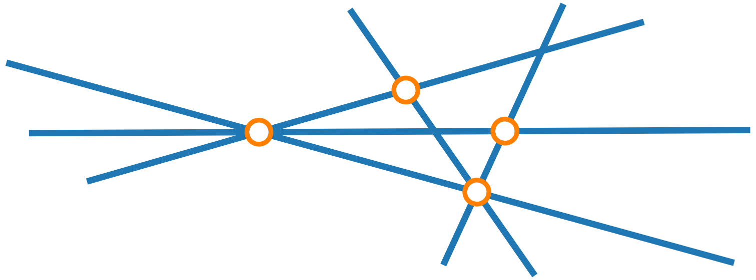
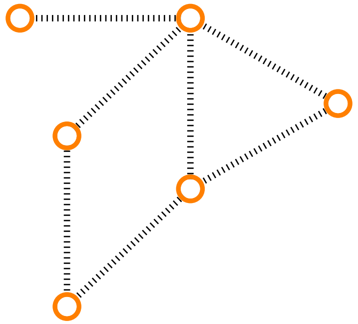
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 9: The Crossing Lemma

and Its Applications





Alexander Wolff

Summer term 2026

Crossing Number and Topological Graphs

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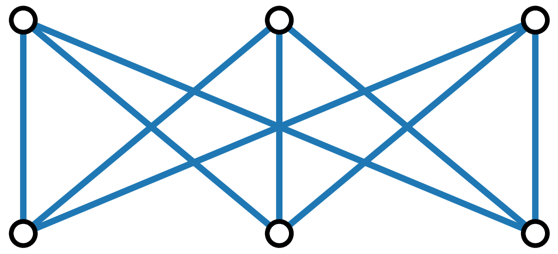

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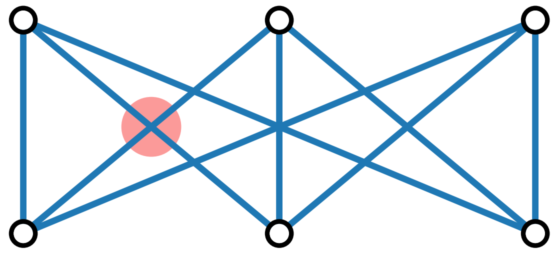

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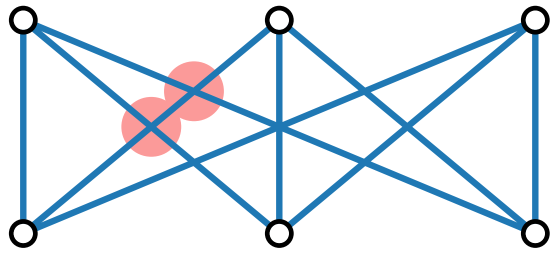

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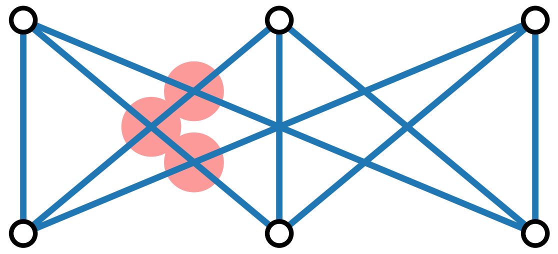

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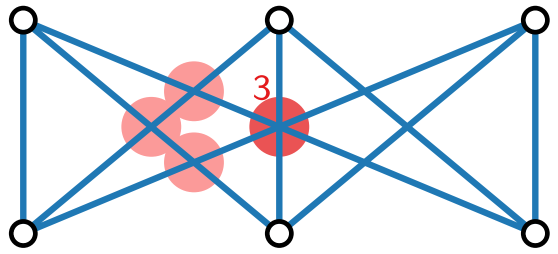

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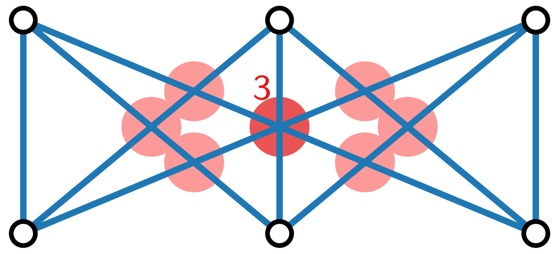

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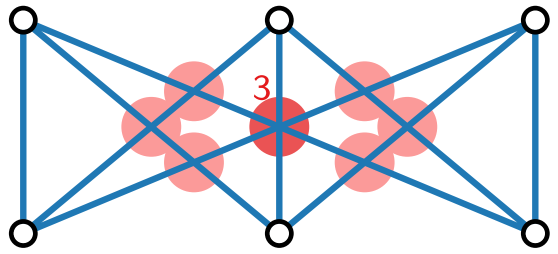

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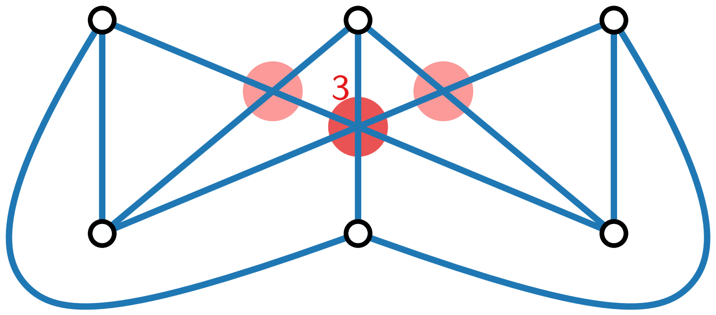

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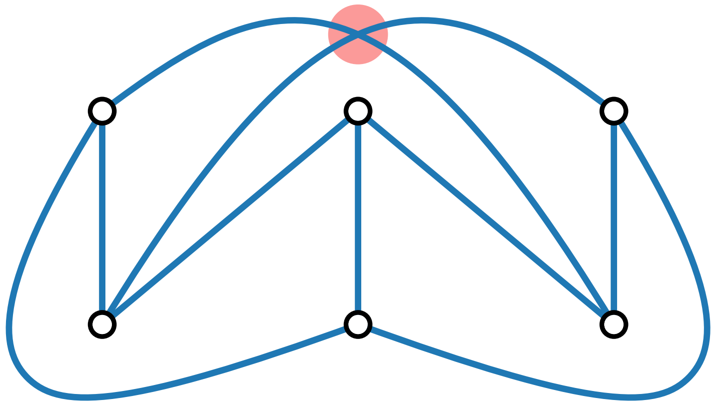

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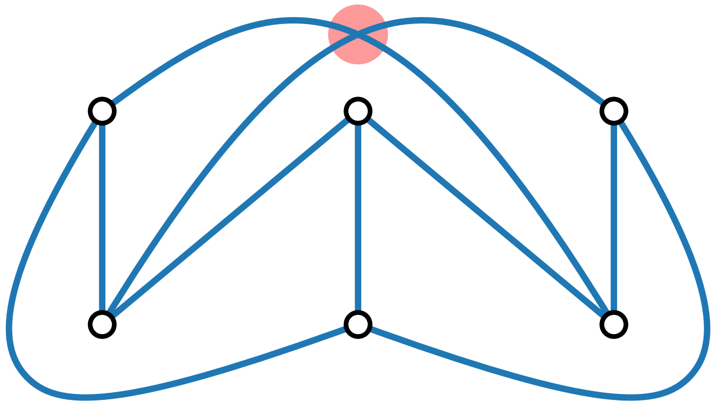

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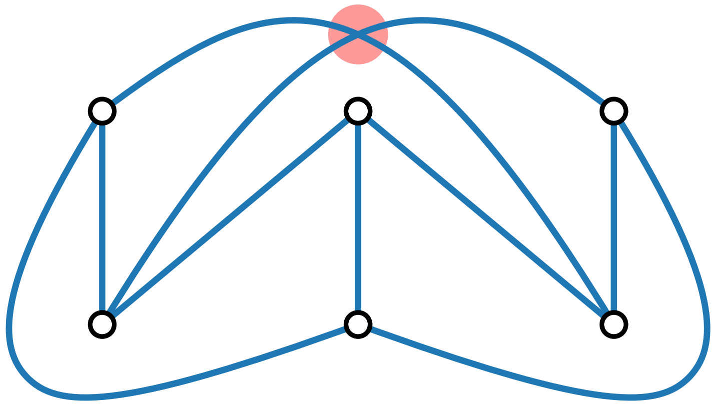

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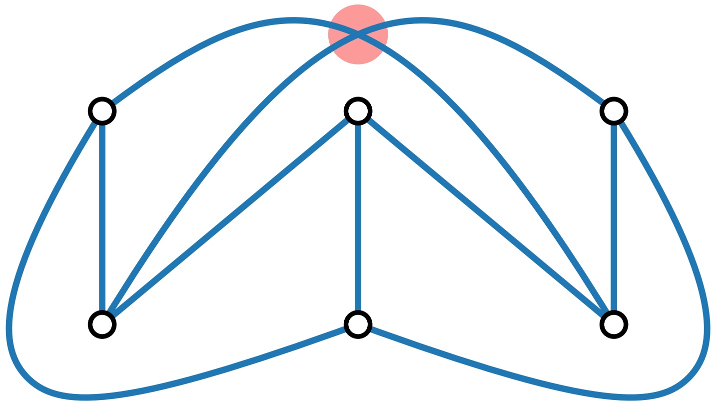

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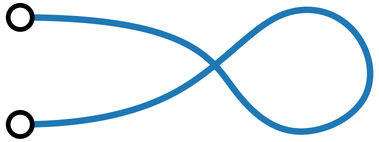

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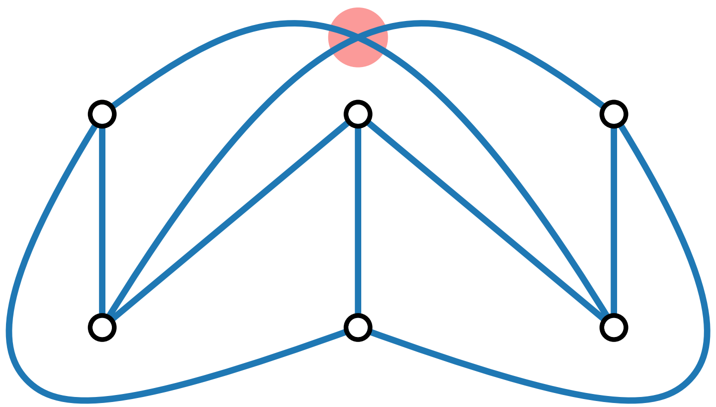

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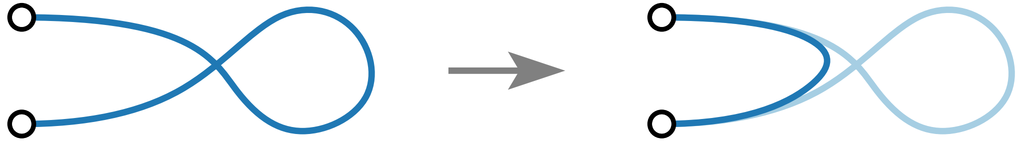

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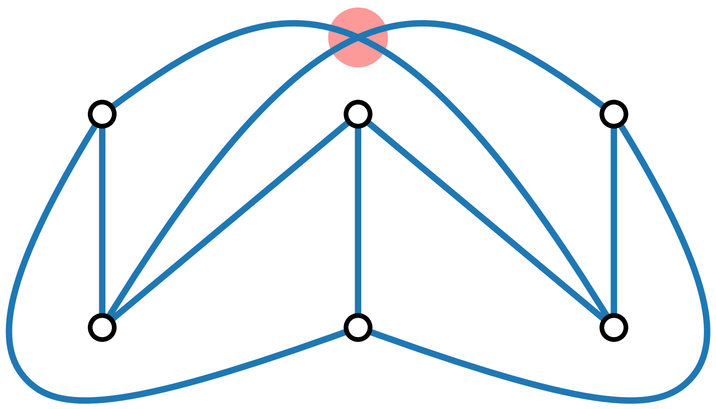

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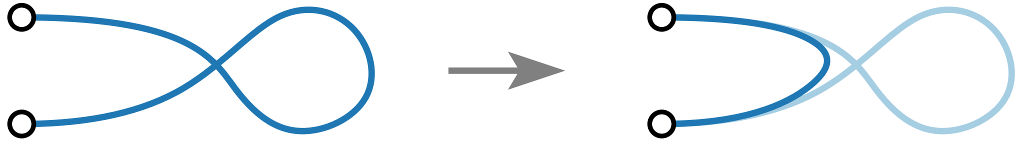

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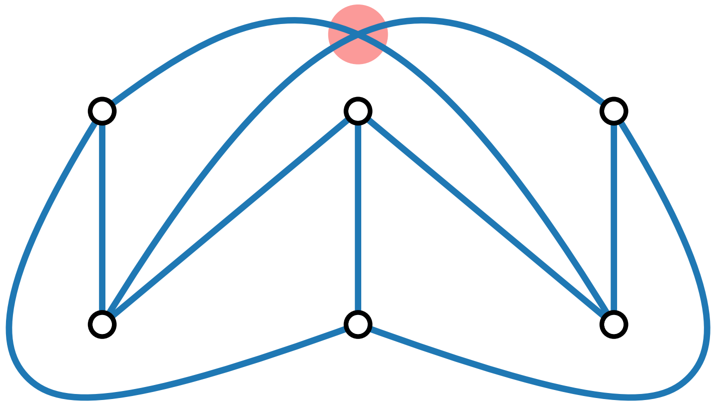

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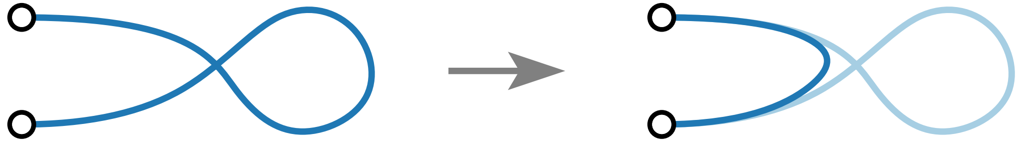

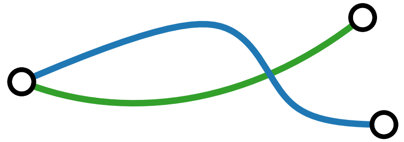

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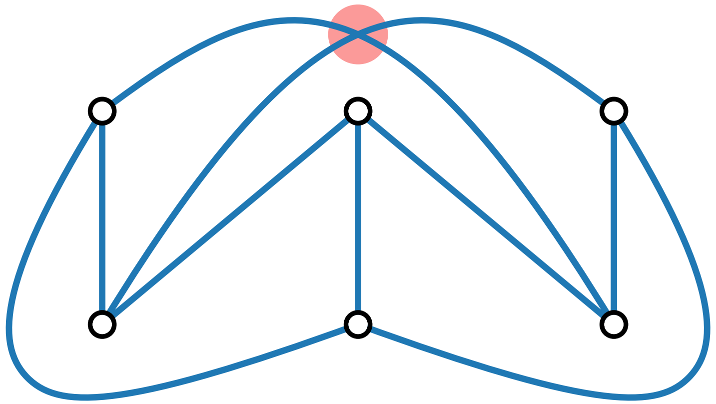

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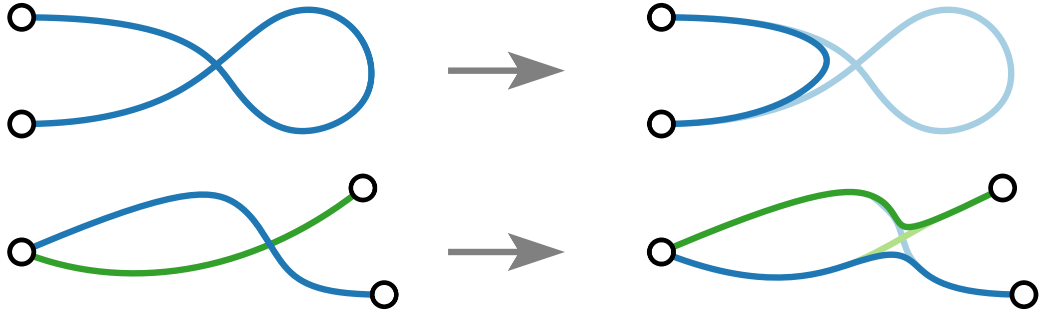

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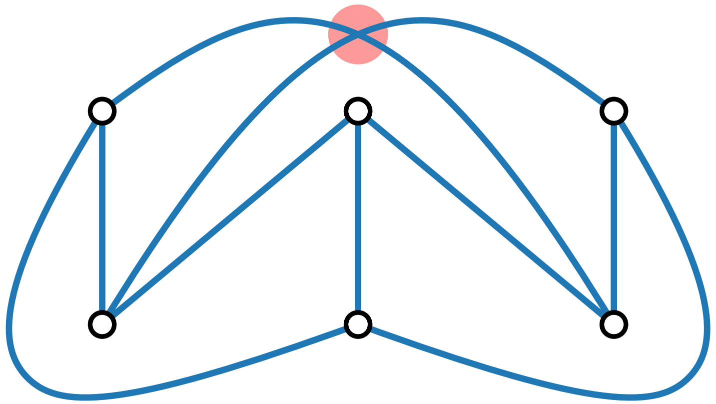

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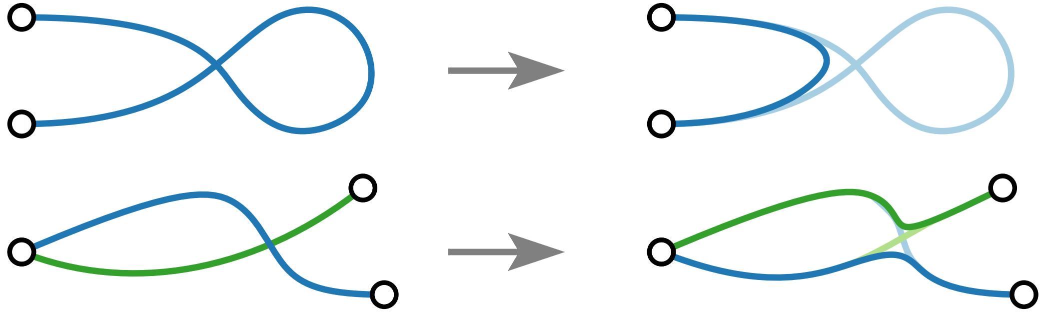

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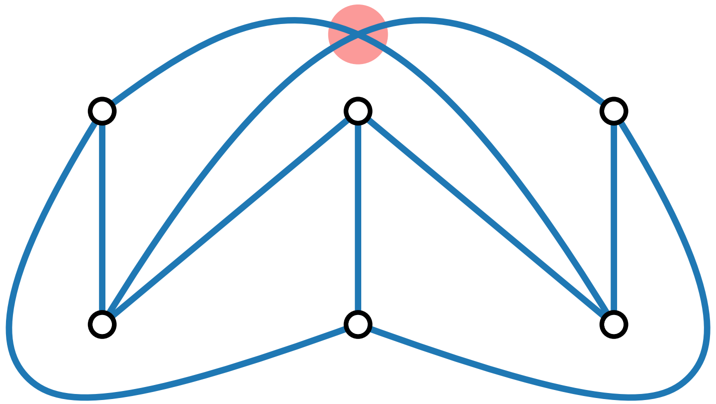

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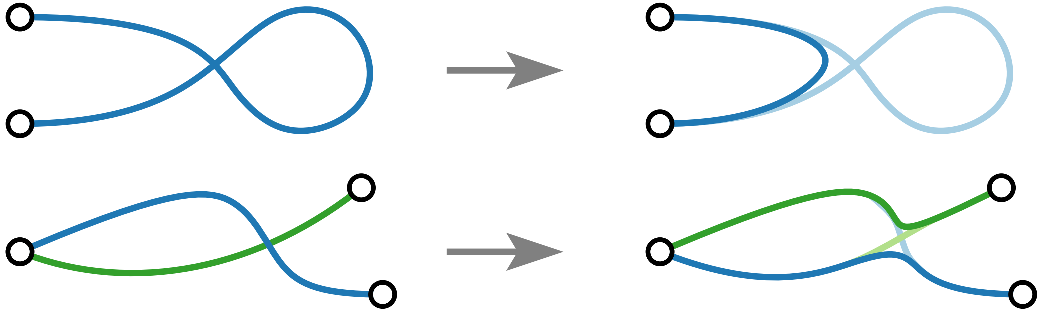

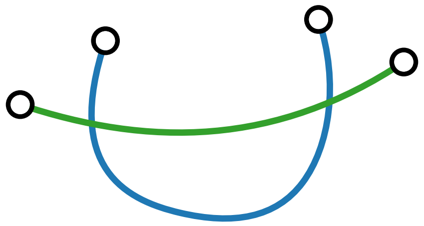

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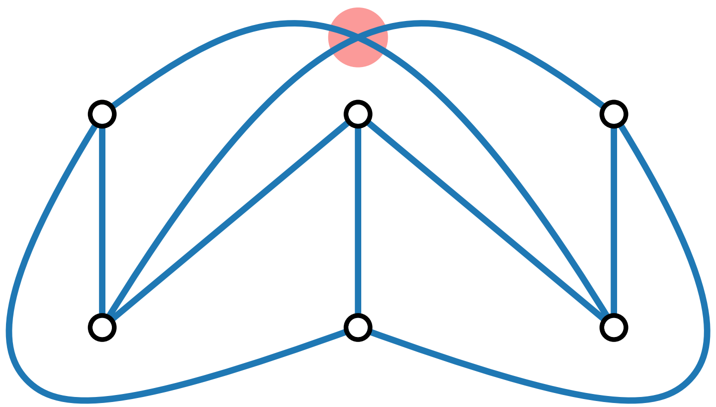

