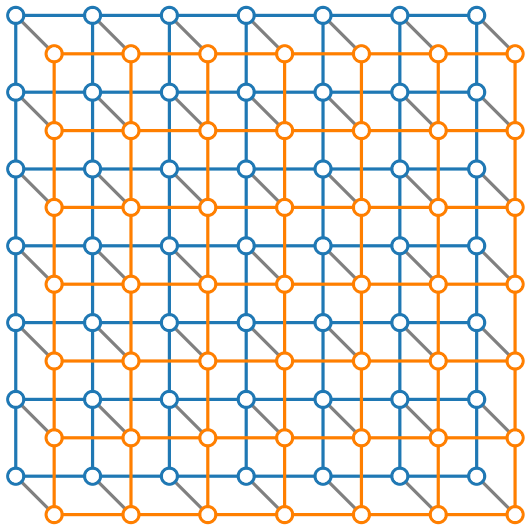
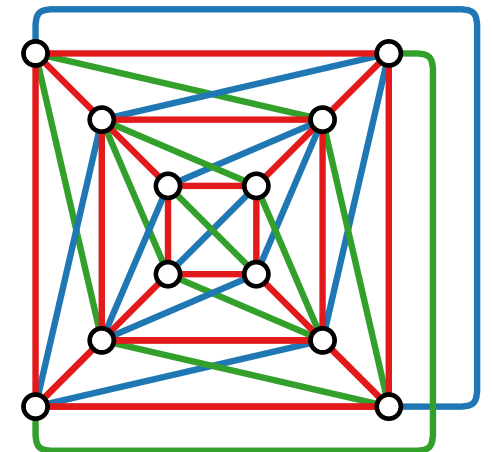
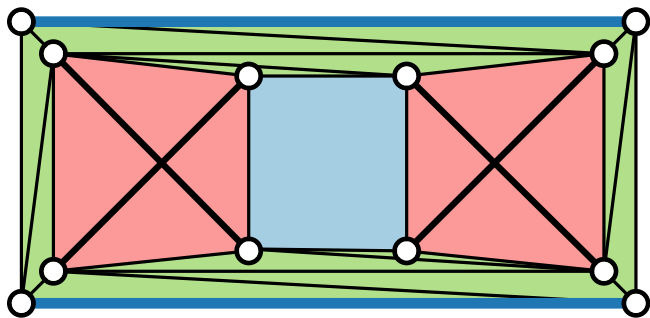
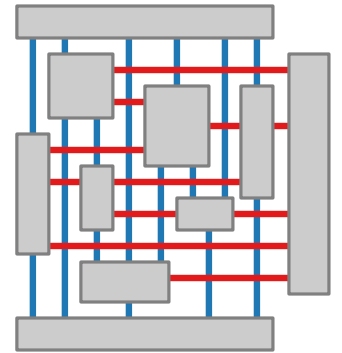


Visualization of Graphs



Lecture 11: Beyond Planarity Drawing Graphs with Crossings



Alexander Wolff

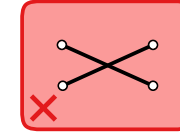
Summer semester 2026

Planar Graphs

Planar graphs admit drawings in the plane without crossings.

Planar Graphs

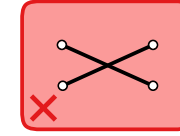
Planar graphs admit drawings in the plane without crossings.



Planar Graphs

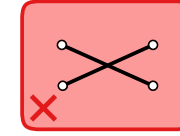
Planar graphs admit drawings in the plane without crossings.

Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).



Planar Graphs

Planar graphs admit drawings in the plane without crossings.

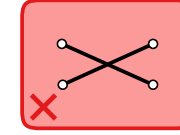


Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

Planar Graphs

Planar graphs admit drawings in the plane without crossings.



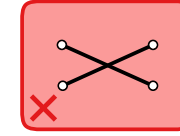
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

Different drawing styles . . .

Planar Graphs

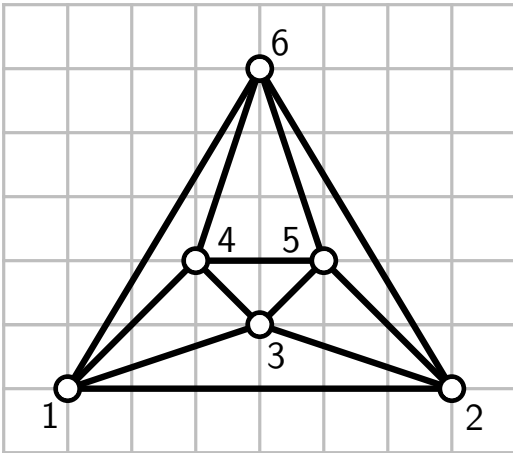
Planar graphs admit drawings in the plane without crossings.



Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

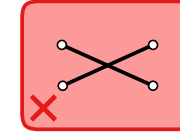
Different drawing styles ...



straight-line drawing

Planar Graphs

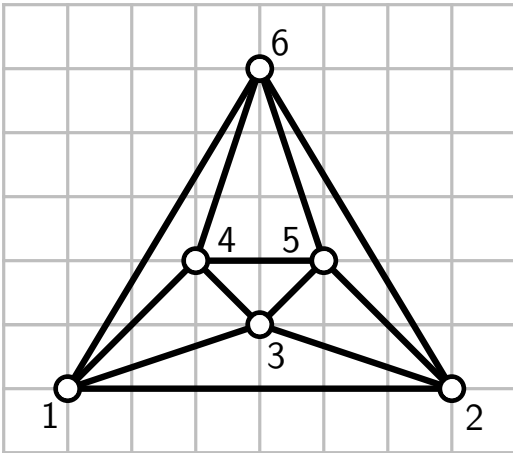
Planar graphs admit drawings in the plane without crossings.



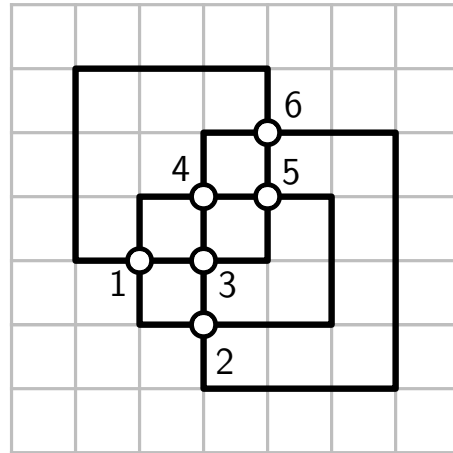
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

Different drawing styles ...



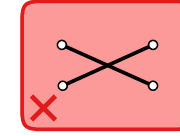
straight-line drawing



orthogonal drawing

Planar Graphs

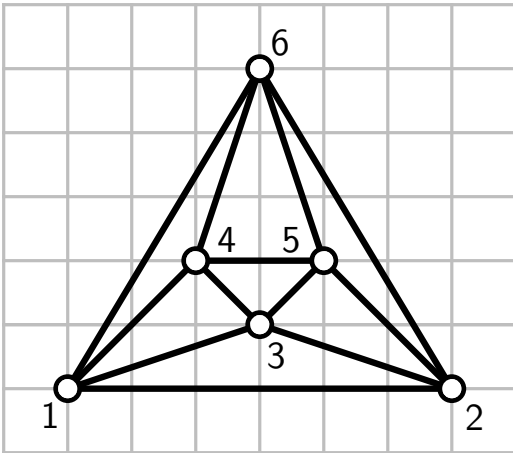
Planar graphs admit drawings in the plane without crossings.



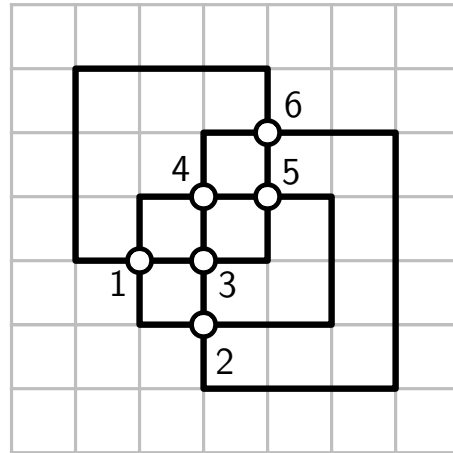
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

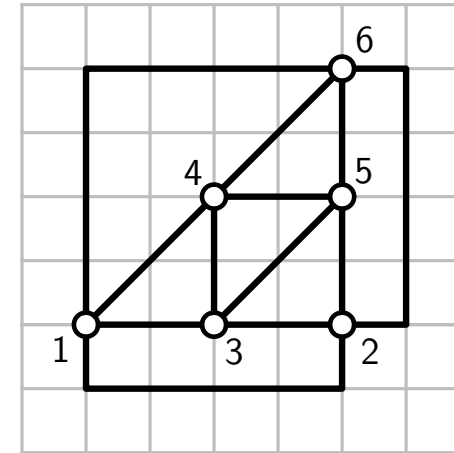
Different drawing styles ...



straight-line drawing



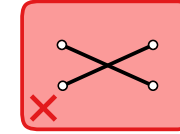
orthogonal drawing



grid drawing with
bends & 3 slopes

Planar Graphs

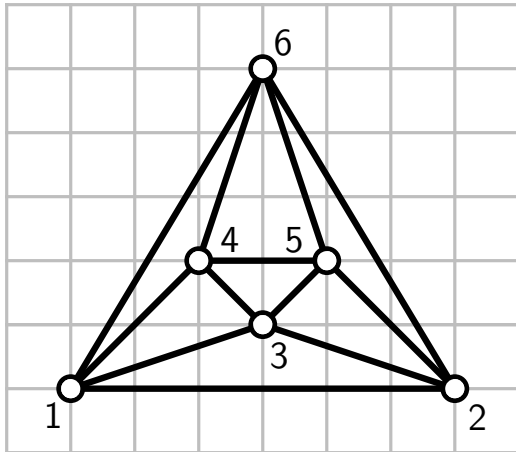
Planar graphs admit drawings in the plane without crossings.



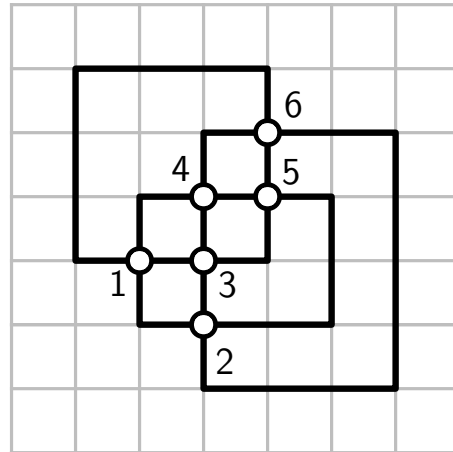
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

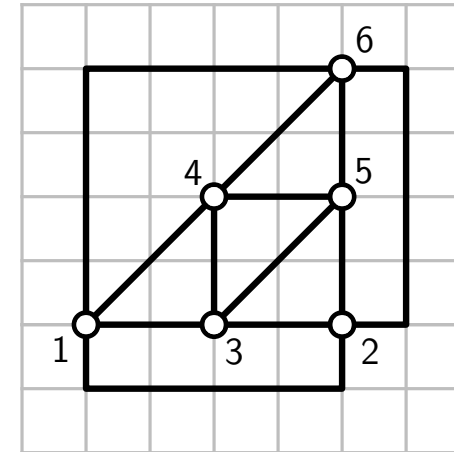
Different drawing styles ...



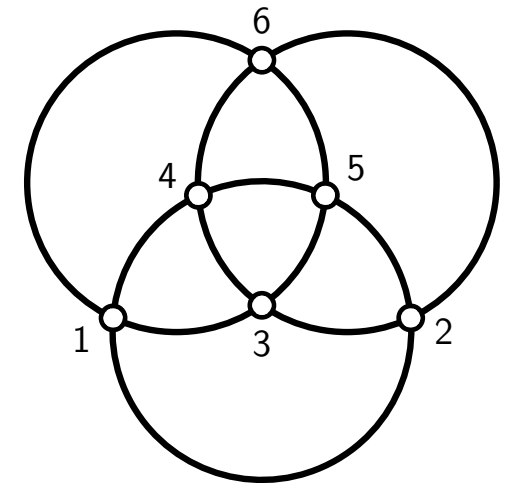
straight-line drawing



orthogonal drawing



grid drawing with
bends & 3 slopes



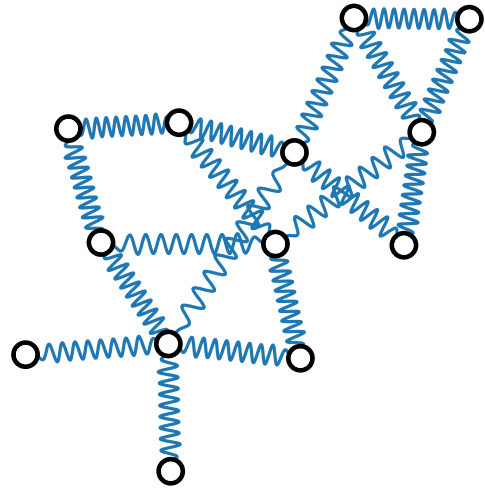
circular-arc drawing

And Non-Planar Graphs?

We have seen a few drawing styles:

And Non-Planar Graphs?

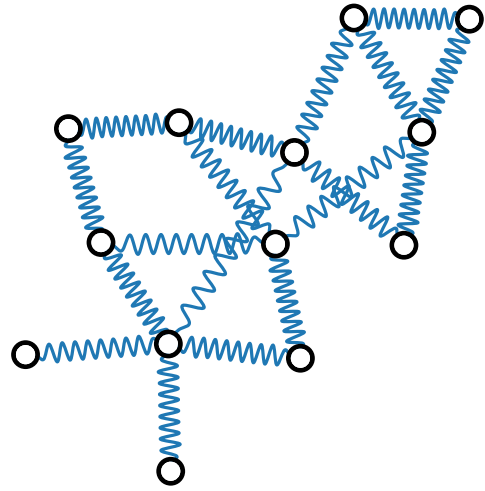
We have seen a few drawing styles:



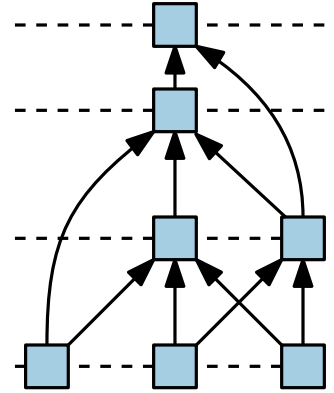
force-directed drawing

And Non-Planar Graphs?

We have seen a few drawing styles:



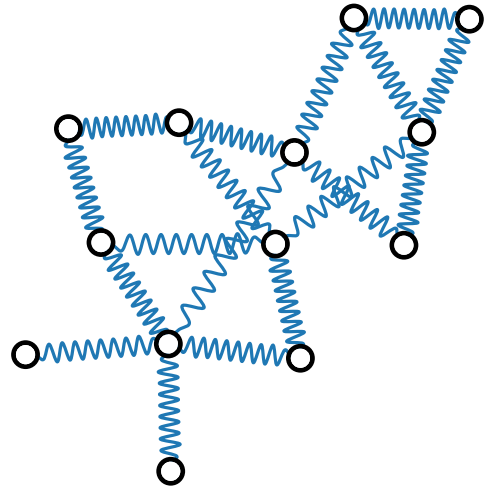
force-directed drawing



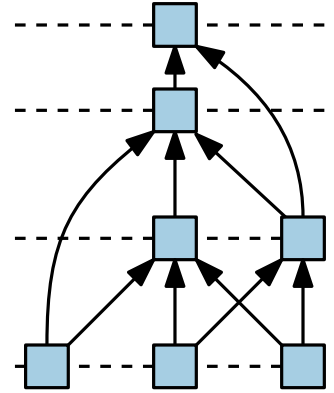
hierarchical drawing

And Non-Planar Graphs?

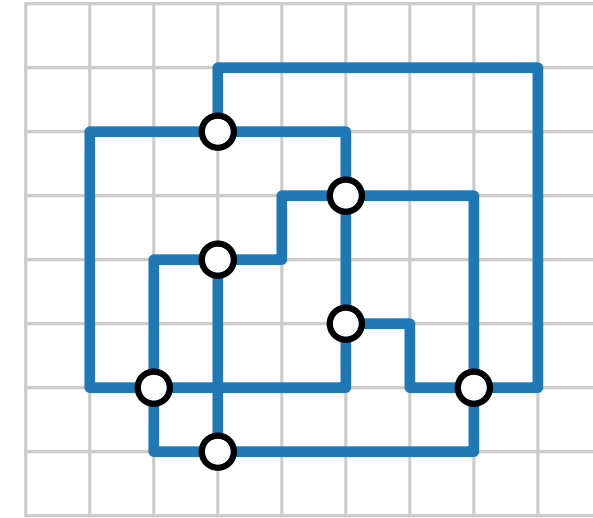
We have seen a few drawing styles:



force-directed drawing



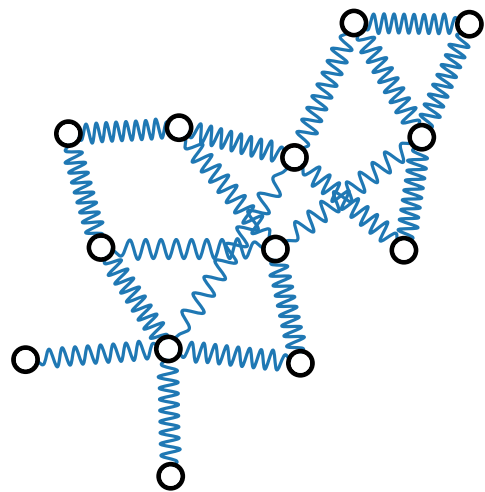
hierarchical drawing



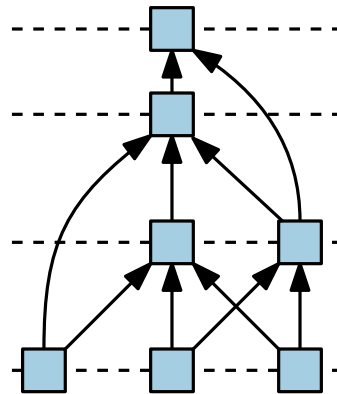
orthogonal layouts
(via planarization)

And Non-Planar Graphs?

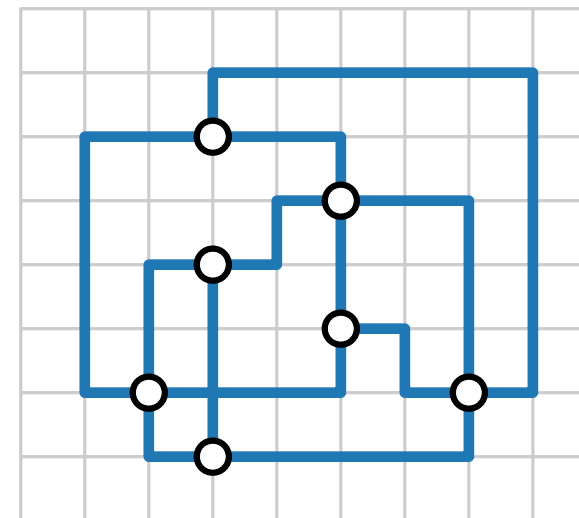
We have seen a few drawing styles:



force-directed drawing



hierarchical drawing

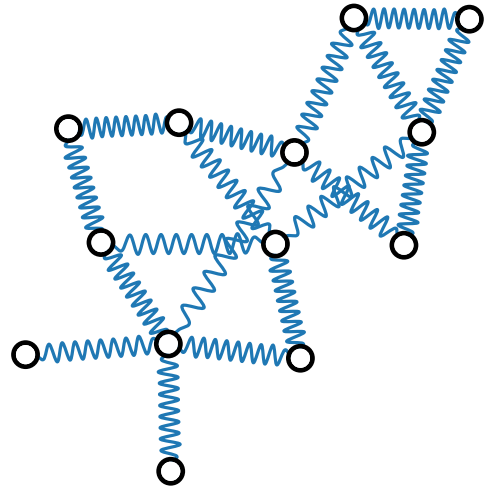


orthogonal layouts
(via planarization)

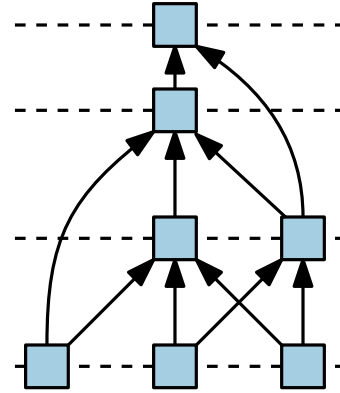
Maybe not all crossings are equally bad?

And Non-Planar Graphs?

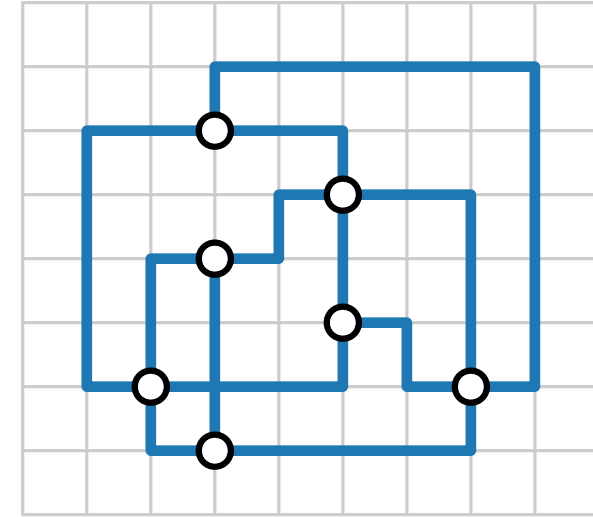
We have seen a few drawing styles:



force-directed drawing

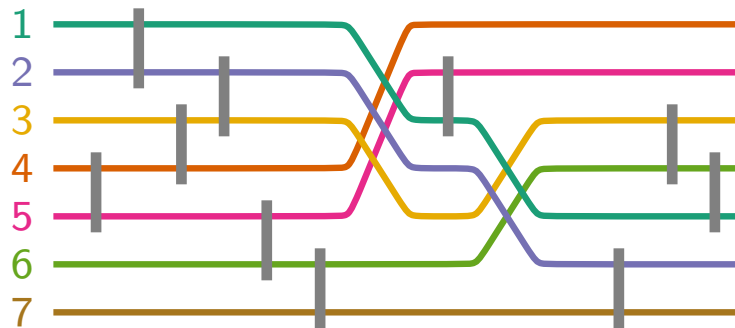


hierarchical drawing



orthogonal layouts
(via planarization)

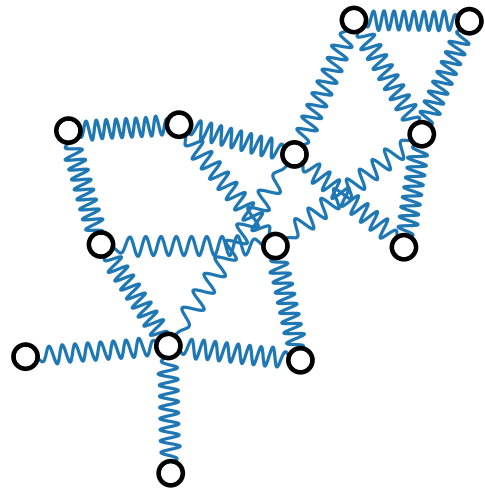
Maybe not all crossings are equally bad?



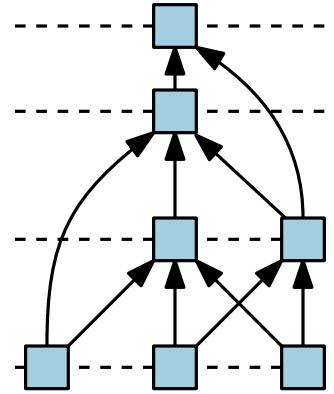
block crossings

And Non-Planar Graphs?

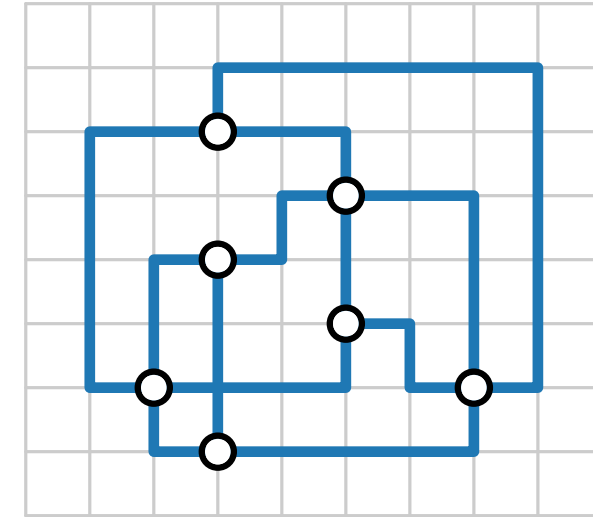
We have seen a few drawing styles:



force-directed drawing

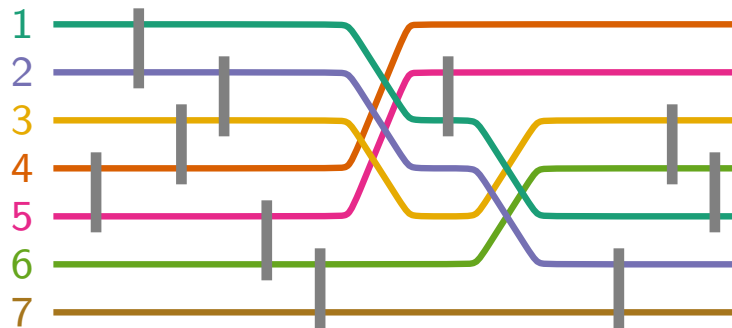


hierarchical drawing

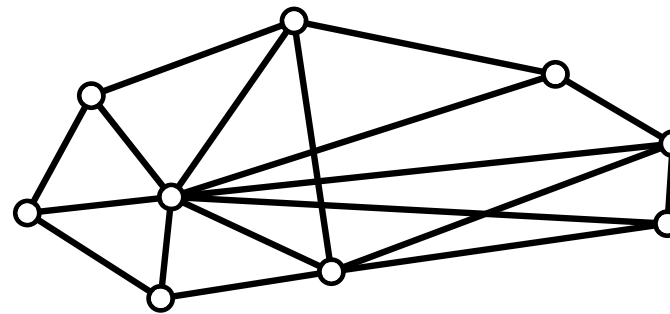


orthogonal layouts
(via planarization)

Maybe not all crossings are equally bad?



block crossings

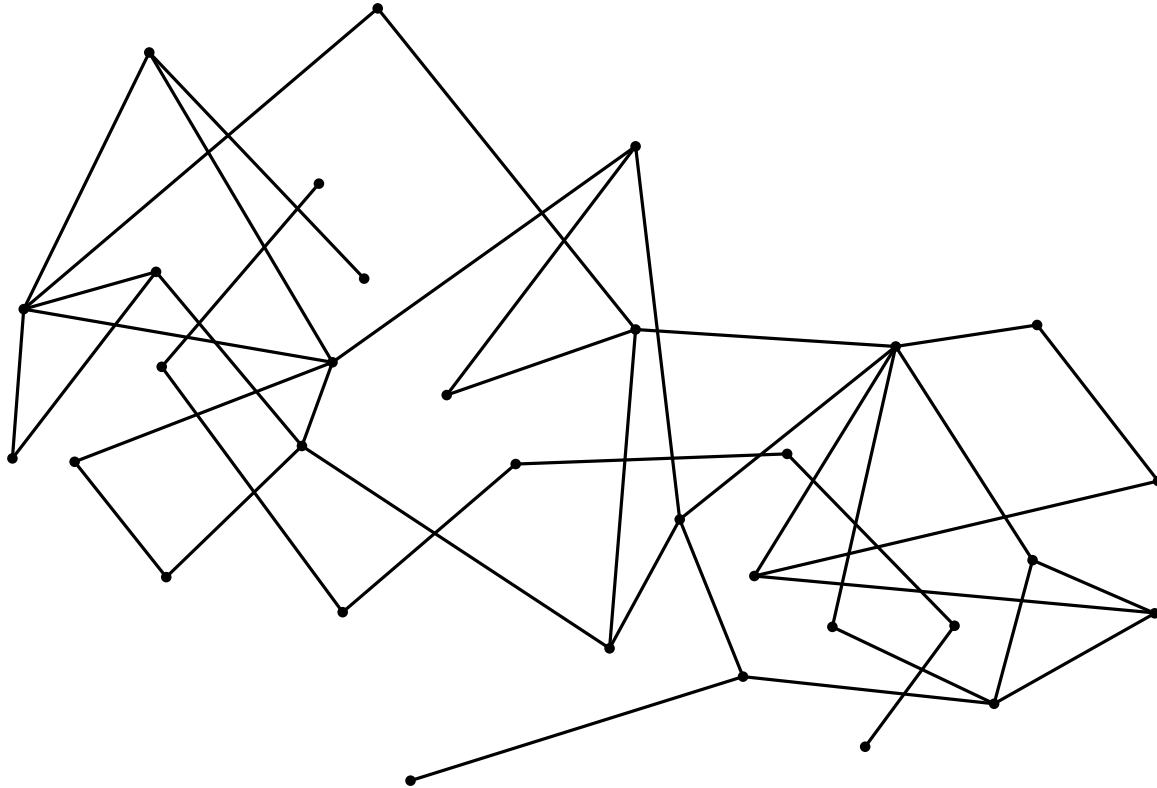


Which crossings feel worse?

Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

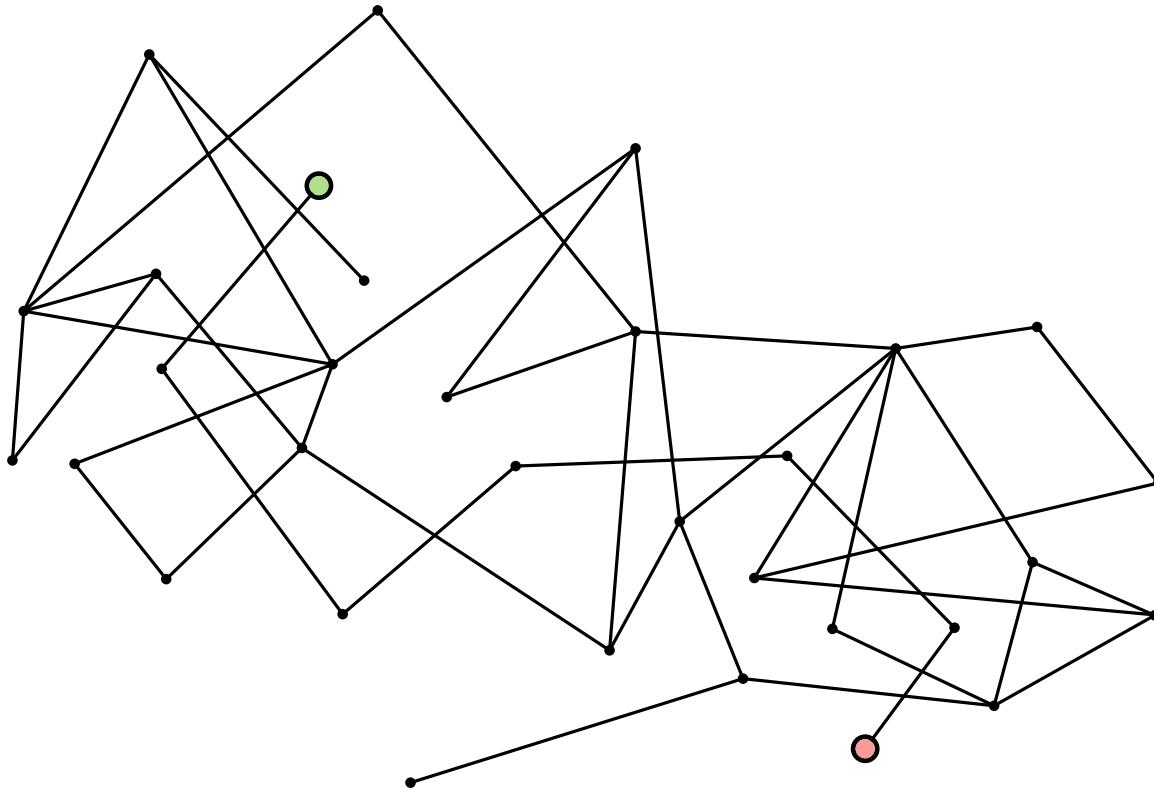
Input: A graph drawing and designated path.



Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

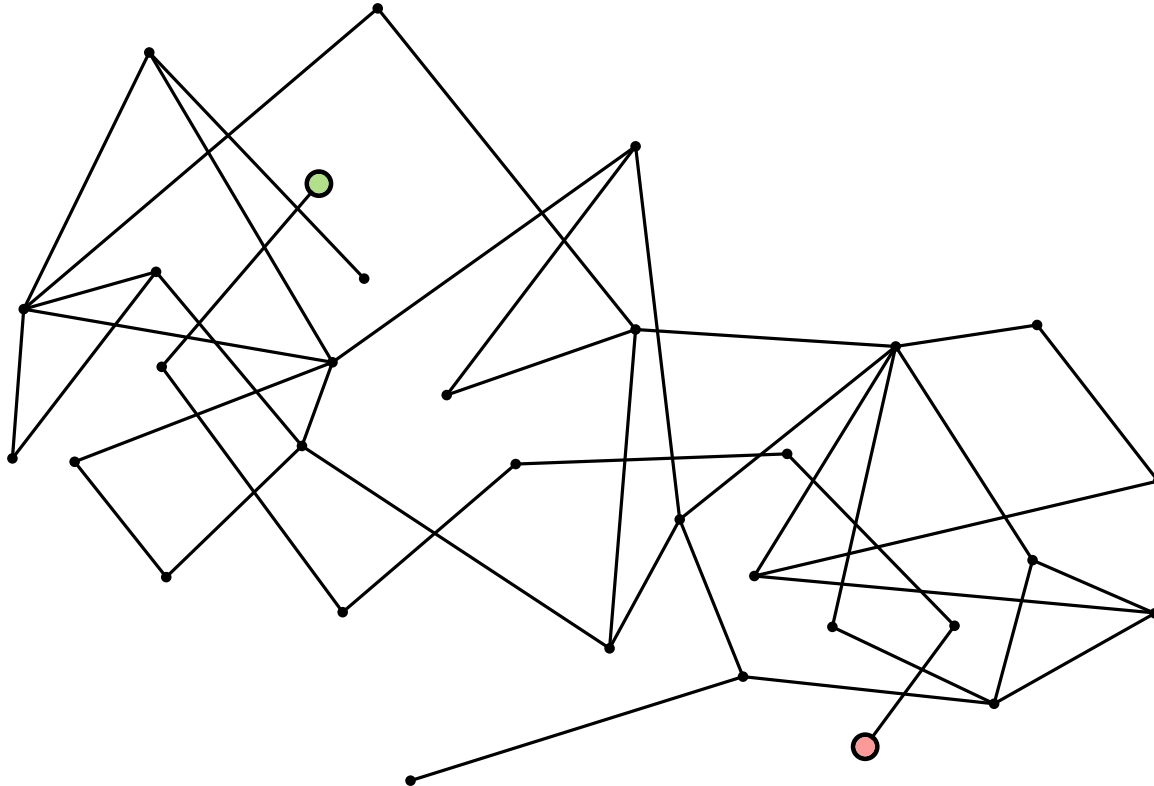


Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

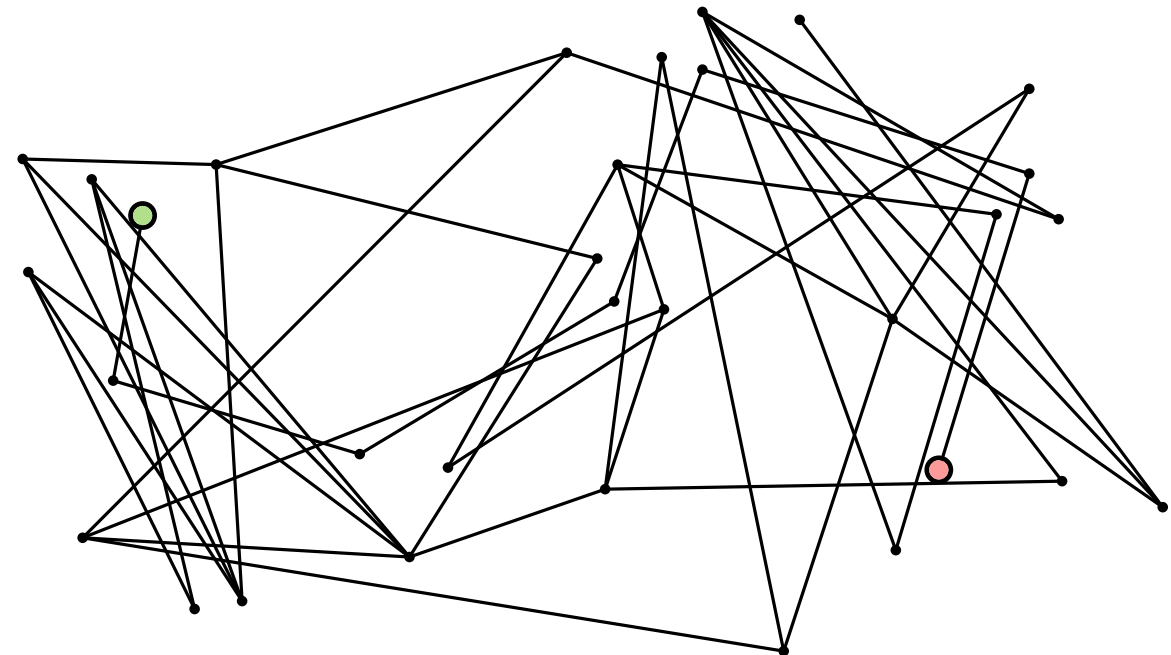
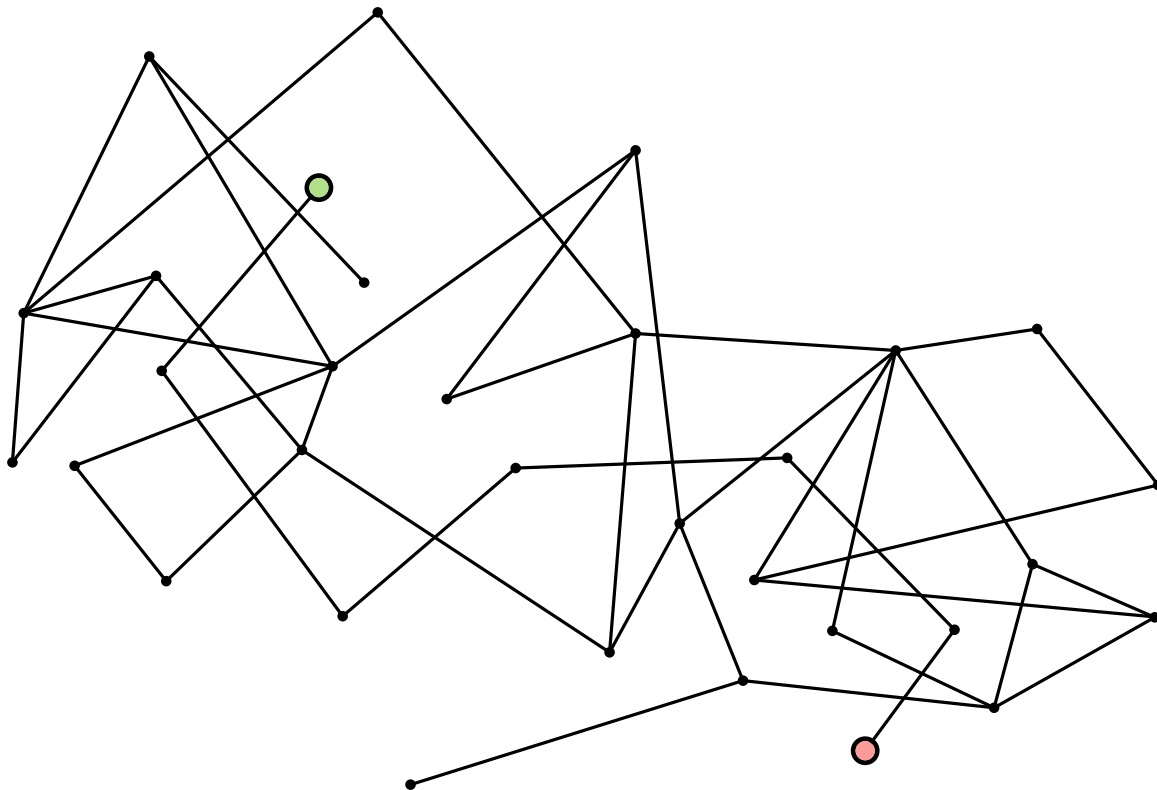


Eye-Tracking Experiment

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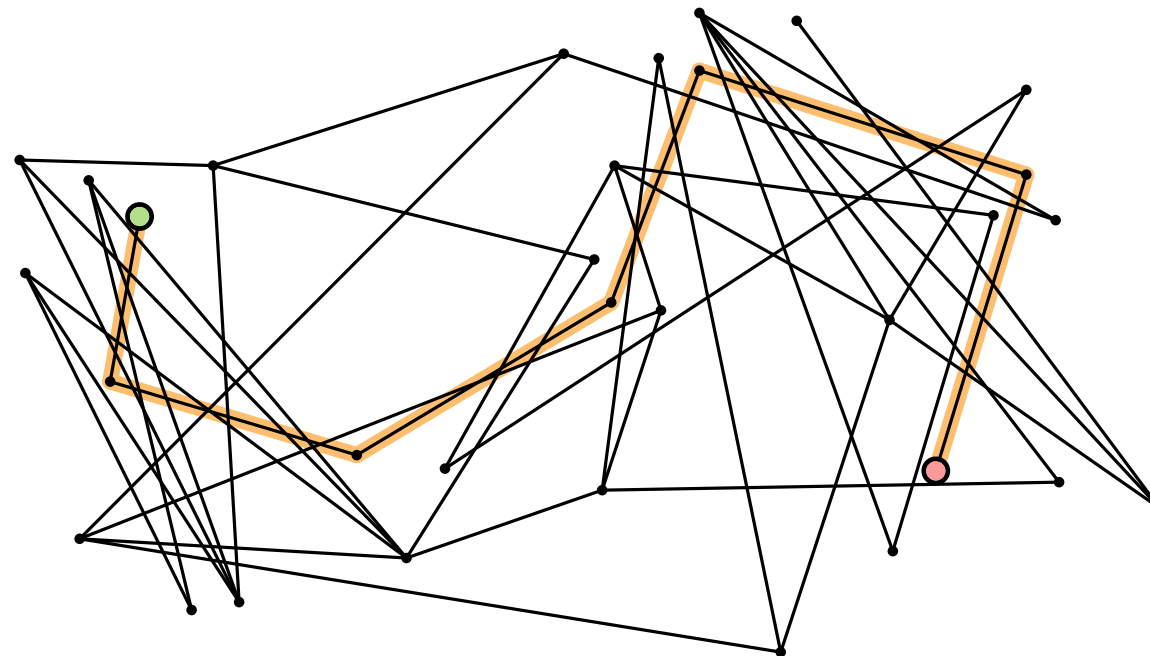
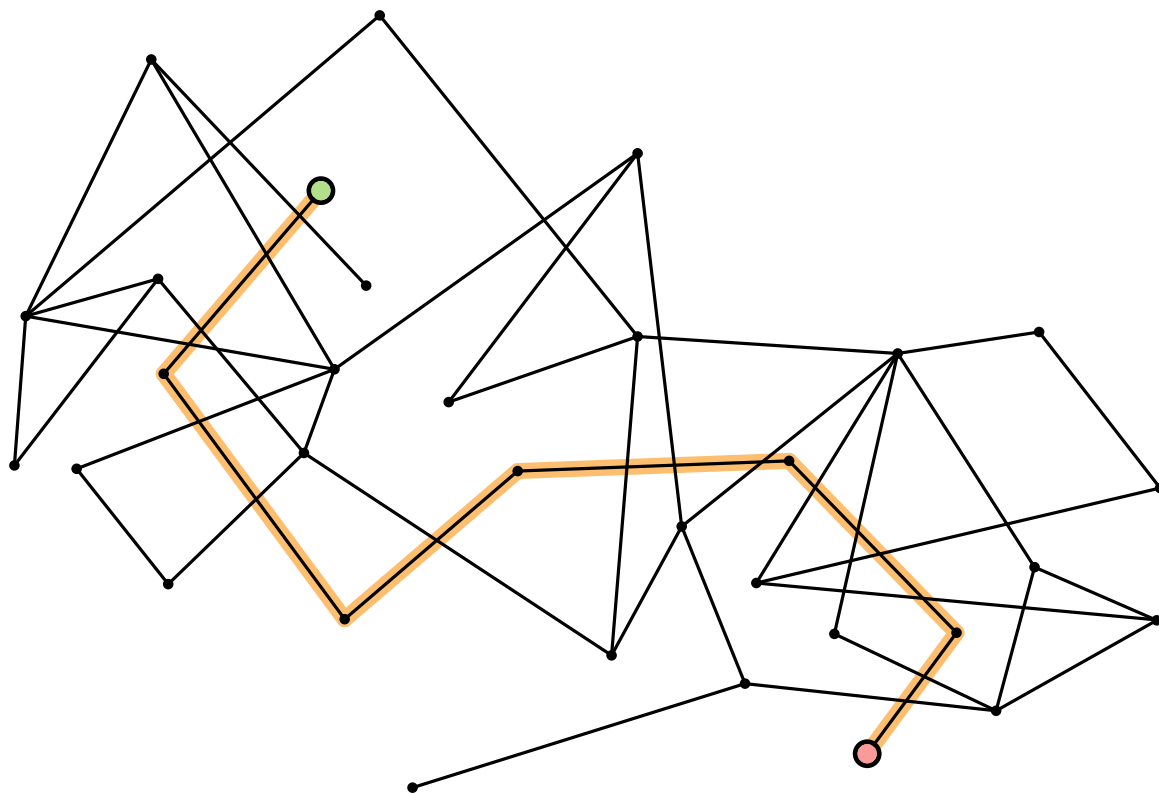


Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.



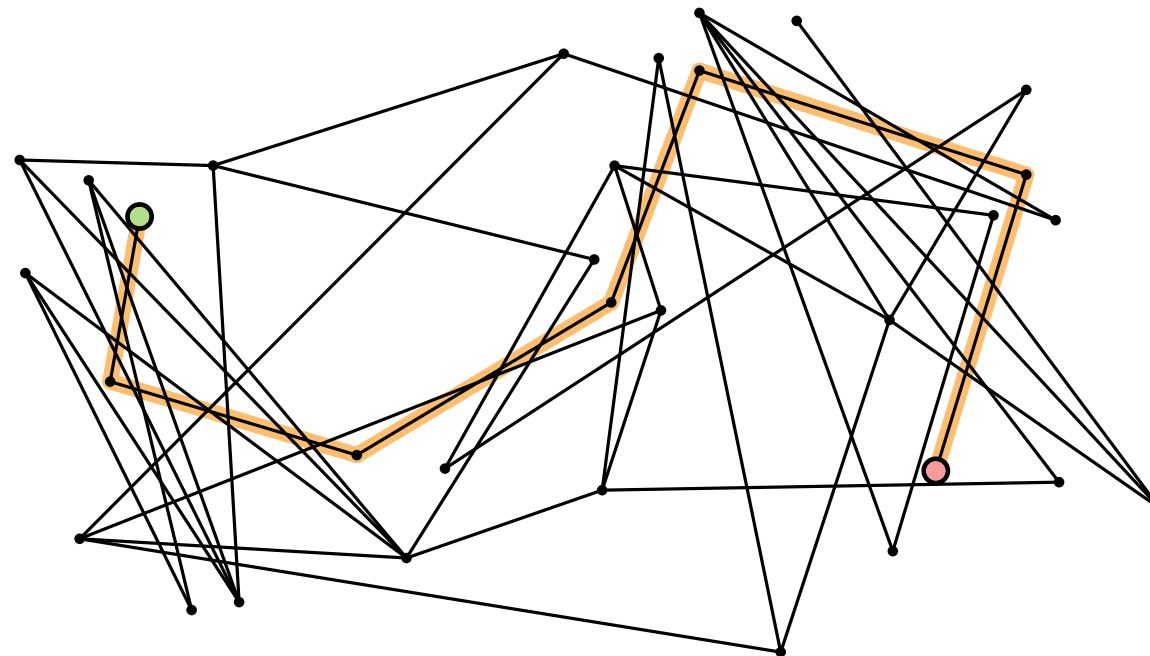
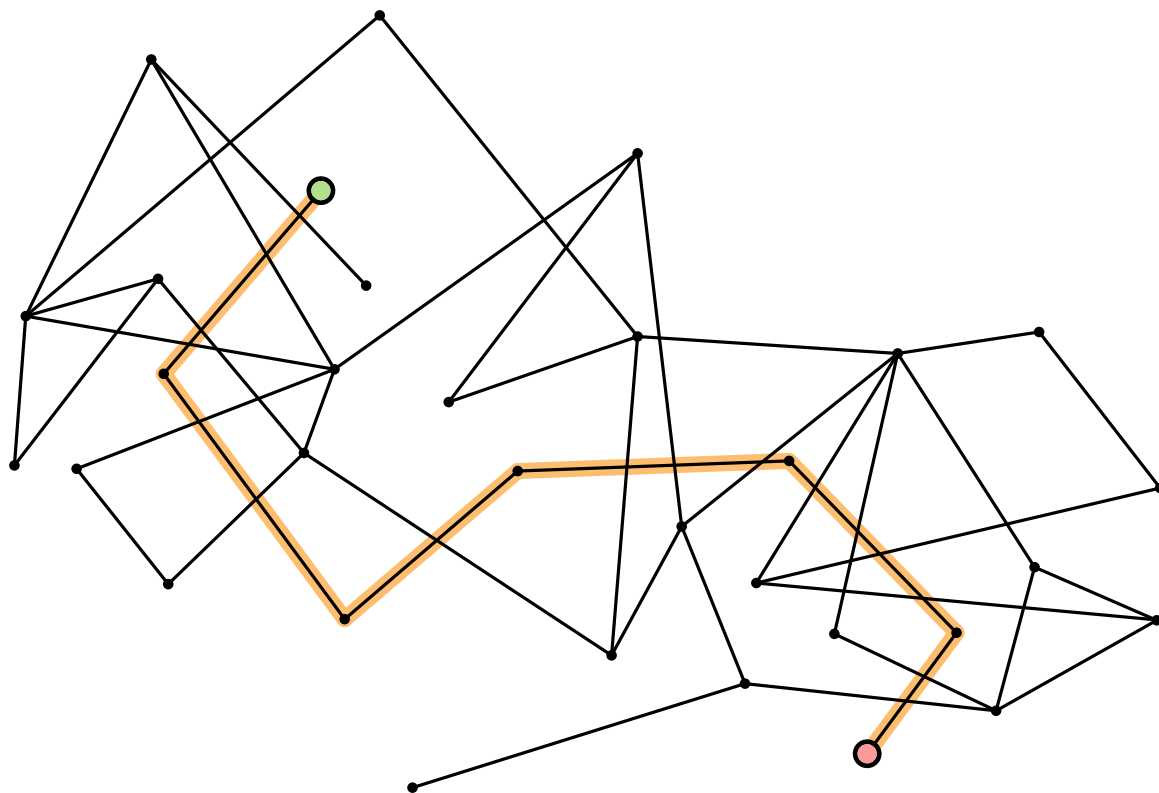
Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results:



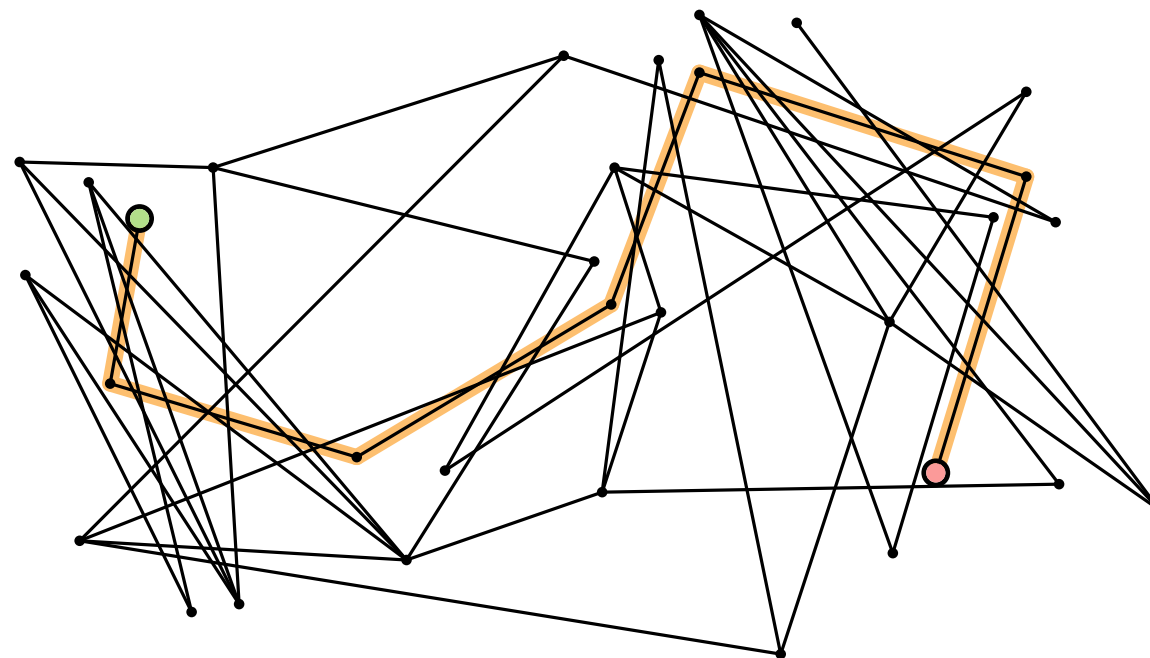
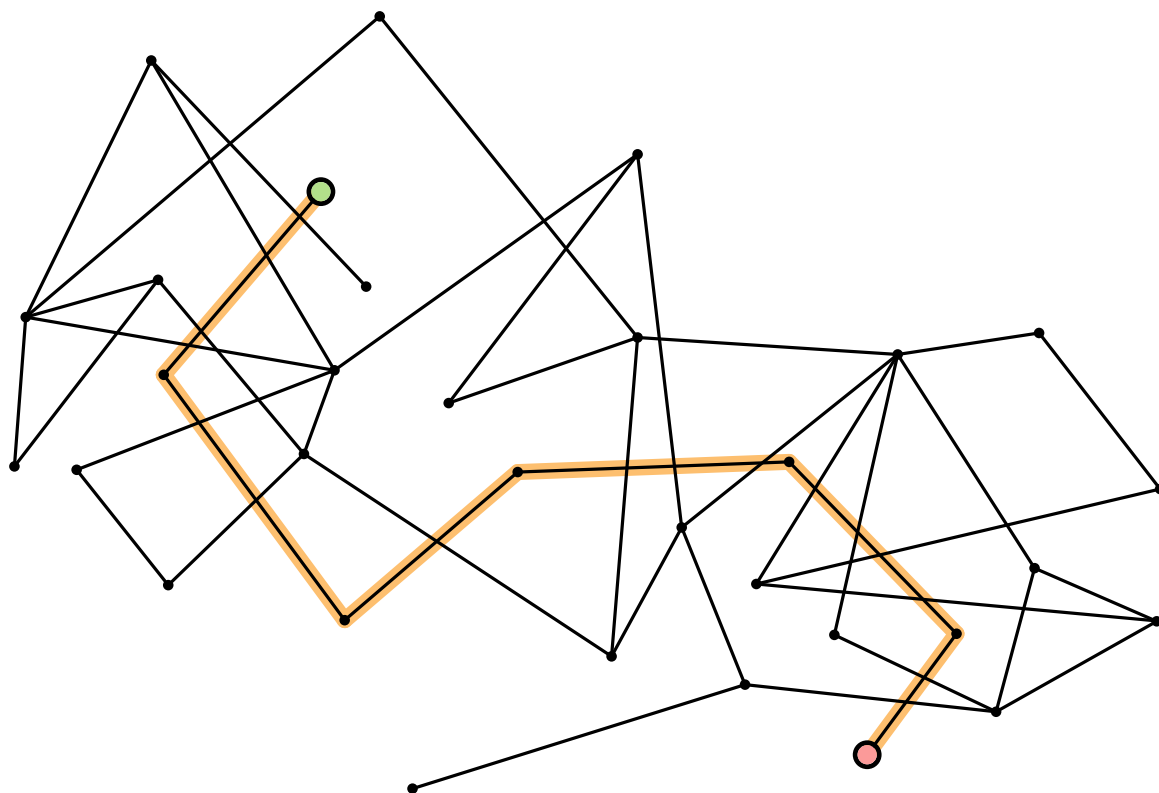
Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results: no crossings eye movements smooth and fast



Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

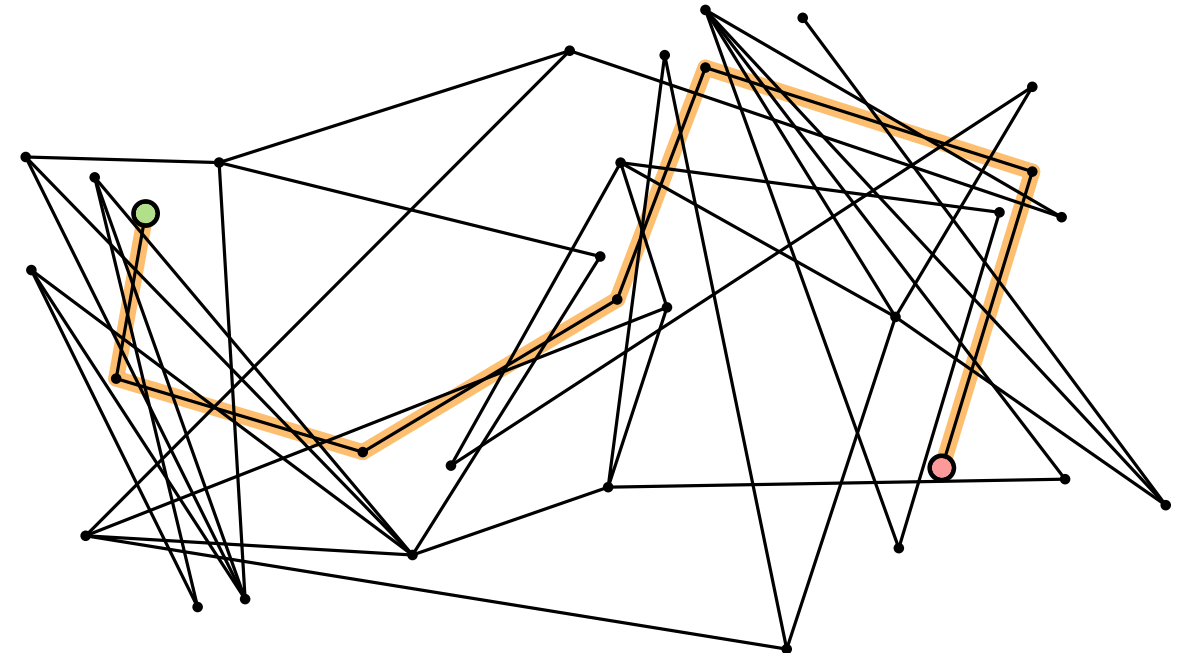
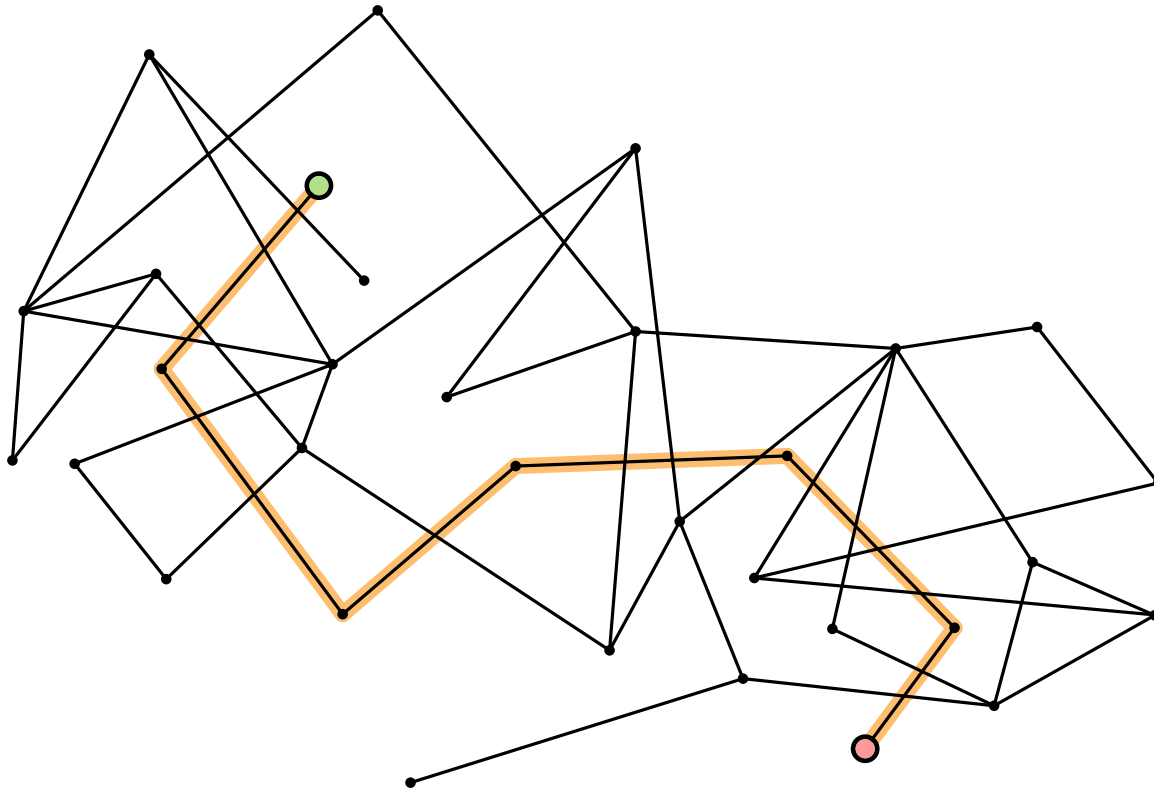
Task: Trace path and count number of edges.

Results: no crossings

eye movements smooth and fast

large crossing angles

eye movements smooth but slightly slower



Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results: no crossings

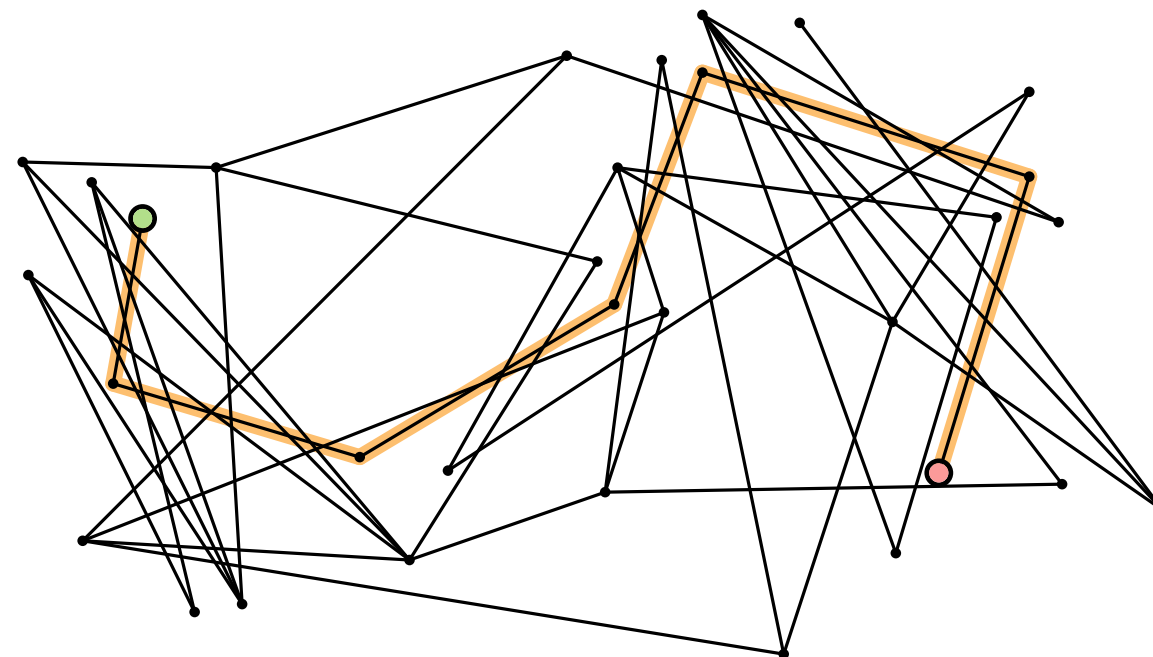
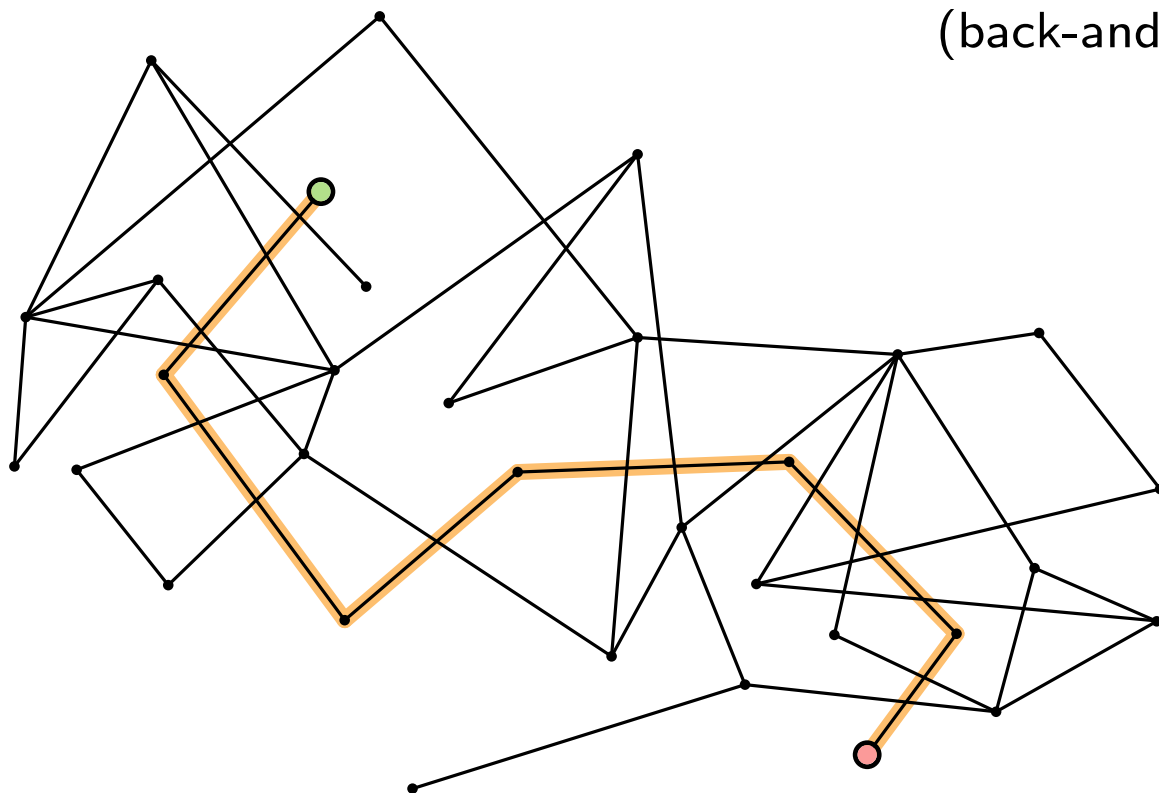
large crossing angles

small crossing angles

eye movements smooth and fast

eye movements smooth but slightly slower

eye movements no longer smooth and very slow
(back-and-forth movements at crossing points)

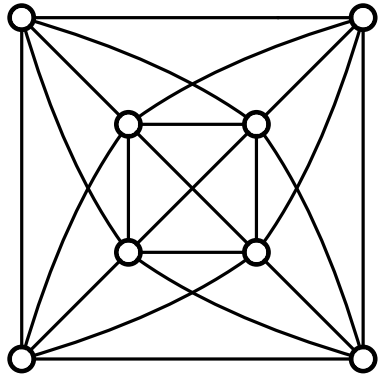


Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.

Some Beyond-Planar Graph Classes

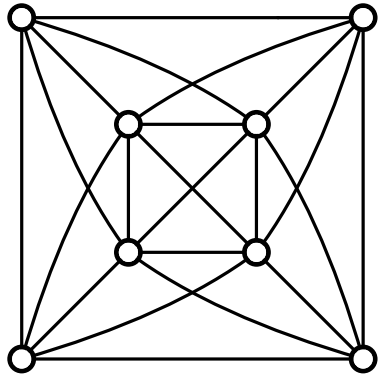
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



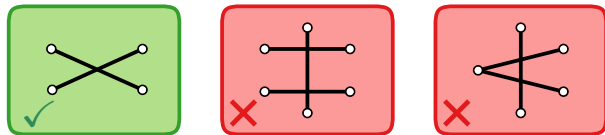
k -planar ($k = 1$)

Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.

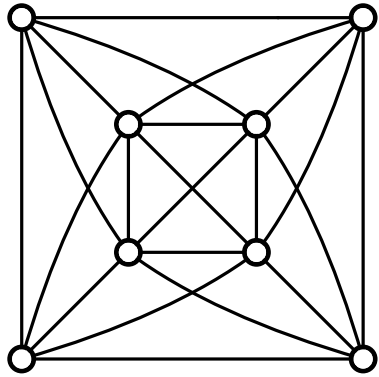


k -planar ($k = 1$)

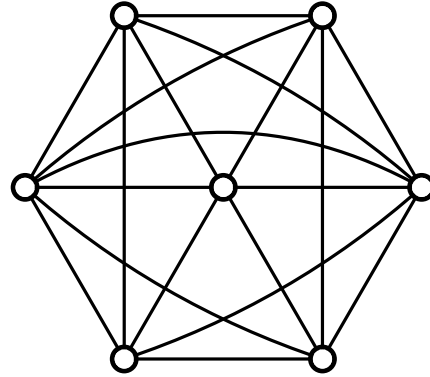


Some Beyond-Planar Graph Classes

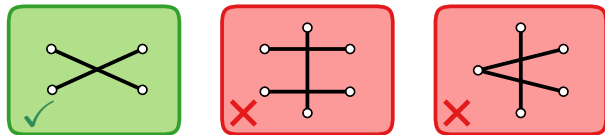
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k -planar ($k = 1$)

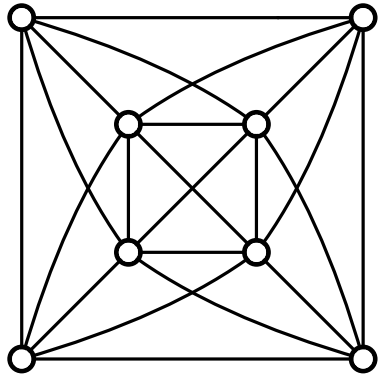


k -quasi-planar ($k = 3$)

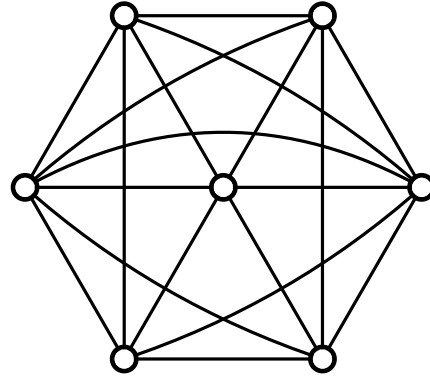
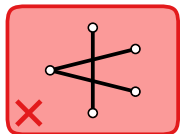
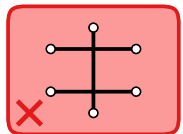
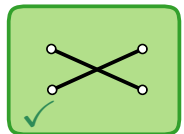


Some Beyond-Planar Graph Classes

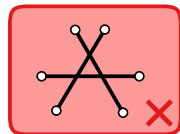
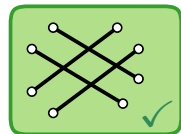
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k -planar ($k = 1$)

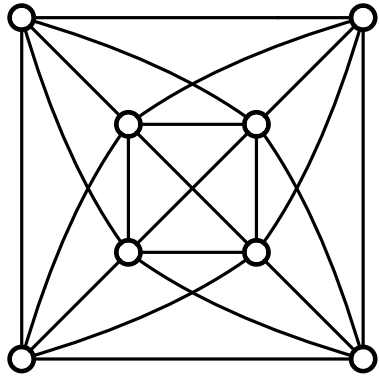


k -quasi-planar ($k = 3$)

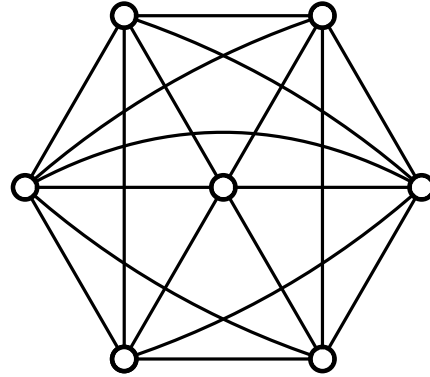
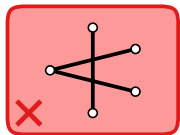
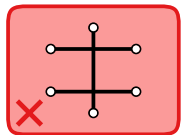
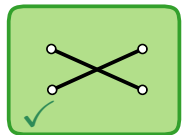


Some Beyond-Planar Graph Classes

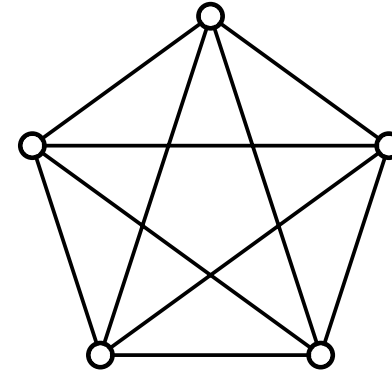
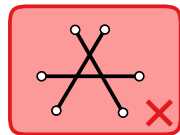
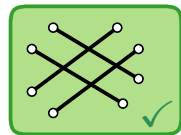
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



k -planar ($k = 1$)



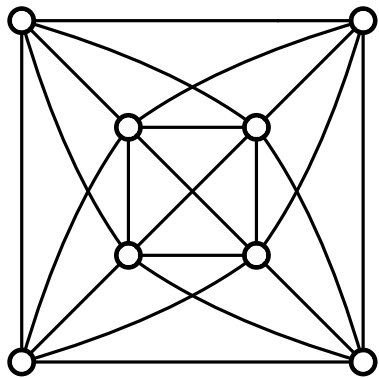
k -quasi-planar ($k = 3$)



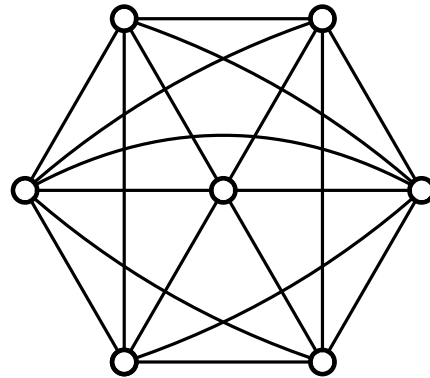
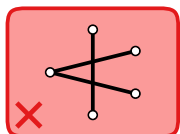
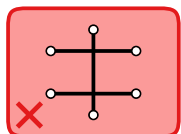
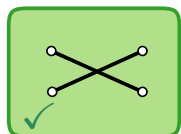
fan-planar

Some Beyond-Planar Graph Classes

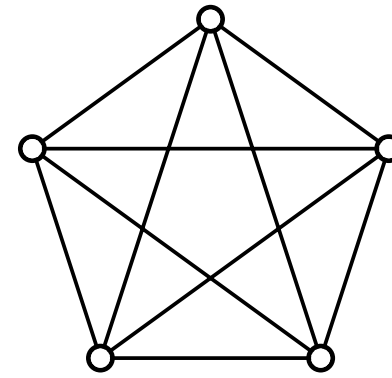
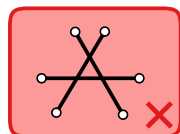
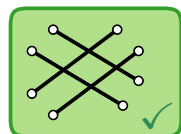
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



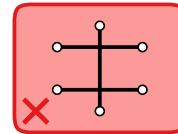
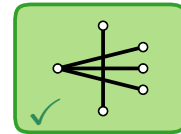
k -planar ($k = 1$)



k -quasi-planar ($k = 3$)

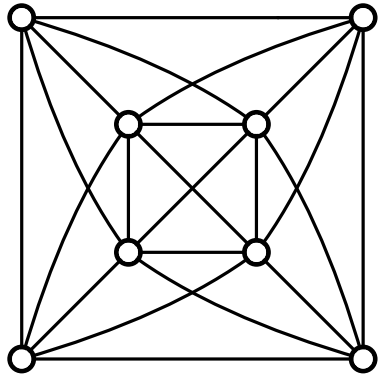


fan-planar

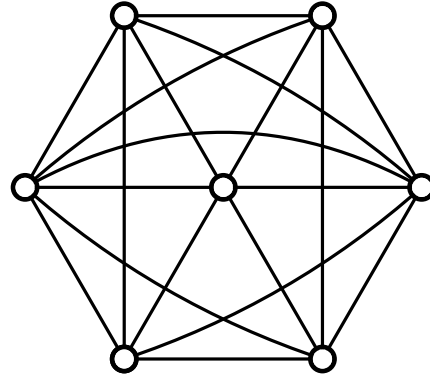
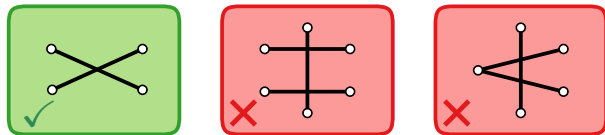


Some Beyond-Planar Graph Classes

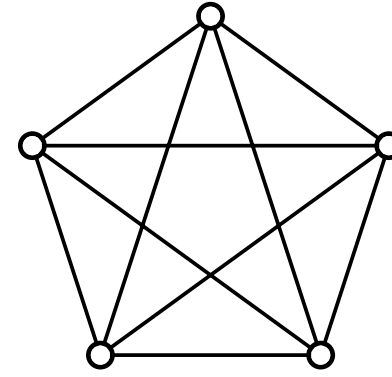
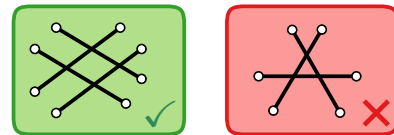
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



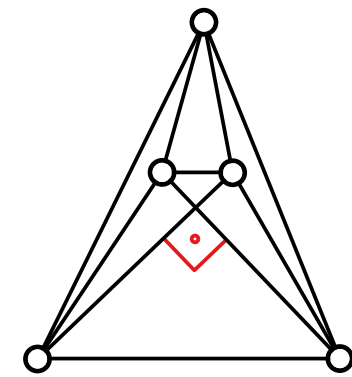
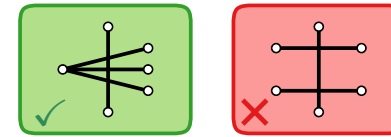
k -planar ($k = 1$)



k -quasi-planar ($k = 3$)



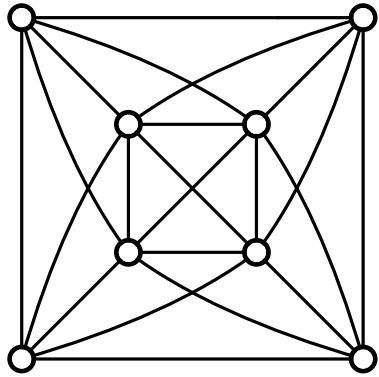
fan-planar



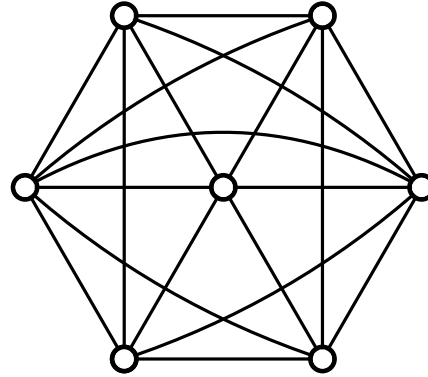
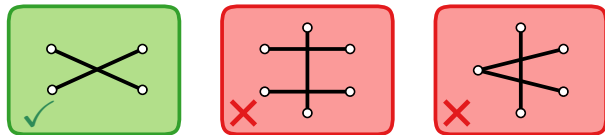
RAC

Some Beyond-Planar Graph Classes

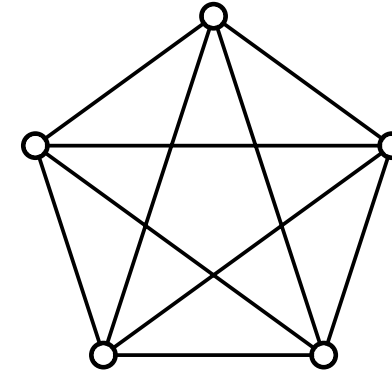
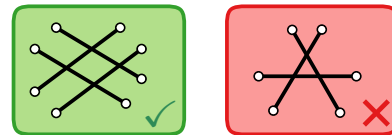
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



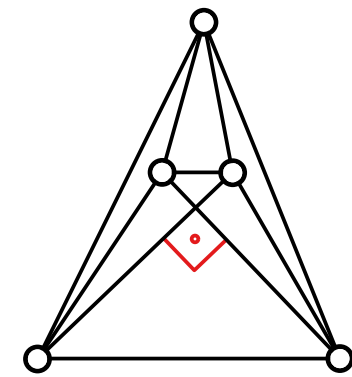
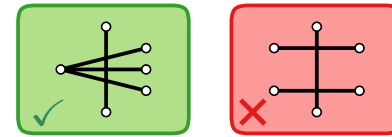
k -planar ($k = 1$)



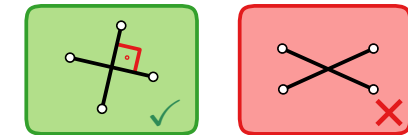
k -quasi-planar ($k = 3$)



fan-planar

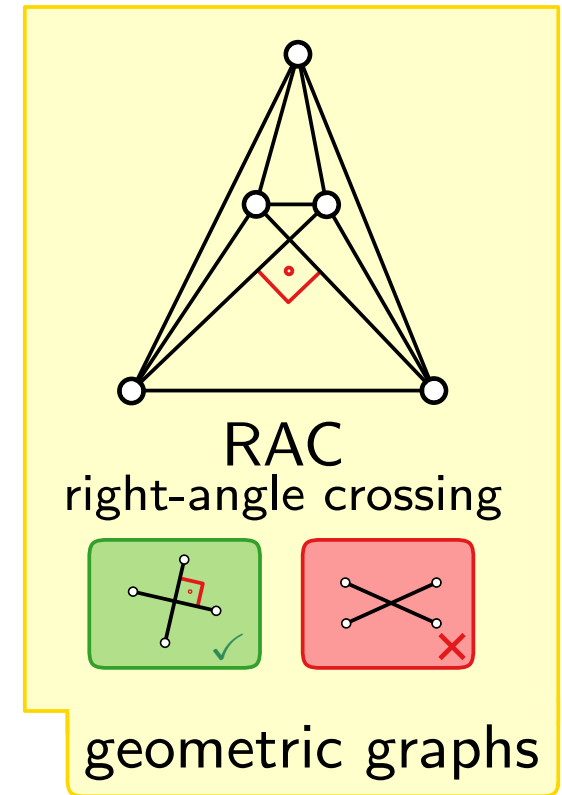
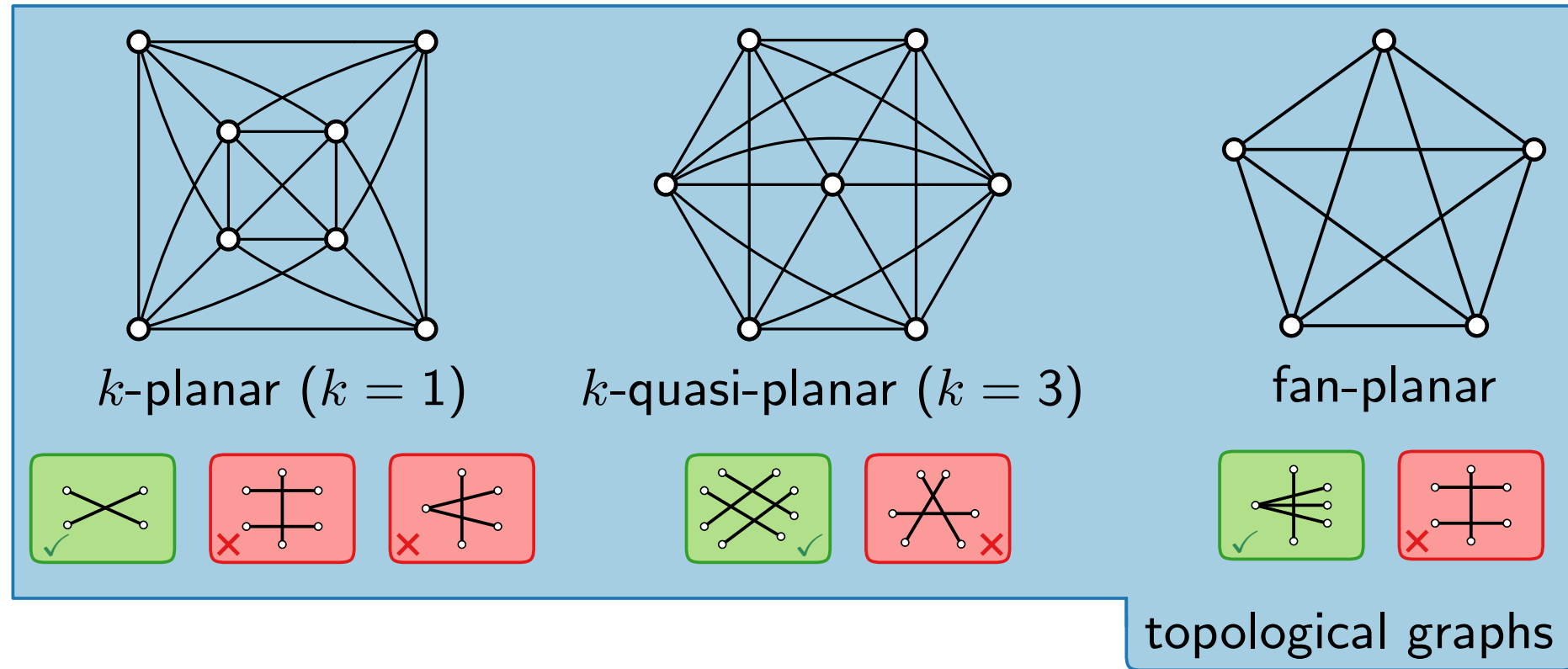


RAC
right-angle crossing



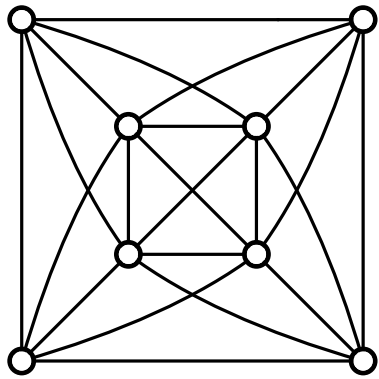
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.

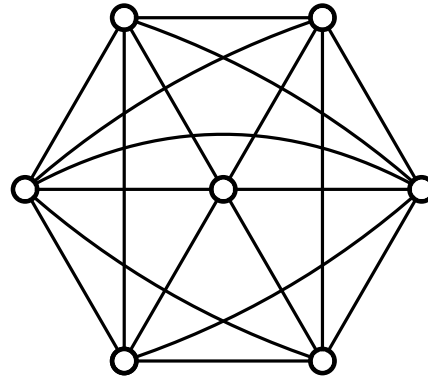
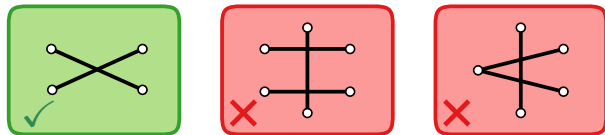


Some Beyond-Planar Graph Classes

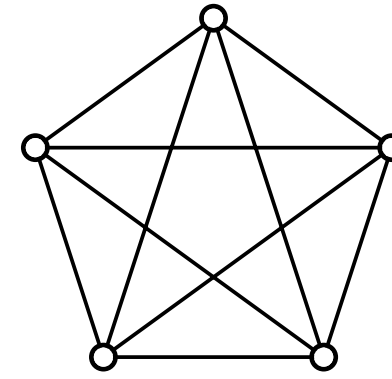
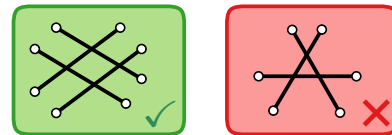
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



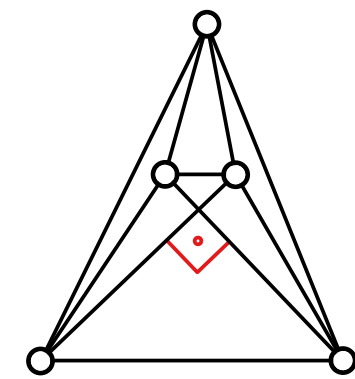
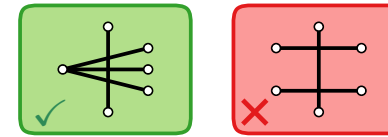
k -planar ($k = 1$)



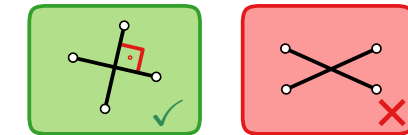
k -quasi-planar ($k = 3$)



fan-planar



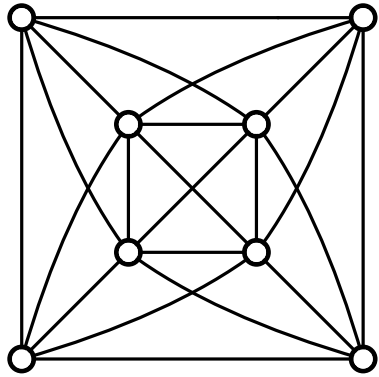
RAC
right-angle crossing



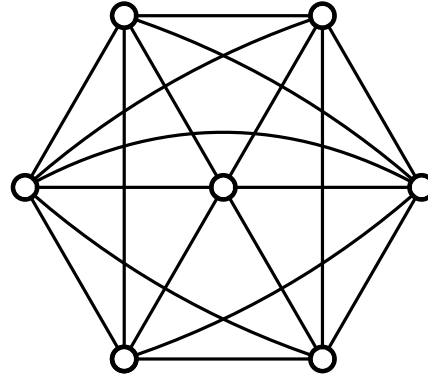
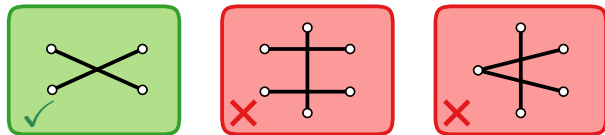
There are many more beyond-planar graph classes...

Some Beyond-Planar Graph Classes

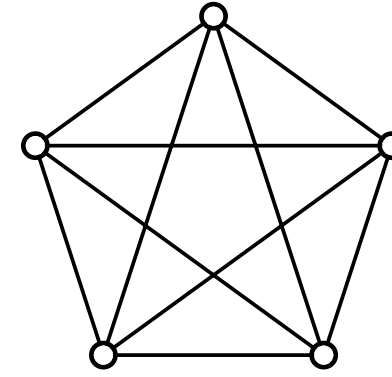
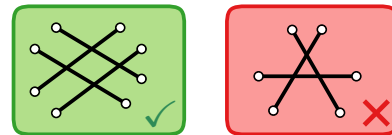
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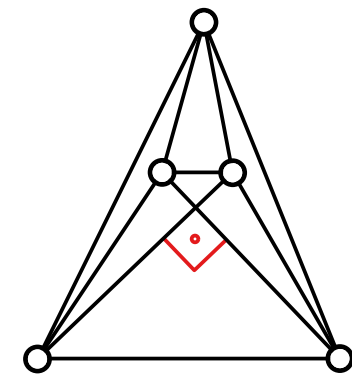
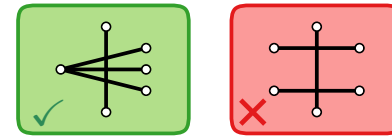
k -planar ($k = 1$)



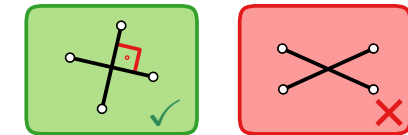
k -quasi-planar ($k = 3$)



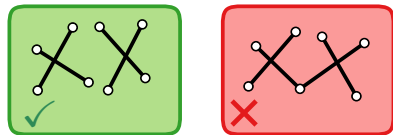
fan-planar



RAC
right-angle crossing



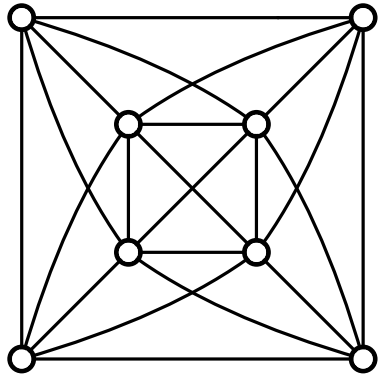
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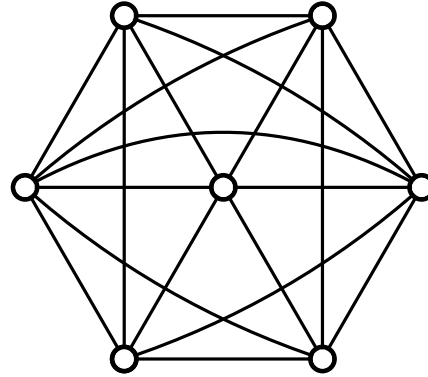
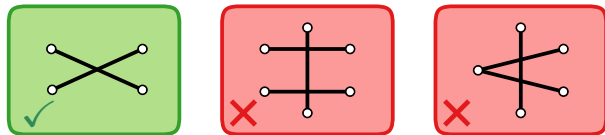
IC (independent crossing)

Some Beyond-Planar Graph Classes

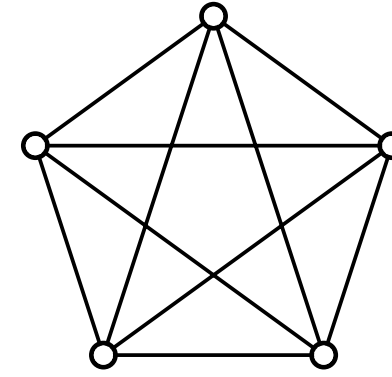
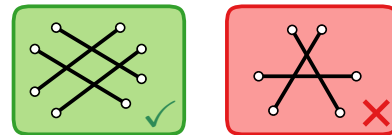
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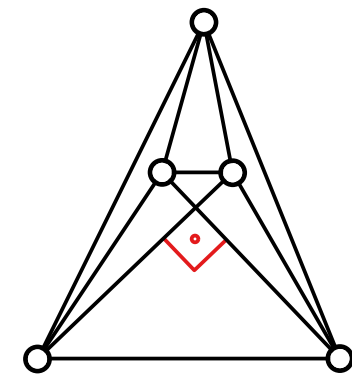
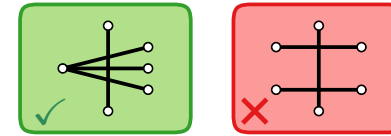
k -planar ($k = 1$)



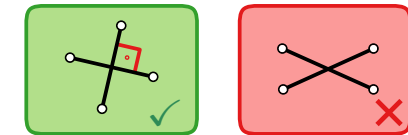
k -quasi-planar ($k = 3$)



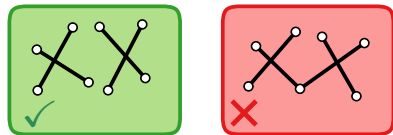
fan-planar



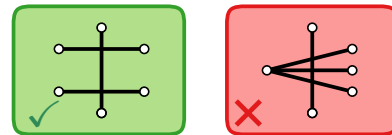
RAC
right-angle crossing



There are many more beyond-planar graph classes...



