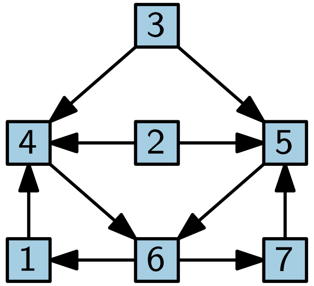
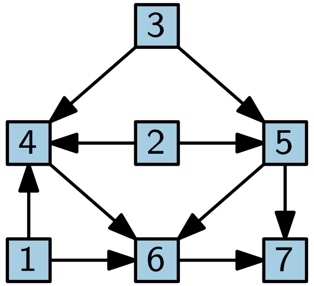
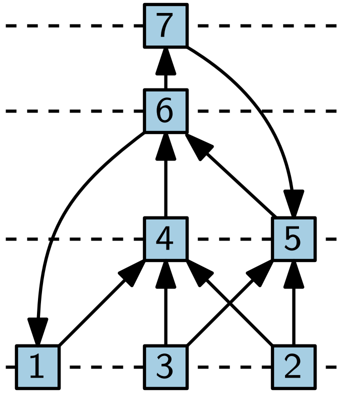
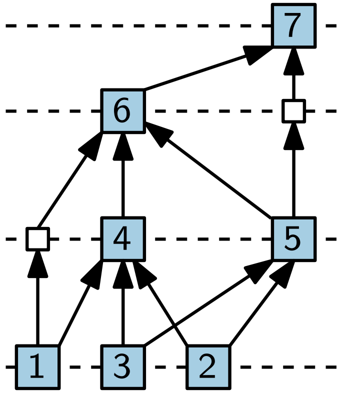
Visualization of Graphs

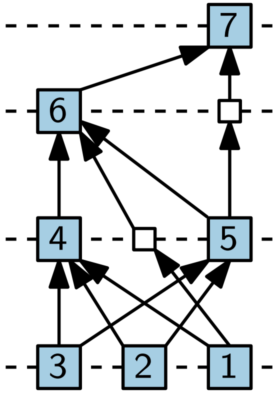


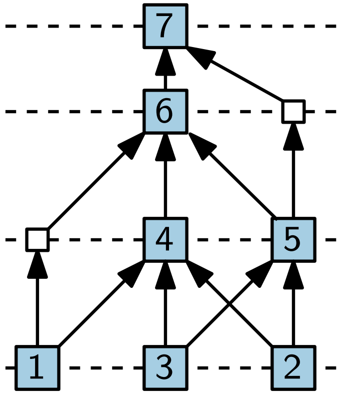




Lecture 8:
Hierarchical Layouts:
Sugiyama Framework



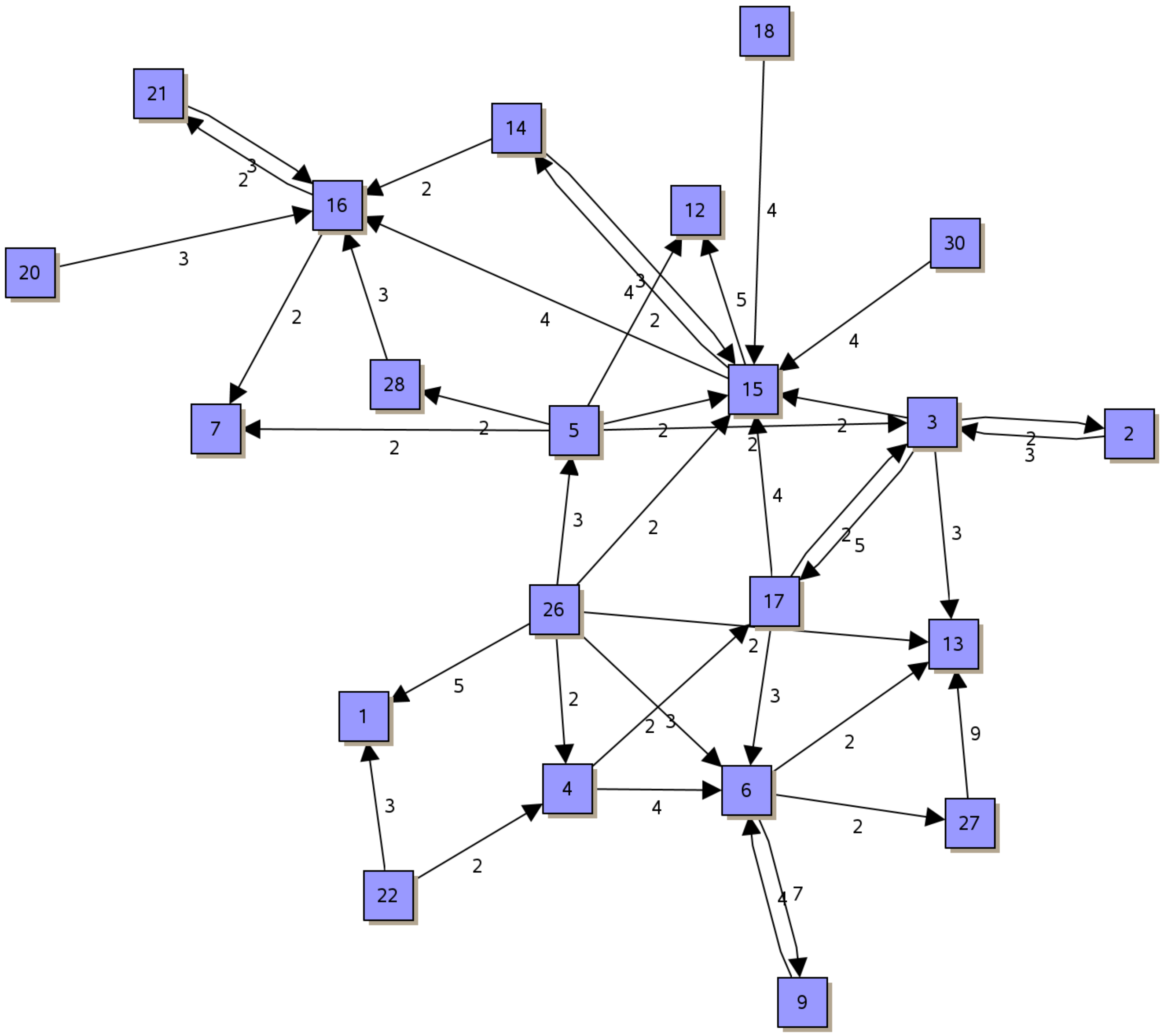




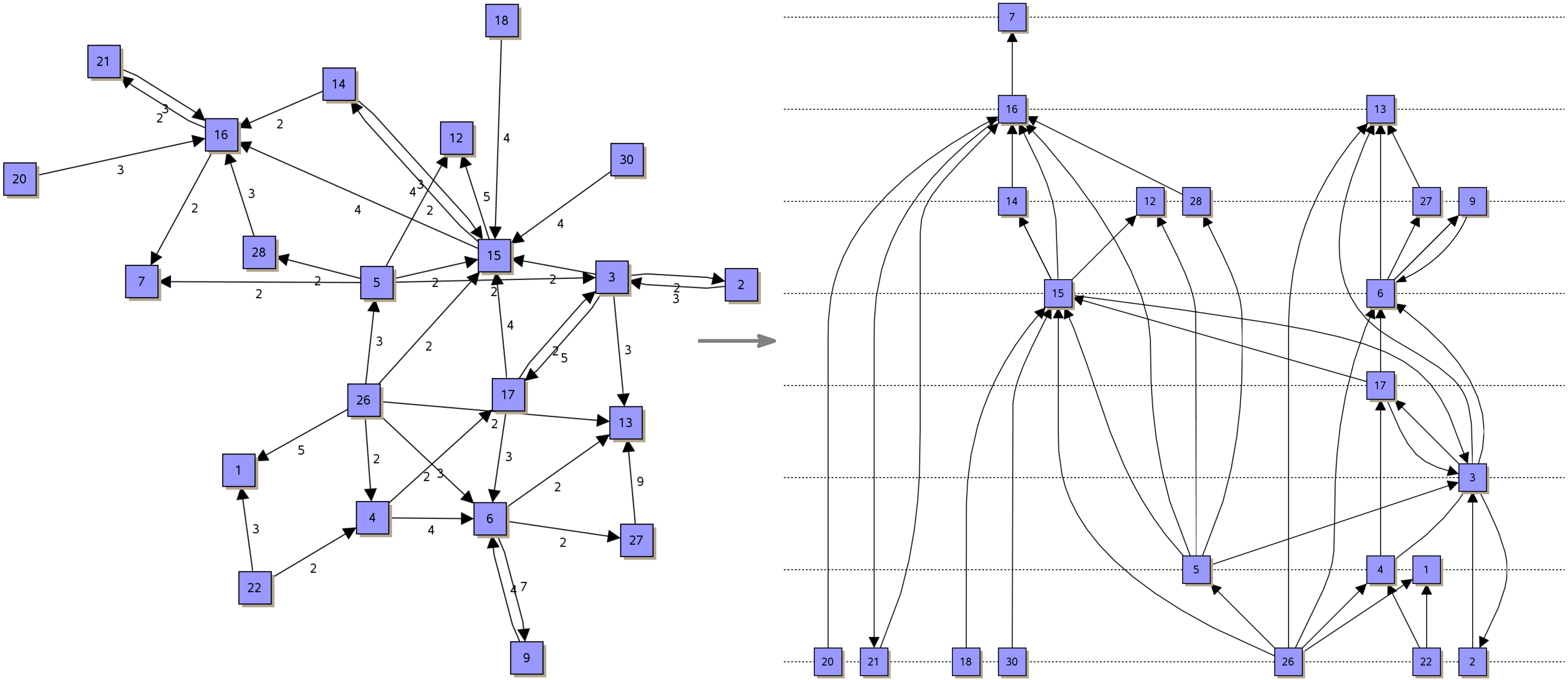
Alexander Wolff

Summer term 2026

Hierarchical Drawings – Motivation



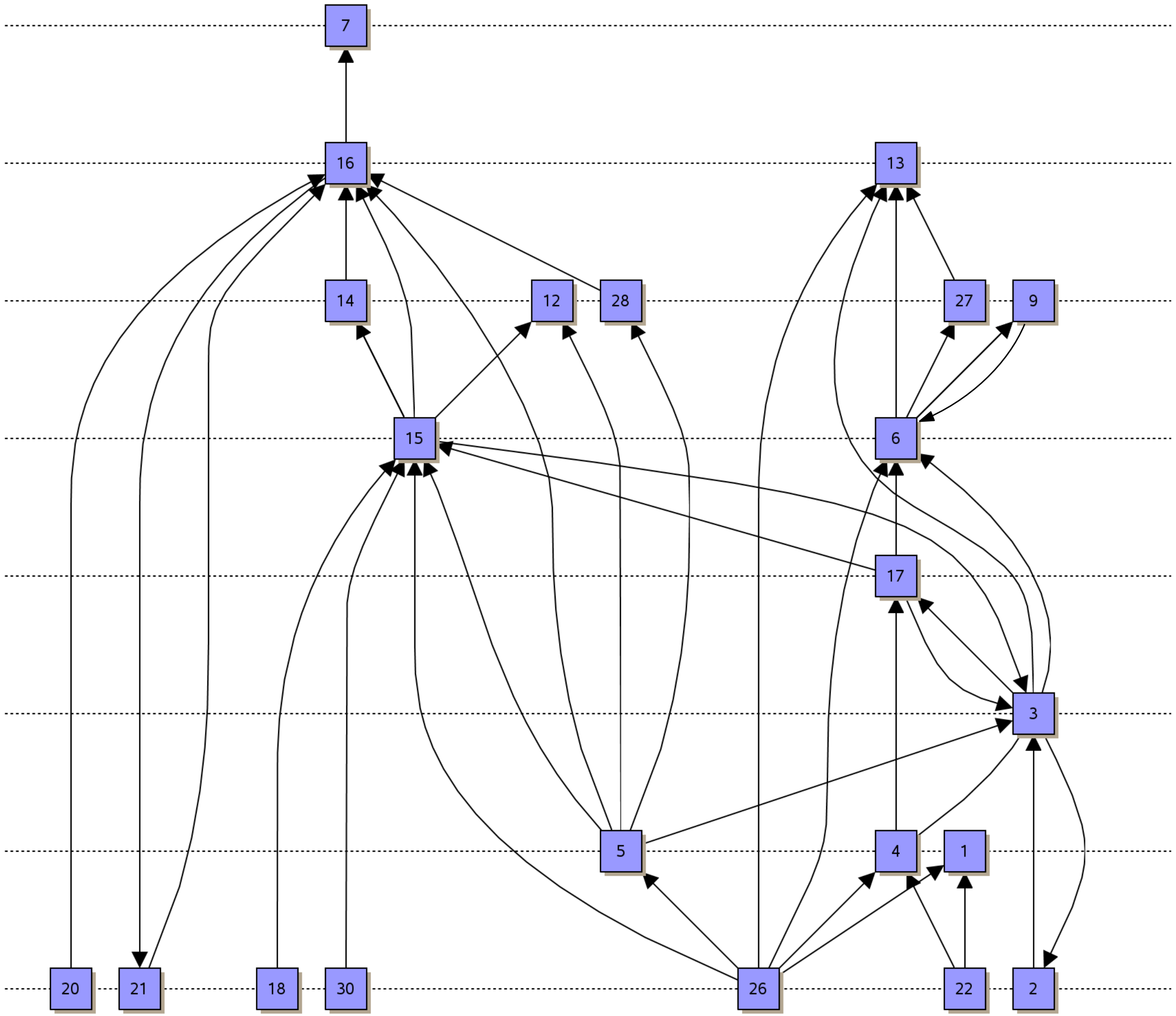
Hierarchical Drawings – Motivation



Hierarchical Drawing

Problem Statement:

- Input: digraph G
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



Hierarchical Drawing

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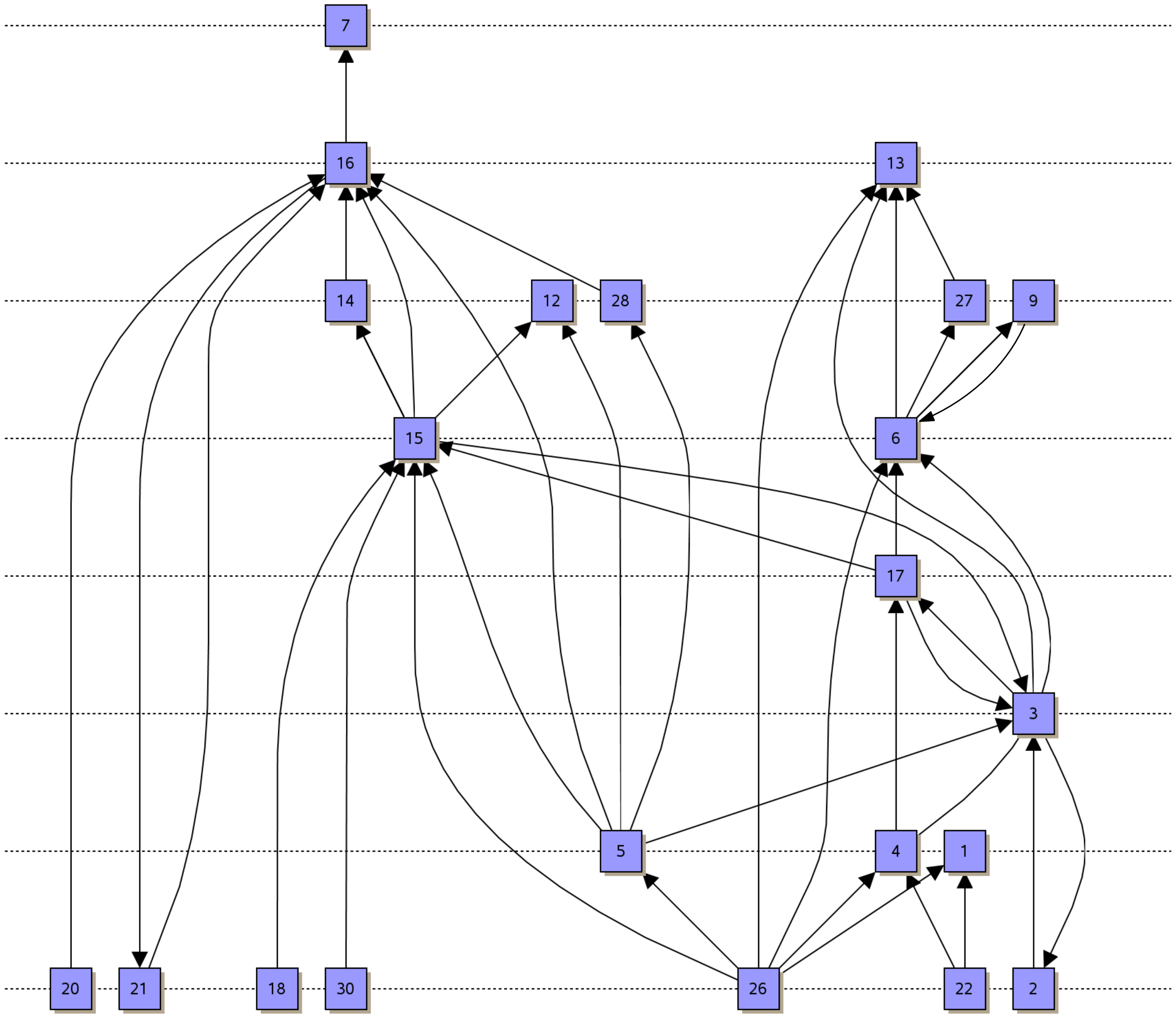
Desirable Properties:



Hierarchical Drawing

Problem Statement:

- Input: digraph G
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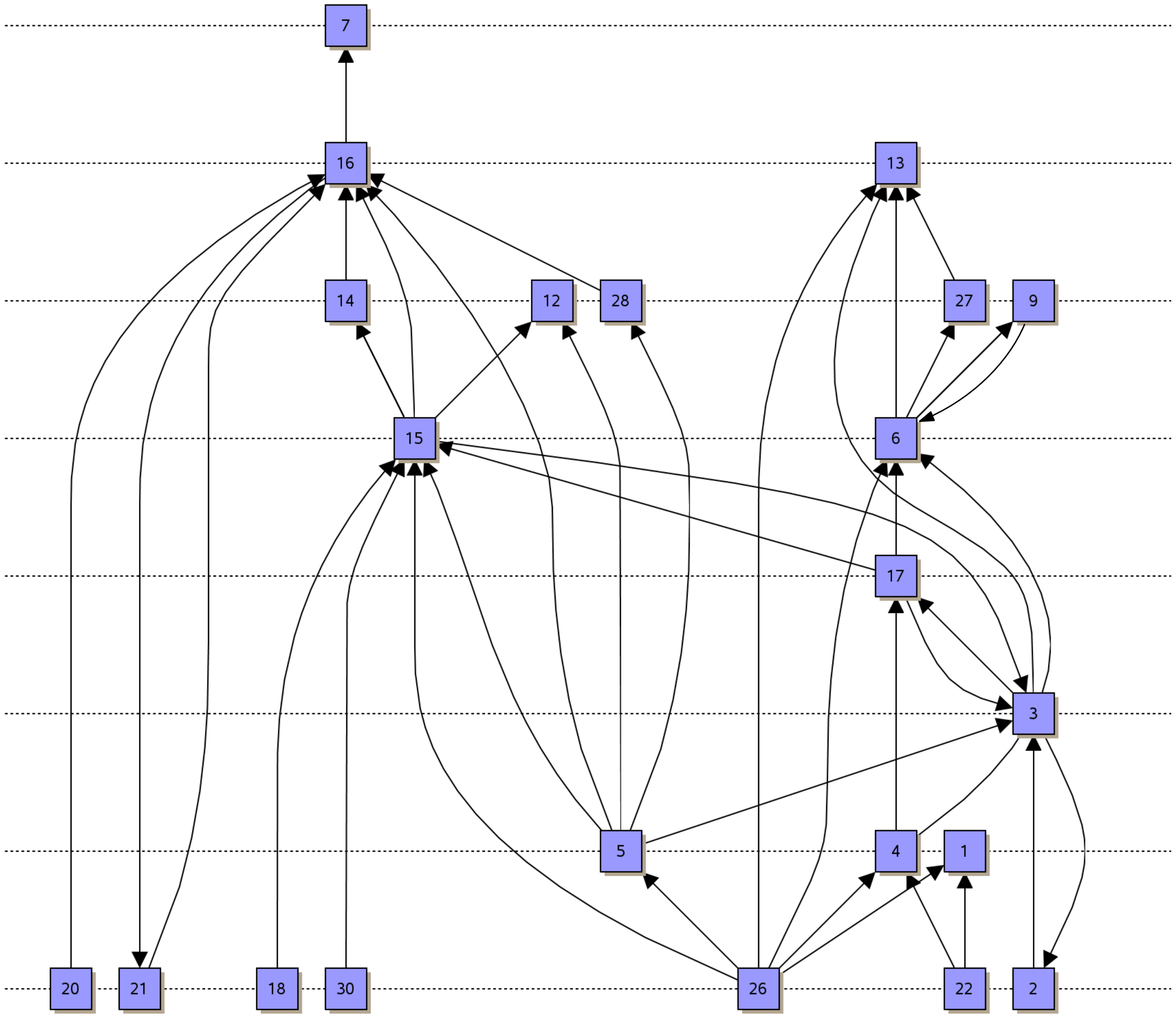
Desirable Properties:

- edges are directed upwards,



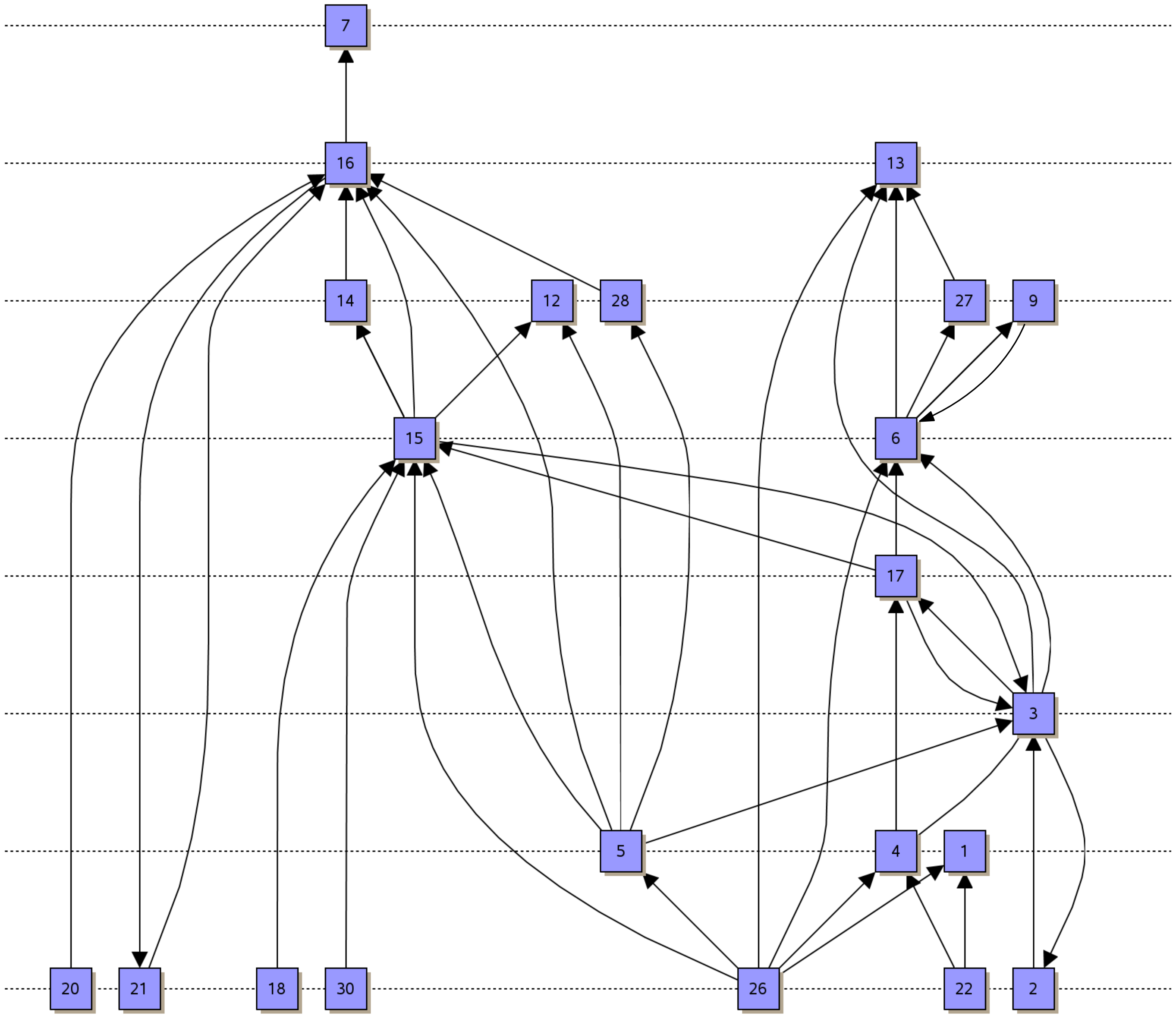
Hierarchical Drawing

Problem Statement:

- Input: digraph G
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties:

- edges are directed upwards,
- vertices lie on (few) horizontal lines,



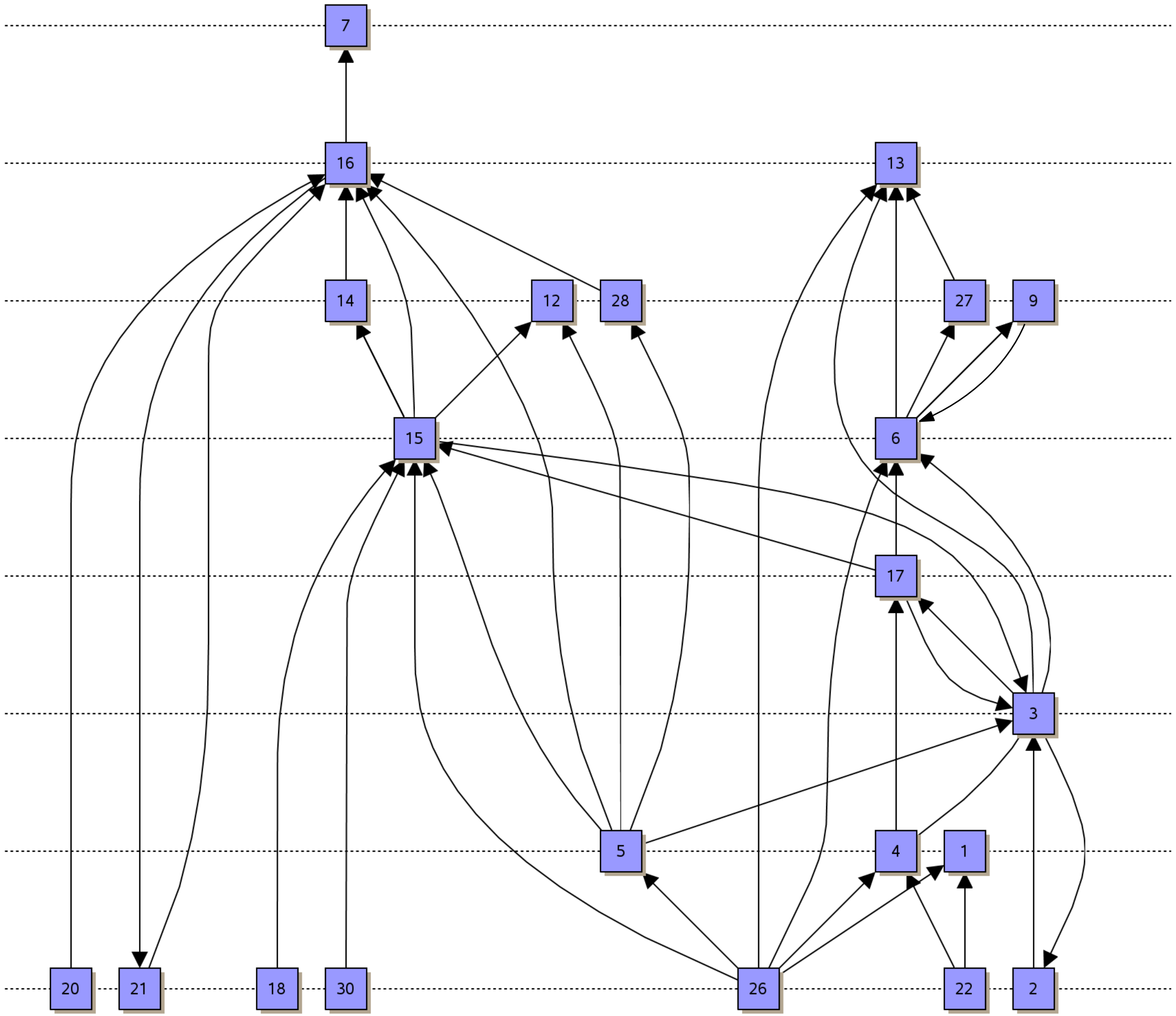
Hierarchical Drawing

Problem Statement:

- Input: digraph G
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

Desirable Properties:

- edges are directed upwards,
- vertices lie on (few) horizontal lines,
- few pairs of edges cross,



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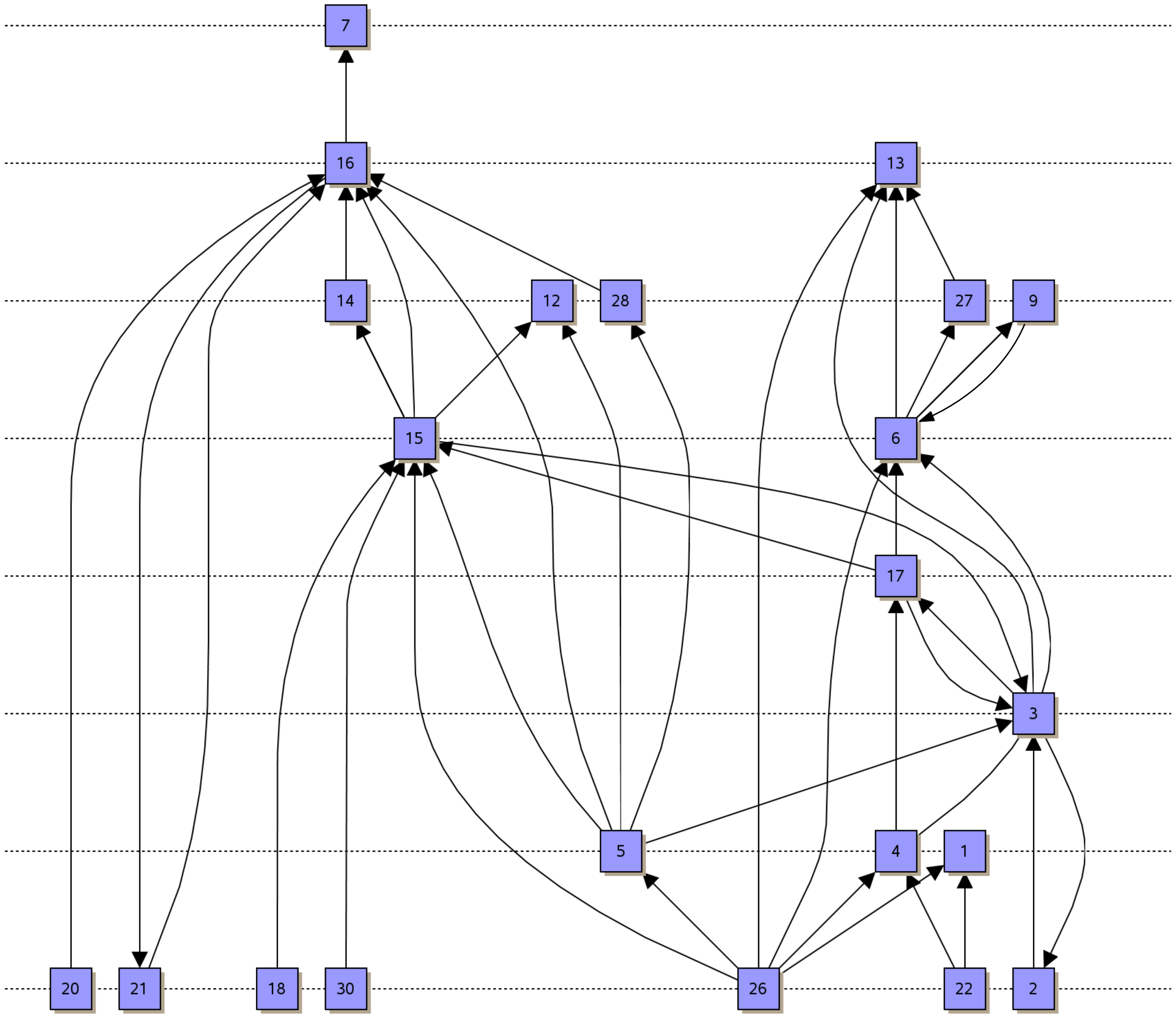

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Desirable Properties:

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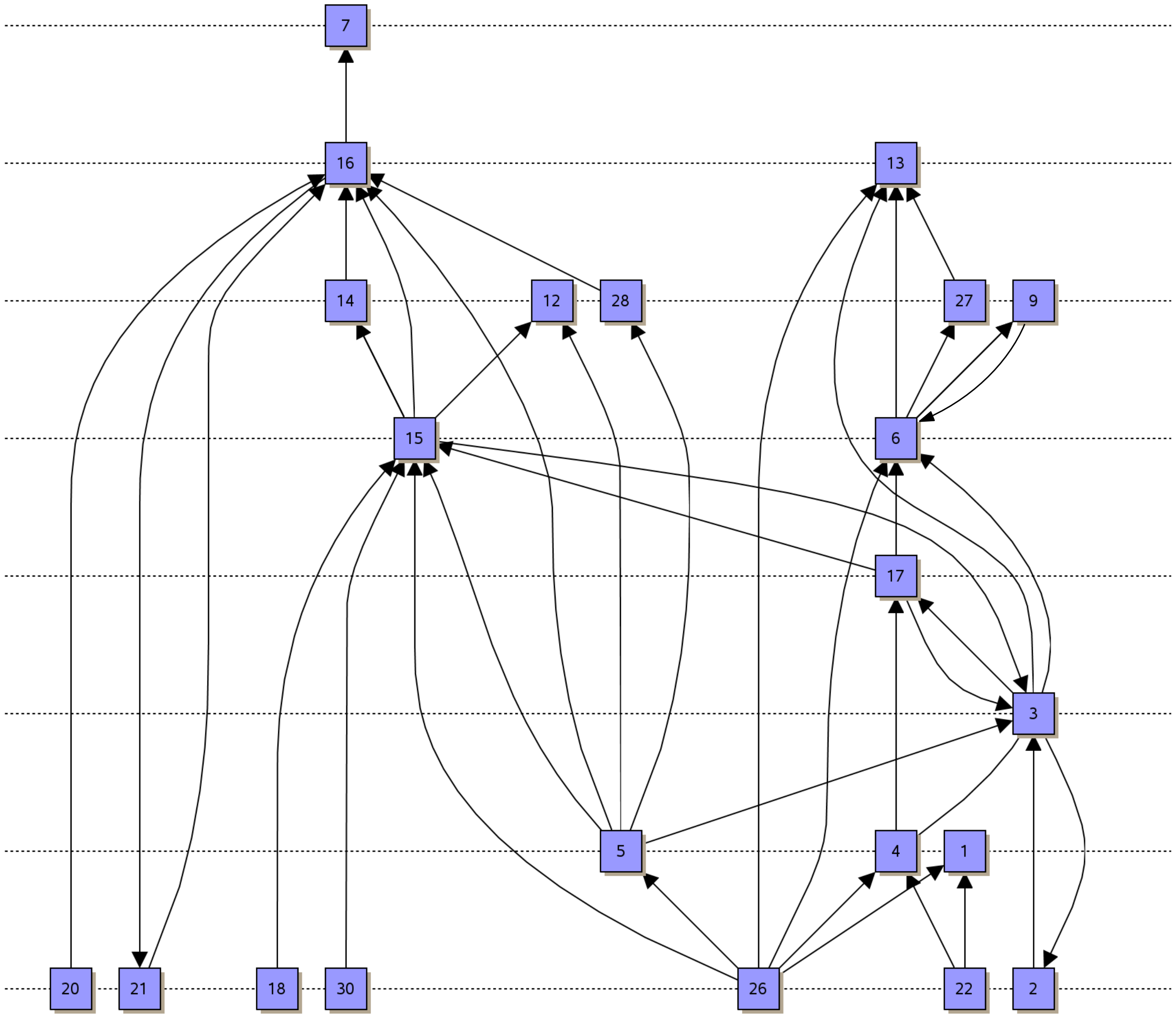
- vertices are evenly spaced.



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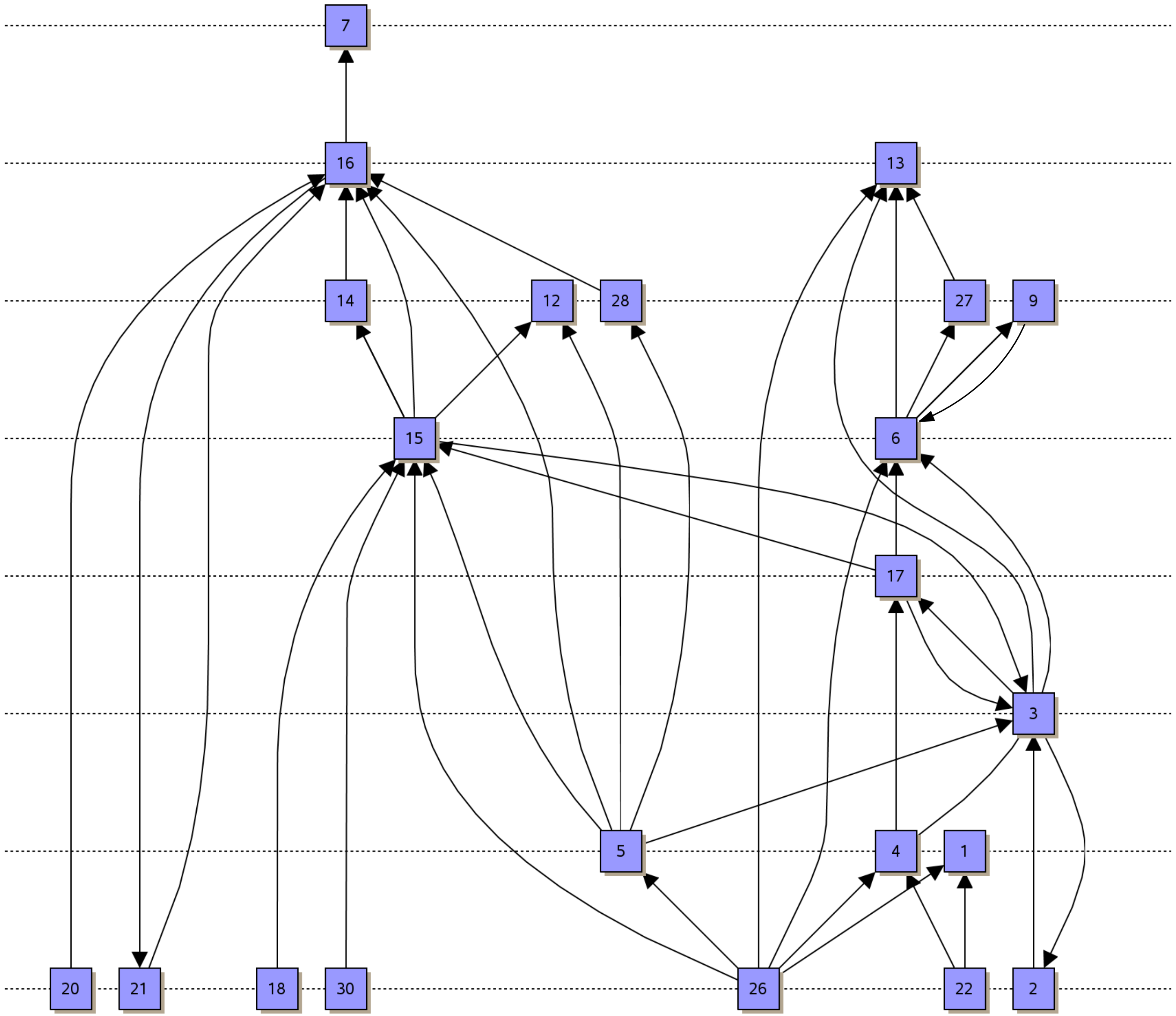
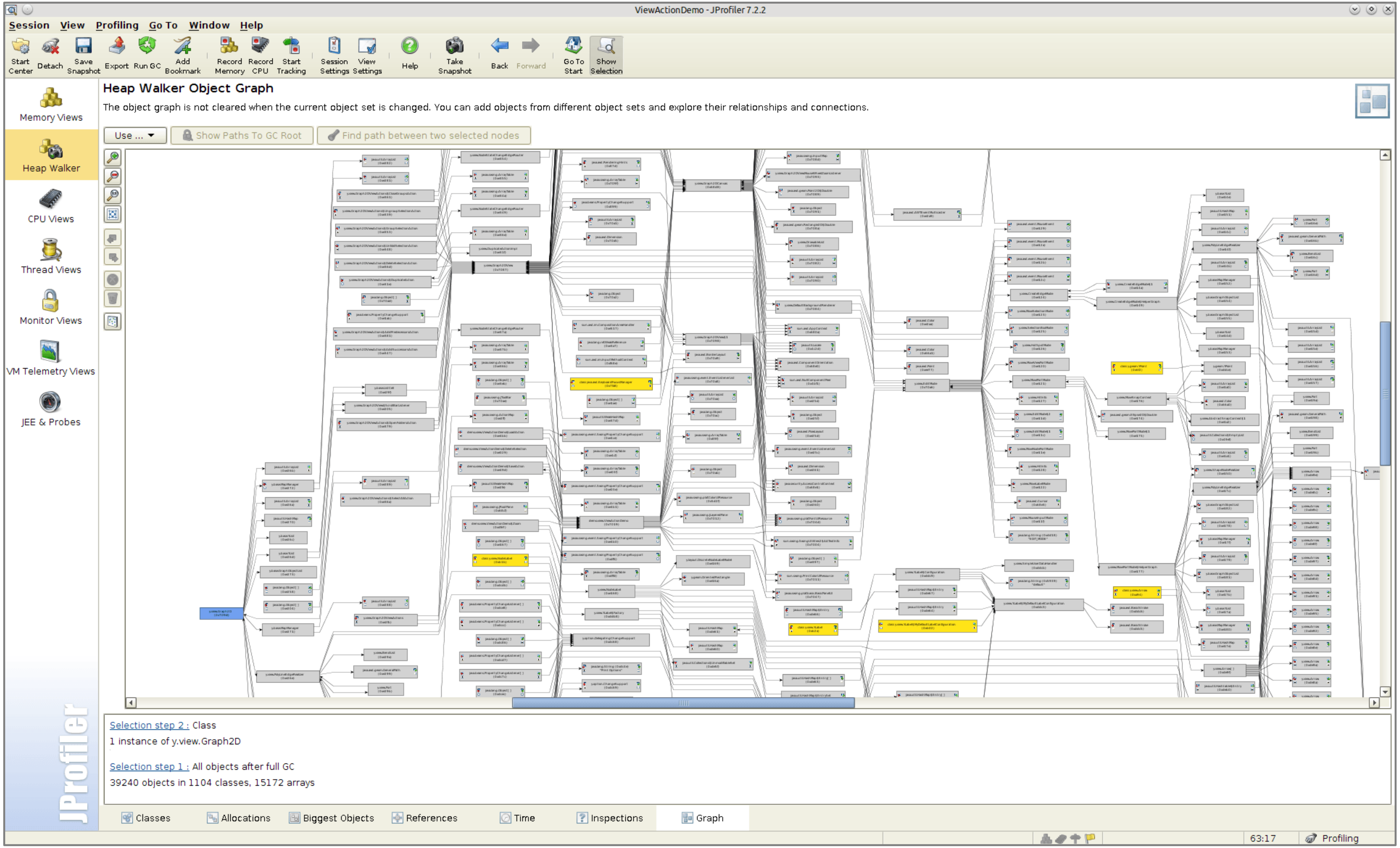
- edges are short,
- vertices are evenly spaced.

Criteria can be contradictory!

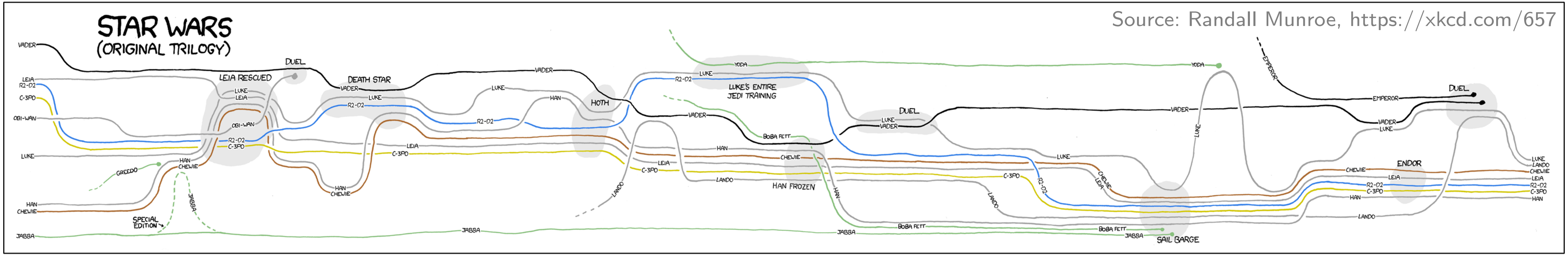


Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles

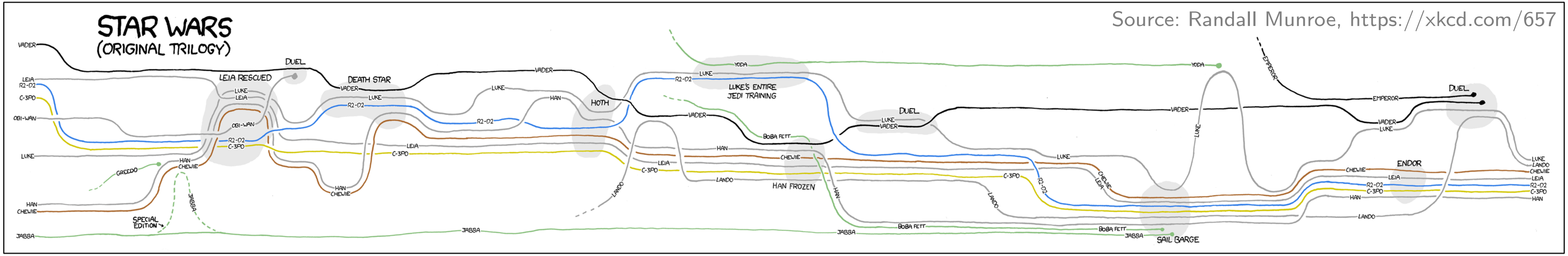


Hierarchical Drawing – Applications



Source: Randall Munroe, https://xkcd.com/657

Hierarchical Drawing – Applications



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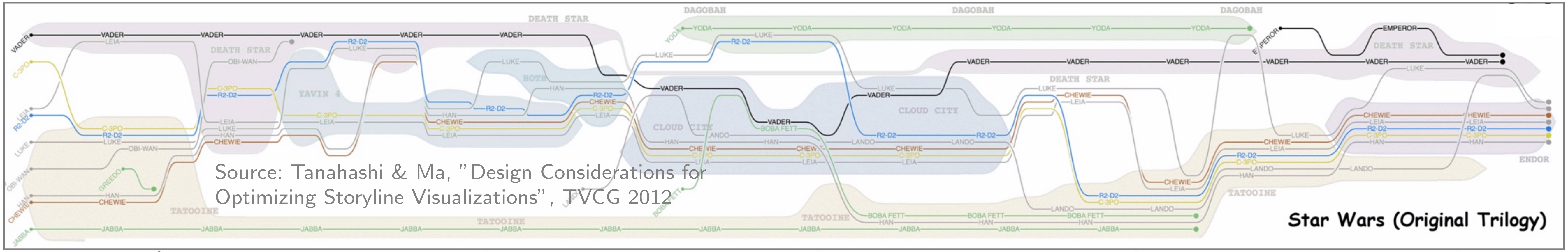
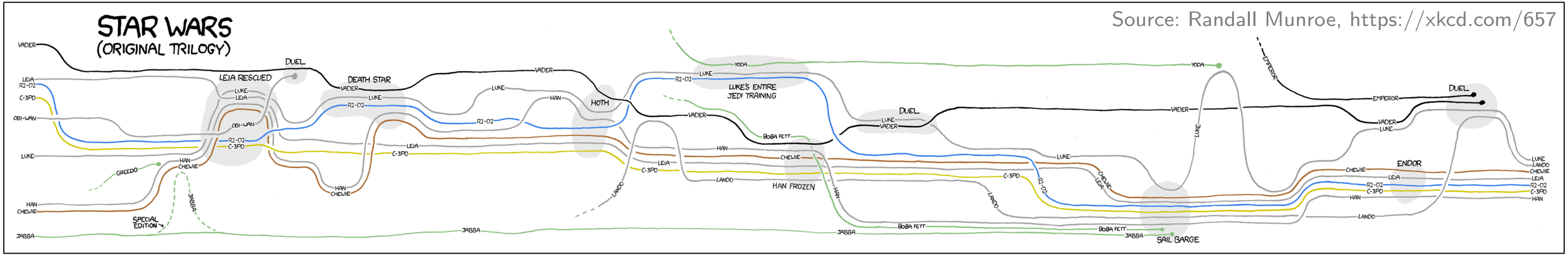


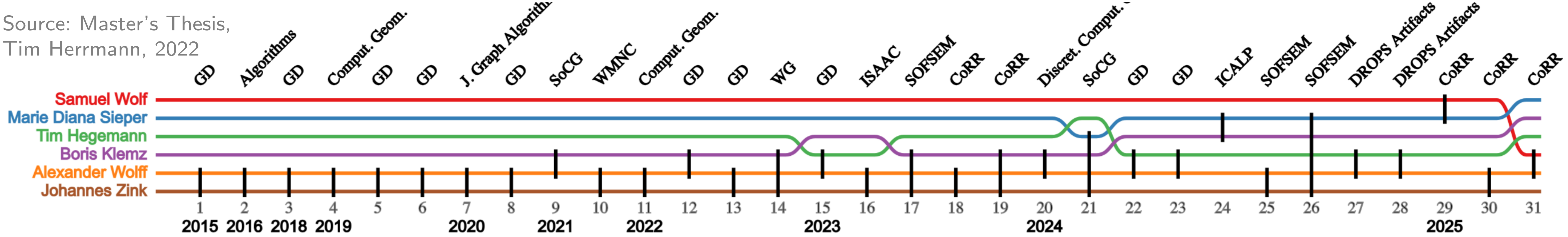
Source: Tanahashi & Ma, "Design Considerations for Optimizing Storyline Visualizations", TVCG 2012

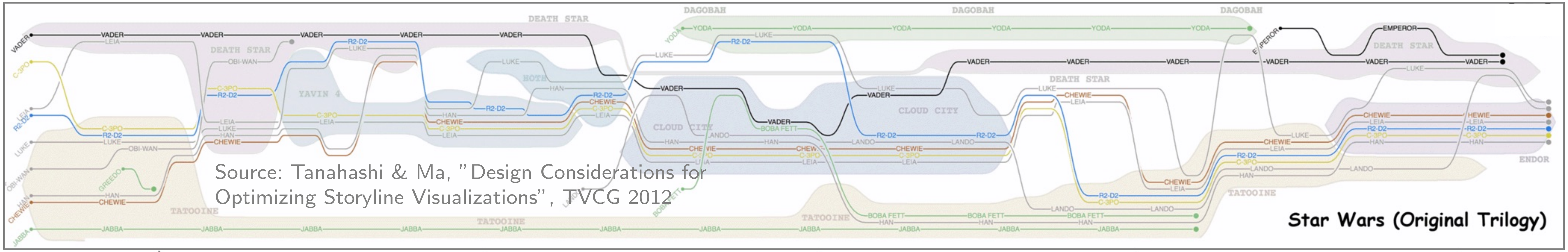
Hierarchical Drawing – Applications



Source: Randall Munroe, https://xkcd.com/657

Source: Master's Thesis, Tim Herrmann, 2022





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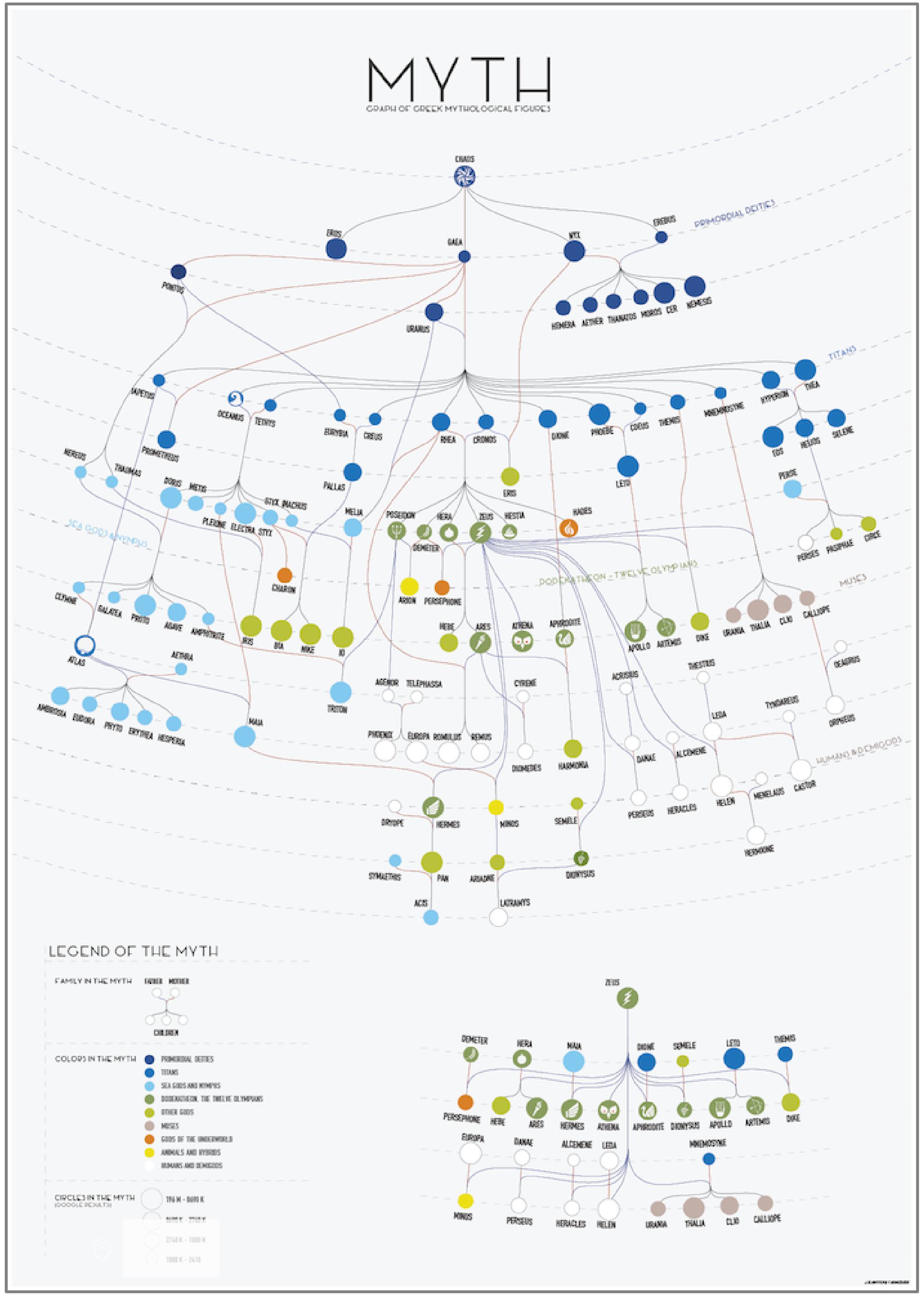
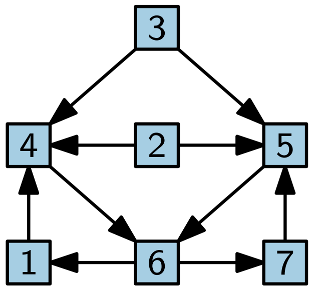
Hierarchical Drawing – Applications