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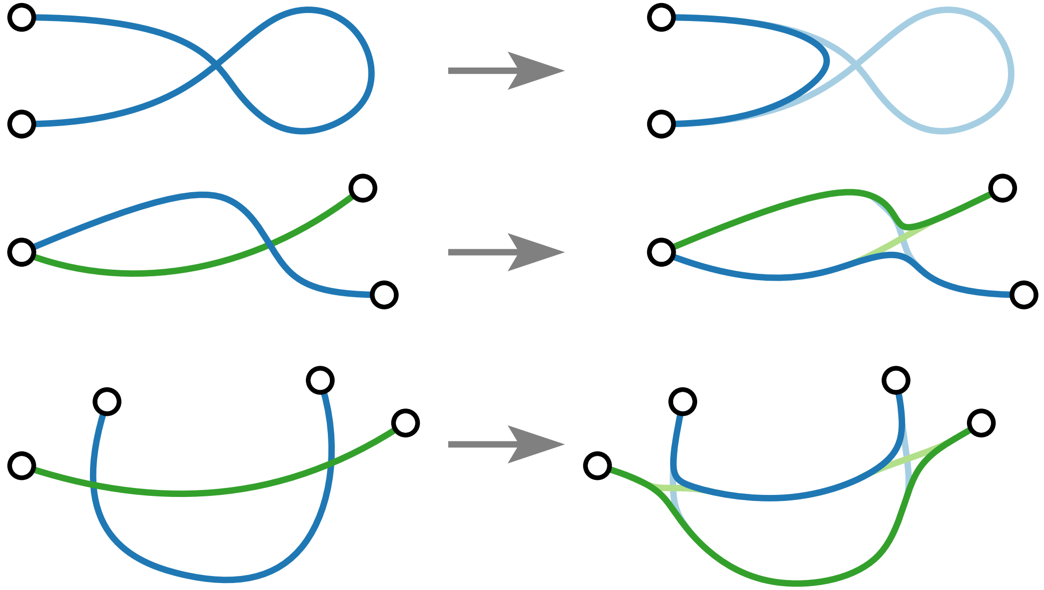

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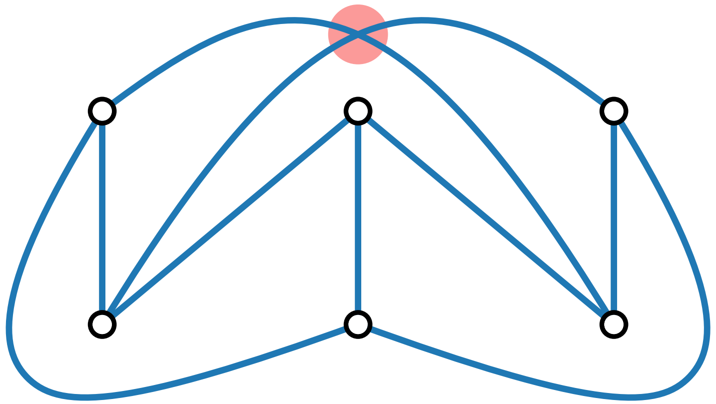

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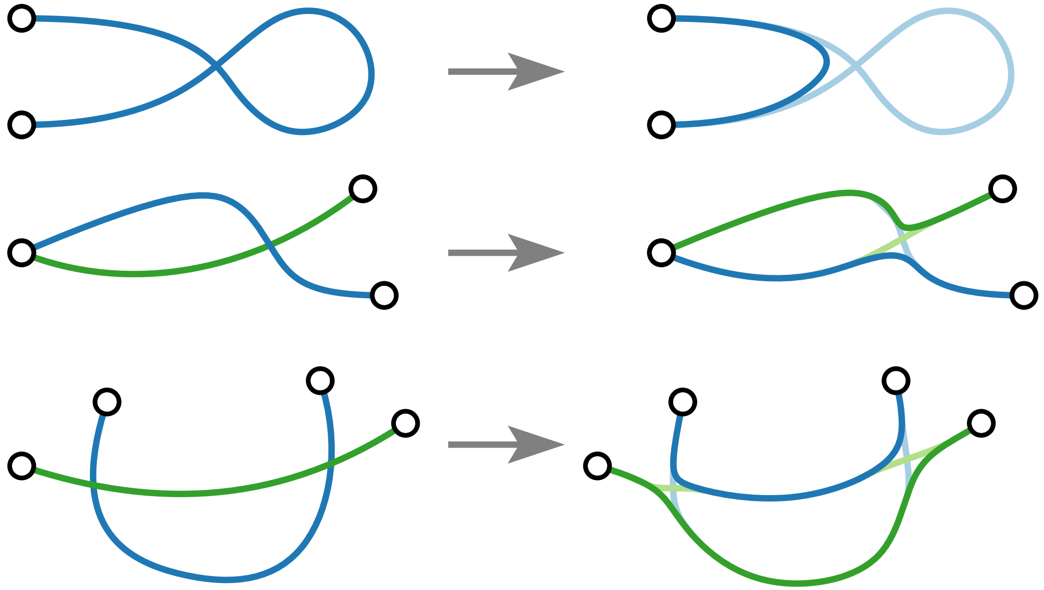

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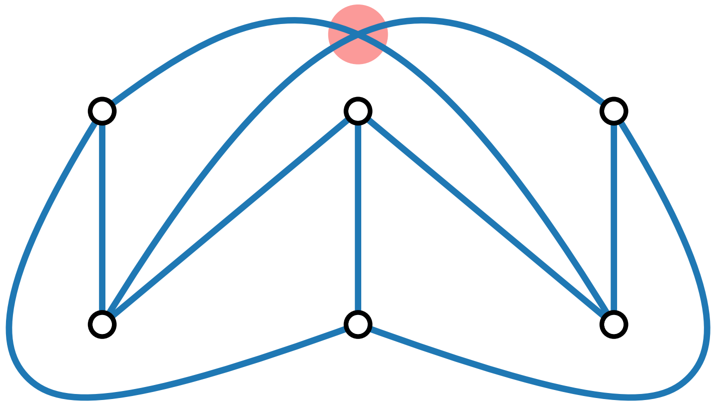

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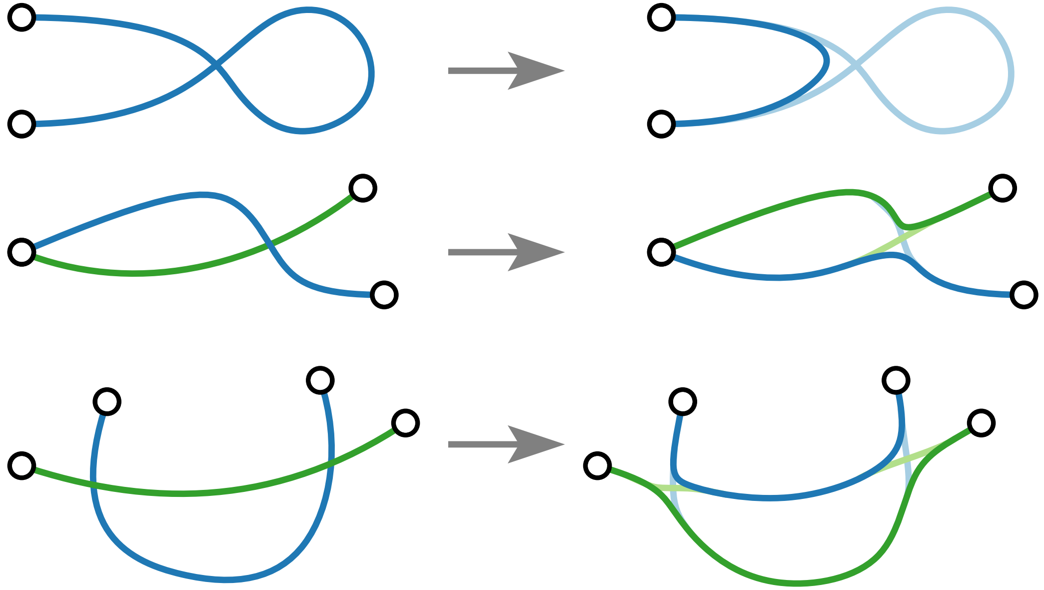

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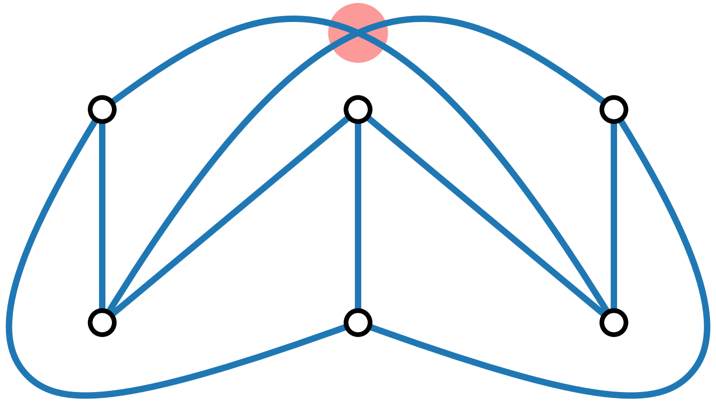

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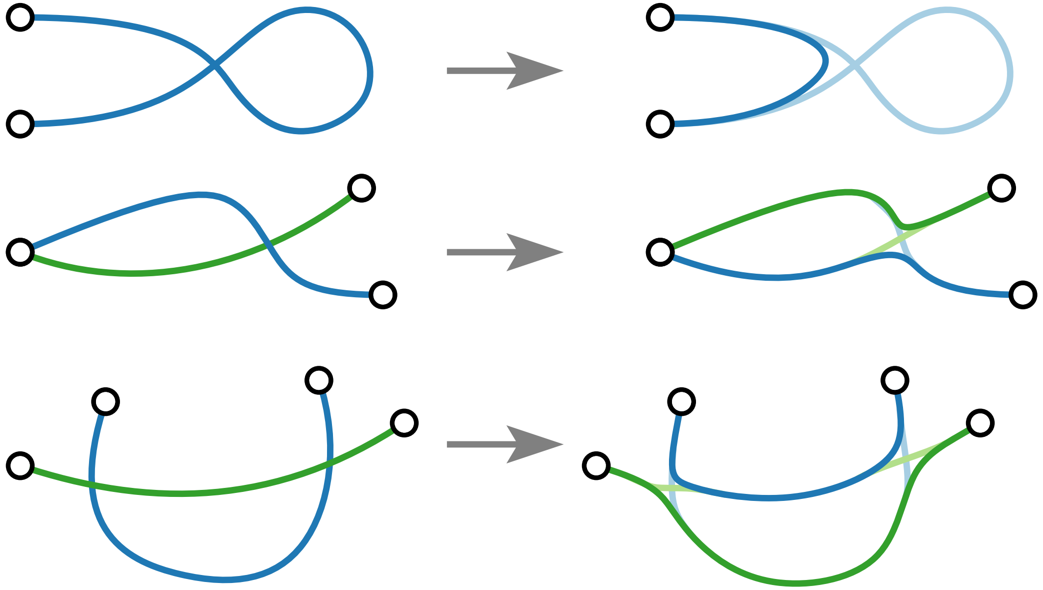

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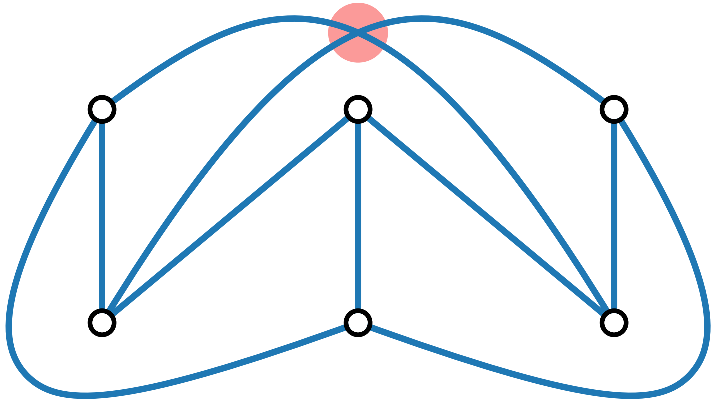

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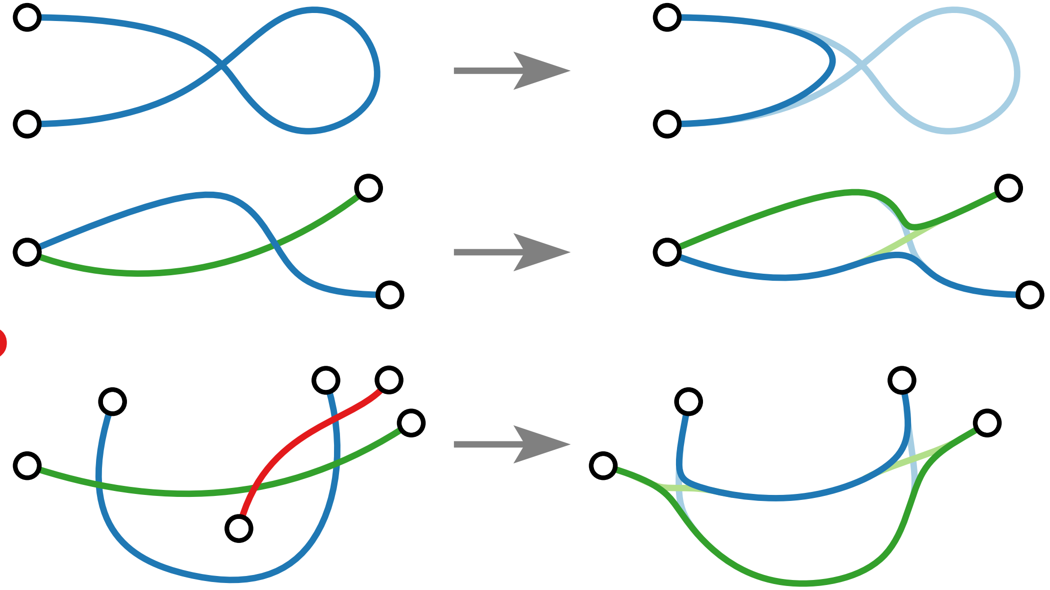

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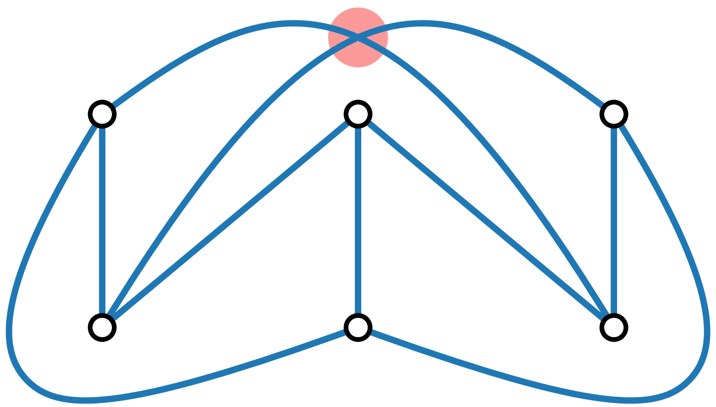

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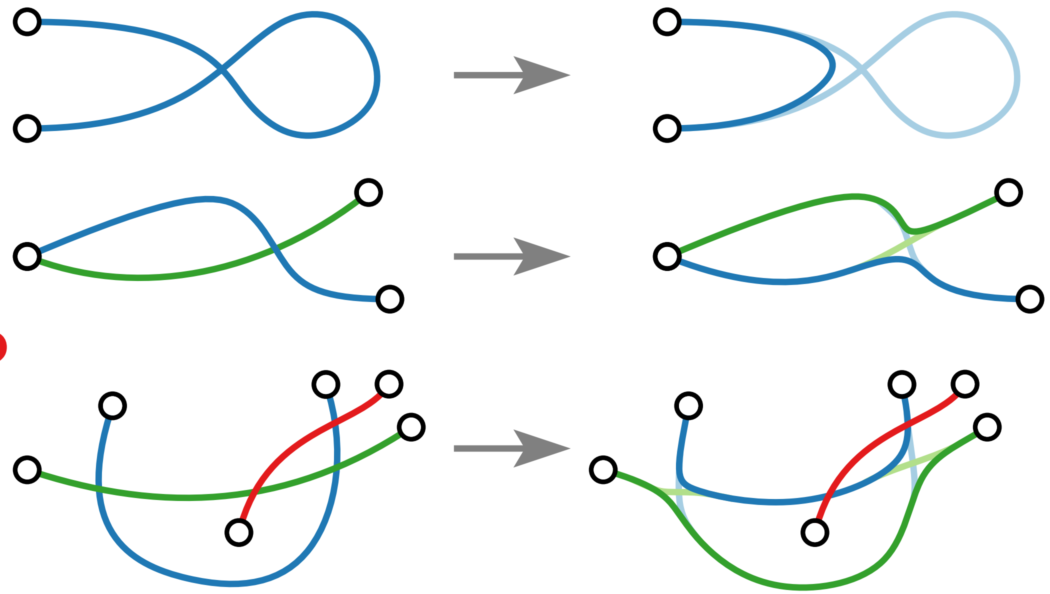

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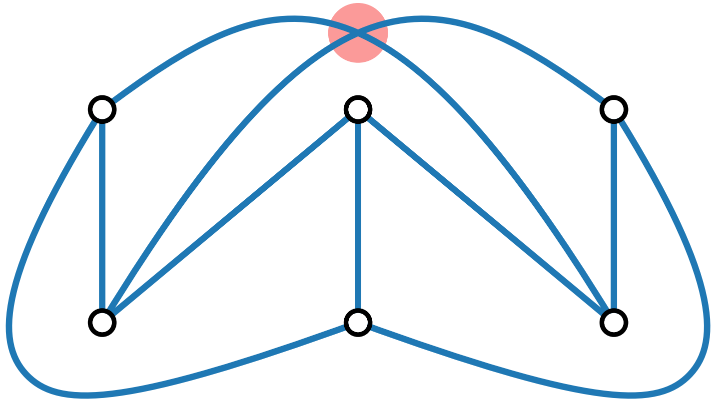

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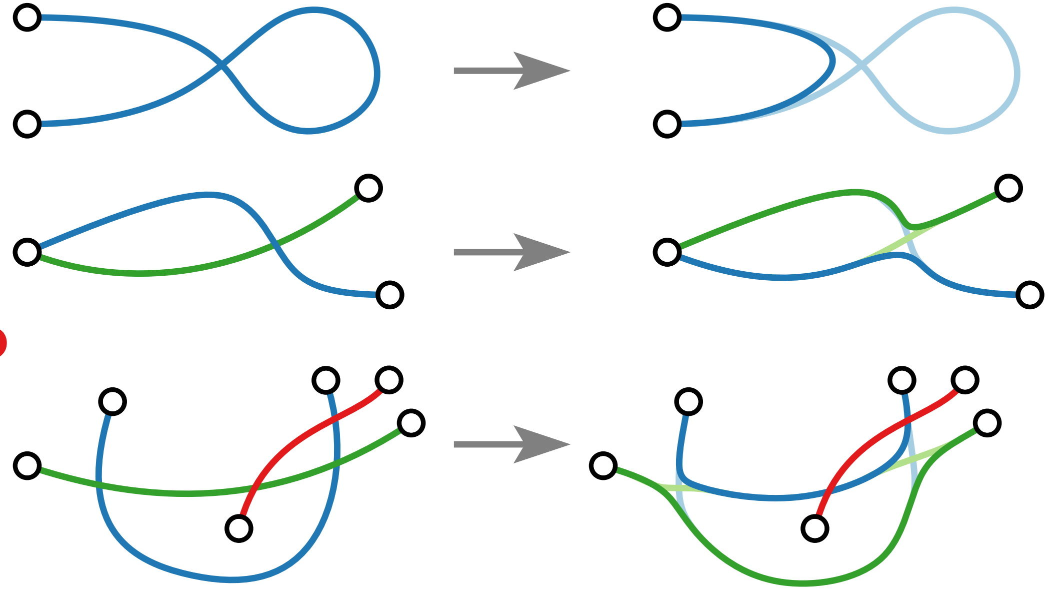

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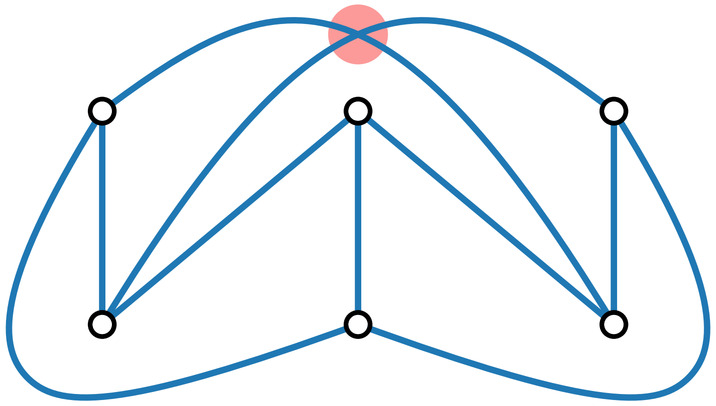

