 3-planar

Density of k -Planar Graphs

Theorem.

A k -planar graph with n vertices has at most:

k	number of edges	
0	$3(n - 2)$	Euler's formula
1	$4(n - 2)$	[Ringel 1965]
2	$5(n - 2)$	[Pach and Tóth 1997]
3	$5.5(n - 2)$	[Pach et al. 2006]



optimal 3-planar

Planar structure:

$$\frac{3}{2}(n - 2) \text{ edges}$$

$$\frac{1}{2}(n - 2) \text{ faces}$$

Edges per face: 8 edges

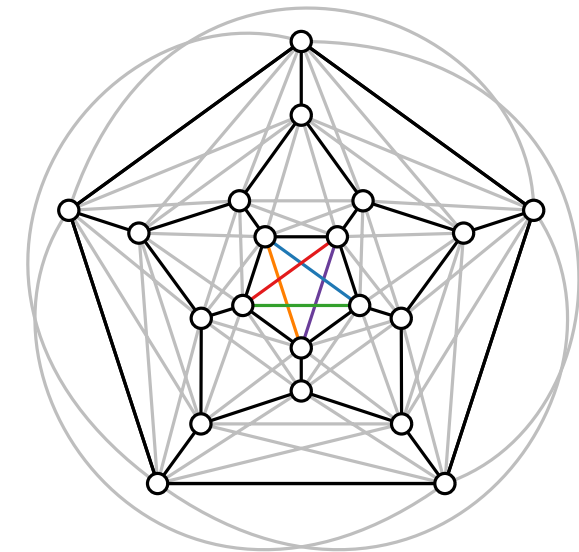
Total: $5.5(n - 2)$ edges

Density of k -Planar Graphs

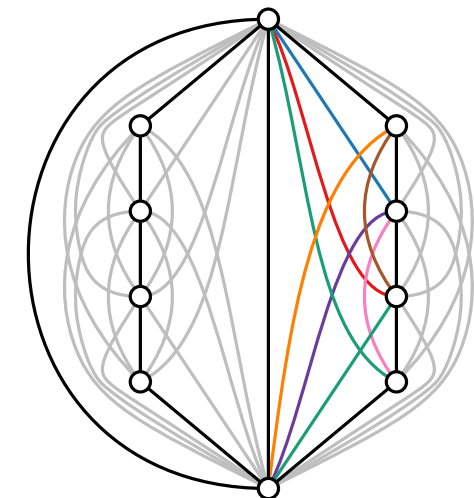
Theorem.

A k -planar graph with n vertices has at most:

k	number of edges	
0	$3(n - 2)$	Euler's formula
1	$4(n - 2)$	[Ringel 1965]
2	$5(n - 2)$	[Pach and Tóth 1997]
3	$5.5(n - 2)$	[Pach et al. 2006]
4	$6(n - 2)$	[Ackerman 2015]
> 4	$4.108\sqrt{k}n$	[Pach and Tóth 1997]

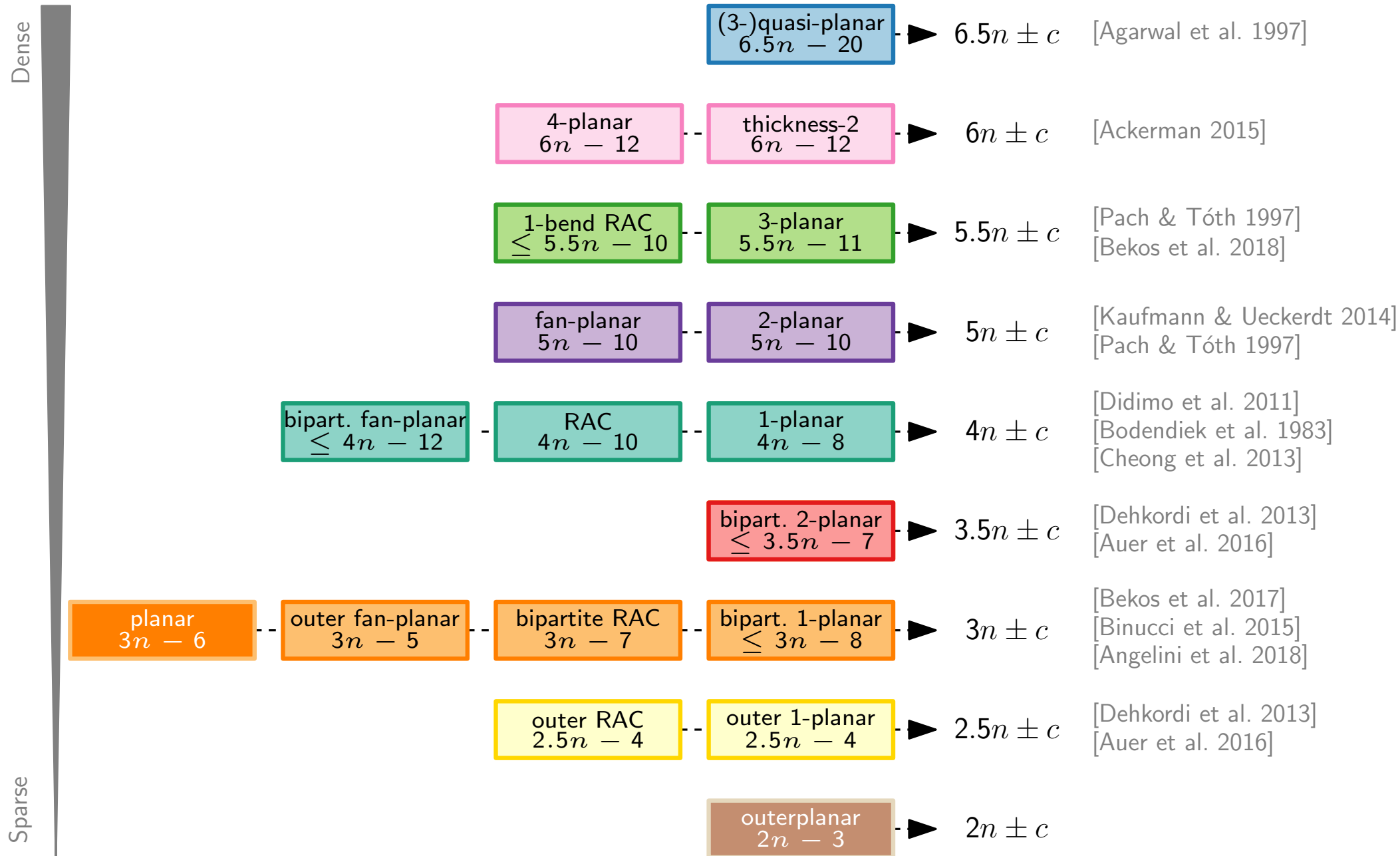


optimal 2-planar



optimal 3-planar

GD Beyond Planarity: a Hierarchy



Crossing Numbers

The **k -planar crossing number** $cr_{k\text{-pl}}(G)$ of a k -planar graph G is the number of crossings required in any k -planar drawing of G .

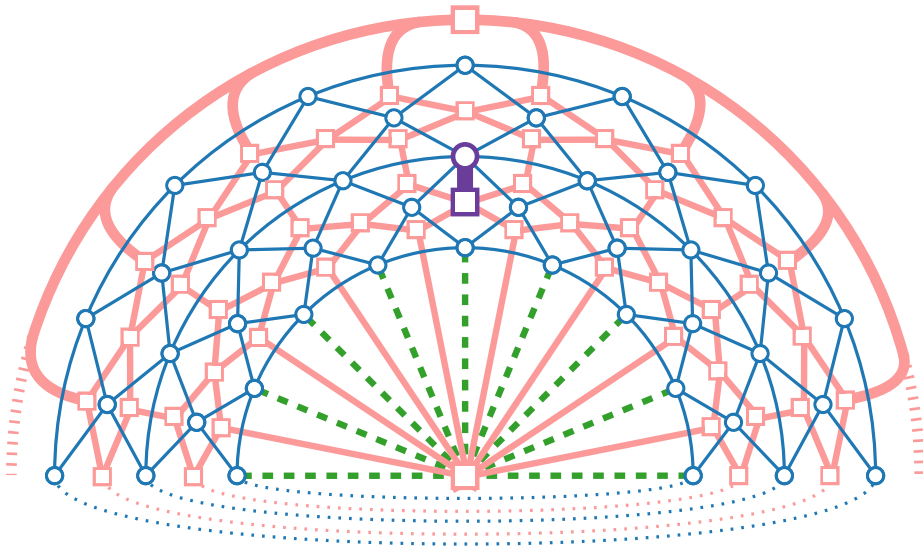
- $cr_{1\text{-pl}}(G) \leq n - 2$ (there are at most $n - 2$ green edges in the coloring of Theorem 1)
- $cr(G) = 1 \Rightarrow cr_{1\text{-pl}}(G) = 1$

Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]

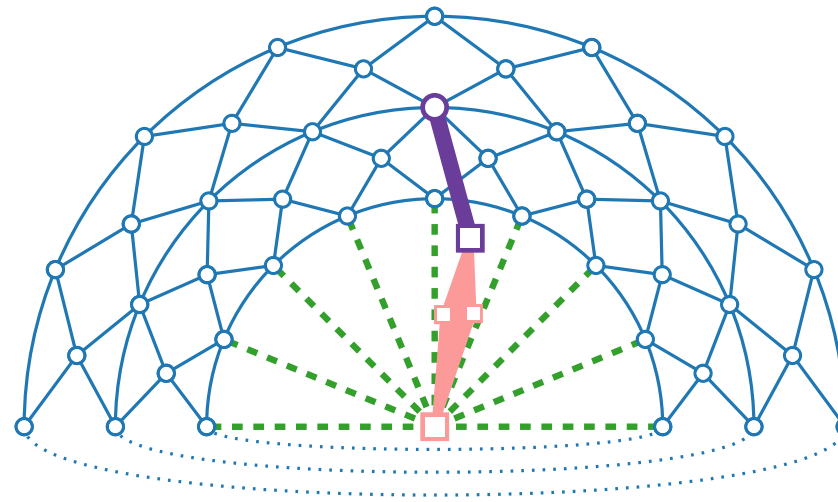
For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $cr(G) = 2$ and $cr_{1\text{-pl}}(G) = n - 2$.

Crossing ratio

$$\rho_{1\text{-pl}}(n) = (n - 2)/2$$



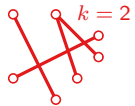
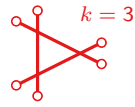
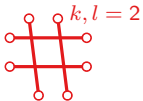
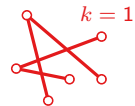
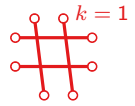
$$cr_{1\text{-pl}}(G) = n - 2$$



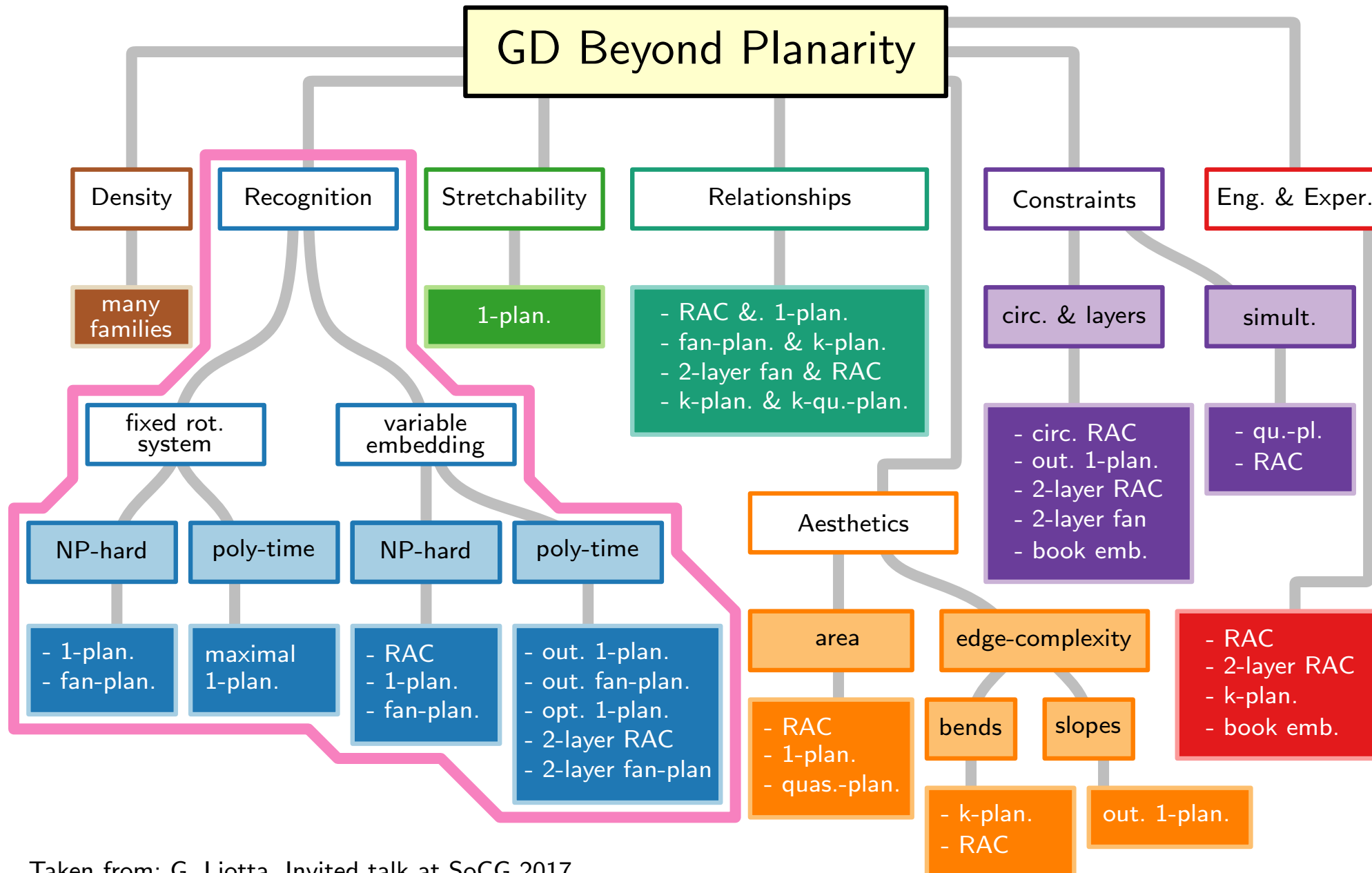
$$cr(G) = 2$$

Crossing Ratios

Table from “Crossing Numbers of Beyond-Planar Graphs Revisited”
[van Beusekom, Parada & Speckmann 2021]

Family	Forbidden Configurations		Lower	Upper
k -planar	An edge crossed more than k times		$\Omega(n/k)$	$O(k\sqrt{kn})$
k -quasi-planar	k pairwise crossing edges		$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different “side”		$\Omega(n)$	$O(n^2)$
(k, l) -grid-free	Set of k edges such that each edge crosses each edge from a set of l edges.		$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k, l)n^2$
k -gap-planar	More than k crossings mapped to an edge in an optimal mapping		$\Omega(n/k^3)$	$O(k\sqrt{kn})$
Skewness- k	Set of crossings not covered by at most k edges		$\Omega(n/k)$	$O(kn + k^2)$
k -apex	Set of crossings not covered by at most k vertices		$\Omega(n/k)$	$O(k^2n^2 + k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge		$\Omega(n^2)$	$O(n^2)$
k -fan-crossing-free	An edge that crosses k adjacent edges		$\Omega(n^2/k^3)$	$O(k^2n^2)$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$		$\Omega(n^2)$	$O(n^2)$

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

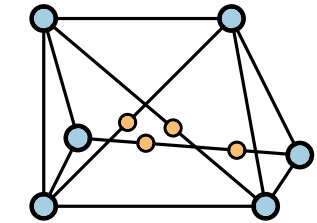
Minors of 1-Planar Graphs

Theorem.

[Kuratowski 1930]

G planar $\Leftrightarrow$ neither K_5 nor $K_{3,3}$ minor of G

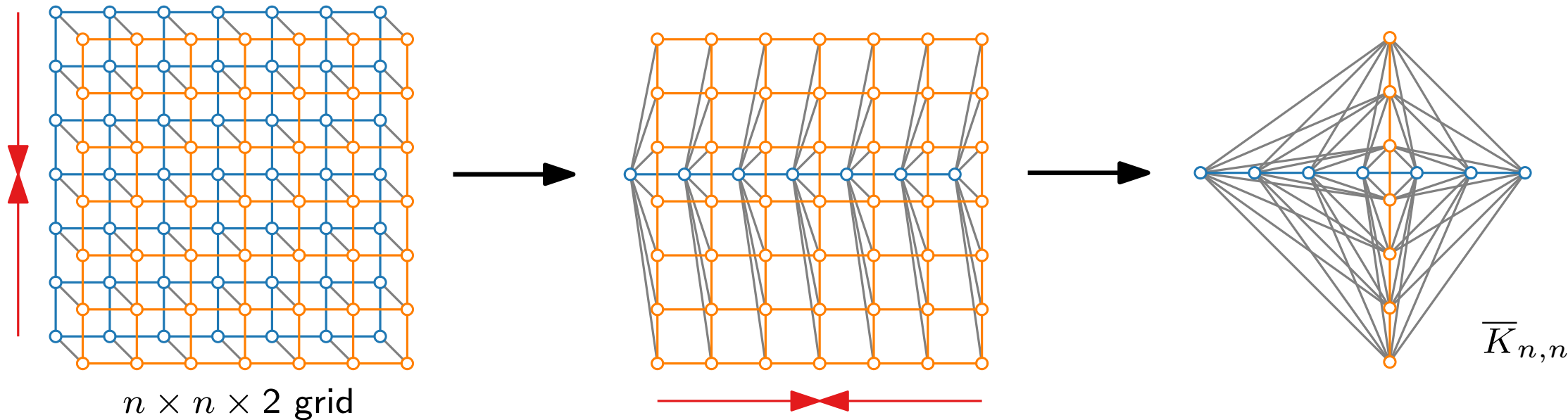
For every graph there is a 1-planar subdivision.