Source: Visualization that won the Graph Drawing Contest, Creative Track, 2016. Klawitter & Mchedlidze

Classical Approach: Sugiyama Framework [Sugiyama, Tagawa, Toda '81]

Input



Classical Approach: Sugiyama Framework [Sugiyama, Tagawa, Toda '81]

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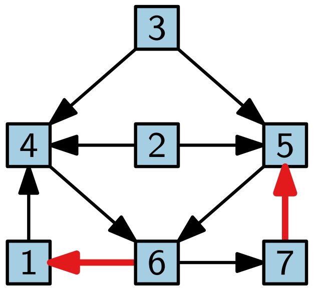

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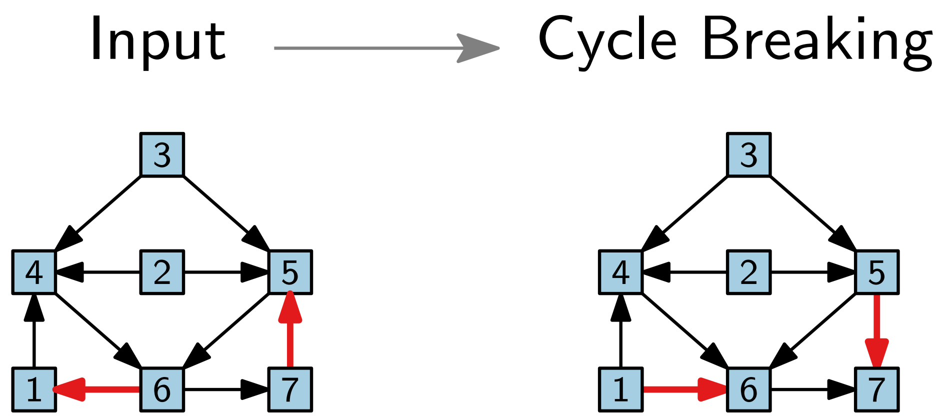

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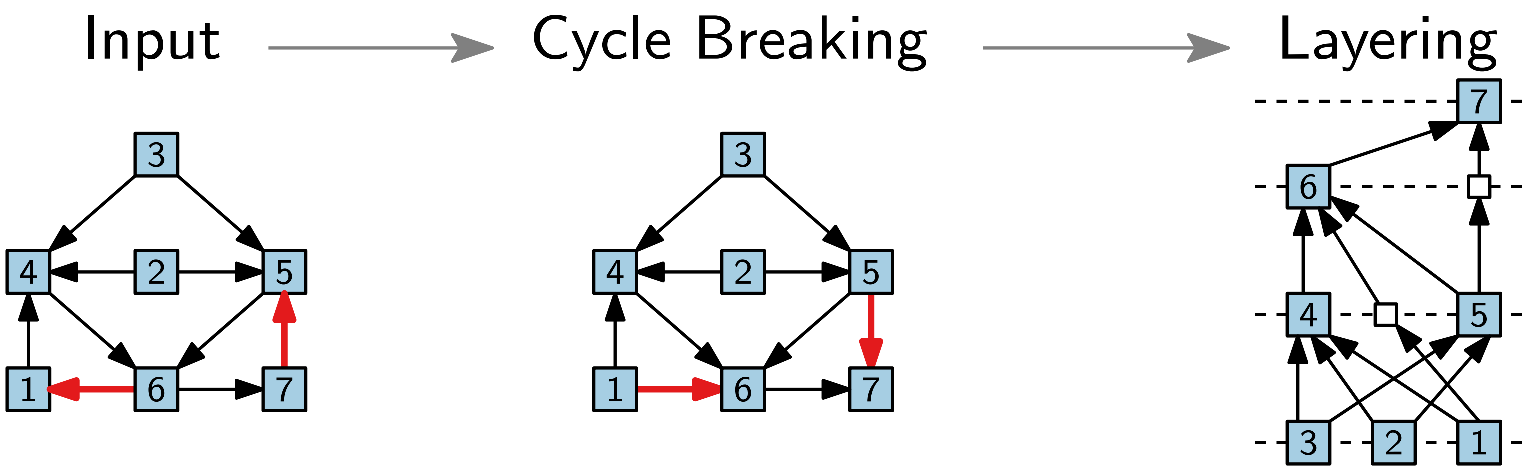

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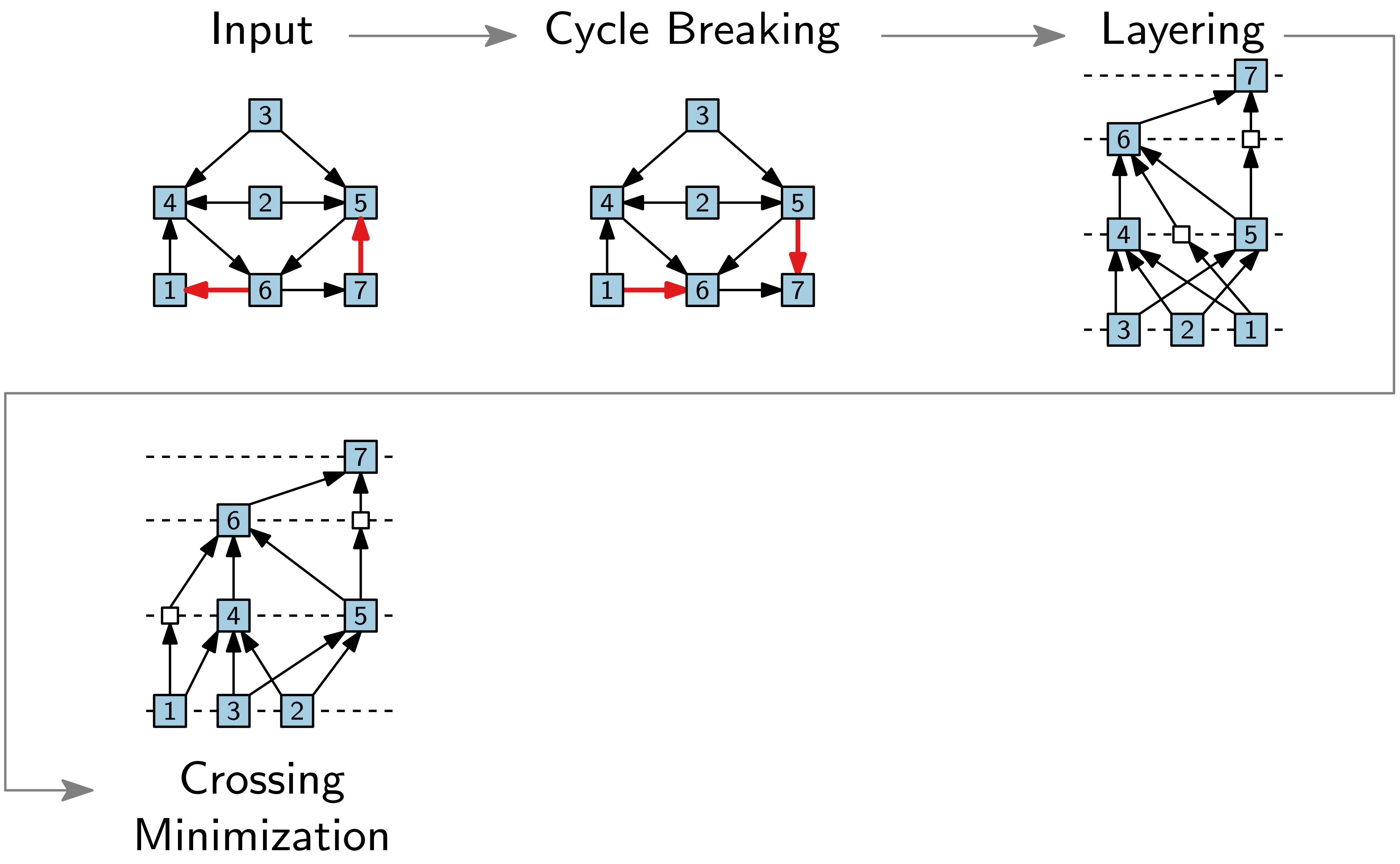

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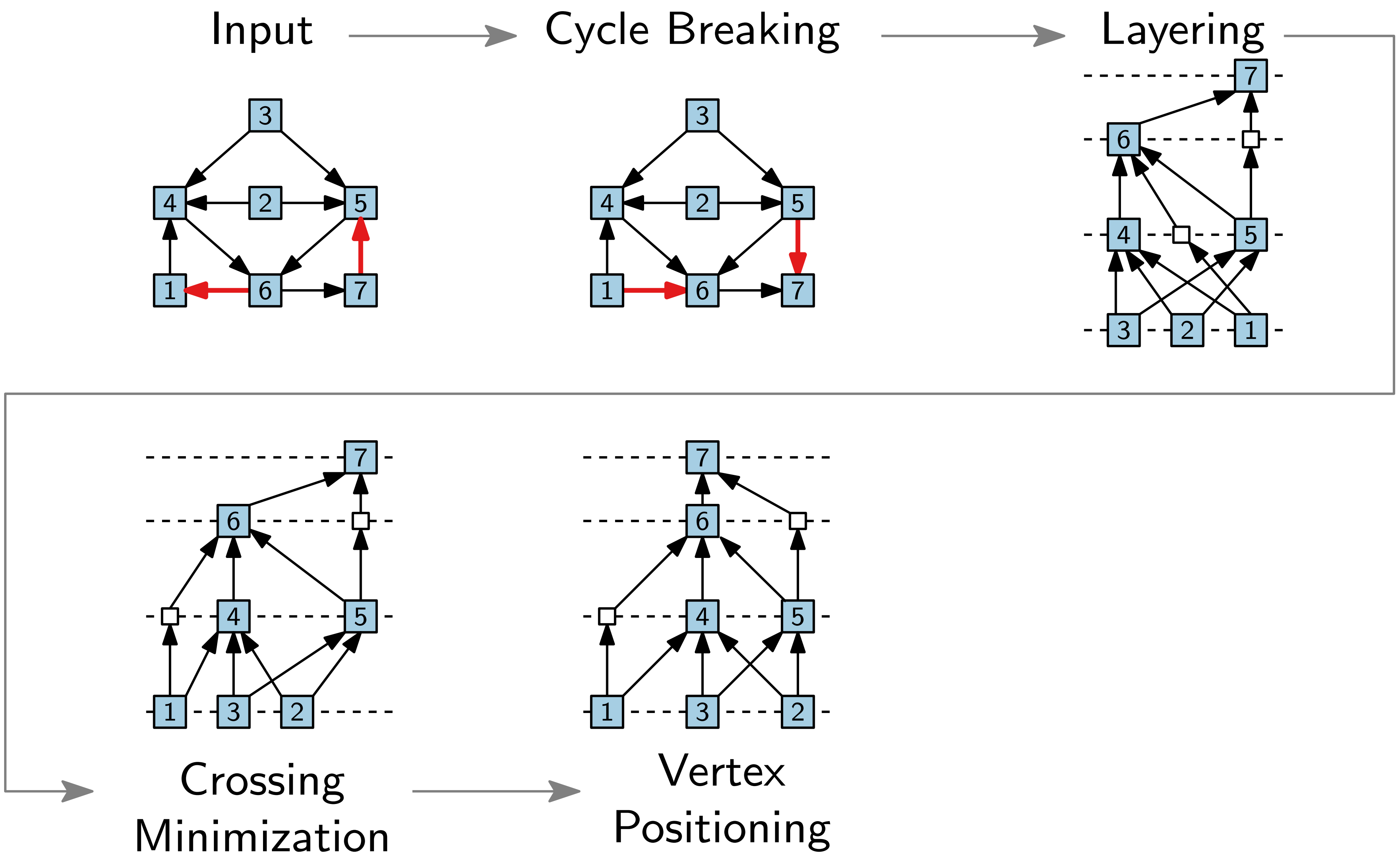

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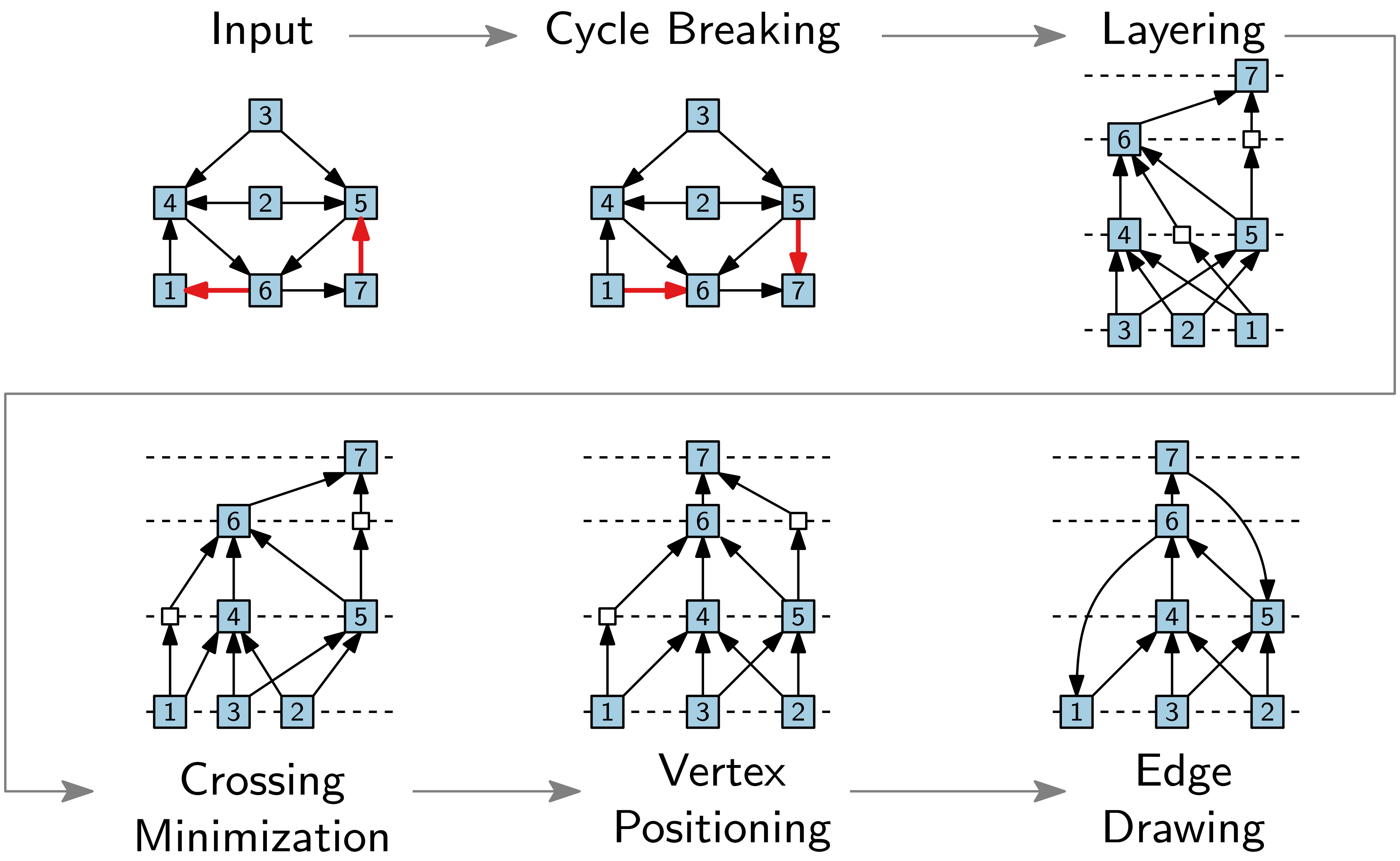
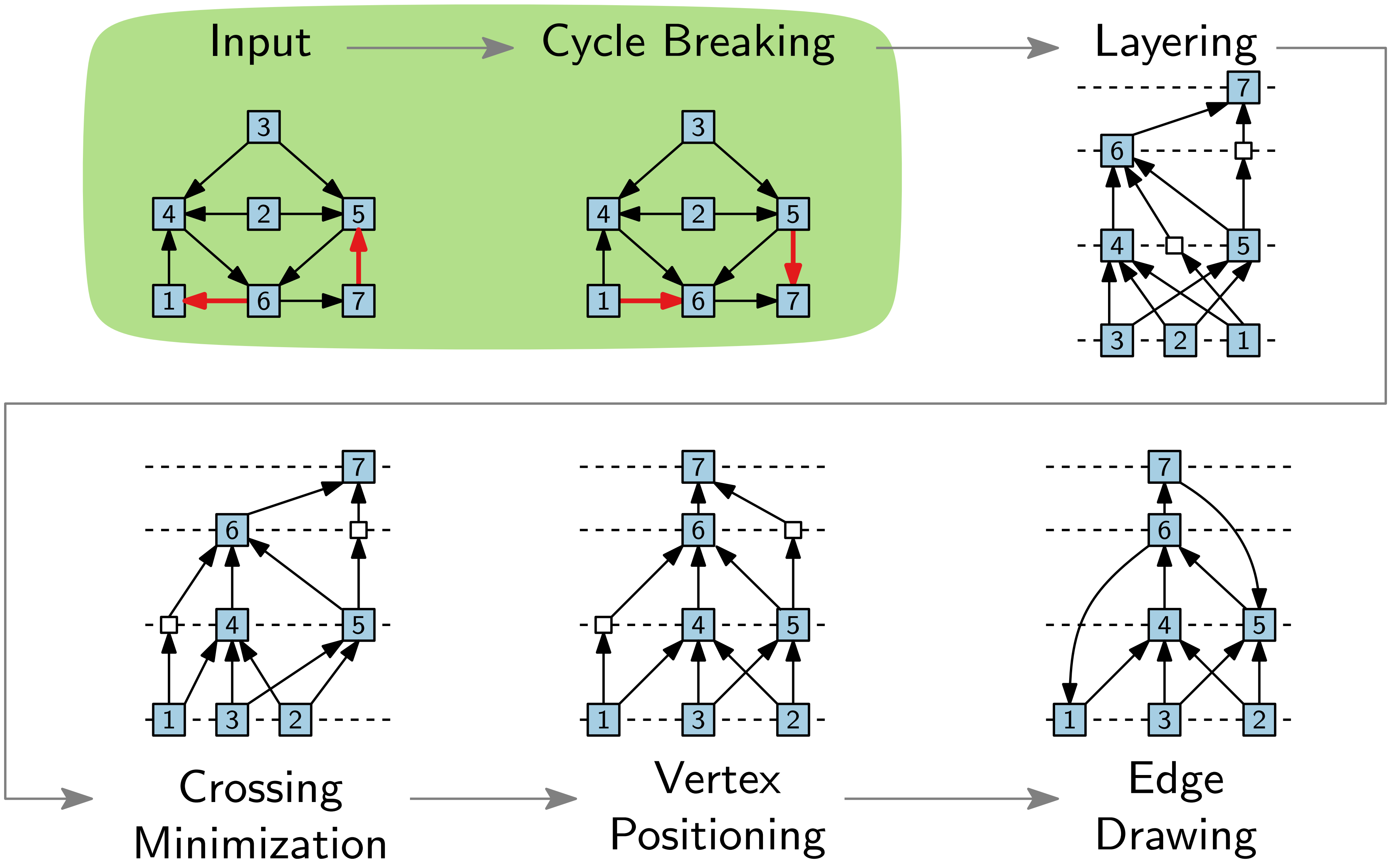


Step 1: Cycle Breaking



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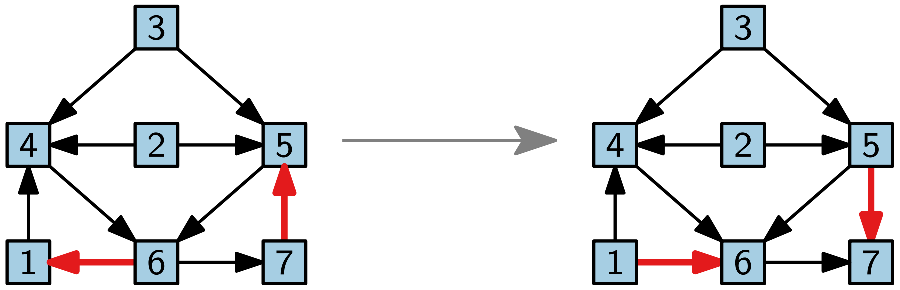
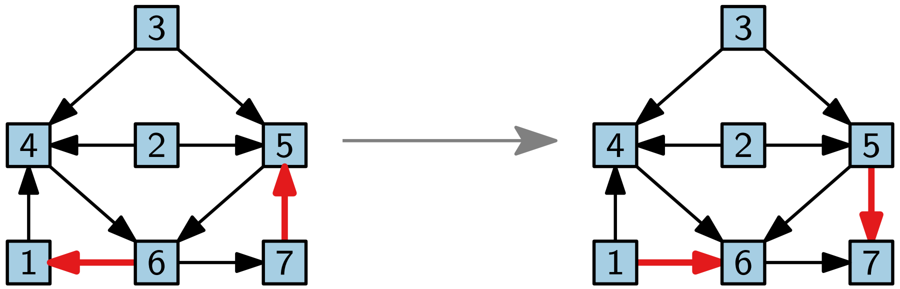


Approach.

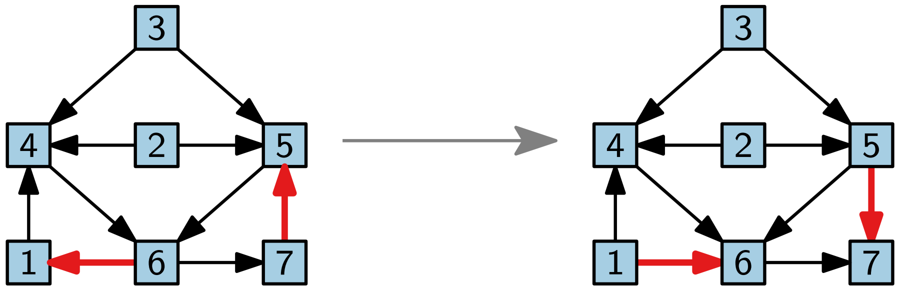
Step 1: Cycle Breaking



Approach.

- Find minimum-size set $E^{\star}$ of edges that are not upward.

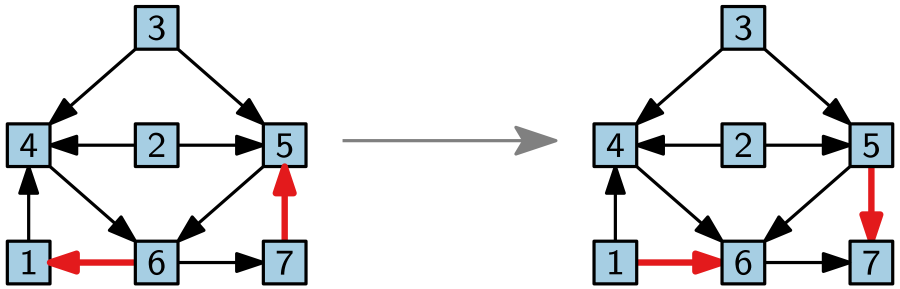
Step 1: Cycle Breaking



Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
- Remove $E^\star$ and insert reversed edges.

Step 1: Cycle Breaking



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Problem Minimum Feedback Arc Set **(FAS).**

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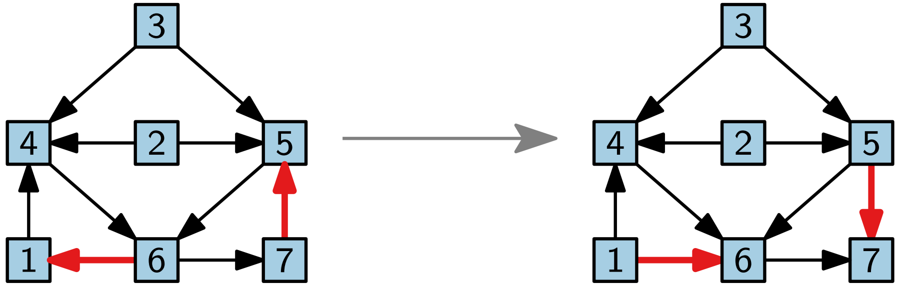

Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
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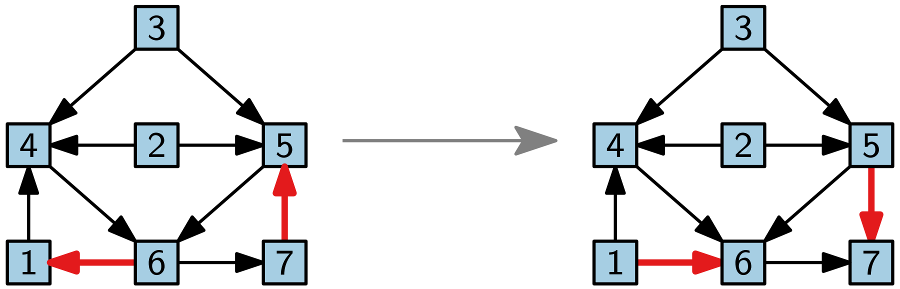
Problem $\textsc{Minimum Feedback Arc Set}$ **(FAS).**

- Input: directed graph G
- Output:

Step 1: Cycle Breaking



Approach.

- Find minimum-size set $E^\star$ of edges that are not upward.
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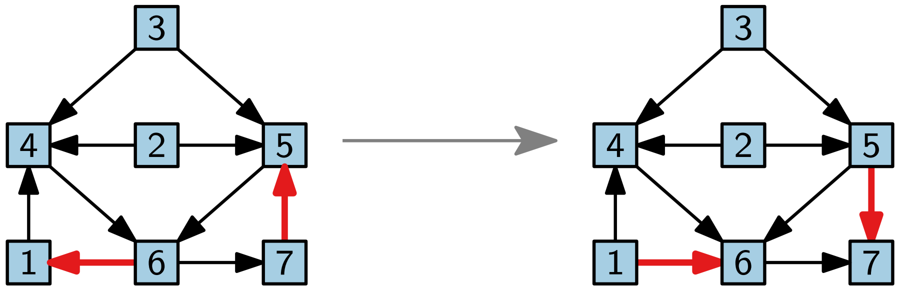
Problem Minimum Feedback Arc Set (FAS).

- Input: directed graph G
- Output: minimum-size set $E^\star \subseteq E(G)$ such that $G^\star = (V(G), E(G) \setminus E^\star)$ acyclic

Step 1: Cycle Breaking



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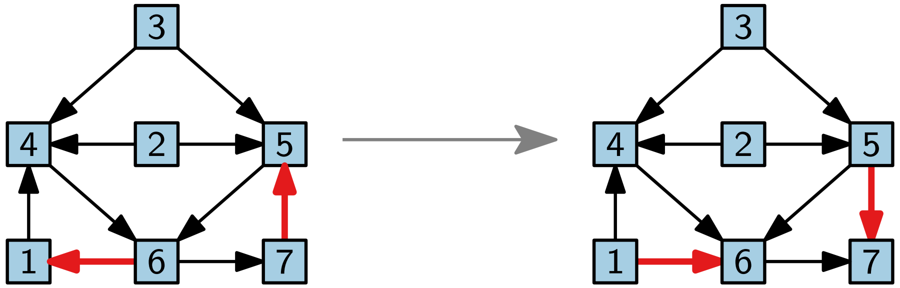
$$(E(G) \setminus E^\star) \cup E^\star_{\text{rev}}$$

Step 1: Cycle Breaking



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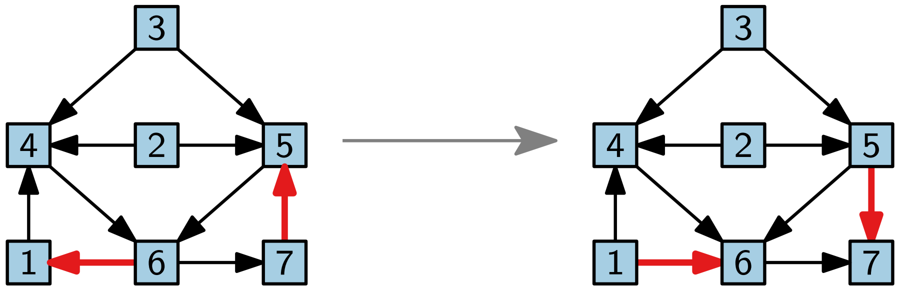
edges in $E^\star$ but reversed

$$(E(G) \setminus E^\star) \cup E^\star_{\text{rev}}$$

Step 1: Cycle Breaking



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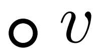
edges in $E^\star$ but reversed

$$(E(G) \setminus E^\star) \cup \boxed{E^\star_{\text{rev}}}$$

. . . NP-hard ☹

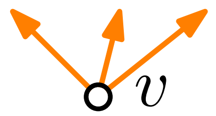
Heuristic 1

[Berger, Shor '90]



Heuristic 1

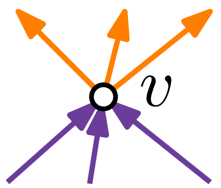
[Berger, Shor '90]



$$E^{\rightarrow}(v) \quad := \quad \{(v,u)\colon (v,u) \in E(G)\}$$

Heuristic 1

[Berger, Shor '90]



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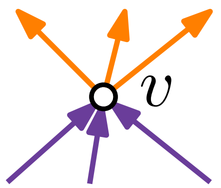

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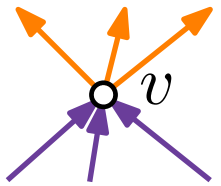
Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph G):



$$
\begin{aligned}
E^{\rightarrow}(v) &:= \{(v,u)\colon (v,u) \in E(G)\} \\
E^{\leftarrow}(v) &:= \{(u,v)\colon (u,v) \in E(G)\} \\
E(v) &:= E^{\rightarrow}(v) \cup E^{\leftarrow}(v)
\end{aligned}
$$

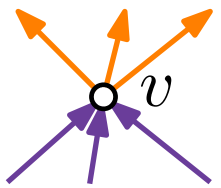
Heuristic 1

[Berger, Shor '90]

GreedyMakeAcyclic(Digraph G):

$E' \leftarrow \emptyset$

return $G' = (V(G), E')$



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Heuristic 1

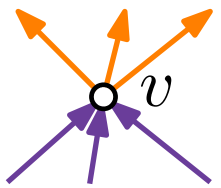
[Berger, Shor '90]

GreedyMakeAcyclic(Digraph G):

$E' \leftarrow \emptyset$

foreach $v \in V(G)$ **do**

return $G' = (V(G), E')$



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Heuristic 1

[Berger, Shor '90]

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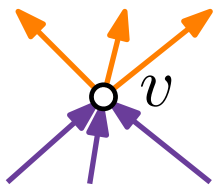
return $G' = (V(G), E')$



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[Berger, Shor '90]

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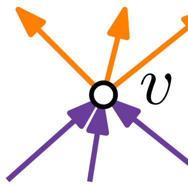
foreach $v \in V(G)$ **do**

  if $|E^{\rightarrow}(v)| \geq |E^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup E^{\rightarrow}(v)$

return $G' = (V(G), E')$



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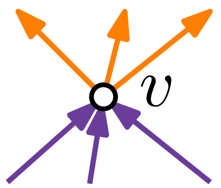

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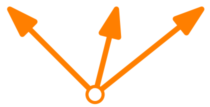

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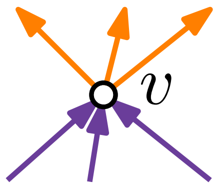

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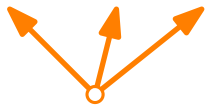

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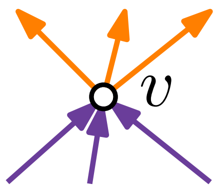

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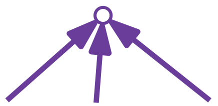

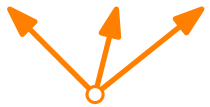

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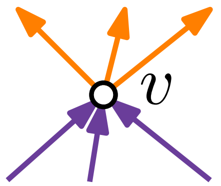

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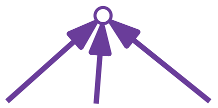

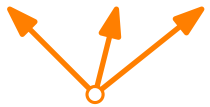

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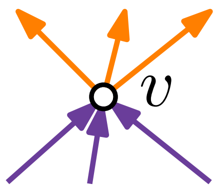

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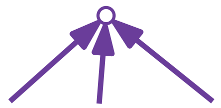

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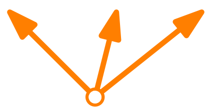

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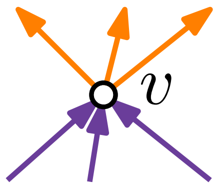

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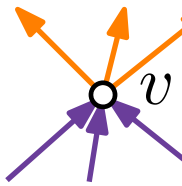

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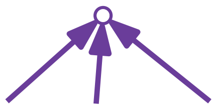

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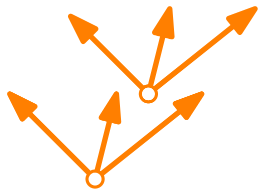

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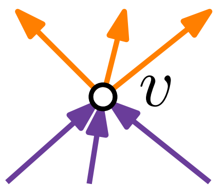

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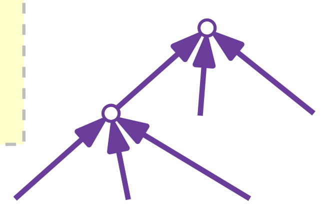

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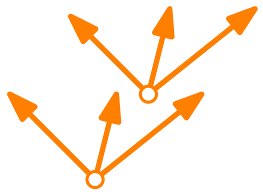

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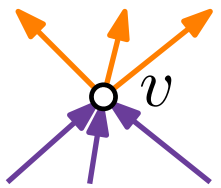

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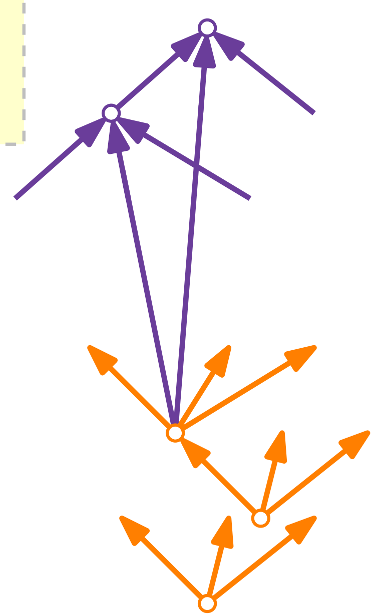

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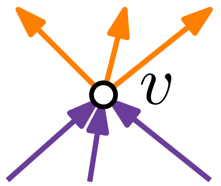

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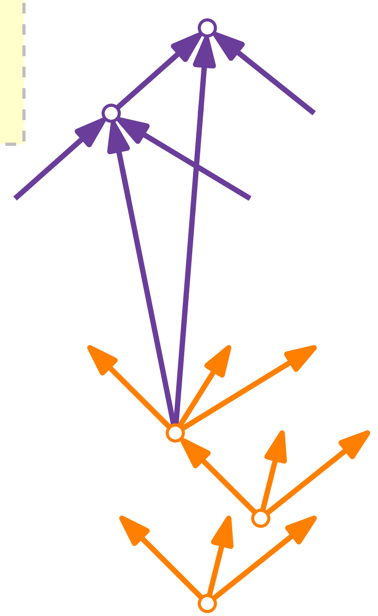
Place the vertices on distinct y-coordinates.



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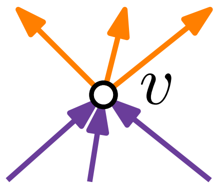

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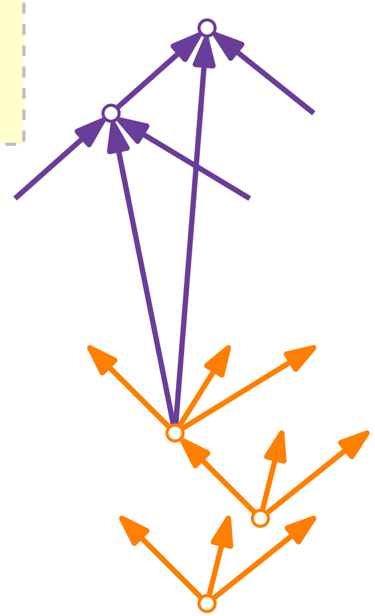

Place the vertices on distinct y-coordinates.

y-coordinates increase/decrease towards the middle.

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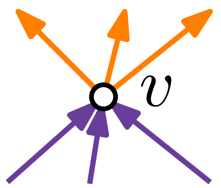

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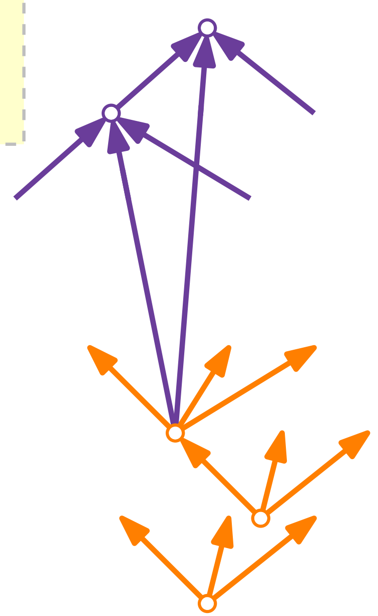

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All edges point upwards.

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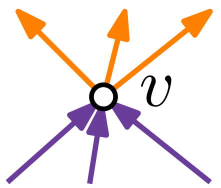

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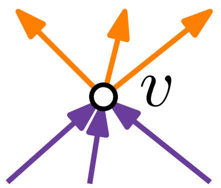

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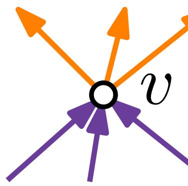

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Use the vertex order from before (edges in E' upwards).

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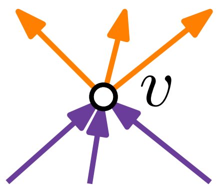

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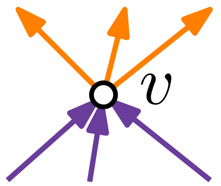

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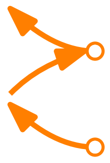

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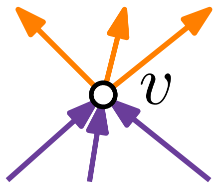

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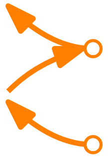
- G' is a DAG.
- $E(G) \setminus E'$ is a feedback set.



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[Berger, Shor '90]

GreedyMakeAcyclic(Digraph G):

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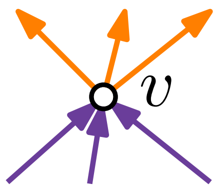

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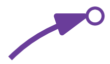

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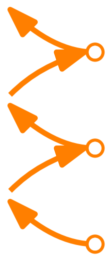

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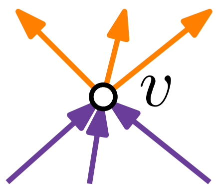

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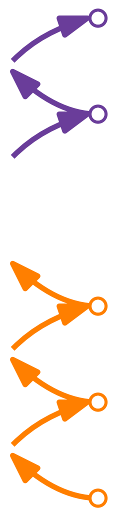

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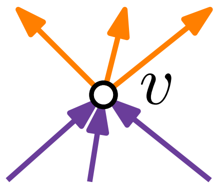

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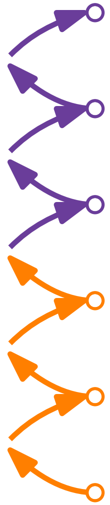

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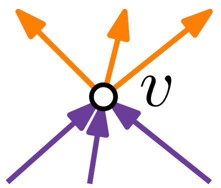

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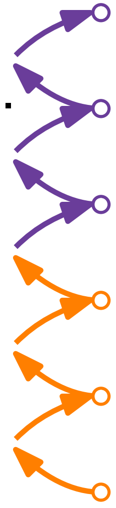
Use the vertex order from before (edges in E' upwards).

In this order, add the edges of $E(G) \setminus E'$ in rev. direction.



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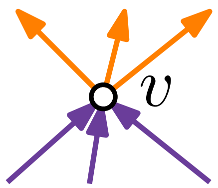

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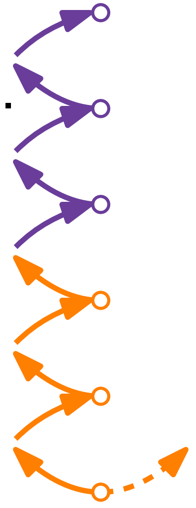

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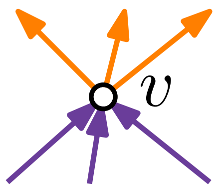

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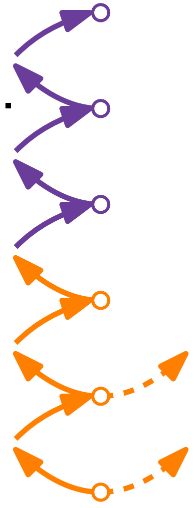

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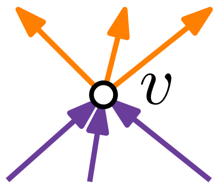

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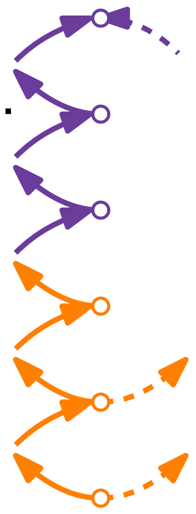

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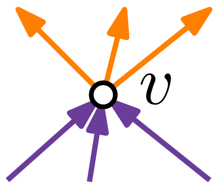

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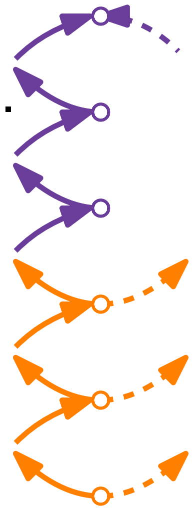

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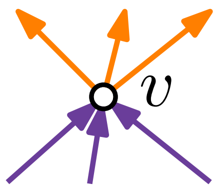

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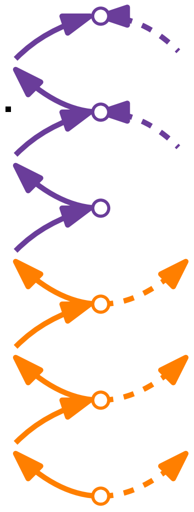

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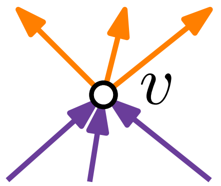

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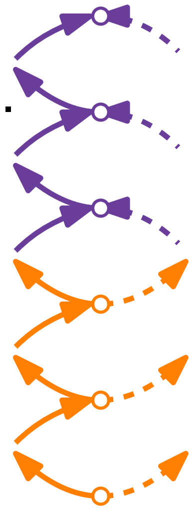

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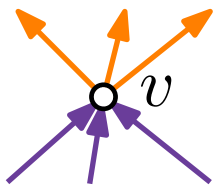

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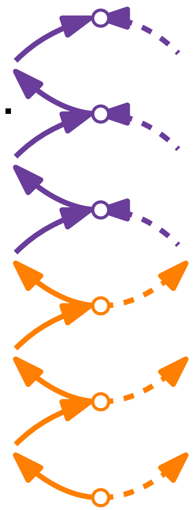
Added edges have other endpoint more in the middle.

$\rightarrow$ All edges point upwards.



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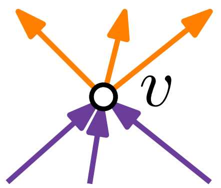

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- Runtime:

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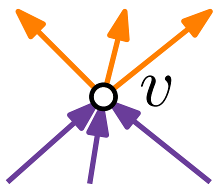

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- Runtime: $\mathcal{O}(|V(G)| + |E(G)|)$

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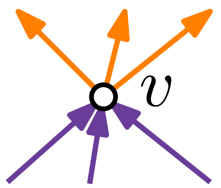

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- Runtime: $\mathcal{O}(|V(G)| + |E(G)|)$
- Quality guarantee: $|E'| \geq$

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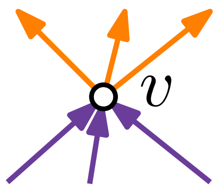

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- Runtime: $\mathcal{O}(|V(G)| + |E(G)|)$
- Quality guarantee: $|E'| \geq |E(G)|/2$

- G' is a DAG.
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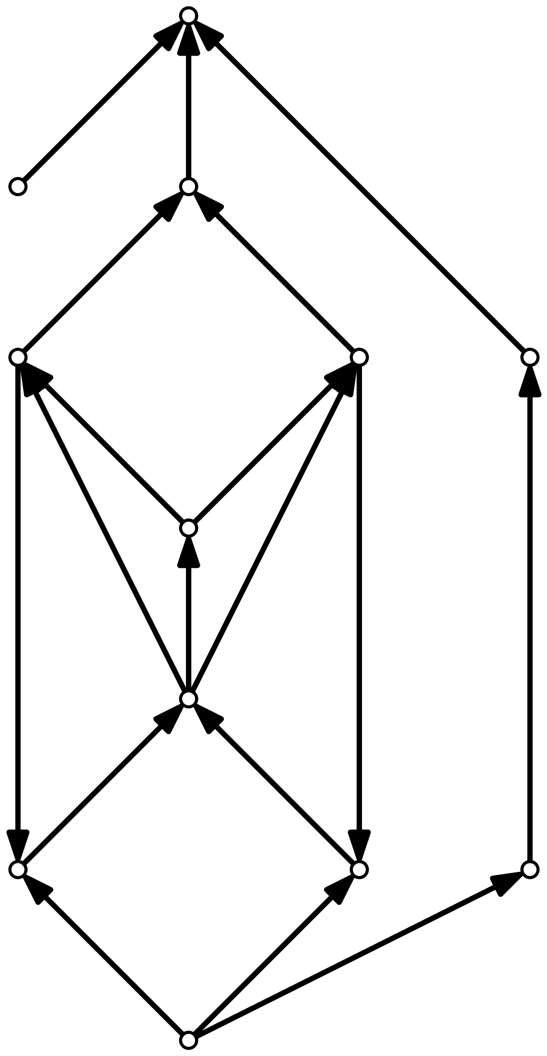
Heuristic 2

[Eades, Lin, Smyth '93]



Heuristic 2

[Eades, Lin, Smyth '93]

GreedyMakeAcyclic2(Digraph G):

$E' \leftarrow \emptyset$

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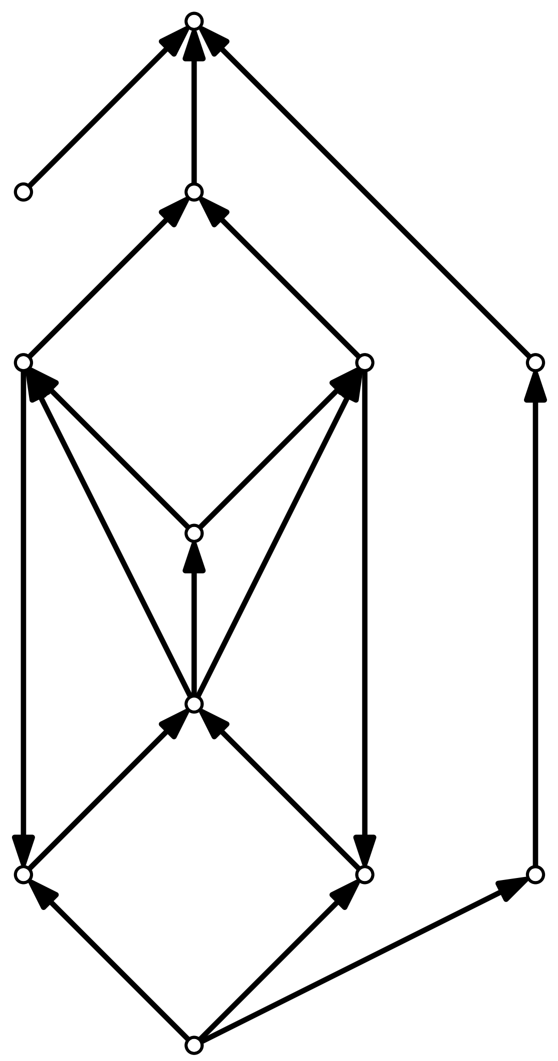


Heuristic 2

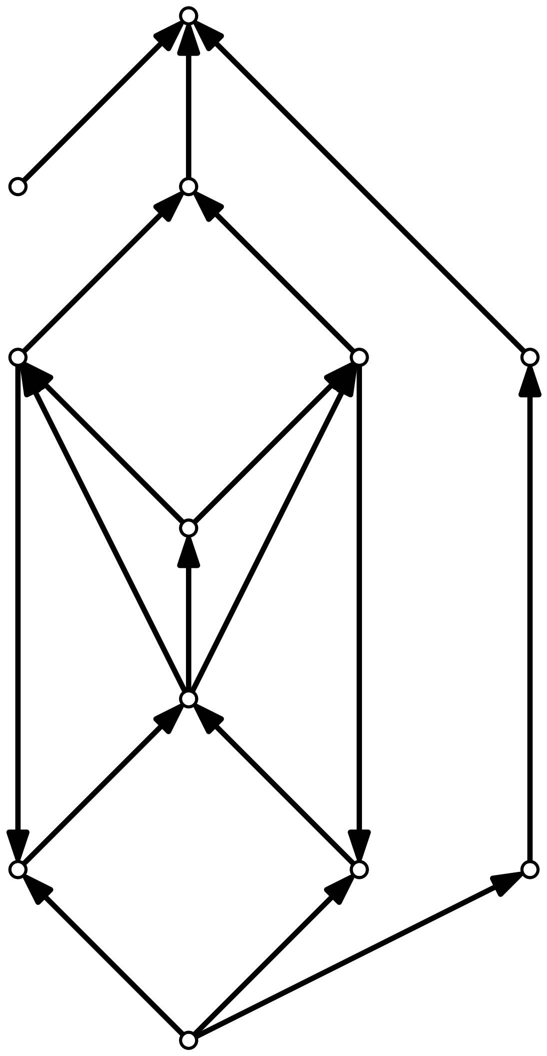
[Eades, Lin, Smyth '93]

GreedyMakeAcyclic2(Digraph G):

$E' \leftarrow \emptyset$

while $V(G) \neq \emptyset$ **do**

return $G' = (V(G), E')$



Heuristic 2

[Eades, Lin, Smyth '93]

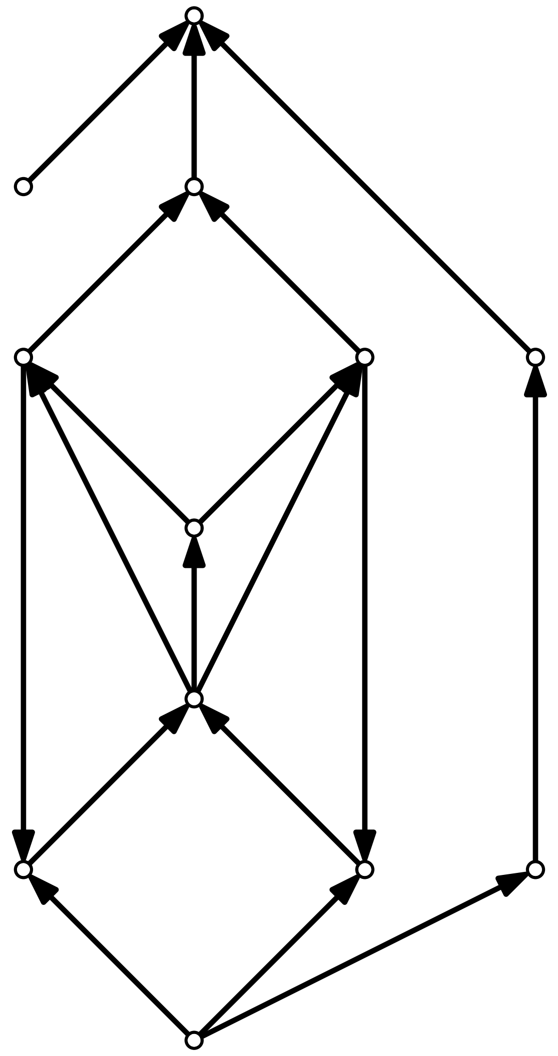
GreedyMakeAcyclic2(Digraph G):

