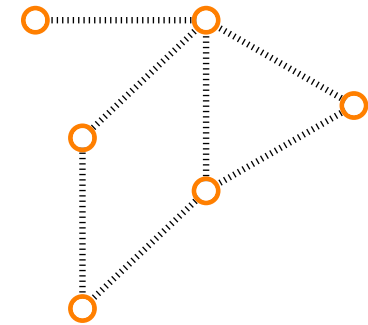
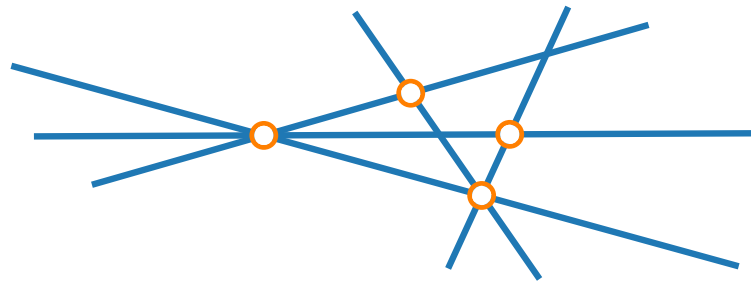


Visualization of Graphs

Lecture 9: The Crossing Lemma and Its Applications



Alexander Wolff

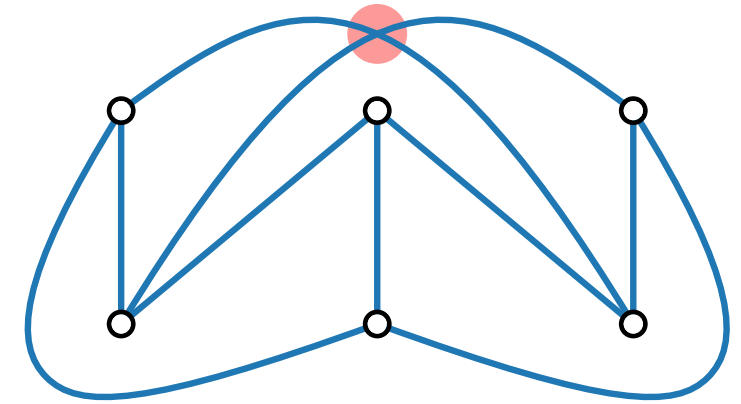
Summer term 2026

Crossing Number and Topological Graphs

For a graph G , the **crossing number** $\text{cr}(G)$ is the smallest number of pairwise edge crossings in a drawing of G (in the plane).

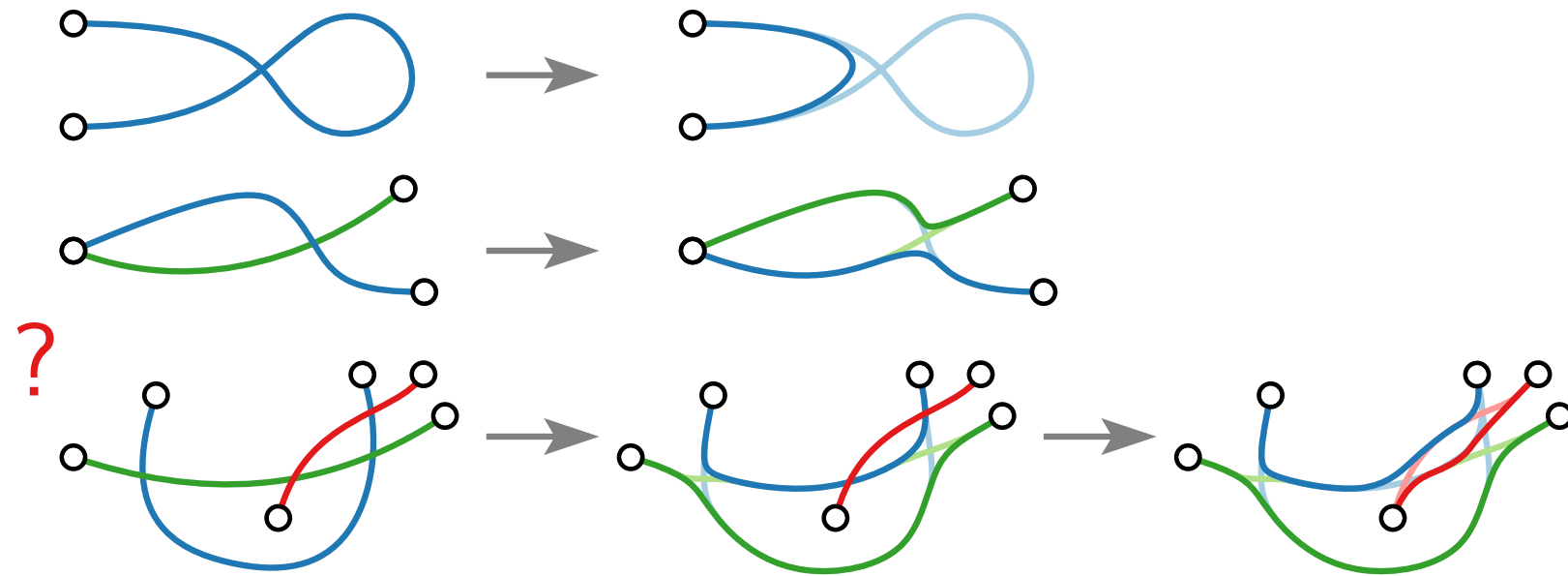
Example.