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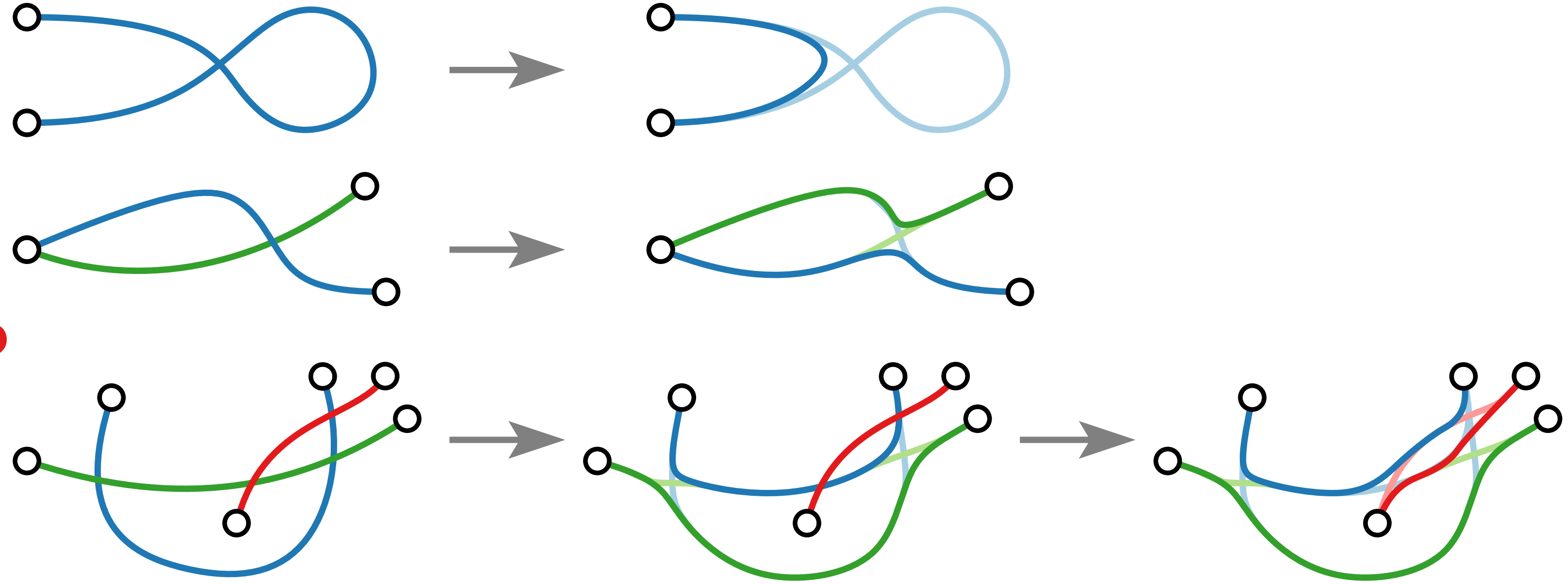

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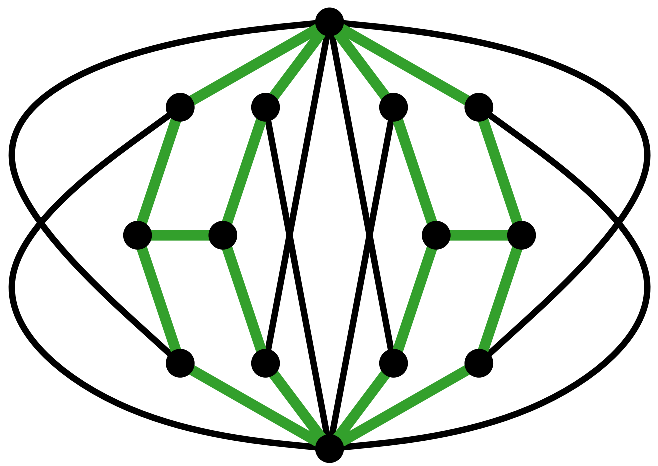
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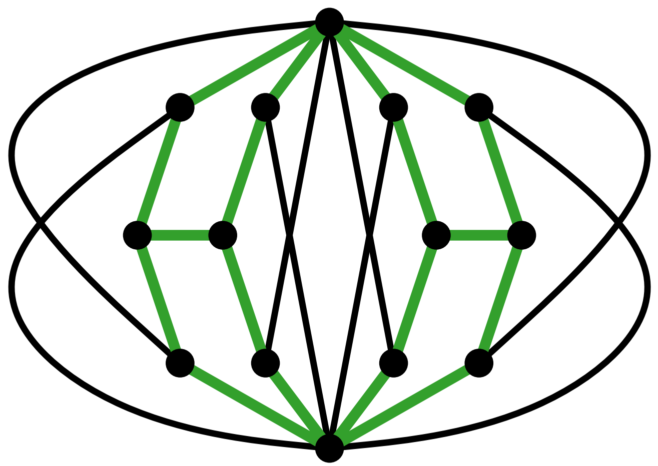
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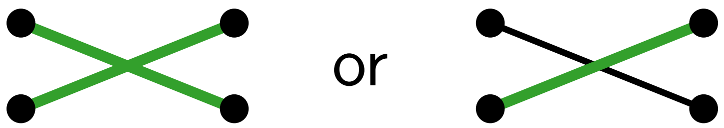

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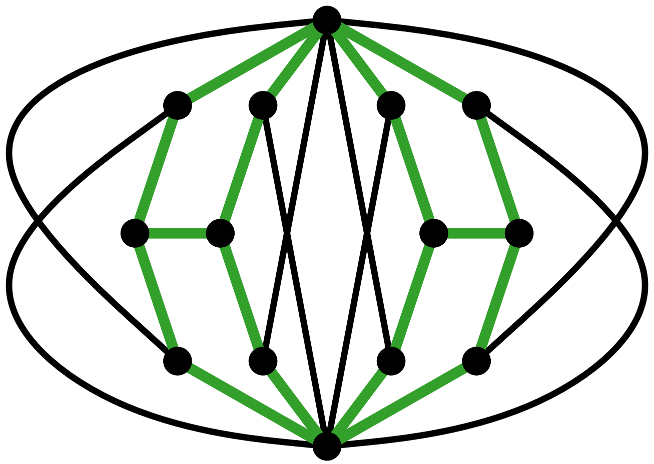

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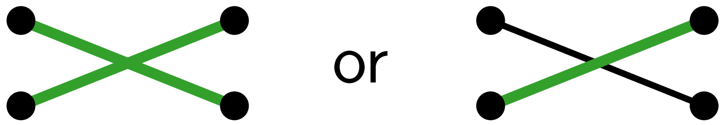

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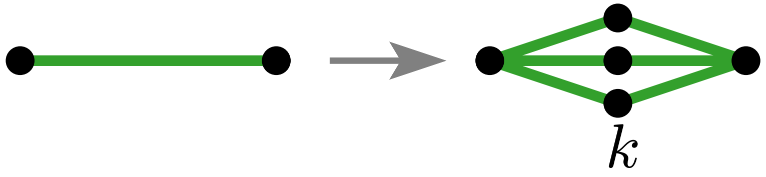

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Theorem. [Guy '60]

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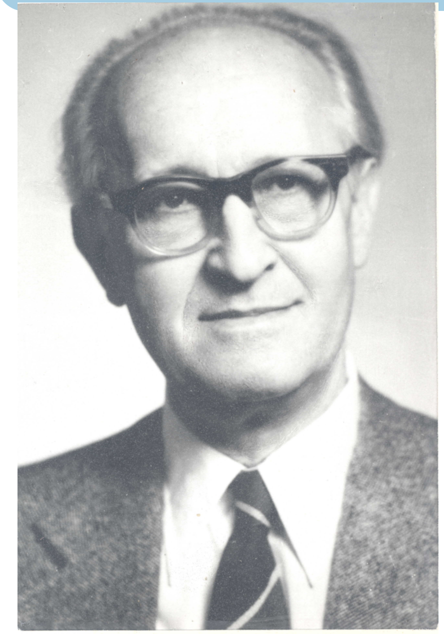

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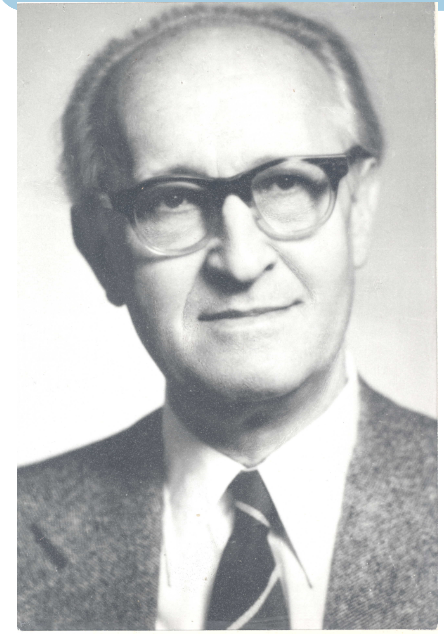

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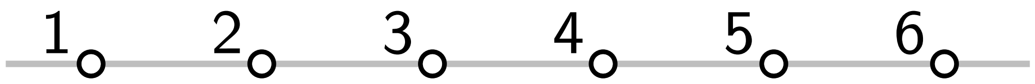

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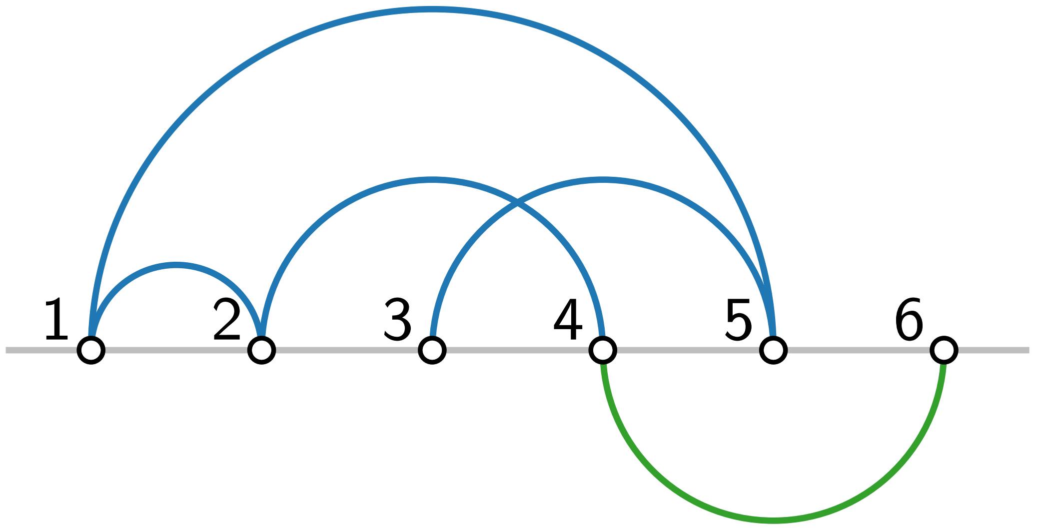

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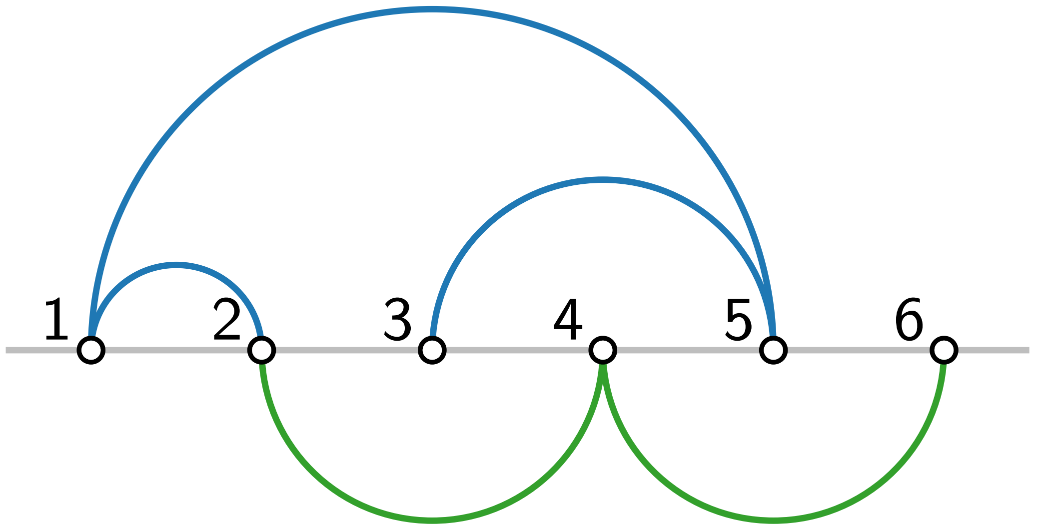

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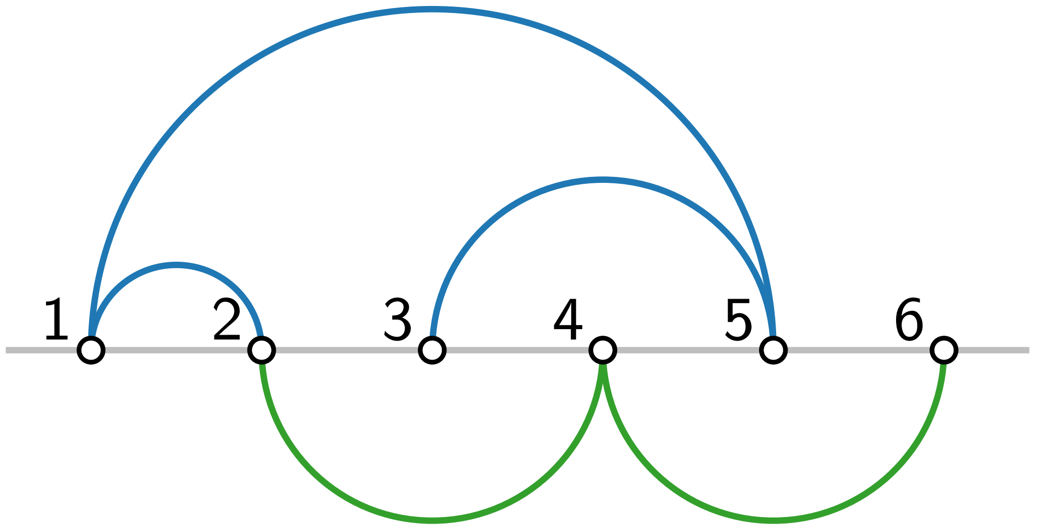

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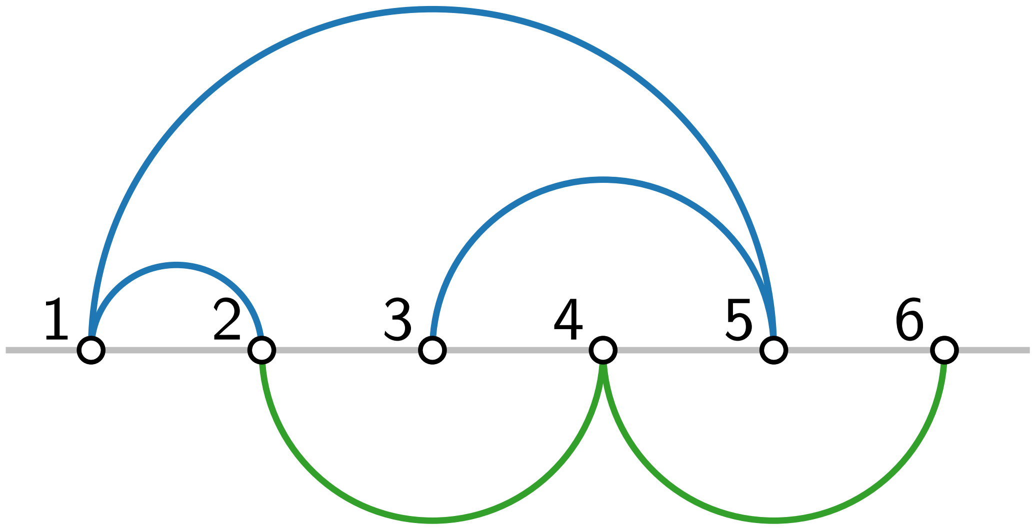

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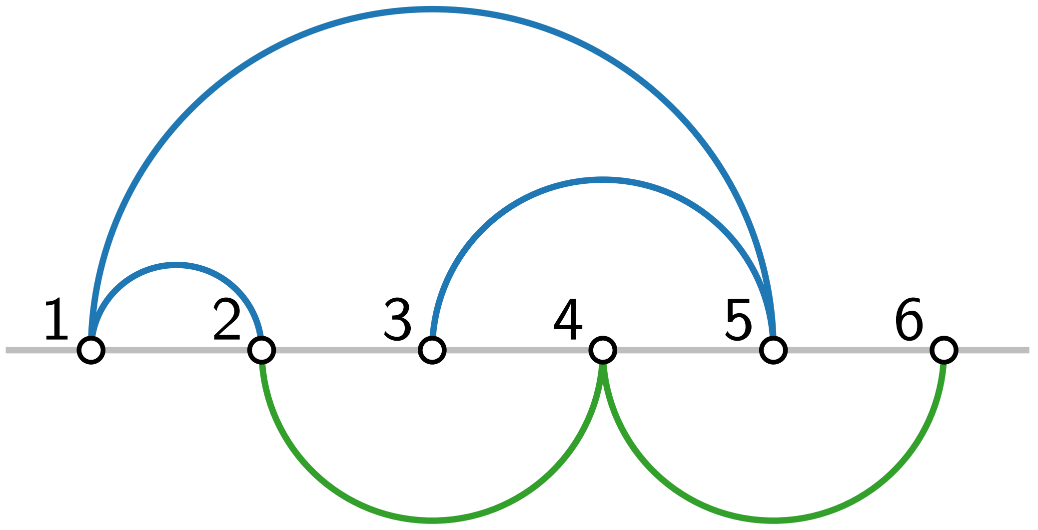

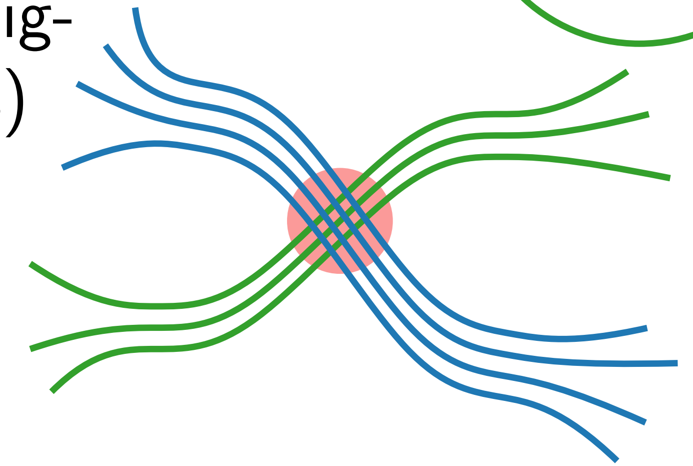

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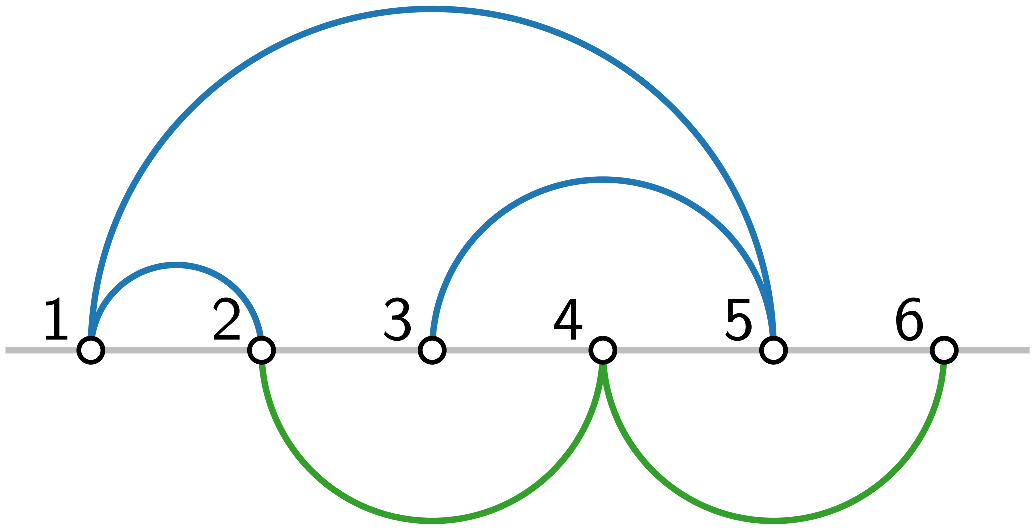