$E' \leftarrow \emptyset$

while $V(G) \neq \emptyset$ **do**

  while $V(G)$ contains a sink v **do**

return $G' = (V(G), E')$



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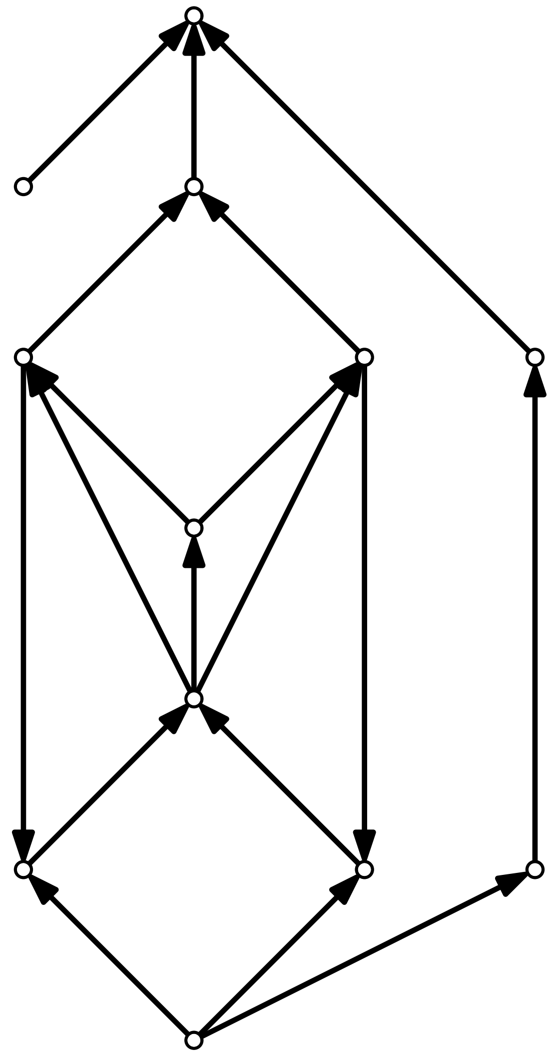

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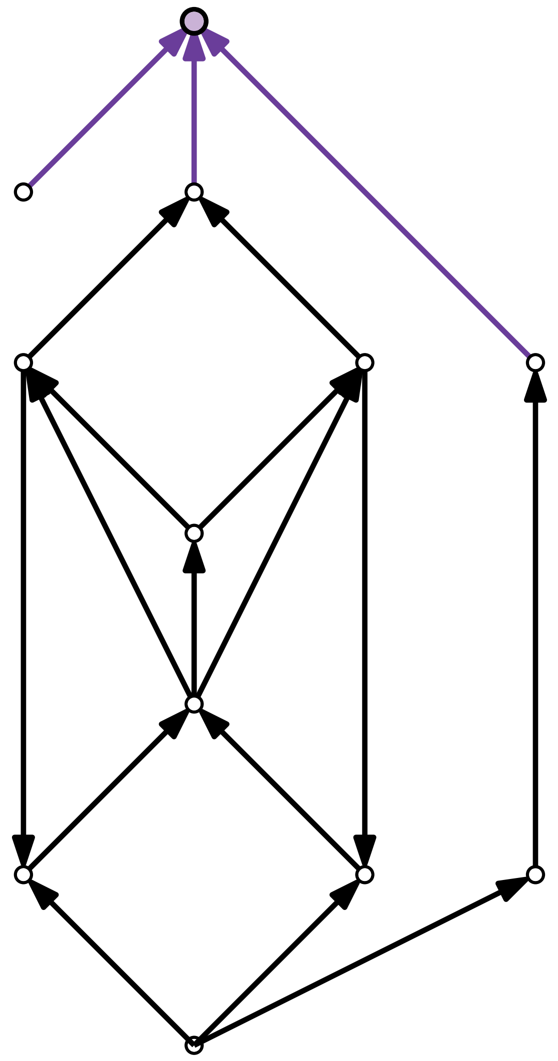

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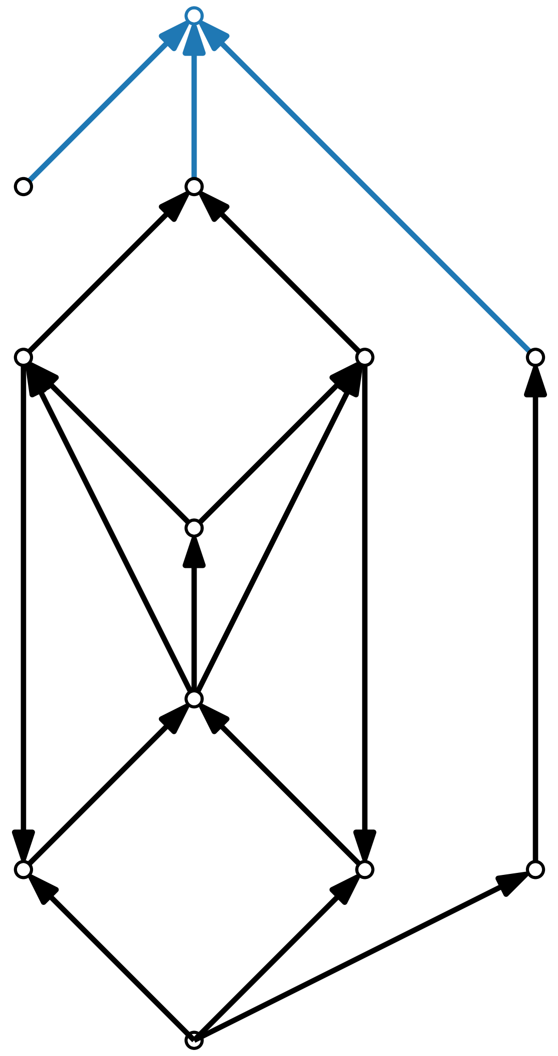

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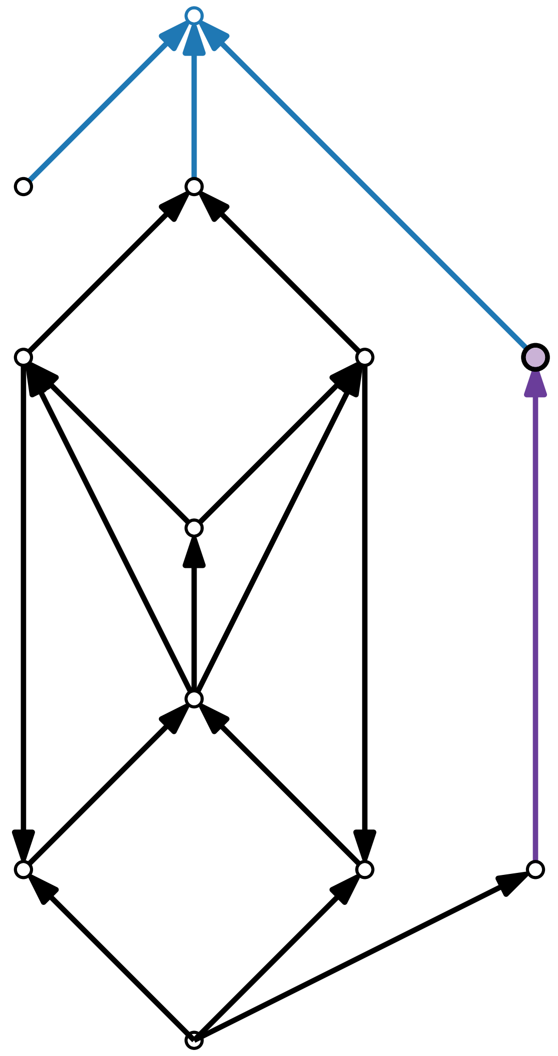

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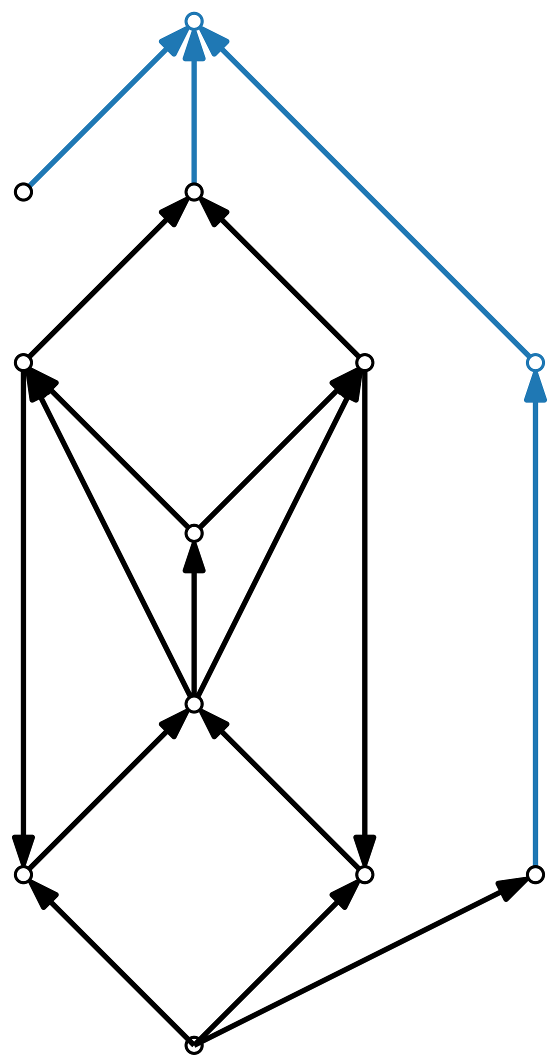

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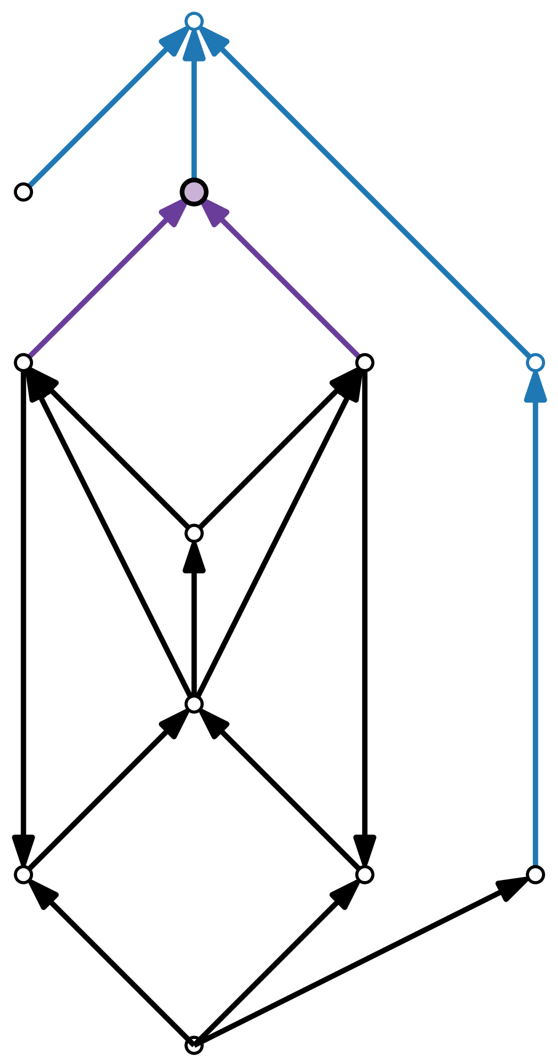

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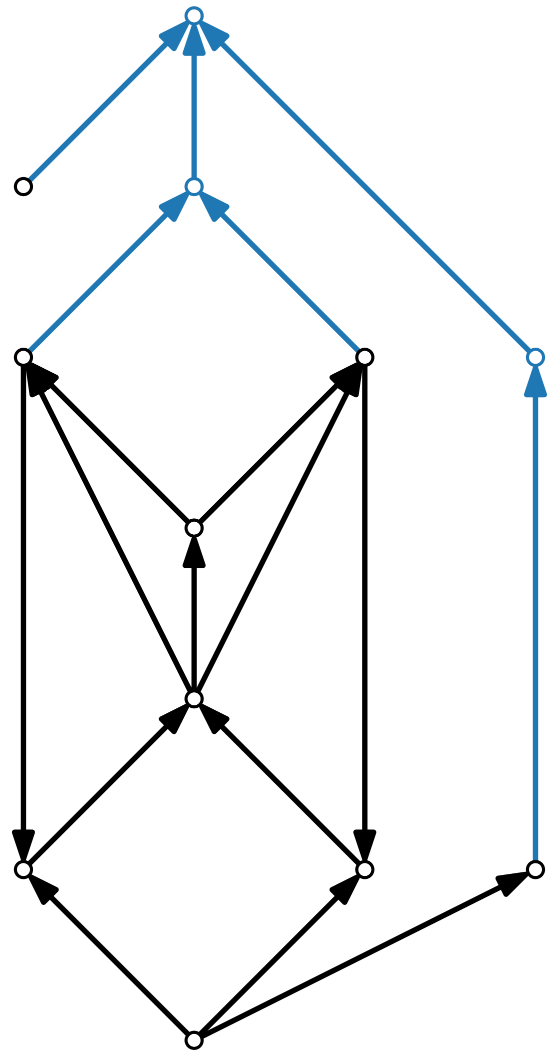

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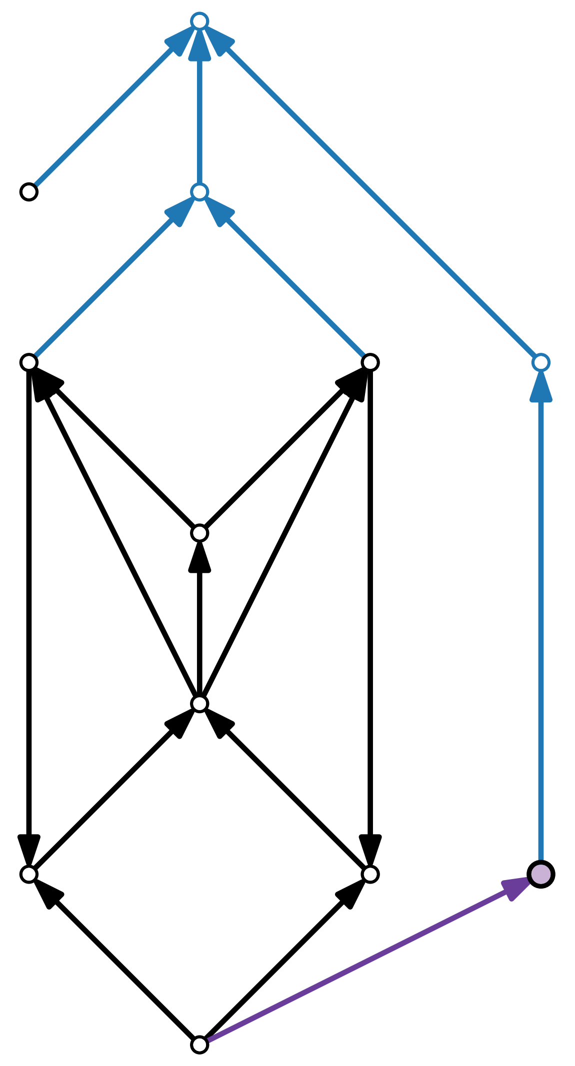

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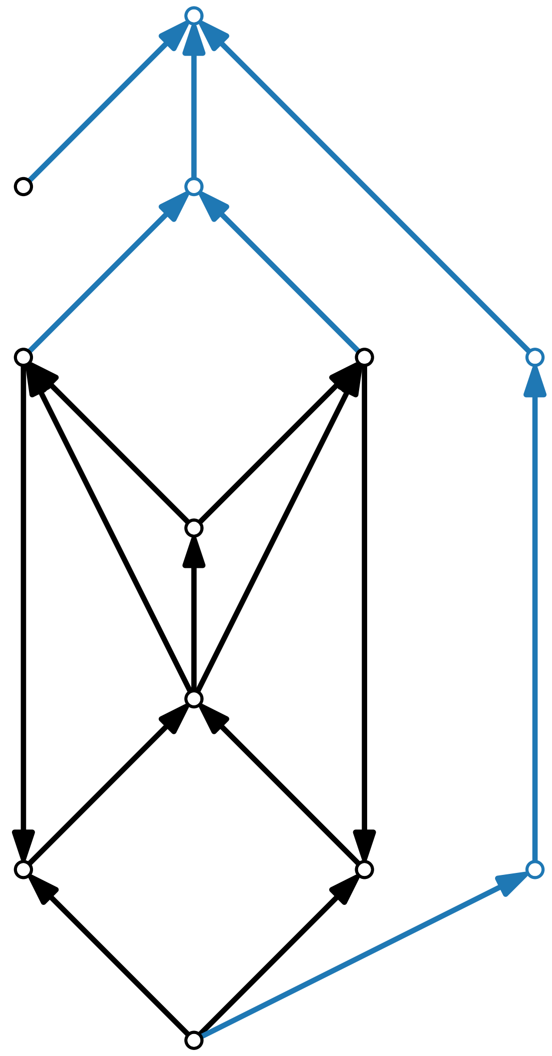

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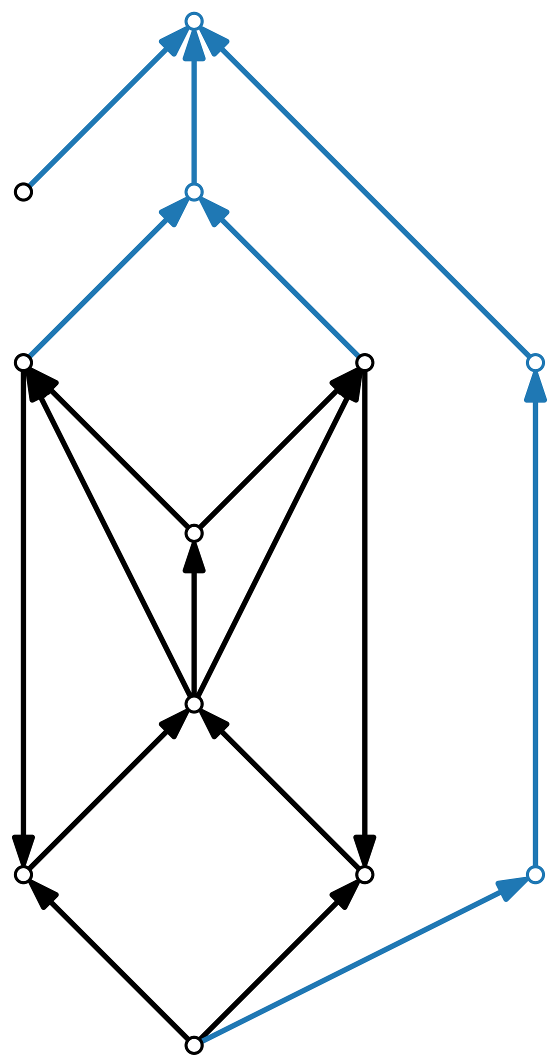

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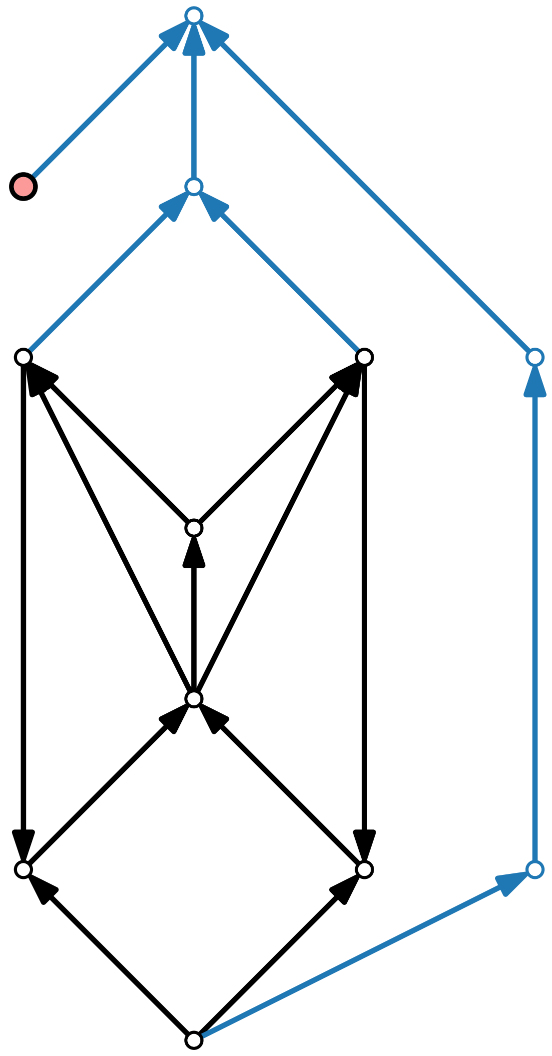

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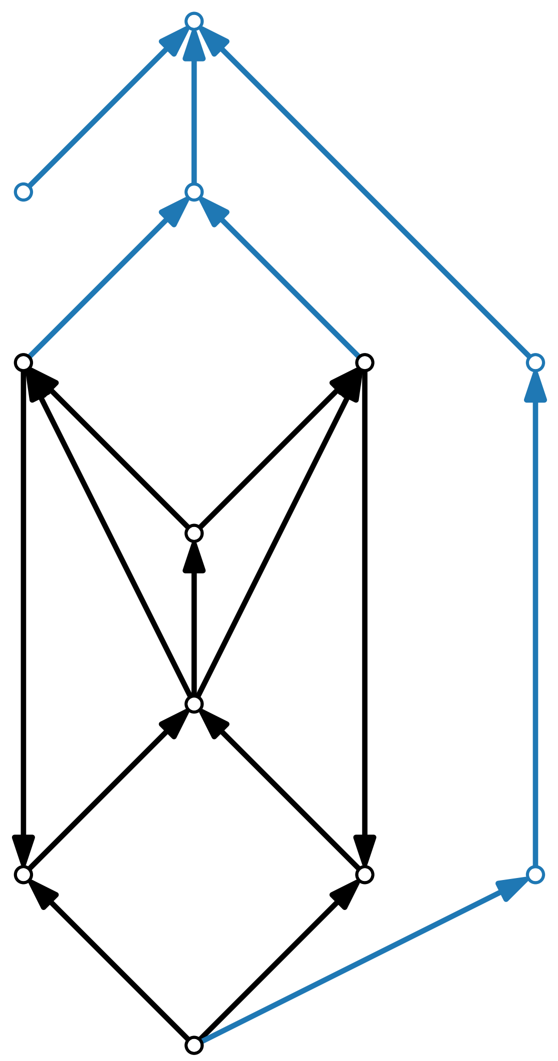

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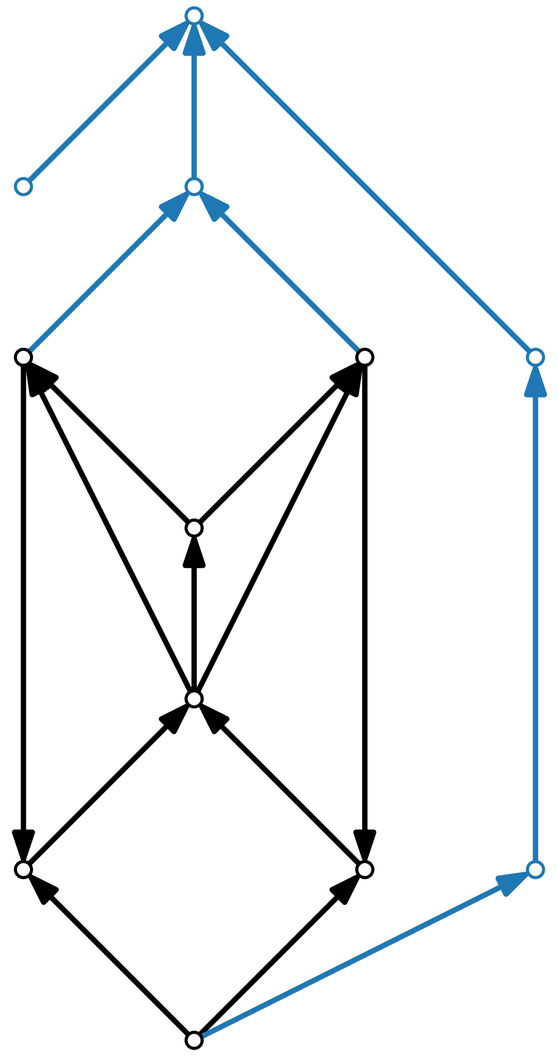

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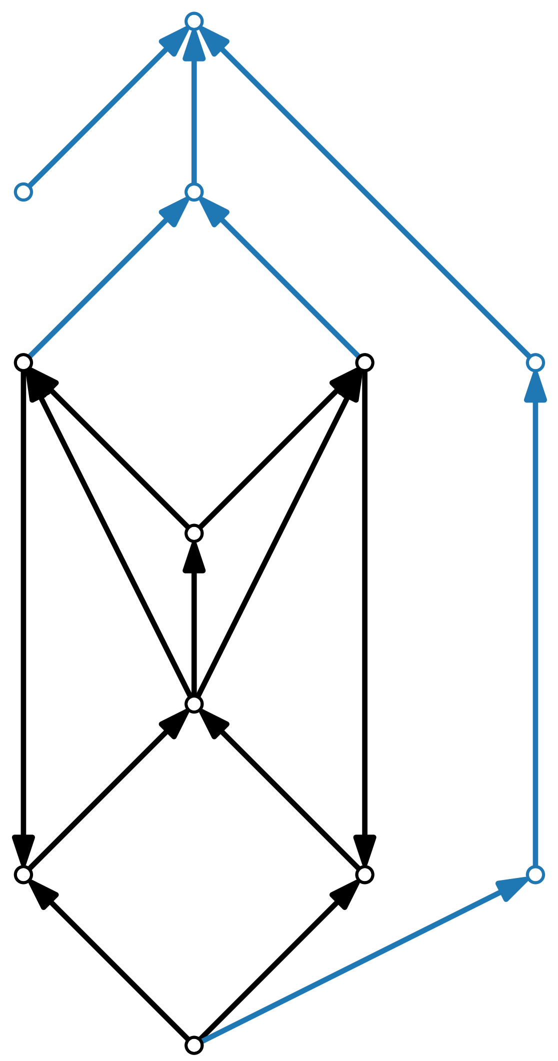

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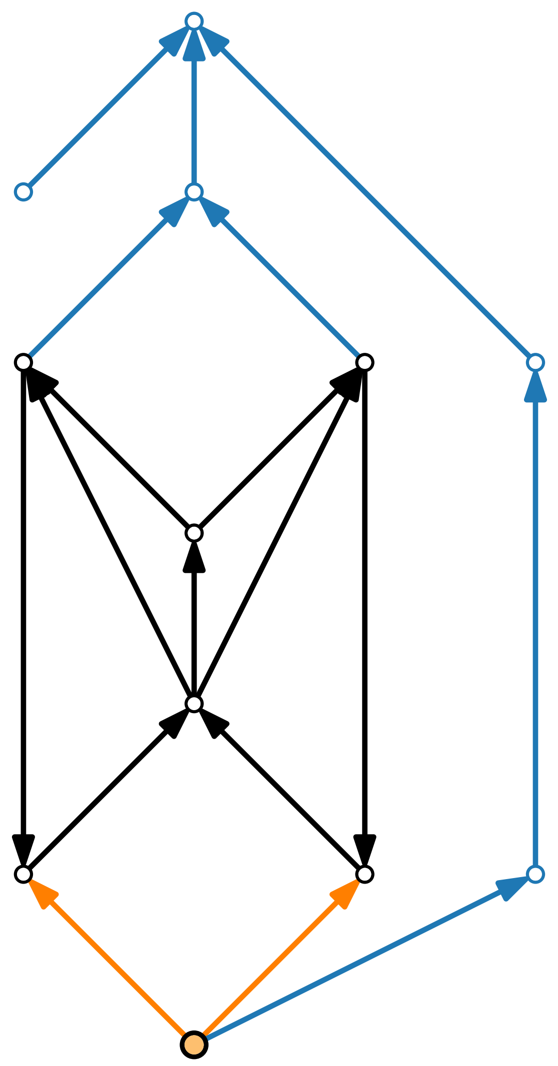

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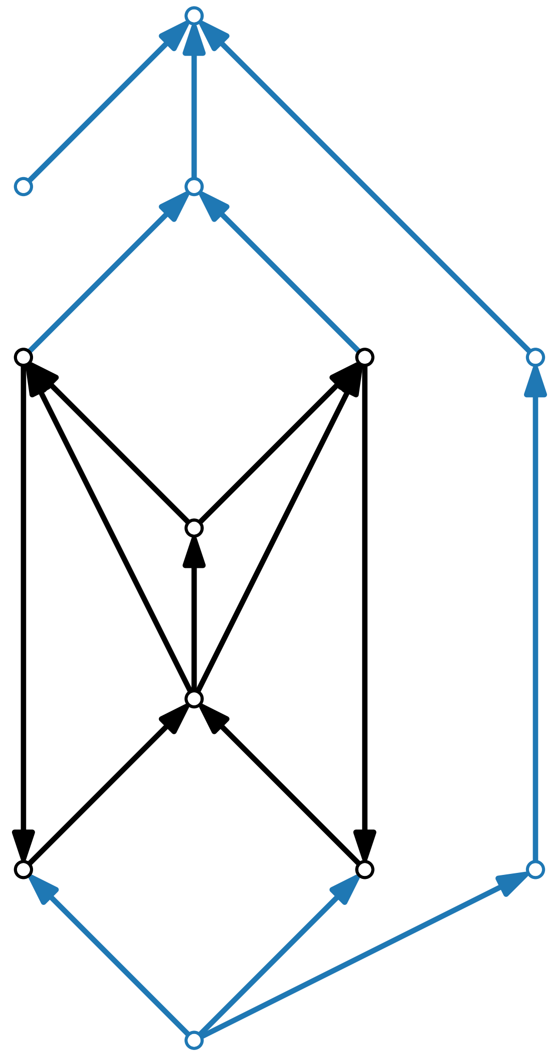

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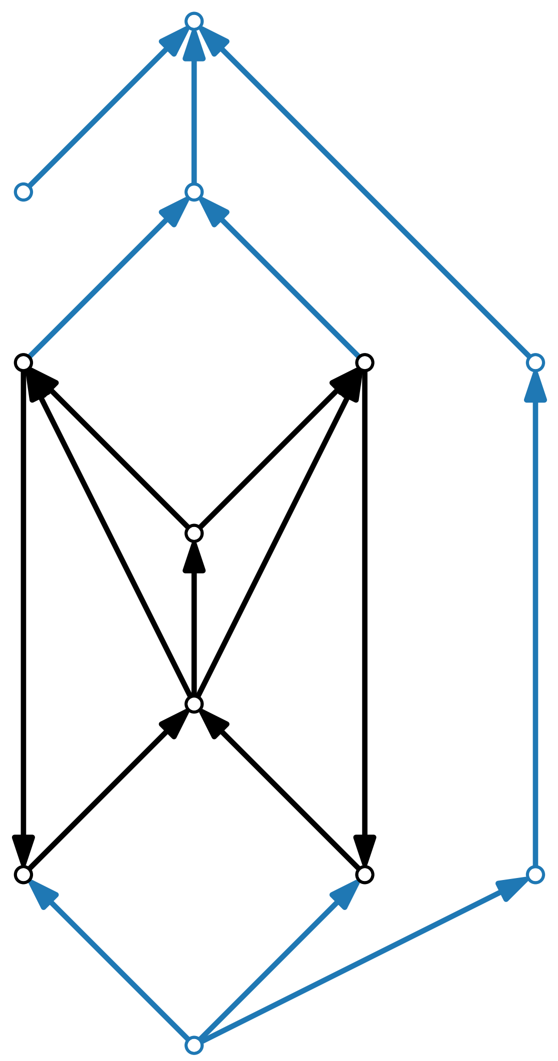

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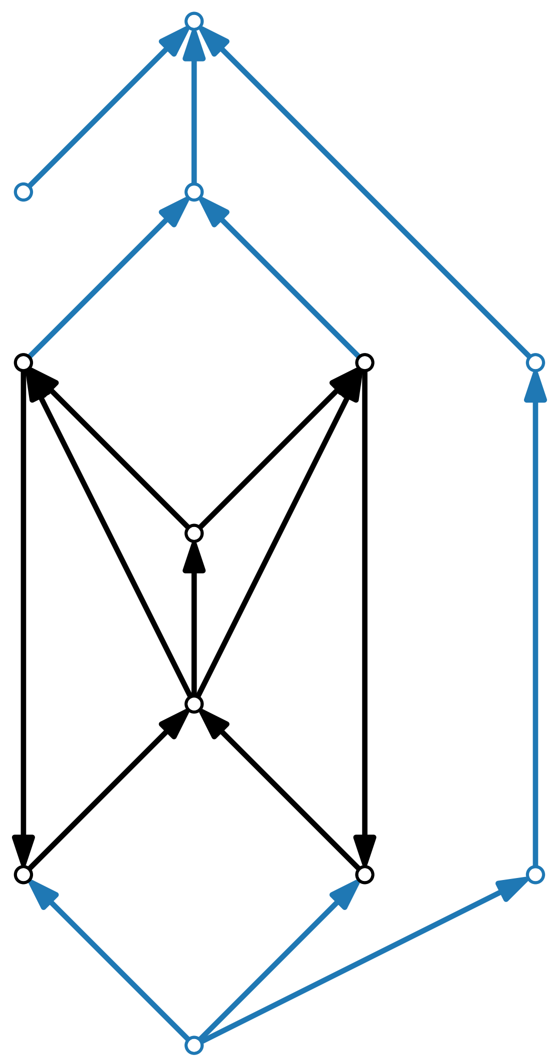

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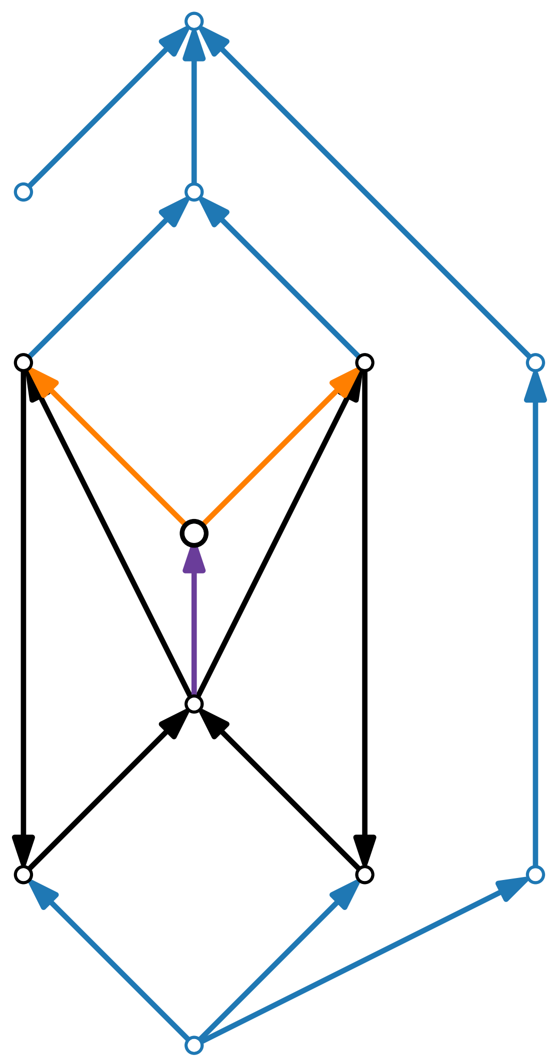

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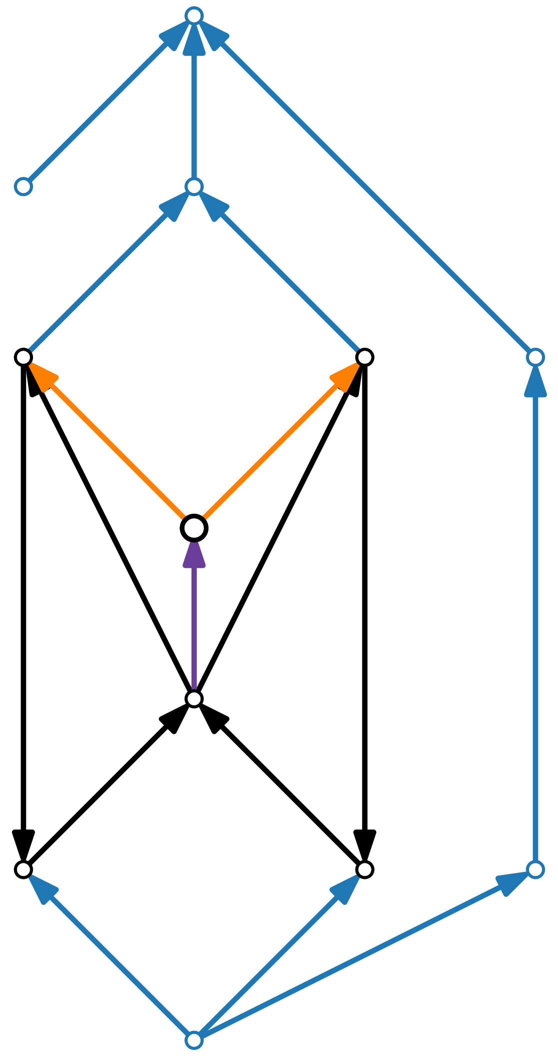

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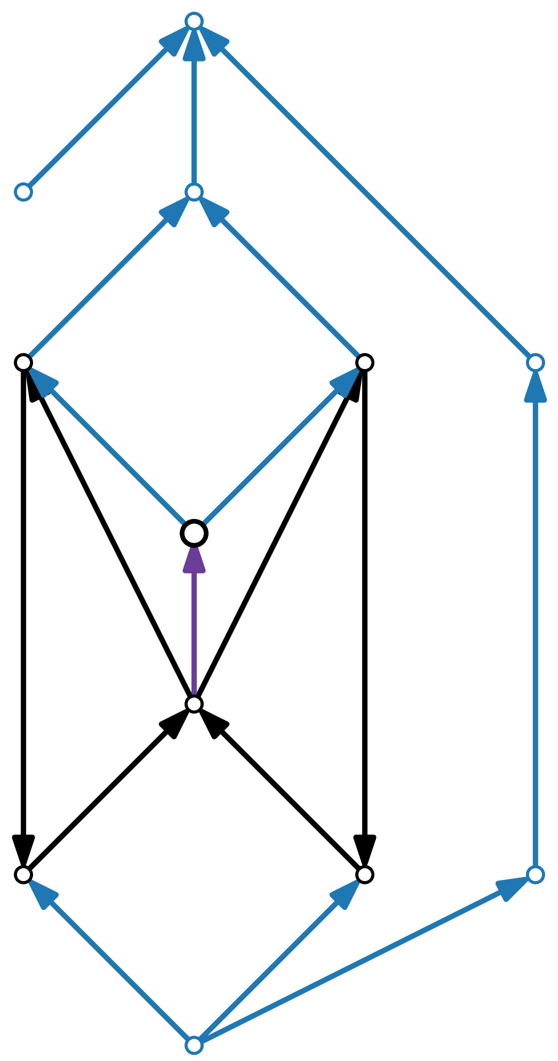

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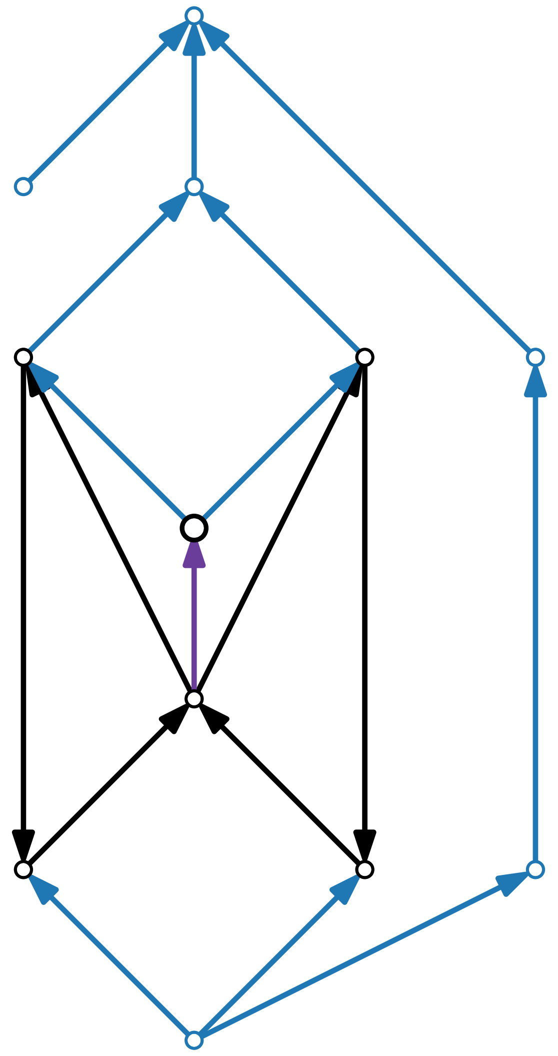

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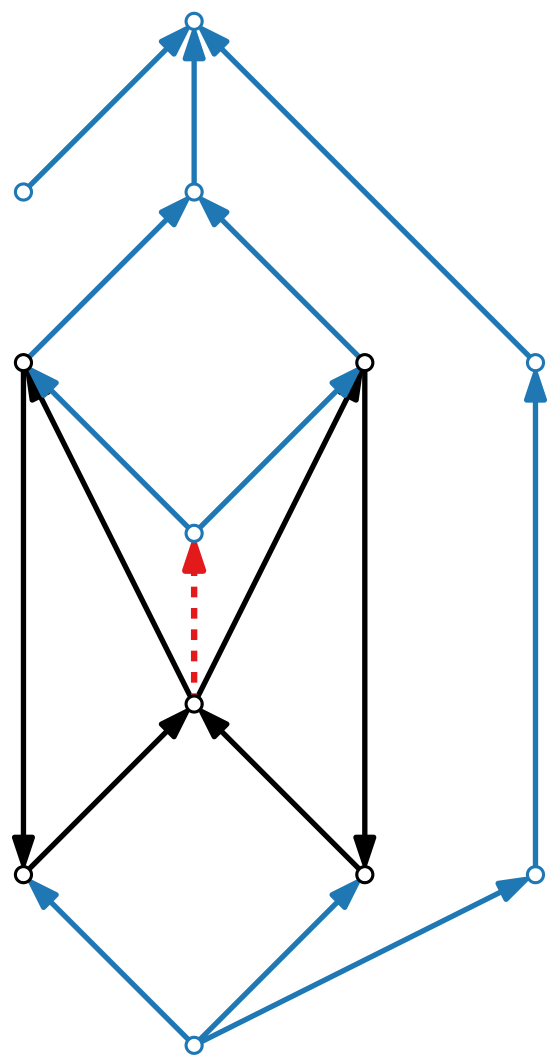

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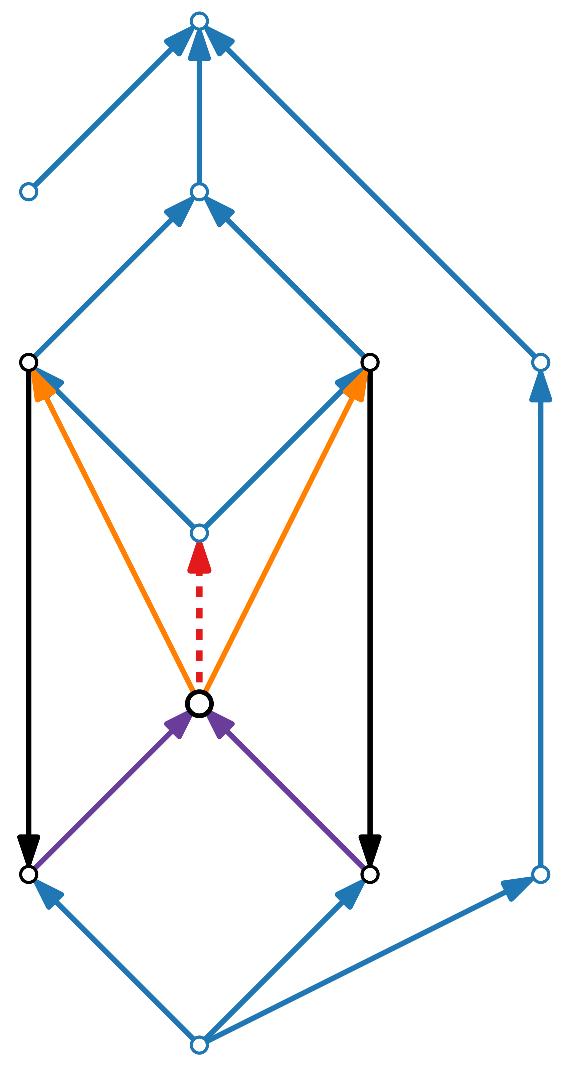

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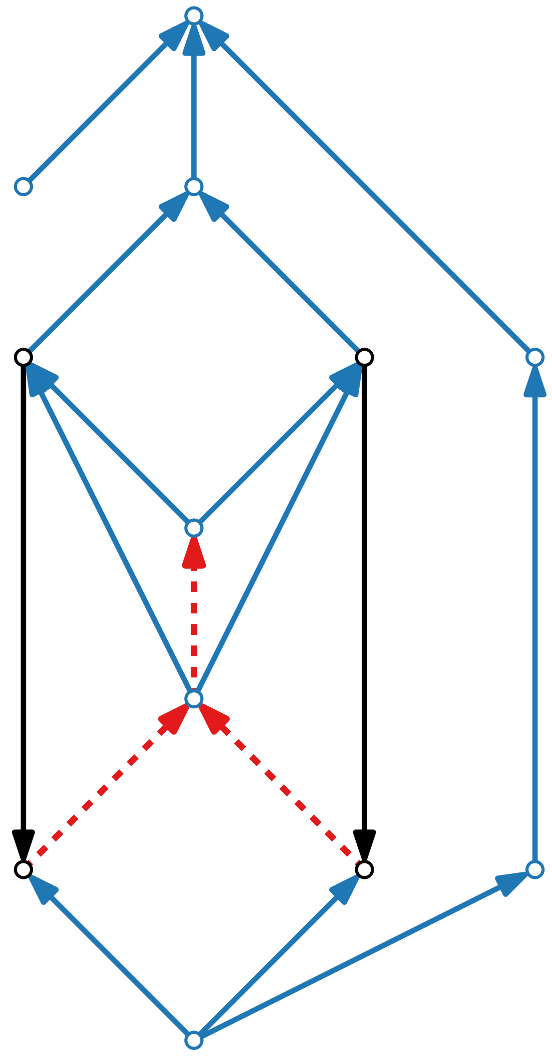

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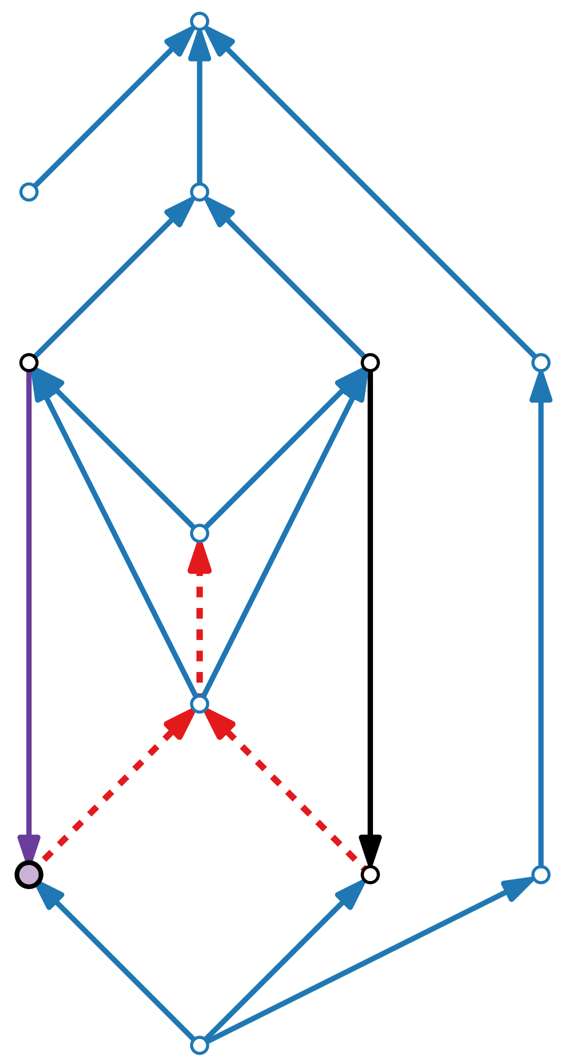

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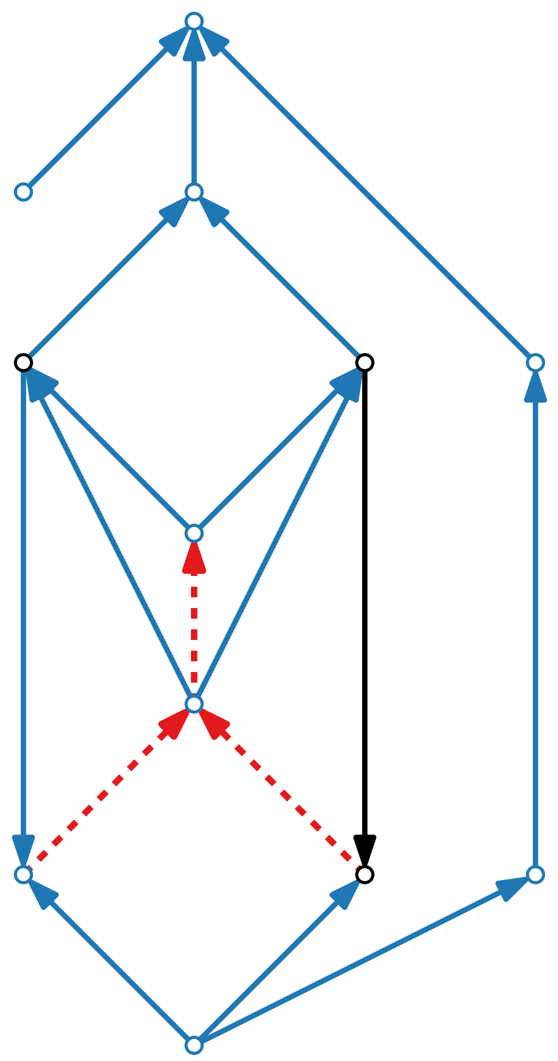

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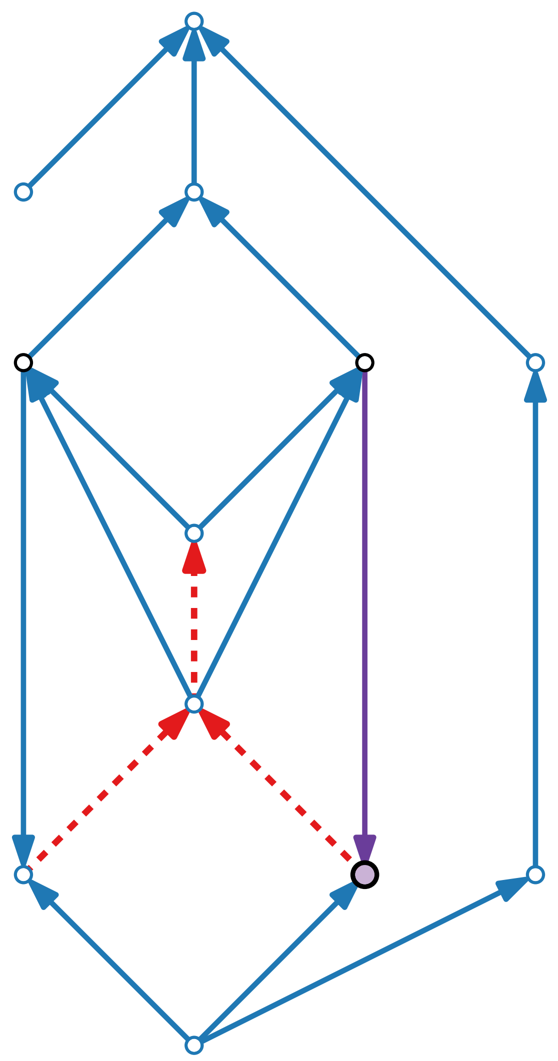

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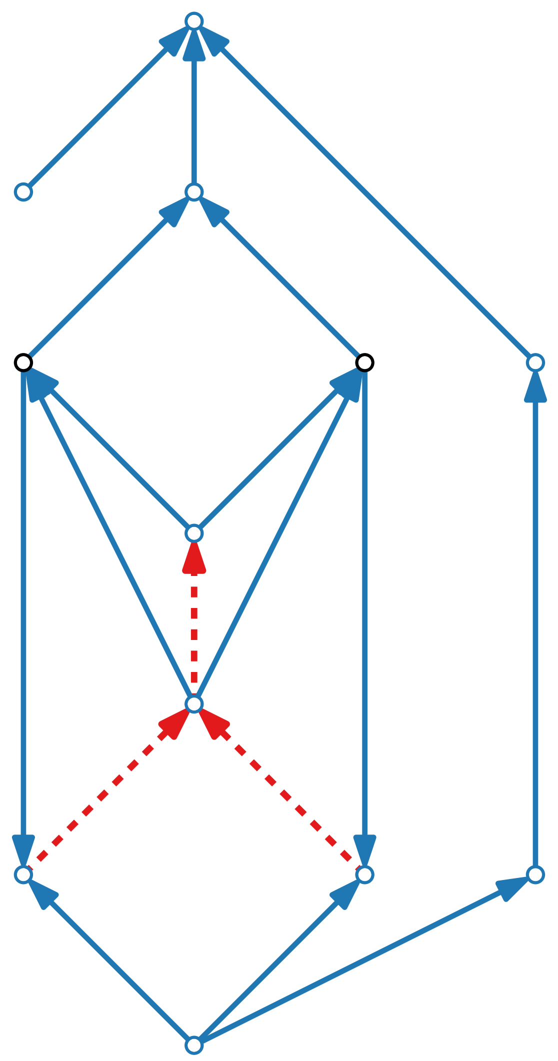

