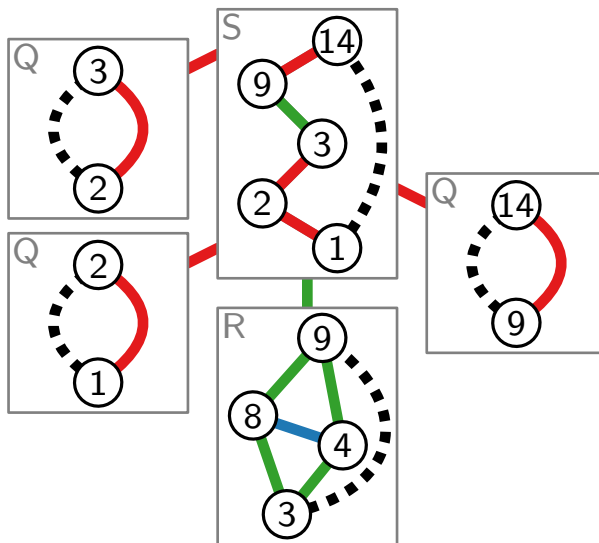


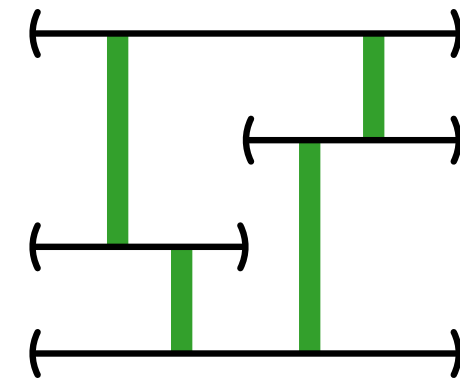
Visualization of Graphs

Lecture 10: Partial Visibility Representation Extension



Alexander Wolff

Summer term 2026



Partial Representation Extension Problem

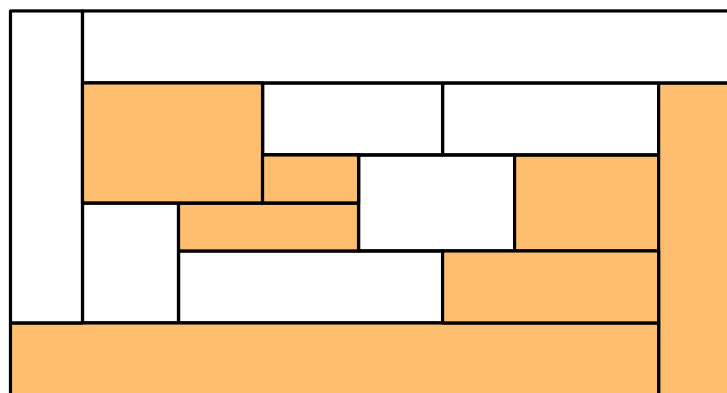
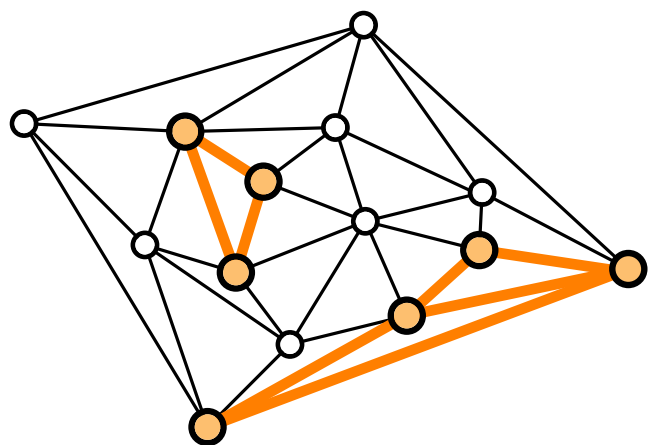
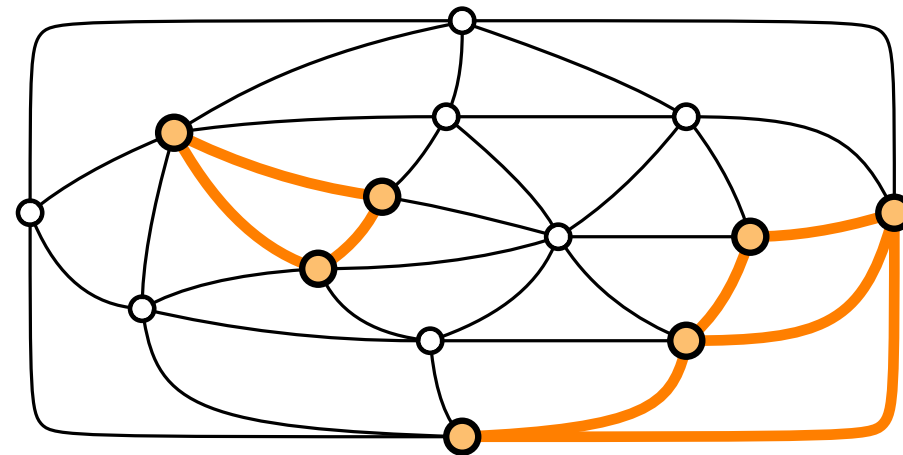
Let G be a graph.

Let $V' \subseteq V(G)$ and $H = G[V']$

Let Γ_H be a representation of H .

Find a representation Γ_G of G that *extends* Γ_H .

induced subgraph of G w.r.t. V' :
 V' and all edges among V'

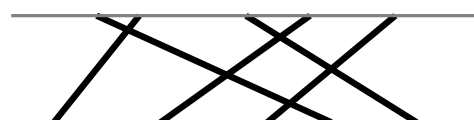


Polytime for:

■ (unit) interval graphs



■ permutation graphs



■ circle graphs



NP-hard for:

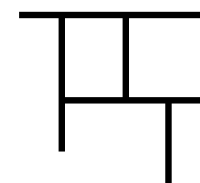
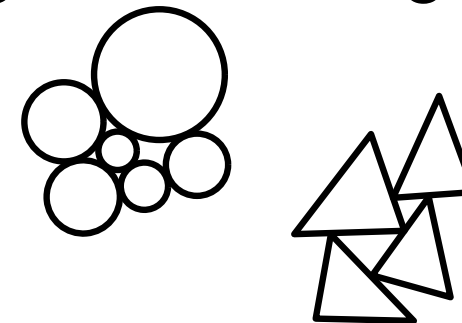
■ planar straight-line drawings

■ contacts of

■ disks

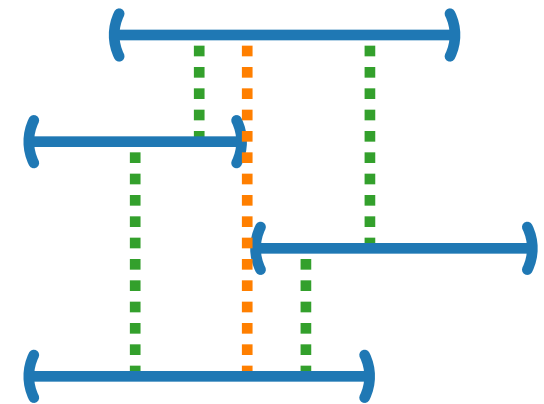
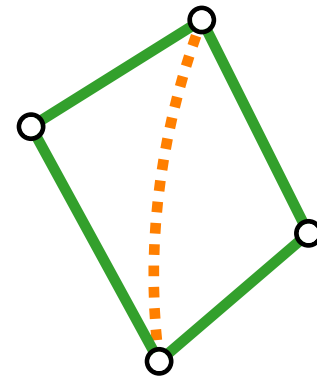
■ triangles

■ orthogonal segments



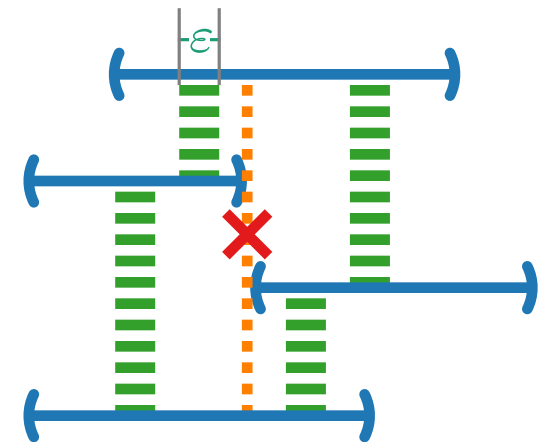
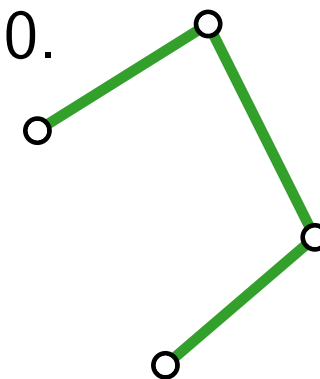
Bar Visibility Representation

- Vertices correspond to horizontal (open) line segments called **bars**.
- **Edges** correspond to unobstructed vertical lines of sight.
- What about unobstructed **0-width** vertical lines of sight? Do all visibilities induce edges?

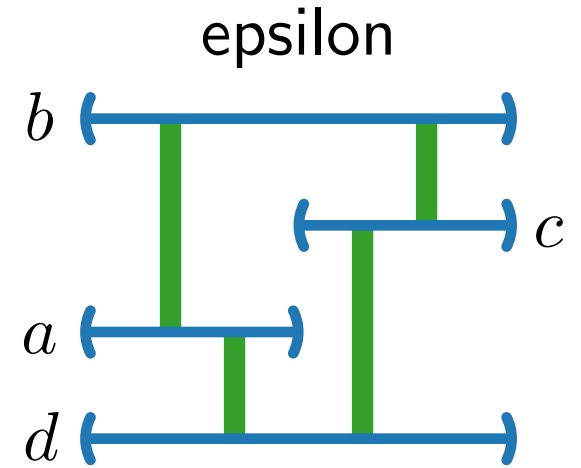
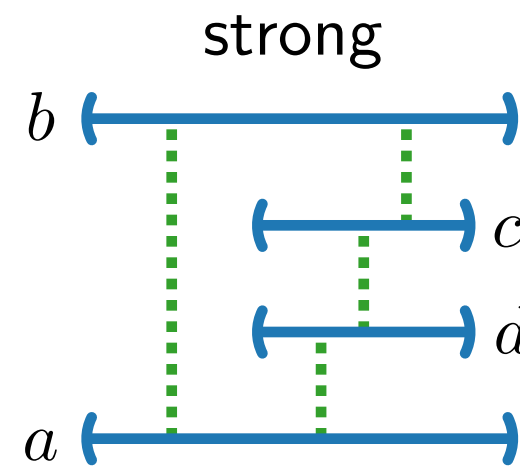
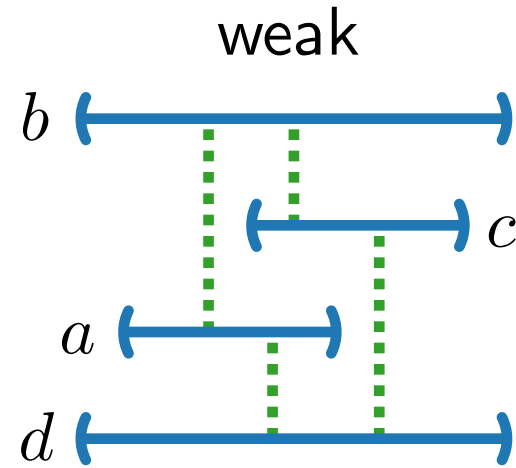
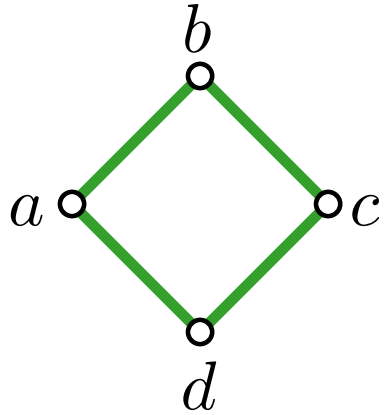


Models.