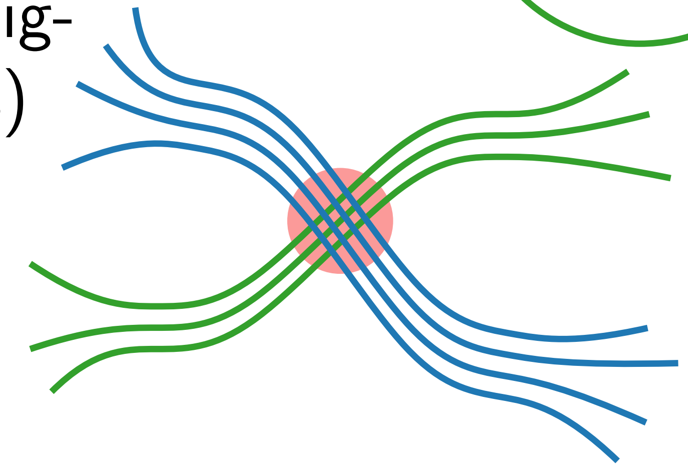

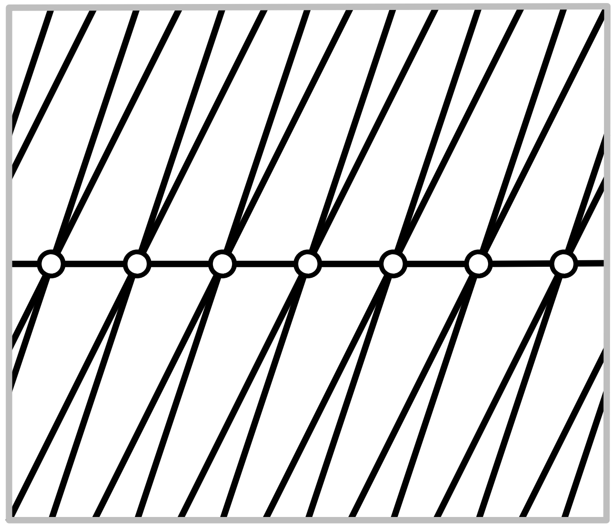

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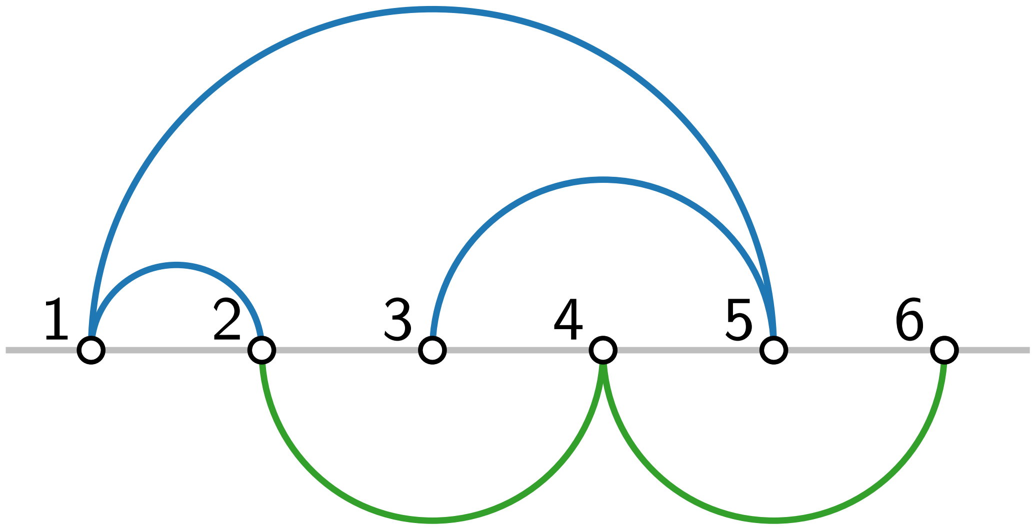

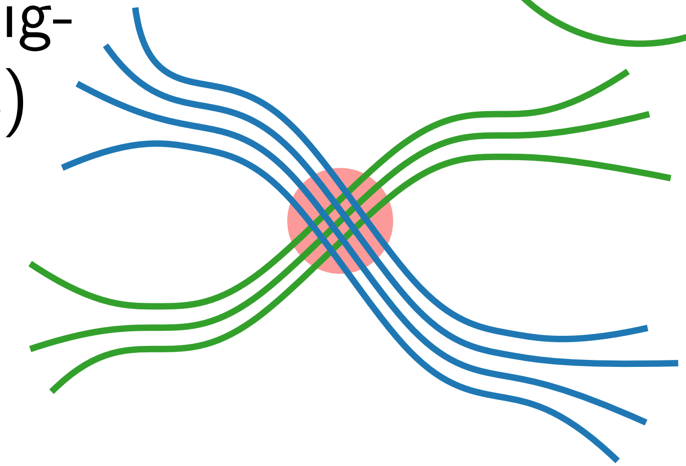

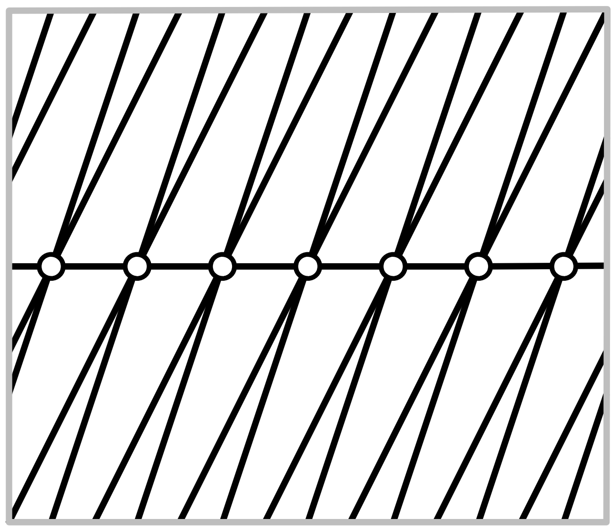

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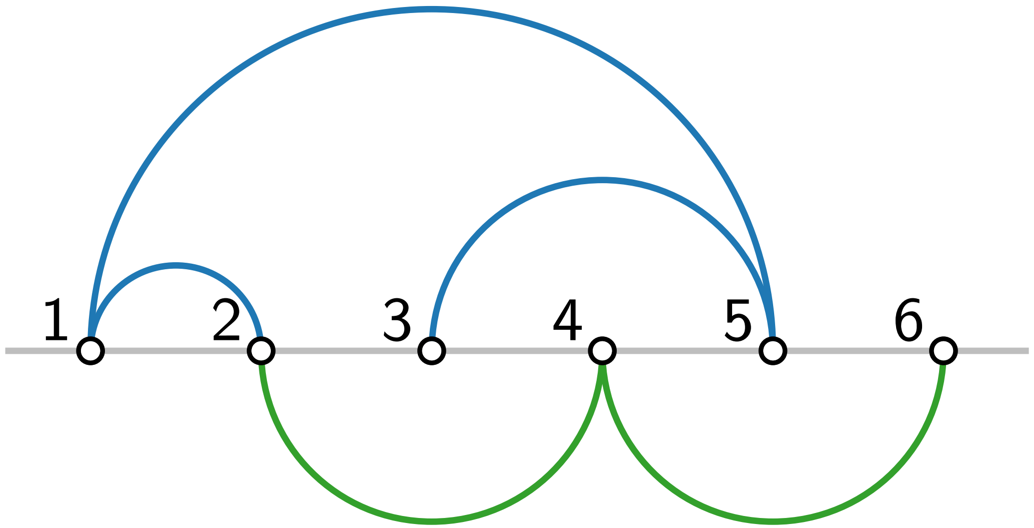

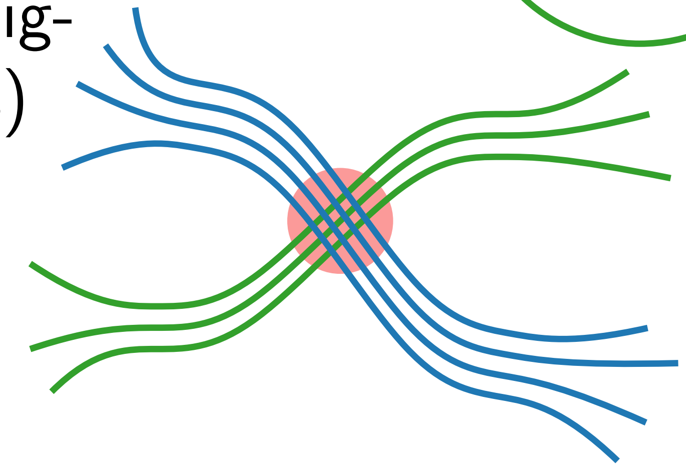

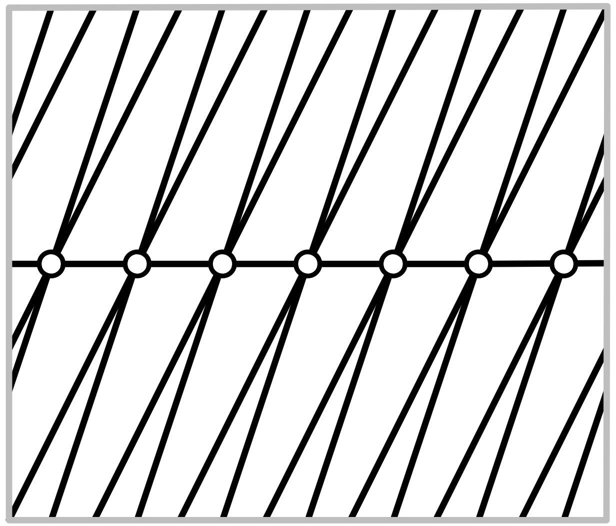

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Hence, there must be two edges on these paths that cross an odd number of times. $\square$

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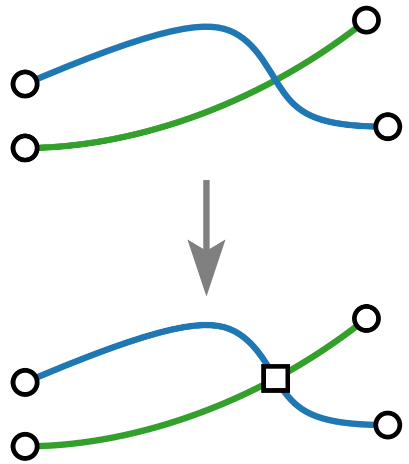

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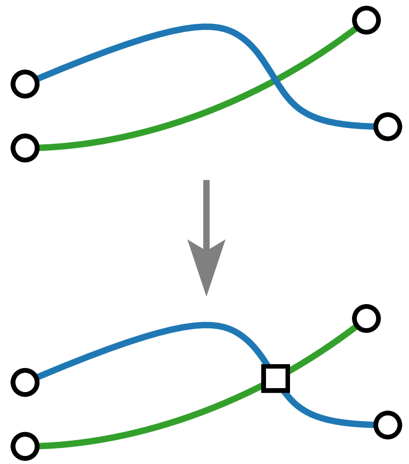

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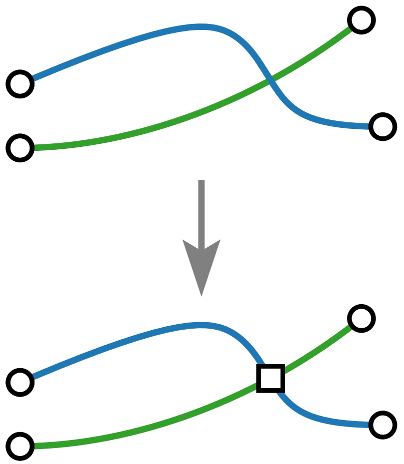

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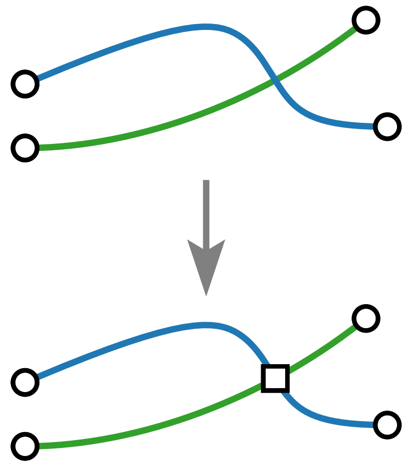

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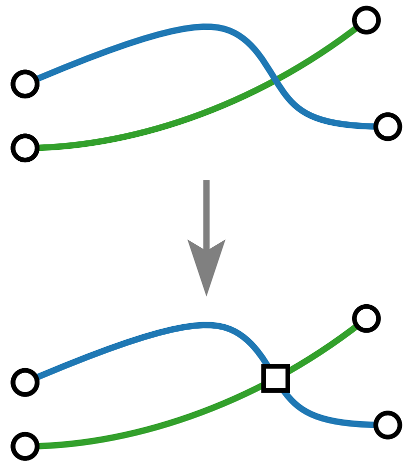

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- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{27.61}$ [Büngener, Kaufmann 2024].

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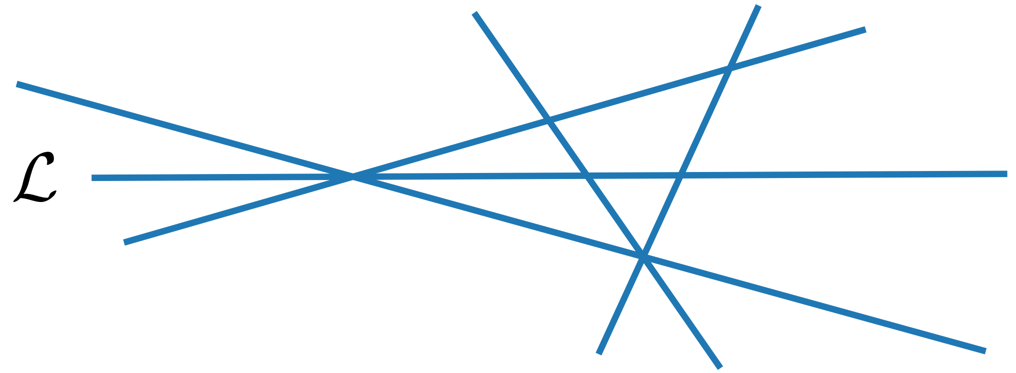

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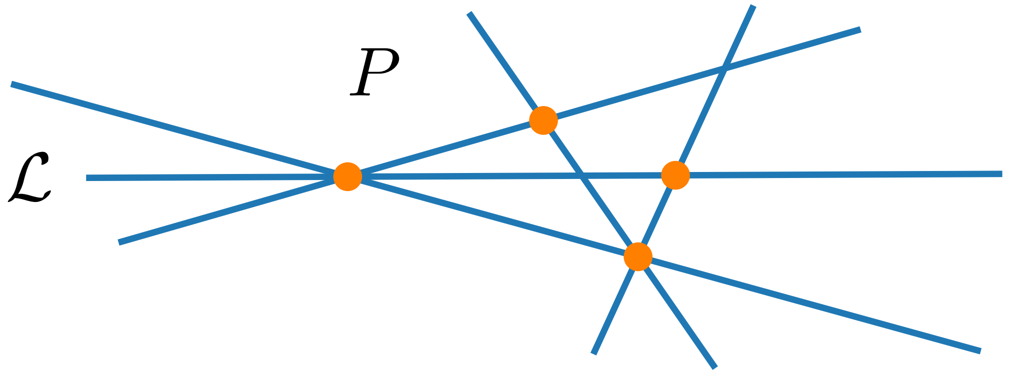

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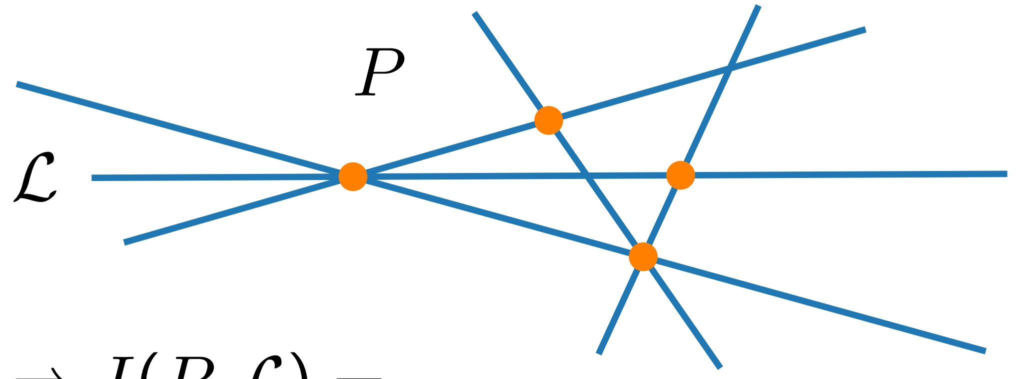

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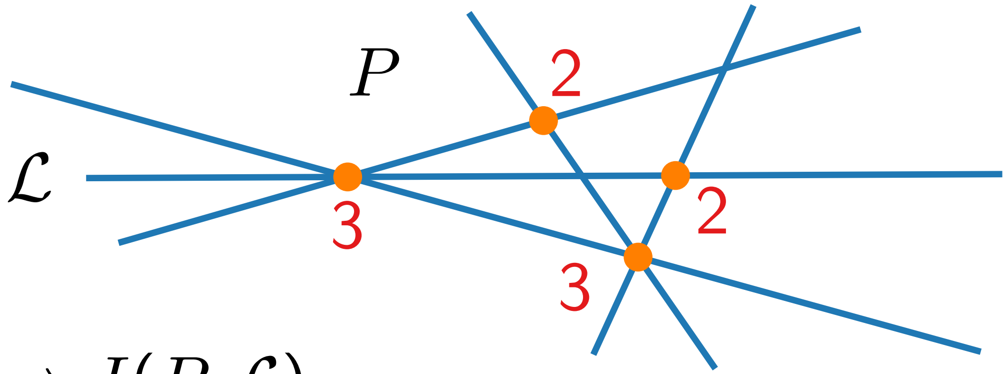

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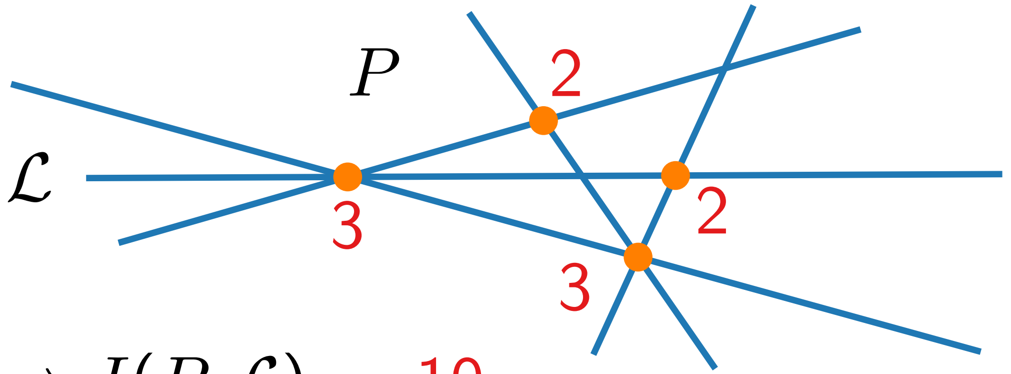

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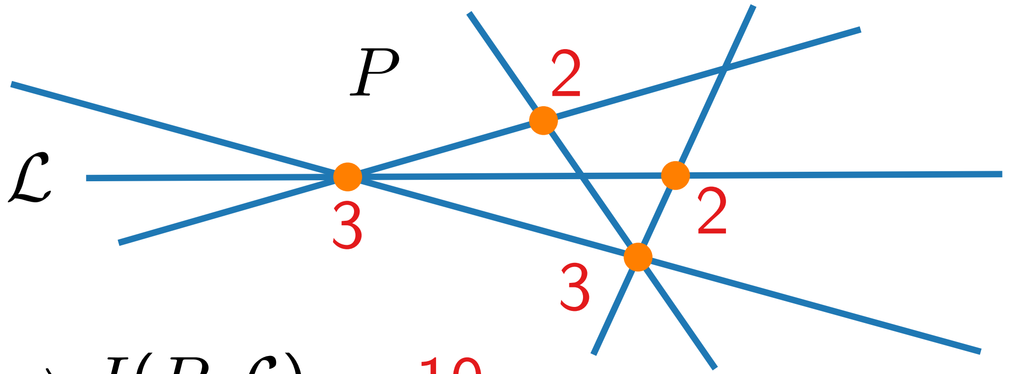

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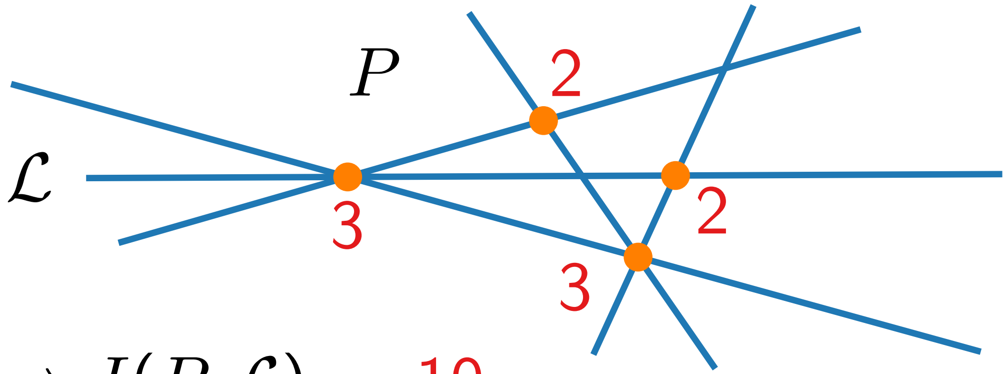

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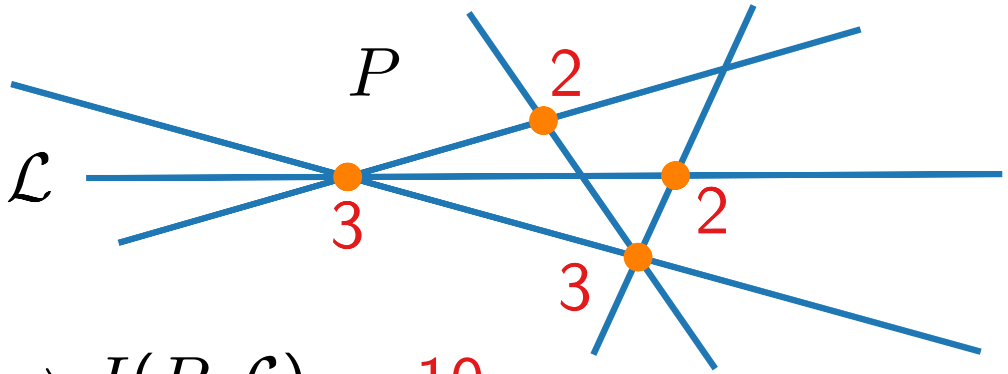