Theorem.

[Chen & Kouno 2005]

The class of 1-planar graphs is not closed under edge contraction.



Theorem.

[Korzhik & Mohar 2013]

For any n , there exist $\Omega(2^n)$ distinct n -vertex graphs that are not 1-planar but all their proper subgraphs are 1-planar.

Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

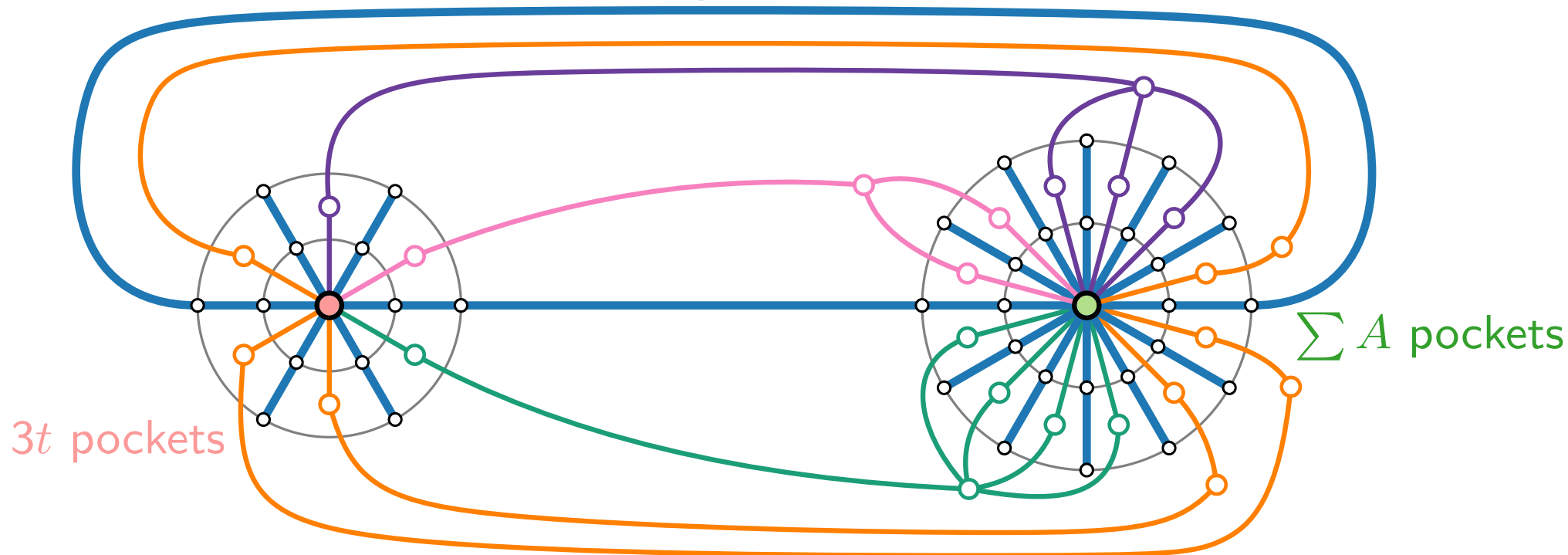
Proof Idea.

Reduction from 3-Partition.

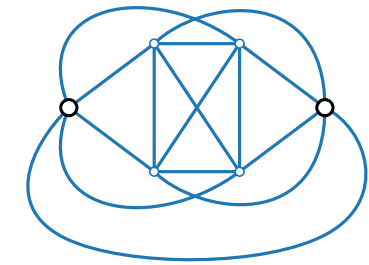
Given a multiset $A = \{a_1, a_2, \dots, a_{3t}\}$ of $3t$ numbers, partition the numbers into t triplets such that the sum of every triplet is the same.

$$A = \{\overbrace{1, 3, 2}^6, \overbrace{4, 1, 1}^6\}$$

t "big" faces



The only 1-planar embedding of K_6 :



(cannot be crossed)

Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

Theorem. [Cabello & Mohar 2013]
Testing 1-planarity is NP-complete –
even for almost planar graphs, i.e., planar graphs plus one edge.

Theorem. [Bannister, Cabello & Eppstein 2018]
Testing 1-planarity is NP-complete –
even for graphs of bounded bandwidth (pathwidth, treewidth).

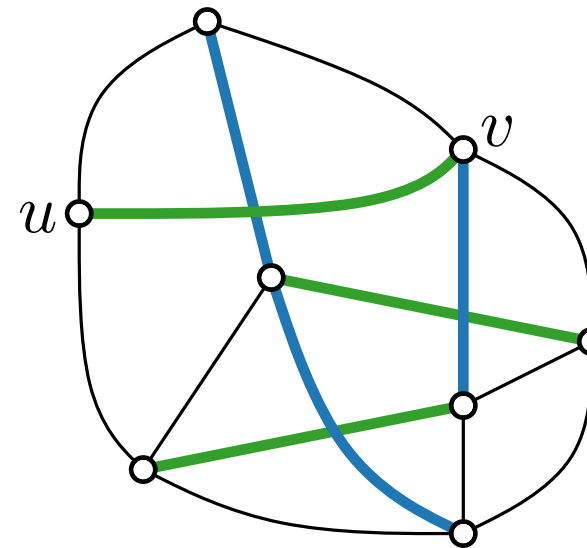
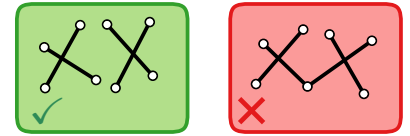
Theorem. [Auer, Brandenburg, Gleißner & Reislhuber 2015]
Testing 1-planarity is NP-complete –
even for 3-connected graphs with a fixed rotation system.

Recognition of IC-Planar Graphs

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.

Proof.

Reduction from 1-planarity testing.

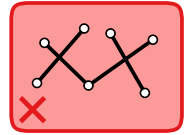
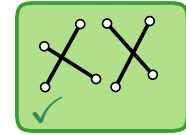
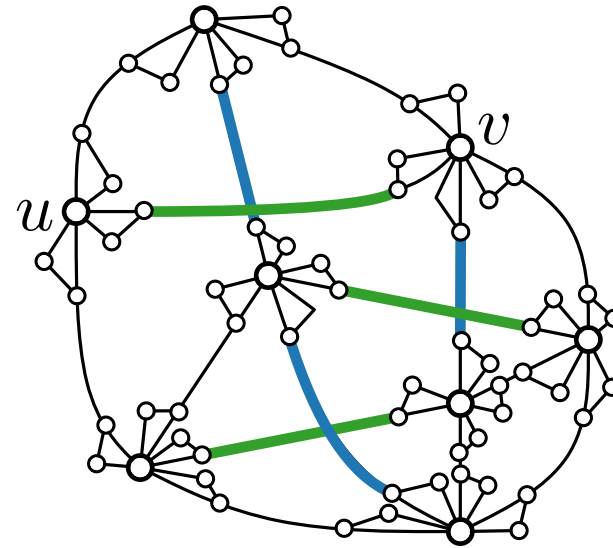
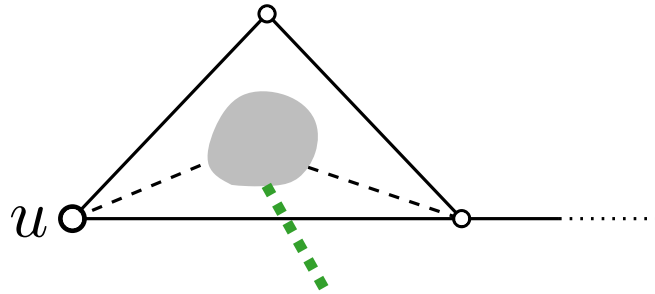


Recognition of IC-Planar Graphs

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.

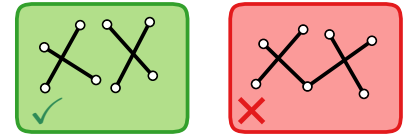
Proof.

Reduction from 1-planarity testing.



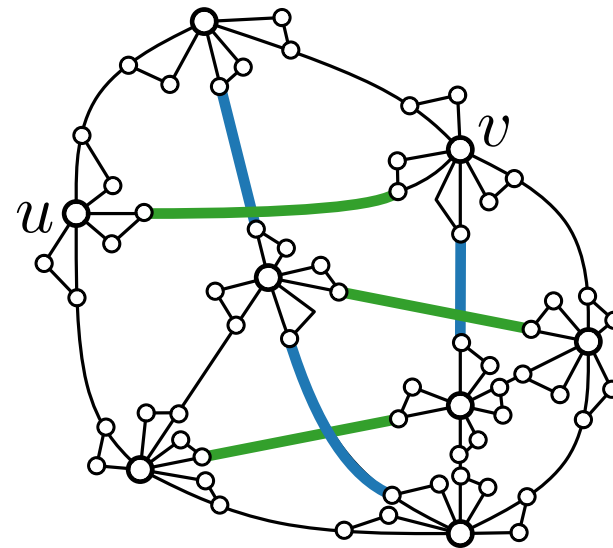
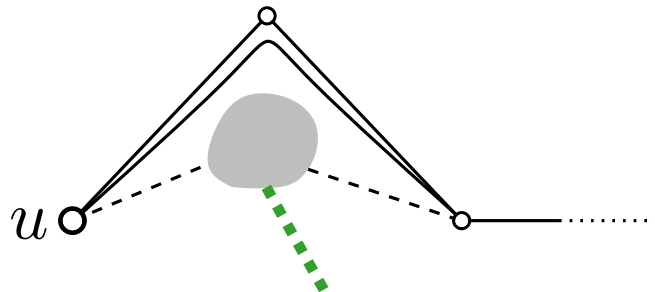
Recognition of IC-Planar Graphs

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.



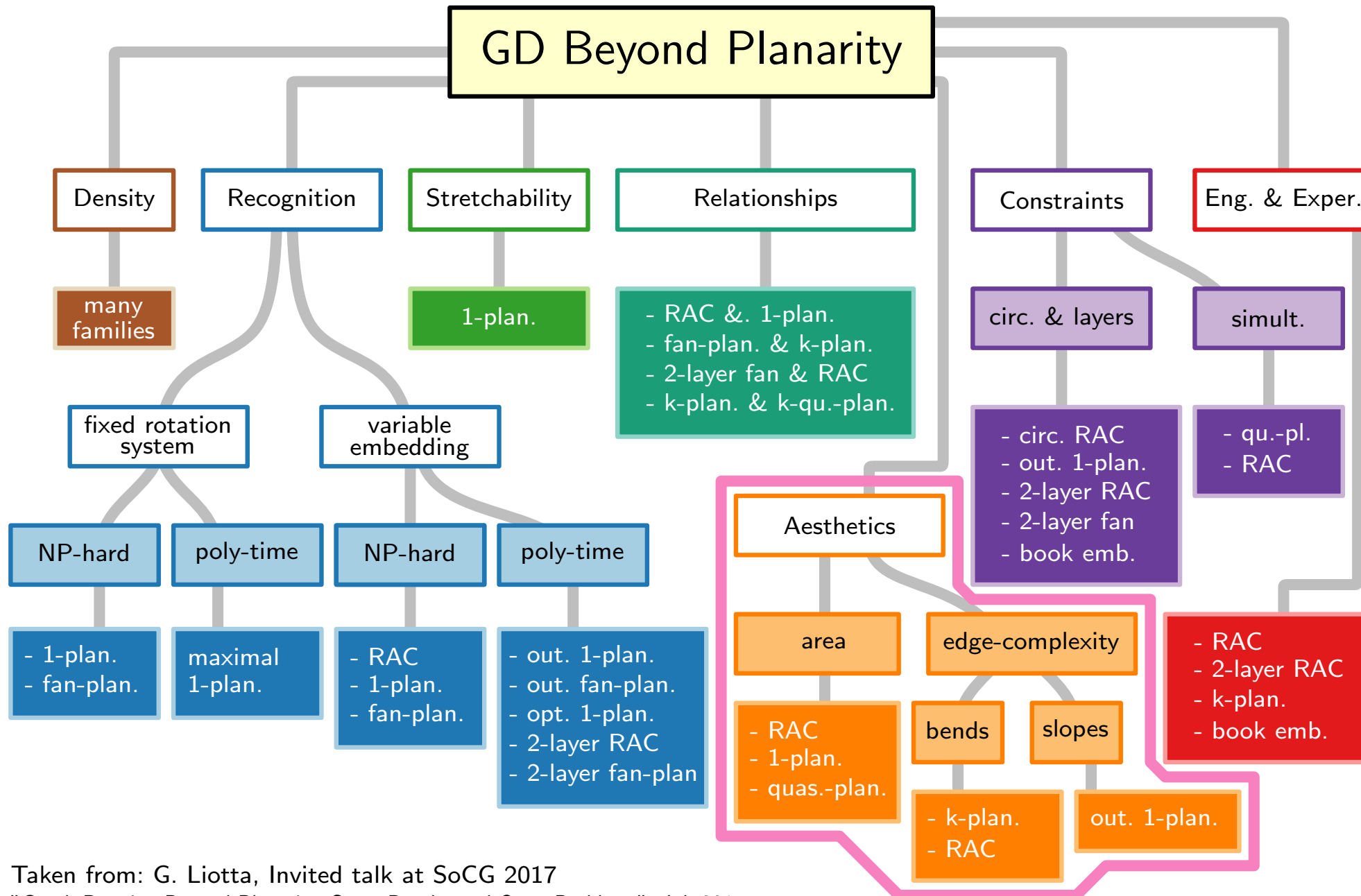
Proof.

Reduction from 1-planarity testing.



Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete,
even if the rotation system is given.

GD Beyond Planarity: a Taxonomy



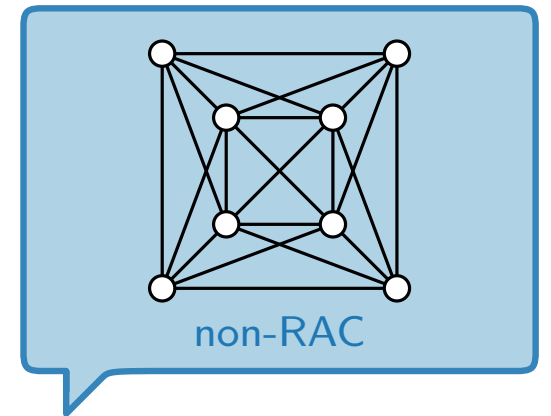
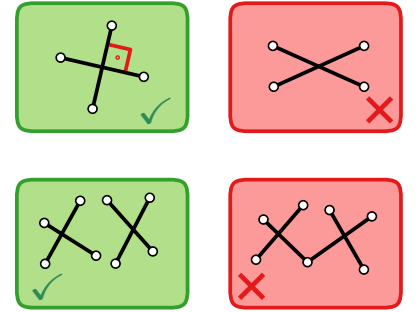
Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

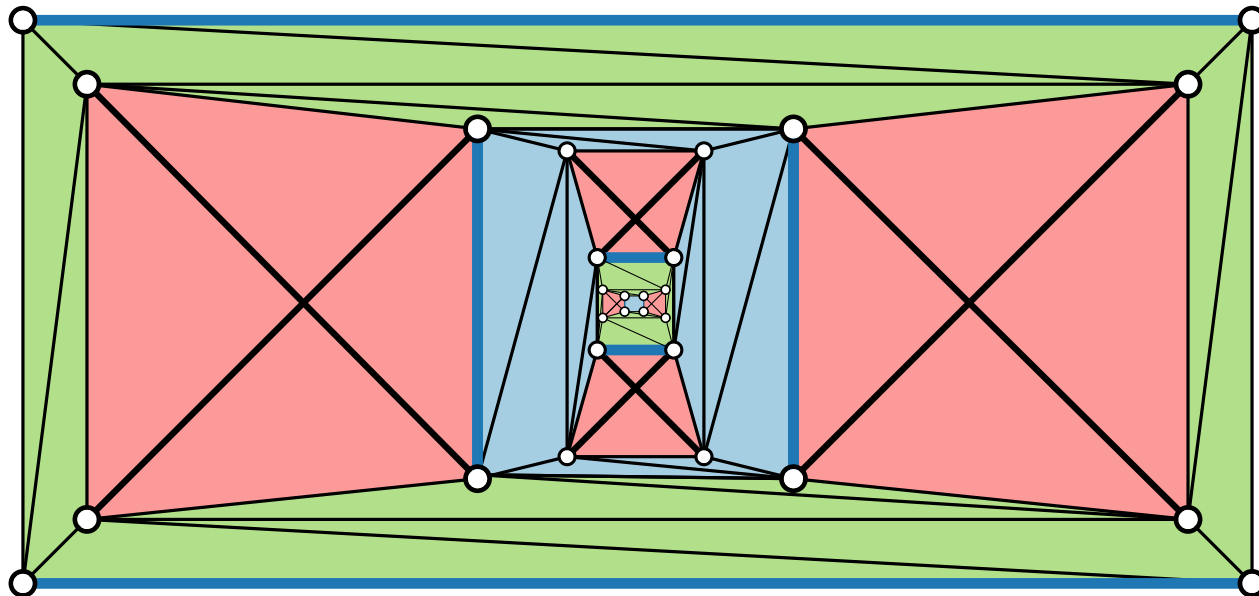
Area of Straight-Line RAC Drawings

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Some IC-planar straight-line RAC drawings require exponential area.