$$\text{cr}(K_{3,3}) = 1$$



In a crossing-minimal drawing of G

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once,
- and, w.l.o.g., at most two edges intersect at the same point.



crossings reduced; so, an iterative procedure terminates

Such a drawing is called a **topological drawing** of G .

Computing the Crossing Number

- Computing $\text{cr}(G)$ is NP-hard. [Garey & Johnson '83]
... even if G is a planar graph plus one edge! [Cabello & Mohar '08]
- $\text{cr}(G)$ often unknown, only conjectures exist
(for K_n exact number are only known for up to 14 vertices)
- In practice, $\text{cr}(G)$ is often not computed directly but rather drawings of G are optimized with
 - force-based methods,
 - multidimensional scaling,
 - other heuristics, ...
- $\text{cr}(G)$ is a measure of how far G is from being planar.
- For planarization, where we replace crossings with dummy vertices, also only heuristic approaches are known.

For exact computations,
check out <http://crossings.uos.de>!

Rectilinear Crossing Number

Definition.

For a graph G , the **rectilinear (straight-line) crossing number** $\overline{\text{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of G .

Even more ...

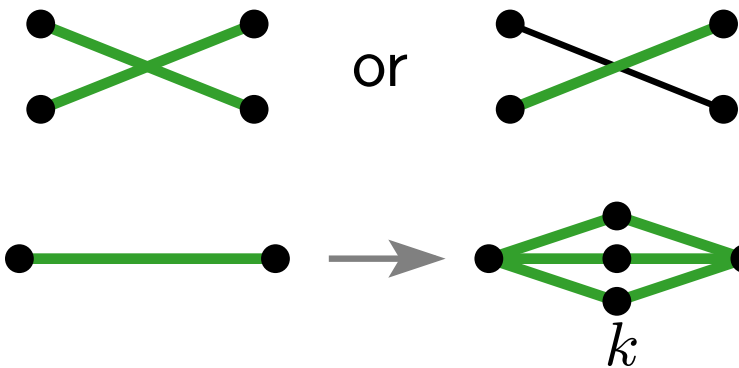
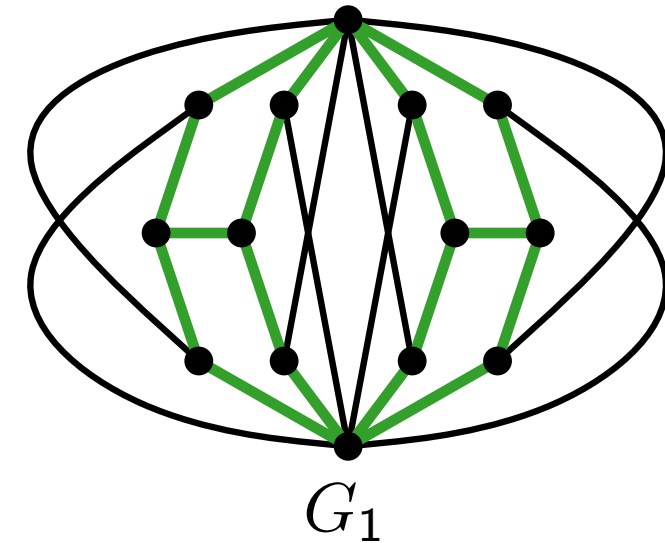
Lemma 1. [Bienstock, Dean '93]

For every $k \geq 4$, there exists a graph G_k with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$.