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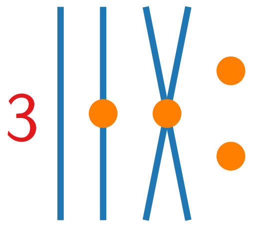

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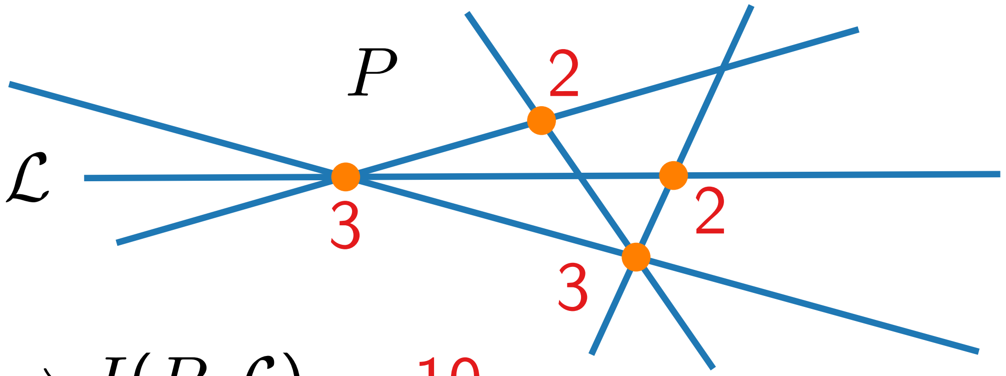

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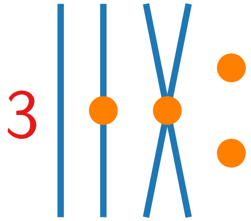

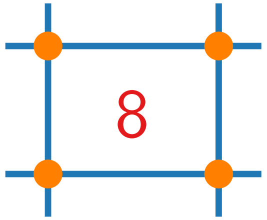

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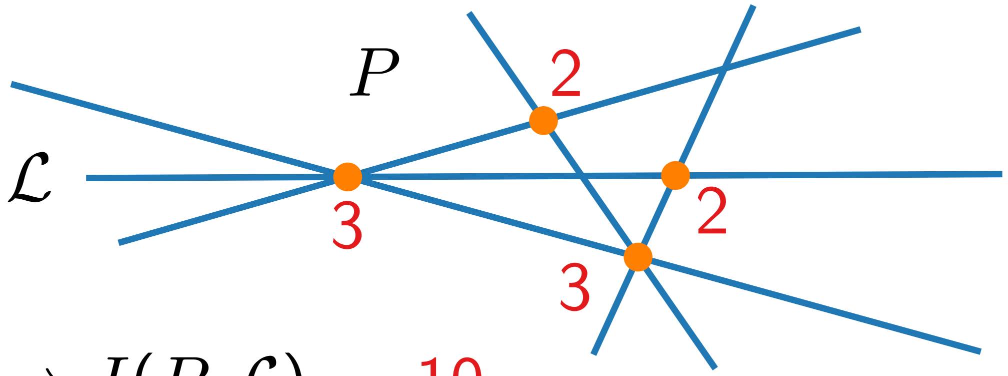

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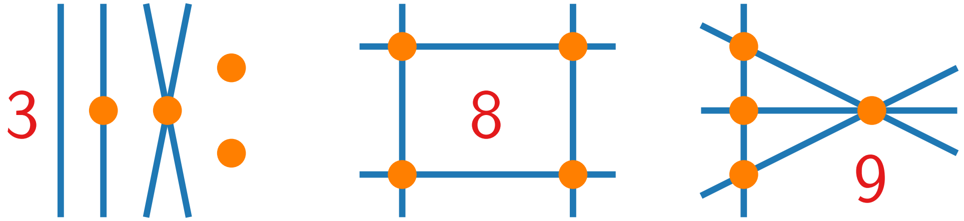

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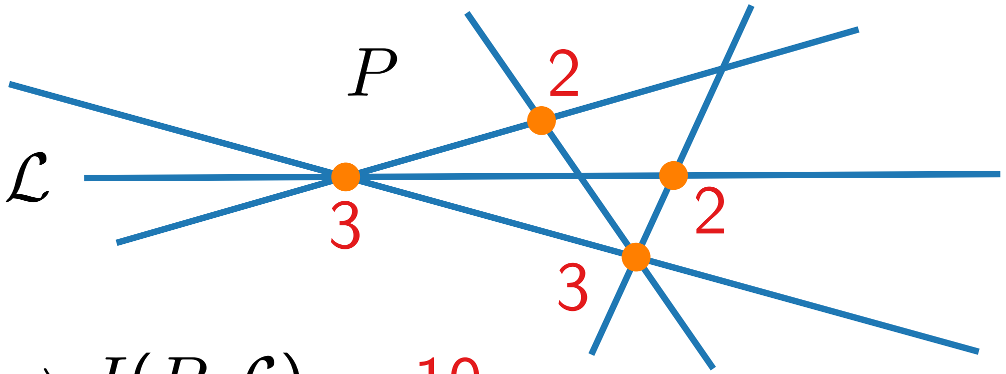

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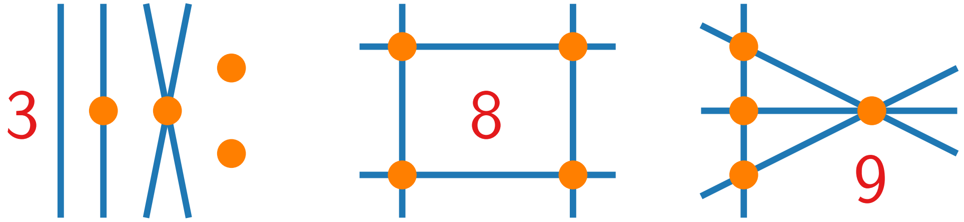

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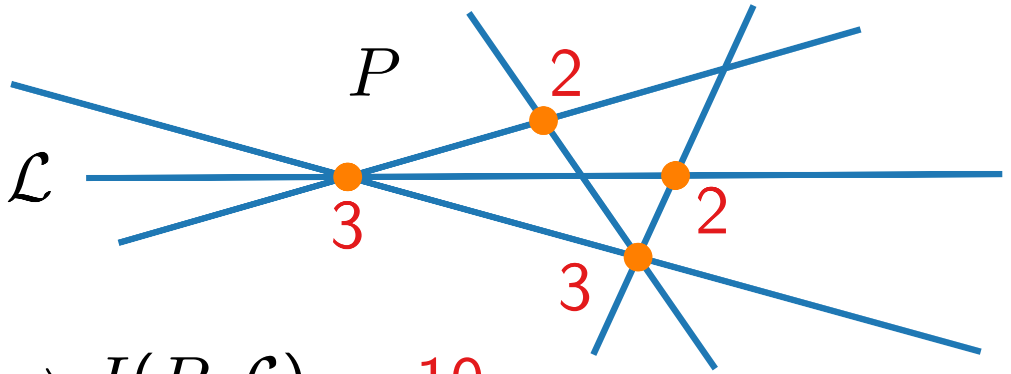

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Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n, k) \leq 2.7 n^{2/3} k^{2/3} + 6n + 2k.$

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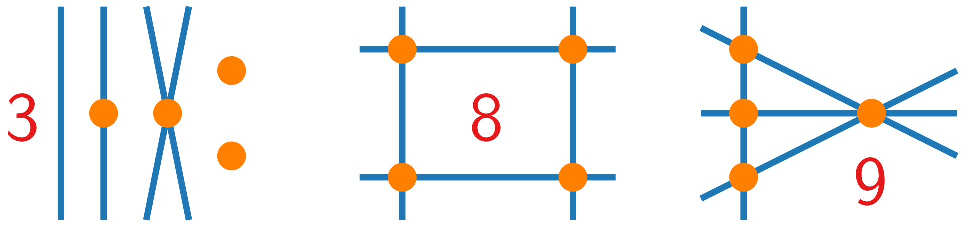

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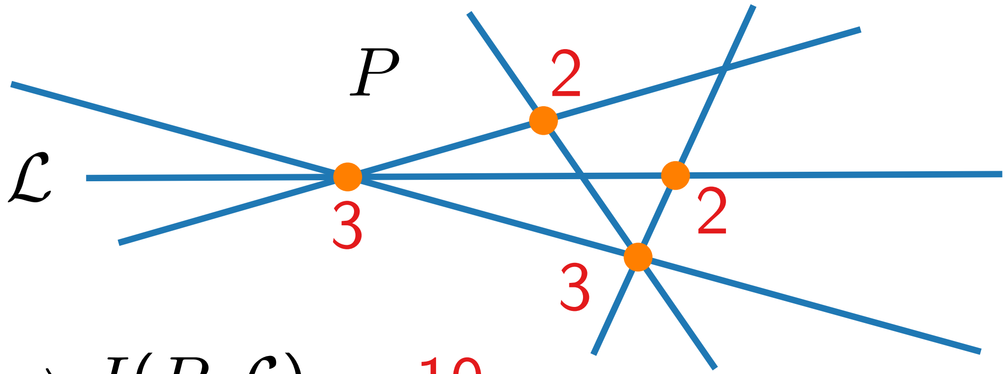

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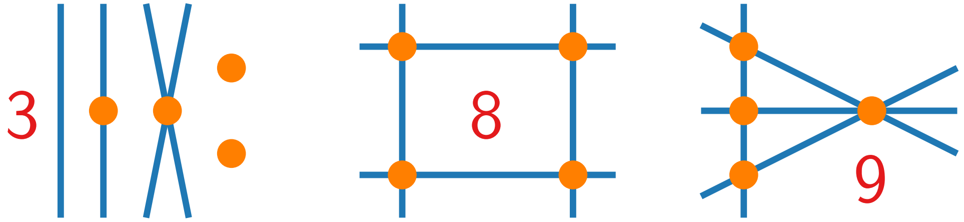

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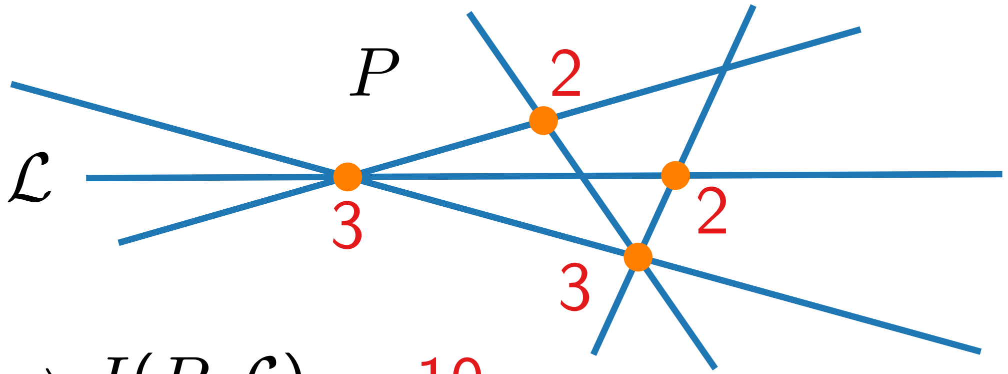

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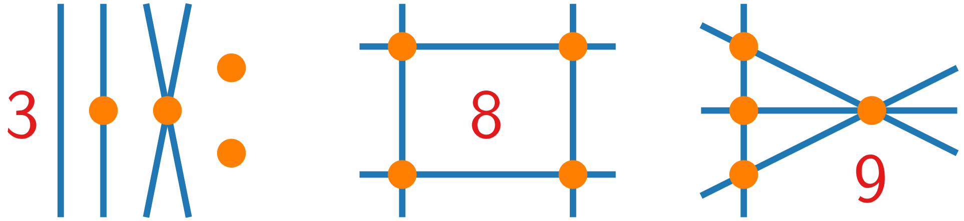

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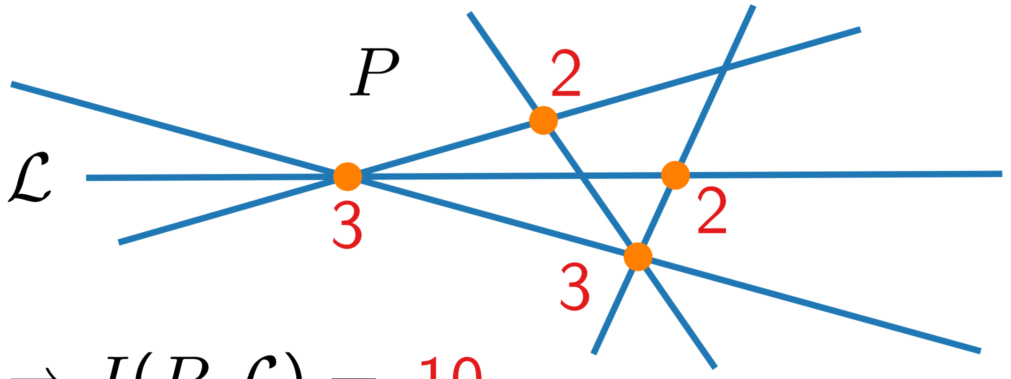

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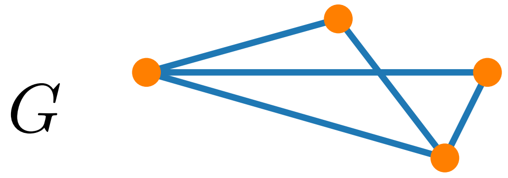

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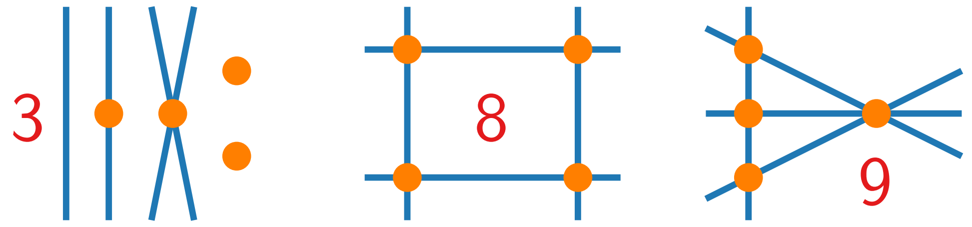

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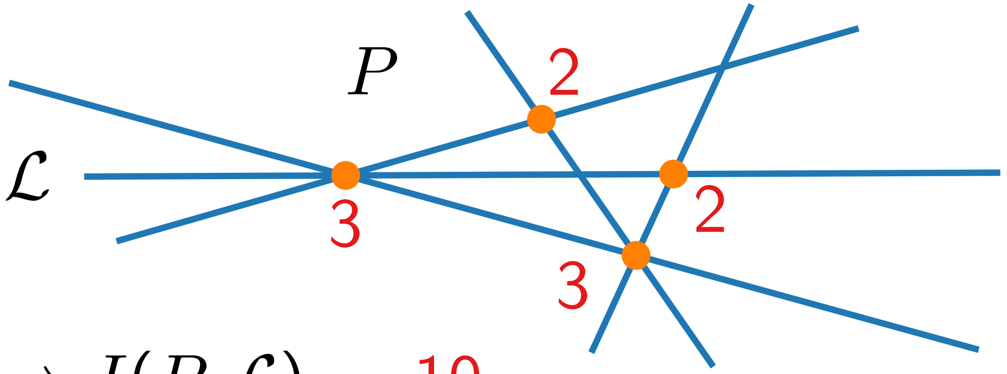

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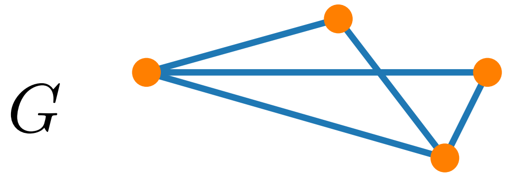

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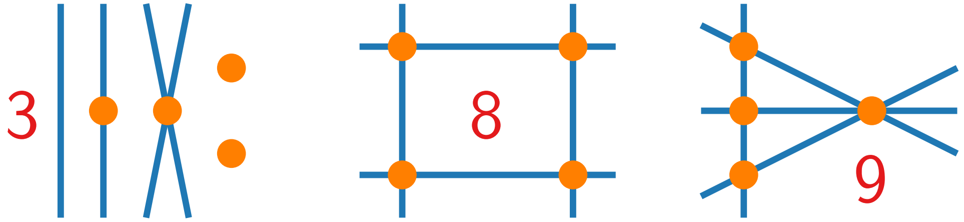

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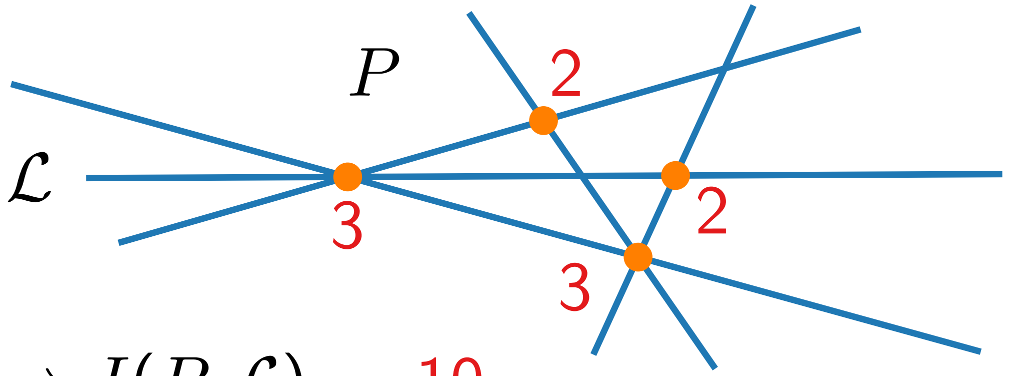

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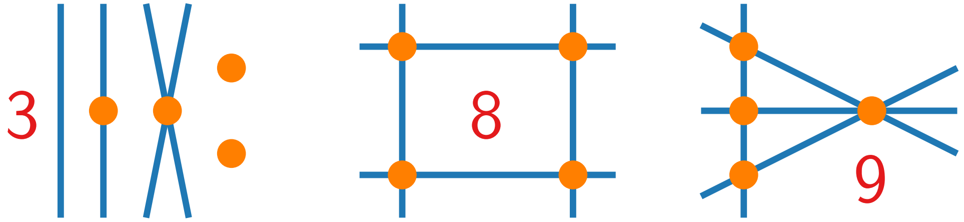

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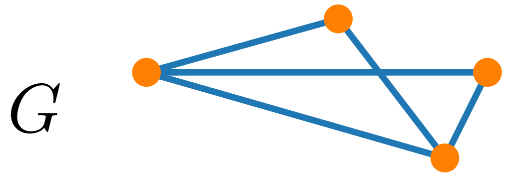

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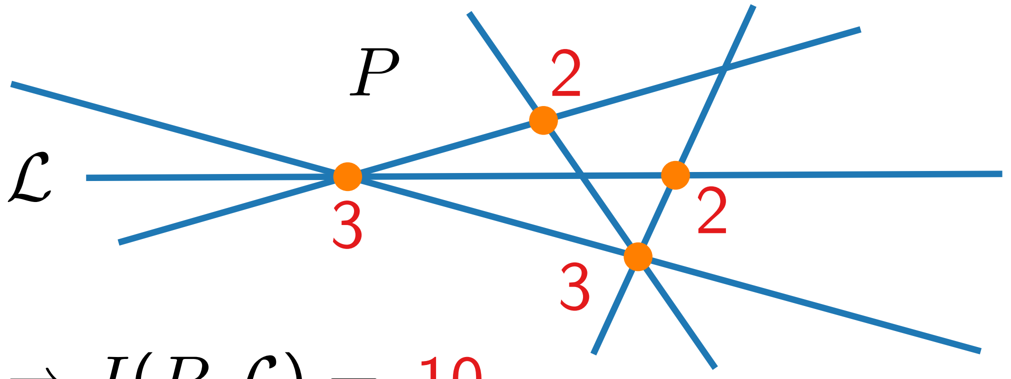

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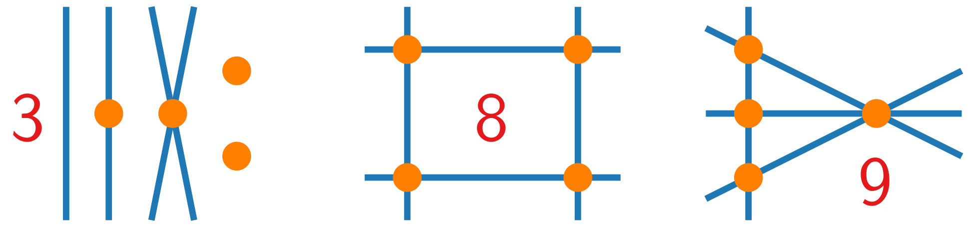

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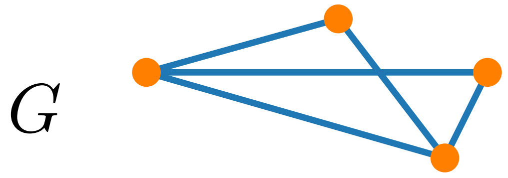