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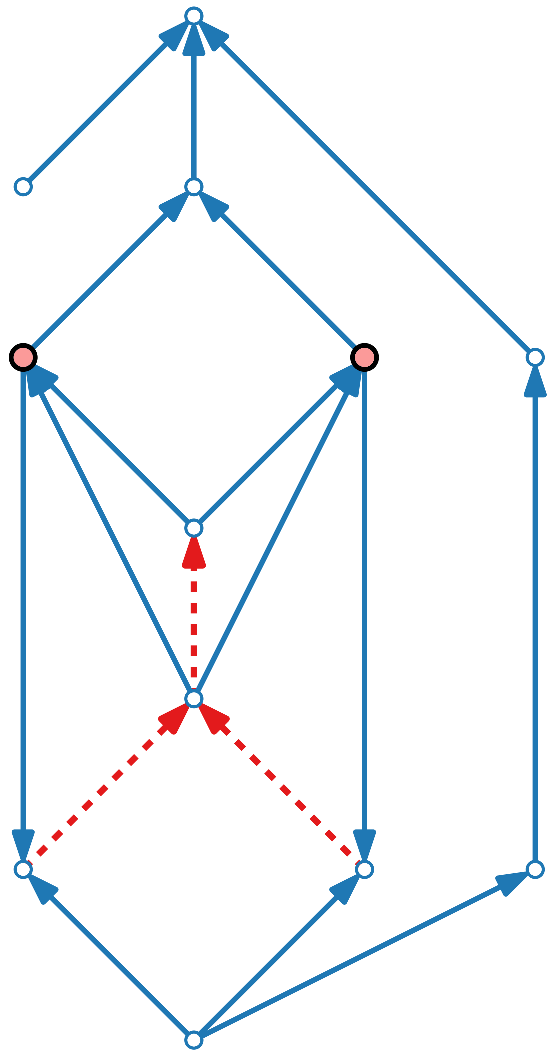

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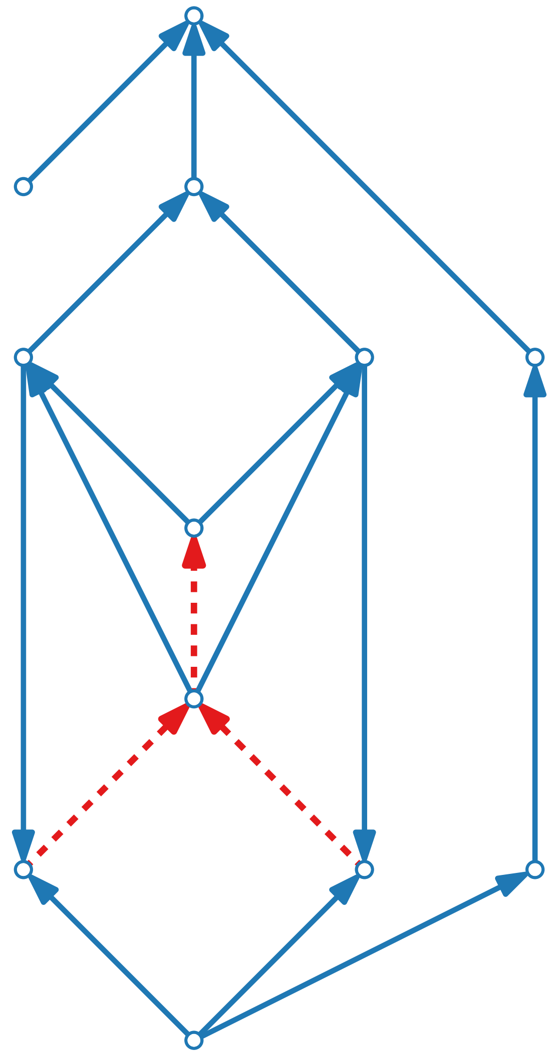

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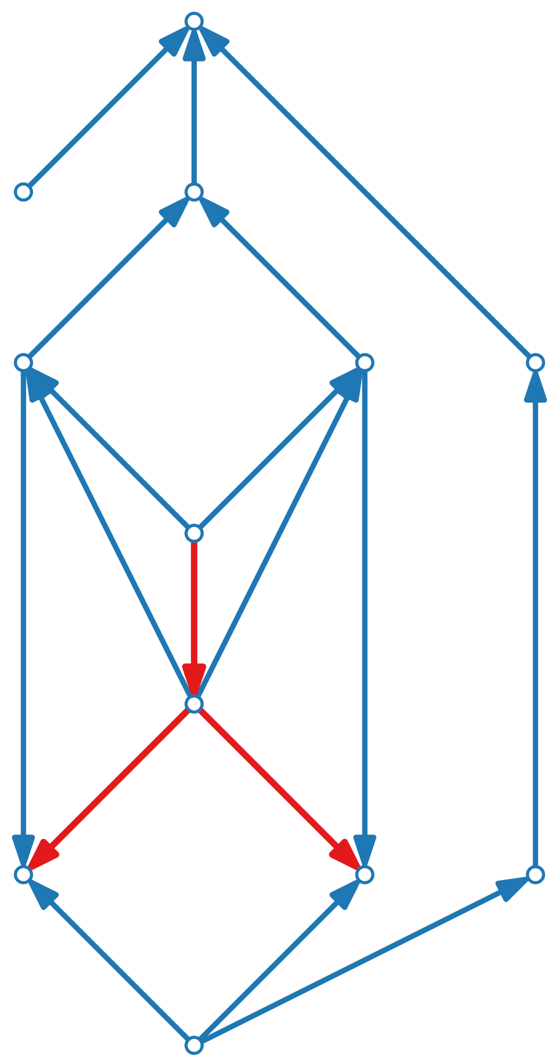

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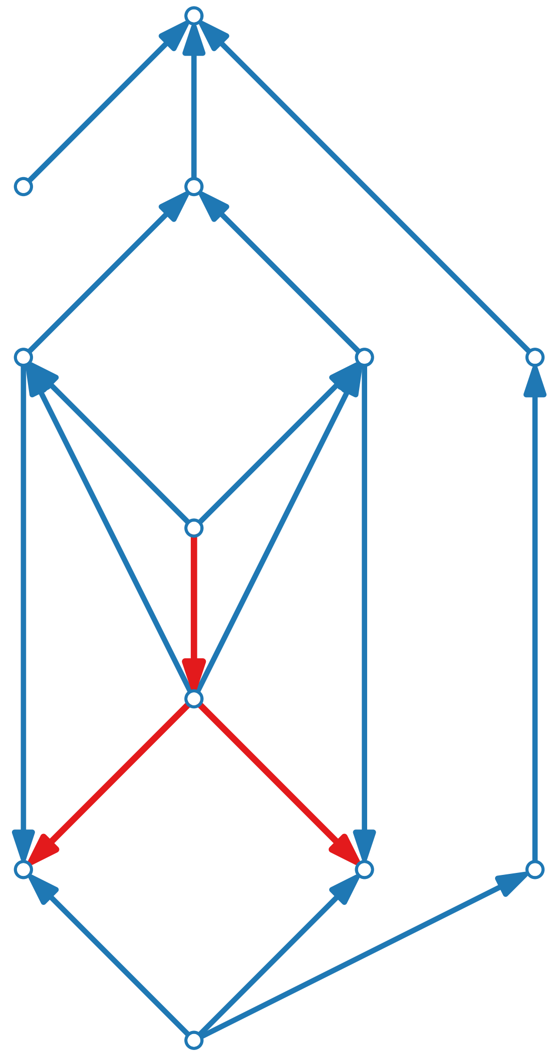

- Time:

Heuristic 2

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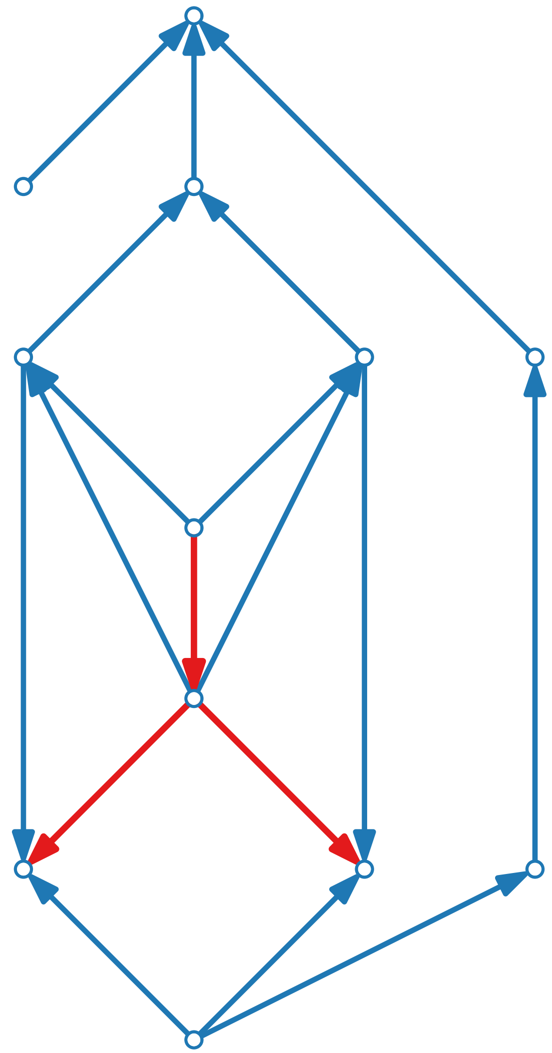

- Time: $\mathcal{O}(|V(G)| + |E(G)|)$

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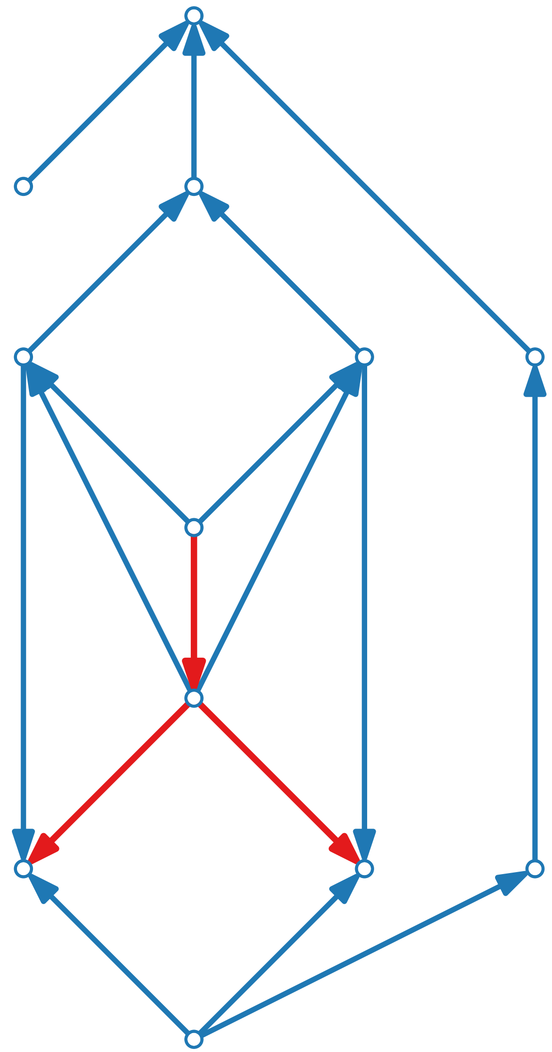

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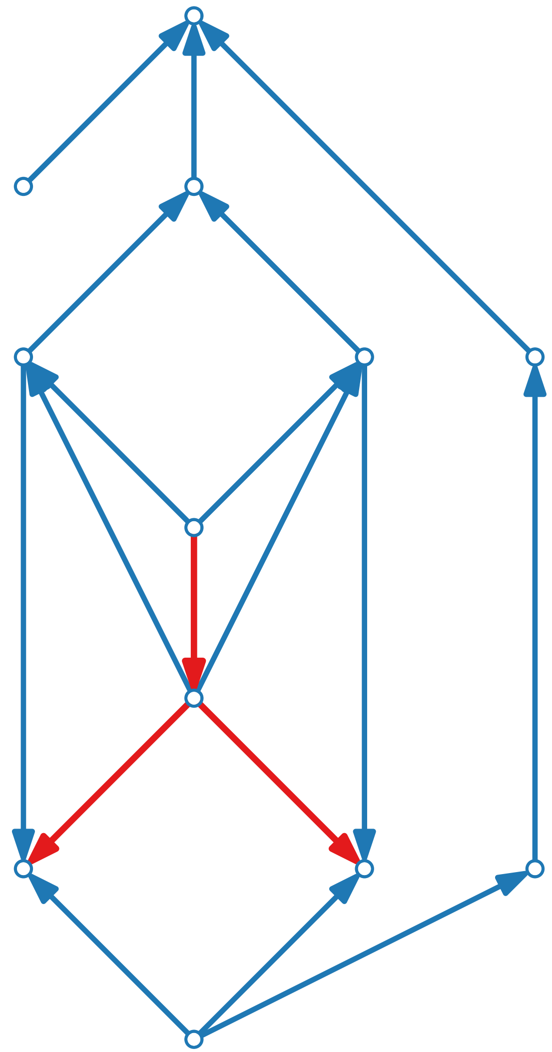
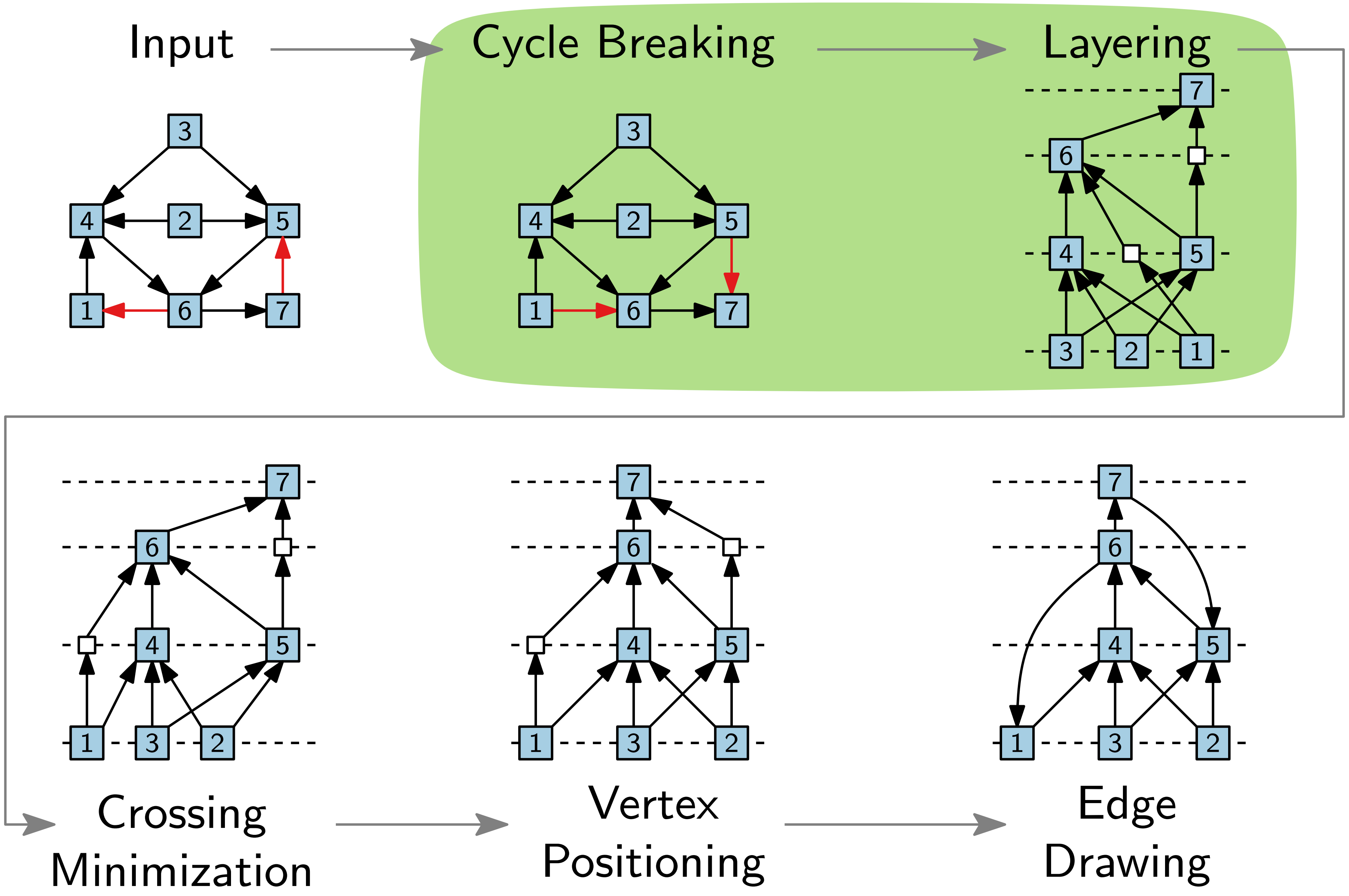


- Time: $\mathcal{O}(|V(G)| + |E(G)|)$ [Main idea: Use bins for sinks and sources, and a bin for each $|E^{\rightarrow}(v)| - |E^{\leftarrow}(v)|$]
- Quality guarantee: $|E'| \geq |E(G)|/2 + |V(G)|/6$

Step 2: Layering



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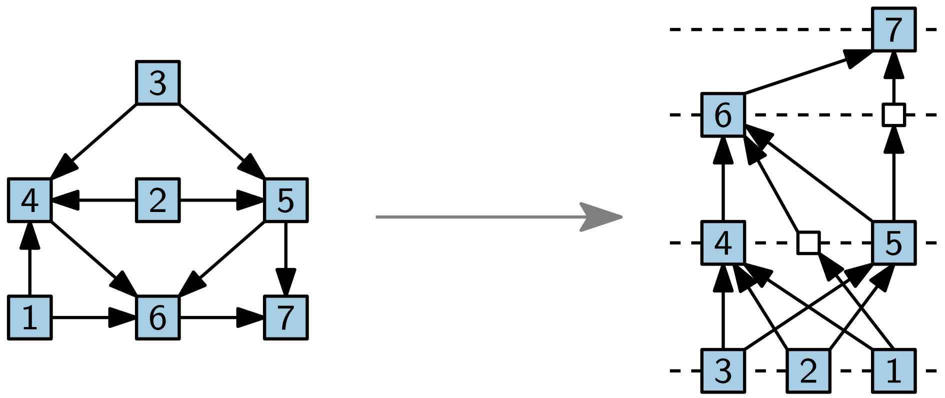
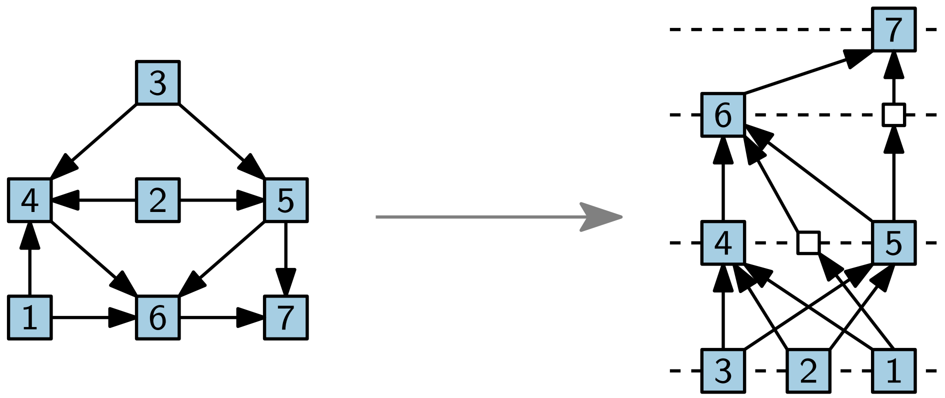


Problem.

Step 2: Layering



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- Input: Acyclic digraph G.

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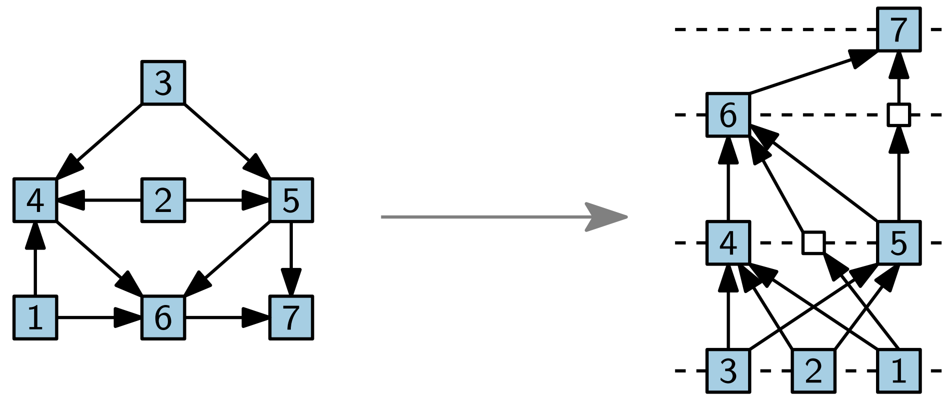

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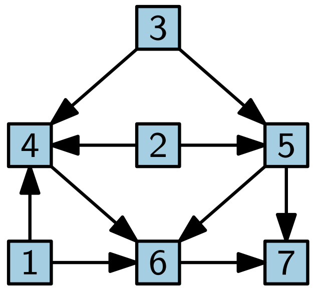

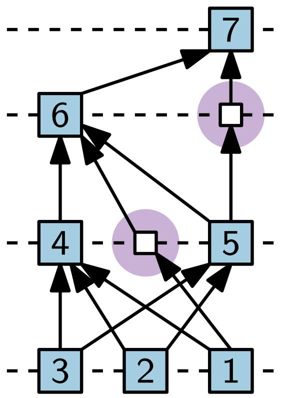

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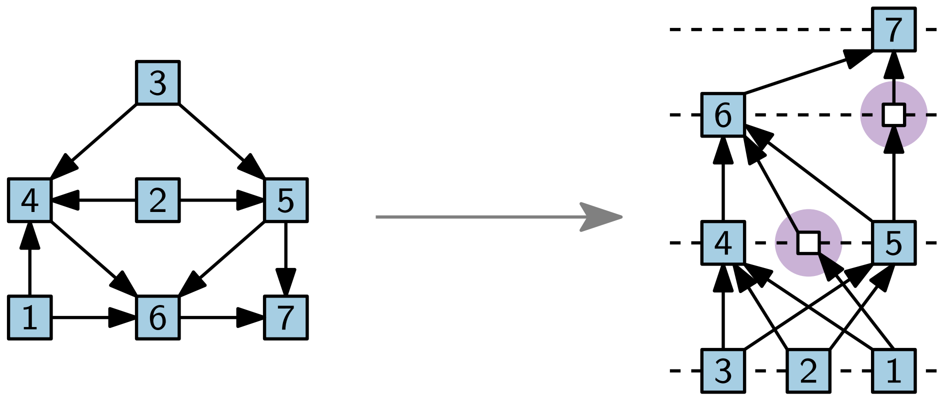

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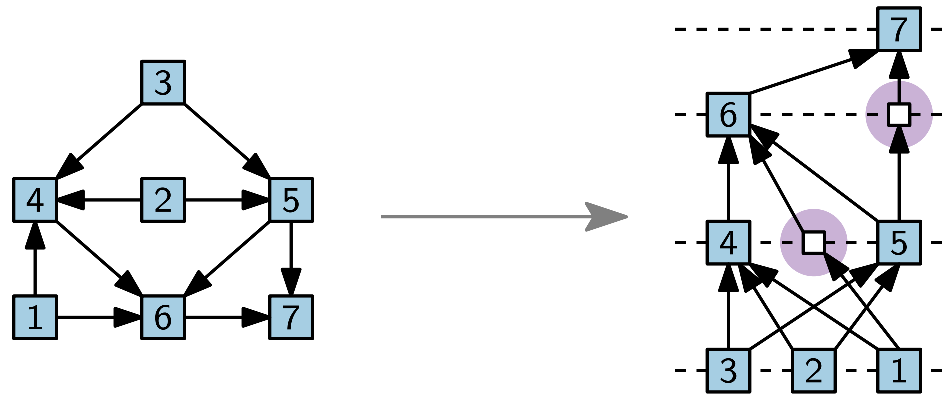

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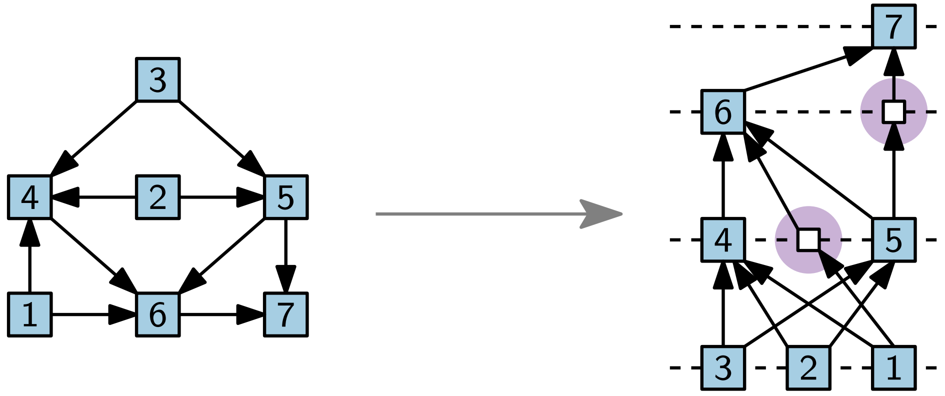

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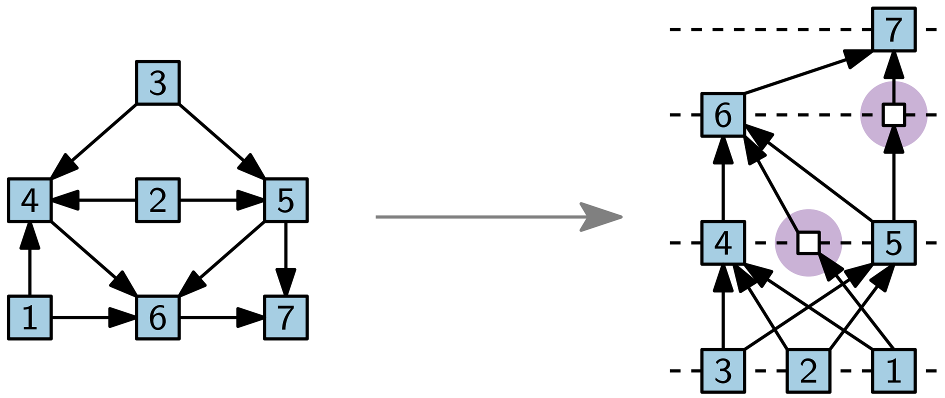

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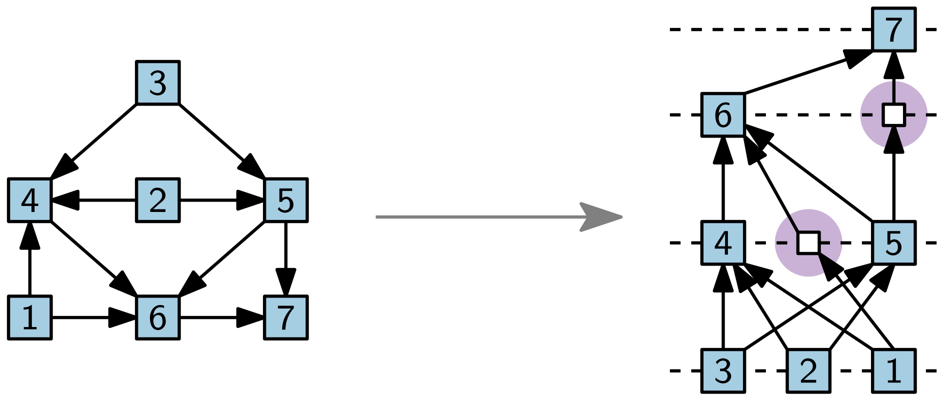

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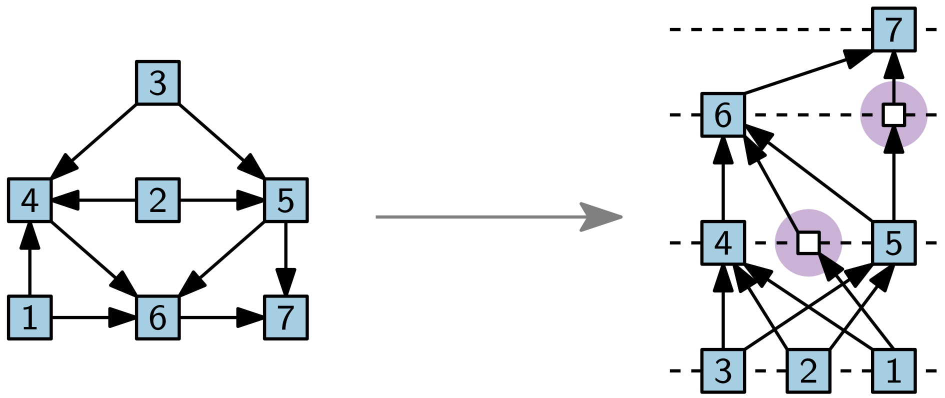

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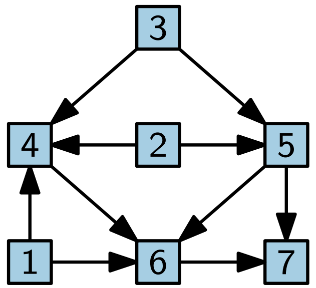
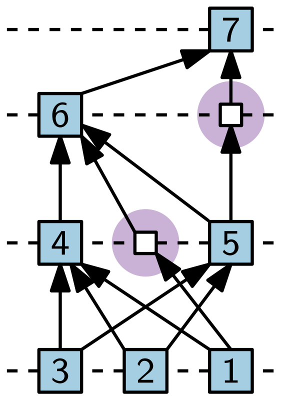
Step 2: Layering





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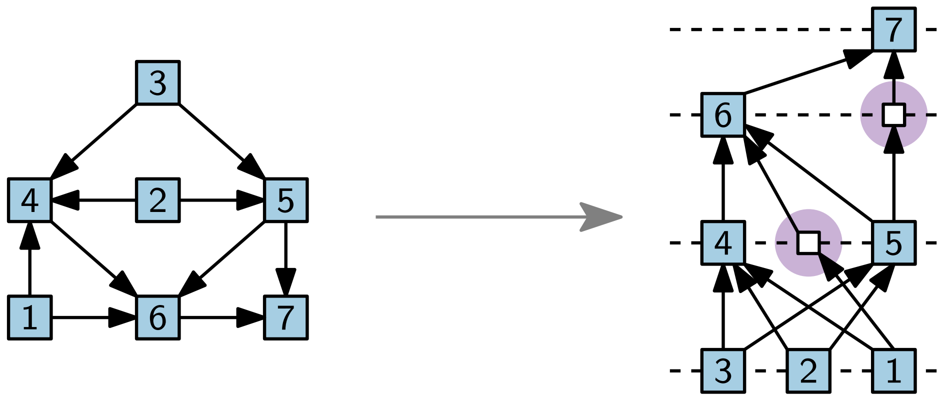

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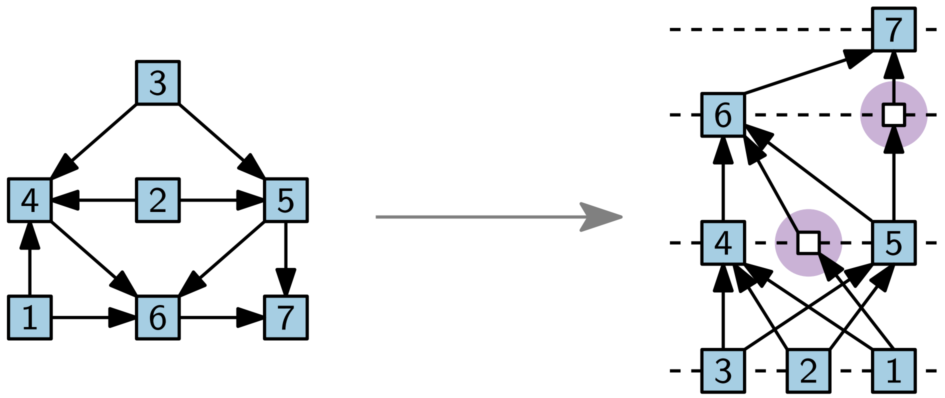

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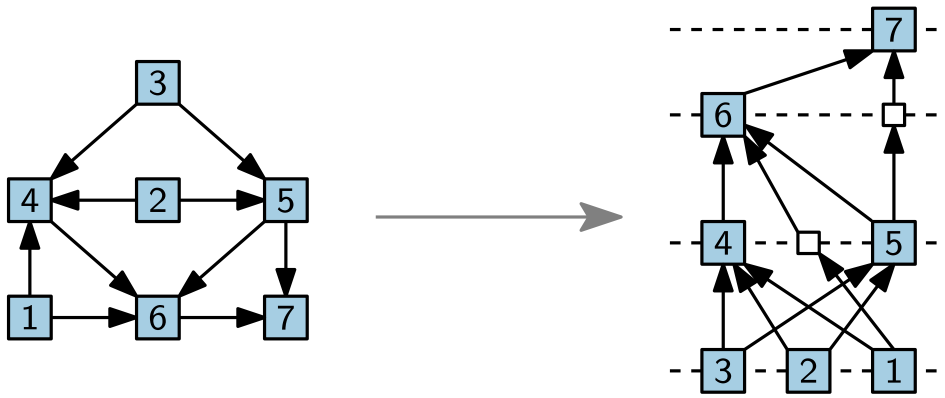
Minimize Number of Layers

Algorithm.



Minimize Number of Layers

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- for each source q,
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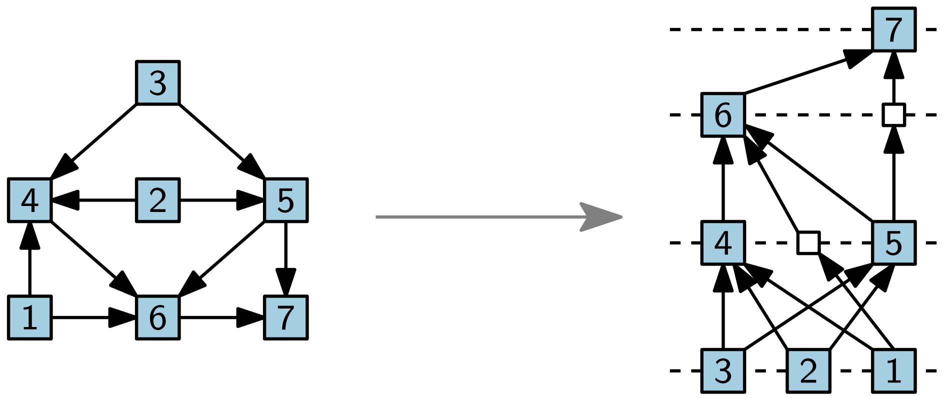

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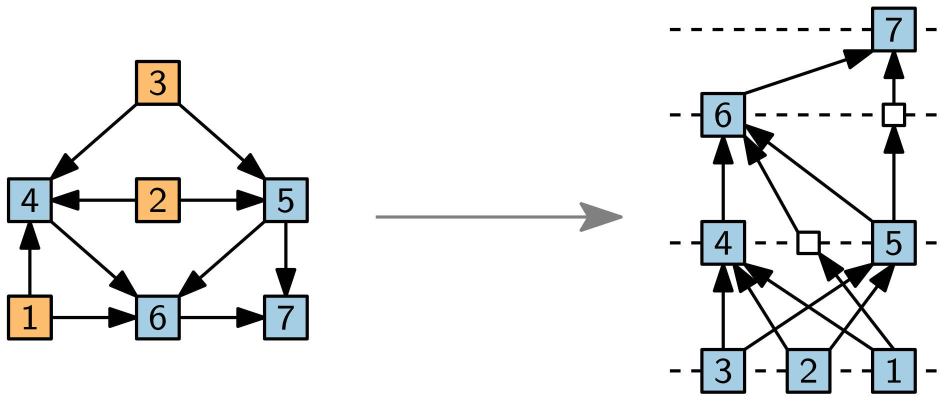

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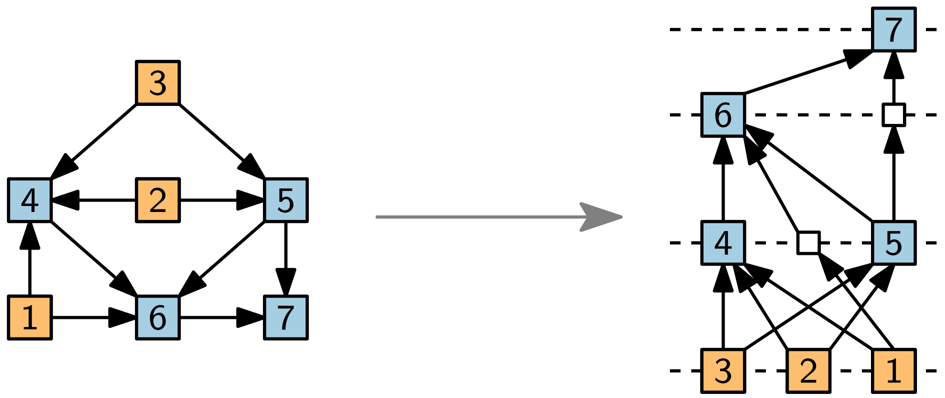

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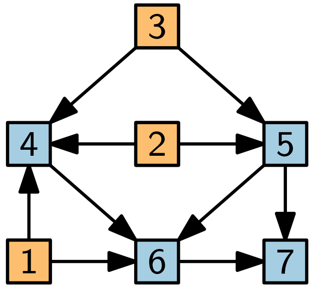

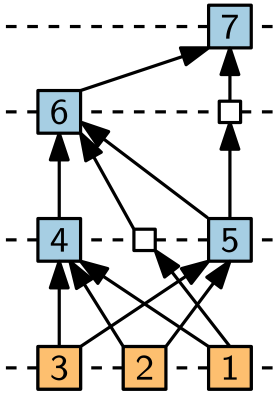

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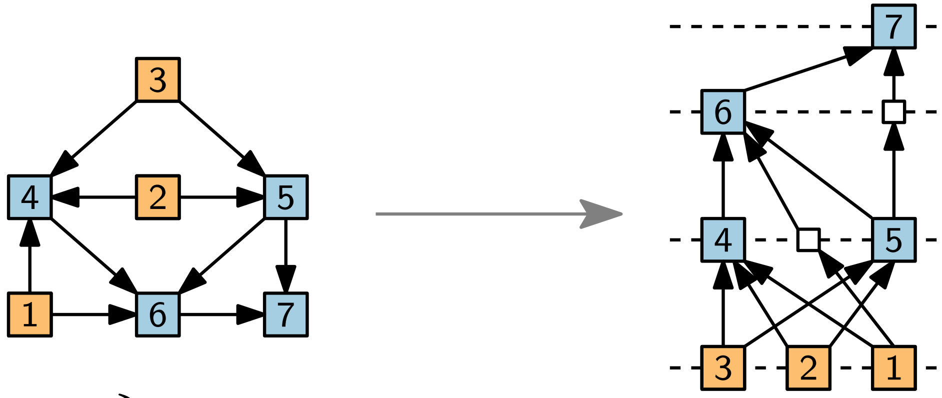

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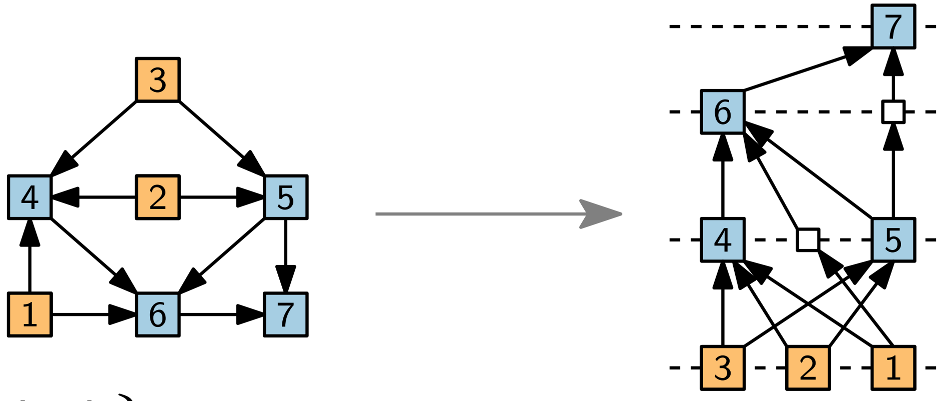

Observation.

- $y(v)$ is the length of the longest path from a source to v plus 1.

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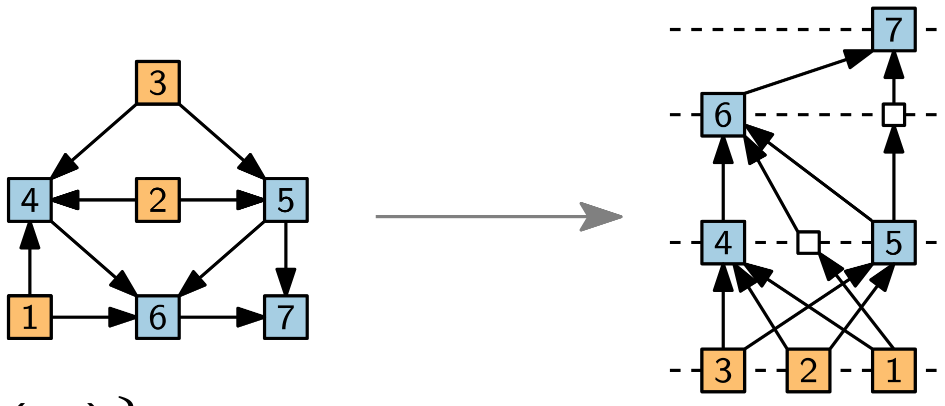

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  . . . which is optimal!

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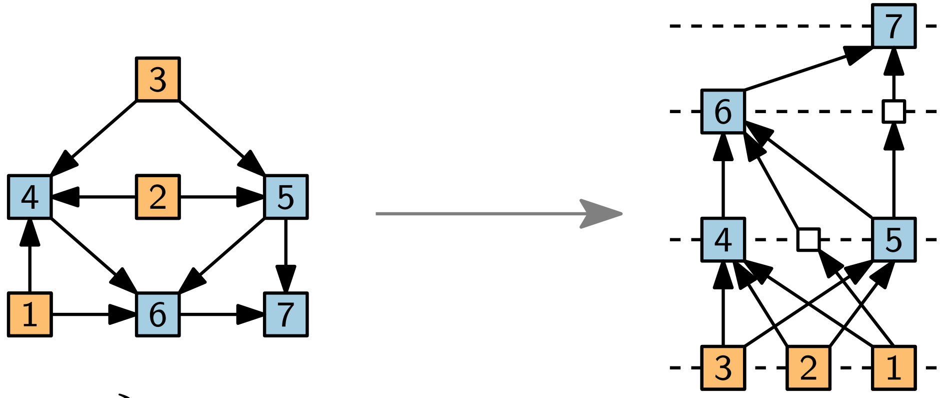

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- Can be implemented in linear time, for example, using a recursive algorithm.

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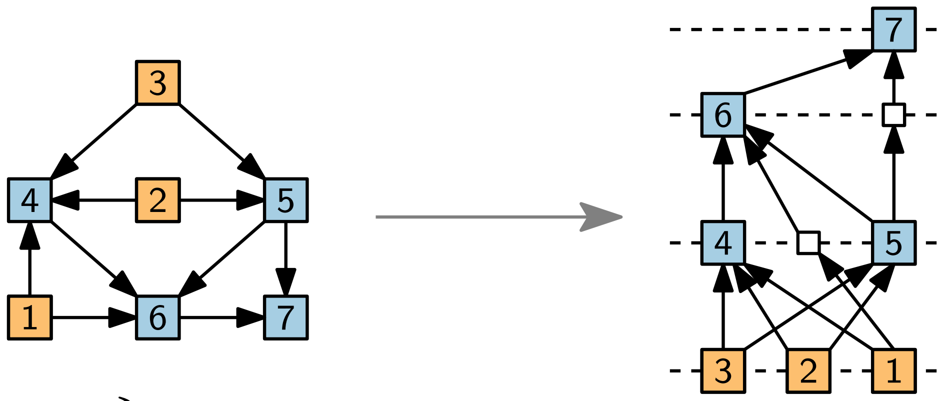

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- $y(v)$ is the length of the longest path from a source to v plus 1.

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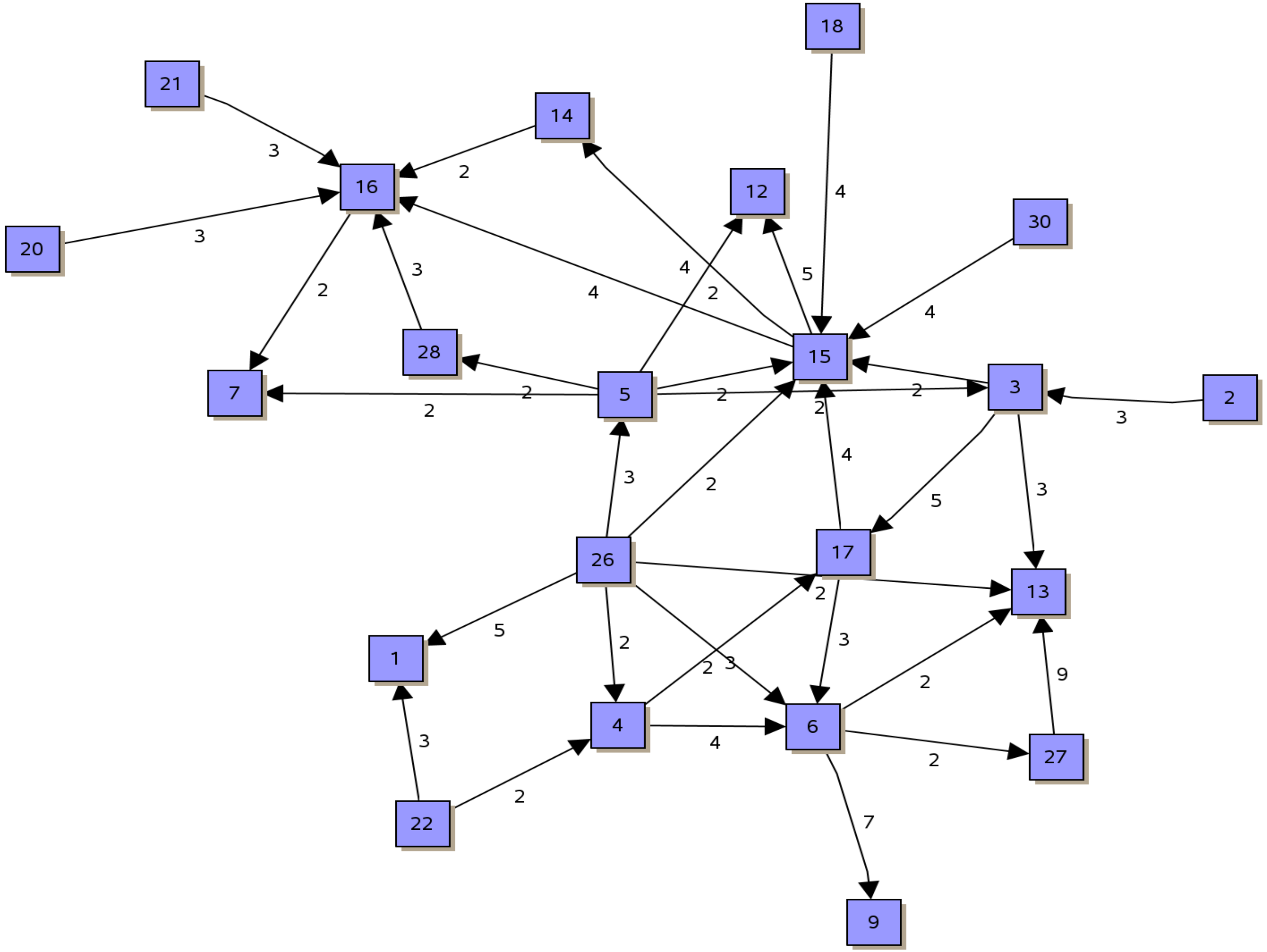
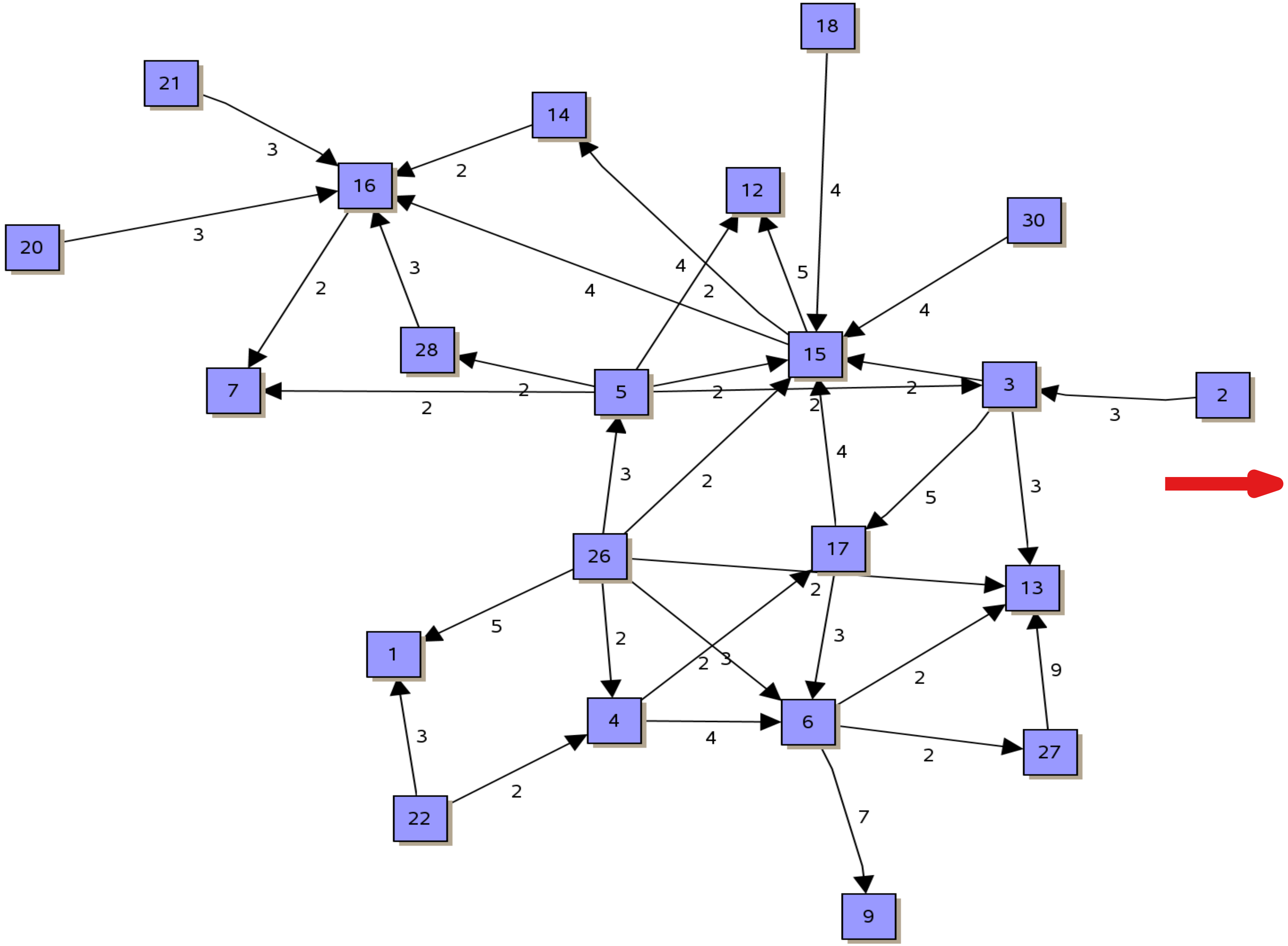
- Can be implemented in linear time, for example, using a recursive algorithm.
- Closely related to topological sorting.

