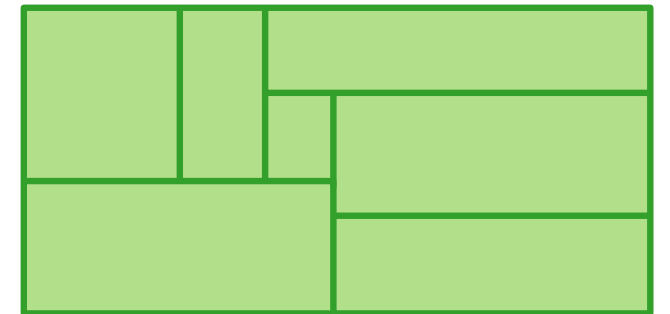
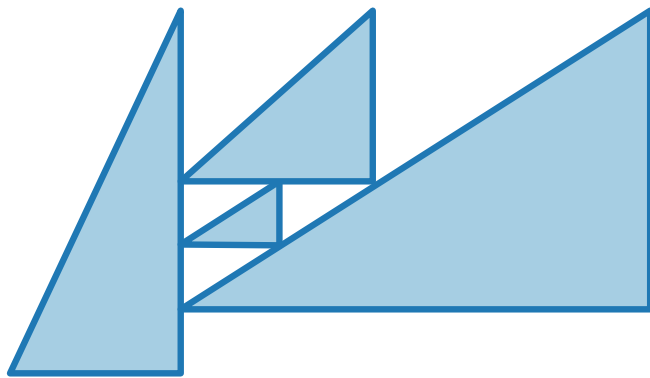


Visualization of Graphs

Lecture 7:

Contact Representations of Planar Graphs: Triangle Contacts and Rectangular Duals



Alexander Wolff

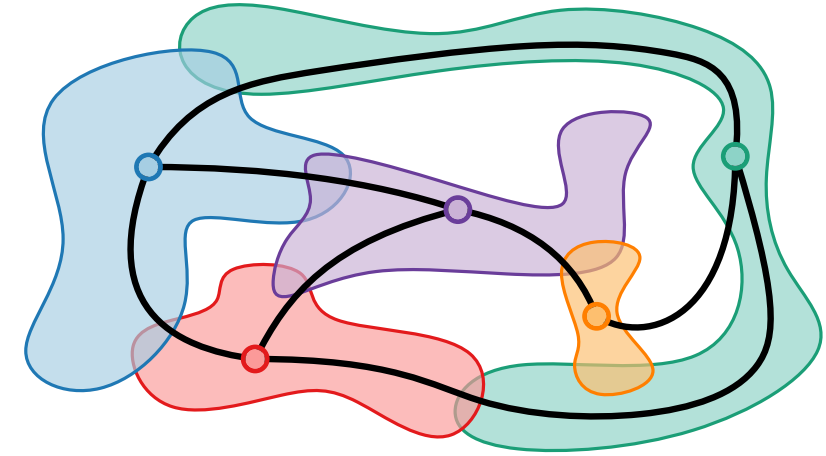
Summer term 2026

Intersection Representation of Graphs

In an **intersection representation** of a graph,

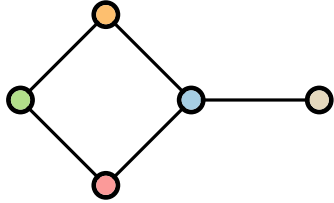
- each vertex is represented by a set
- such that two sets intersect $\Leftrightarrow$ the corresponding vertices are adjacent.

For a collection $\mathcal{S}$ of sets,
the **intersection graph** $G(\mathcal{S})$ of $\mathcal{S}$
has vertex set $\mathcal{S}$ and edge set
 $\{\{S, S'\} : S, S' \in \mathcal{S}, S \neq S', \text{ and } S \cap S' \neq \emptyset\}$.



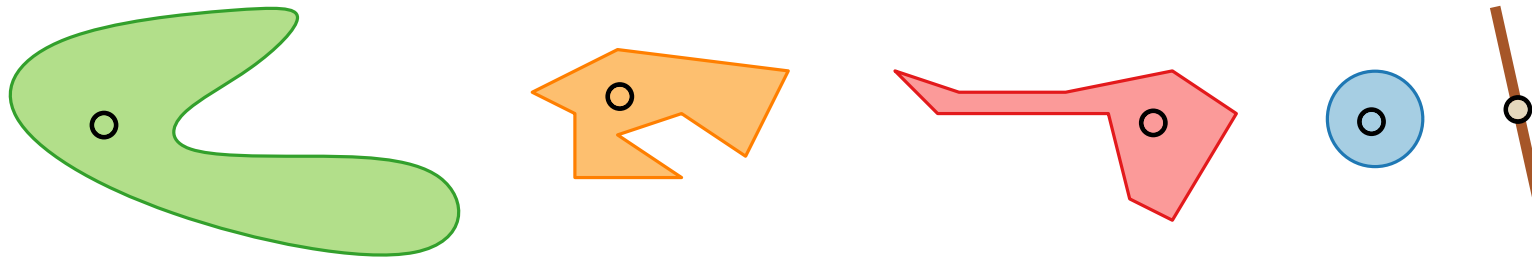
Contact Representation of Graphs

Let G be a graph.

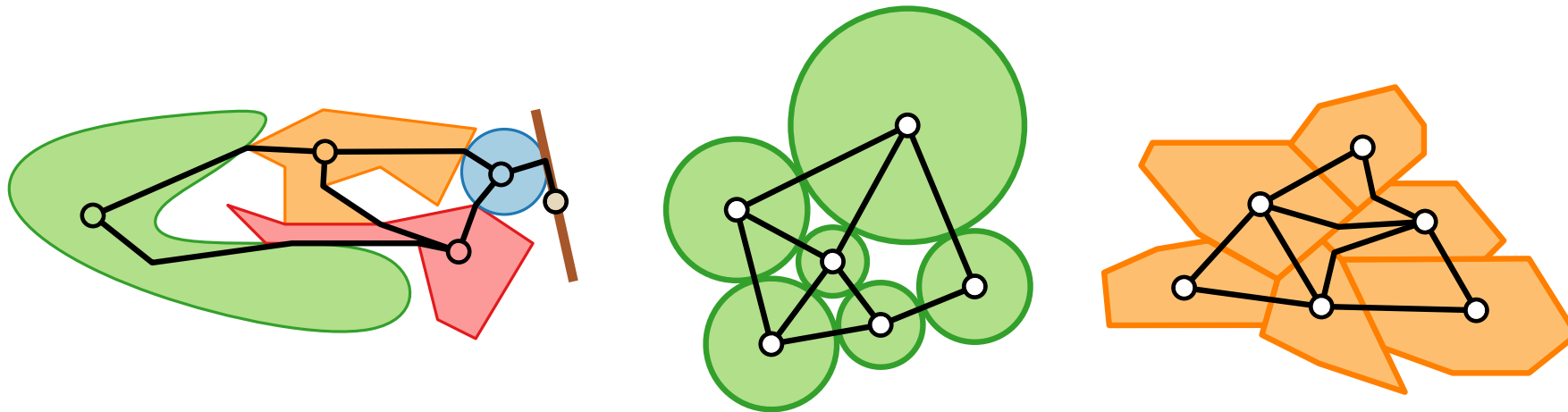


Let $\mathcal{S}$ be a family of geometric objects (e.g., disks).

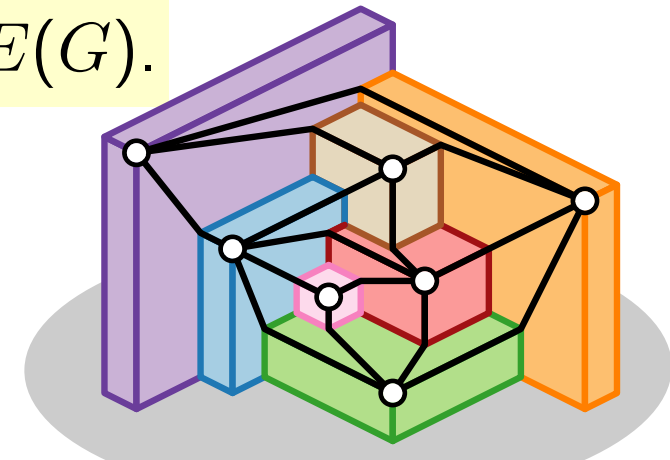
Represent each vertex v by a geometric object $S(v) \in \mathcal{S}$



In an **$\mathcal{S}$ -contact representation** of G , $S(u)$ and $S(v)$ *touch* iff $uv \in E(G)$.



G is planar $\xrightarrow{\text{[Koebe 1936]}}$ disks $\longrightarrow$ polygons



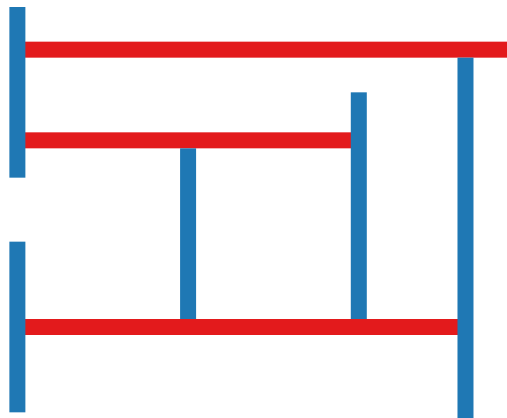
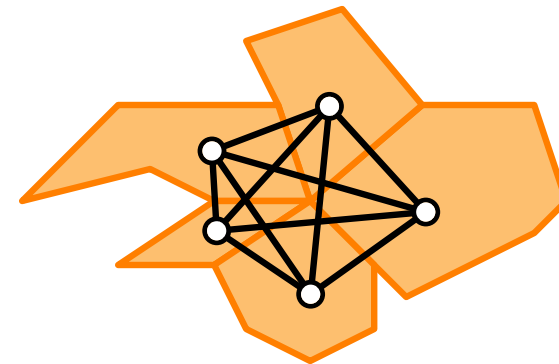
A contact representation is an intersection representation with interior-disjoint sets.

Contact Representation of Planar Graphs

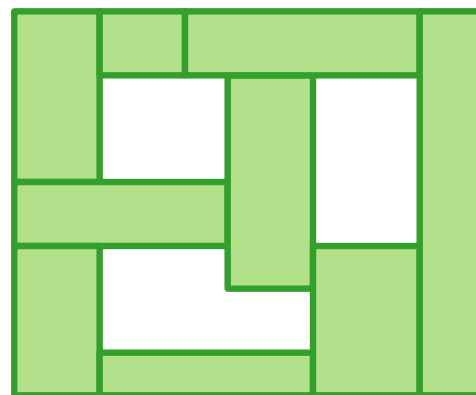
Is the intersection graph of a contact representation always planar?

- No, not even for connected object types in the plane.

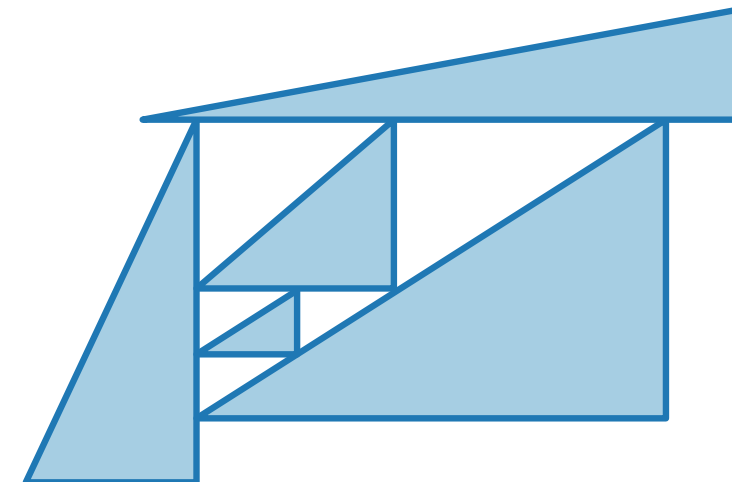
Some object types imply restrictions to **special classes** of planar graphs:



bipartite planar graphs



max. triangle-free planar graphs

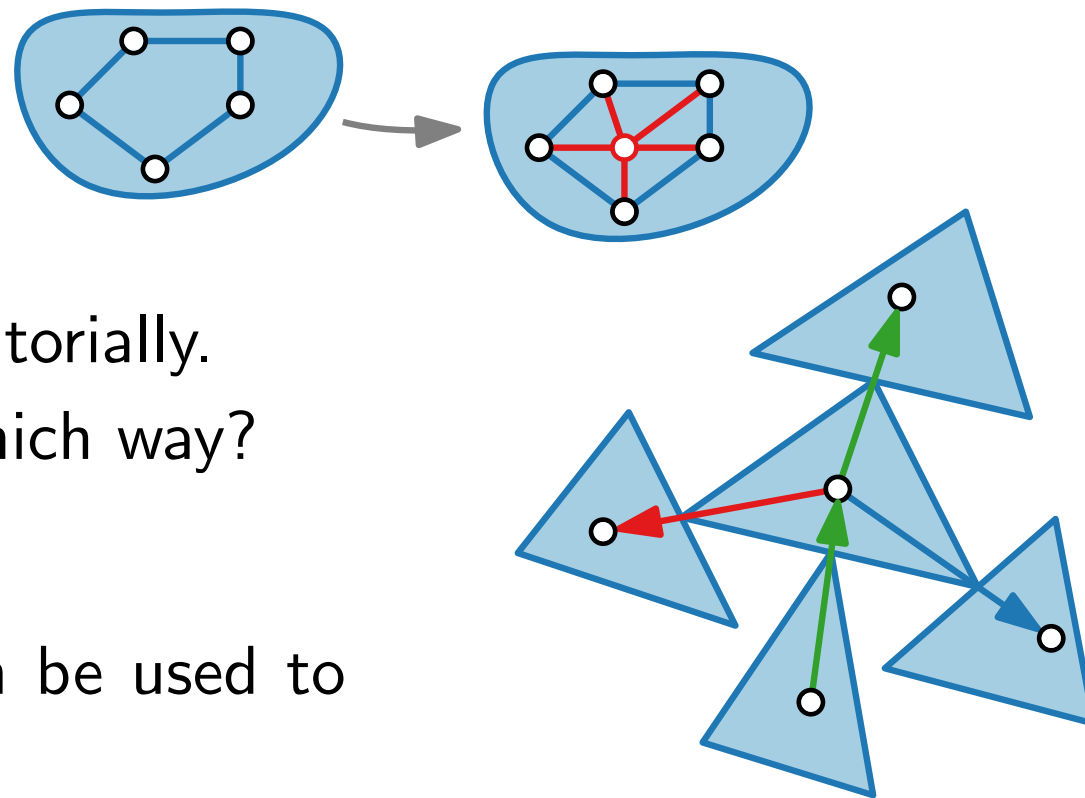


planar triangulations

General Approach

How to compute a contact representation of a given graph G ?

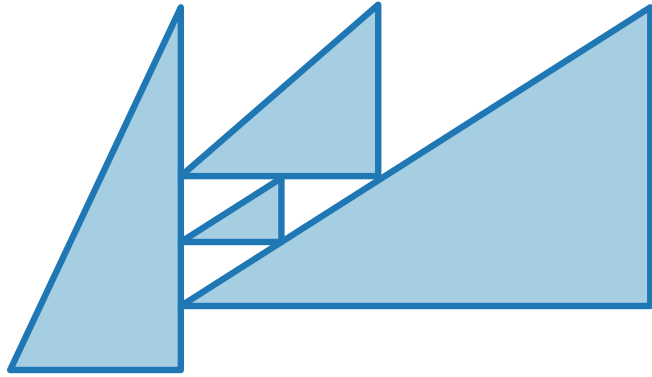
- Consider only inner triangulations (or maximal bipartite graphs, etc.)
 - Triangulate by adding vertices, not by adding edges
- Describe contact representation combinatorially.
 - Which objects touch each other in which way?
- Compute combinatorial description.
- Show that combinatorial description can be used to construct drawing.



This Lecture

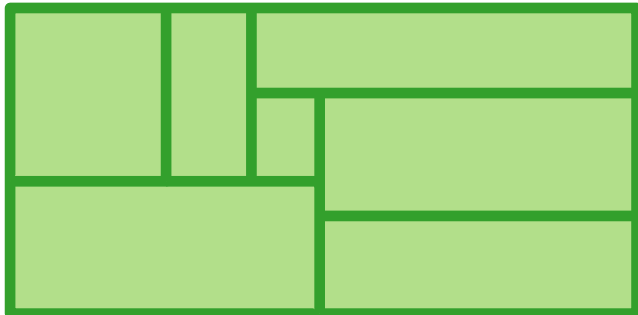
Representation with right-angled triangles and corner contact:

- Use Schnyder realizer to describe contacts between triangles.
- Use canonical order to compute drawing.