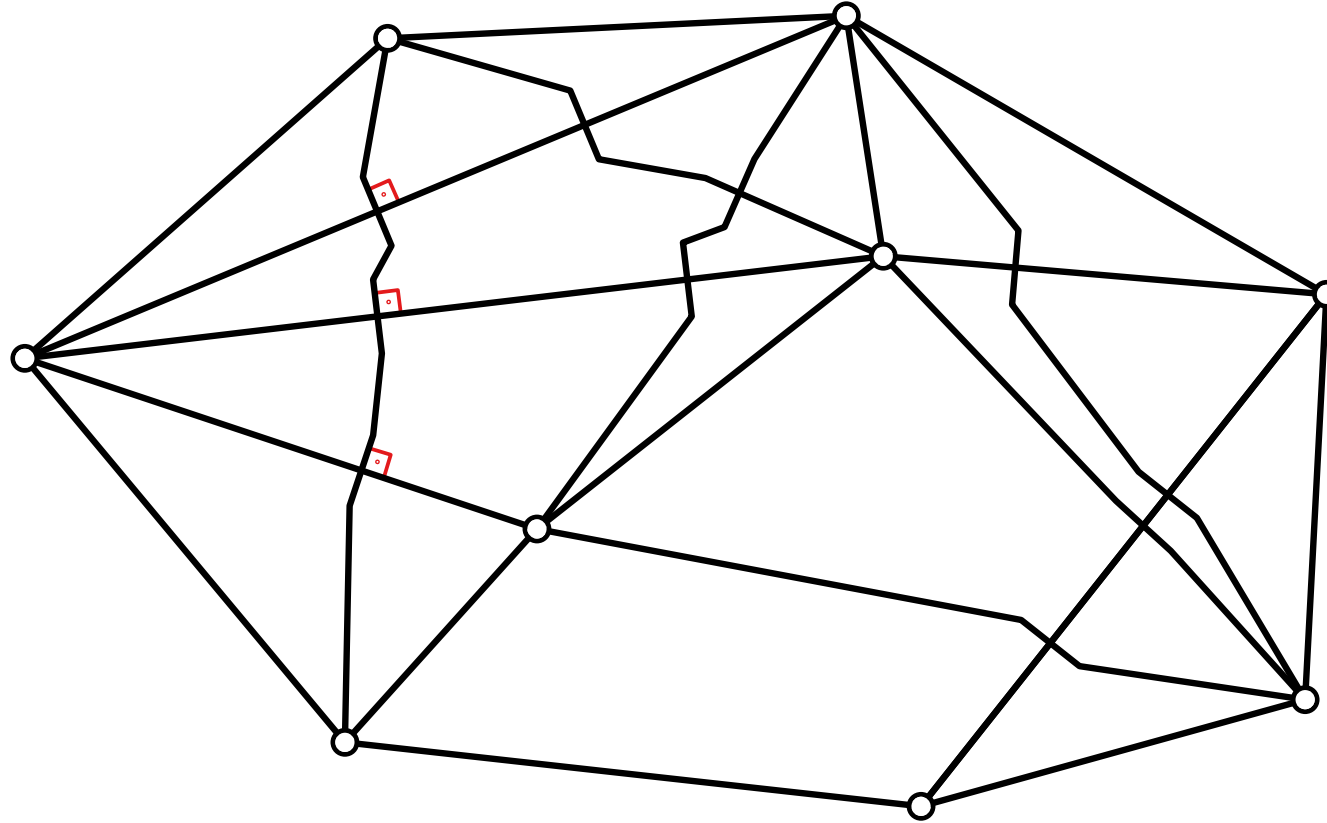
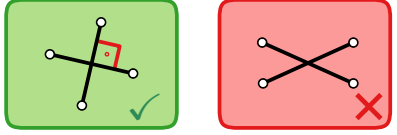
Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Every IC-planar graph has an IC-planar straight-line RAC drawing, and such a drawing can be found in polynomial time.



In contrast:
not every 1-planar graph
admits a straight-line
RAC drawing



RAC Drawings With Enough Bends



Every graph admits a RAC drawing ...
...if we use enough bends.

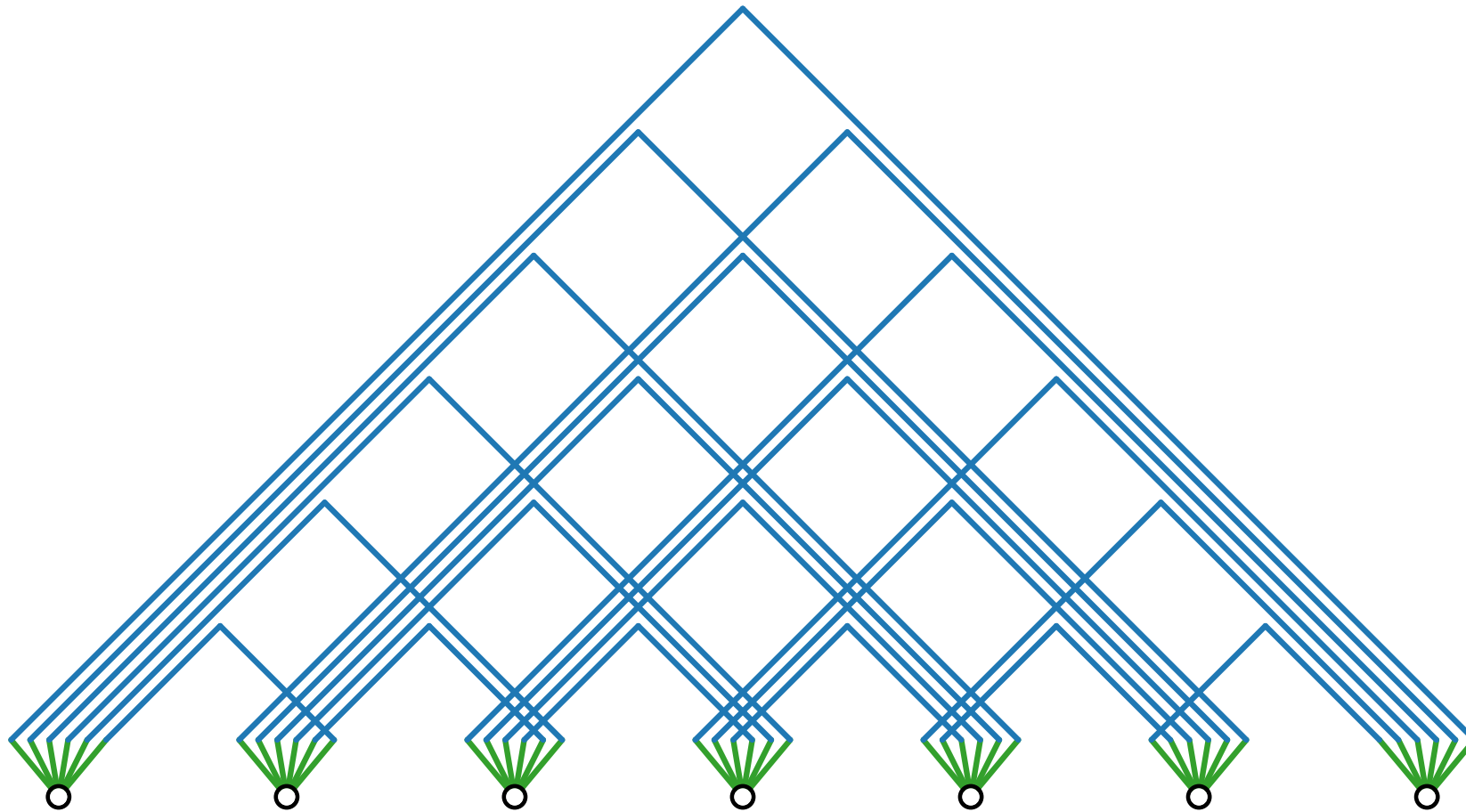
How many do we need – in total or per edge?

3-Bend RAC Drawings

Theorem.

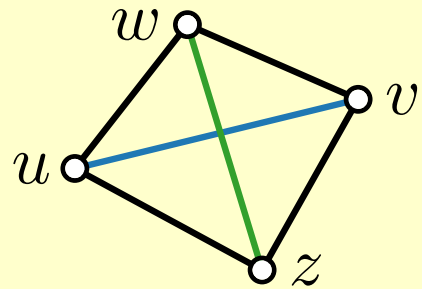
[Didimo, Eades & Liotta 2017]

Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most three bends.

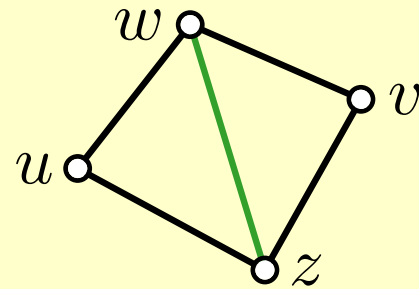


Kite Triangulations

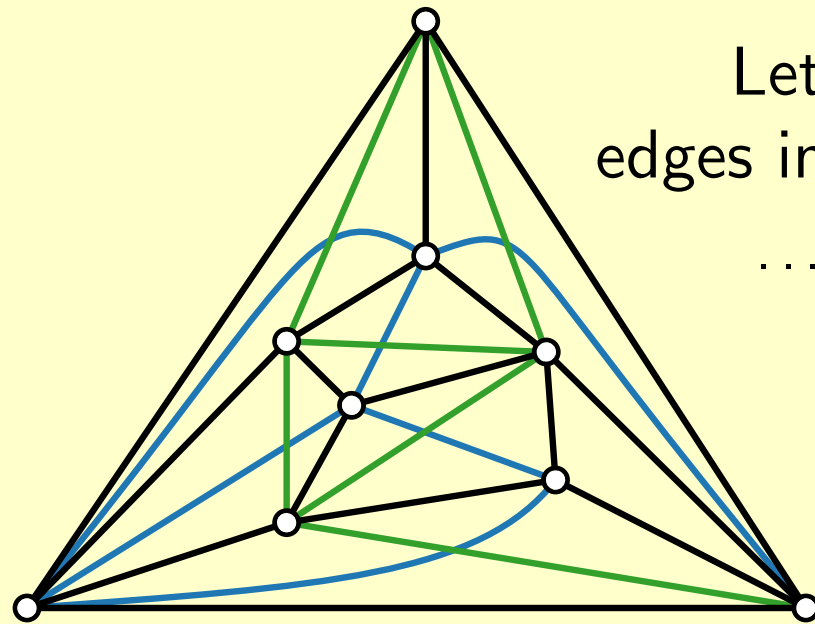
This is a **kite**:



u and v are **opposite**
w.r.t. $\{z, w\}$



Let G' be a plane triangulation.



Let $S \subseteq E(G')$ s.t. no two
edges in S lie on the same face
... and their opposite vertices do
not have an edge in $E(G')$.

Add set T of edges
connecting
opposite vertices.

The resulting graph G is a **kite-triangulation**.

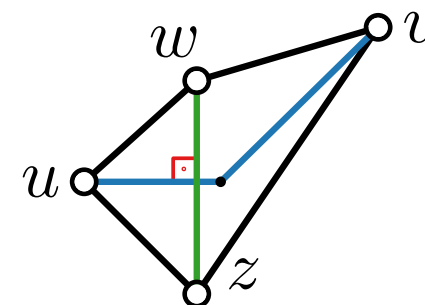
Note: optimal 1-planar graphs $\subsetneq$ kite-triangulations.

Theorem. [Angelini et al. 2011]

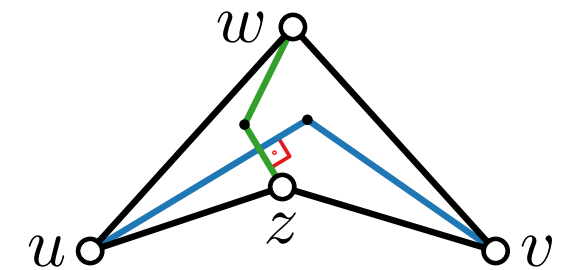
Every **kite**-triangulation G admits a
1-planar 1-bend RAC drawing,
which can be constructed in linear
time.

Proof.

Let G' be the underlying plane trian-
gulation of G . Let $G'' = G' - S$.
Construct straight-line drawing of G'' .
Fill faces as follows:



strictly convex face



otherwise

1-Planar 1-Bend RAC Drawings

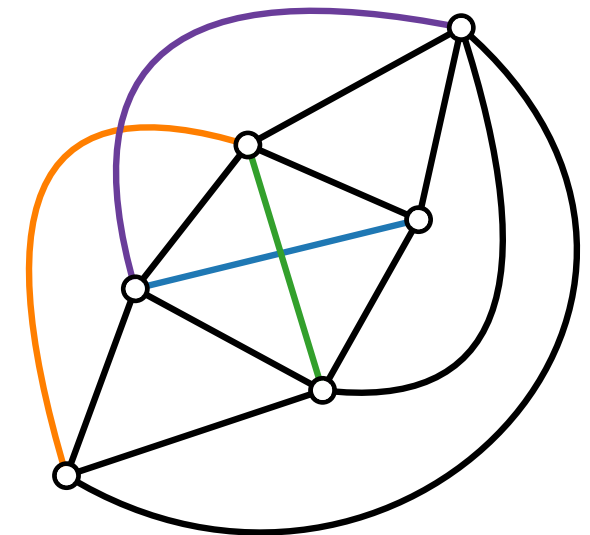
Theorem. [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]

Every 1-planar graph G admits a 1-planar 1-bend RAC drawing.

If a 1-planar embedding of G is given as part of the input, such a drawing can be computed in linear time.

Observation.

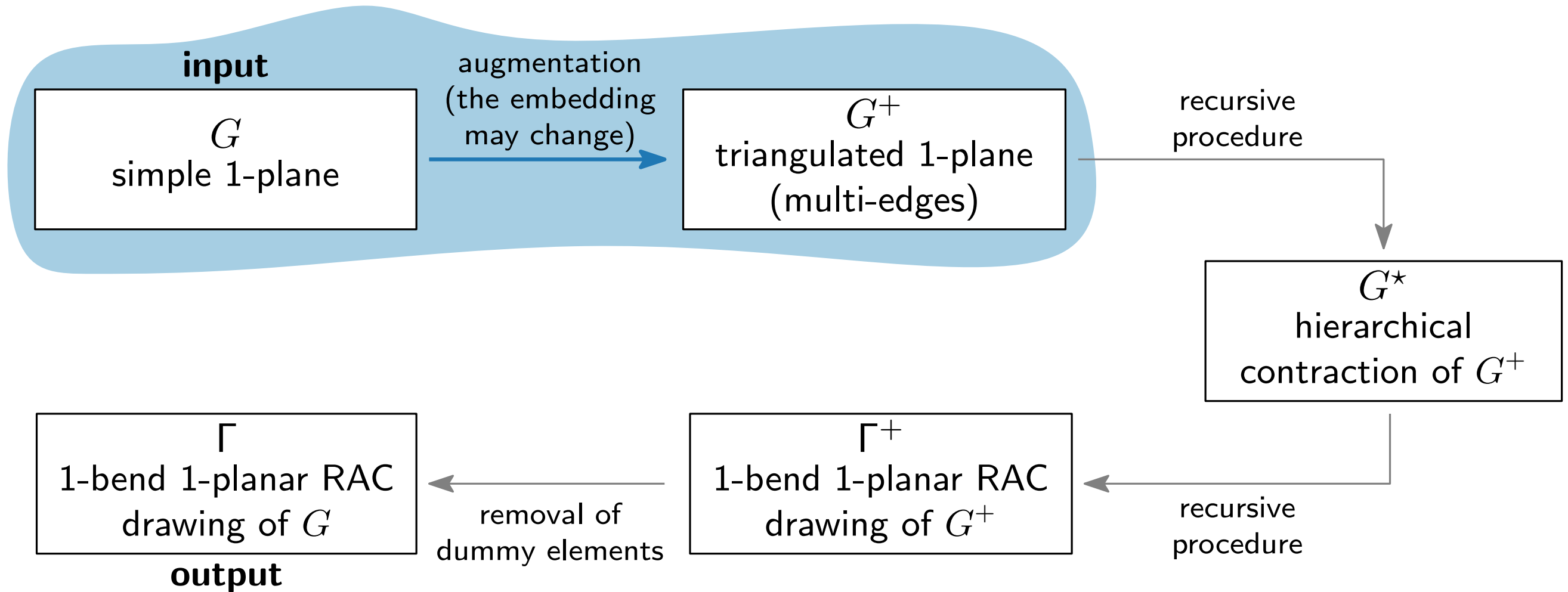
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of G forms an empty **kite**, except for at most one pair if their crossing point is on the outer face of G .