- **Strong:**
Edge $uv \Leftrightarrow$ unobstructed **0-width** vertical lines of sight.
- **Epsilon:**
Edge $uv \Leftrightarrow \varepsilon$ -wide vertical lines of sight for some $\varepsilon > 0$.
- **Weak:**
Edge $uv \Rightarrow$ unobstructed vertical lines of sight exists, i.e., any subset of *visible* pairs



Problems



Recognition Problem.

Given a graph G , **decide** whether there exists a weak/strong/ ϵ -bar visibility representation ψ of G .

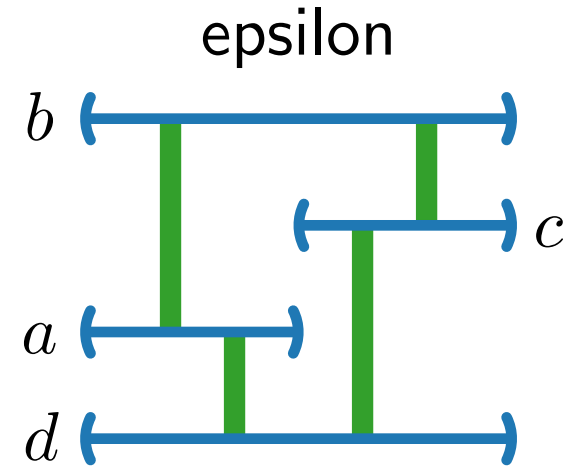
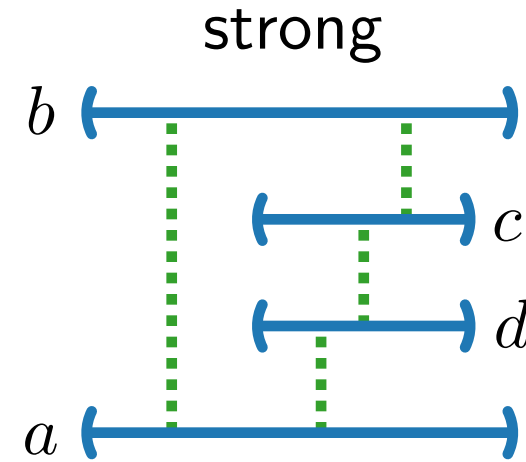
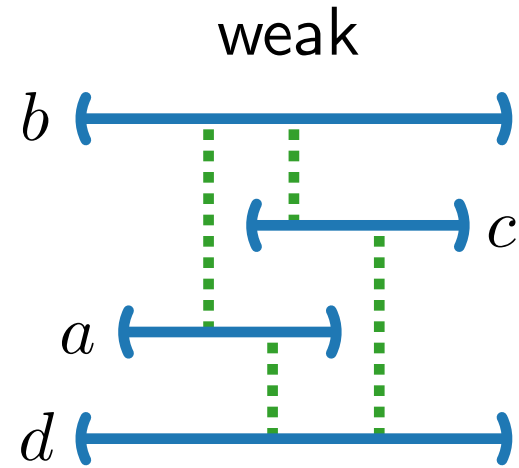
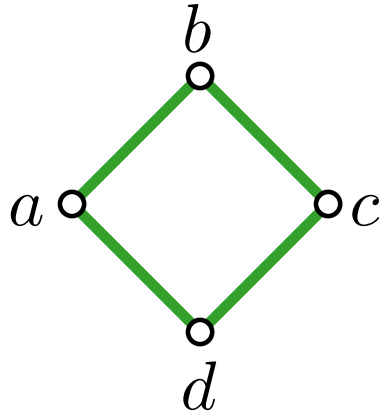
Construction Problem.

Given a graph G , **construct** a weak/strong/ ϵ -bar visibility representation ψ of G – if one exists.

Partial Representation Extension Problem.

Given a graph G and a **set of bars** ψ' of $V' \subseteq V(G)$, **decide** whether there exists a weak/strong/ ϵ -bar visibility representation ψ of G **where** $\psi|_{V'} = \psi'$ (and **construct** ψ if a representation exists).

Background



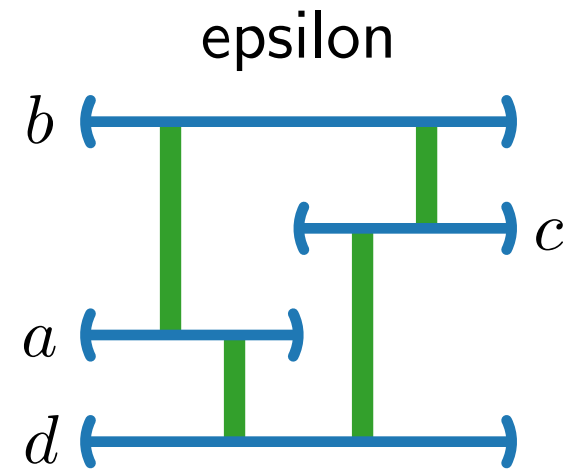
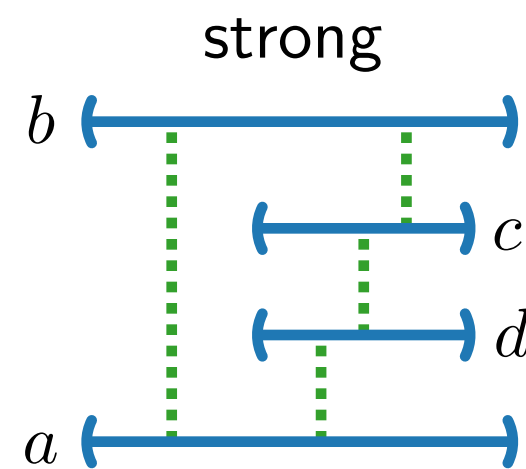
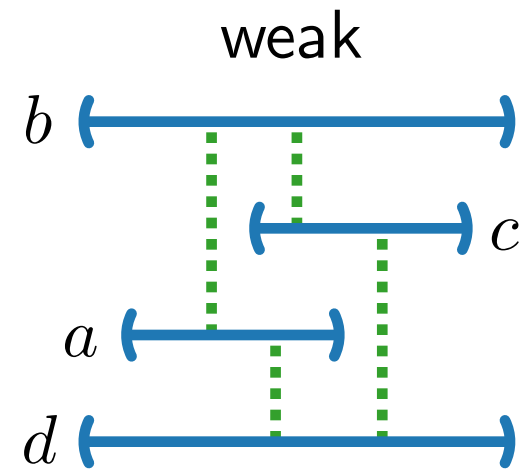
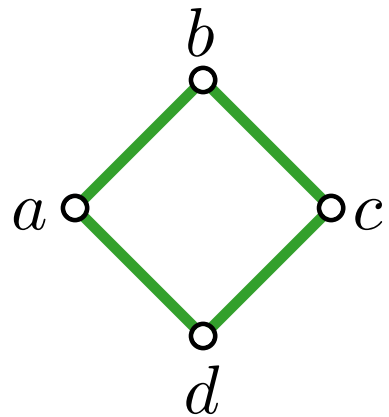
Weak Bar Visibility.

- Exactly all planar graphs [Tamassia & Tollis '86; Wismath '85]
- Linear-time recognition and construction [T&T '86]
- Representation extension is NP-complete [Chaplick et al. '14]

Strong Bar Visibility.

- NP-complete to recognize [Andreae '92]

Background

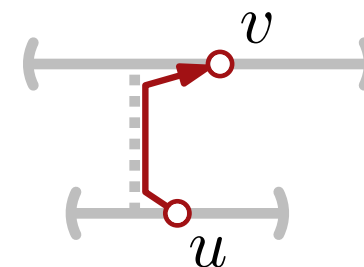
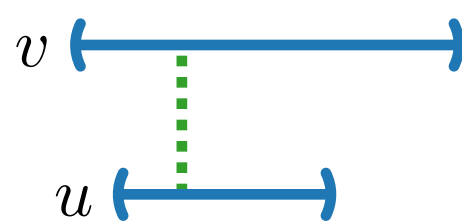
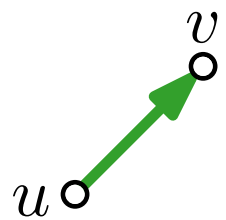


ϵ -Bar Visibility.

- Exactly all planar graphs that can be embedded with all **cut vertices** on the outer face [T&T '86, Wismath '85]
- Linear-time recognition and construction [T&T '86]
- Representation extension? **This Lecture!**

Bar Visibility Representation of Digraphs

- Instead of an undirected graph, we are given a directed graph G .
- The task is to construct a weak/strong/ ε -bar visibility representation of G such that ...
- ... for each directed edge uv , the bar representing u is below the bar representing v .



Weak Bar Visibility.

- NP-complete for directed (acyclic planar) graphs!
- This is because upward planarity testing is NP-complete. [Garg & Tamassia '01]

Strong/ ε -Bar Visibility.

- Open for directed graphs!

Next, we consider ε -bar visibility representations of specific directed graphs ($\rightarrow$ st-graphs)

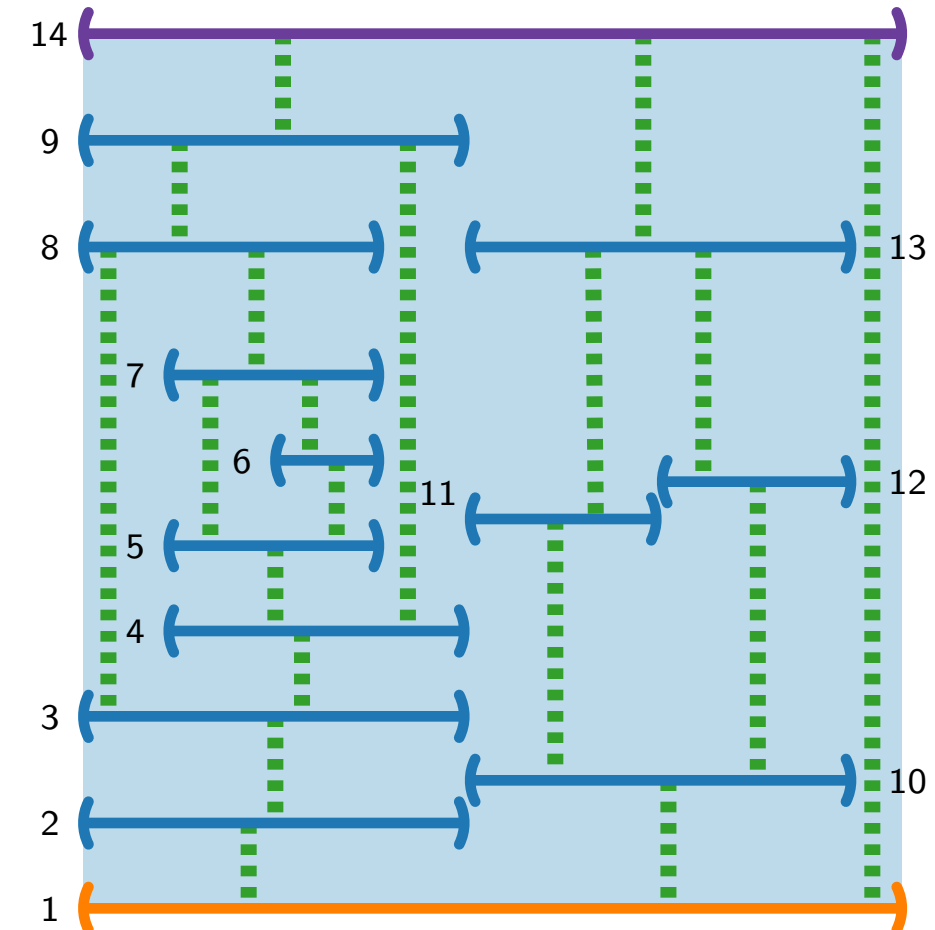
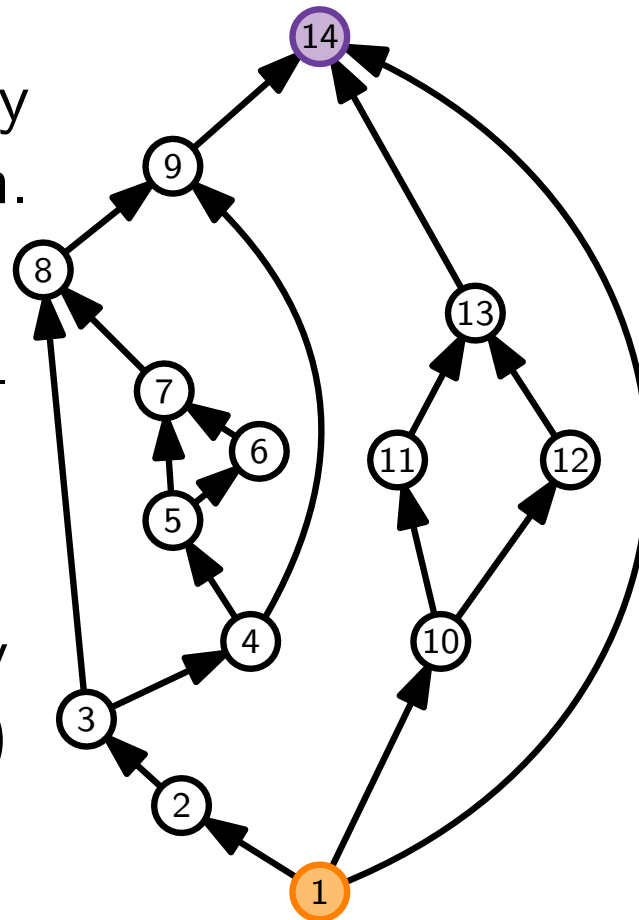
ε -Bar Visibility and st-Graphs