Representation with dissection of a rectangle, called **rectangular dual**:

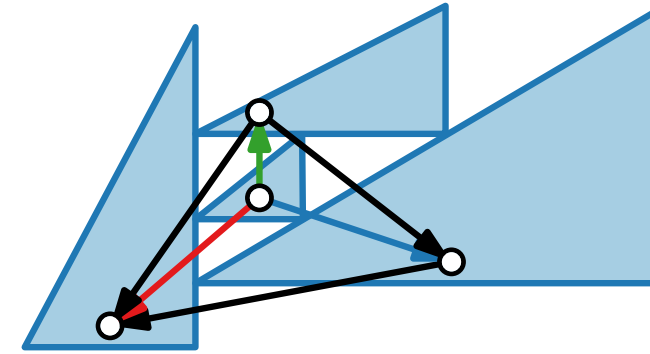
- Find a description similar to a Schnyder realizer for rectangles.
- Construct drawing via st-digraphs, duals, and topological sorting.



Triangle Corner Contact Representation

Main Idea.

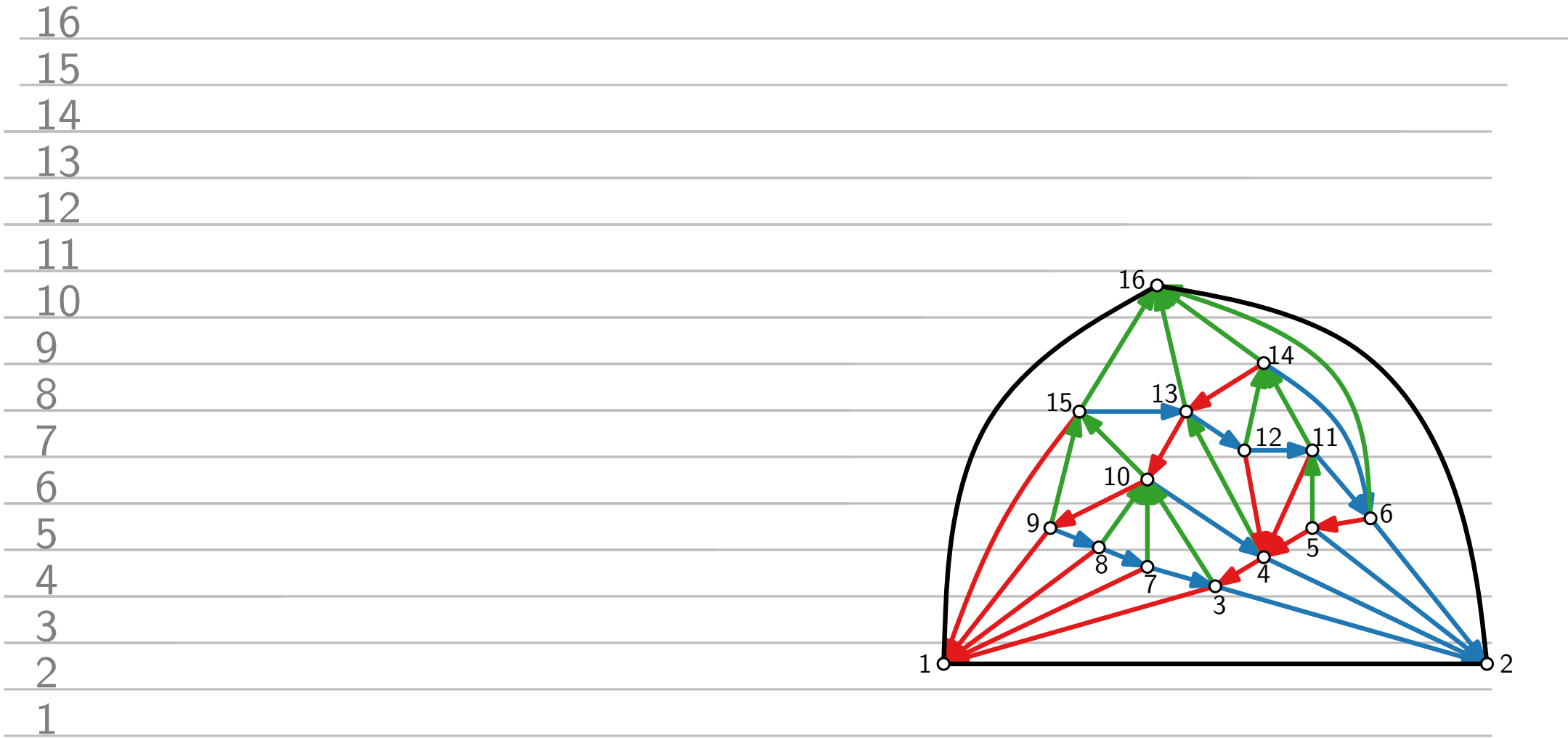
Use canonical order and Schnyder realizer to find coordinates for triangles.



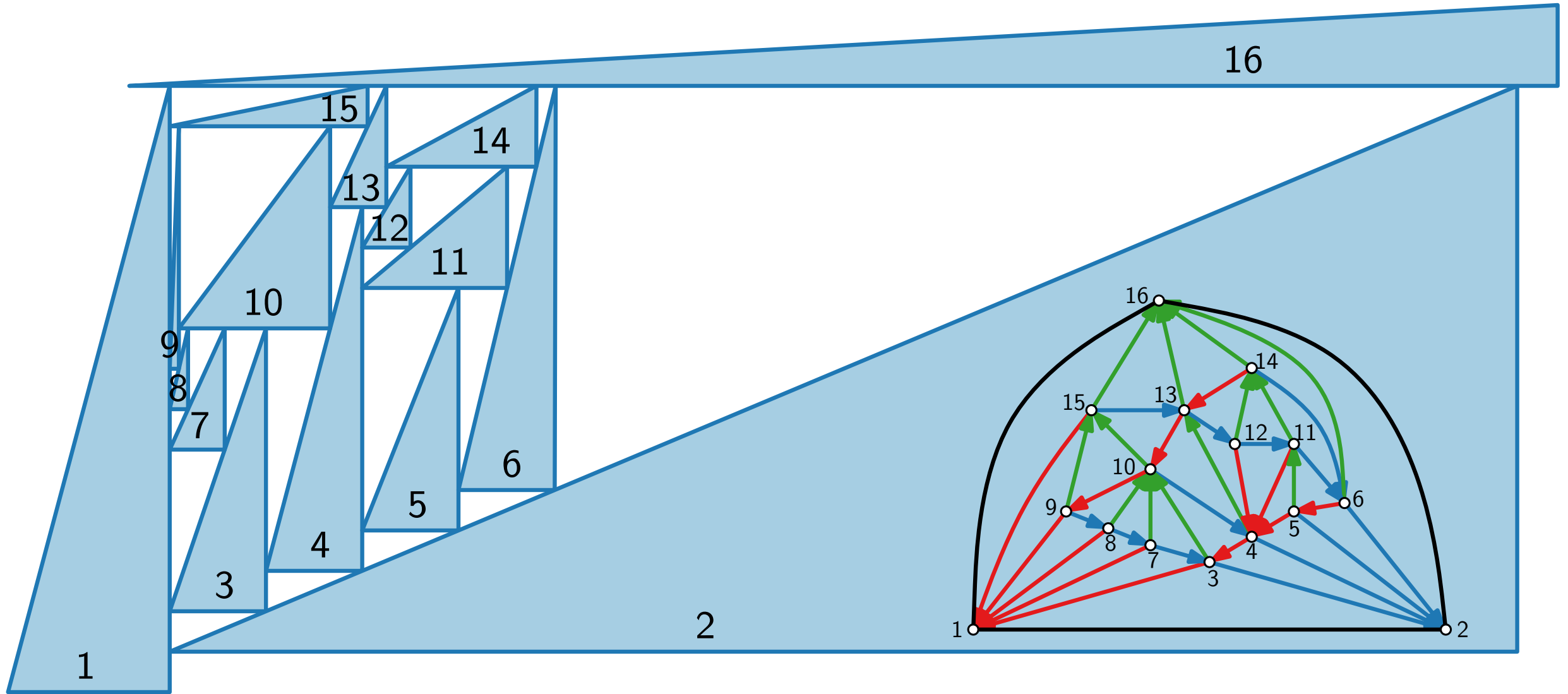
Detailed Idea.

- Place base of triangle at height equal to position in canonical order.
- Triangle tip is precisely at base of triangle corresponding to cover neighbor.
- Outgoing edges in Schnyder forest indicate corner contacts.