Theorem. [Chiba, Yamanouchi & Nishizeki 1984]

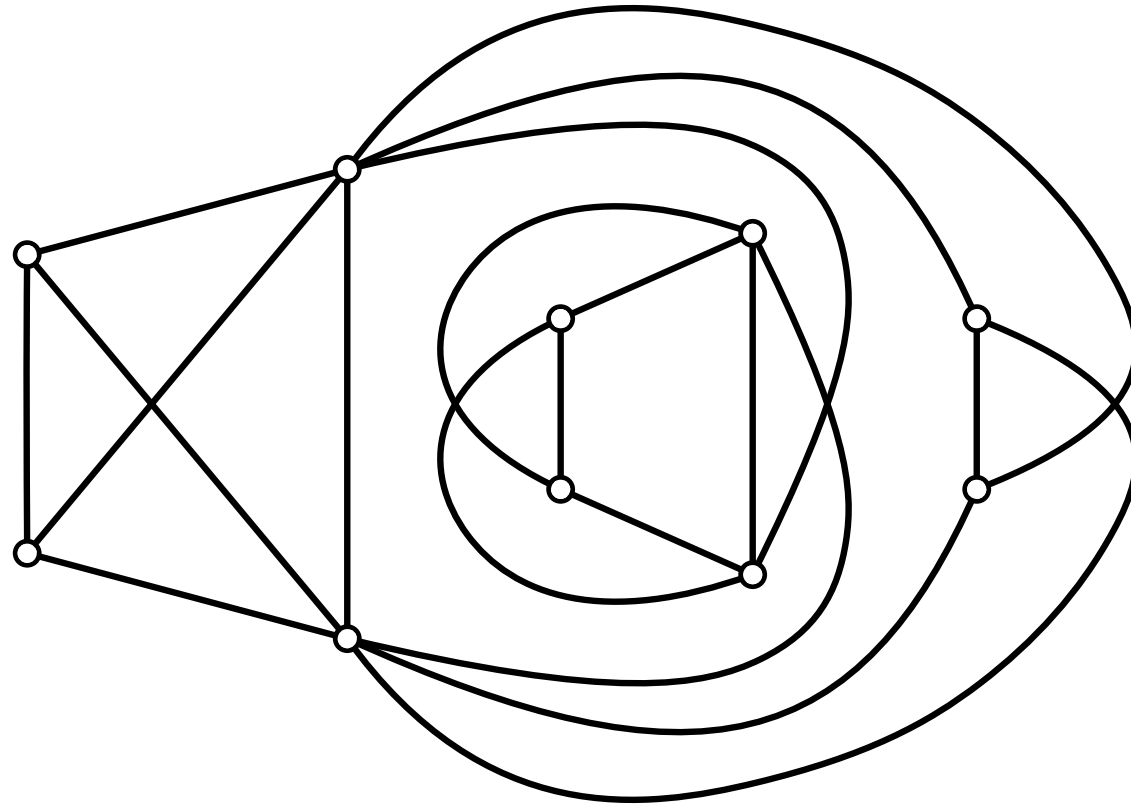
For every 2-connected plane graph G with outer face C_k and every convex k -gon P , there is a strictly convex planar straight-line drawing of G whose outer face coincides with P . Such a drawing can be computed in linear time.

Algorithm Outline



Algorithm Step 1: Augmentation

G : simple 1-plane graph



Algorithm Step 1: Augmentation

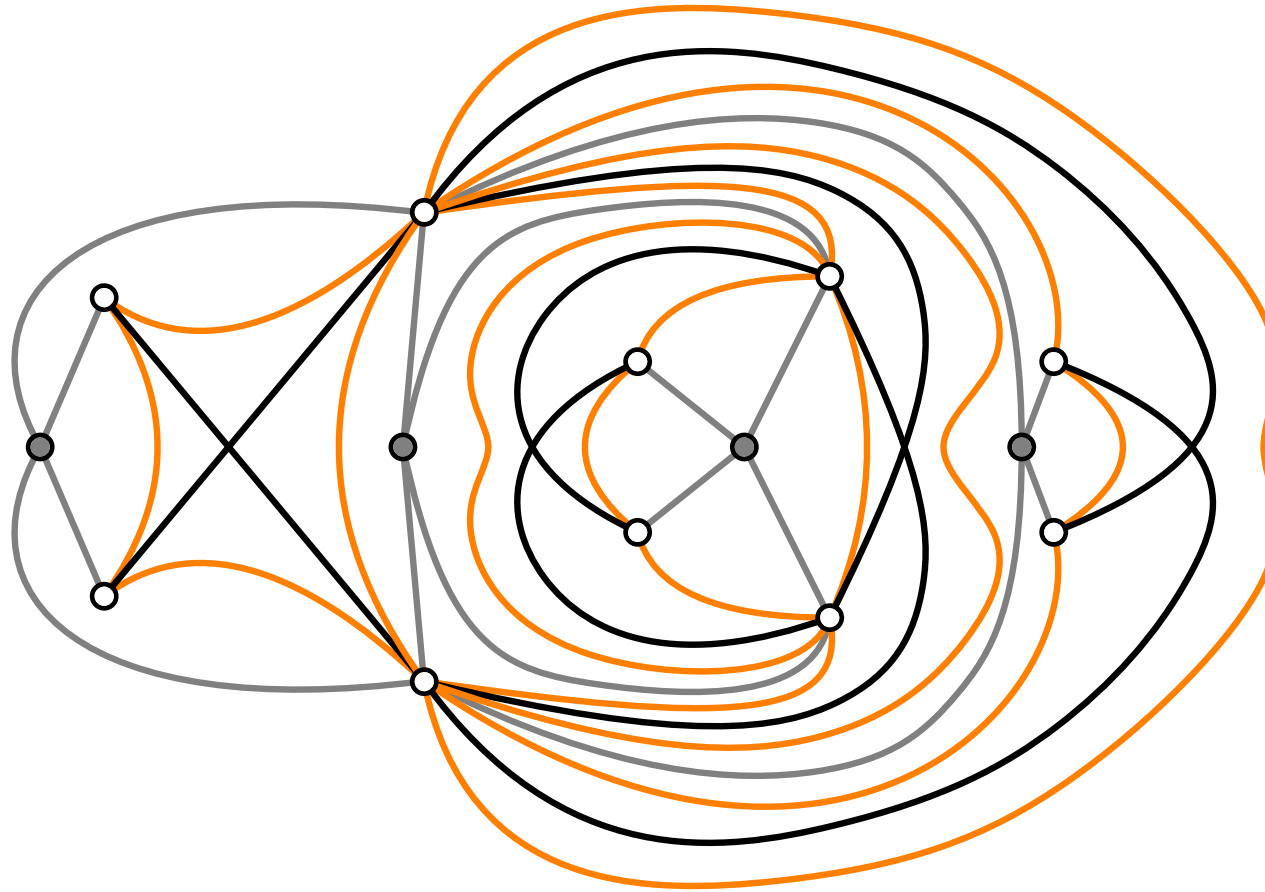
1. For each **pair of crossing edges** add an **enclosing 4-cycle**.

2. Remove those multiple edges that belong to G .

3. Remove one (multiple) edge from each face of degree two (if any). 

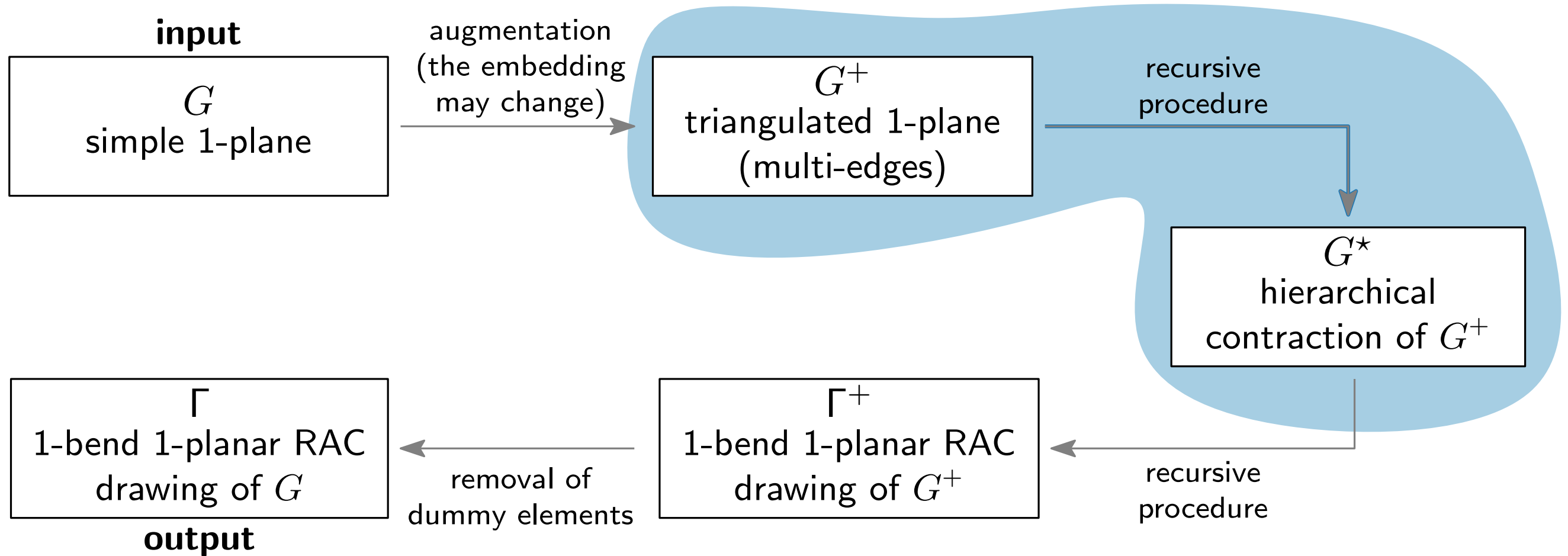
4. Triangulate faces of degree > 3 by inserting a **star** inside them.

G : simple 1-plane graph $\longrightarrow$ G^+ : triangulated 1-plane (possibly with multi-edges)



Note that we can still have parallel (**orange**) edges

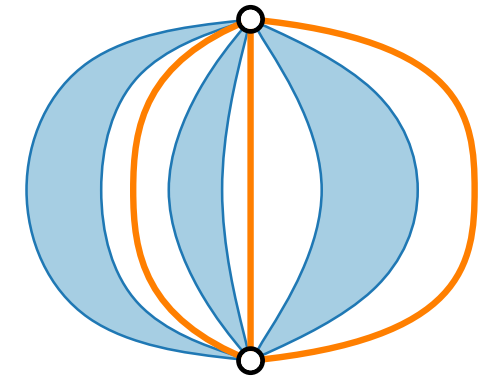
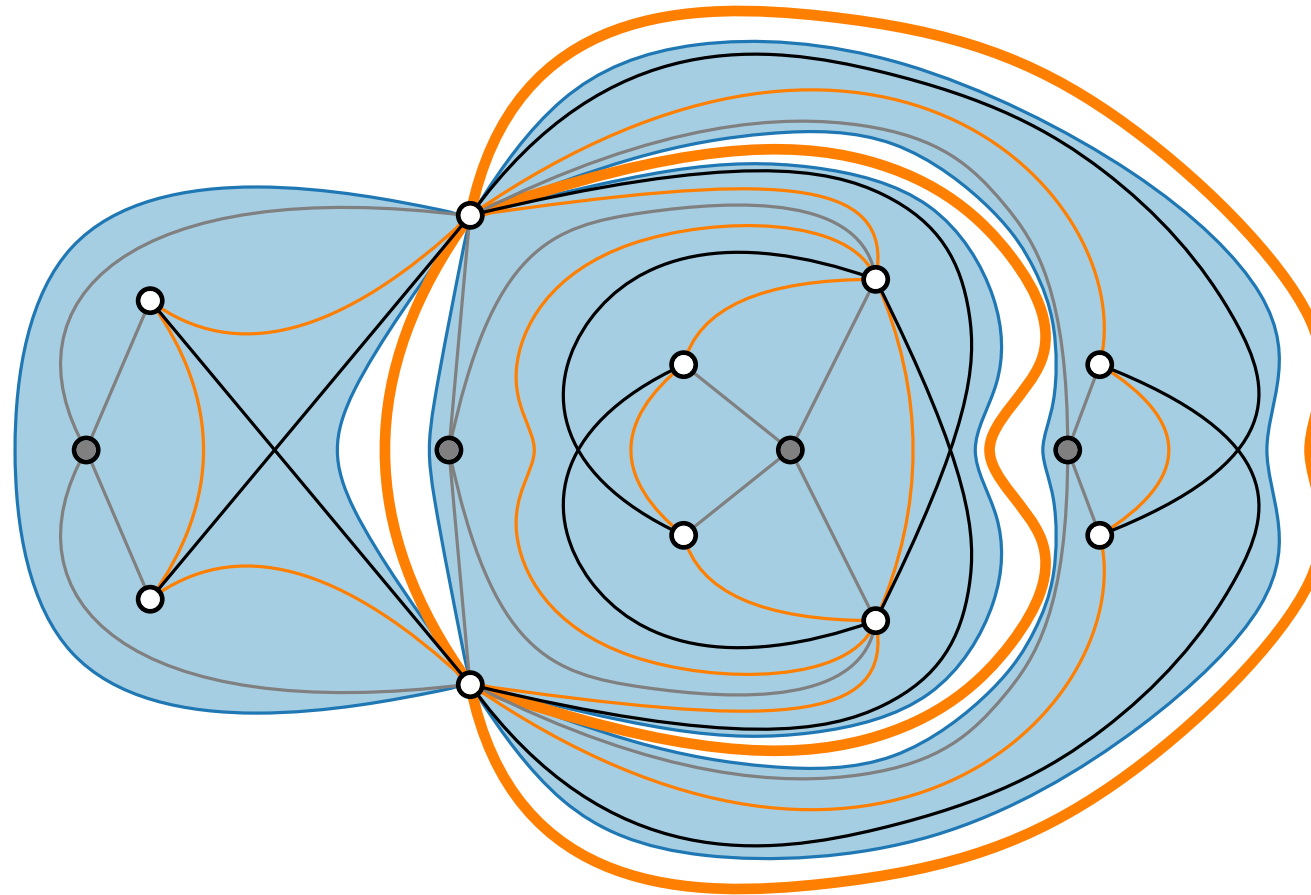
Algorithm Outline



Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites

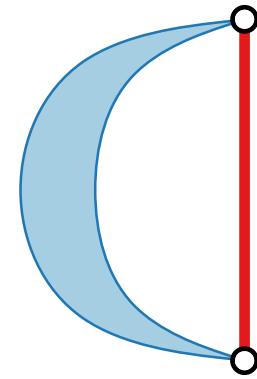
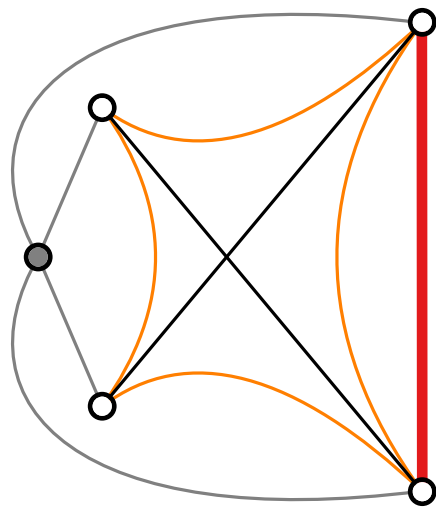