Example



Example



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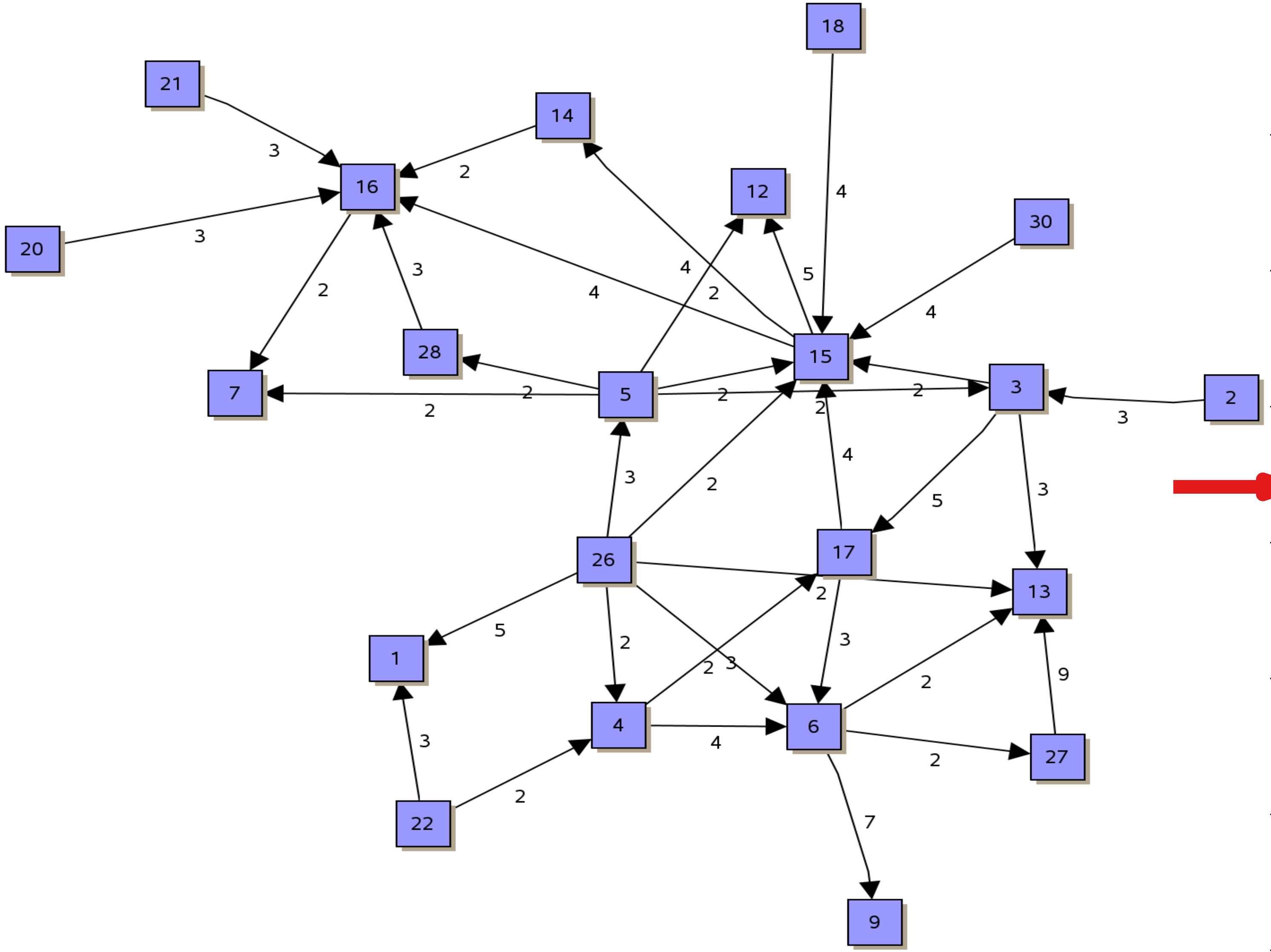

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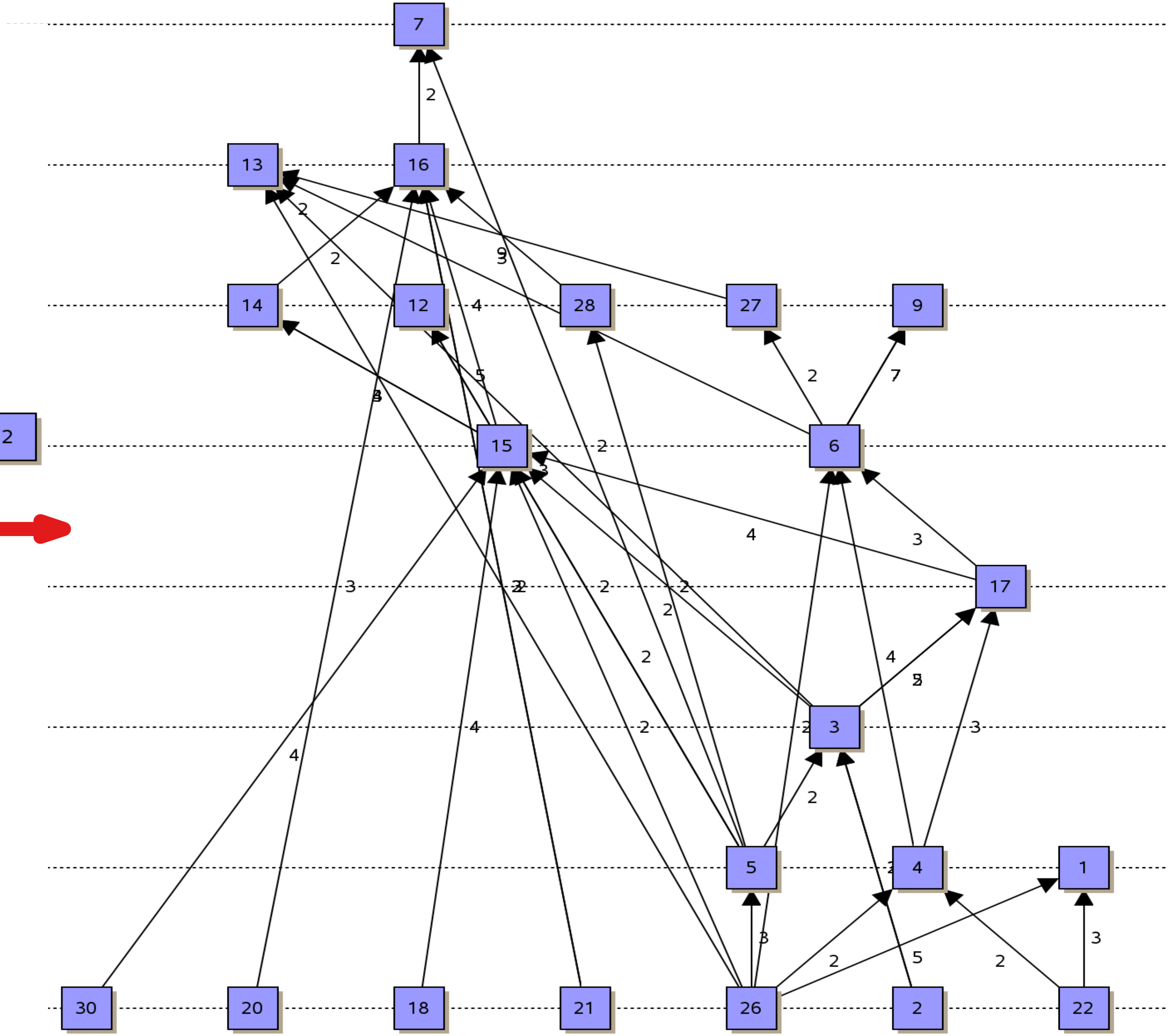

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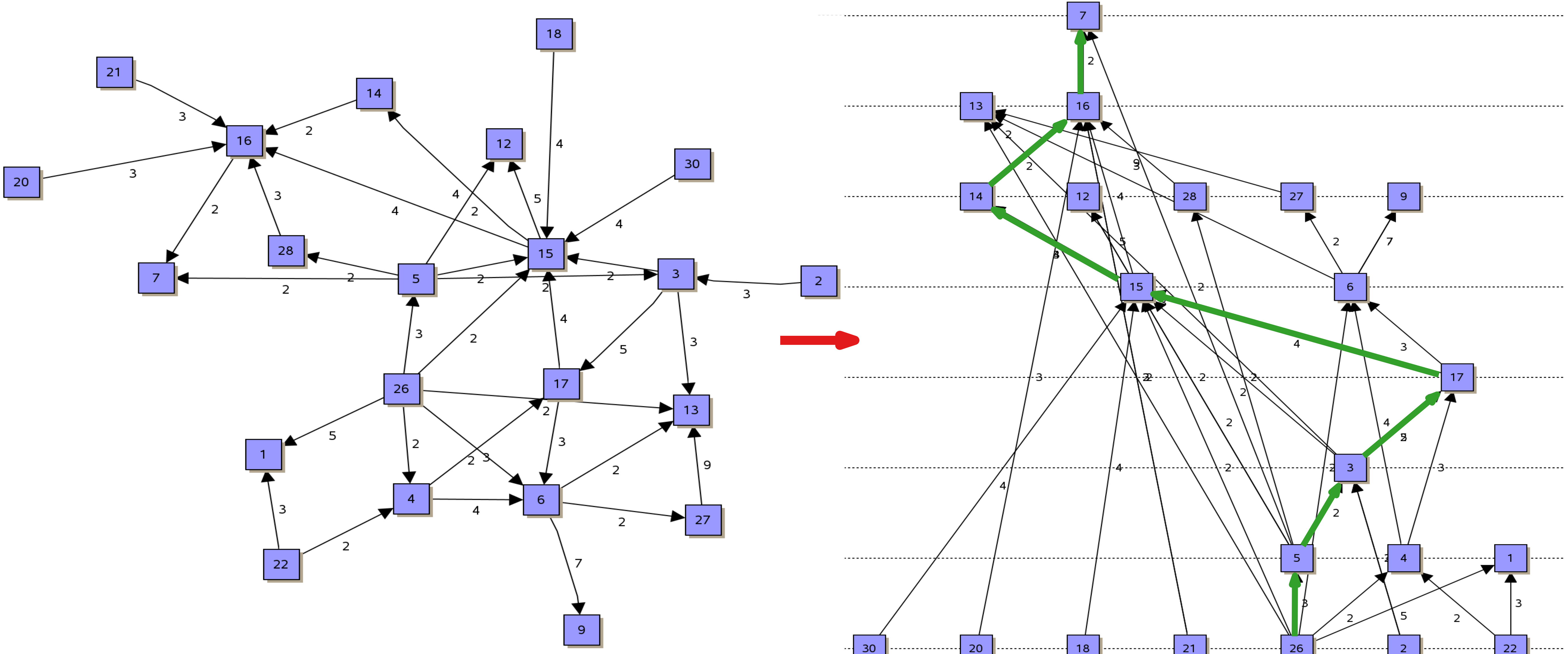

Minimize Total Edge Length – ILP

Can be formulated as an integer linear program:

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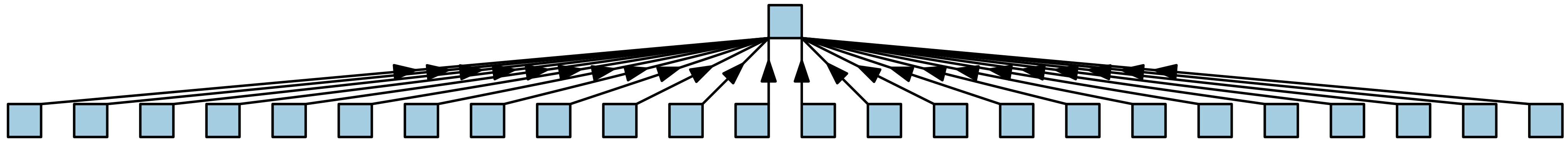
- The total edge length can be minimized in polynomial time.

Width



Drawings can be very wide.

Narrower Layer Assignment

Problem: Layering with a given maximum width.

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Approximating Precedence-Const. Multi-Processor Scheduling

- Jobs stored in a list L, which is topologically sorted.

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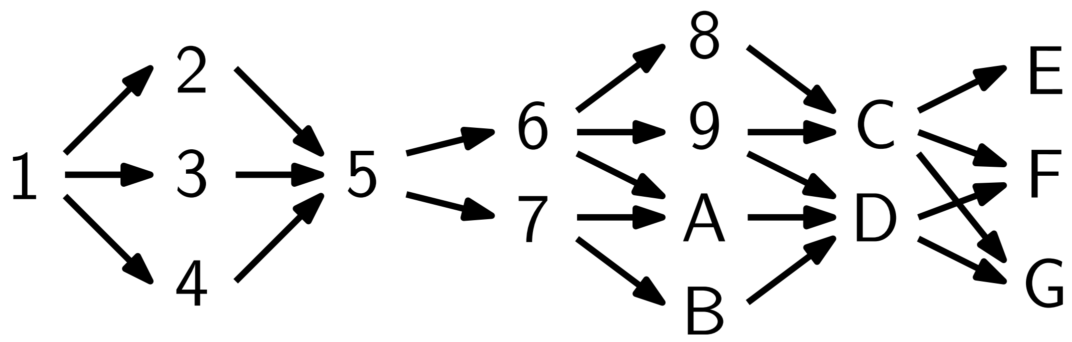
- A job in L is *available* if all its predecessors have been scheduled.
- For each point in time $t = 1, 2, \ldots,$ we can schedule $\leq W$ available jobs.
- As long as there are free machines and available jobs,
  take the first available job and assign it to a free machine.

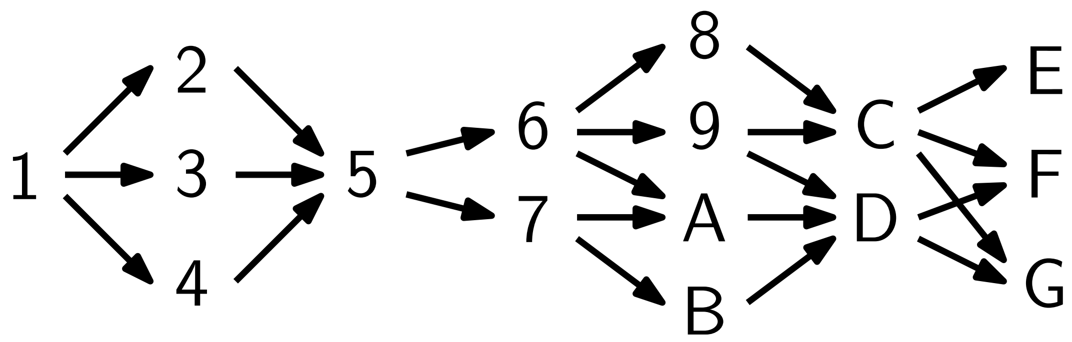
Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



Approximating Precedence-Const. Multi-Processor Scheduling

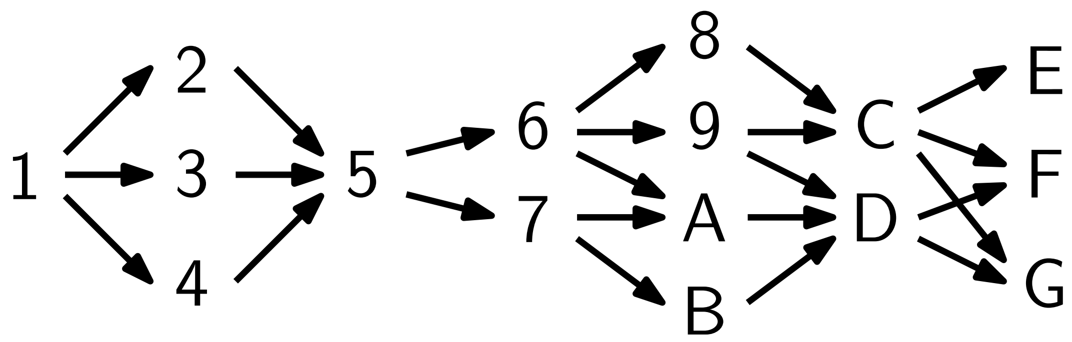
Input: Precedence graph (divided into layers of arbitrary width)



Number of machines is $W = 2$.

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



Number of machines is $W = 2$.

Output: Schedule

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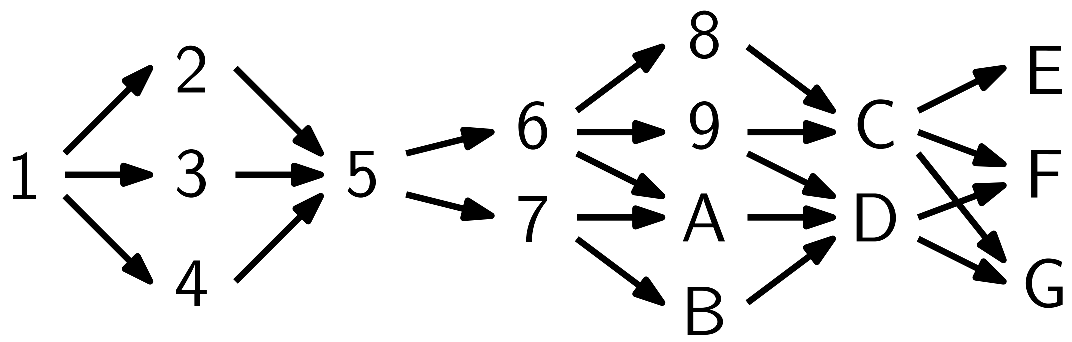


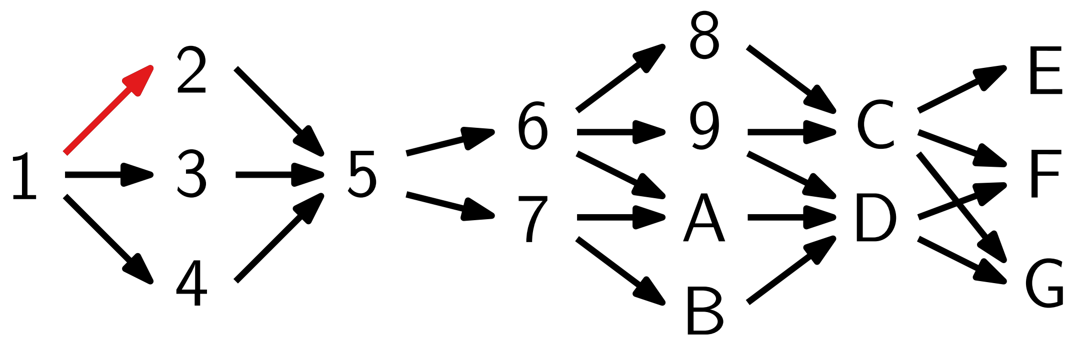
Number of machines is $W = 2$.

Output: Schedule

M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



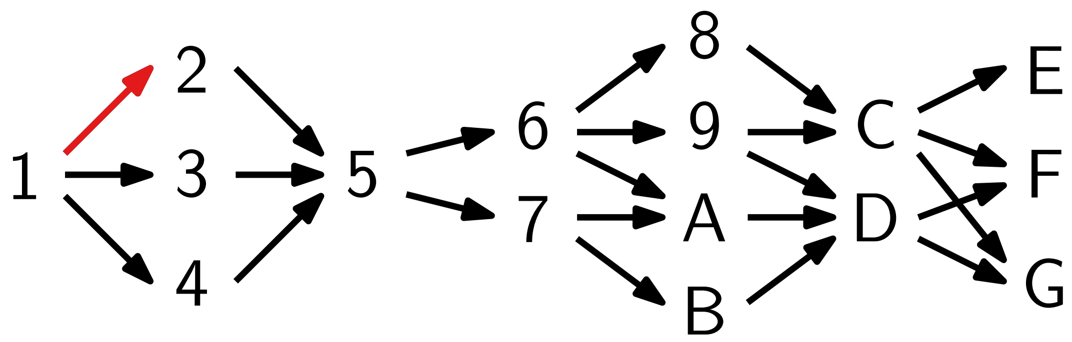
Number of machines is $W = 2$.

Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



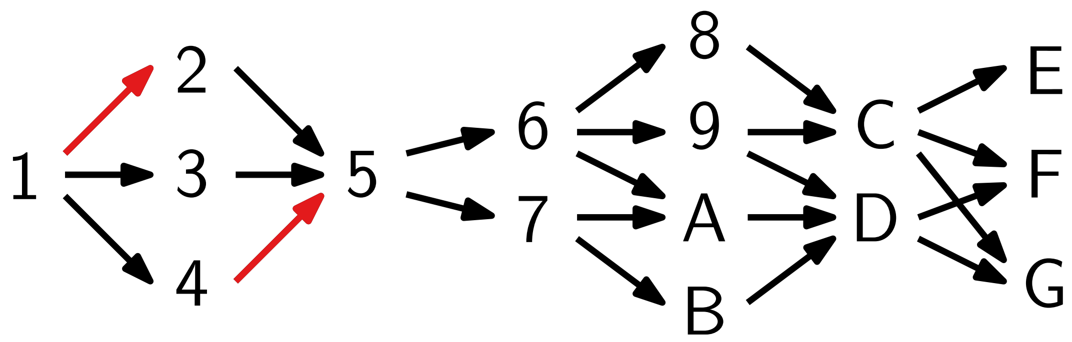
Number of machines is $W = 2$.

Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



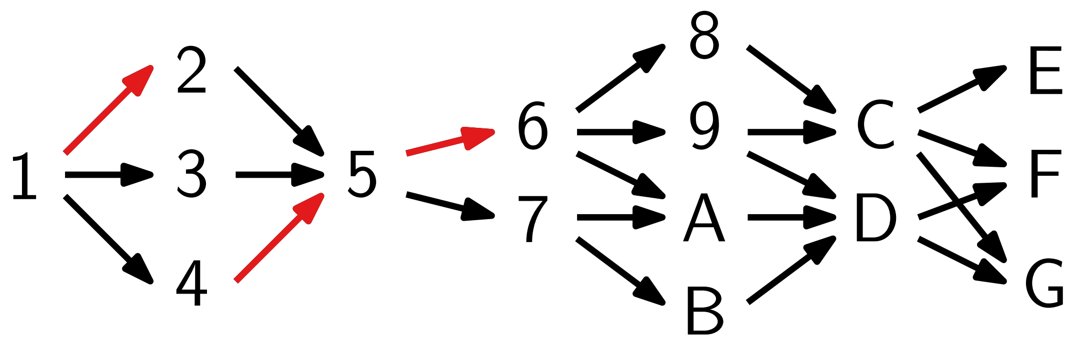
Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



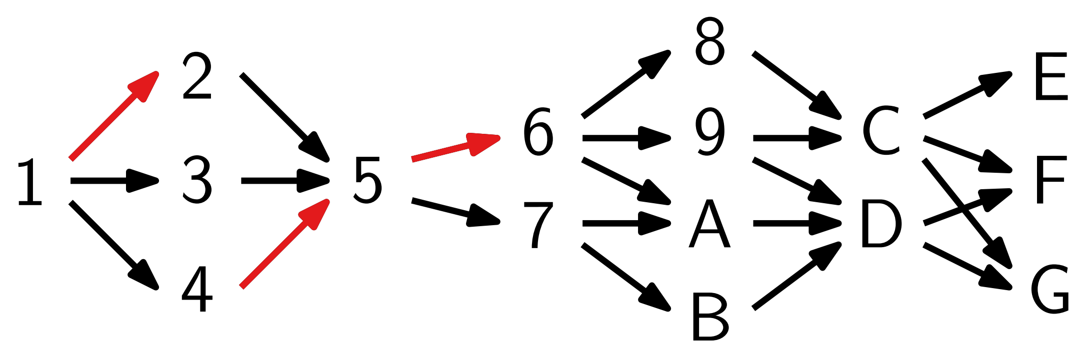
Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



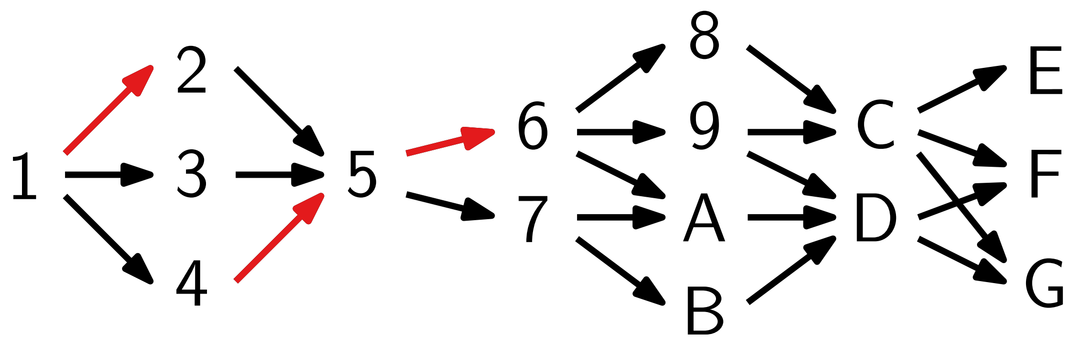
Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

Input: Precedence graph (divided into layers of arbitrary width)



Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

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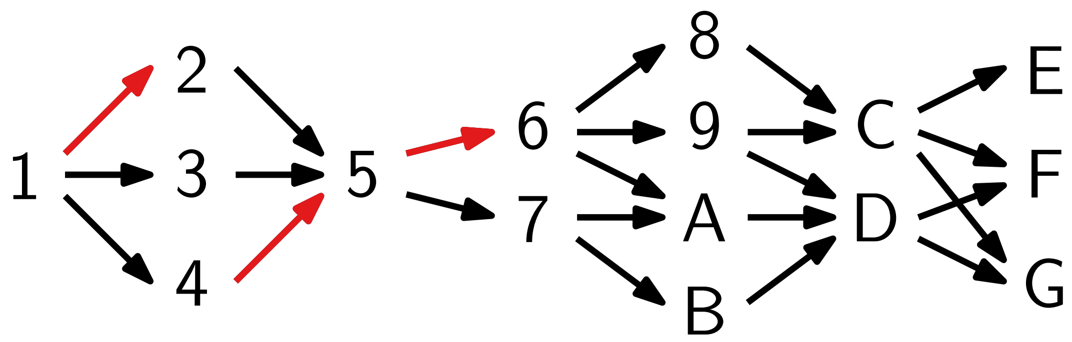

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

Approximating Precedence-Const. Multi-Processor Scheduling

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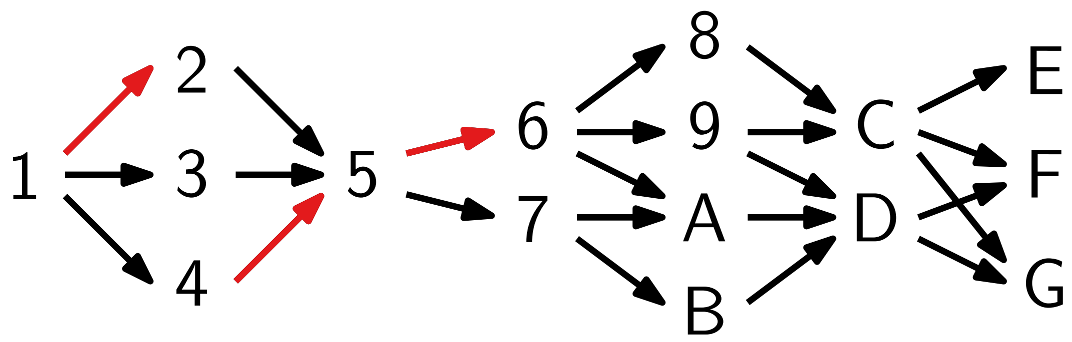

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

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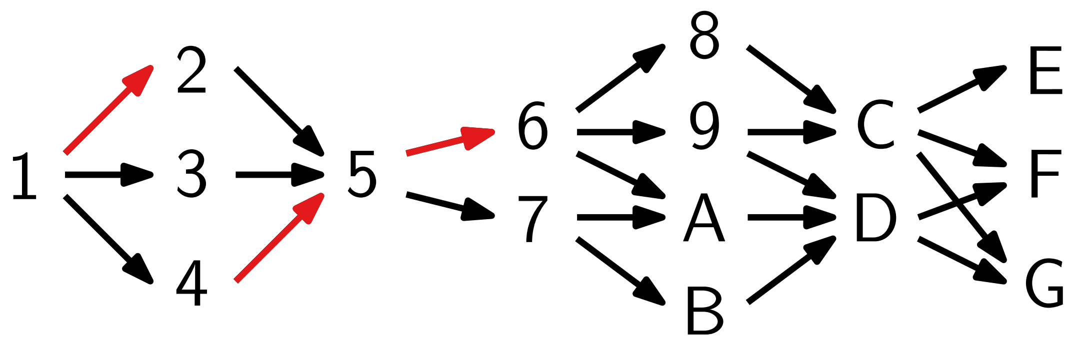

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	
M_2	–	3	–	–	7	9	B	D	F	
t	1	2	3	4	5	6	7	8	9	10

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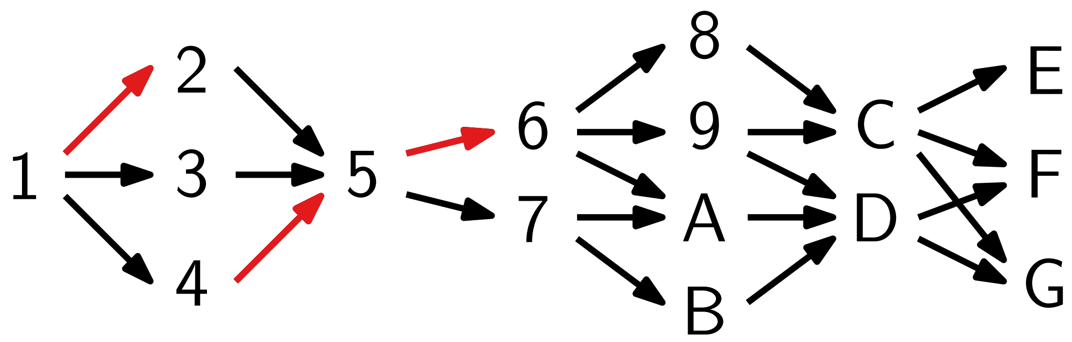

Number of machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
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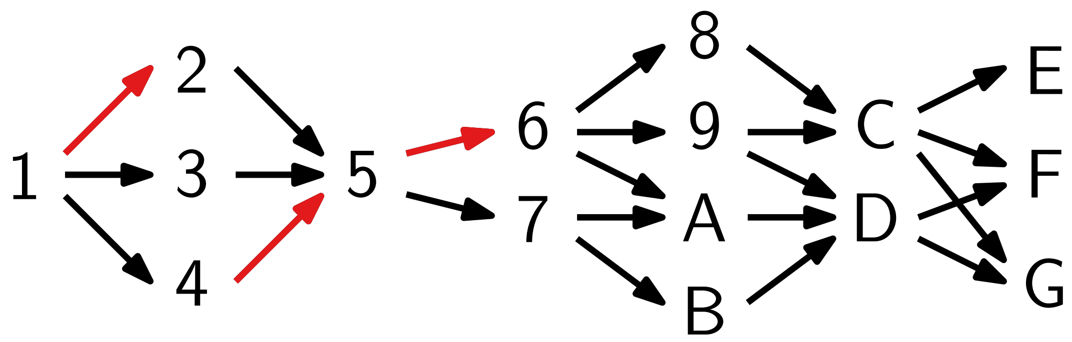

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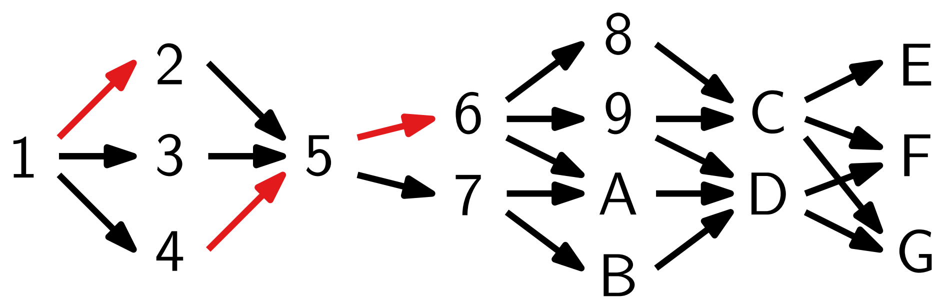
t	1	2	3	4	5	6	7	8	9	10

Question: Good approximation factor?

Approximating PCMPS – Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

Approximating PCMPS – Analysis for $W = 2$

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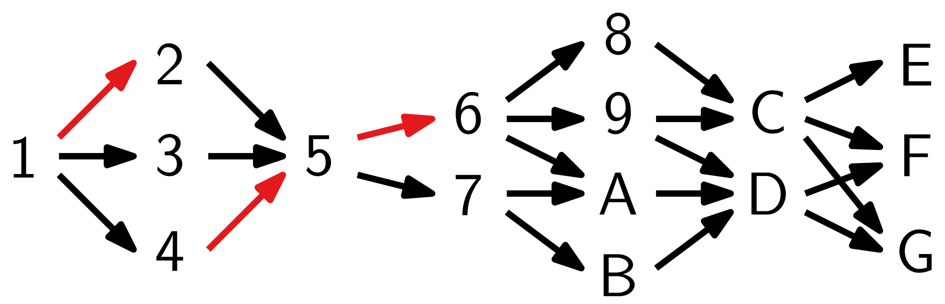

Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq$

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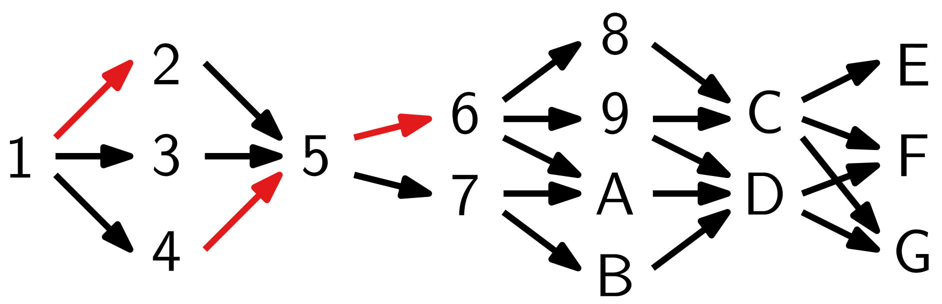
Precedence graph $G_<$



Schedule

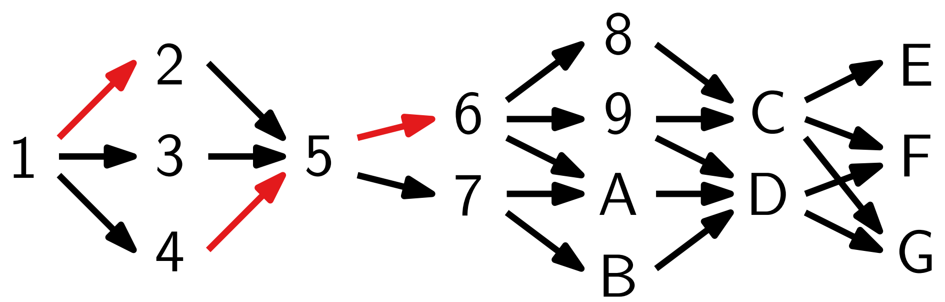
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq \lceil n/2 \rceil$

Approximating PCMPS – Analysis for $W = 2$

Precedence graph $G_<$



Schedule

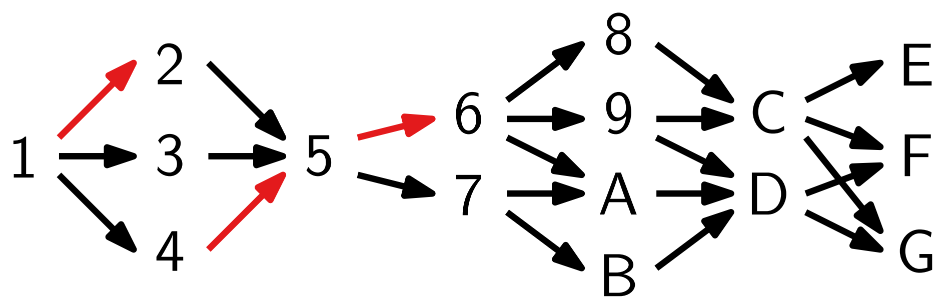
M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

OPT $\geq$ $\lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS – Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$ ($=$ length of longest path in $G_<$)

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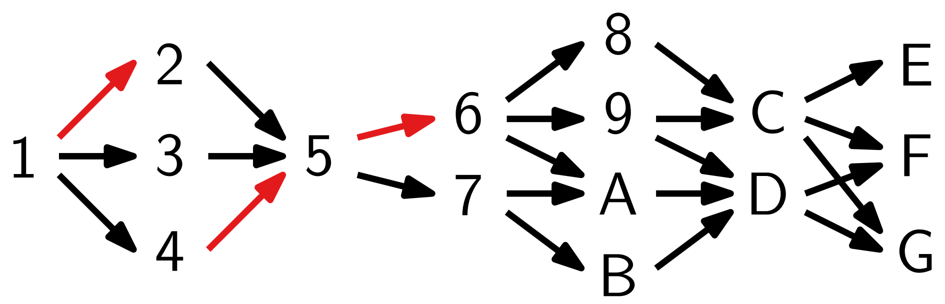
Precedence graph $G_<$



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M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

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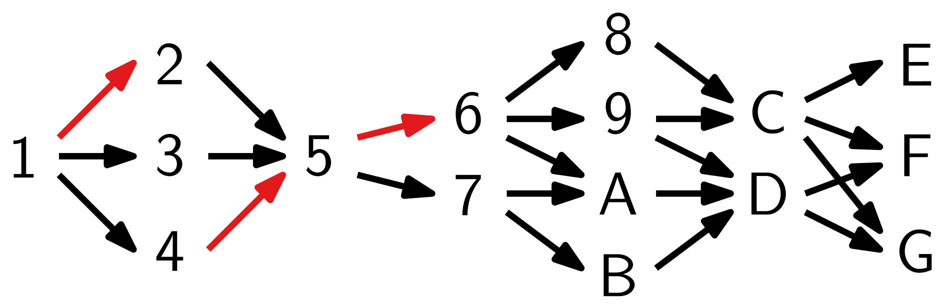
OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$ ($=$ length of longest path in $G_<$)

Goal: measure the quality of our algorithm using the lower bounds

Approximating PCMPS – Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound"

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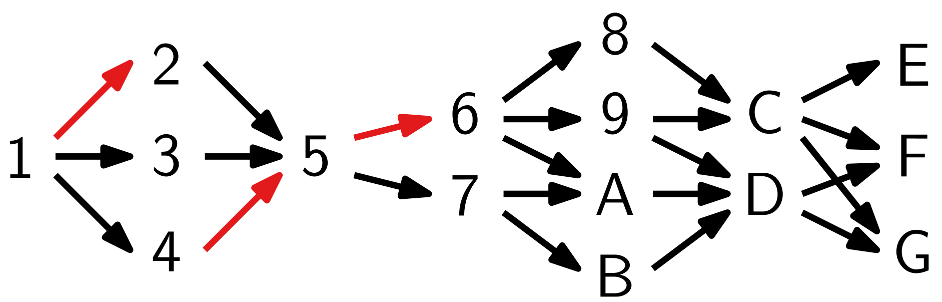

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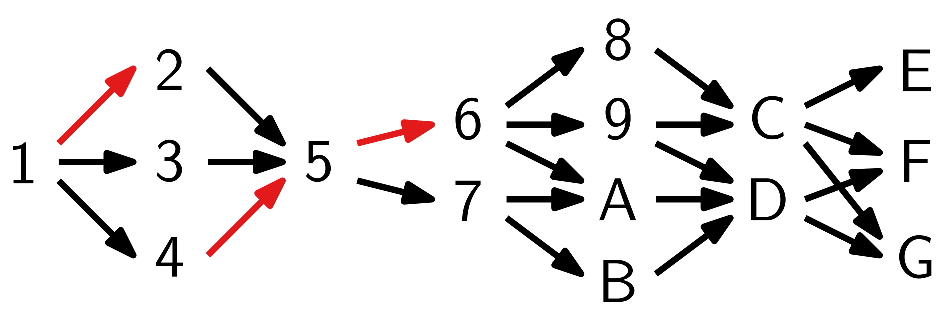

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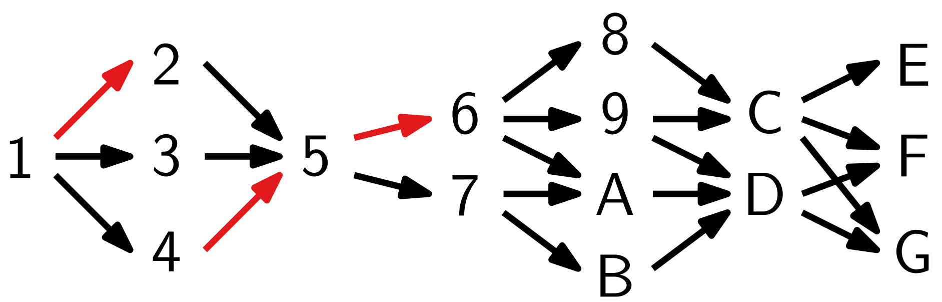

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Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil$

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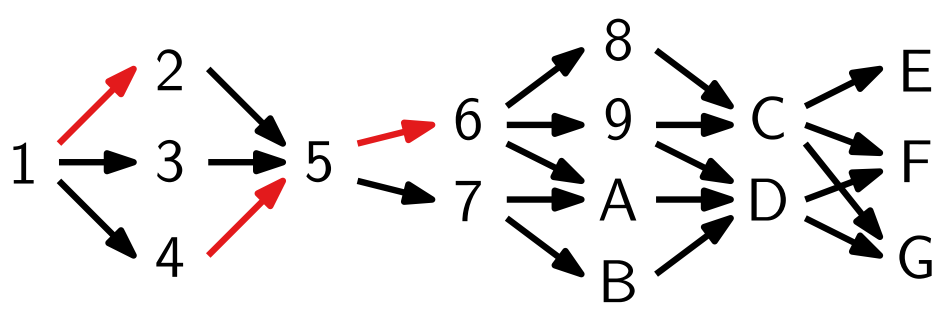

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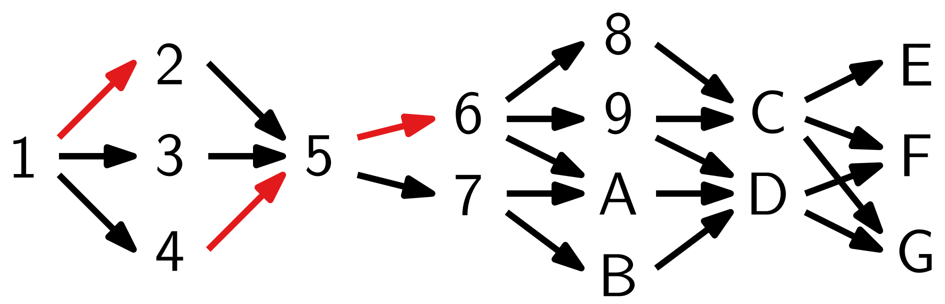

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Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2$

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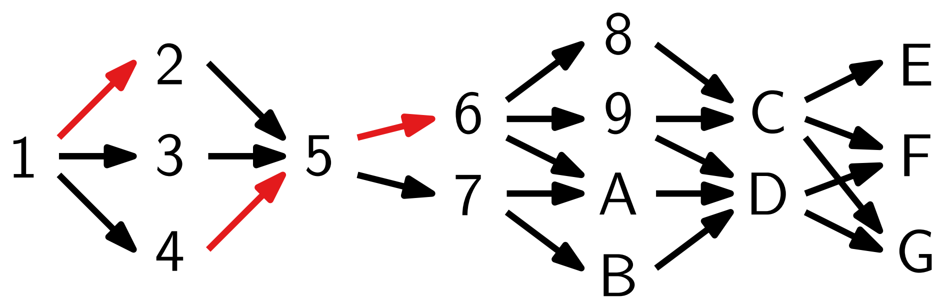

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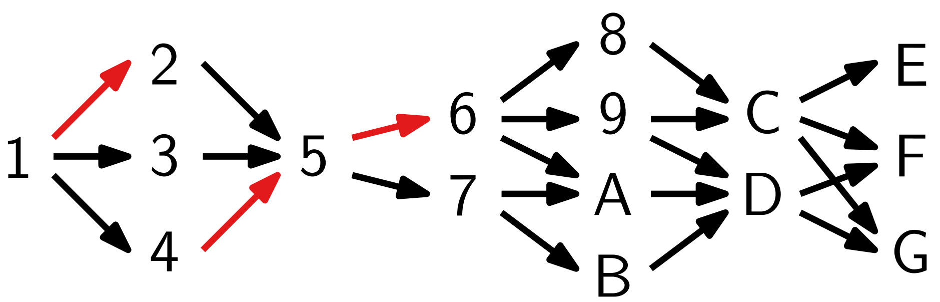

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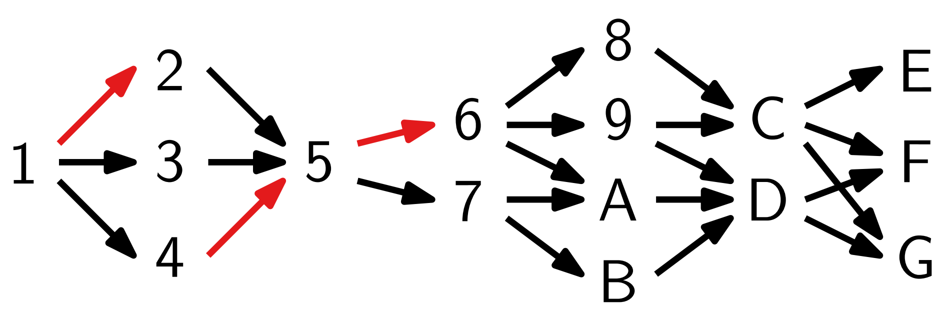

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$\leq (2 - 1/W) \cdot$ OPT in general case

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

insertion of pauses (–) in the schedule
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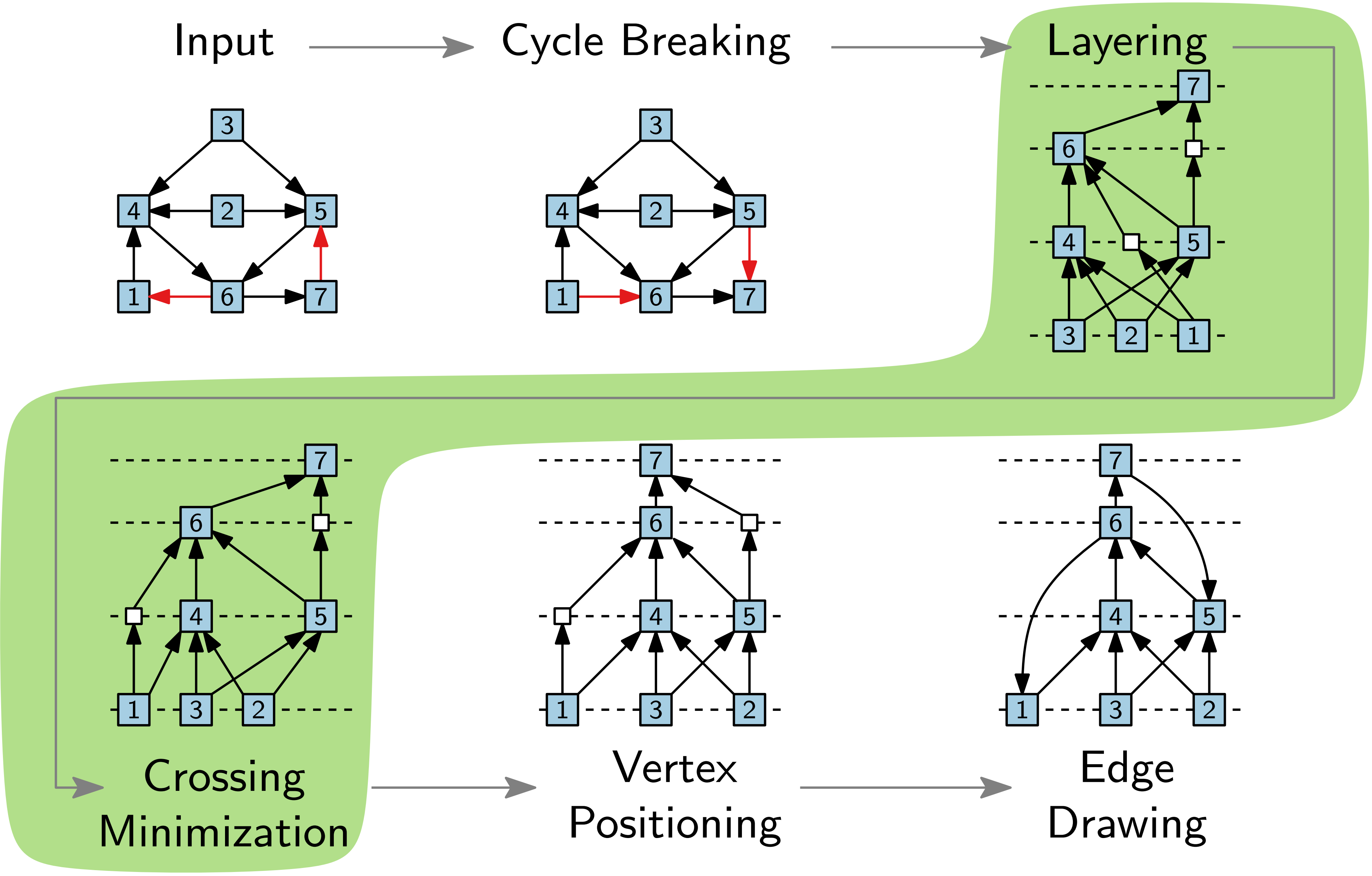
Step 3: Crossing Minimization



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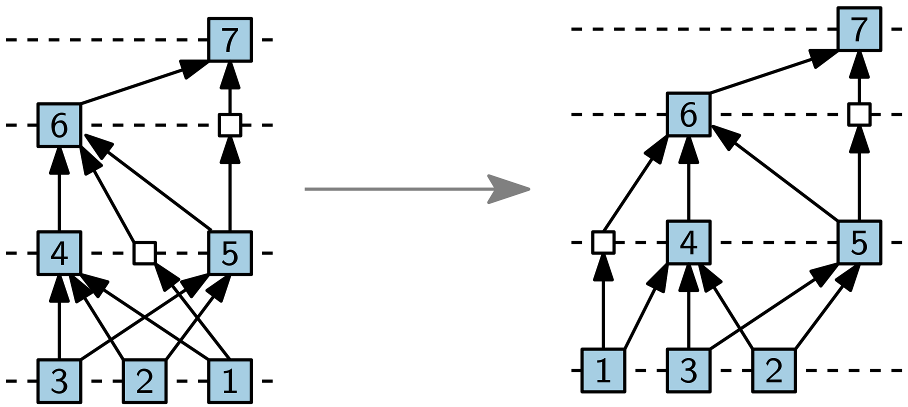
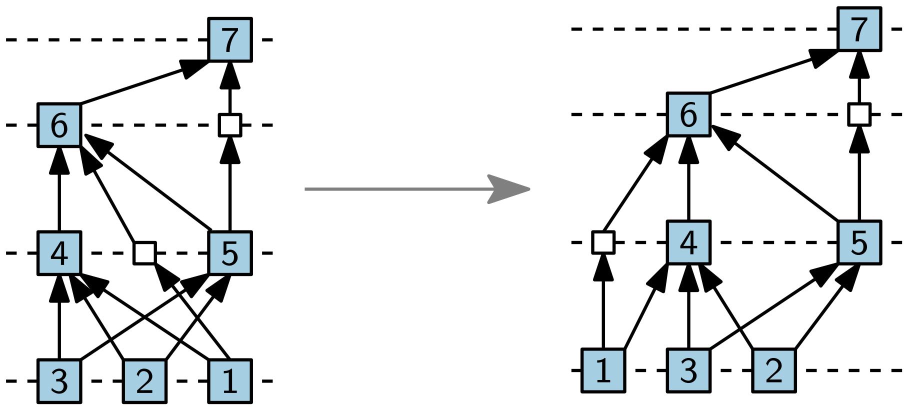


Problem.

Step 3: Crossing Minimization



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- Input: Graph G, layering $y\colon V(G) \to \{1, \ldots, n\}$

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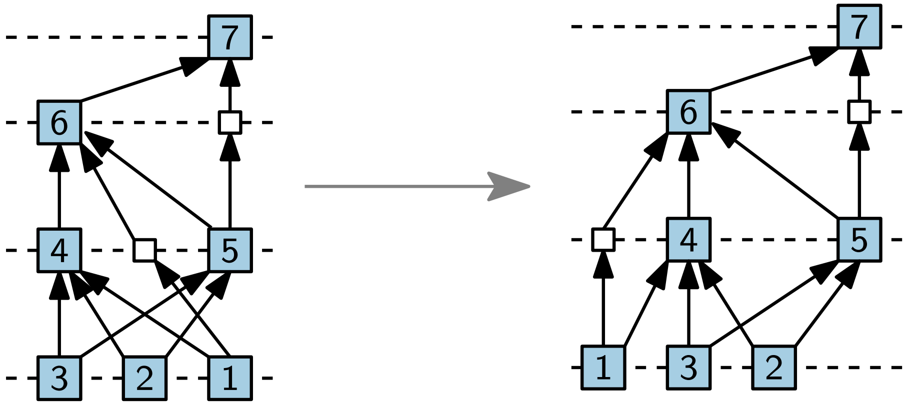

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- Input: Graph G, layering $y \colon V(G) \to \{1, \ldots, n\}$
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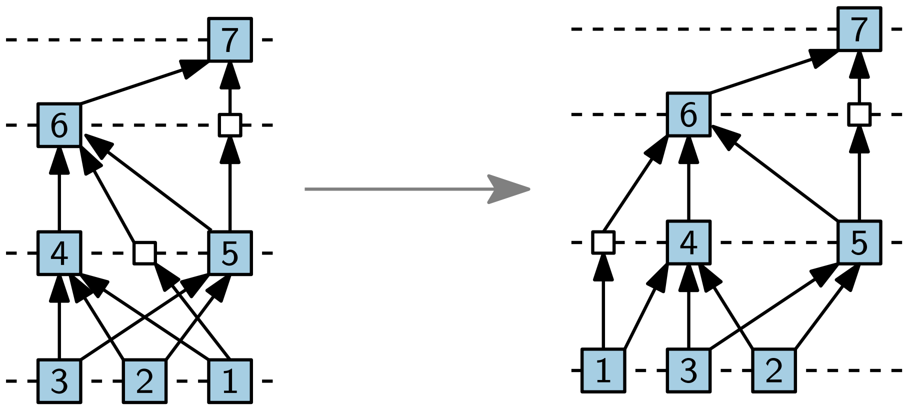

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- NP-hard, even for two layers. [Garey & Johnson '83]

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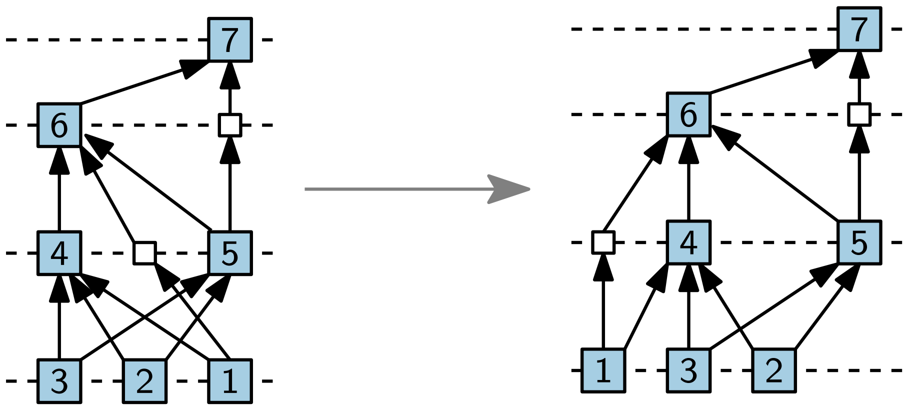

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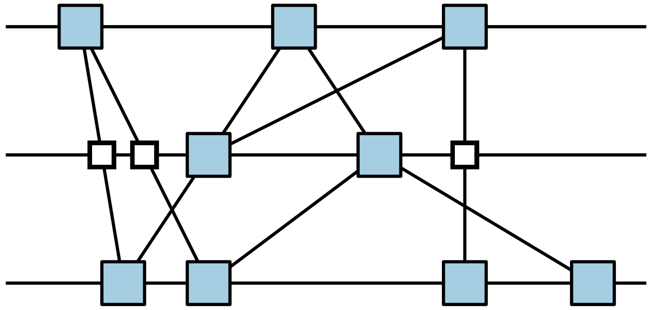
- Hardly any approaches optimize over multiple layers. ☹

Iterative Crossing Reduction

Observation. The number of crossings depends only on permutations of adjacent layers.