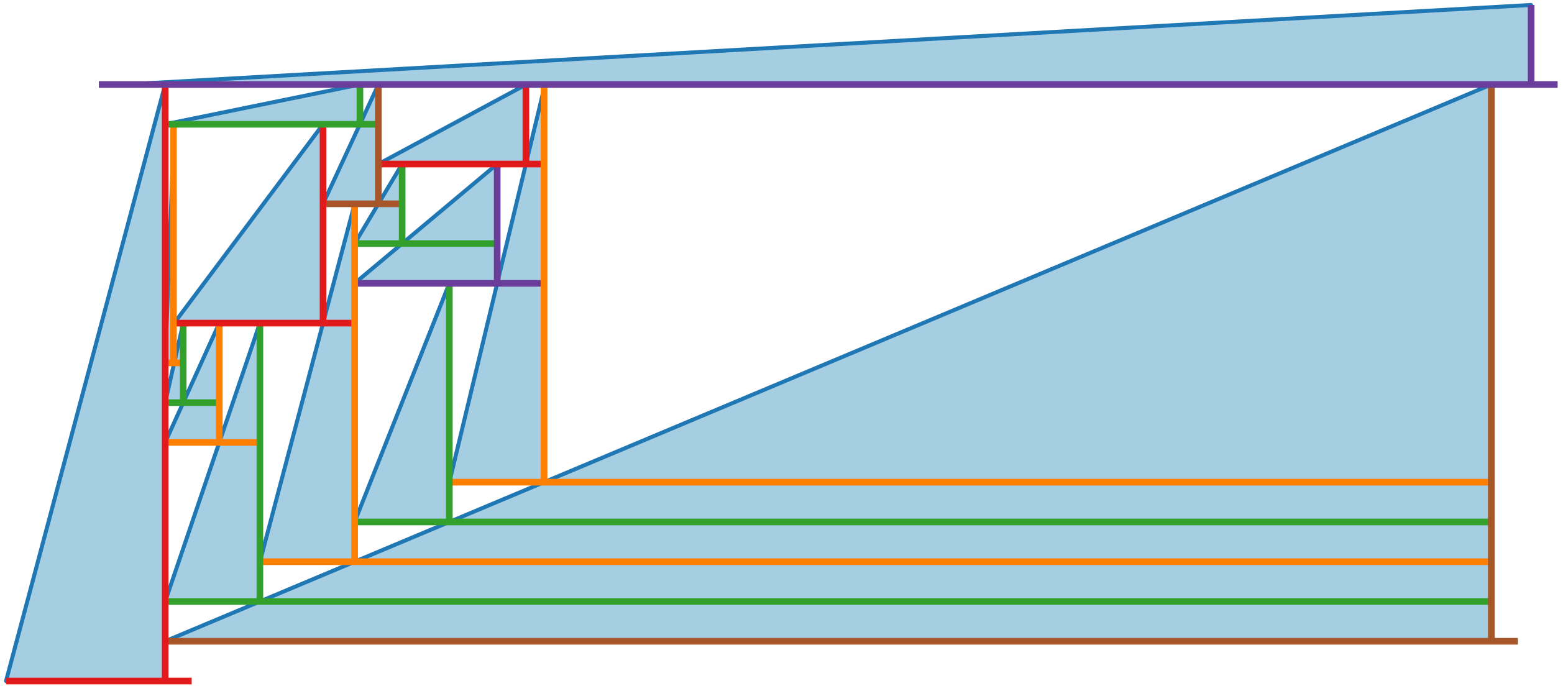
Triangle Contact Representation Example



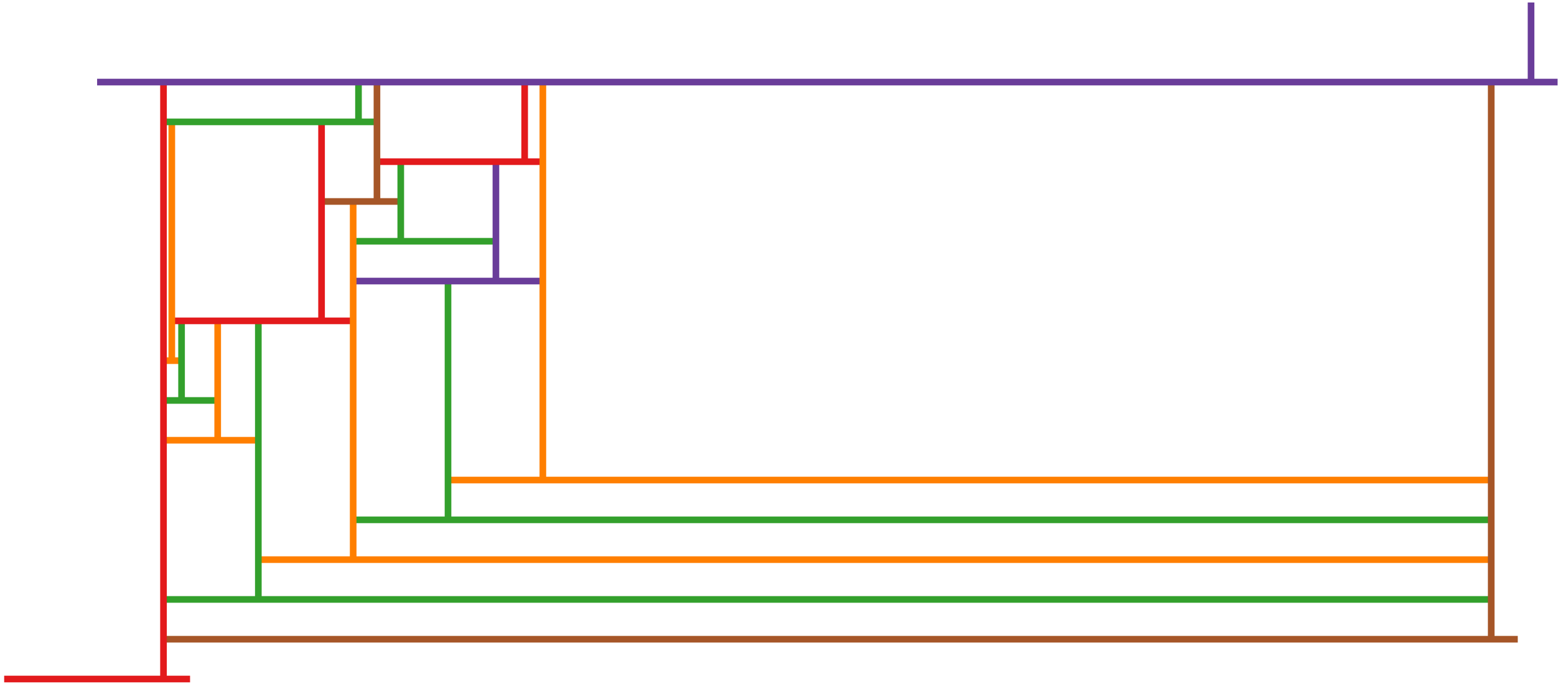
Triangle Contact Representation Example



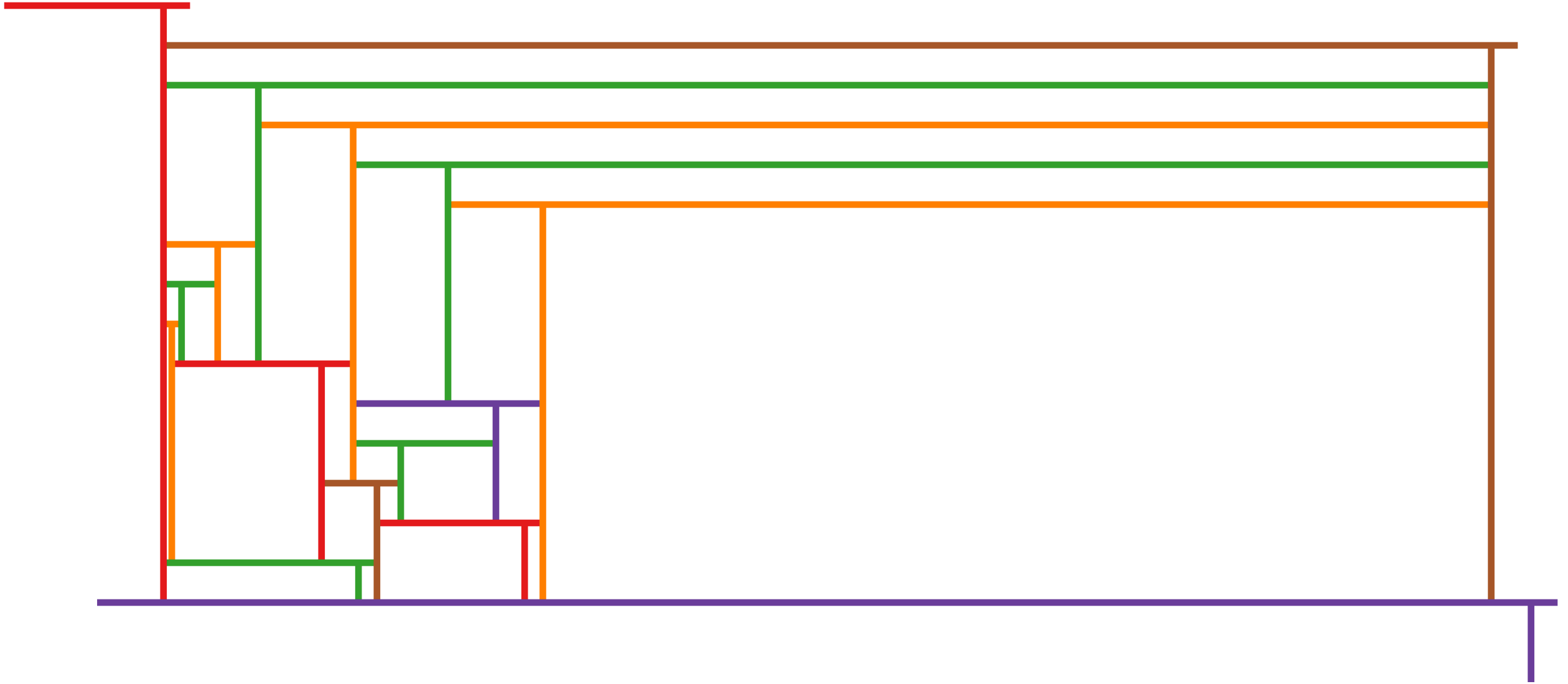
T-shape Contact Representation



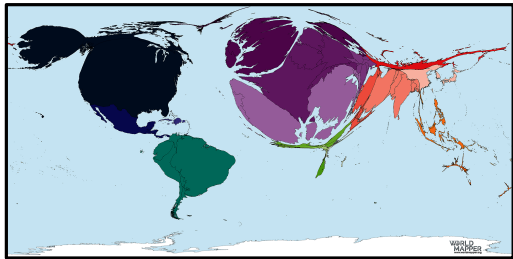
T-shape Contact Representation



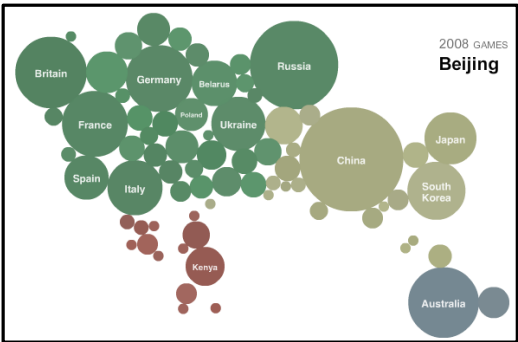
T-shape Contact Representation



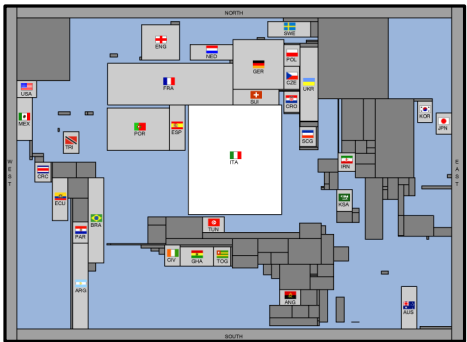
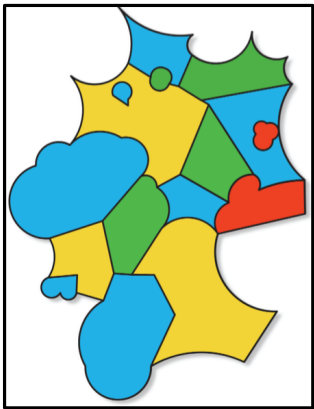
Cartograms



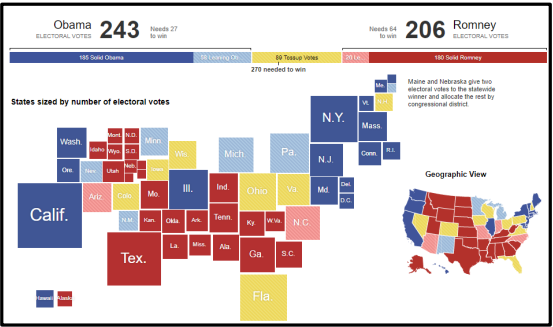
© worldmapper.org



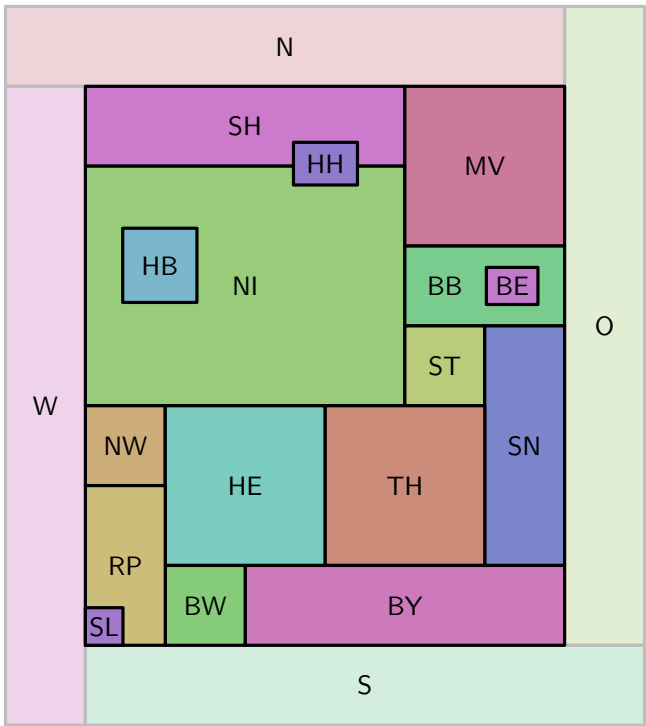
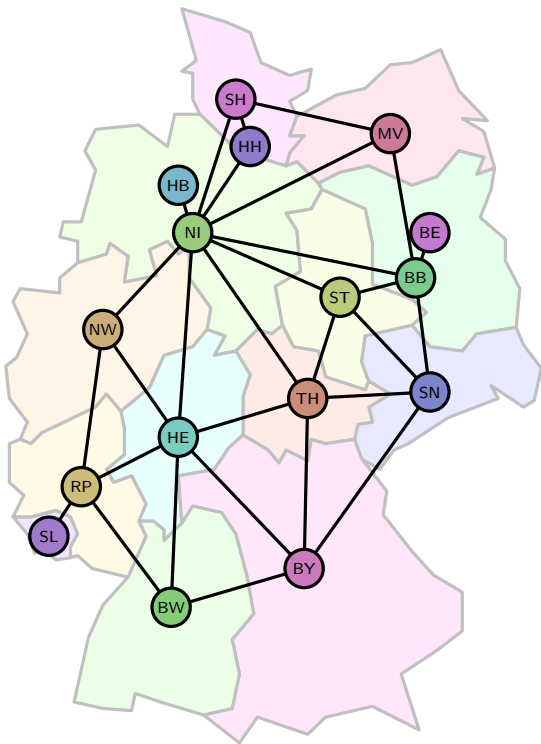
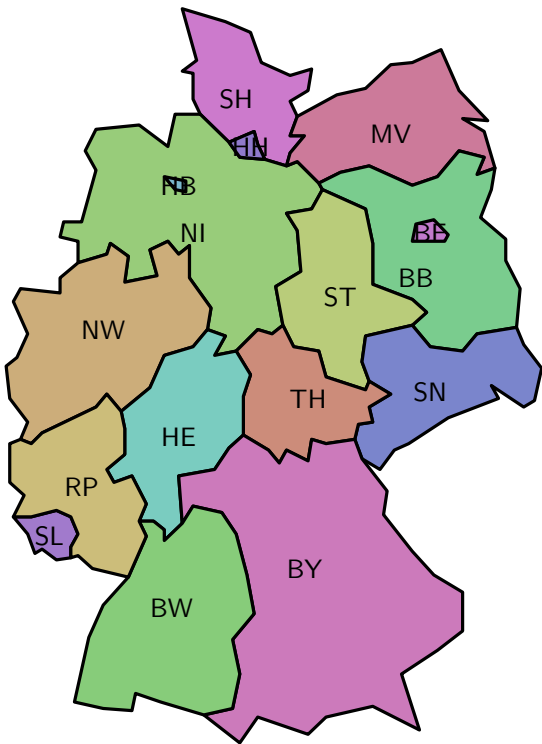
© New York Times



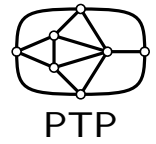
© Bettina Speckmann



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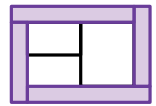


Rectangular Dual



PTP

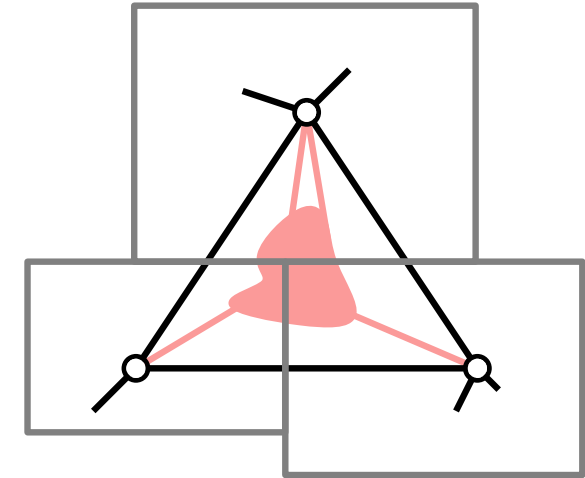
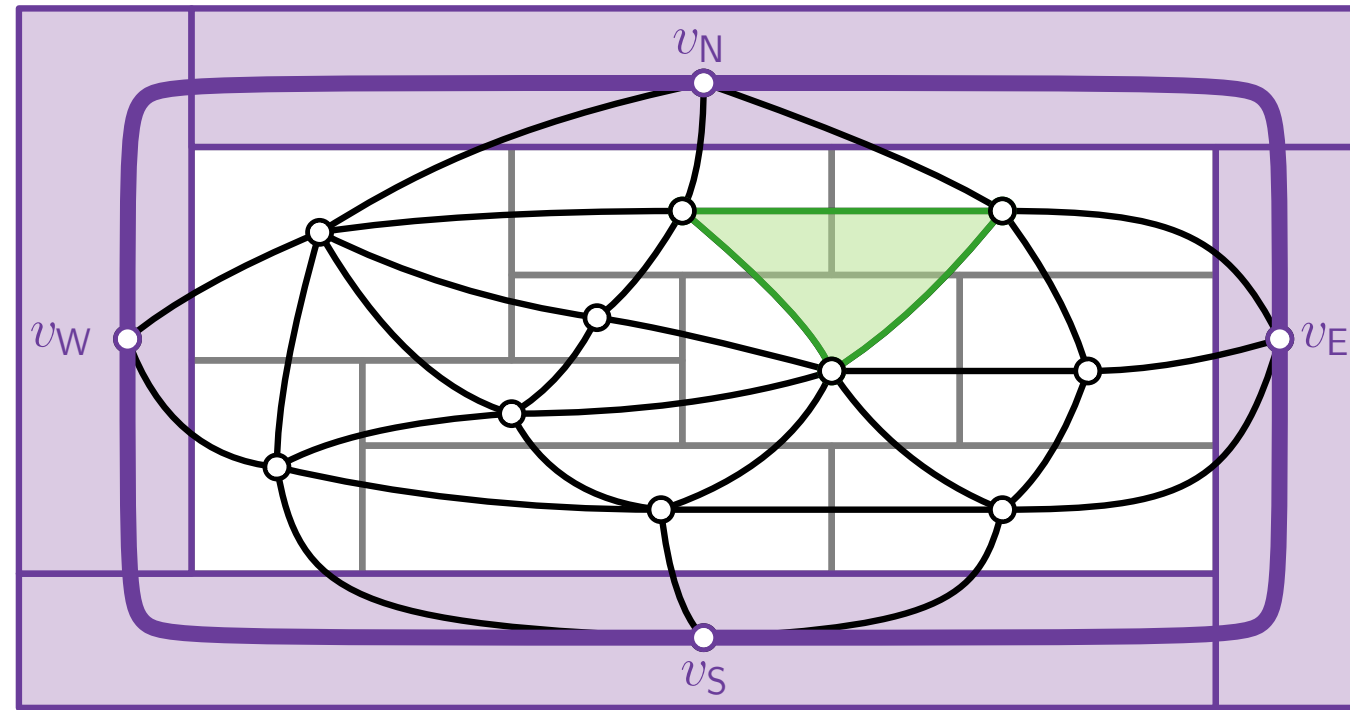
Properly Triangulated
Planar Graph G



RD

Rectangular Dual $\mathcal{R}$

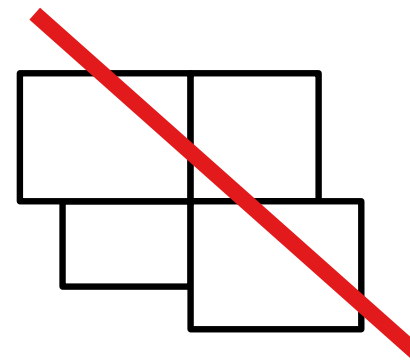
Exactly four vertices on the outer face.



No separating
triangle!

A **rectangular dual** of a graph G is a contact representation with axis-aligned rectangles s.t.

- no four rectangles share a point, and
- the union of all rectangles is a rectangle

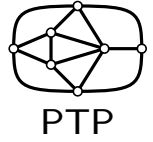


Theorem.

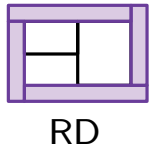
A graph G has a rectangular dual if and only if G is a PTP graph.

[Koźmiński, Kinnen '85]