structure of each
separation pair

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
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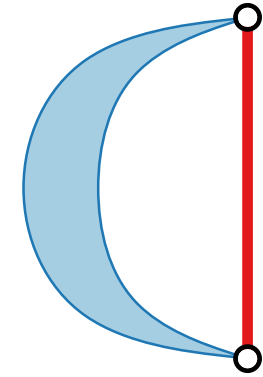
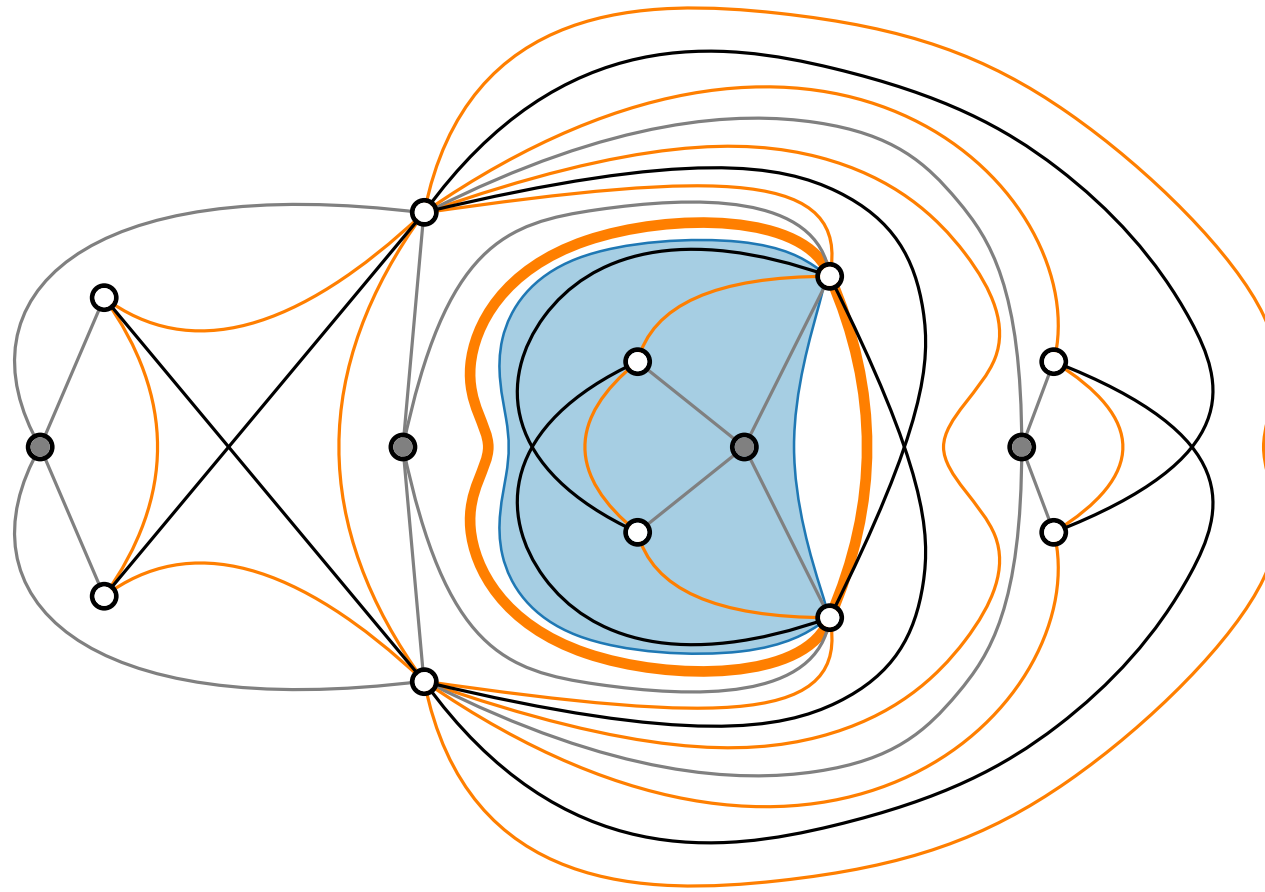
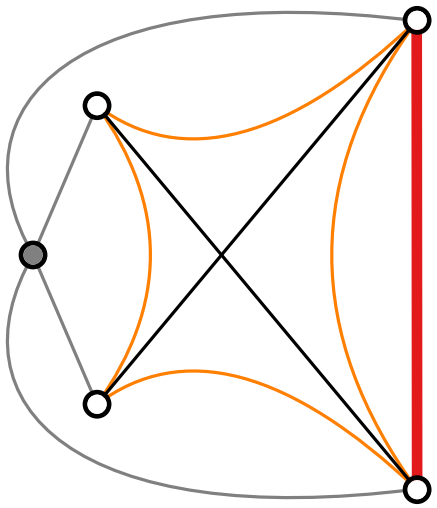
structure of each
separation pair

Contract all inner
components of each
separation pair into
a **thick edge**.

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



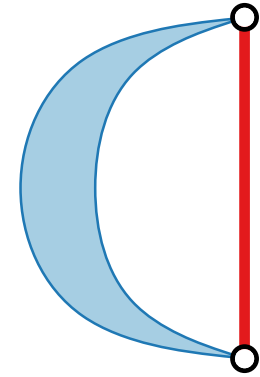
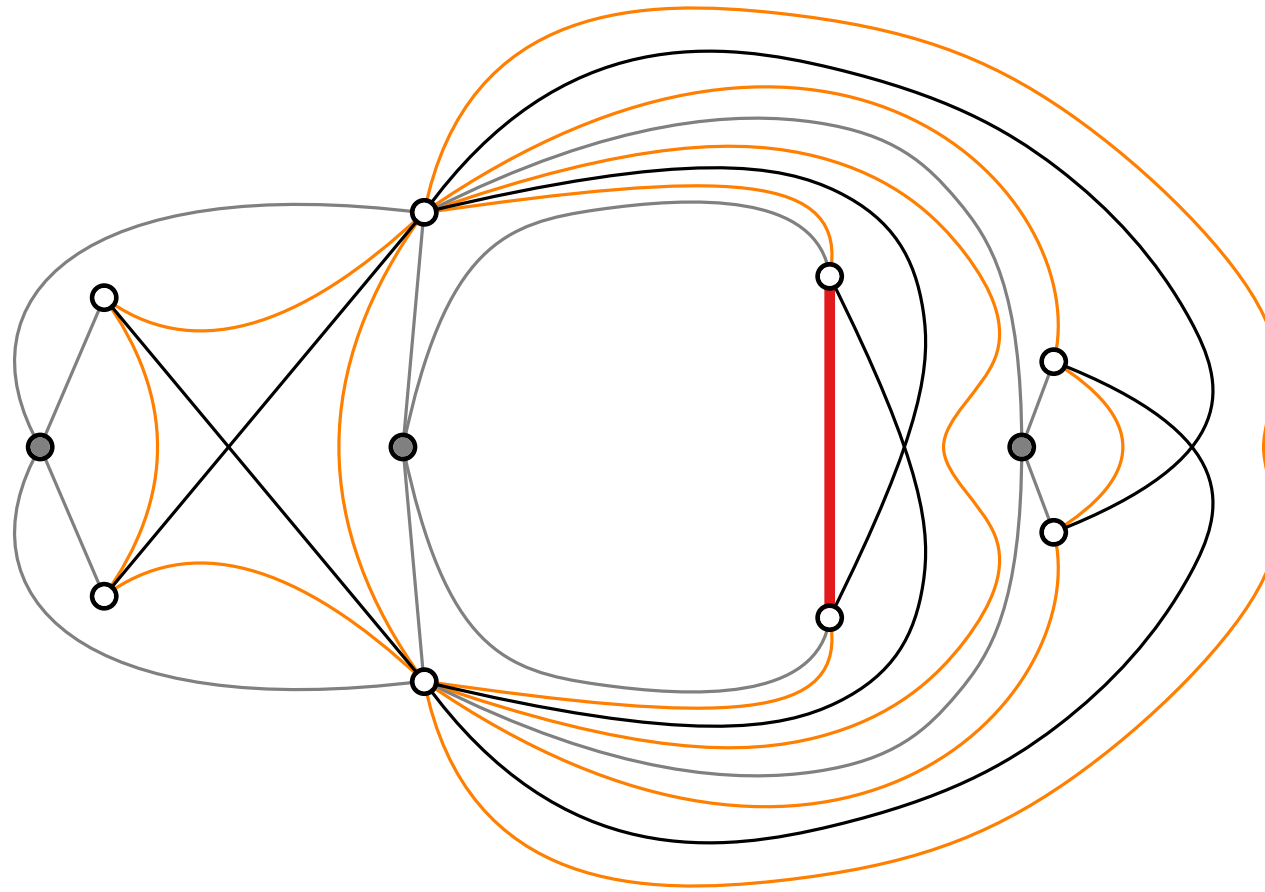
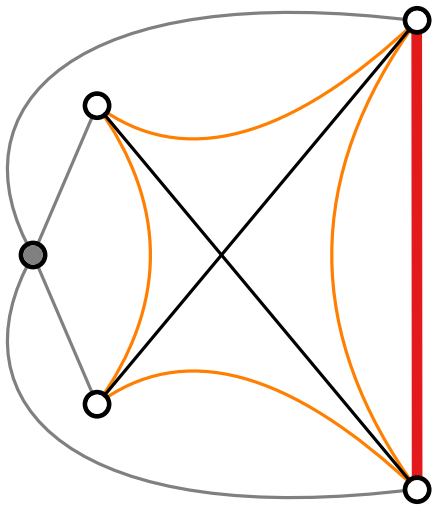
structure of each
separation pair

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Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites

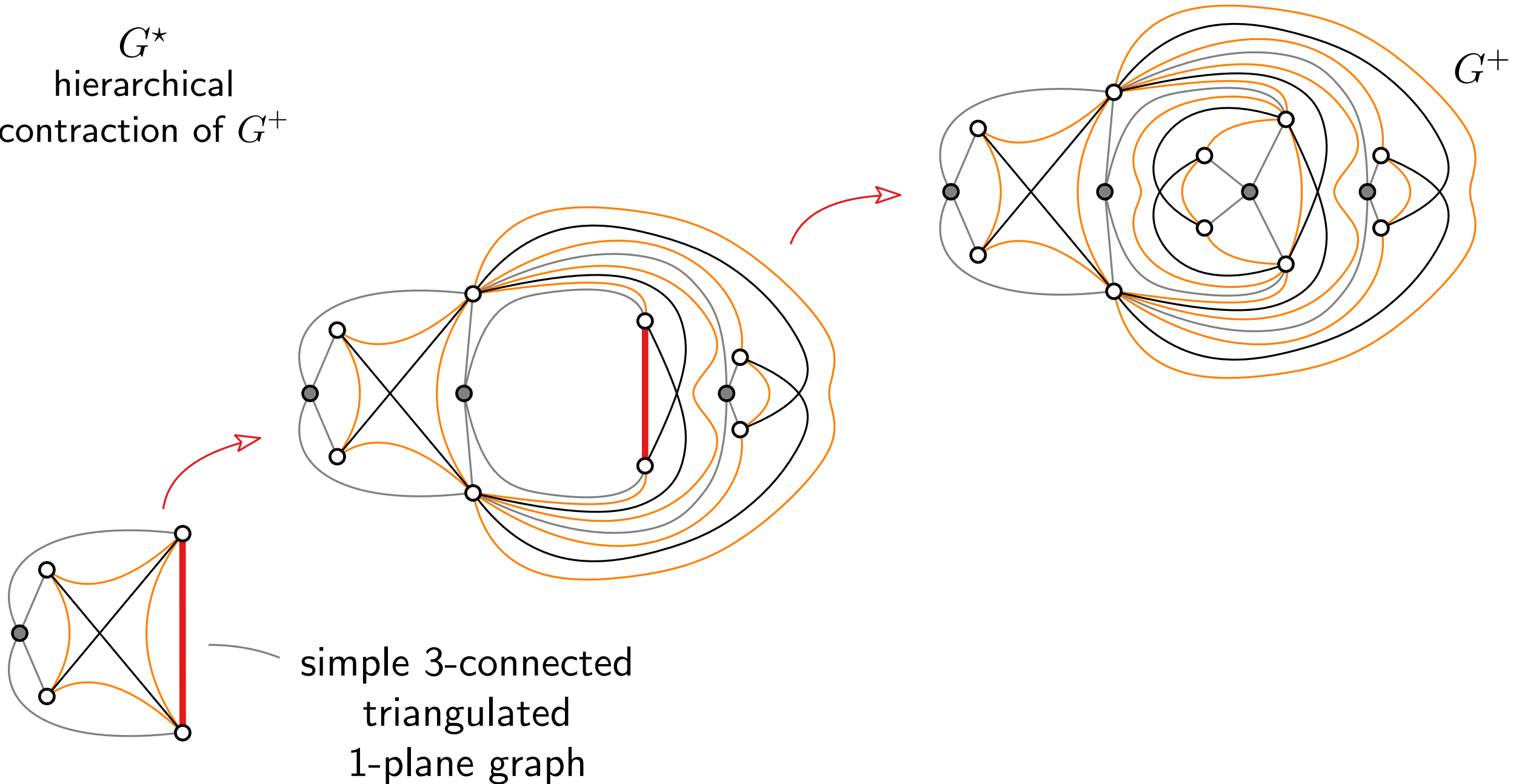


structure of each
separation pair

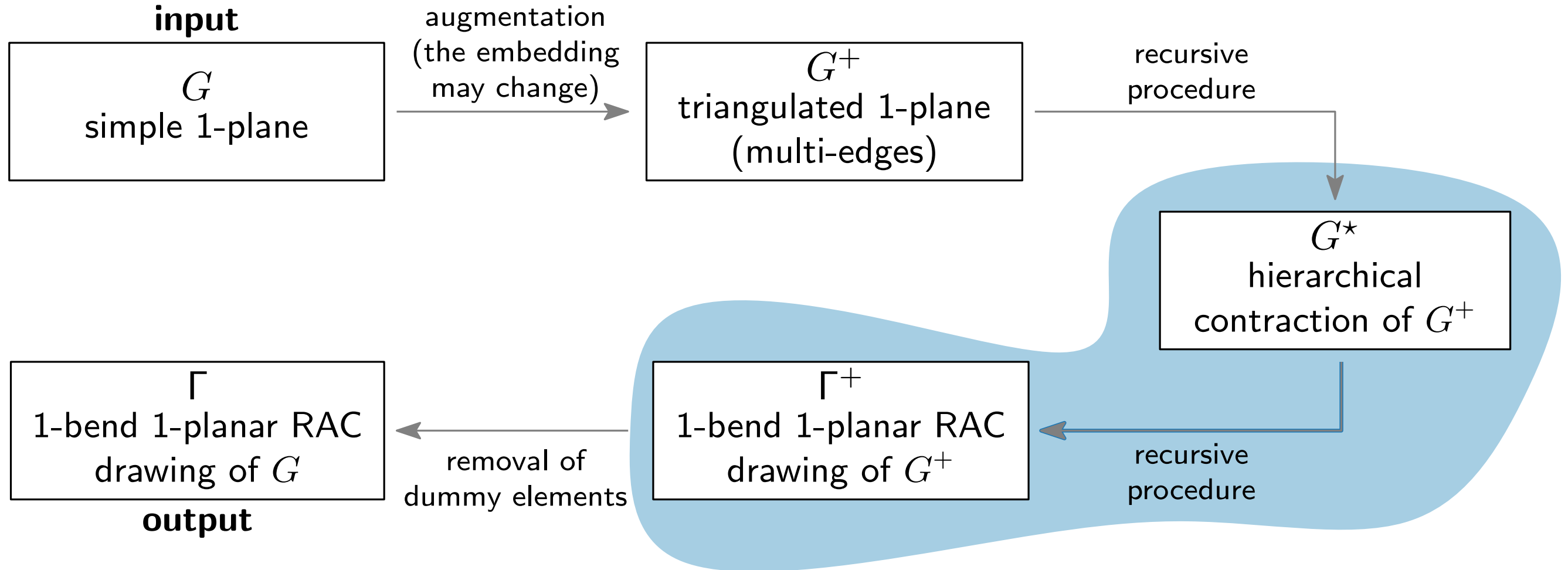
Contract all inner
components of each
separation pair into
a **thick edge**.

Algorithm Step 2: Hierarchical Contractions

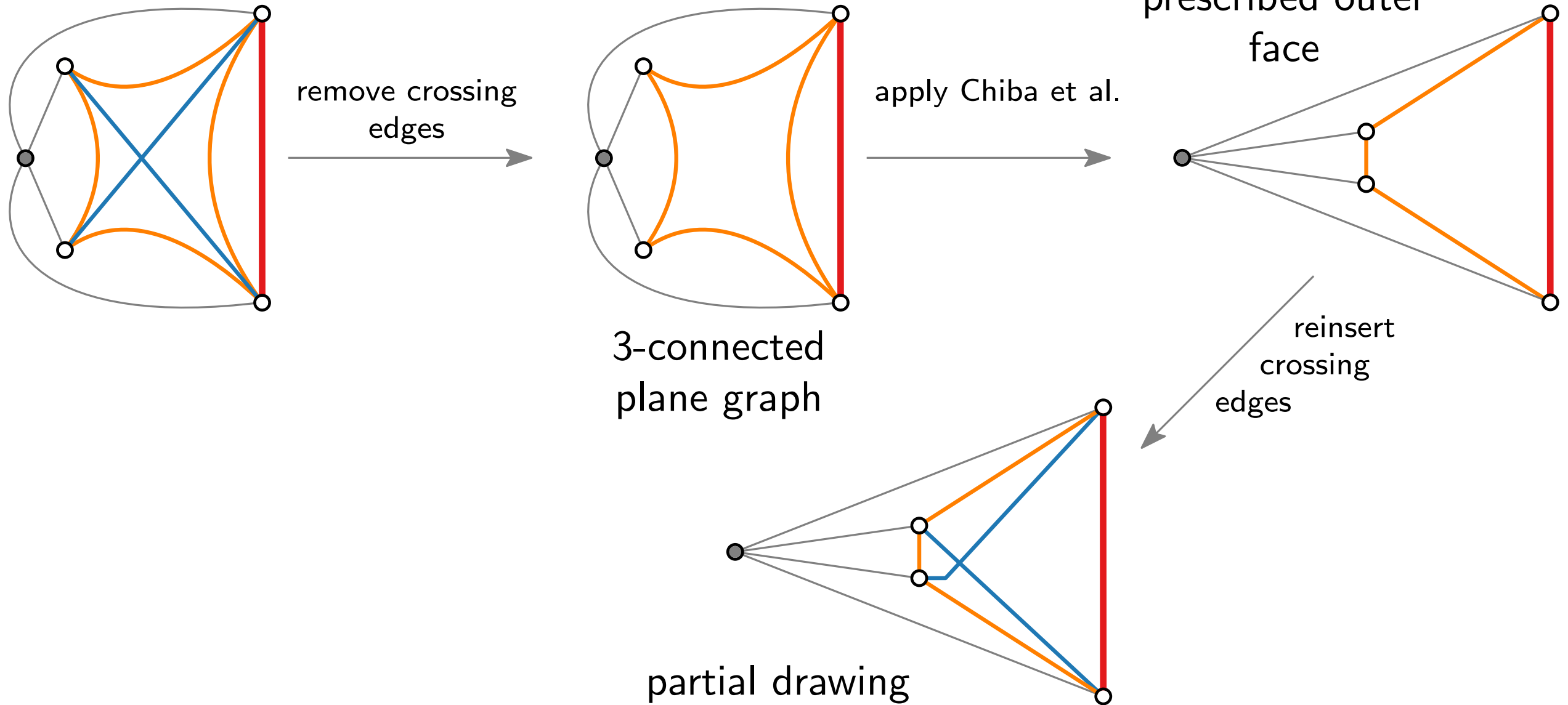
G^*
hierarchical
contraction of G^+



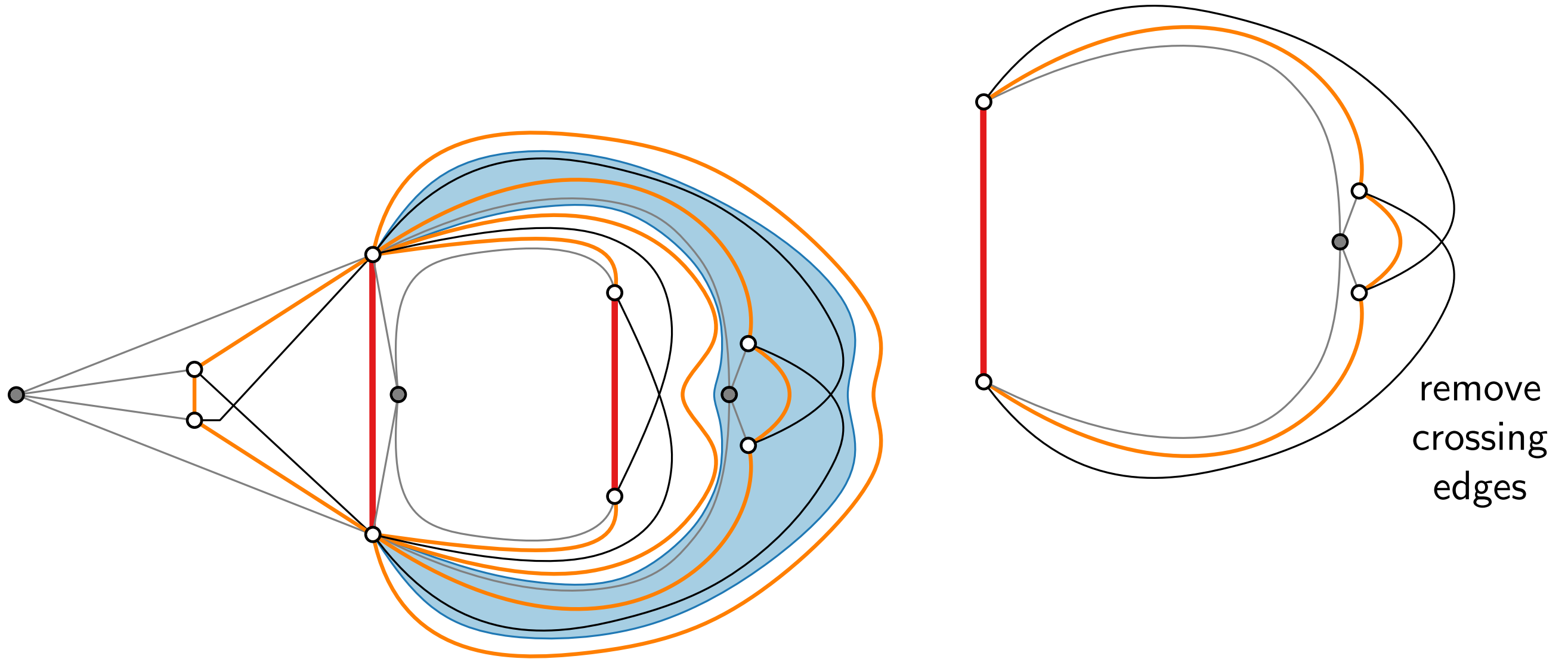
Algorithm Outline



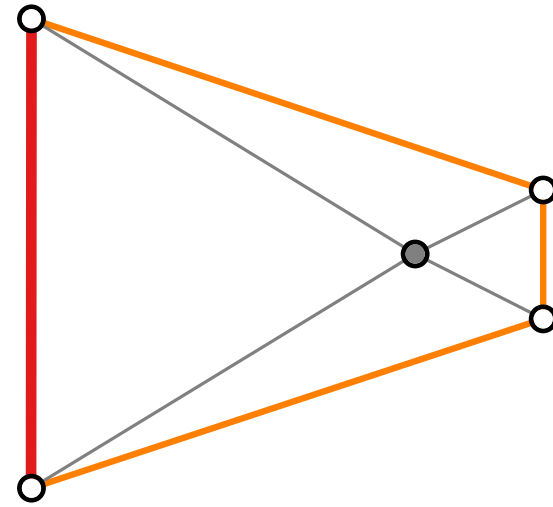
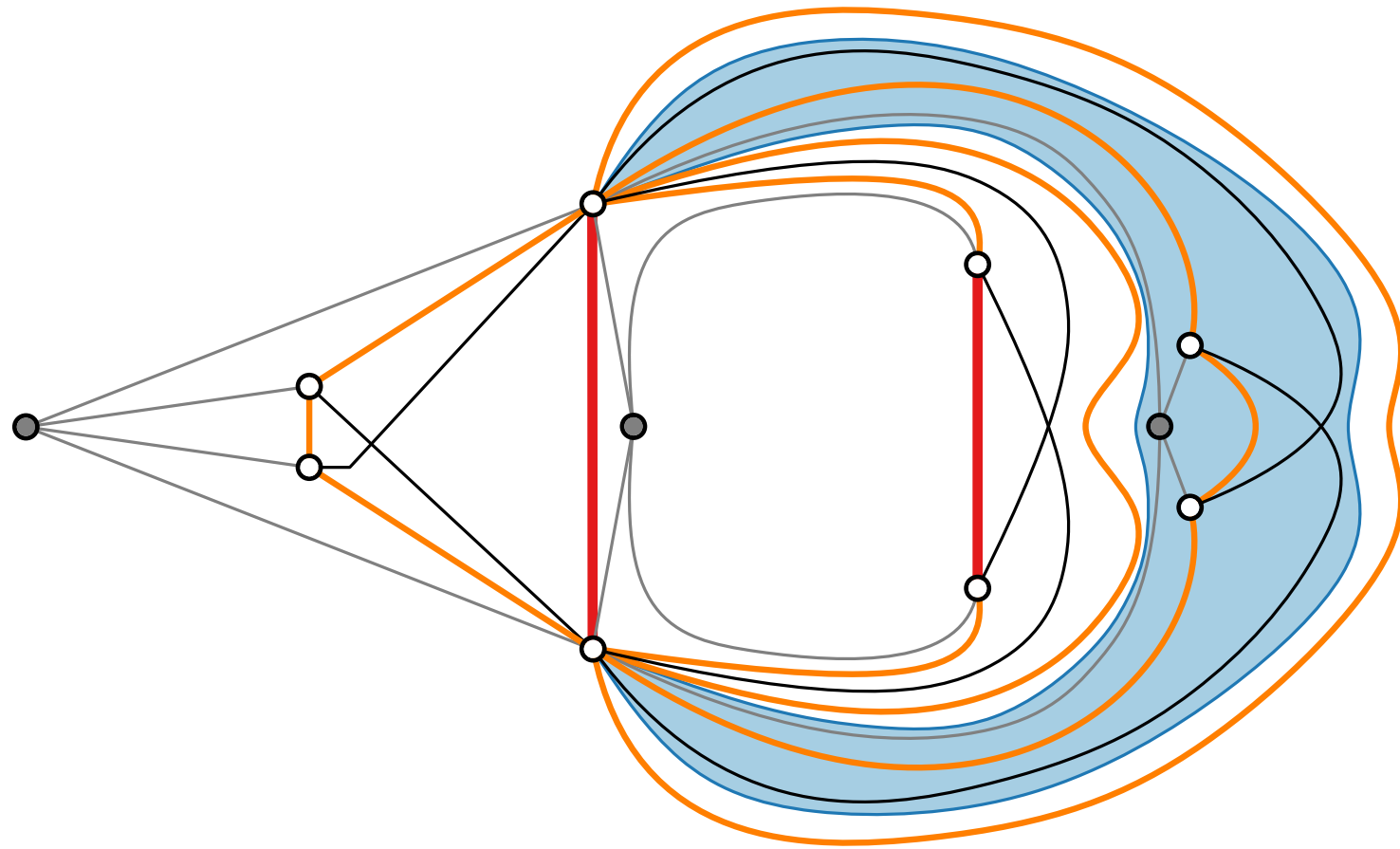
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

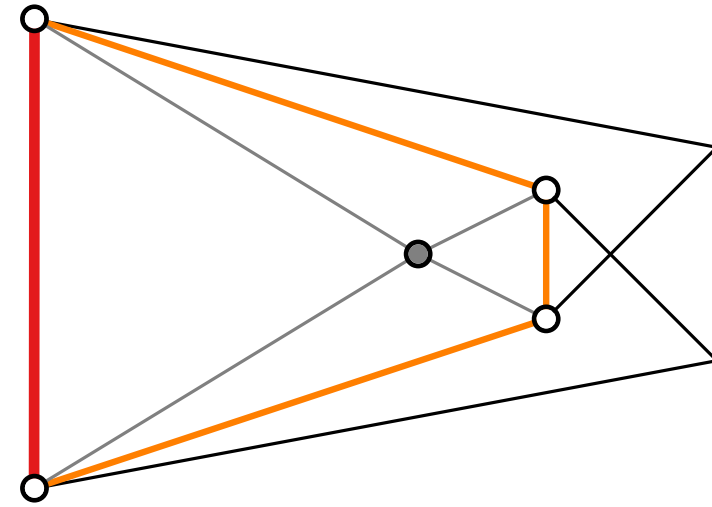
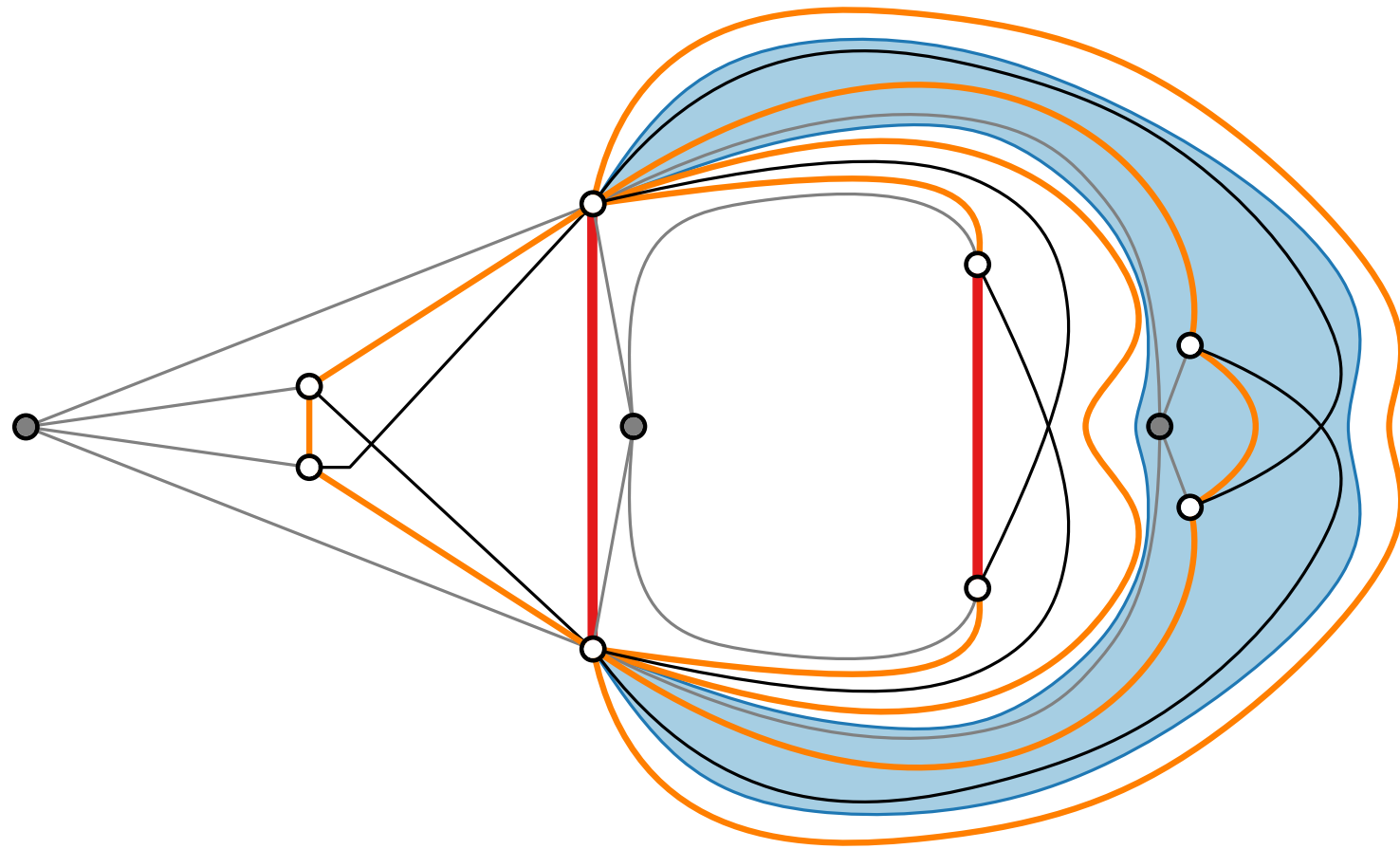


Algorithm Step 3: Drawing Procedure



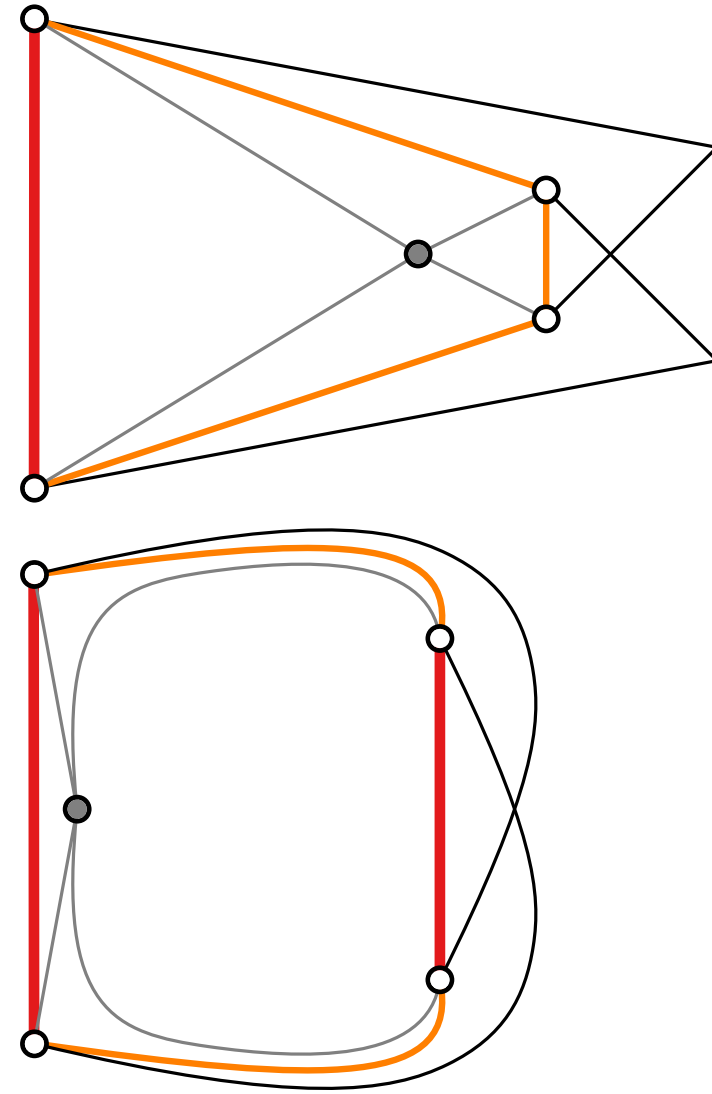
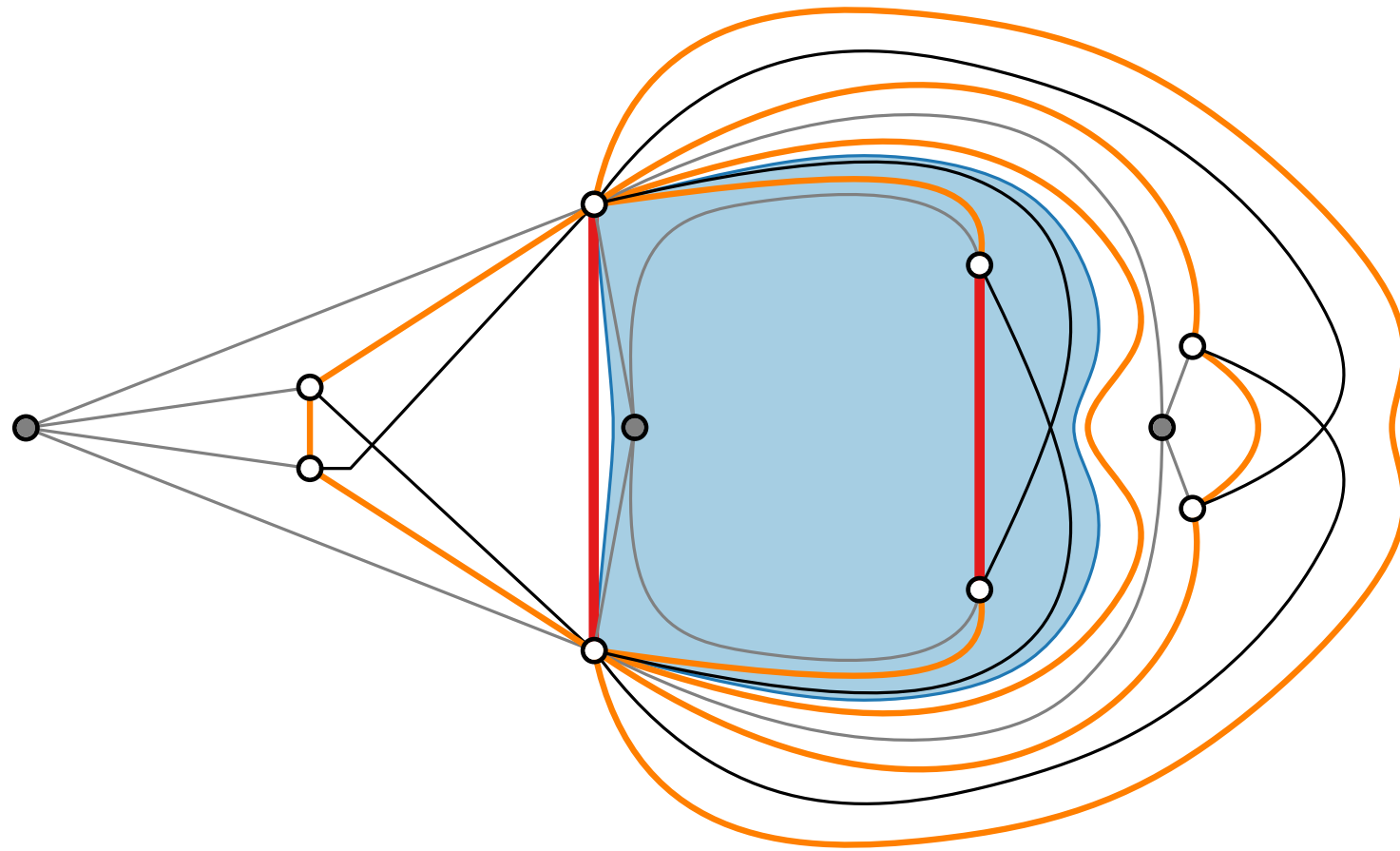
apply Chiba et al.