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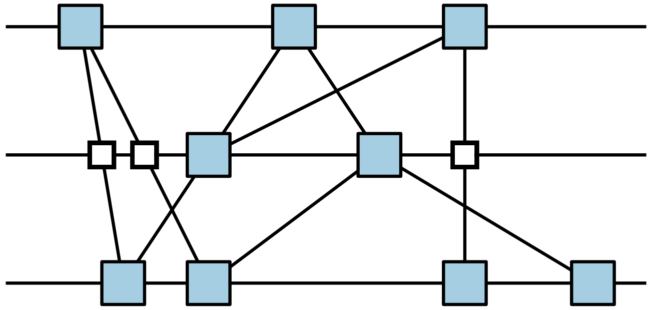
Idea.

- Permute one layer after the other.
- Treat dummy vertices as “regular” vertices.



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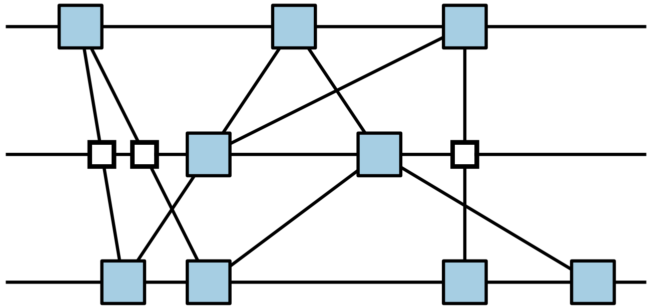

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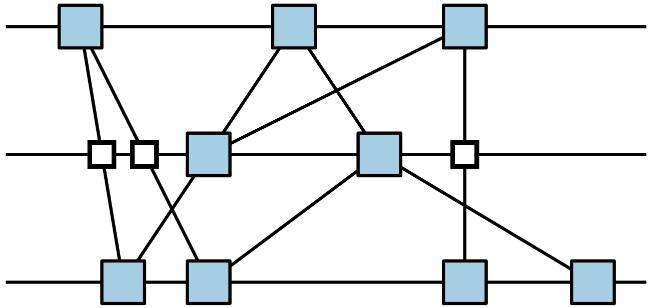

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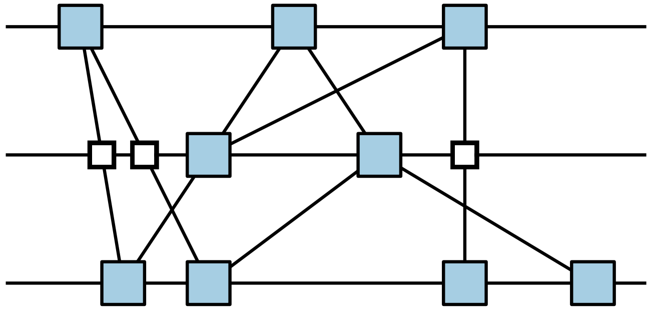

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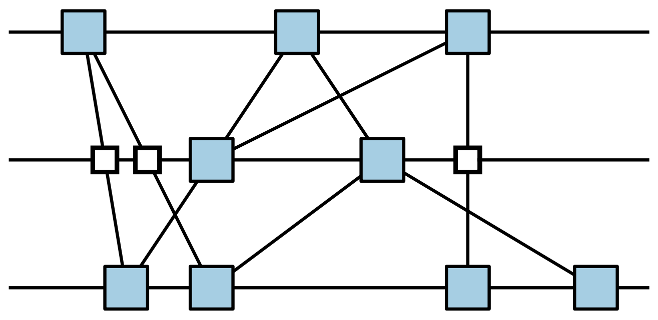

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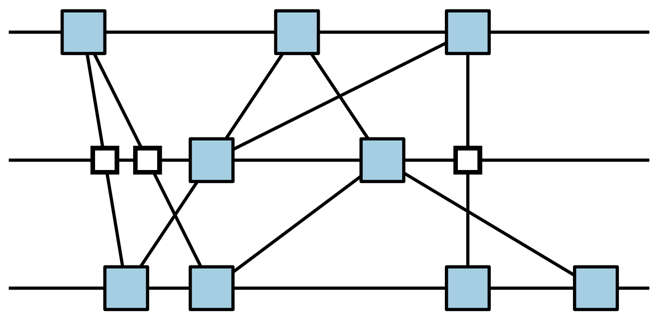

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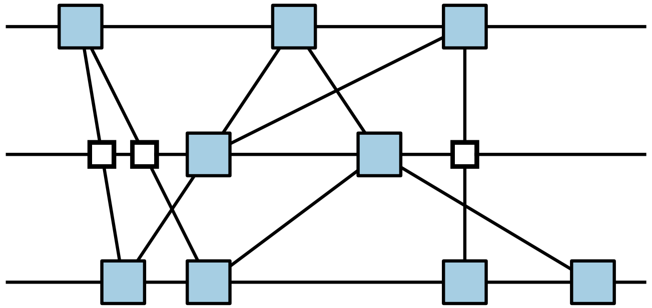

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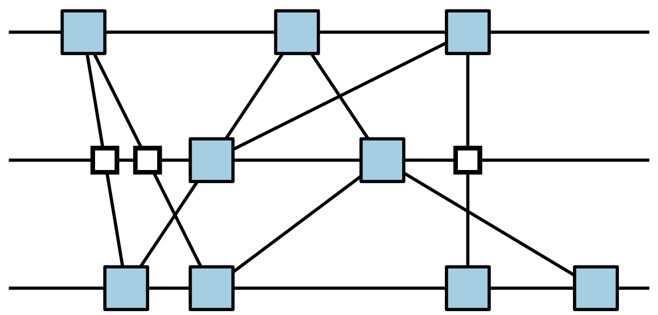

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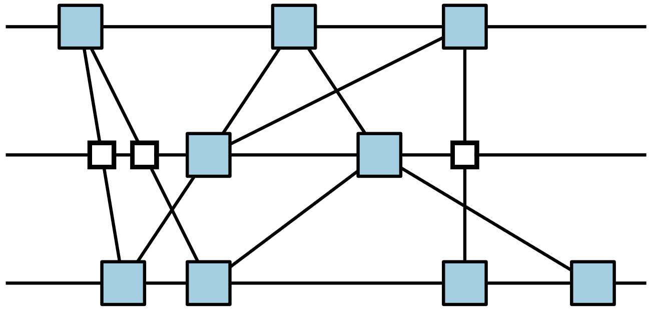

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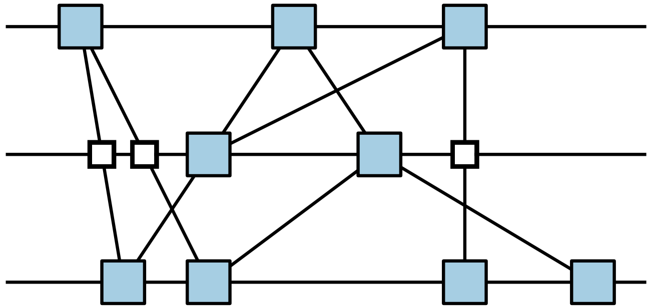

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one-sided crossing minimization

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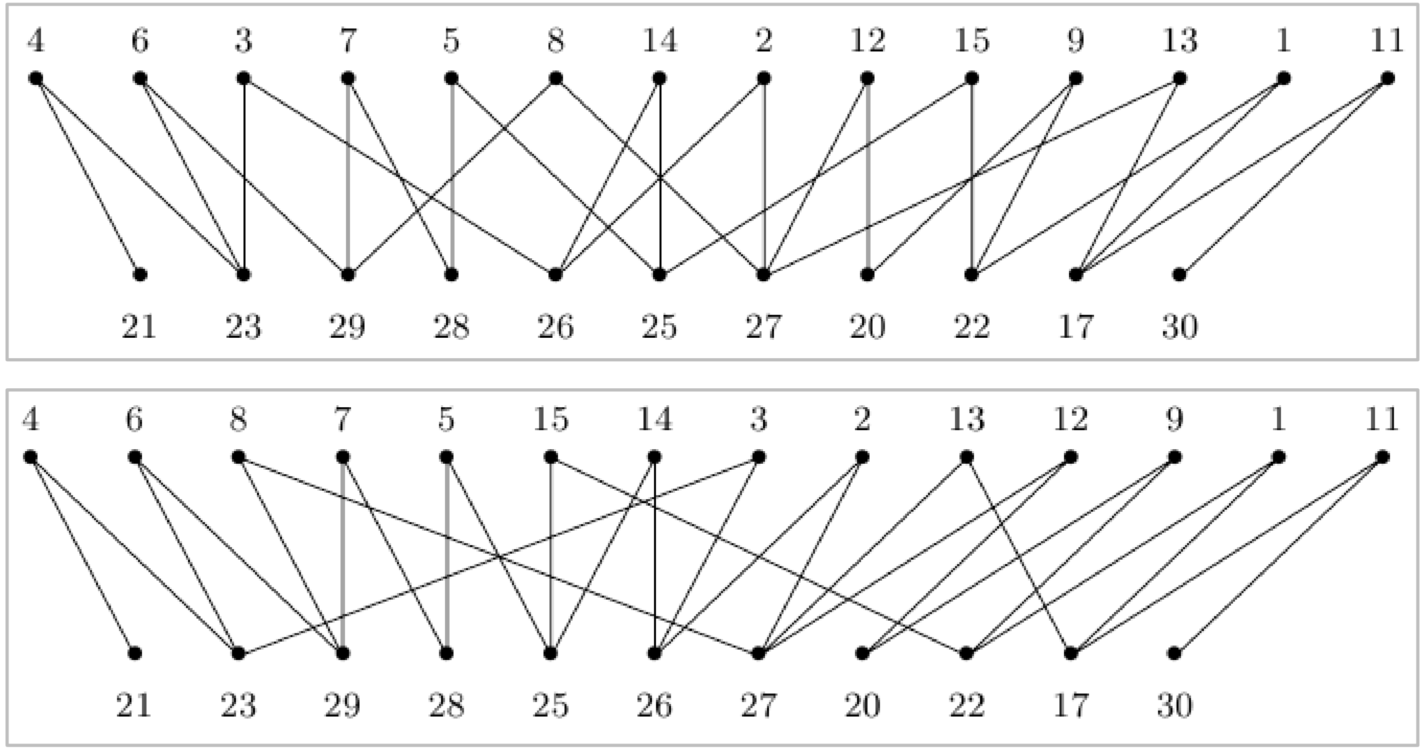
One-Sided Crossing Minimization

Problem.



[Kaufmann & Wagner: Drawing Graphs]

One-Sided Crossing Minimization

Problem.

- Input: – bipartite graph G with $V(G) = L_1 \cup L_2$,
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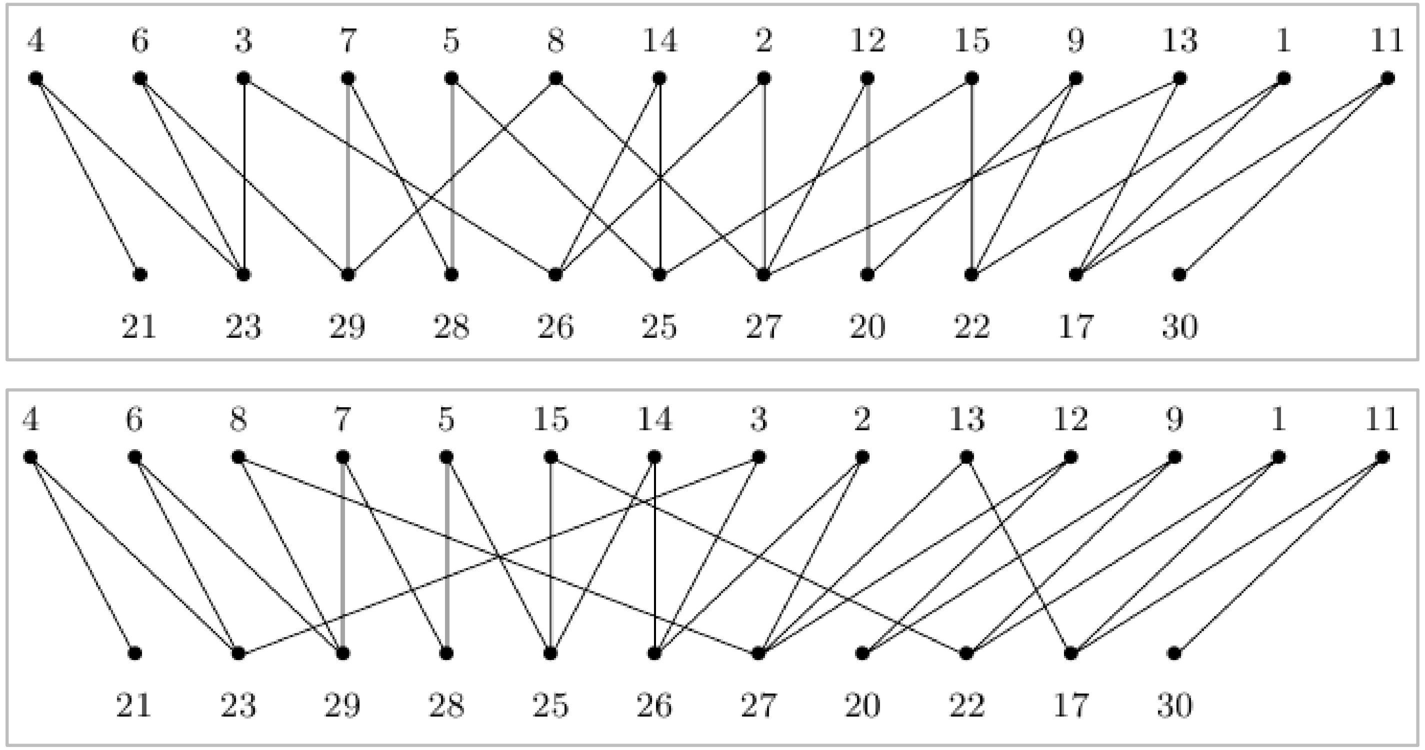

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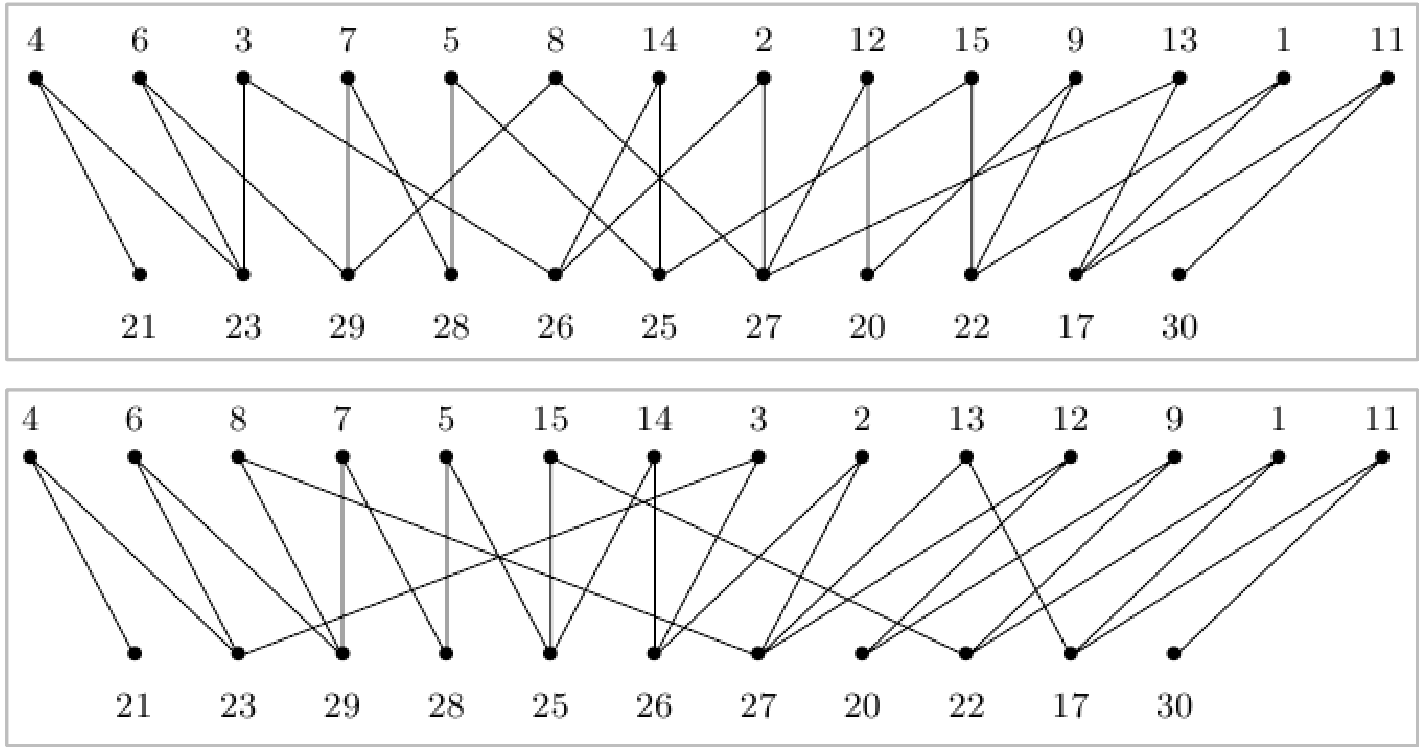

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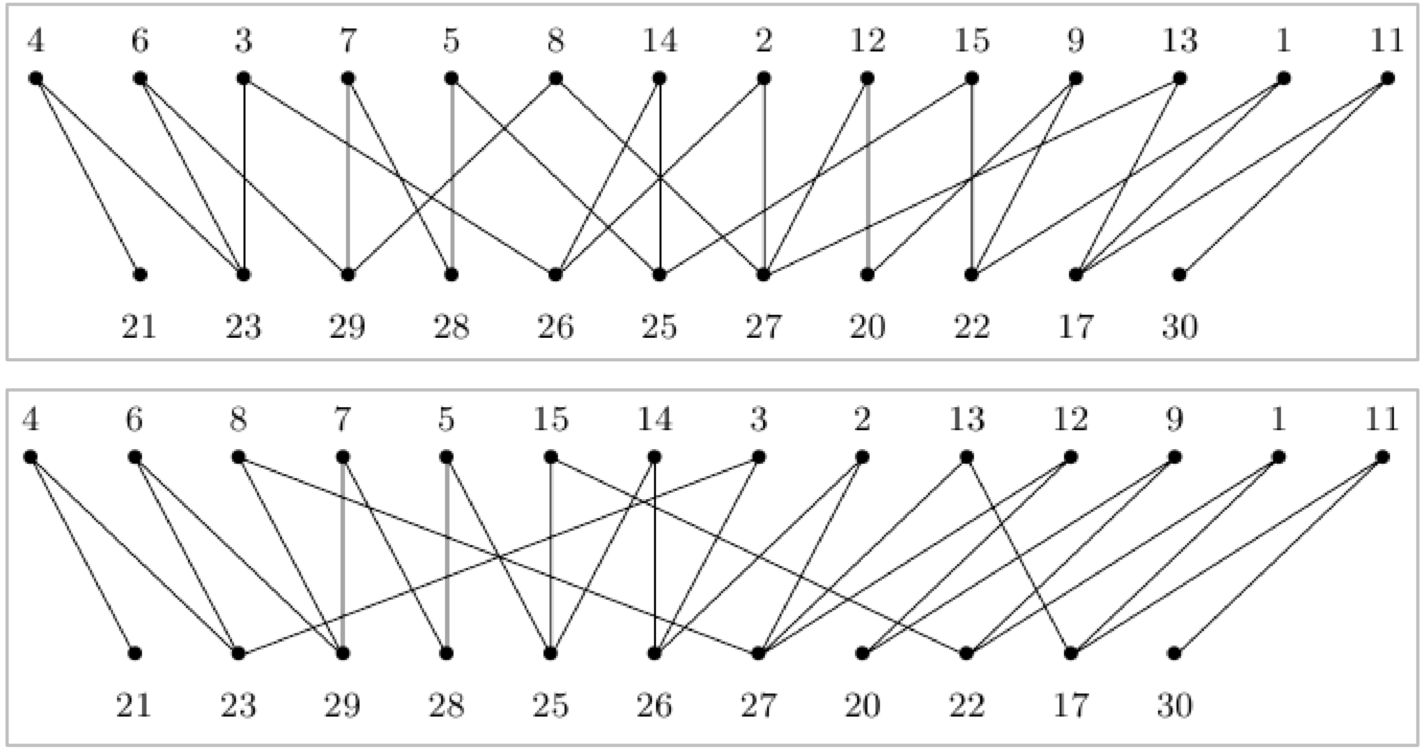
One-sided crossing minimization is NP-hard. [Eades & Whitesides '94]



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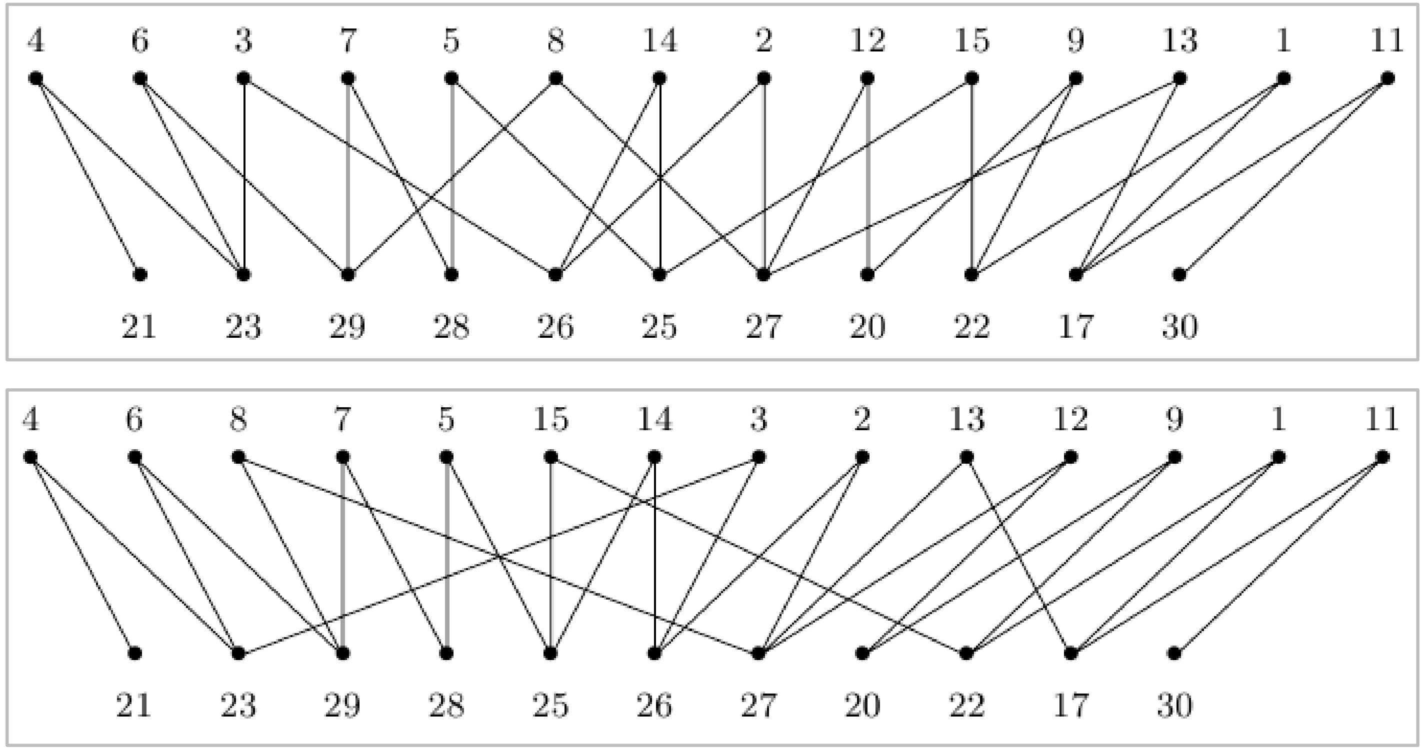

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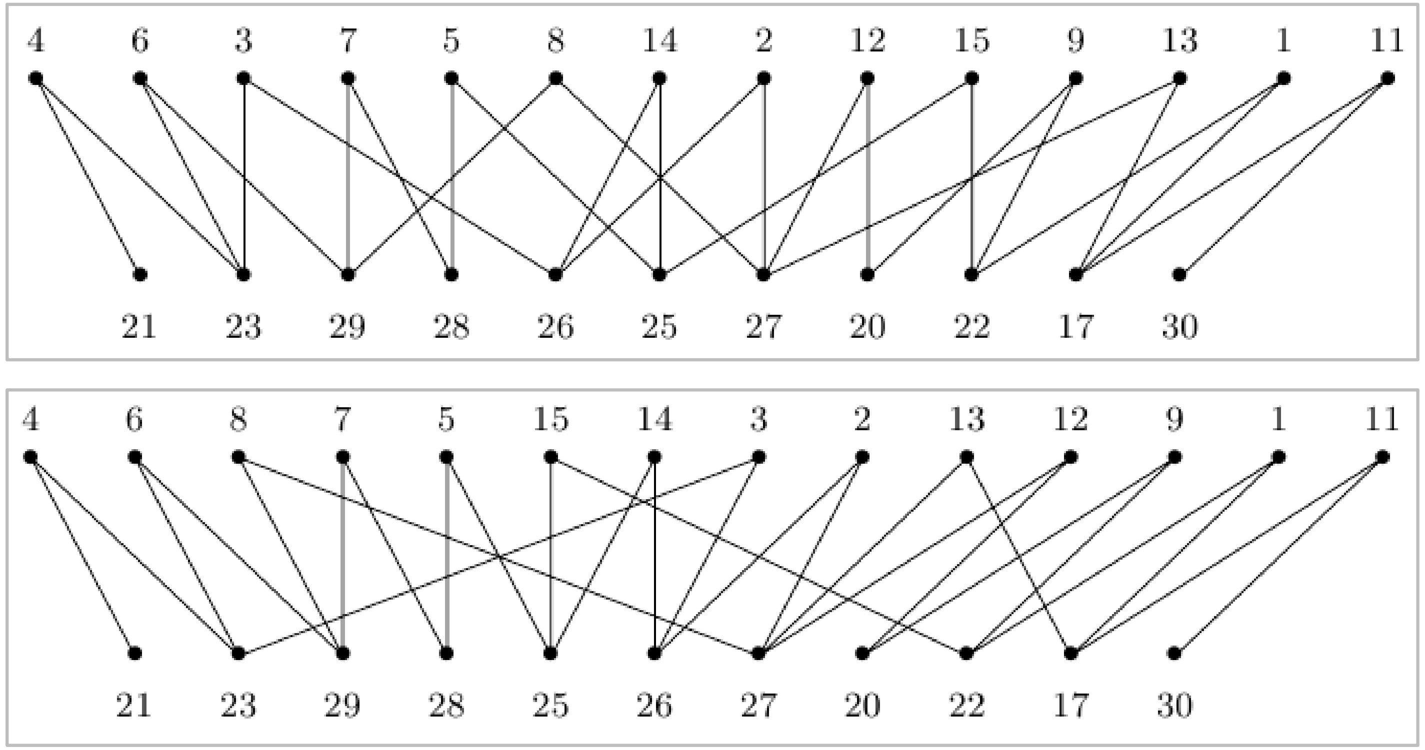

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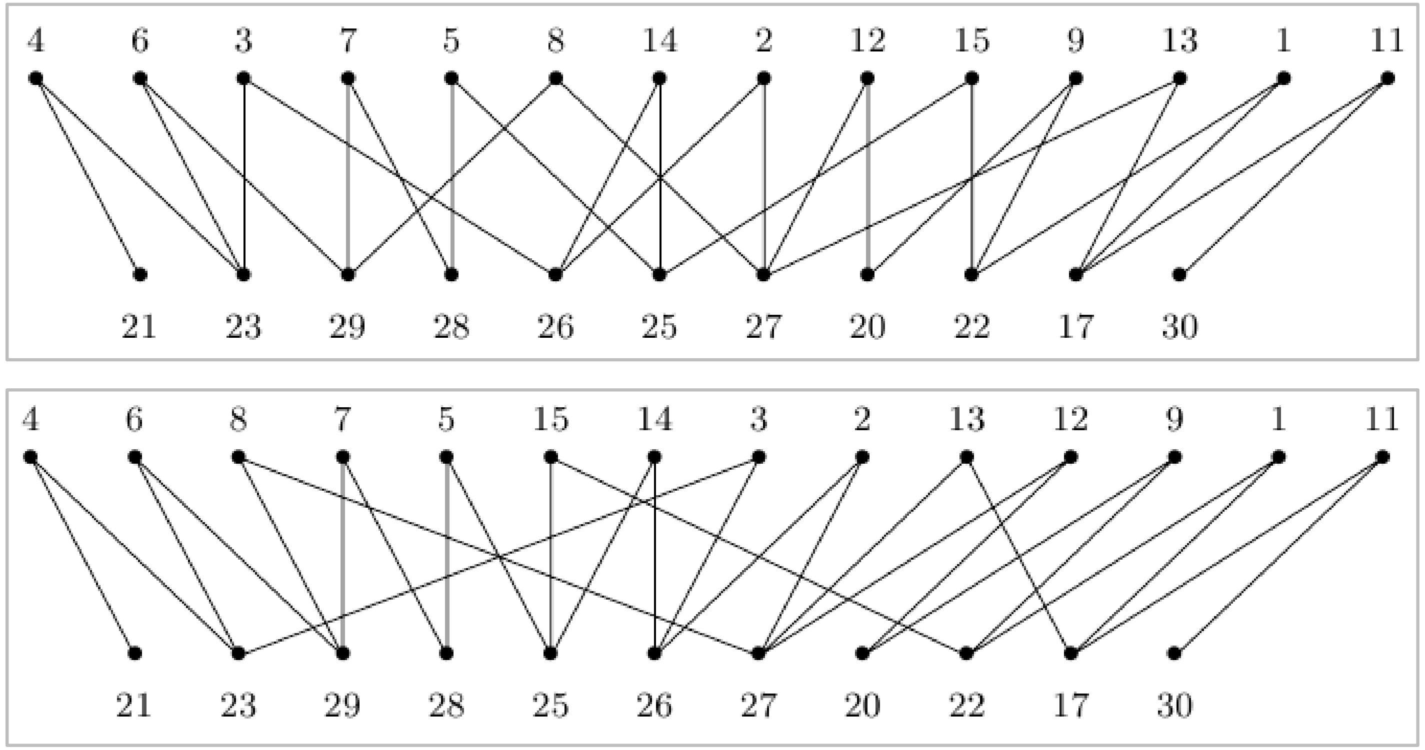

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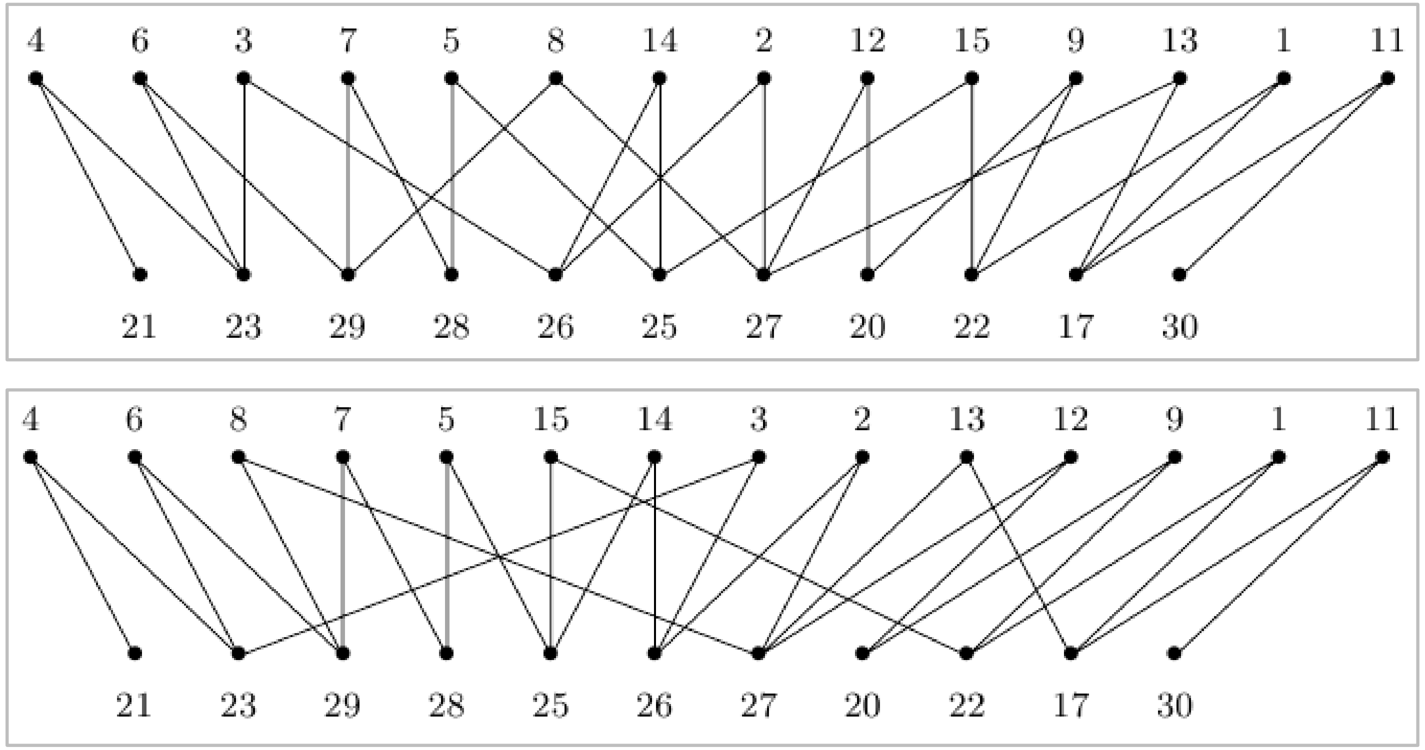

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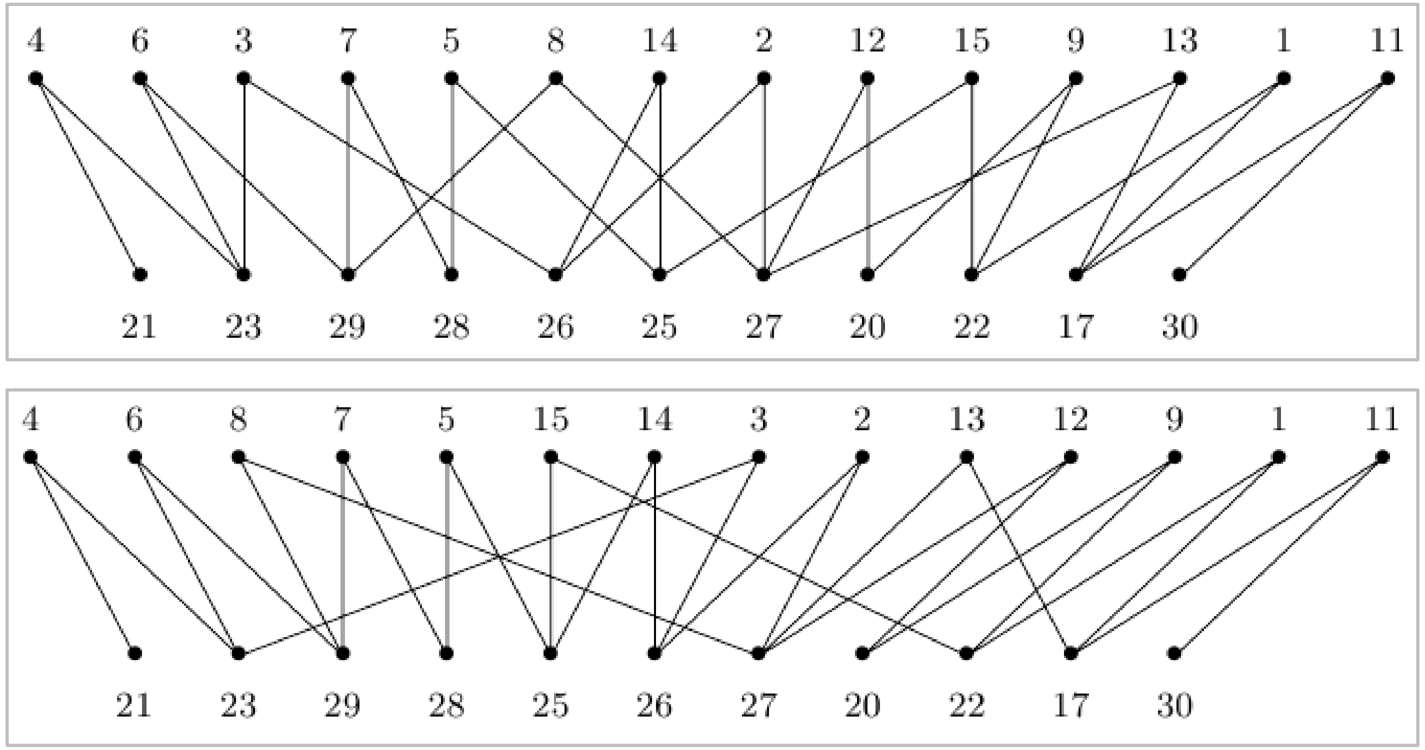

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One-Sided Crossing Minimization

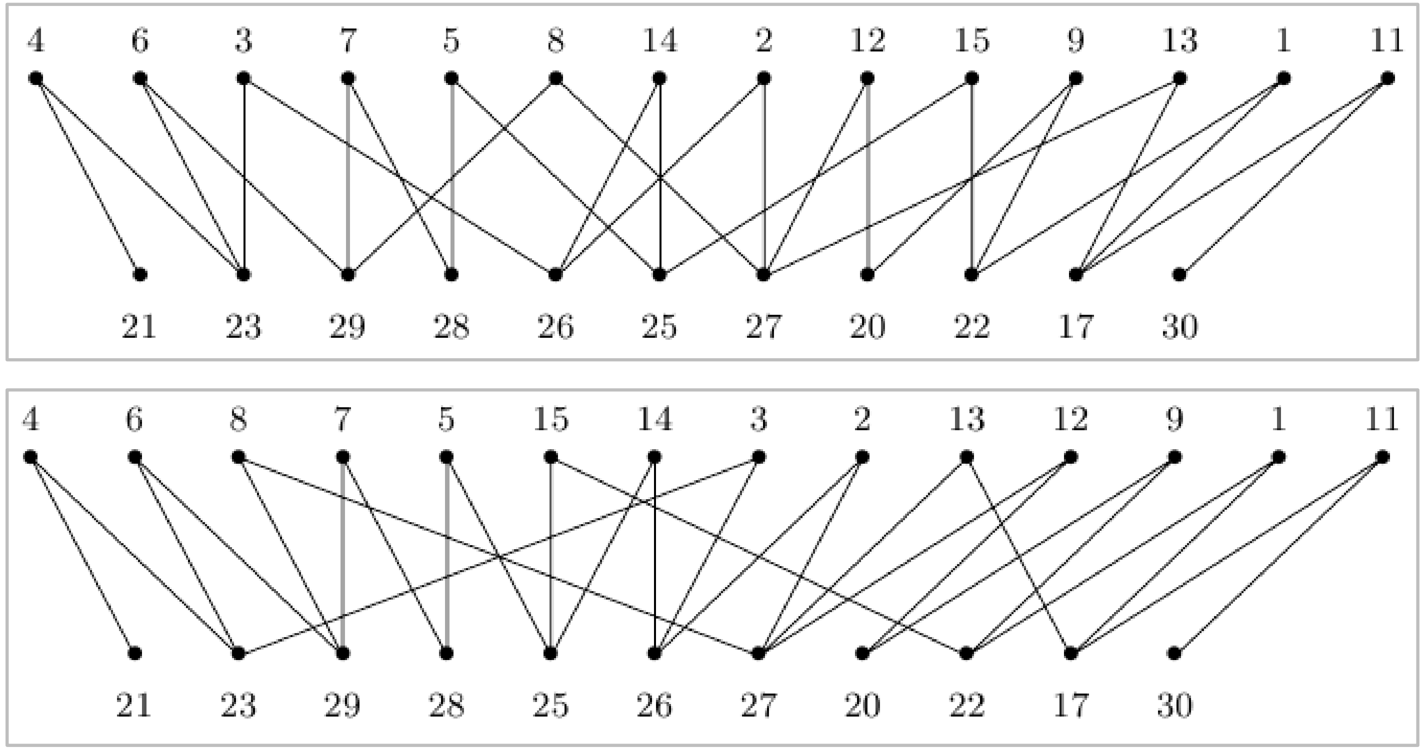
Problem.

- Input: – bipartite graph G with $V(G) = L_1 \cup L_2$,
  – permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.

One-sided crossing minimization is NP-hard. [Eades & Whitesides '94]

Algorithms.

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP
- FPT



[Kaufmann & Wagner: Drawing Graphs]

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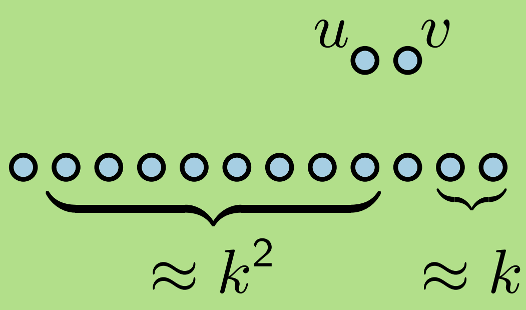

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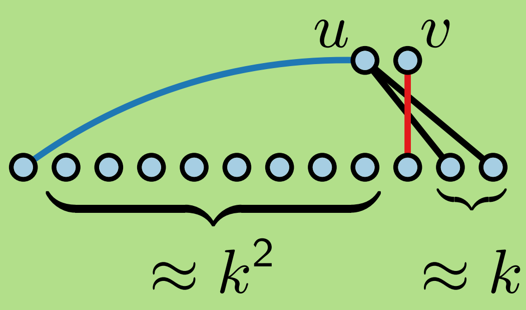

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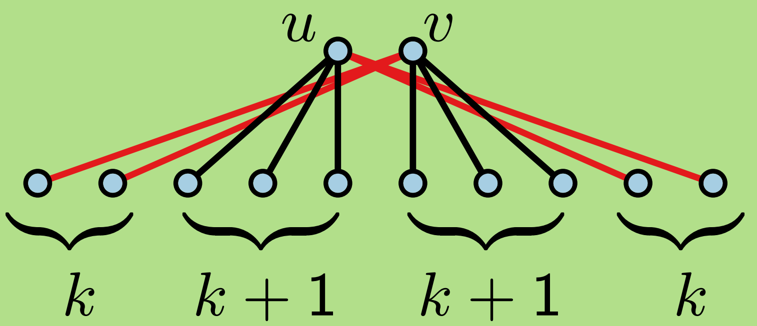

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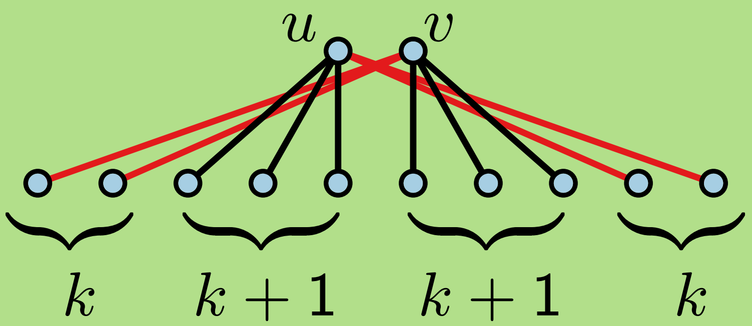

crossings: $2k(k+1) + k^2$ vs. $(k+1)^2$

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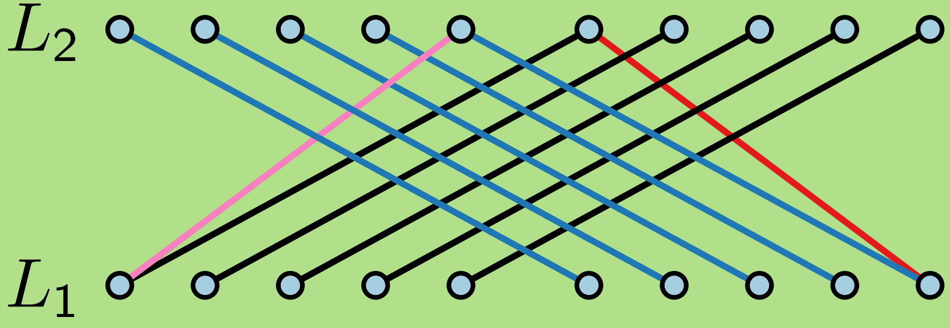

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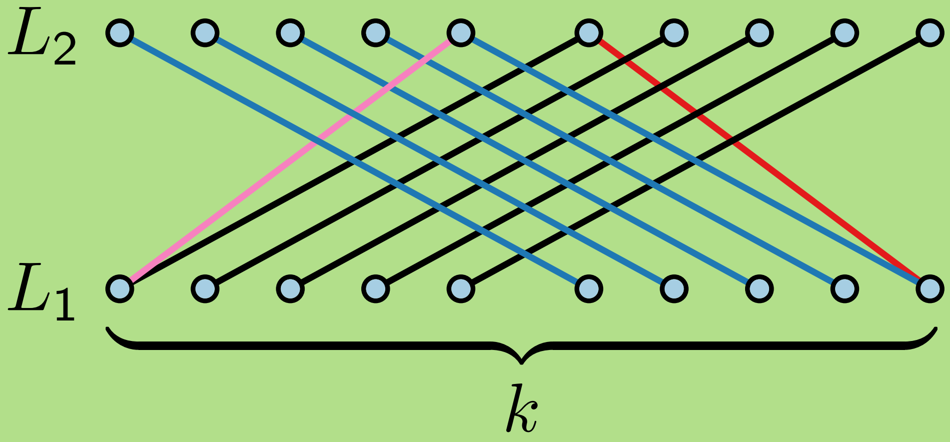

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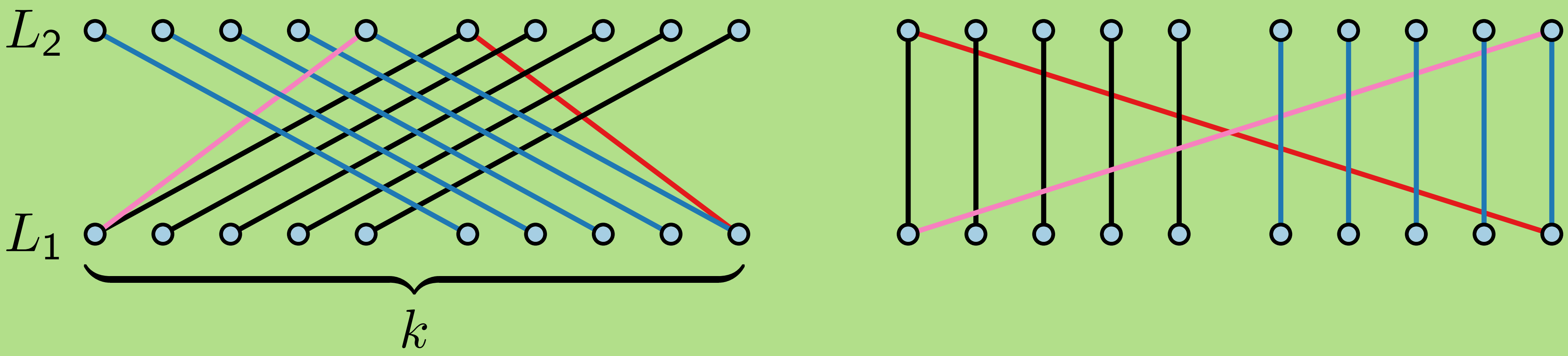

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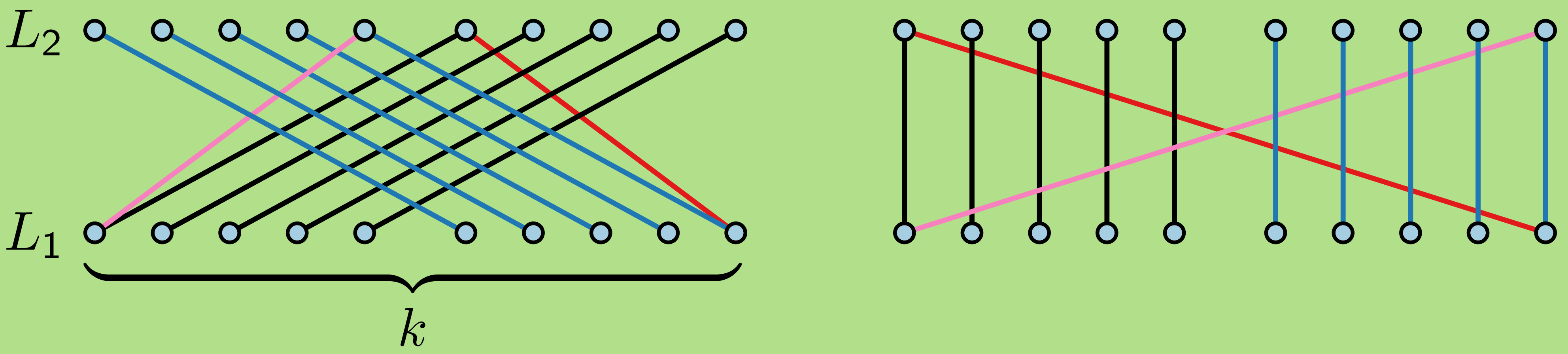
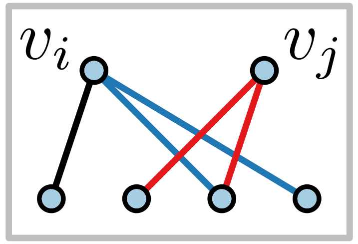


crossings: $\approx k^2/4$ $\approx 2k$

Integer Linear Program (ILP)

[Jünger & Mutzel, '97]

- Constant $c_{ij} := \#$ crossings between edges incident to v_i and v_j if $\pi_2(v_i) < \pi_2(v_j)$



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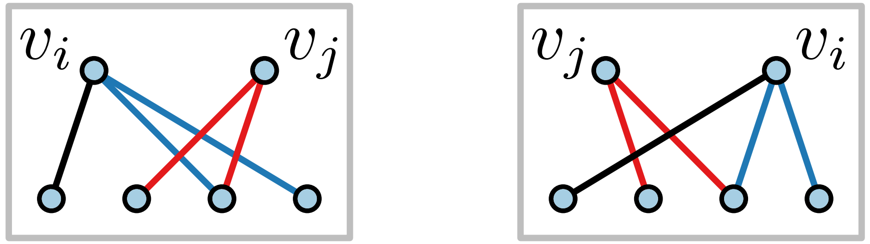

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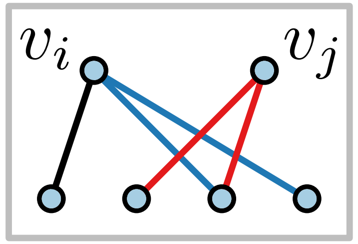

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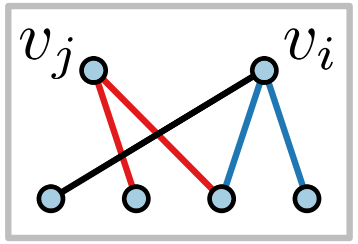

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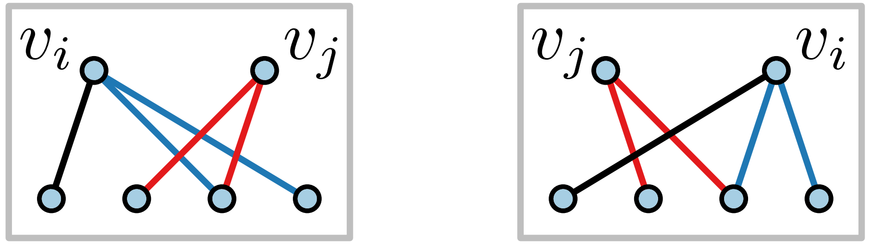

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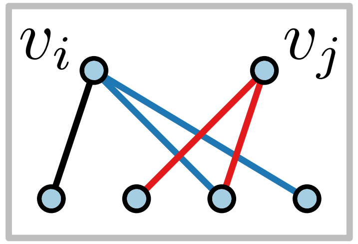

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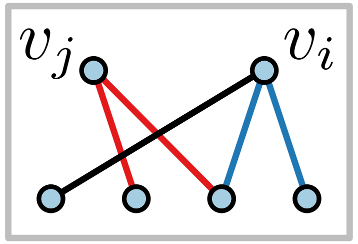

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$$\text{cross}(\pi_2) =$$

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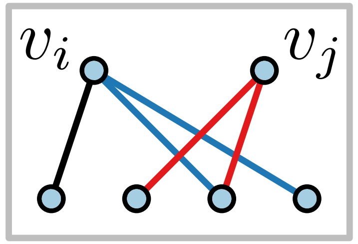

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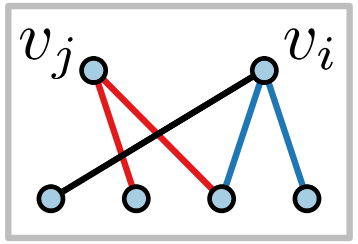

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Integer Linear Program (ILP)

[Jünger & Mutzel, '97]

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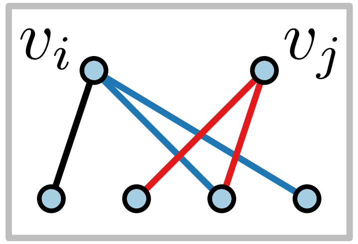

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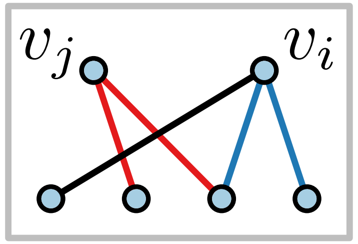

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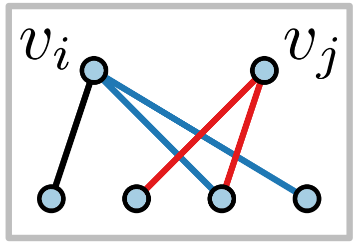