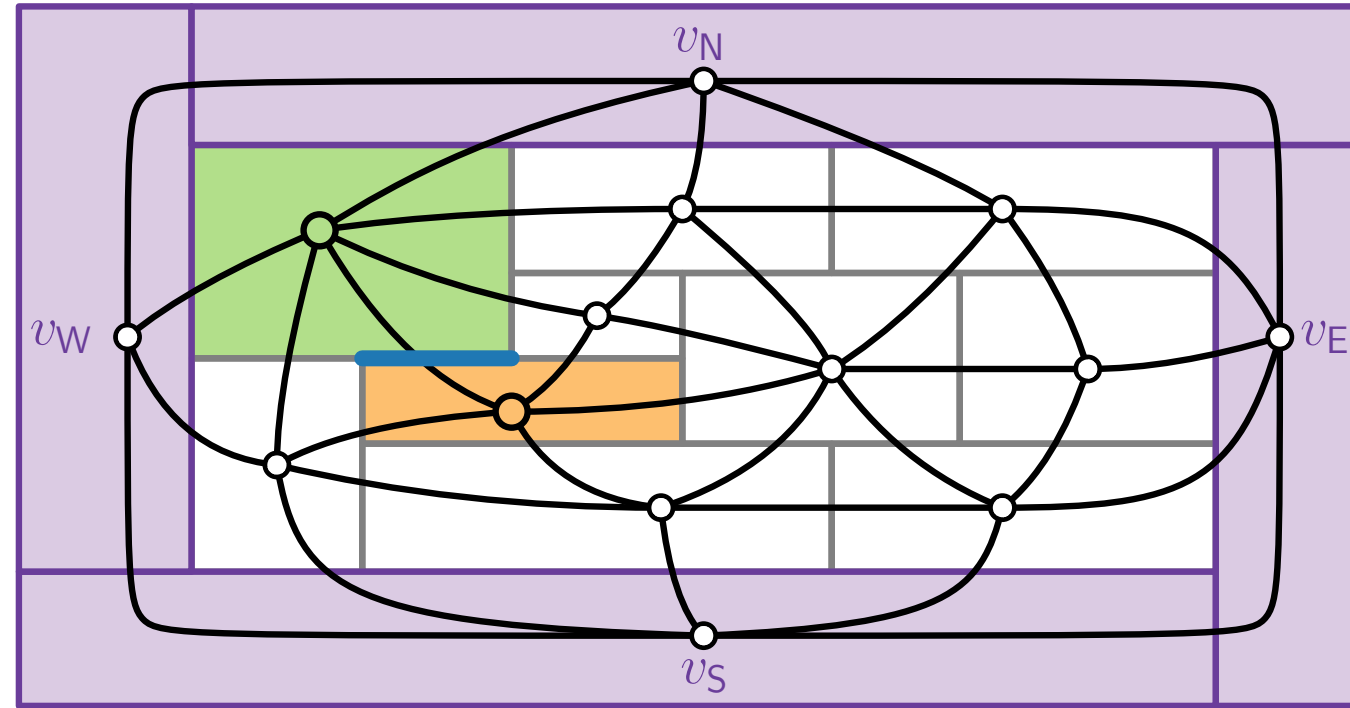
Regular Edge Labeling



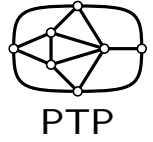
Properly Triangulated
Planar Graph G



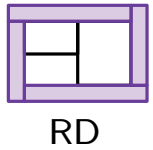
Rectangular Dual $\mathcal{R}$



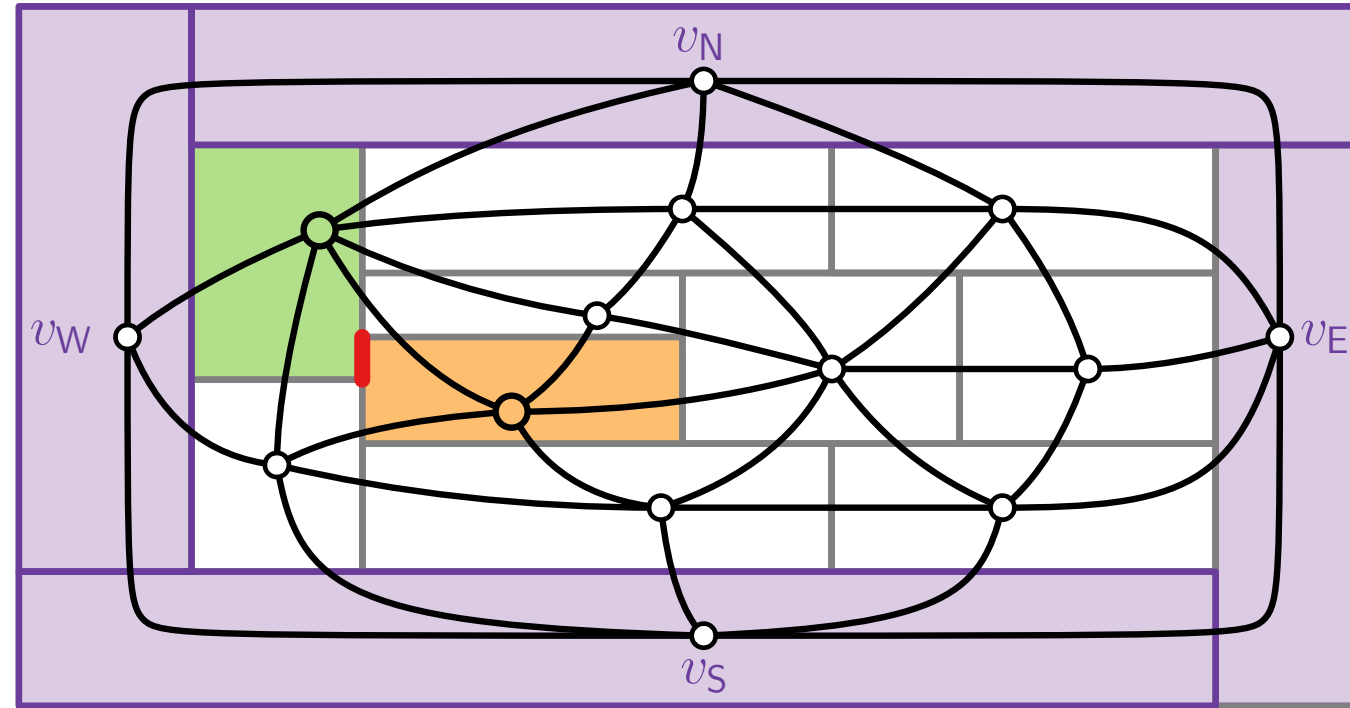
Regular Edge Labeling



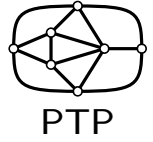
Properly Triangulated
Planar Graph G



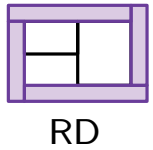
Rectangular Dual $\mathcal{R}$



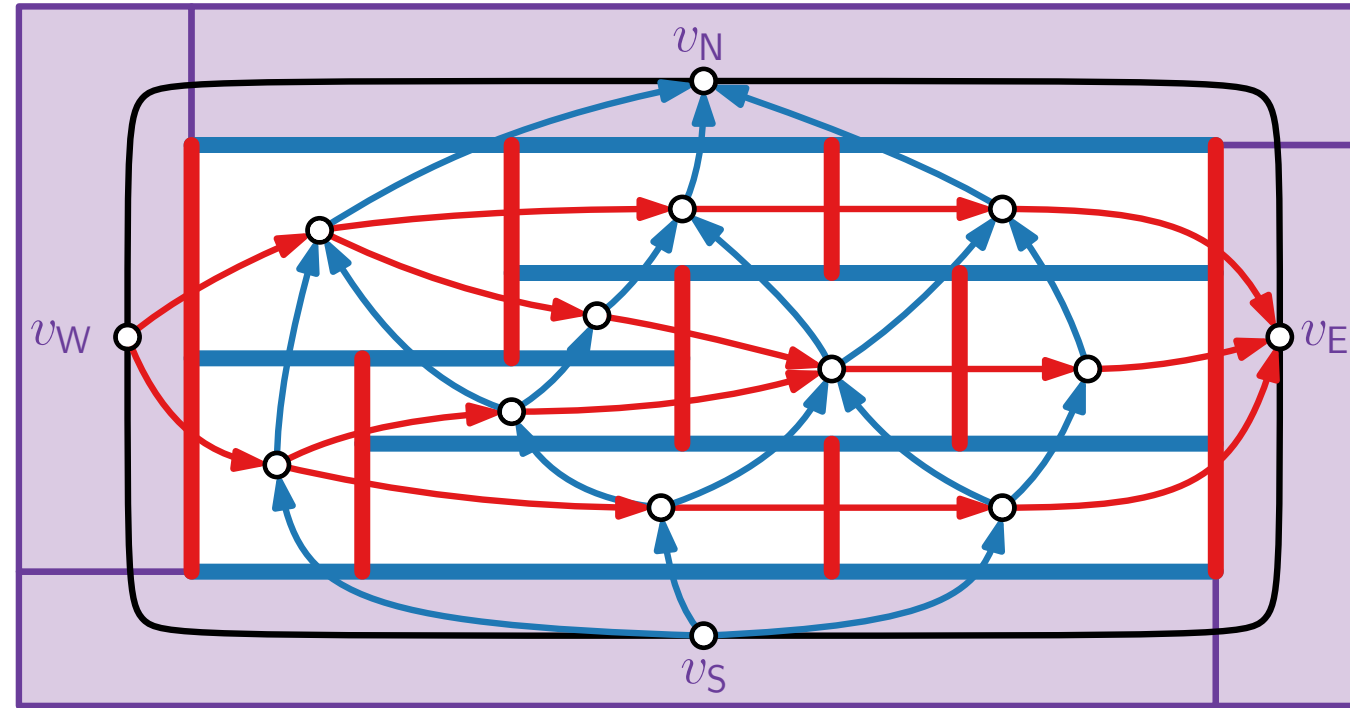
Regular Edge Labeling



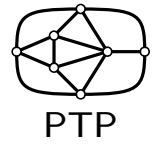
Properly Triangulated
Planar Graph G



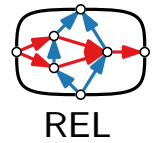
Rectangular Dual $\mathcal{R}$



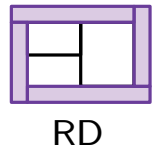
Regular Edge Labeling



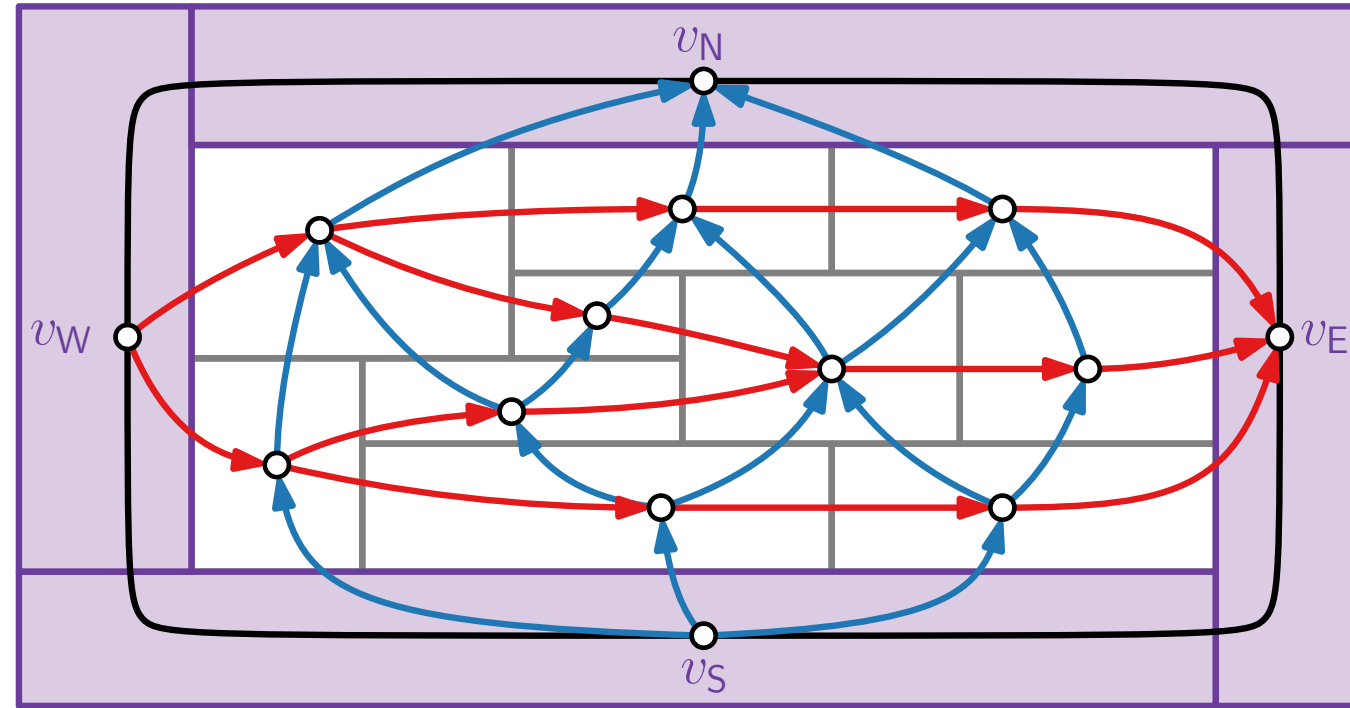
Properly Triangulated
Planar Graph G



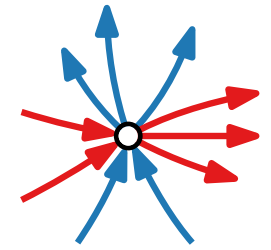
Regular Edge Labeling



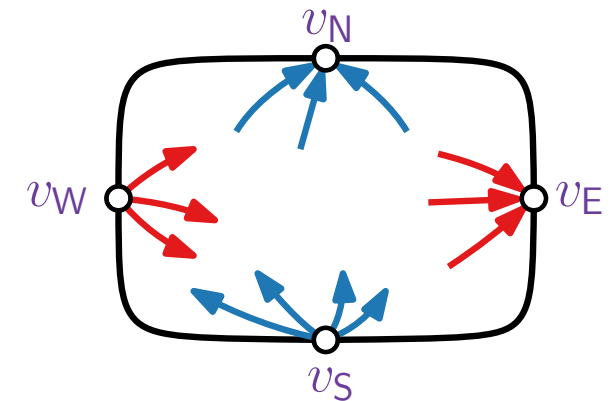
Rectangular Dual $\mathcal{R}$



Properties:

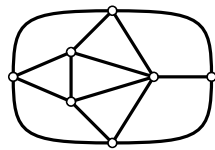


for every
inner vertex

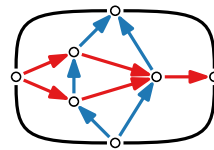
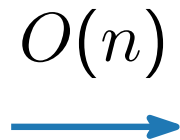


for four
outer vertices

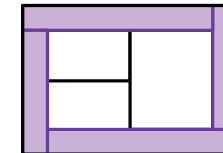
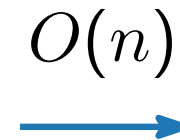
[Kant, He '94]:



PTP



REL



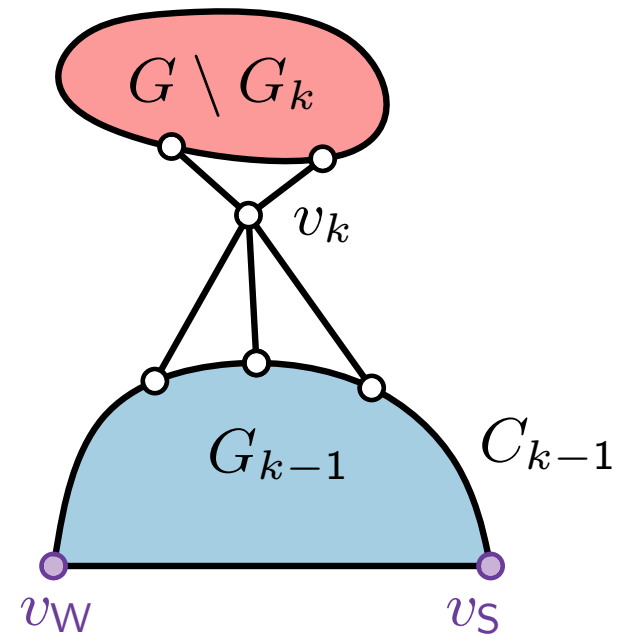
RD

Refined Canonical Order

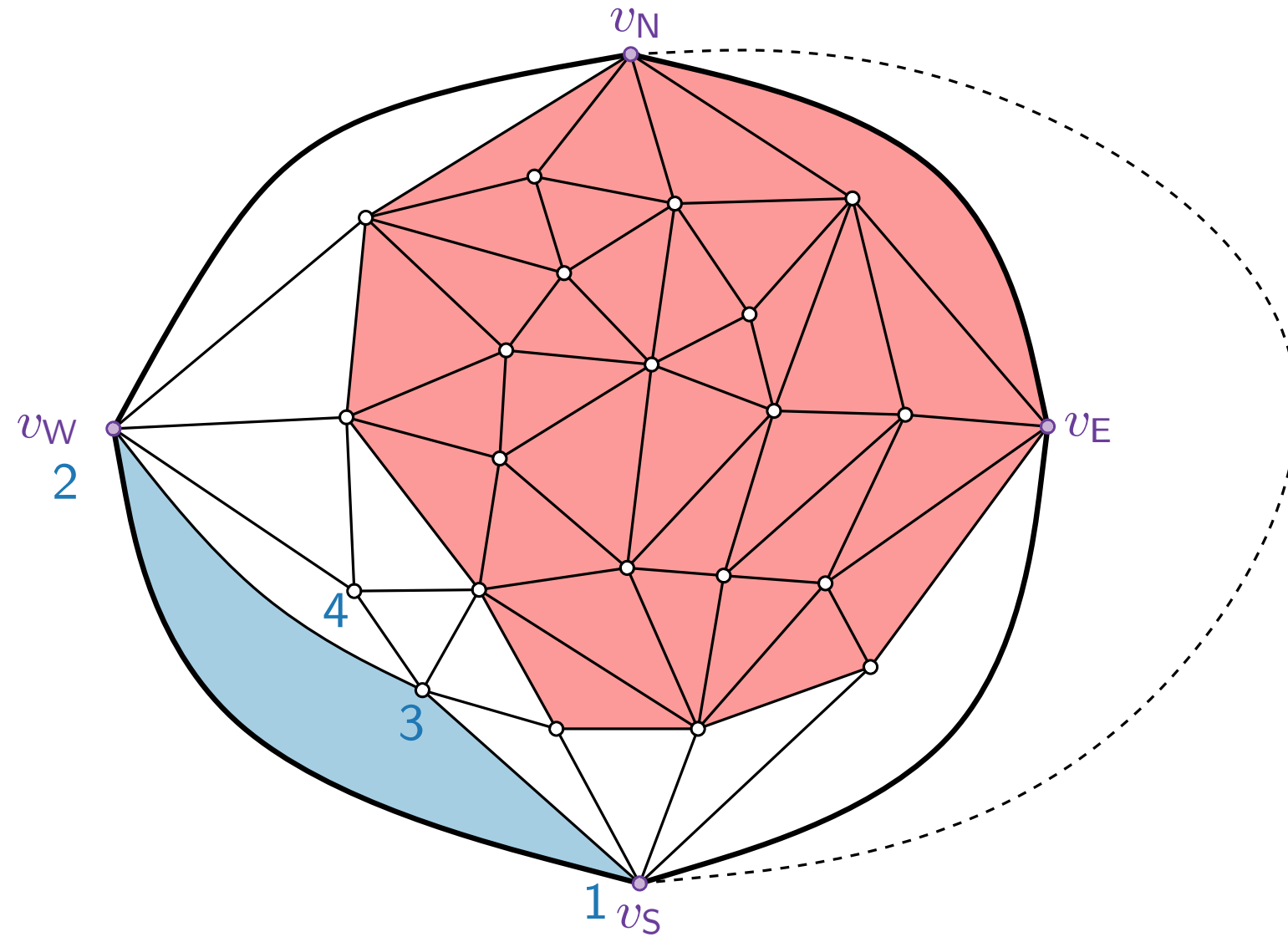
Theorem.

Let G be a PTP graph that is embedded in its unique planar embedding with counter-clockwise outer face $\langle v_W, v_S, v_E, v_N \rangle$. There exists a labeling $v_1 = v_S, v_2 = v_W, v_3, \dots, v_n = v_N$ of the vertices of G such that for every $4 \leq k \leq n$:

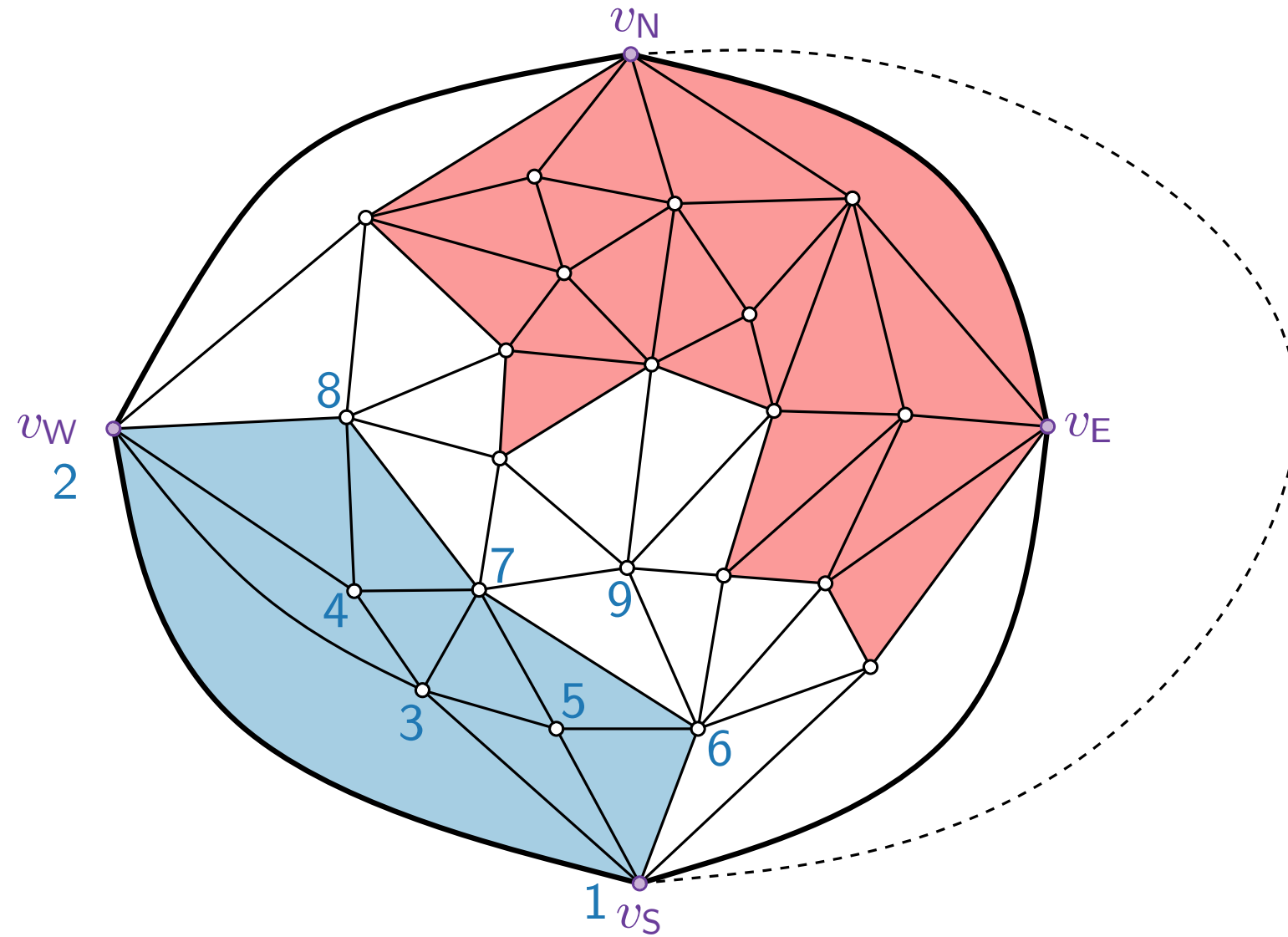
- The subgraph G_{k-1} induced by $v_1, \dots, v_{k-1}$ is biconnected and the boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W) .
- v_k is in the outer face of G_{k-1} , and its neighbors in G_{k-1} form an (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$.
- If $k \leq n - 2$, then v_k has at least two neighbors in $G \setminus G_k$.



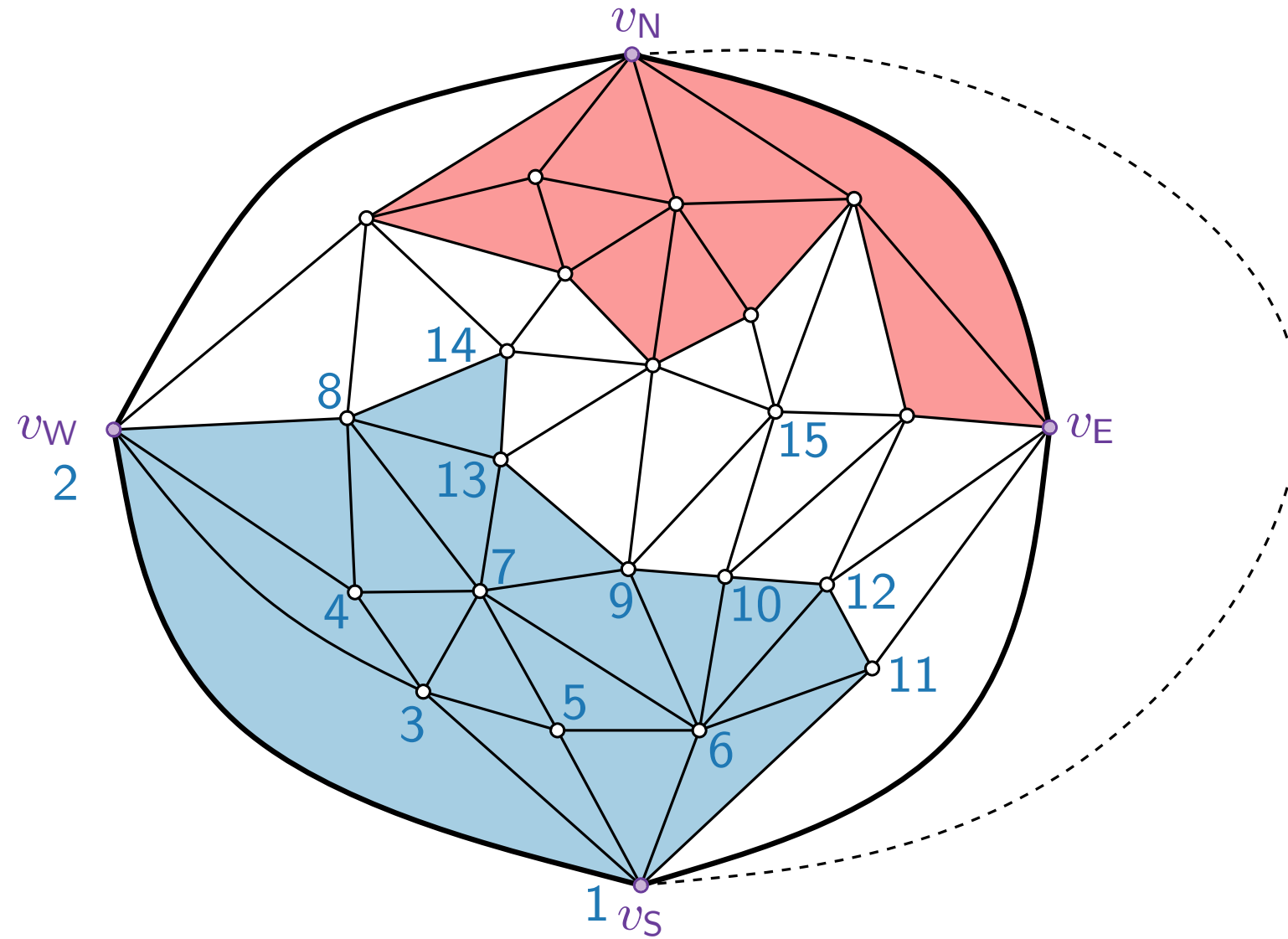
Refined Canonical Order Example



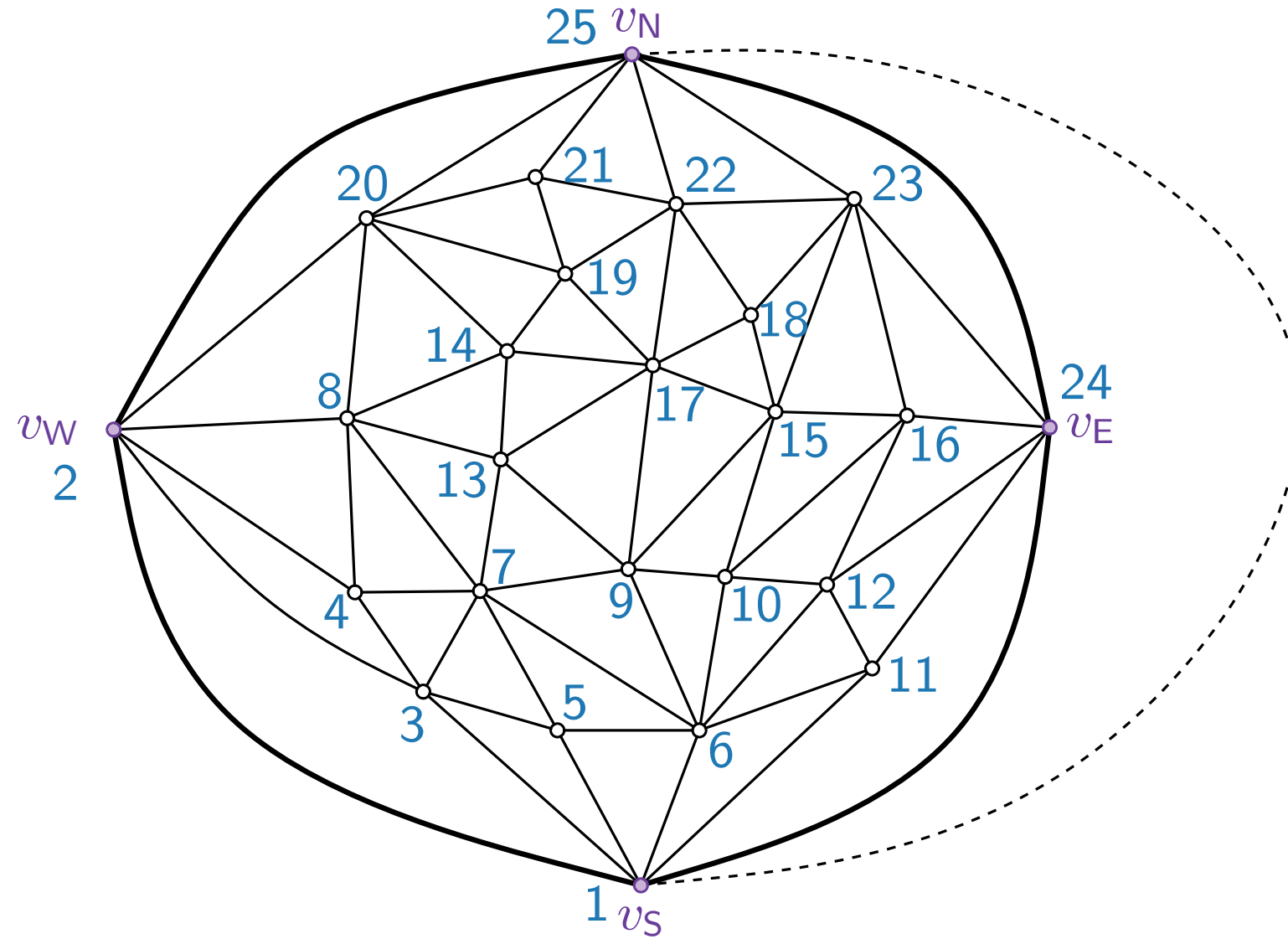
Refined Canonical Order Example



Refined Canonical Order Example



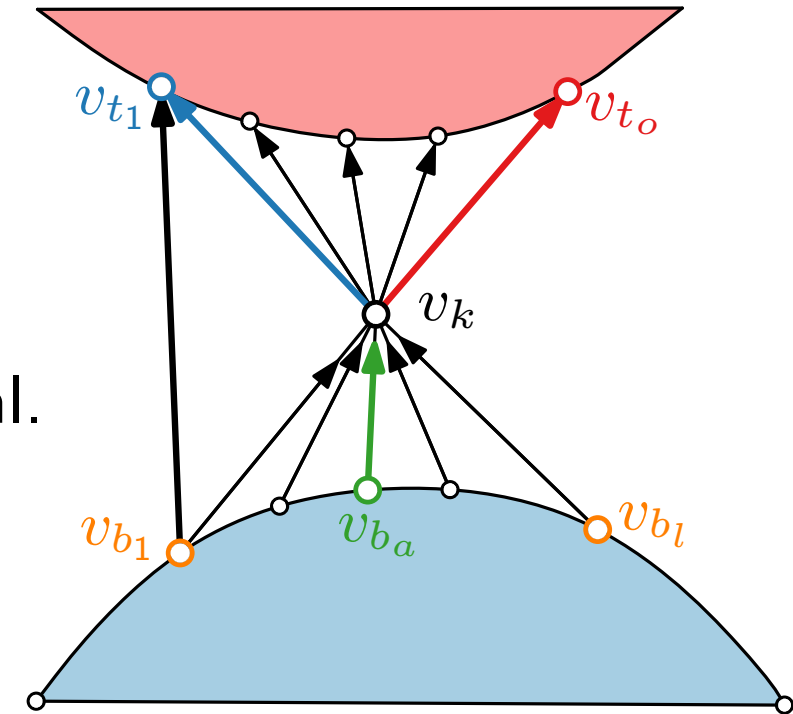
Refined Canonical Order Example



Refined Canonical Order $\rightarrow$ REL

We construct a REL as follows:

- For $i < j$, orient (v_i, v_j) from v_i to v_j ;
- If v_k has incoming edges from $v_{b_1}, \dots, v_{b_l}$, we say that v_{b_1} is the **left point** of v_k and v_{b_l} is the **right point** of v_k .
- **Base edge** of v_k is (v_{b_a}, v_k) , where $b_a \in \{b_1, \dots, b_l\}$ is minimal.
- If $v_{t_1}, \dots, v_{t_o}$ are the higher numbered neighbors of v_k , then we call (v_k, v_{t_1}) **left edge** and (v_k, v_{t_o}) **right edge** of v_k .



Lemma 1.

A **left edge** or **right edge** cannot be a **base edge**.

Proof. Suppose that the **left edge** (v_k, v_{t_1}) is the **base edge** of v_{t_1} . Since G is triangulated, $(v_{b_1}, v_{t_1}) \in E(G)$. Contradiction since $k > b_1$.

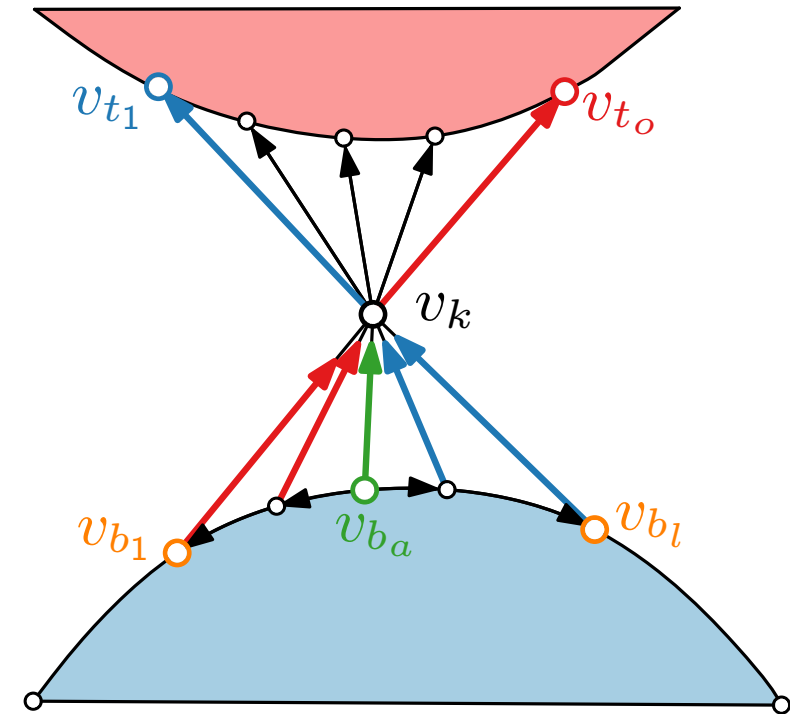
Refined Canonical Order $\rightarrow$ REL

Lemma 2.

Every edge is either a **left edge**, a **right edge** or a **base edge**.

Proof.

- The exclusivity of the “or” follows from Lemma 1.
- Let (v_{b_a}, v_k) be the **base edge** of v_k .
- v_{b_a} is the **right point** of $v_{b_{a-1}}$. Generally, for $1 \leq i \leq l$,
 - v_{b_i} has at least two higher-numbered neighbors.
 - One of them is v_k ; the other one is $v_{b_{i-1}}$ or $v_{b_{i+1}}$.
 - For $1 \leq i < a - 1$, it is $v_{b_{i-1}}$. Thus, v_{b_i} is the right point of $v_{b_{i-1}}$.
 - Analogously, v_{b_i} is the **left point** of $v_{b_{i+1}}$ for $i \geq a$.
- Edges (v_{b_i}, v_k) , $1 \leq i < a - 1$, are **right edges**.
- Similarly, (v_{b_i}, v_k) , for $a + 1 \leq i \leq l$, are **left edges**.



Refined Canonical Order $\rightarrow$ REL

Coloring.

- Color **right** (**left**) edges in **red** (**blue**).
- Color a **base edge** (v_{b_i}, v_k) **red** if $i = 1$ and **blue** if $i = l$ and otherwise arbitrarily.

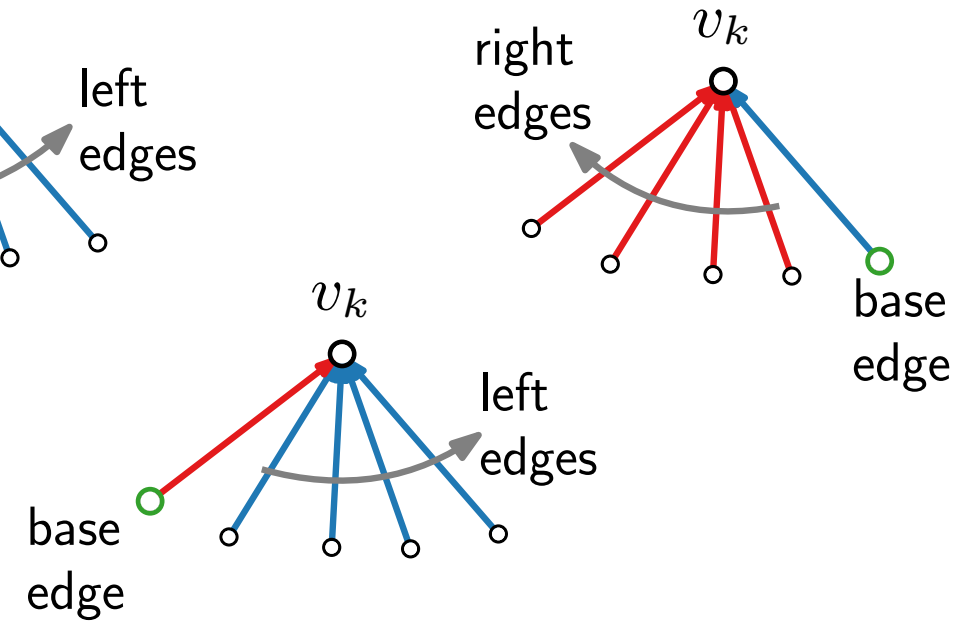
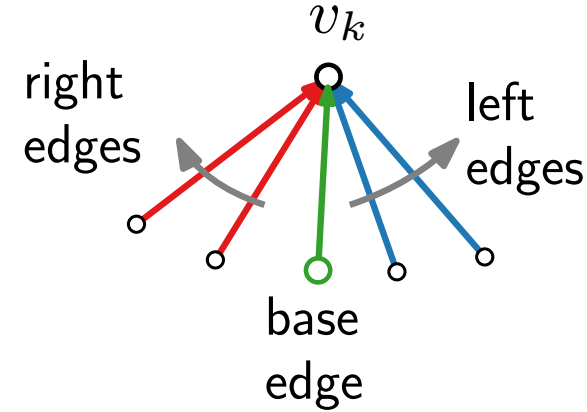
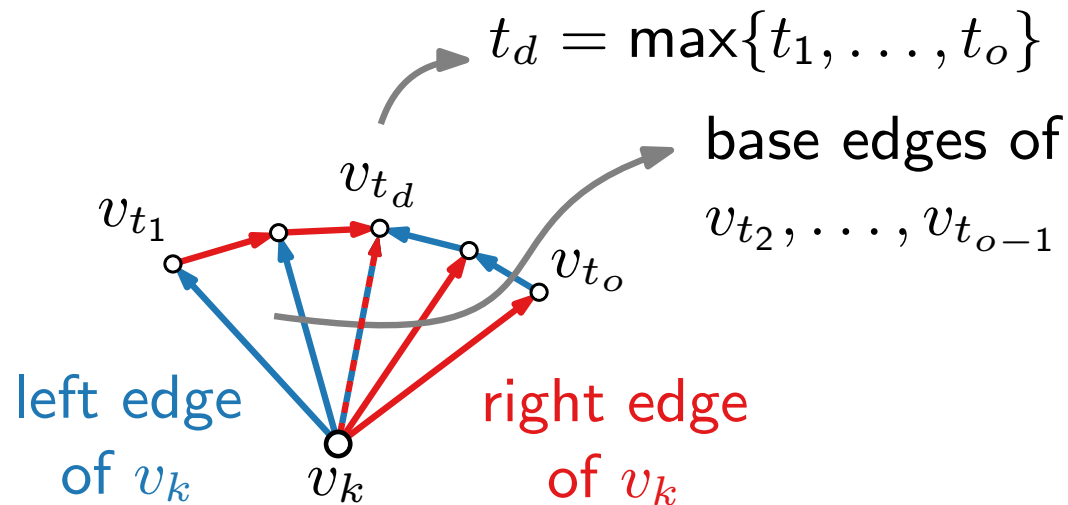
Let T_r be the red edges and T_b the blue edges.

Lemma 3.

$\{T_r, T_b\}$ is a regular edge labeling.

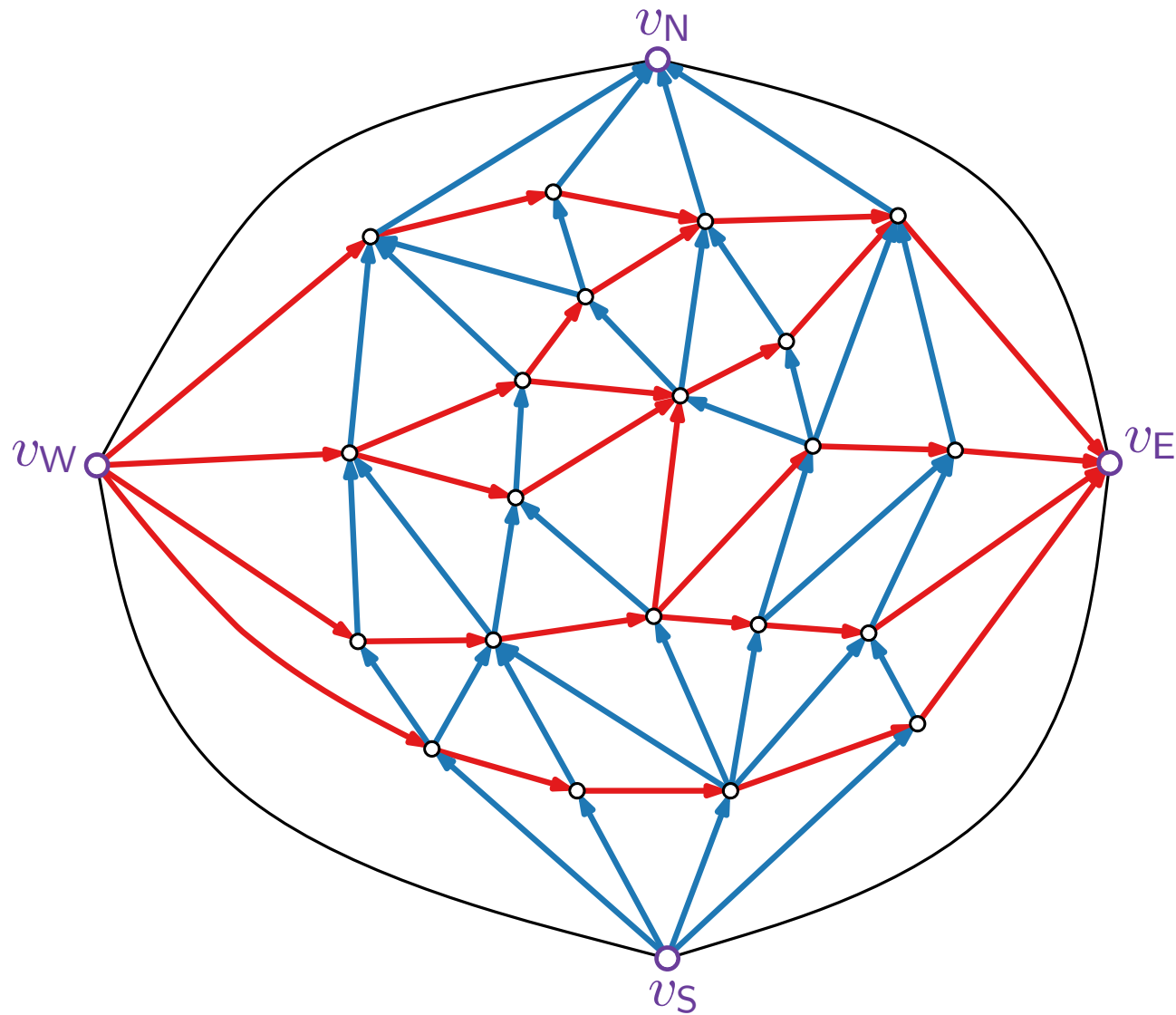
Proof.

$$t_o \geq 2$$

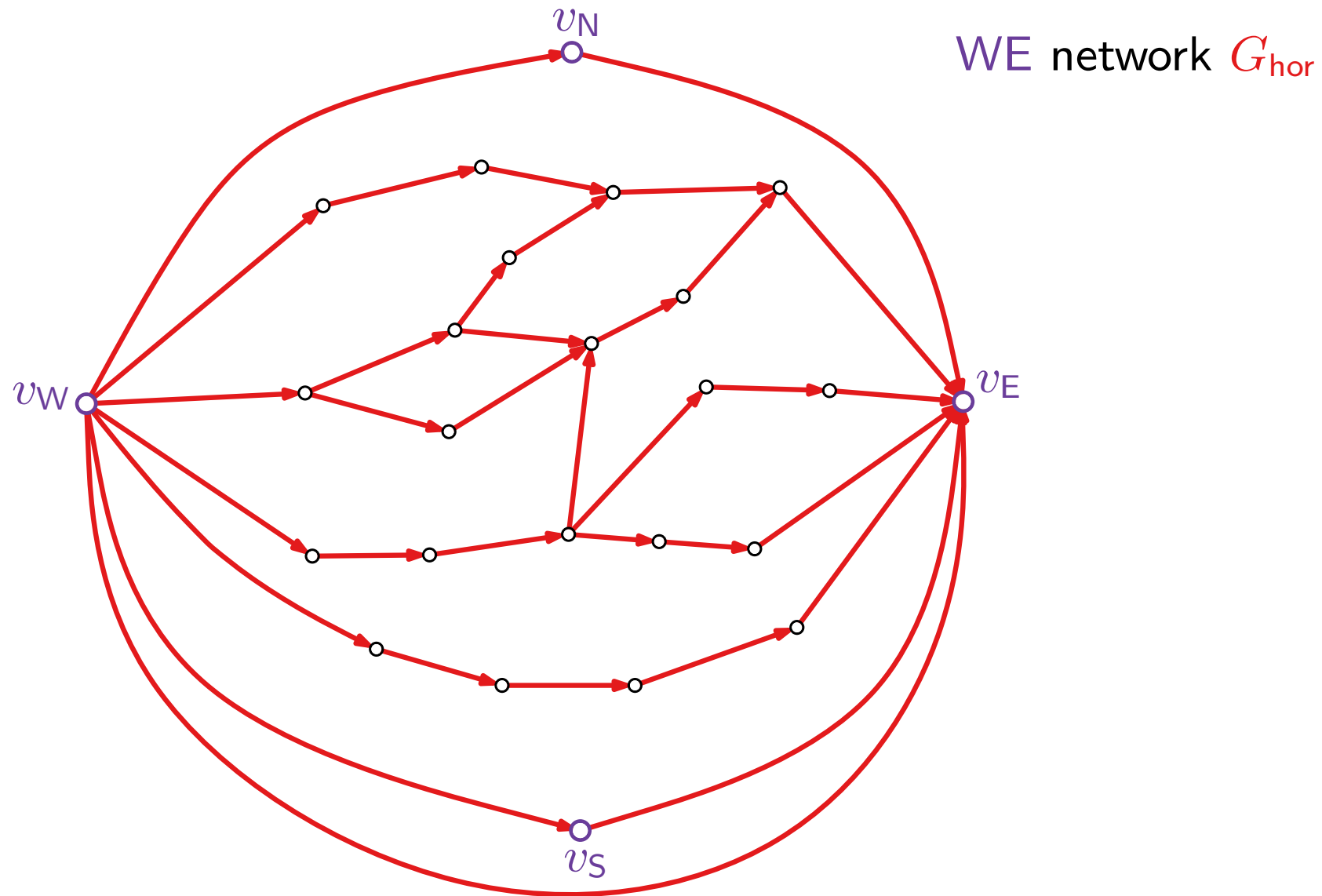


- $t_1 < t_2 < \dots < t_d$ and $t_d > t_{d+1} > \dots > t_o$
 - $(v_k, v_{t_i}), 2 \leq i \leq d-1$ are **blue**
 - $(v_k, v_{t_i}), d+1 \leq i \leq o-1$ are **red**
 - (v_k, v_{t_d}) is either **red** or **blue**
- $\Rightarrow$ Circular order of outgoing edges at v_k correct.