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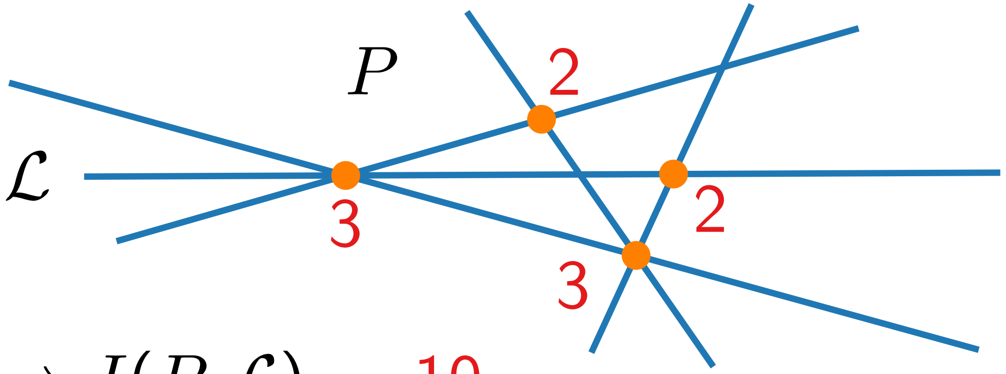

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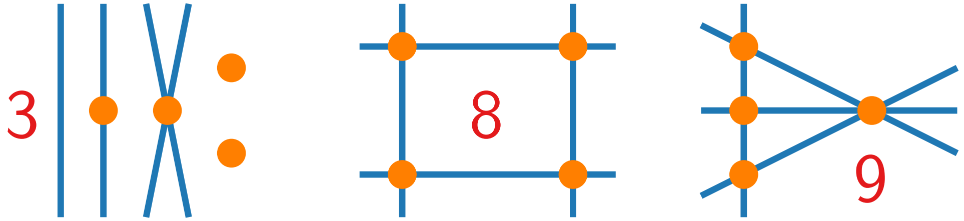

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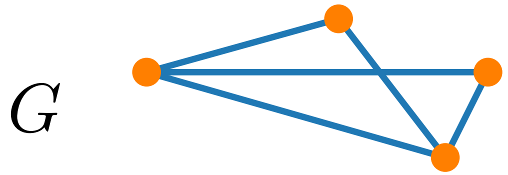

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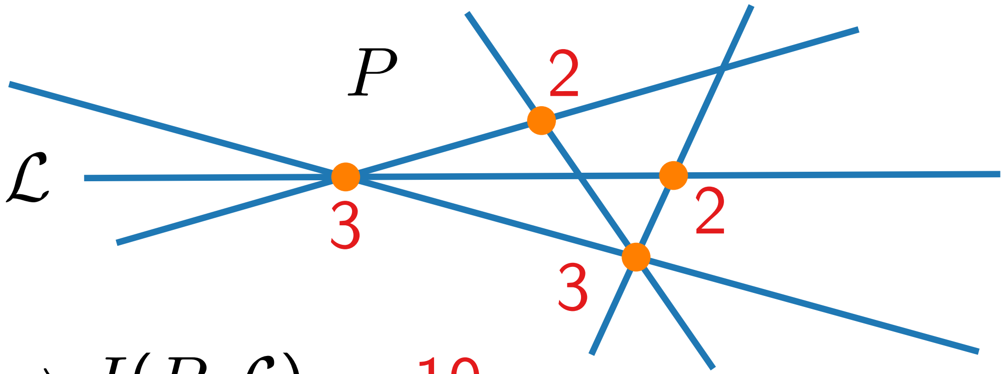

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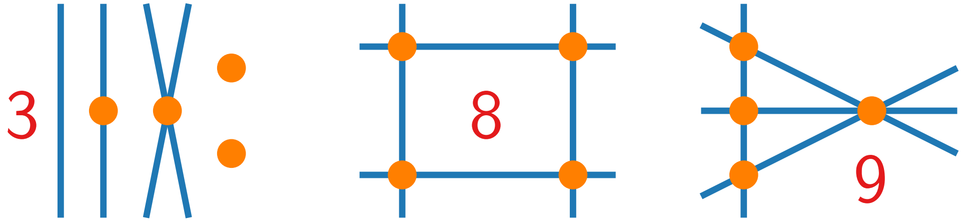

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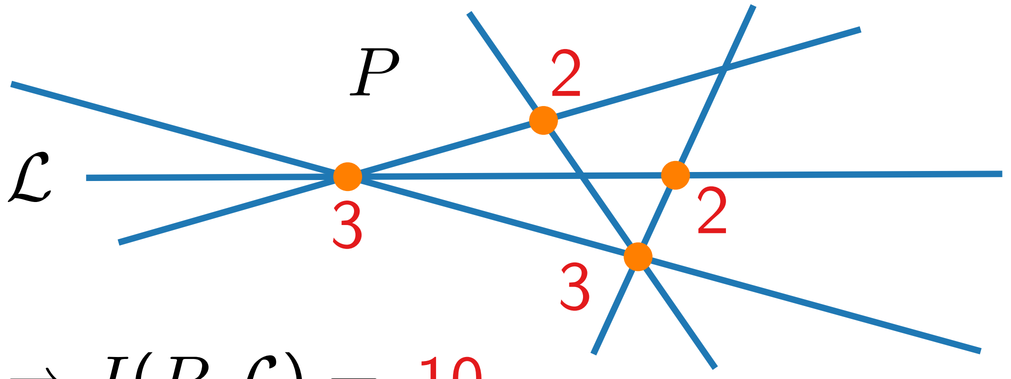

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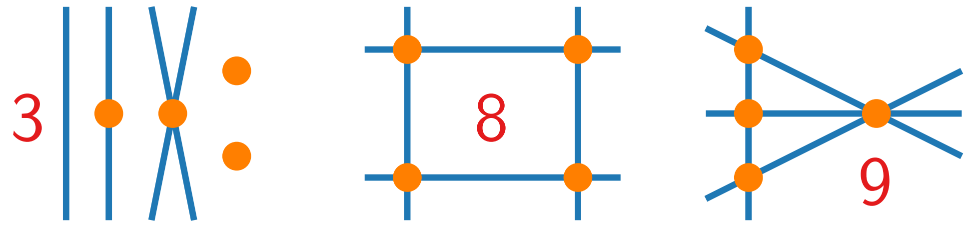

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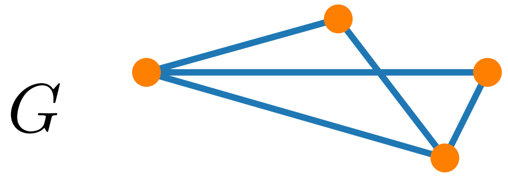

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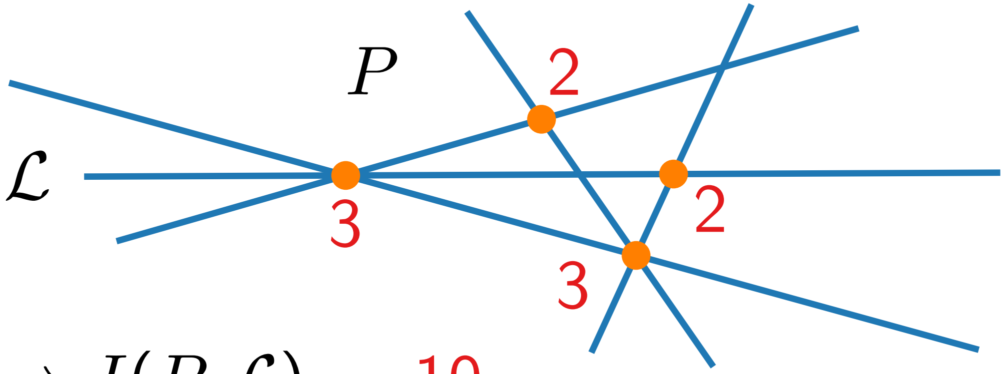

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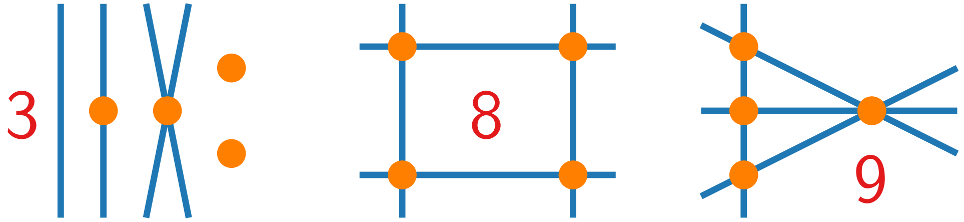

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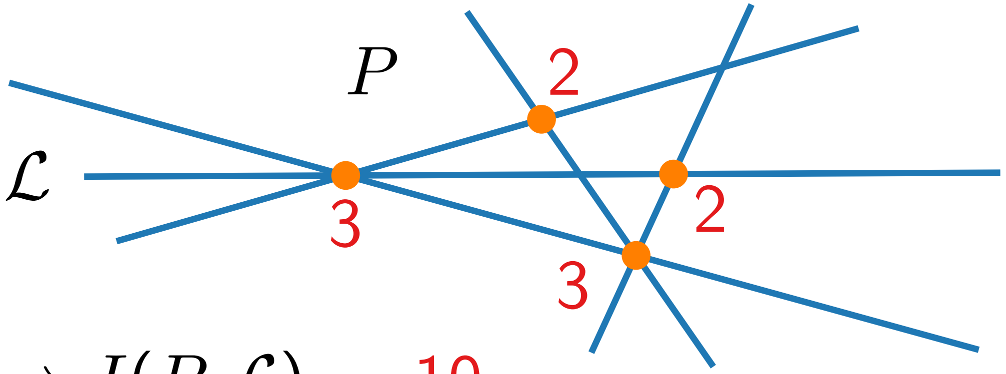

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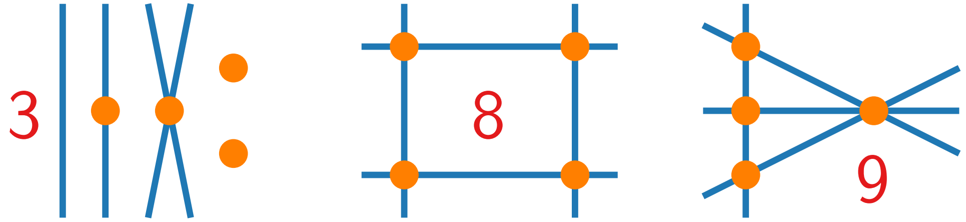

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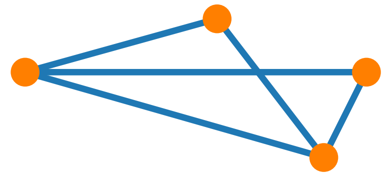

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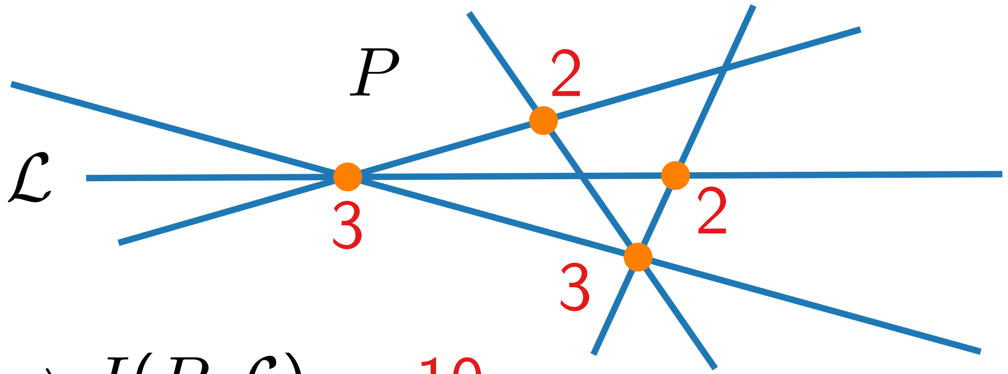

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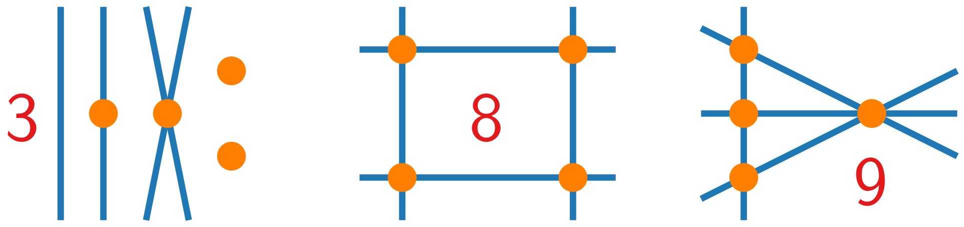

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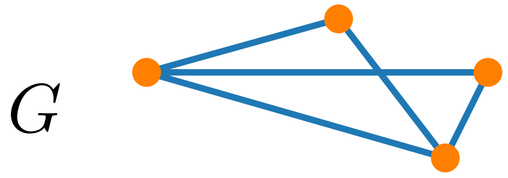

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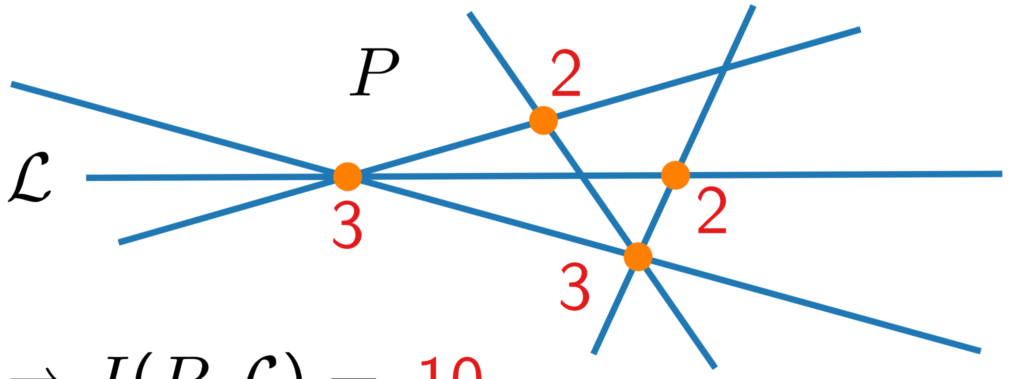

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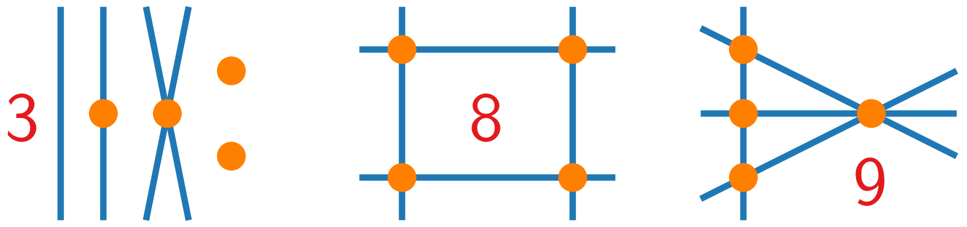

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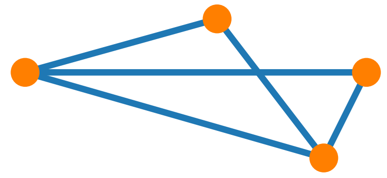

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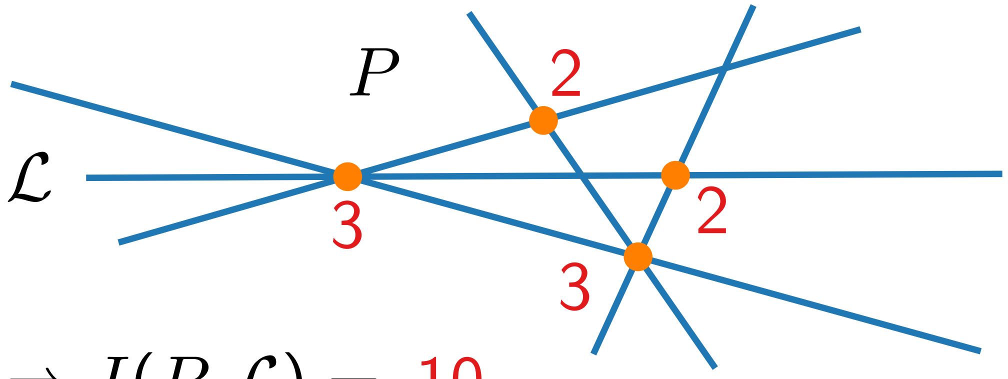

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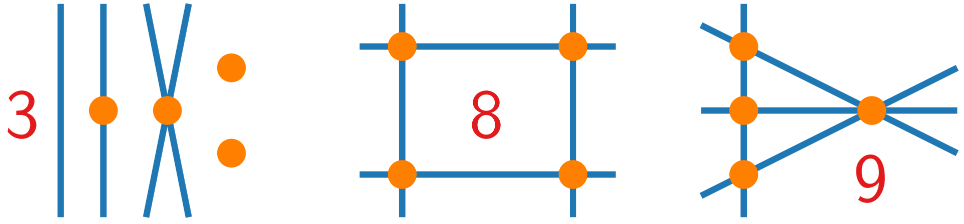

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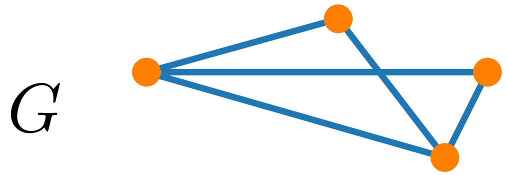

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- $\#(\text{points on a line } \ell) - 1 = \#(\text{edges on } \ell)$
  $\Rightarrow I(n, k) - k = m$ (sum up over $\mathcal{L}$ in an "optimal" instance)

- If $m \leq 4n$, then $I(n, k) - k = m \leq 4n$.
  $\Rightarrow I(n, k) \leq 4n + k \leq c(n + k + n^{2/3}k^{2/3})$

- Otherwise, employ the Crossing Lemma:

$$\frac{m^3}{64n^2} \leq \mathrm{cr}(G) \leq k^2/2 \quad \Rightarrow \frac{(I(n,k)-k)^3}{64n^2} \leq k^2/2$$

$$\Leftrightarrow I(n, k) \leq c(n^{2/3}k^{2/3} + k)$$
$$\leq c(n^{2/3}k^{2/3} + k + n).$$

$\square$

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For a set $P \subset \mathbb{R}^2$ of points, define

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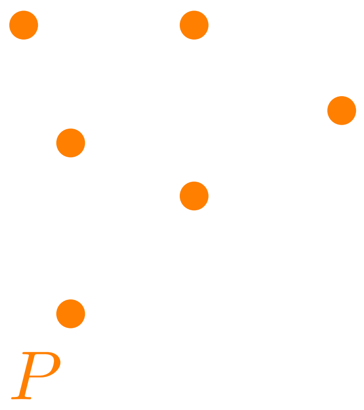

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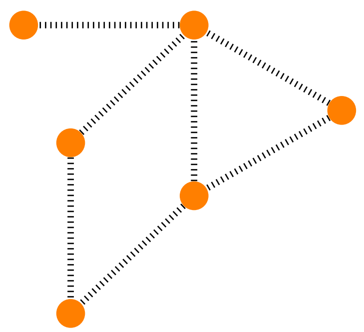

P

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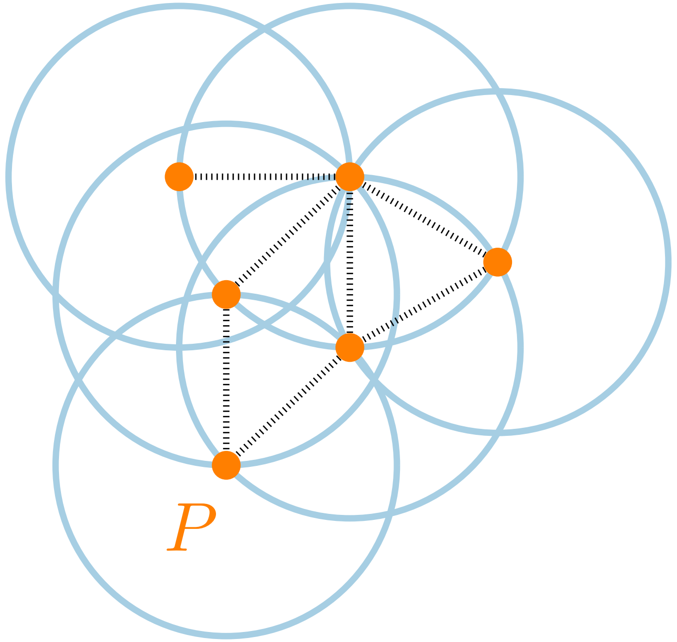

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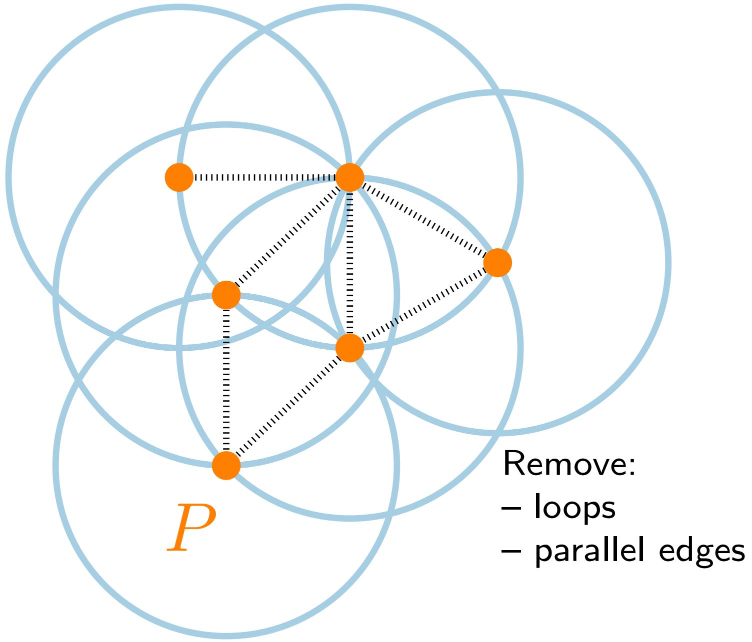