Algorithm Step 3: Drawing Procedure



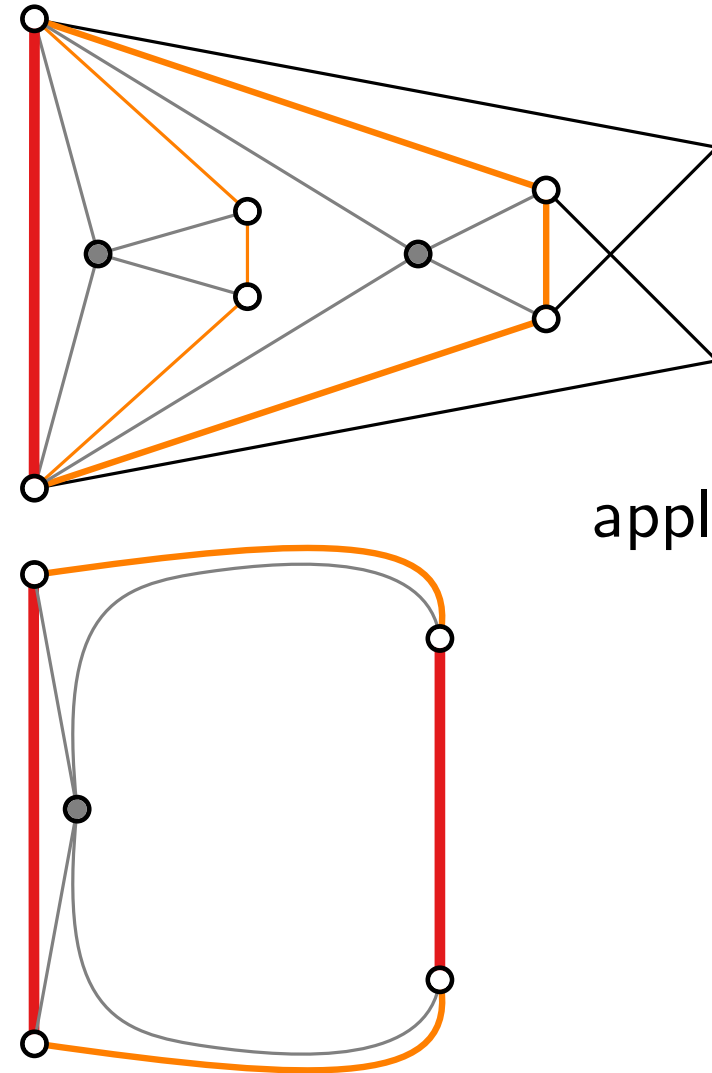
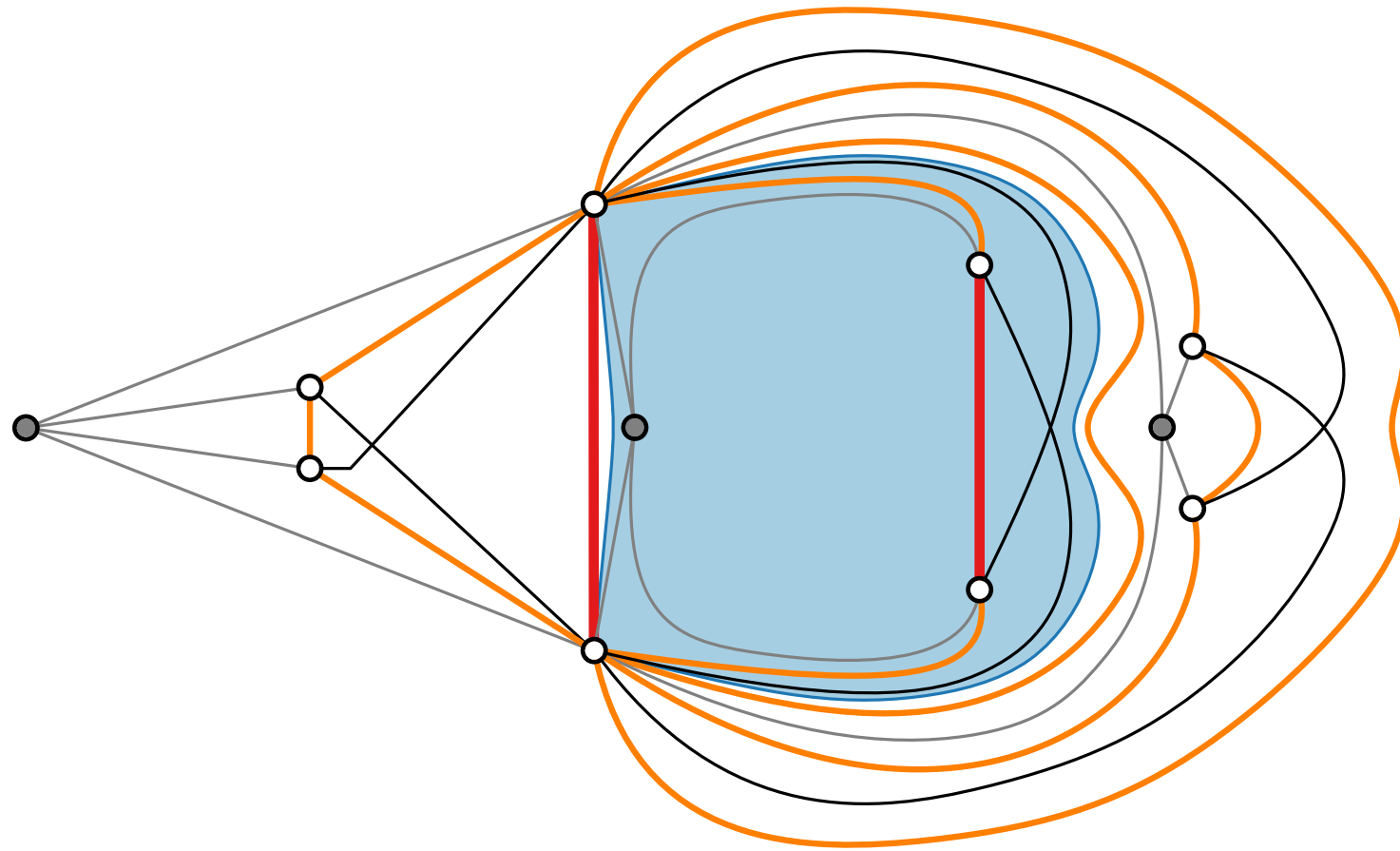
reinsert
crossing
edges

Algorithm Step 3: Drawing Procedure



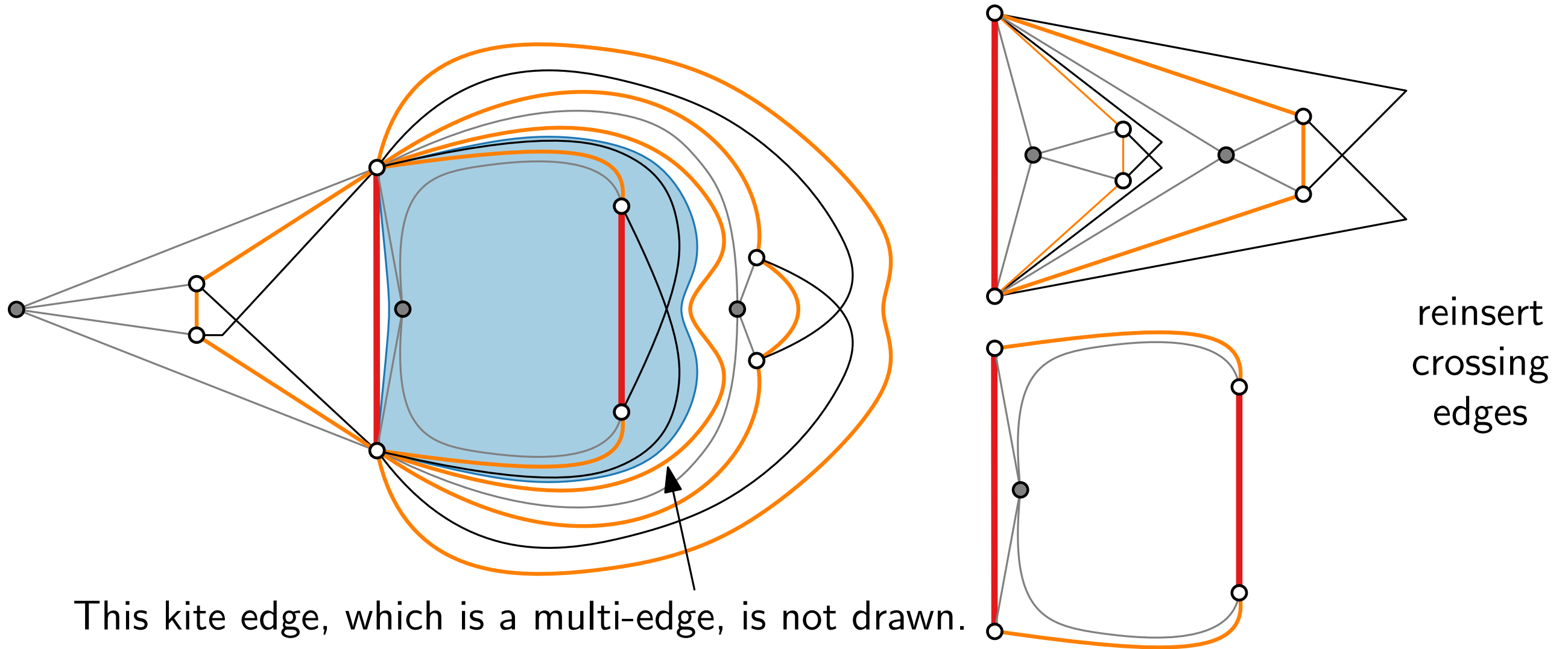
remove
crossing
edges

Algorithm Step 3: Drawing Procedure

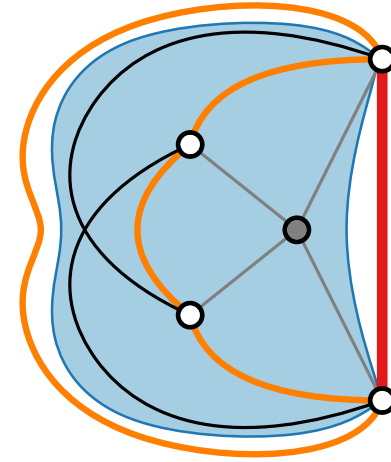
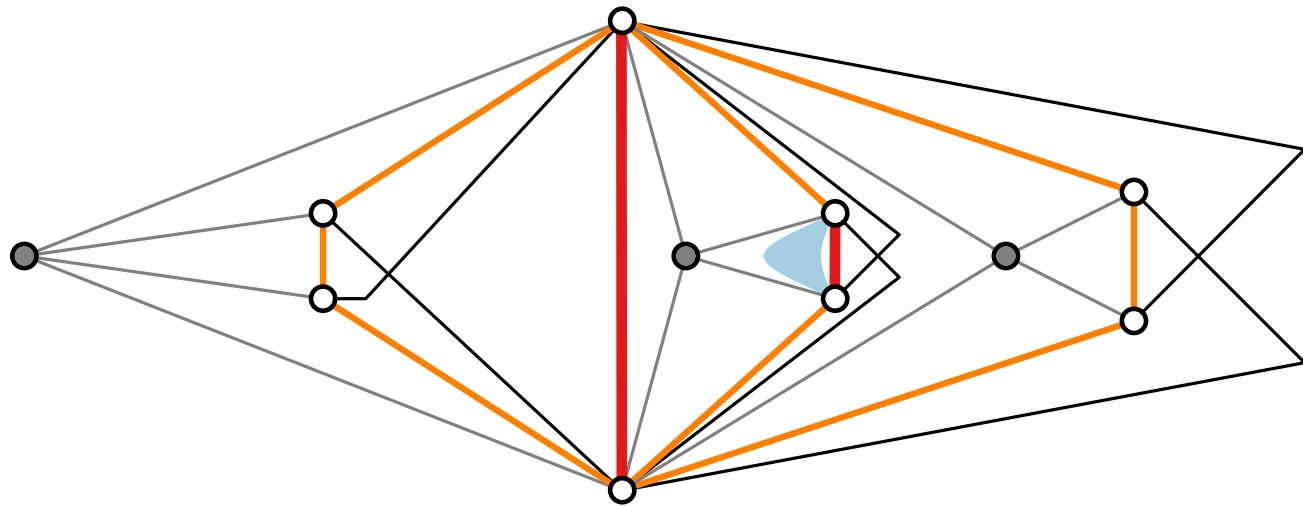


apply Chiba et al.