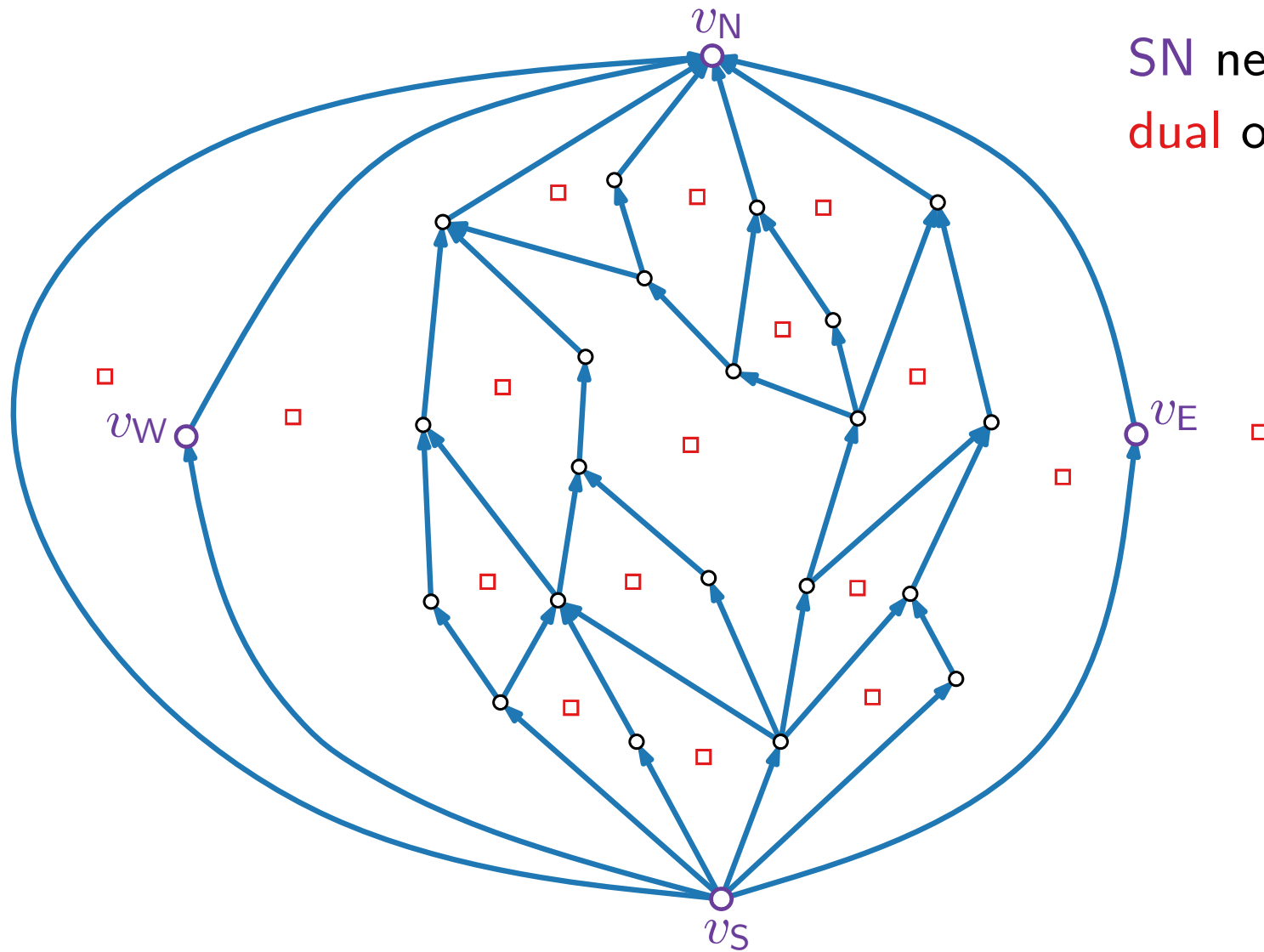
From REL to st-Digraphs to Coordinates



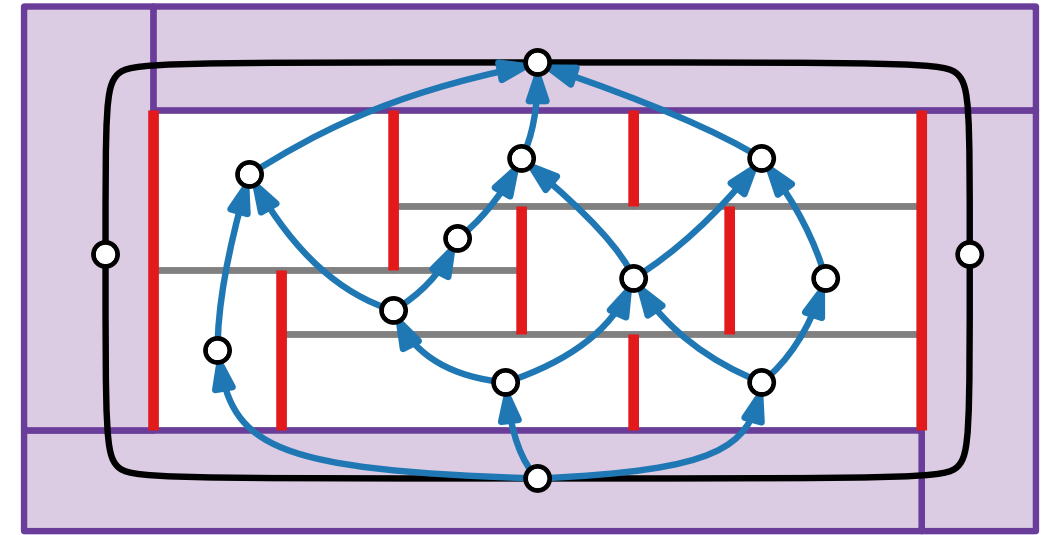
From REL to st-Digraphs to Coordinates



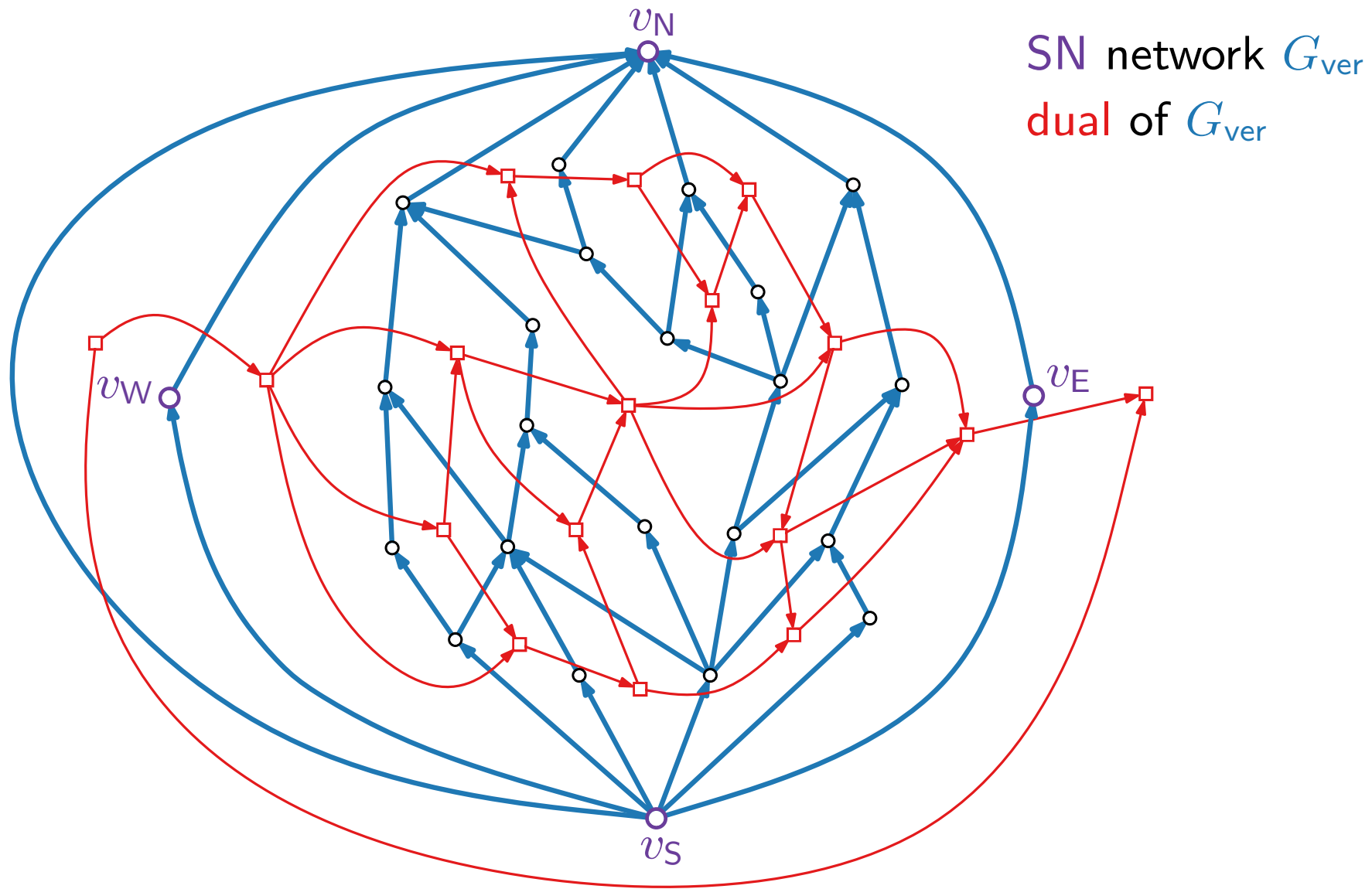
From REL to st-Digraphs to Coordinates



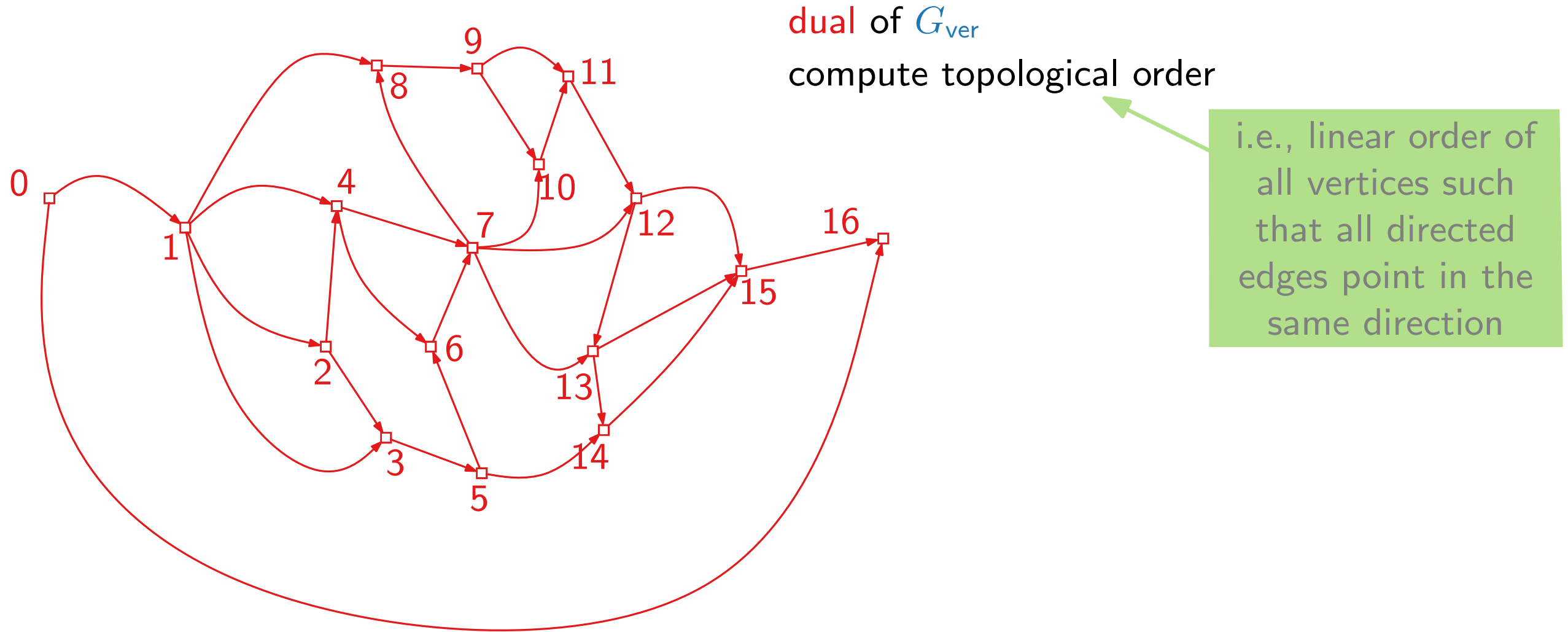
SN network G_{ver}
 dual of G_{ver}



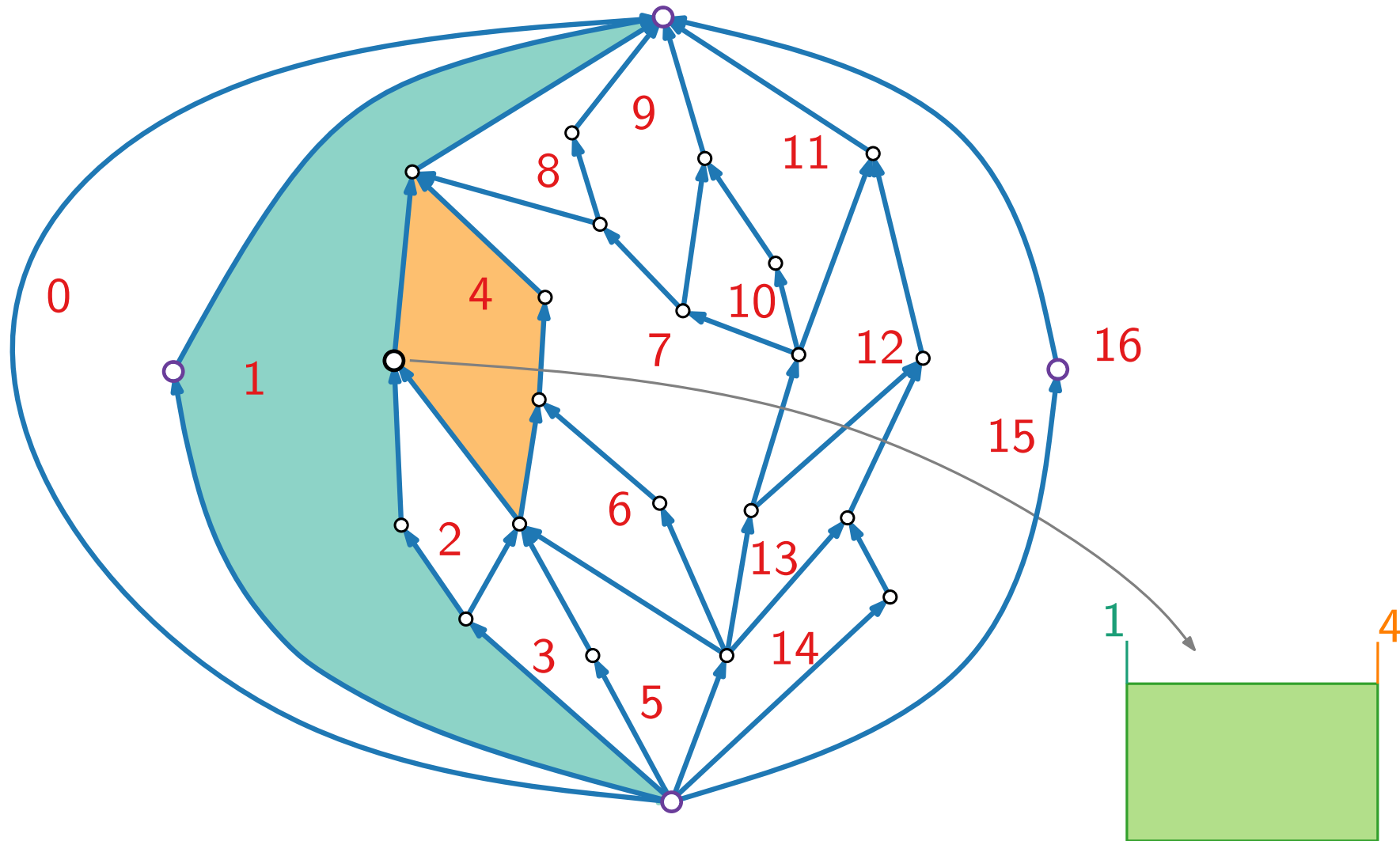
From REL to st-Digraphs to Coordinates



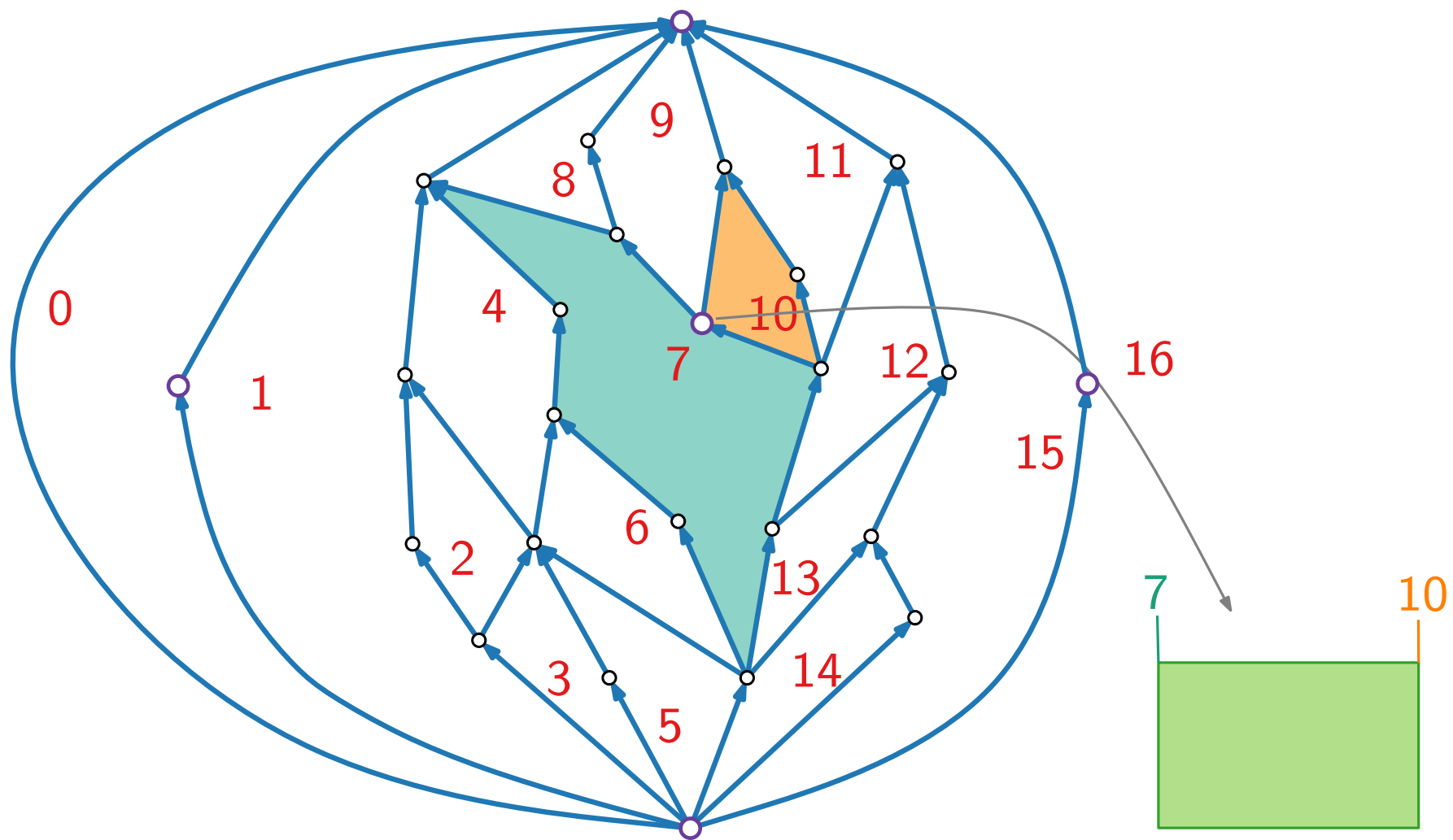
From REL to st-Digraphs to Coordinates



From REL to st-Digraphs to Coordinates



From REL to st-Digraphs to Coordinates

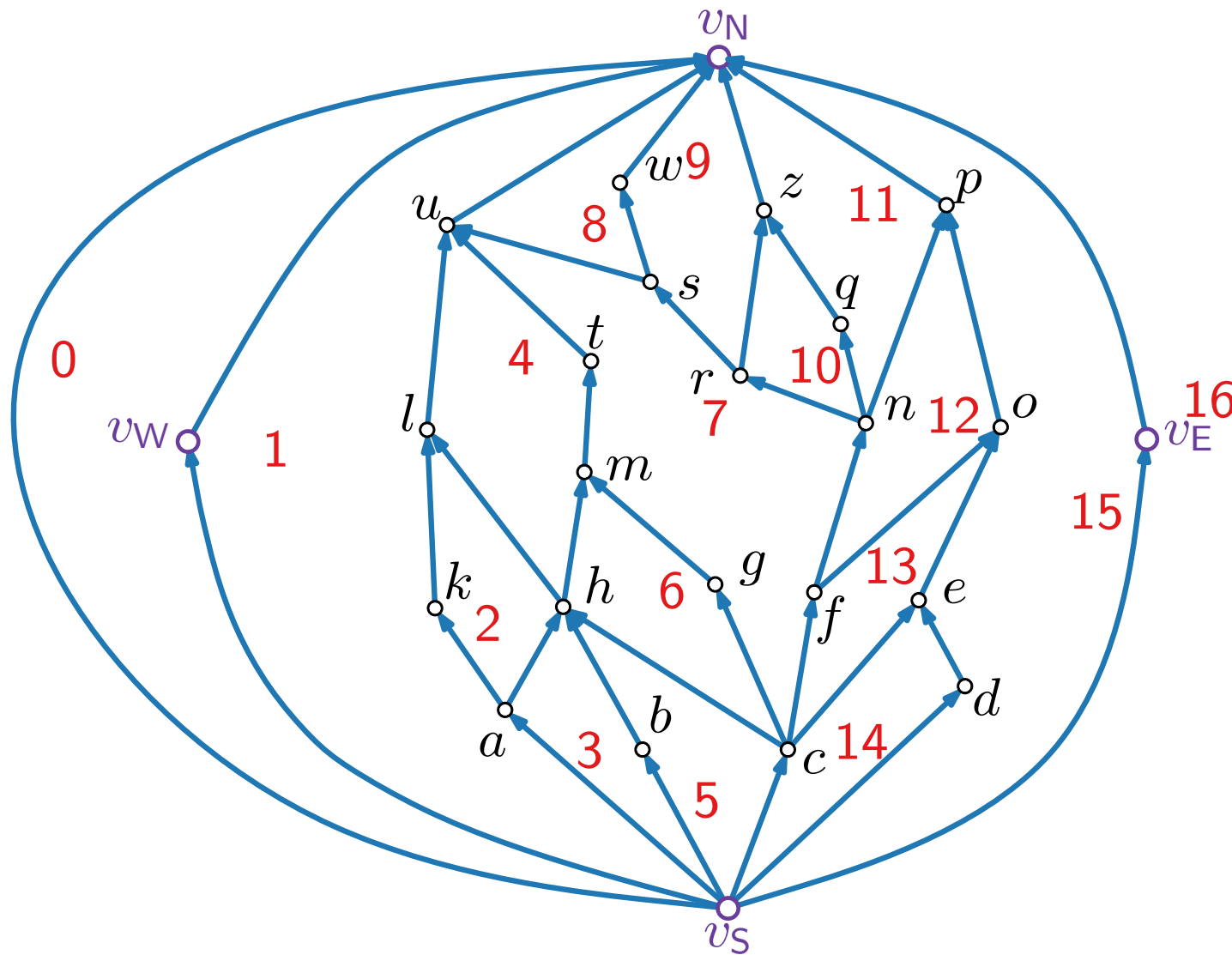


Rectangular Dual Algorithm

For a PTP graph G :

- Find a REL $\{T_r, T_b\}$ of G .
- Construct a SN network G_{ver} of G (consists of T_b plus outer edges).
- Construct the dual G_{ver}^* of G_{ver} and compute a topological ordering f_{ver} of G_{ver}^* .
- For each vertex v of G , let g_v and h_v be the face on the left and face on the right of v .
Set $x_1(v) = f_{\text{ver}}(g_v)$ and $x_2(v) = f_{\text{ver}}(h_v)$.
- Define $x_1(v_N) = 0$, $x_1(v_S) = 1$ and $x_2(v_S) = \max_{v \in V(G)} f_{\text{ver}}(h_v)$, $x_2(v_N) = x_2(v_S) - 1$.
- Analogously compute $y_1(\cdot)$ and $y_2(\cdot)$ using G_{hor} .
- For each vertex v of G , let $R(v) = [x_1(v), x_2(v)] \times [y_1(v), y_2(v)]$.

Reading off Coordinates to Get the Rectangular Dual



$$x_1(v_N) = 0, \quad x_2(v_N) = 15$$

$$x_1(v_S) = 1, \quad x_2(v_S) = 16$$

$$x_1(v_W) = 0, \quad x_2(v_W) = 1$$

$$x_1(v_E) = 15, \quad x_2(v_E) = 16$$

$$x_1(a) = 1, \quad x_2(a) = 3$$

$$x_1(b) = 3, \quad x_2(b) = 5$$

$$x_1(c) = 5, \quad x_2(c) = 14$$

$$x_1(d) = 14, \quad x_2(d) = 15$$

$$x_1(e) = 13, \quad x_2(e) = 15$$

...

$$y_1(v_W) = 0, \quad y_2(v_W) = 9$$

$$y_1(v_E) = 1, \quad y_2(v_E) = 10$$

$$y_1(v_N) = 9, \quad y_2(v_N) = 10$$

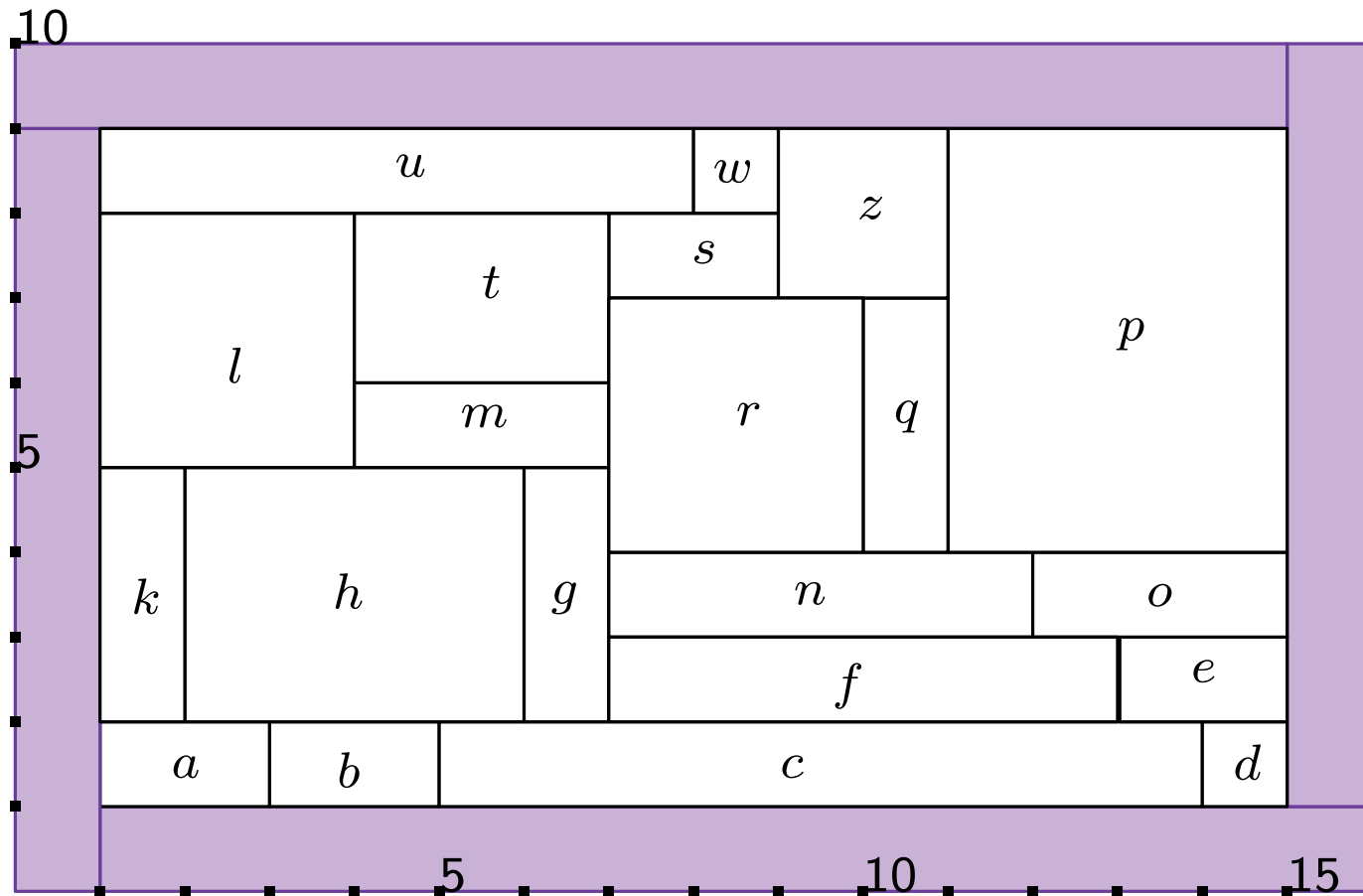
$$y_1(v_S) = 0, \quad y_2(v_S) = 1$$

$$y_1(a) = 1, \quad y_2(a) = 2$$

$$y_1(b) = 1, \quad y_2(b) = 2$$

...

Reading off Coordinates to Get the Rectangular Dual



$$x_1(v_N) = 0, \quad x_2(v_N) = 15$$

$$x_1(v_S) = 1, \quad x_2(v_S) = 16$$

$$x_1(v_W) = 0, \quad x_2(v_W) = 1$$

$$x_1(v_E) = 15, \quad x_2(v_E) = 16$$

$$x_1(a) = 1, \quad x_2(a) = 3$$

$$x_1(b) = 3, \quad x_2(b) = 5$$

$$x_1(c) = 5, \quad x_2(c) = 14$$

$$x_1(d) = 14, \quad x_2(d) = 15$$