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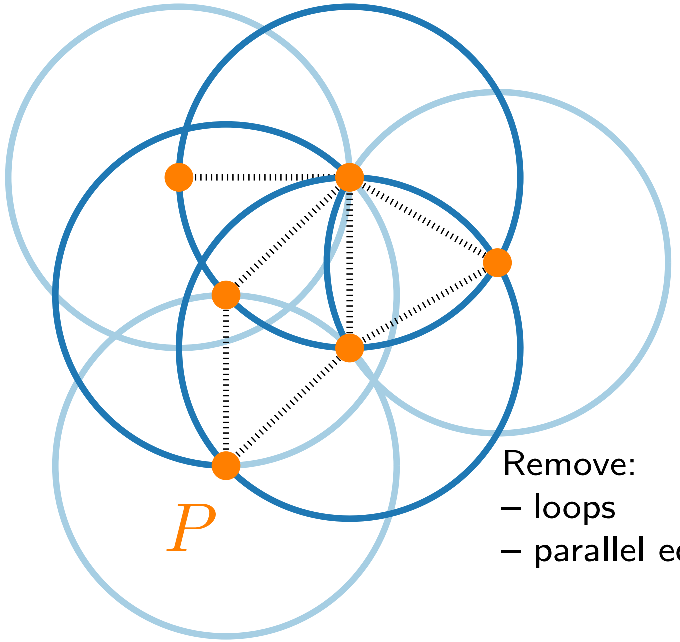

Remove:
– loops
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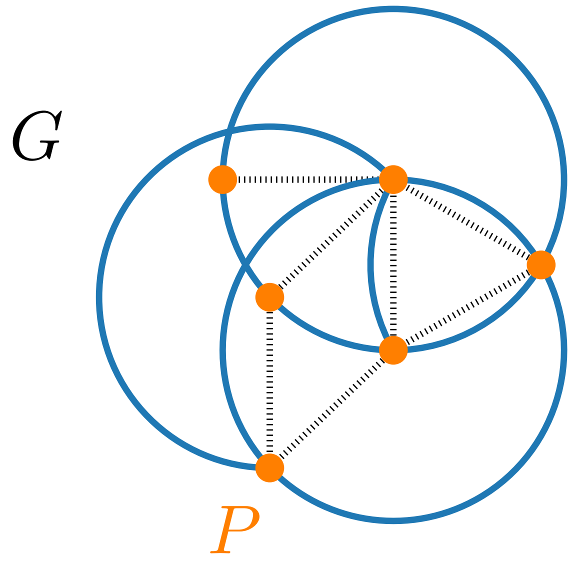

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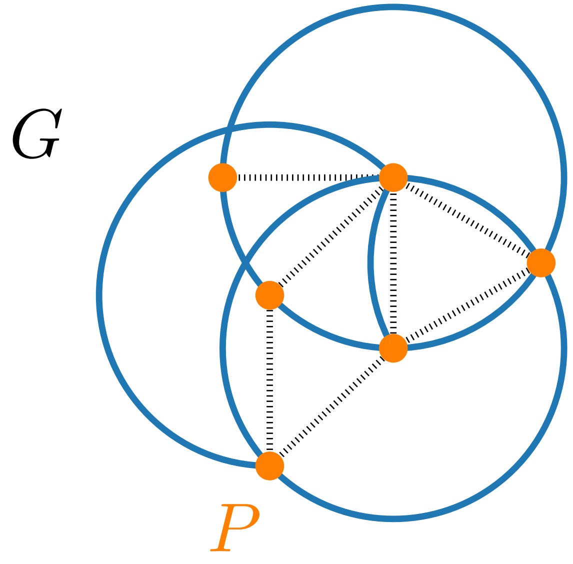

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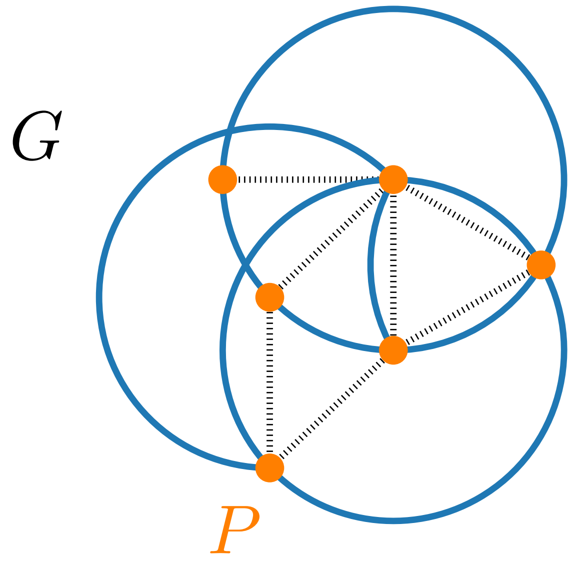

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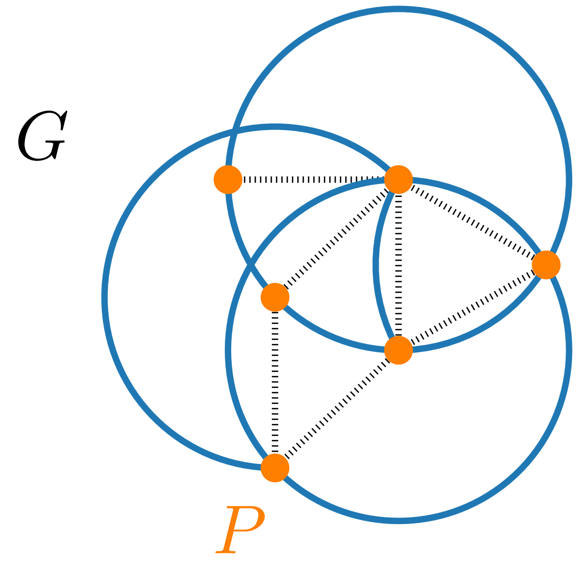

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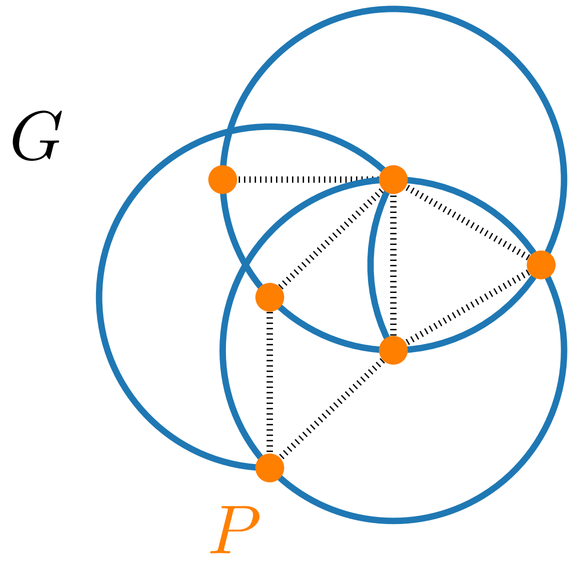

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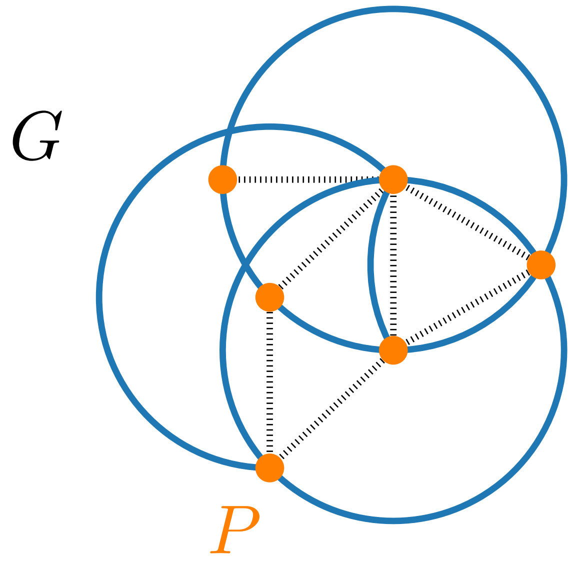

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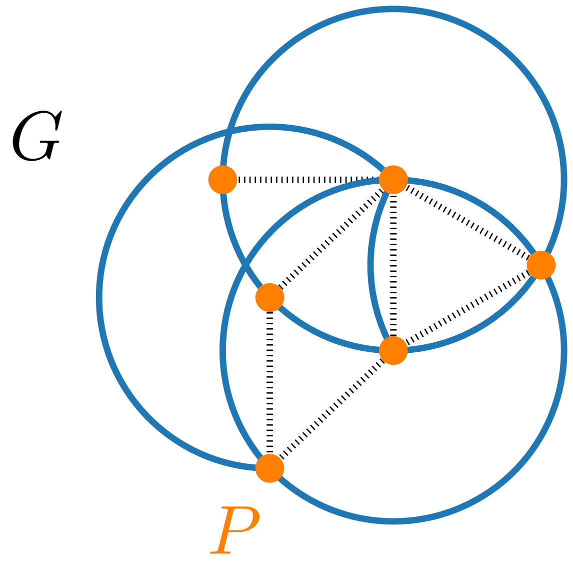

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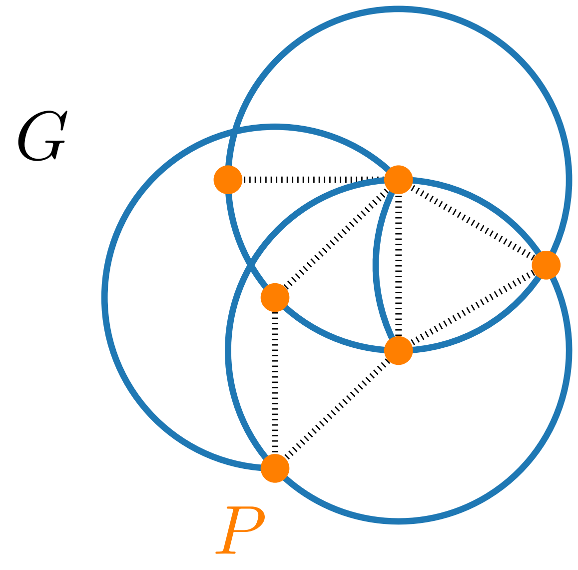

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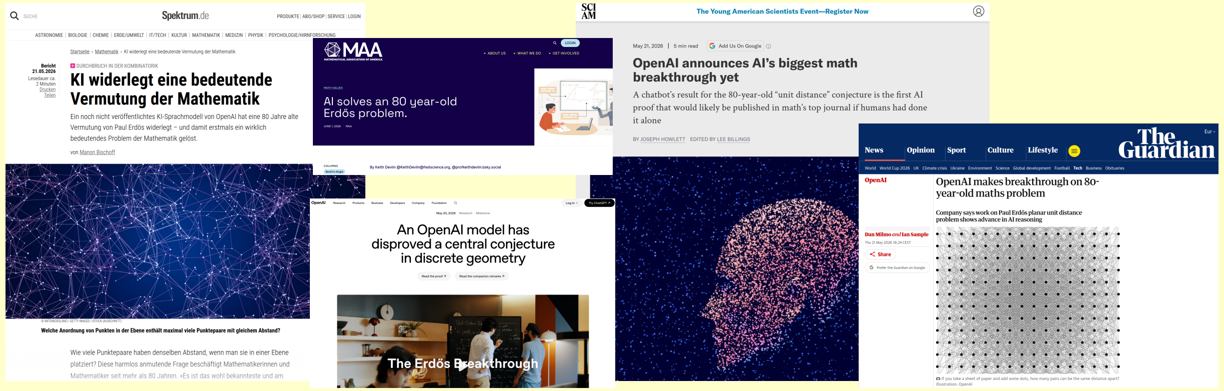

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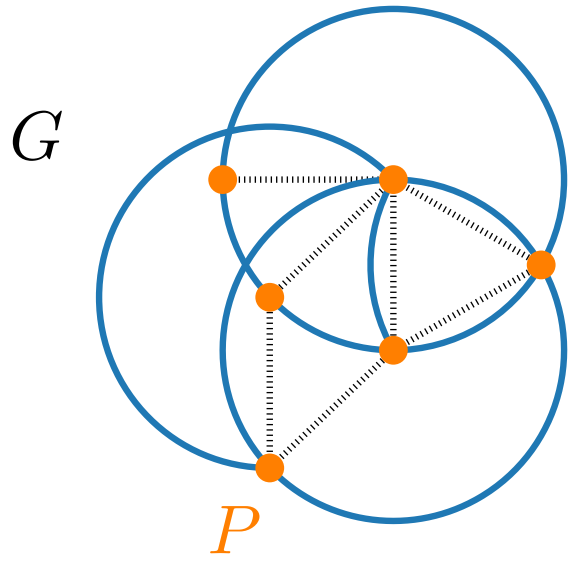

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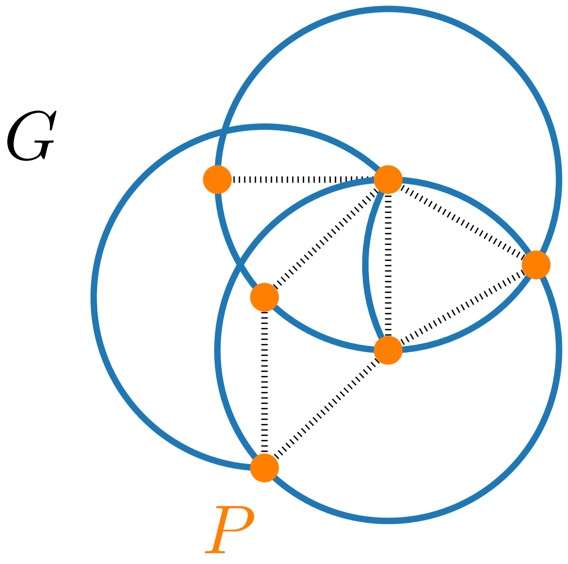

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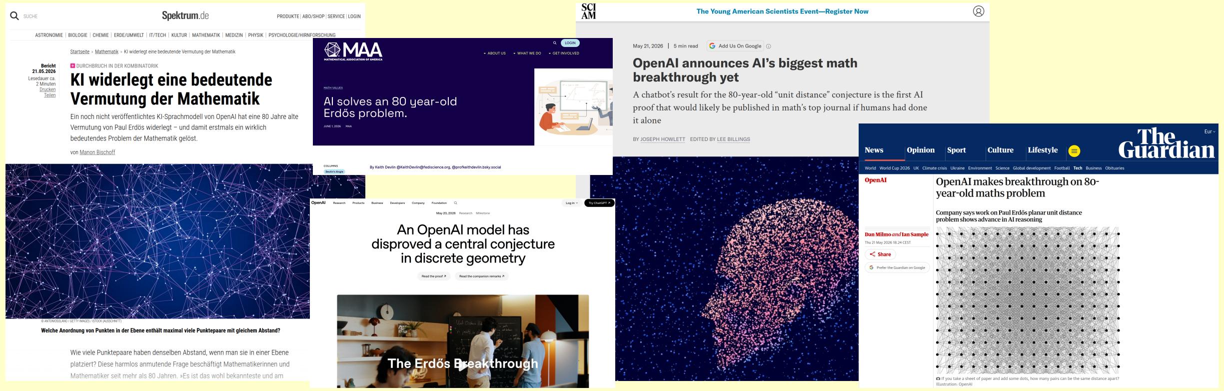

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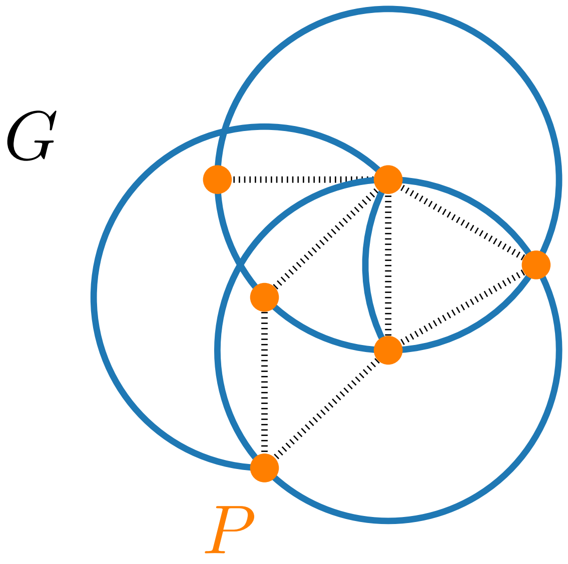

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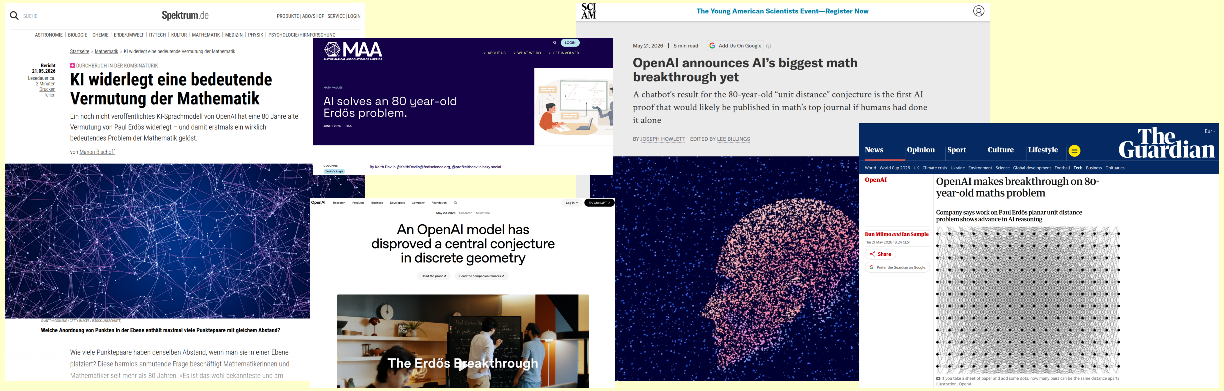

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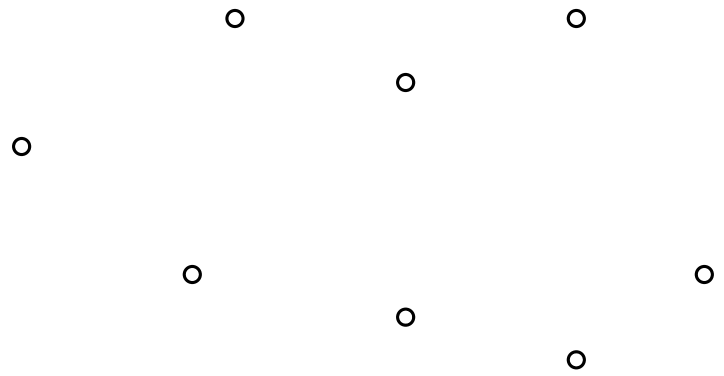
https://cdn.openai.com/pdf/74c24085-19b0-4534-9c90-465b8e29ad73/unit-distance-proof.pdf

https://www.erdosproblems.com/forum/thread/90

https://teorth.github.io/optimizationproblems/constants/84a.html

Application 3: Expected Number of Crossings in a Matching

Given a set of n points (in general position, n even) –



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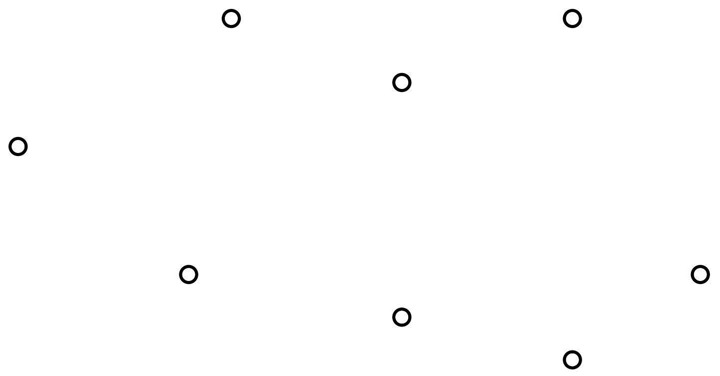

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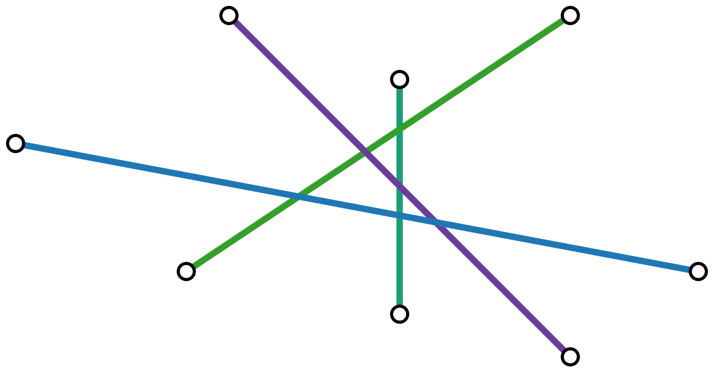

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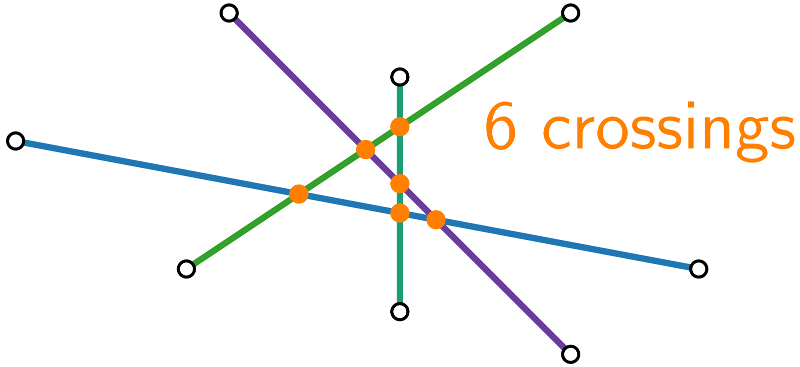

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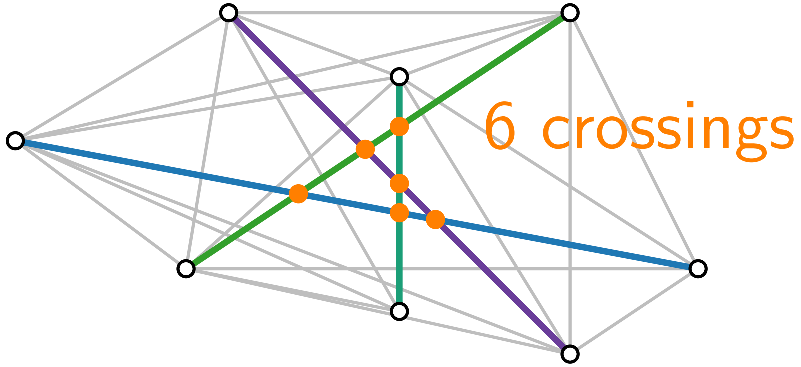

Point set spans drawing Γ of K_n.