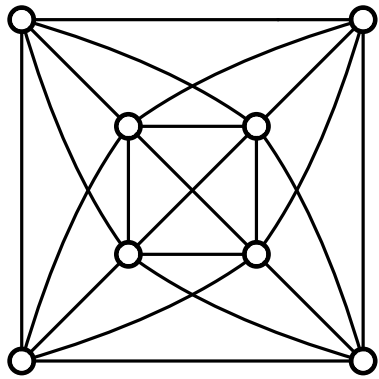
IC (independent crossing)



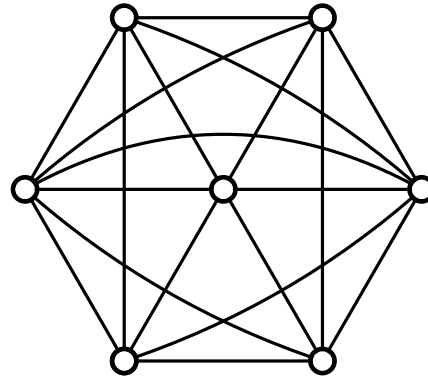
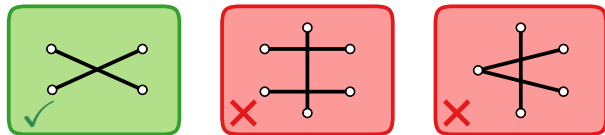
fan-crossing-free

Some Beyond-Planar Graph Classes

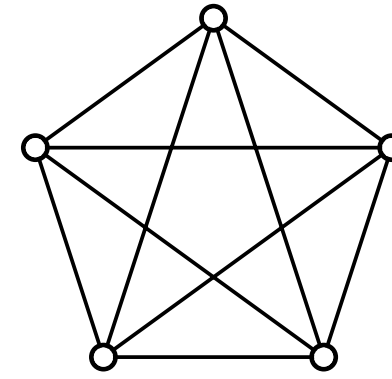
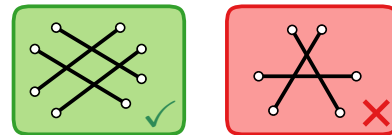
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



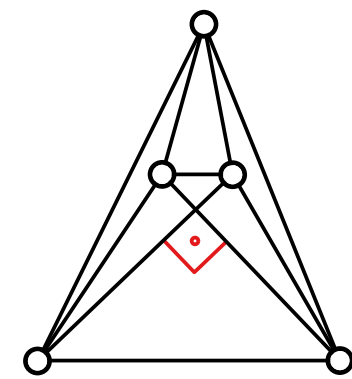
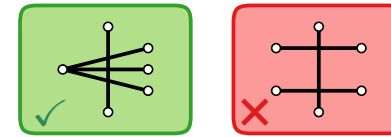
k -planar ($k = 1$)



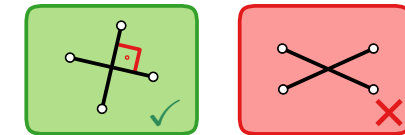
k -quasi-planar ($k = 3$)



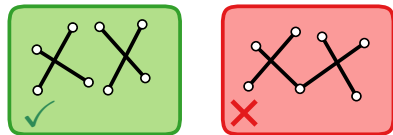
fan-planar



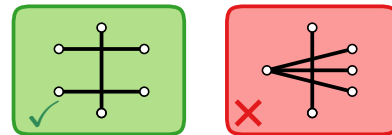
RAC
right-angle crossing



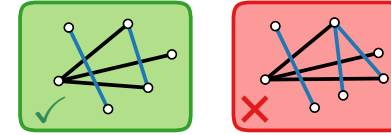
There are many more beyond-planar graph classes...



IC (independent crossing)



fan-crossing-free

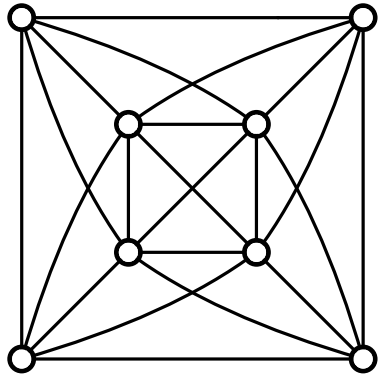


skewness- k ($k = 2$)

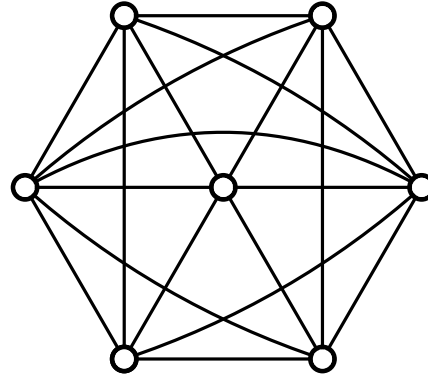
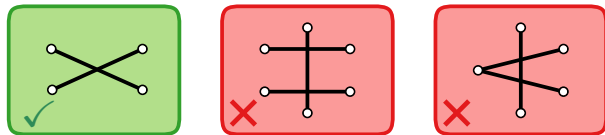
remove $\leq k$ edges to make it planar

Some Beyond-Planar Graph Classes

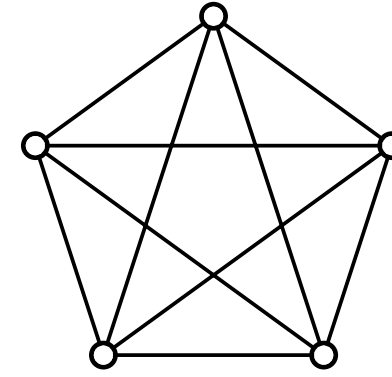
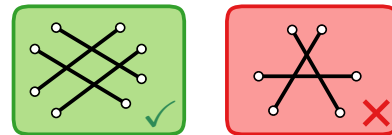
We define **aesthetics** for edge crossings and avoid/minimize “**bad**” crossing configurations.



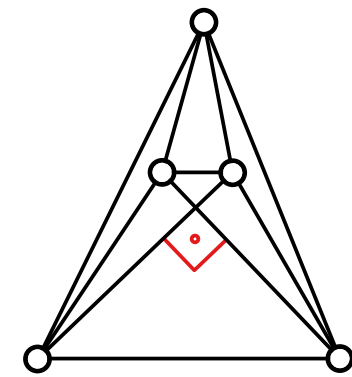
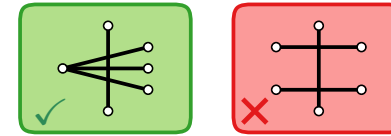
k -planar ($k = 1$)



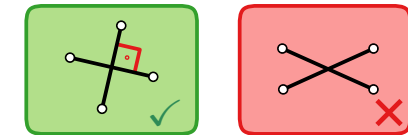
k -quasi-planar ($k = 3$)



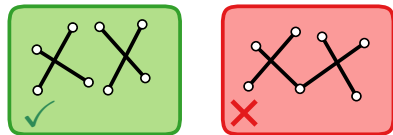
fan-planar



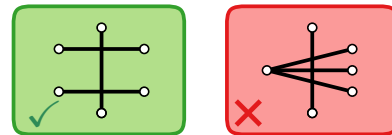
RAC
right-angle crossing



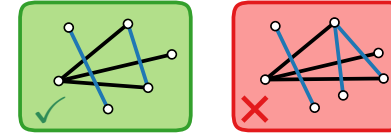
There are many more beyond-planar graph classes...



IC (independent crossing)



fan-crossing-free

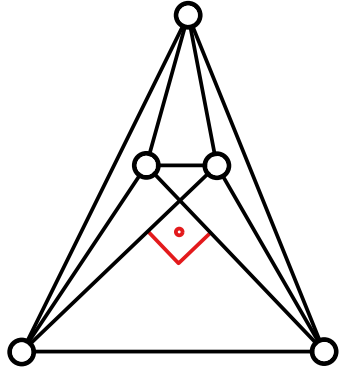


skewness- k ($k = 2$)

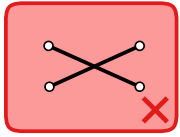
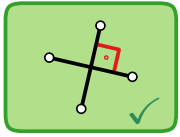
remove $\leq k$ edges to make it planar

combinations, ...

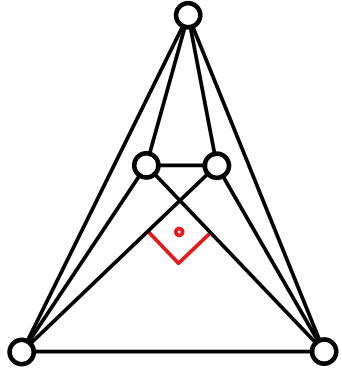
Drawing Styles for Crossings



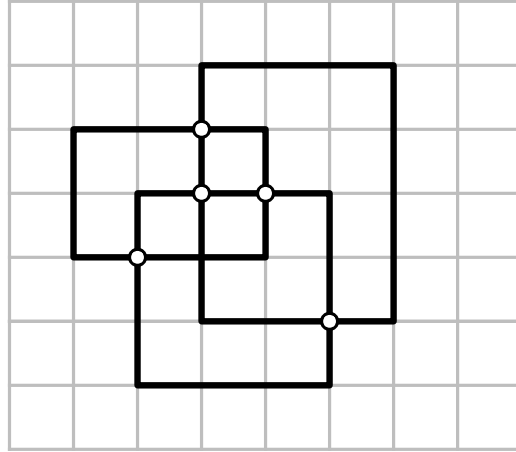
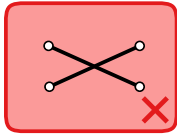
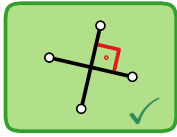
RAC
right-angle crossing



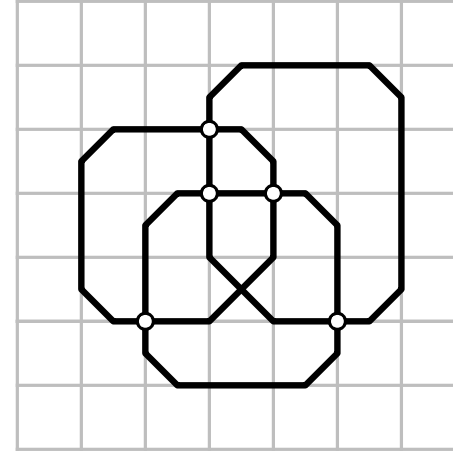
Drawing Styles for Crossings



RAC
right-angle crossing

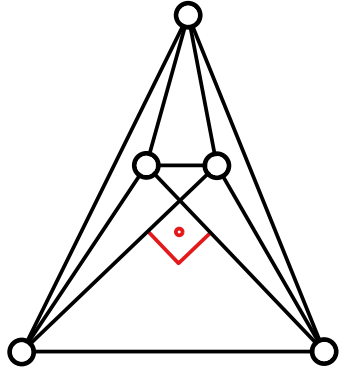


orthogonal

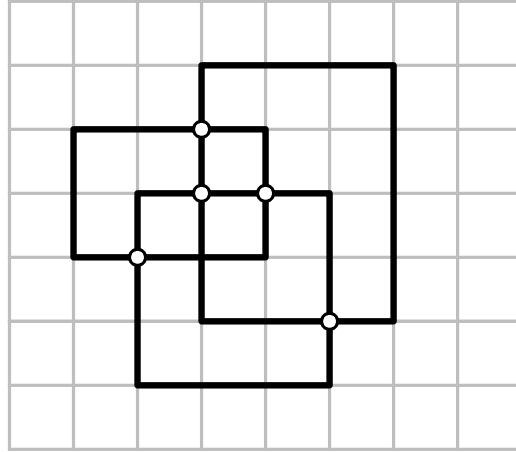
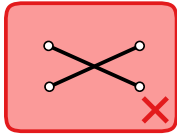
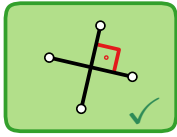


slanted orthogonal

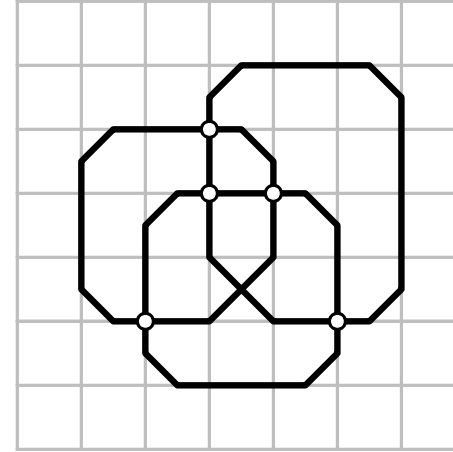
Drawing Styles for Crossings



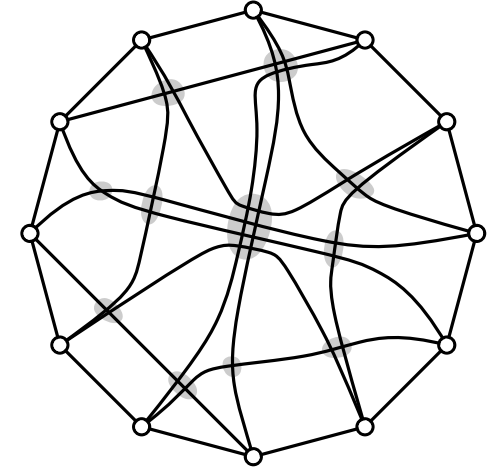
RAC
right-angle crossing



orthogonal



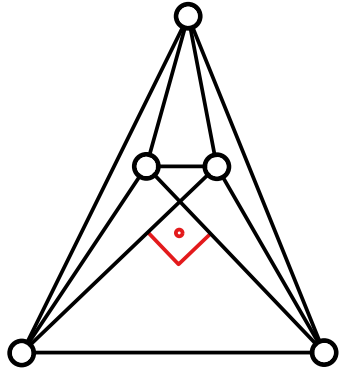
slanted orthogonal



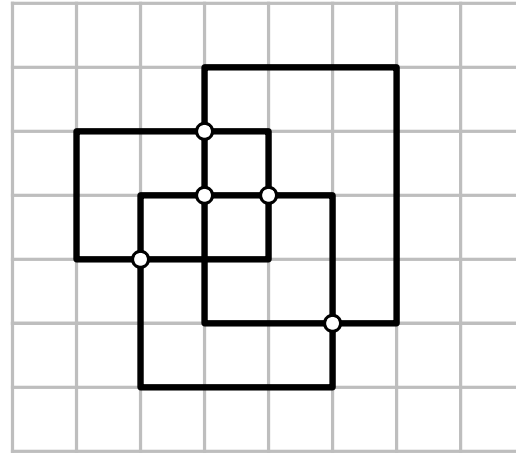
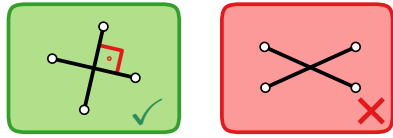
block / bundled crossings

circular layout: 28 individual
vs. 12 bundle crossings

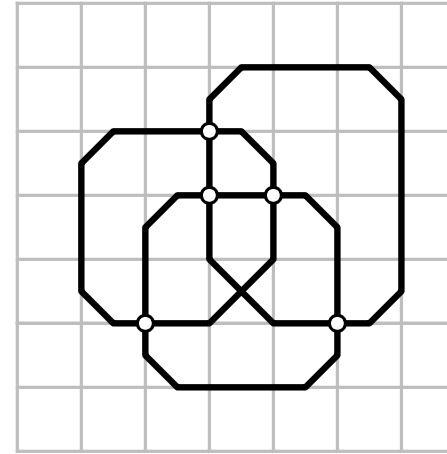
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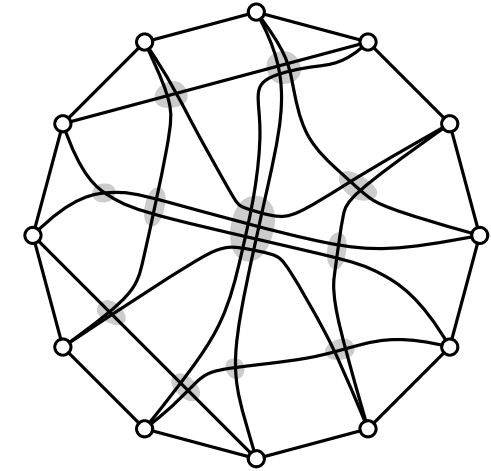
RAC
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orthogonal



slanted orthogonal

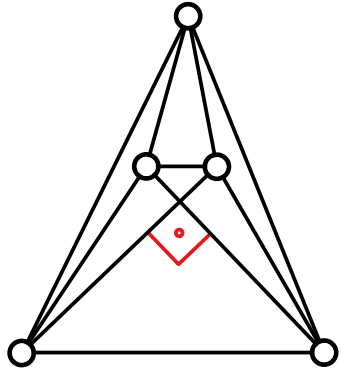


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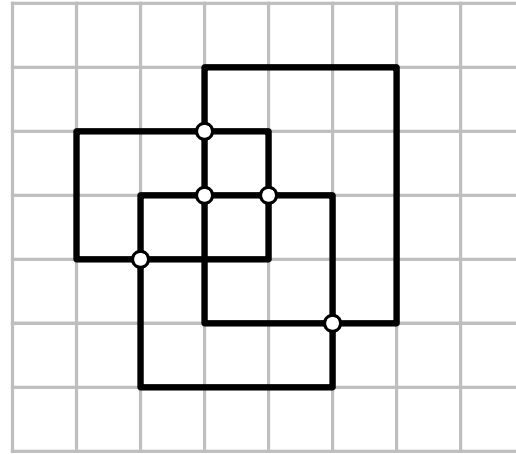
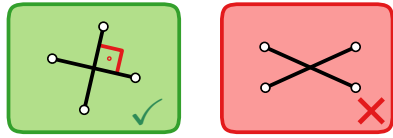
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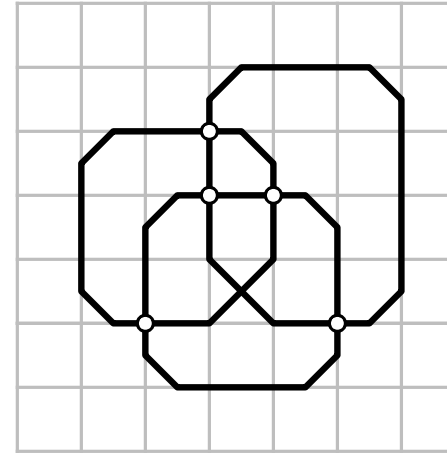
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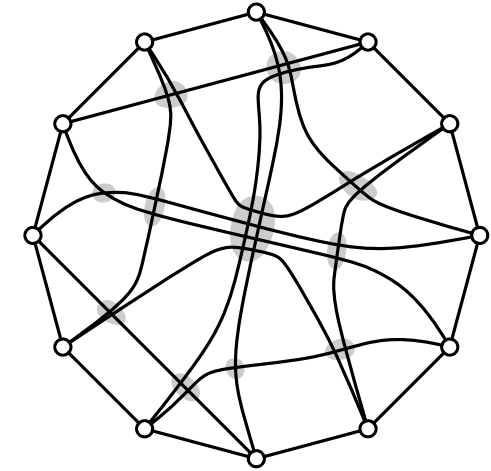
RAC
right-angle crossing



orthogonal

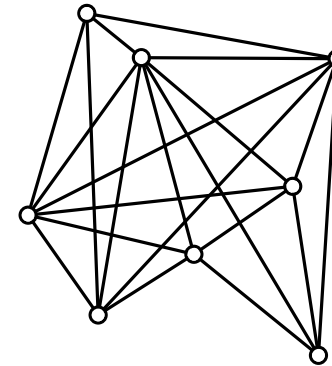


slanted orthogonal

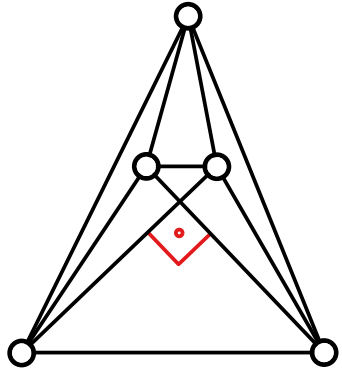


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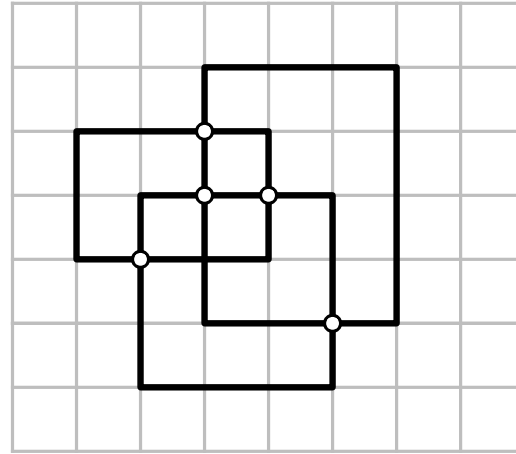
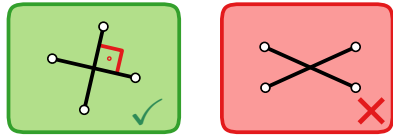
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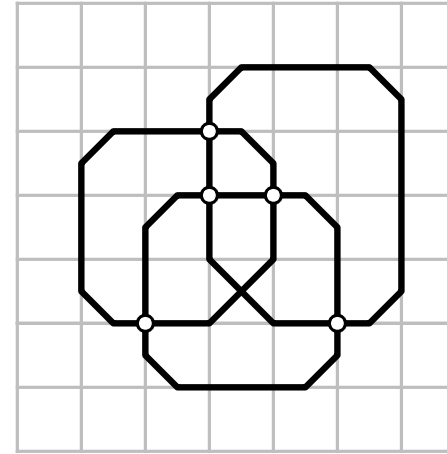
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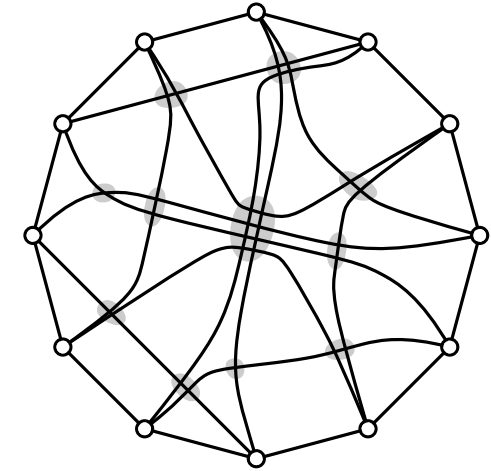
RAC
right-angle crossing



orthogonal

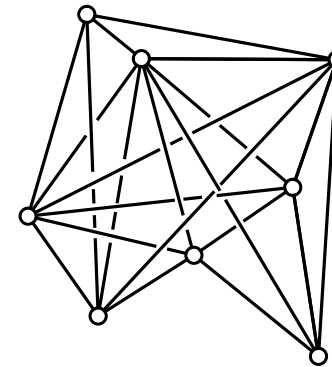


slanted orthogonal



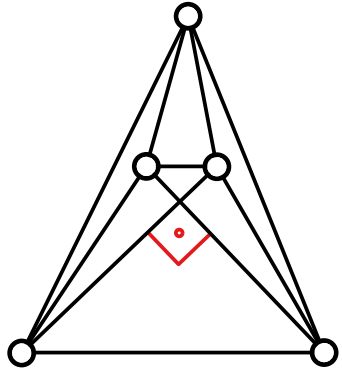
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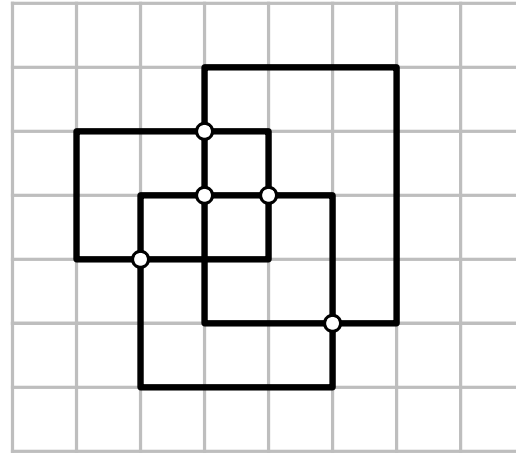
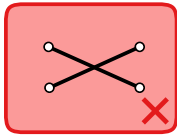
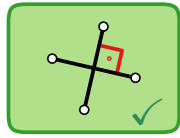


cased crossings

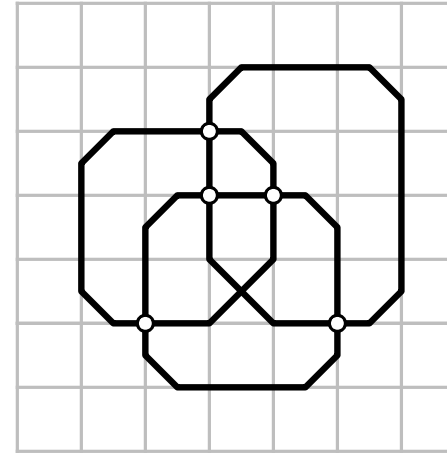
Drawing Styles for Crossings



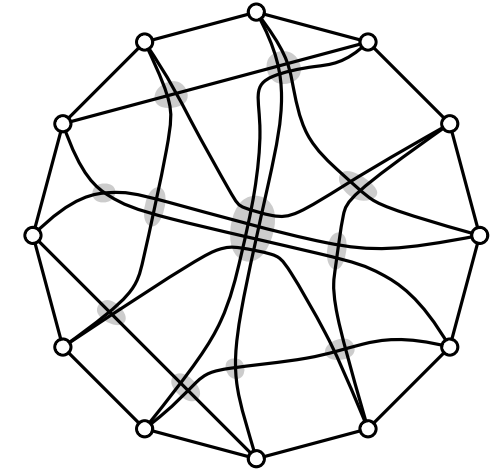
RAC
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orthogonal

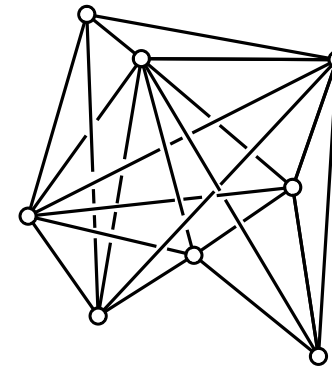


slanted orthogonal

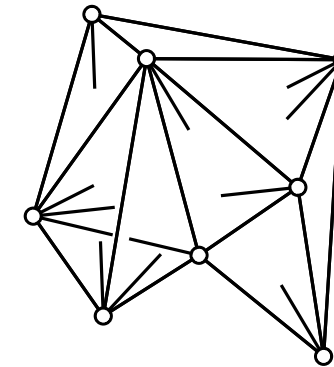


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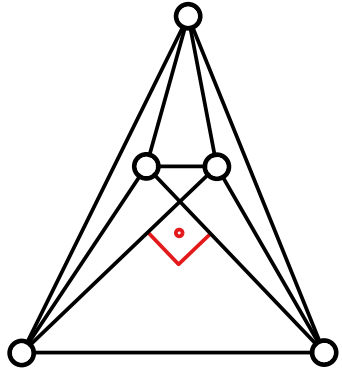


cased crossings

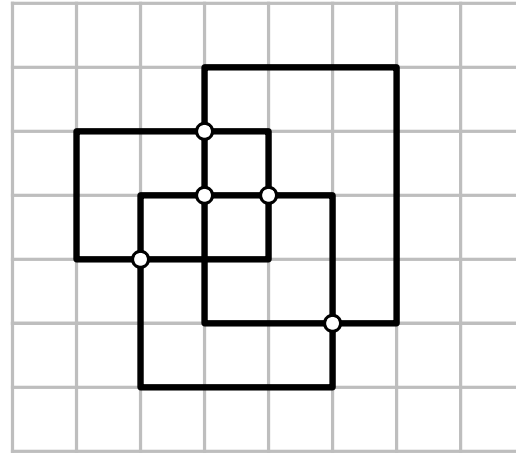
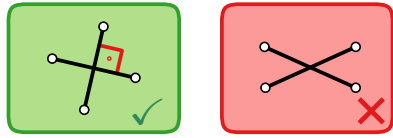


symmetric partial
edge drawing

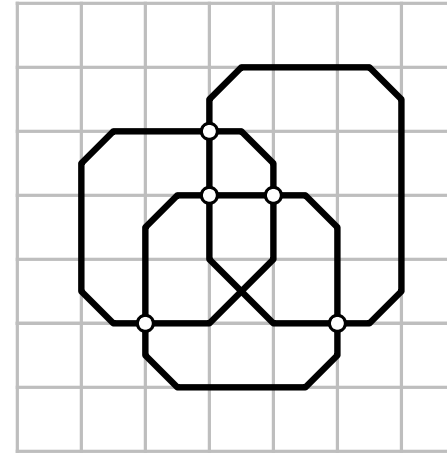
Drawing Styles for Crossings



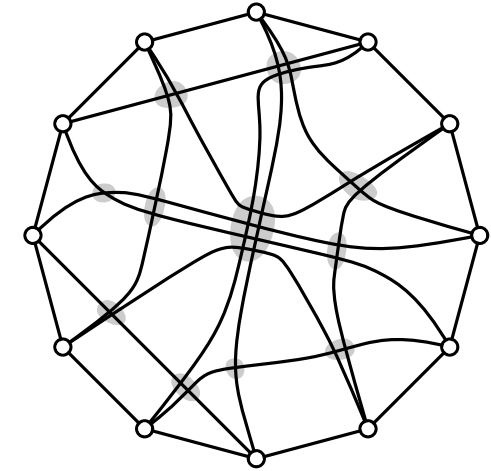
RAC
right-angle crossing



orthogonal

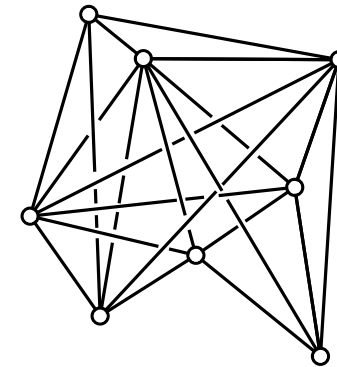


slanted orthogonal

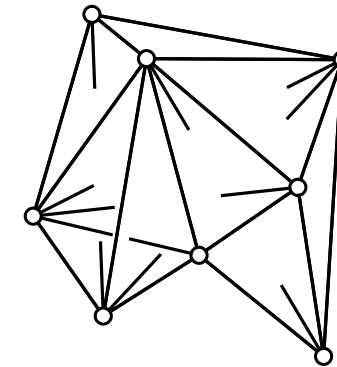


block / bundled crossings

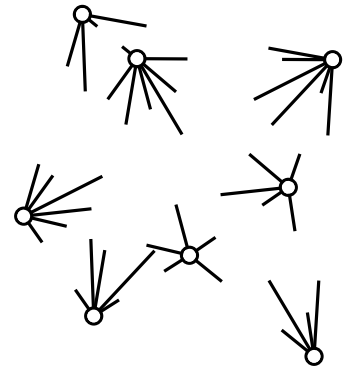
circular layout: 28 individual
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cased crossings

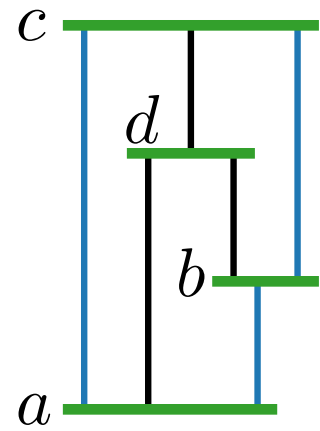
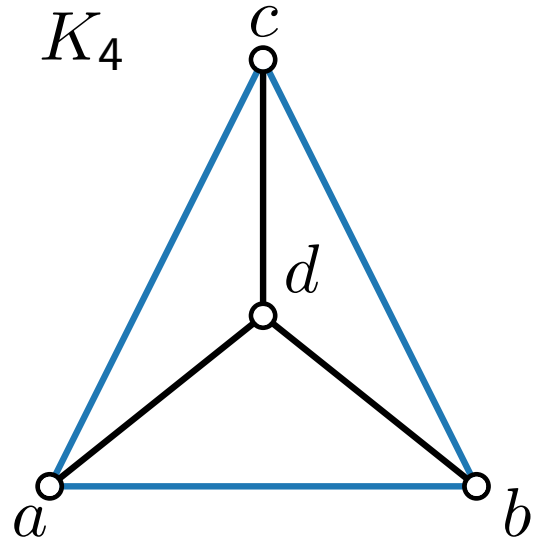


symmetric partial
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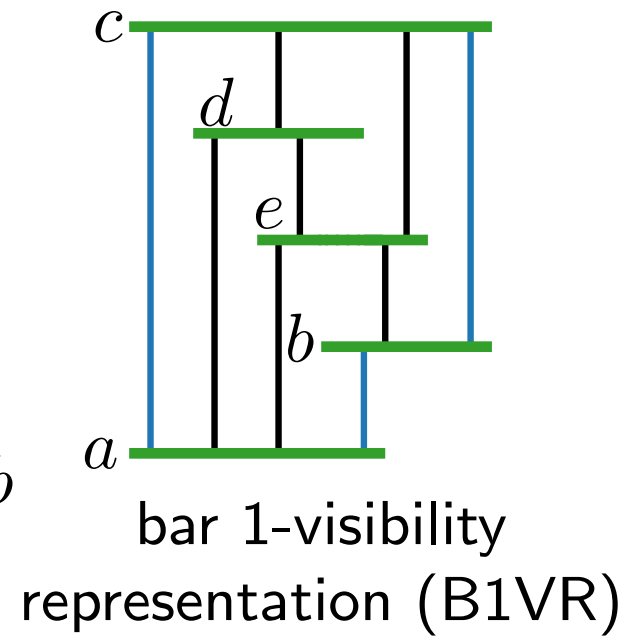
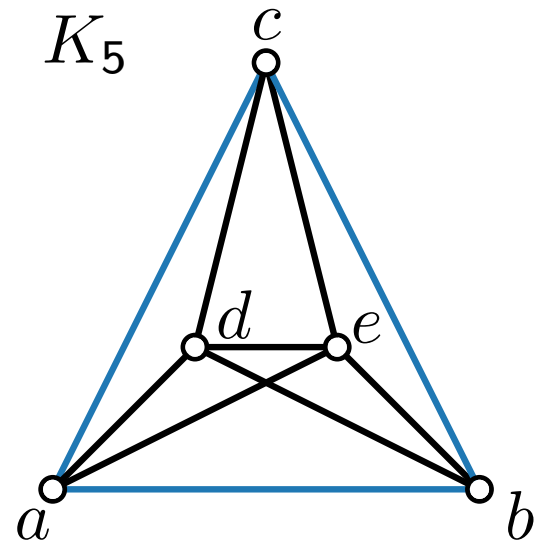
1/4-SHPED
symmetric homogenous
partial edge drawing

Geometric Representations



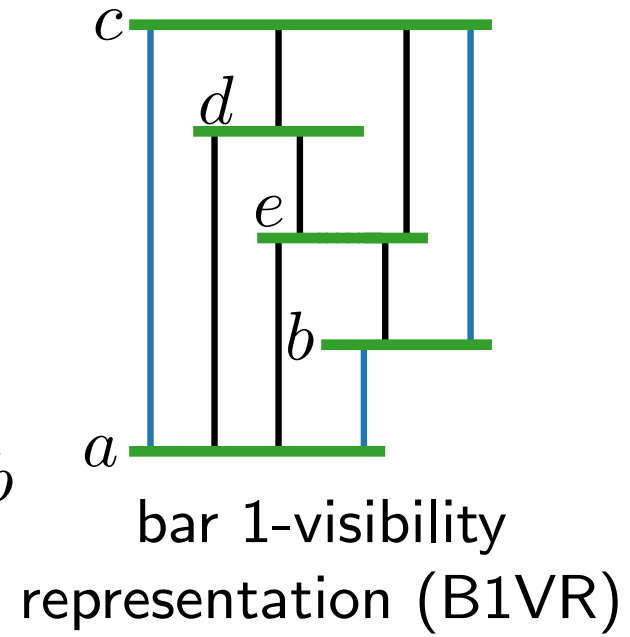
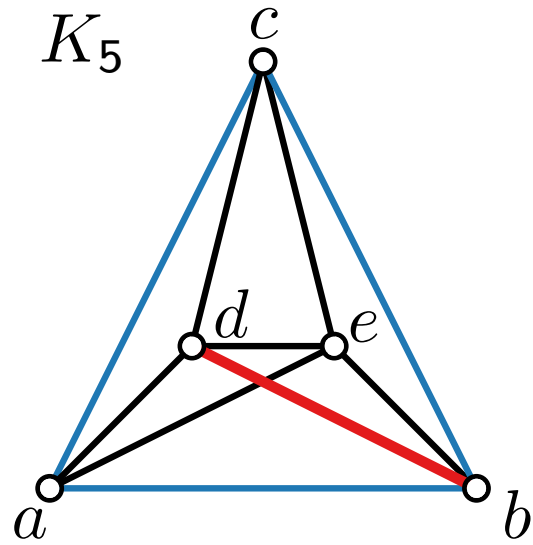
bar visibility
representation

Geometric Representations



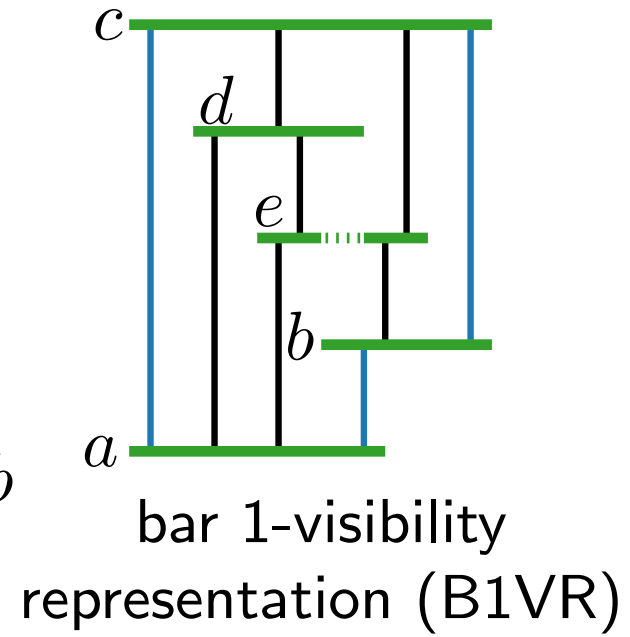
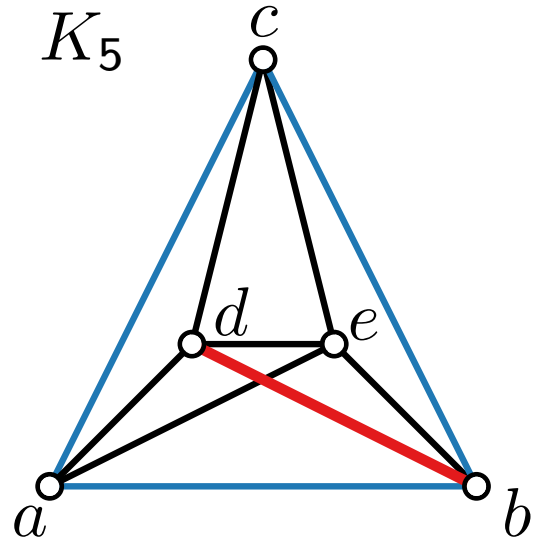
lines of sight through ≤ 1 bars

Geometric Representations



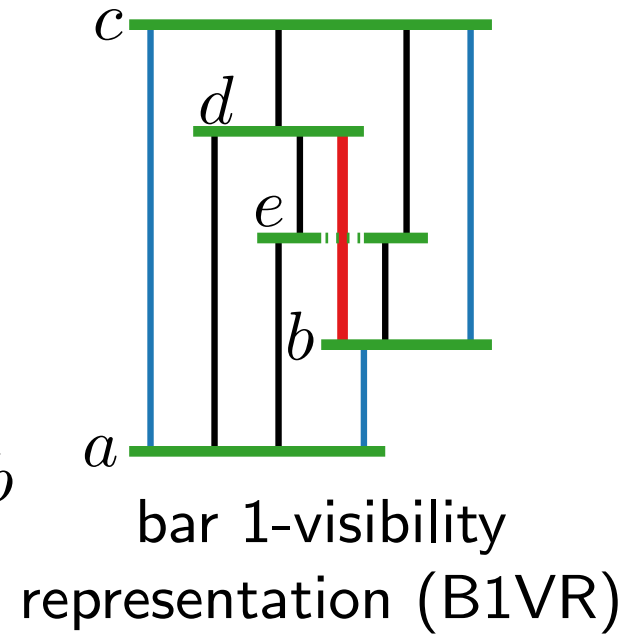
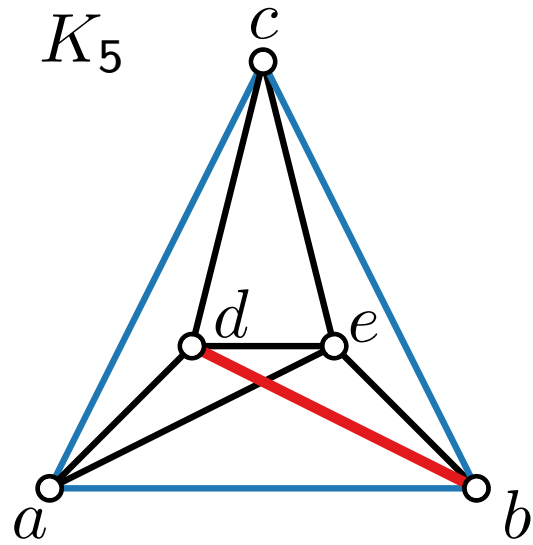
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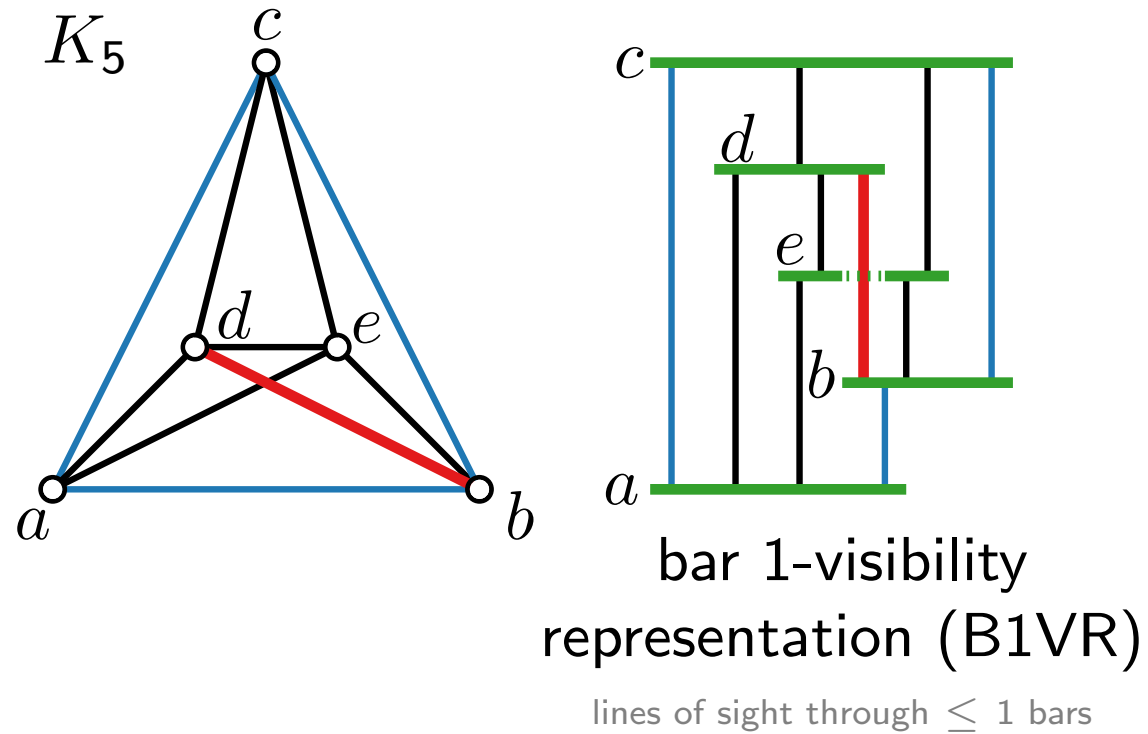
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Geometric Representations



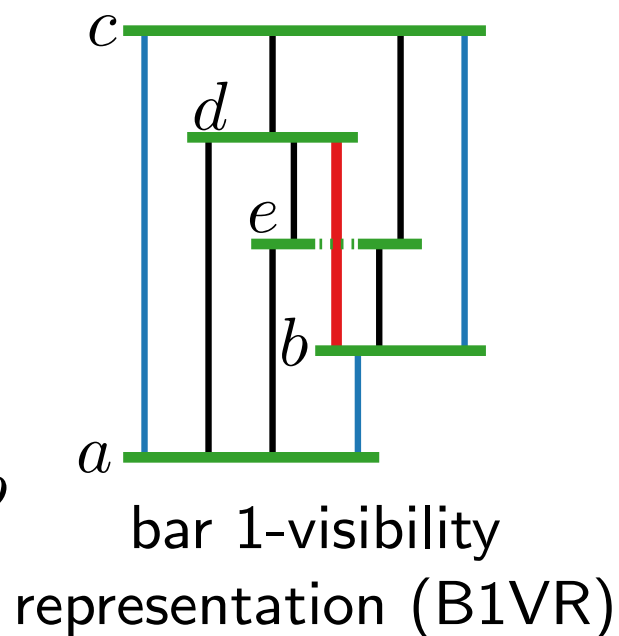
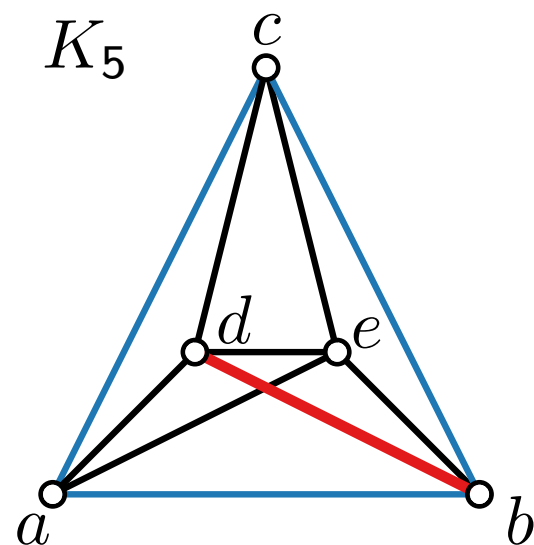
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Geometric Representations

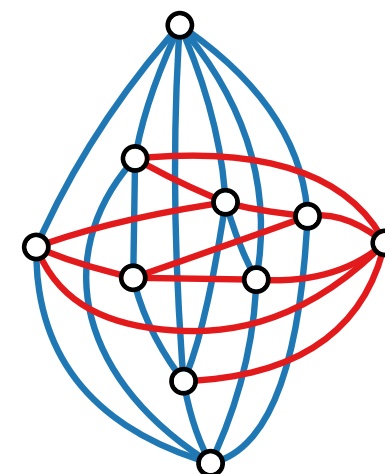


- Every 1-planar graph admits a B1VR.
[Brandenburg 2014; Evans et al. 2014;
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Geometric Representations



lines of sight through ≤ 1 bars

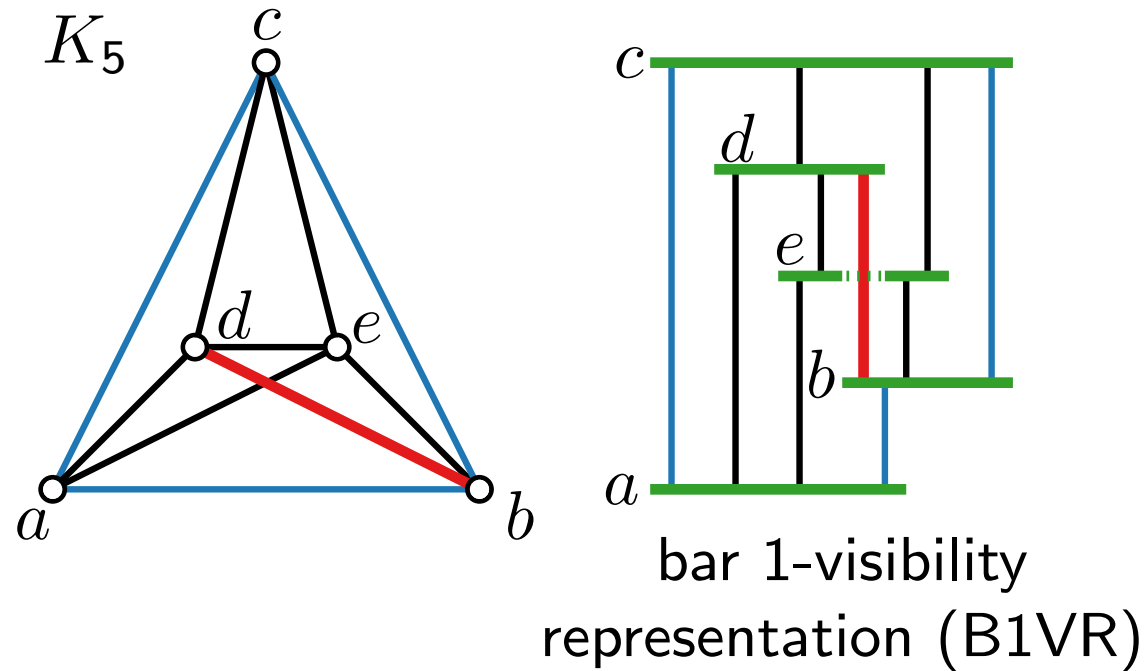


thickness-2
graph

decompose into 2 planar graphs

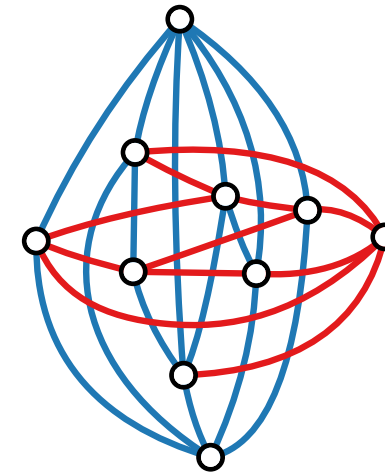
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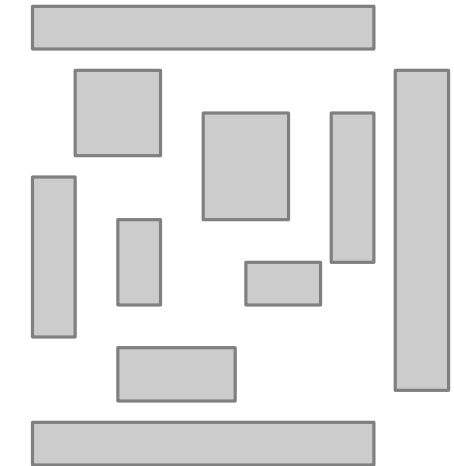


lines of sight through ≤ 1 bars

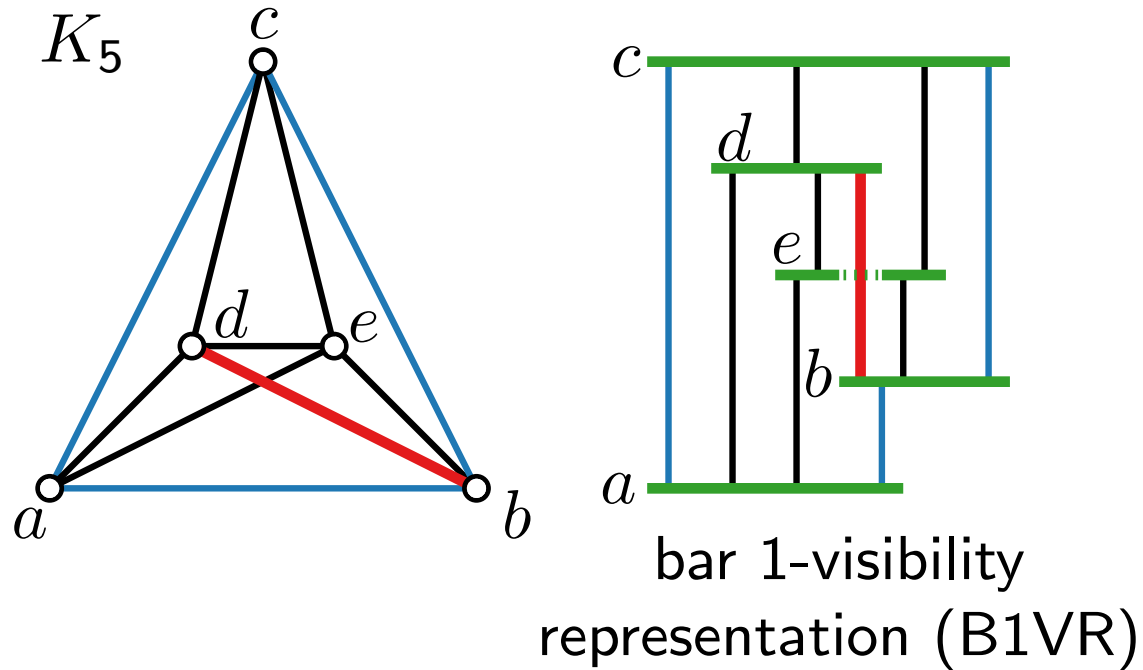
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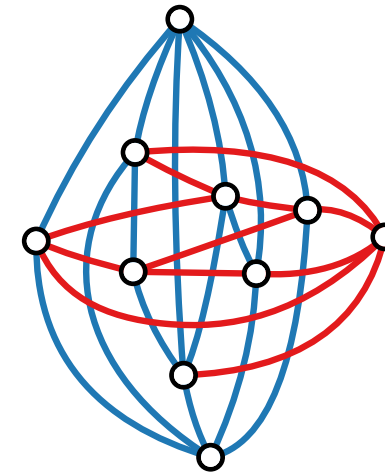


Geometric Representations

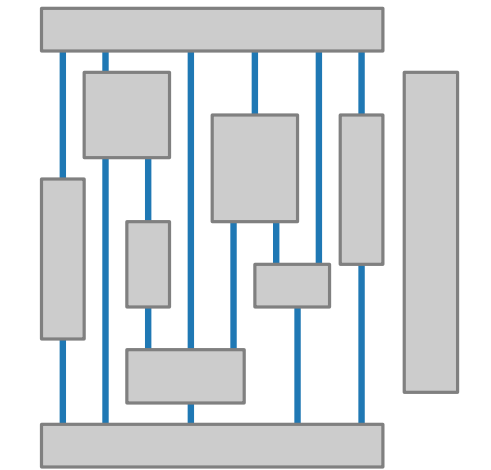


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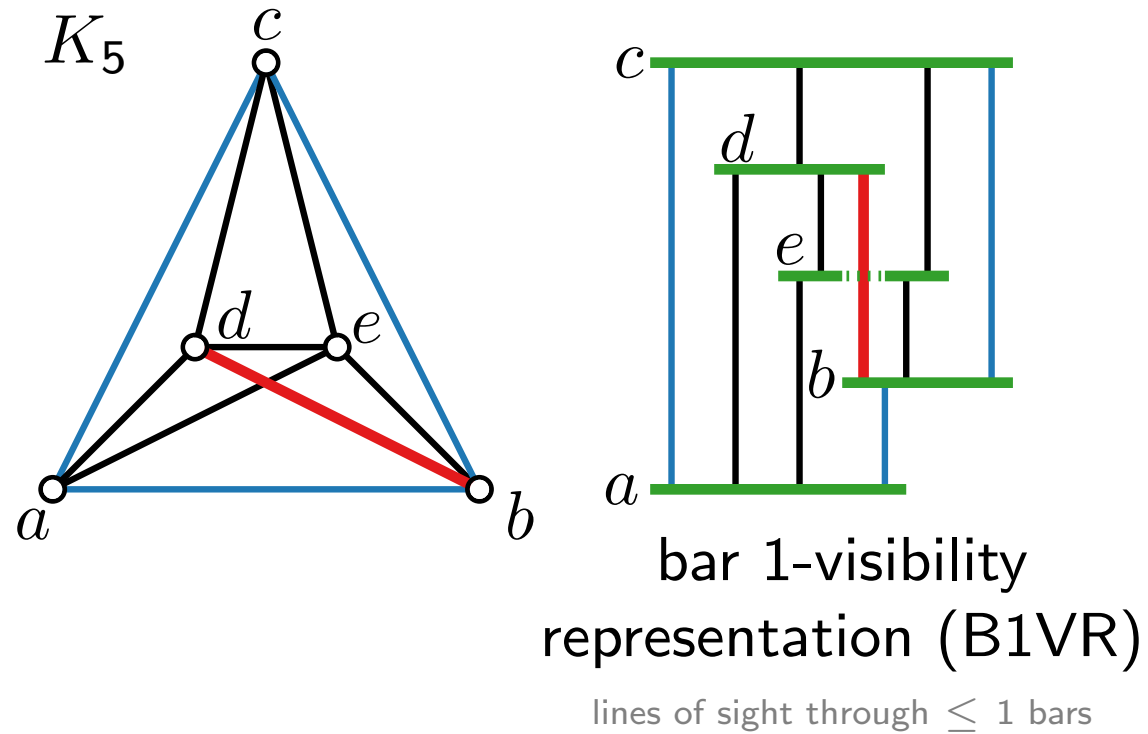
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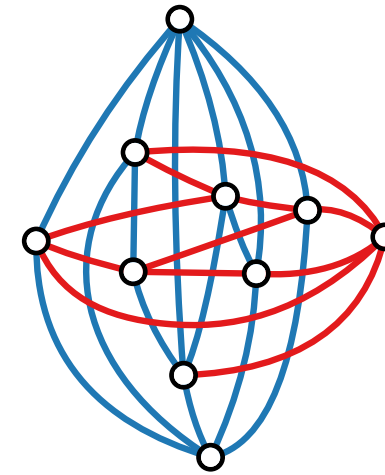
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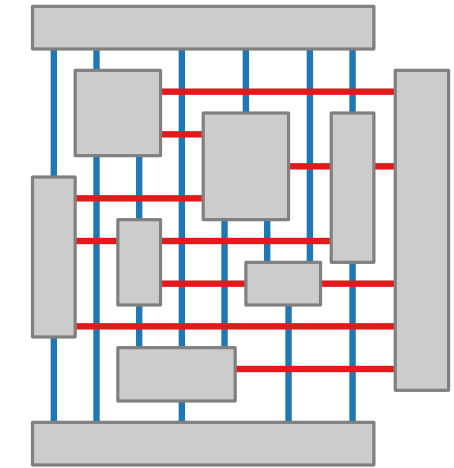
Geometric Representations



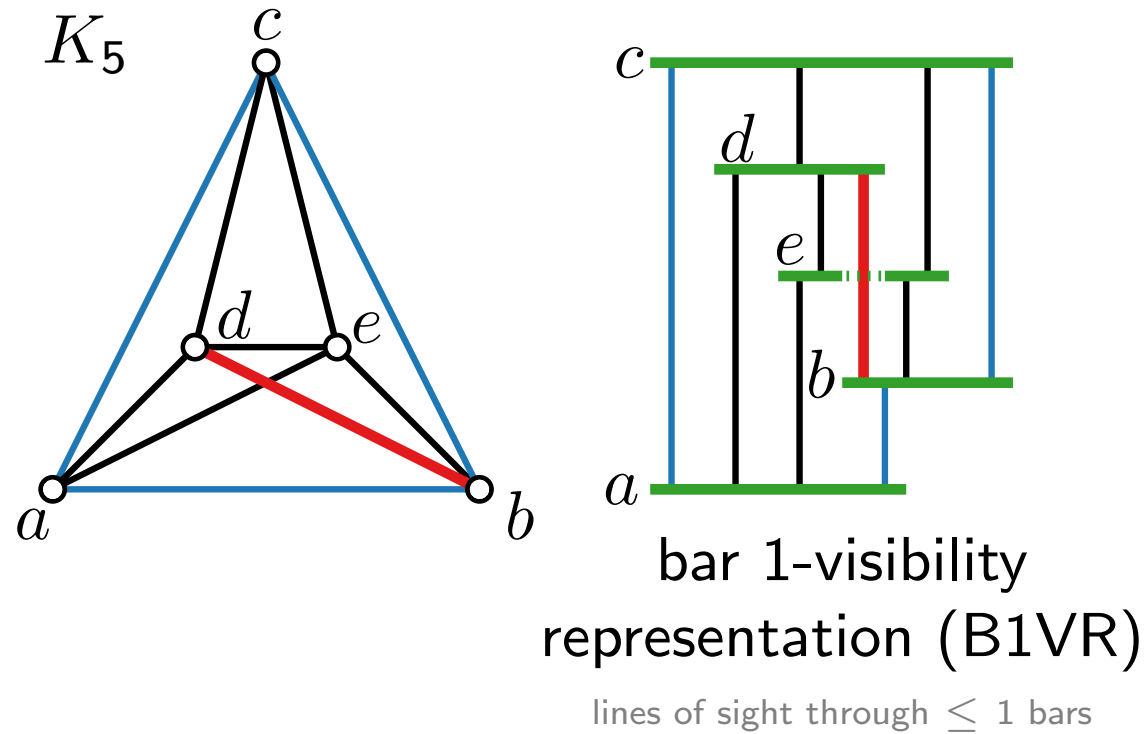
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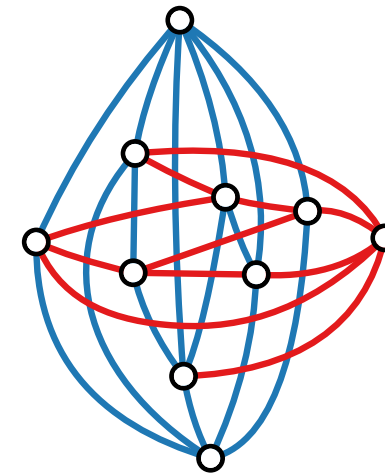
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Geometric Representations

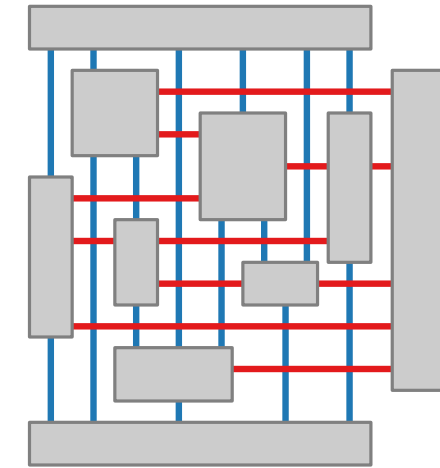


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thickness-2
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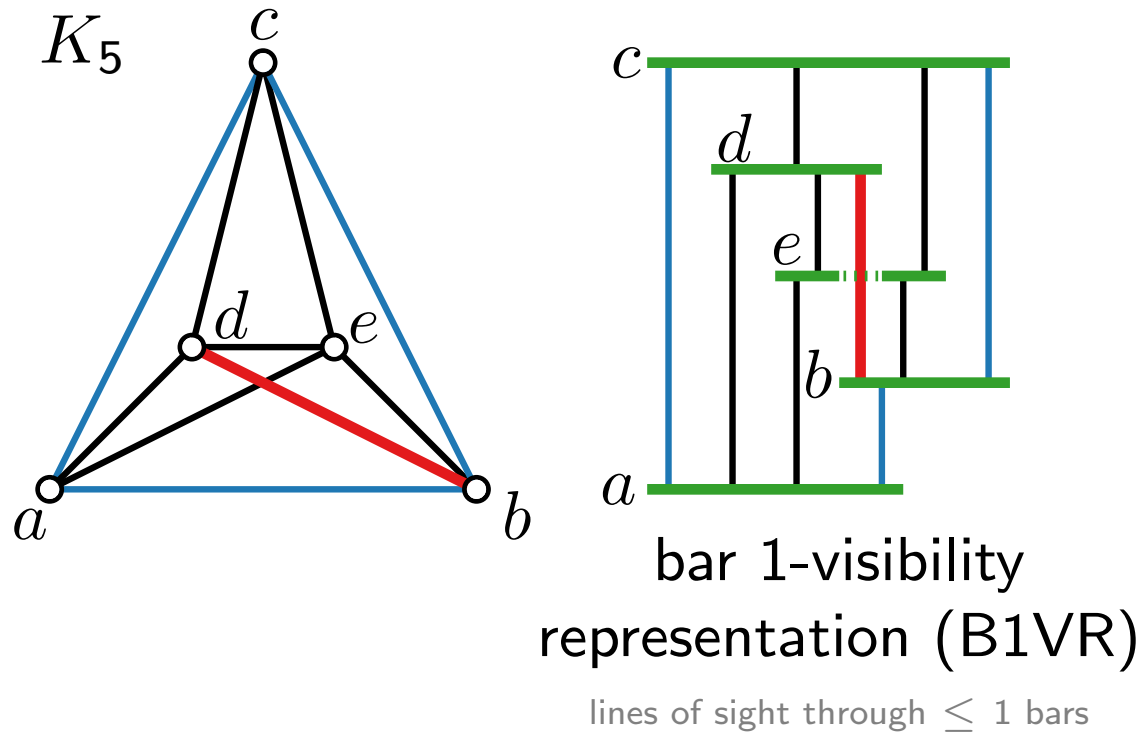
decompose into 2 planar graphs



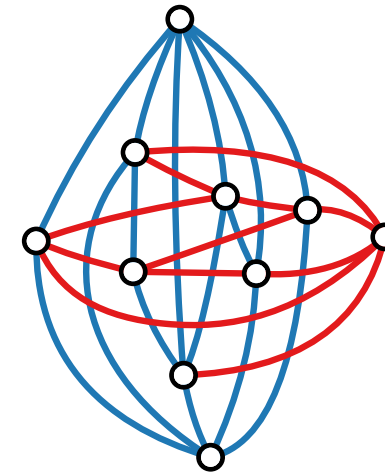
rectangle visibility
representation

- Rectangle visibility graphs (RVGs) have $\leq 6n - 20$ edges. [Hutchinson, Shermer, Vince 1996]

Geometric Representations

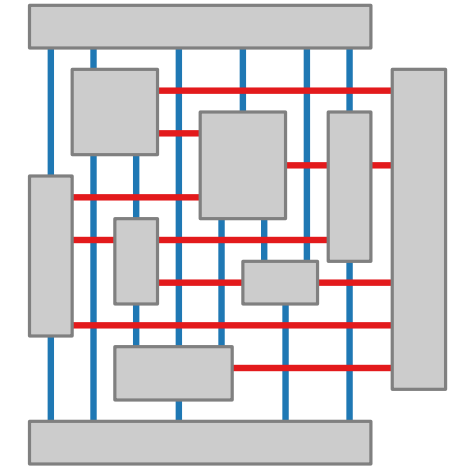


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thickness-2 graph

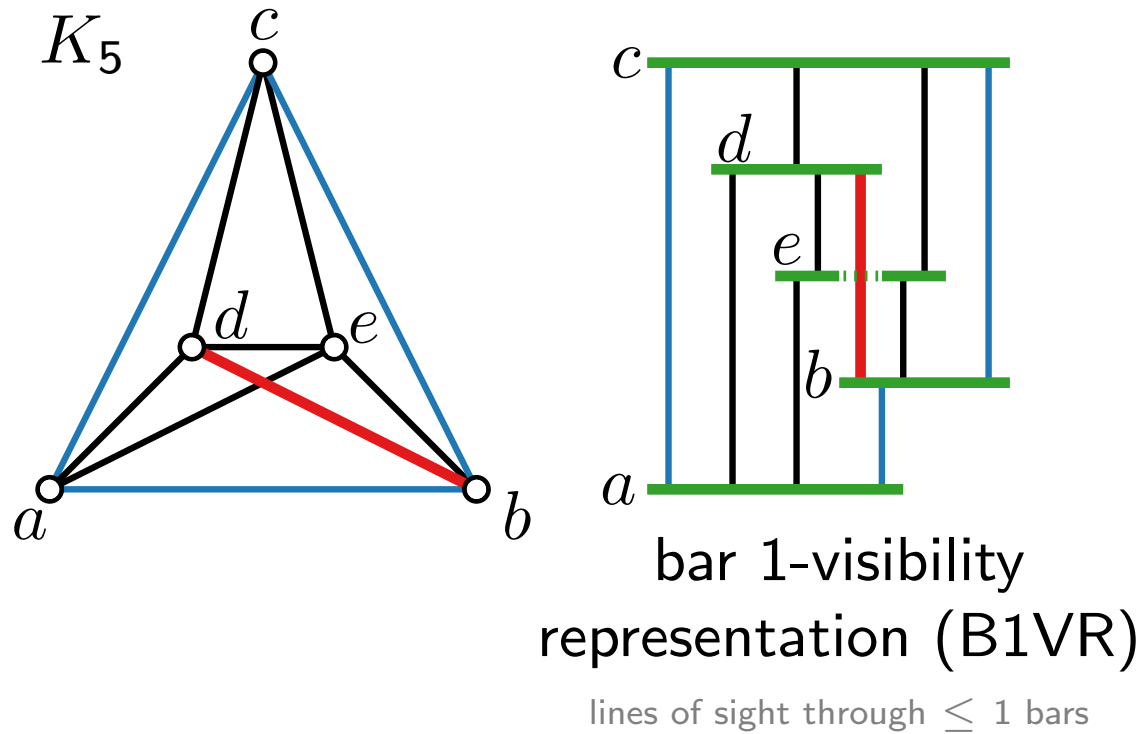
decompose into 2 planar graphs



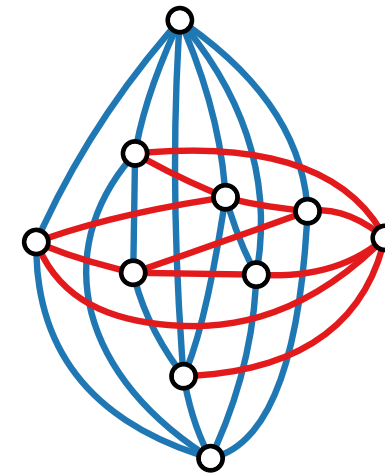
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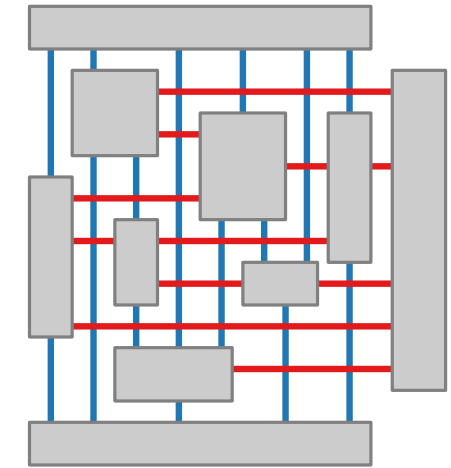
Geometric Representations



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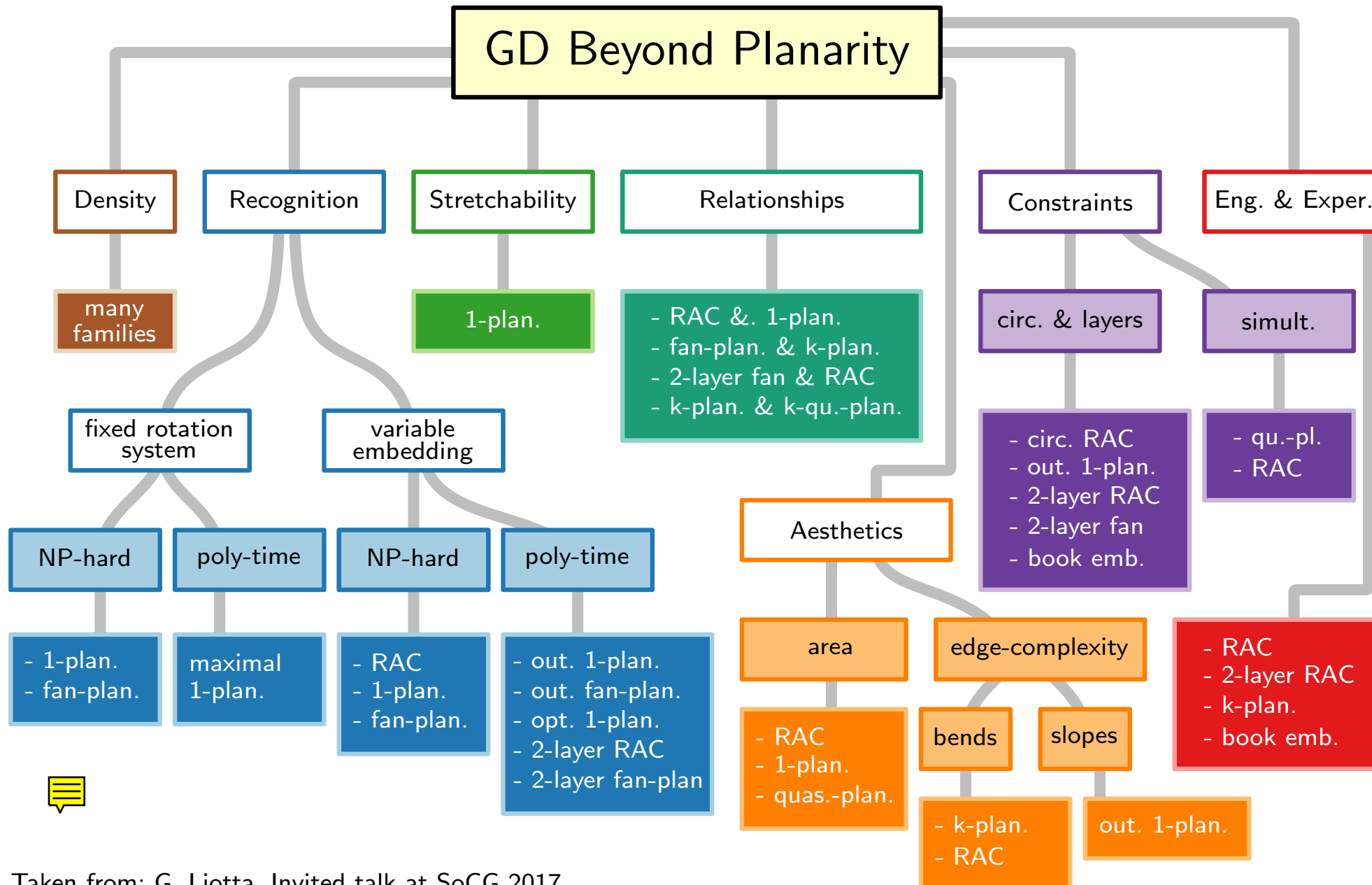


decompose into 2 planar graphs



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- RVGs can be recognized efficiently if embedding is fixed. [Biedl, Liotta, Montecchiani 2018]

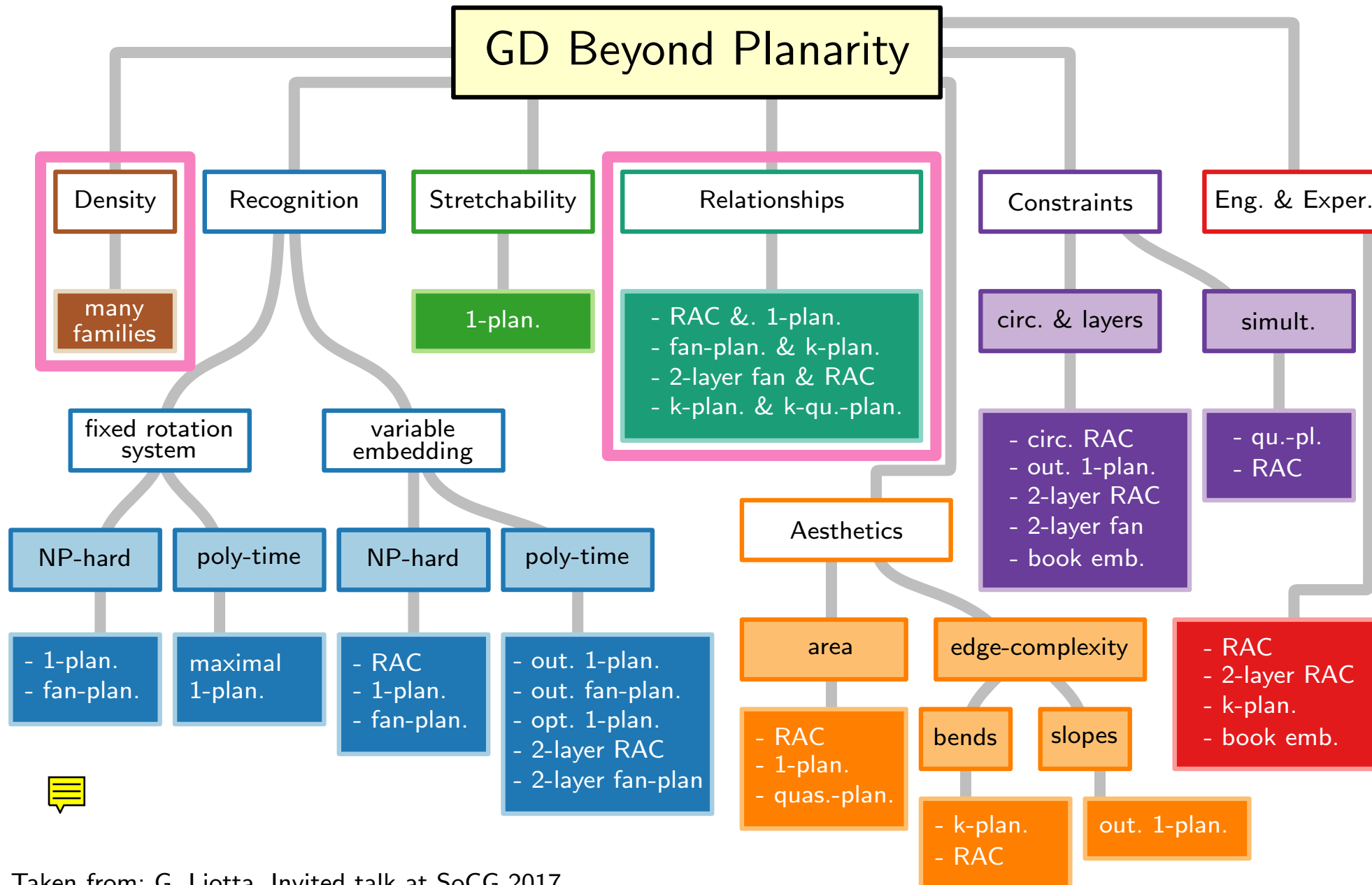
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

GD Beyond Planarity: a Taxonomy



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Density of 1-Planar Graphs

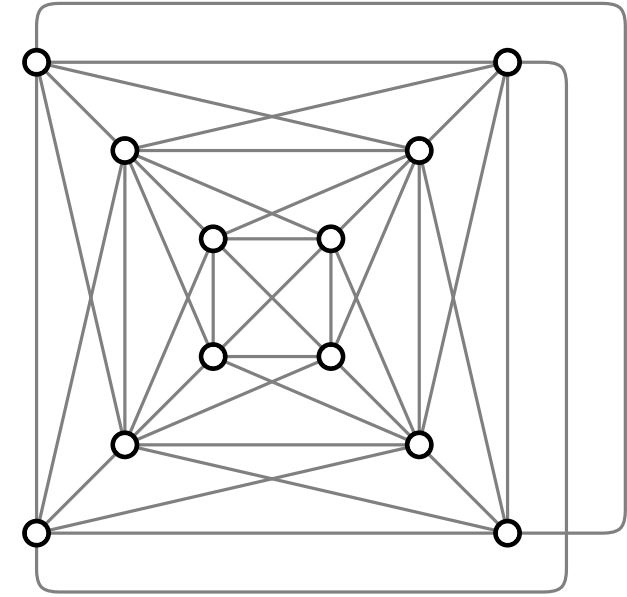
Theorem. [Ringel 1965, Pach & Tóth 1997]

A 1-planar graph with n vertices has at most $4n - 8$ edges.

Density of 1-Planar Graphs

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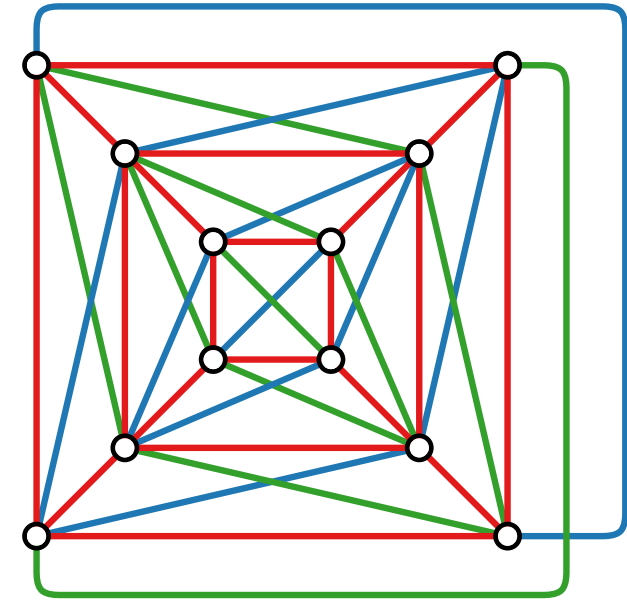
Proof sketch.



Density of 1-Planar Graphs

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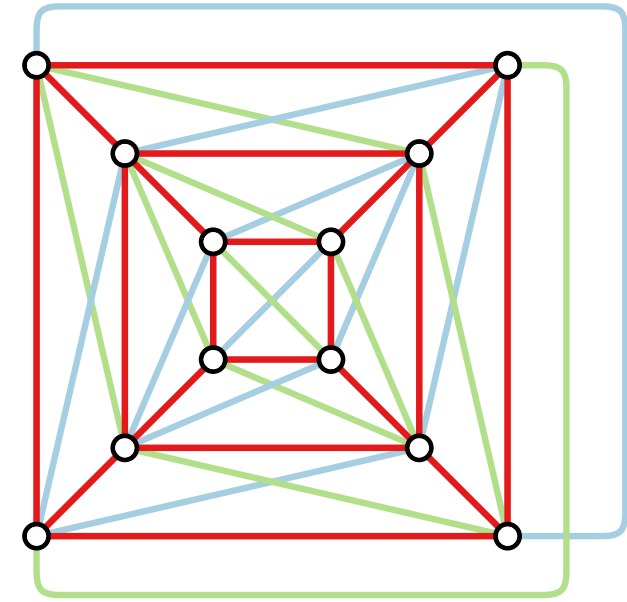
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- Let the **red** edges be those that do not cross.

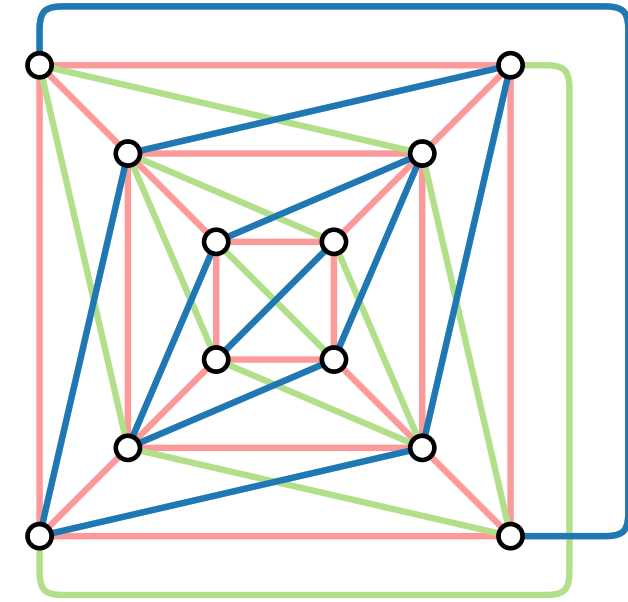


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Proof sketch.

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- Each **blue** edge



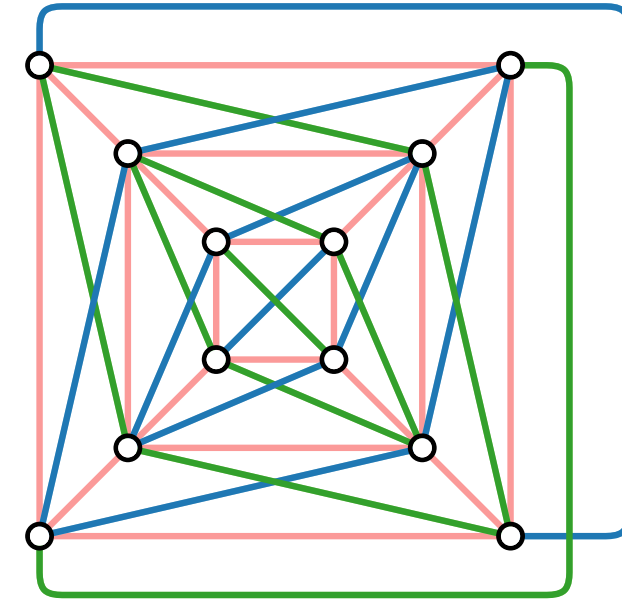
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Proof sketch.

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- Each **blue** edge crosses a **green** edge.



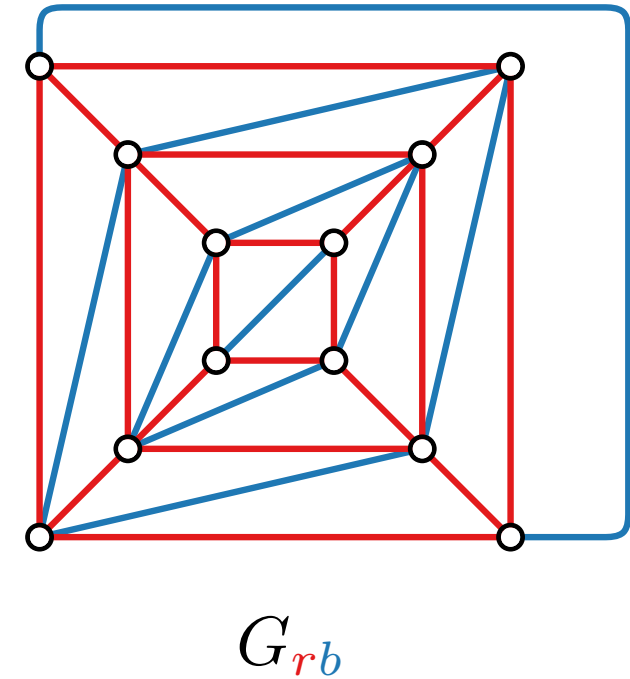
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A 1-planar graph with n vertices has at most $4n - 8$ edges.