$$x_1(e) = 13, \quad x_2(e) = 15$$

...

$$y_1(v_W) = 0, \quad y_2(v_W) = 9$$

$$y_1(v_E) = 1, \quad y_2(v_E) = 10$$

$$y_1(v_N) = 9, \quad y_2(v_N) = 10$$

$$y_1(v_S) = 0, \quad y_2(v_S) = 1$$

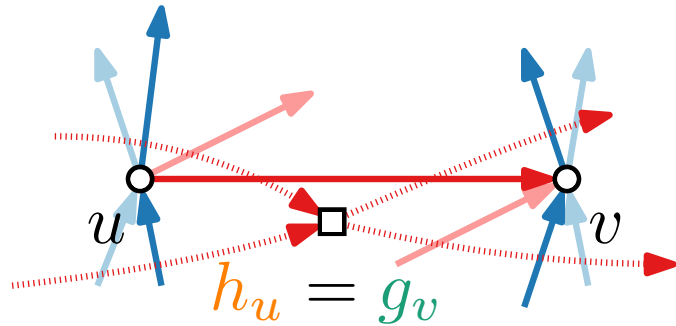
$$y_1(a) = 1, \quad y_2(a) = 2$$

$$y_1(b) = 1, \quad y_2(b) = 2$$

...

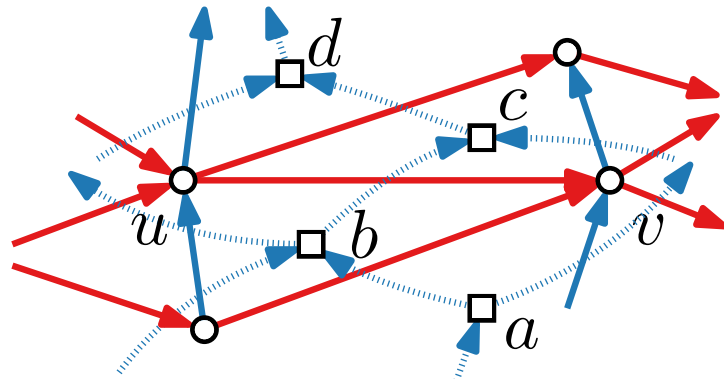
Correctness of Algorithm (Sketch)

- If edge (u, v) exists, then $x_2(u) = x_1(v)$



$$x_2(u) = f_{\text{ver}}(h_u) = f_{\text{ver}}(g_v) = x_1(v)$$

- ... and the vertical segments of their rectangles overlap.



$$\begin{aligned} y_1(v) &= f_{\text{hor}}(a) \leq y_1(u) = f_{\text{hor}}(b) \\ &< y_2(v) = f_{\text{hor}}(c) \leq y_2(u) = f_{\text{hor}}(d) \end{aligned}$$

- If the path from u to v in red is at least two edges long, then $x_2(u) < x_1(v)$.
- No two boxes overlap.
- For details, see [He '93].

Rectangular Dual Result

Theorem.

Every PTP graph G has a rectangular dual.

A rectangular dual can be computed in linear time.

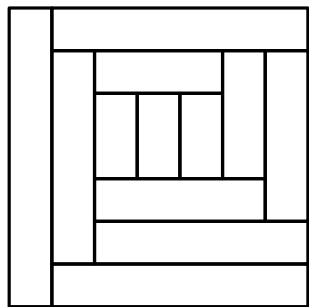
Proof.

- Compute a planar embedding of G .
- Compute a refined canonical ordering of G .
- Traverse the graph and color the edges. $\rightarrow$ REL
- Construct G_{ver} and G_{hor} .
- Construct their duals G_{ver}^* and G_{hor}^* .
- Compute topological orderings of G_{ver}^* and G_{hor}^* .
- Assign coordinates to the rectangles that represent the vertices.

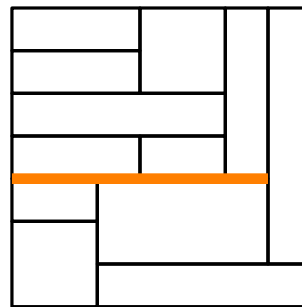
Discussion

- A layout is **area-universal** if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.
- A rectangular layout is **area-universal** if and only if it is **one-sided**.
[Eppstein et al., SIAM J. Comp. 2012]

one-sided



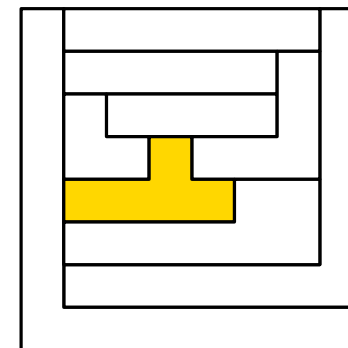
s



not one-sided

i.e., every segment belongs to exactly one rectangle

- Area-universal **rectilinear** representation: possible for all planar graphs.
- [Alam et al. 2013]: 8 sides (matches the lower bound)



Literature

Construction of triangle contact representations based on

- [de Fraysseix, Ossona de Mendez, Rosenstiehl '94] On Triangle Contact Graphs

Construction of rectangular dual based on

- [He '93] On Finding the Rectangular Duals of Planar Triangulated Graphs
- [Kant, He '94] Two algorithms for finding rectangular duals of planar graphs

and originally from

- [Koźmiński, Kinnen '85] Rectangular Duals of Planar Graphs