- Every straight-line drawing of G_1 has at least one crossing of the following types:
- From G_1 to G_k do

Separation.

$\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.



Bounds for Complete Graphs

Theorem. Conjecture.

[Guy '60]

$$\text{cr}(K_n) \stackrel{?}{=} \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor = \frac{3}{8} \binom{n}{4} + O(n^3)$$

Bound is tight for $n \leq 14$.

complete bipartite graph with $m \times n$ edges

Theorem. Conjecture.

[Zarankiewicz '54, Urbaník '55]

$$\text{cr}(K_{m,n}) \stackrel{?}{=} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor$$



Pál Turán
*1910 – 1976
Budapest, Hungary

Turán's brick factory problem (1944)



© TruckinTim

Bounds for Complete Graphs

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[Guy '60]

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Theorem.

[Lovász et al. '04, Aichholzer et al. '06]

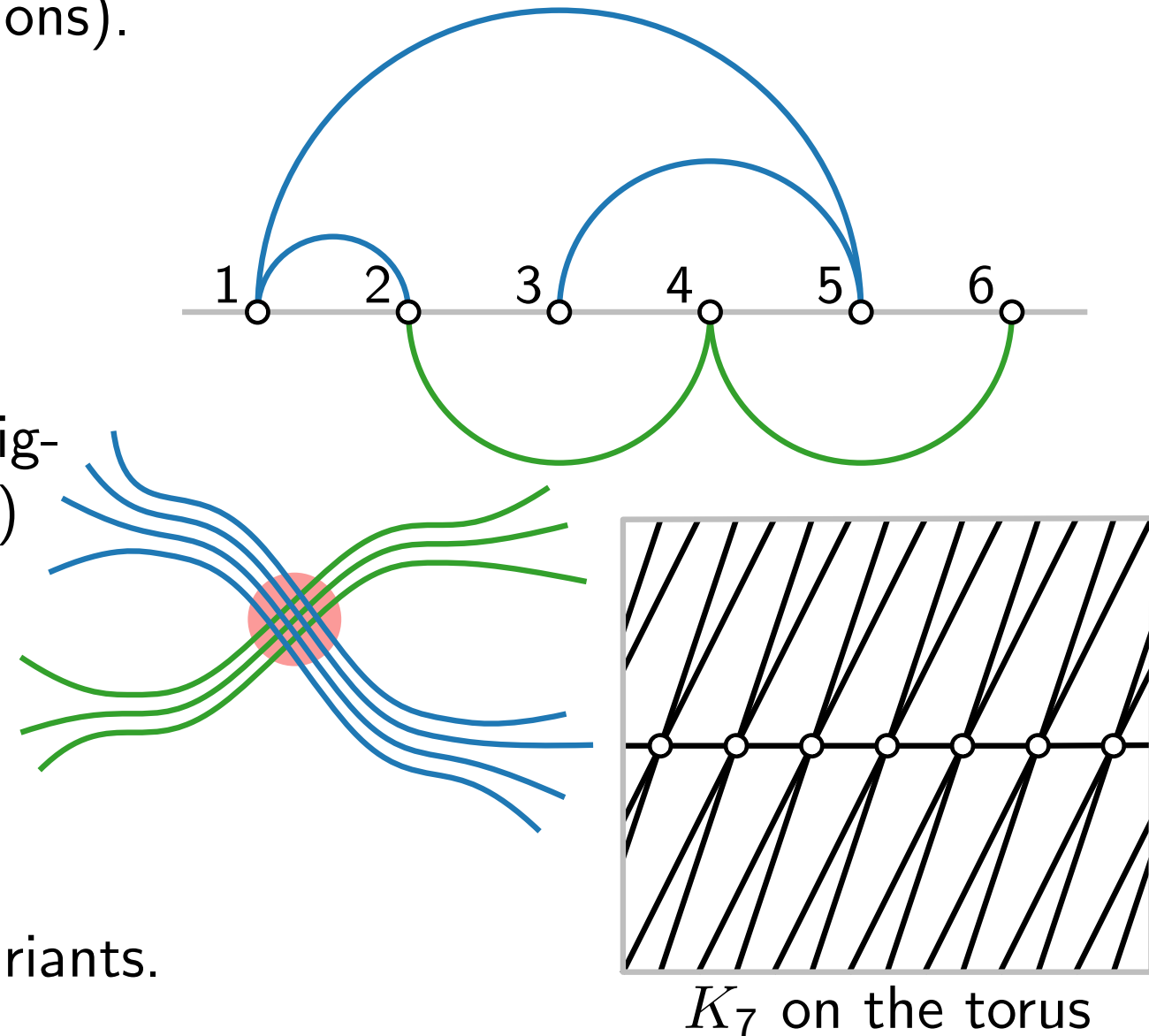
$$\left(\frac{3}{8} + \varepsilon \right) \binom{n}{4} + O(n^3) < \overline{\text{cr}}(K_n) < 0.3807 \binom{n}{4} + O(n^3)$$