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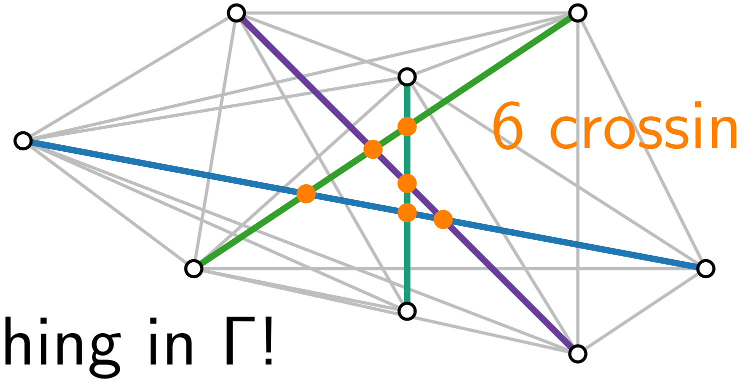

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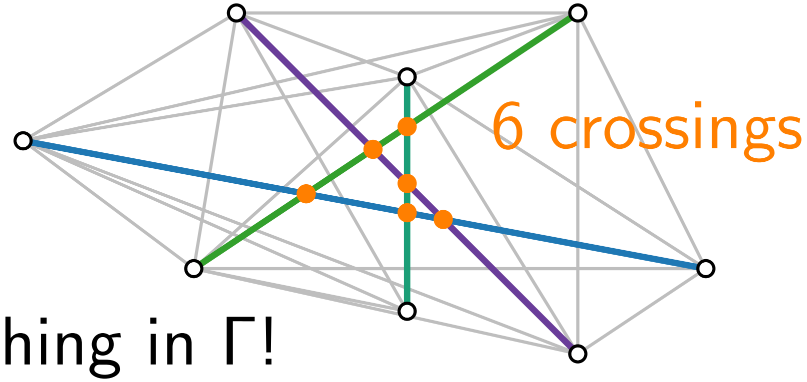

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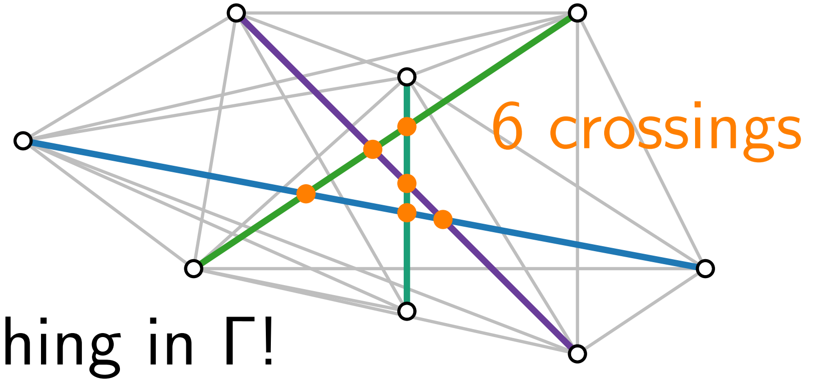

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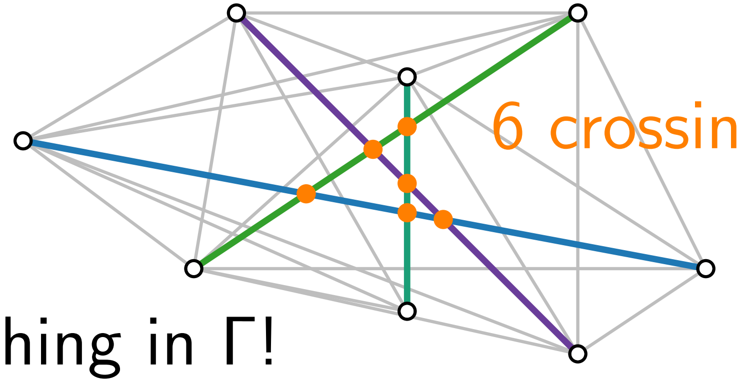

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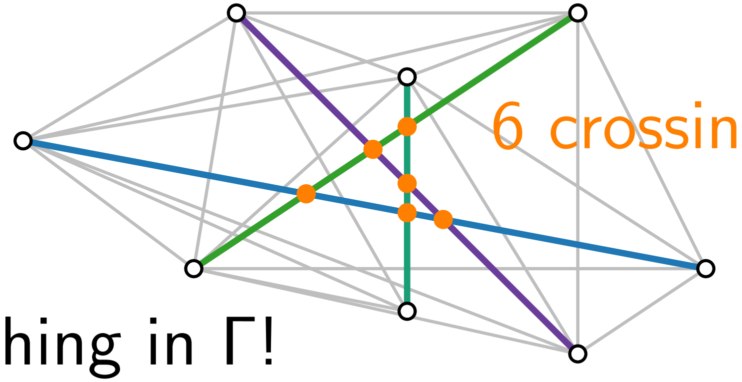

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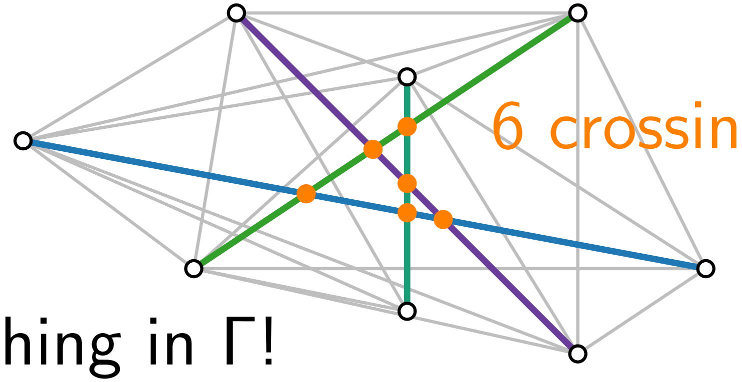

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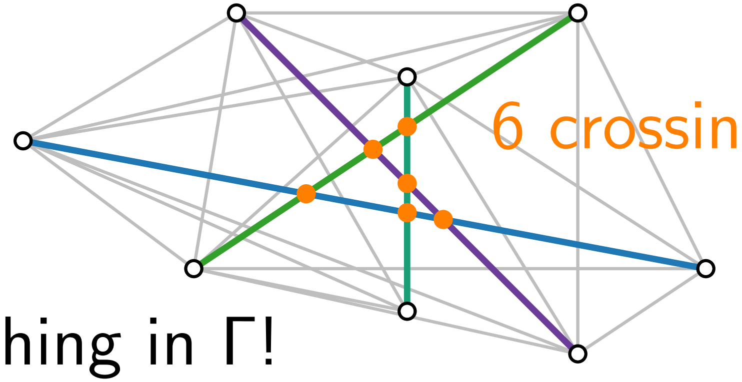

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Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) =$

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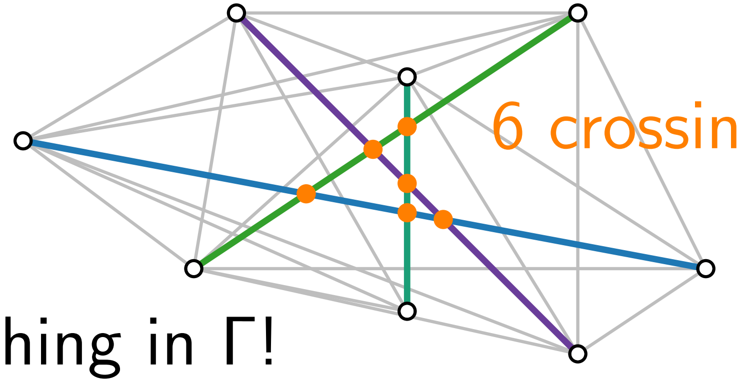

Point set spans drawing Γ of K_n.
We will analyze the number of crossings in a **random** perfect matching in Γ!

Number of crossings in $\Gamma \geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$ [Lovász et al. '04, Aichholzer et al. '06]

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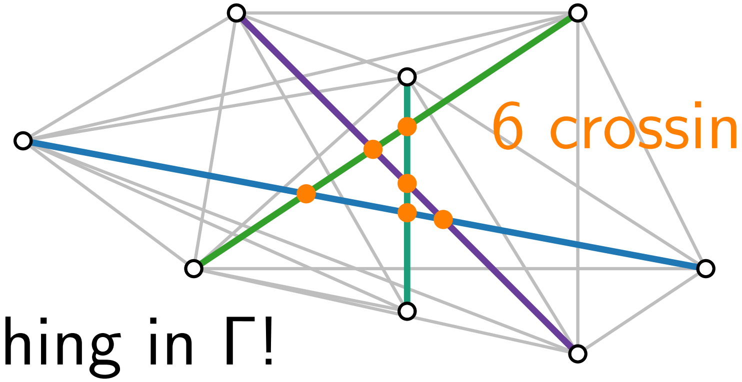

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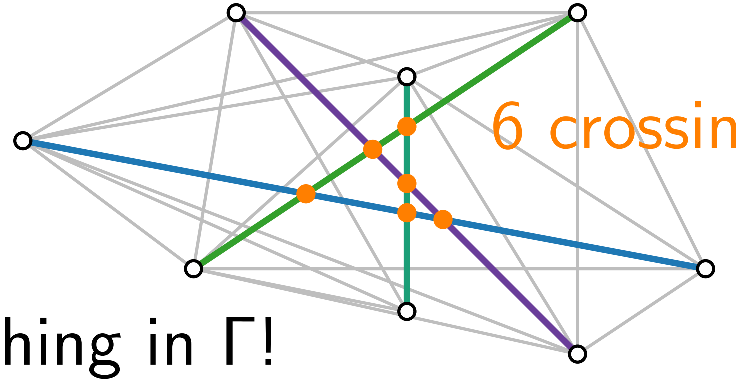

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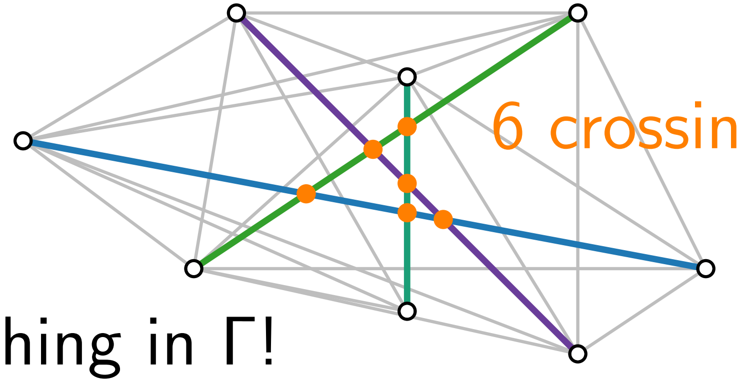

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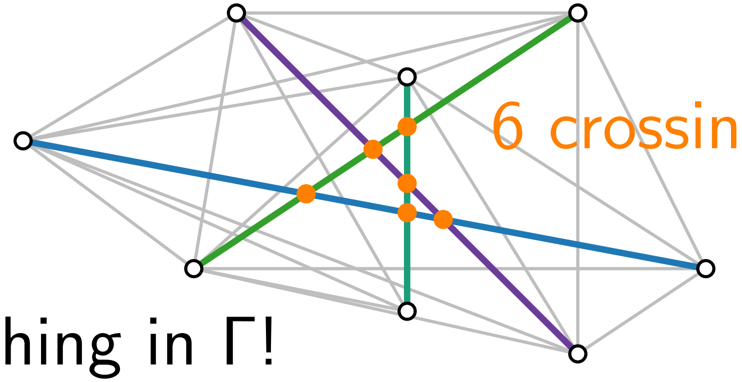

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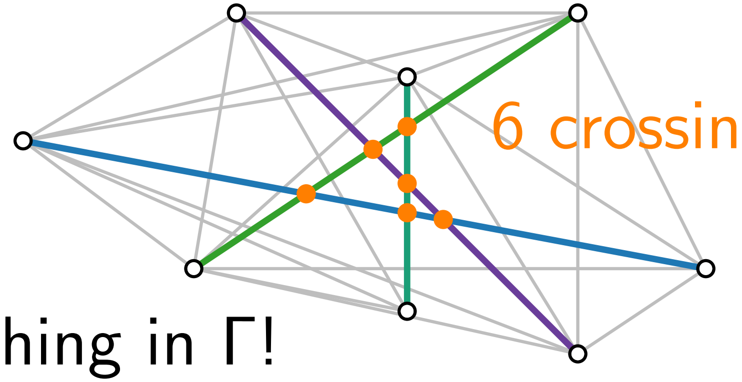

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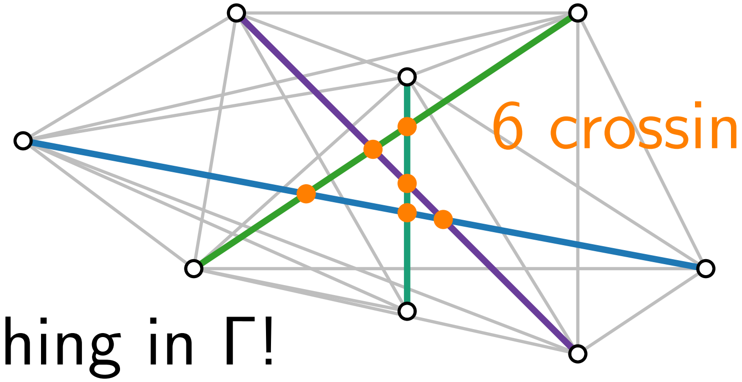

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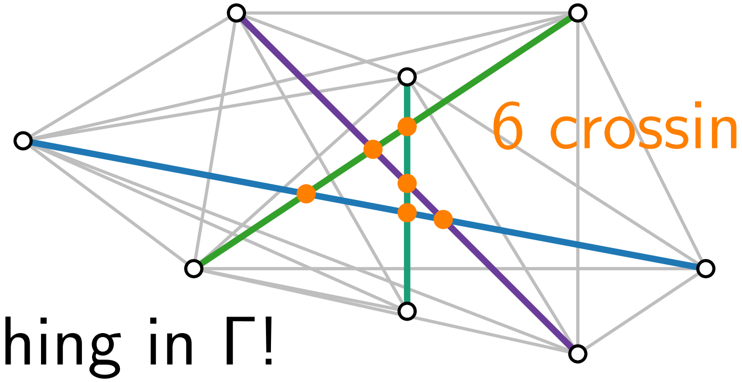

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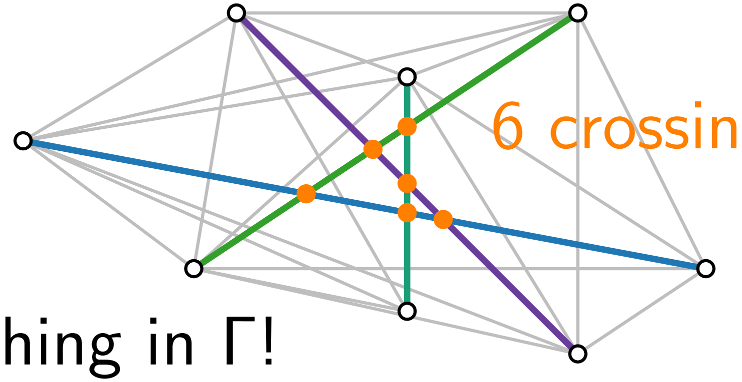

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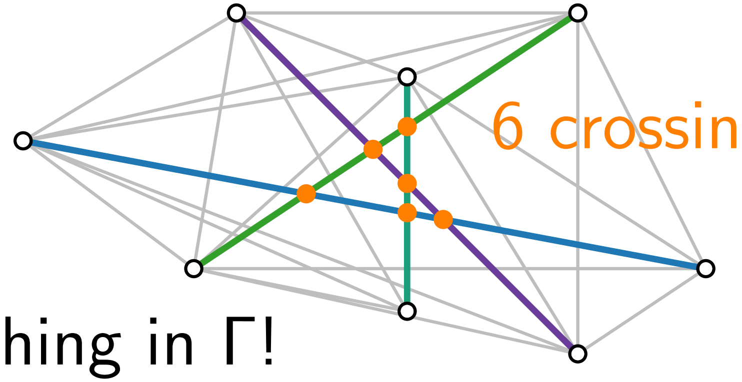

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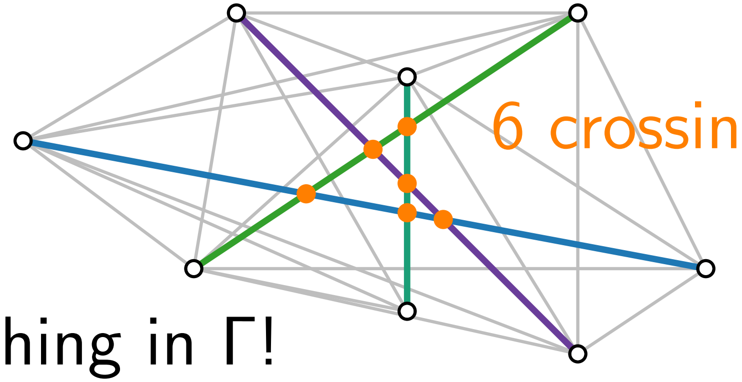

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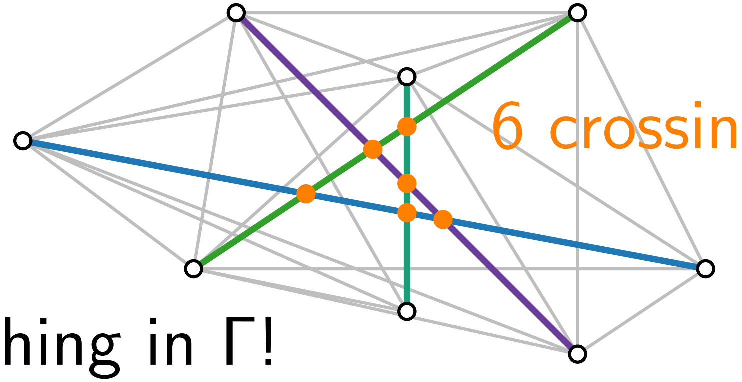

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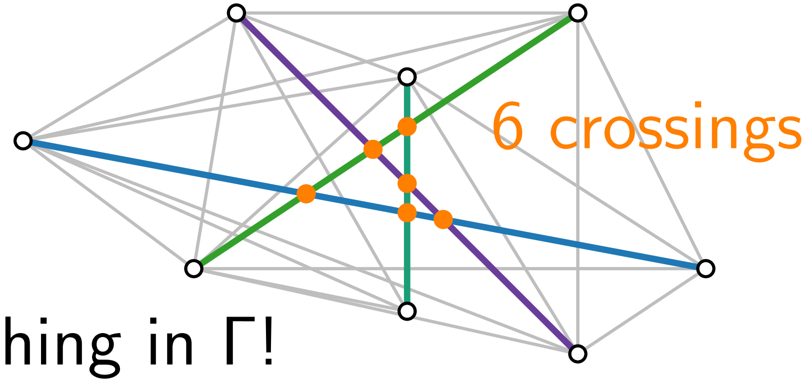

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Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao's blog post "The crossing number inequality" from 2007
- [Hanani '43] Über wesentlich unplättbare Kurven im dreidimensionalen Raume
- [Tutte '70] Toward a theory of crossing numbers
- [Pach & Tóth '00] Which crossing number is it anyway?
- [Pelsmajer, Schaefer & Štefankovič '07] Removing even crossings
- [Pelsmajer, Schaefer & Štefankovič '08] Odd Crossing Number and Crossing Number Are Not the Same
- [Tóth '08] Note on the Pair-crossing Number and the Odd-crossing Number
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Lovász, Vesztergombi, Wagner, Welzl '04] Convex quadrilaterals and k-sets
- [Aichholzer, Aurenhammer, Krasser '06] On the Crossing Number of Complete Graphs
- [Székely '97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdős"
- Exact computations of crossing numbers: `http://crossings.uos.de`