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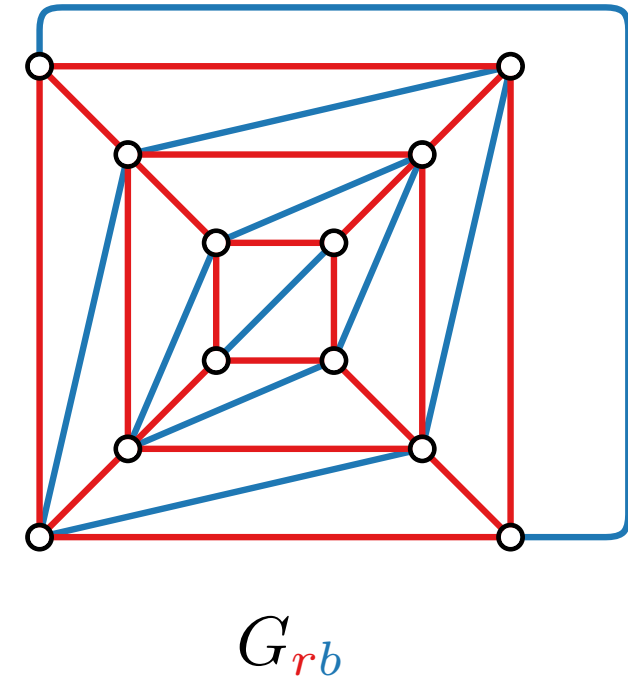
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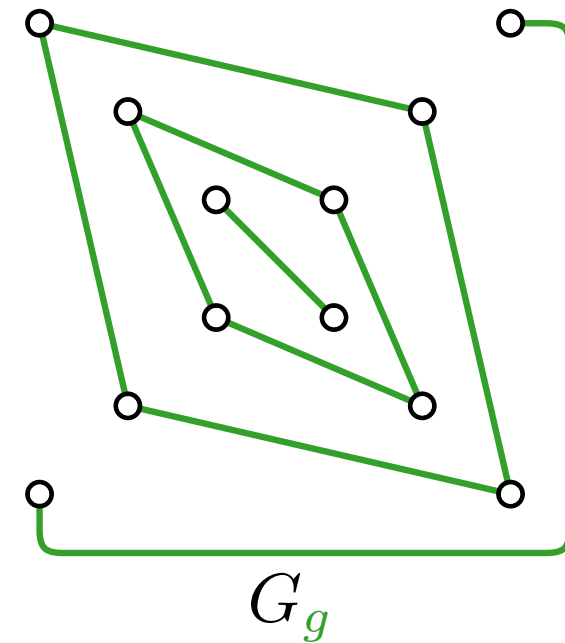
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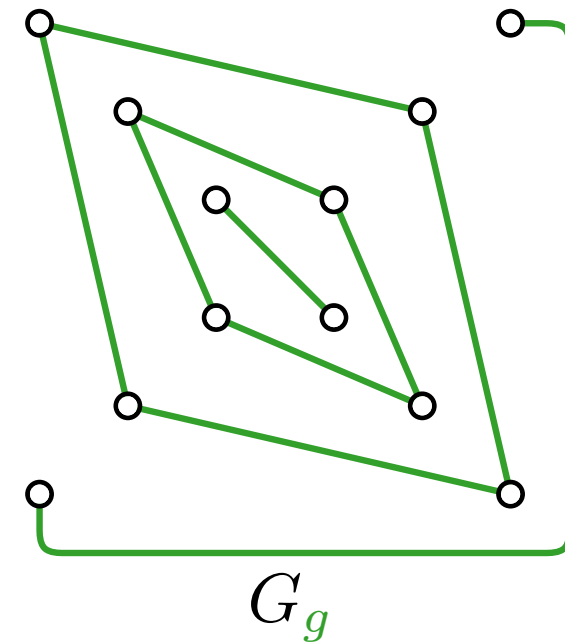
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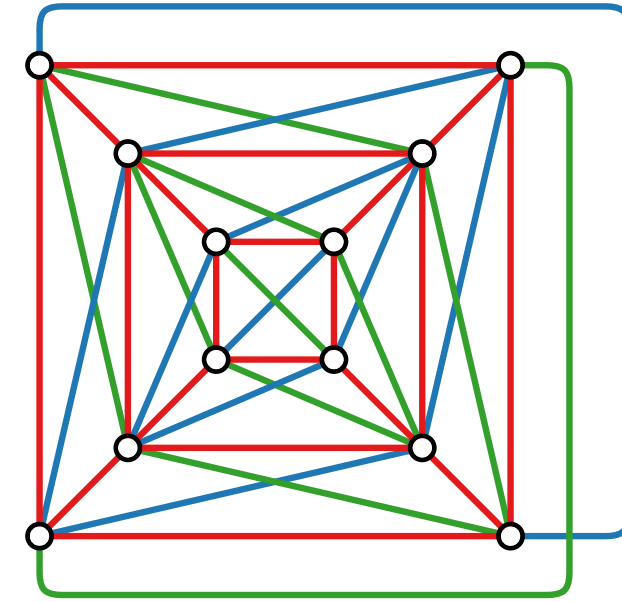
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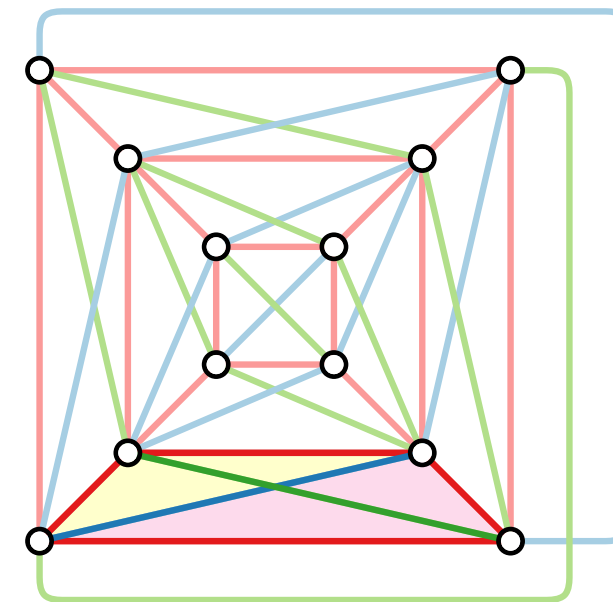
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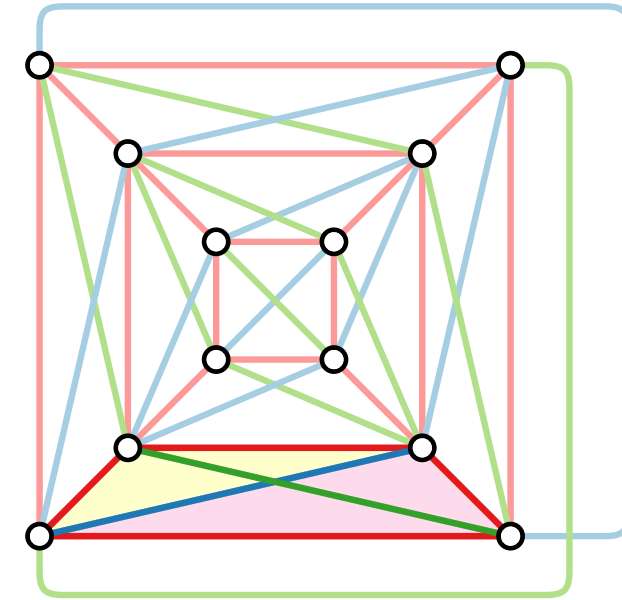
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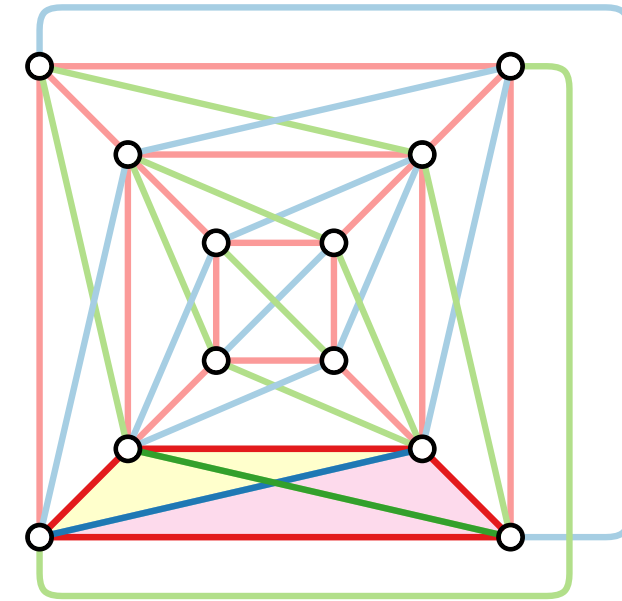
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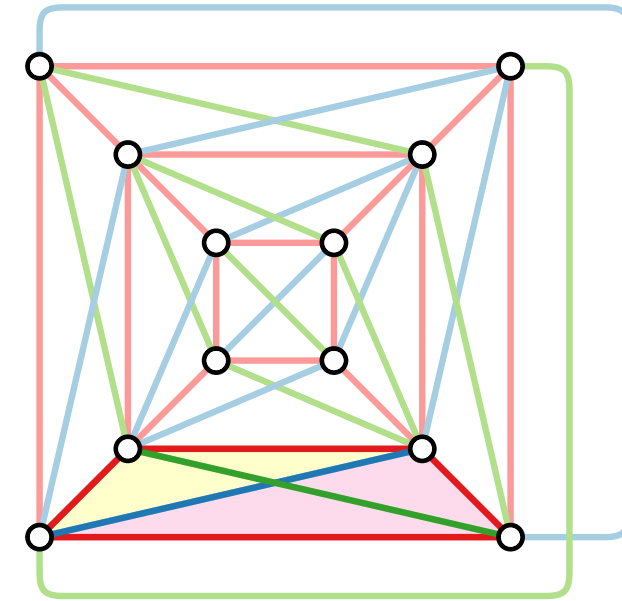
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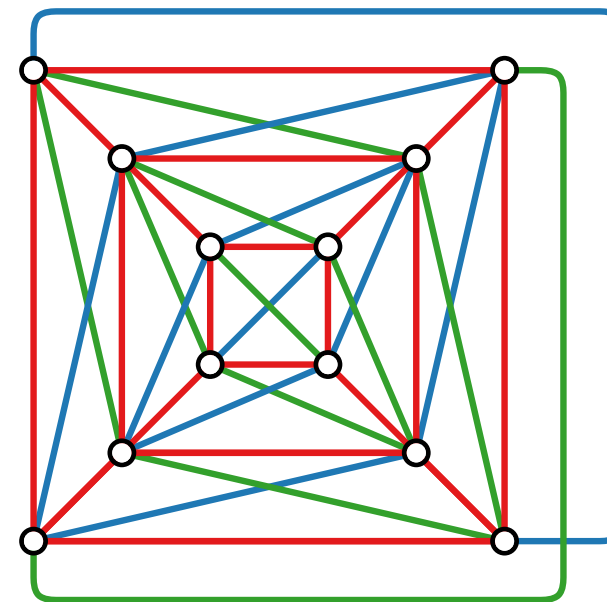
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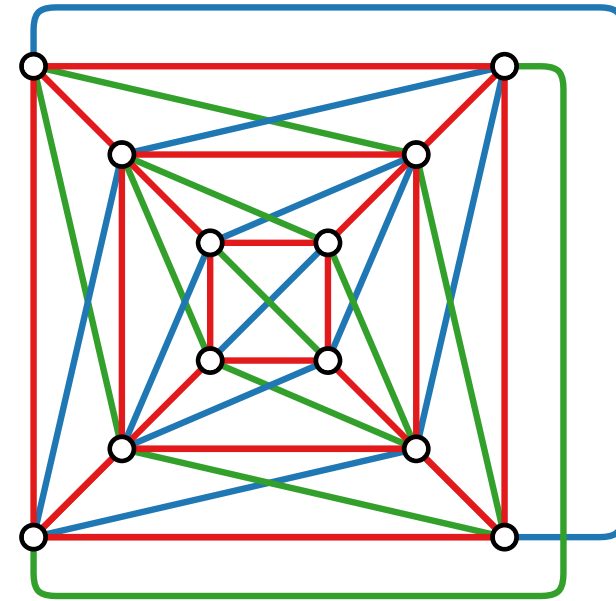
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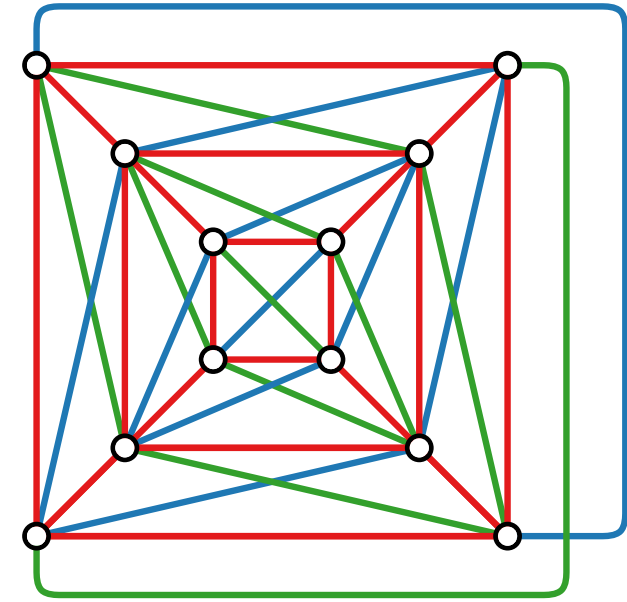
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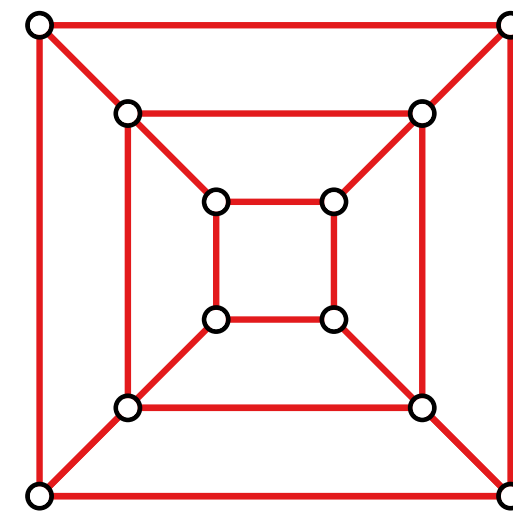
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Lower-bound construction:

$2n - 4$ edges

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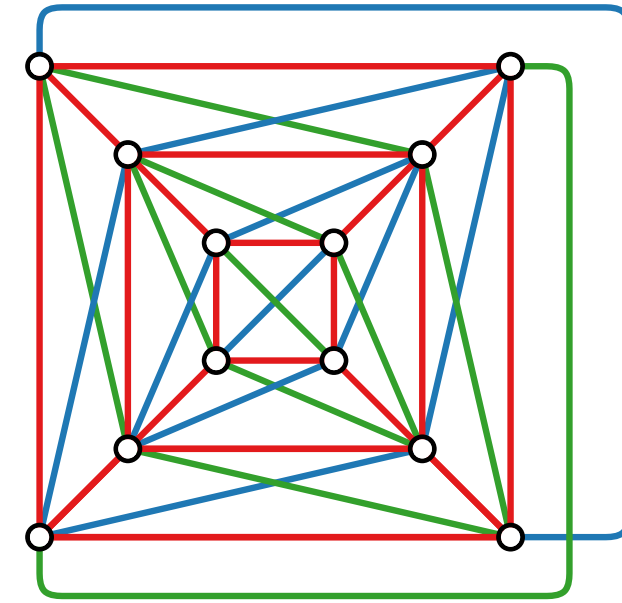
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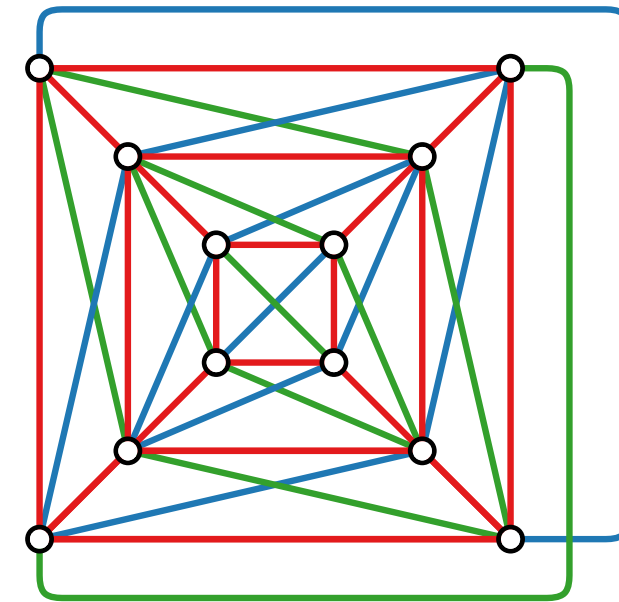
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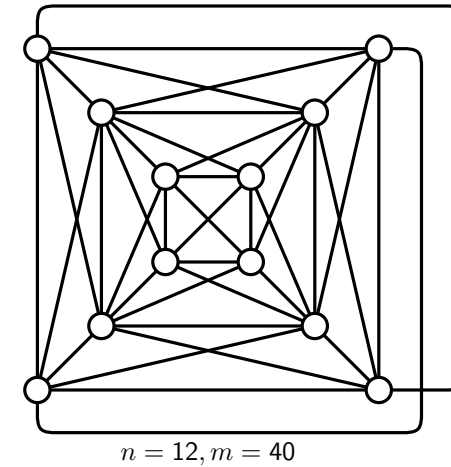
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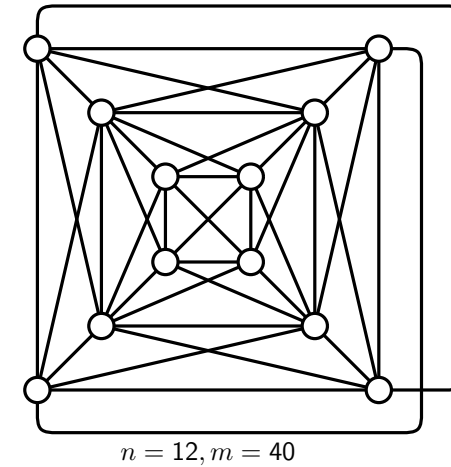


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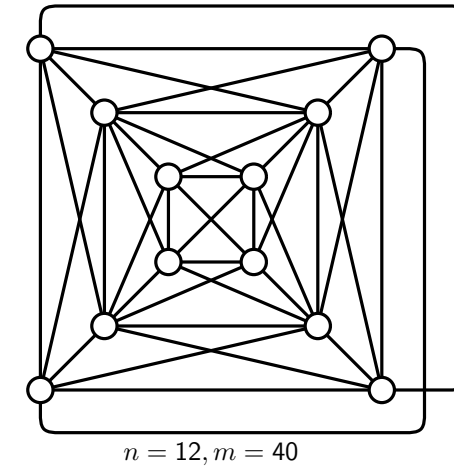
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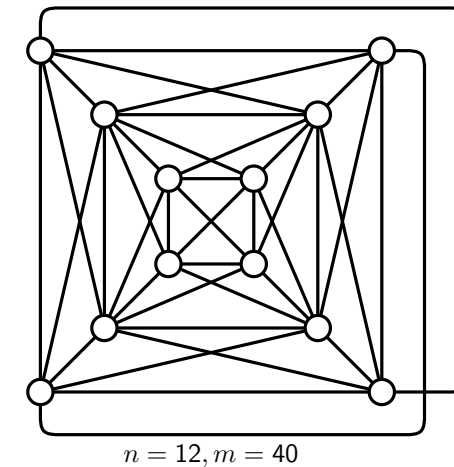
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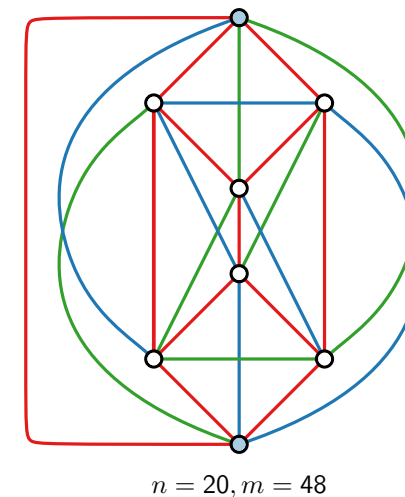
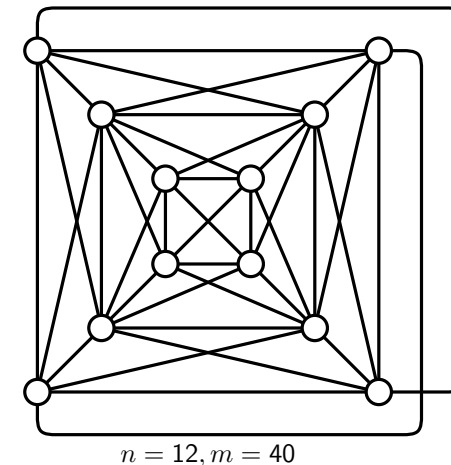
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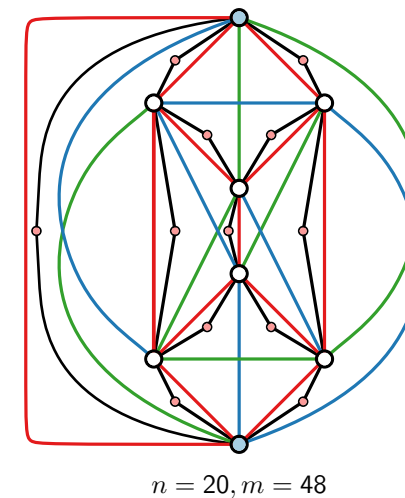
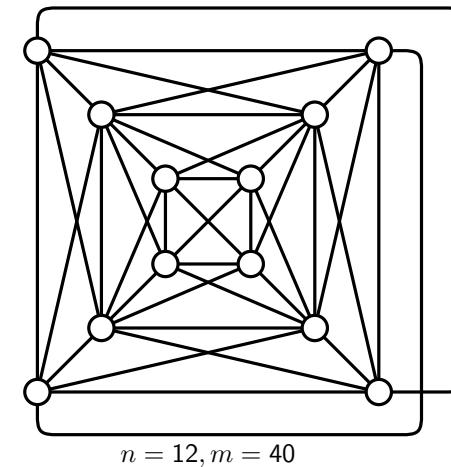
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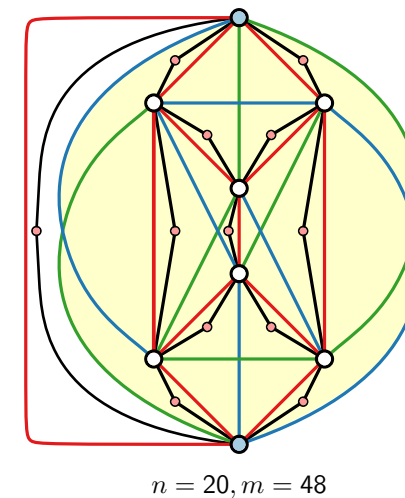
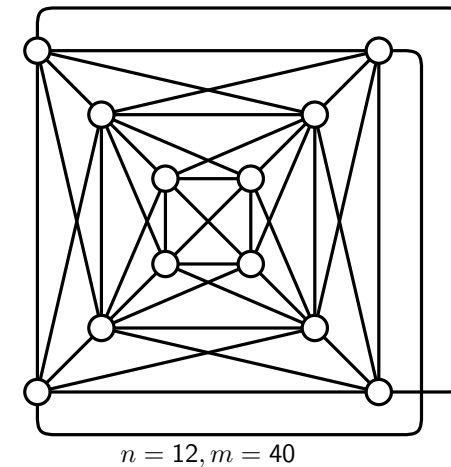
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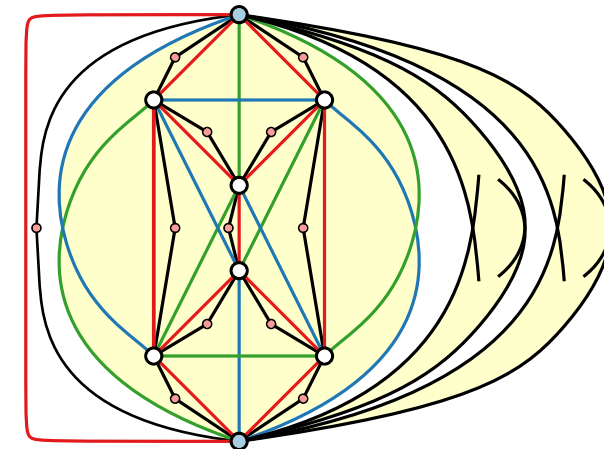
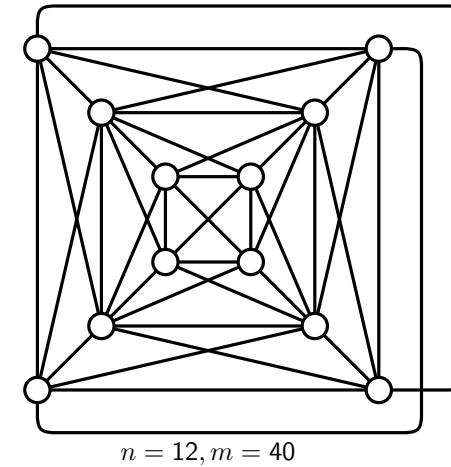
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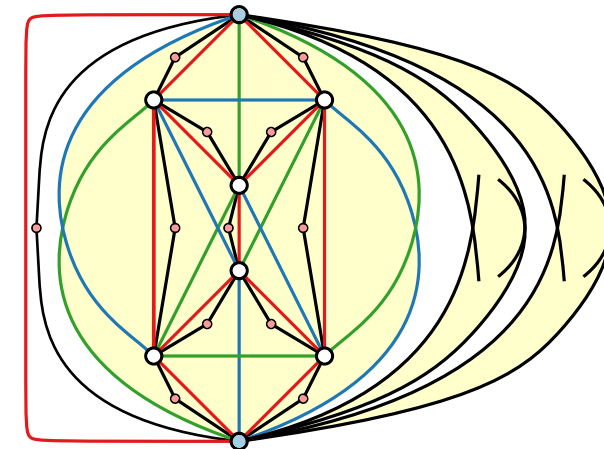
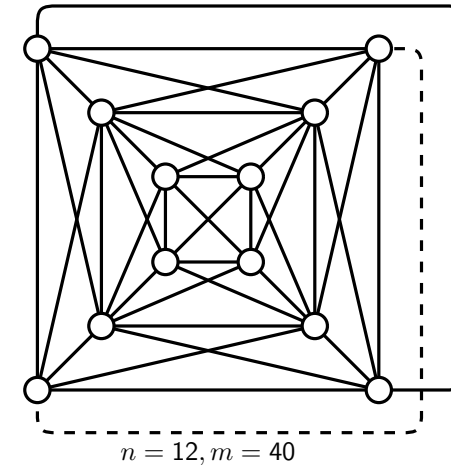
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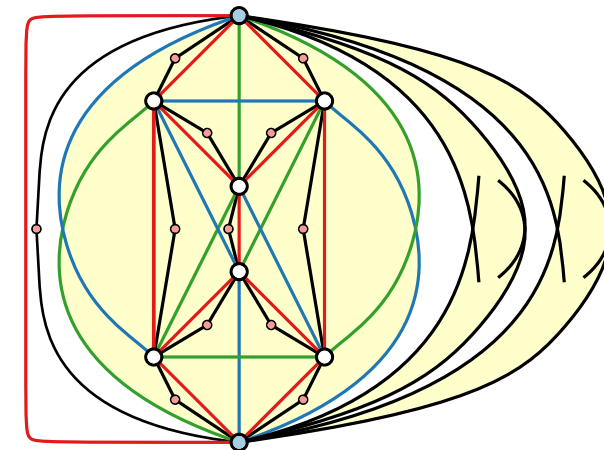
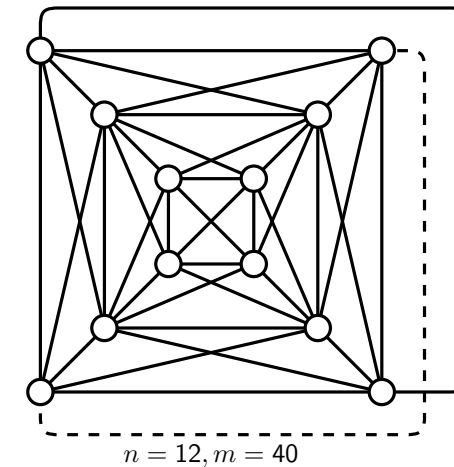
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Idea: in a drawing of an optimal 1-planar graph, we cannot realize the crossing on the outer face with two straight-line edges.

Density of k -Planar Graphs

Theorem.

A k -planar graph with n vertices has at most:

k number of edges

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0

Euler's formula

Density of k -Planar Graphs

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0 $3(n - 2)$ Euler's formula

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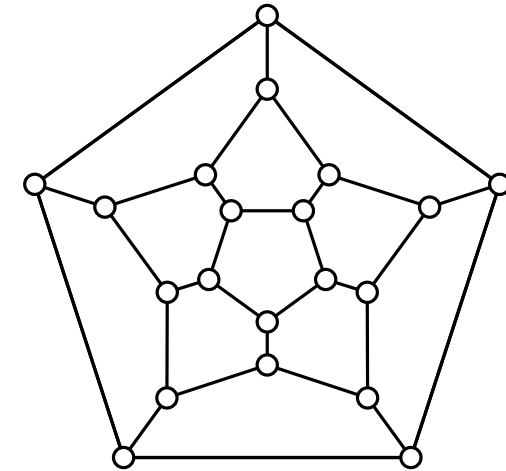
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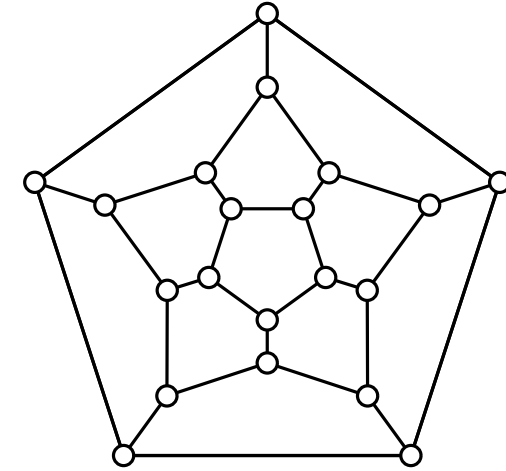
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Planar structure:

Edges per face:

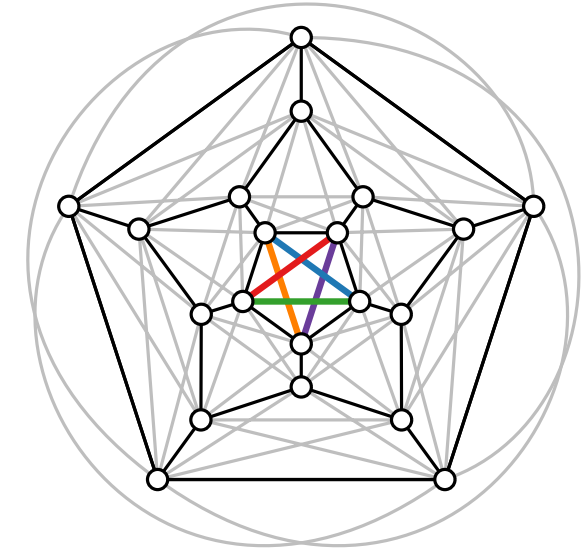
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optimal 2-planar

Planar structure:

Edges per face:

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Density of k -Planar Graphs

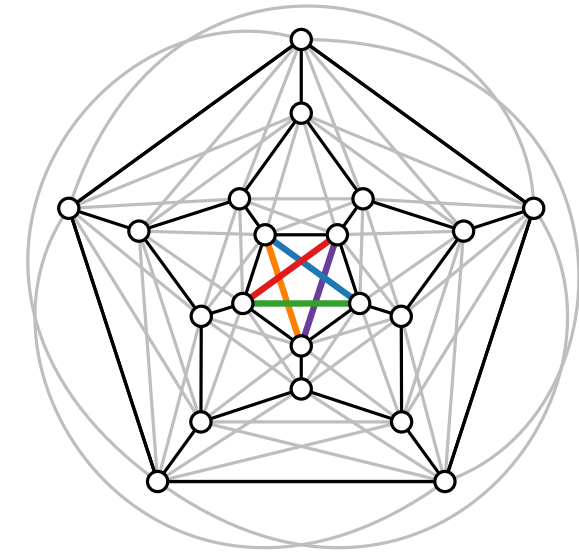
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$$n - m + f = 2$$

$$m = c \cdot f ?$$



optimal 2-planar

Planar structure:

Edges per face:

Total:

Density of k -Planar Graphs

Theorem.

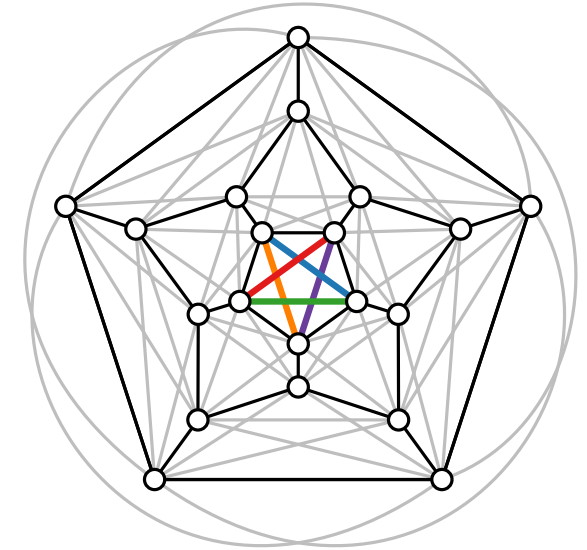
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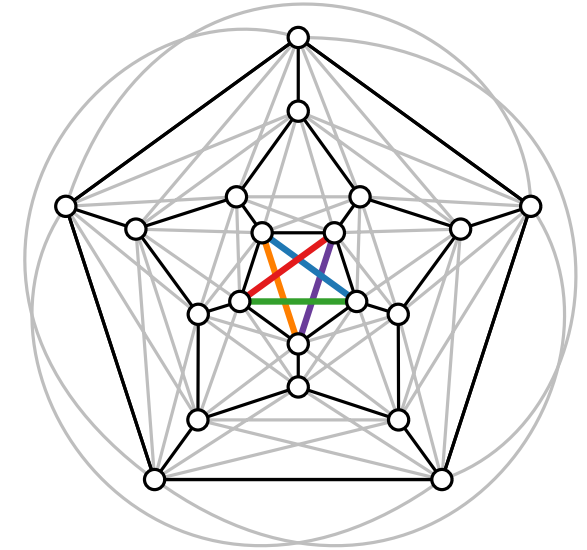
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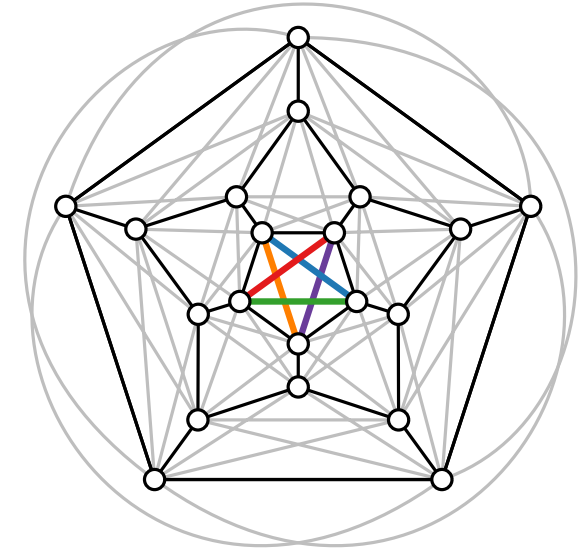
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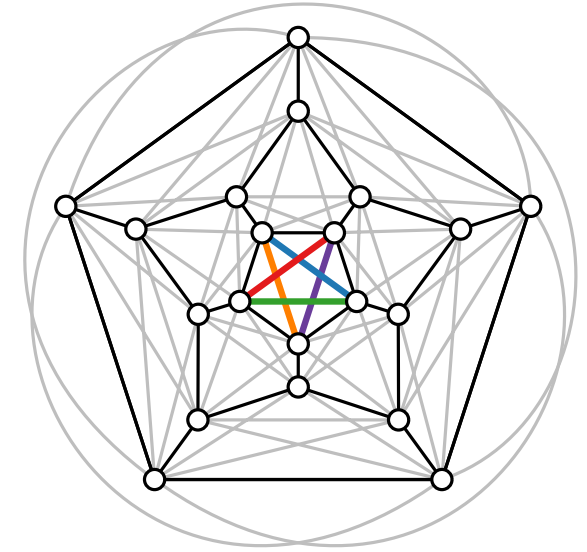
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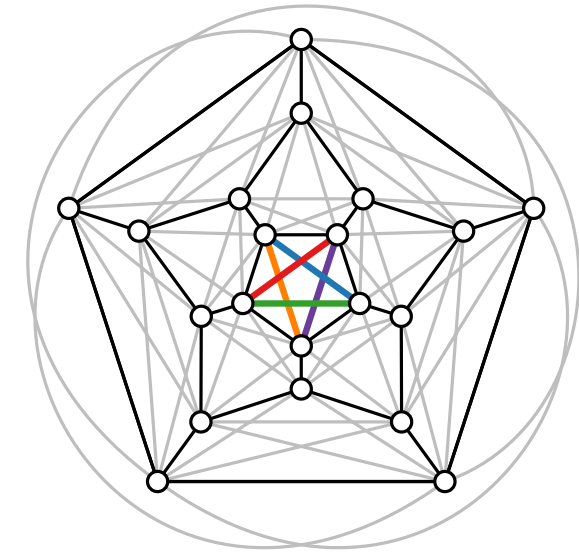
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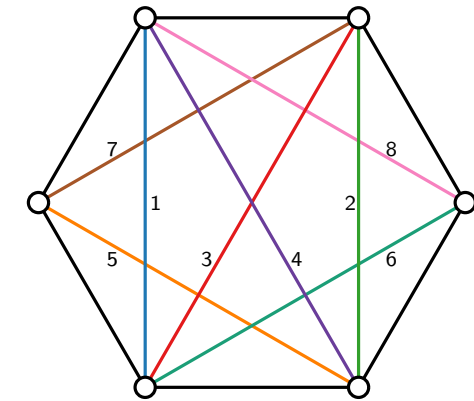
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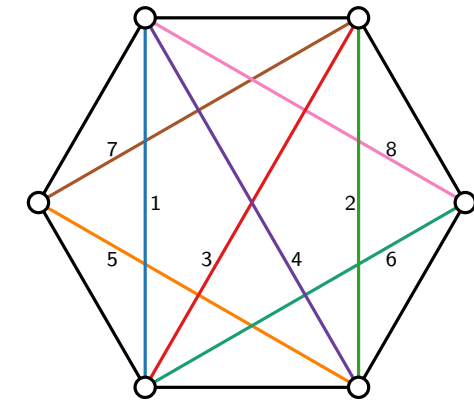
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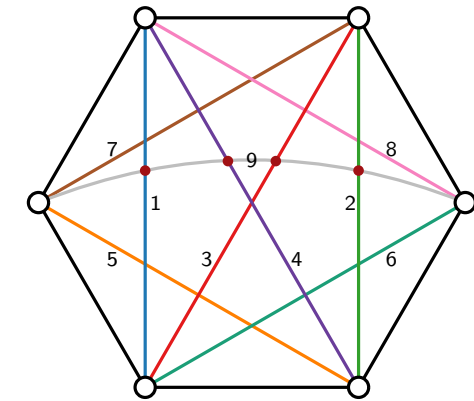
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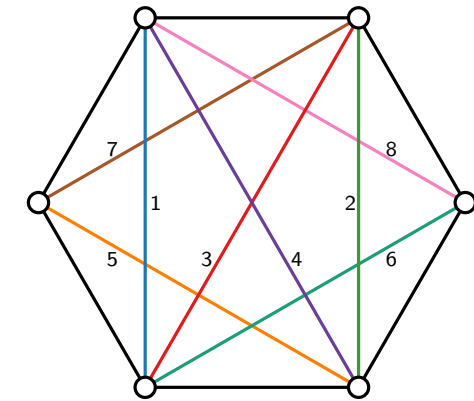
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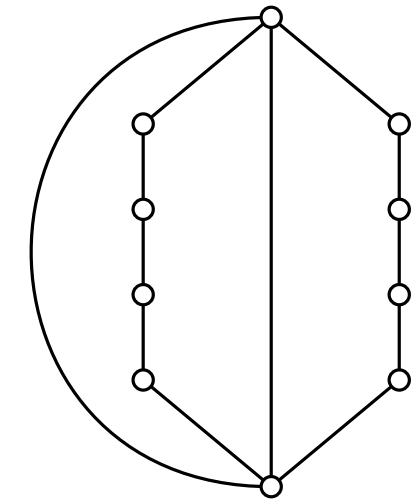
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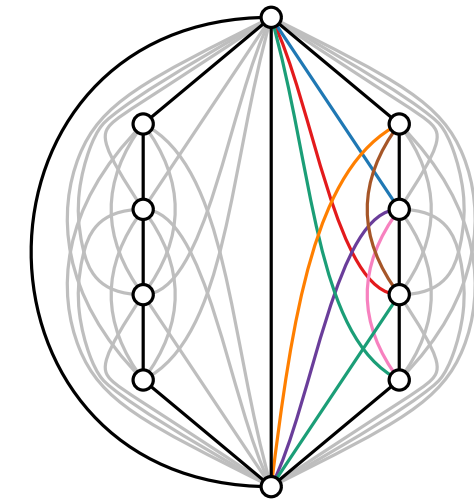
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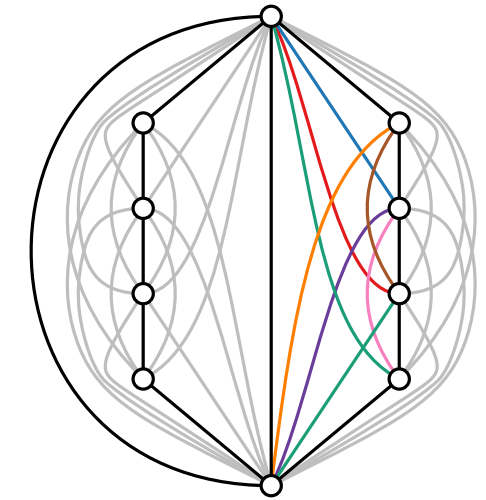
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optimal 3-planar

Planar structure:

$$\frac{3}{2}(n - 2) \text{ edges}$$

$$\frac{1}{2}(n - 2) \text{ faces}$$

Edges per face: 8 edges

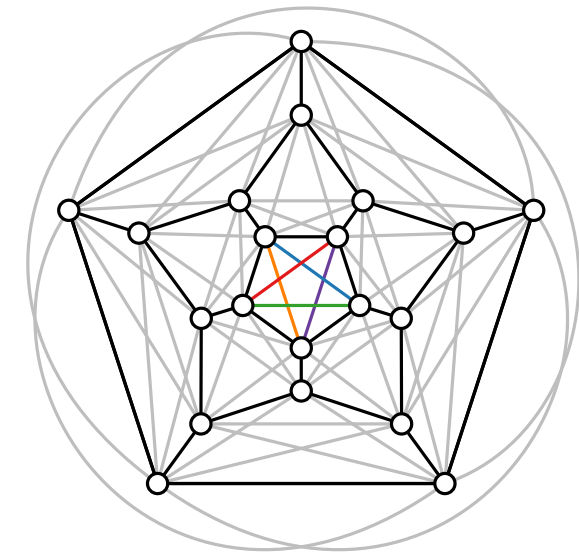
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Density of k -Planar Graphs

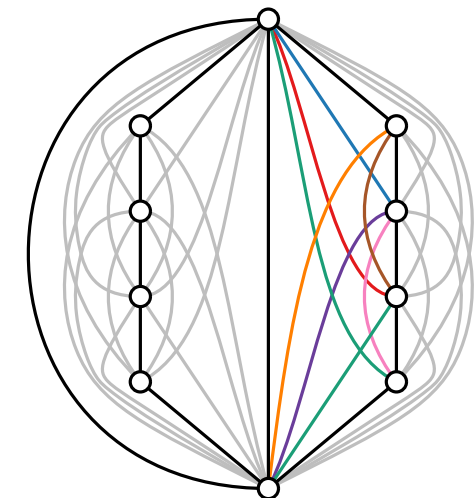
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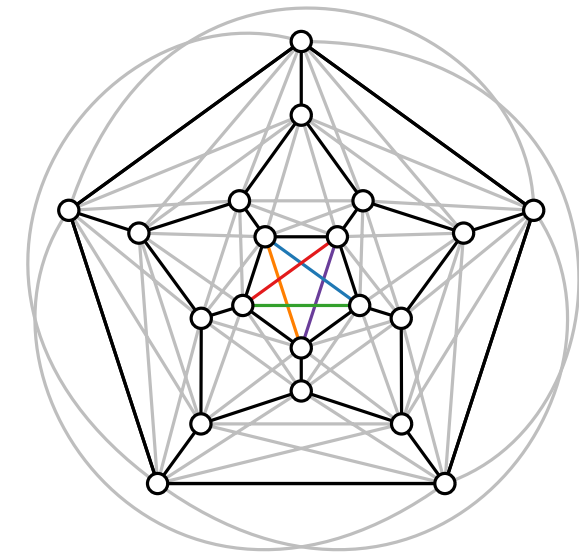
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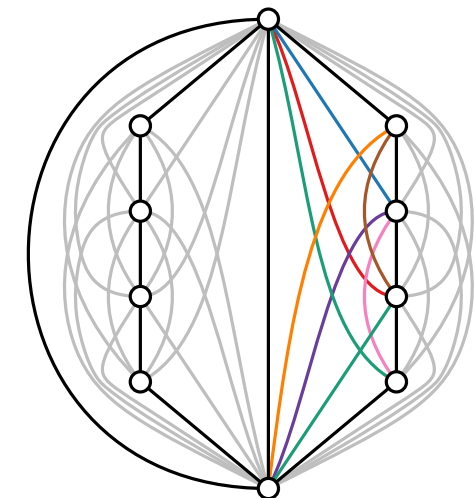
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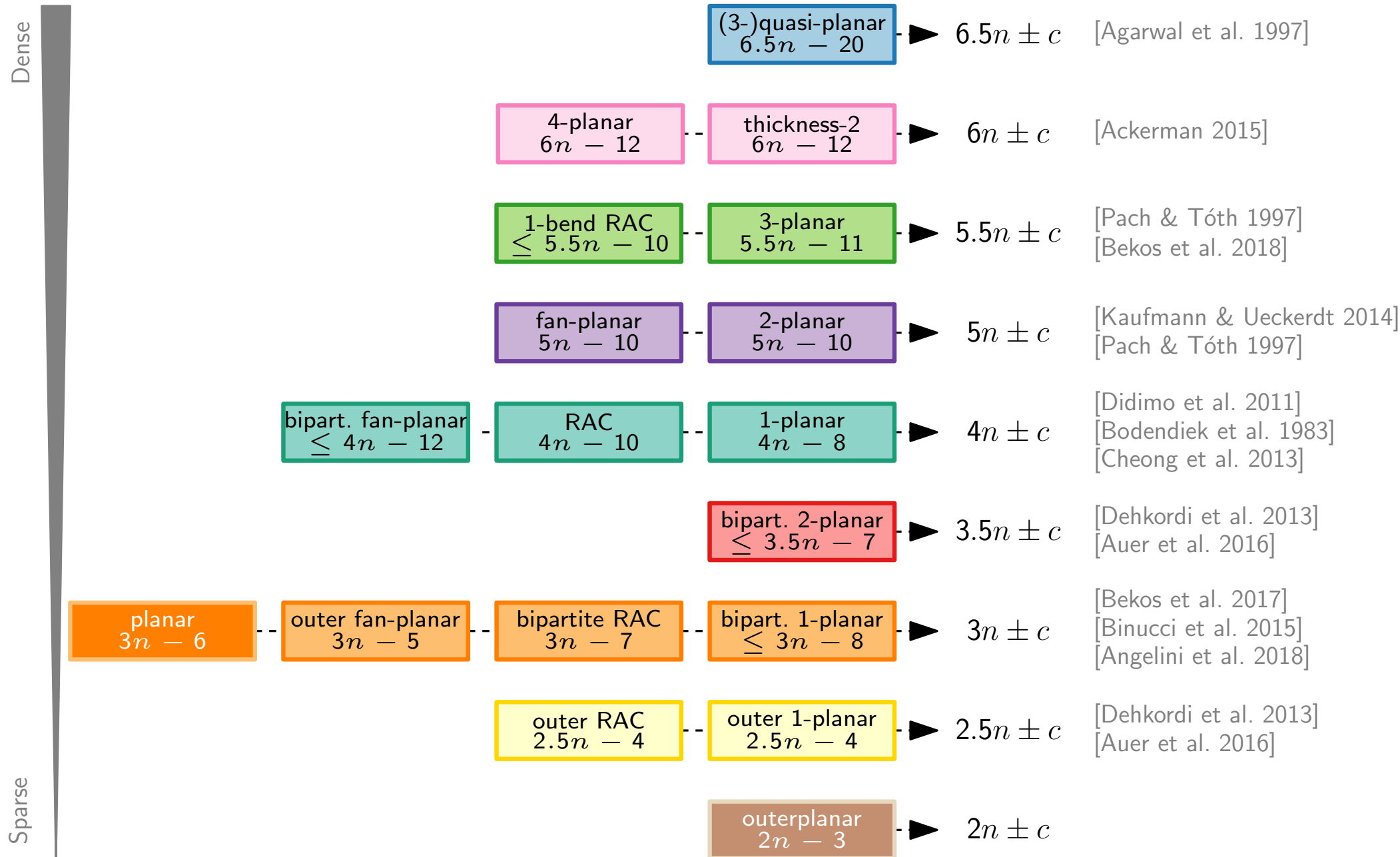


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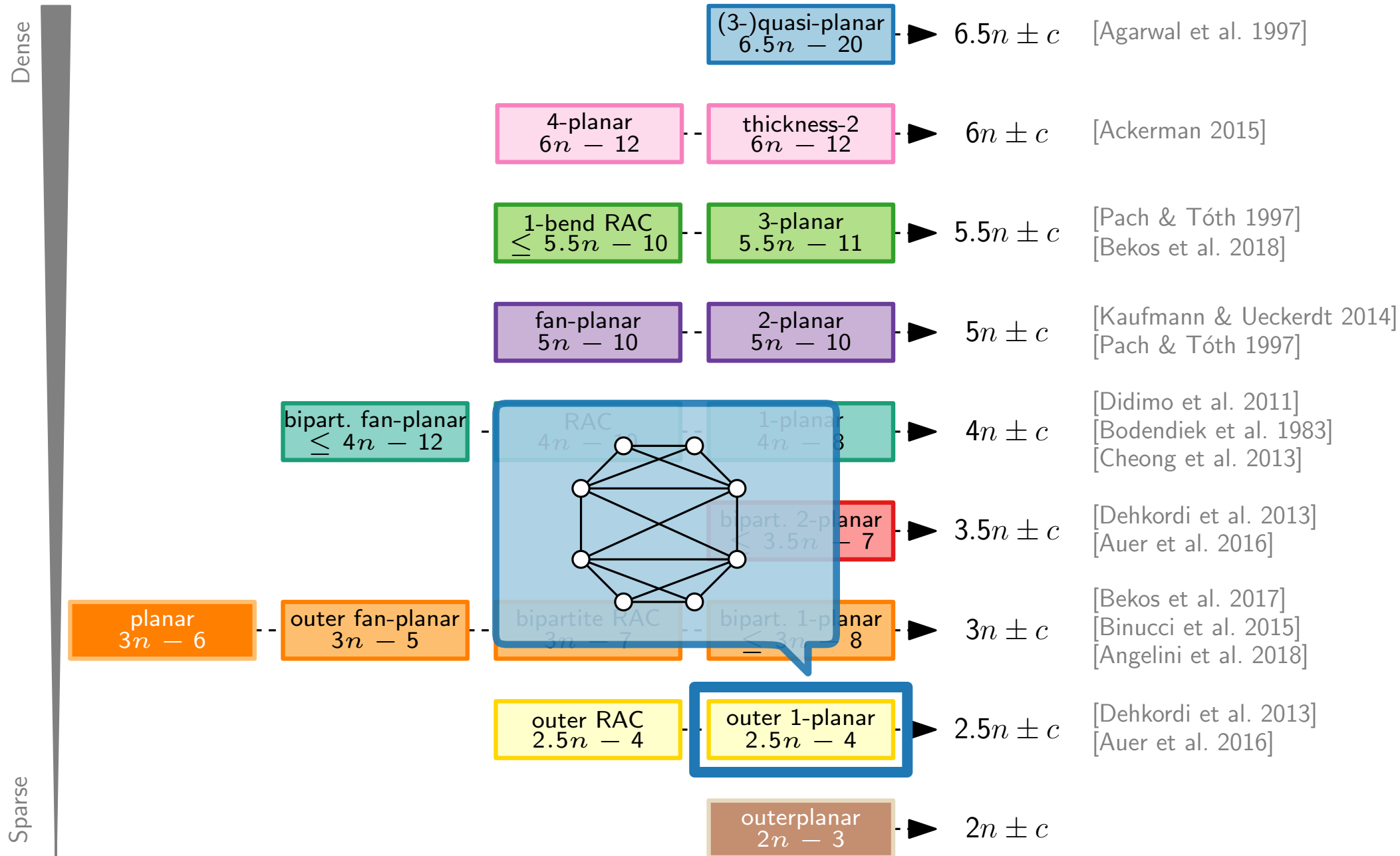


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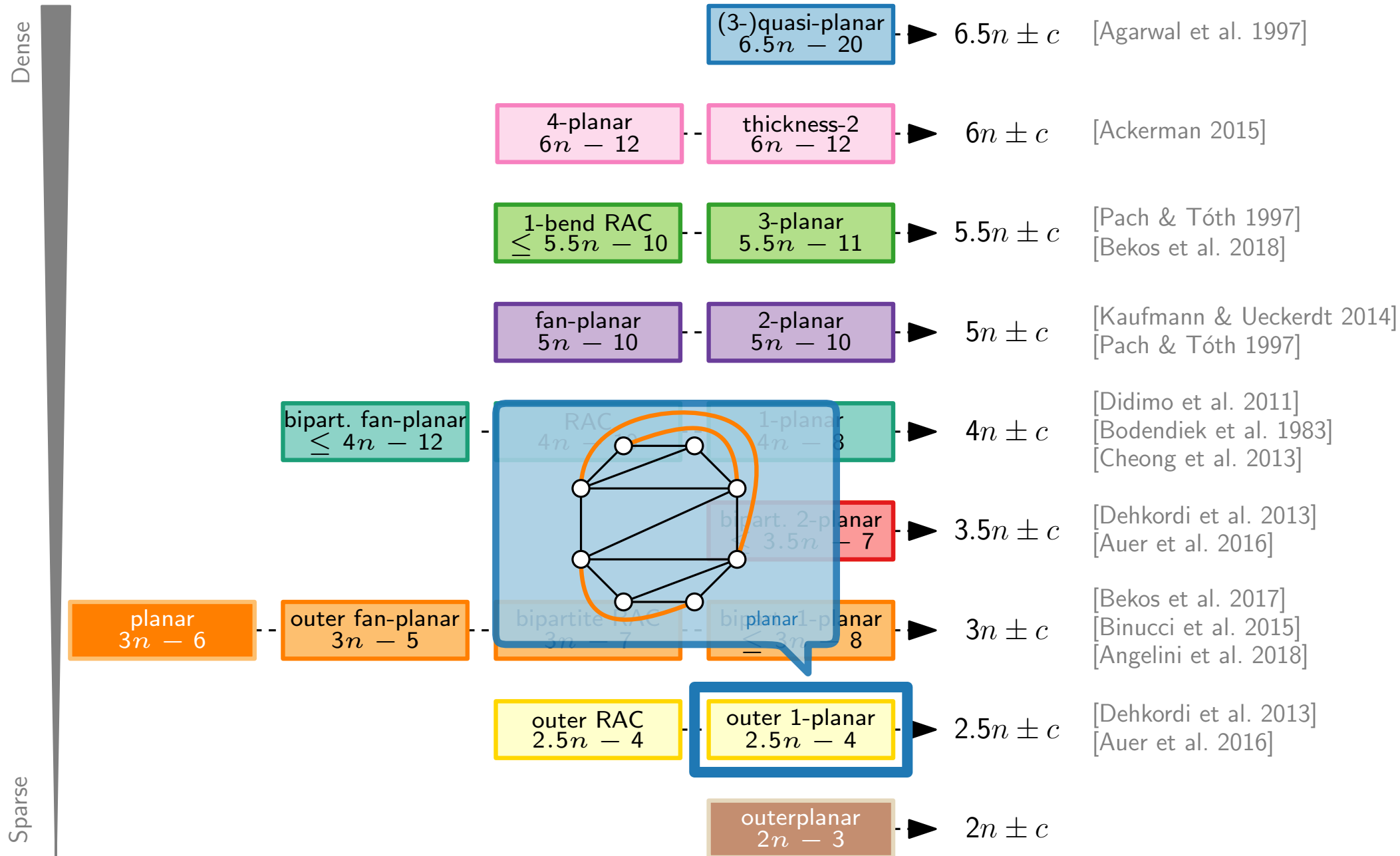
GD Beyond Planarity: a Hierarchy



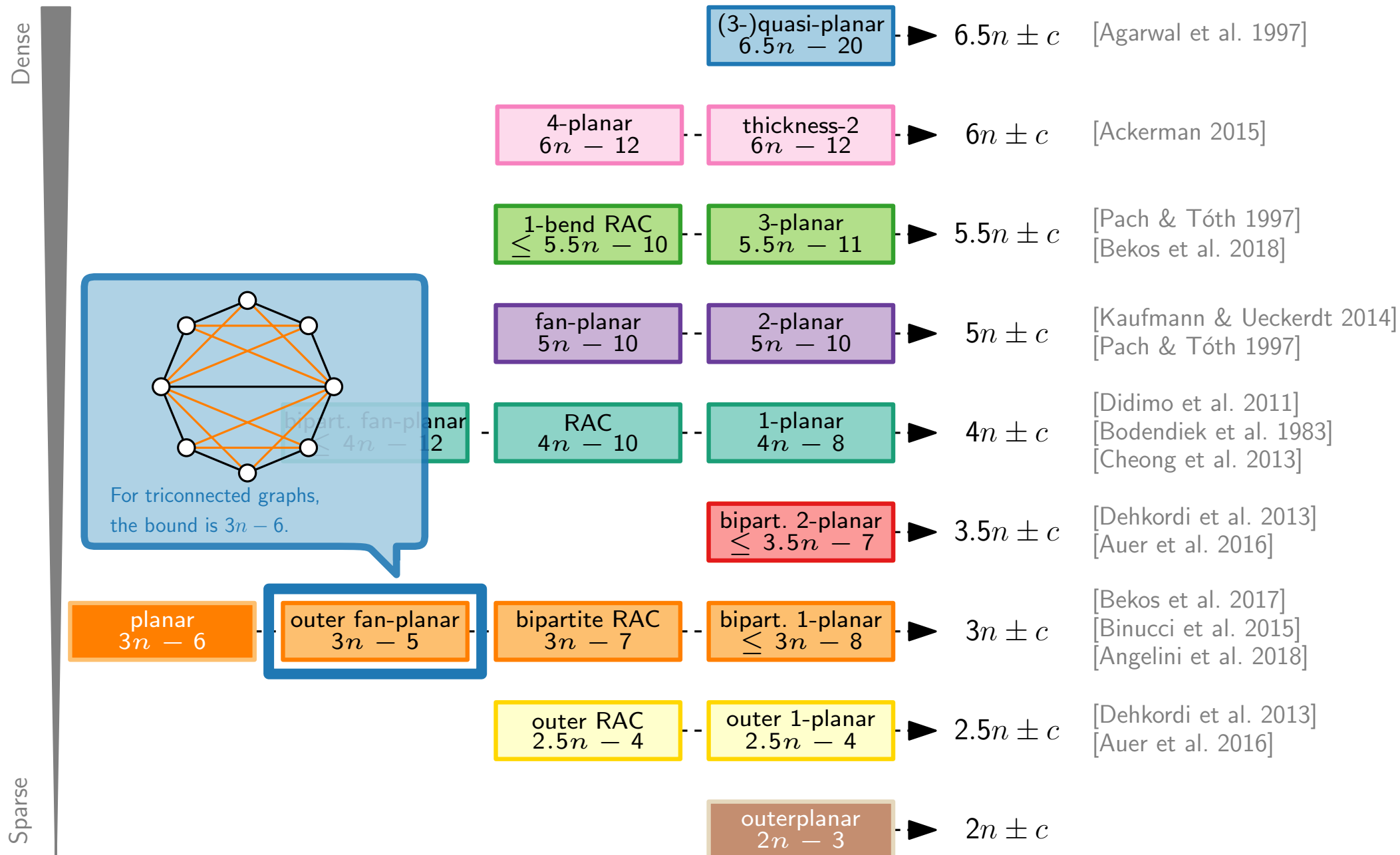
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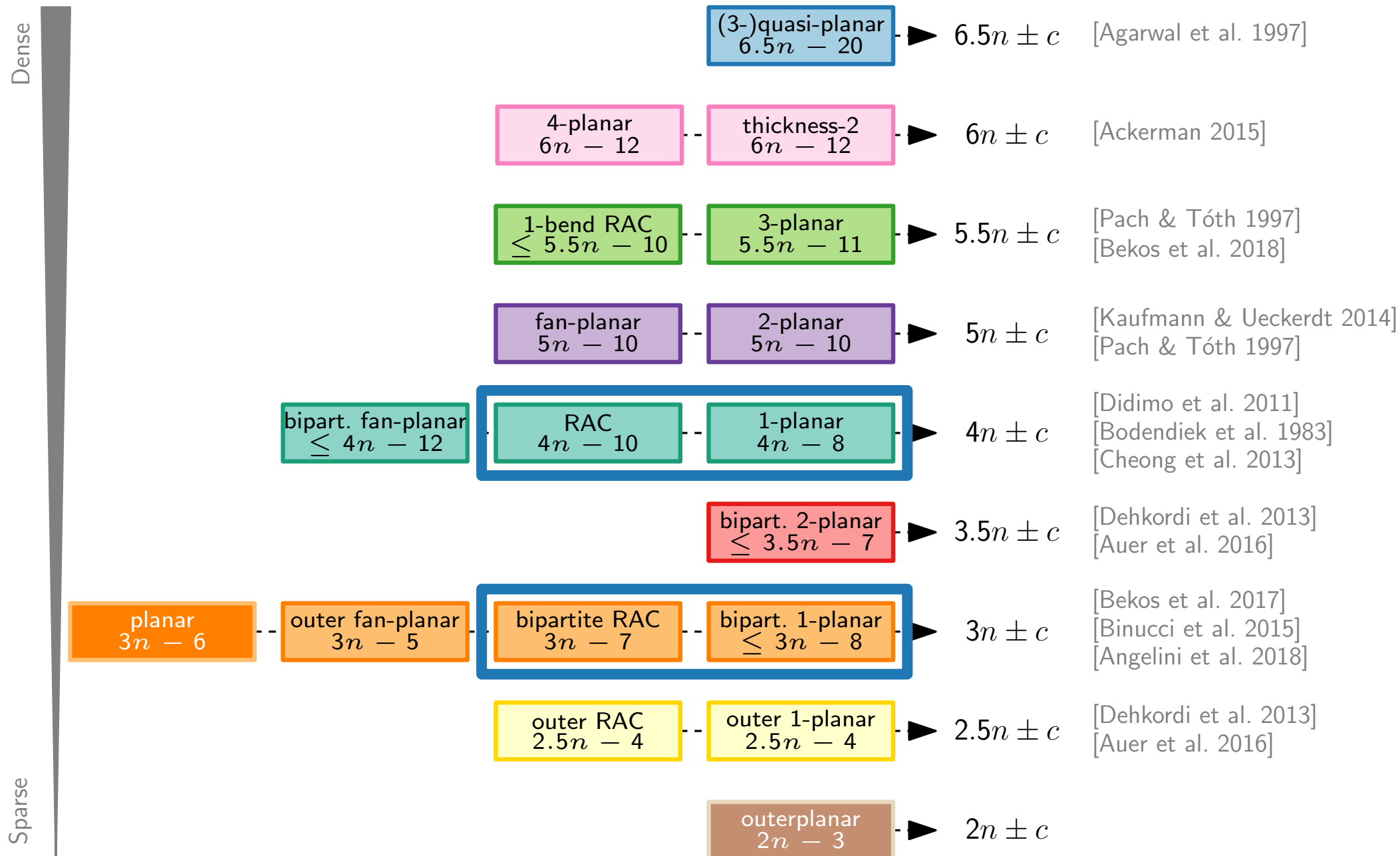
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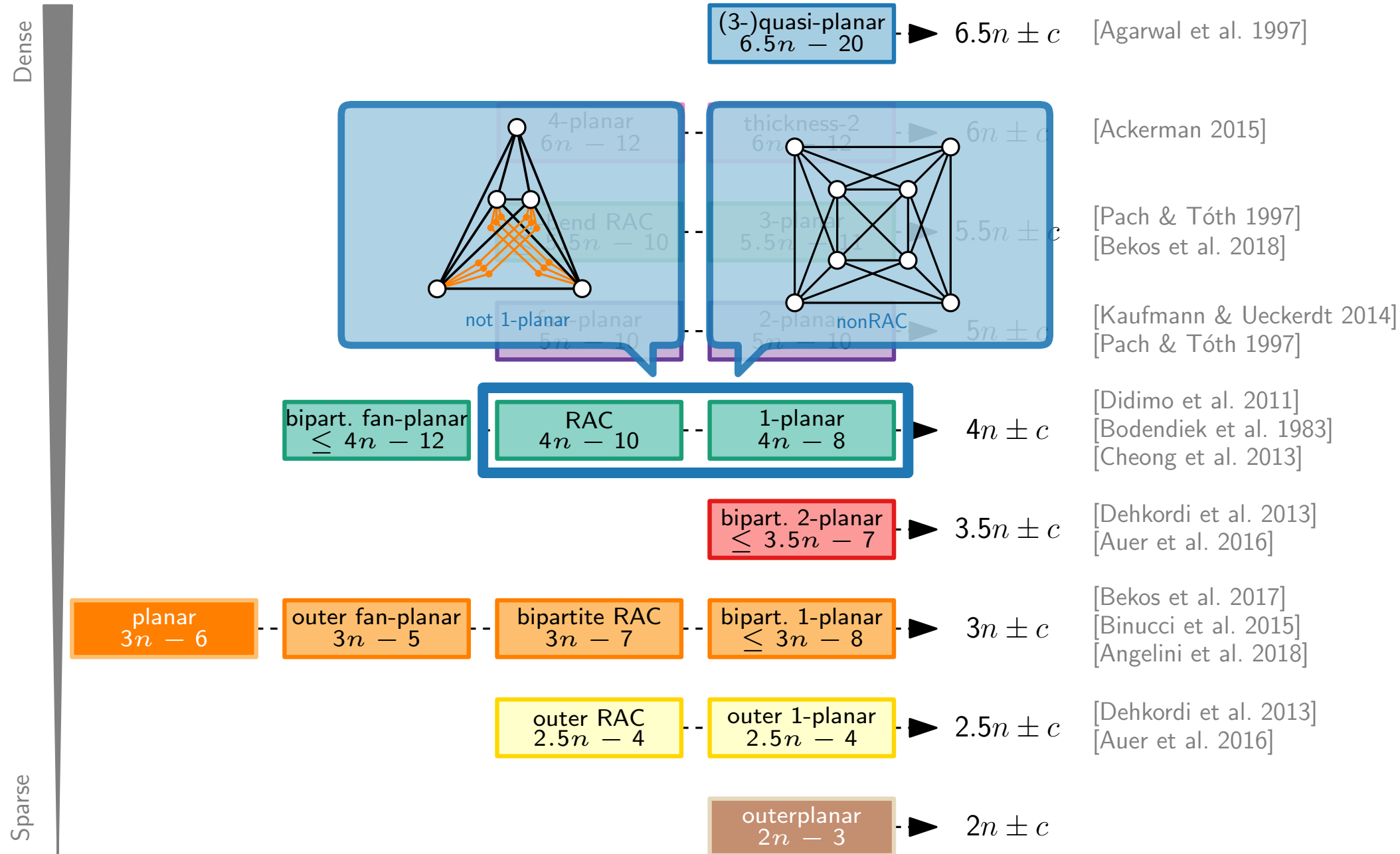
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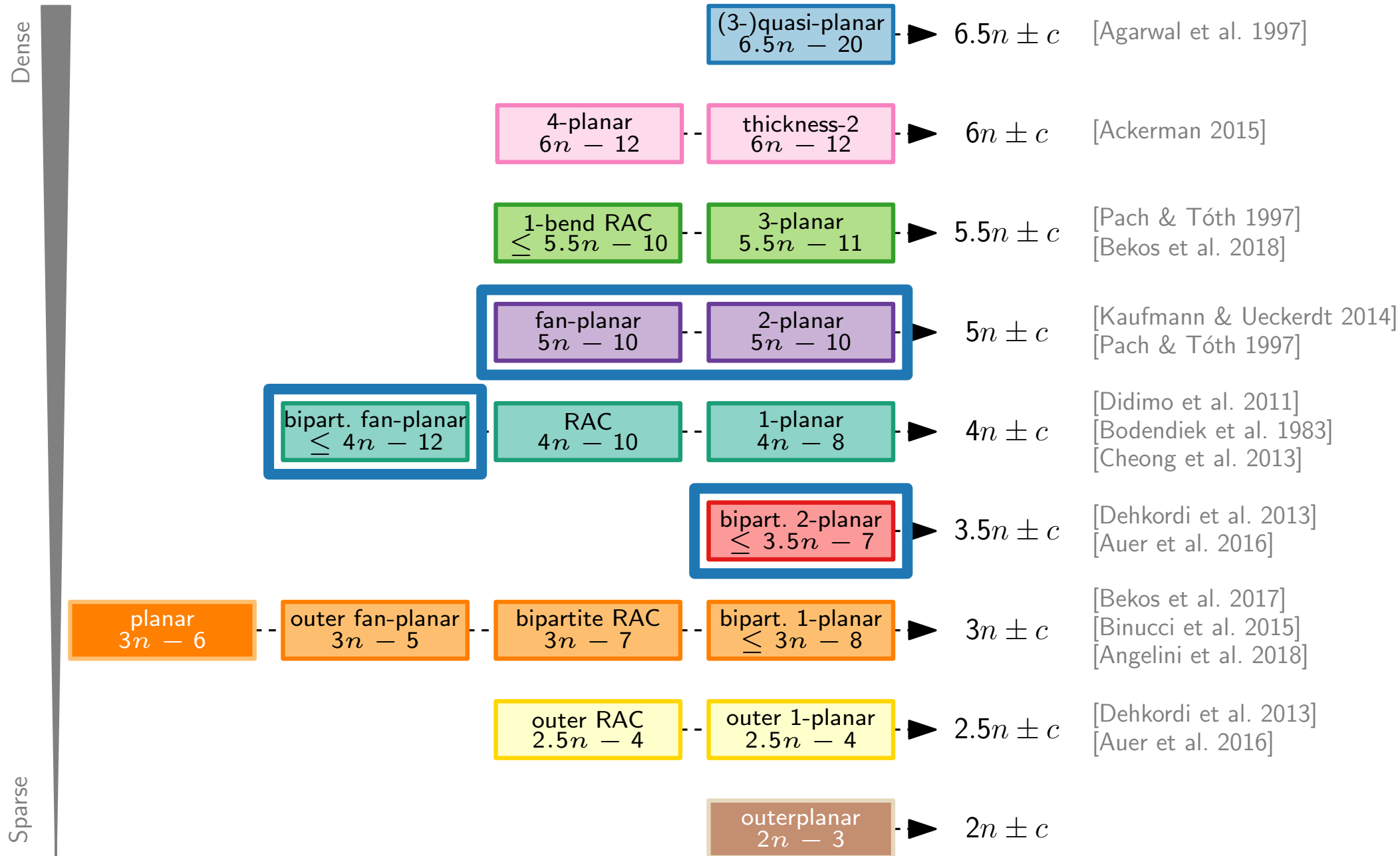
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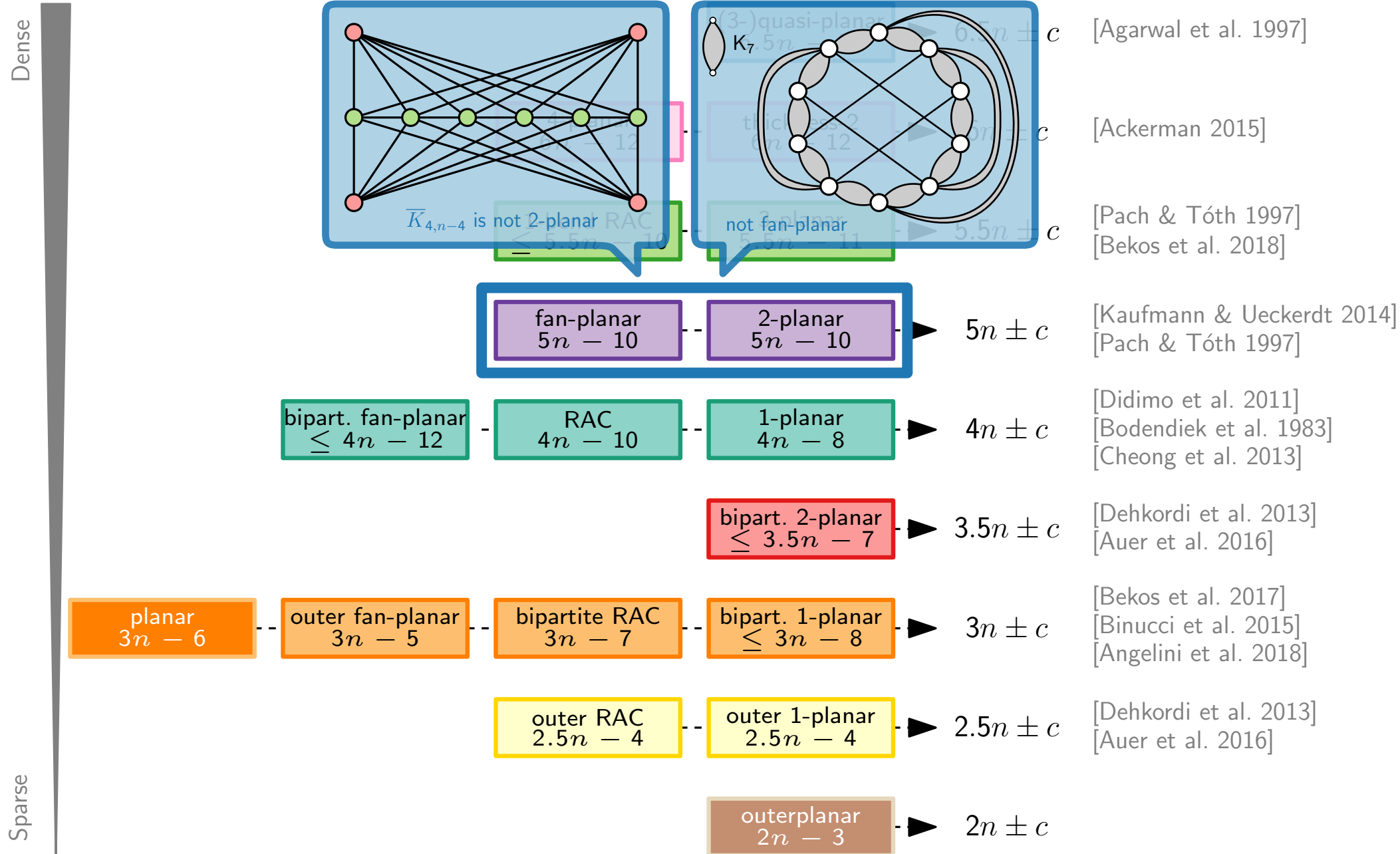
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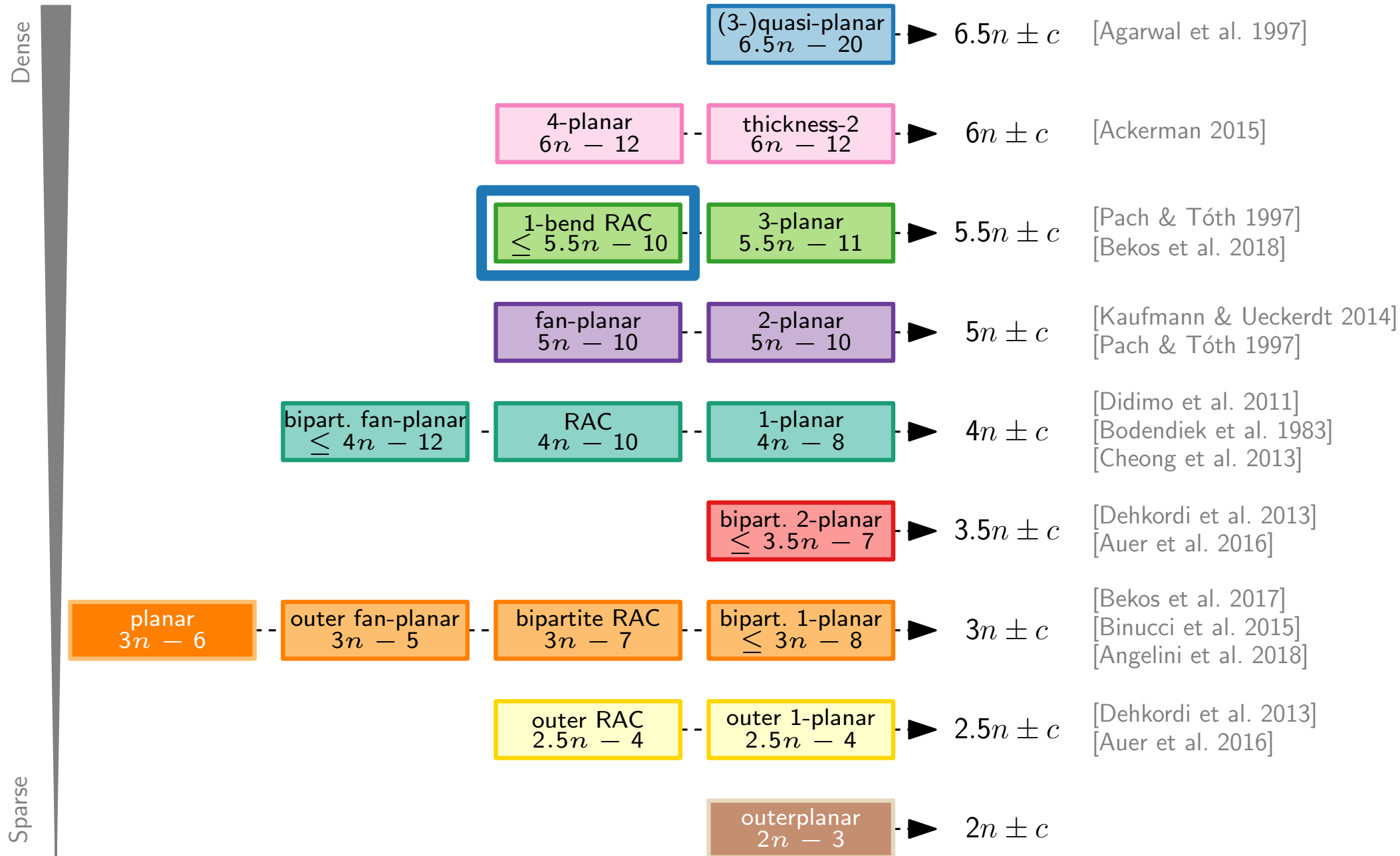
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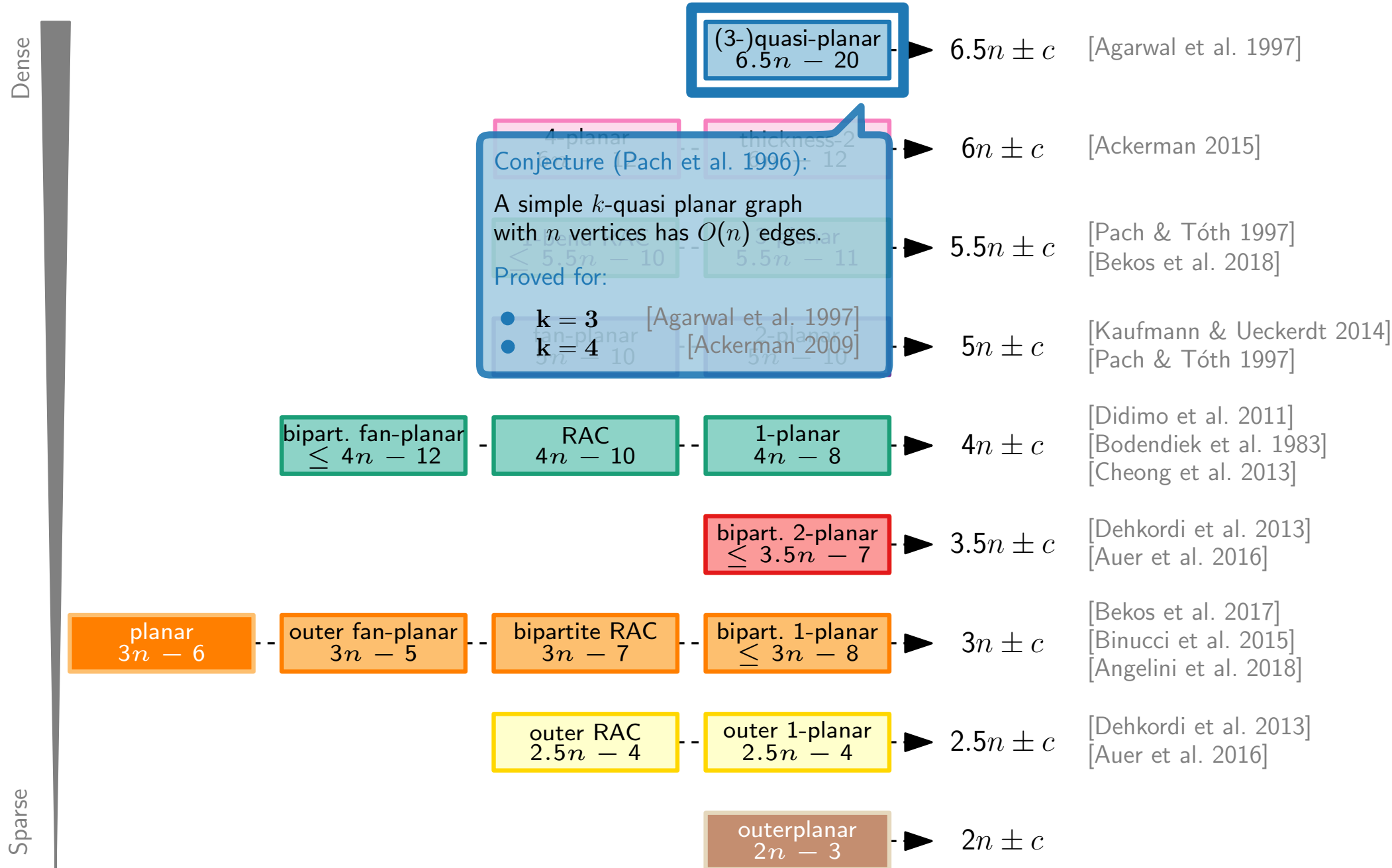
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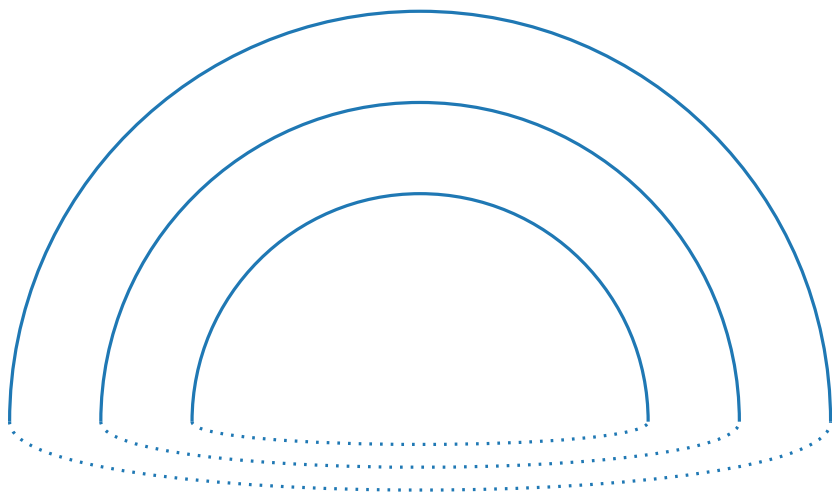
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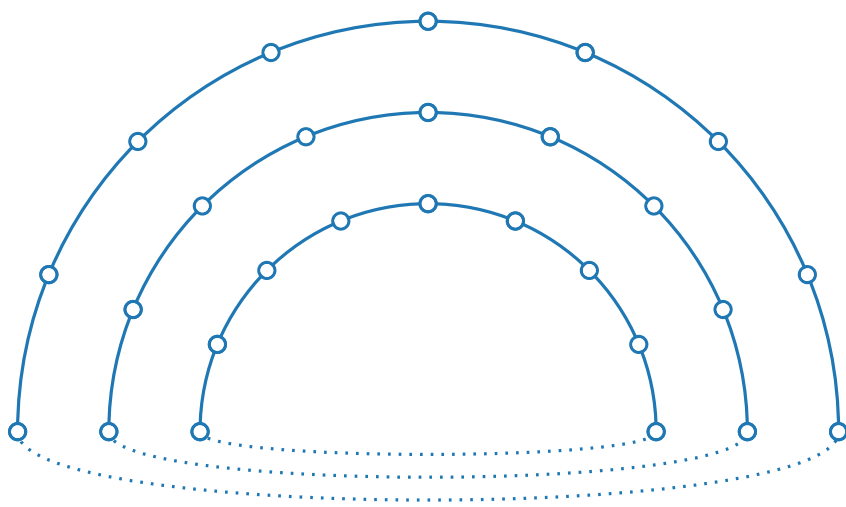
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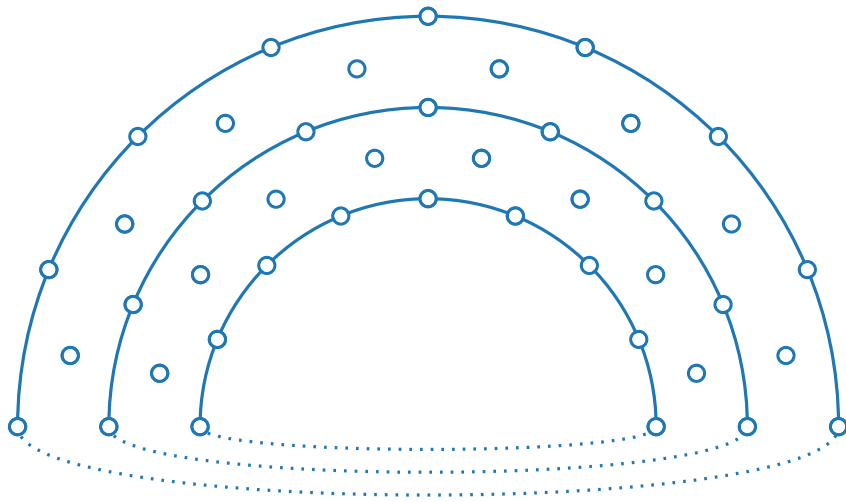
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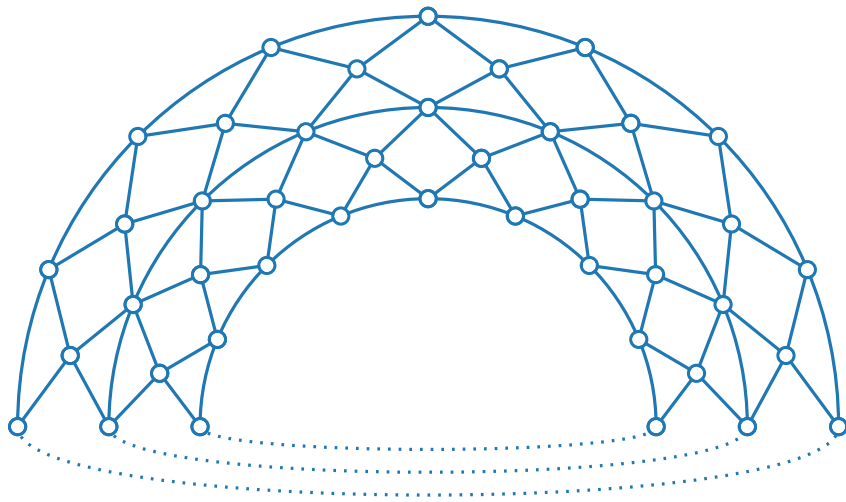
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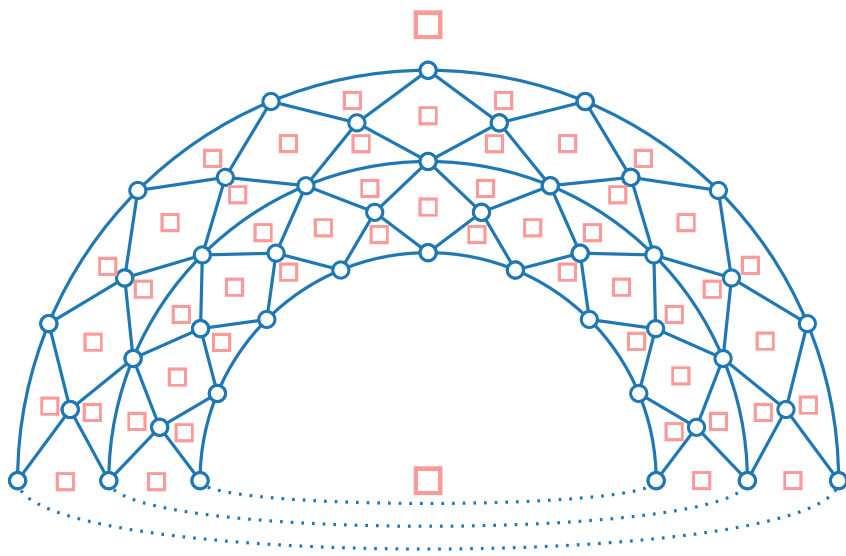
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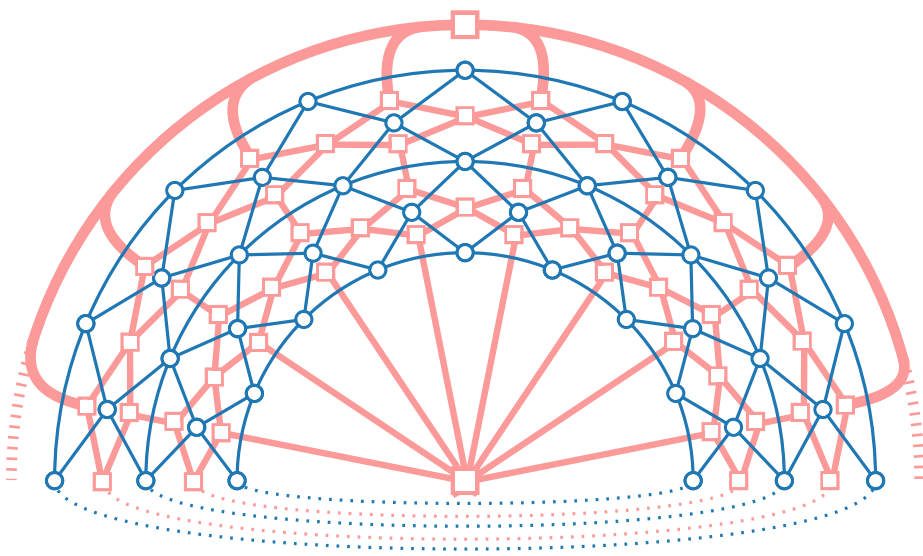
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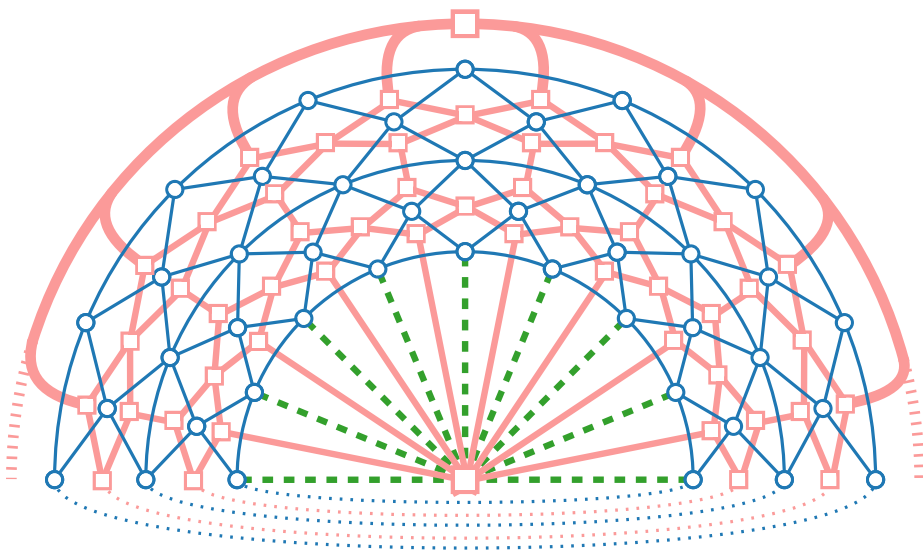
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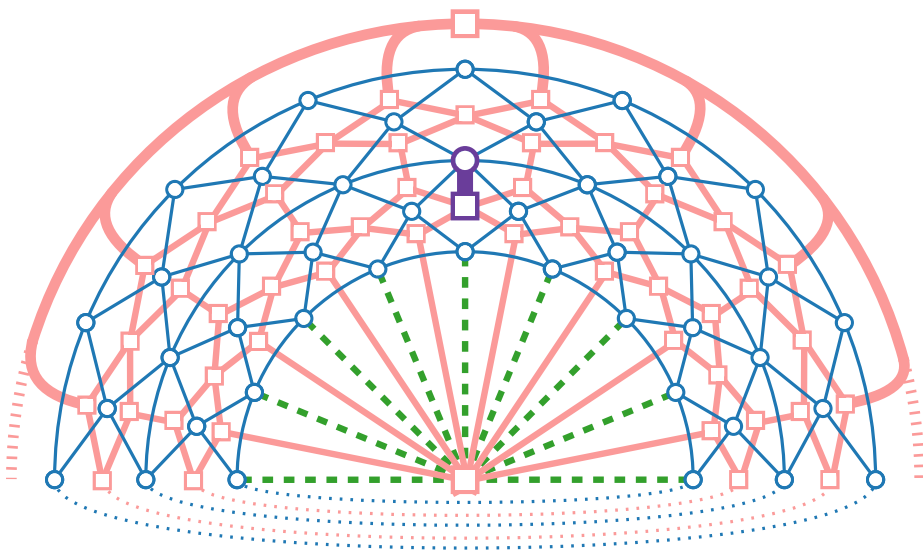
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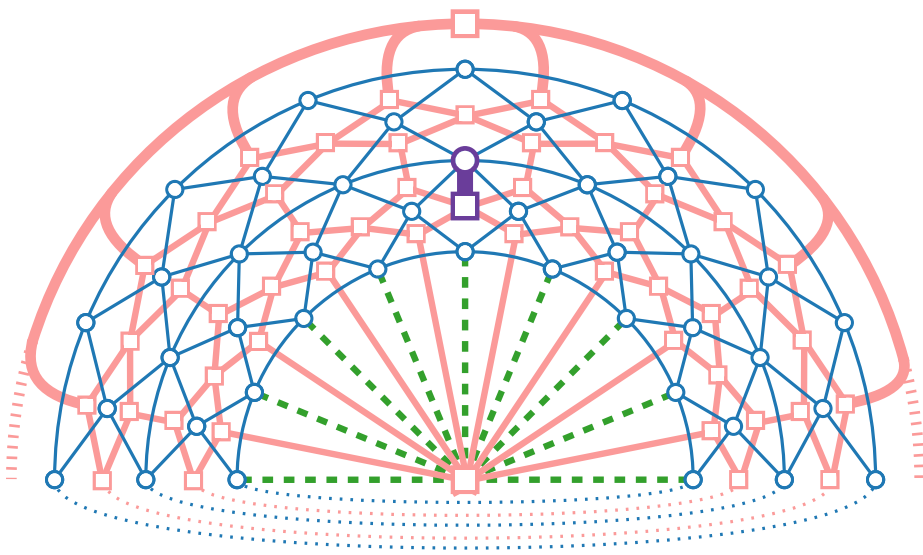
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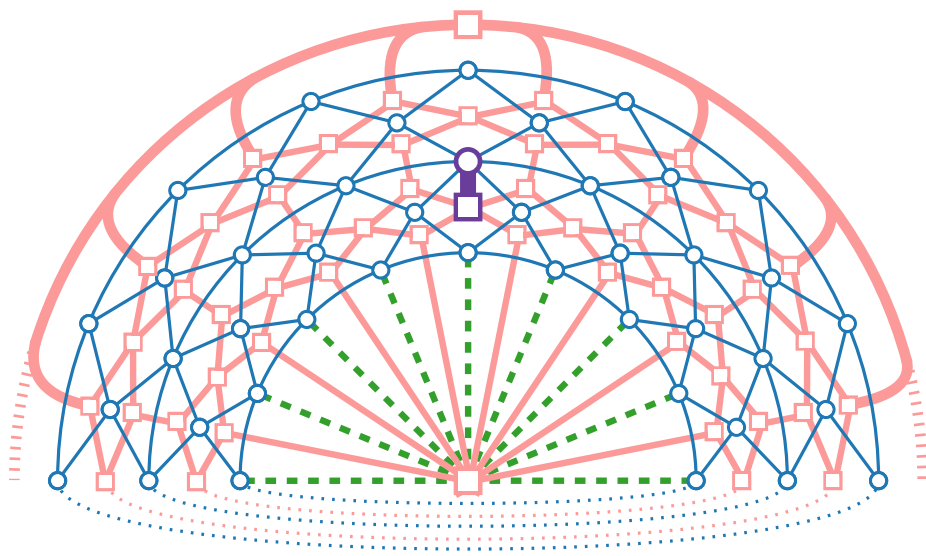
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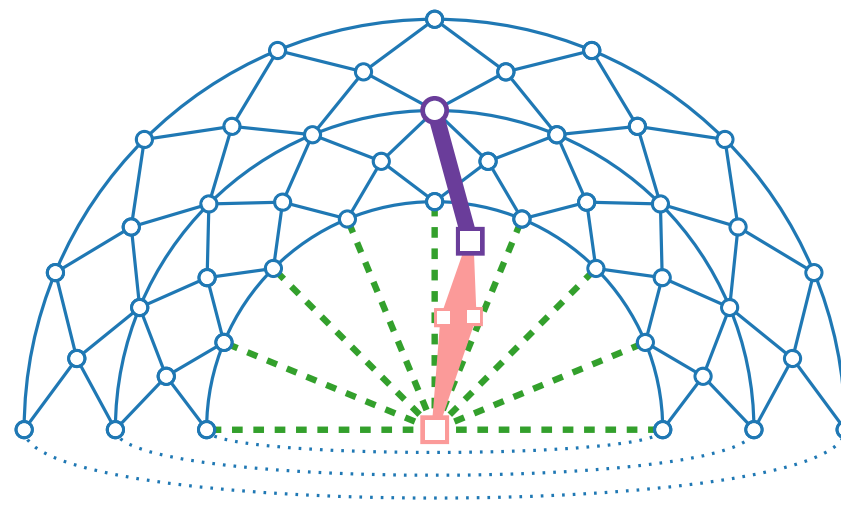
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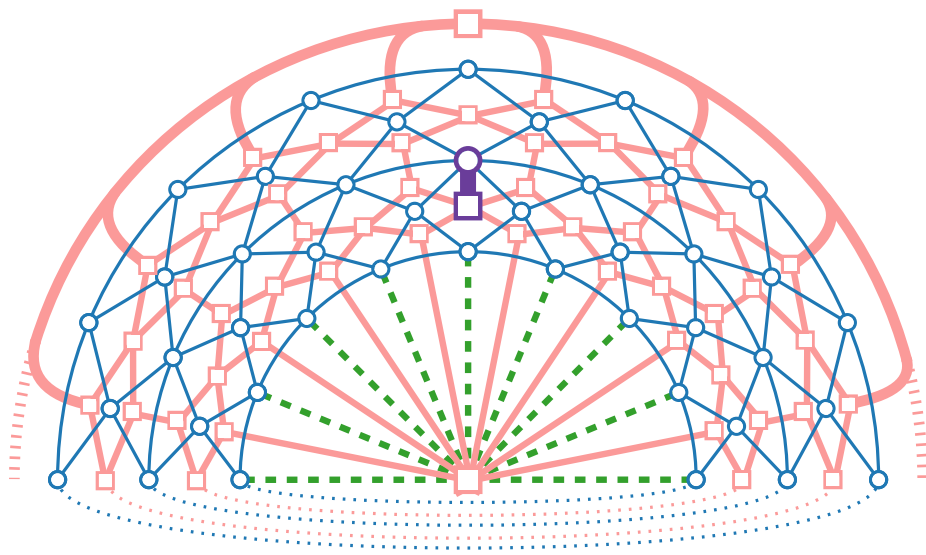
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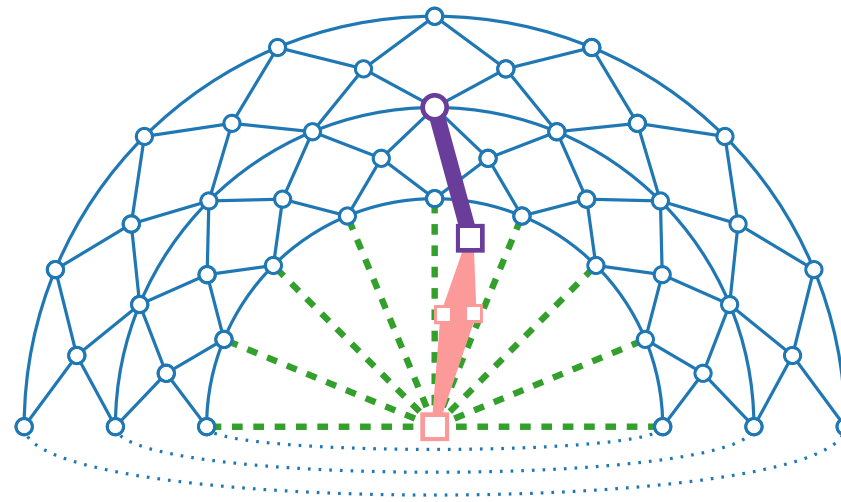
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For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $cr(G) = 2$ and $cr_{1\text{-pl}}(G) = n - 2$.



$$cr_{1\text{-pl}}(G) = n - 2$$



$$cr(G) = 2$$

Crossing Numbers

The **k -planar crossing number** $cr_{k\text{-pl}}(G)$ of a k -planar graph G is the number of crossings required in any k -planar drawing of G .