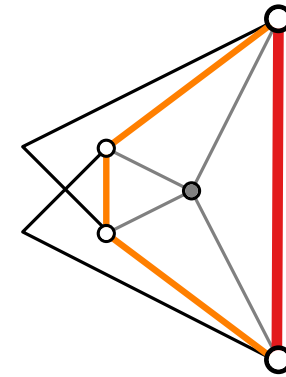
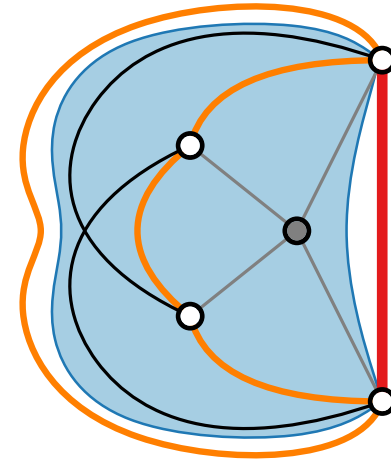
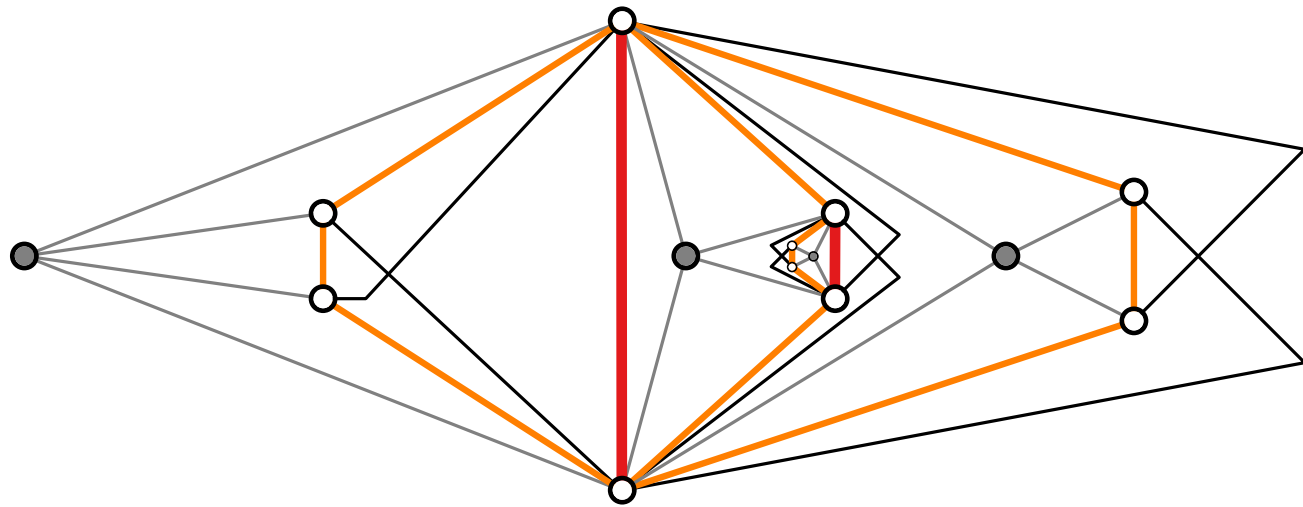
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

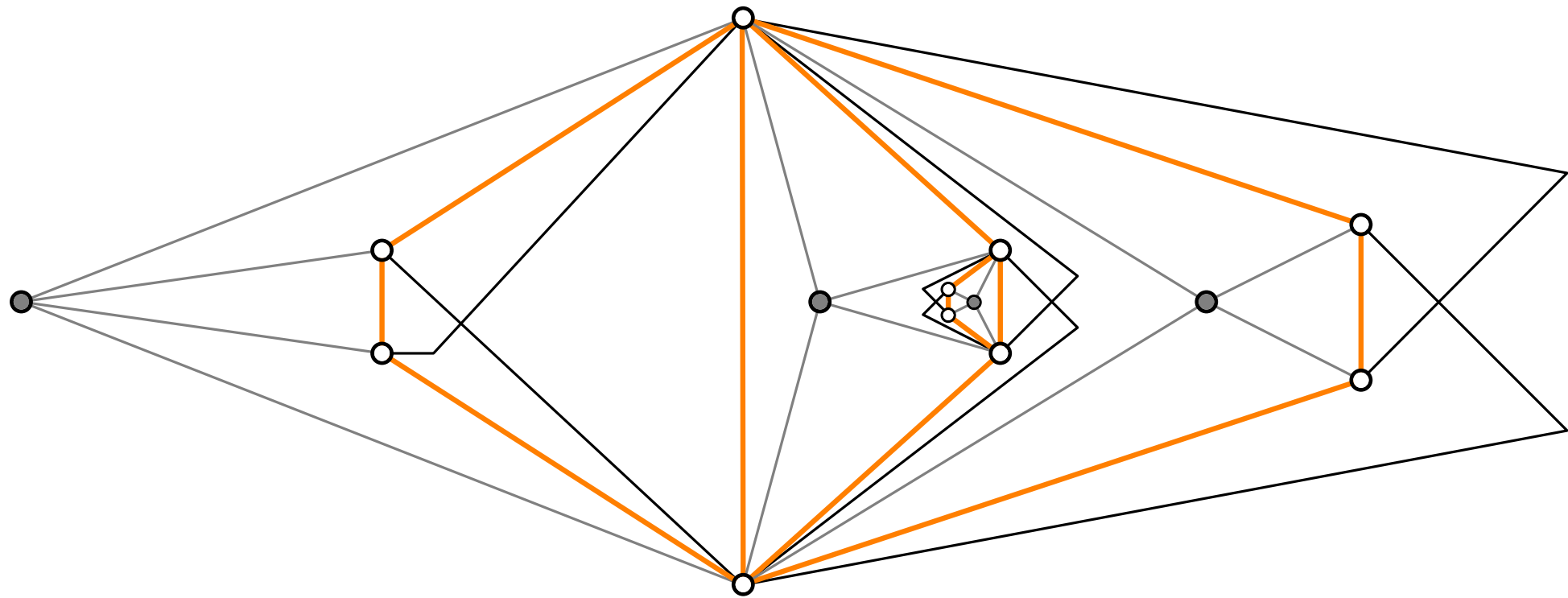


Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

Γ^+ : 1-bend 1-planar RAC drawing of G^+



Algorithm Outline

input

G
simple 1-plane

augmentation
(the embedding
may change)

G^+
triangulated 1-plane
(multi-edges)

recursive
procedure

G^*
hierarchical
contraction of G^+

Γ^+
1-bend 1-planar RAC
drawing of G^+

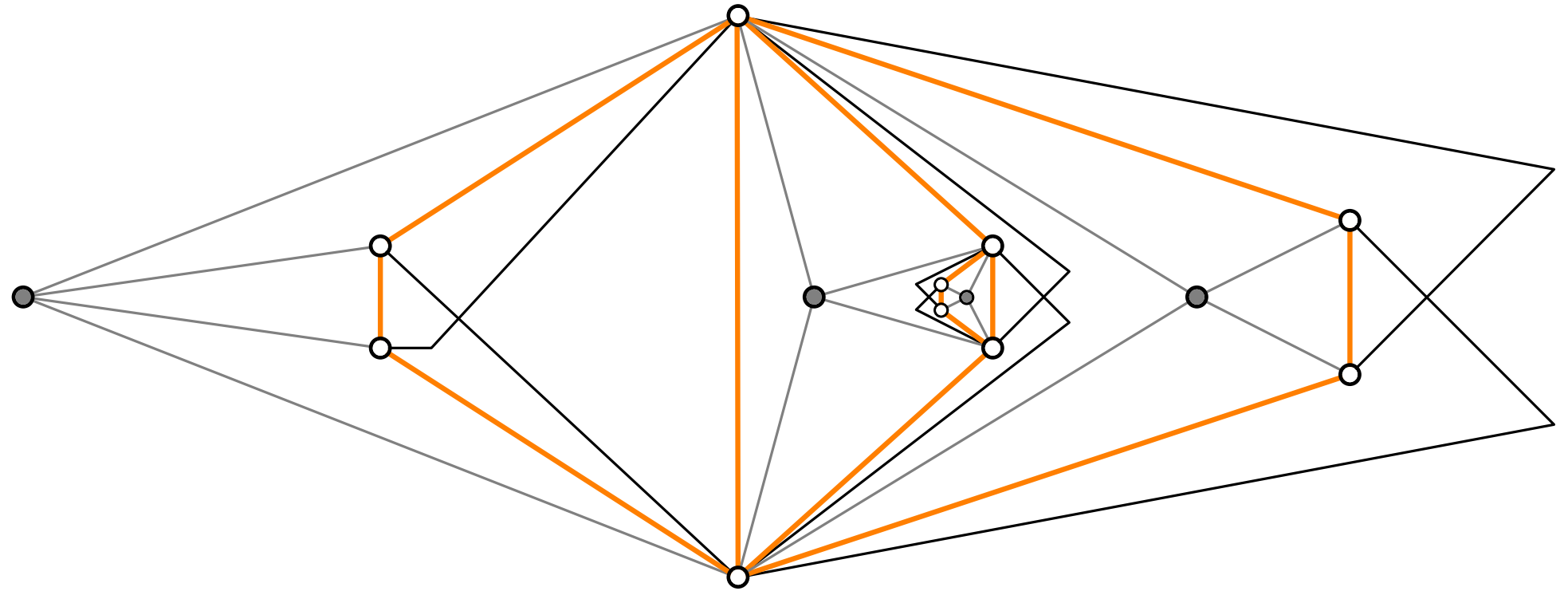
recursive
procedure

Γ
1-bend 1-planar RAC
drawing of G

removal of
dummy elements

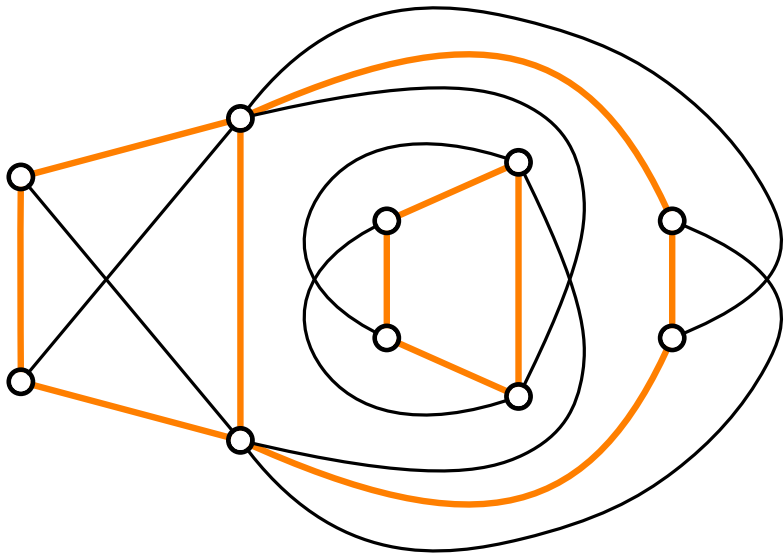
output

Algorithm Step 4: Removal of Dummy Vertices



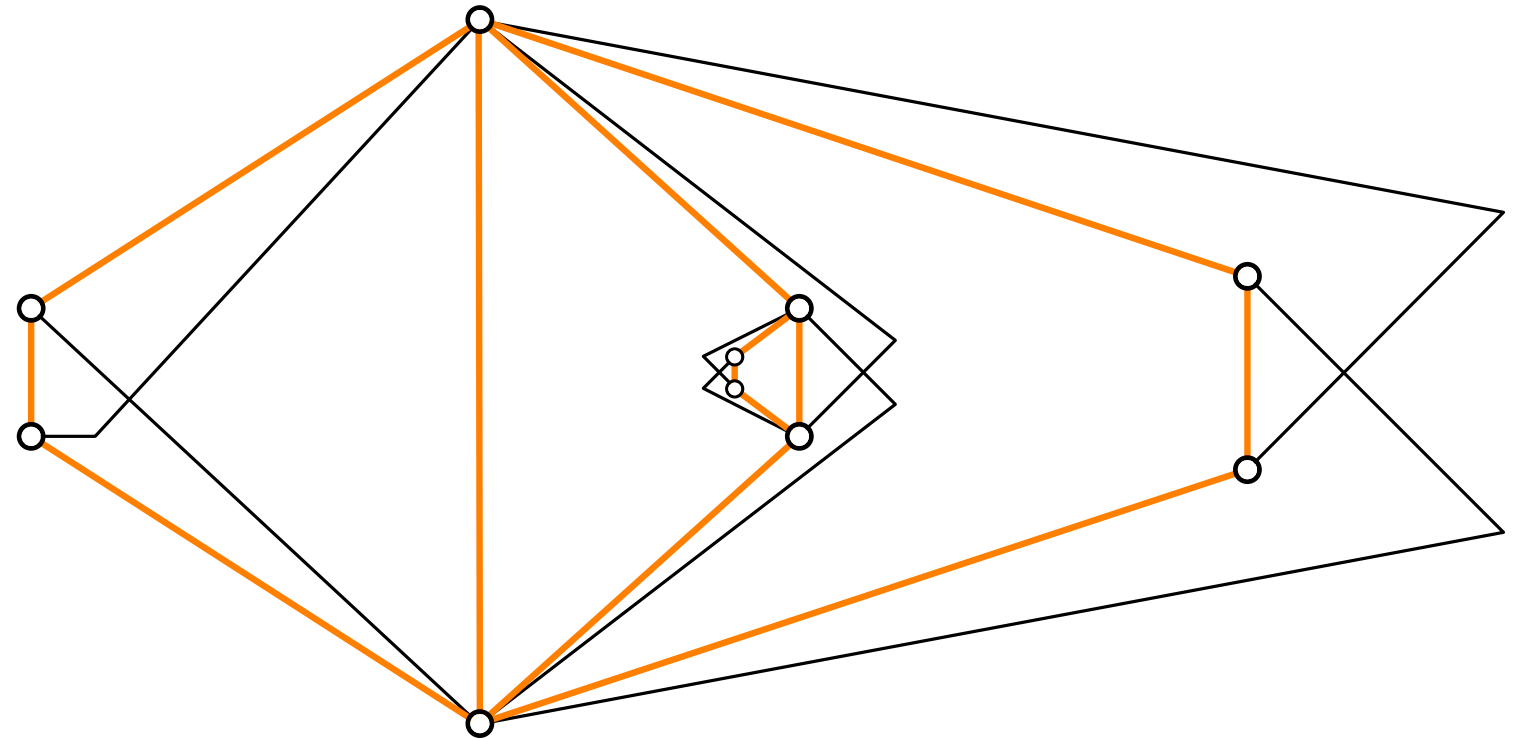
Algorithm Step 4: Removal of Dummy Vertices

G : simple 1-plane graph



Γ : 1-bend 1-planar RAC drawing of G

(embedding may differ)

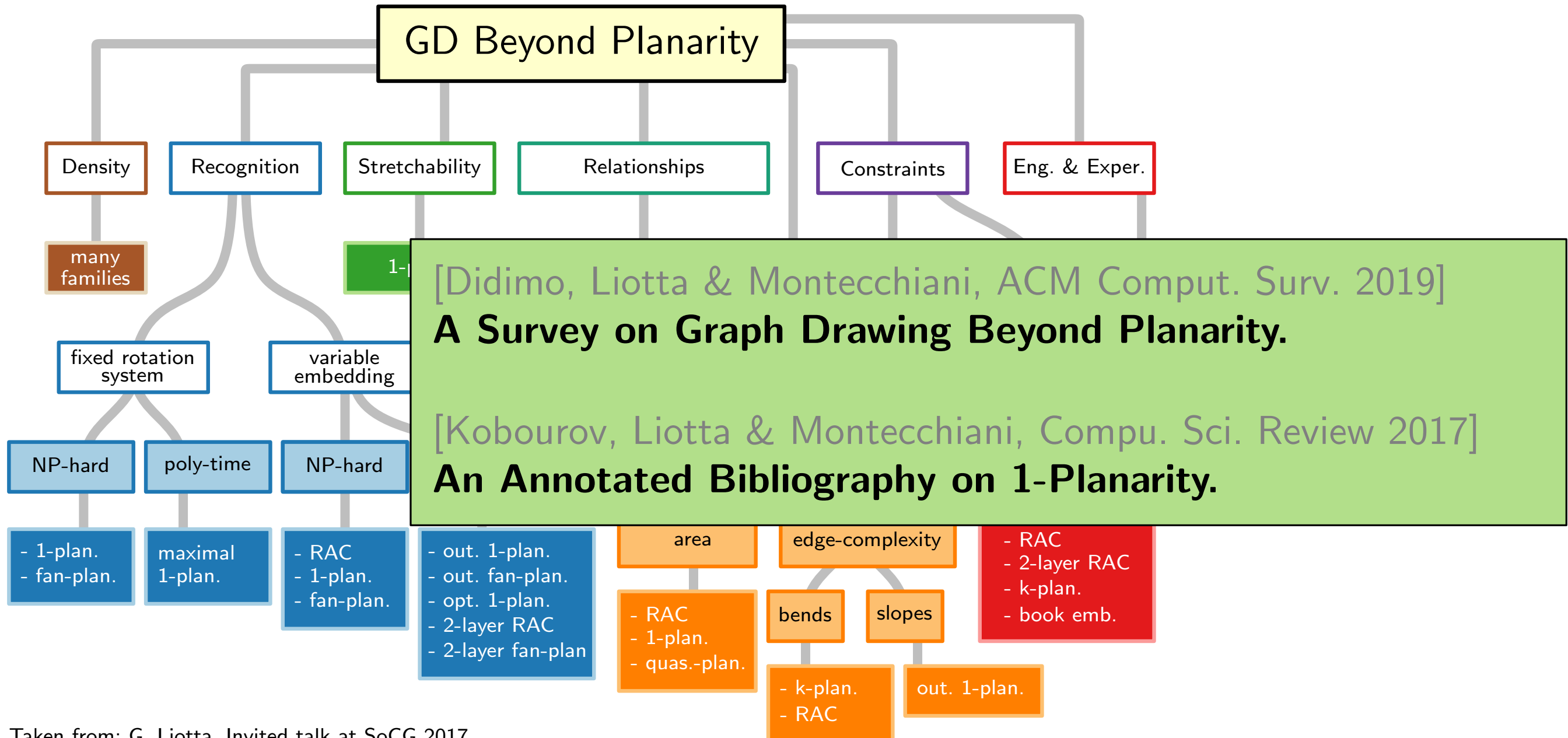


Remark.

By modifying the algorithm slightly, the given input embedding can be preserved.

[Chaplick, Lipp, Wolff, Zink 2019]

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Hong and Tokuyama, editors '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchiani, Valtr '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
- [Didimo, Eades, Liotta '17] Drawing graphs with right angle crossings
- [Bekos et al. '17] On RAC drawings of 1-planar graphs