Exact numbers on $\overline{\text{cr}}(K_n)$ are known for $n \leq 27$.

Check out <http://www.ist.tugraz.at/staff/aichholzer/crossings.html>

Other Crossing Numbers

- Schaefer [Sch20] wrote a survey on many variants of crossing numbers (including precise definitions).
- One-sided crossing minimization (see lecture 8 on hierarchical layouts)
- Fixed linear crossing number → Exercise
- Book embeddings (vertices on a line, edges assigned to few “pages” where edges do not cross)
- Crossings of edge bundles
- On other surfaces, such as donuts
- Weighted crossings
- Crossing minimization is **NP-hard** for most variants.



Hanani–Tutte Theorem

Theorem.

[Hanani '43, Tutte '70]

A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Proof sketch.

Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

Every non-planar graph has K_5 or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

Hence, there must be two edges on these paths that cross an odd number of times. □

Hanani–Tutte Theorem

Theorem.

[Hanani '43, Tutte '70]

A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Corollary.

$\text{ocr}(G) = 0 \Rightarrow \text{pcr}(G) = 0 \Rightarrow \text{cr}(G) = 0$

Theorem.

[Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]

There is a graph G with $\text{ocr}(G) < \text{cr}(G) \leq 10$

Theorem.

[Pelsmajer, Schaefer & Štefankovič '07] [Pach & Tóth '00]

If Γ is a drawing of G and E_0 is the set of edges that cross any other edge an even number of times in Γ , then G can be drawn such that no edge in E_0 is involved in any crossings **and no new pairs of edges cross**.

The **pairwise crossing number** $\text{pcr}(G)$ of G is the smallest number of pairs of edges that cross in a drawing of G .

By definition $\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$

The **odd crossing number**

$\text{ocr}(G)$ of G is the smallest number of pairs of edges that cross an odd number of times in a drawing of G .

Is $\text{ocr}(G) = \text{cr}(G)$? **No!**

Is $\text{ocr}(G) = \text{pcr}(G)$? **No!**

Is $\text{pcr}(G) = \text{cr}(G)$? **Open!**

Note that, in the resulting drawing of G , an edge might cross some edges an odd number of times and some other edges an even number of times. So, no implications on $\text{ocr}(G) = \text{pcr}(G)$.

Theorem. [Pelsmajer, S. & Š.'08, Tóth'08]
There exist graphs where $\text{ocr}(G) < \text{pcr}(G)$.

First Lower Bounds on $\text{cr}(G)$

Lemma 2.

For a graph G with n vertices and m edges,

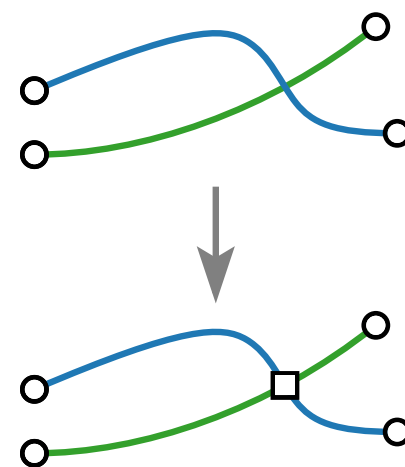
$$\text{cr}(G) \geq m - 3n + 6.$$

Proof.

- Consider a drawing of G with $\text{cr}(G)$ crossings.
- Obtain a graph H by turning crossings into dummy vertices.
- H has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- H is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6 \quad \Leftrightarrow \quad \text{cr}(G) \geq m - 3n + 6. \quad \square$$

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.



First Lower Bounds on $\text{cr}(G)$

Lemma 3.

For a non-planar graph G with n vertices and m edges,

$$\text{cr}(G) \geq r \cdot \binom{\lfloor m/r \rfloor}{2} \in \Omega\left(\frac{m^2}{n}\right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of G .

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

Proof sketch.

- Take $\lfloor m/r \rfloor$ edge-disjoint subgraphs $G_1, G_2, \dots, G_{\lfloor m/r \rfloor}$ of G with (at least) r edges.
- In the best case, they are all planar.
- For every $i < j$, any edge of G_j induces at least one crossings with G_i . (Otherwise, we could add an edge to G_i and obtain a planar subgraph of G with $r + 1$ edges.)
- So, for each of the $\binom{\lfloor m/r \rfloor}{2}$ pairs of subgraphs, there are at least r crossings.

The Crossing Lemma

- In 1973 Erdős and Guy conjectured that $\text{cr}(G) \in \Omega(m^3/n^2)$.
- In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that

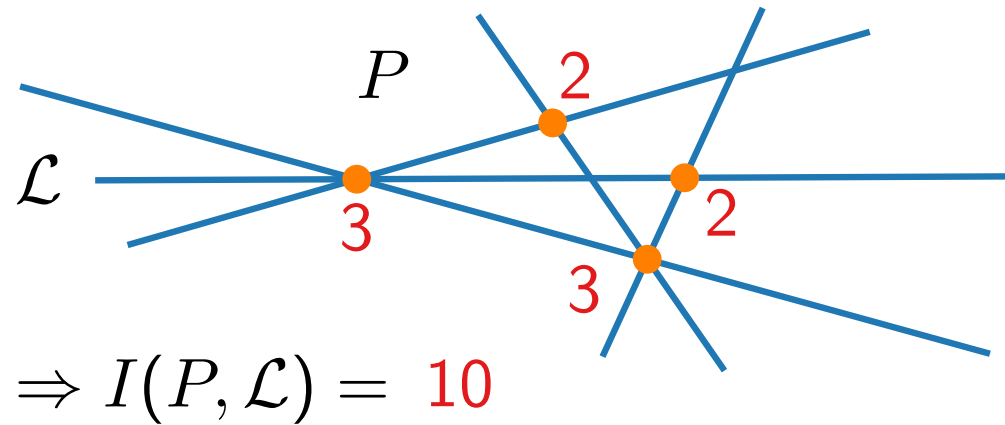
$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

- Bound is asymptotically tight.
- Result stayed hardly known until Székely demonstrated its usefulness in 1997.
- We go through the proof of Chazelle, Sharir, and Welzl (see “THE BOOK”).
- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{27.61}$ [Büngener, Kaufmann 2024].

Application 1: Point-Line Incidences

For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \#$ point-line incidences in $(P, \mathcal{L})$.



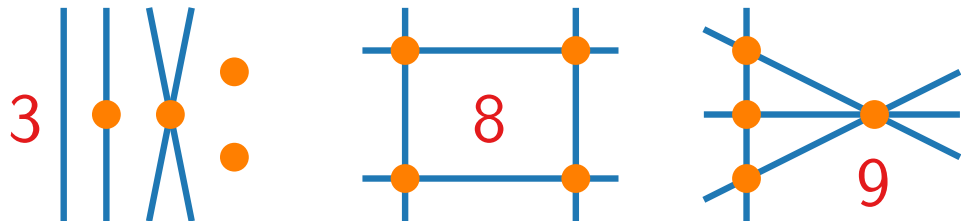
Theorem 1.

[Szemerédi, Trotter '83, Székely '97]

$$I(n, k) \leq 2.7n^{2/3}k^{2/3} + 6n + 2k.$$

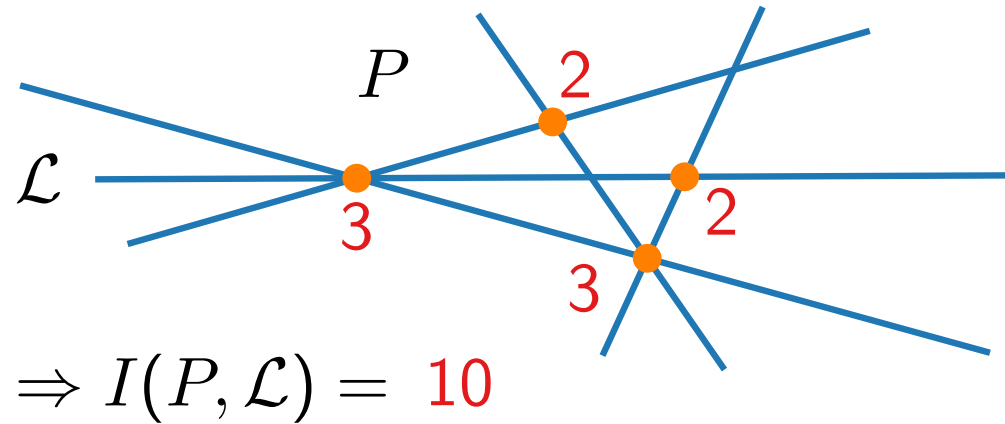
■ Define $I(n, k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.

■ For example: $I(4, 4) = 9$



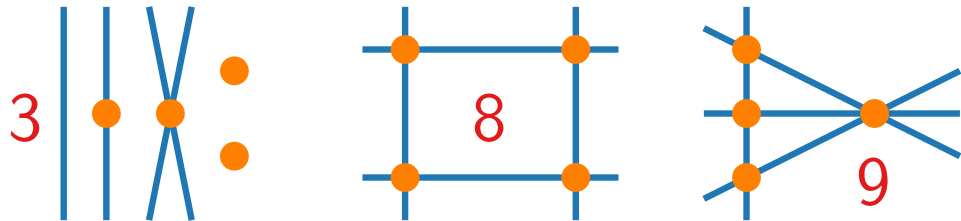
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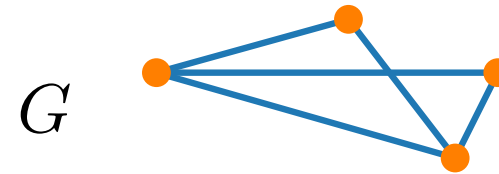


Theorem 1.

[Szemerédi, Trotter '83, Székely '97]

$$I(n, k) \leq c(n^{2/3}k^{2/3} + n + k).$$

Proof.



■ $\text{cr}(G) \leq k^2/2$

■ $\#(\text{points on a line } \ell) - 1 = \#(\text{edges on } \ell)$
 $\Rightarrow I(n, k) - k = m$ (sum up over $\mathcal{L}$ in an “optimal” instance)

■ If $m \leq 4n$, then $I(n, k) - k = m \leq 4n$.
 $\Rightarrow I(n, k) \leq 4n + k \leq c(n + k + n^{2/3}k^{2/3})$

■ Otherwise, employ the Crossing Lemma:

$$\frac{m^3}{64n^2} \leq \text{cr}(G) \leq k^2/2 \Rightarrow \frac{(I(n, k) - k)^3}{64n^2} \leq k^2/2$$

$$\Leftrightarrow I(n, k) \leq c(n^{2/3}k^{2/3} + k)$$

$$\leq c(n^{2/3}k^{2/3} + k + n).$$

□

Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

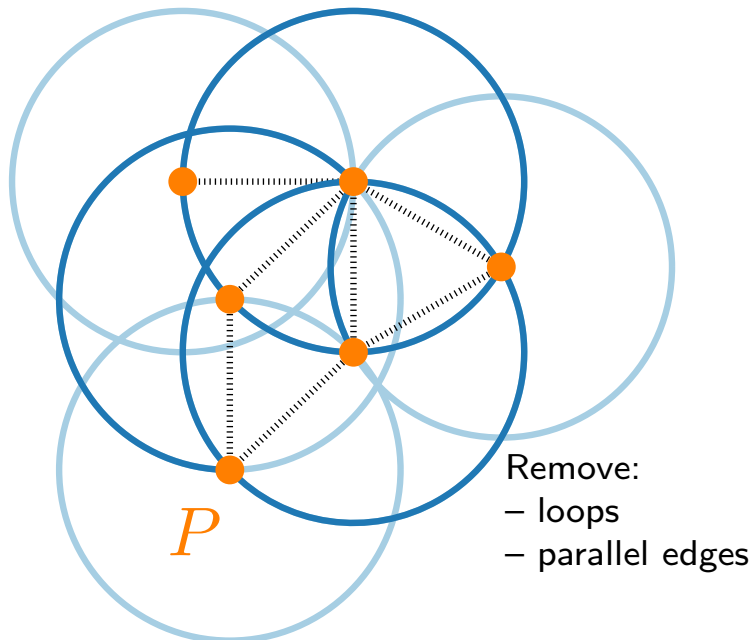
- $U(P)$ = number of pairs in P at unit distance
- $U(n) = \max_{|P|=n} U(P)$.

Theorem 2.

[Spencer, Szemerédi, Trotter '84, Székely '97]

$$U(n) < 6.7n^{4/3}$$

Proof sketch.



Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

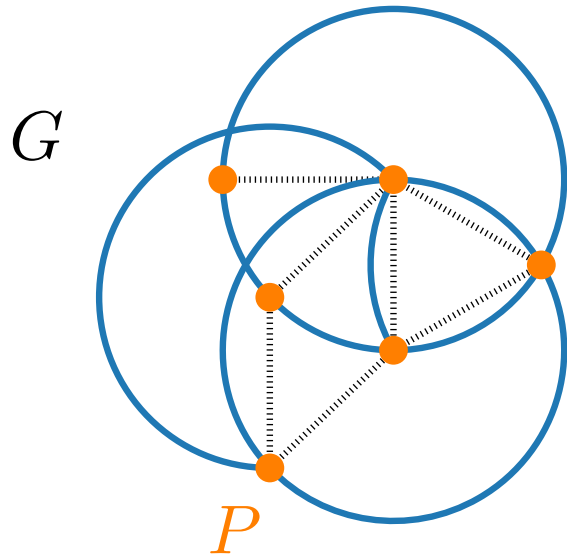
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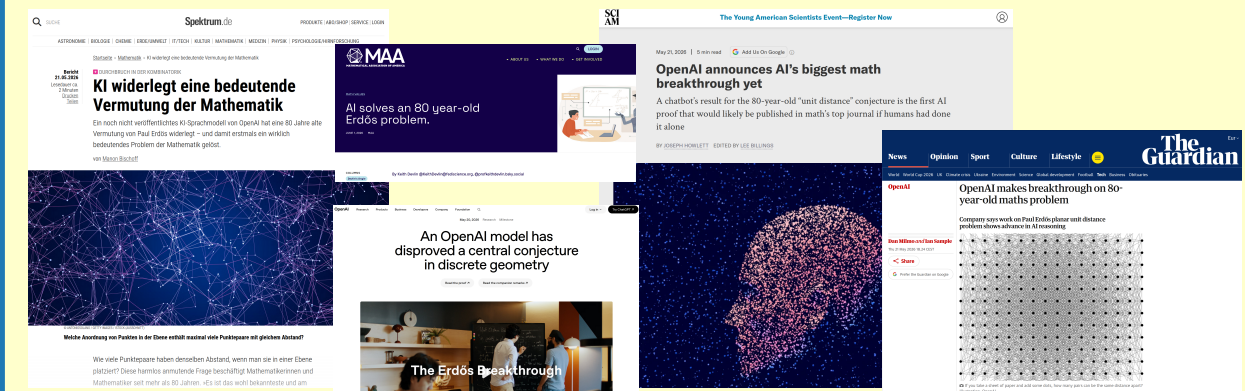
$$U(n) < 6.7n^{4/3}$$

Proof sketch.