Recall that an **st-graph** is a planar acyclic digraph G with exactly one **source** s and one **sink** t where s and t occur on the outer face of an embedding of G .

Observation.

st-orientations correspond to ε -bar visibility representations.

- ε -bar visibility testing is easily done via st-graph recognition.
- Strong bar visibility recognition... open!
- In a **rectangular** bar visibility representation $\psi(s)$ and $\psi(t)$ span an enclosing rectangle.



Results and Outline

[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]

Theorem 1.

Rectangular ε -bar visibility representation extension can be solved in $\mathcal{O}(n \log^2 n)$ time for st-graphs.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2.

ε -bar visibility representation extension is NP-complete.

- Reduction from PLANAR MONOTONE 3-SAT

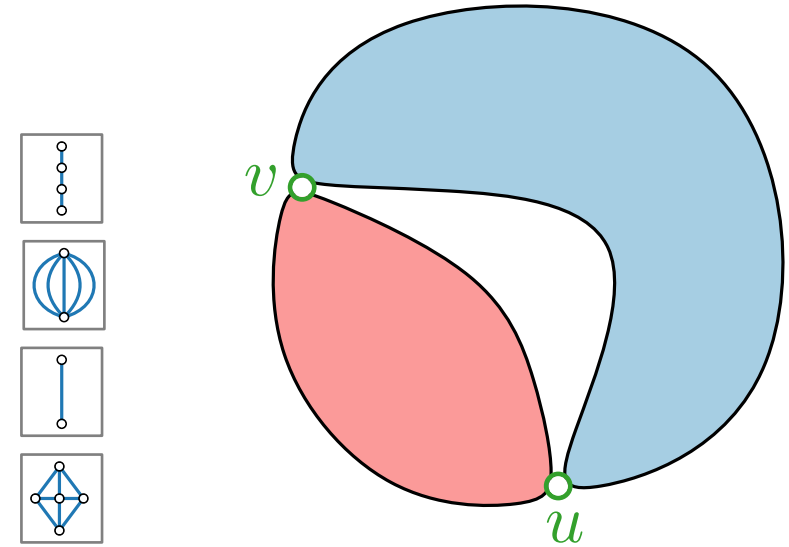
Theorem 3.

ε -bar visibility representation extension is NP-complete even for (series-parallel) st-graphs when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

- Reduction from 3-PARTITION

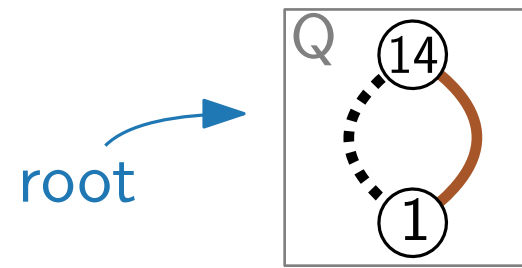
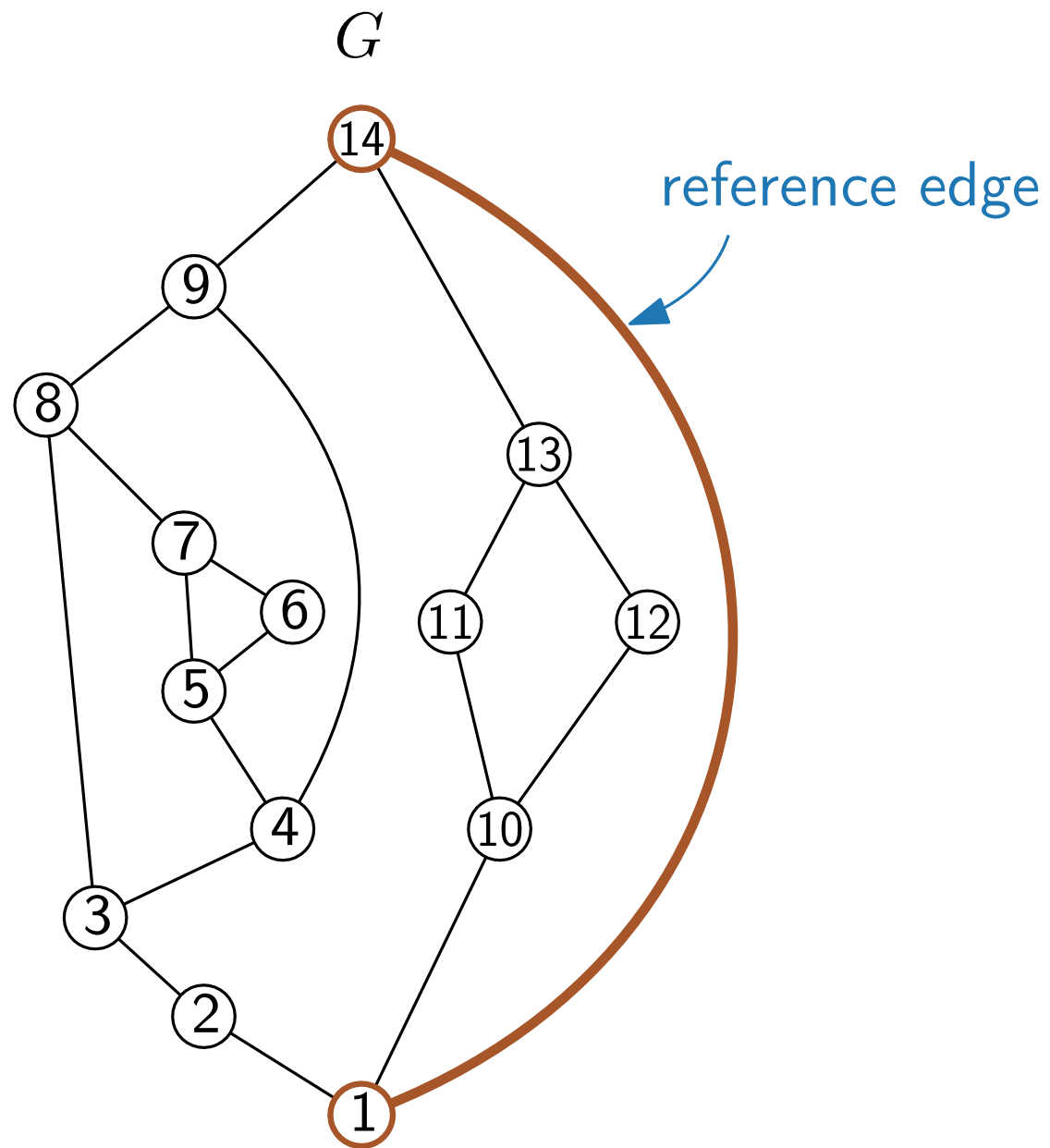
SPQR-Tree

- An **SPQR-tree** T is a decomposition of a planar graph G by **separation pairs**.
- The nodes of T are of four types:
 - **S**-nodes represent a series composition
 - **P**-nodes represent a parallel composition
 - **Q**-nodes represent a single edge
 - **R**-nodes represent 3-connected (*rigid*) subgraphs
- A decomposition tree of a series-parallel graph is an SPQR-tree without **R**-nodes.
- T represents all planar embeddings of G .
- T can be computed in time linear in the size of G .