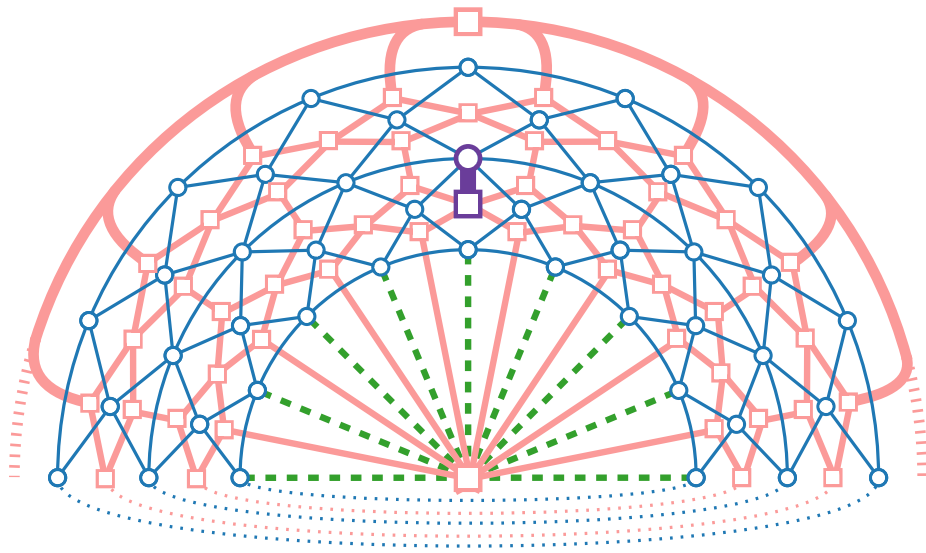
- $cr_{1\text{-pl}}(G) \leq n - 2$ (there are at most $n - 2$ green edges in the coloring of Theorem 1)
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Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]

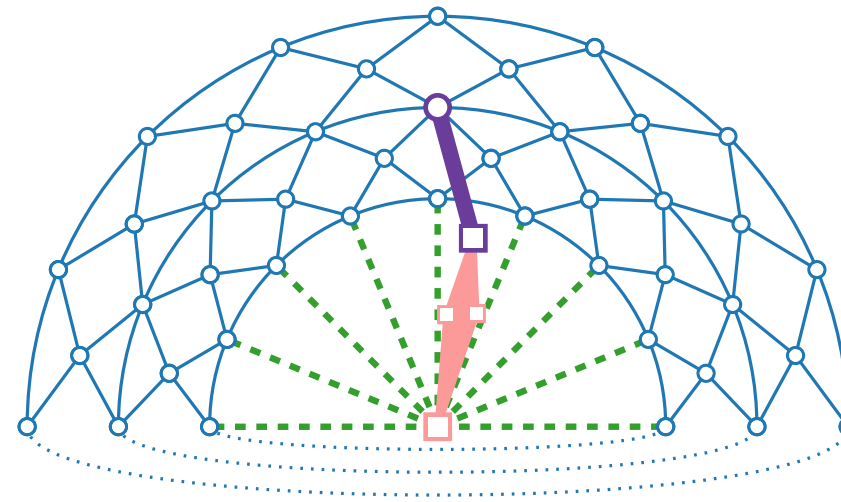
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Crossing ratio

$$\rho_{1\text{-pl}}(n) = (n - 2)/2$$



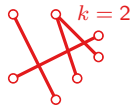
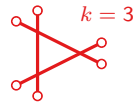
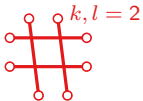
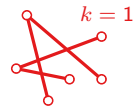
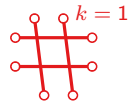
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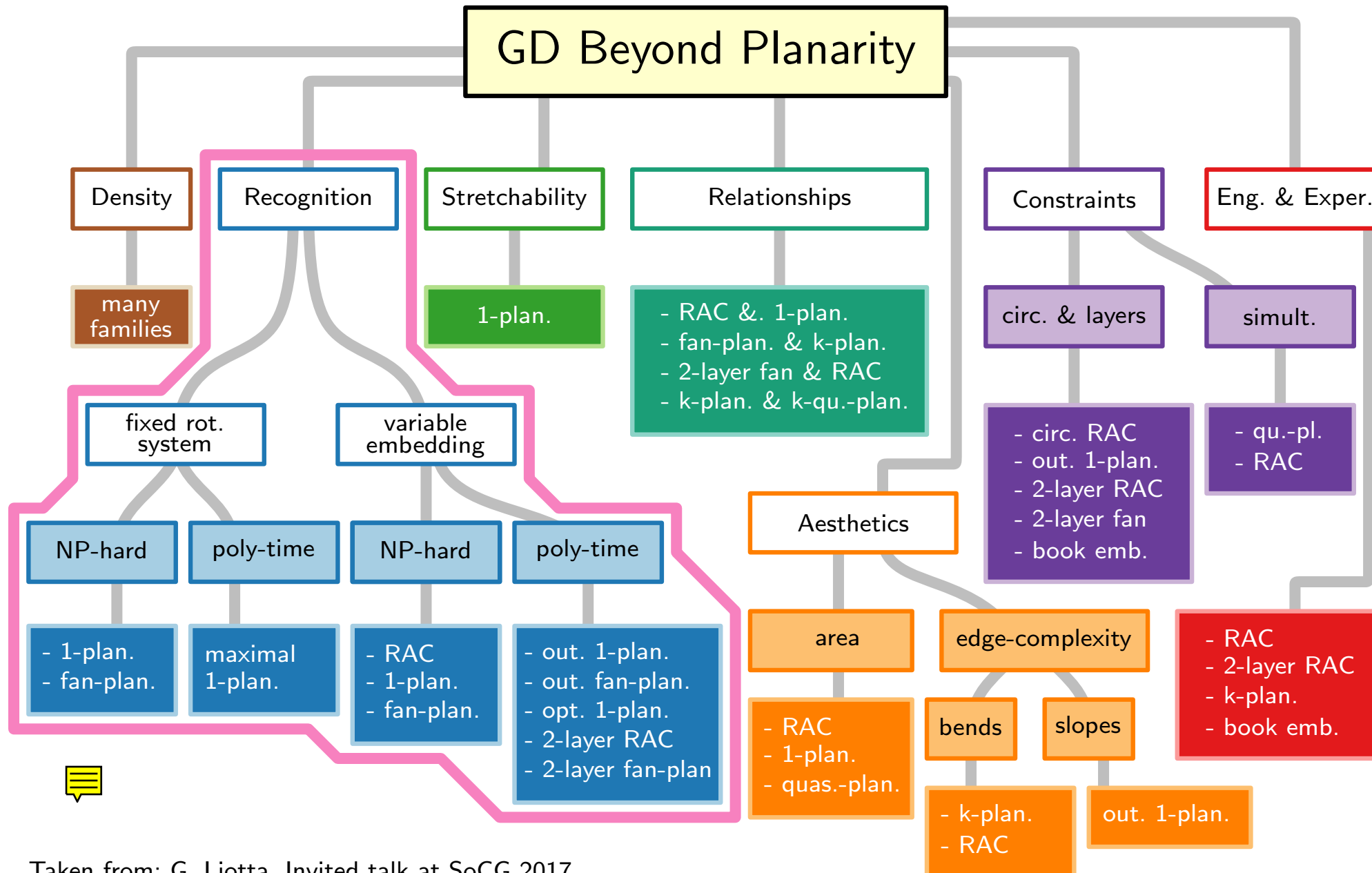
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Crossing Ratios

Table from “Crossing Numbers of Beyond-Planar Graphs Revisited”
[van Beusekom, Parada & Speckmann 2021]

Family	Forbidden Configurations		Lower	Upper
k -planar	An edge crossed more than k times		$\Omega(n/k)$	$O(k\sqrt{kn})$
k -quasi-planar	k pairwise crossing edges		$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different “side”		$\Omega(n)$	$O(n^2)$
(k, l) -grid-free	Set of k edges such that each edge crosses each edge from a set of l edges.		$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k, l)n^2$
k -gap-planar	More than k crossings mapped to an edge in an optimal mapping		$\Omega(n/k^3)$	$O(k\sqrt{kn})$
Skewness- k	Set of crossings not covered by at most k edges		$\Omega(n/k)$	$O(kn + k^2)$
k -apex	Set of crossings not covered by at most k vertices		$\Omega(n/k)$	$O(k^2n^2 + k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge		$\Omega(n^2)$	$O(n^2)$
k -fan-crossing-free	An edge that crosses k adjacent edges		$\Omega(n^2/k^3)$	$O(k^2n^2)$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$		$\Omega(n^2)$	$O(n^2)$

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Minors of 1-Planar Graphs

Theorem.

[Kuratowski 1930]

G planar $\Leftrightarrow$ neither K_5 nor $K_{3,3}$ minor of G

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The class of 1-planar graphs is not closed under edge contraction.

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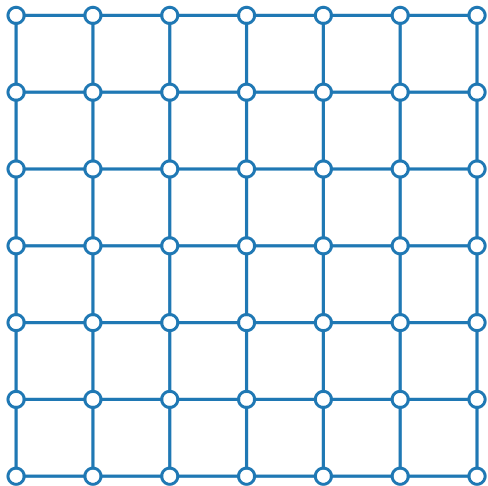
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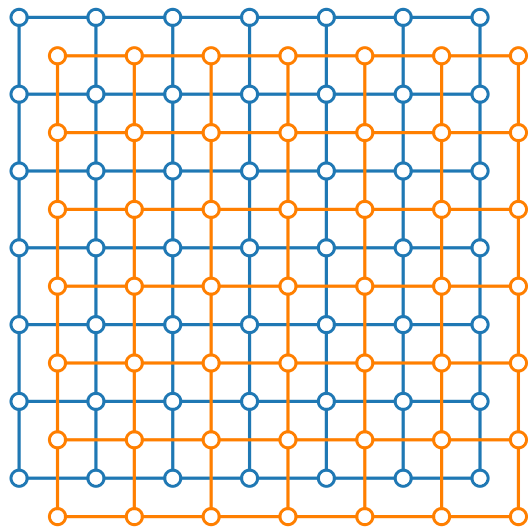
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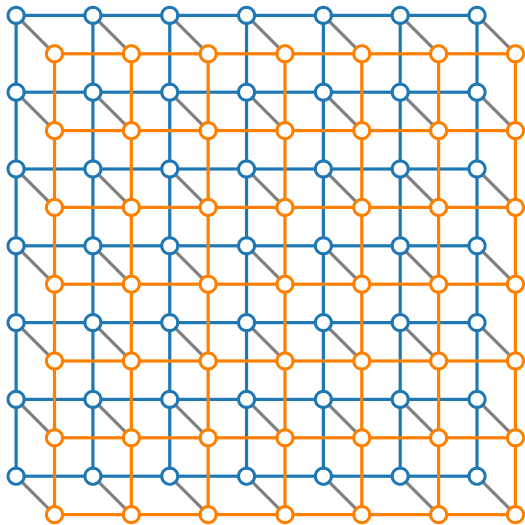
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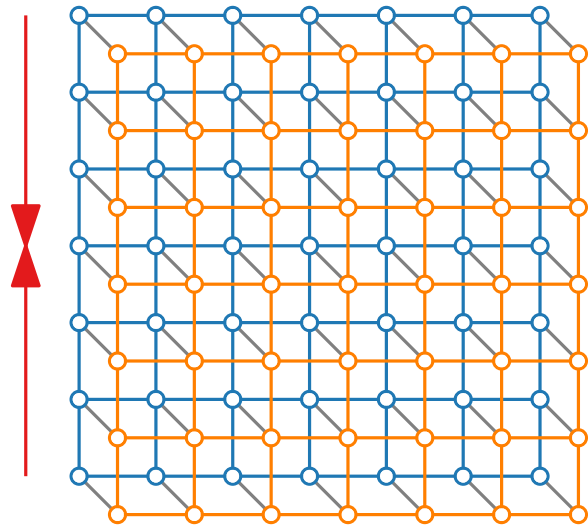
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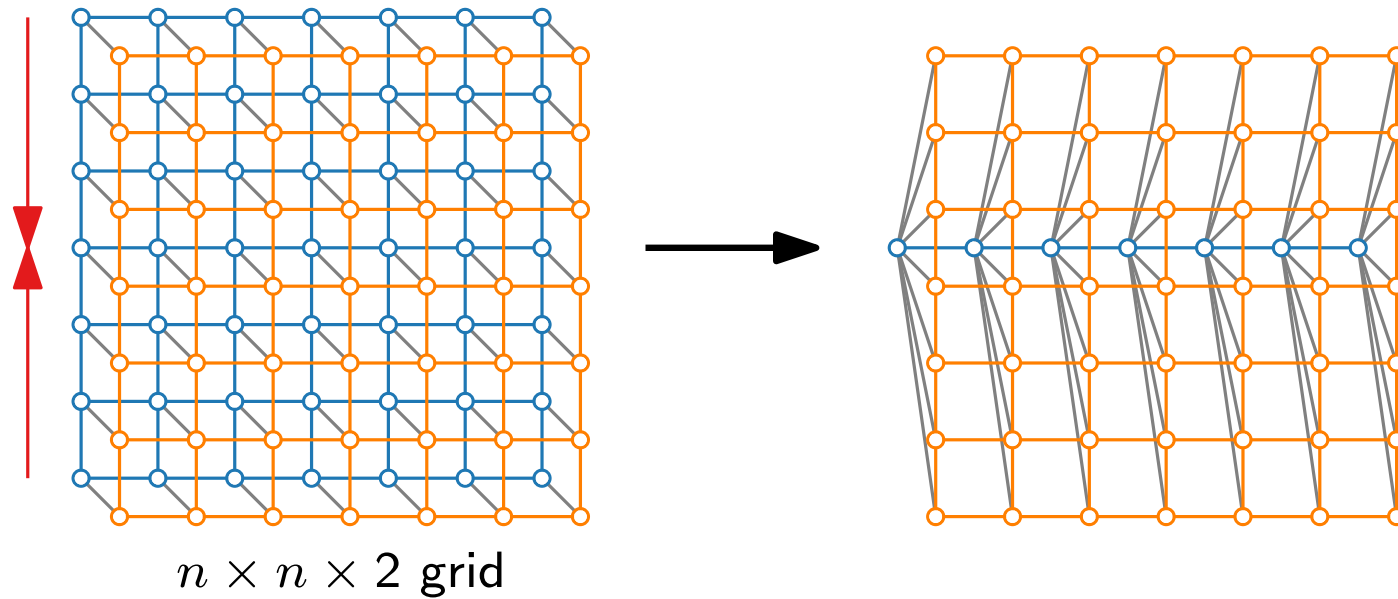
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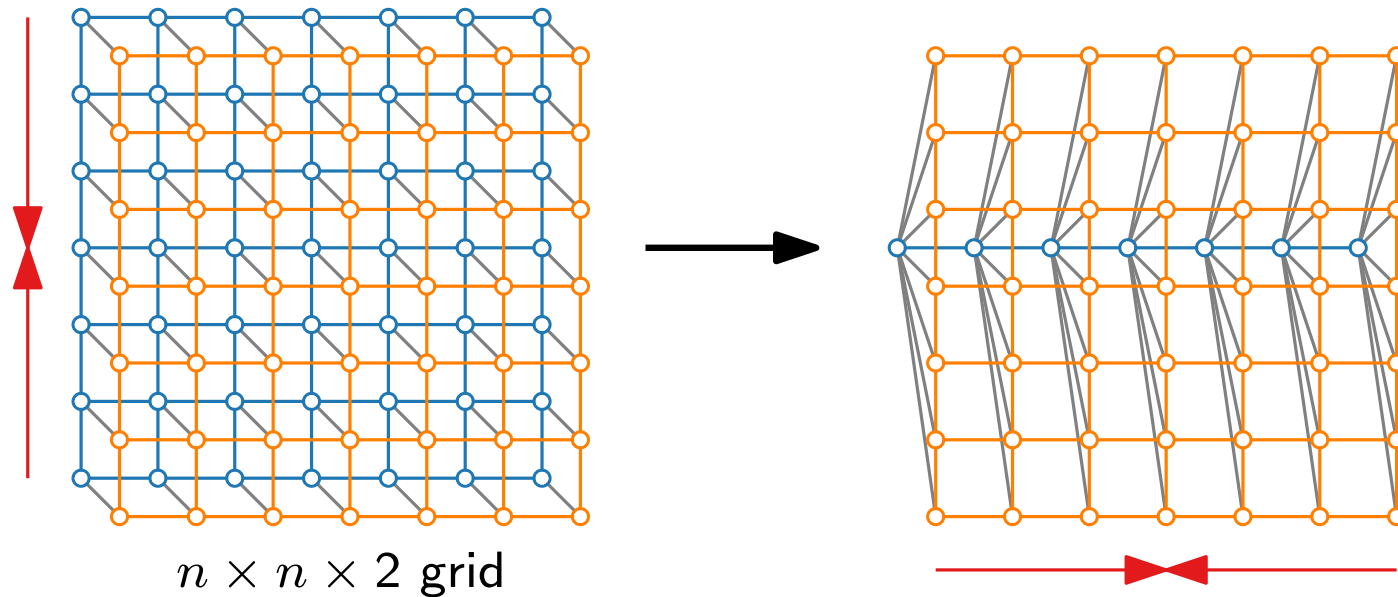
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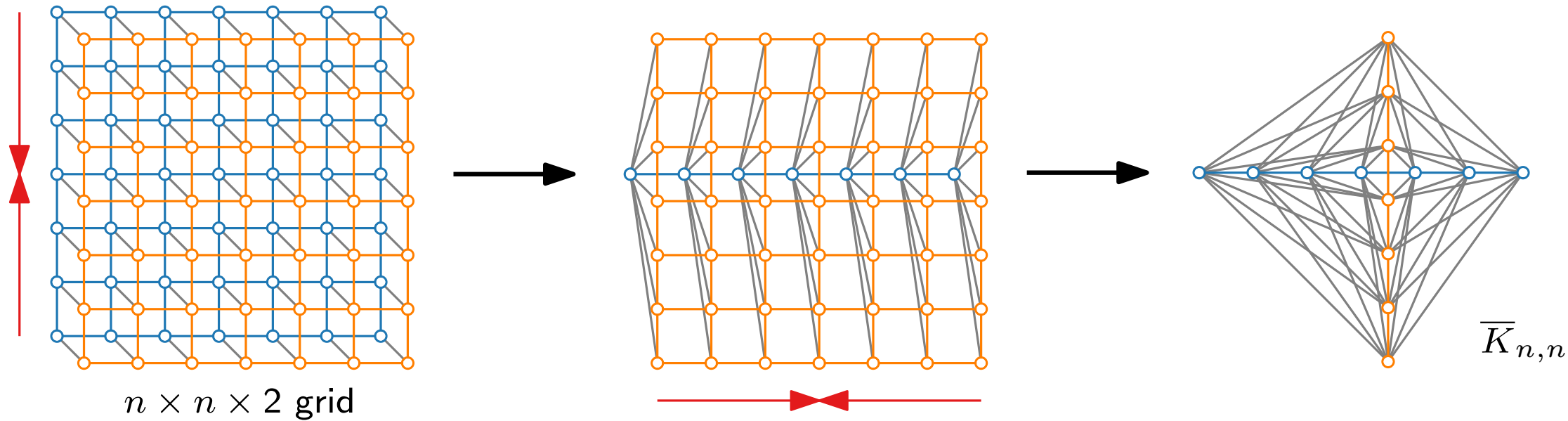
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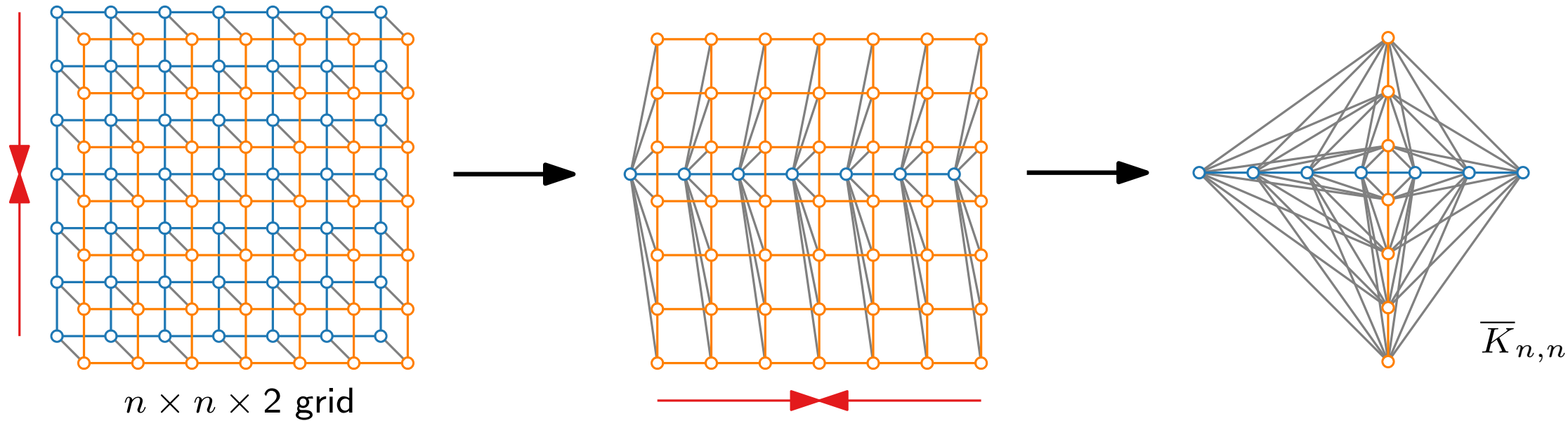
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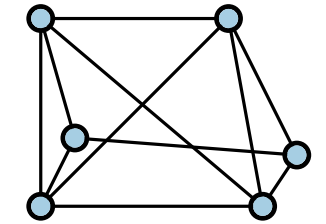
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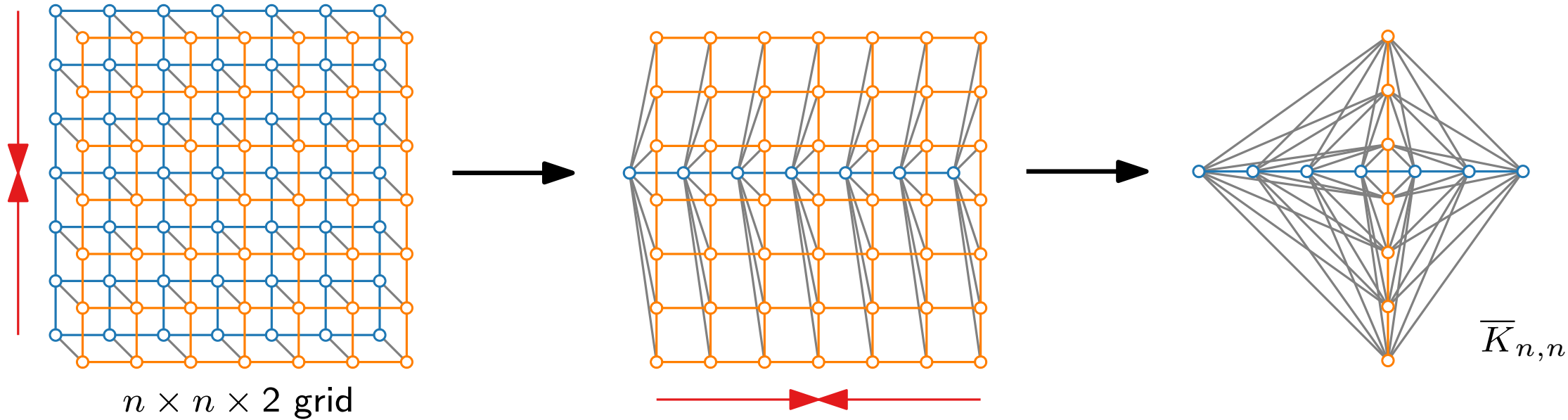
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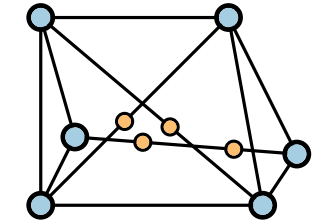
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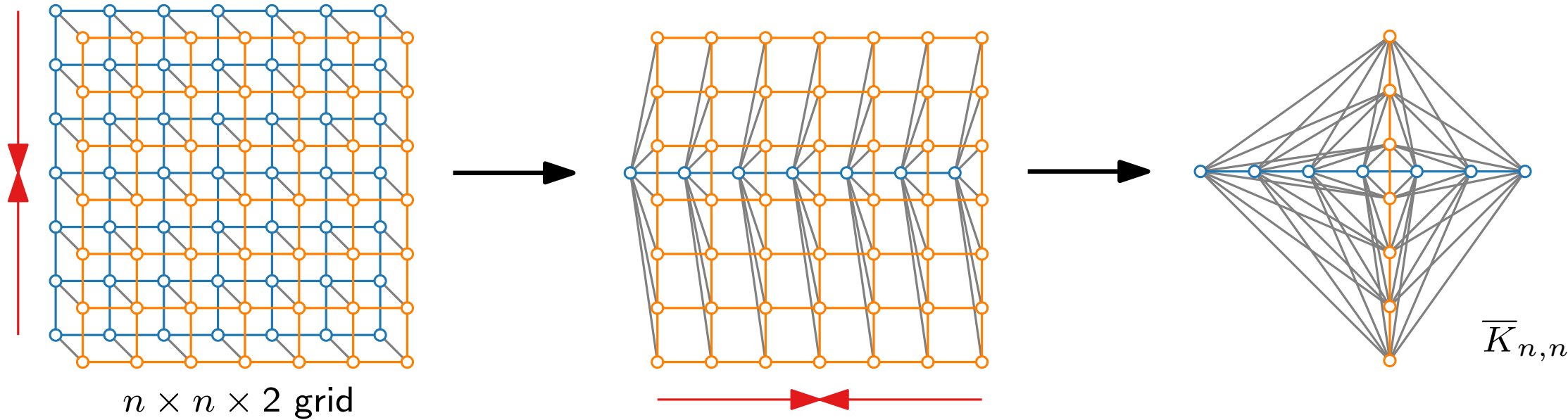
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For any n , there exist $\Omega(2^n)$ distinct n -vertex graphs that are not 1-planar but all their proper subgraphs are 1-planar.

Recognition of 1-Planar Graphs

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Testing 1-planarity

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Recognition of 1-Planar Graphs

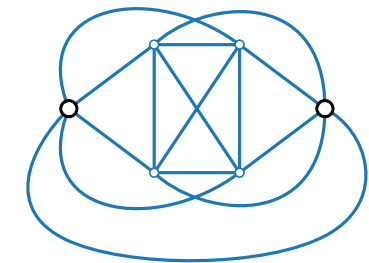
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The only 1-planar embedding of K_6 :



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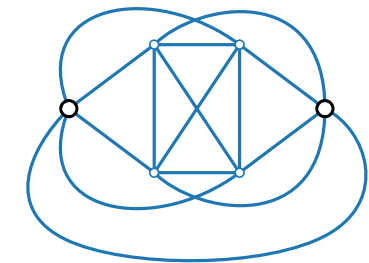
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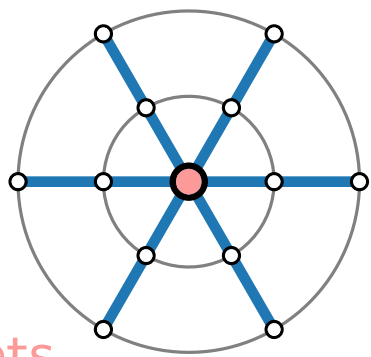
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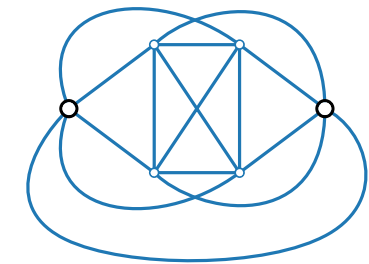
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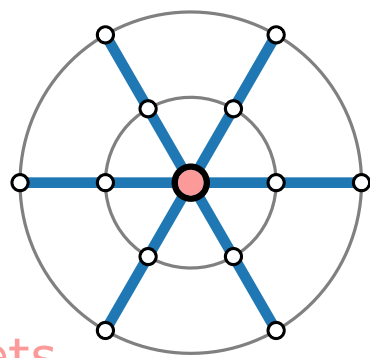
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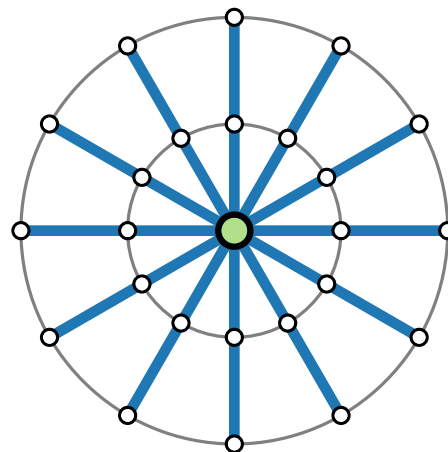
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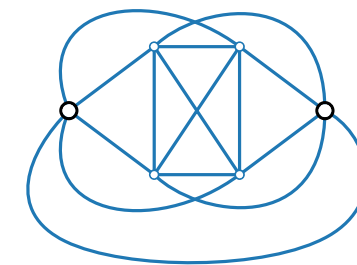


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$\sum A$ pockets

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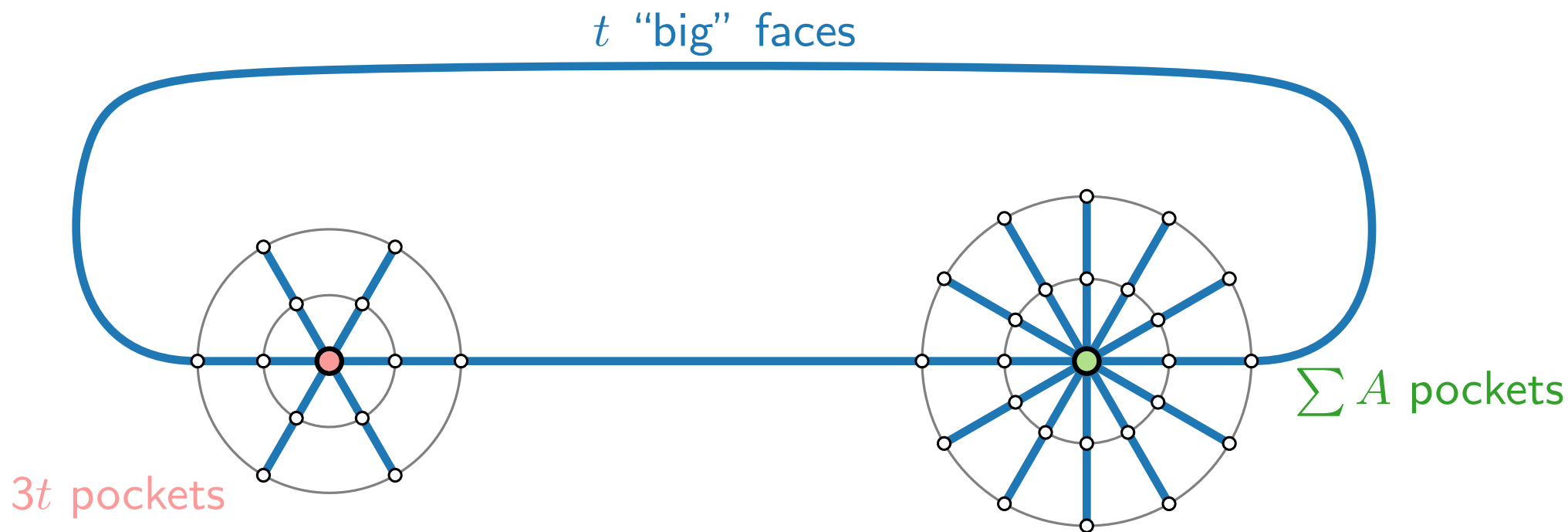
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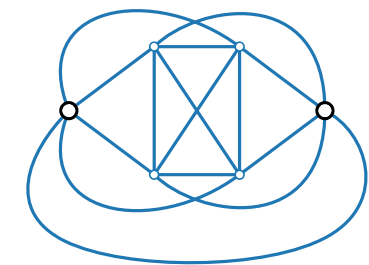
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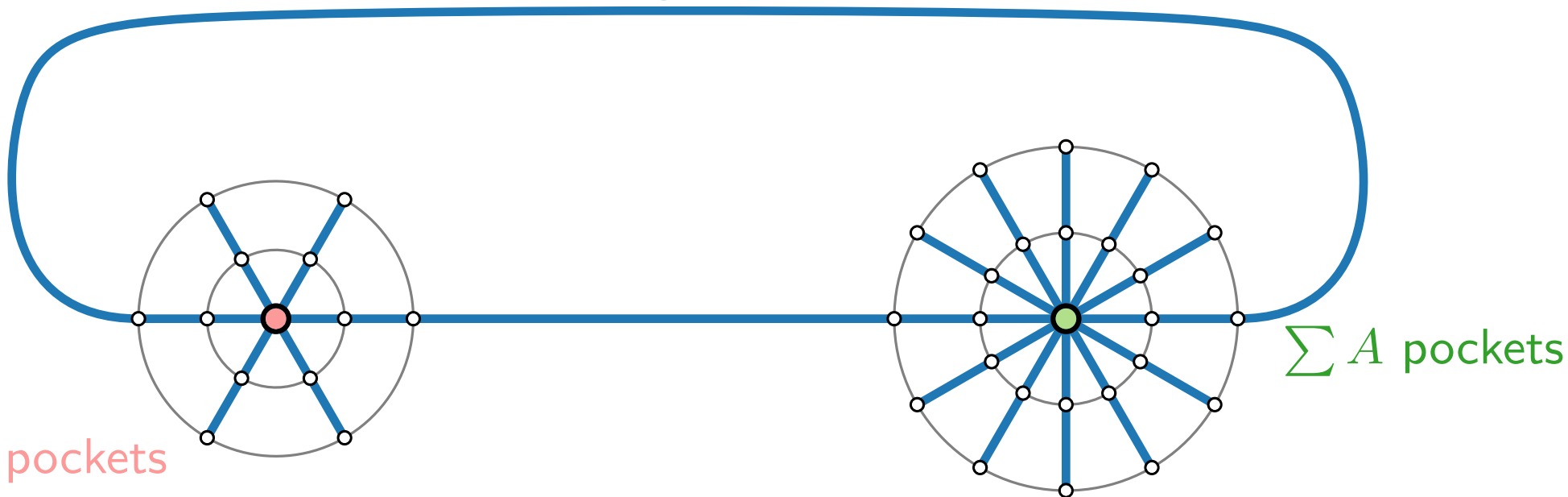
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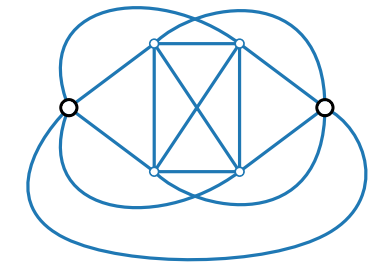
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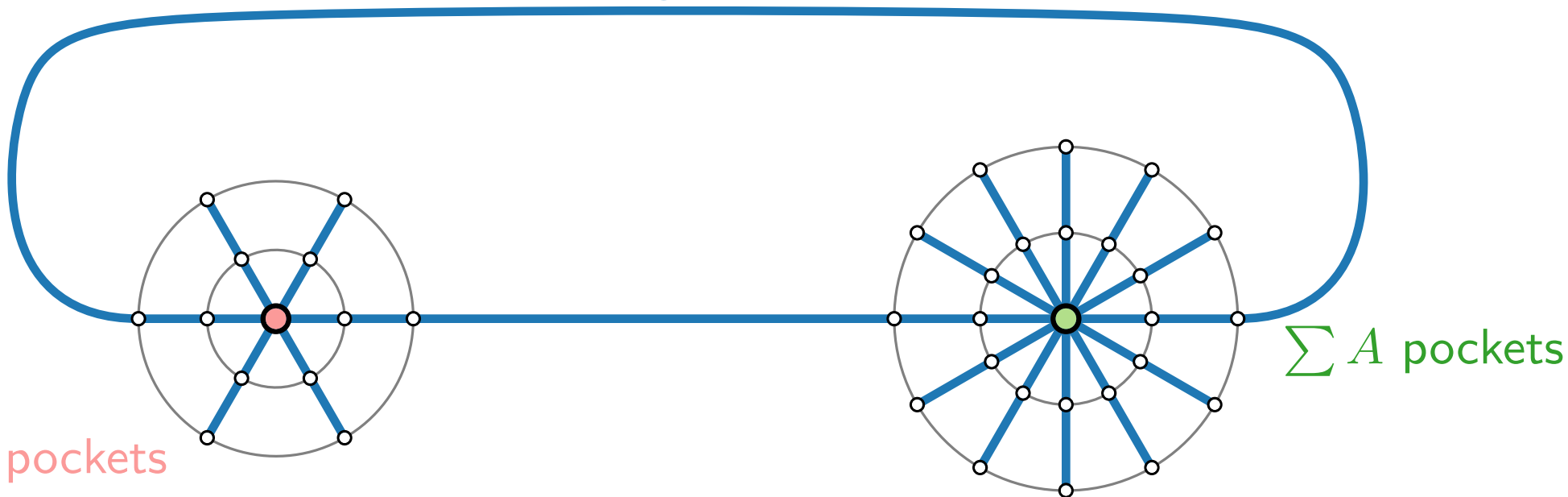
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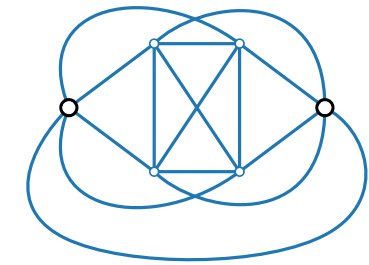
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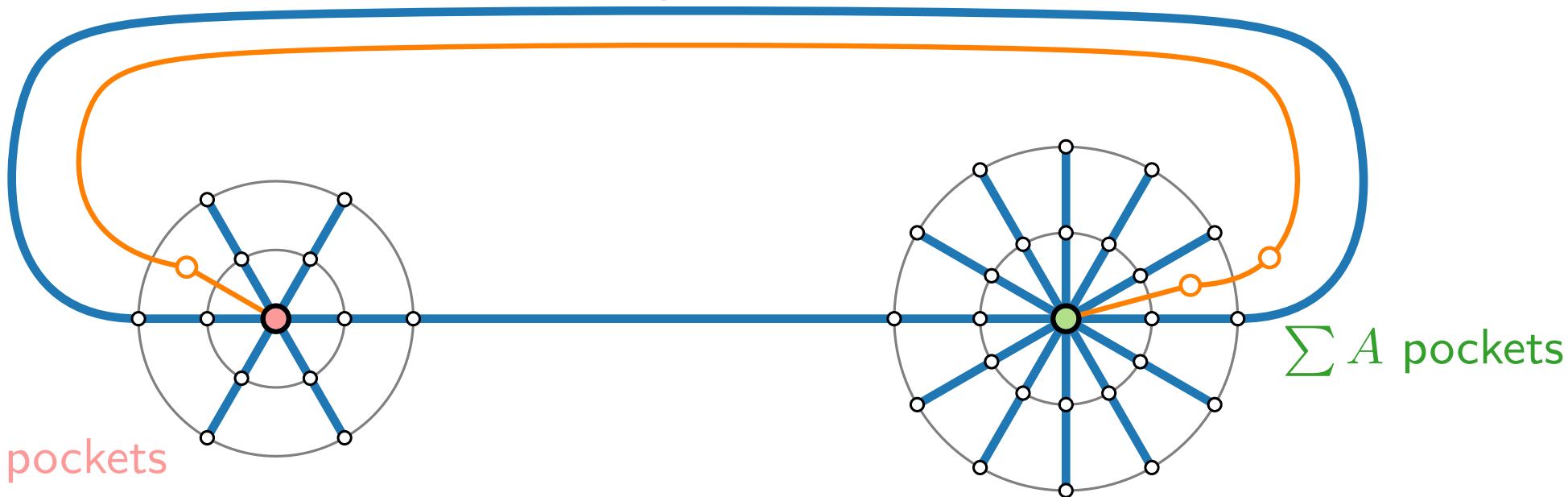
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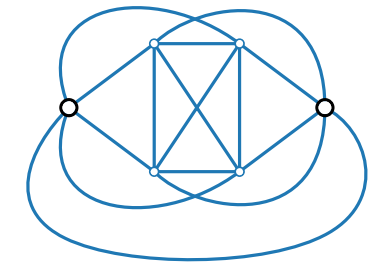
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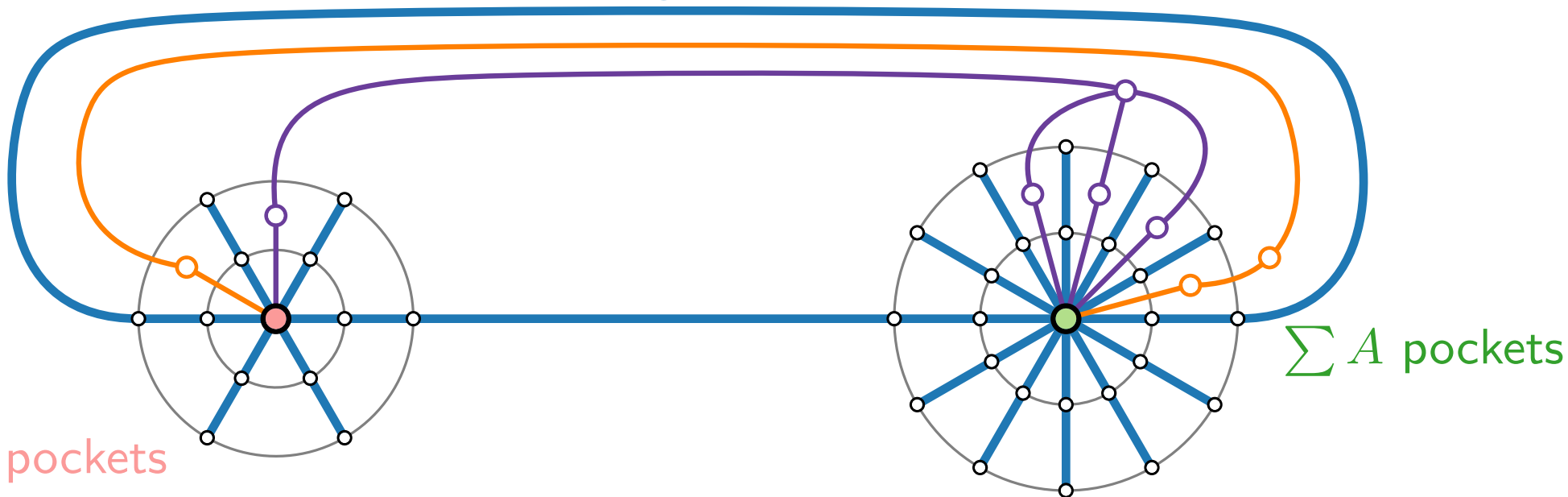
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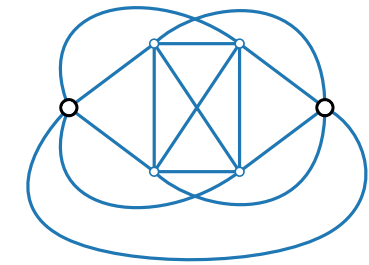
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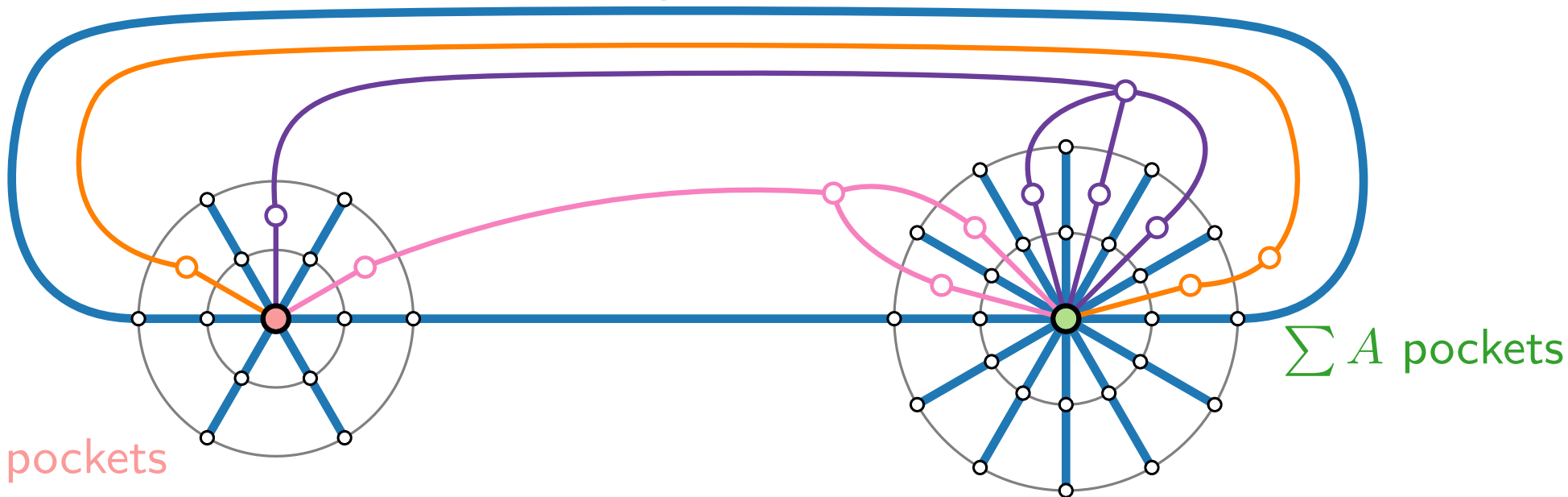
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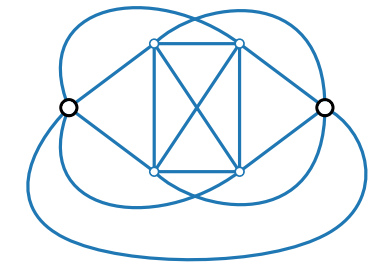
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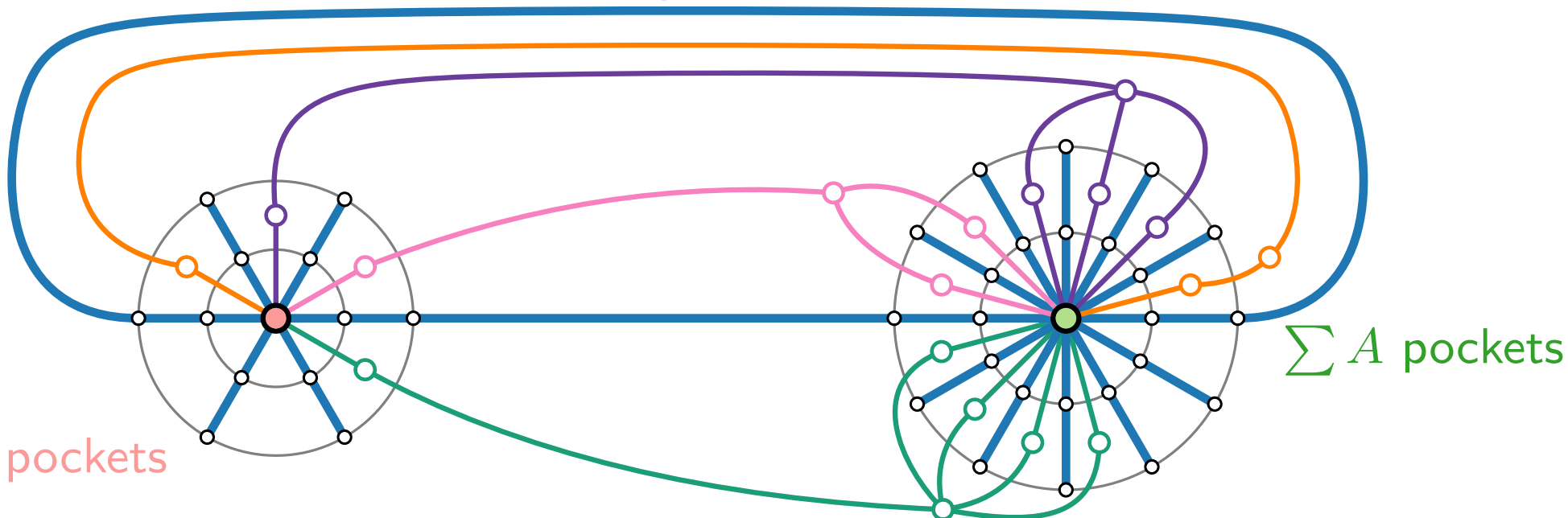
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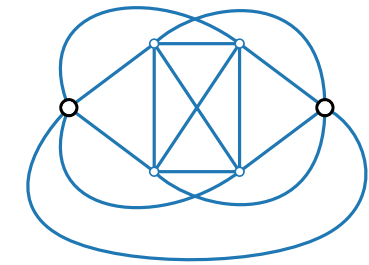
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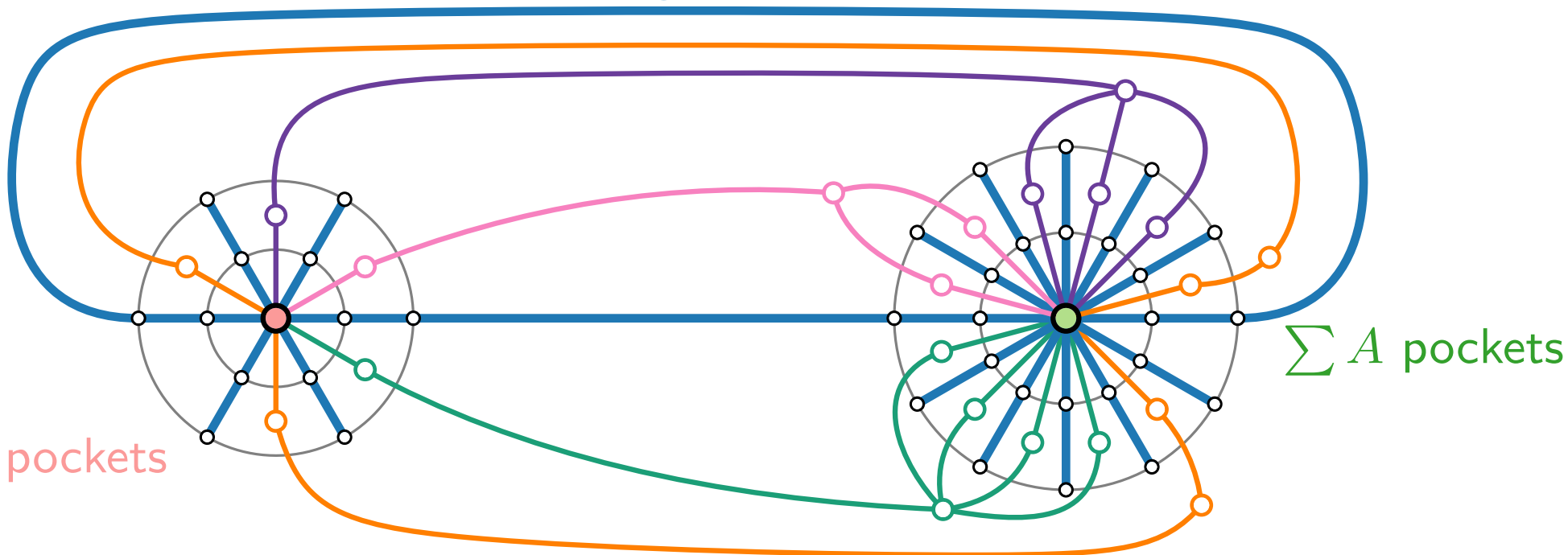
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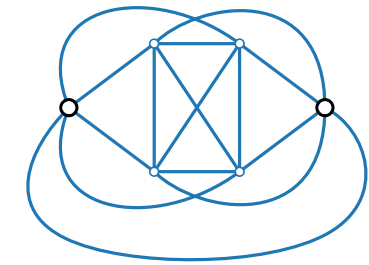
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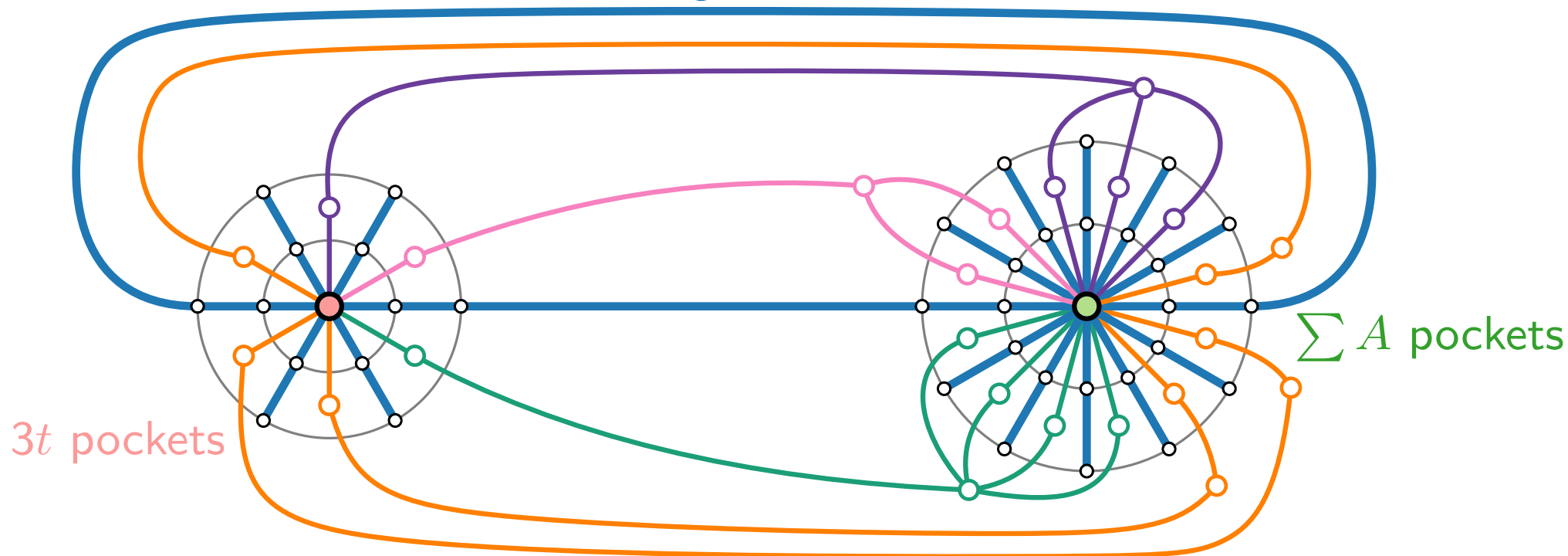
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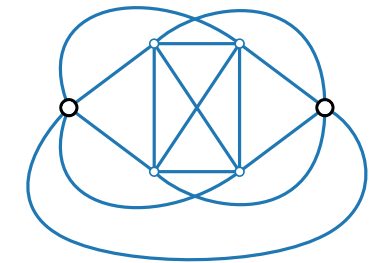
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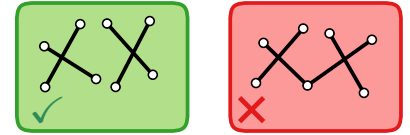
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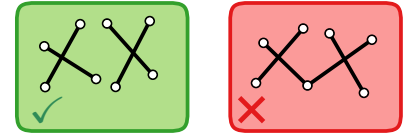


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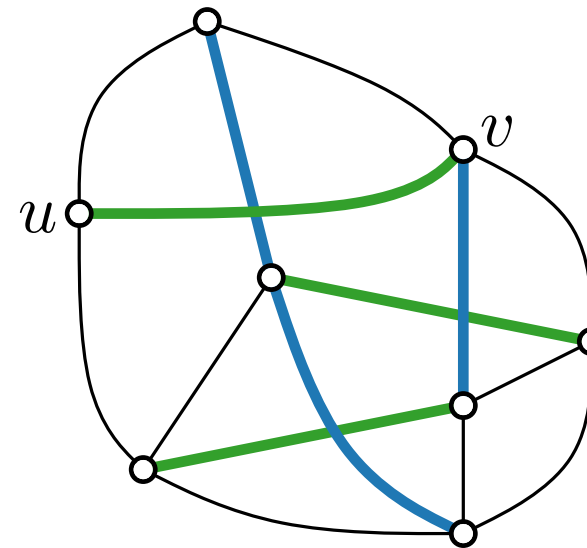
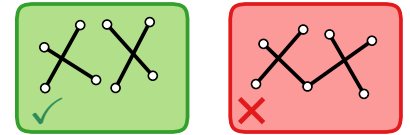


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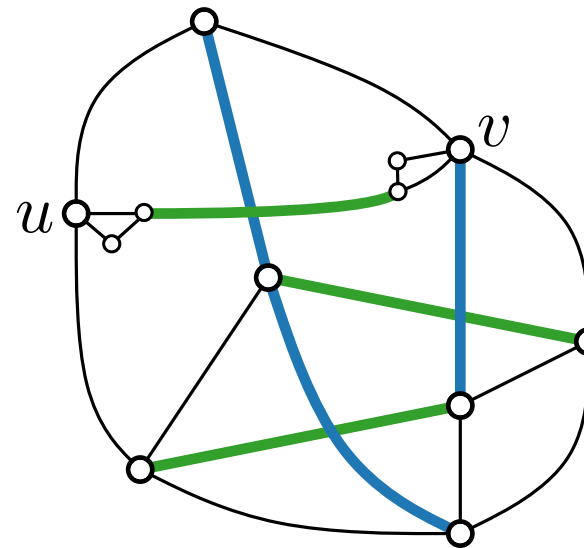
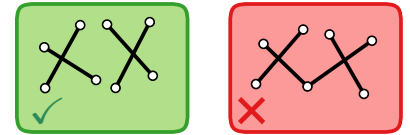


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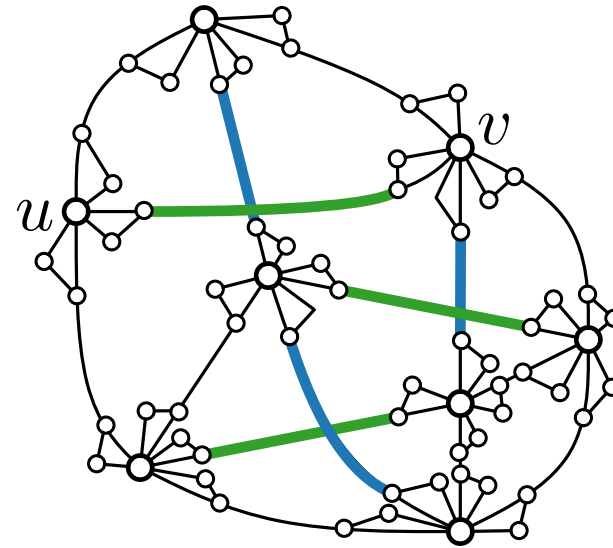
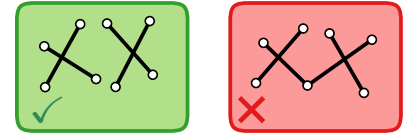


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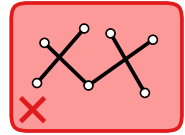
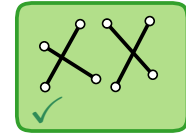
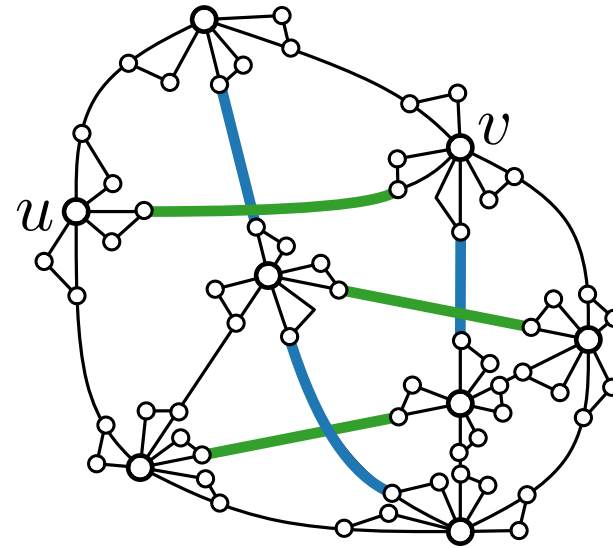
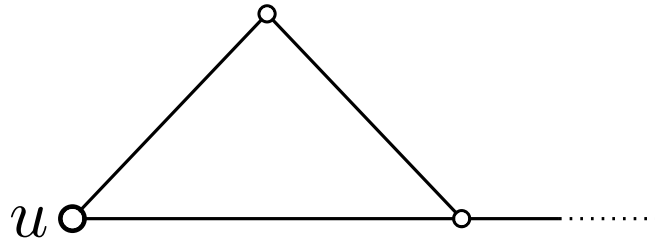


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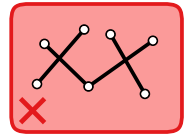
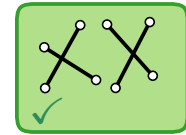
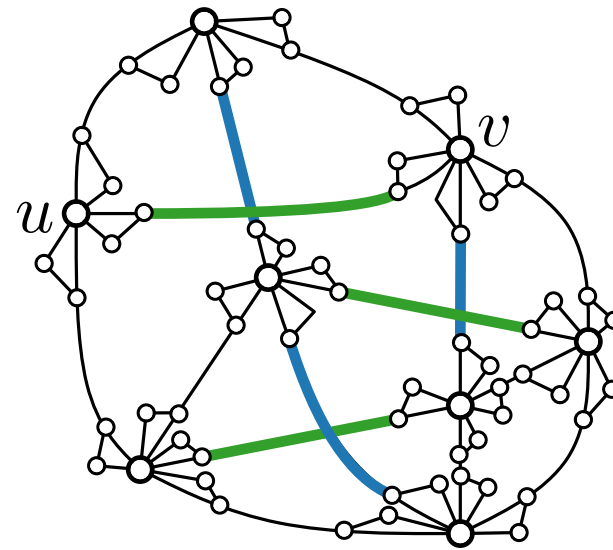
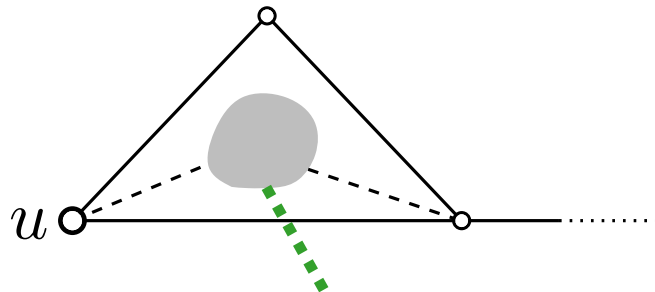


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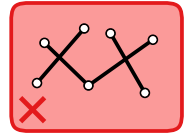
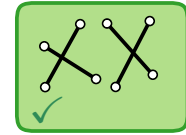
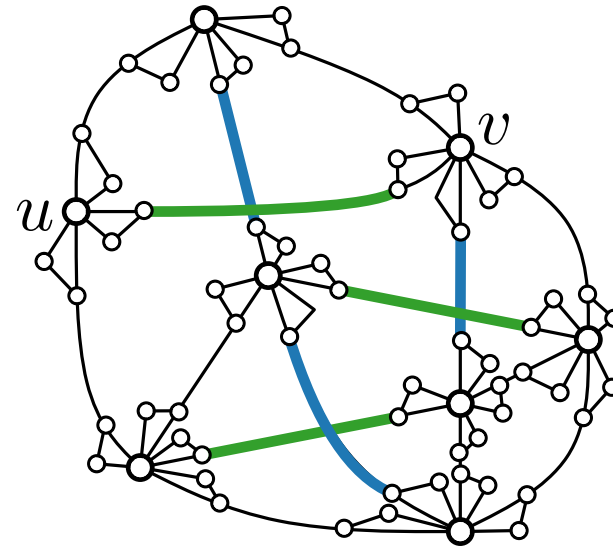
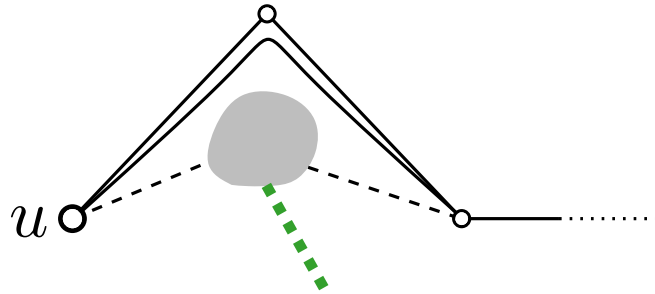


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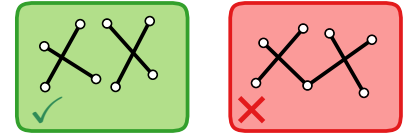
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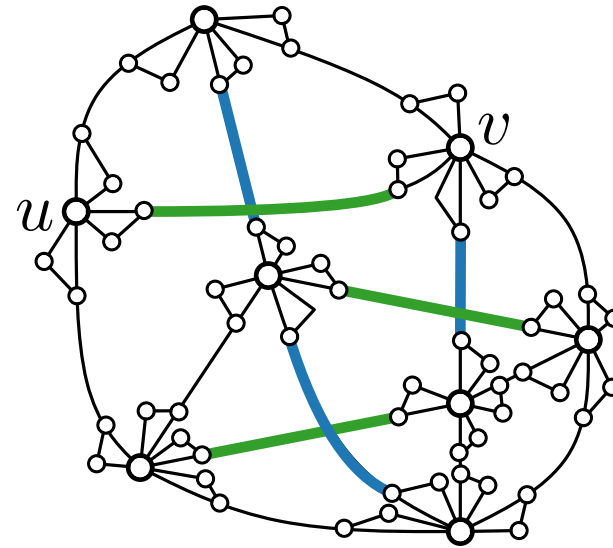
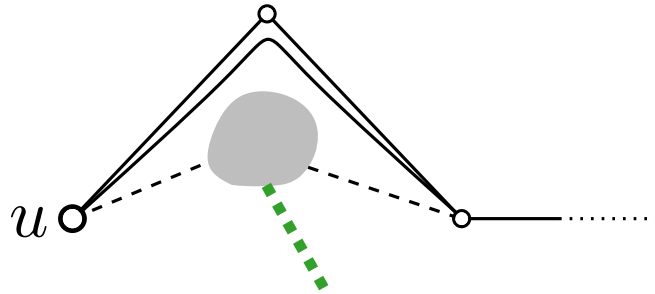
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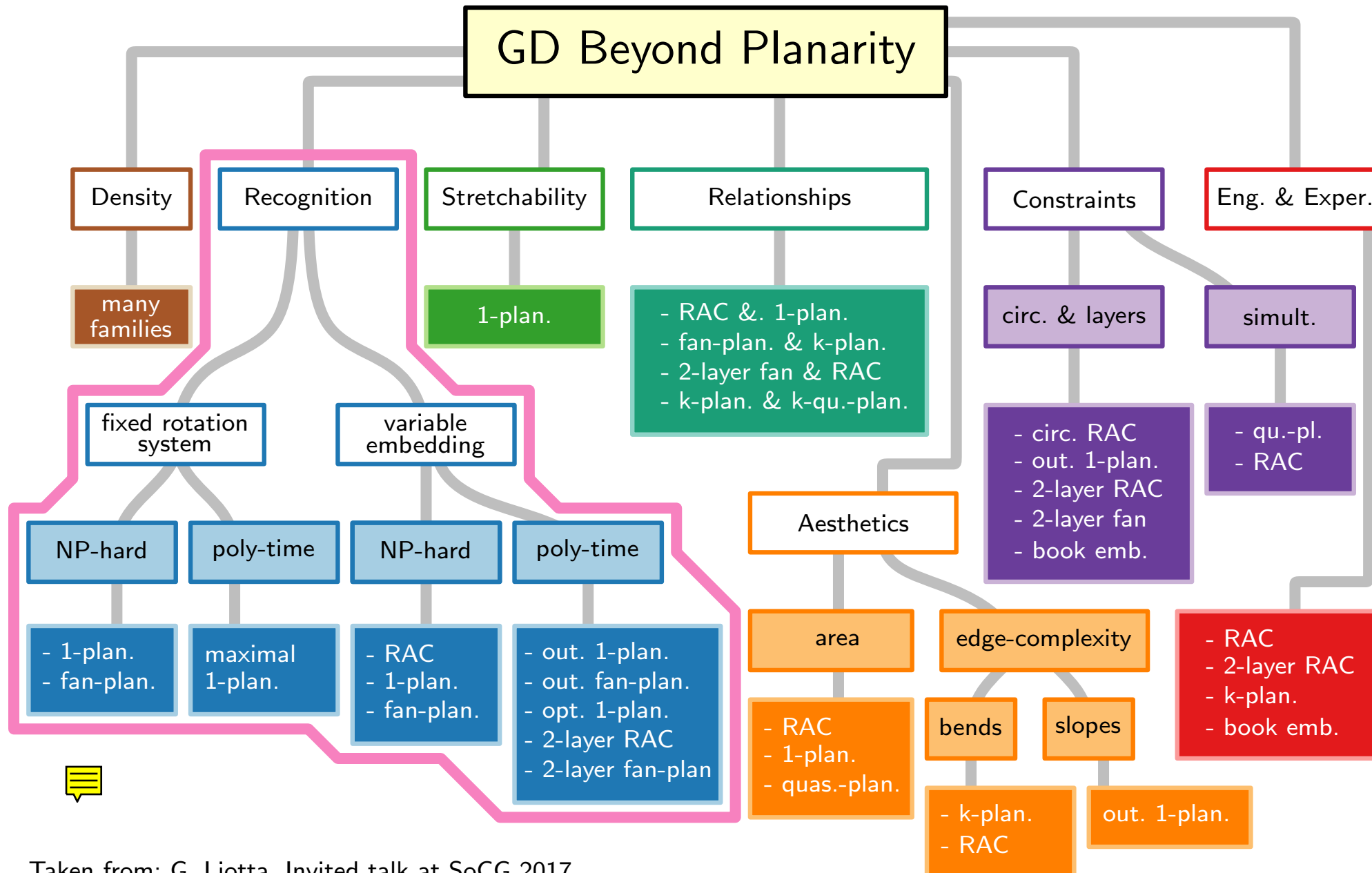
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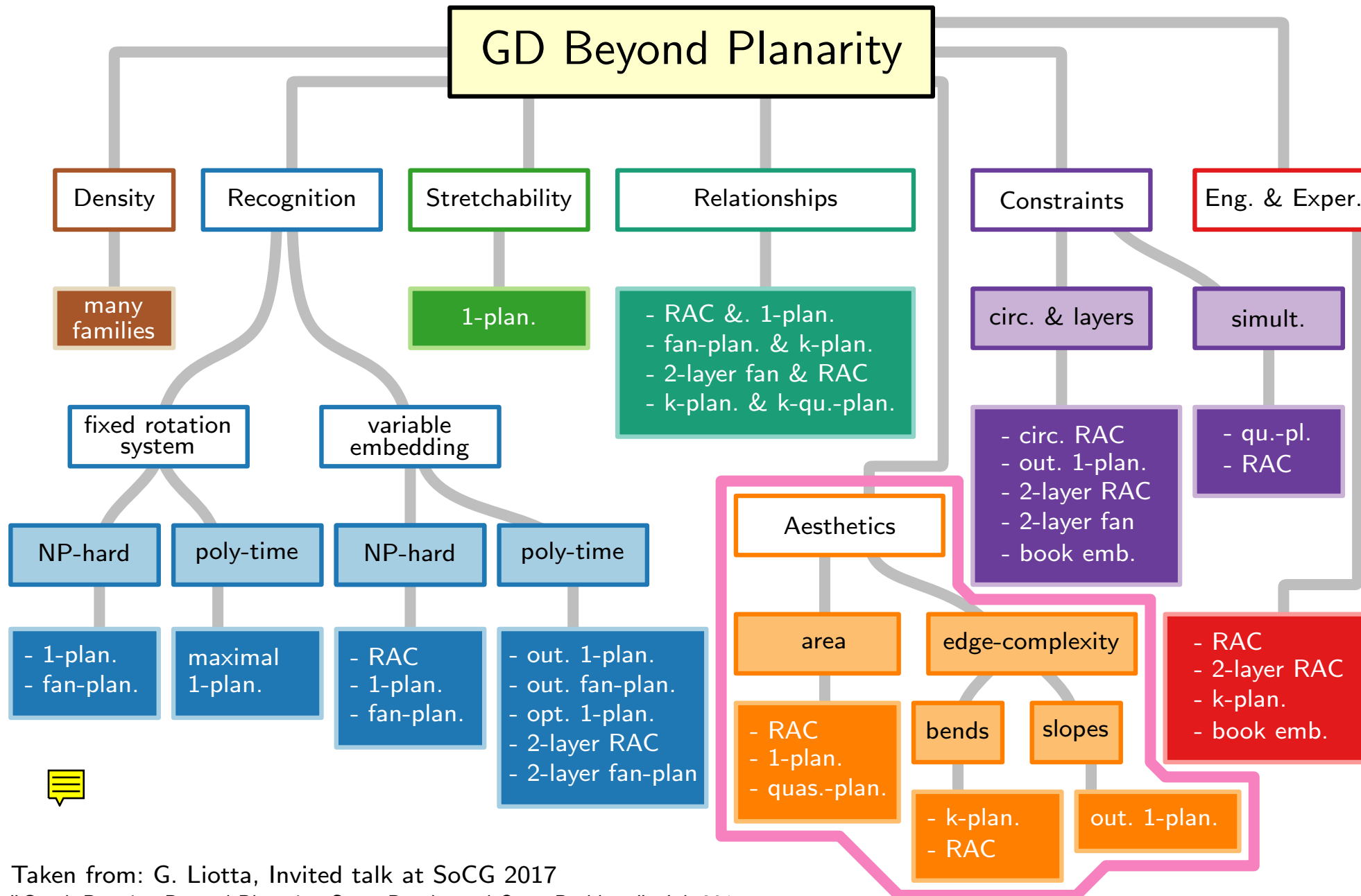
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

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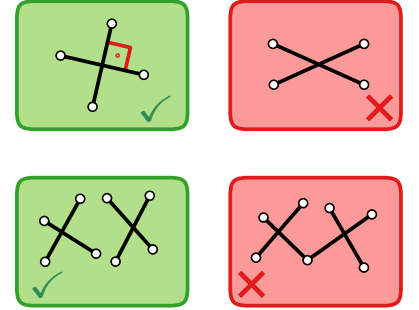