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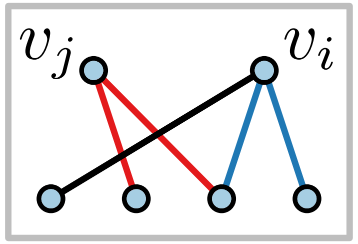

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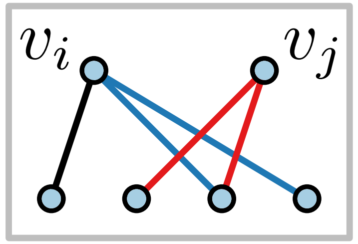

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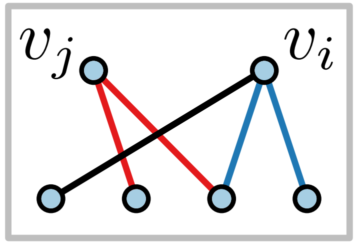

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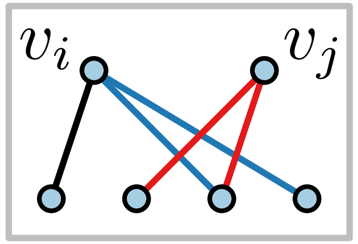

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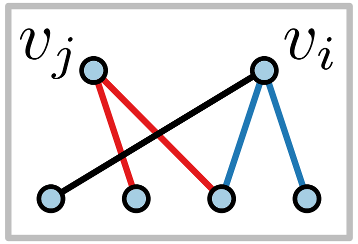

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i.e., if $x_{ij} = 1$ and $x_{jk} = 1$, then $x_{ik} = 1$

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- finds optimal solution

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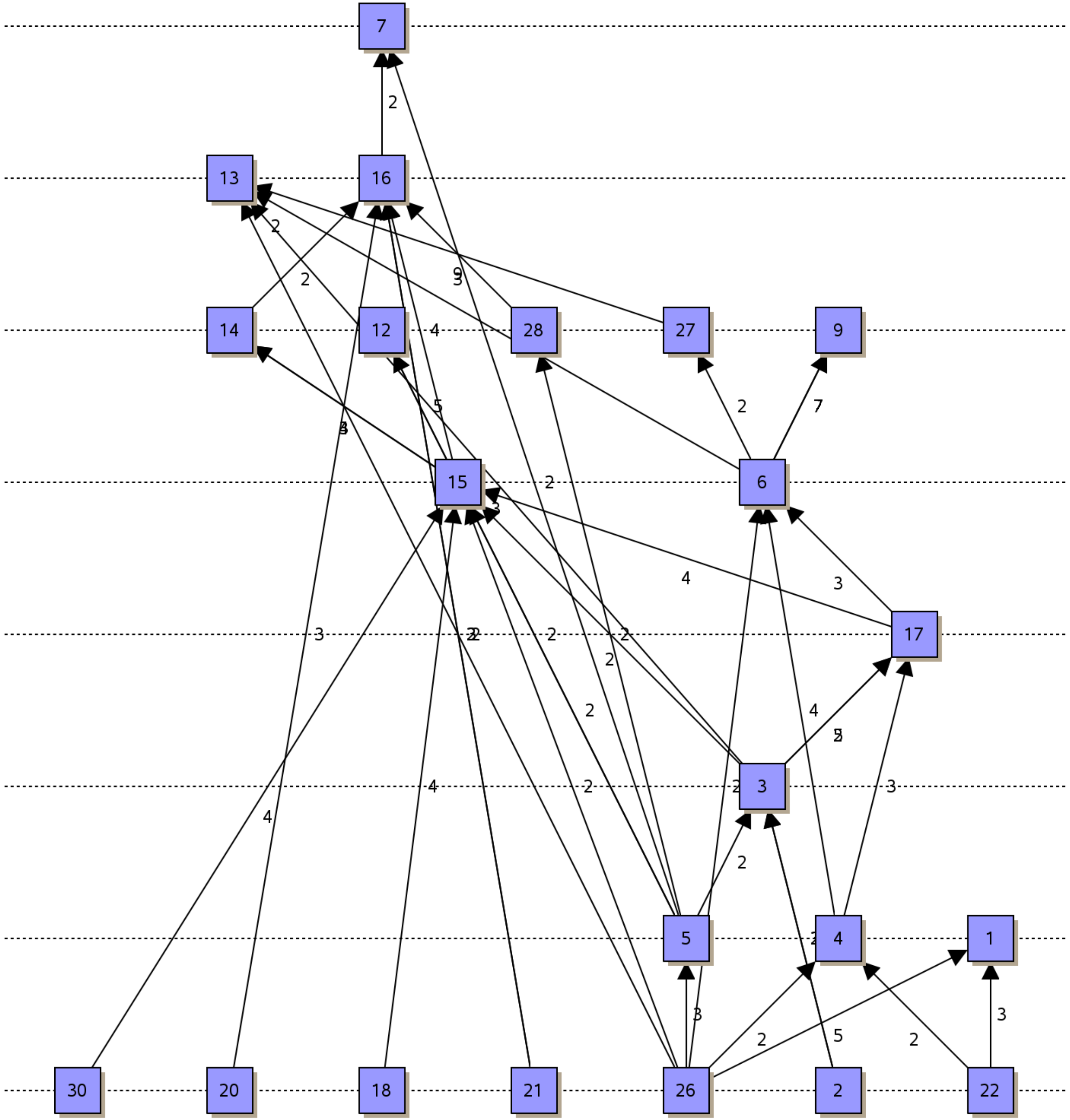
- branch-and-cut technique applicable for this ILP
- useful for graphs of small to medium size
- finds optimal solution
- solution in polynomial time is not guaranteed

Iterations on Example



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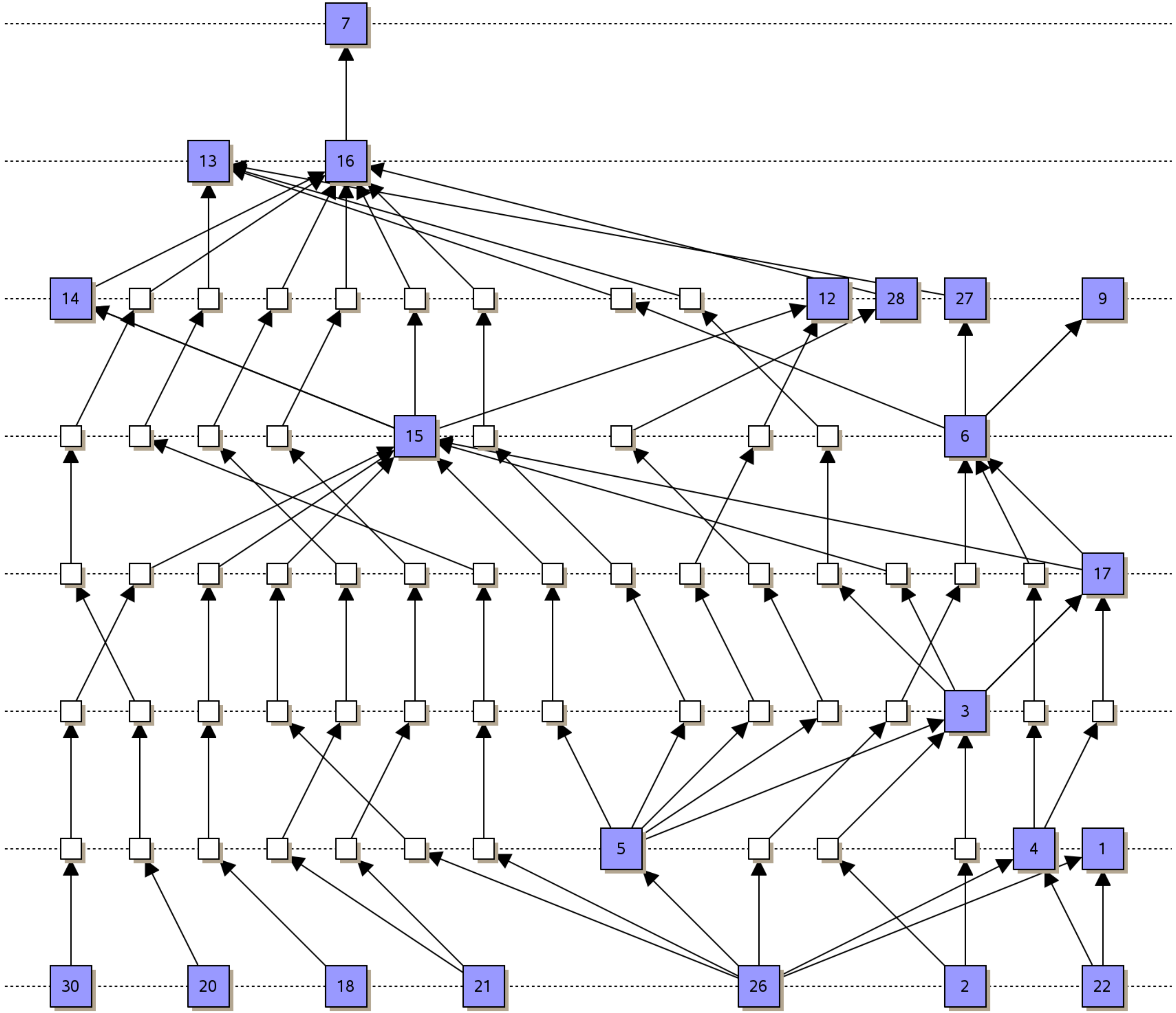

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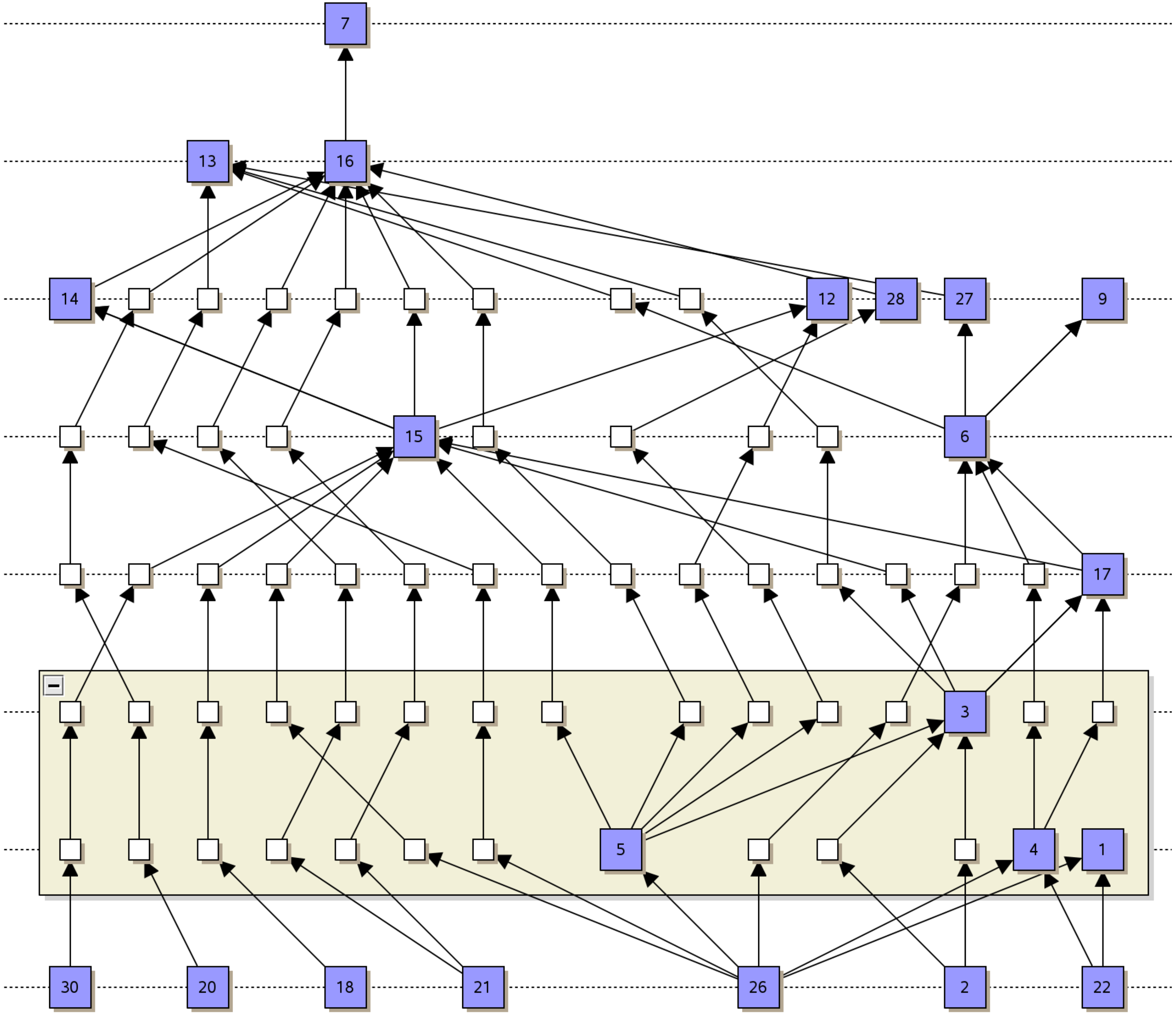

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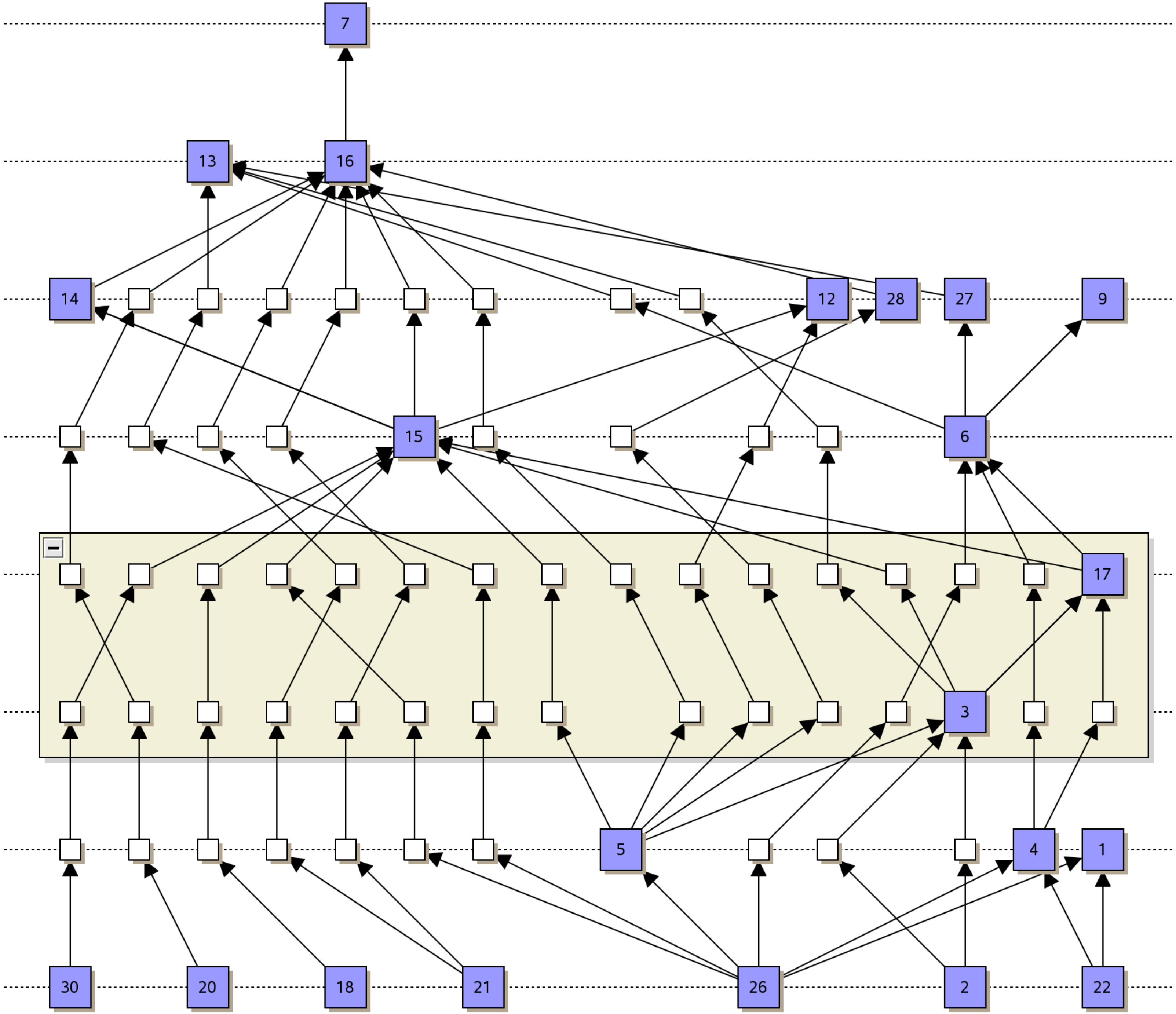

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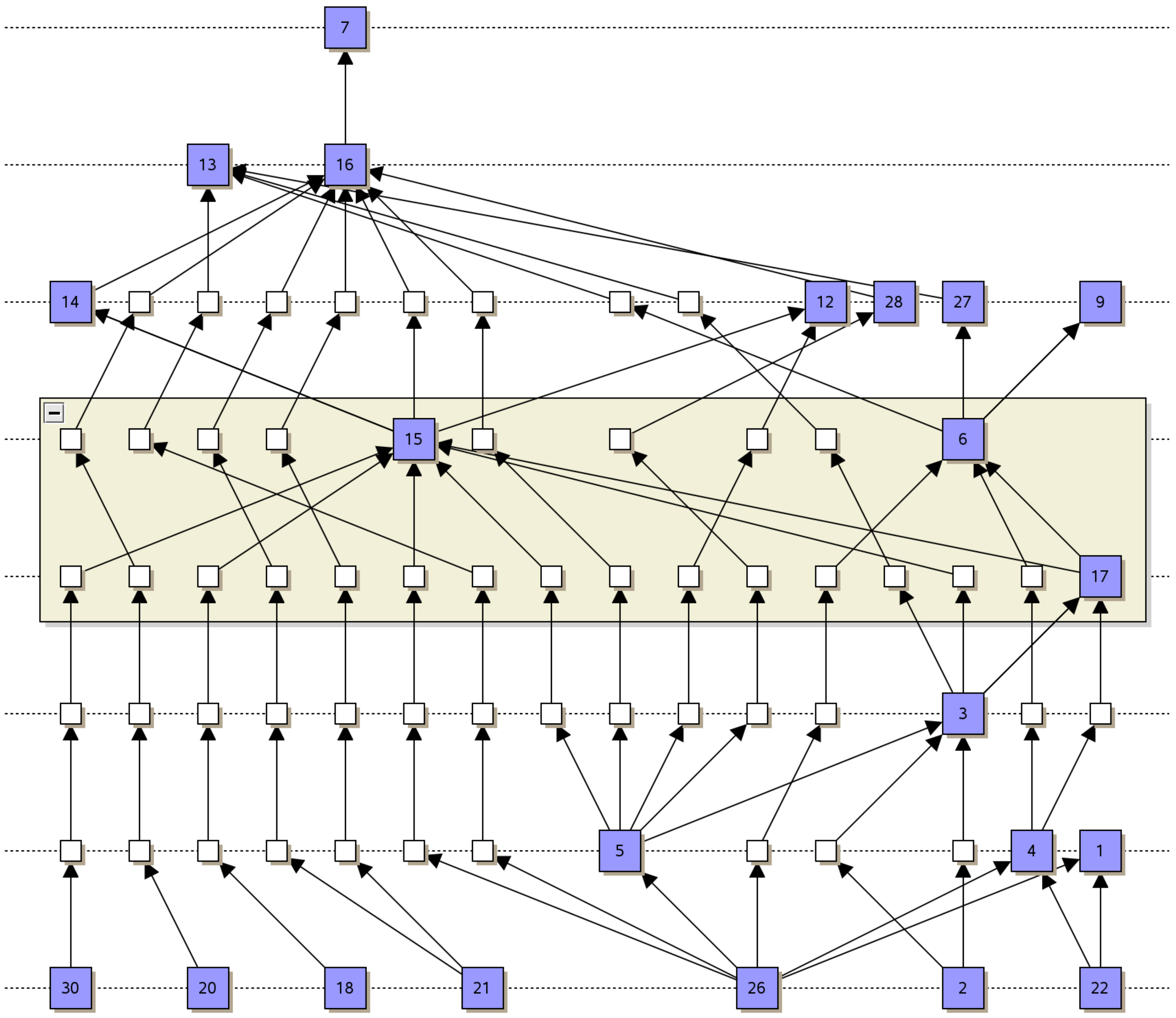

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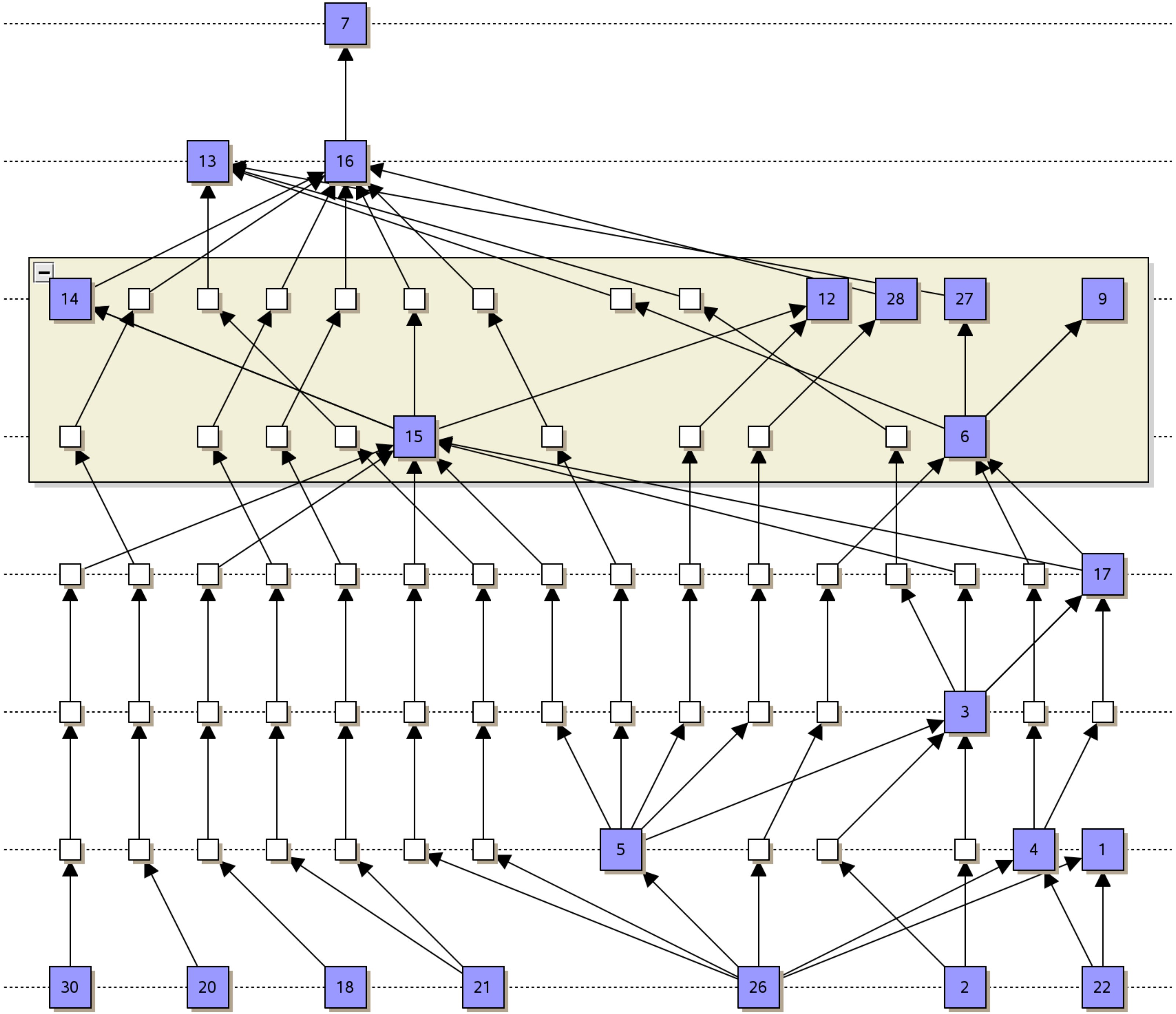

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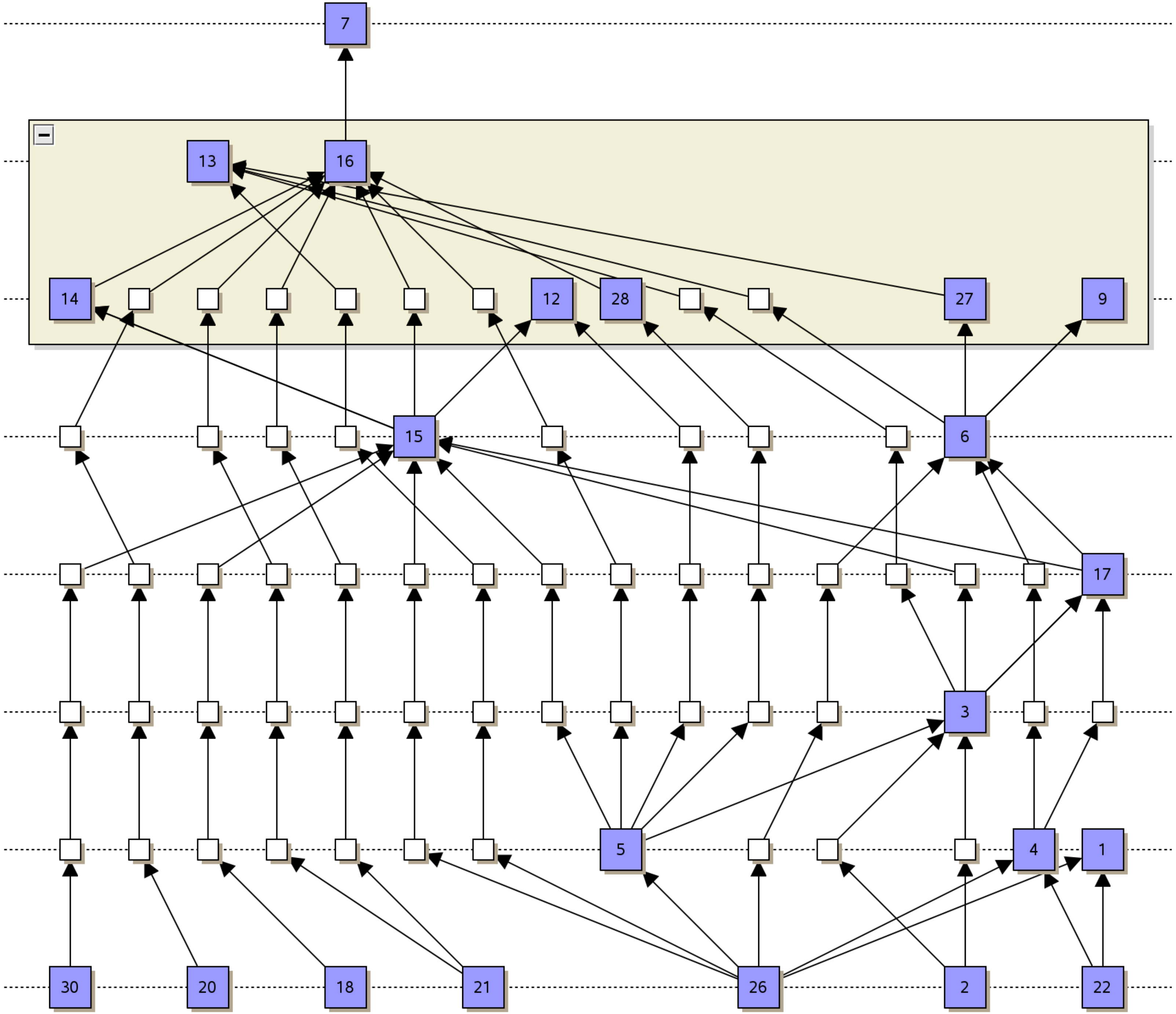

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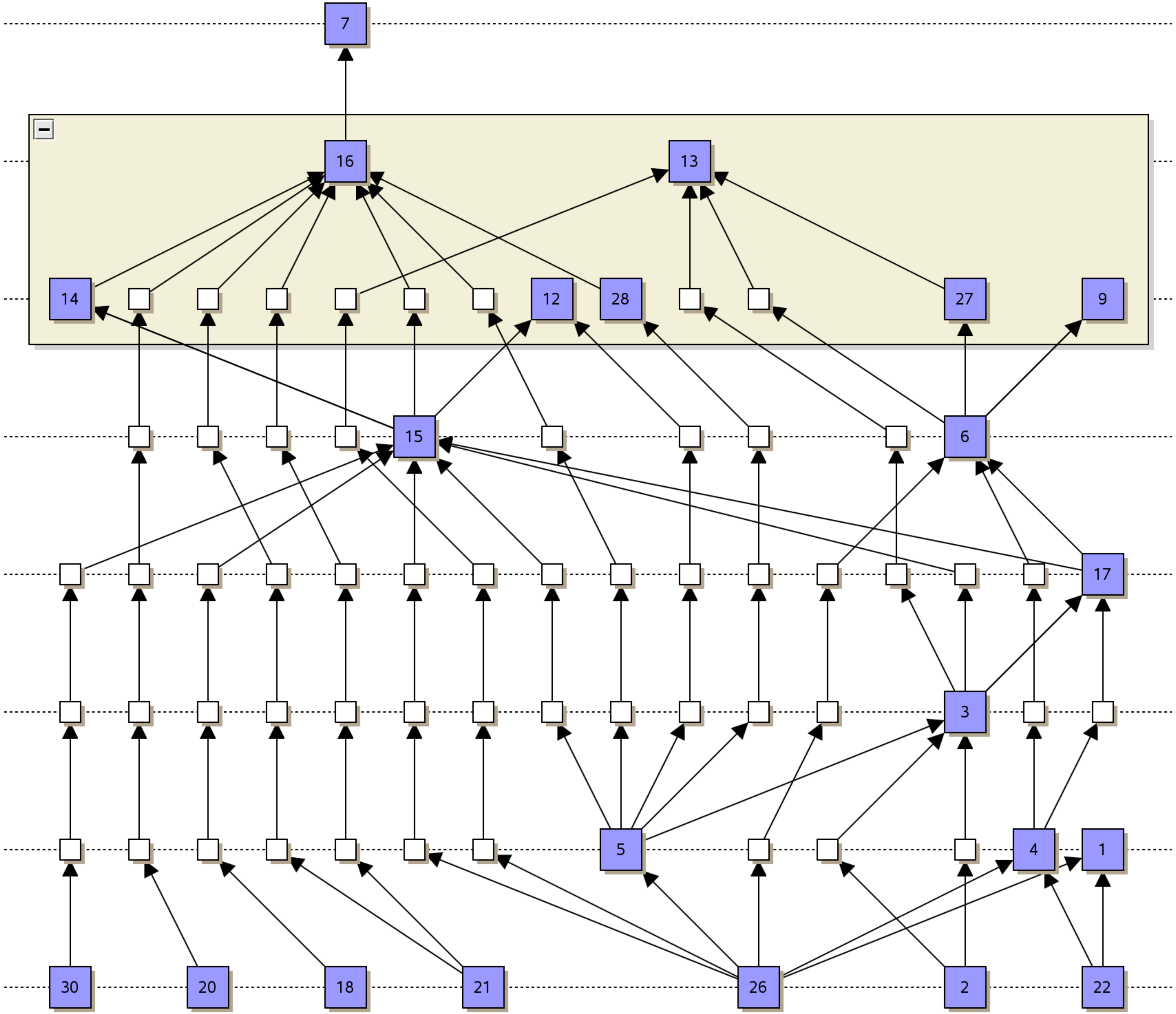

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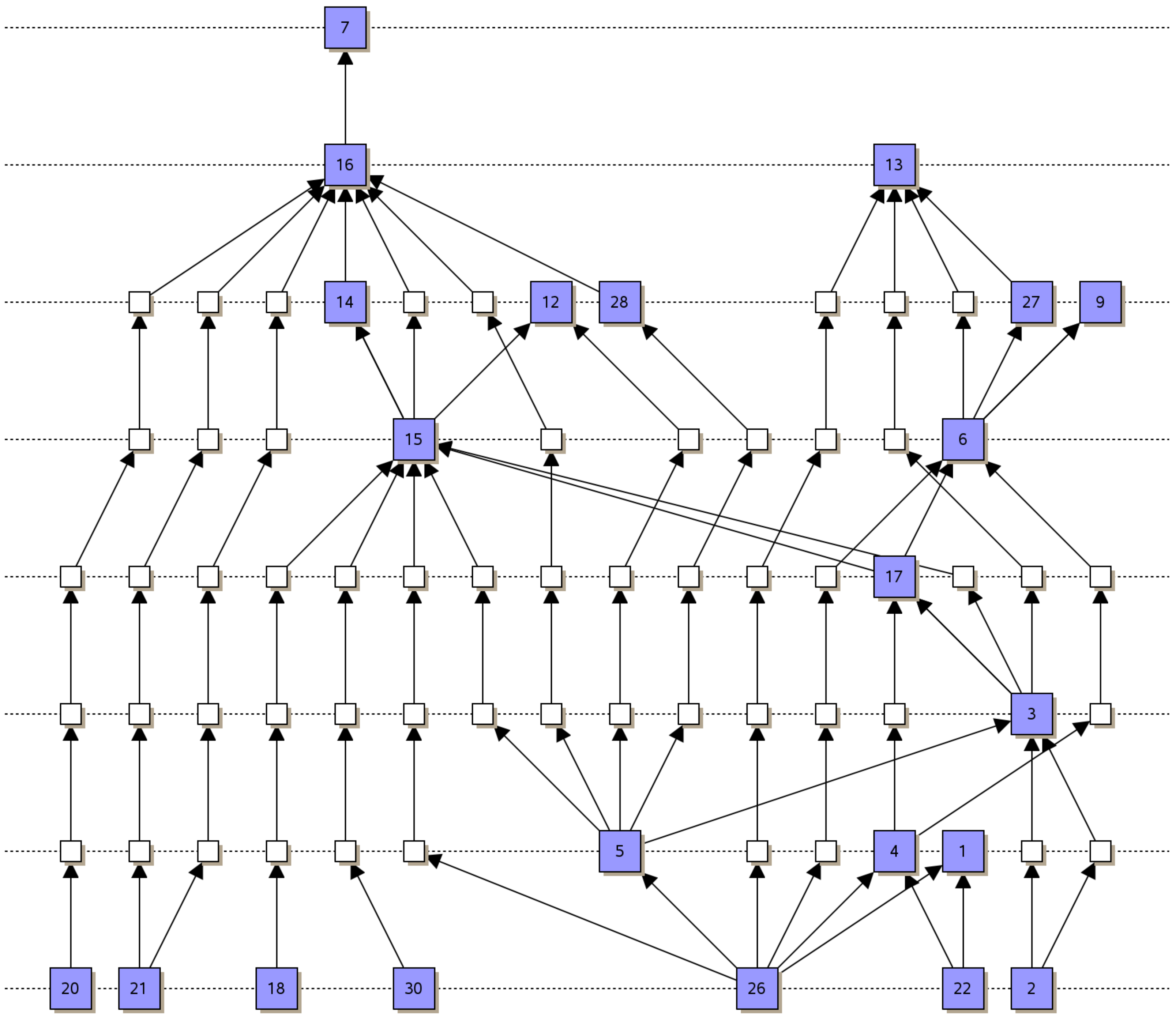
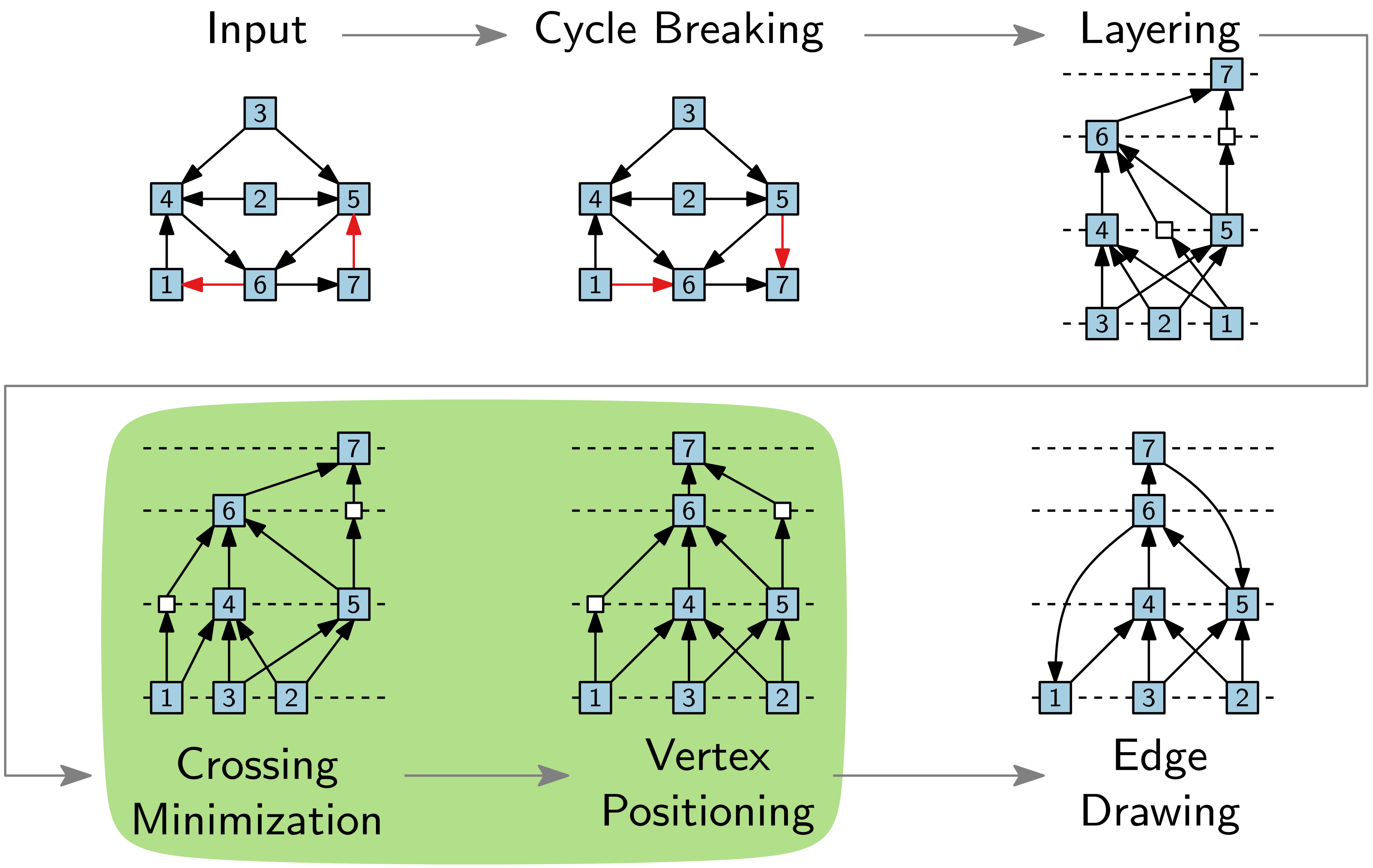


Step 4: Vertex Positioning



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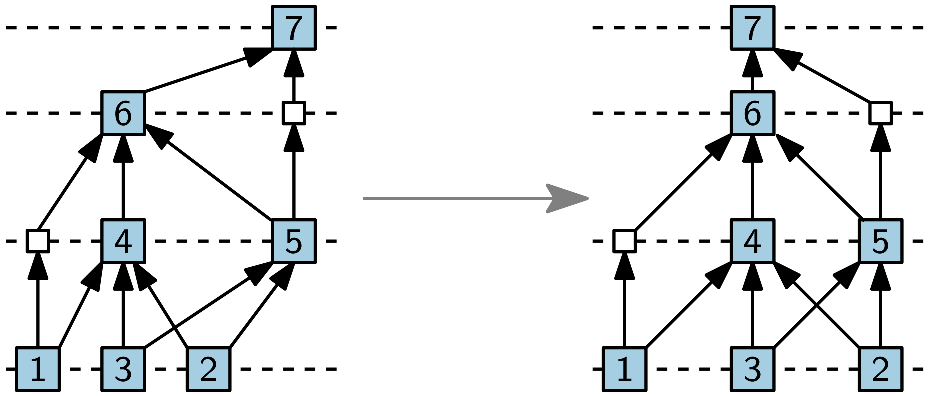
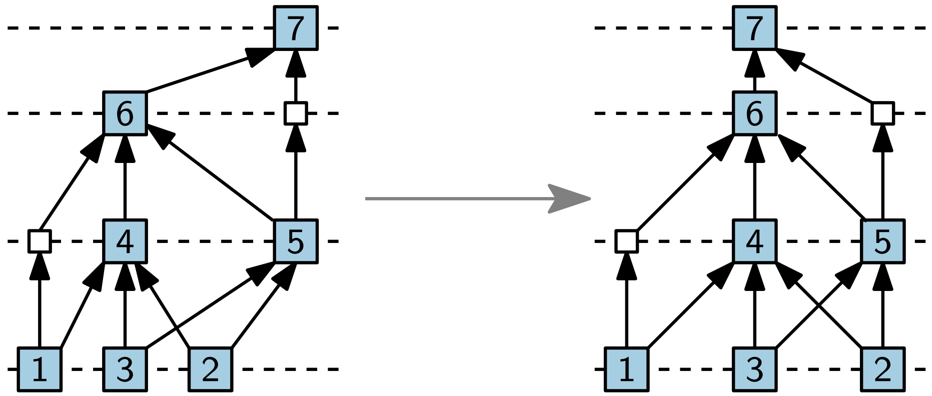


Step 4: Vertex Positioning

Goals.

- paths of a single edge should be (close to) straight
- vertices on a layer evenly spaced
- perfer vertical edges



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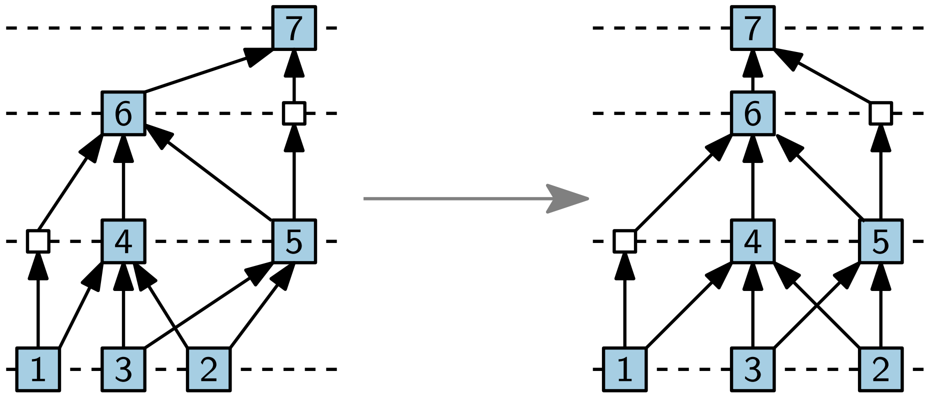

- **Exact:** Quadratic Program (QP)

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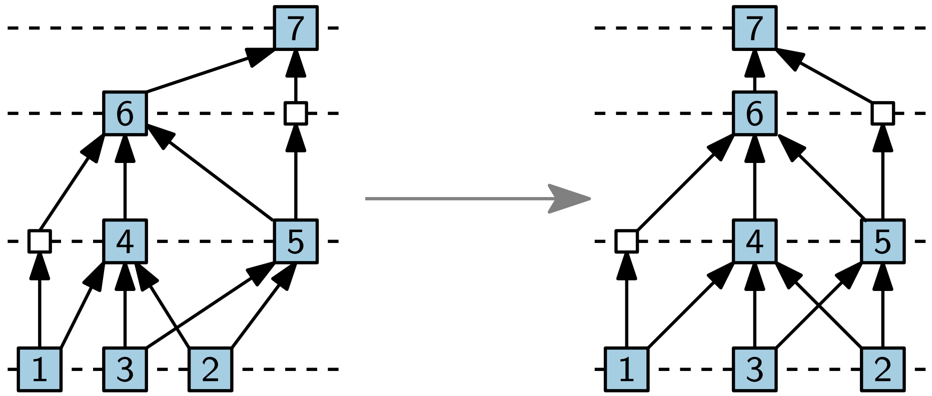
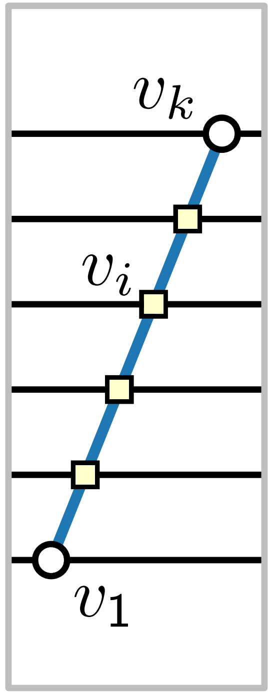


- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach

Quadratic Program

- Let $e = (v_1, v_k)$ be an edge of G, and let $p_e = (v_1, \ldots, v_k)$ be the corresponding path with dummy vertices $v_2, \ldots, v_{k-1}$.



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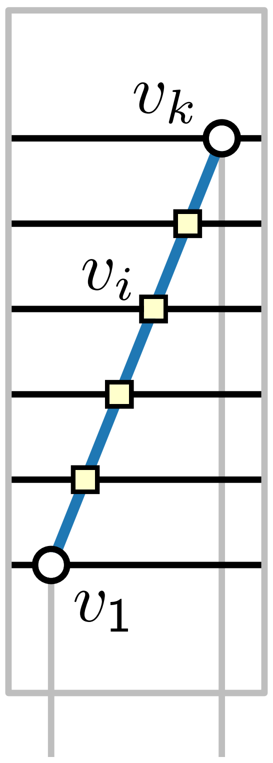

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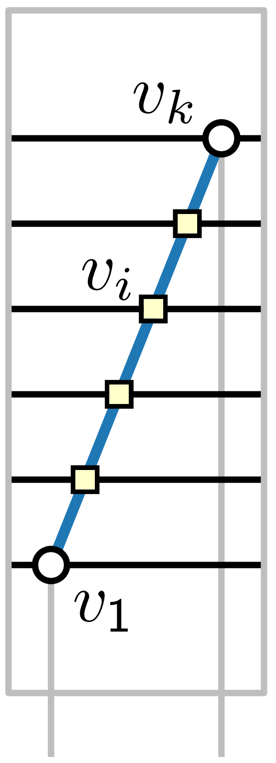

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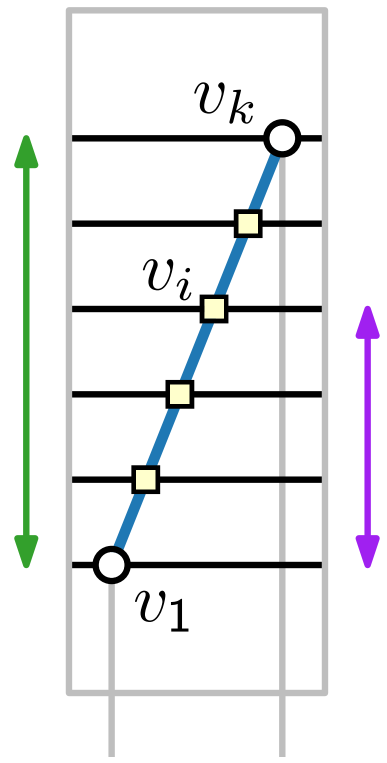

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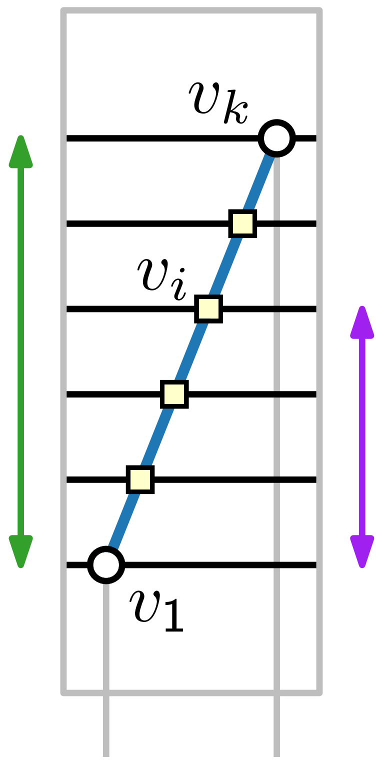

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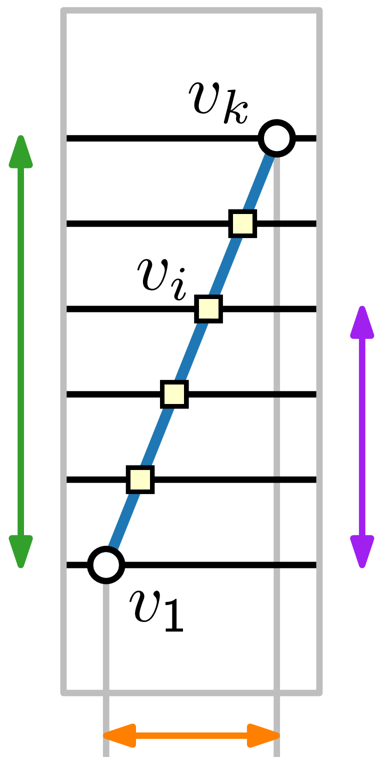

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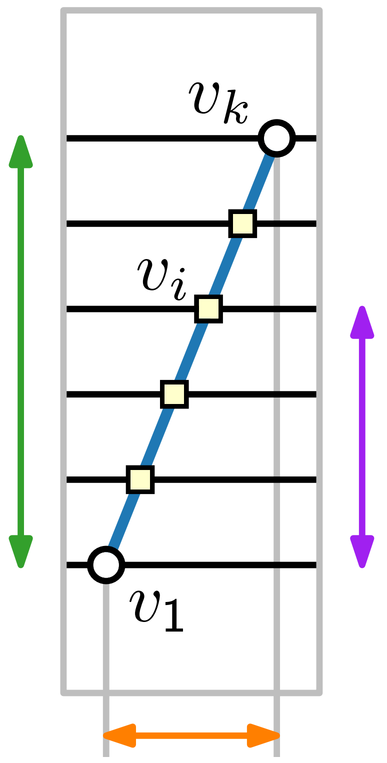

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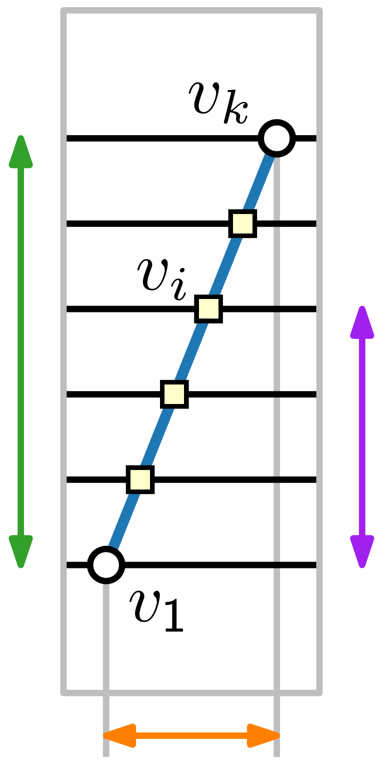
- Define the deviation from the line



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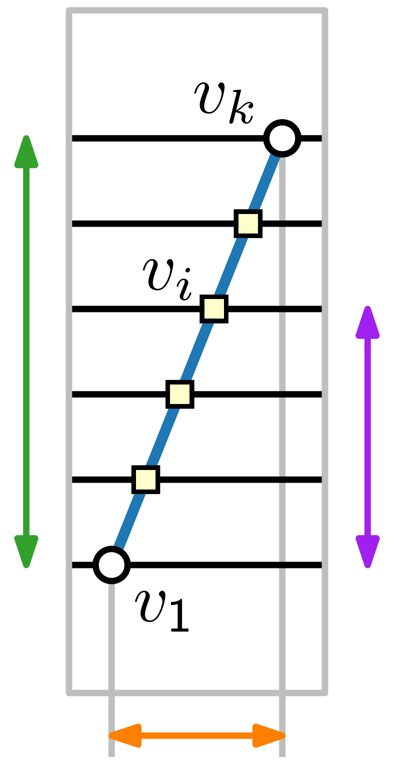

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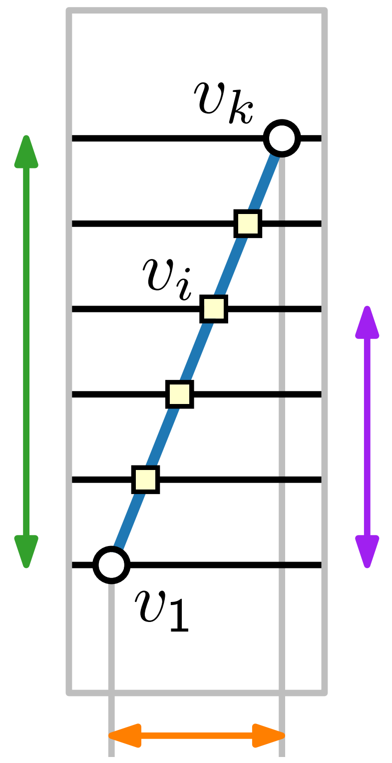
$$\operatorname{dev}(p_e) := \sum_{i=2}^{k-1}$$



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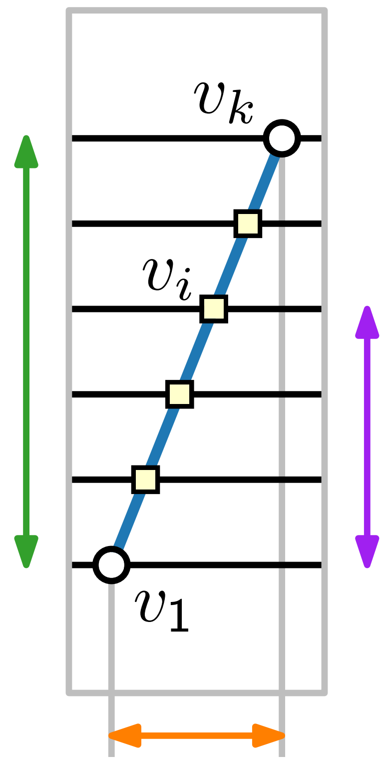