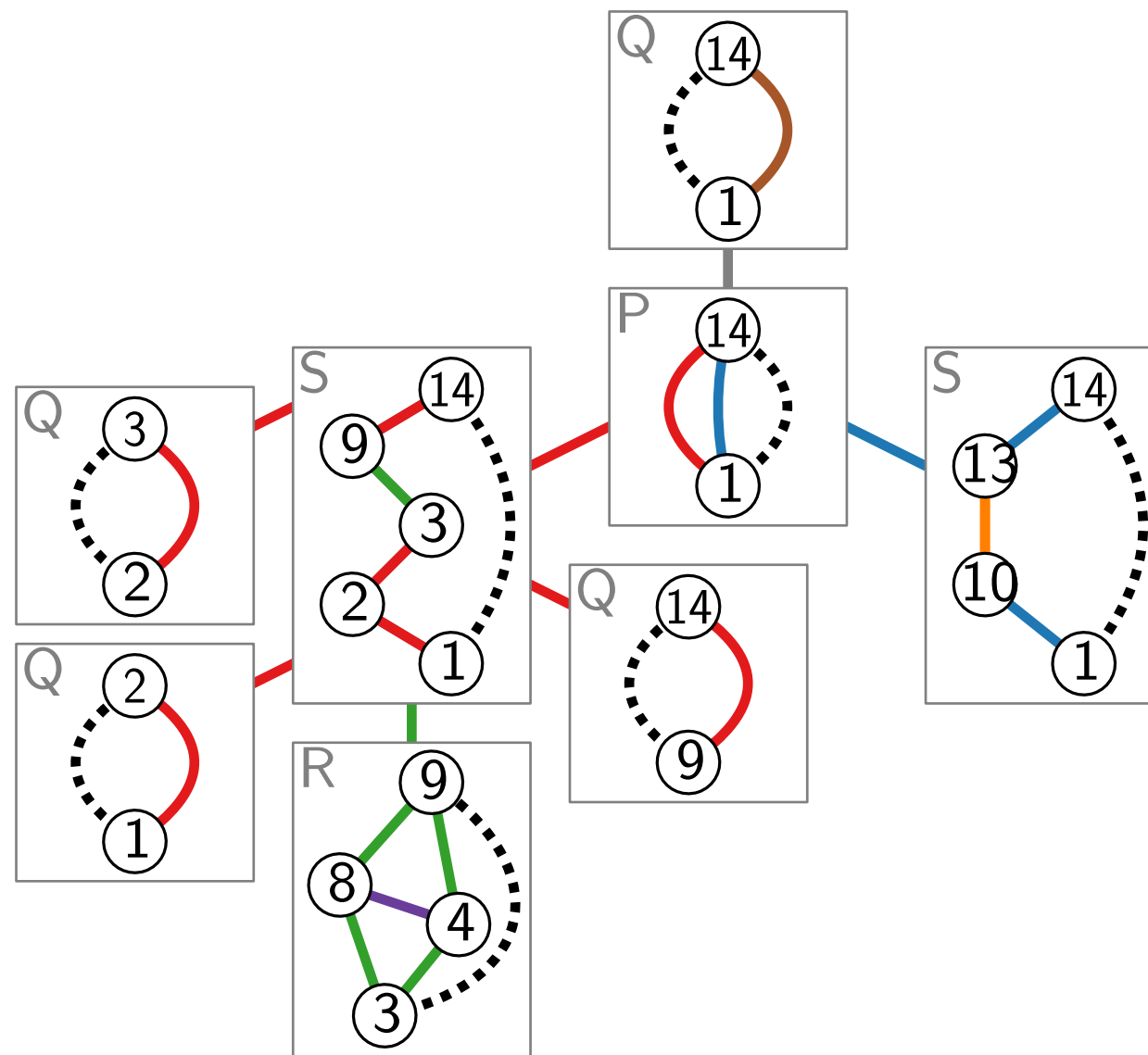
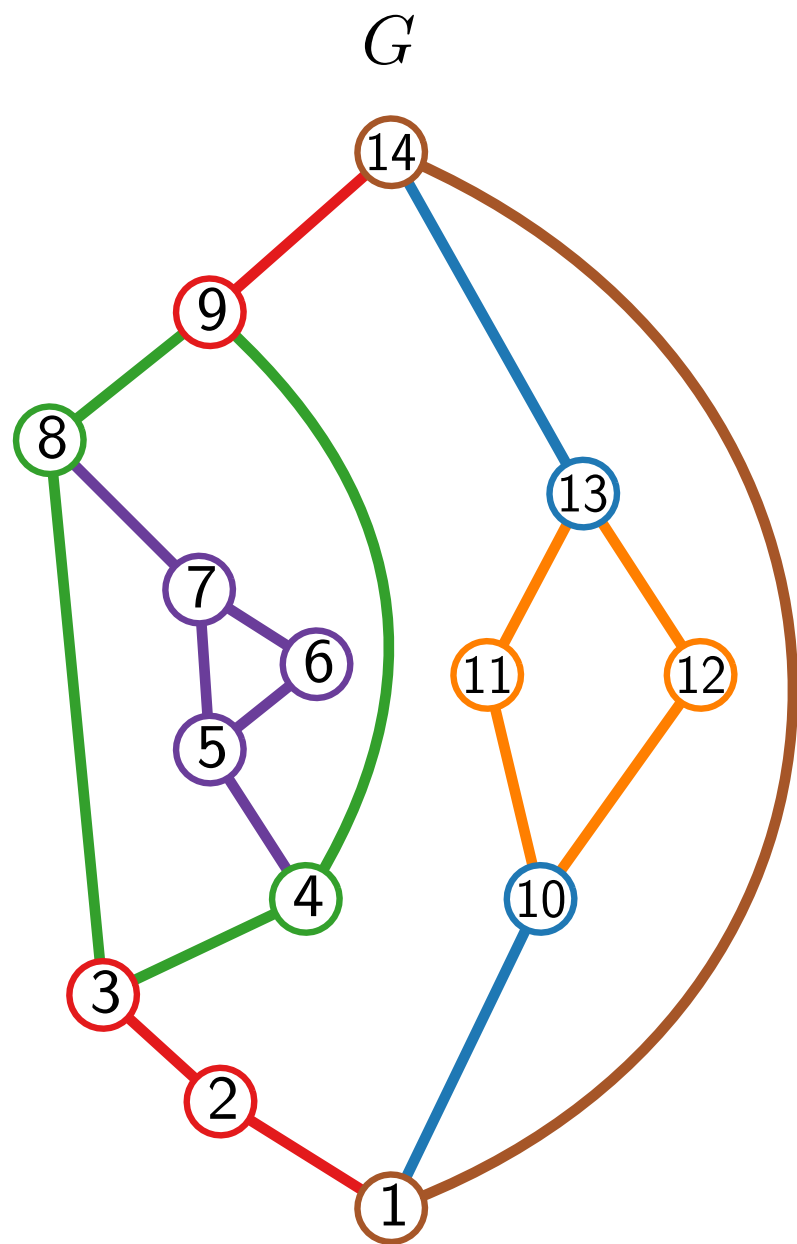


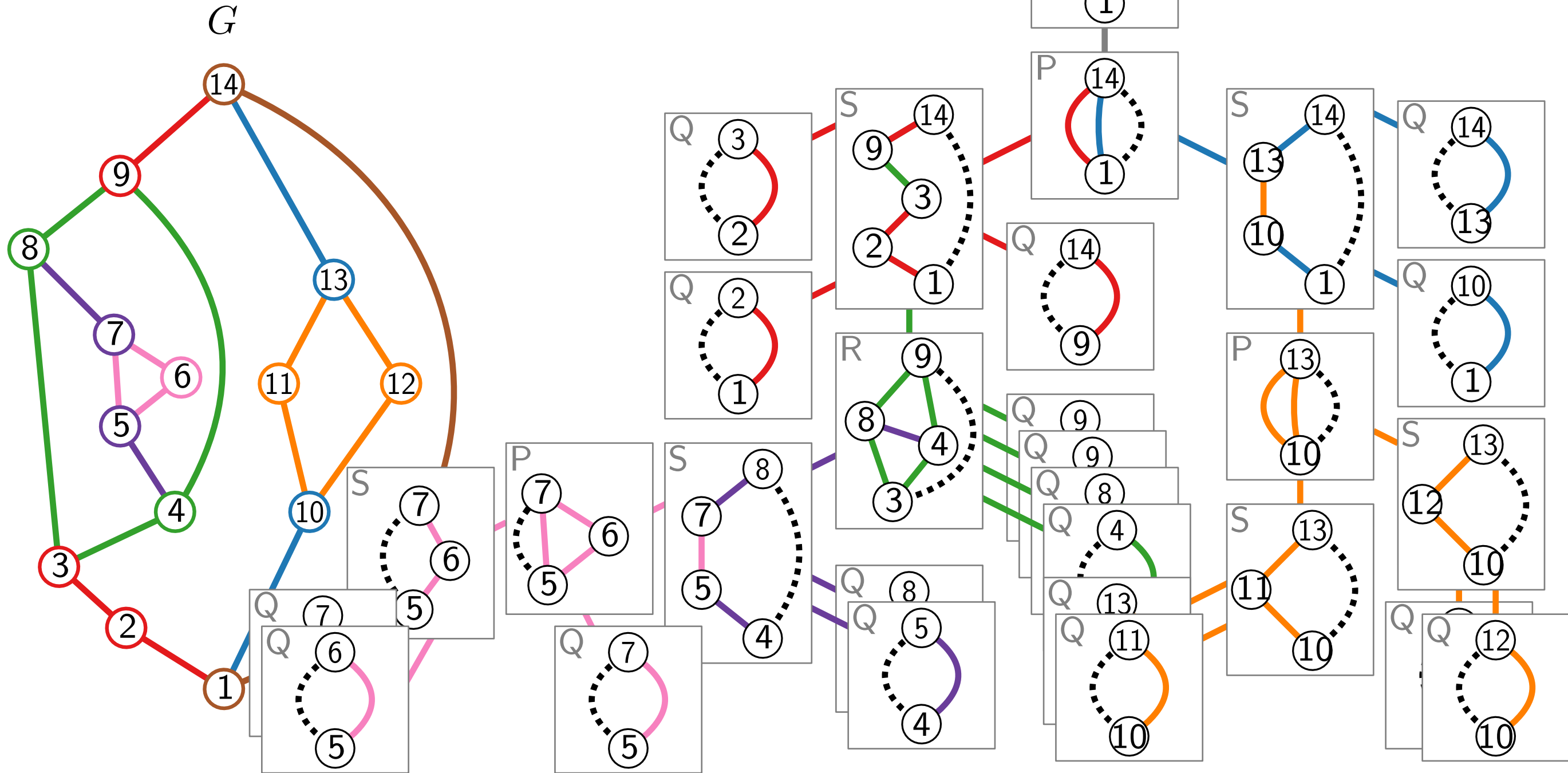
[Gutwenger, Mutzel '01]

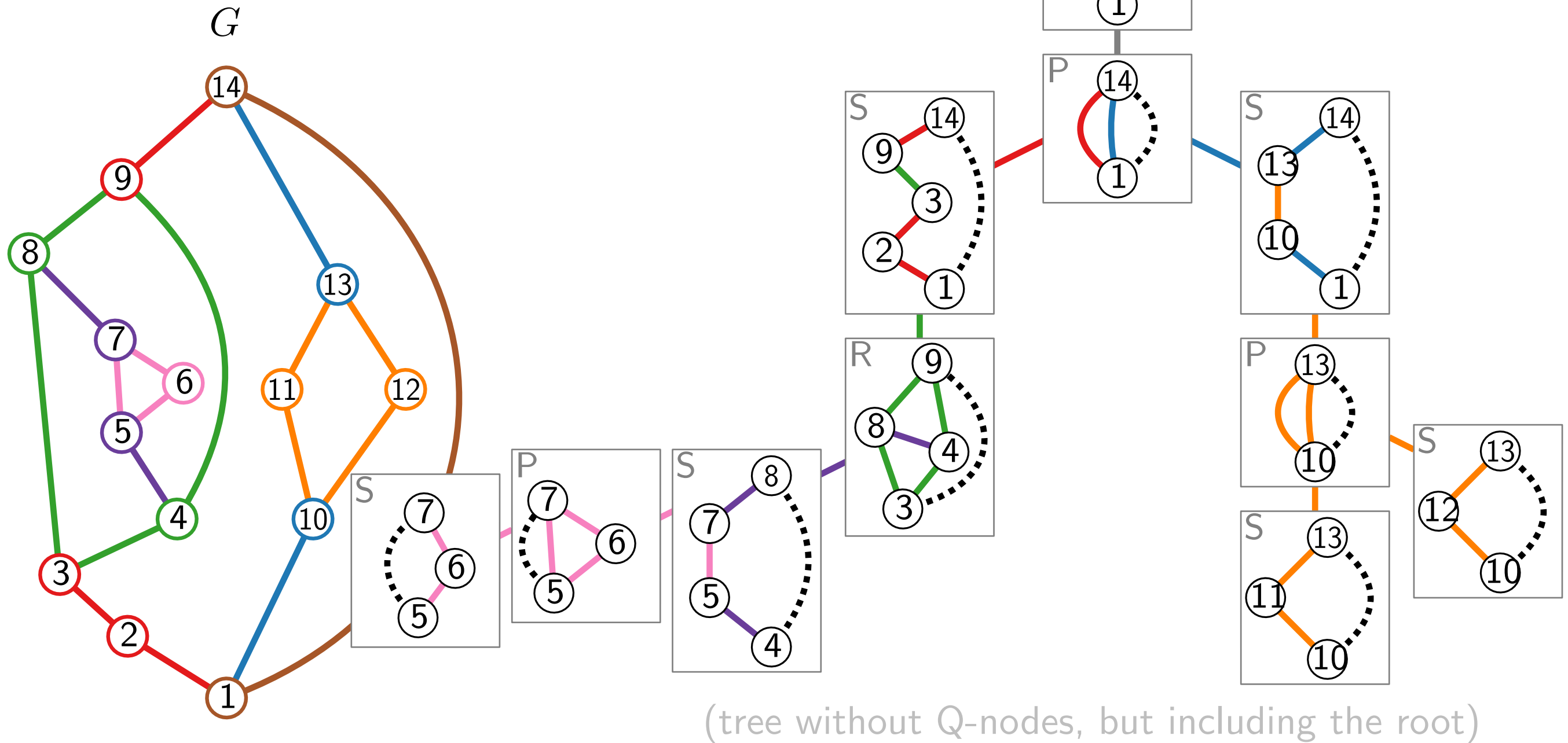
SPQR-Tree – Example



SPQR-Tree – Example



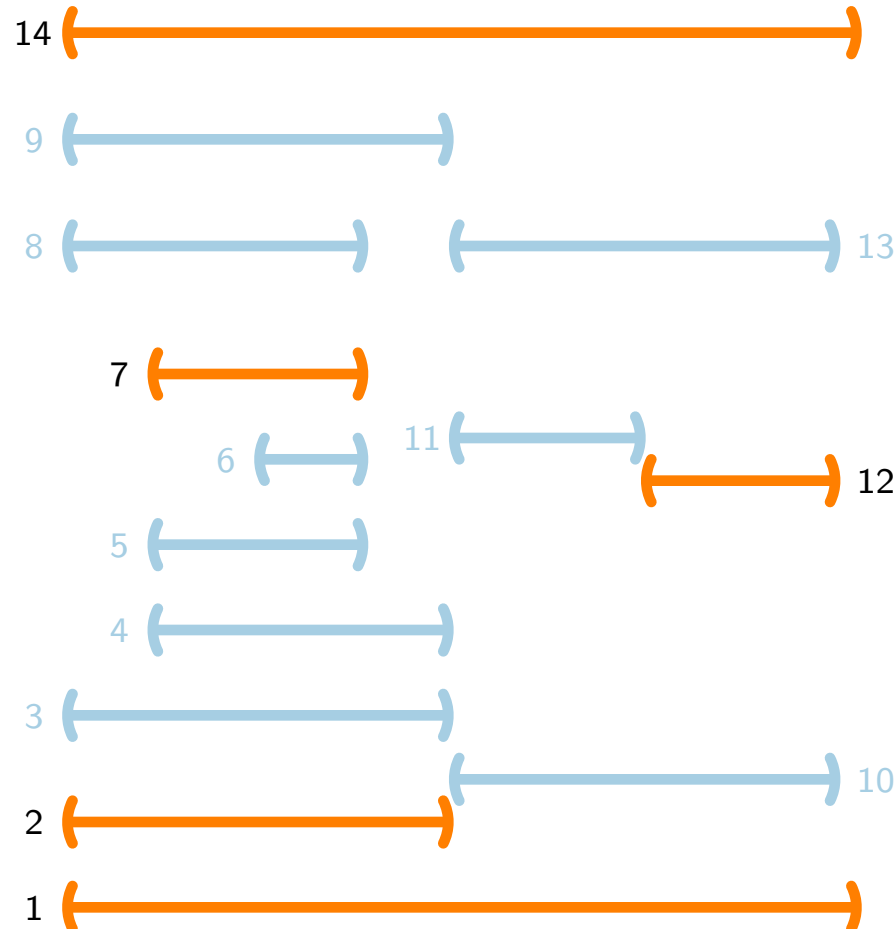
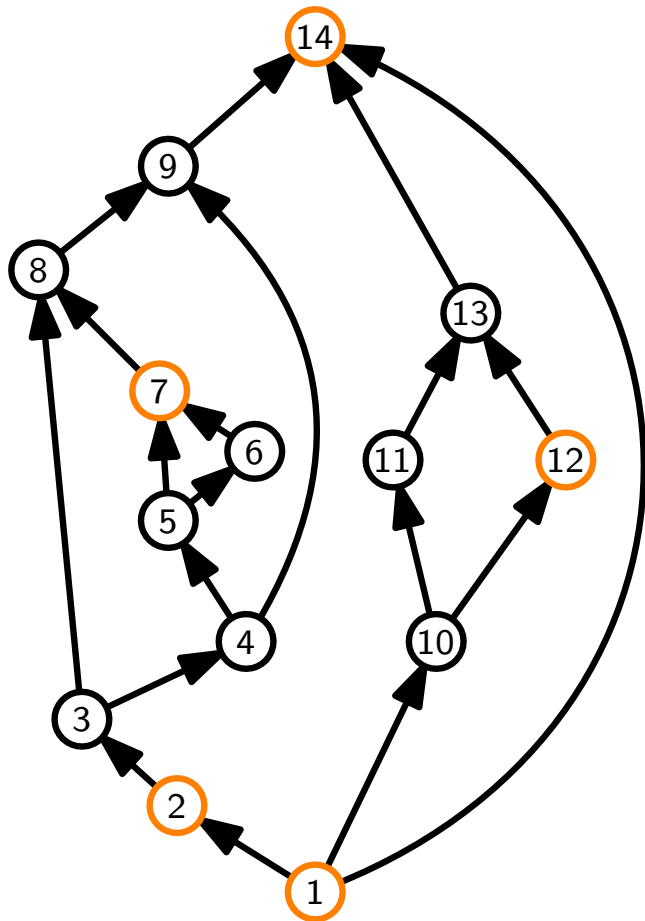




Representation Extension for st-Graphs

Theorem 1'.

Rectangular ε -bar visibility representation extension can be solved in $\mathcal{O}(n^2)$ time for st-graphs.



- Simplify problem via assumption regarding y-coordinates
- Exploit connection between SPQR-trees and rectangle tiling
- Solve problems for **S**-, **P**-, and **R**-nodes
- Dynamic program via structure of SPQR-tree

y-Coordinate Invariant

- Let G be an st-graph, and let ψ' be a representation of $V' \subseteq V(G)$.
- Let $y: V(G) \rightarrow \mathbb{R}$ such that
 - for every $v \in V'$, $y(v)$ = y-coordinate of $\psi'(v)$.
 - for every $(u, v) \in E(G)$, it holds that $y(u) < y(v)$.

Lemma 1.

G has a representation extending $\psi' \Leftrightarrow$
 G has a representation extending ψ'
 where the y-coordinates of the bars are as in y .

We can now assume that all
y-coordinates are given!

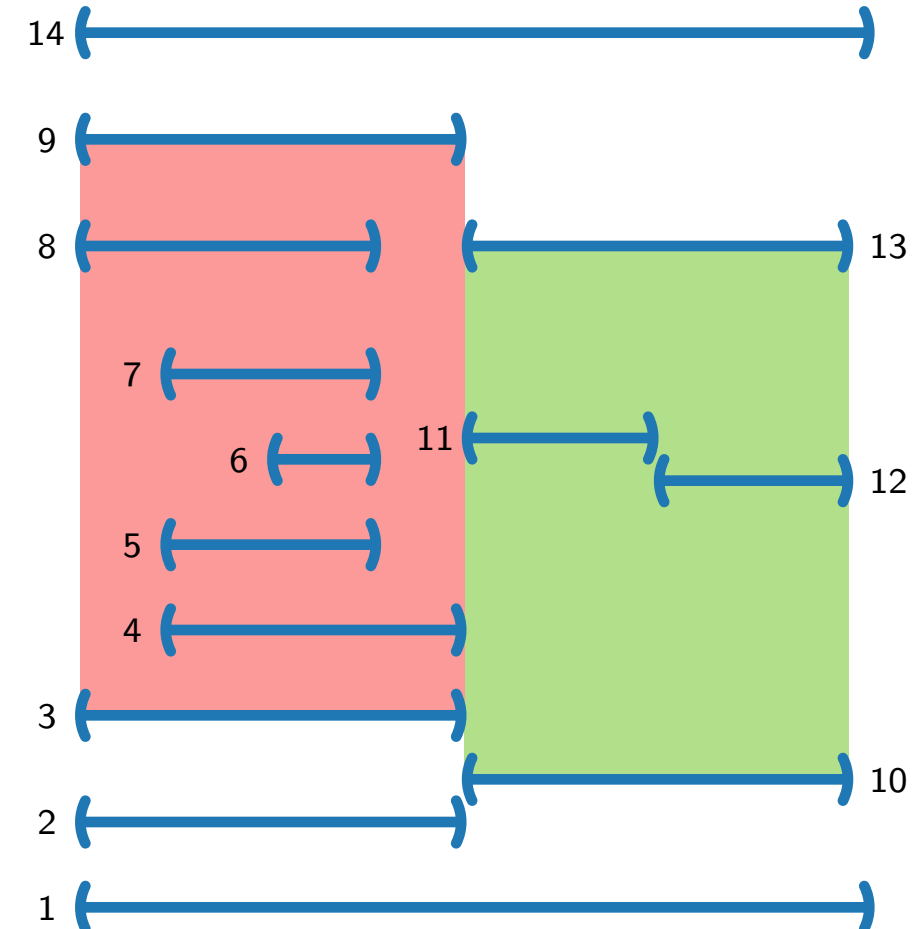
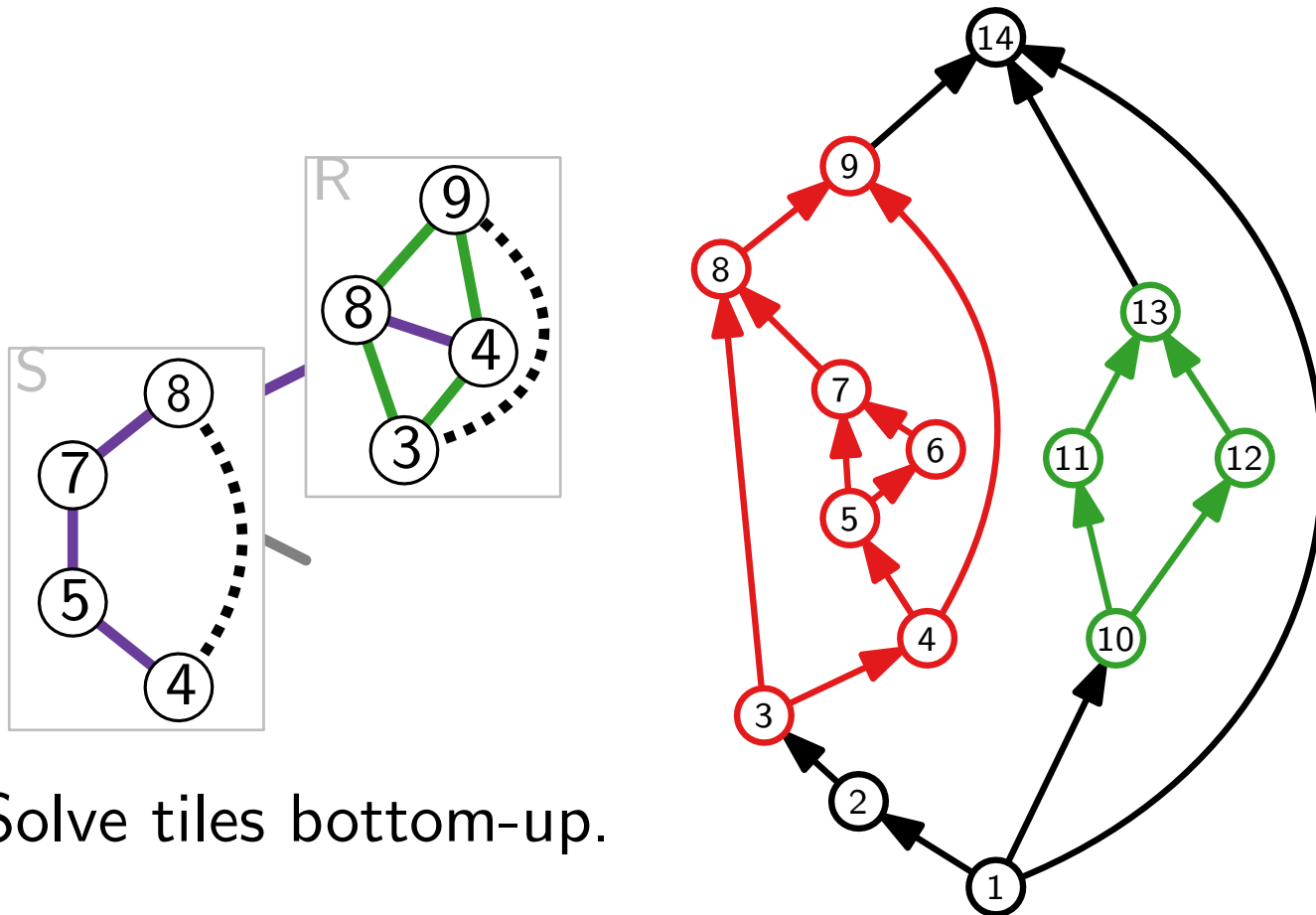
Proof idea. The relative positions of **adjacent** bars must match the order given by y .

So, we can adjust the y-coordinates of any solution to be as in y by sweeping from bottom to top.

But Why Do SPQR-Trees Help?

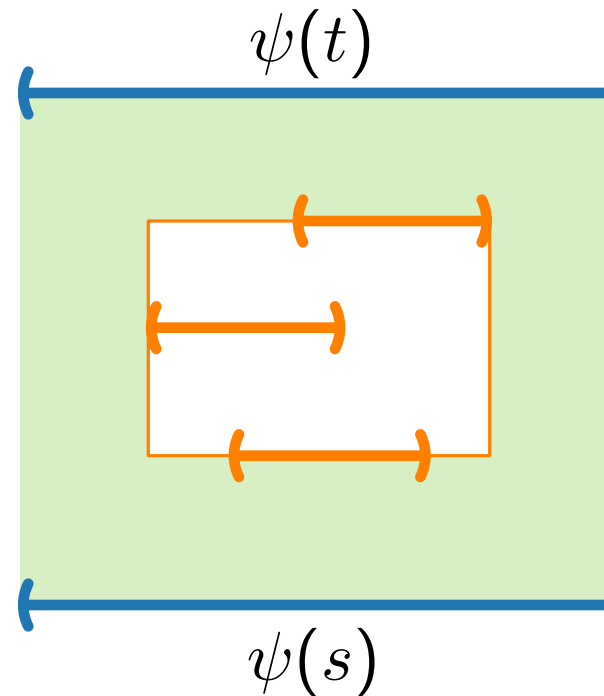
Lemma 2.

The SPQR-tree of an st-graph G induces a recursive **tiling** of any ε -bar visibility representation of G .



Tiles

Convention. Orange bars are from the given partial representation.

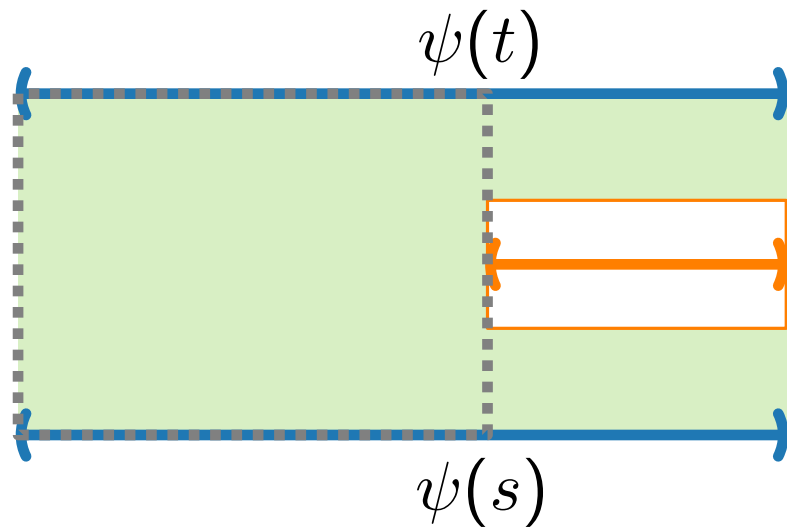


Observation.

The bounding box (tile) of any solution ψ **contains** the bounding box of the partial representation.

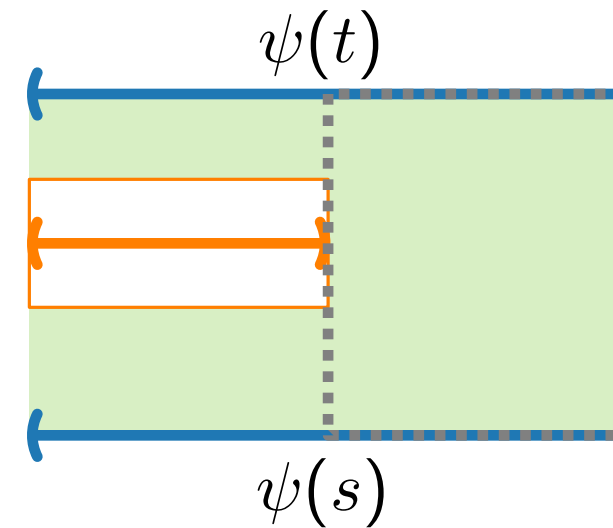
How many different **types** of tiles are there?

Types of Tiles



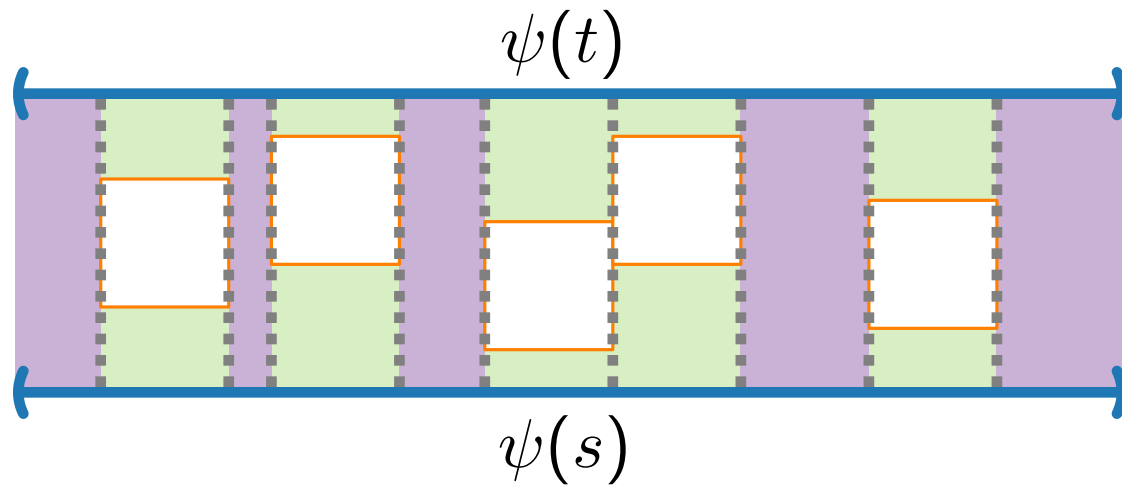
- Left **L**oose
- Right **F**ixed

- Left **F**ixed
- Right **L**oose



Four different types: **FF**, **FL**, **LF**, **LL**

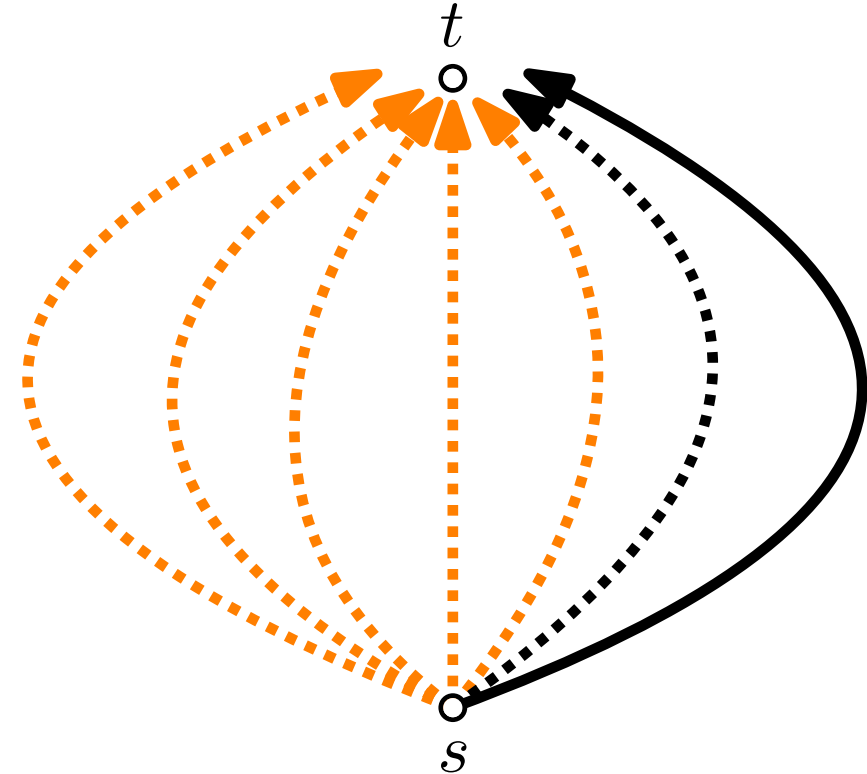
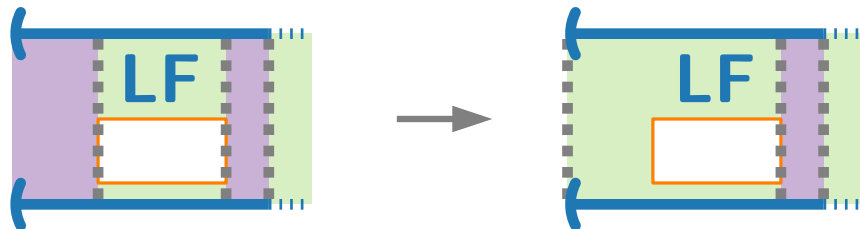
P-Nodes



- Children of **P**-node with **prescribed bars** occur in given left-to-right order
- But there might be some **gaps**...

Idea.

Greedy *fill* the **gaps** by preferring to “stretch” the children with prescribed bars.



Outcome.

After processing, we must know the valid types for the corresponding subgraphs.

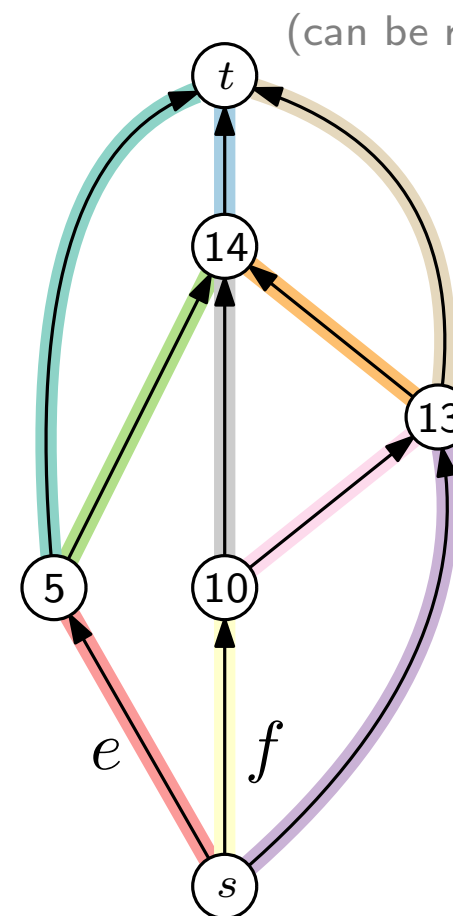
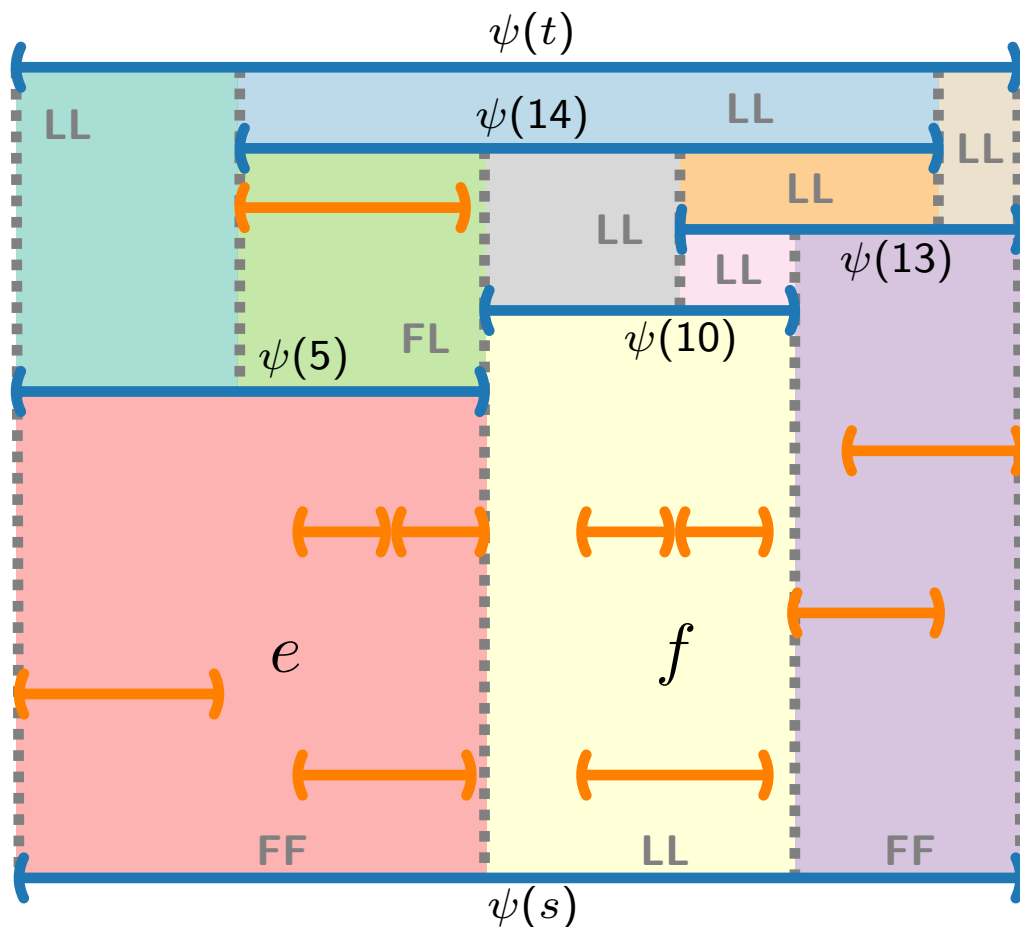


This **fixed vertex** means we can only make a **Fixed-Fixed** representation!

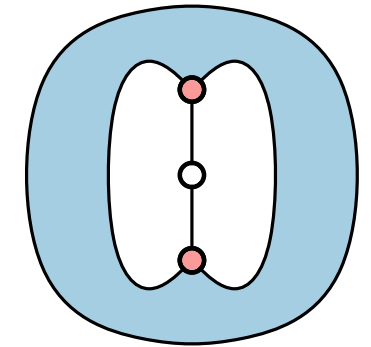
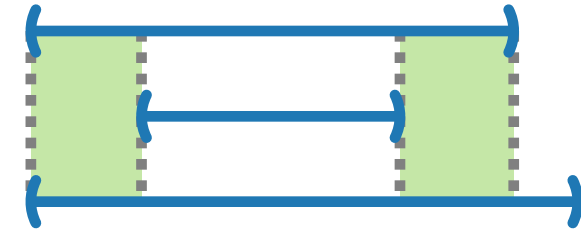


R-Nodes with 2-SAT Formulation

- For each child (edge) e :
 - Find all types of $\{\mathbf{FF}, \mathbf{FL}, \mathbf{LF}, \mathbf{LL}\}$ that admit a drawing.
 - Use two variables (l_e and r_e) to encode the type of its tile ($\mathbf{F} = 0$).
 - Add *consistency clauses*: e.g., $\neg(\neg r_e \wedge \neg l_f) \rightarrow O(n^2)$ many.



(can be reduced to $O(n \log^2 n)$)



Separation pair!
($\nexists$ in $\mathbf{R}$ -component.)

- Finding a satisfying assignment of a 2-SAT formula can be done in linear time!

$\Rightarrow O(n^2)$ time in total
or $O(n \log^2 n)$

Results and Outline

[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]

Theorem 1.

Rectangular ε -bar visibility representation extension can be solved in $\mathcal{O}(n \log^2 n)$ time for st-graphs.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2.

ε -bar visibility representation extension is NP-complete.

- Reduction from PLANAR MONOTONE 3-SAT

Theorem 3.

ε -bar visibility representation extension is NP-complete even for (series-parallel) st-graphs when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

- Reduction from 3-PARTITION

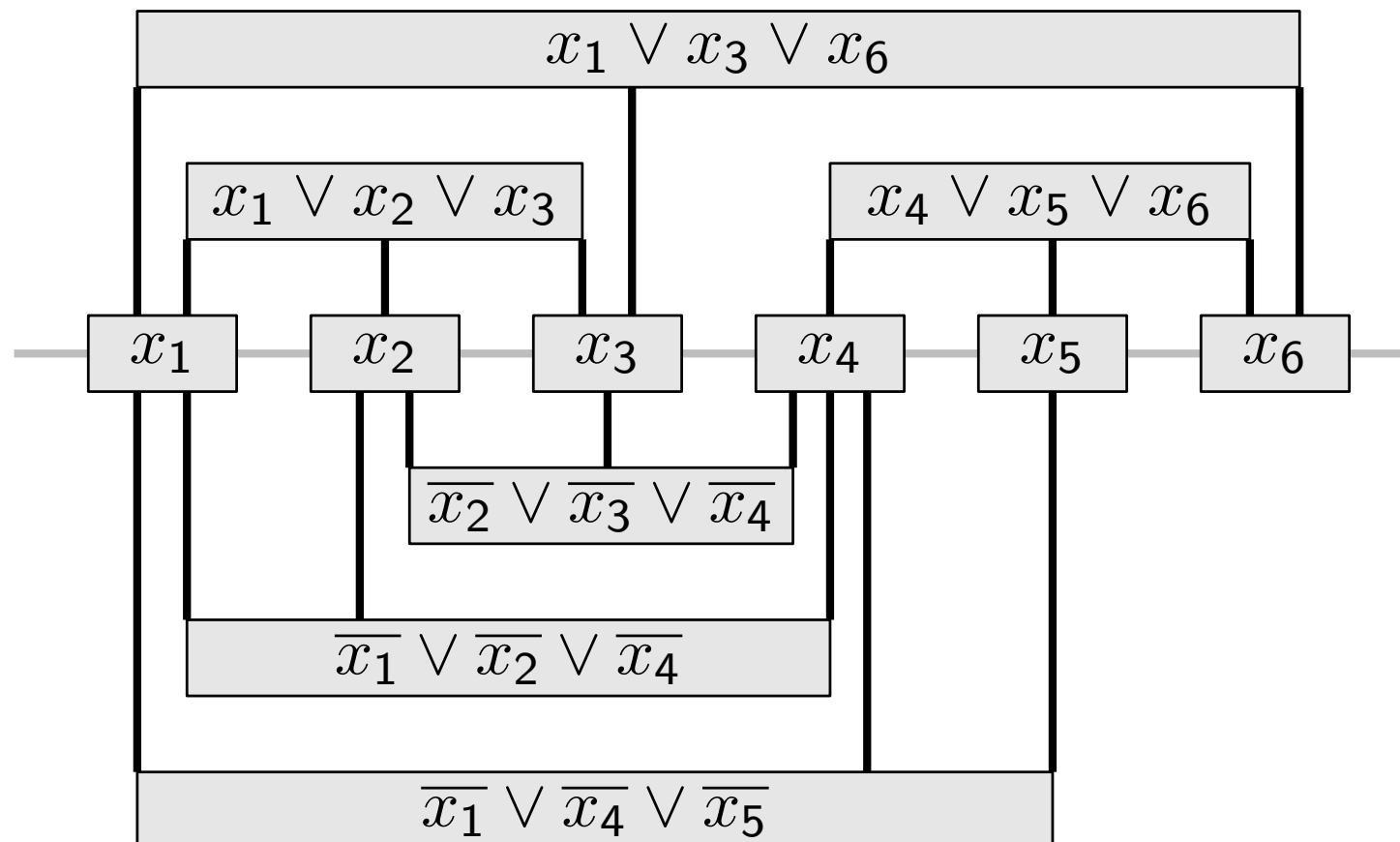
NP-Hardness of RepExt in the General Case

Theorem 2.

ε -Bar visibility representation extension is NP-complete.

■ Membership in NP?

■ NP-hard: Reduction from Planar Monotone 3-SAT



■ NP-complete
[de Berg & Khosravi '10]

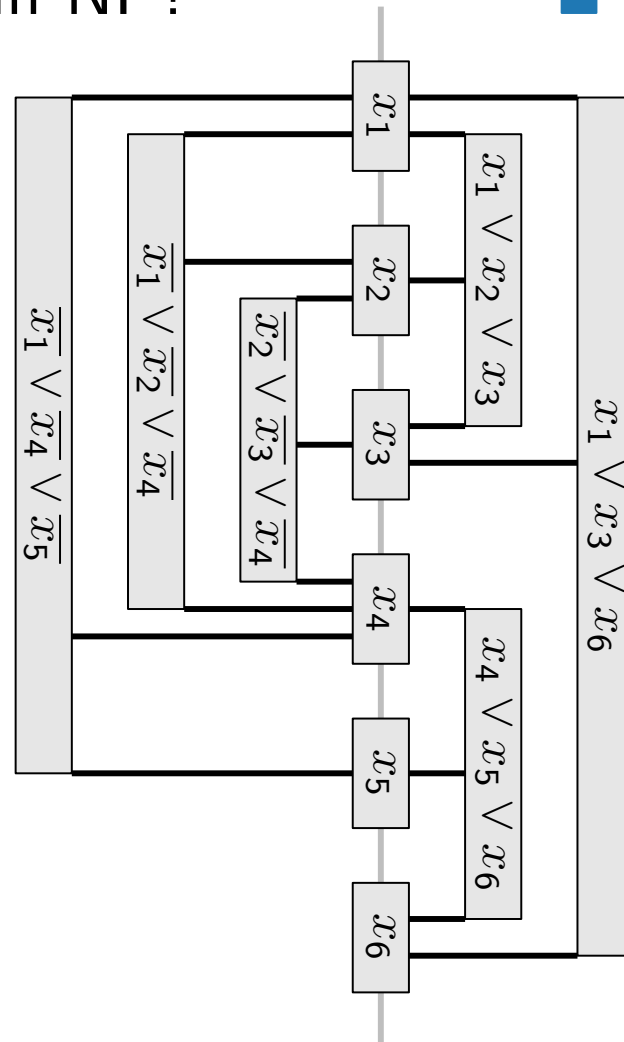
NP-Hardness of RepExt in the General Case

Theorem 2.

ε -Bar visibility representation extension is NP-complete.

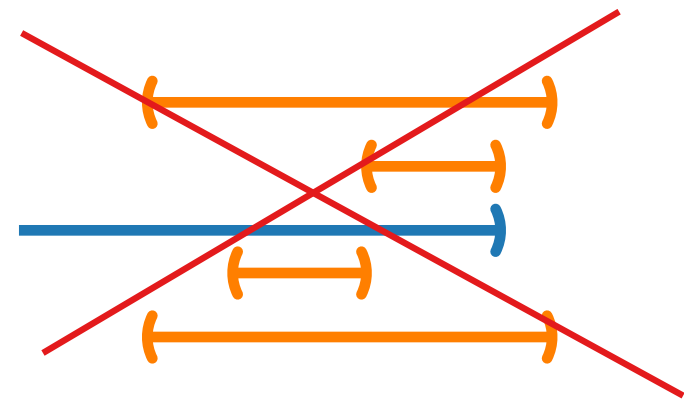
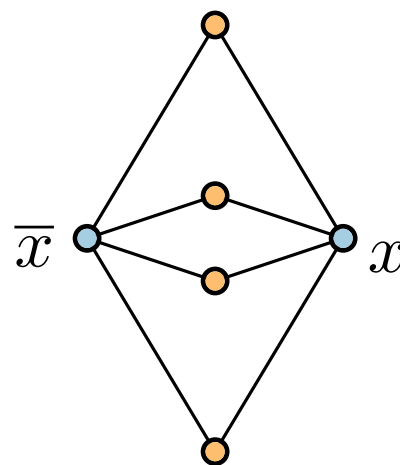
■ Membership in NP?