- $U(P) \leq \overset{\text{some constant}}{c''} m$ ← number of edges in G
- $\text{cr}(G) \leq 2\binom{n}{2} \leq n^2$ (Circles intersect each other at most twice.)
- $n^2 \geq \text{cr}(G) \geq \frac{m^3}{64n^2} \geq c' \frac{U(P)^3}{n^2}$ by the Crossing Lemma.
 $\Rightarrow U(P) \leq cn^{4/3}$

Remark: This problem made recently (May 2026) news in popular media to be solved by AI.



The team from OpenAI found a lower bound example with $U(n) > n^{1+\varepsilon}$ for some constant $\varepsilon > 0$, disproving an 80-year old conjecture by Erdős. Later, various researchers fixed the constant to $\varepsilon > 0.03583$.

[https://cdn.openai.com/pdf/](https://cdn.openai.com/pdf/74c24085-19b0-4534-9c90-465b8e29ad73/unit-distance-proof.pdf)

[74c24085-19b0-4534-9c90-465b8e29ad73/unit-distance-proof.pdf](https://cdn.openai.com/pdf/74c24085-19b0-4534-9c90-465b8e29ad73/unit-distance-proof.pdf)

<https://www.erdosproblems.com/forum/thread/90>

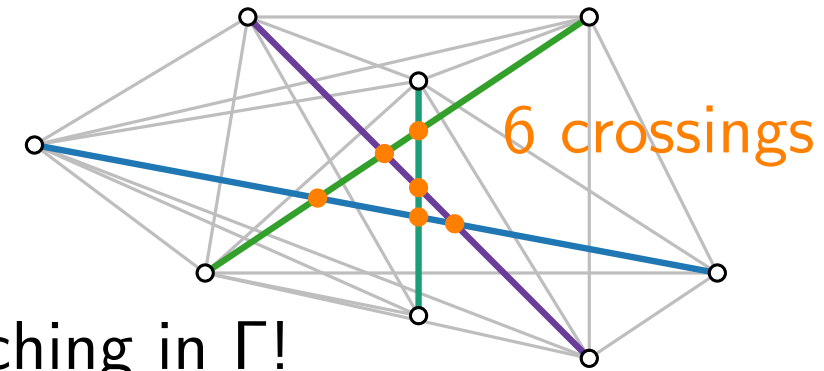
<https://teorth.github.io/optimizationproblems/constants/84a.html>

Application 3: Expected Number of Crossings in a Matching

Given a set of n points (in general position, n even) – what is the average number of crossings in a perfect matching?

Point set spans drawing Γ of K_n .

We will analyze the number of crossings in a **random** perfect matching in Γ !



Number of crossings in $\Gamma \geq \overline{\text{cr}}(K_n) > \frac{3}{8} \binom{n}{4}$

[Lovász et al. '04, Aichholzer et al. '06]

Number of edges in K_n : $\binom{n}{2}$

(The constant $3/8$ for K_n is larger than that for general graphs in the Crossing Lem.)

Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3 \binom{n}{4}$

Pick two random edges e_1 and e_2 .

$\Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \overline{\text{cr}}(K_n) / \text{pot}(K_n) > \frac{1}{8}.$

Pick random perfect matching M ; it has $n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges.

By linearity of expectation,

the expected number of crossings in M is $> \frac{1}{8} \binom{n/2}{2} = \frac{1}{64}n(n-2) \in \Omega(n^2).$

□

Literature

- [Aigner, Ziegler] Proofs from THE BOOK [<https://doi.org/10.1007/978-3-662-57265-8>]
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Hanani '43] Über wesentlich unplättbare Kurven im dreidimensionalen Raume
- [Tutte '70] Toward a theory of crossing numbers
- [Pach & Tóth '00] Which crossing number is it anyway?
- [Pelsmayer, Schaefer & Štefankovič '07] Removing even crossings
- [Pelsmayer, Schaefer & Štefankovič '08] Odd Crossing Number and Crossing Number Are Not the Same
- [Tóth '08] Note on the Pair-crossing Number and the Odd-crossing Number
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Lovász, Vesztergombi, Wagner, Welzl '04] Convex quadrilaterals and k -sets
- [Aichholzer, Aurenhammer, Krasser '06] On the Crossing Number of Complete Graphs
- [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography " N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: <http://crossings.uos.de>