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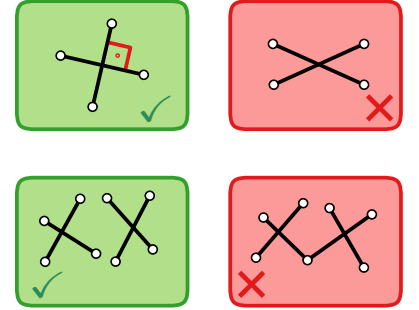
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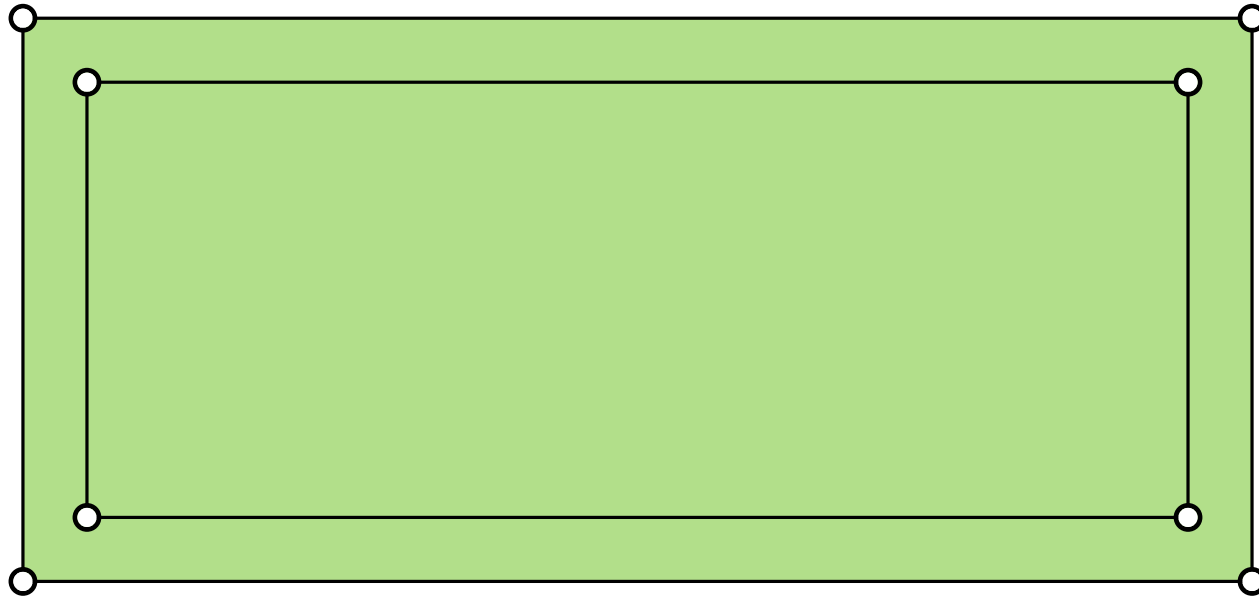
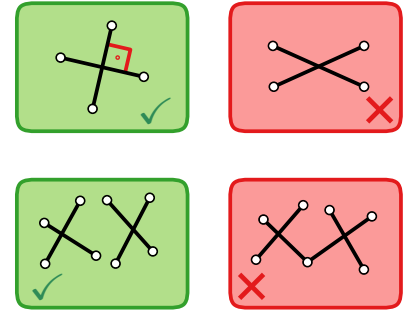
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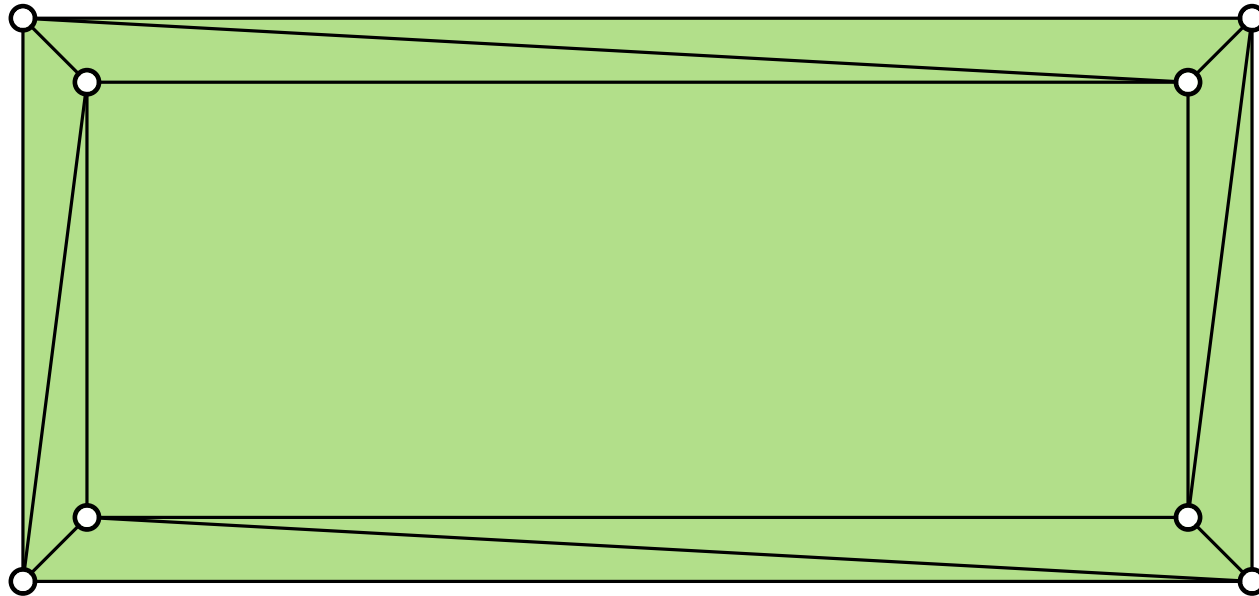
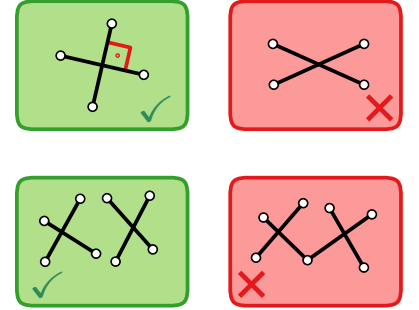
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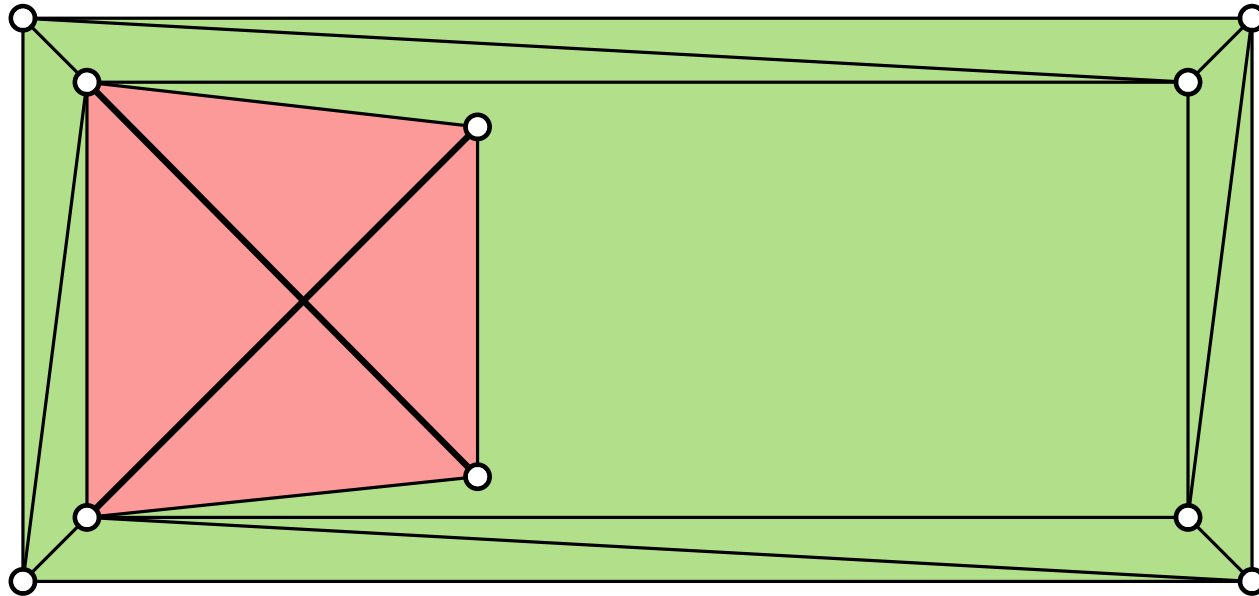
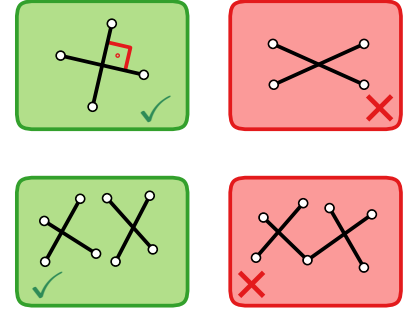
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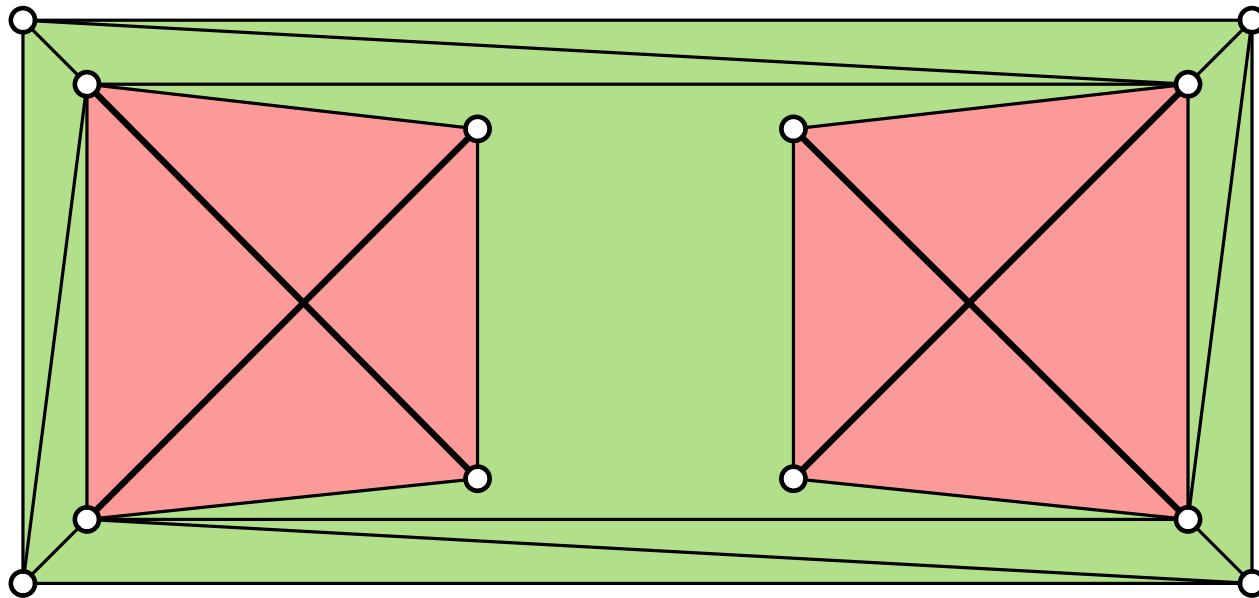
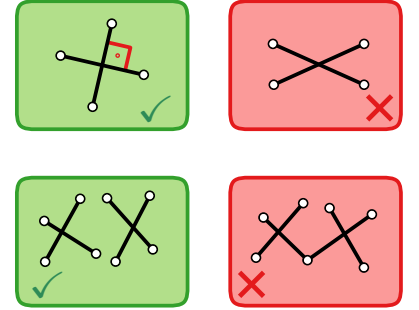
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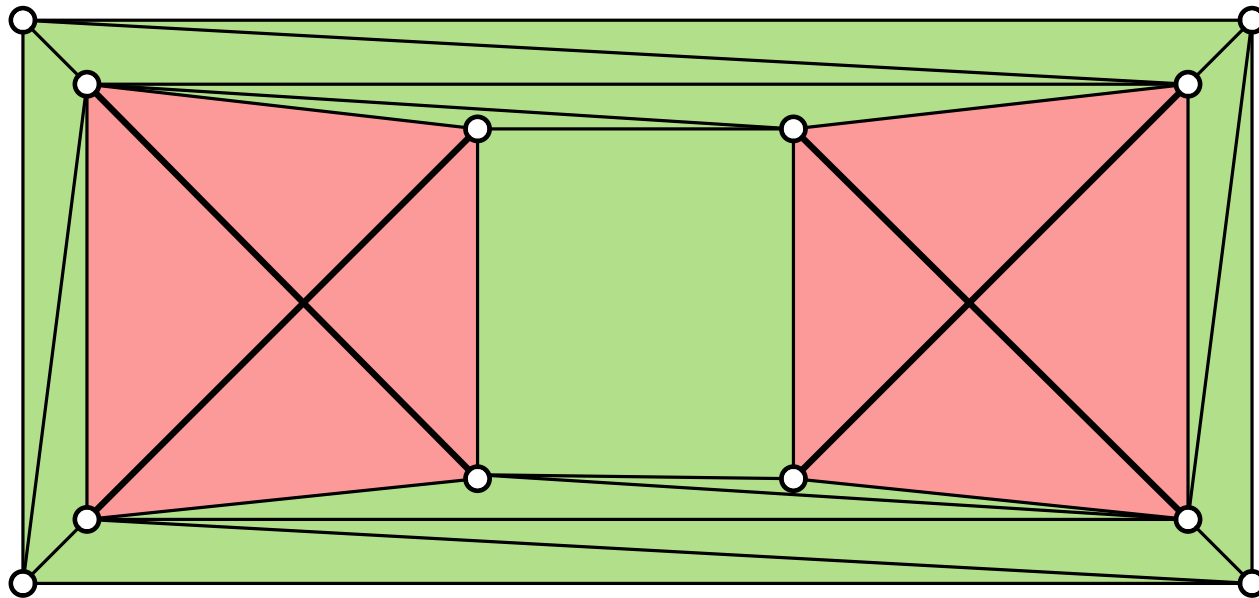
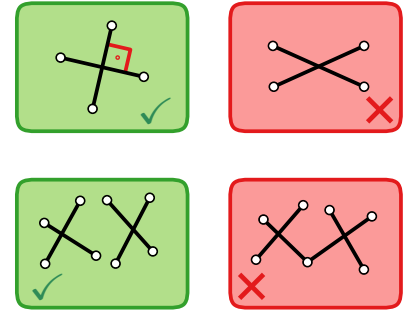
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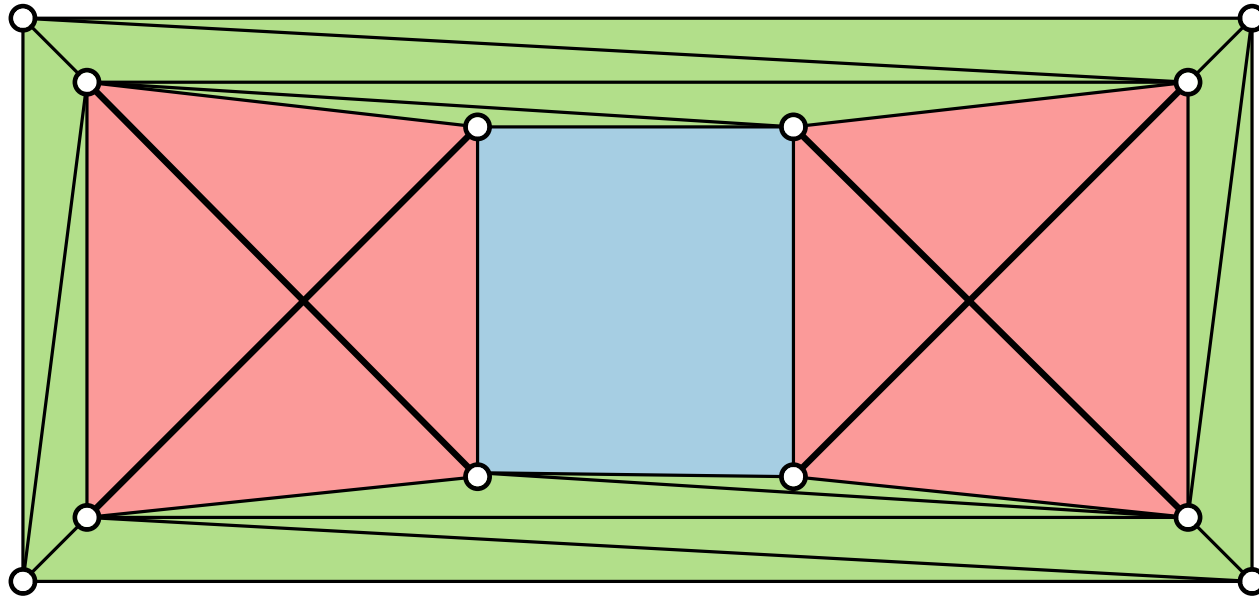
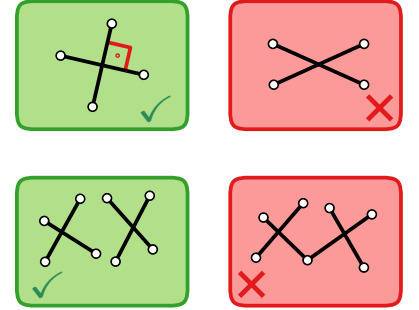
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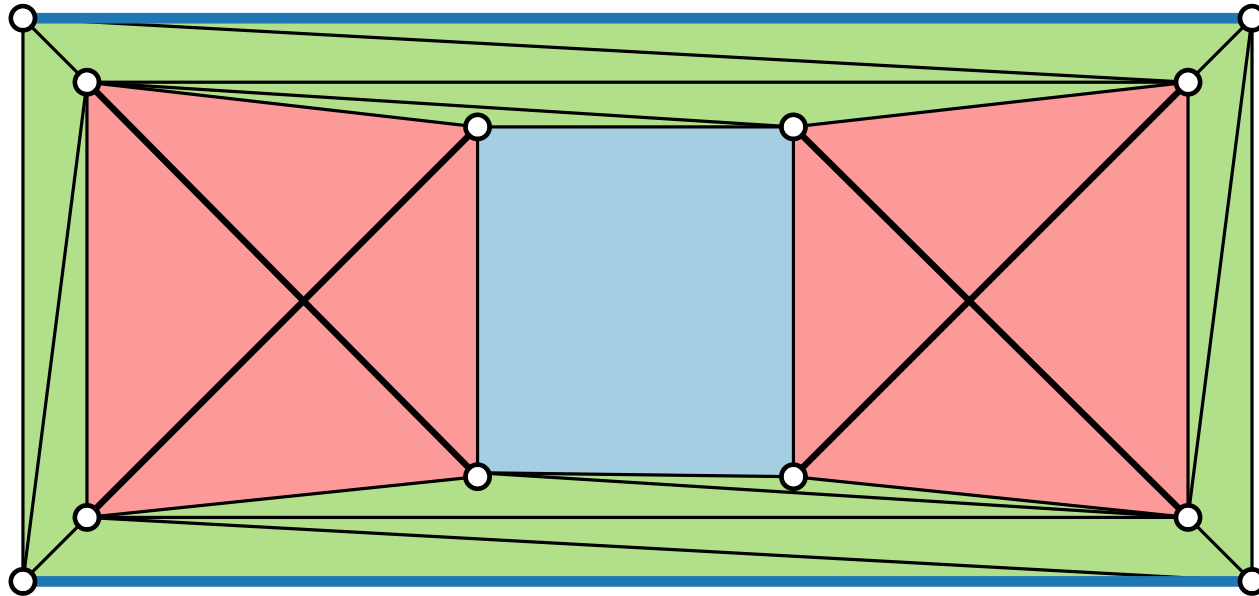
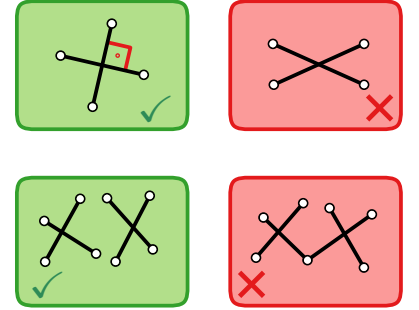
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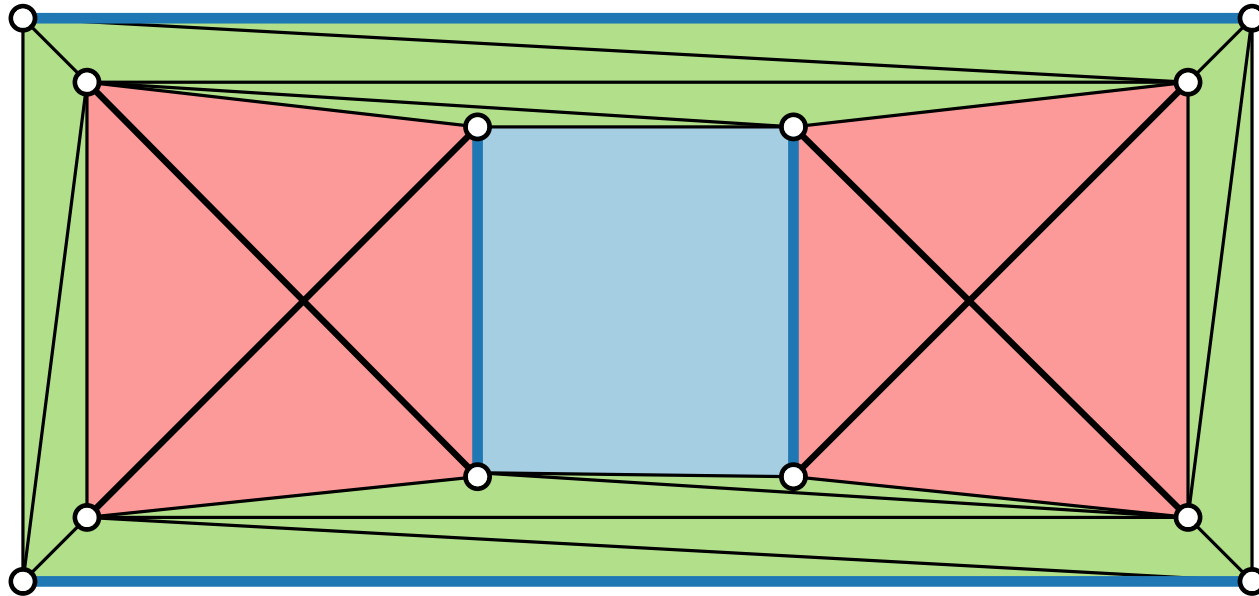
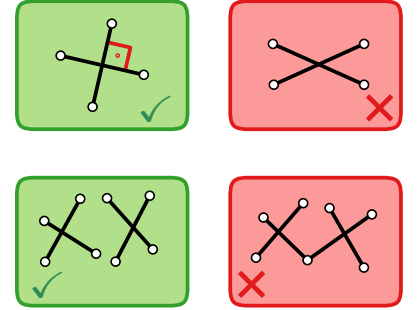
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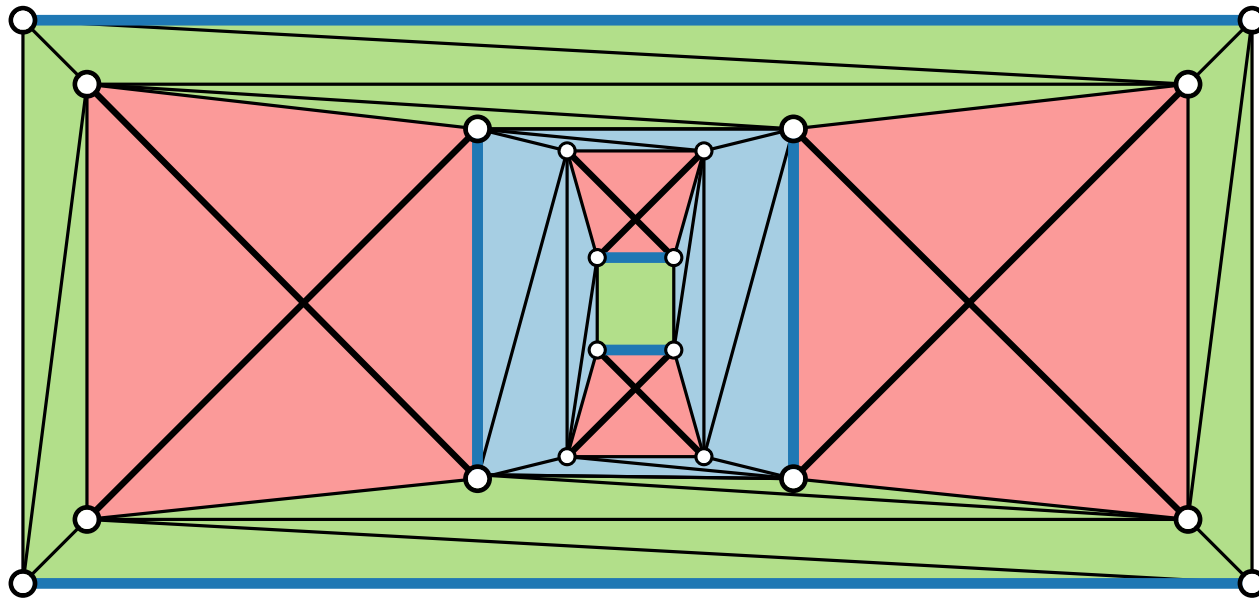
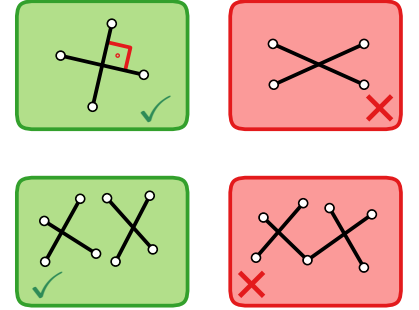
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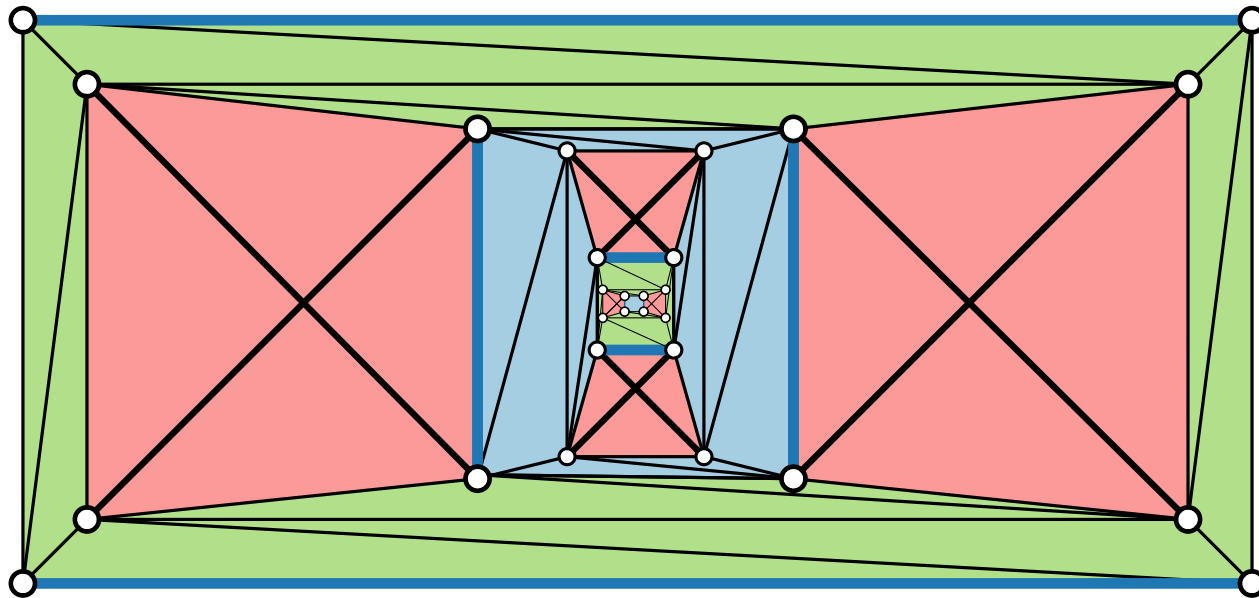
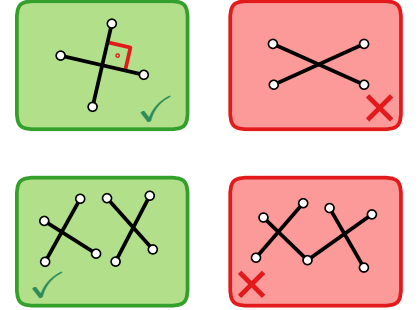
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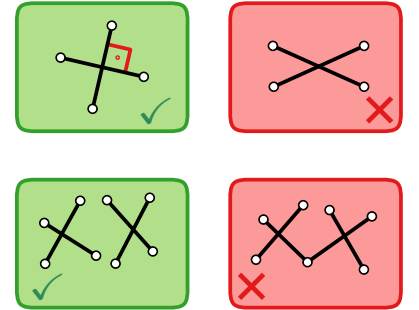
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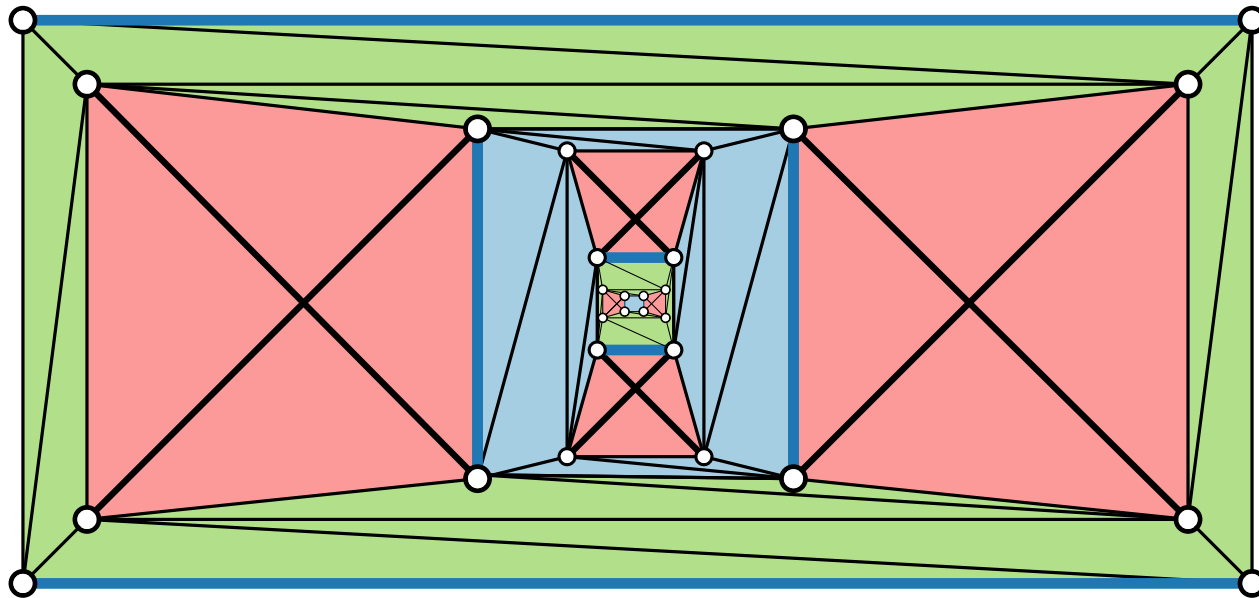


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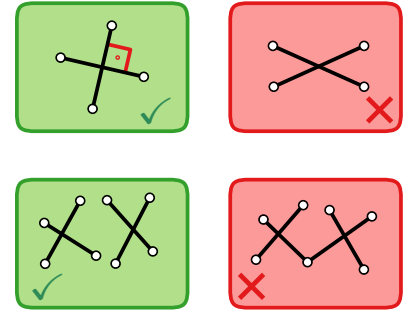


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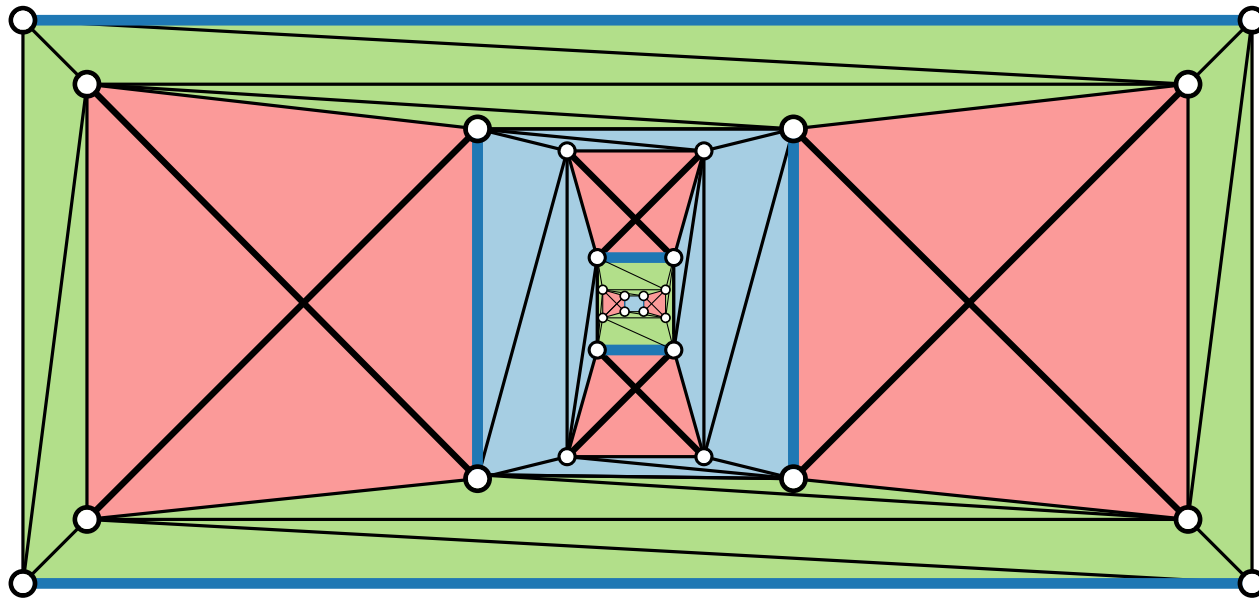


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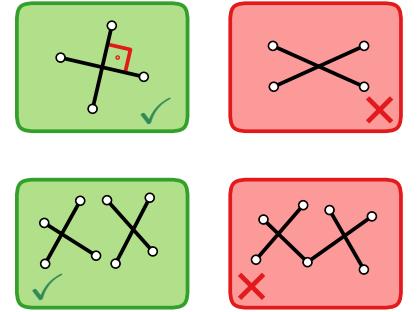
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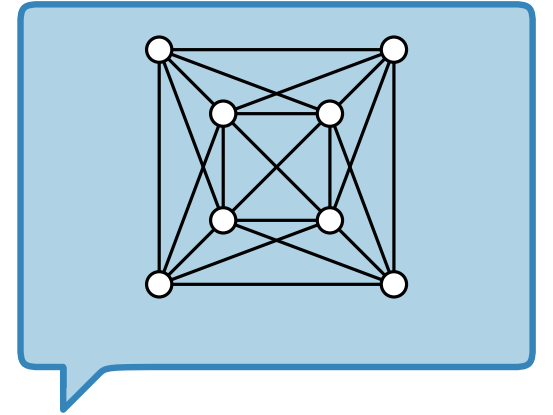
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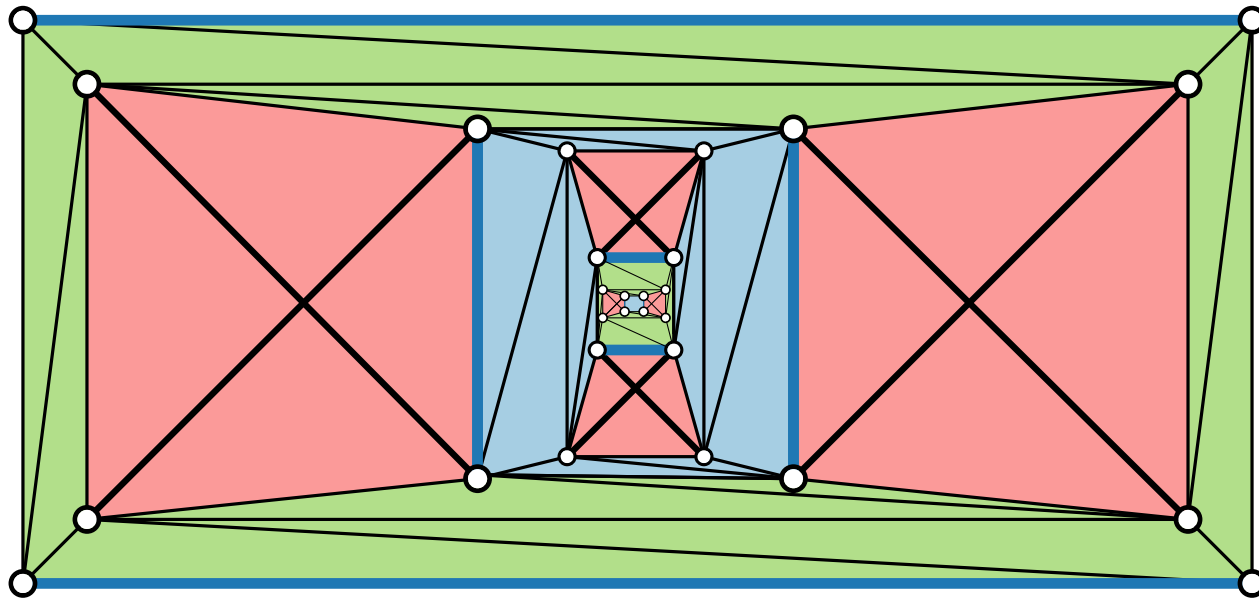
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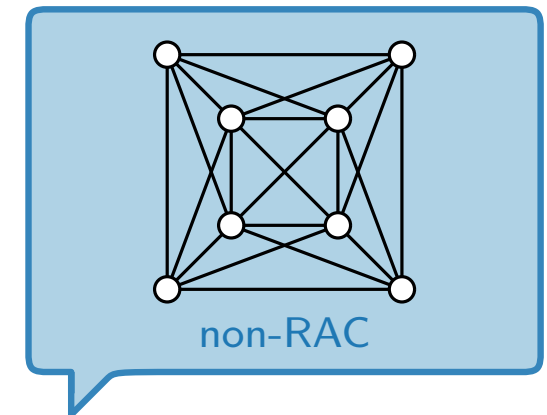
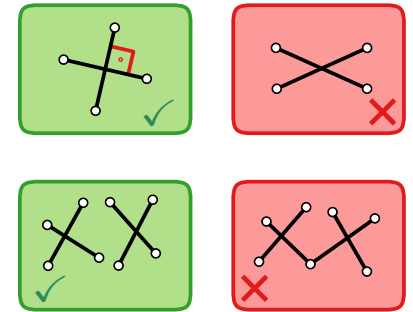
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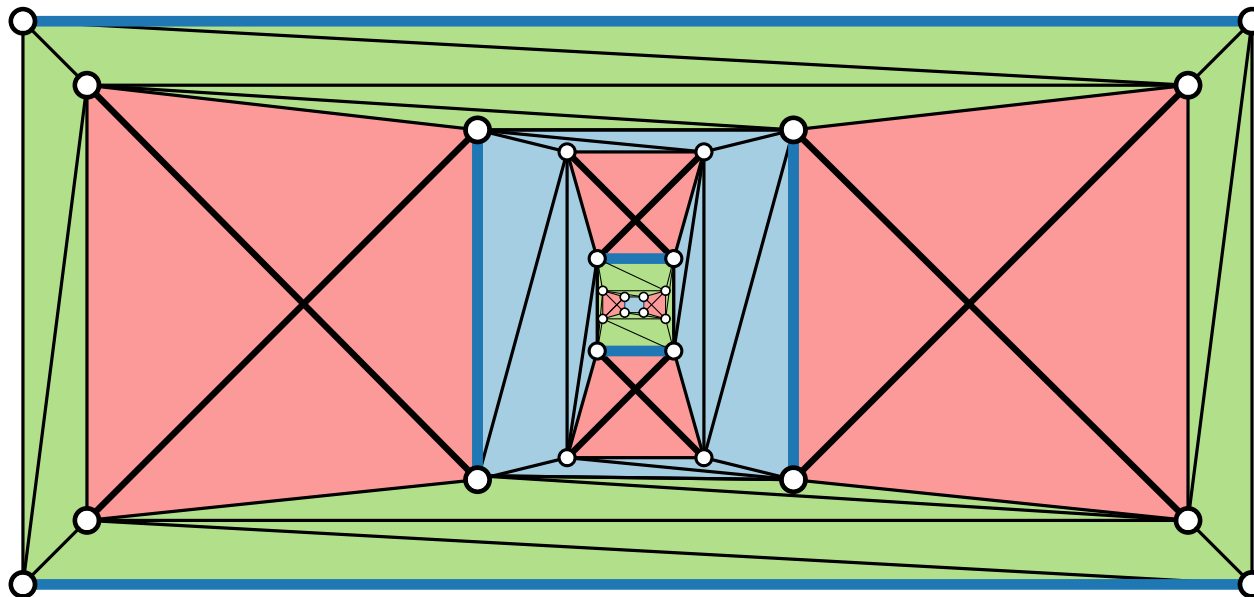
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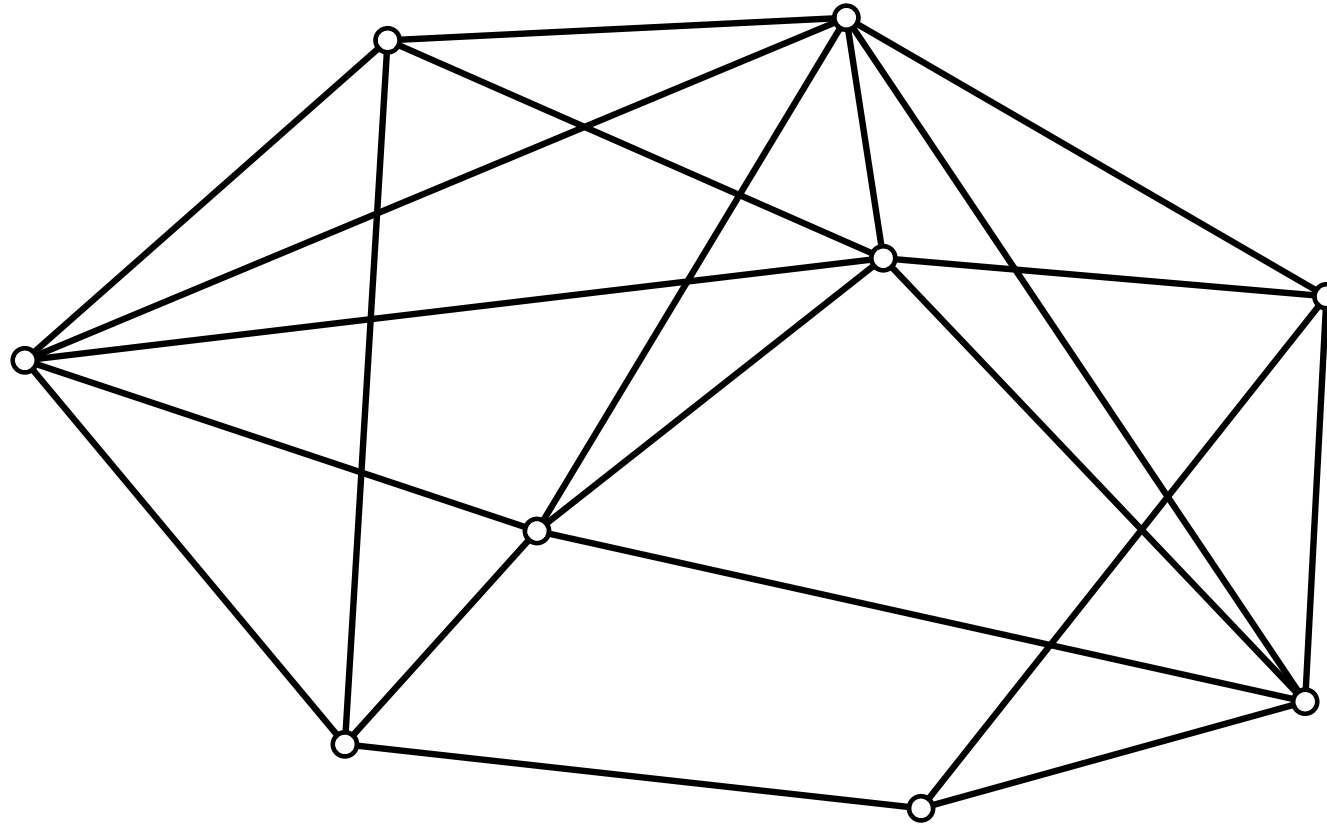
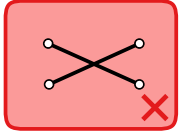
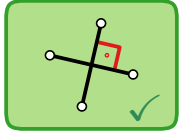
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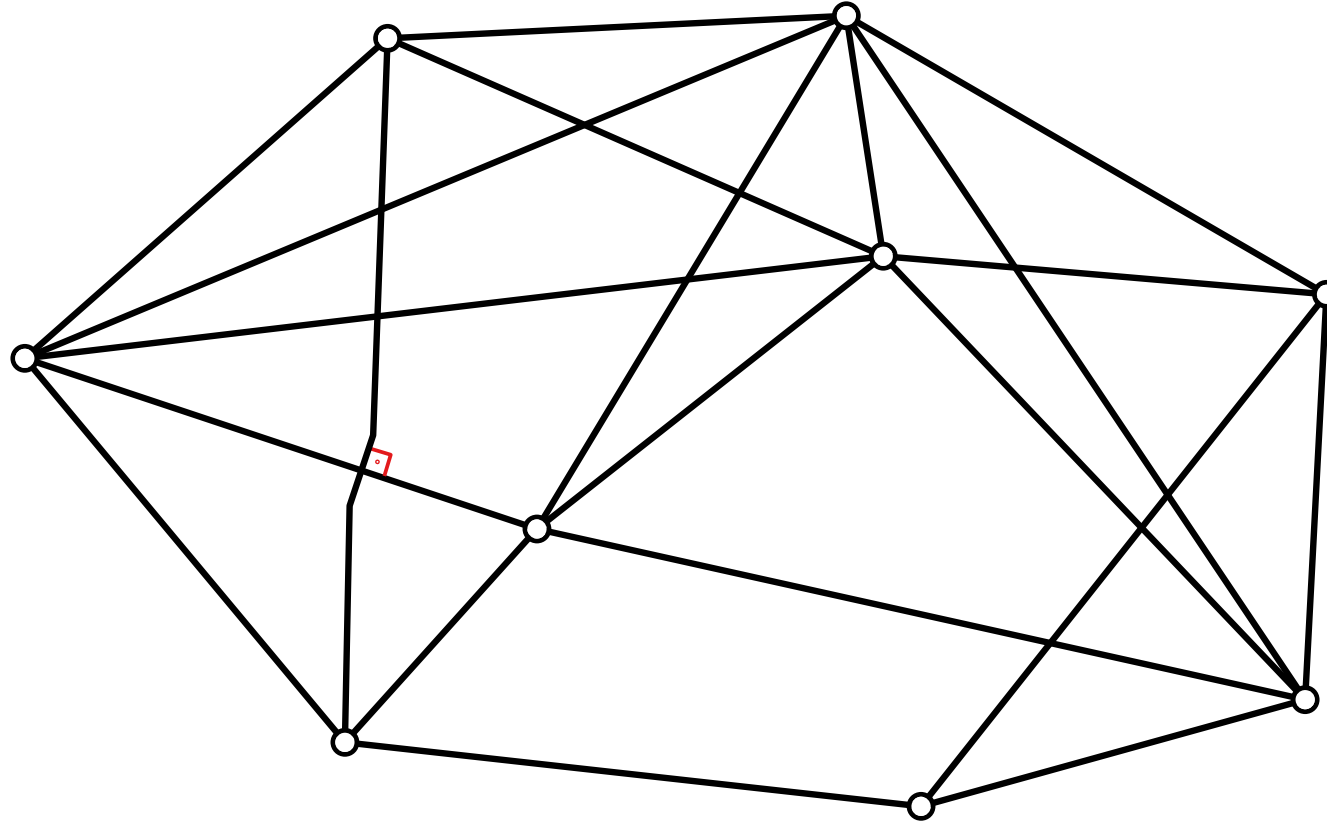
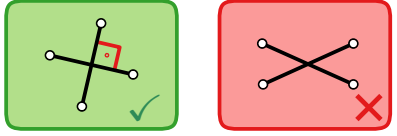
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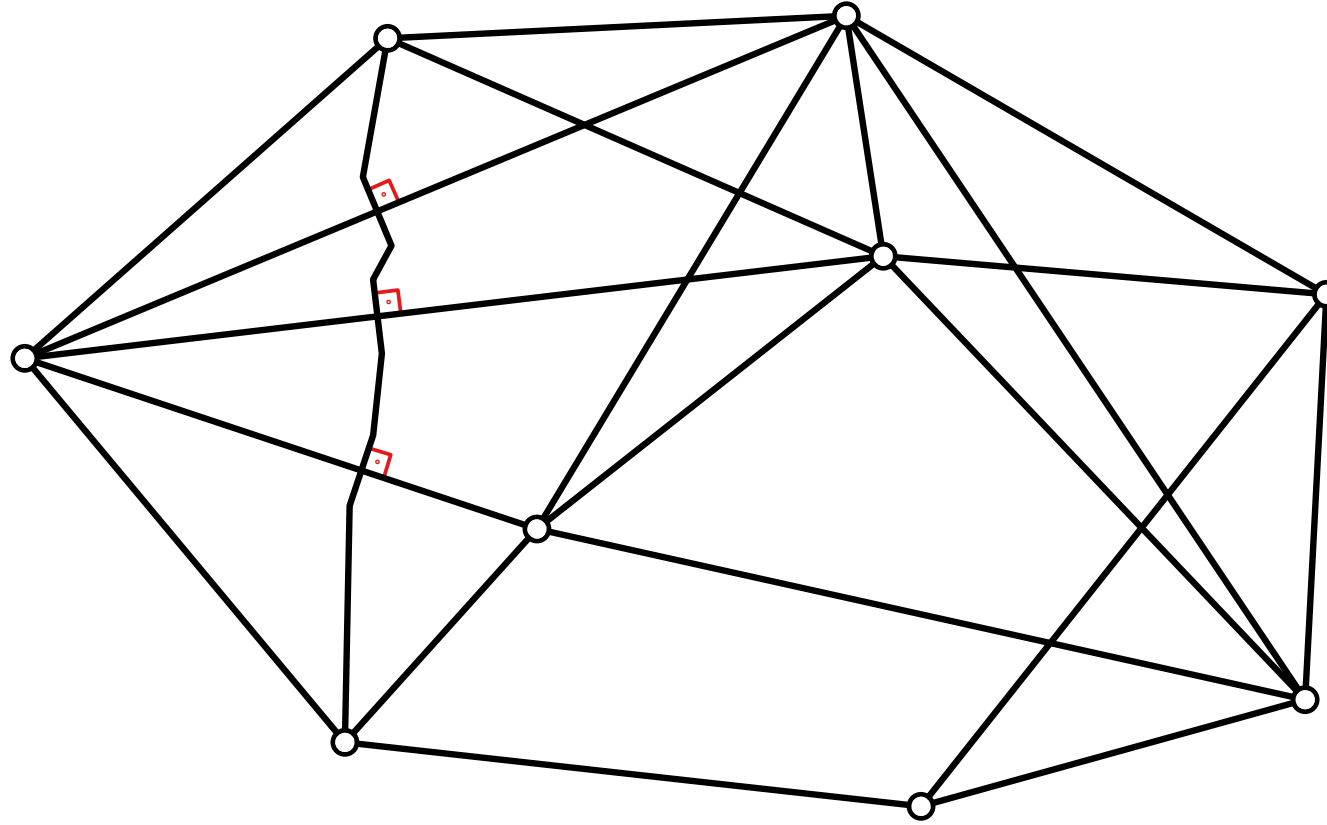
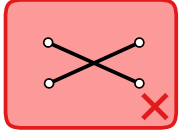
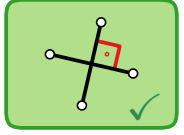
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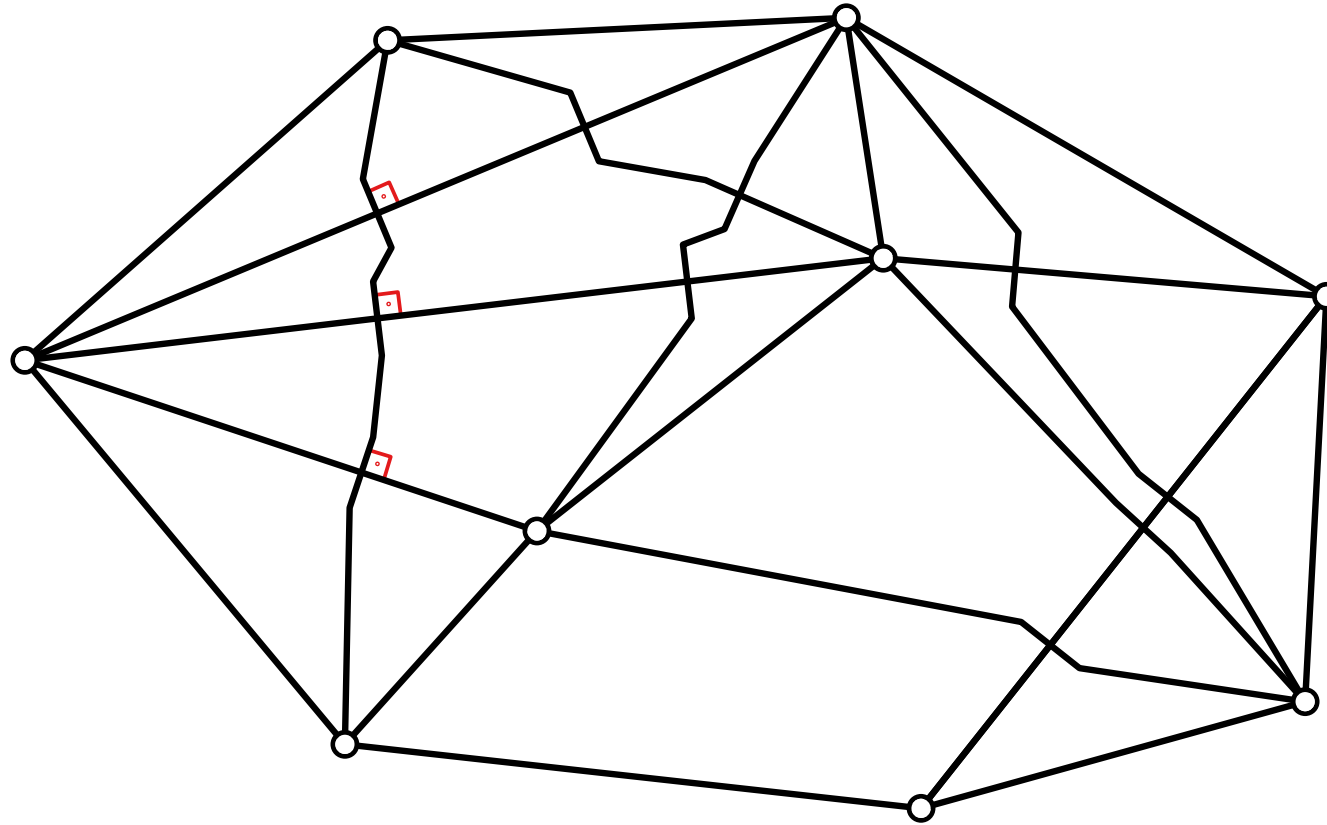
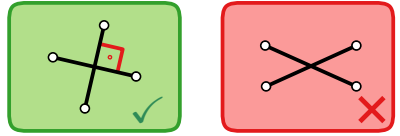
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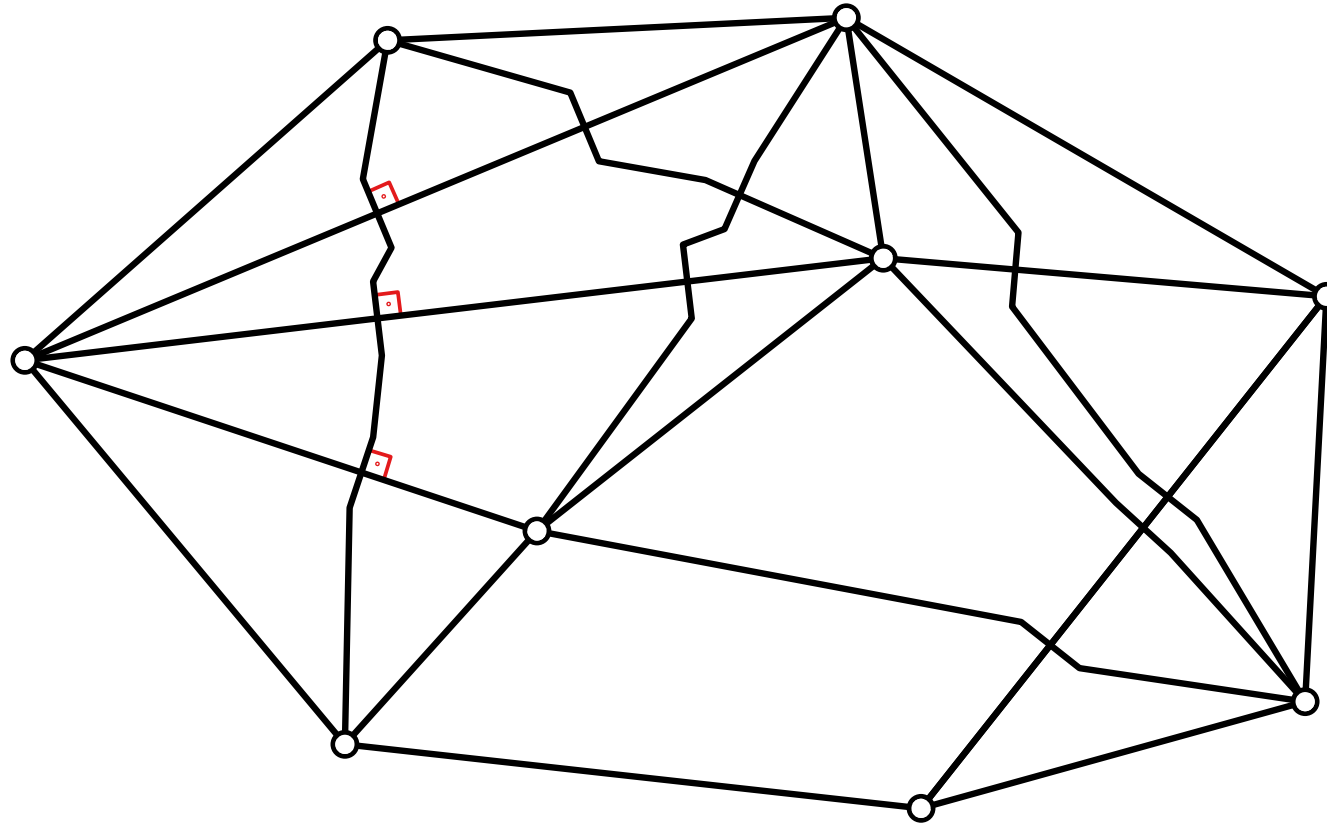
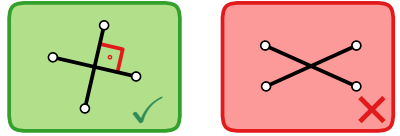
RAC Drawings



RAC Drawings

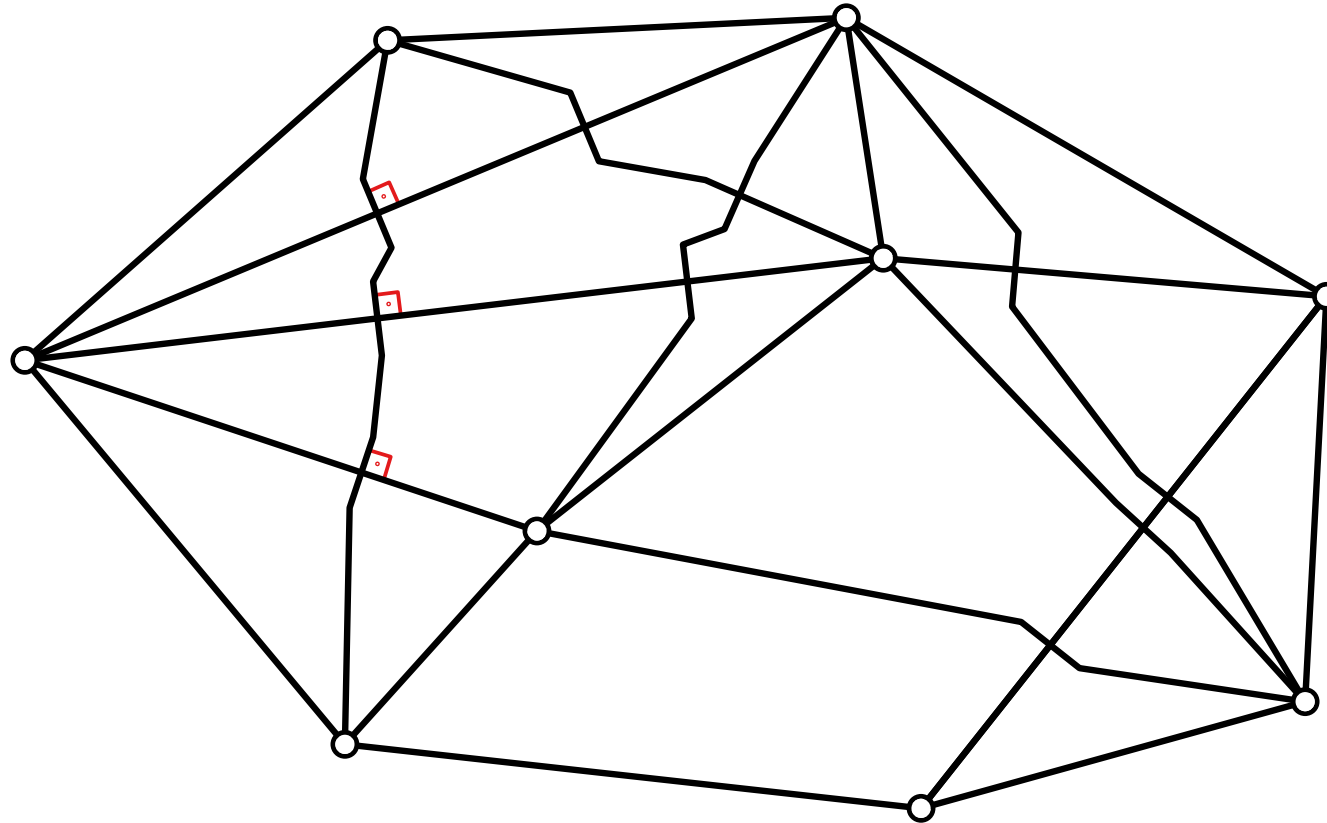
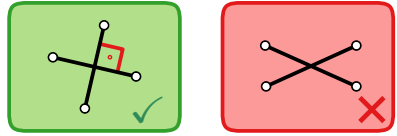


RAC Drawings



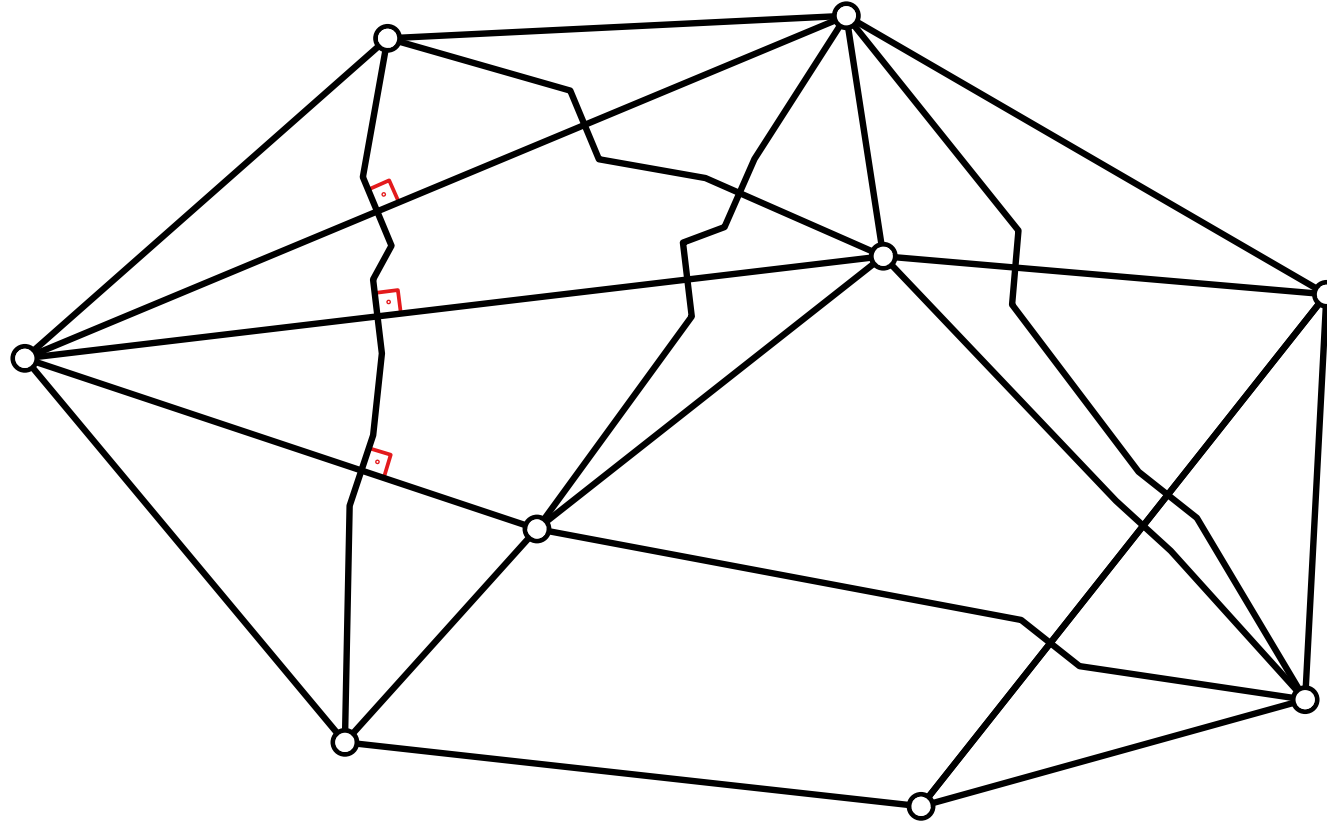
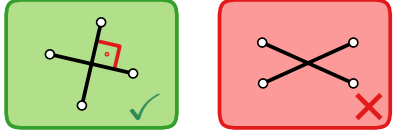
Every graph admits a RAC drawing ...

RAC Drawings With Enough Bends



Every graph admits a RAC drawing ...
...if we use enough bends.

RAC Drawings With Enough Bends



Every graph admits a RAC drawing ...
...if we use enough bends.

How many do we need – in total or per edge?

3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most three bends.

3-Bend RAC Drawings

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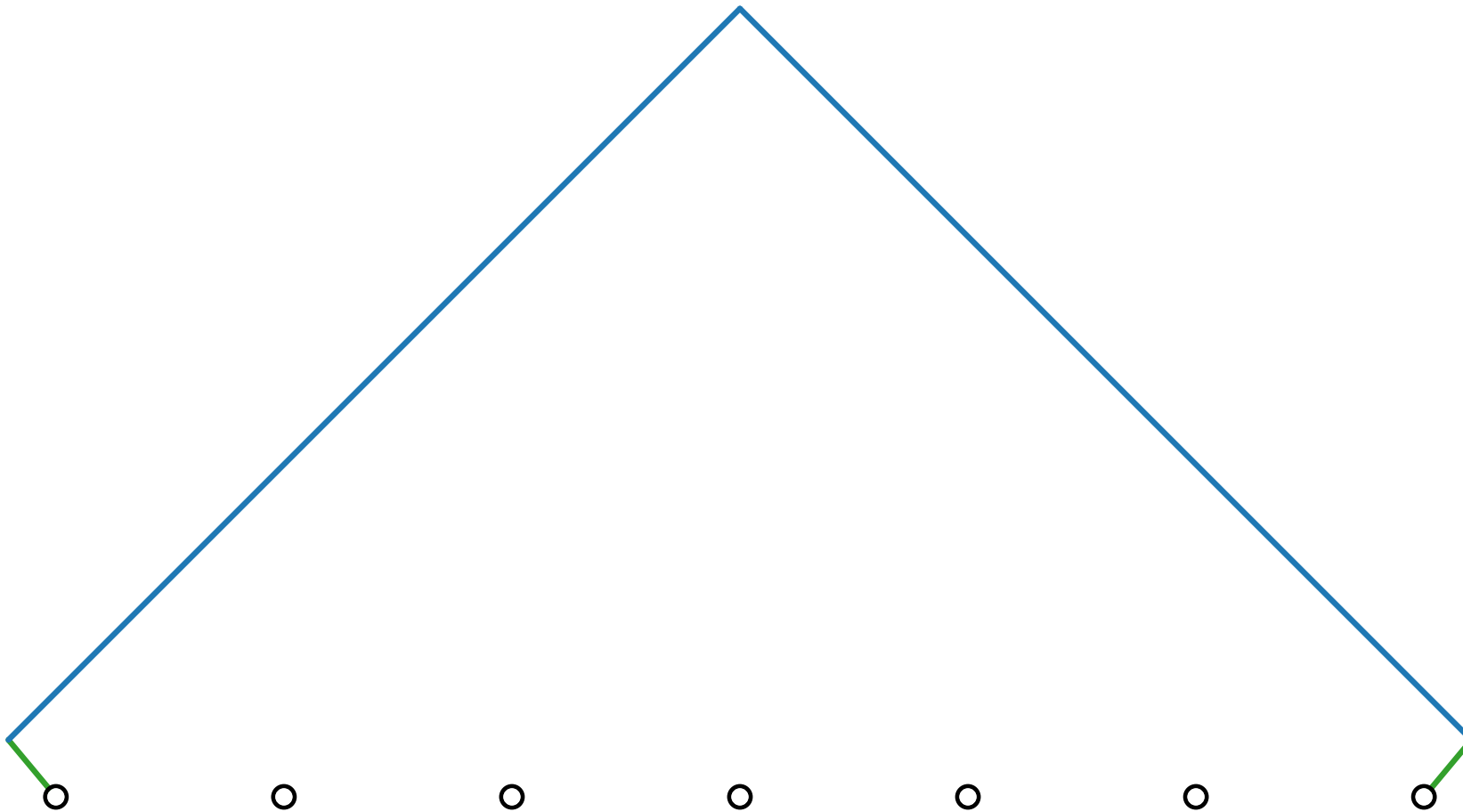
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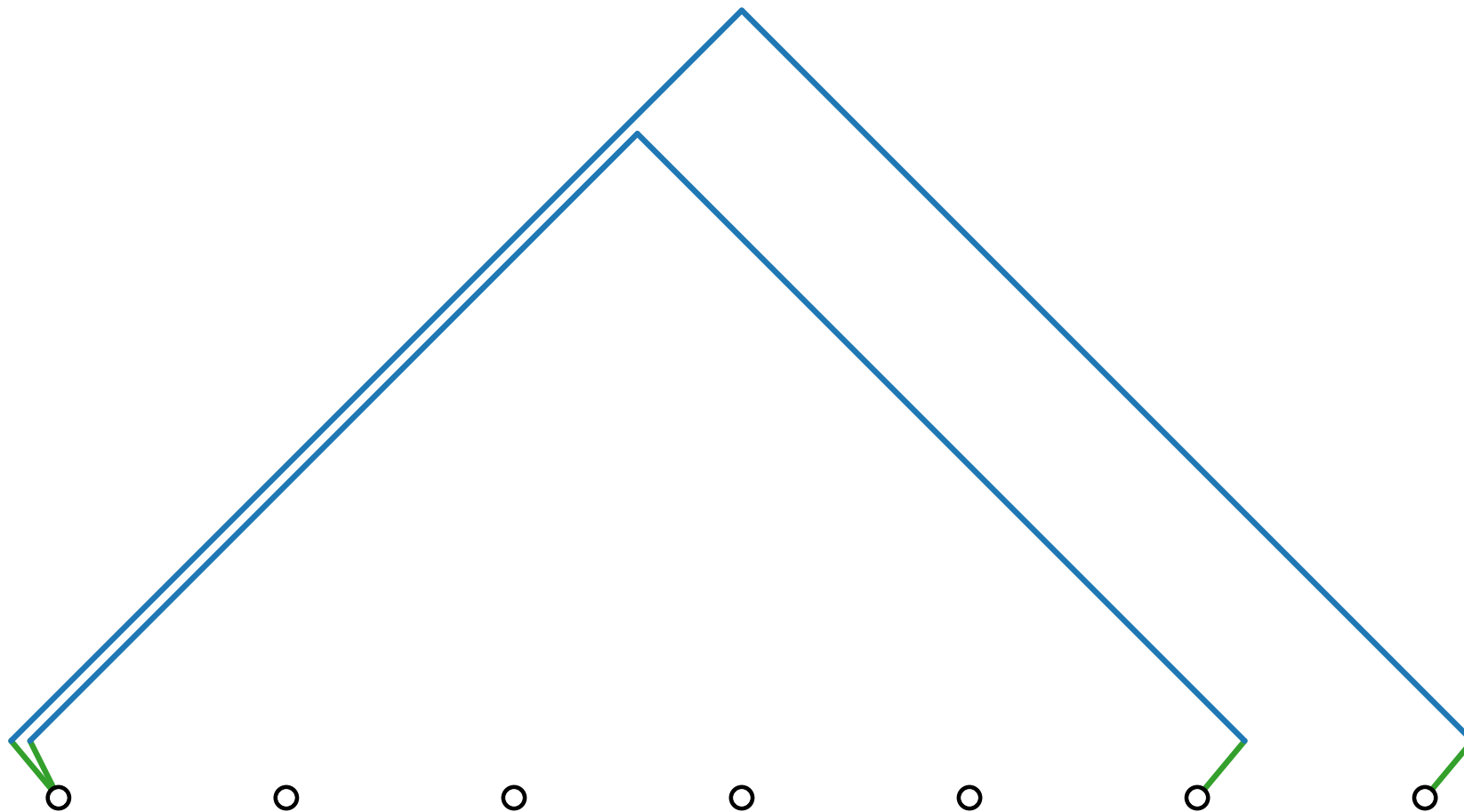


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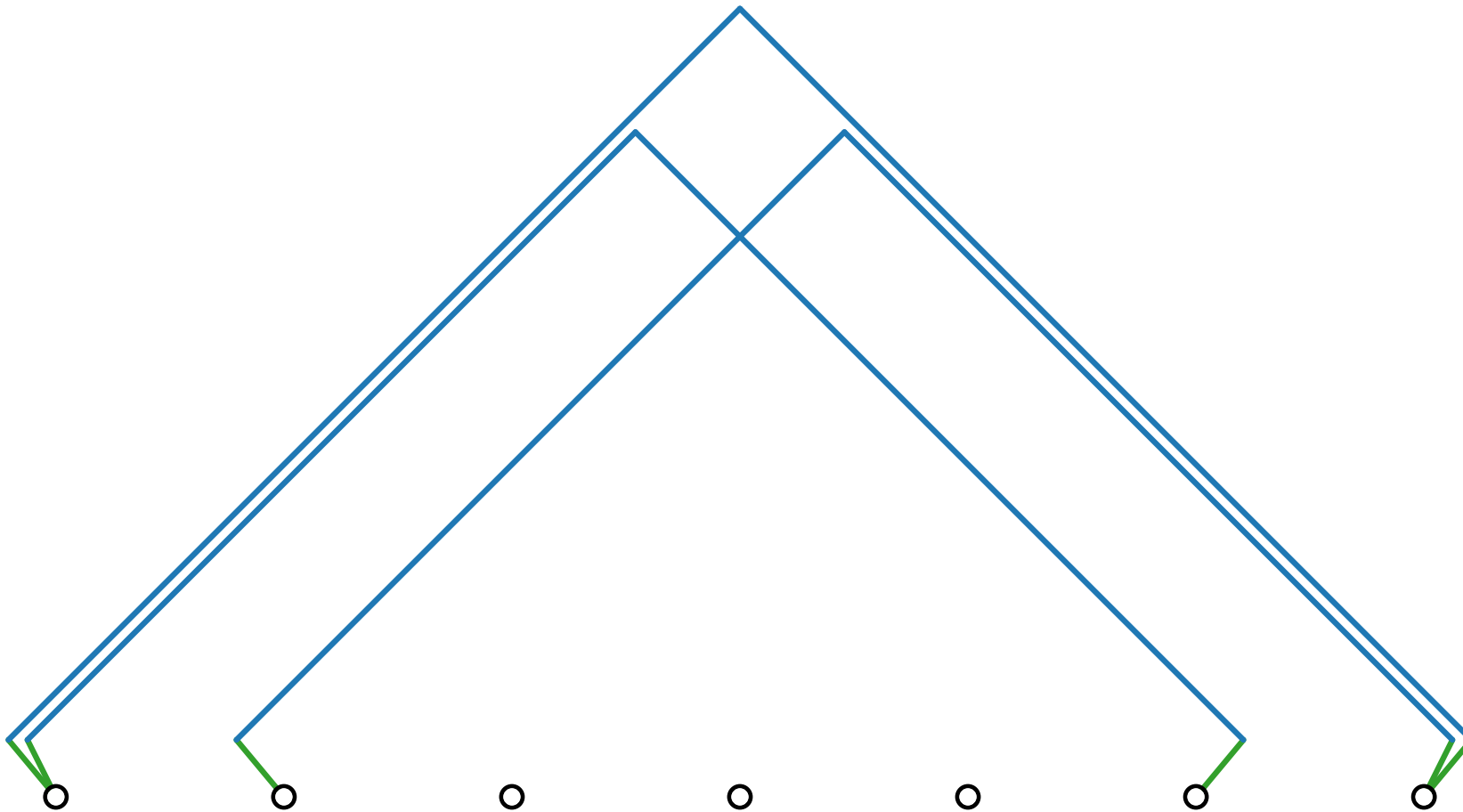


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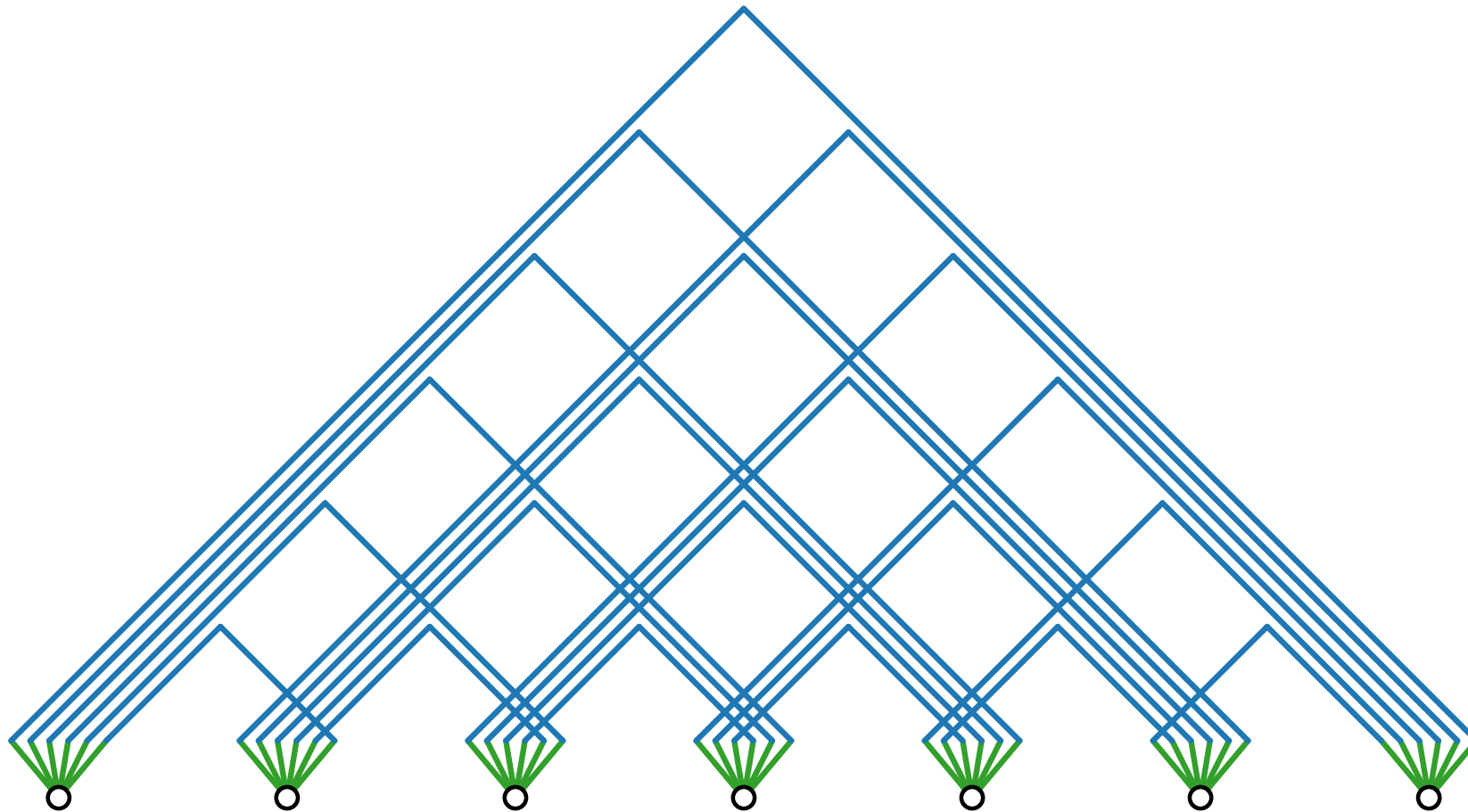


3-Bend RAC Drawings

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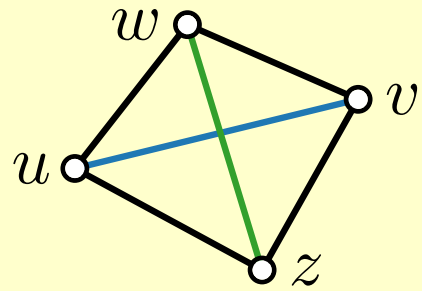
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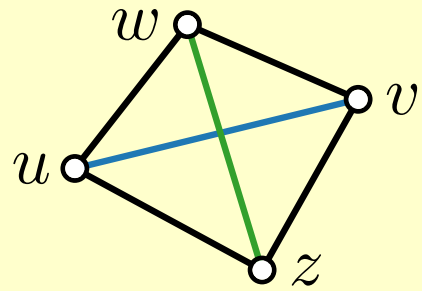
Kite Triangulations

This is a **kite**:

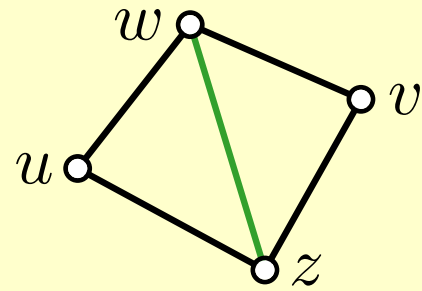


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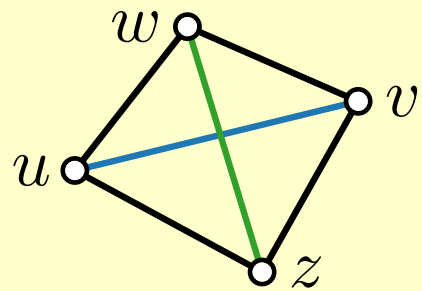


u and v are **opposite**
w.r.t. $\{z, w\}$

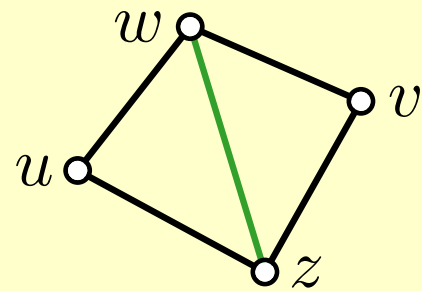


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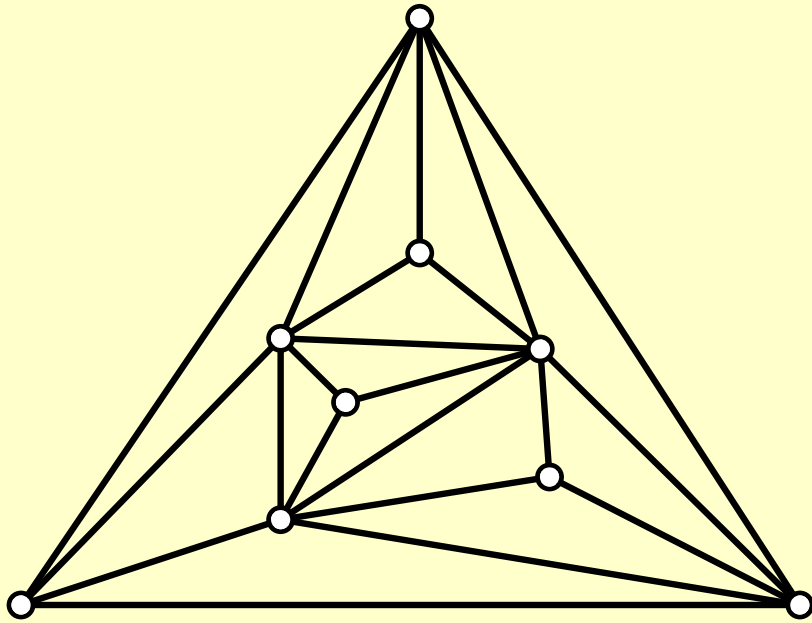
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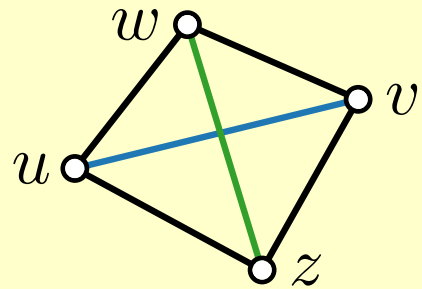


Let G' be a plane triangulation.

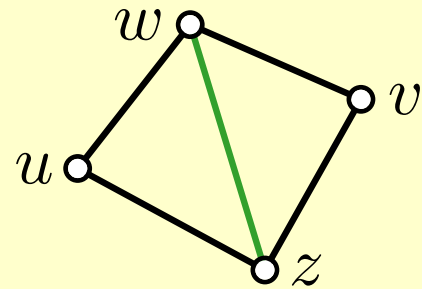


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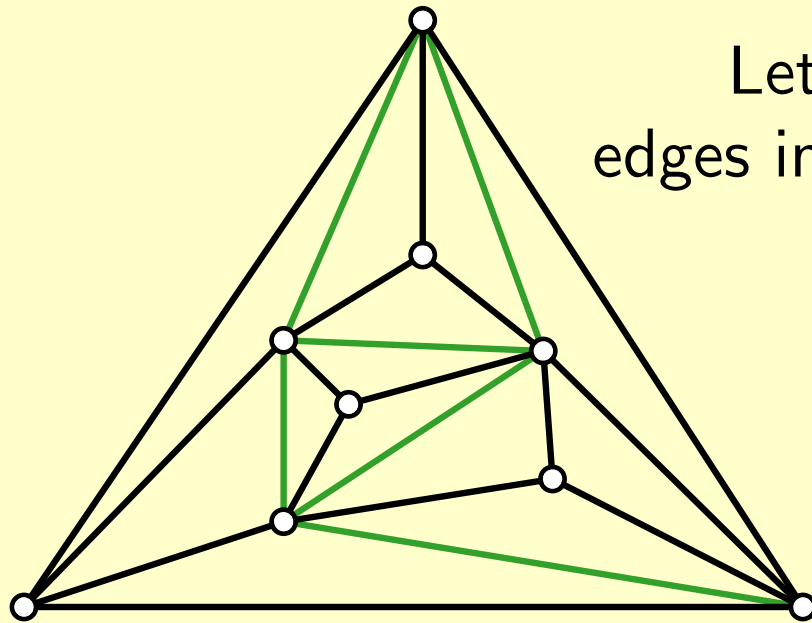
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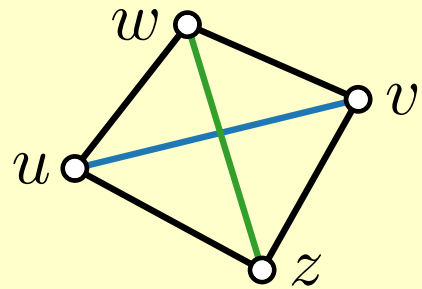
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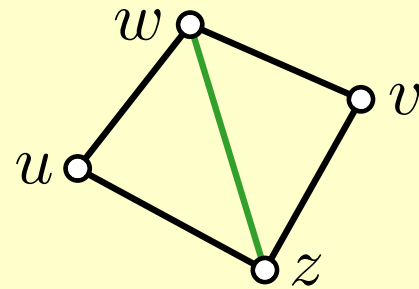
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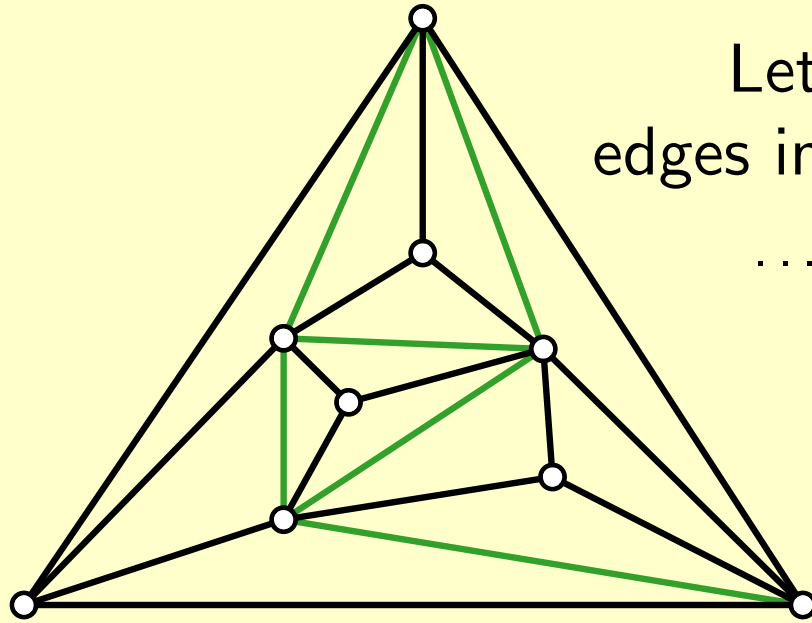
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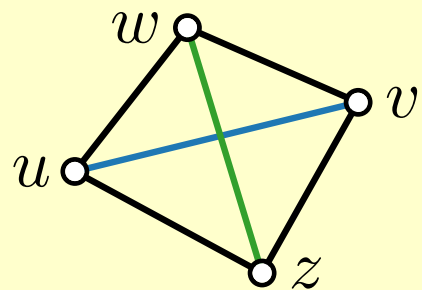


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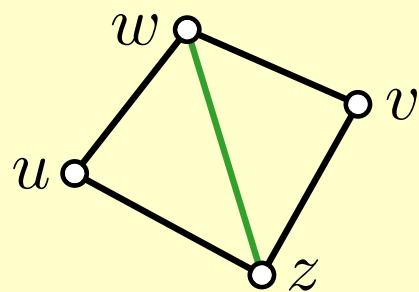
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Kite Triangulations

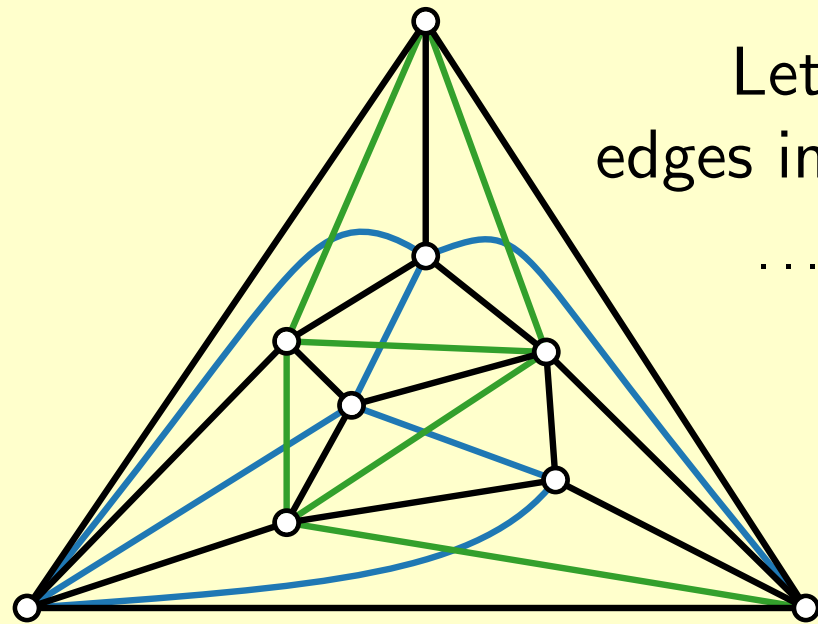
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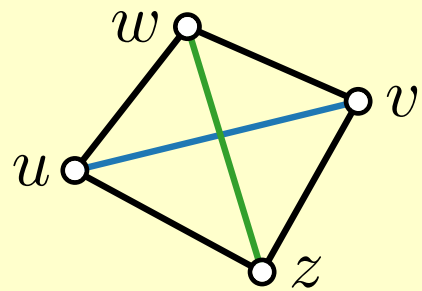
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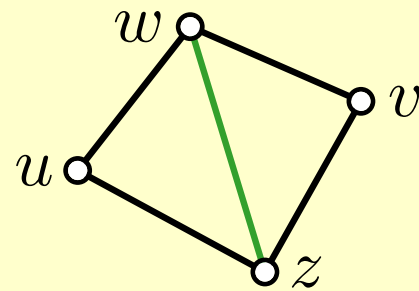
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Kite Triangulations

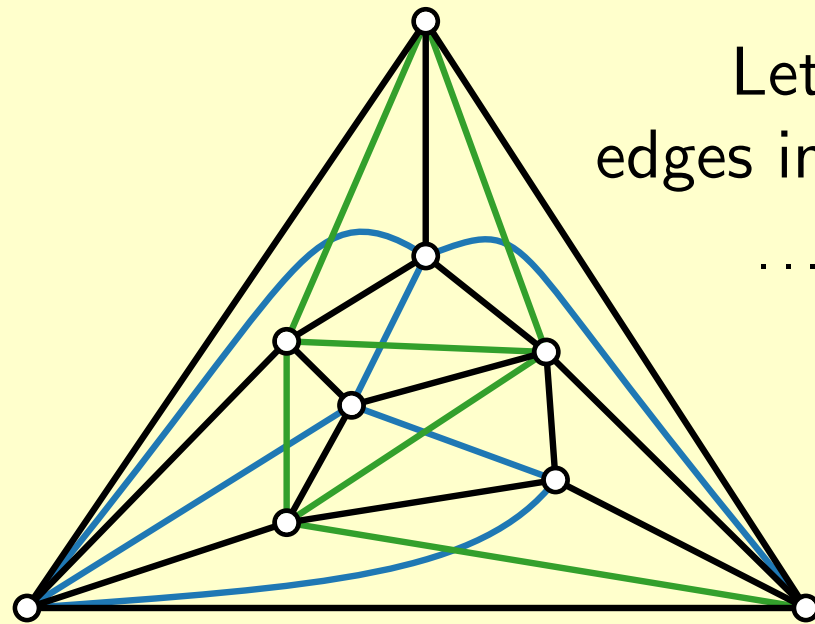
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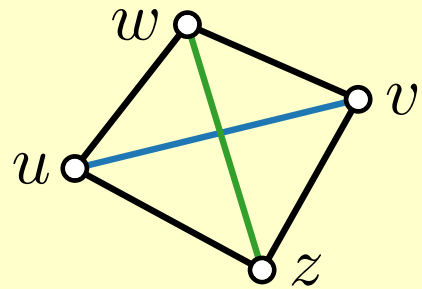
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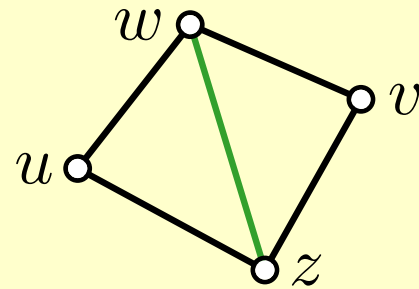
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Kite Triangulations

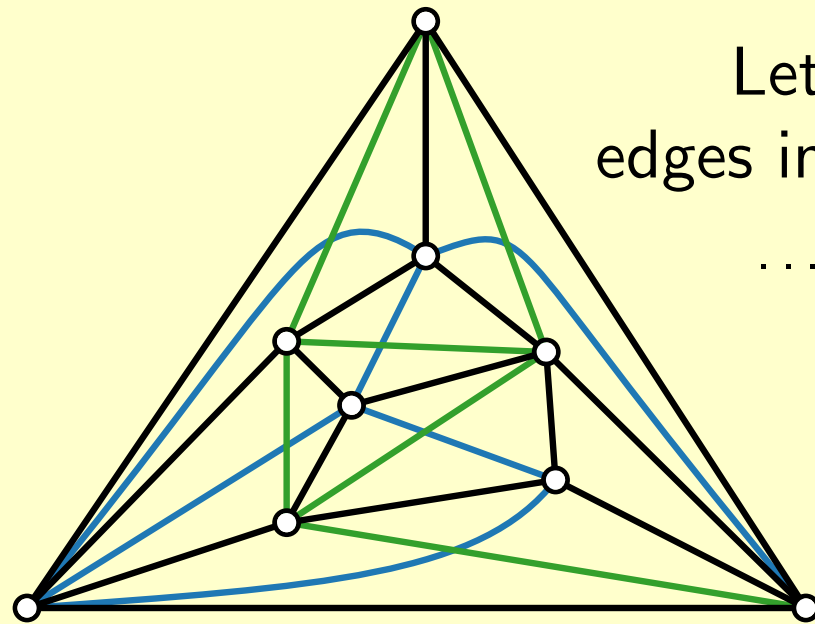
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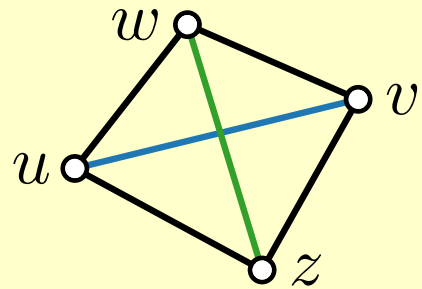
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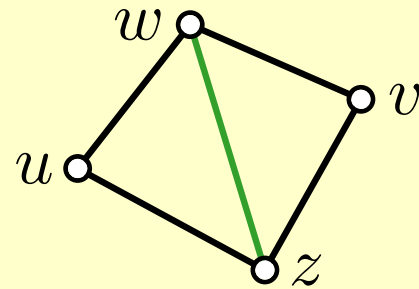
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Kite Triangulations

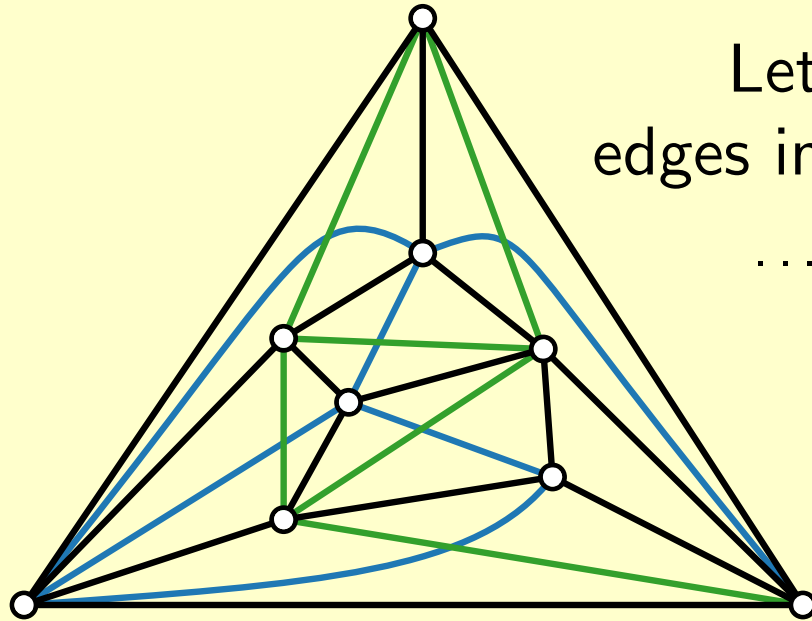
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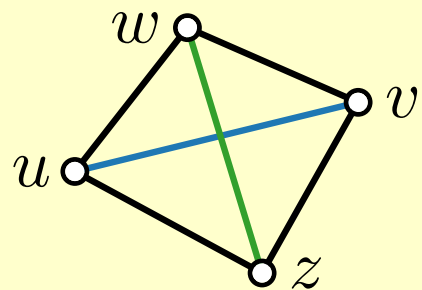
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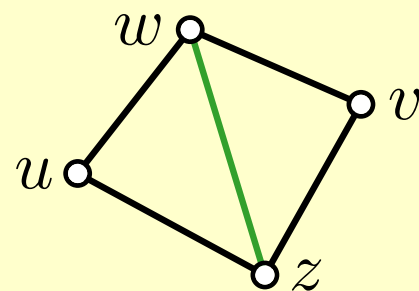
Every kite-triangulation G admits a
1-planar 1-bend RAC drawing

Kite Triangulations

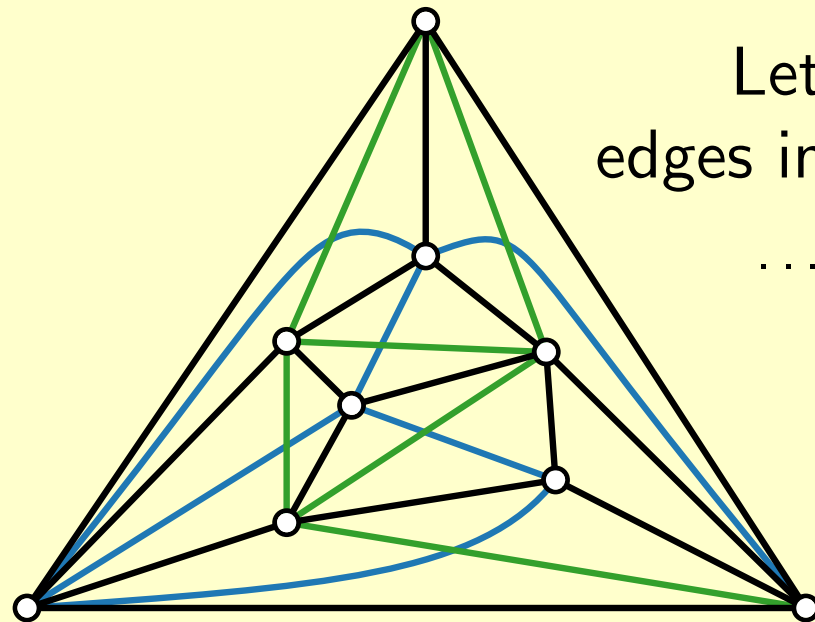
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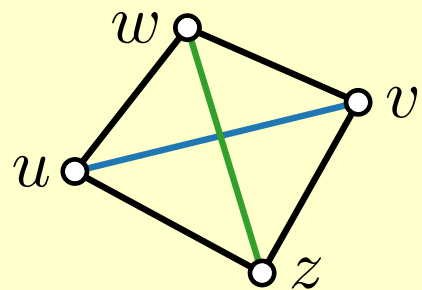
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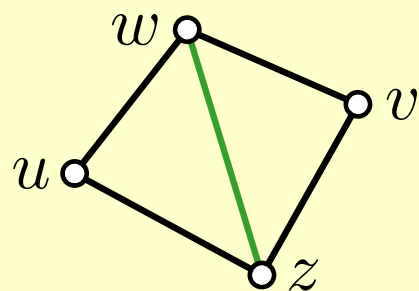
Every kite-triangulation G admits a 1-planar 1-bend RAC drawing, which can be constructed in linear time.

Kite Triangulations

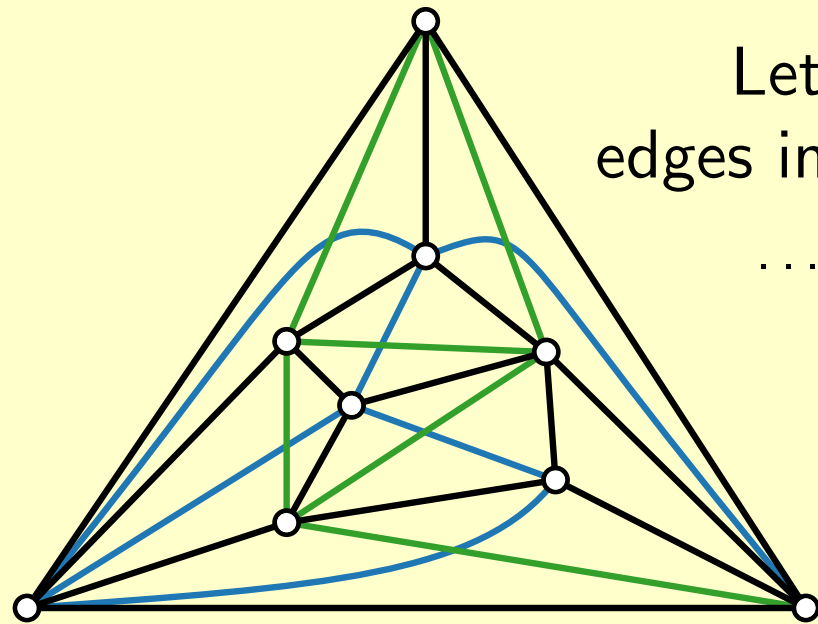
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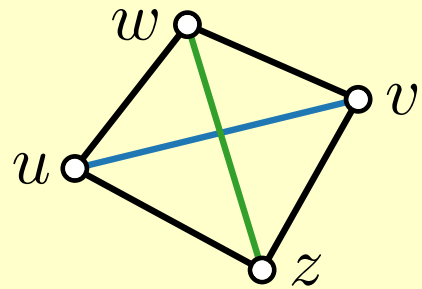
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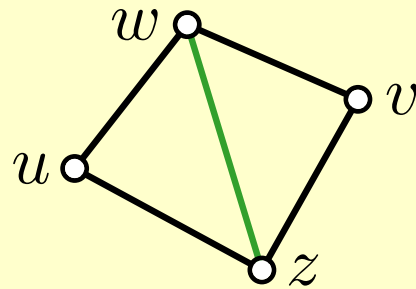
Proof.

Kite Triangulations

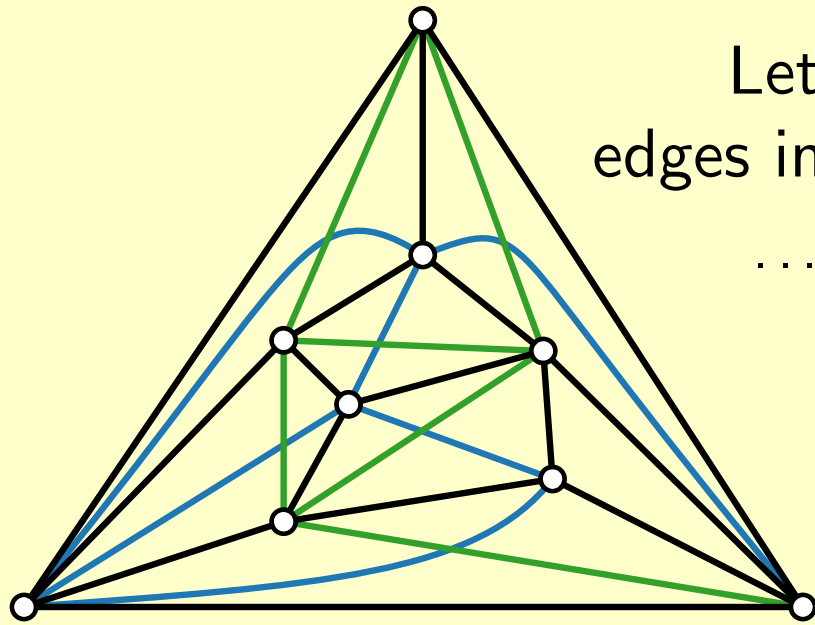
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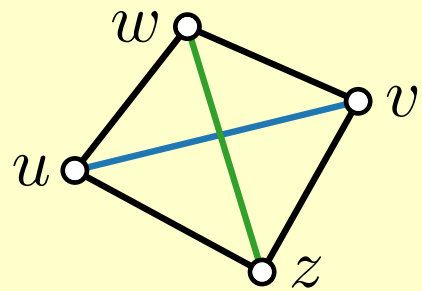
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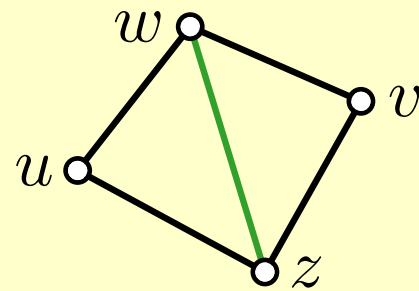
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Kite Triangulations

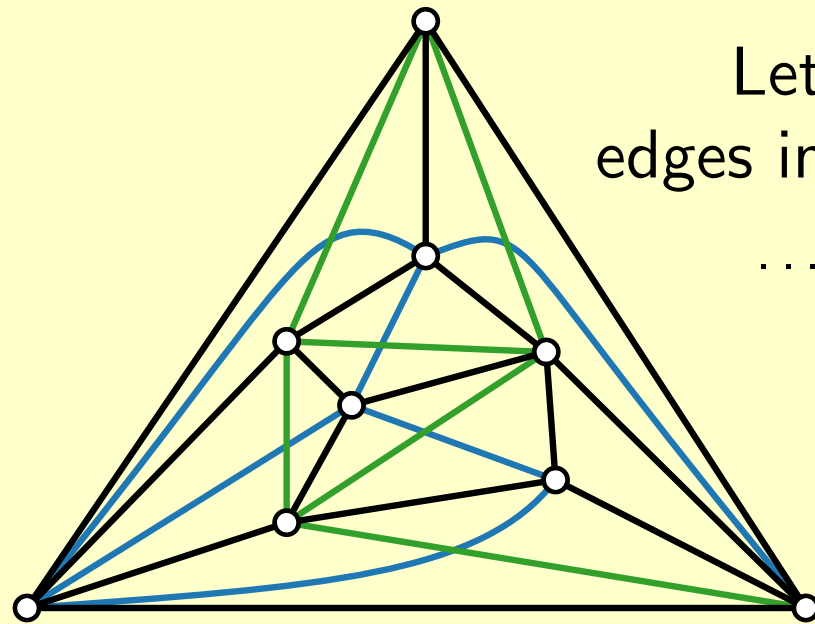
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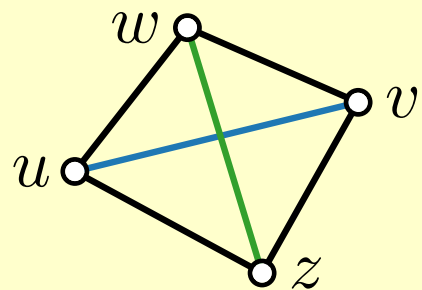
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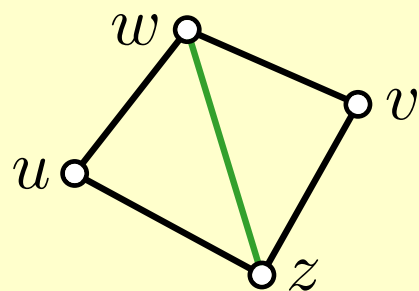
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Kite Triangulations

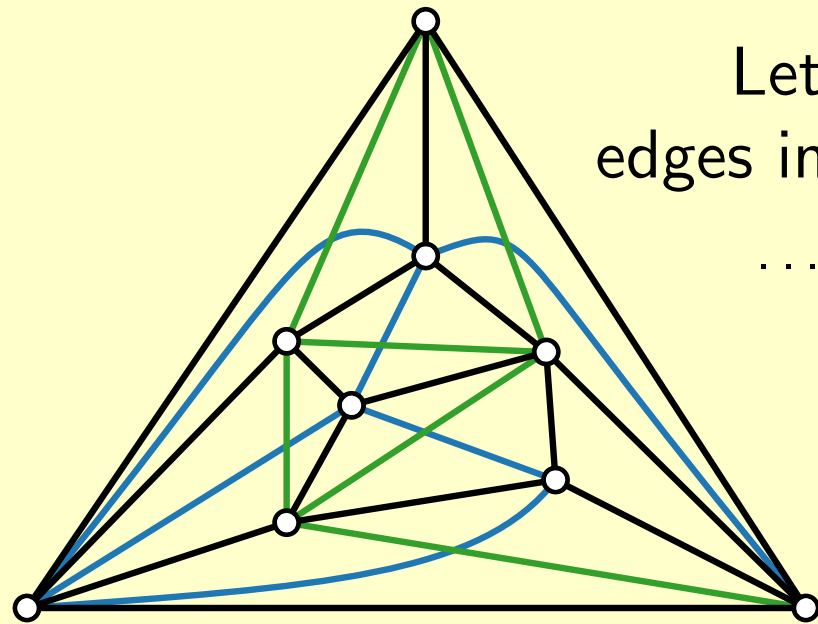
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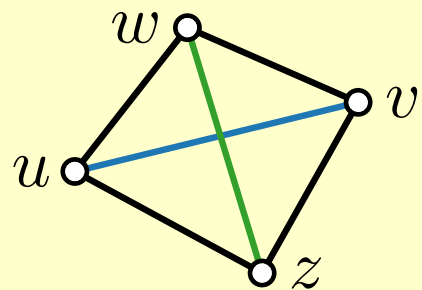
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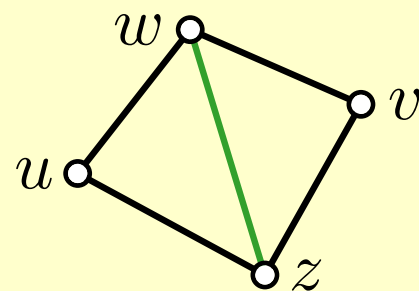
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Kite Triangulations

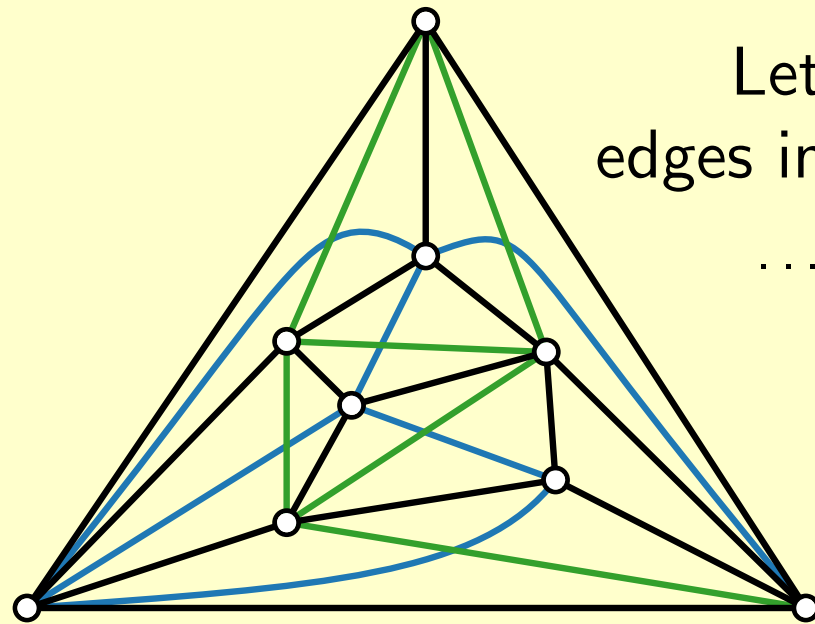
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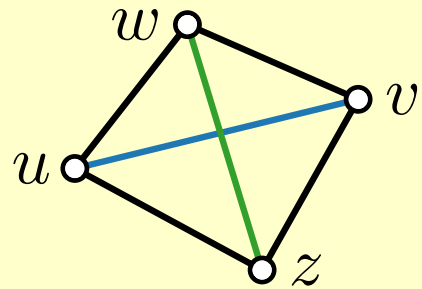
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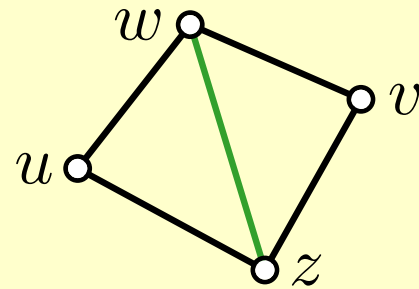
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Construct straight-line drawing of G'' .
Fill faces as follows:

Kite Triangulations

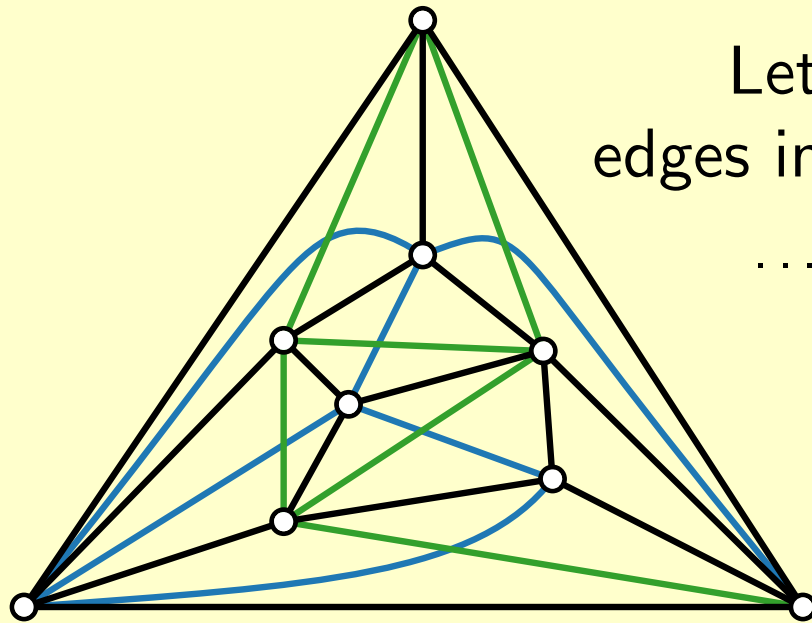
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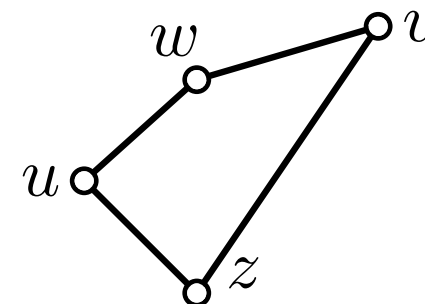
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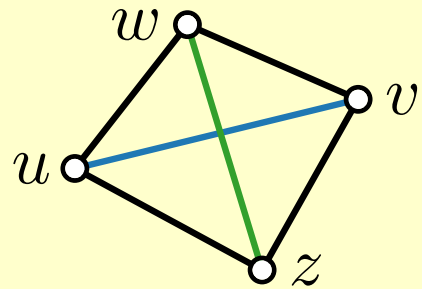
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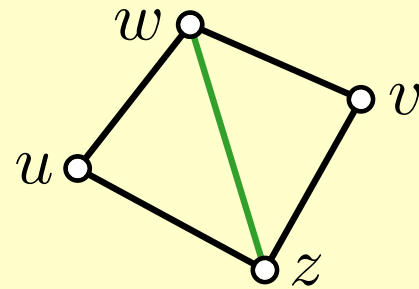
strictly convex face

Kite Triangulations

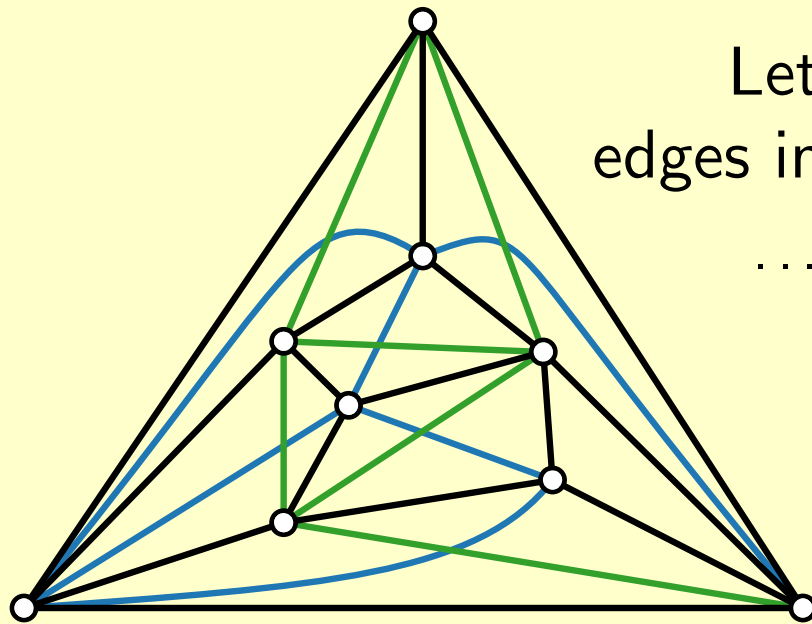
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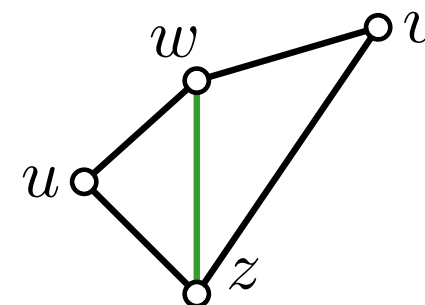
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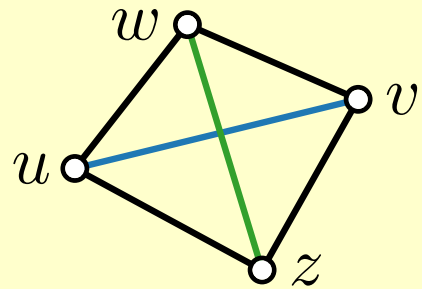
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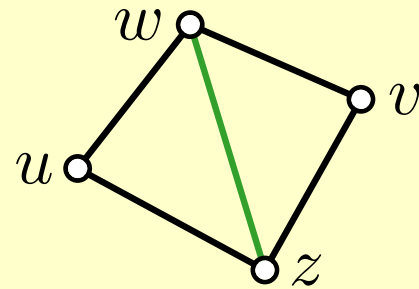
strictly convex face

Kite Triangulations

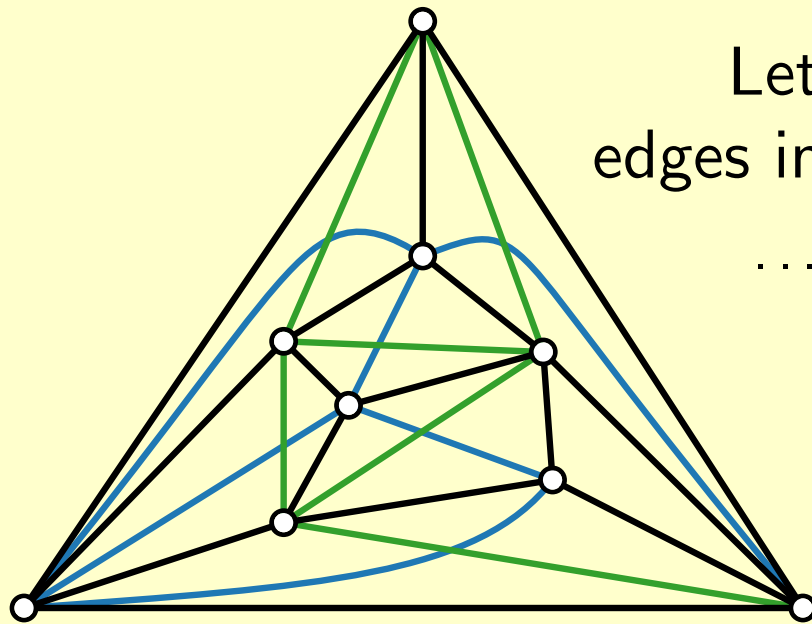
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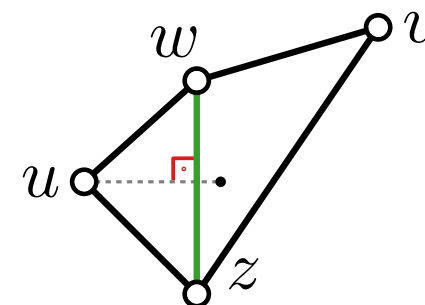
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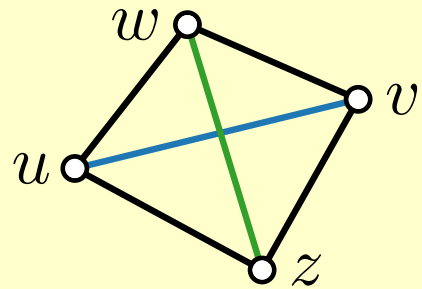
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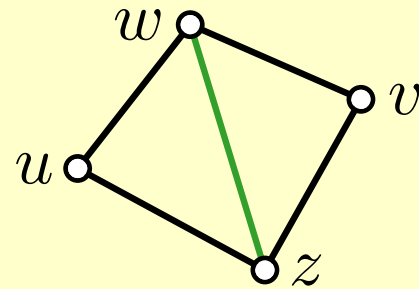
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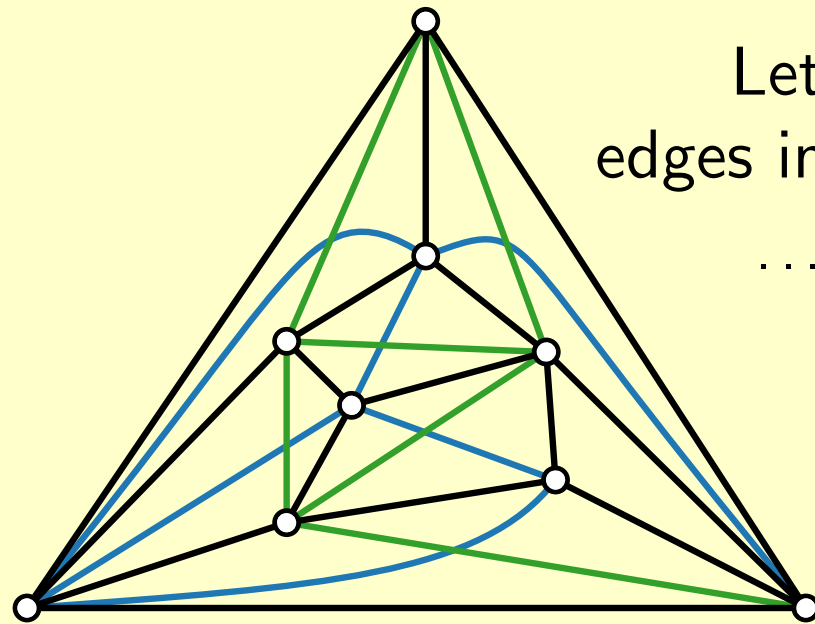
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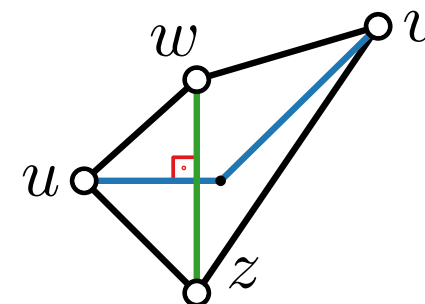
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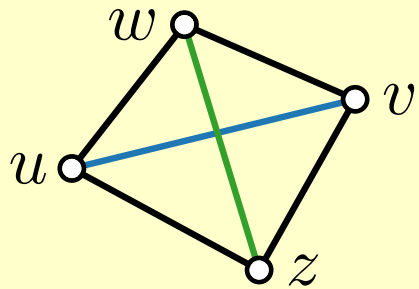
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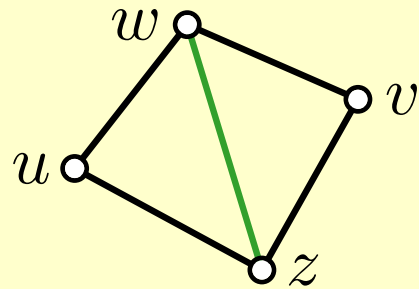
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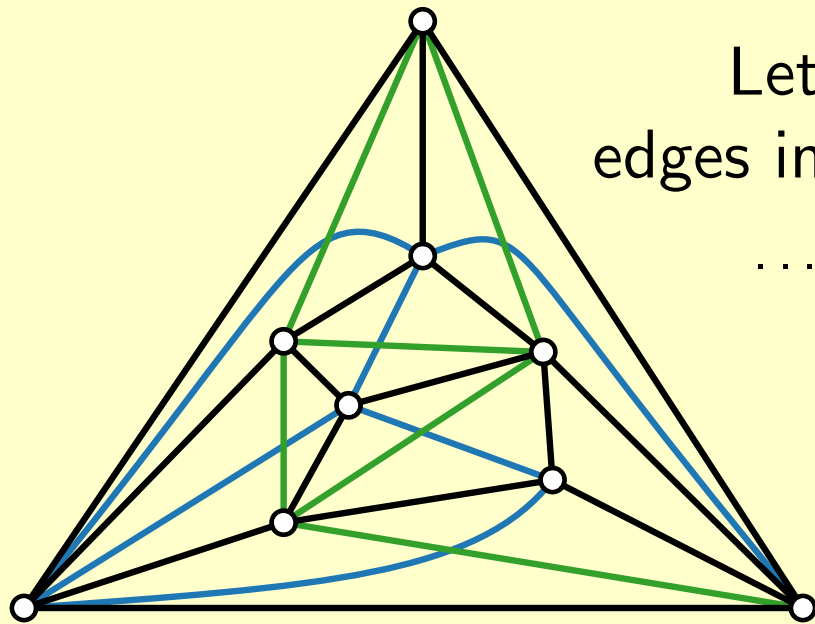
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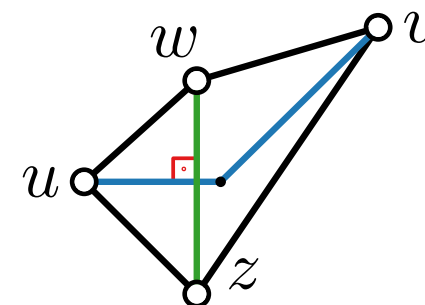
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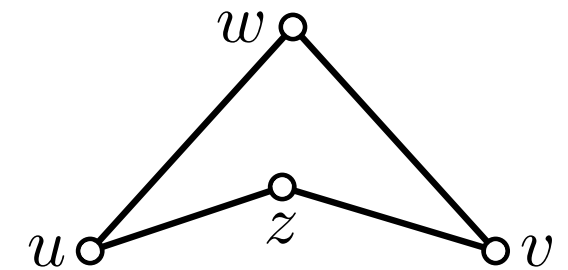
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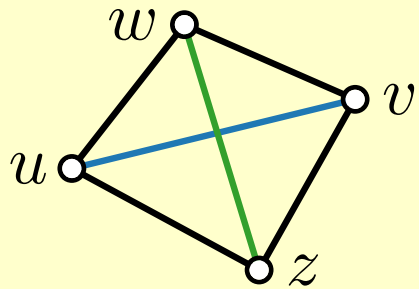
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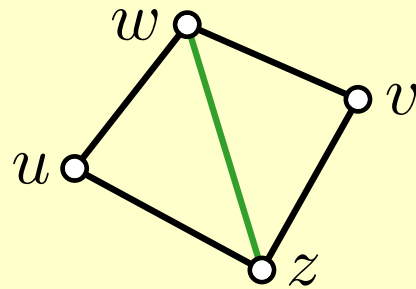
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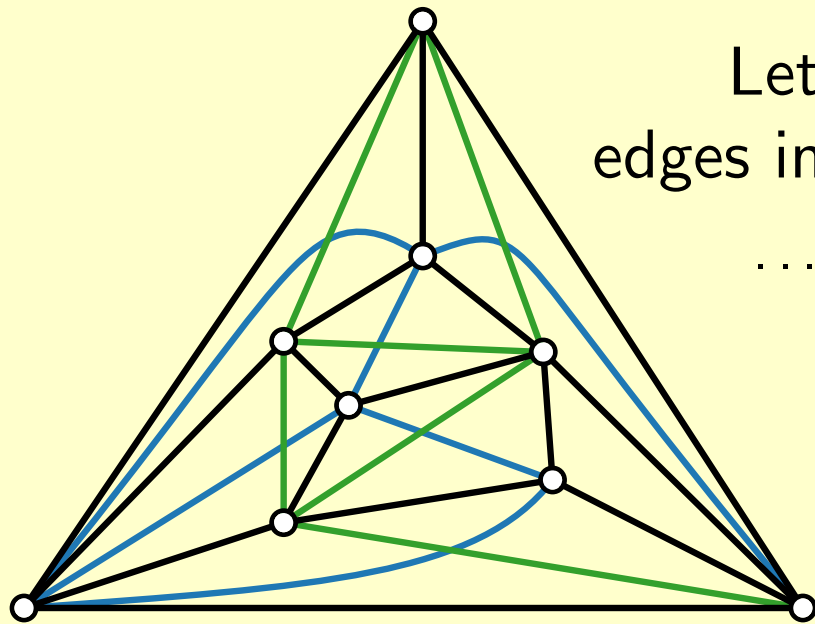
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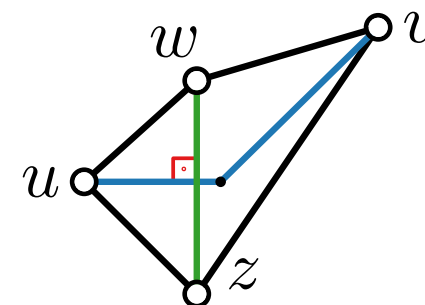
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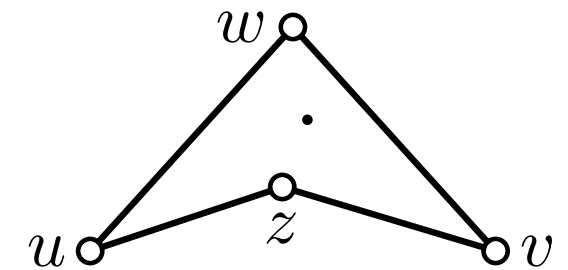
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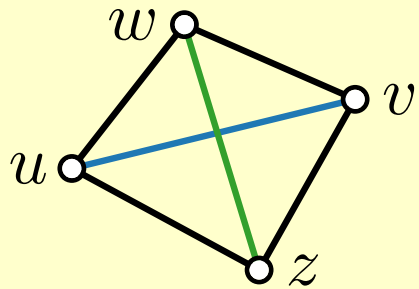
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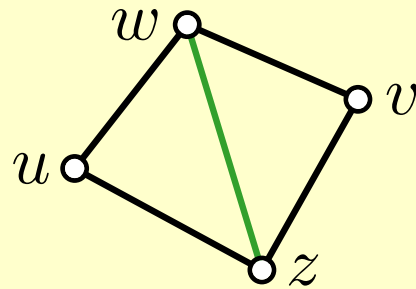
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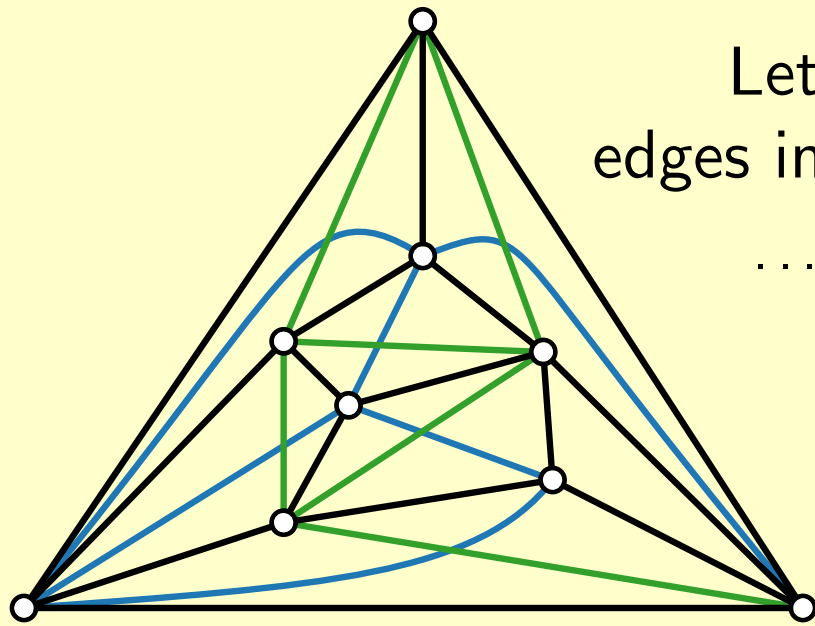
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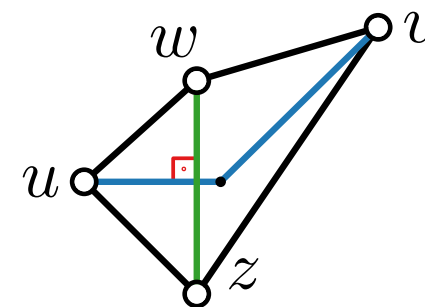
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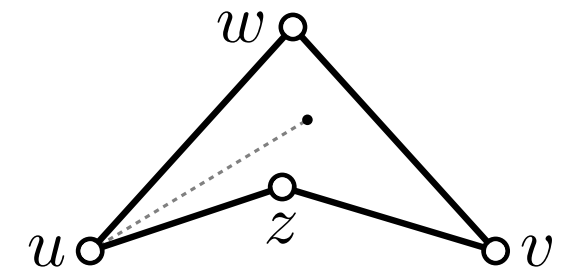
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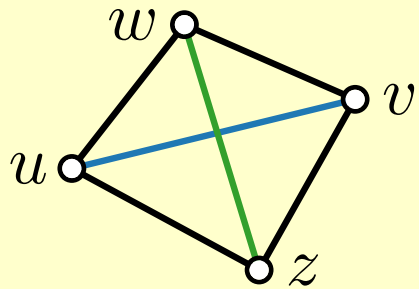
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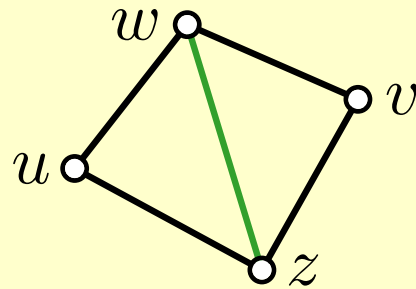
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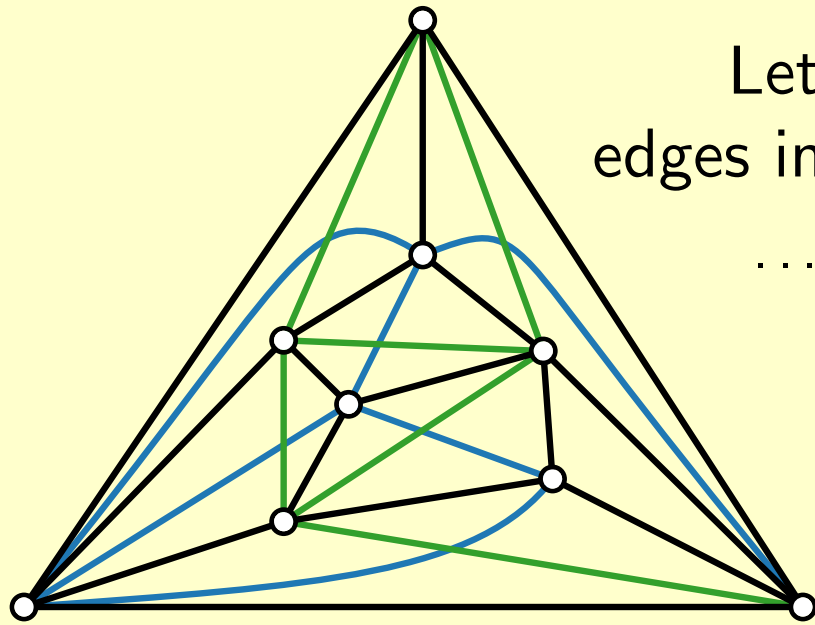
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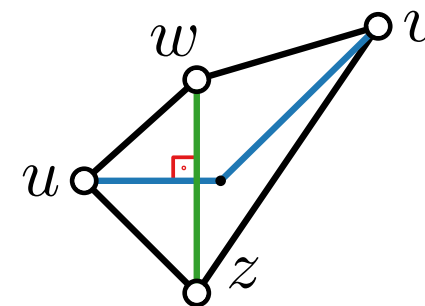
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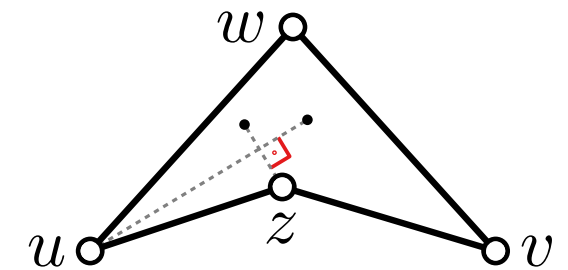
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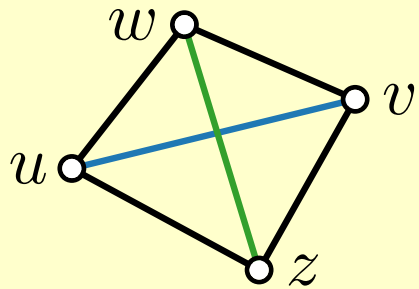
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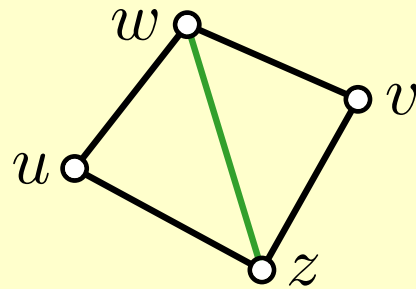
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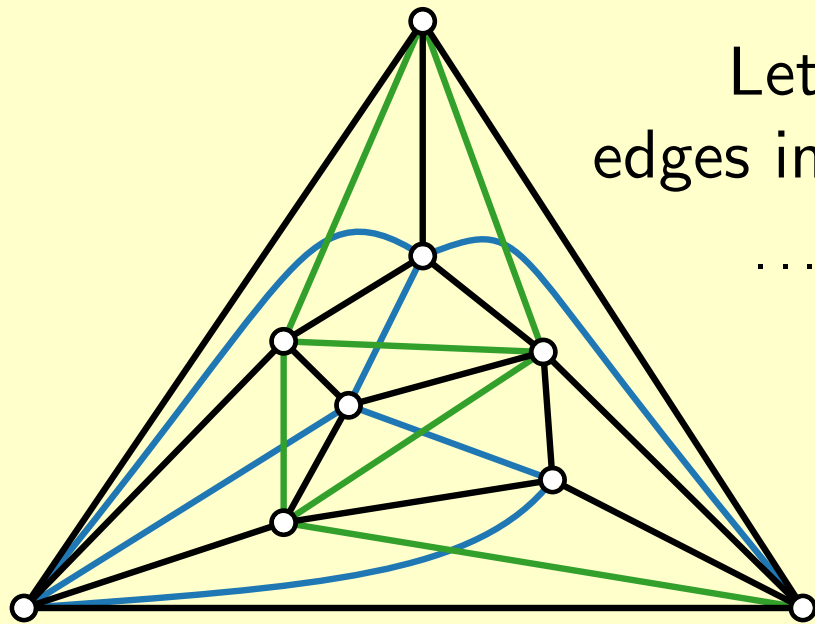
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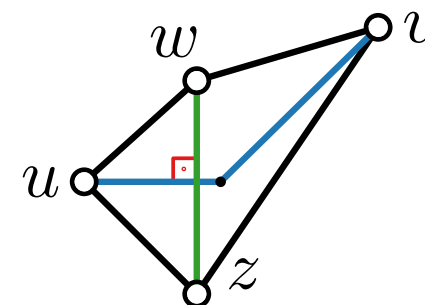
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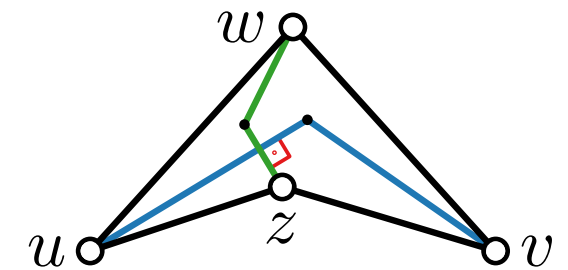
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1-Planar 1-Bend RAC Drawings

Theorem. [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
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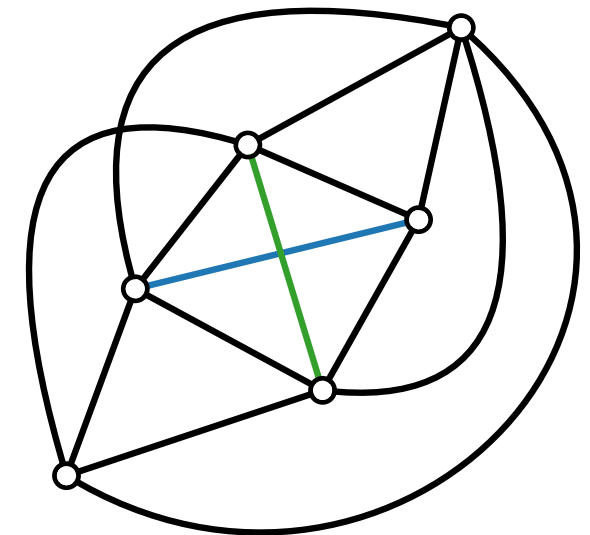
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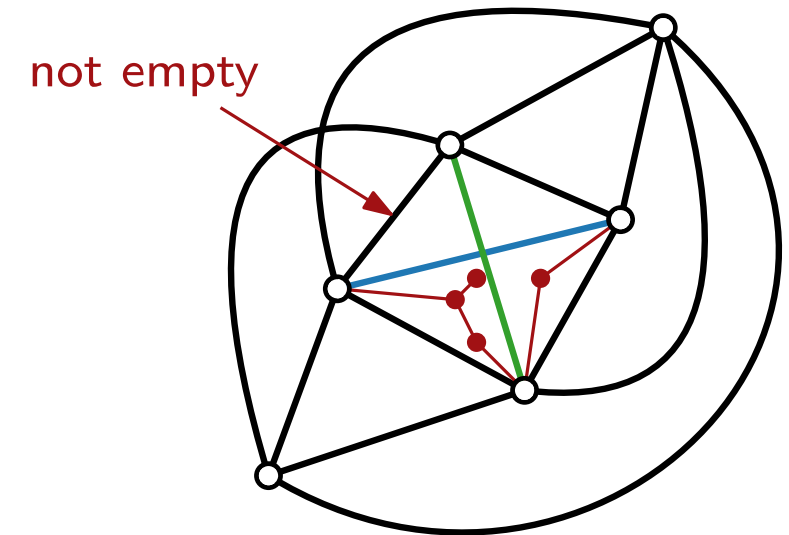
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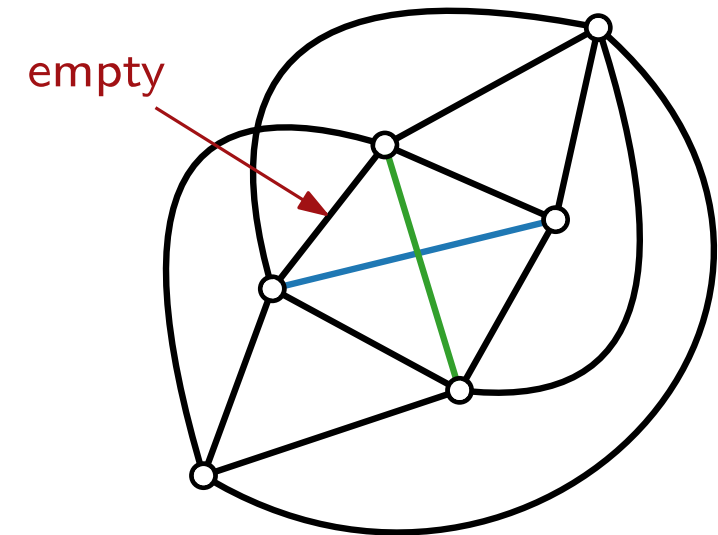


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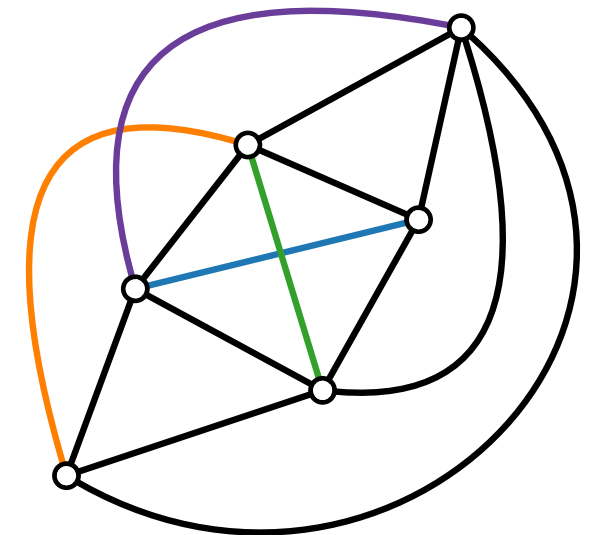
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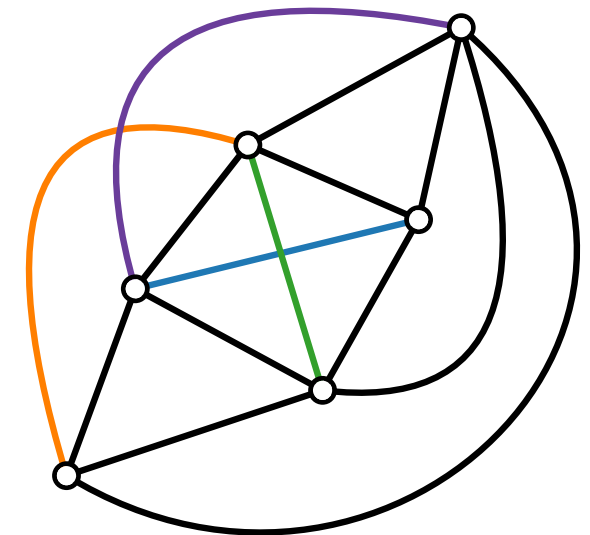
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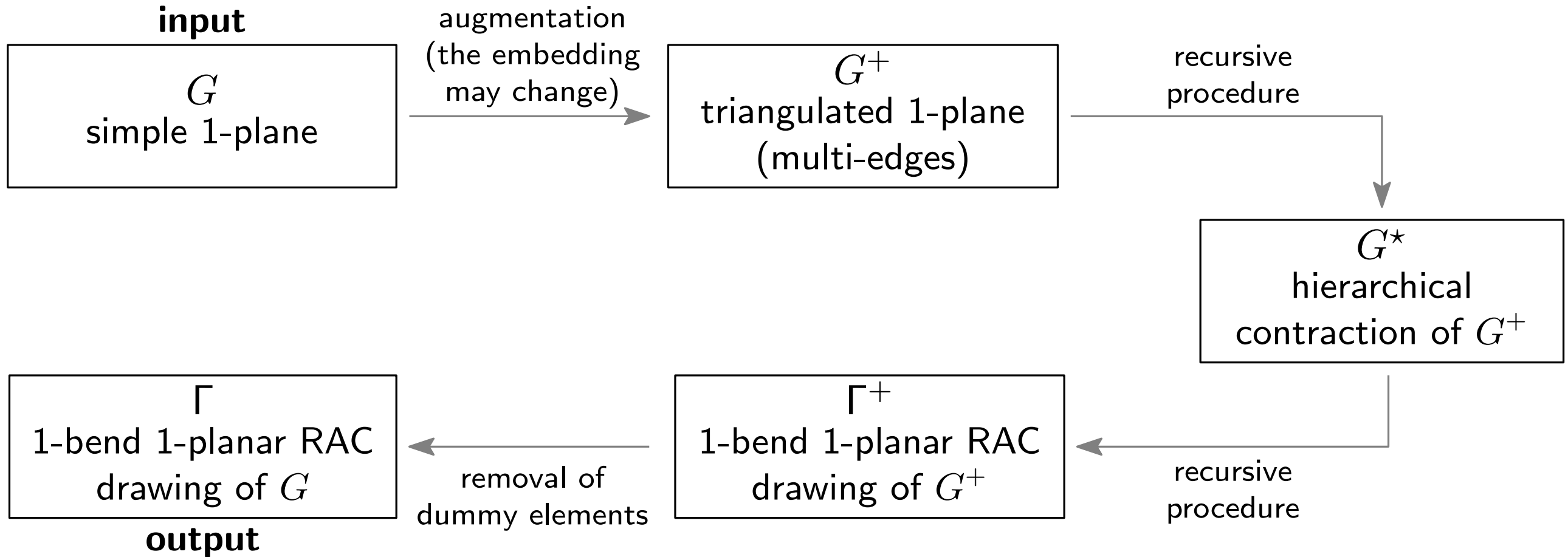
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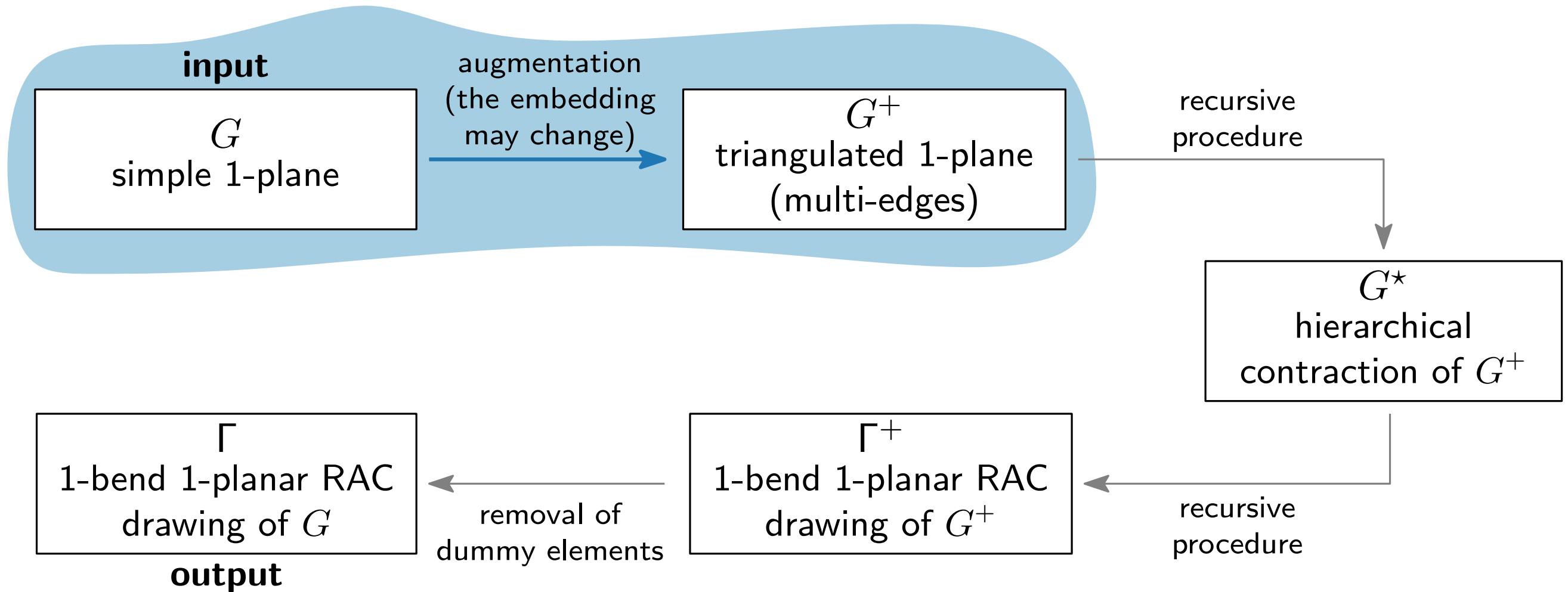
Theorem. [Chiba, Yamanouchi & Nishizeki 1984]

For every 2-connected plane graph G with outer face C_k and every convex k -gon P , there is a strictly convex planar straight-line drawing of G whose outer face coincides with P . Such a drawing can be computed in linear time.

Algorithm Outline

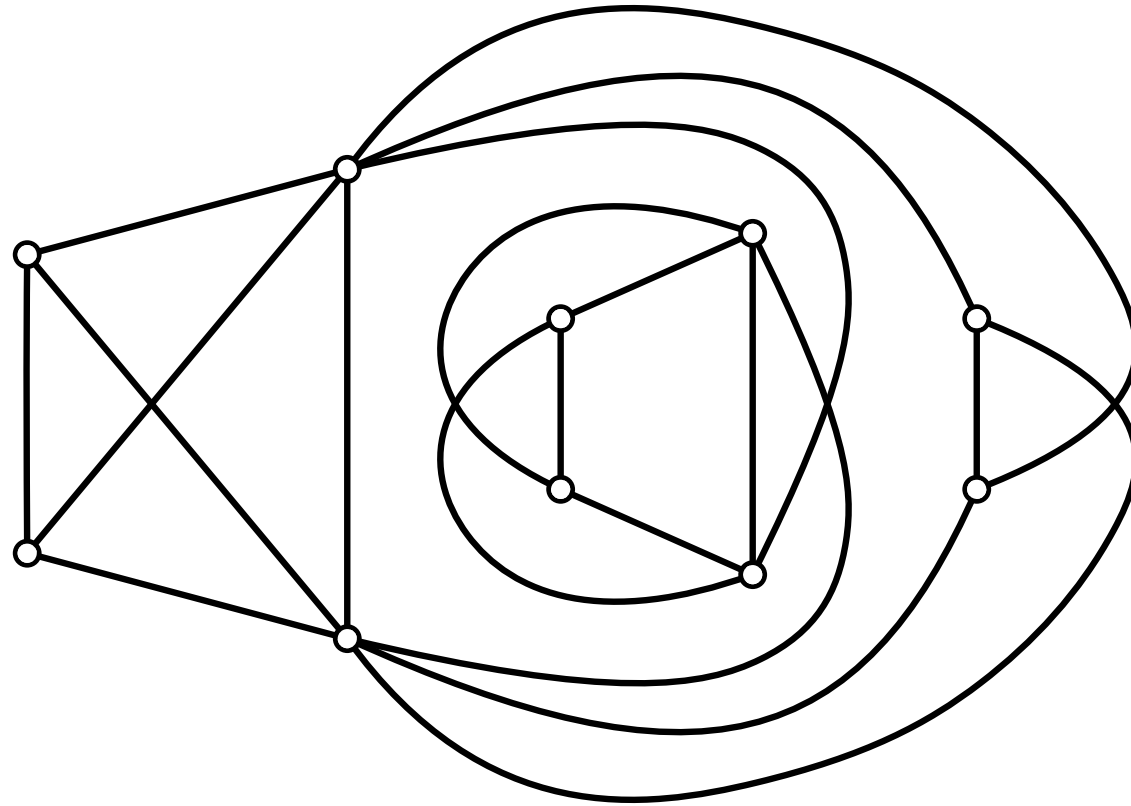


Algorithm Outline



Algorithm Step 1: Augmentation

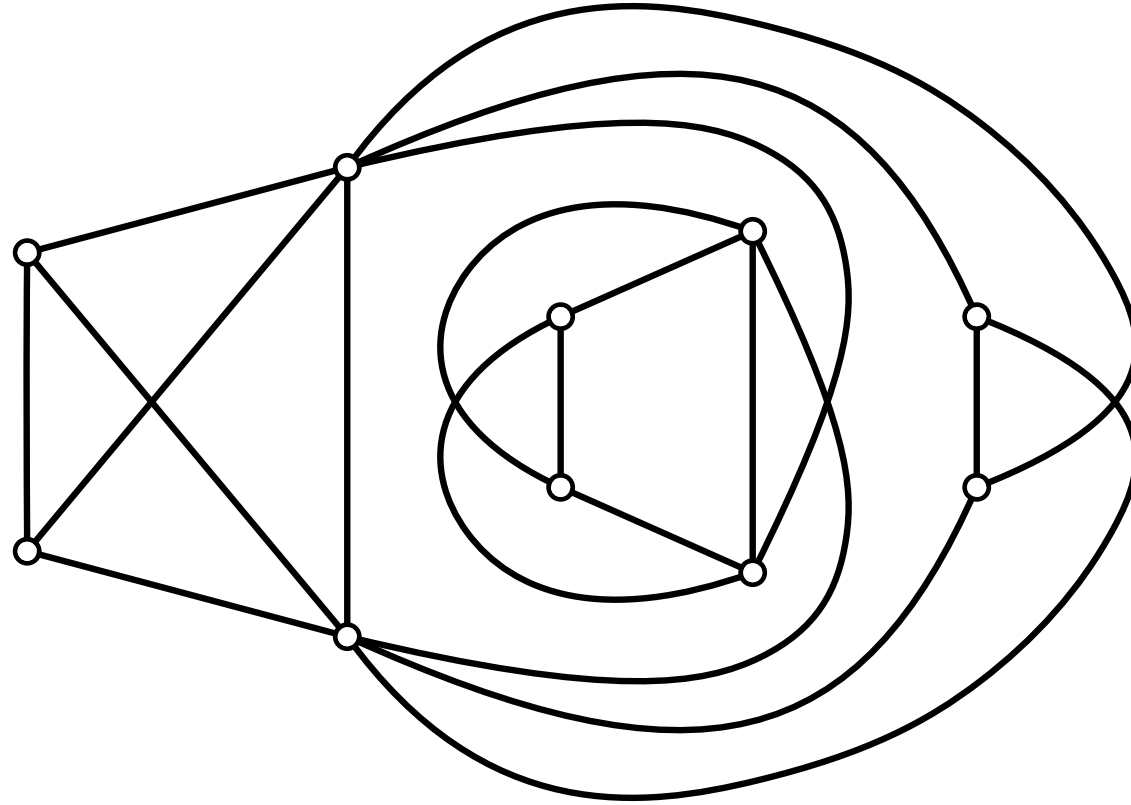
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Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

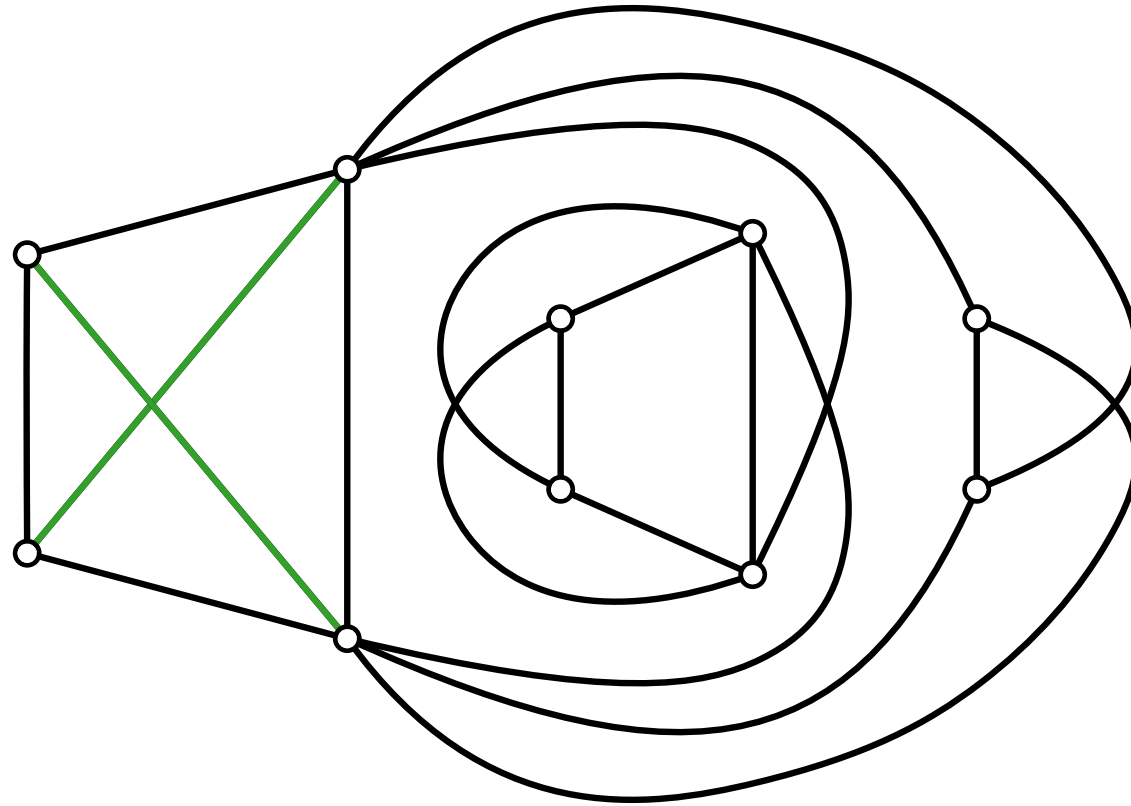
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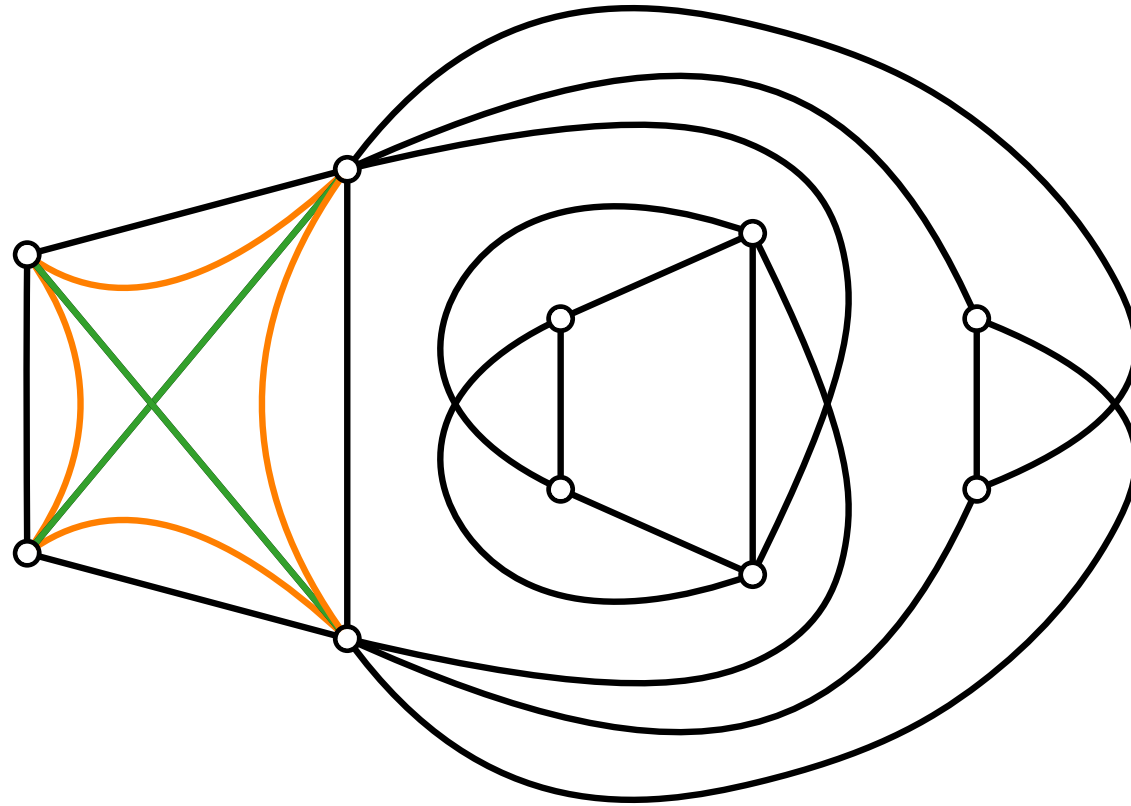
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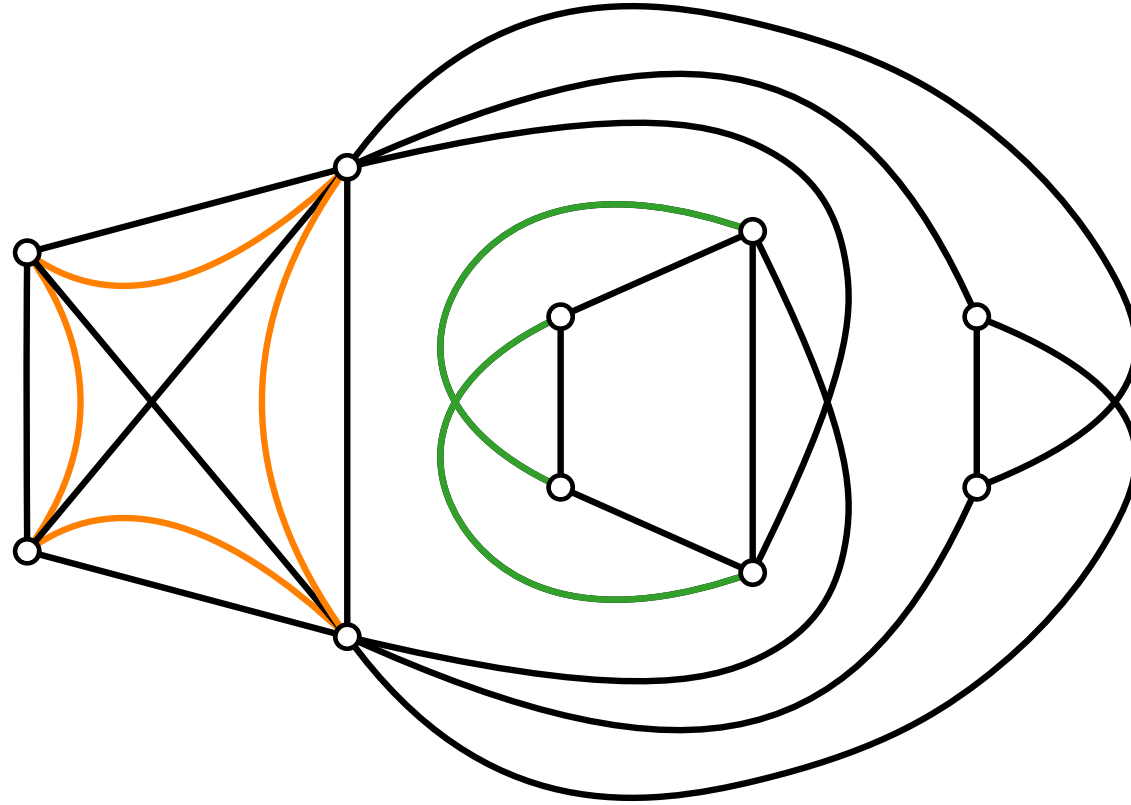
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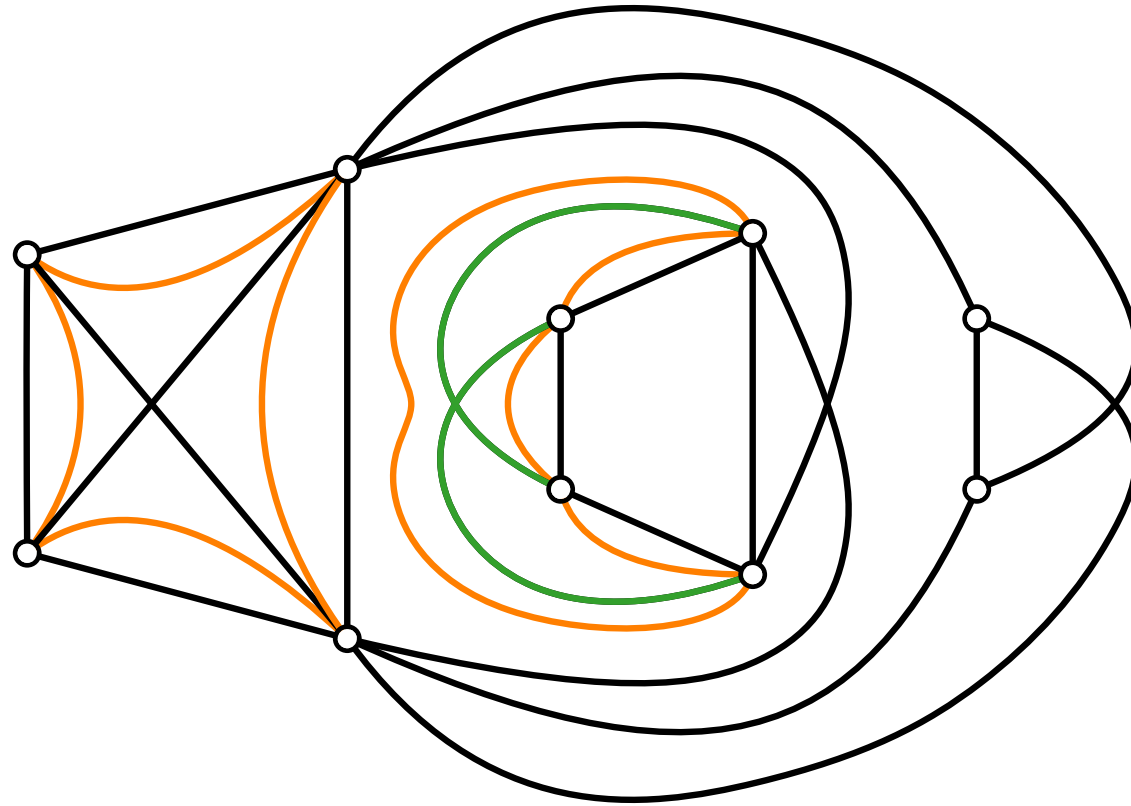
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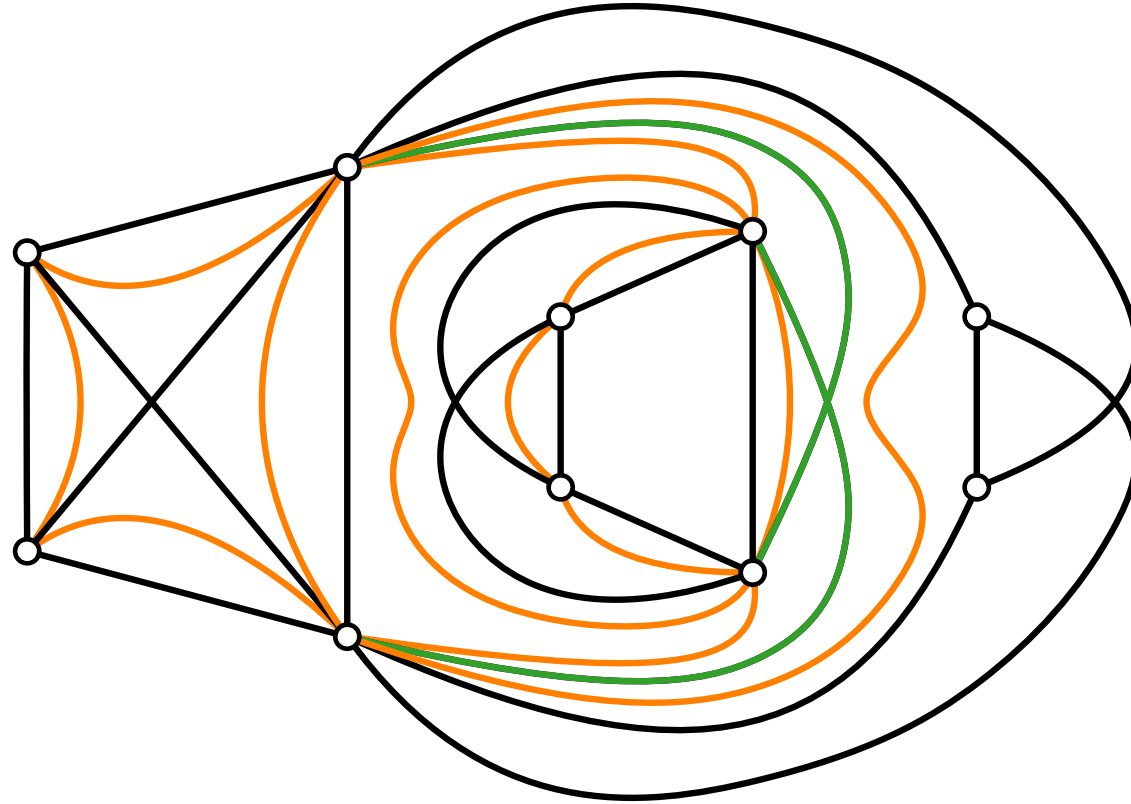
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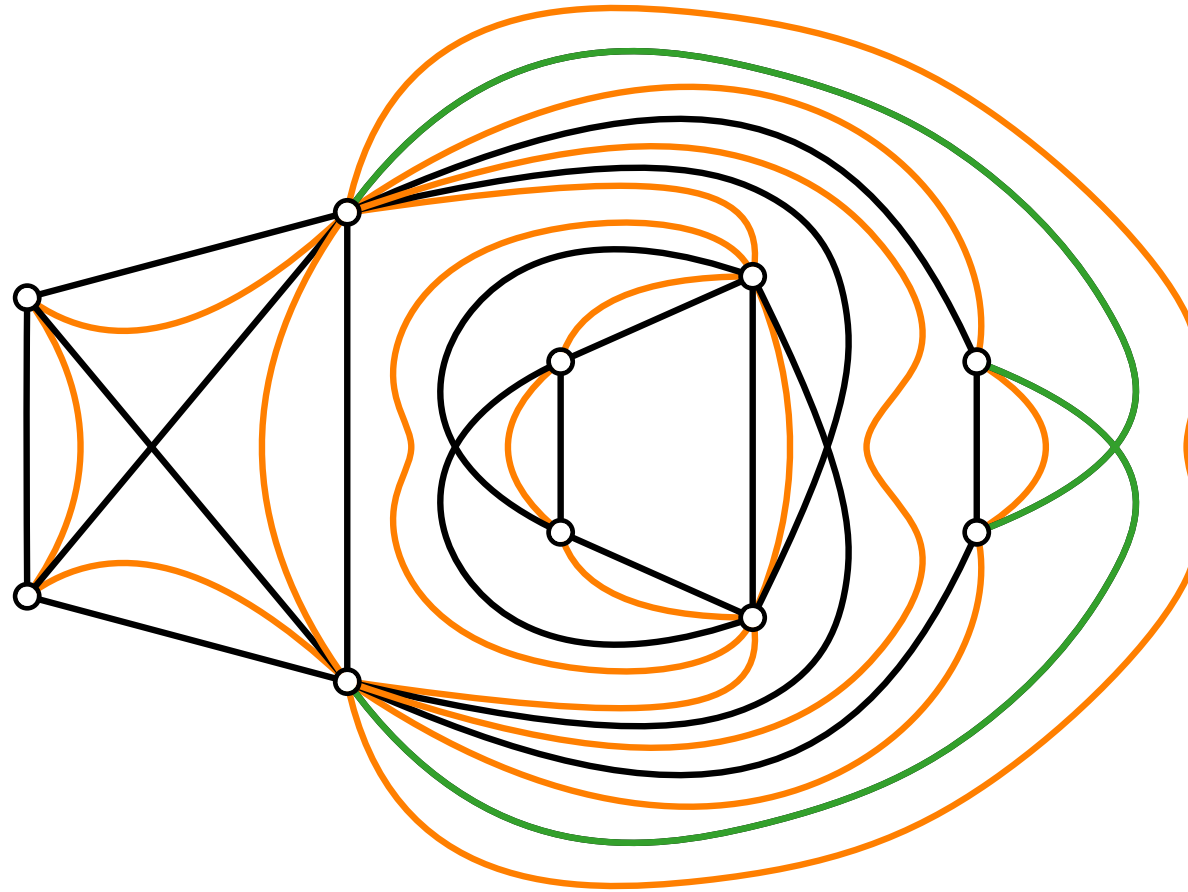
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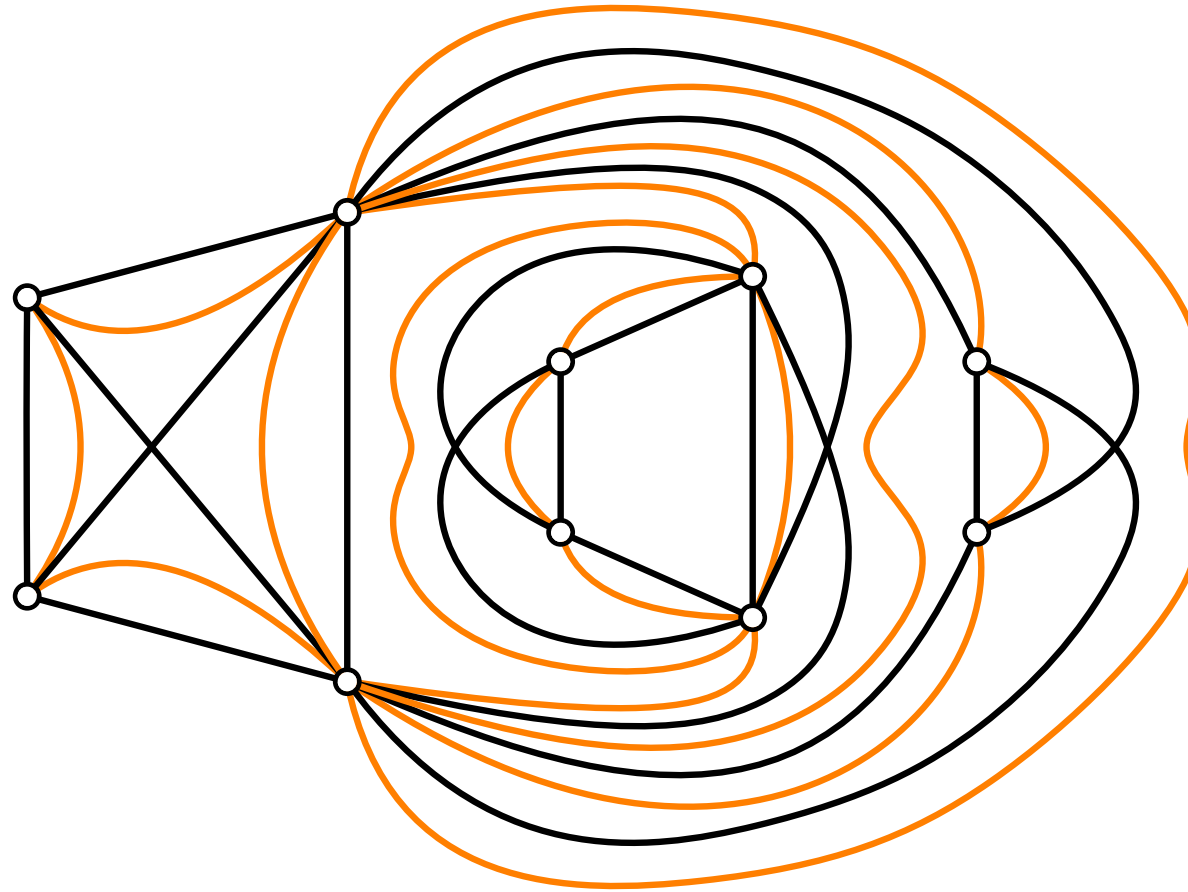


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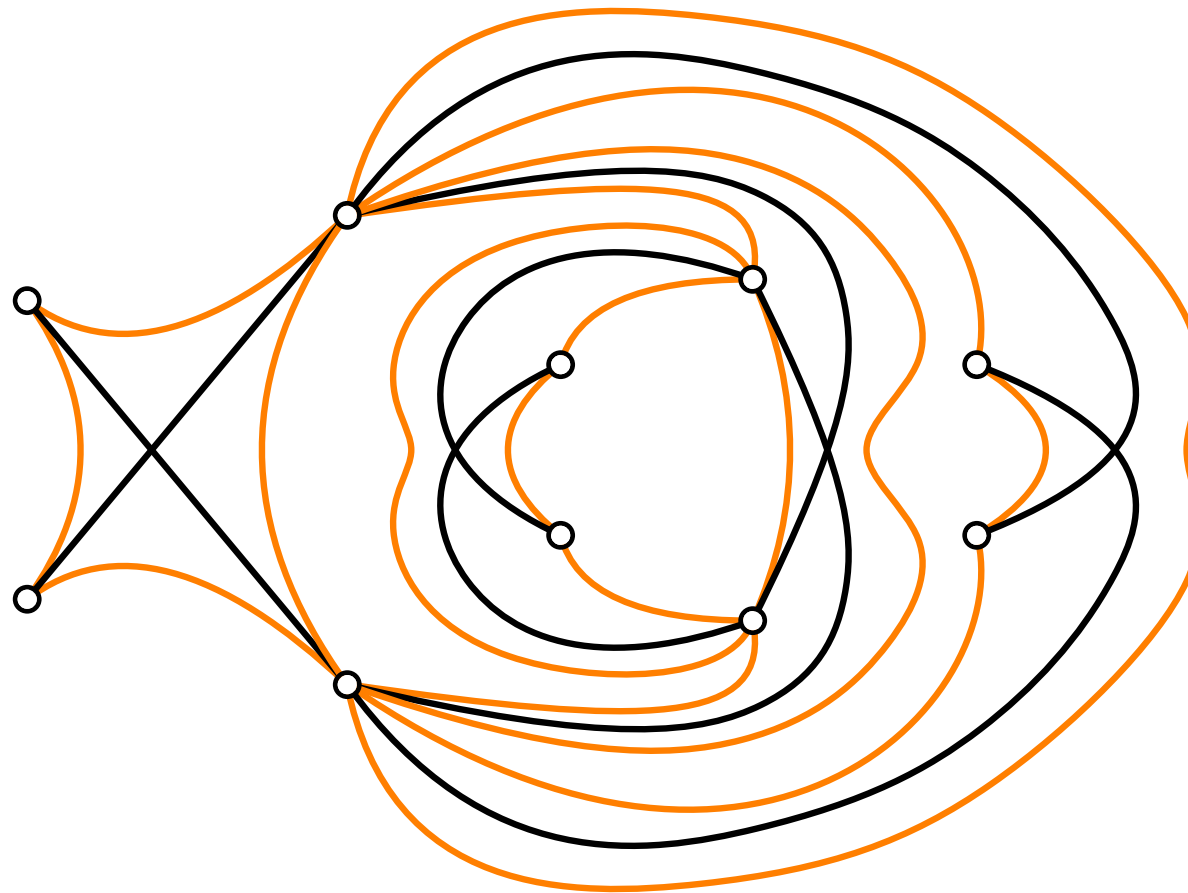


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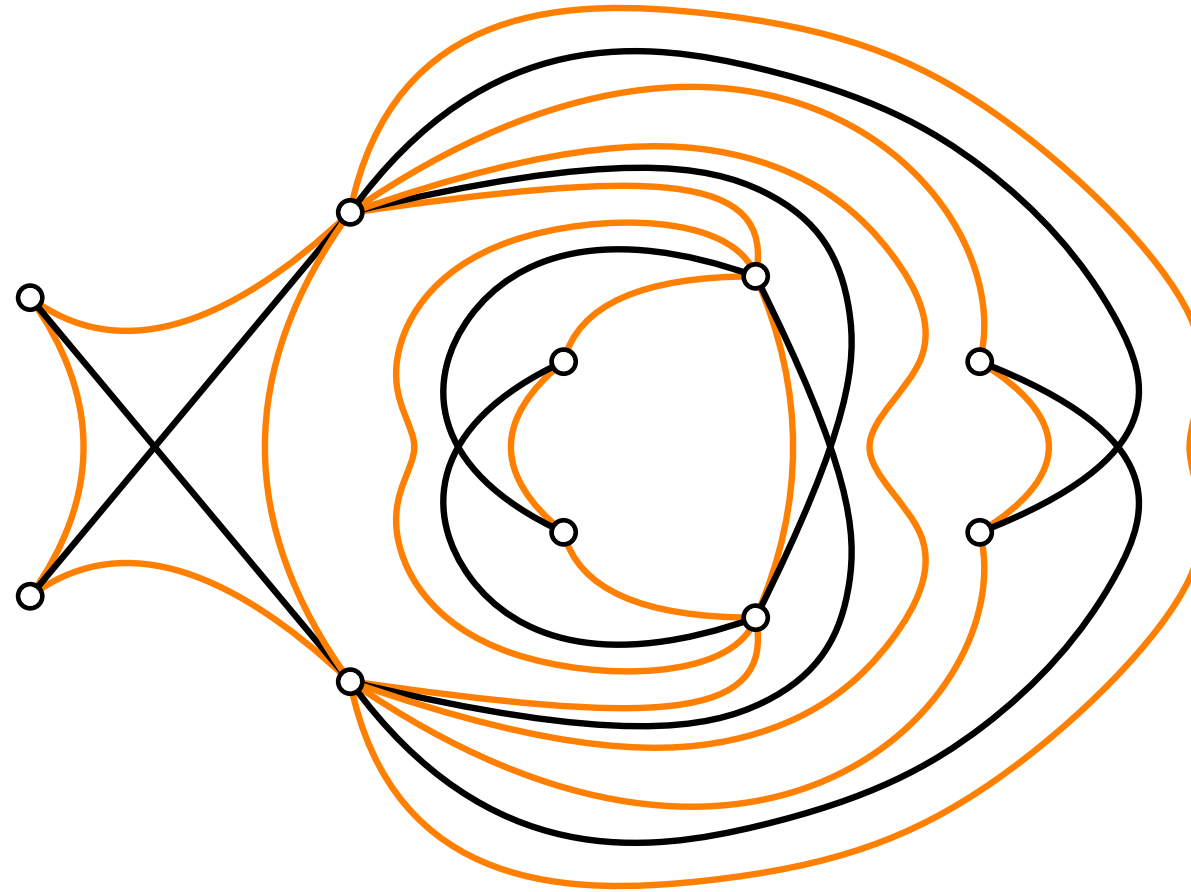
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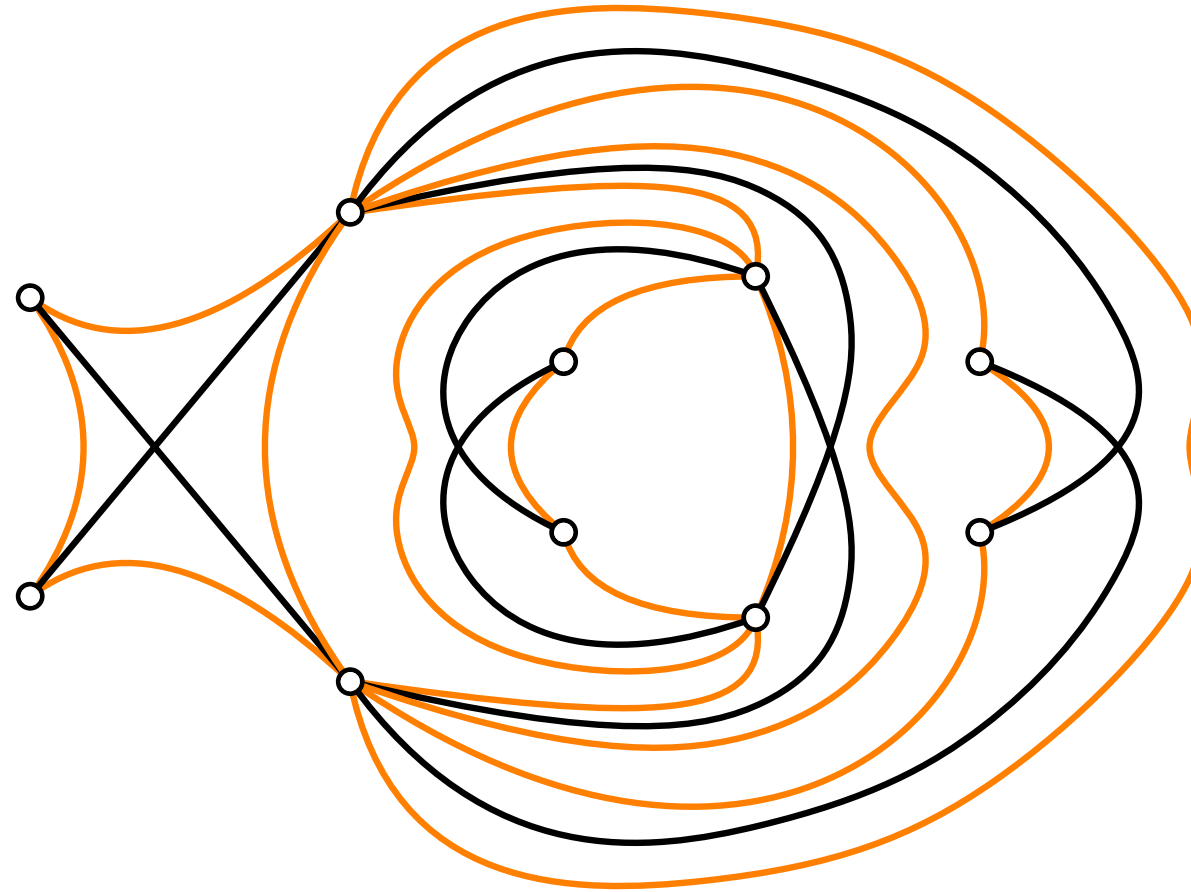
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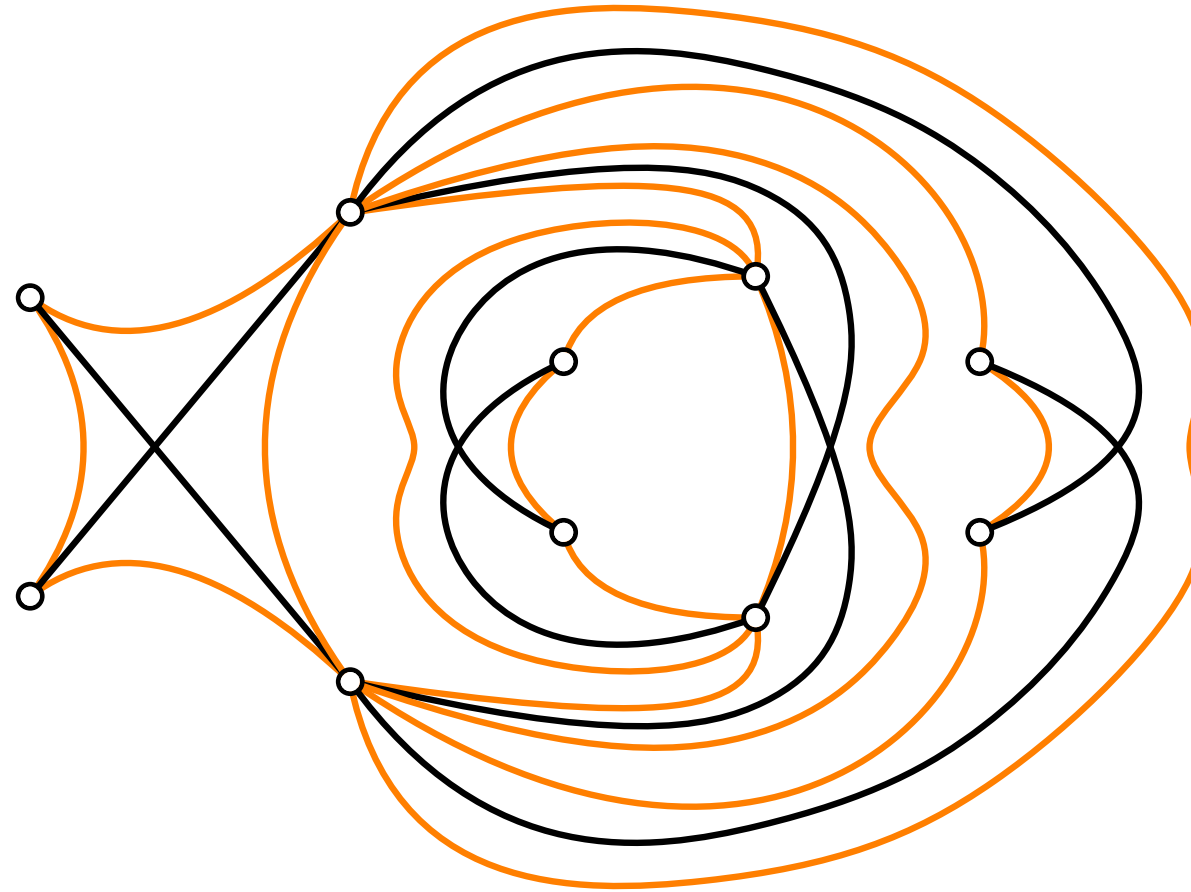
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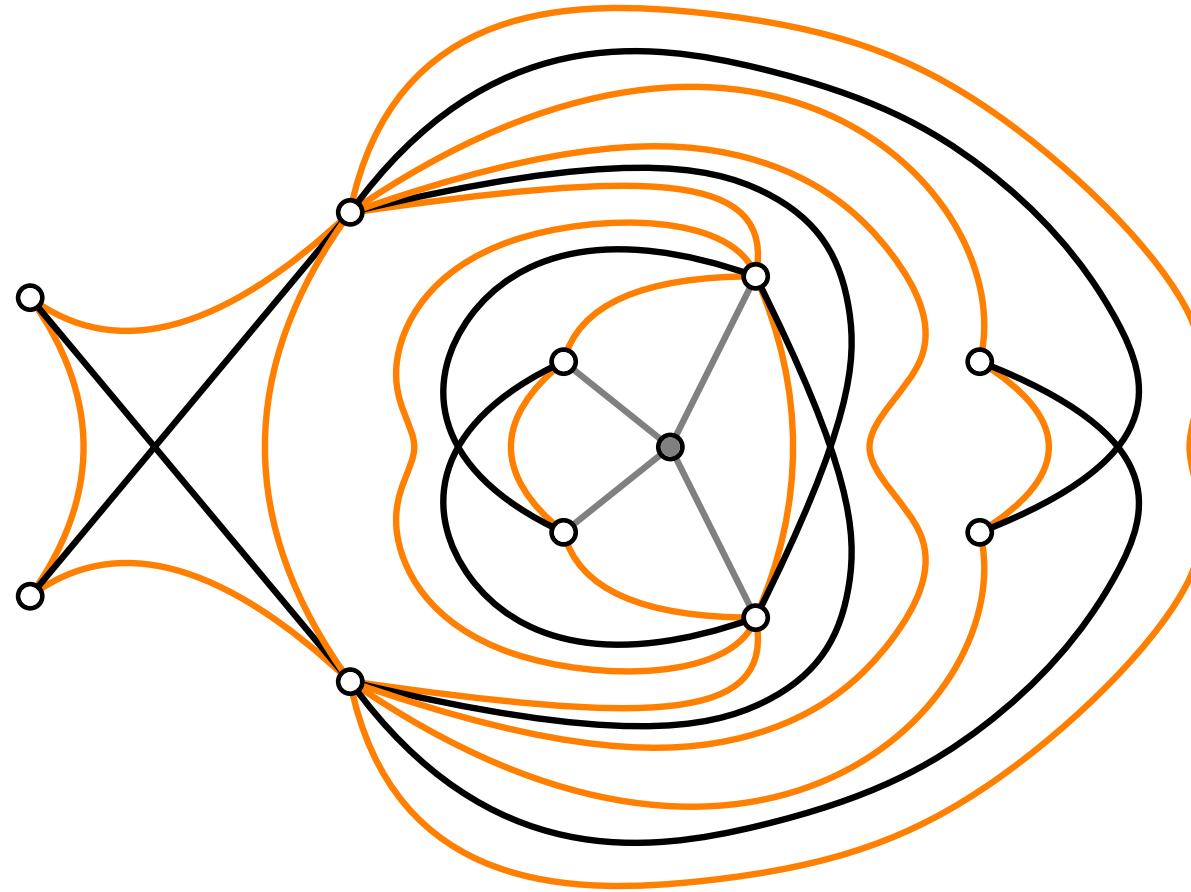
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Algorithm Step 1: Augmentation

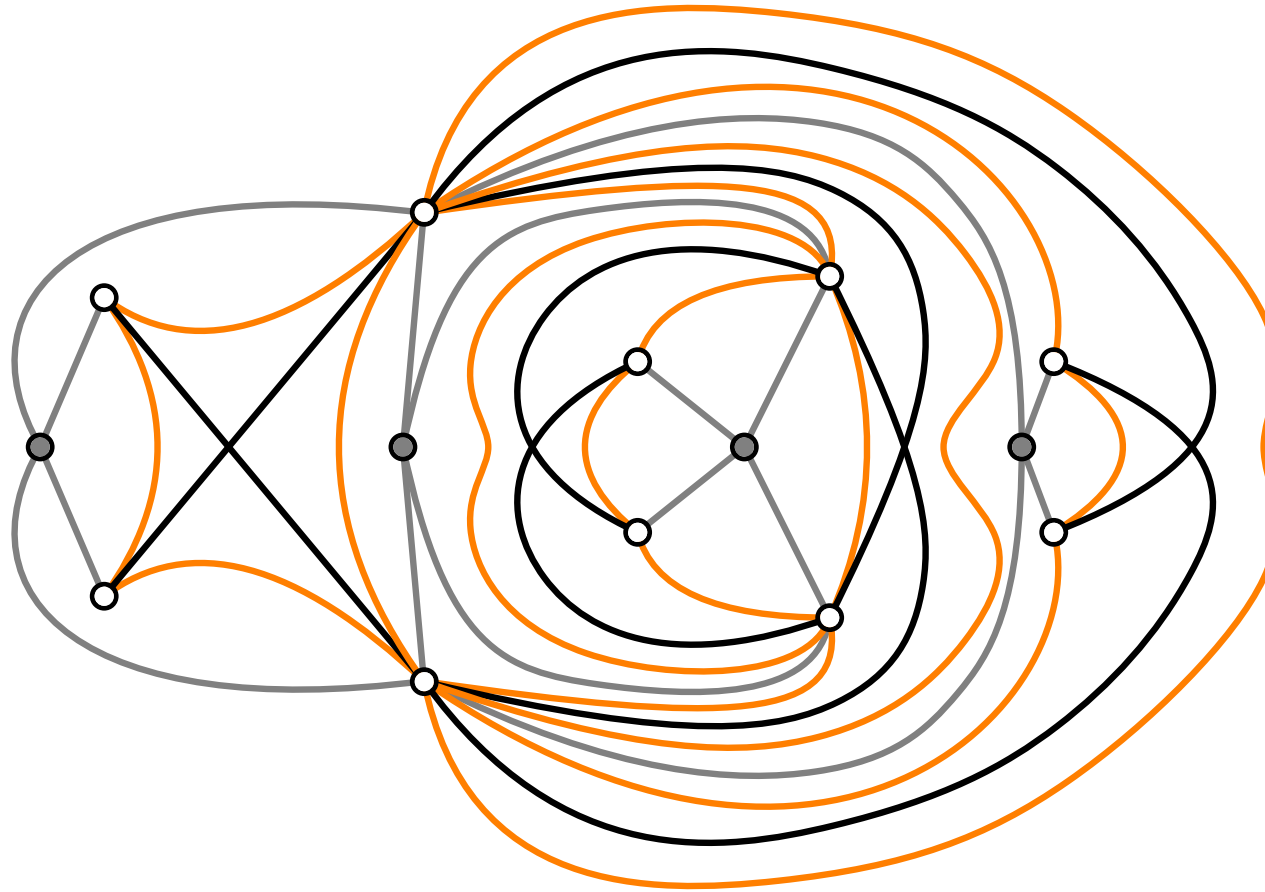
1. For each **pair of crossing edges** add an **enclosing 4-cycle**.

2. Remove those multiple edges that belong to G .

3. Remove one (multiple) edge from each face of degree two (if any). 

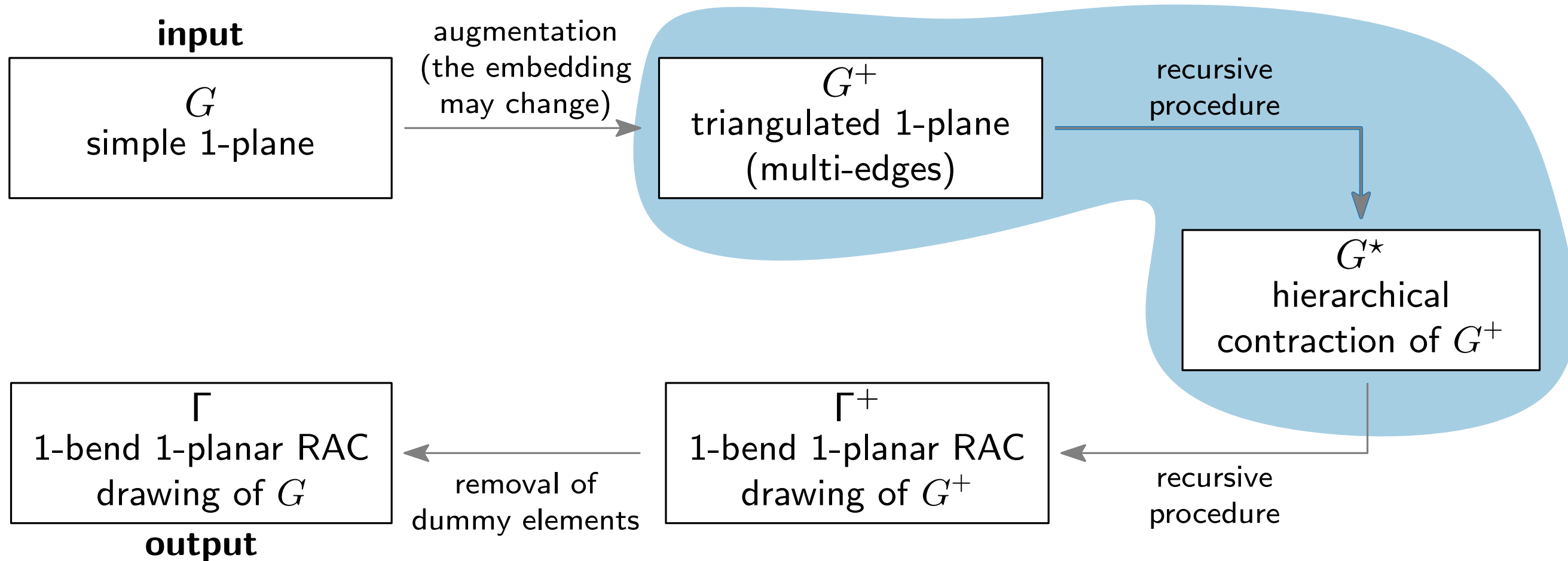
4. Triangulate faces of degree > 3 by inserting a **star** inside them.

G : simple 1-plane graph $\longrightarrow$ G^+ : triangulated 1-plane (possibly with multi-edges)



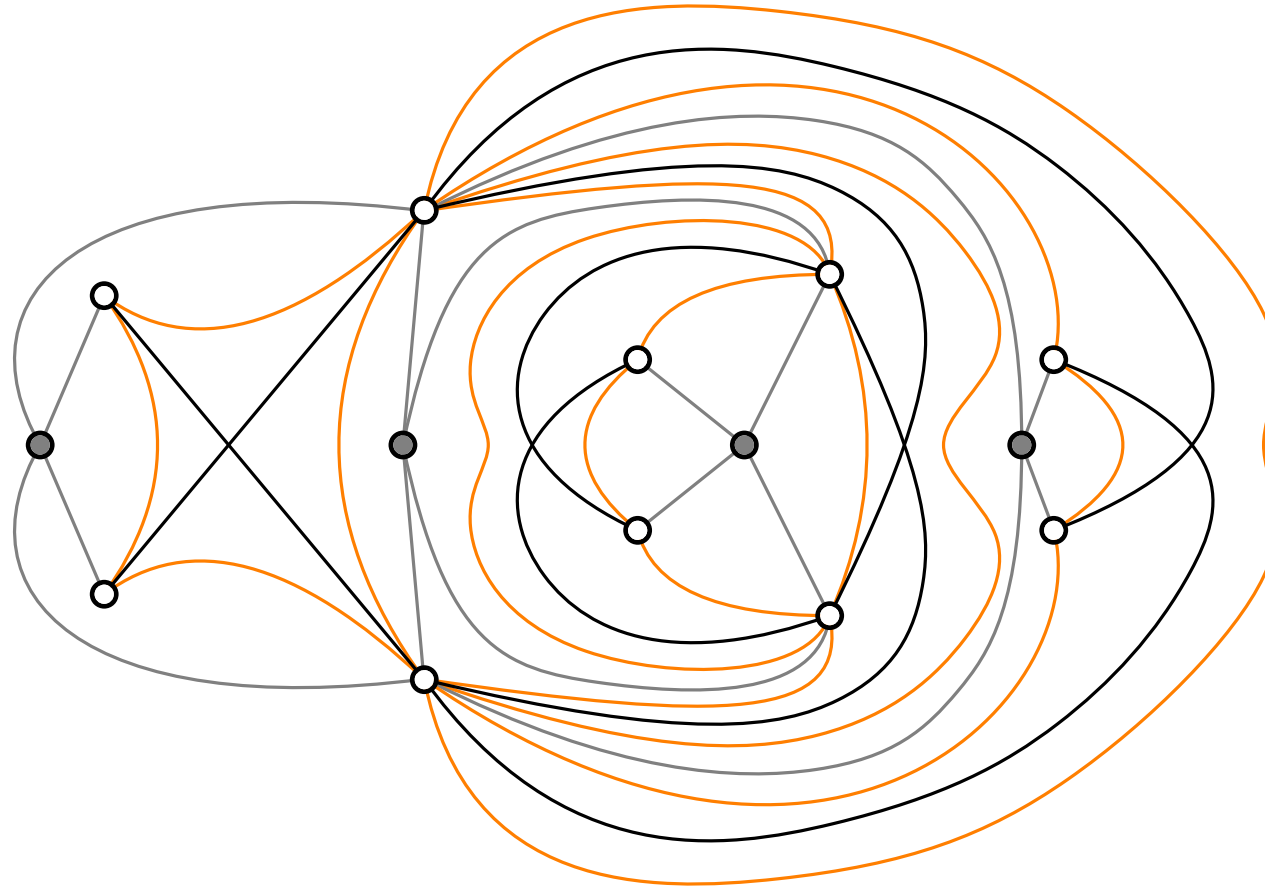
Note that we can still have parallel (**orange**) edges

Algorithm Outline



Algorithm Step 2: Hierarchical Contractions

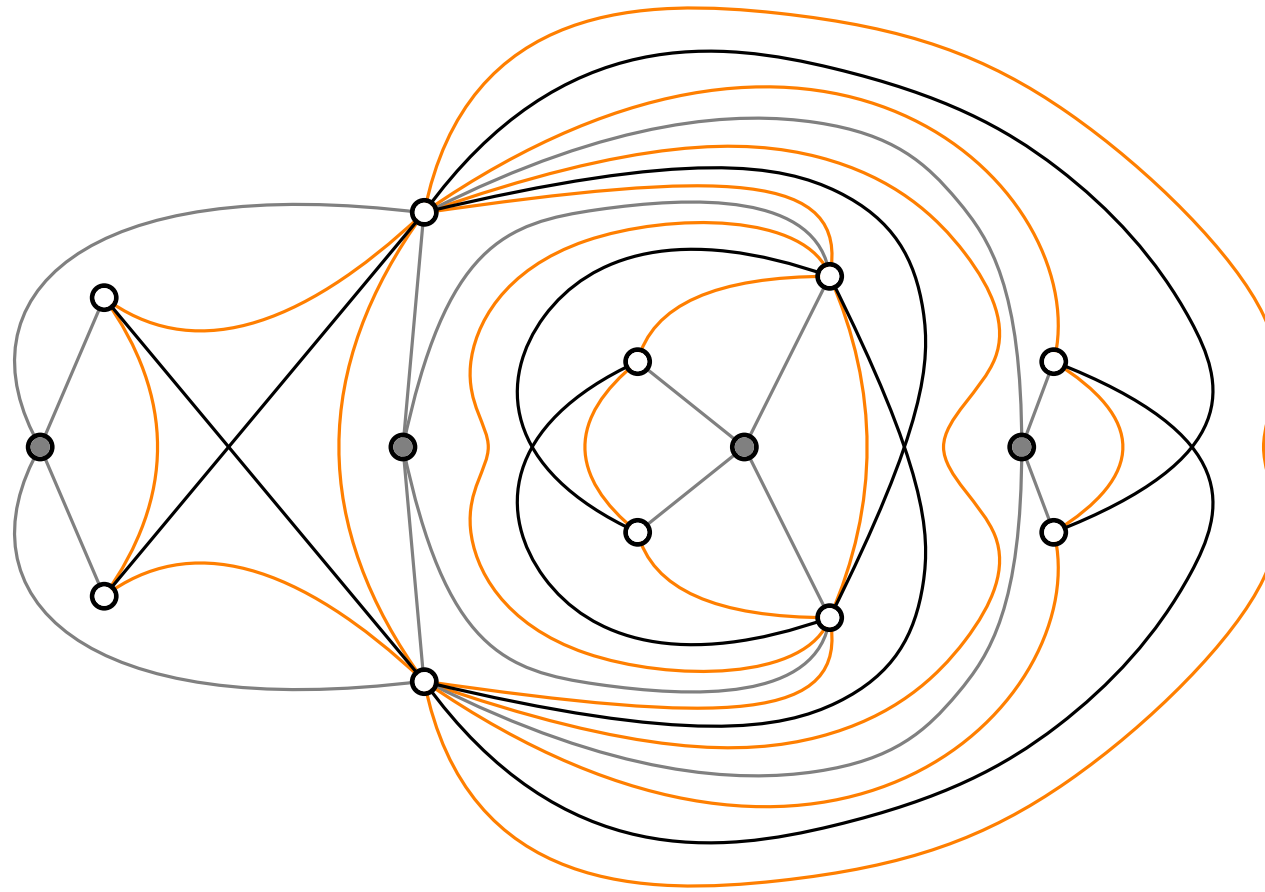
G^+
triangulated 1-plane
(multi-edges)



Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

■ triangular faces

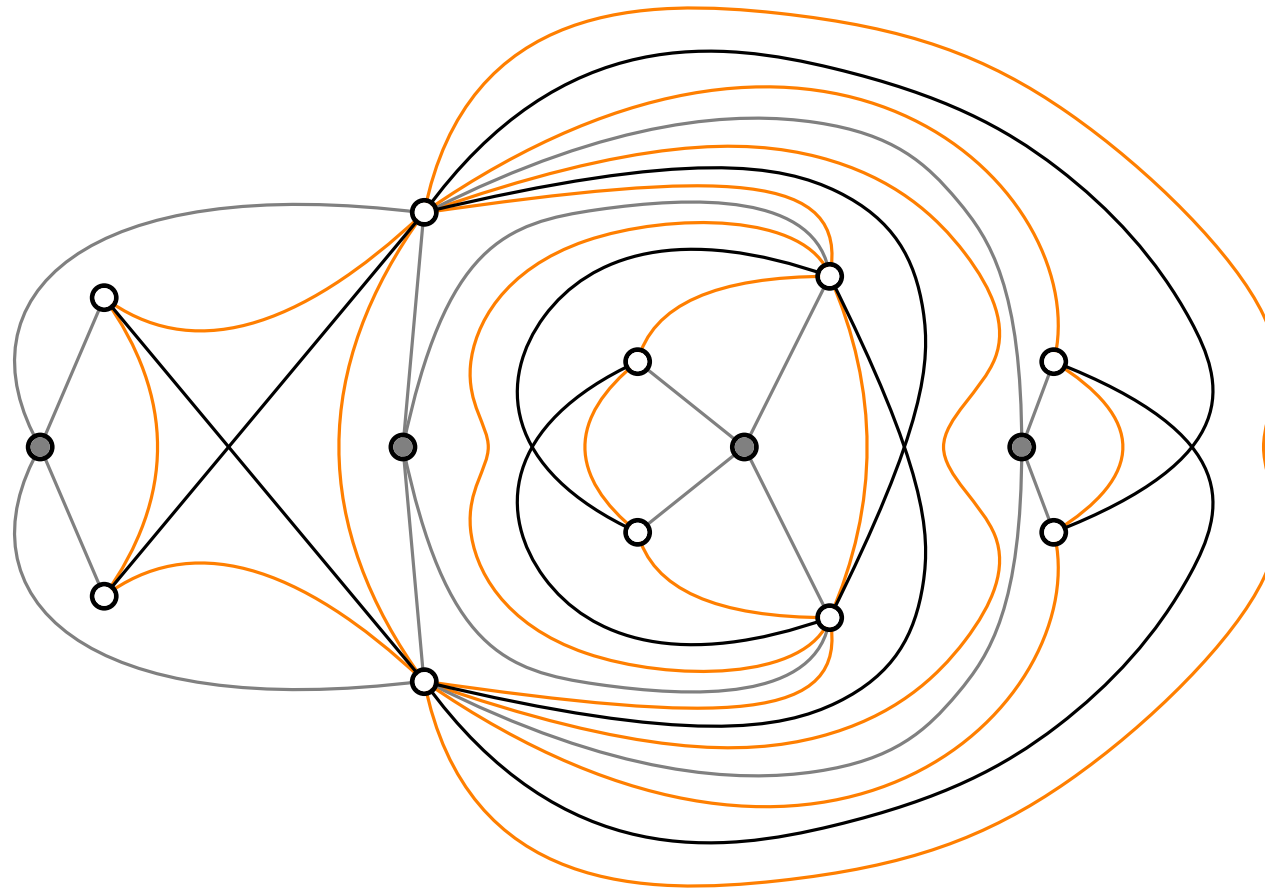


Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

■ triangular faces

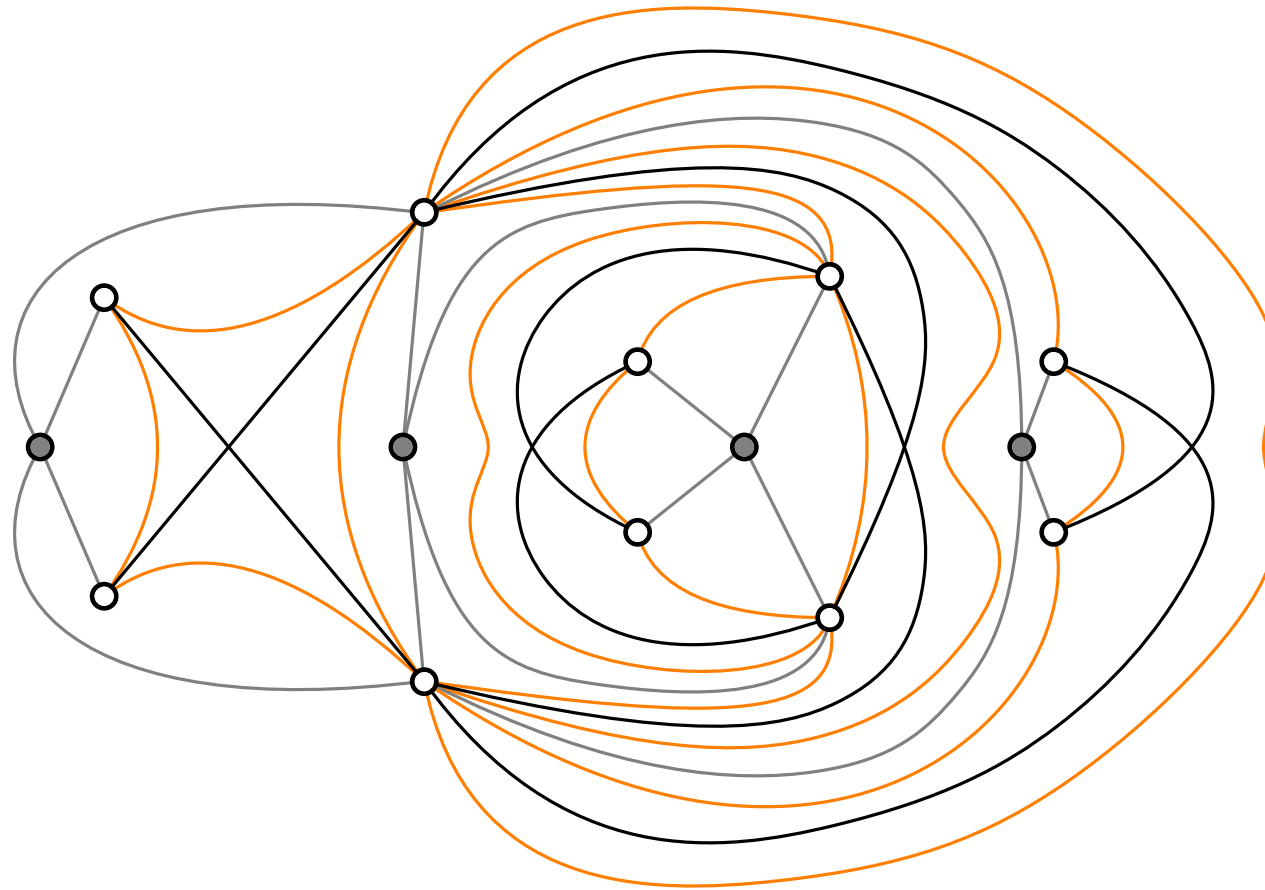
■ multiple edges
never crossed



Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

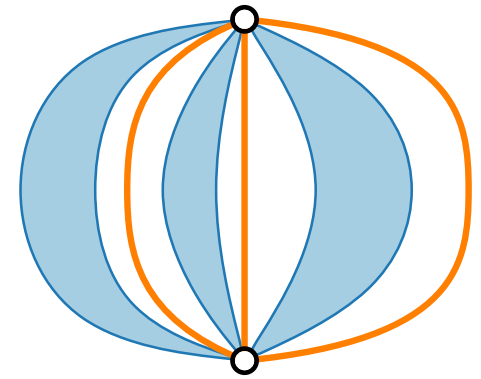
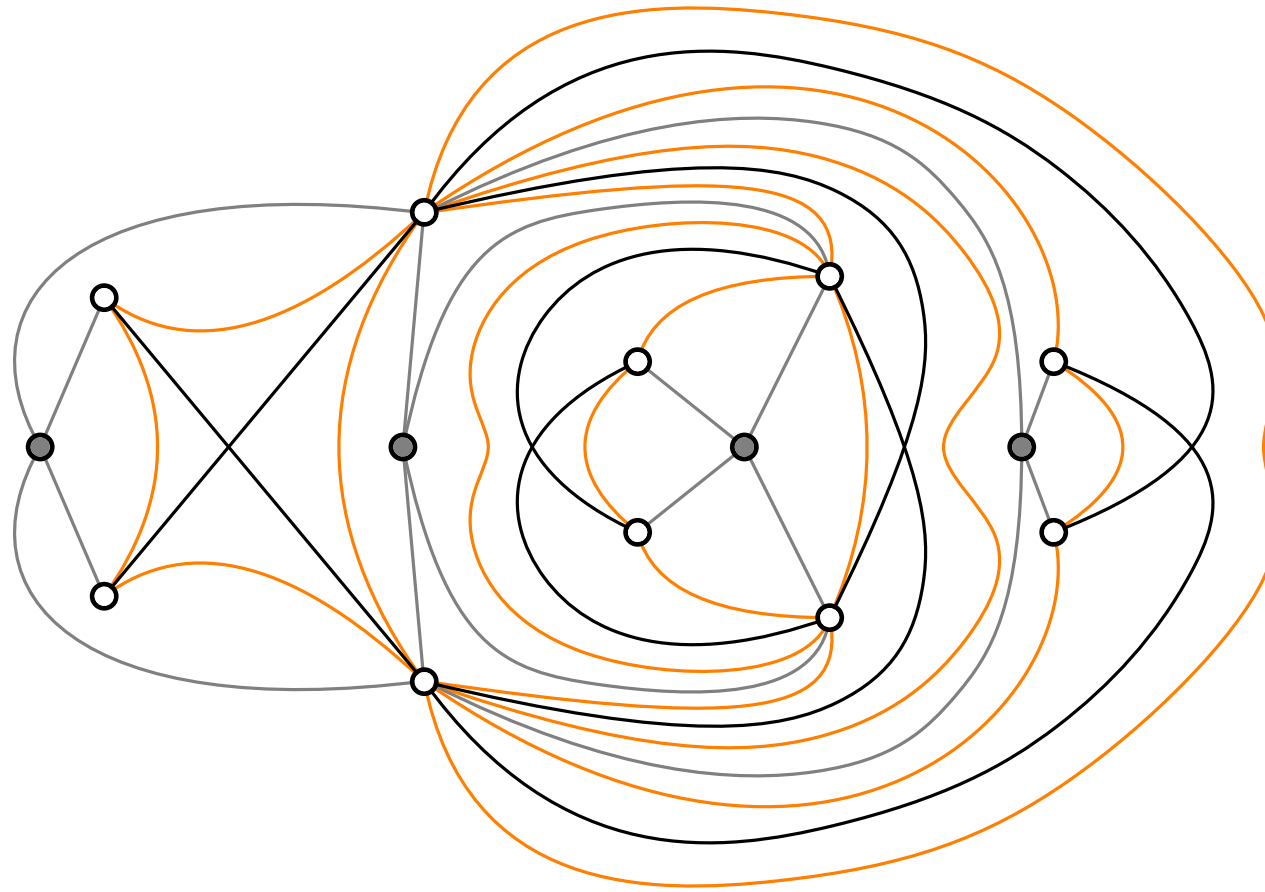
- triangular faces
- multiple edges
never crossed
- only empty kites



Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites

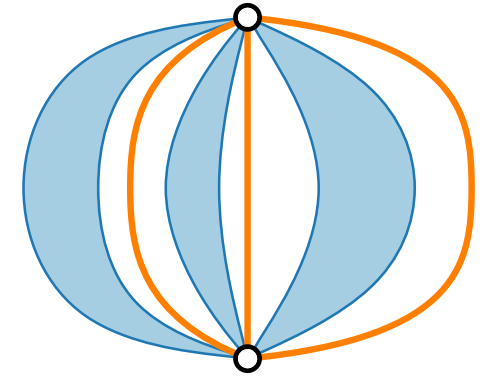
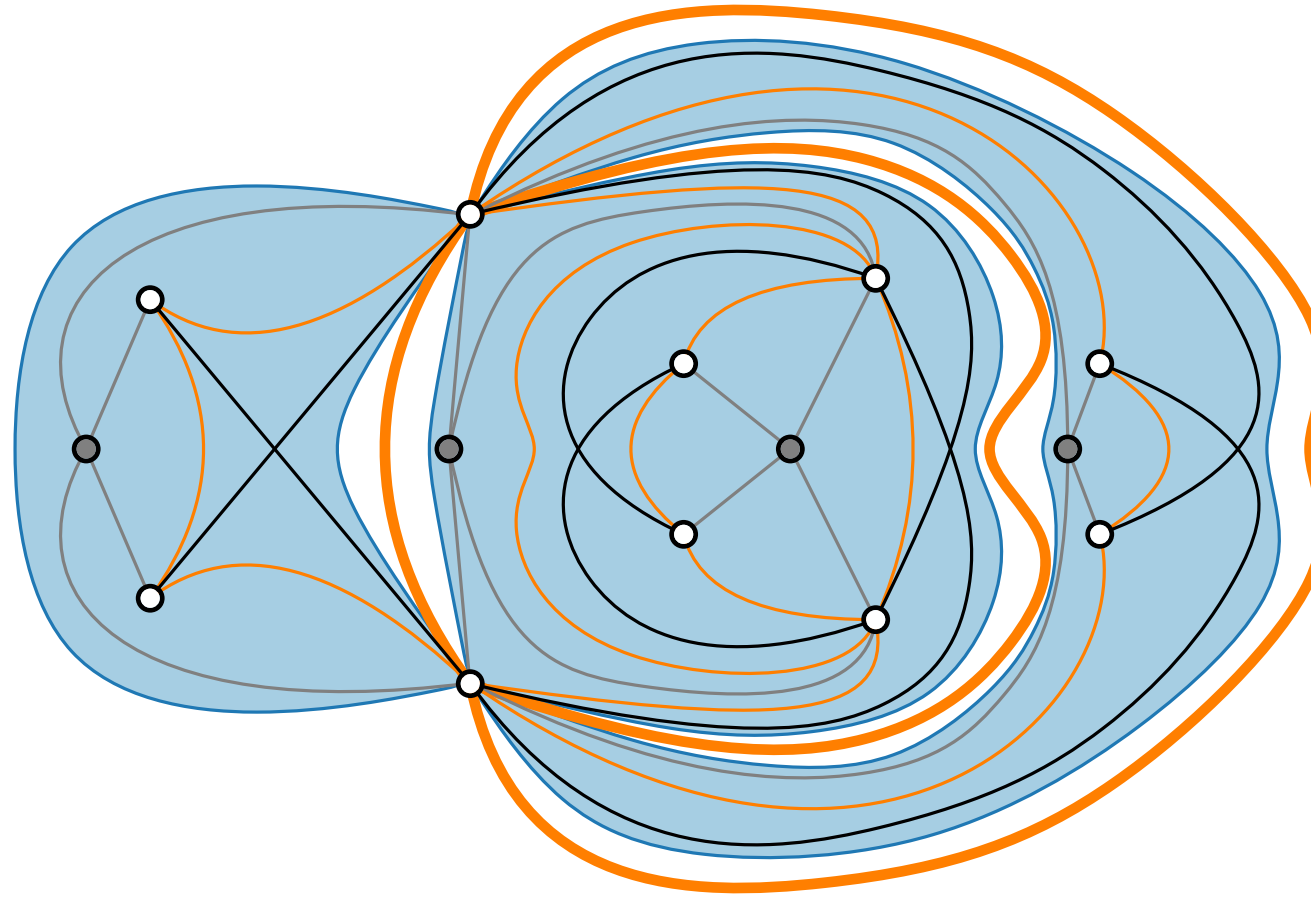


structure of each
separation pair

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites

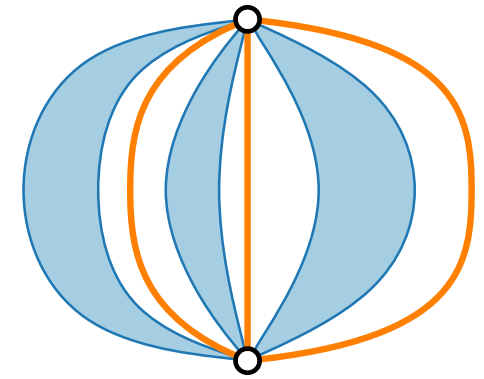
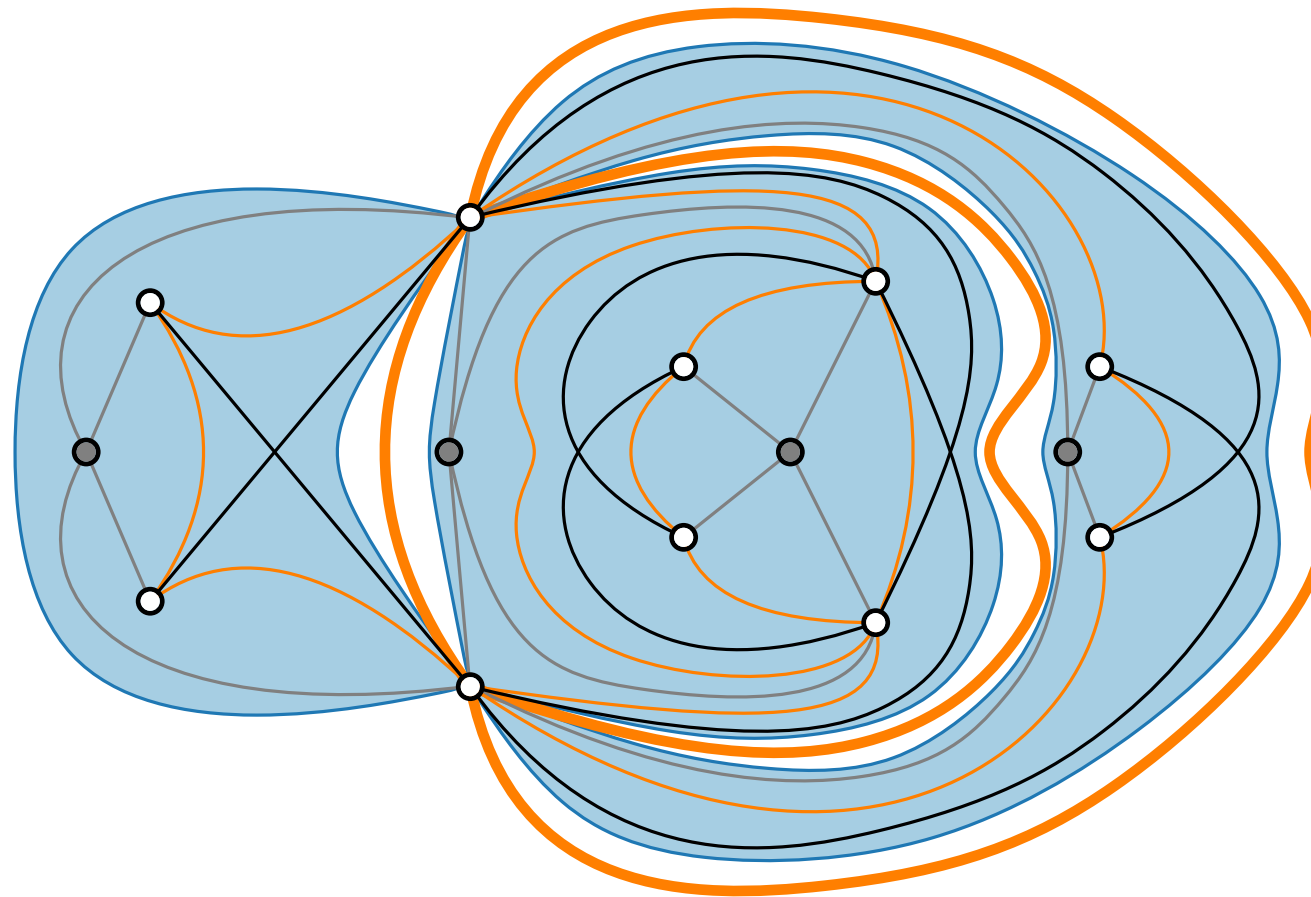


structure of each
separation pair

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



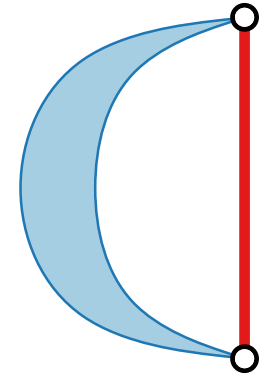
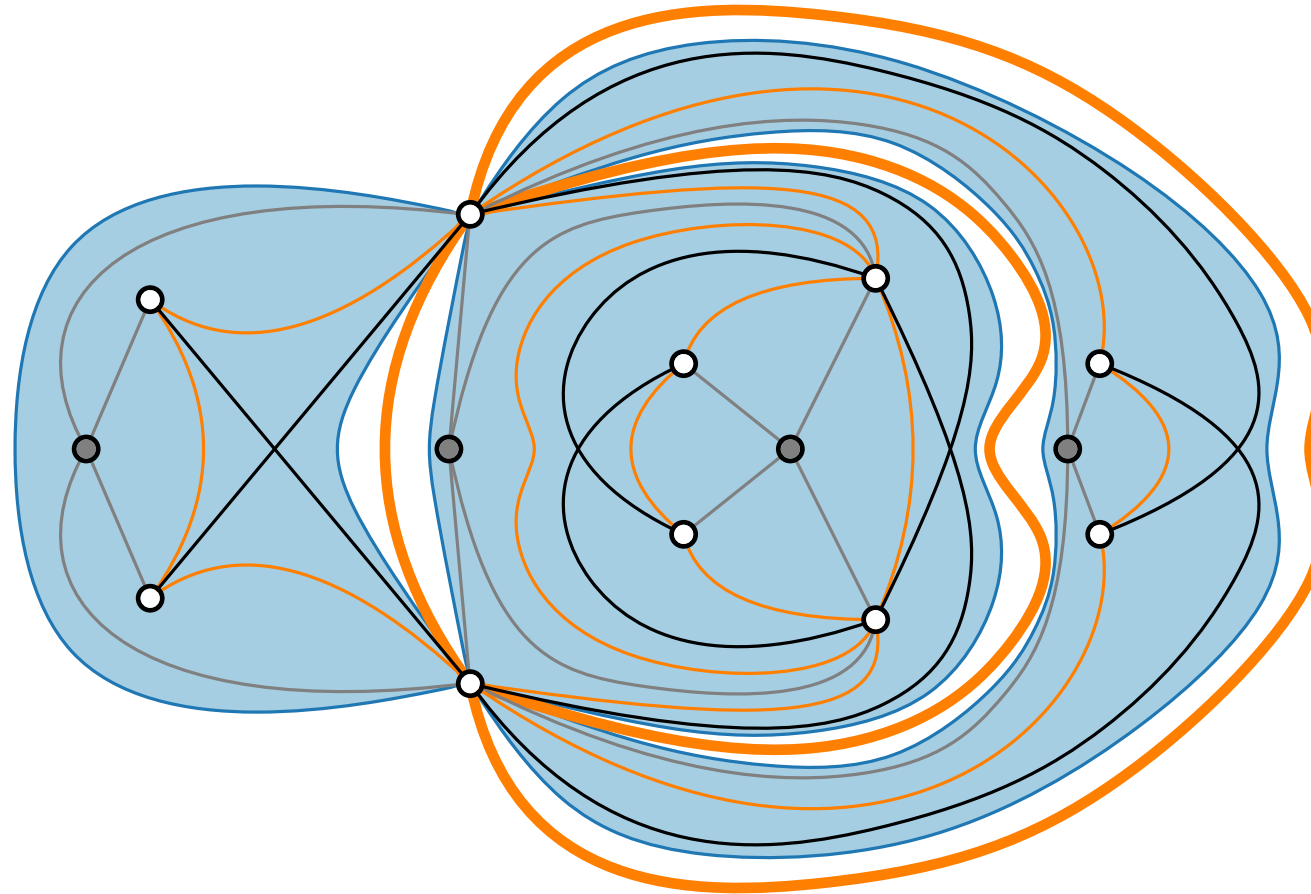
structure of each
separation pair

Contract all inner
components of each
separation pair into
a **thick edge**.

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



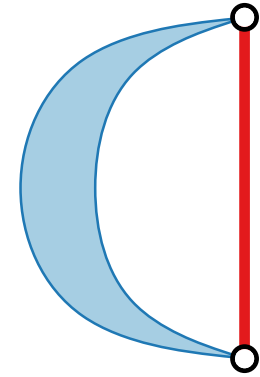
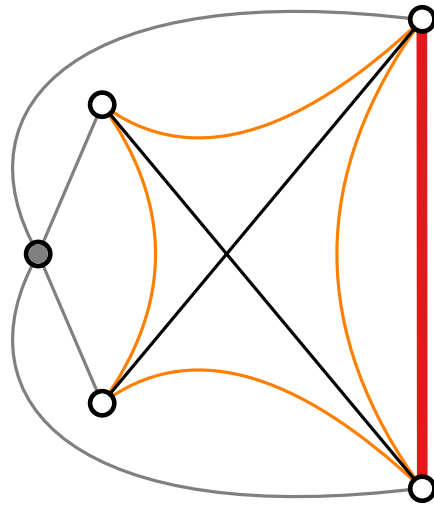
structure of each
separation pair

Contract all inner
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Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



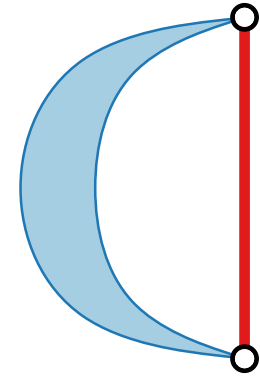
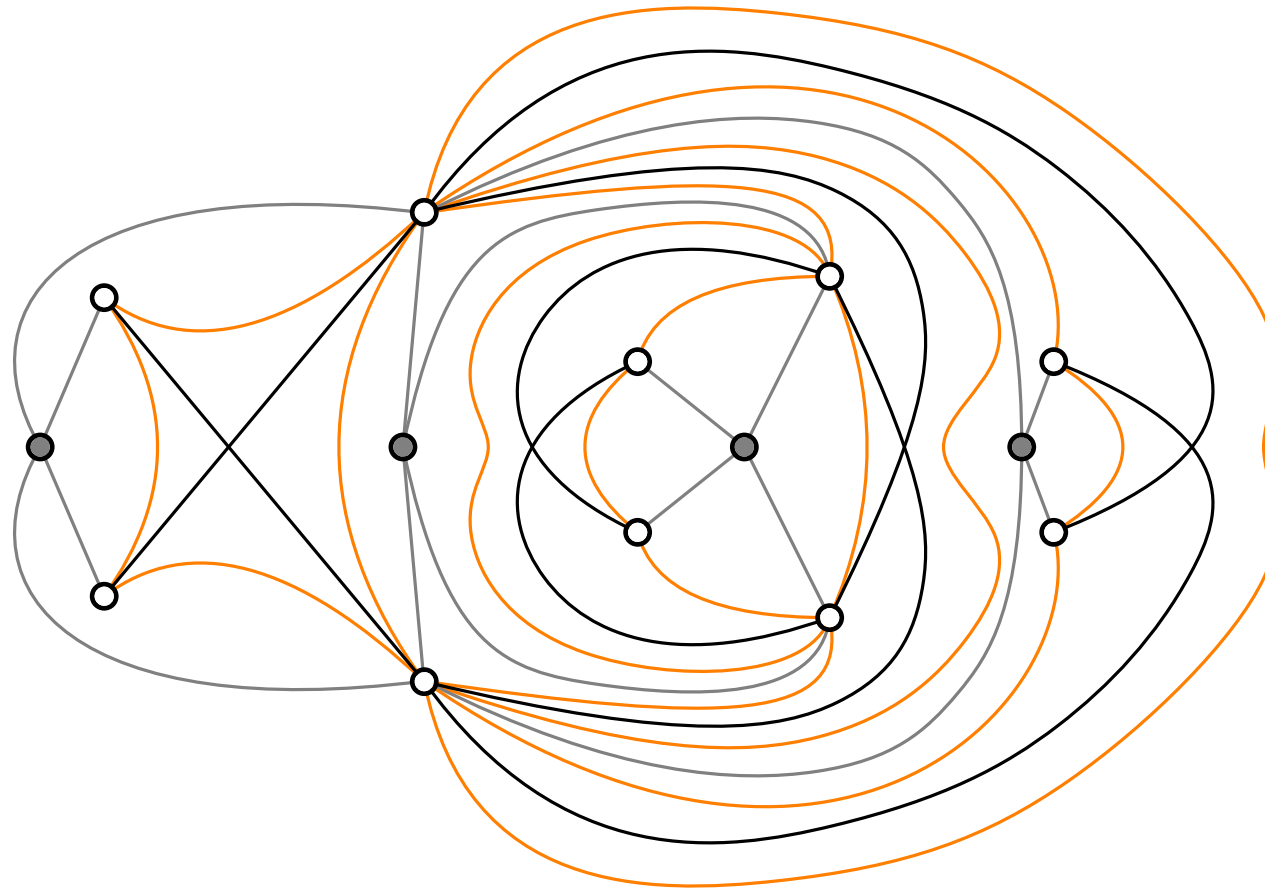
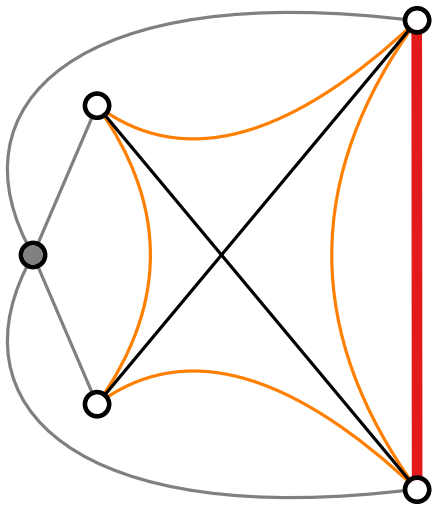
structure of each
separation pair

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components of each
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a **thick edge**.

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



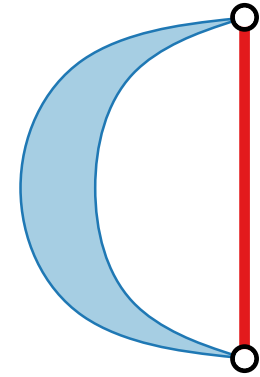
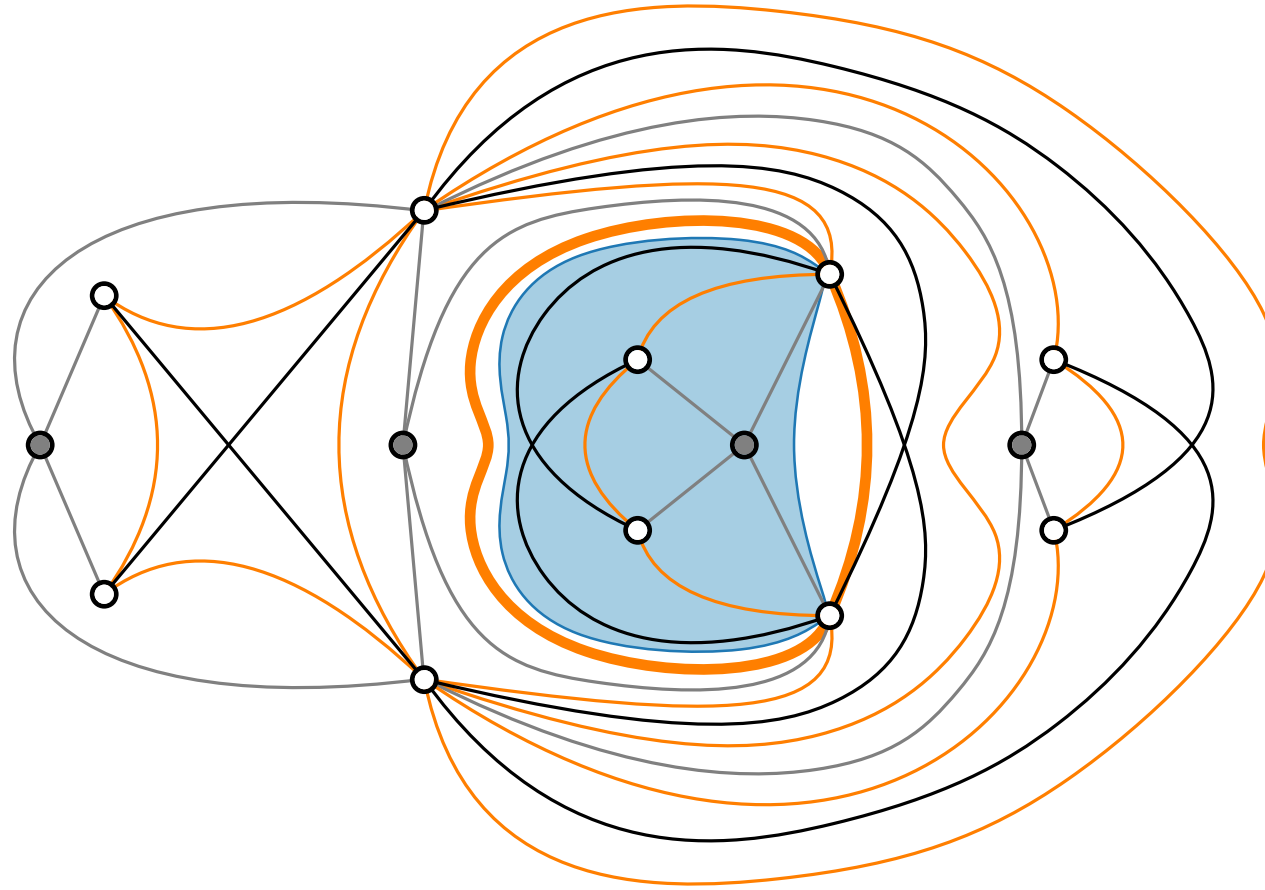
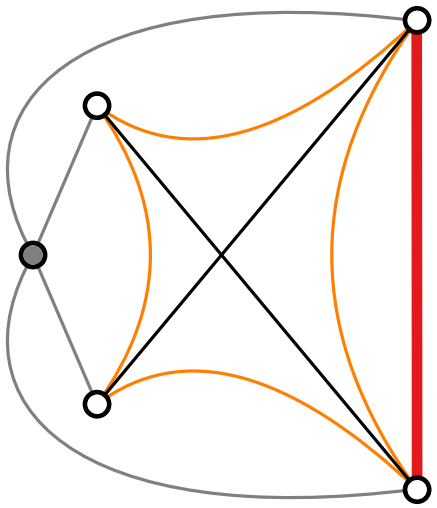
structure of each
separation pair

Contract all inner
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a **thick edge**.

Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites



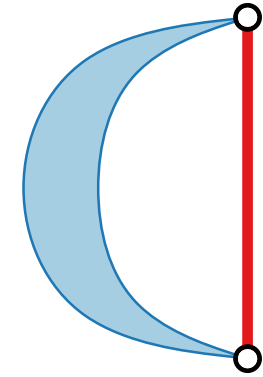
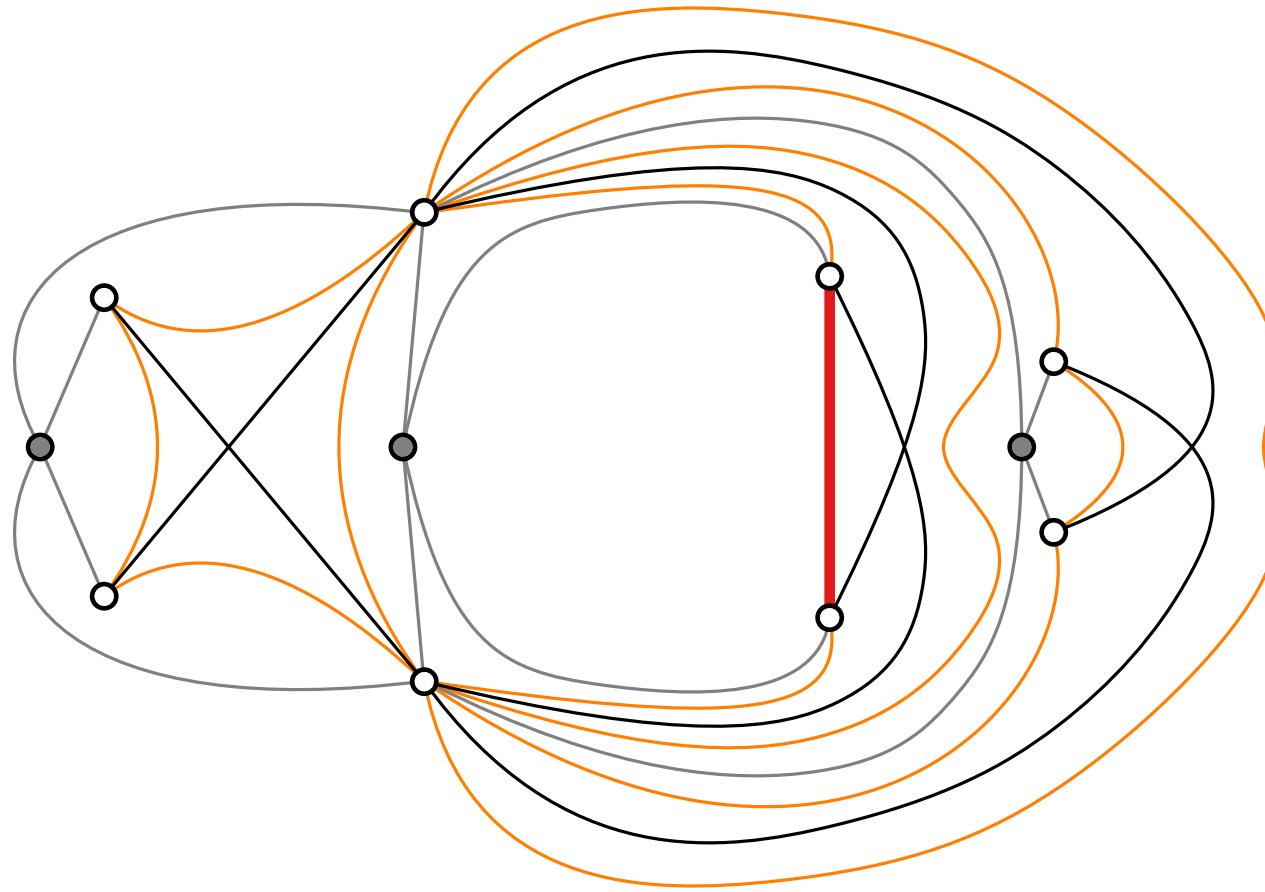
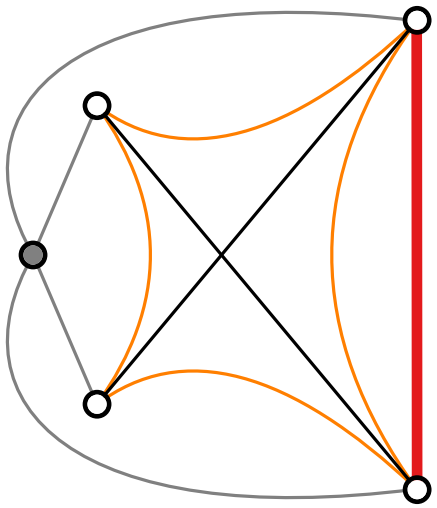
structure of each
separation pair

Contract all inner
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Algorithm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges
never crossed
- only empty kites

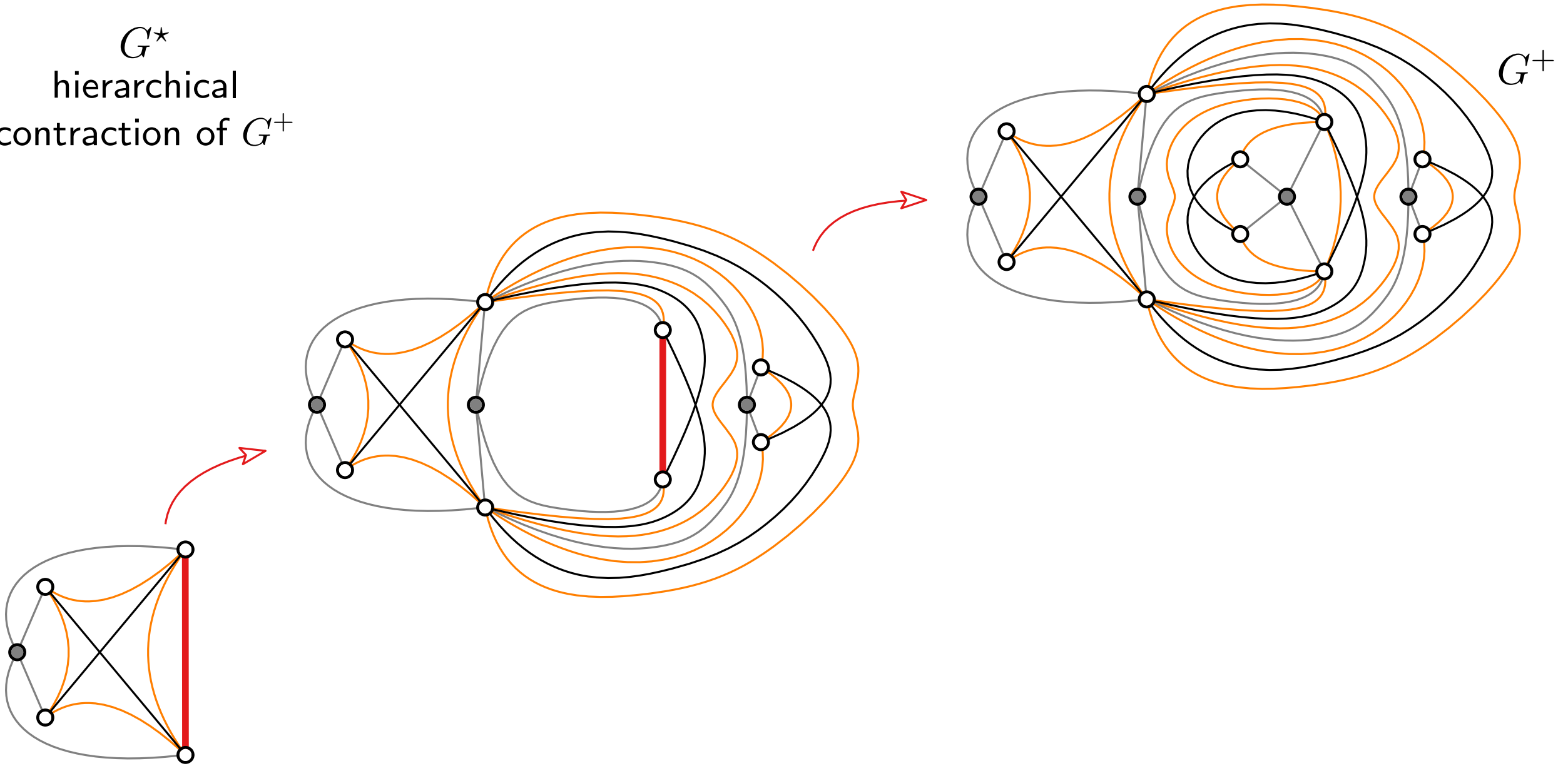


structure of each
separation pair

Contract all inner
components of each
separation pair into
a **thick edge**.

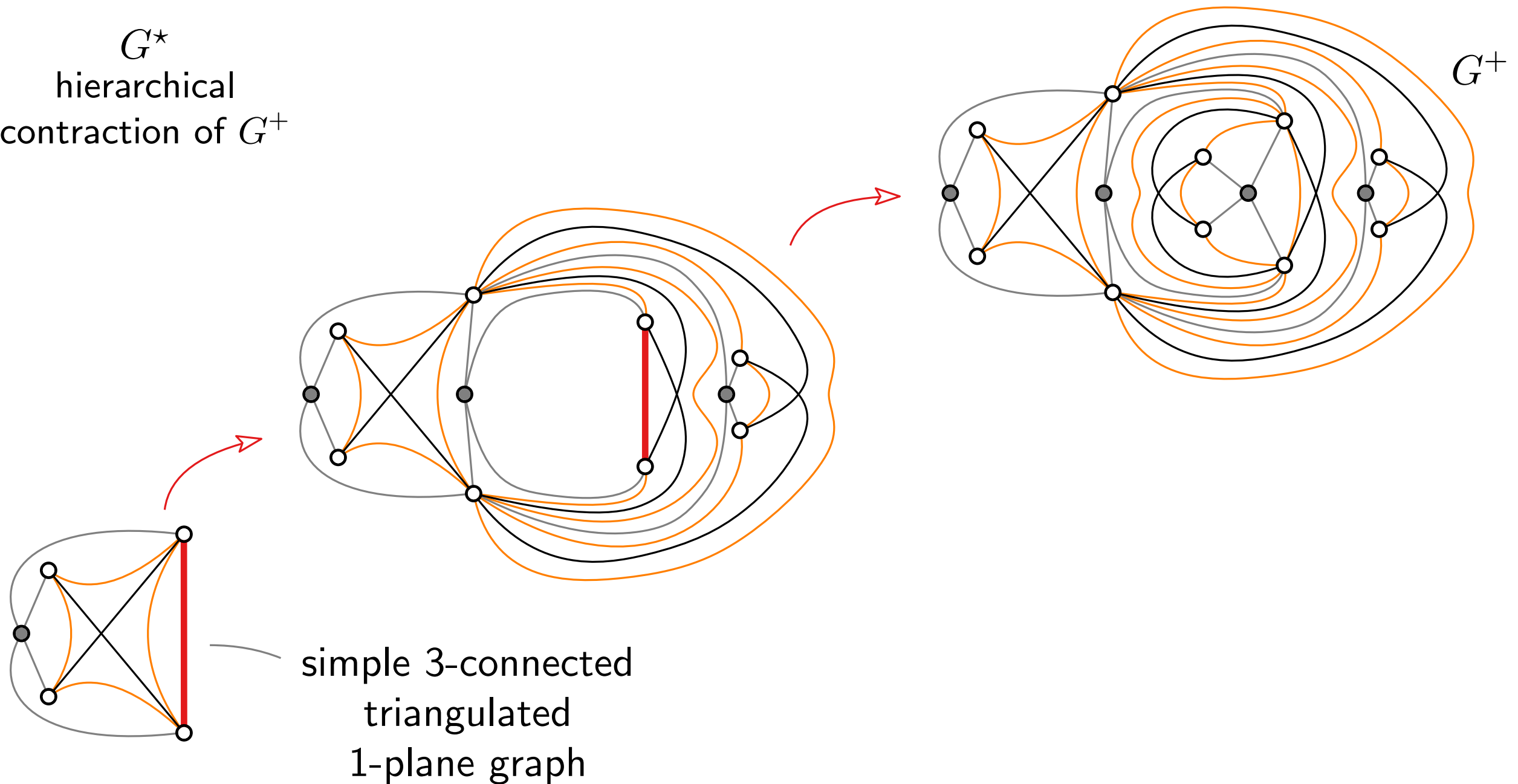
Algorithm Step 2: Hierarchical Contractions

G^*
hierarchical
contraction of G^+

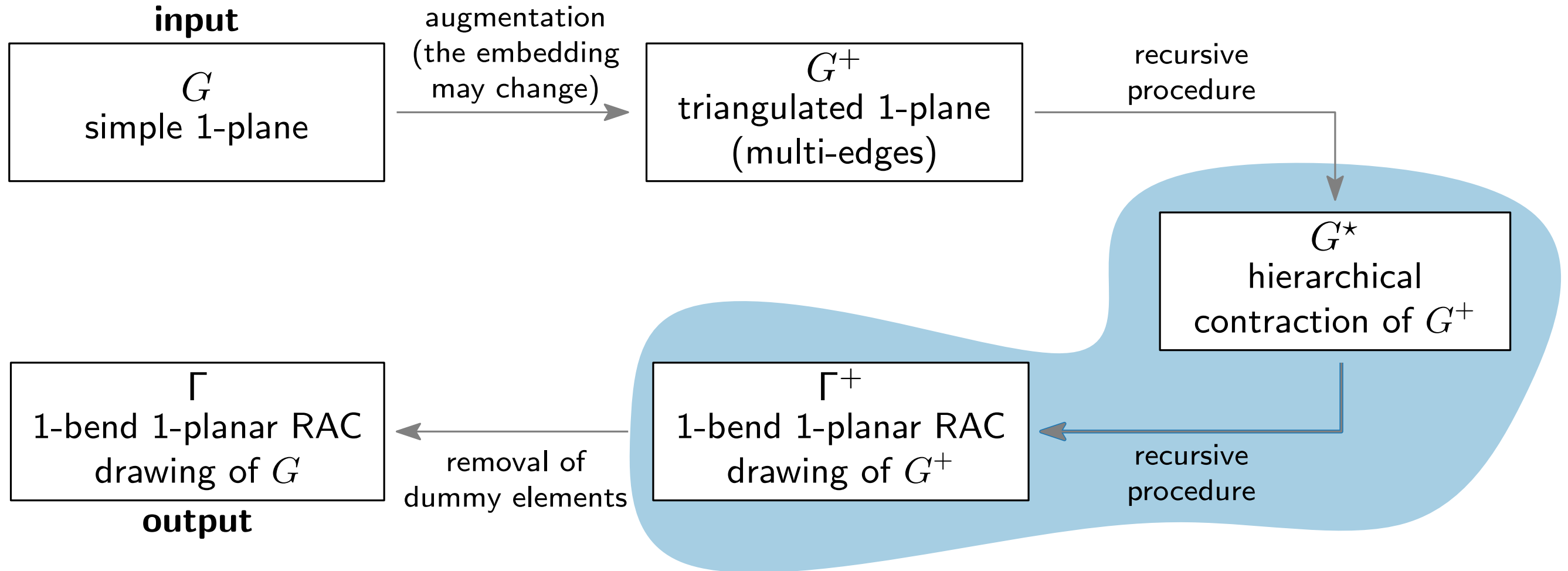


Algorithm Step 2: Hierarchical Contractions

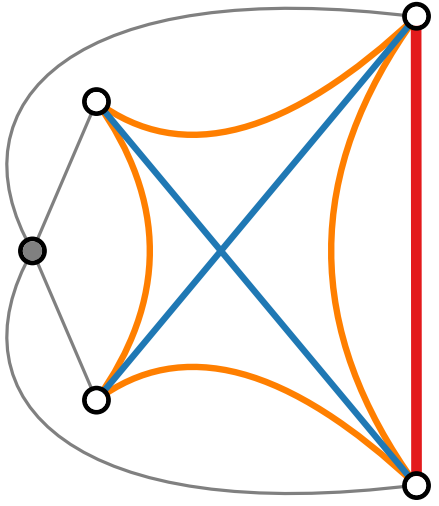
G^*
hierarchical
contraction of G^+



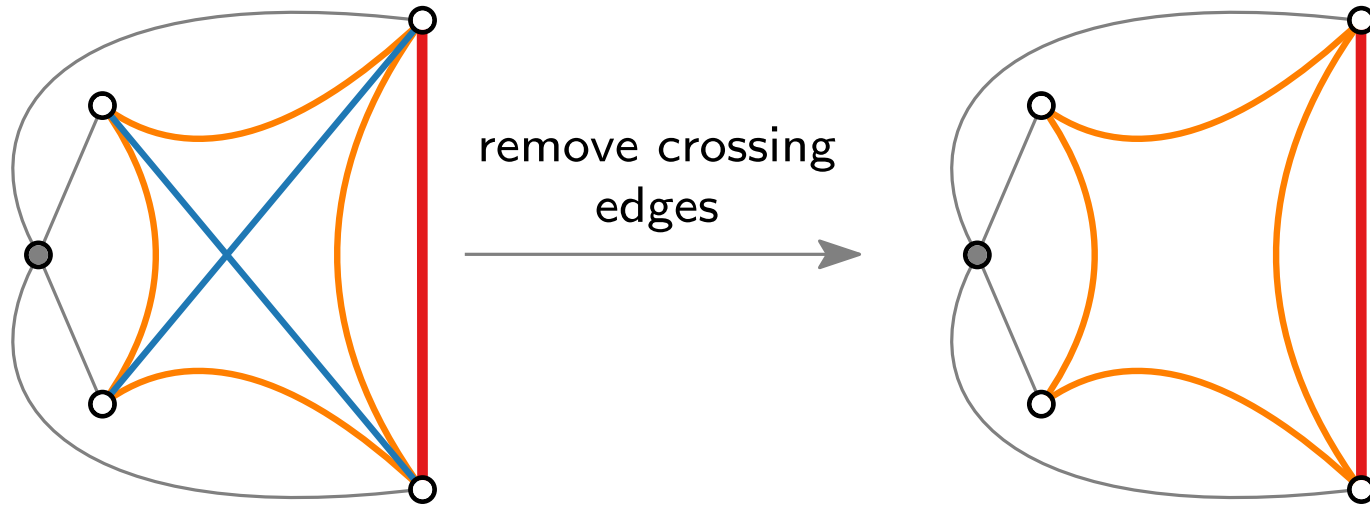
Algorithm Outline



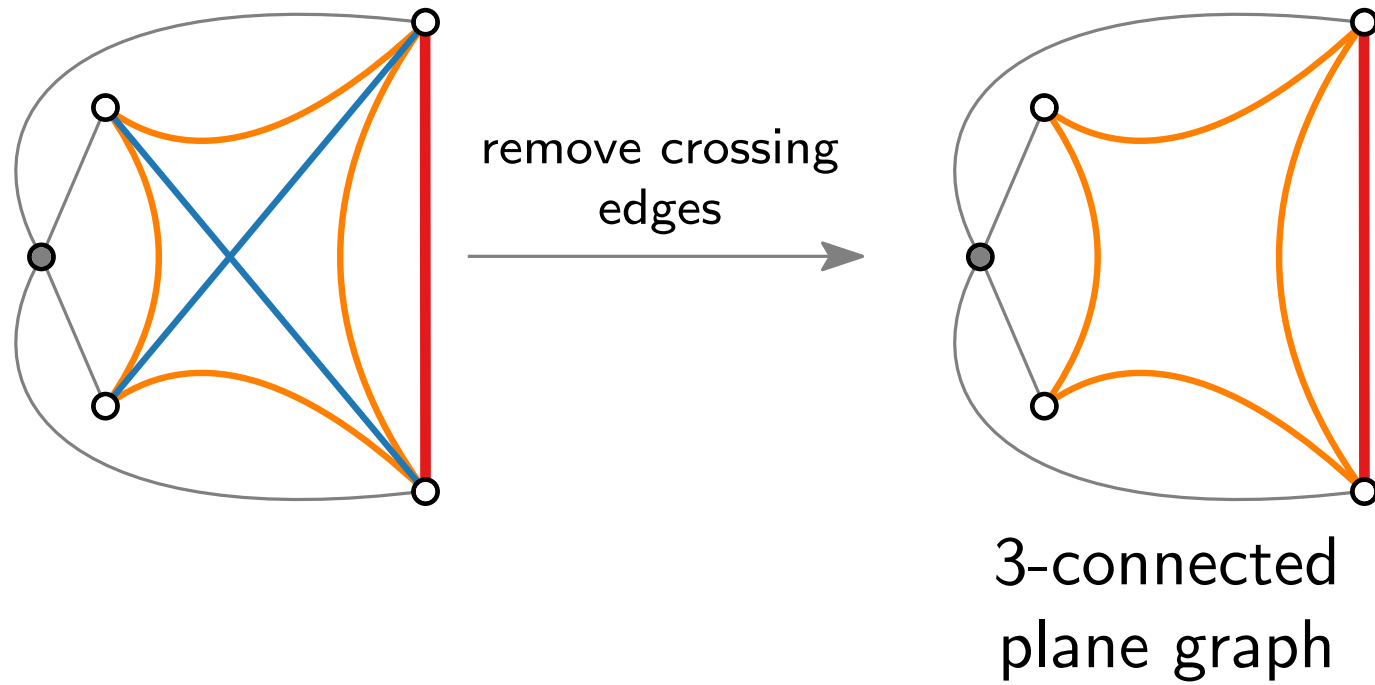
Algorithm Step 3: Drawing Procedure



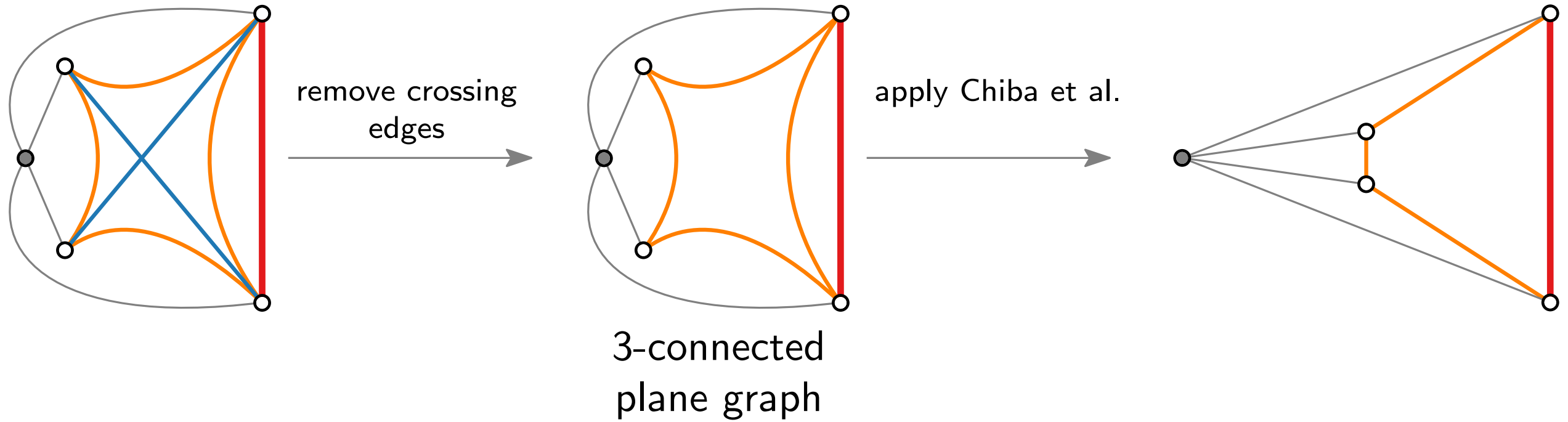
Algorithm Step 3: Drawing Procedure



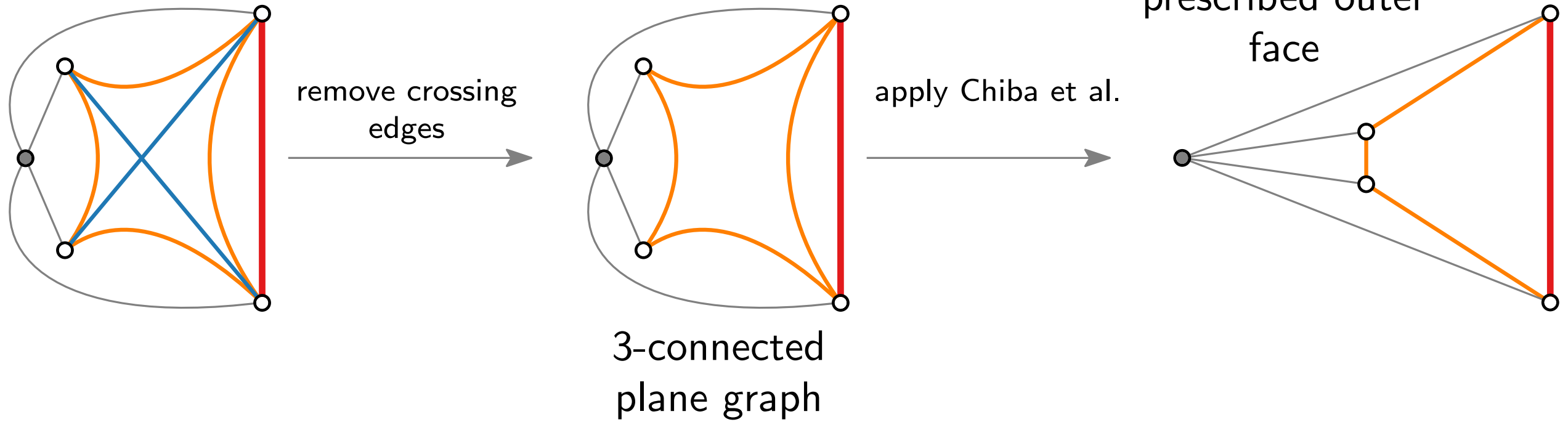
Algorithm Step 3: Drawing Procedure



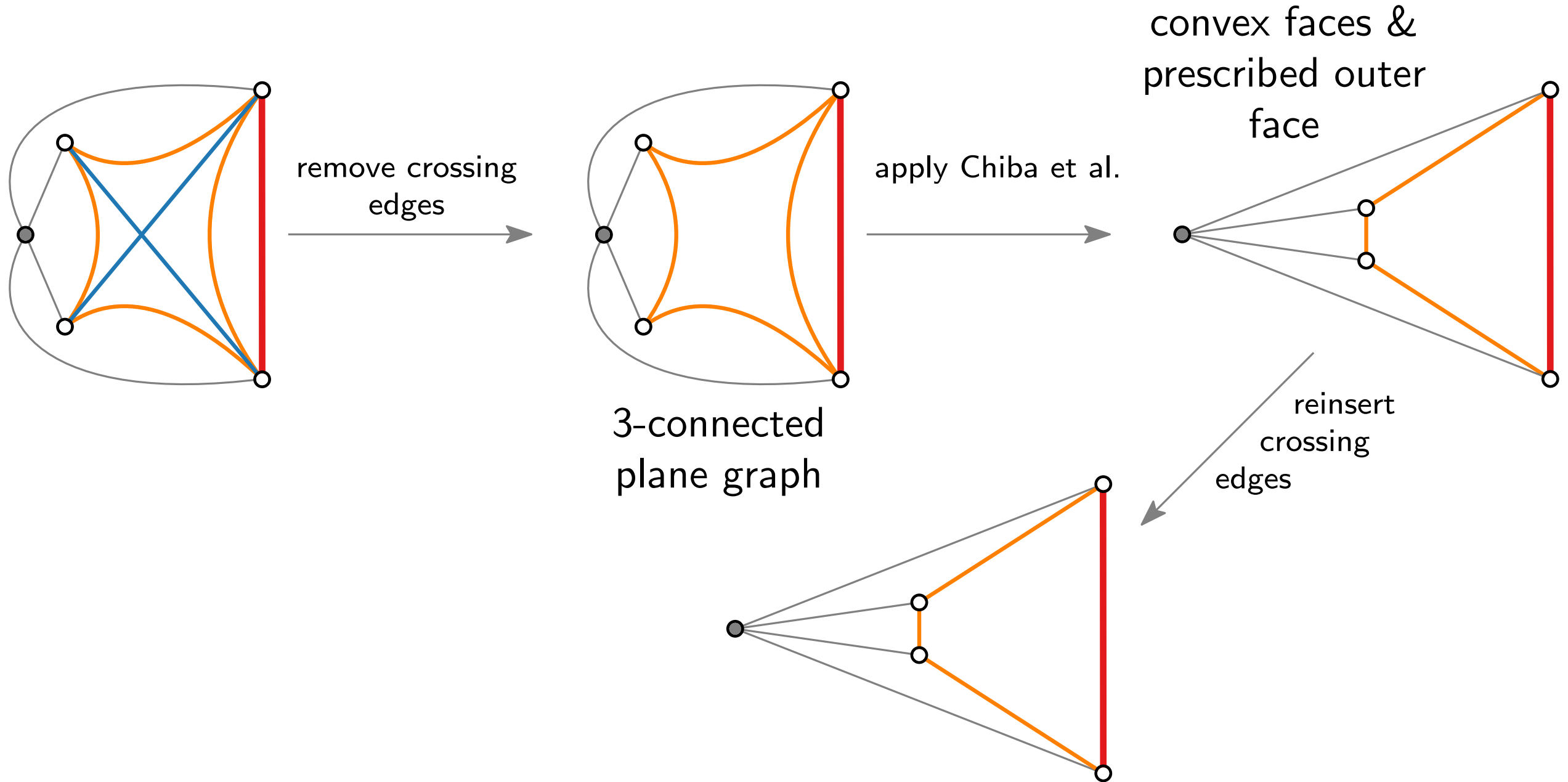
Algorithm Step 3: Drawing Procedure



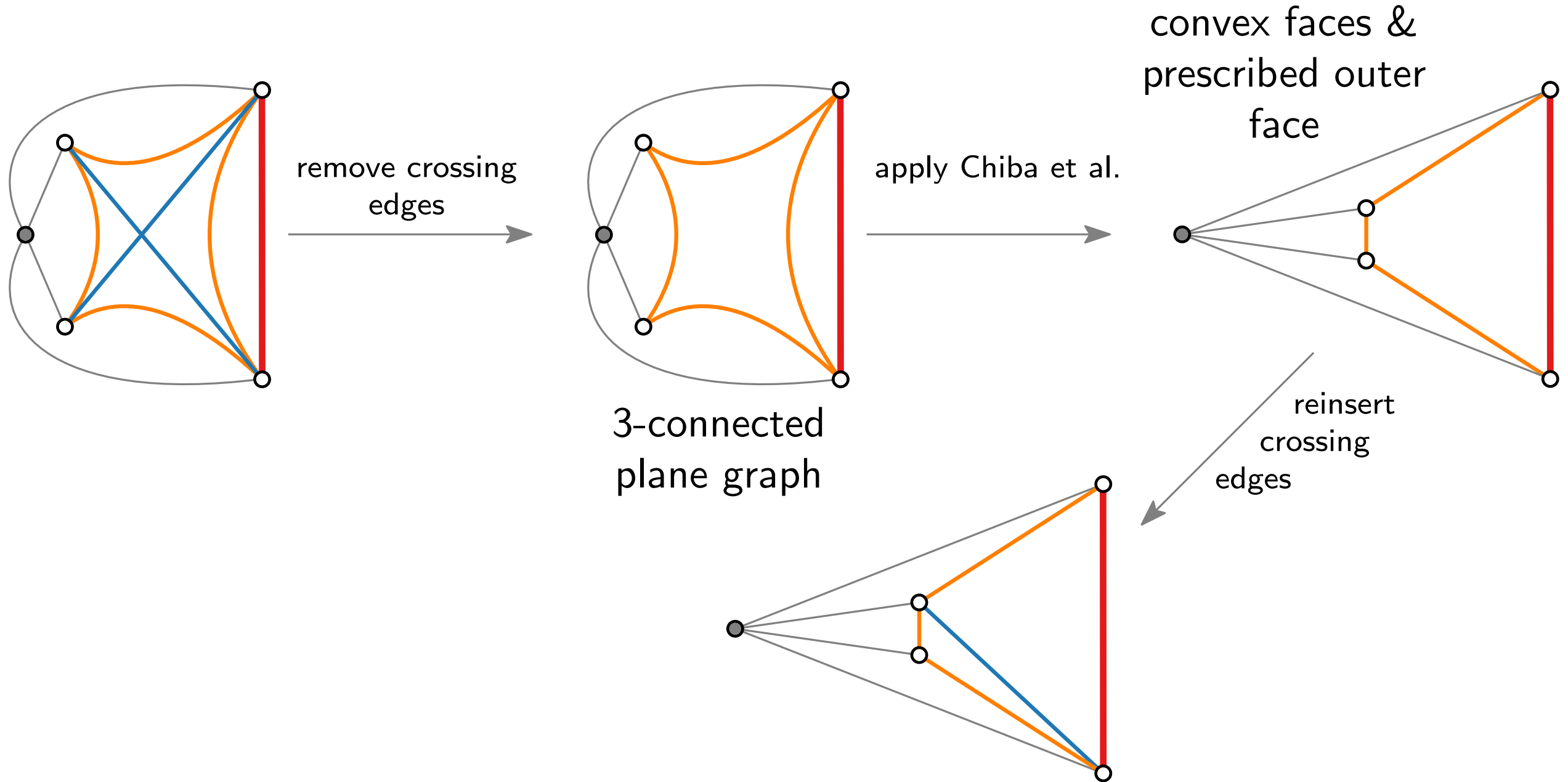
Algorithm Step 3: Drawing Procedure



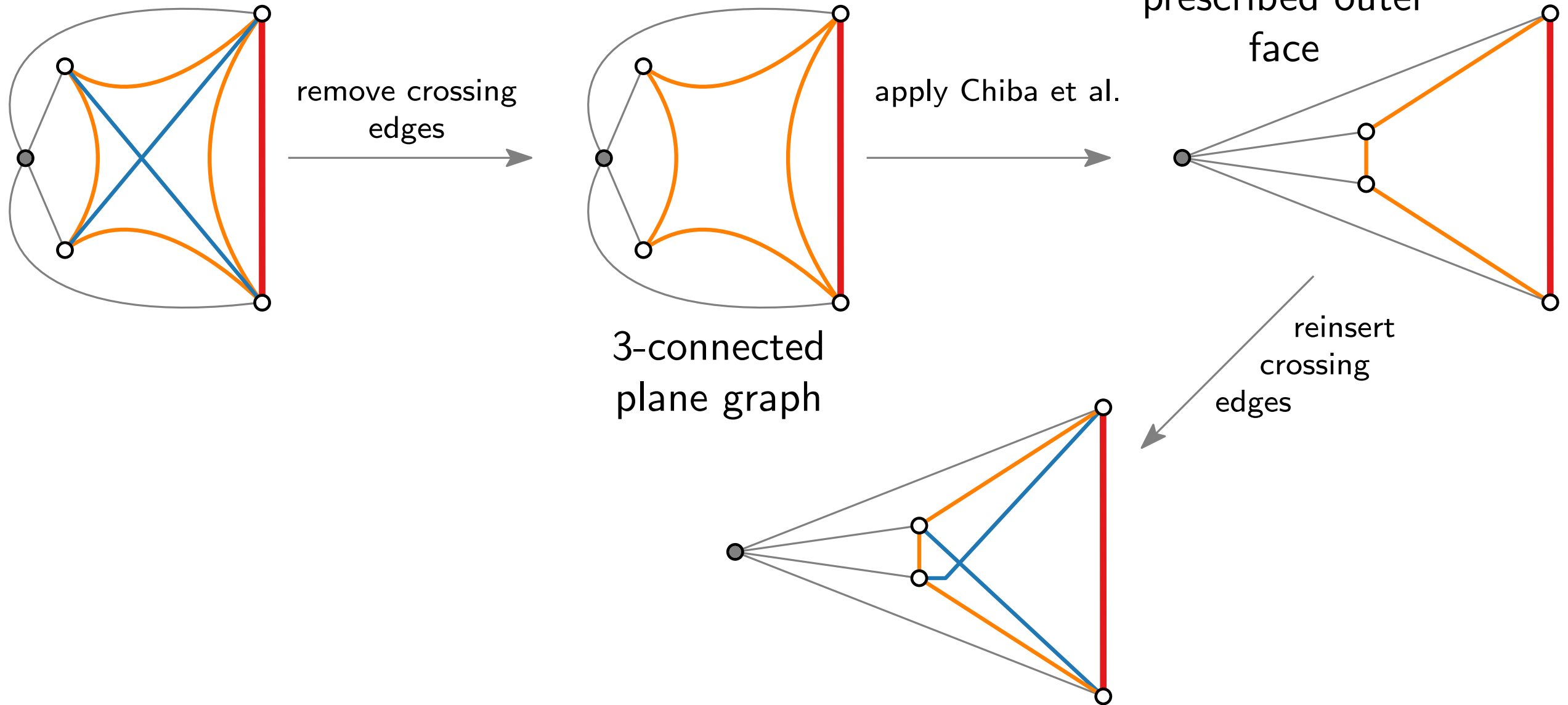
Algorithm Step 3: Drawing Procedure



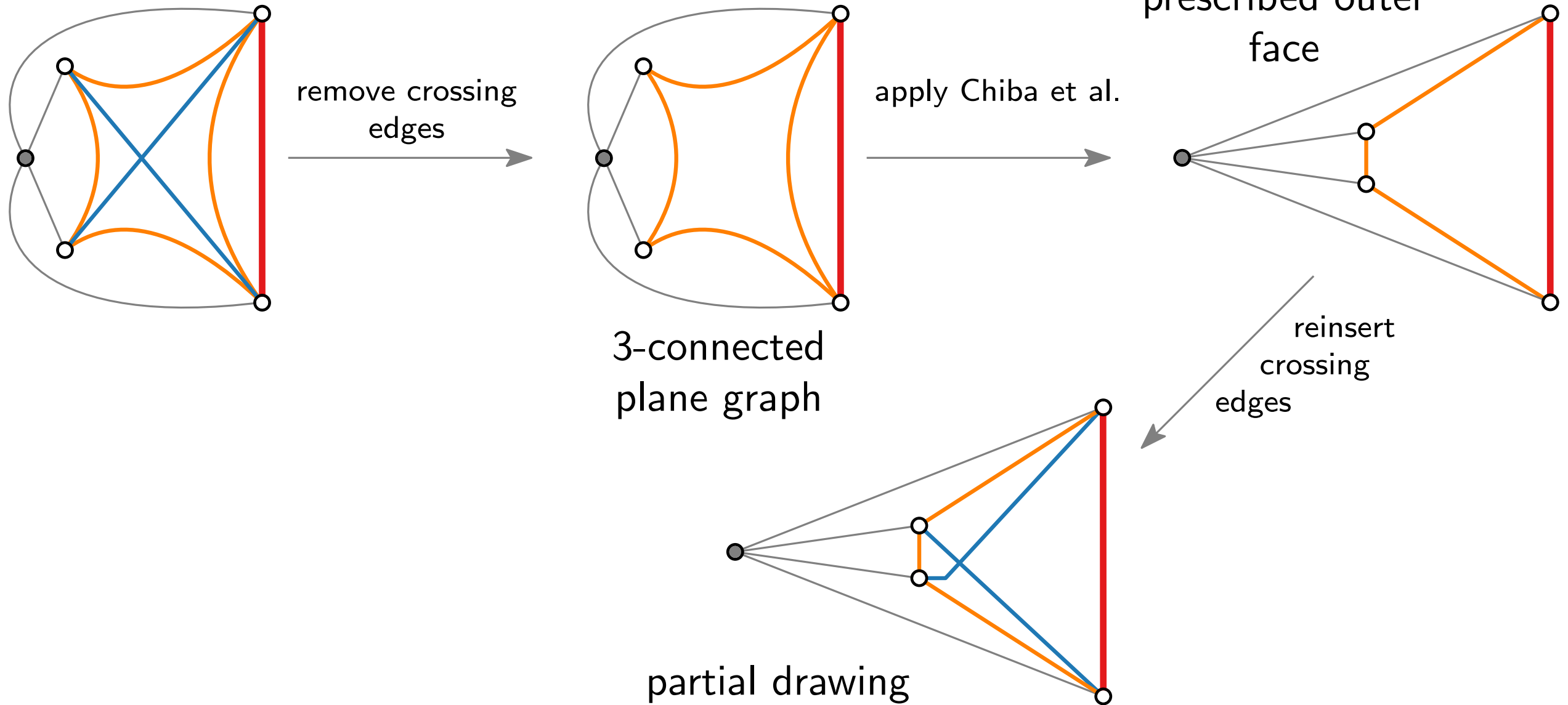
Algorithm Step 3: Drawing Procedure



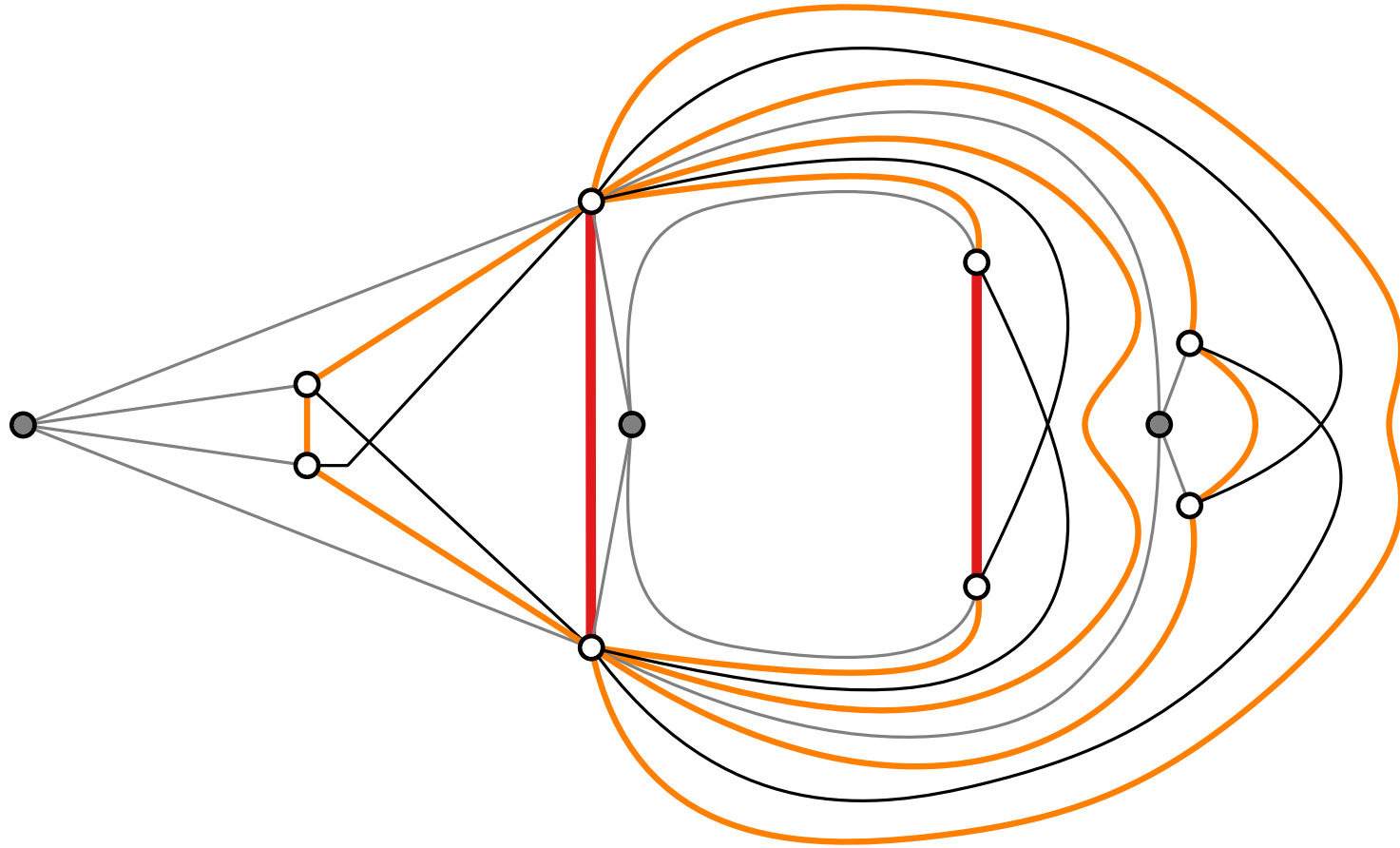
Algorithm Step 3: Drawing Procedure



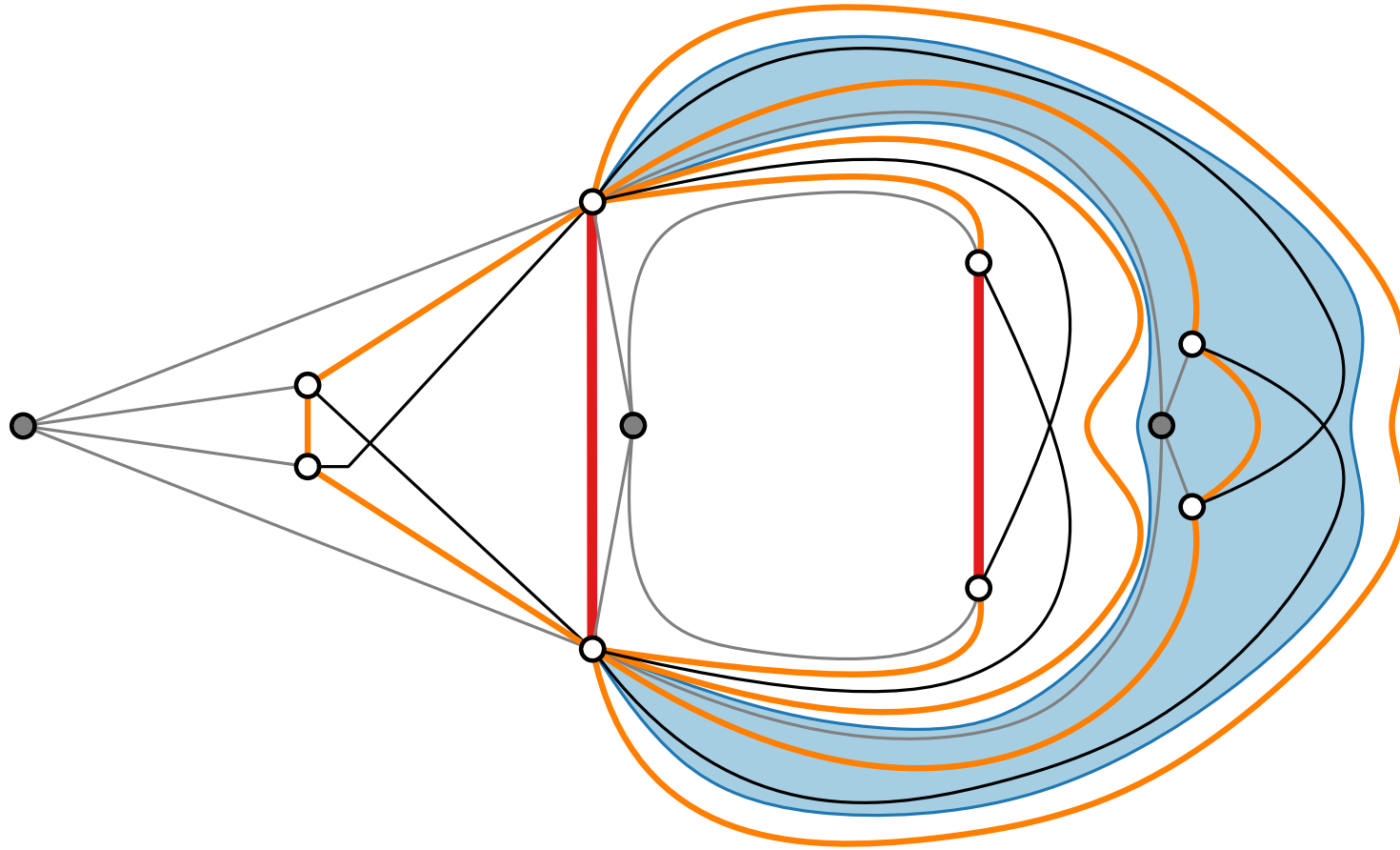
Algorithm Step 3: Drawing Procedure



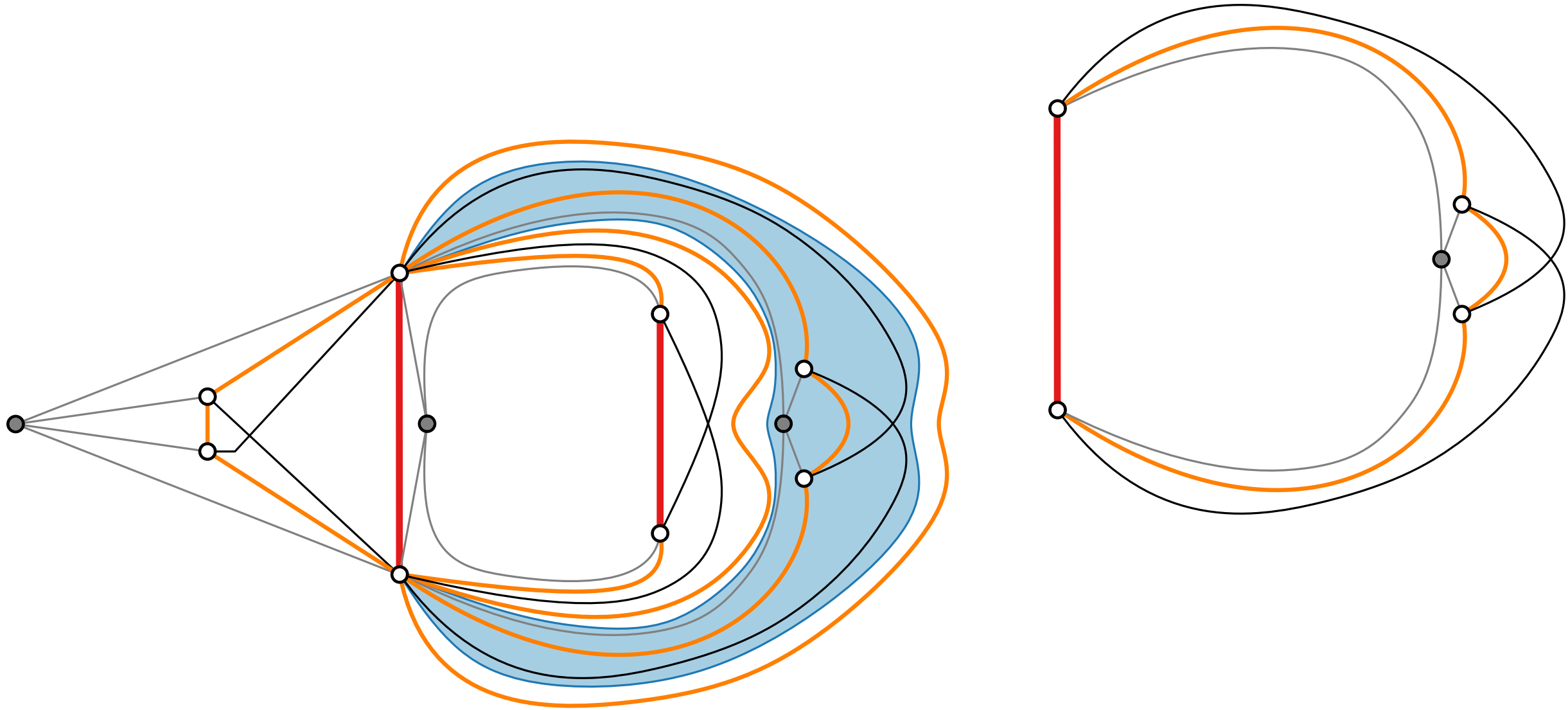
Algorithm Step 3: Drawing Procedure



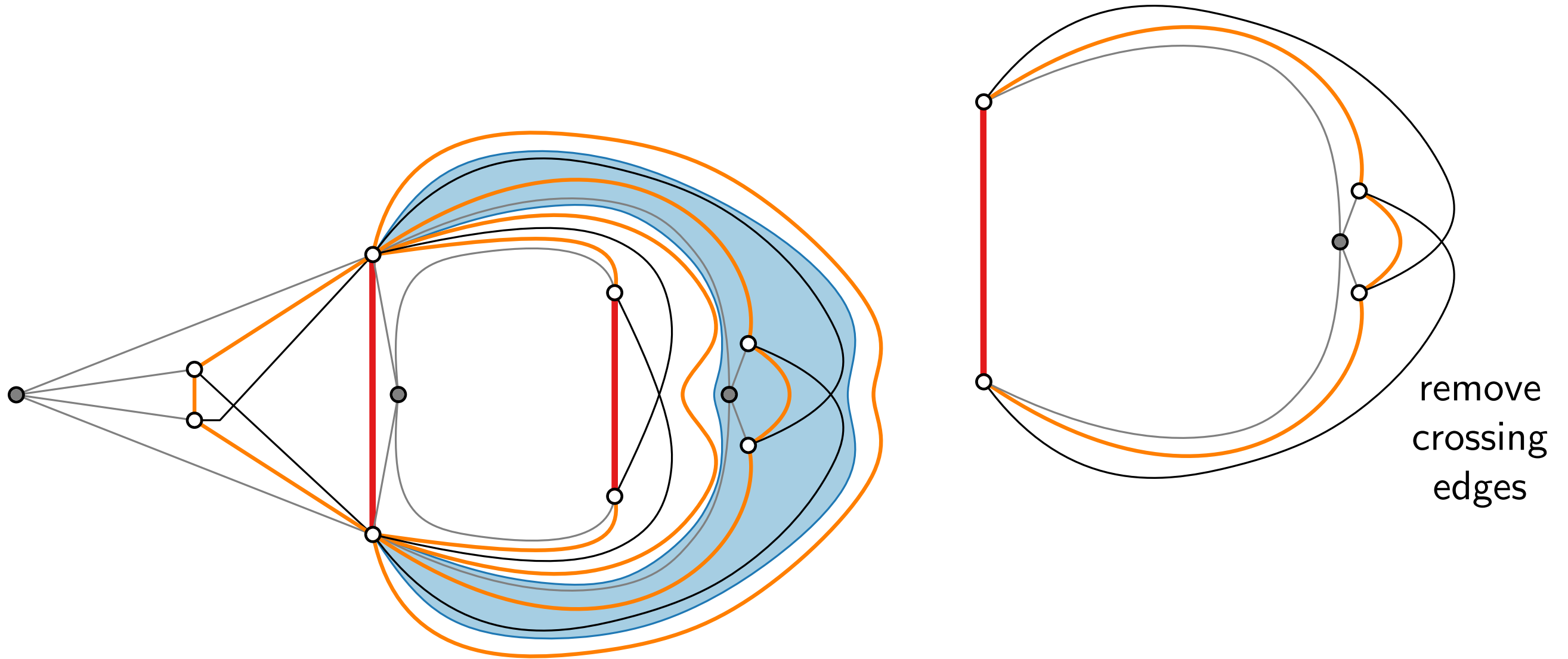
Algorithm Step 3: Drawing Procedure



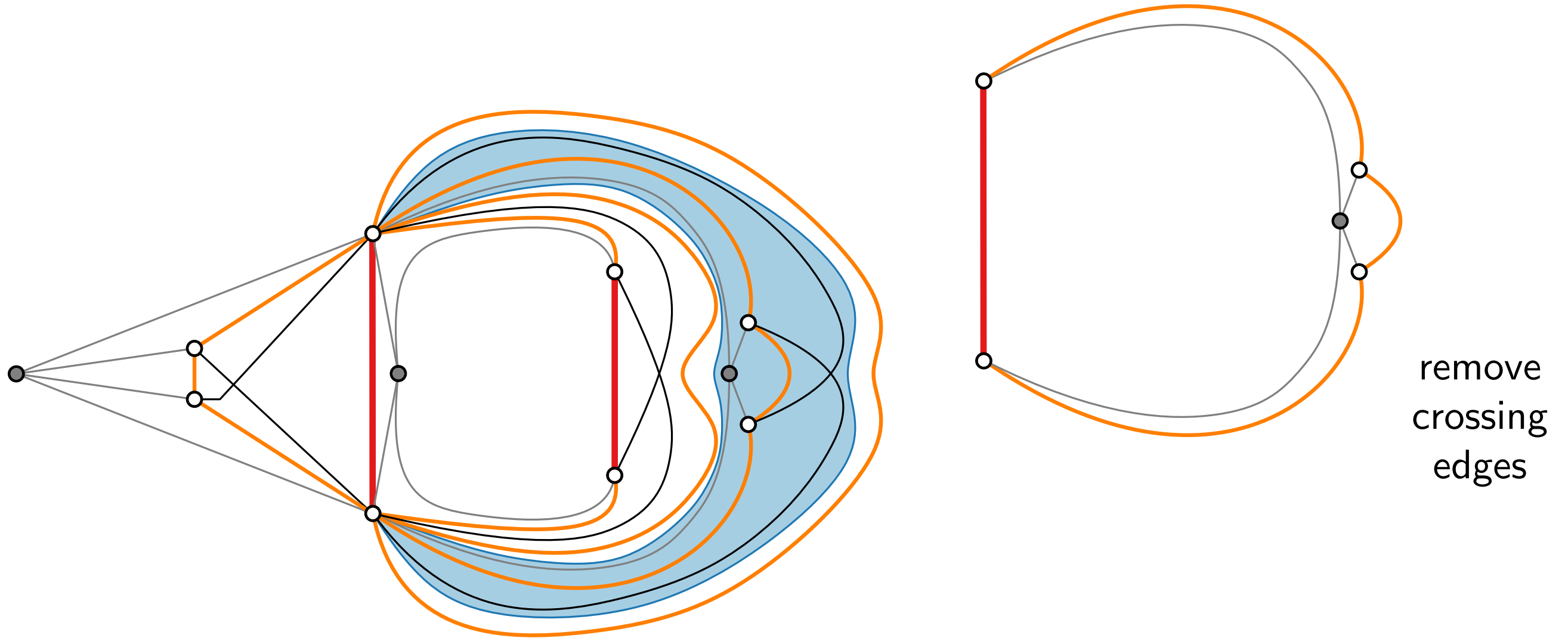
Algorithm Step 3: Drawing Procedure



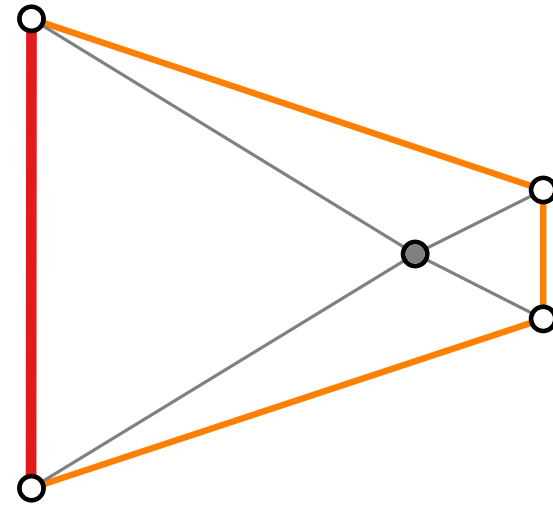
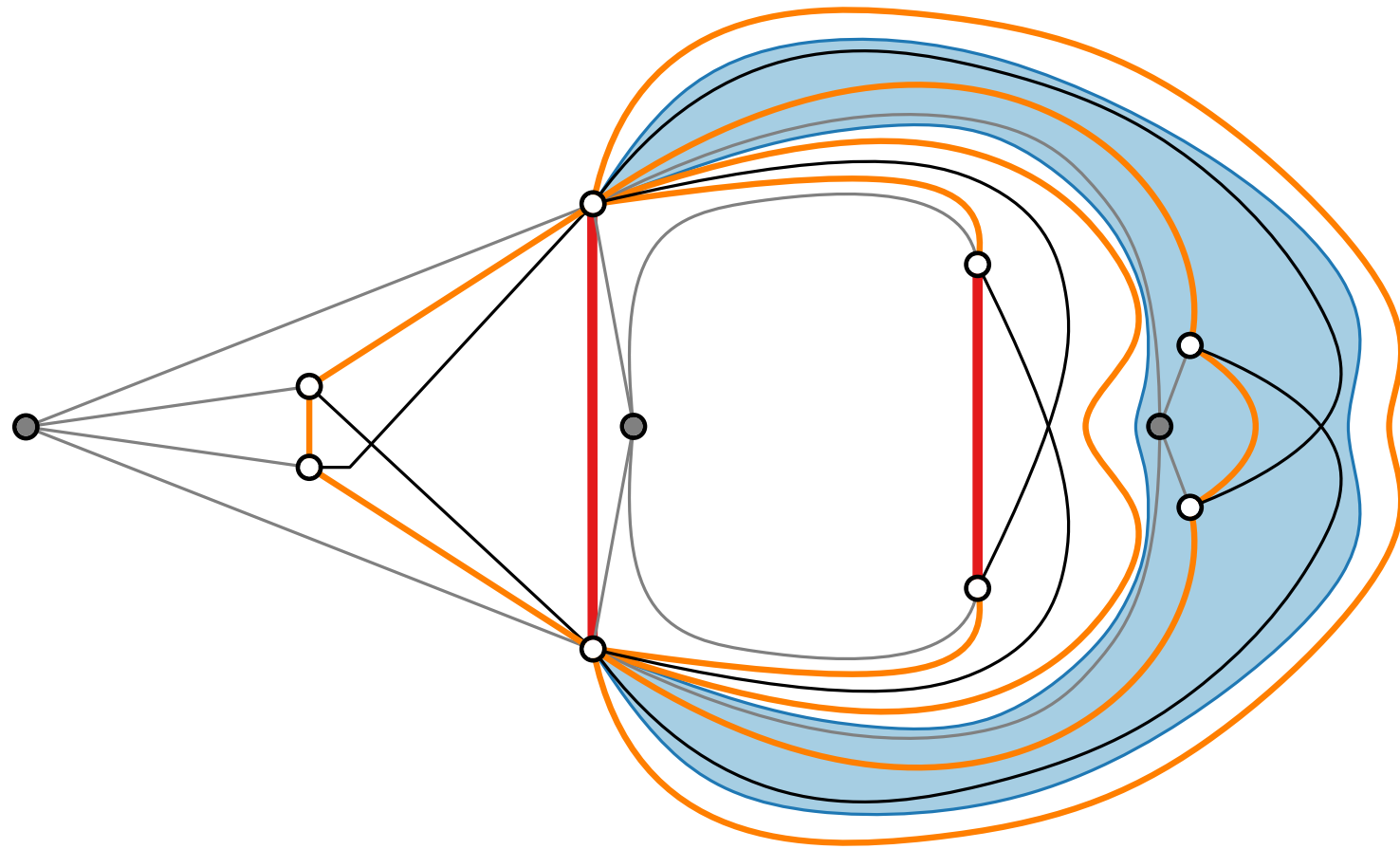
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

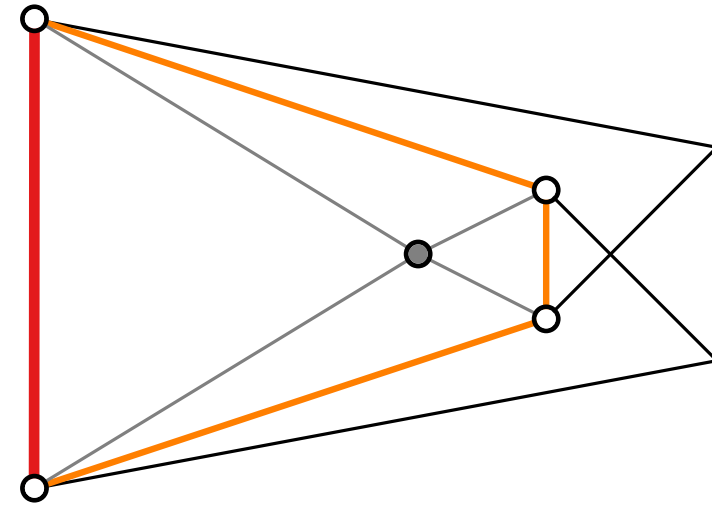
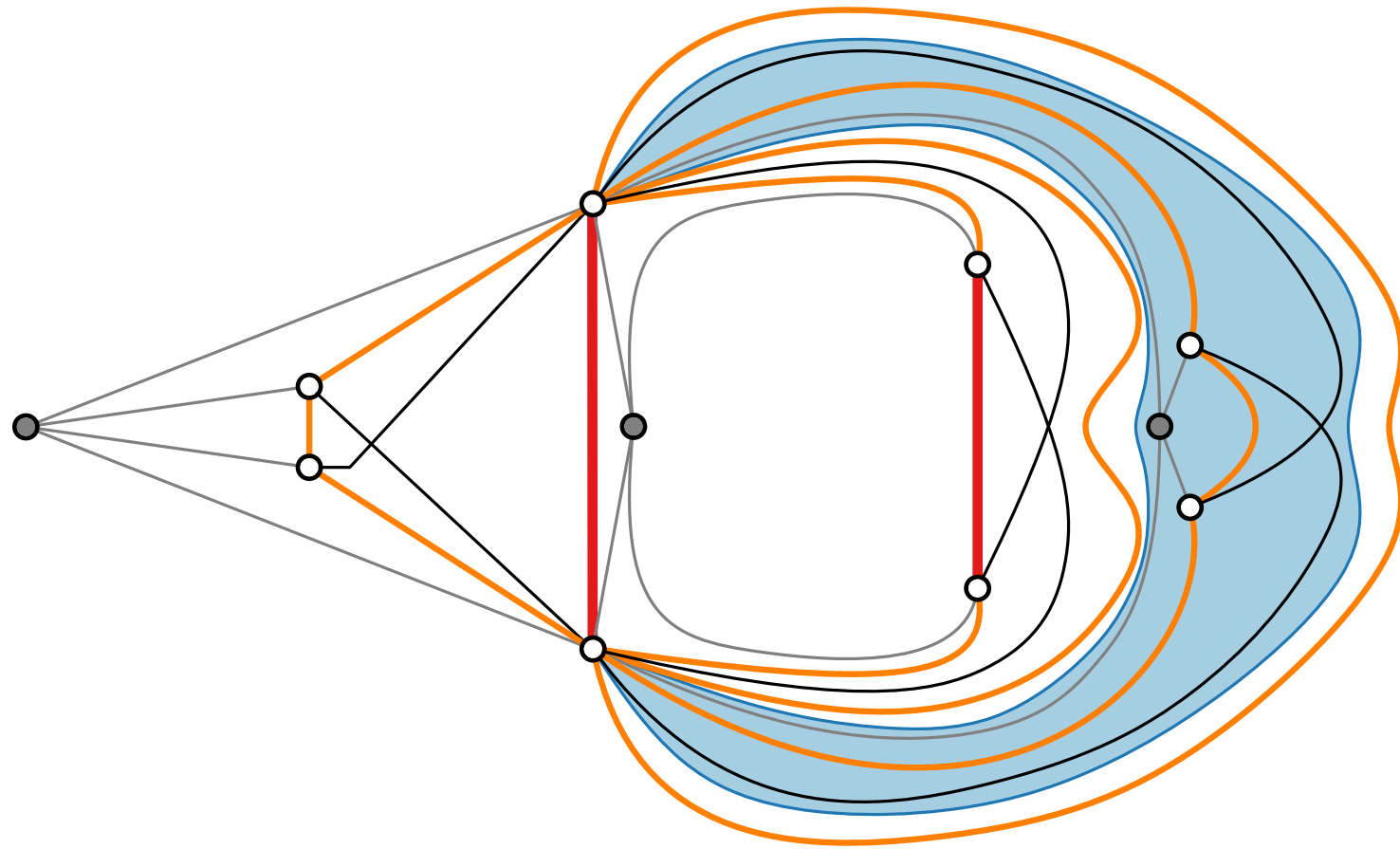


Algorithm Step 3: Drawing Procedure



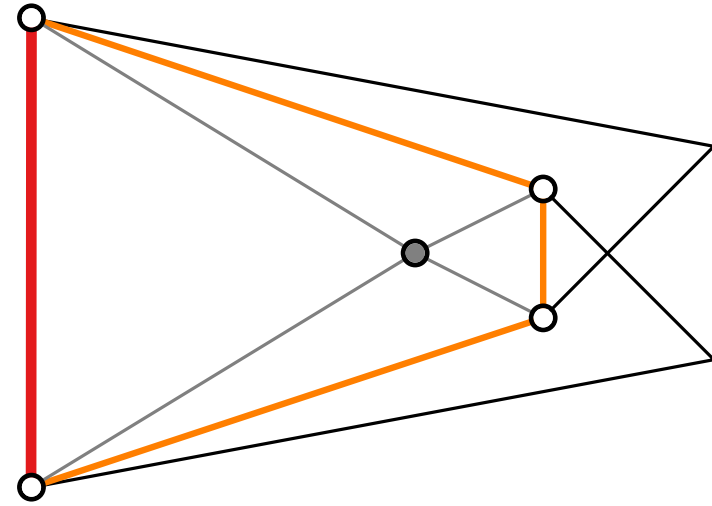
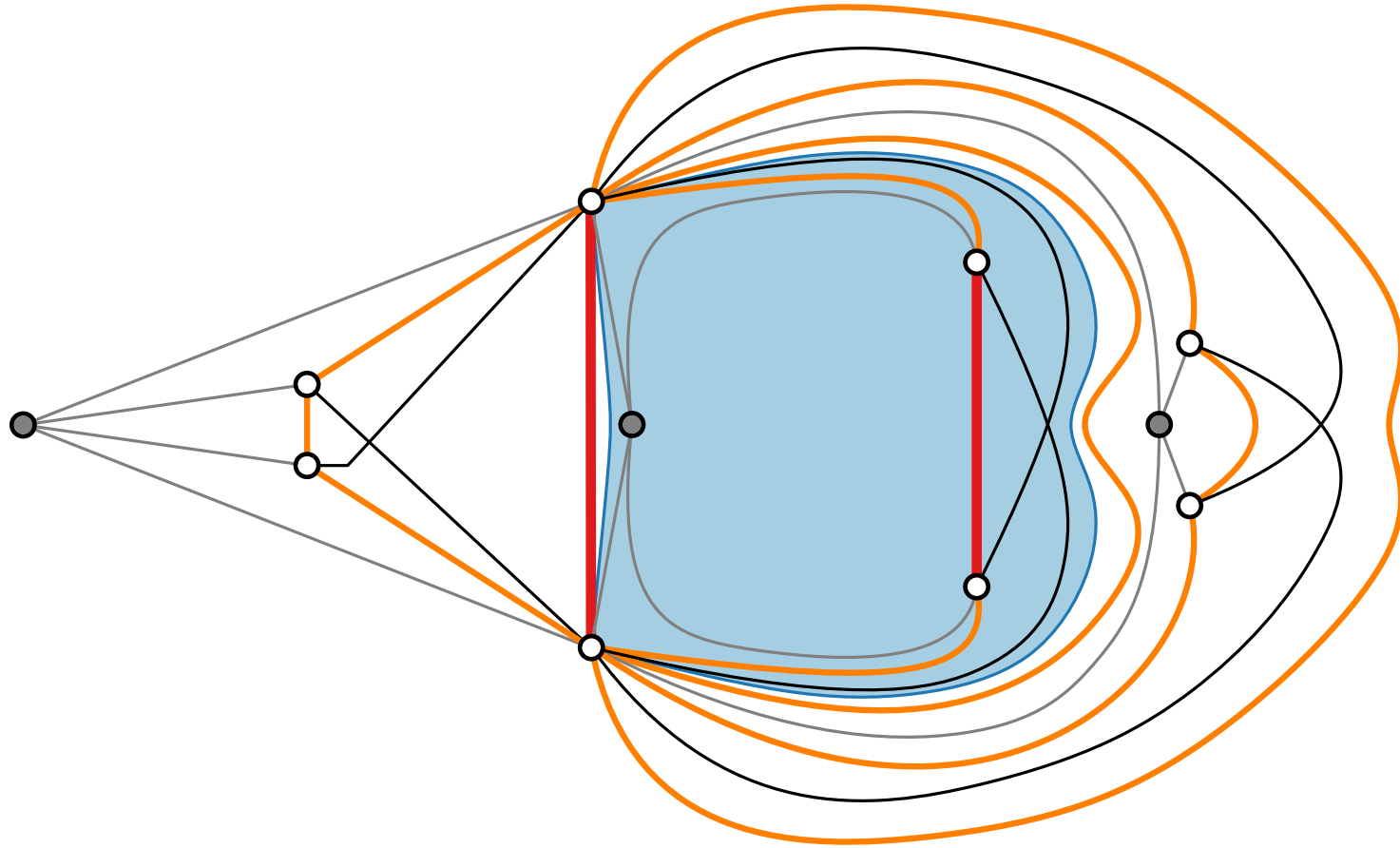
apply Chiba et al.

Algorithm Step 3: Drawing Procedure

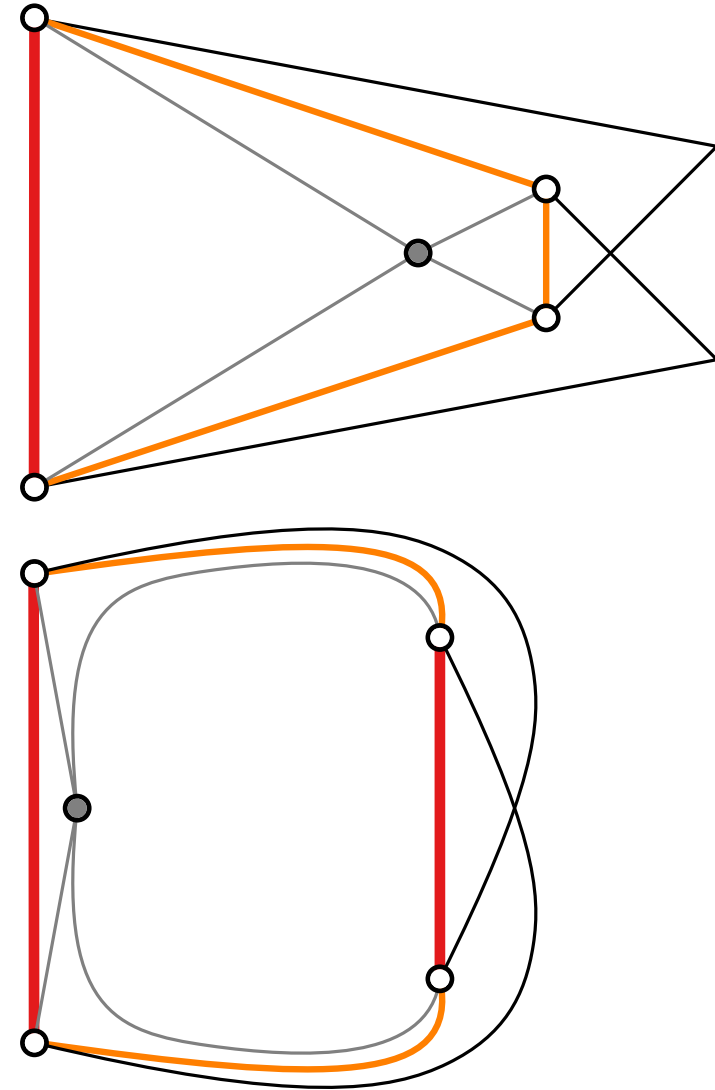
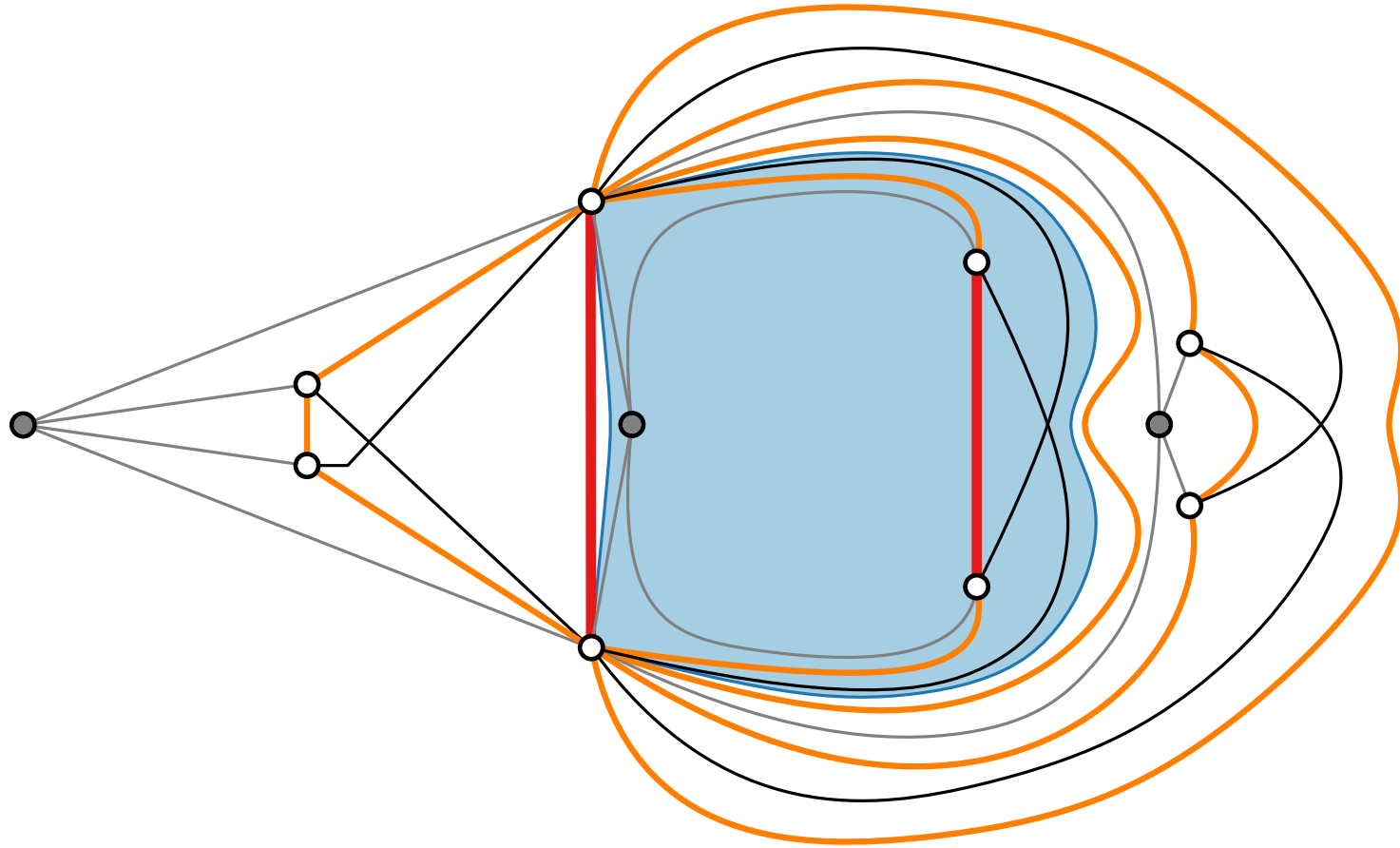


reinsert
crossing
edges

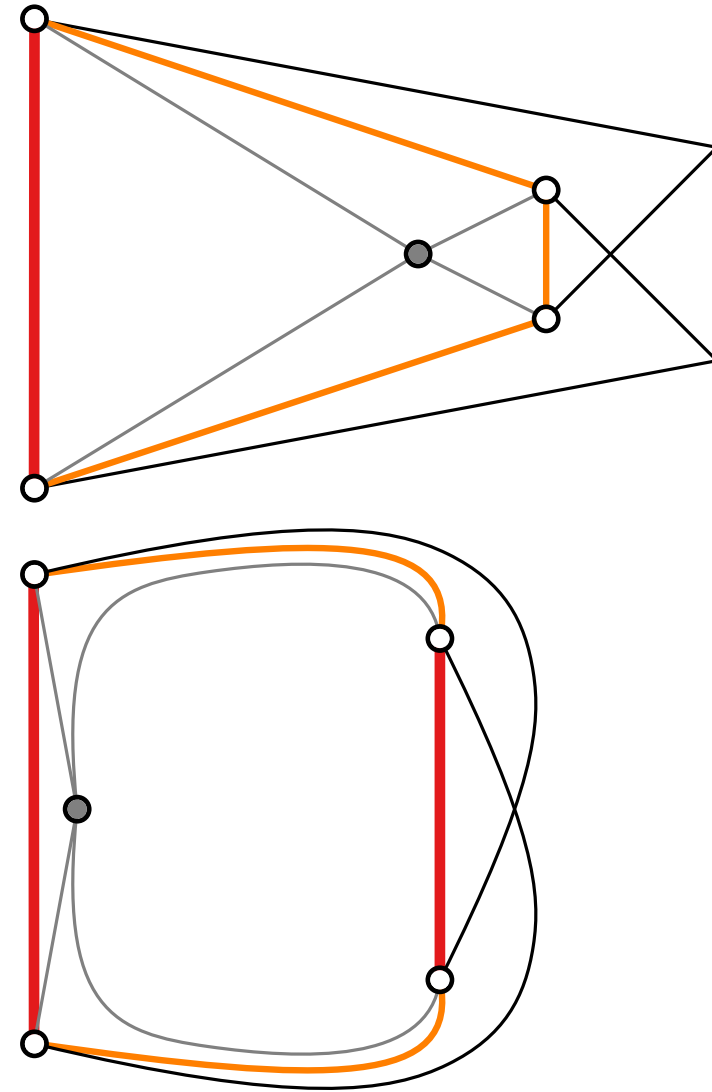
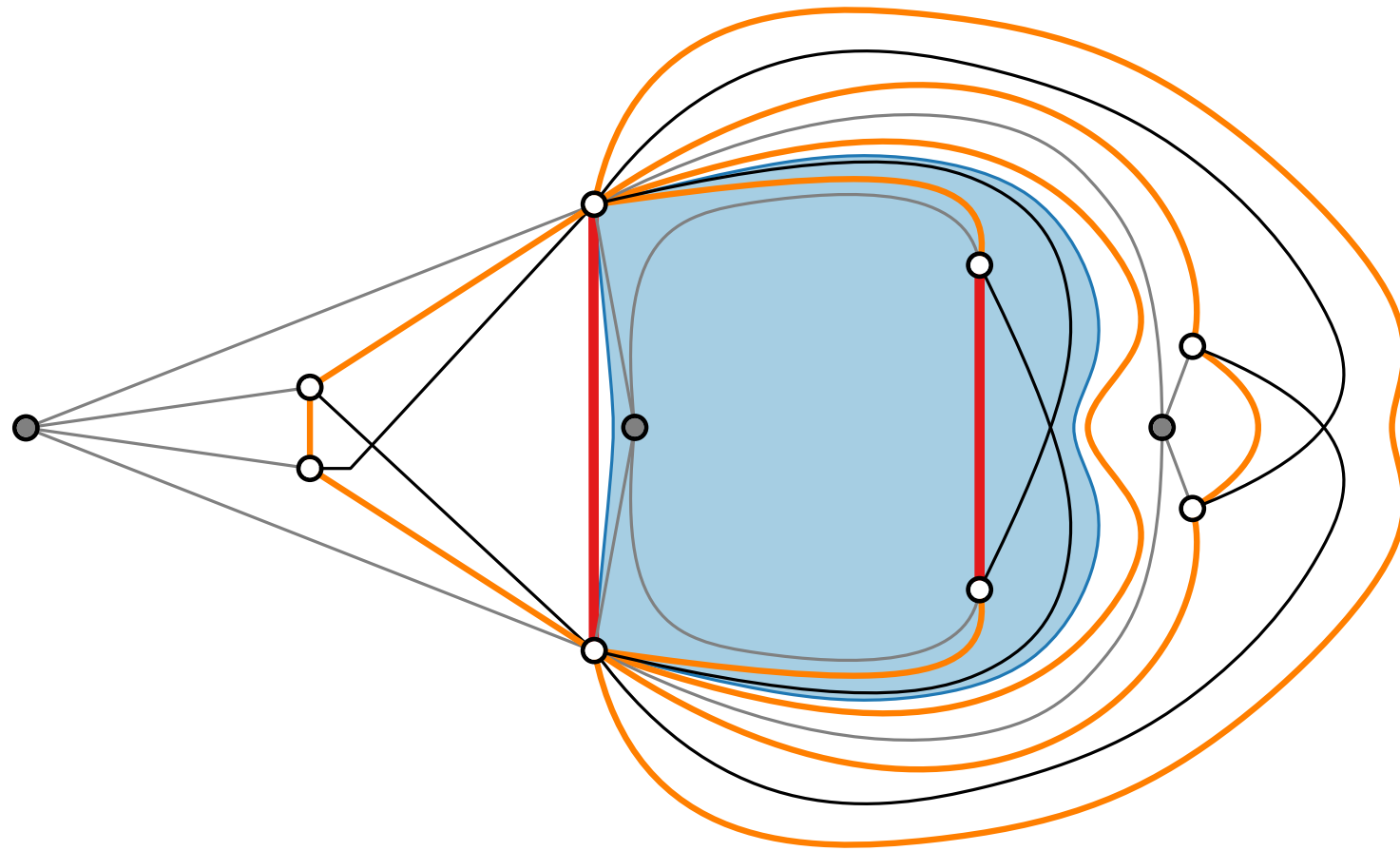
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

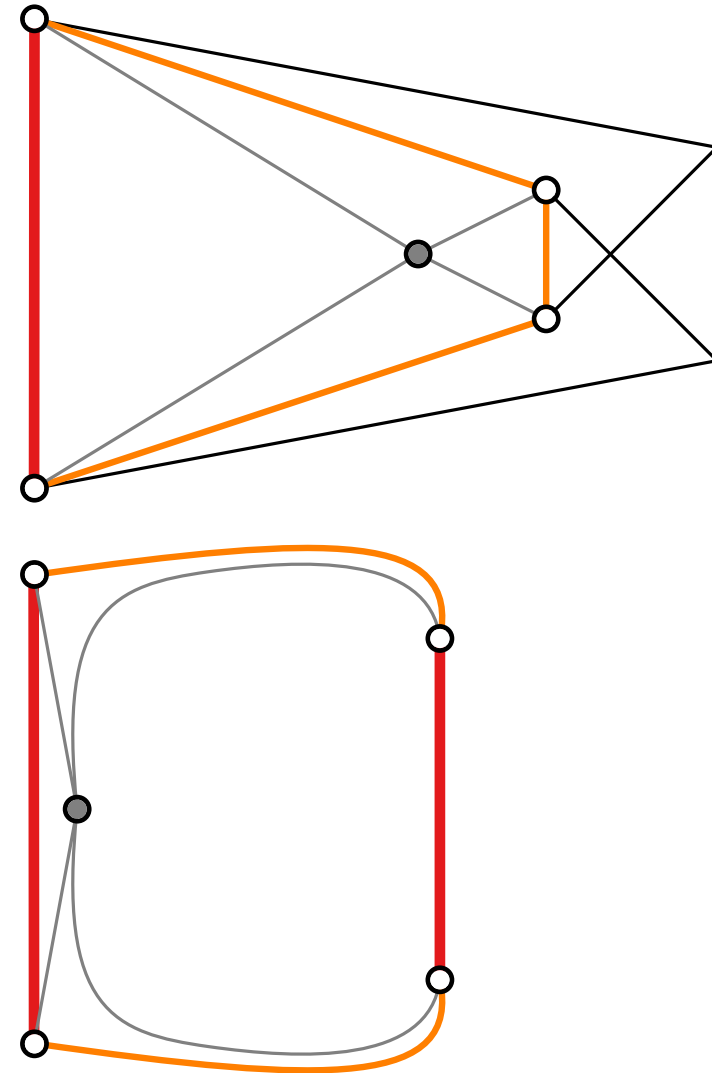
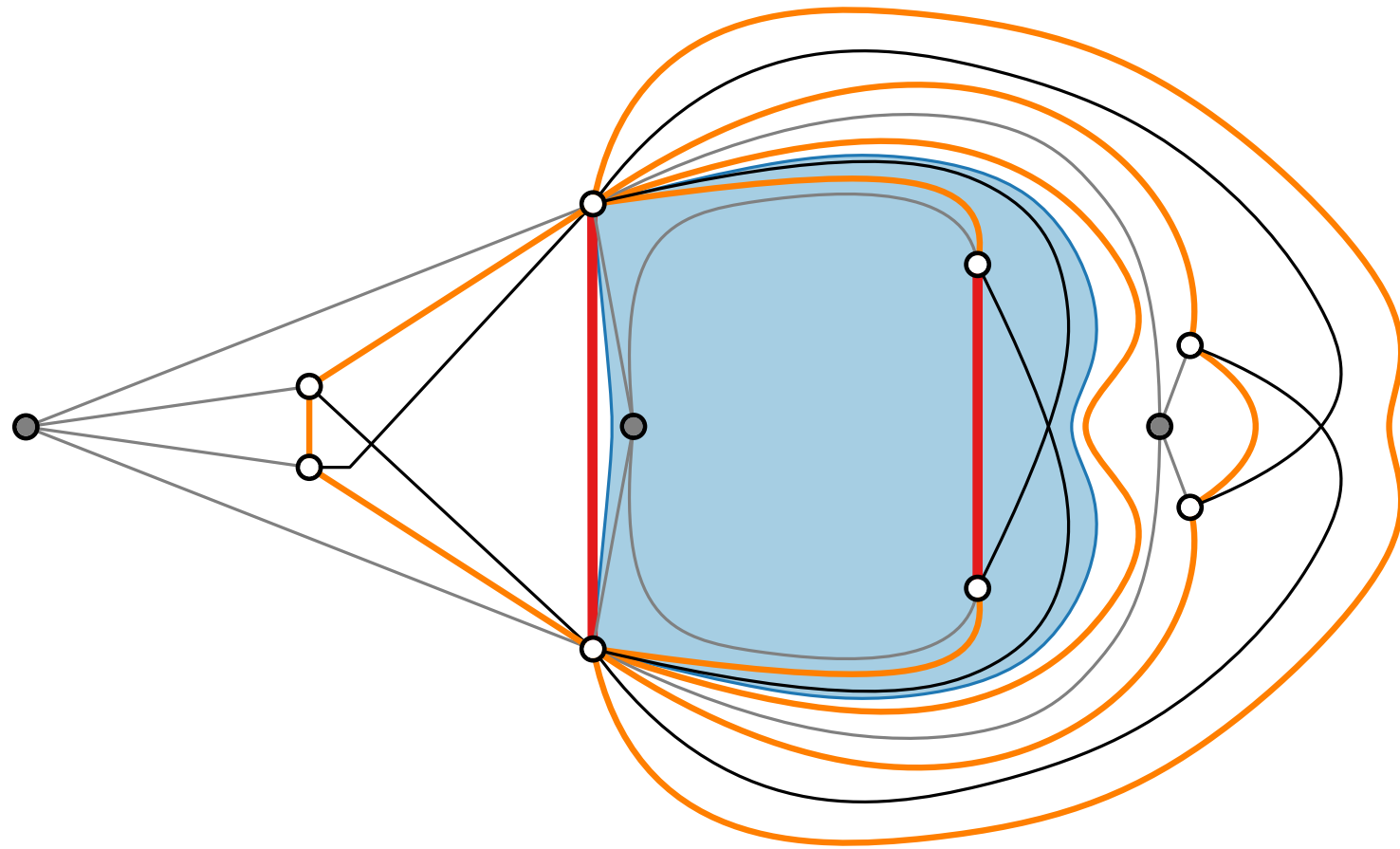


Algorithm Step 3: Drawing Procedure



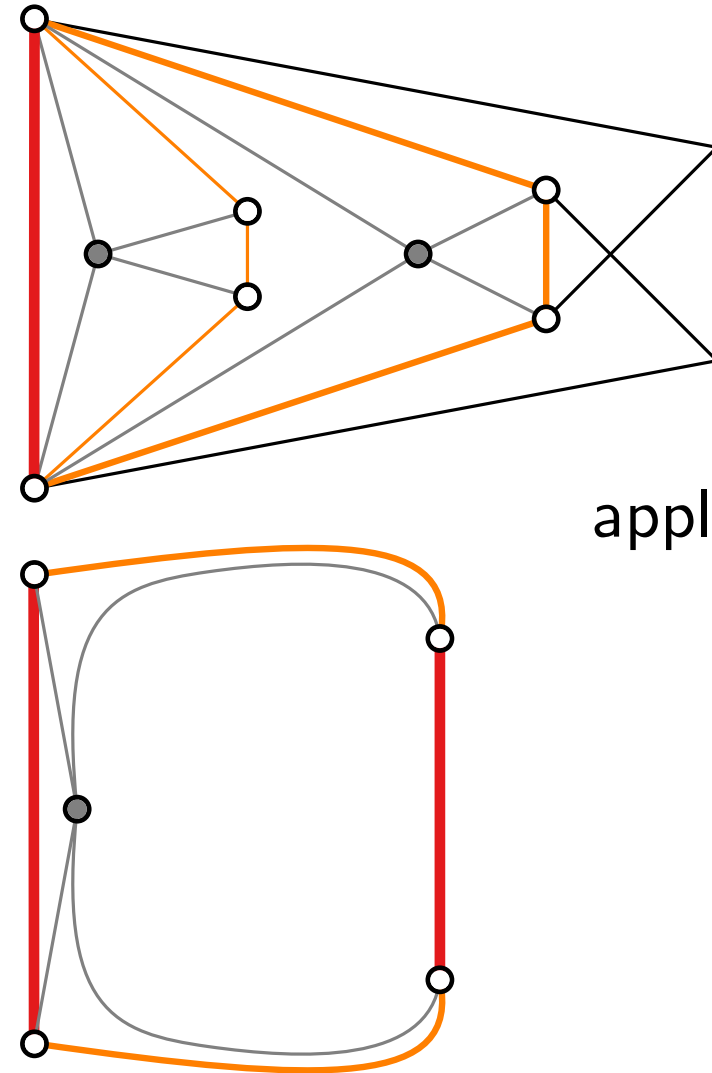
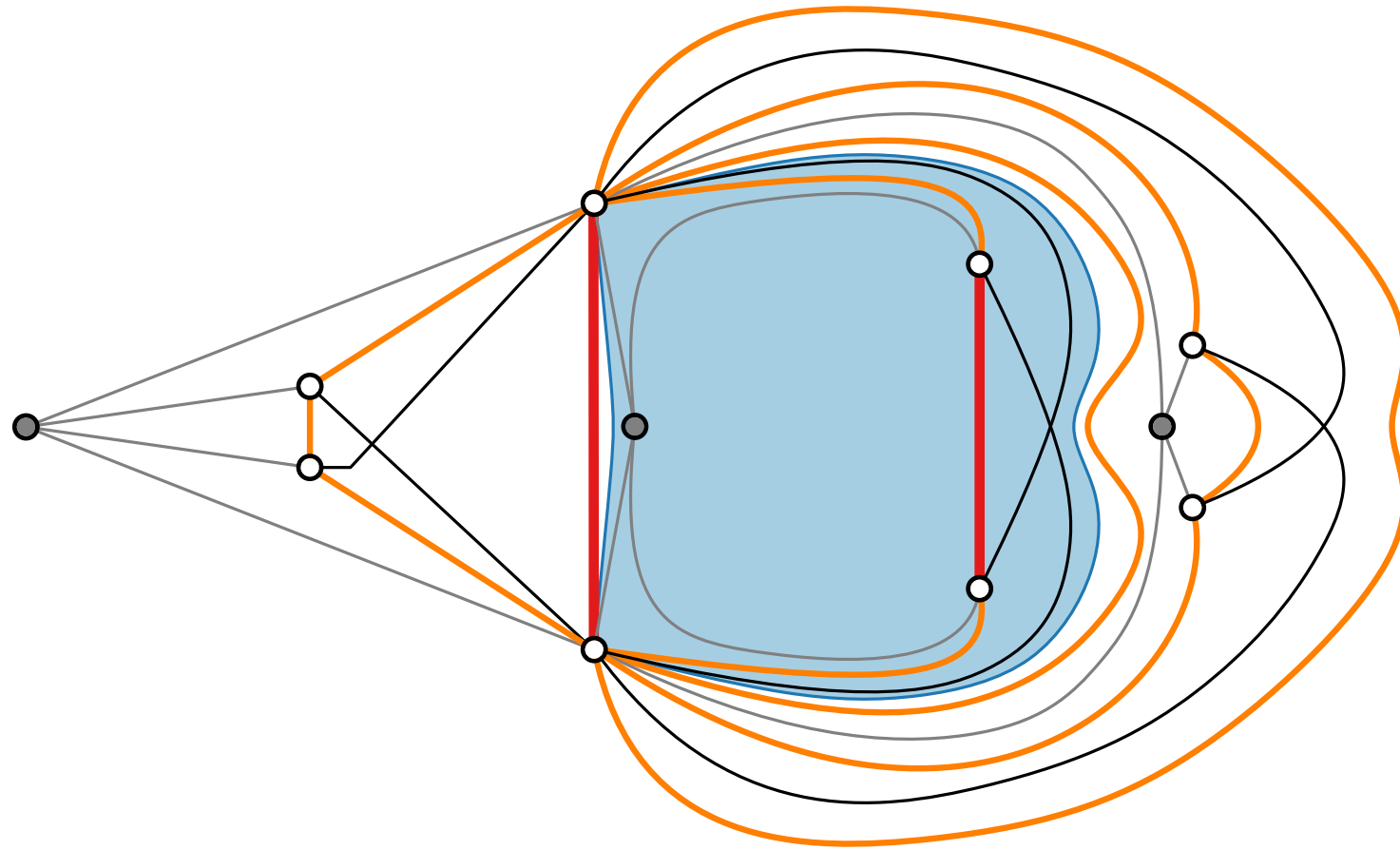
remove
crossing
edges

Algorithm Step 3: Drawing Procedure



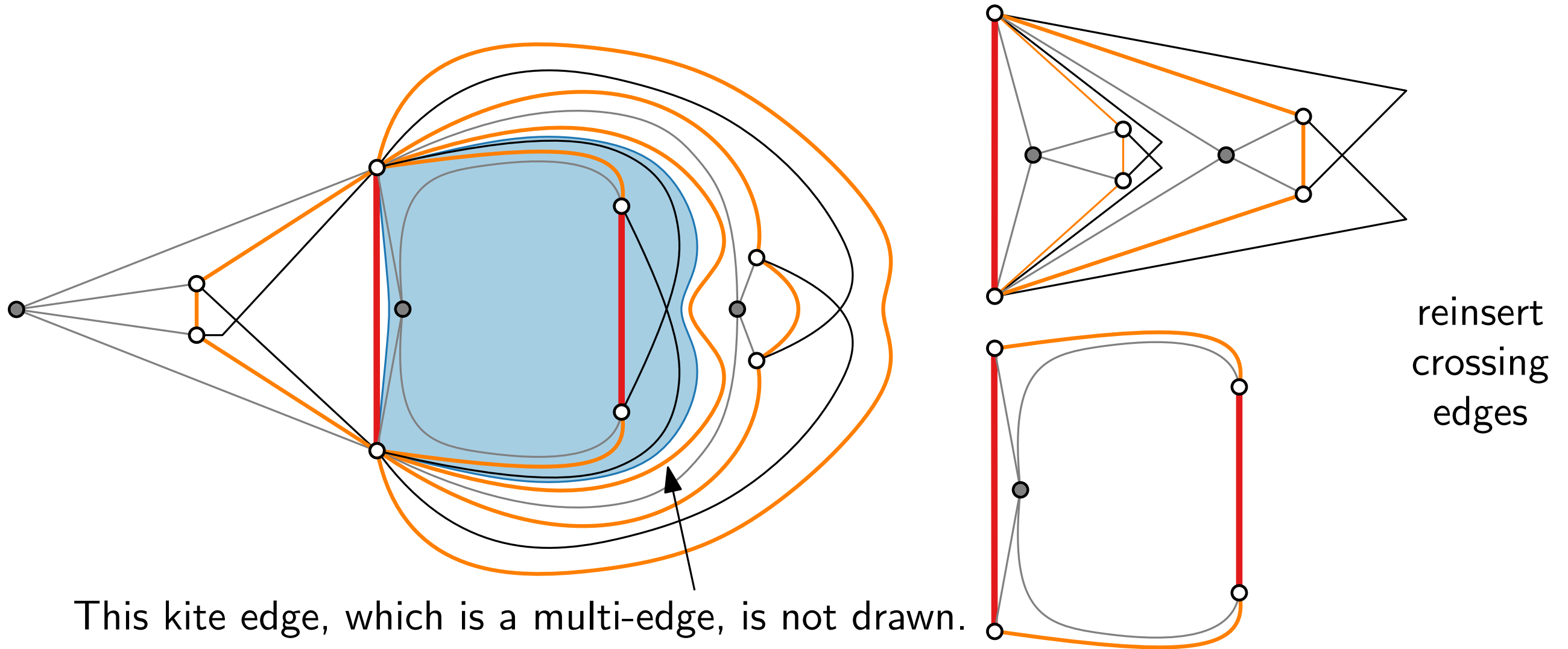
remove
crossing
edges

Algorithm Step 3: Drawing Procedure

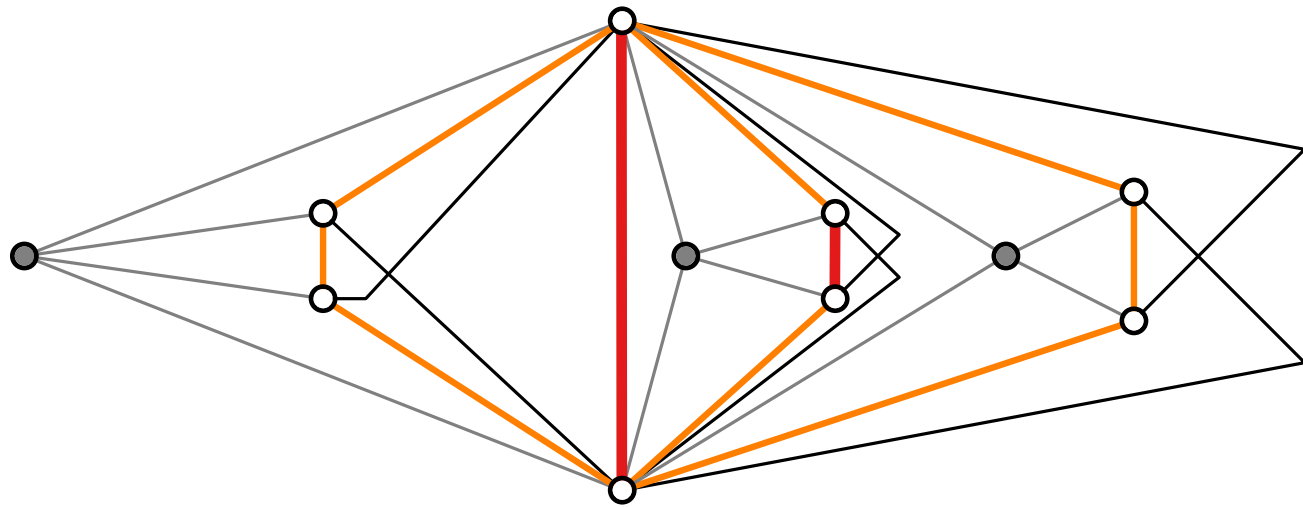


apply Chiba et al.

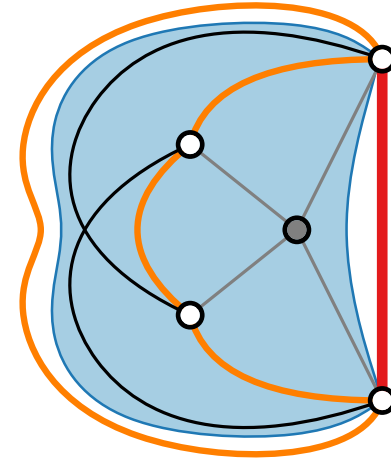
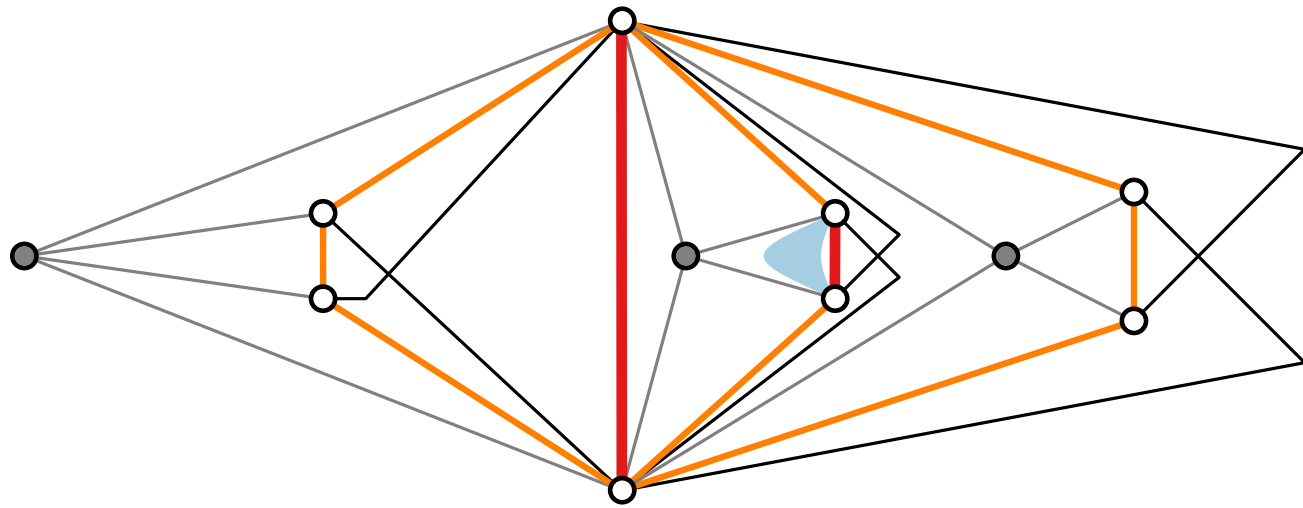
Algorithm Step 3: Drawing Procedure



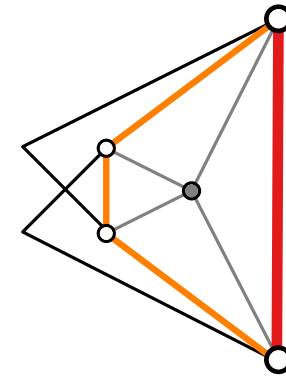
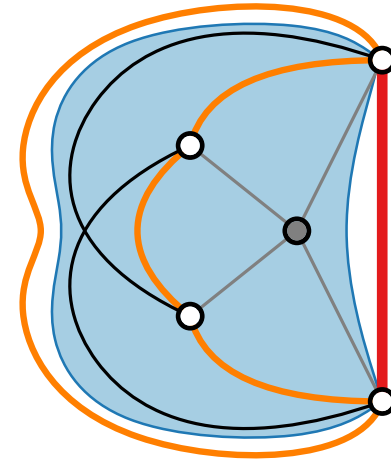
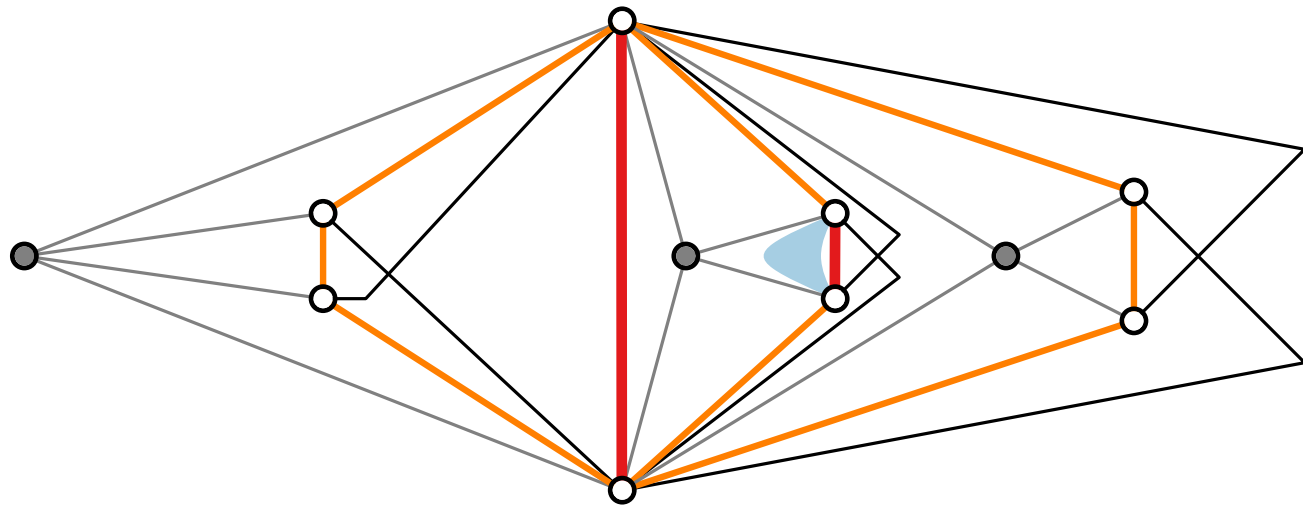
Algorithm Step 3: Drawing Procedure



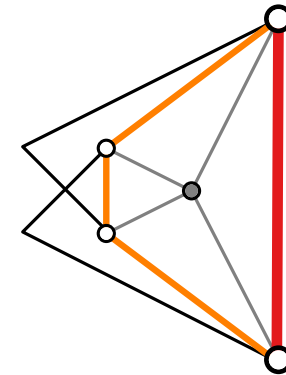
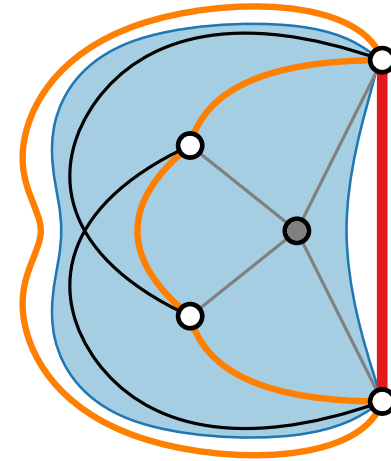
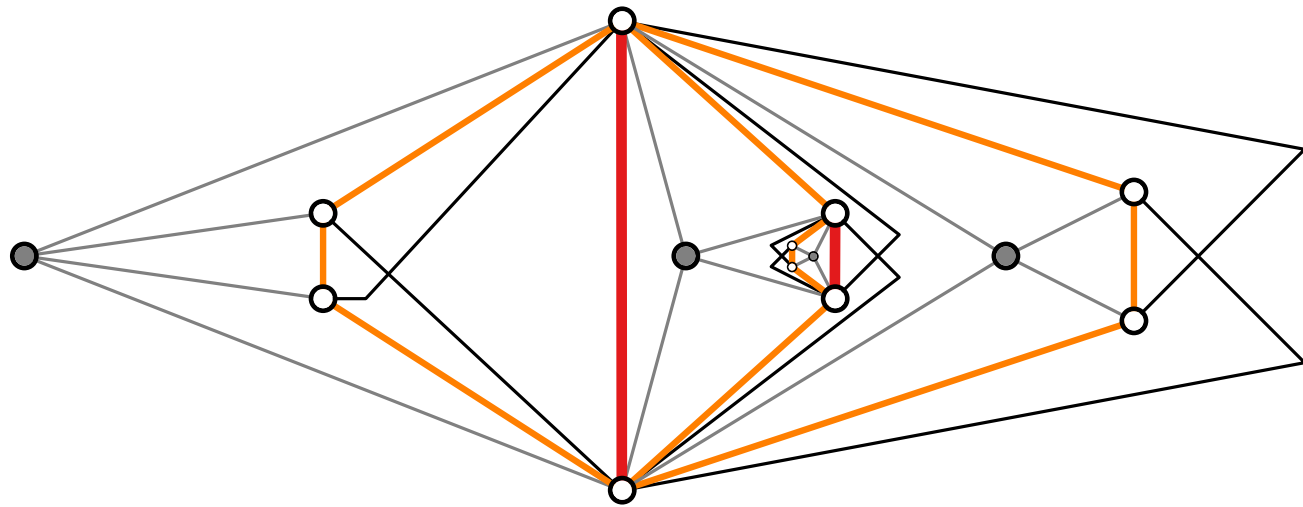
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

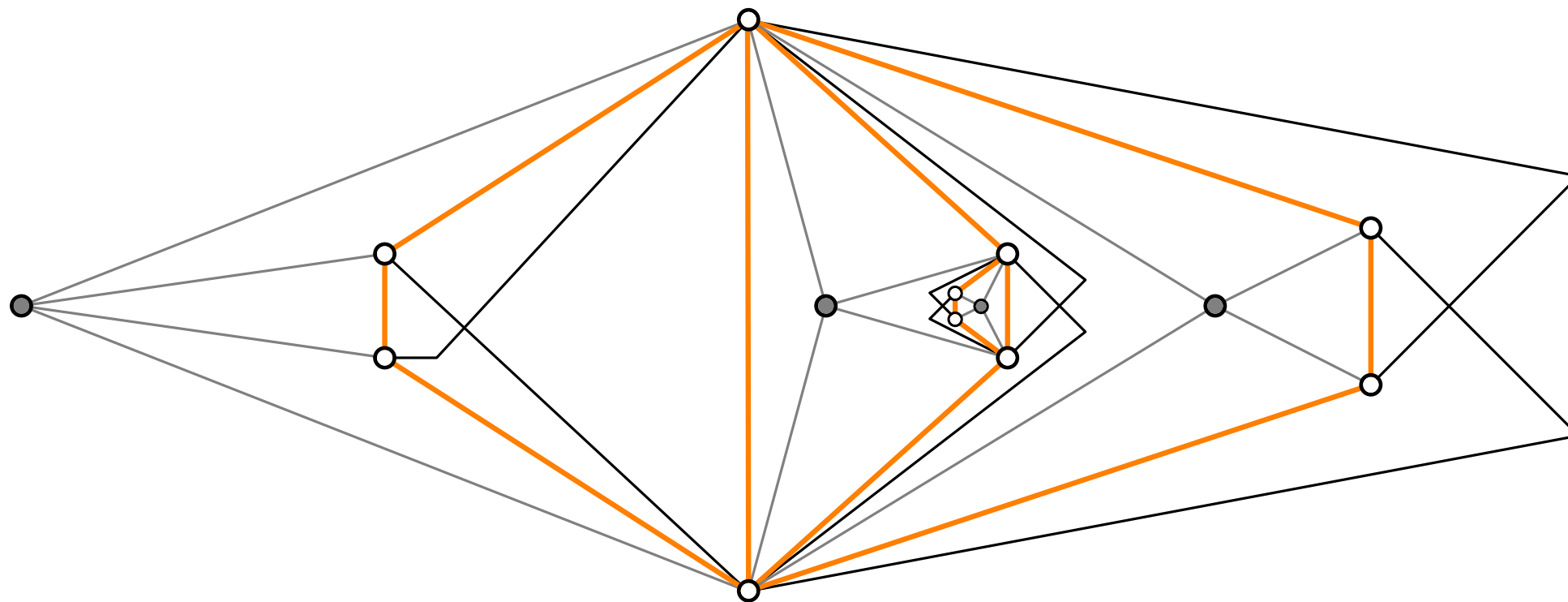


Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

Γ^+ : 1-bend 1-planar RAC drawing of G^+



Algorithm Outline

input

G
simple 1-plane

augmentation
(the embedding
may change)

G^+
triangulated 1-plane
(multi-edges)

recursive
procedure

G^*
hierarchical
contraction of G^+

Γ^+
1-bend 1-planar RAC
drawing of G^+

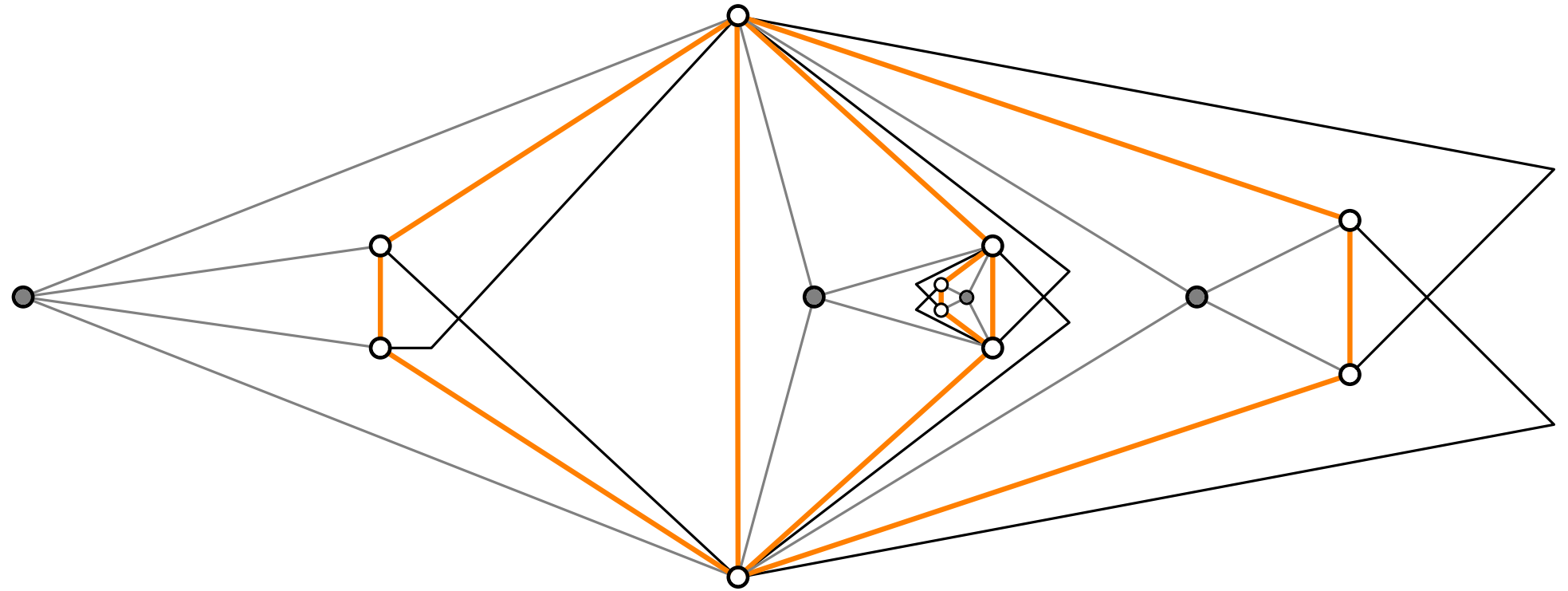
recursive
procedure

Γ
1-bend 1-planar RAC
drawing of G

removal of
dummy elements

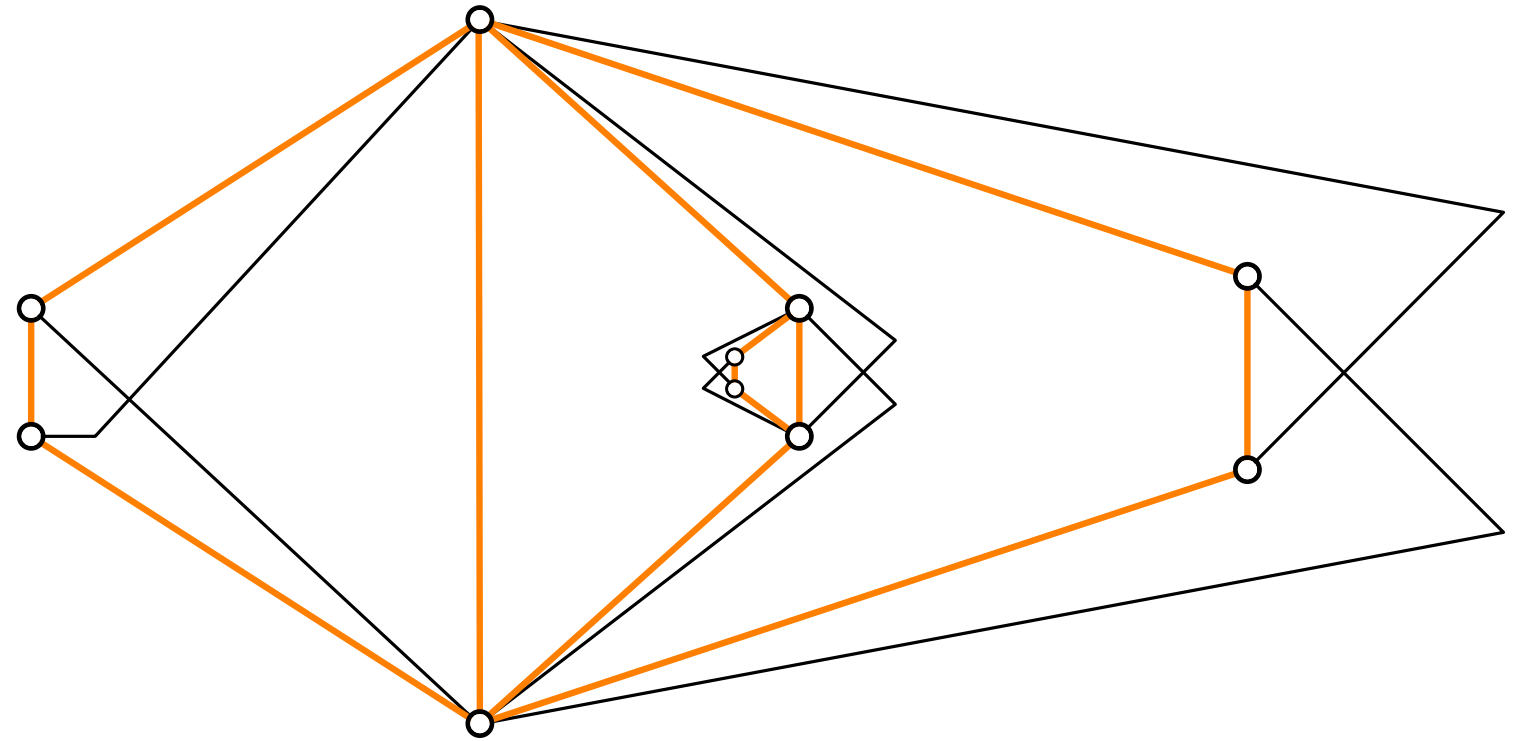
output

Algorithm Step 4: Removal of Dummy Vertices



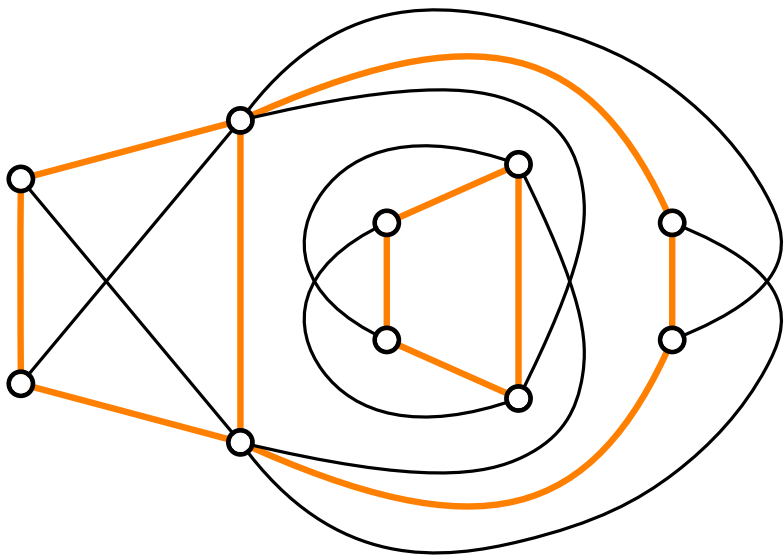
Algorithm Step 4: Removal of Dummy Vertices

Γ : 1-bend 1-planar RAC drawing of G



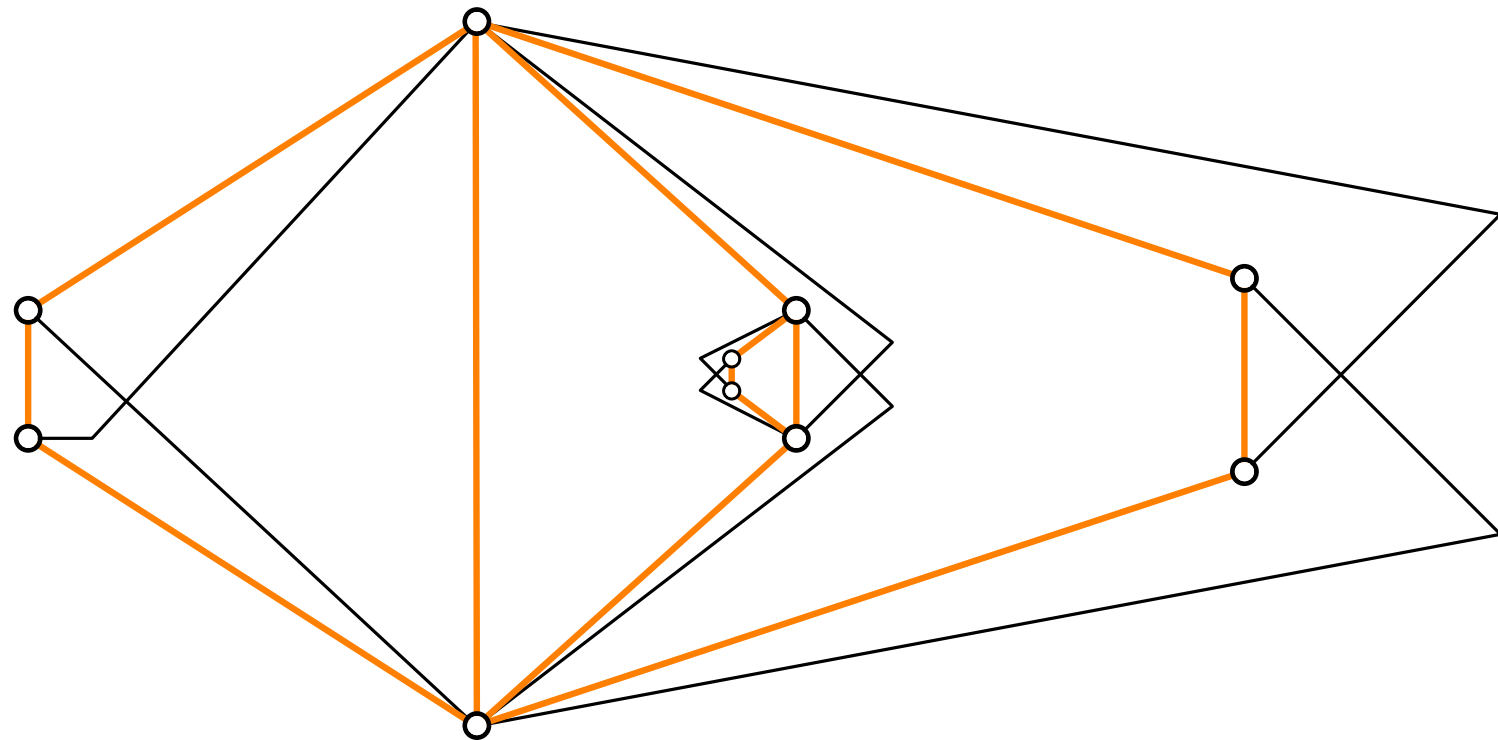
Algorithm Step 4: Removal of Dummy Vertices

G : simple 1-plane graph



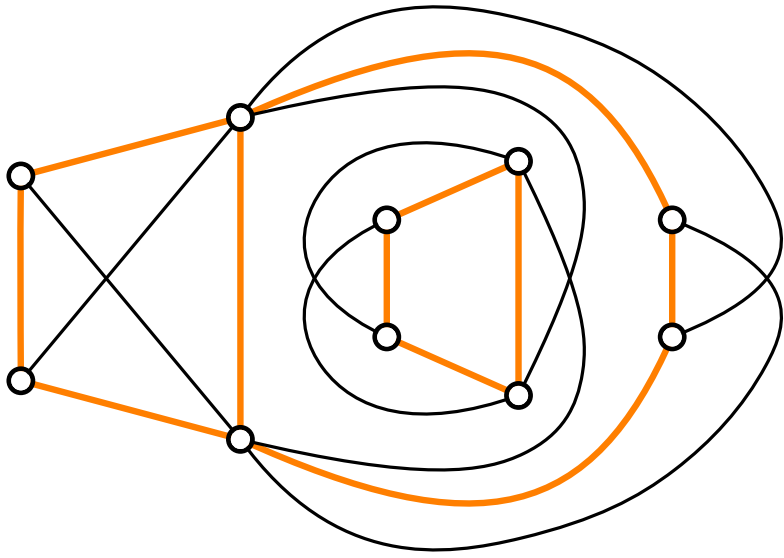
Γ : 1-bend 1-planar RAC drawing of G

(embedding may differ)



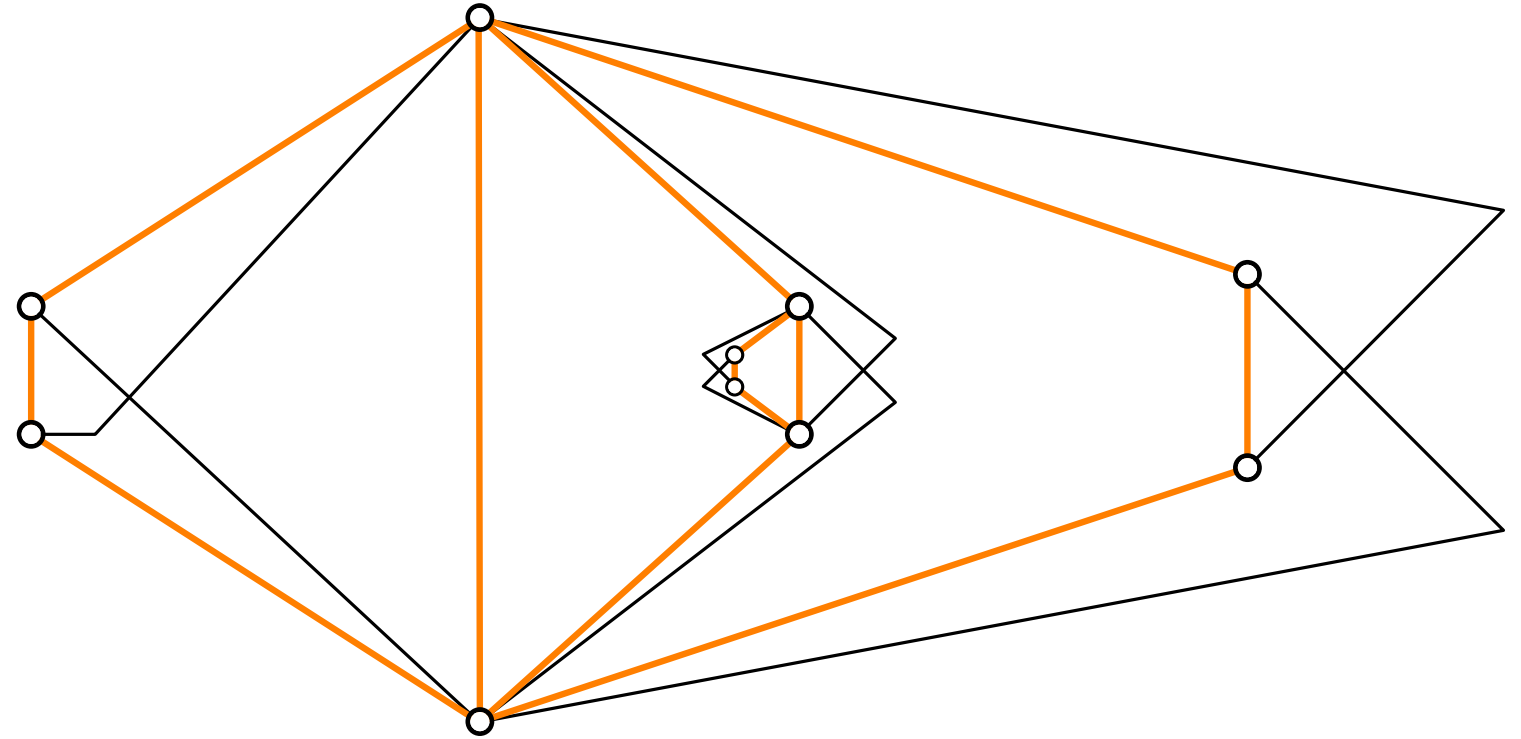
Algorithm Step 4: Removal of Dummy Vertices

G : simple 1-plane graph



Γ : 1-bend 1-planar RAC drawing of G

(embedding may differ)

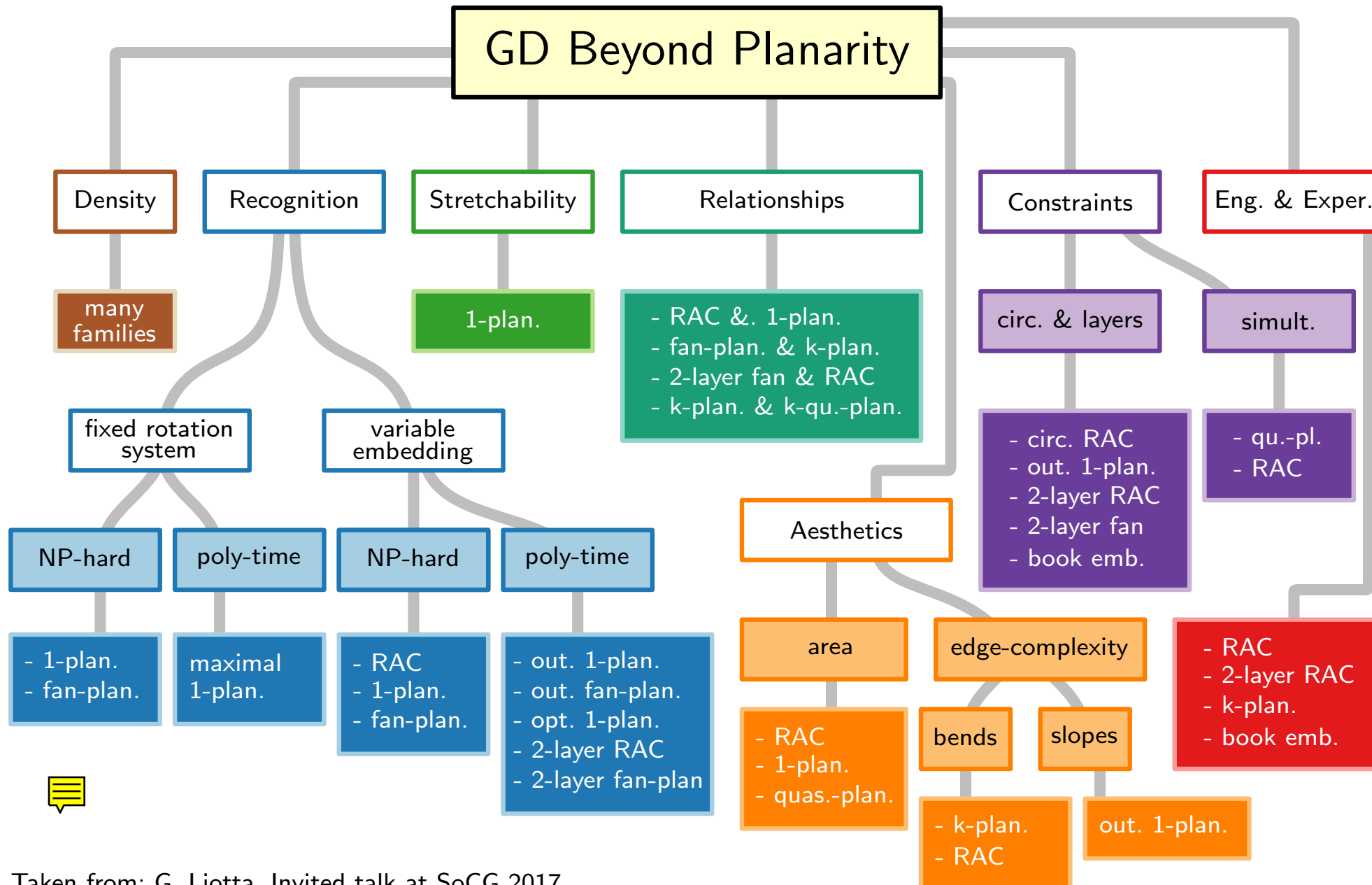


Remark.

By modifying the algorithm slightly, the given input embedding can be preserved.

[Chaplick, Lipp, Wolff, Zink 2019]

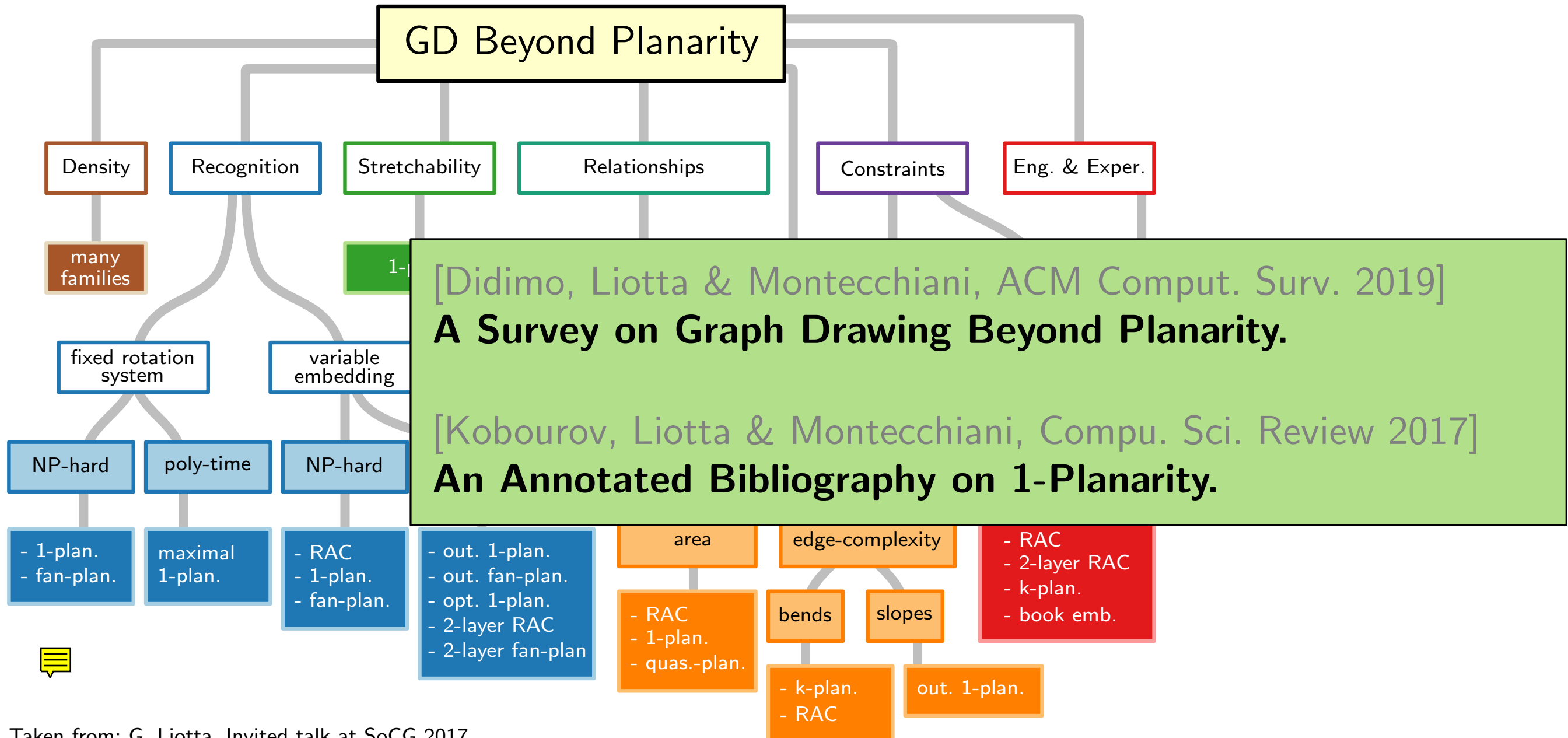
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Hong and Tokuyama, editors '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchiani, Valtr '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
- [Didimo, Eades, Liotta '17] Drawing graphs with right angle crossings
- [Bekos et al. '17] On RAC drawings of 1-planar graphs