■ NP-hard: Reduction from Planar Monotone 3-SAT

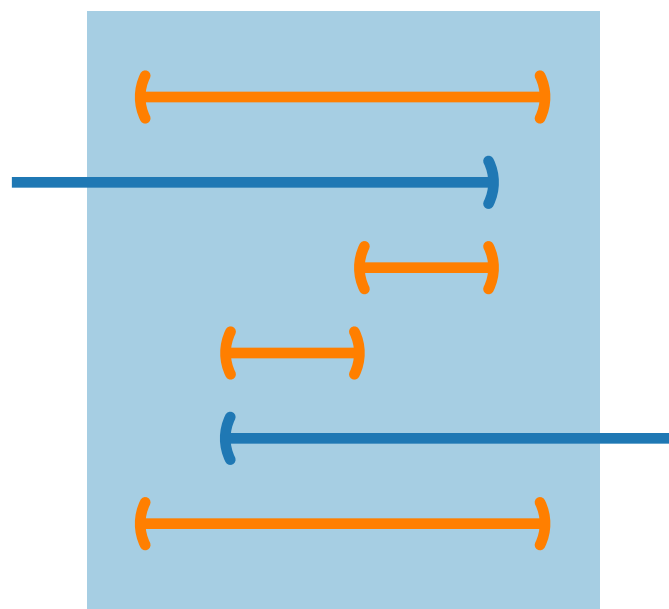


■ NP-complete
[de Berg & Khosravi '10]

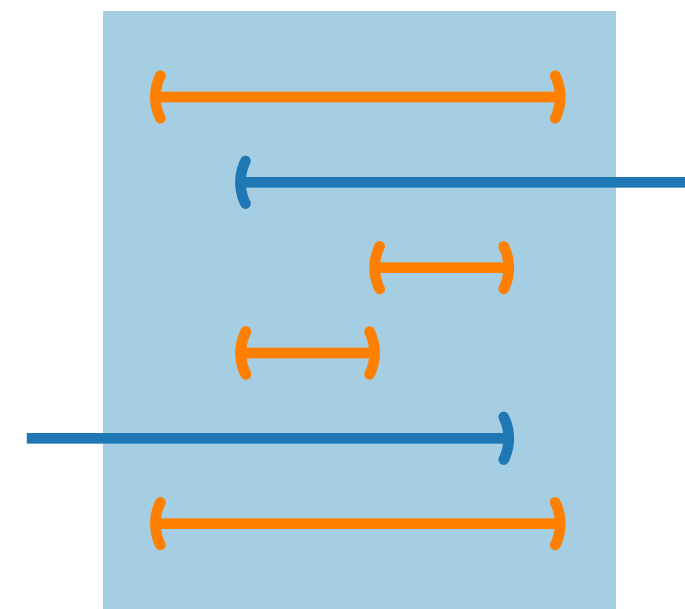
Variable Gadget



$x = \text{FALSE}$

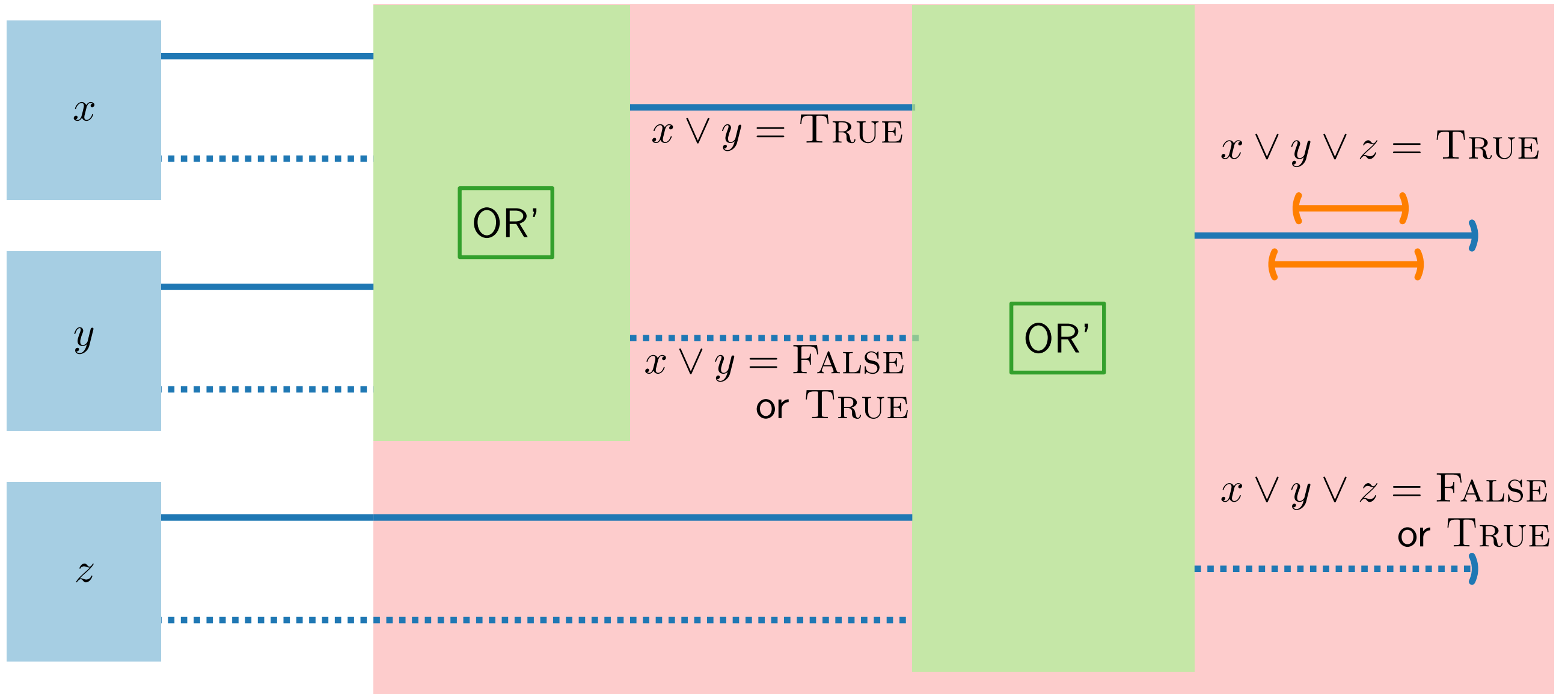


$x = \text{TRUE}$

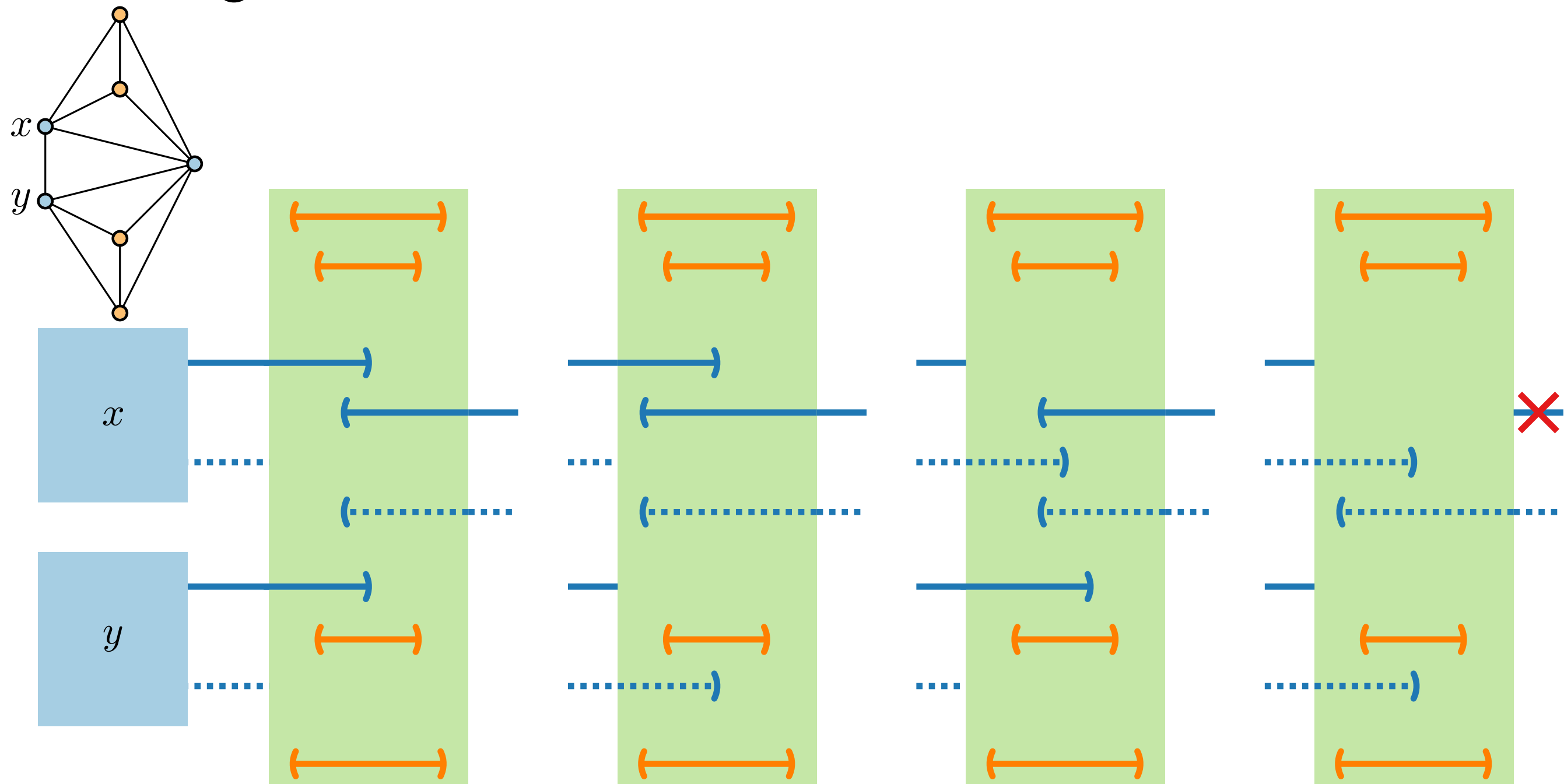


Clause Gadget

$$x \vee y \vee z$$



OR' Gadget



Discussion

- *Rectangular* ε -bar visibility representation extension can be solved in $O(n \log^2 n)$ time for *st-graphs*.
- ε -bar visibility representation extension is NP-complete.
- ε -bar visibility representation extension is NP-complete for (series-parallel) *st-graphs* when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

Open Problems:

- Can ~~*rectangular*~~ ε -bar visibility representation extension be solved in polynomial time for *st-graphs*? For DAGs?
- Can *strong* bar visibility recognition / representation extension be solved in polynomial time for *st-graphs*?

Literature

Main source:

- [Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]
The Partial Visibility Representation Extension Problem

Referenced papers:

- [Tamassia, Tollis '86] Algorithms for visibility representations of planar graphs
- [Wismath '85] Characterizing bar line-of-sight graphs
- [Chaplick, Dorbec, Kratochvíl, Montassier, Stacho '14]
Contact representations of planar graphs: Extending a partial representation is hard
- [Andreae '92] Some results on visibility graphs
- [Garg, Tamassia '01]
On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Gutwenger, Mutzel '01] A Linear Time Implementation of SPQR-Trees
- [de Berg, Khosravi '10] Optimal Binary Space Partitions in the Plane