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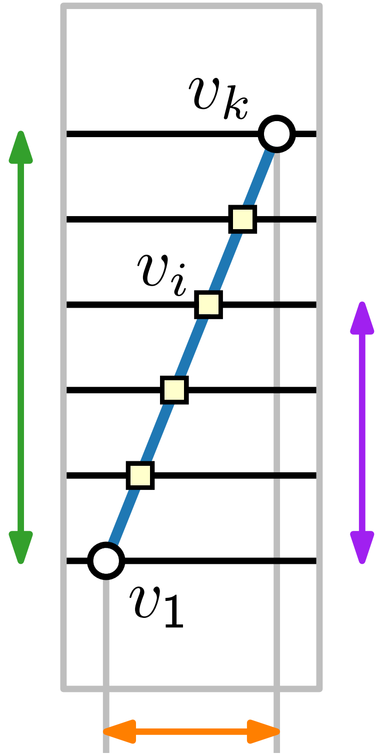

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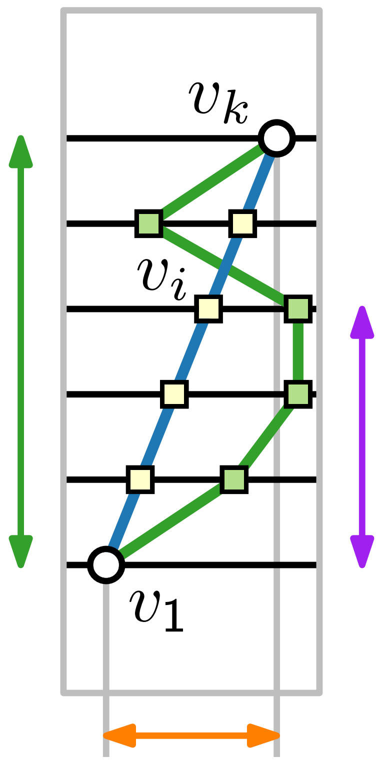

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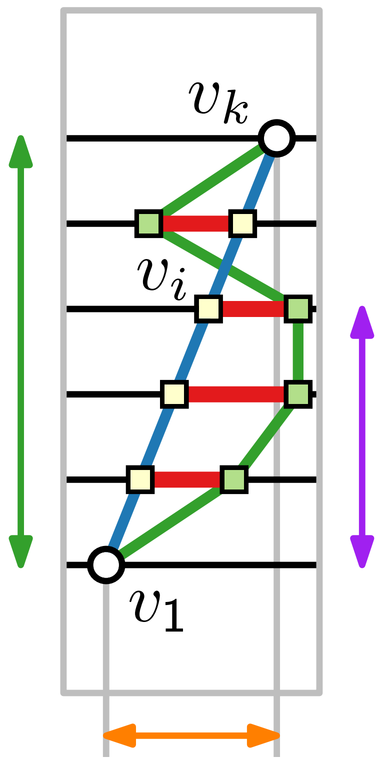

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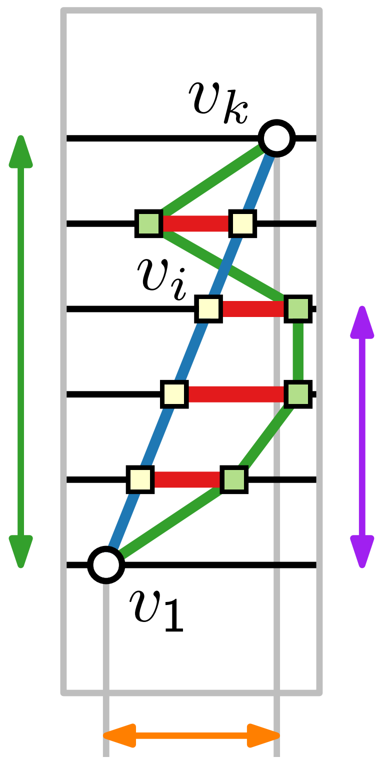

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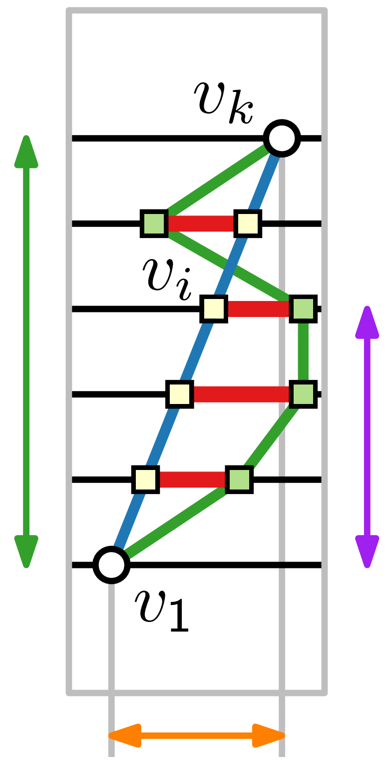

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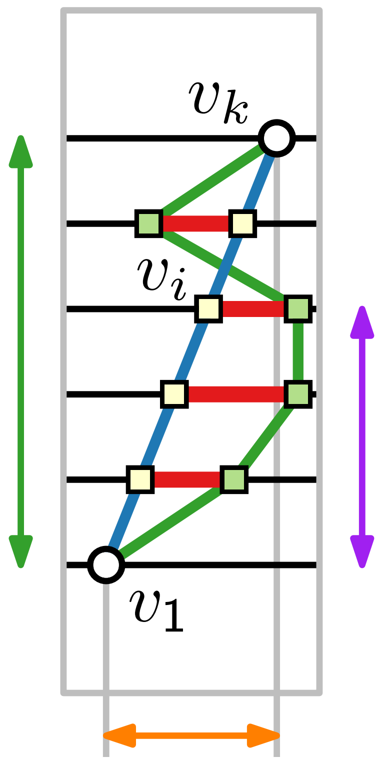

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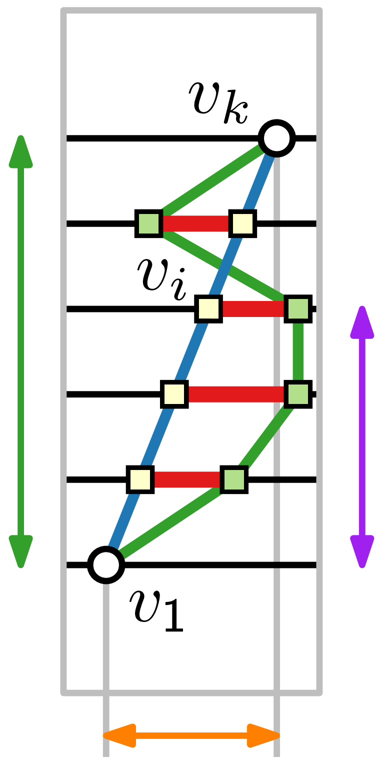

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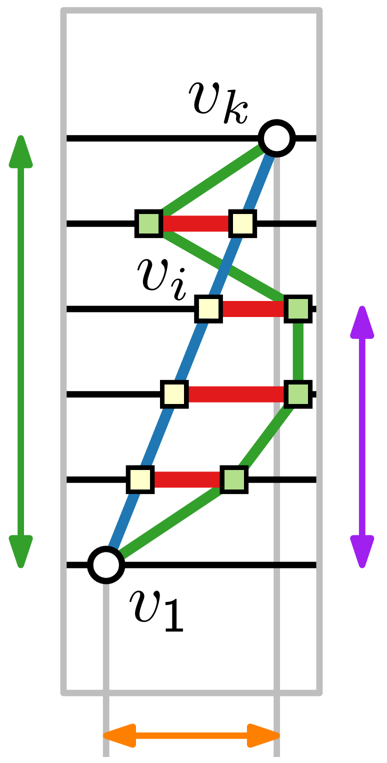

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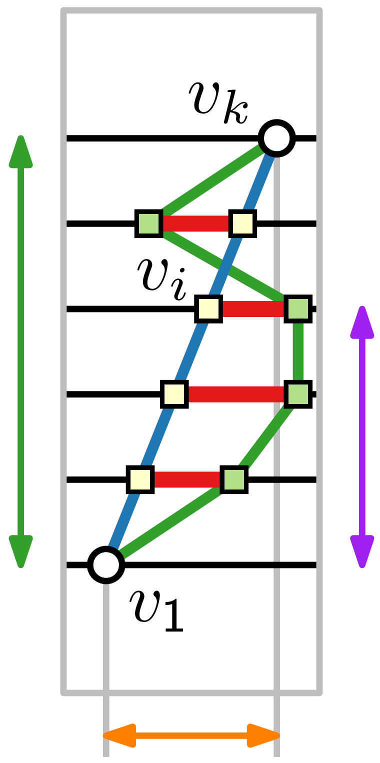

- QP is time-expensive.

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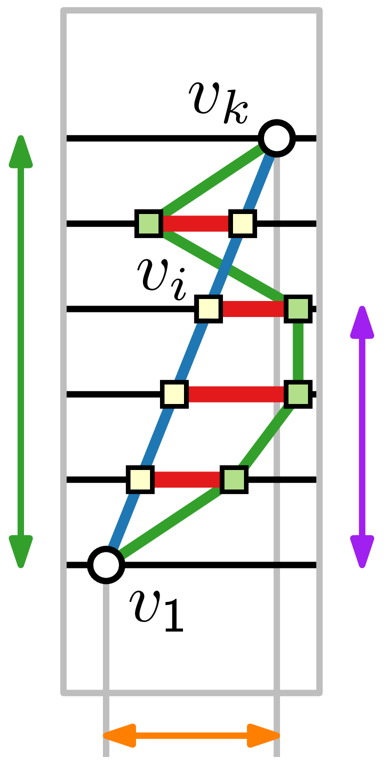

- QP is time-expensive.
- Width can be exponential.

Iterative Heuristic

- Compute an initial layout

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- Apply the following steps as long as improvements can be made:

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  2. edge straightening

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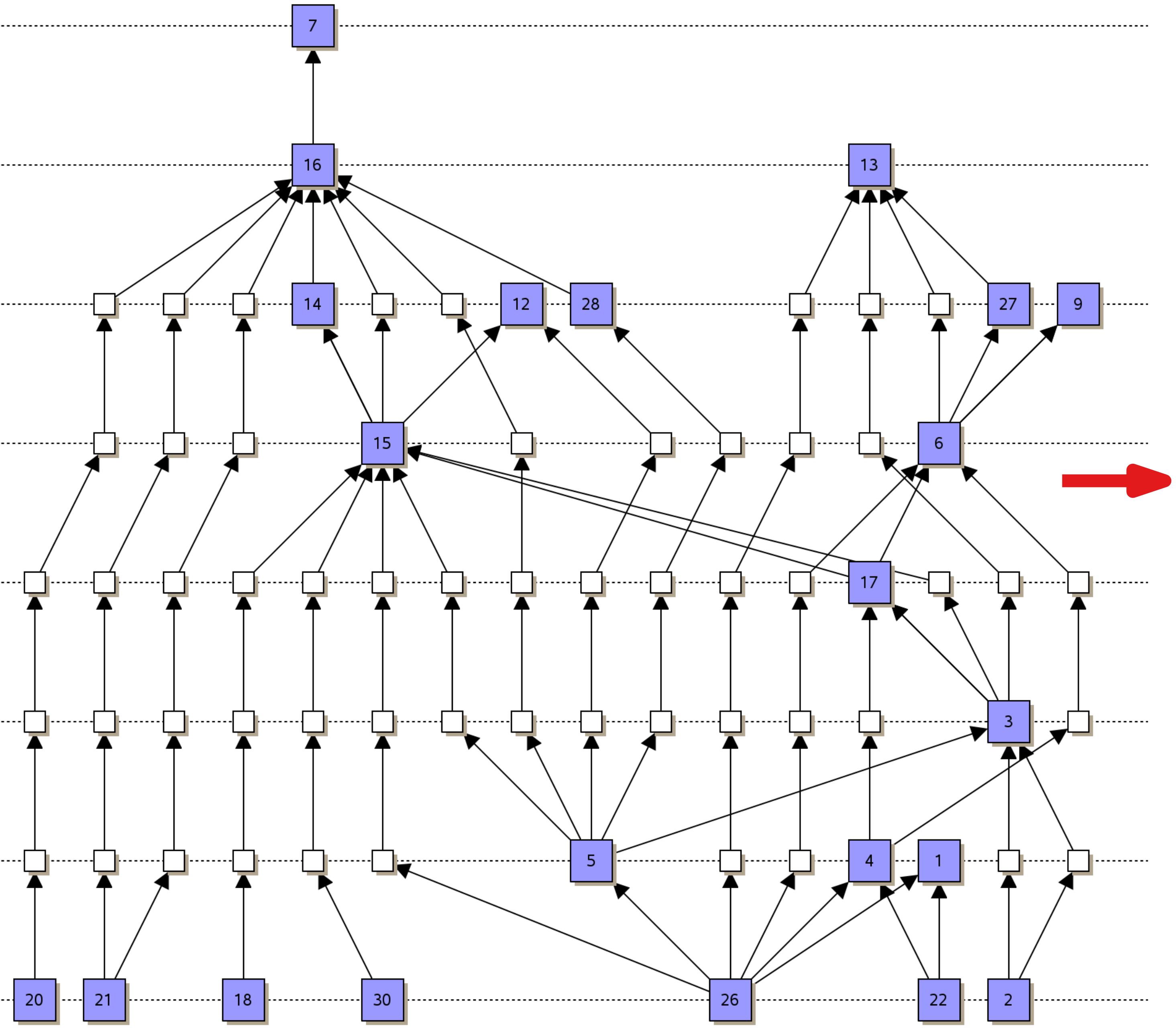
- Other algorithms, e.g., the one of Brandes and Köpf [GD 2002, see also Brandes, Walter, Zink: arXiv 2020]:
  - tries to align vertices vertically
  - does horizontal compaction afterwards
  - linear running time

Example



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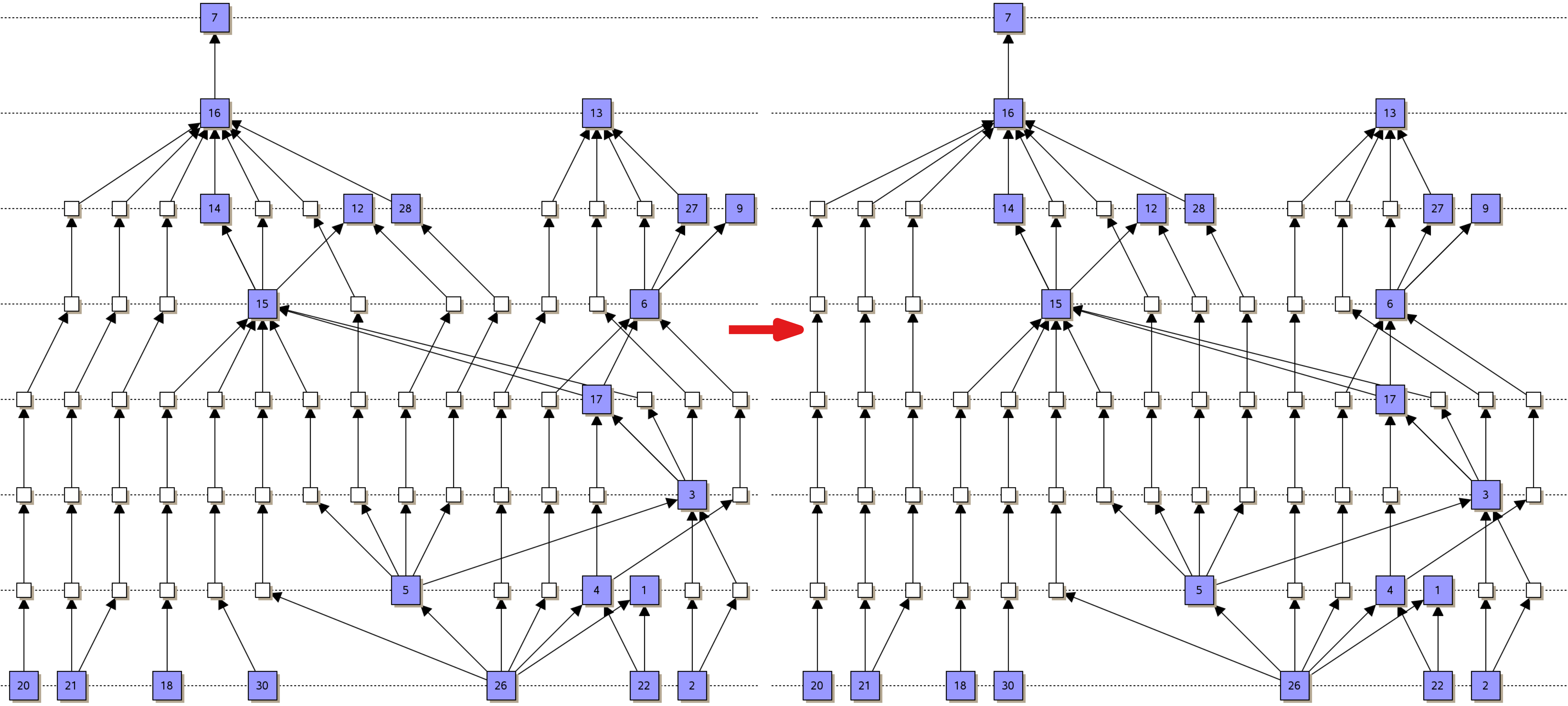
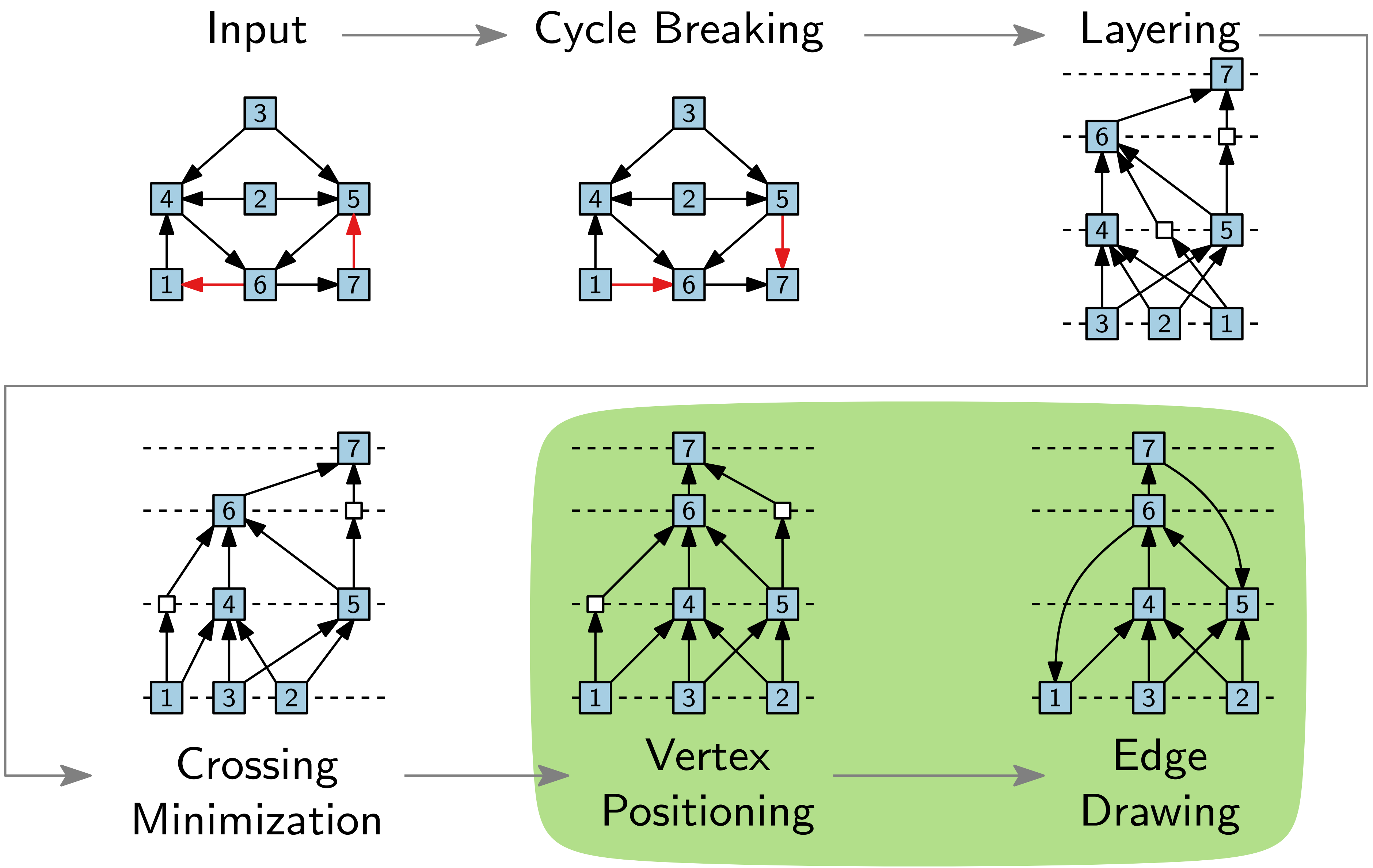
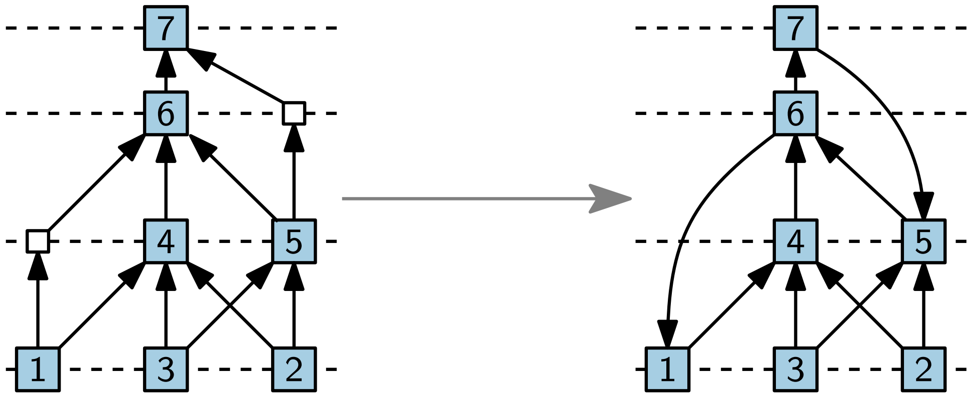


Step 5: Drawing Edges



Step 5: Drawing Edges

Possibility.
Substitute polylines by Bézier curves.



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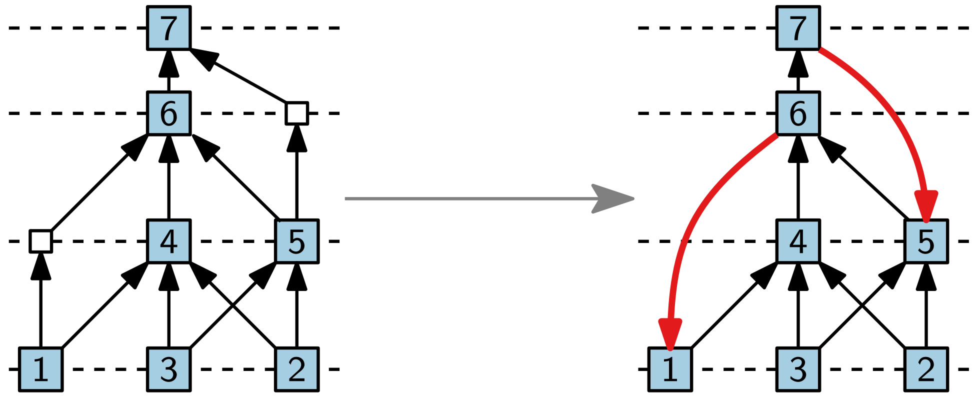
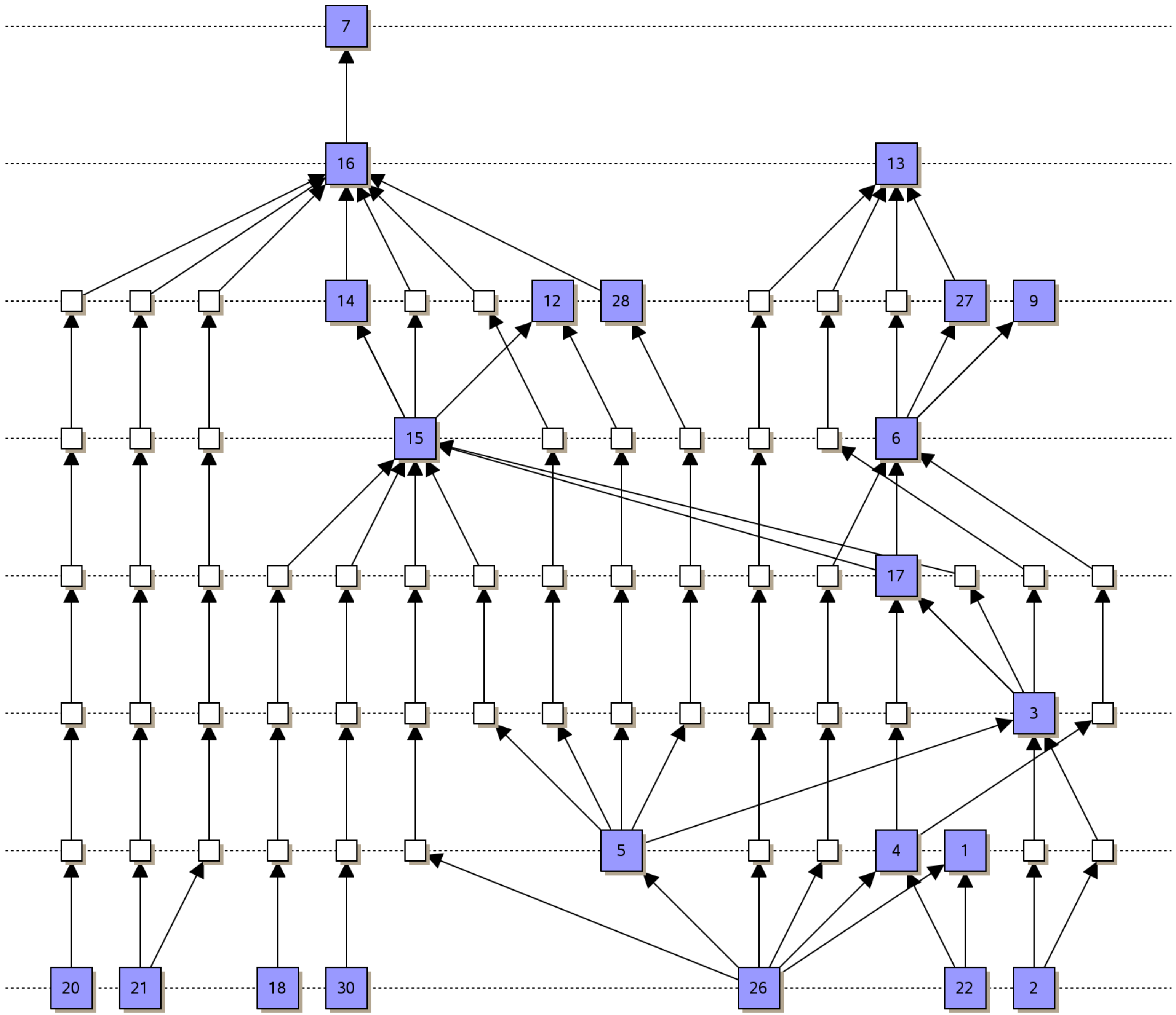


Remark.
Draw reversed edges downwards.

Example



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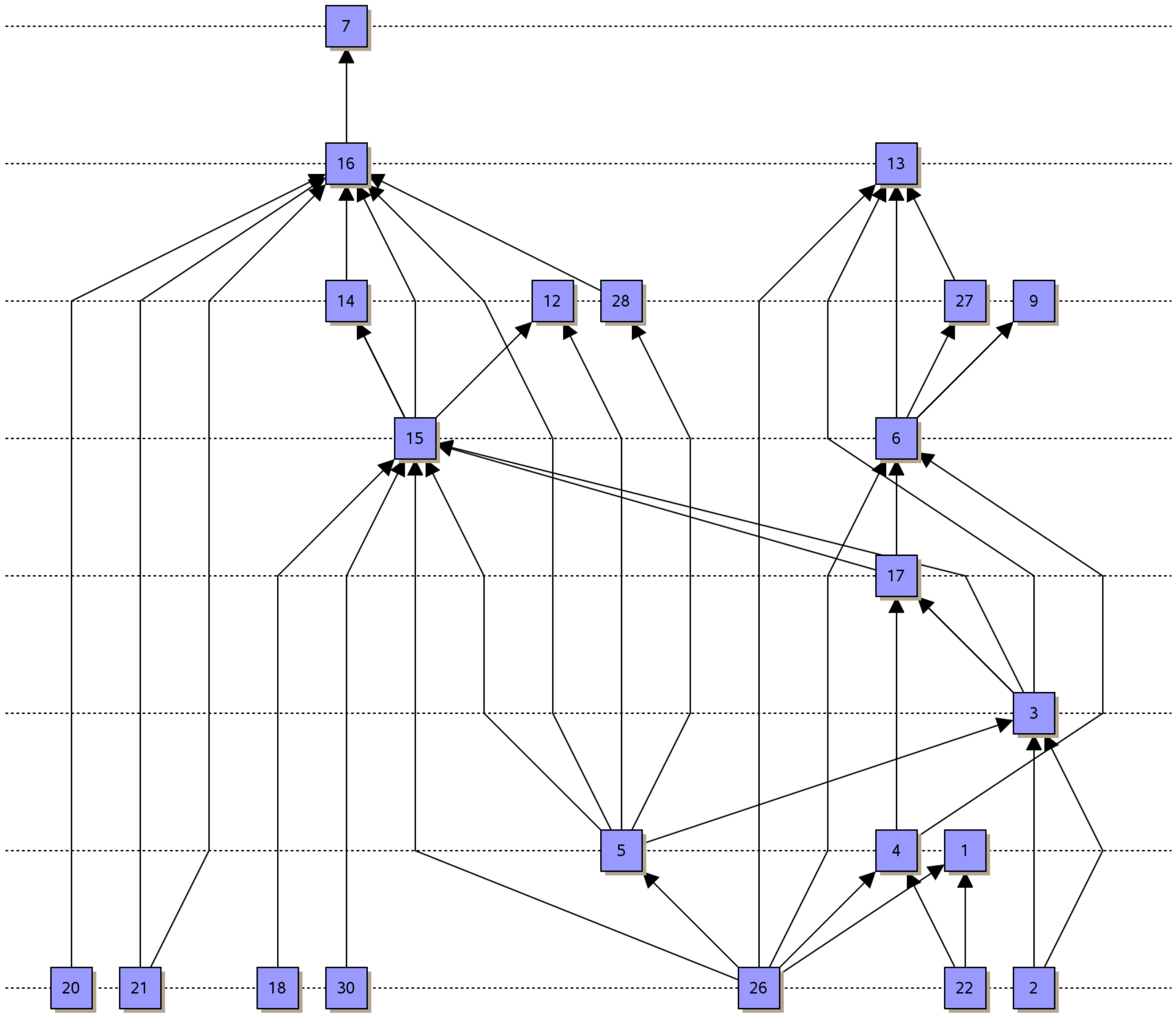

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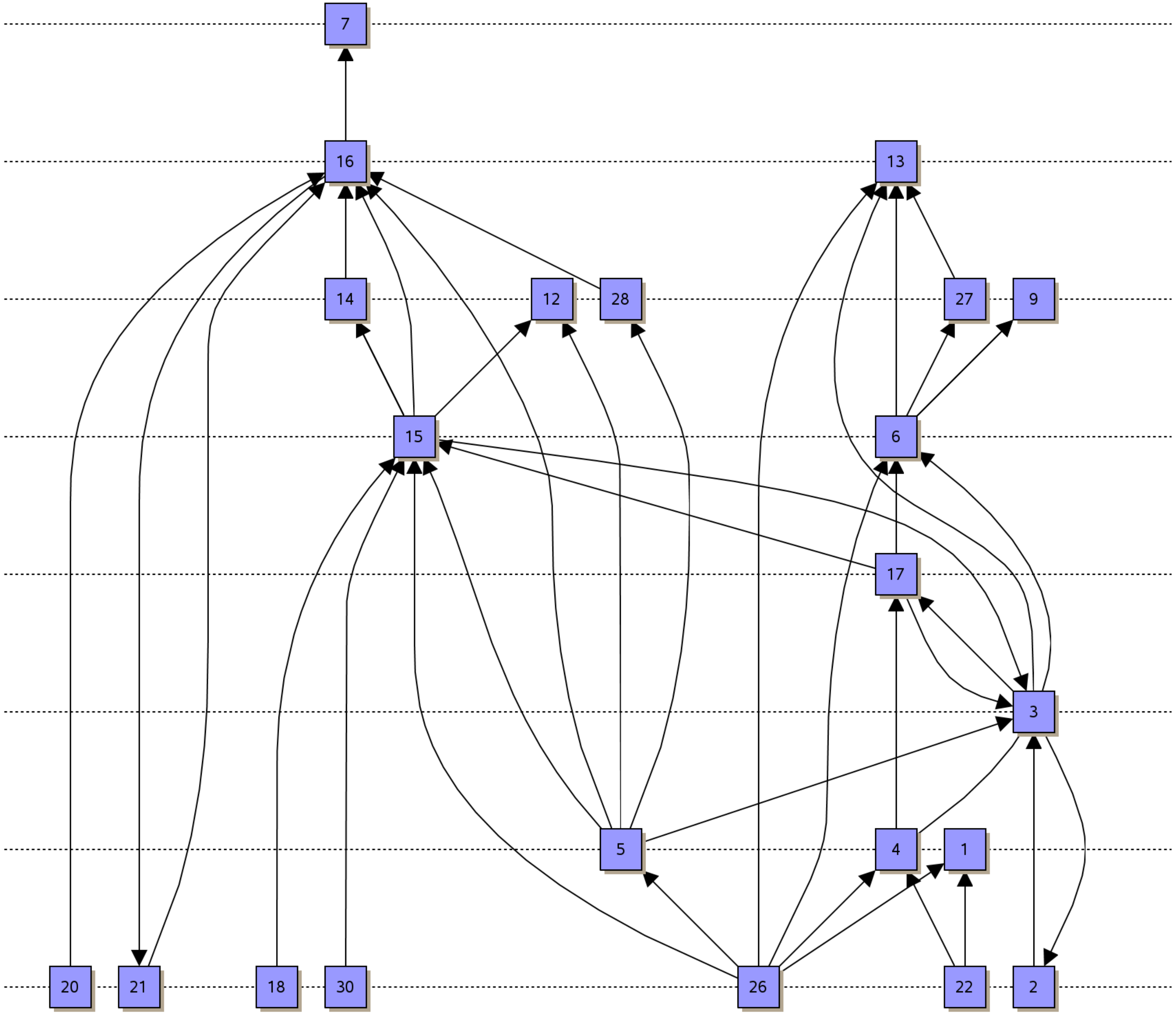
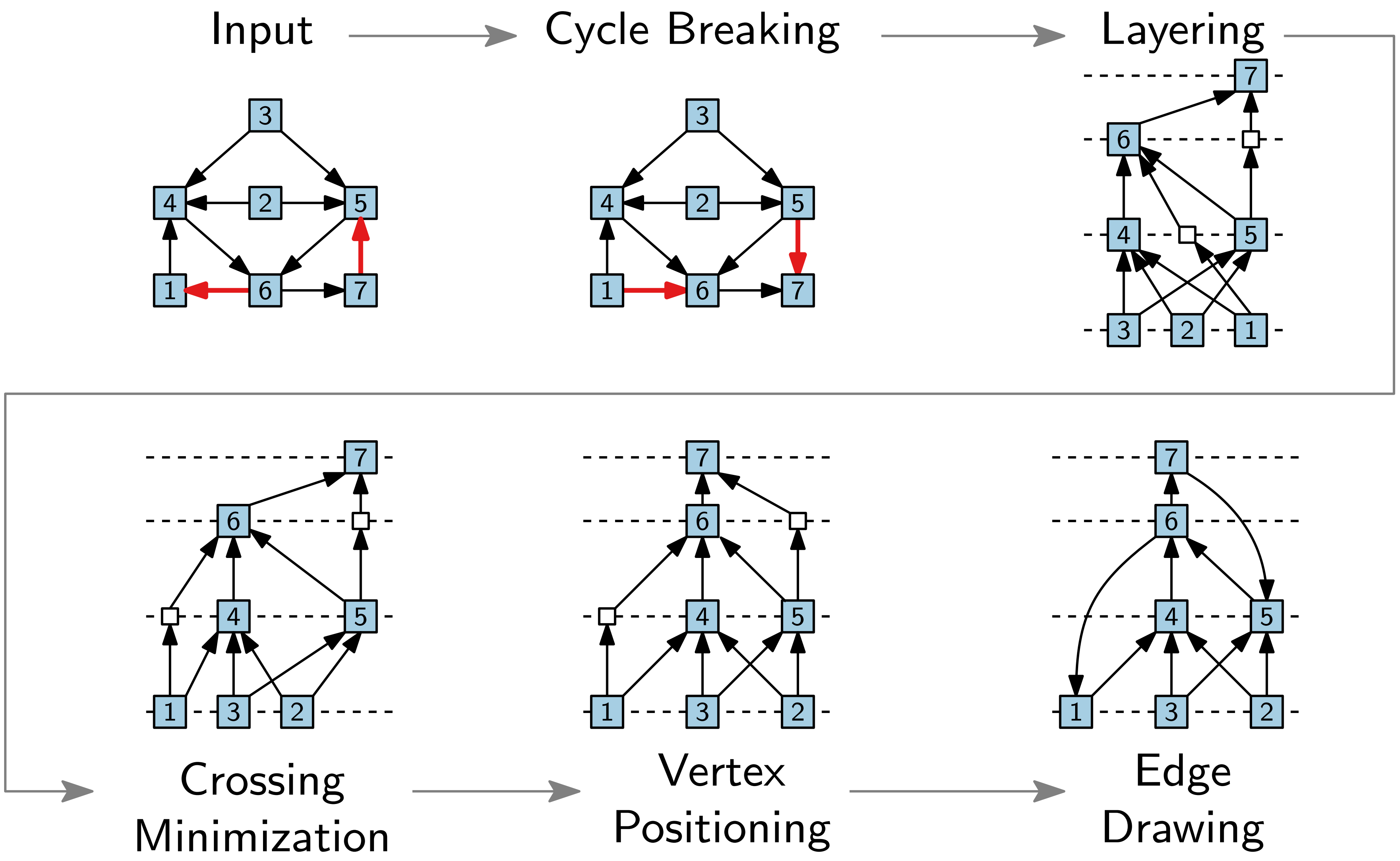


Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]



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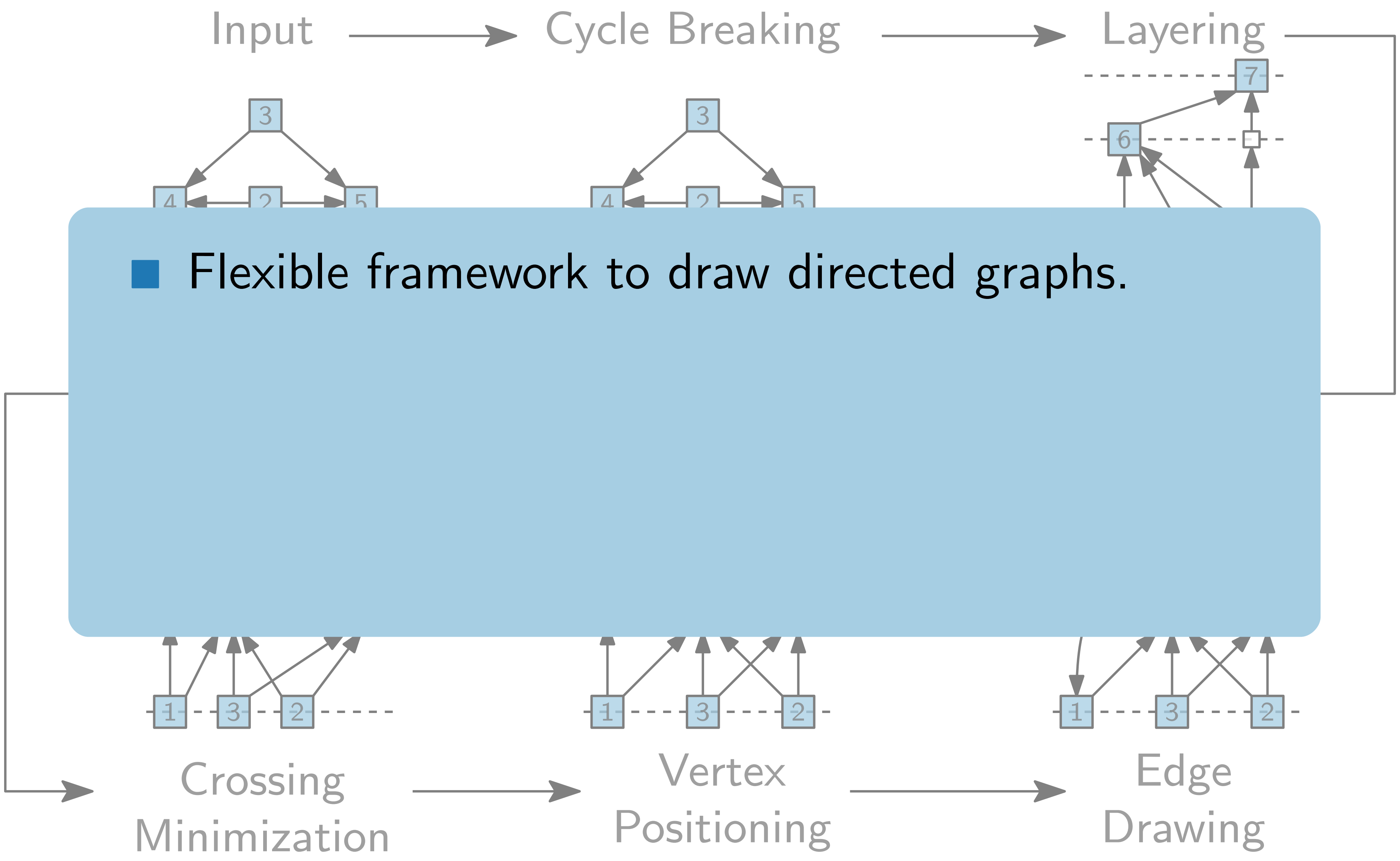

- Flexible framework to draw directed graphs.

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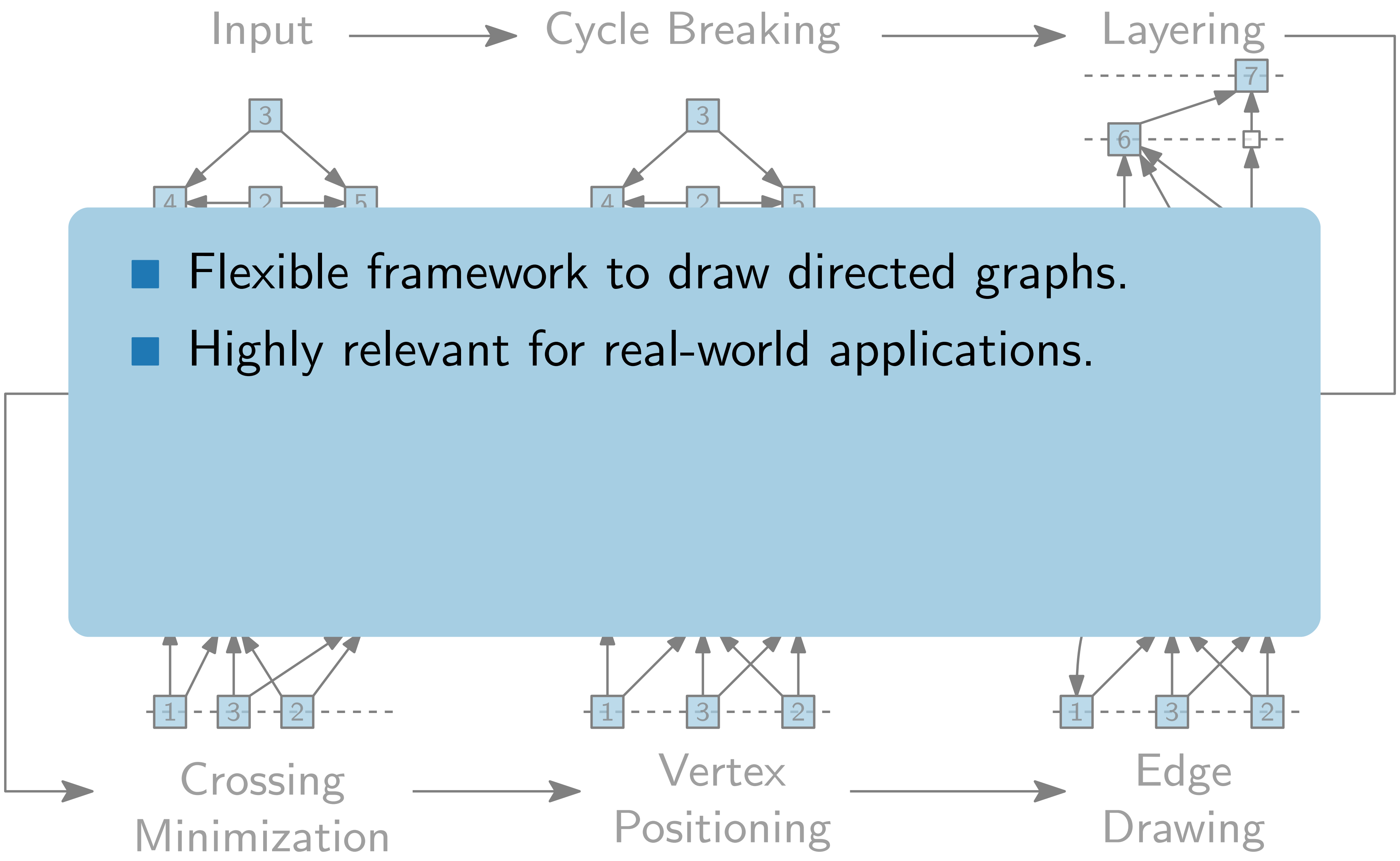

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- Highly relevant for real-world applications.

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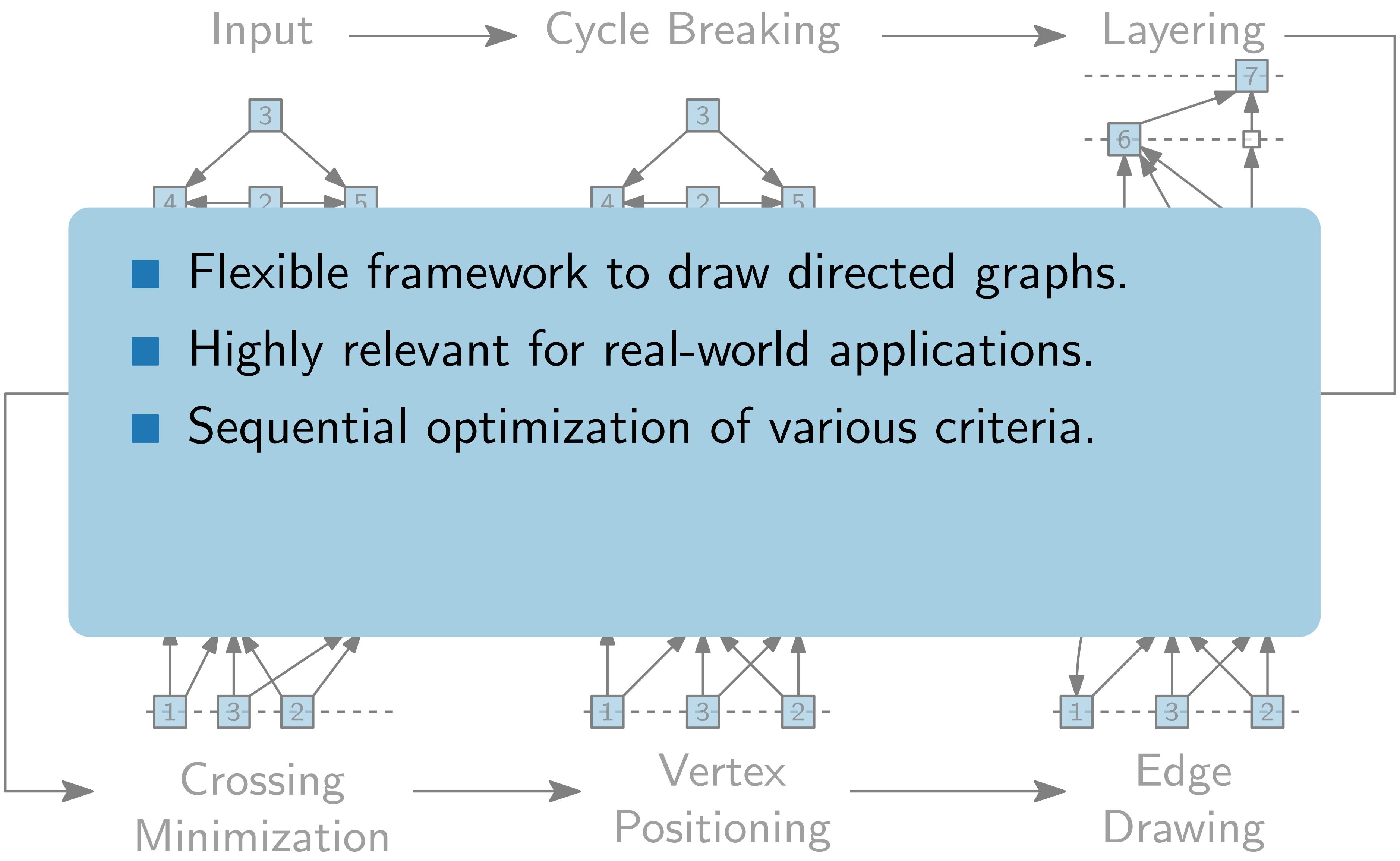

- Flexible framework to draw directed graphs.
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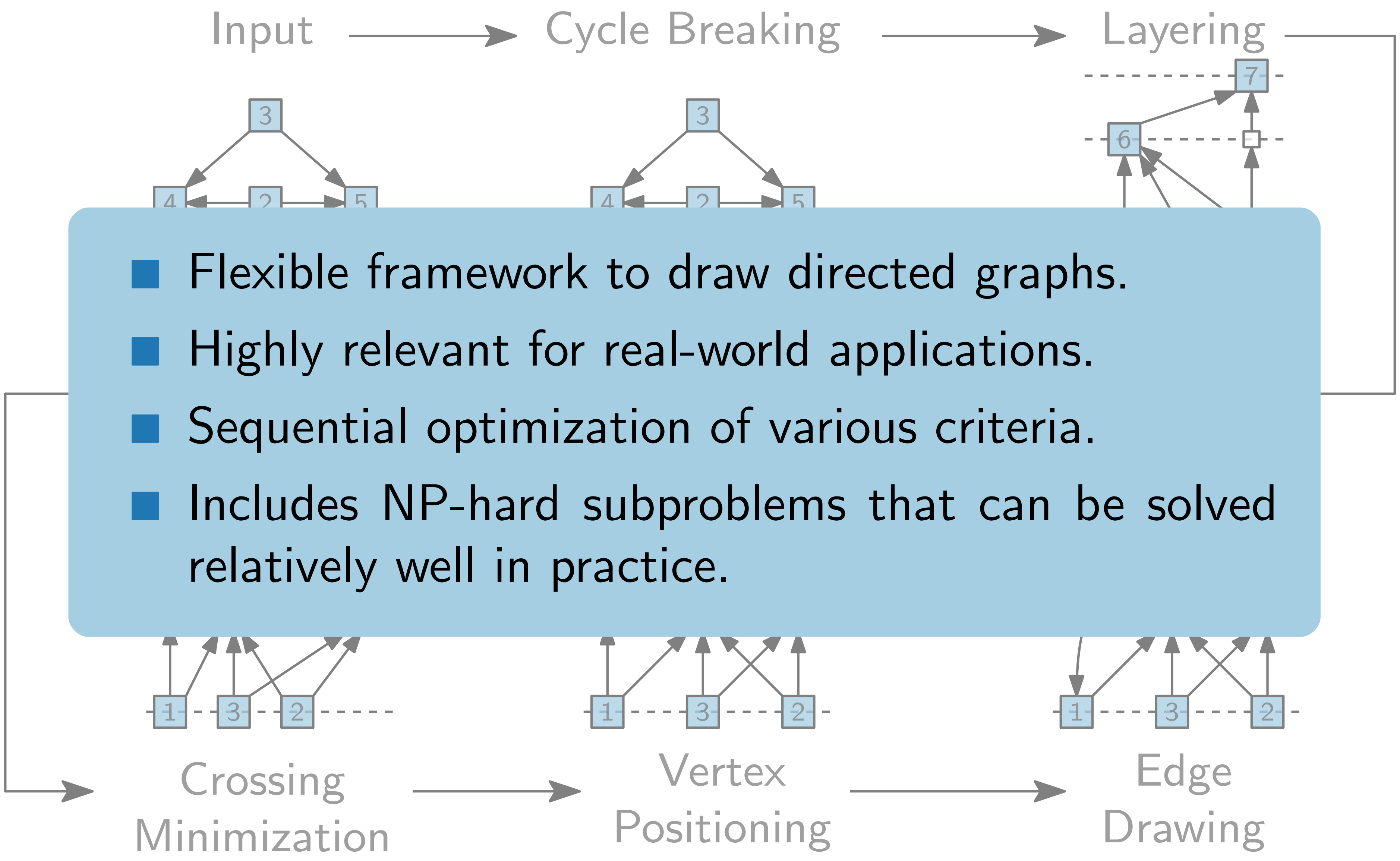

- Flexible framework to draw directed graphs.
- Highly relevant for real-world applications.
- Sequential optimization of various criteria.
- Includes NP-hard subproblems that can be solved relatively well in practice.

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81]
  Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation algorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Kelly '86] Heuristics for reducing crossings in 2-layered networks.
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97]
  2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms