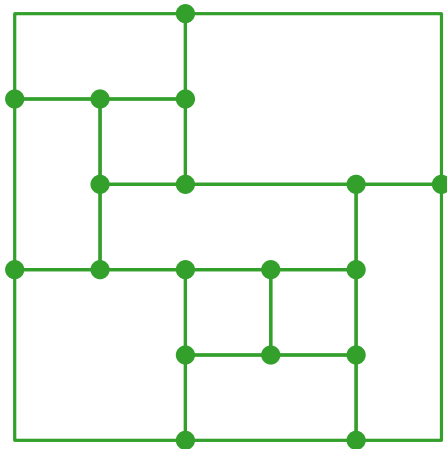
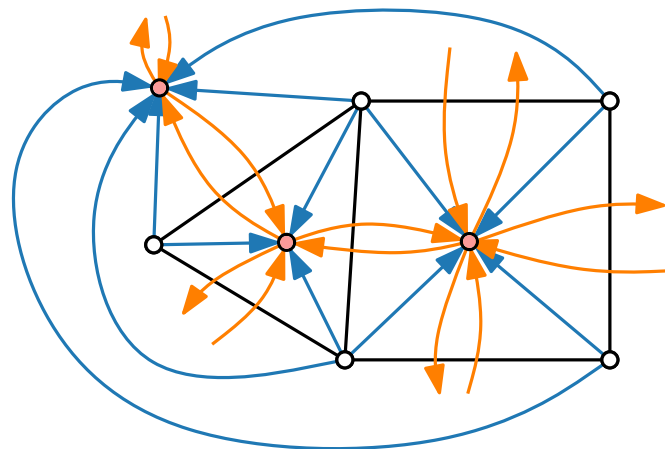
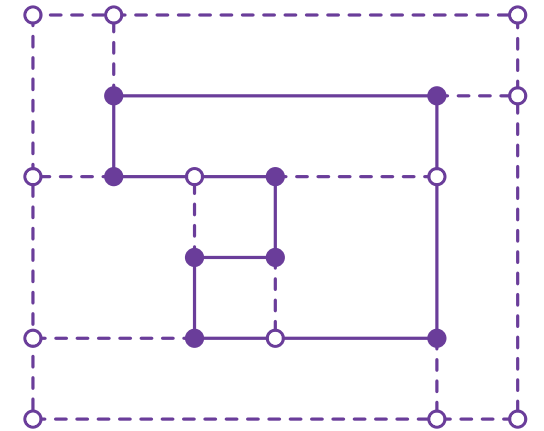


# Visualization of Graphs



## Lecture 6: Orthogonal Layouts

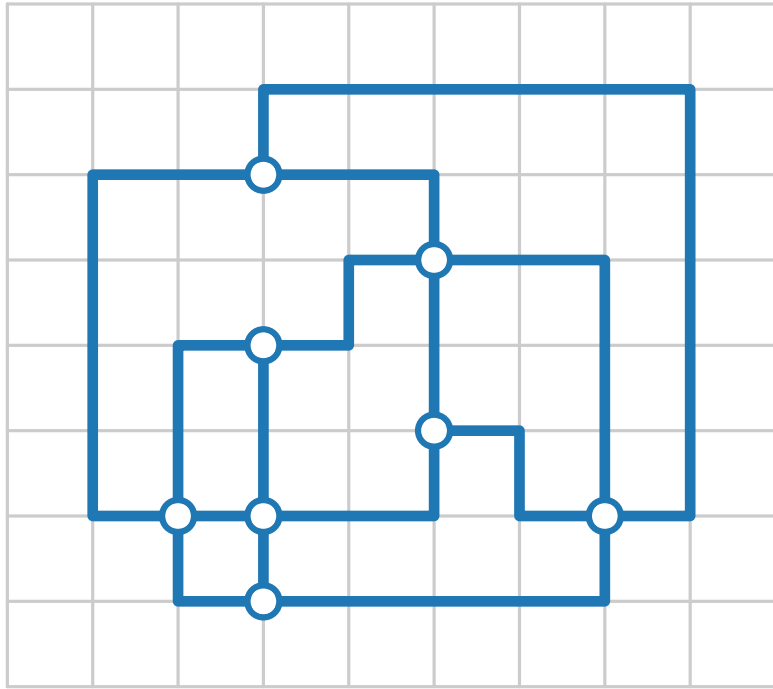


Samuel Wolf

Summer term 2026



# Orthogonal Layout – Definition



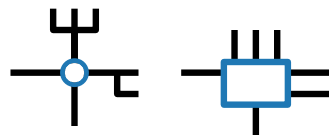
## Definition.

A drawing  $\Gamma$  of a graph  $G$  is called **orthogonal** if

- vertices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical line segments of the grid, and
- pairs of edges are disjoint or cross orthogonally.

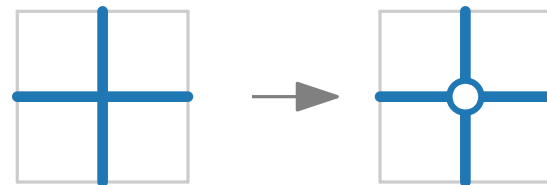
## Observations.

- Edges lie on a grid  $\Rightarrow$  **bends** lie on grid points
- Max. degree of each vertex is at most 4
- Otherwise



## Planarization.

- Fix embedding
- Crossings become vertices



## Aesthetic criteria to optimize.

- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

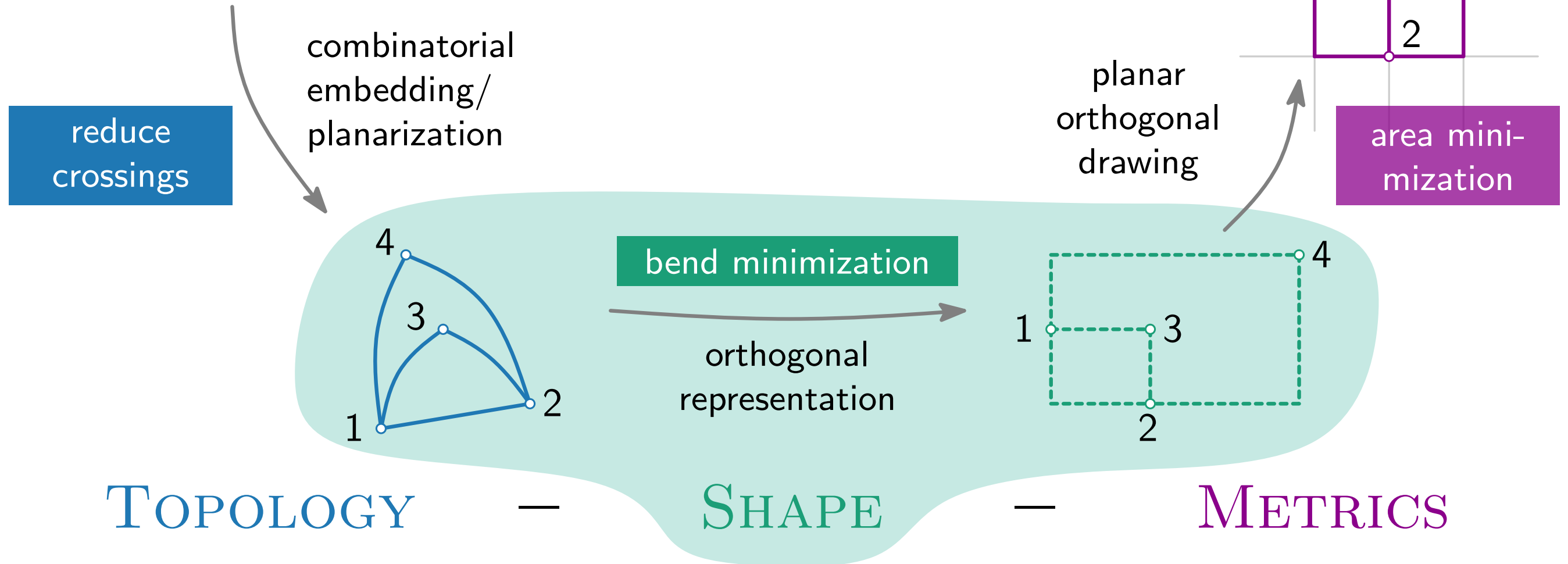
# Topology – Shape – Metrics

Three-step approach:

[Tamassia 1987]

$$V(G) = \{v_1, v_2, v_3, v_4\}$$

$$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$



# Orthogonal Representation

## Idea.

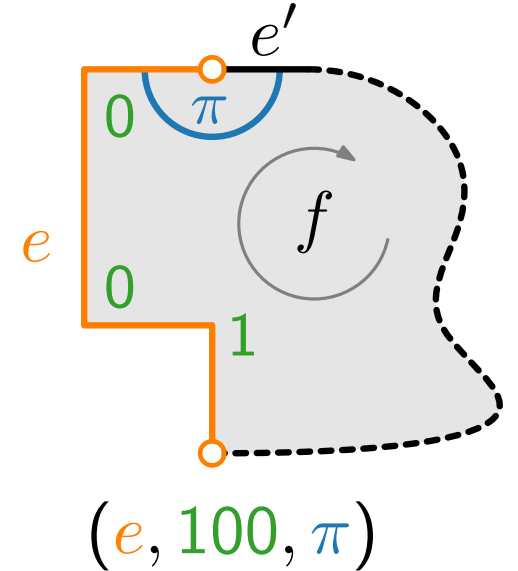
Describe orthogonal drawing combinatorially.

## Definitions.

Let  $G$  be a plane graph with set  $F$  of faces and outer face  $f_0 \in F$ .

- Let  $e$  be an edge with the face  $f$  to the right.
  - An **edge description** of  $e$  w.r.t.  $f$  is a triple  $(e, \delta, \alpha)$  where
    - $\delta \in \{0, 1\}^*$  (where 0 = right bend, 1 = left bend)
    - $\alpha$  is angle  $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$  between  $e$  and next edge  $e'$
- A **face representation**  $H(f)$  of a face  $f$  is a clockwise ordered sequence  $(e_1, \delta_1, \alpha_1), (e_2, \delta_2, \alpha_2), \dots, (e_{\deg(f)}, \delta_{\deg(f)}, \alpha_{\deg(f)})$  of edge descriptions w.r.t.  $f$ .
- An **orthogonal representation**  $H(G)$  of  $G$  is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$

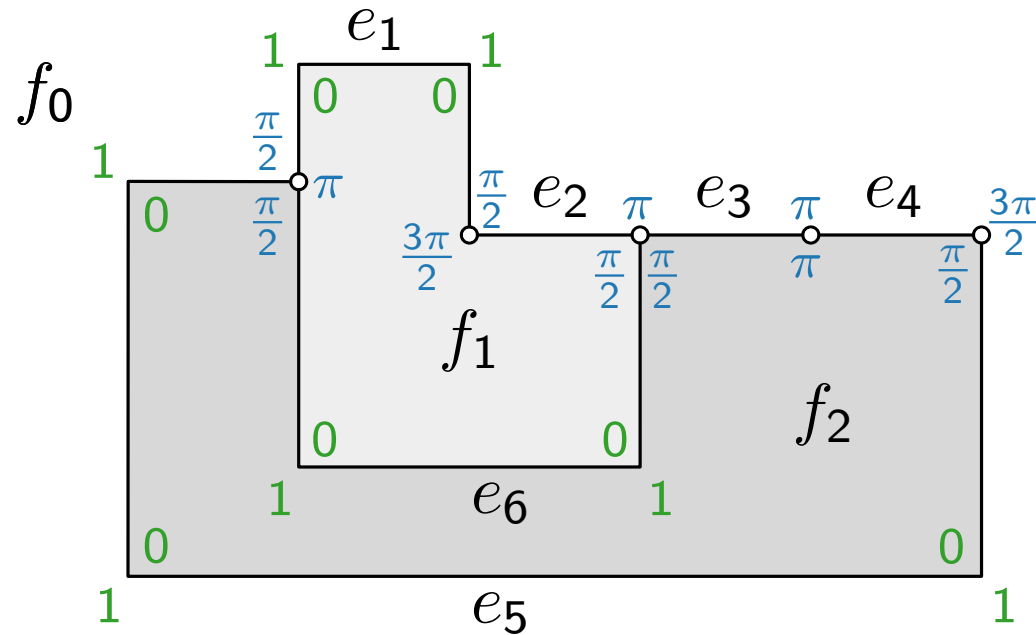
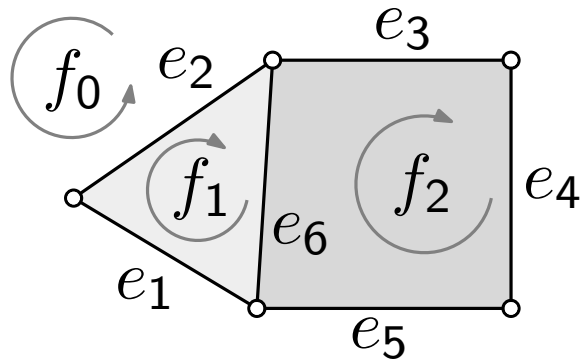


# Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2}))$$



Coordinates are not fixed yet!

# Correctness of an Orthogonal Representation

(H1)  $H(G)$  corresponds to  $F, f_0$ .

(H2) For each **edge**  $\{u, v\}$  shared by faces  $f$  and  $g$  with  $((u, v), \delta_1, \alpha_1) \in H(f)$  and  $((v, u), \delta_2, \alpha_2) \in H(g)$ , the sequence  $\delta_1$  is like  $\delta_2$ , but reversed and inverted.

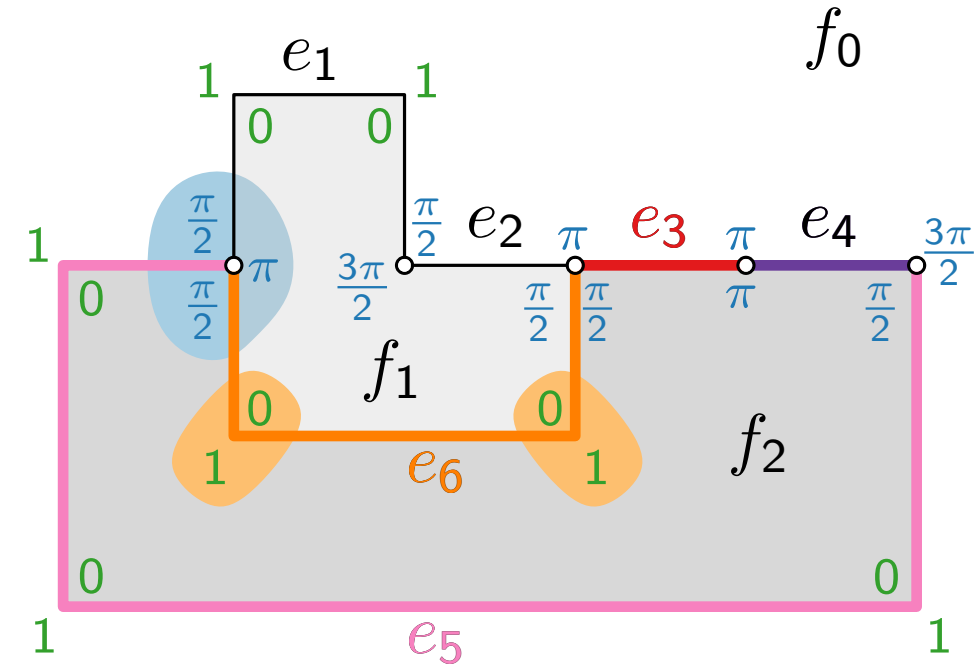
(H3) Let  $|\delta|_0$  (resp.  $|\delta|_1$ ) be the number of zeros (resp. ones) in  $\delta$ , and let  $r = (e, \delta, \alpha)$ .

Let  $C(r) := |\delta|_0 - |\delta|_1 - \alpha/\frac{\pi}{2} + 2$ .

For each **face**  $f$ , it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex**  $v$ , the sum of incident angles is  $2\pi$ .



$$C((e_3, \emptyset, \pi)) = 0 - 0 - 2 + 2 = 0$$

$$C((e_4, \emptyset, \frac{\pi}{2})) = 0 - 0 - 1 + 2 = 1$$

$$C((e_5, 000, \frac{\pi}{2})) = 3 - 0 - 1 + 2 = 4$$

$$C((e_6, 11, \frac{\pi}{2})) = 0 - 2 - 1 + 2 = -1$$

---


$$\sum_{r \in H(f_2)} C(r) = +4$$





# General Flow Network

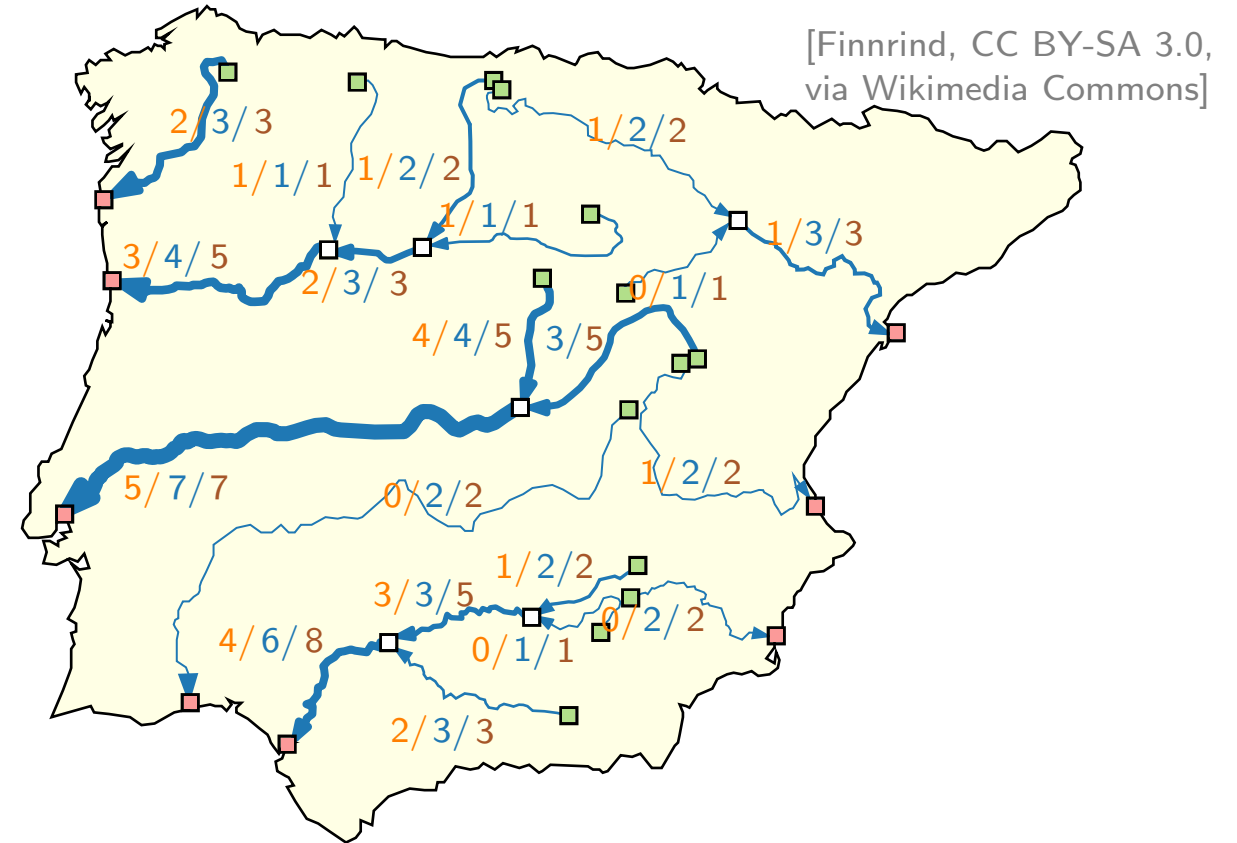
**Flow network**  $(G; S, T; \ell; u)$  with

- directed graph  $G$
- *sources*  $S \subseteq V(G)$ , *sinks*  $T \subseteq V(G)$
- edge *lower bound*  $\ell: E(G) \rightarrow \mathbb{R}_0^+$
- edge *capacity*  $u: E(G) \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$

A function  $X: E(G) \rightarrow \mathbb{R}_0^+$  is called  **$S$ - $T$  flow** if:

$$\begin{aligned} \ell(i, j) &\leq X(i, j) \leq u(i, j) & \forall (i, j) \in E(G) \\ \sum_{(i, j) \in E(G)} X(i, j) - \sum_{(j, i) \in E(G)} X(j, i) &= 0 & \forall i \in V(G) \setminus (S \cup T) \end{aligned}$$

A **maximum  $S$ - $T$  flow** is an  $S$ - $T$  flow where  $\sum_{(i, j) \in E(G), i \in S} X(i, j) - \sum_{(j, i) \in E(G), i \in S} X(j, i)$  is maximized.



# General Flow Network

**Flow network**  $(G; b; \ell; u)$  with

- directed graph  $G$
- node *production/consumption*  $b: V(G) \rightarrow \mathbb{R}$  with  $\sum_{i \in V(G)} b(i) = 0$
- edge *lower bound*  $\ell: E(G) \rightarrow \mathbb{R}_0^+$
- edge *capacity*  $u: E(G) \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$

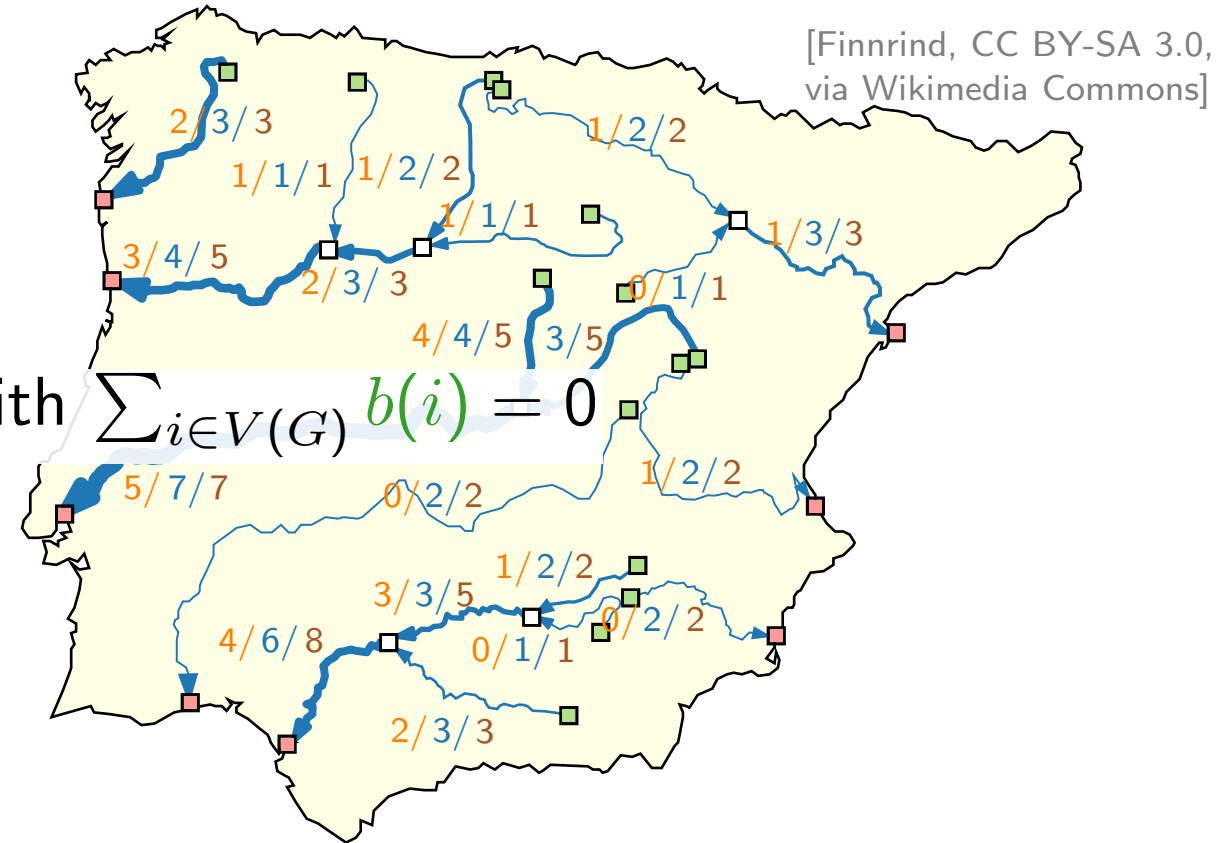
A function  $X: E(G) \rightarrow \mathbb{R}_0^+$  is called **valid flow** if:

$$\ell(i, j) \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in E(G)$$

$$\sum_{(i, j) \in E(G)} X(i, j) - \sum_{(j, i) \in E(G)} X(j, i) = b(i) \quad \forall i \in V(G)$$

- **Cost function**:  $\text{cost}: E(G) \rightarrow \mathbb{R}_0^+$  and  $\text{cost}(X) := \sum_{(i, j) \in E(G)} \text{cost}(i, j) \cdot X(i, j)$

$X$  is a **minimum-cost flow** if  $X$  is a valid flow that minimizes  $\text{cost}(X)$ .



# General Flow Network – Algorithms

$n$ : #vertices

$m$ : #edges

## Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	$O((n + m) \log U S(n, m, nC))$
2	Rock	1980	$O((n + m) \log U S(n, m, nC))$
3	Rock	1980	$O(n \log C M(n, m, U))$
4	Bland and Jensen	1985	$O(m \log C M(n, m, U))$
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	$O(nm \log n \log (nC))$
7	Ahuja, Goldberg, Orlin and Tarjan	1988	$O(nm \log \log U \log (nC))$

## Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
2	Orlin	1984	$O((n + m)^2 \log n S(n, m))$
3	Fujishige	1986	$O((n + m)^2 \log n S(n, m))$
4	Galil and Tardos	1986	$O(n^2 \log n S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	$O((n + m) \log n S(n, m))$

$$S(n, m) = O(m + n \log n)$$

Fredman and Tarjan [1984]

$$S(n, m, C) = O(\min(m + n\sqrt{\log C}, m \log \log C))$$

Ahuja, Mehlhorn, Orlin and Tarjan [1990]

Van Emde Boas, Kaas and Zijlstra [1977]

$$M(n, m) = O(\min(nm + n^{2+\epsilon}, nm \log n))$$

where  $\epsilon$  is any fixed constant.

King, Rao, and Tarjan [1991]

$$M(n, m, U) = O(nm \log(\frac{n}{m} \sqrt{\log U} + 2))$$

Ahuja, Orlin and Tarjan [1989]

## Theorem.

[Orlin 1991]

The minimum-cost flow problem can be solved in  $O(n^2 \log^2 n + m^2 \log n)$  time.

## Theorem.

[Cornelsen & Karrenbauer 2011]

The minimum-cost flow problem for planar graphs with bounded costs and face sizes can be solved in  $O(n^{3/2})$  time.

## Theorem.

[van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]

The minimum-cost flow problem with integral vertex demands, edge capacities, and edge costs can be solved in  $O(m^{1+o(1)} \log U \log C)$  time, where  $U$  is the maximum capacity and  $C$  are the maximum costs.

# Topology – Shape – Metrics

Three-step approach:

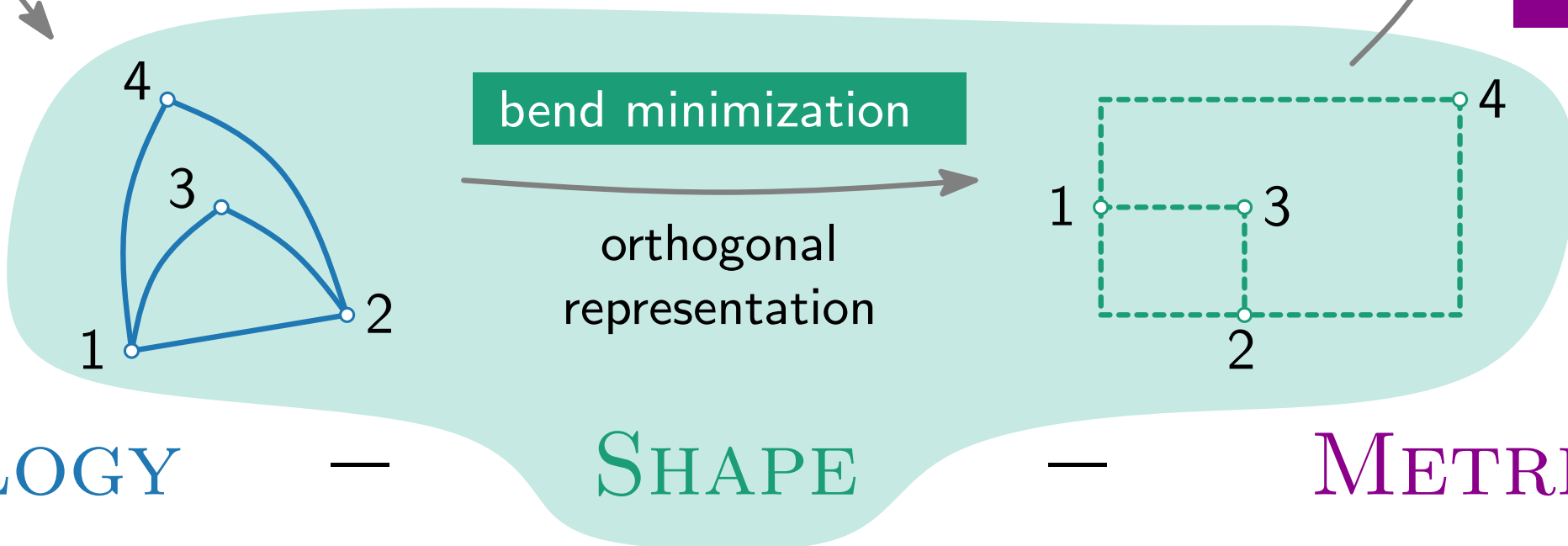
[Tamassia 1987]

$$V(G) = \{v_1, v_2, v_3, v_4\}$$

$$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$

reduce crossings

combinatorial embedding/  
planarization



TOPOLOGY

SHAPE

METRICS

planar  
orthogonal  
drawing

area mini-  
mization

# Bend Minimization with Given Embedding

## Geometric orthogonal bend minimization.

- Given:
- Plane graph  $G$  with maximum degree 4
  - Combinatorial embedding  $F$  and outer face  $f_0$
- Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

Compare with the following variant:

## Combinatorial orthogonal bend minimization.

- Given:
- Plane graph  $G$  with maximum degree 4
  - Combinatorial embedding  $F$  and outer face  $f_0$
- Find: **Orthogonal representation**  $H(G)$  with minimum number of bends that preserves the embedding.

# Bend Minimization with Given Embedding

How to solve the combinatorial orthogonal bend minimization problem?

## Idea.

Formulate as a network-flow problem:

- a unit of flow =  $\sphericalangle \frac{\pi}{2}$
- vertices  $\xrightarrow{\sphericalangle}$  faces ( $\# \sphericalangle \frac{\pi}{2}$  per face)
- faces  $\xrightarrow{\sphericalangle}$  neighboring faces ( $\#$  bends toward the neighbor)

## Combinatorial orthogonal bend minimization.

Given:

- Plane graph  $G$  with maximum degree 4
- Combinatorial embedding  $F$  and outer face  $f_0$

Find: **Orthogonal representation**  $H(G)$  with minimum number of bends that preserves the embedding.

# Flow Network for Bend Minimization

(H1)  $H(G)$  corresponds to  $F, f_0$ .

(H2) For each **edge**  $\{u, v\}$  shared by faces  $f$  and  $g$ , the sequence  $\delta_1$  is reversed and inverted copy of  $\delta_2$ .

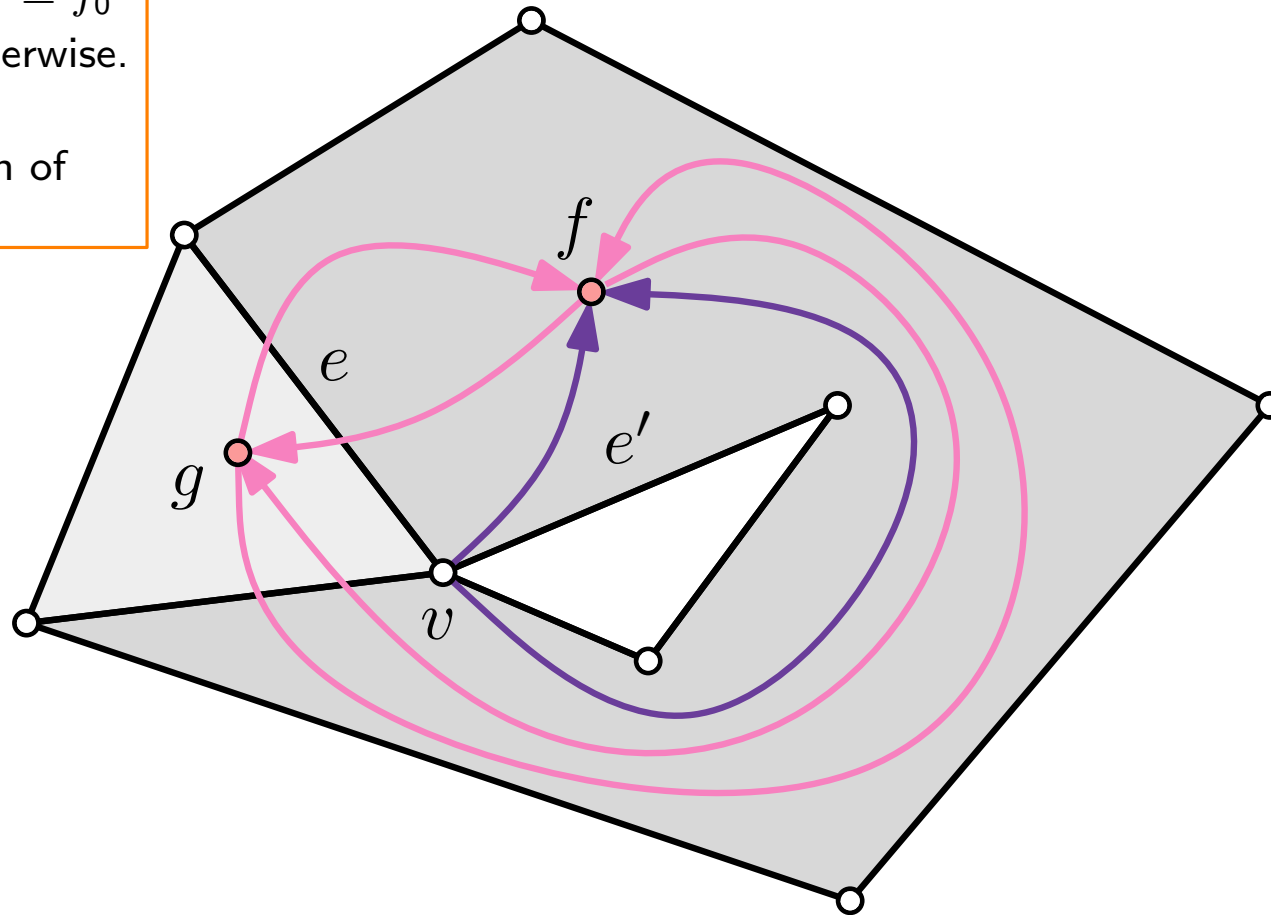
(H3) For each **face**  $f$ , it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex**  $v$ , the sum of incident angles is  $2\pi$ .

Define flow network  $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$ :

$$E' = \{(v, f)_{ee'} \in V(G) \times F \mid v \text{ between edges } e, e' \text{ of } \partial f\} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}$$



*Directed multigraph!*

# Flow Network for Bend Minimization

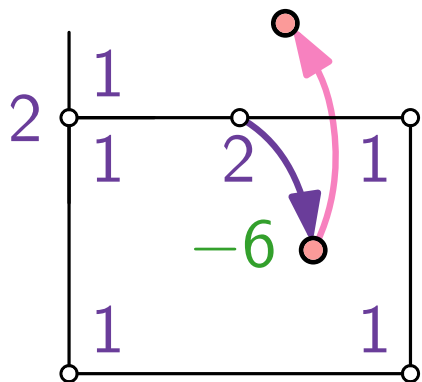
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(H4) For each **vertex**  $v$ , the sum of incident angles is  $2\pi$ .



Define flow network  $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$ :

$$\blacksquare E' = \left\{ (v, f)_{ee'} \in V(G) \times F \mid v \text{ between edges } e, e' \text{ of } \partial f \right\} \cup \left\{ (f, g)_e \in F \times F \mid f, g \text{ have common edge } e \right\}$$

$$\blacksquare b(v) = 4 \quad \forall v \in V(G)$$

$$\blacksquare b(f) = -2 \deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases} \Rightarrow \sum_{w \in V(G) \cup F} b(w) = 0 \quad (\text{Euler})$$

$$\forall (v, f) \in E', v \in V(G), f \in F$$

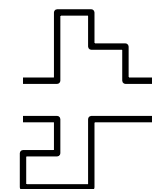
$$\ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$$

$$\text{cost}(v, f) = 0$$

$$\ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$$

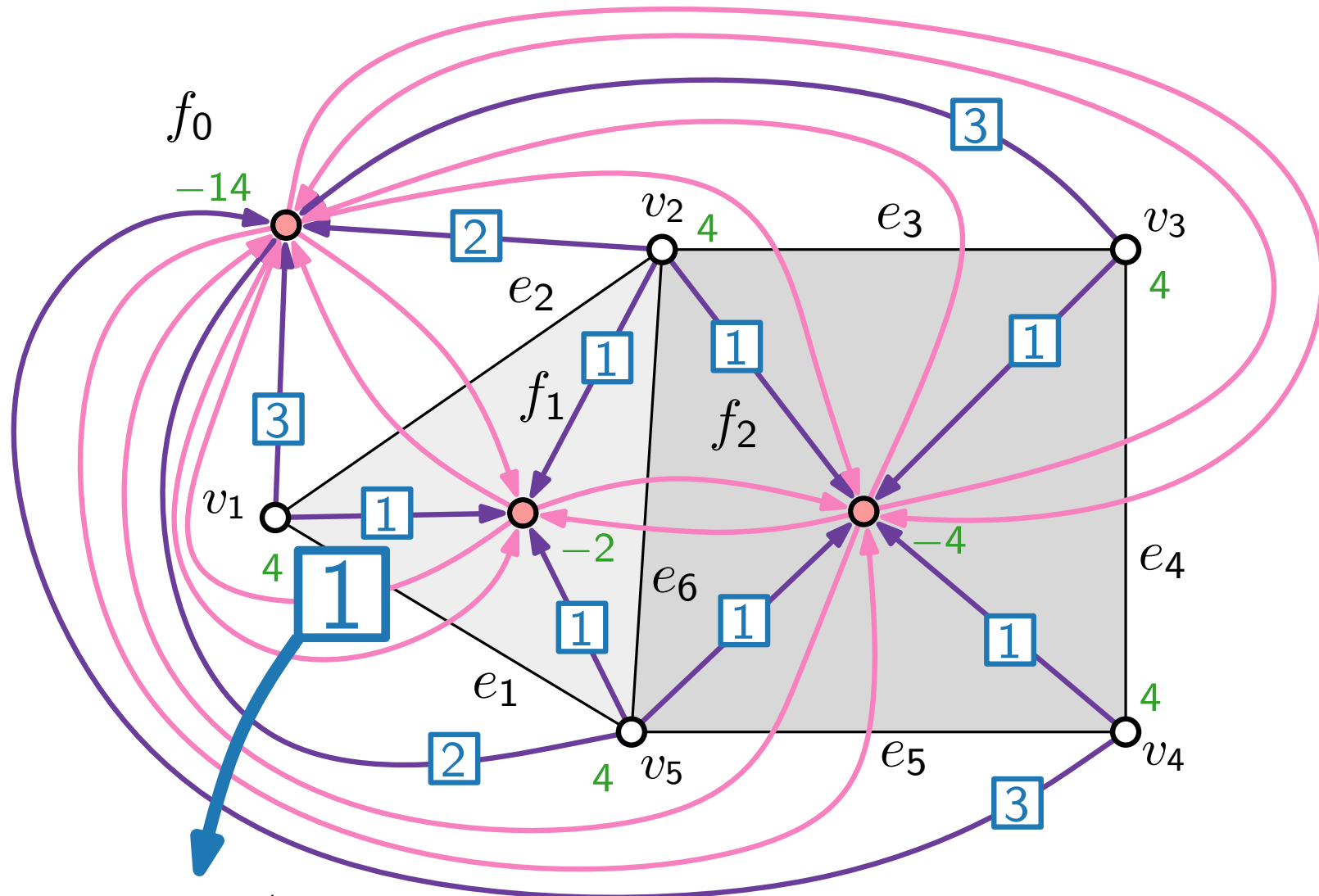
$$\text{cost}(f, g) = 1$$

We model only the number of bends. Why is it enough?



→ Exercise!

# Flow Network Example



Legend

$V(G)$  ○

$F$  ●

$l/u/cost$

$V(G) \times F \supseteq \xrightarrow{1/4/0}$

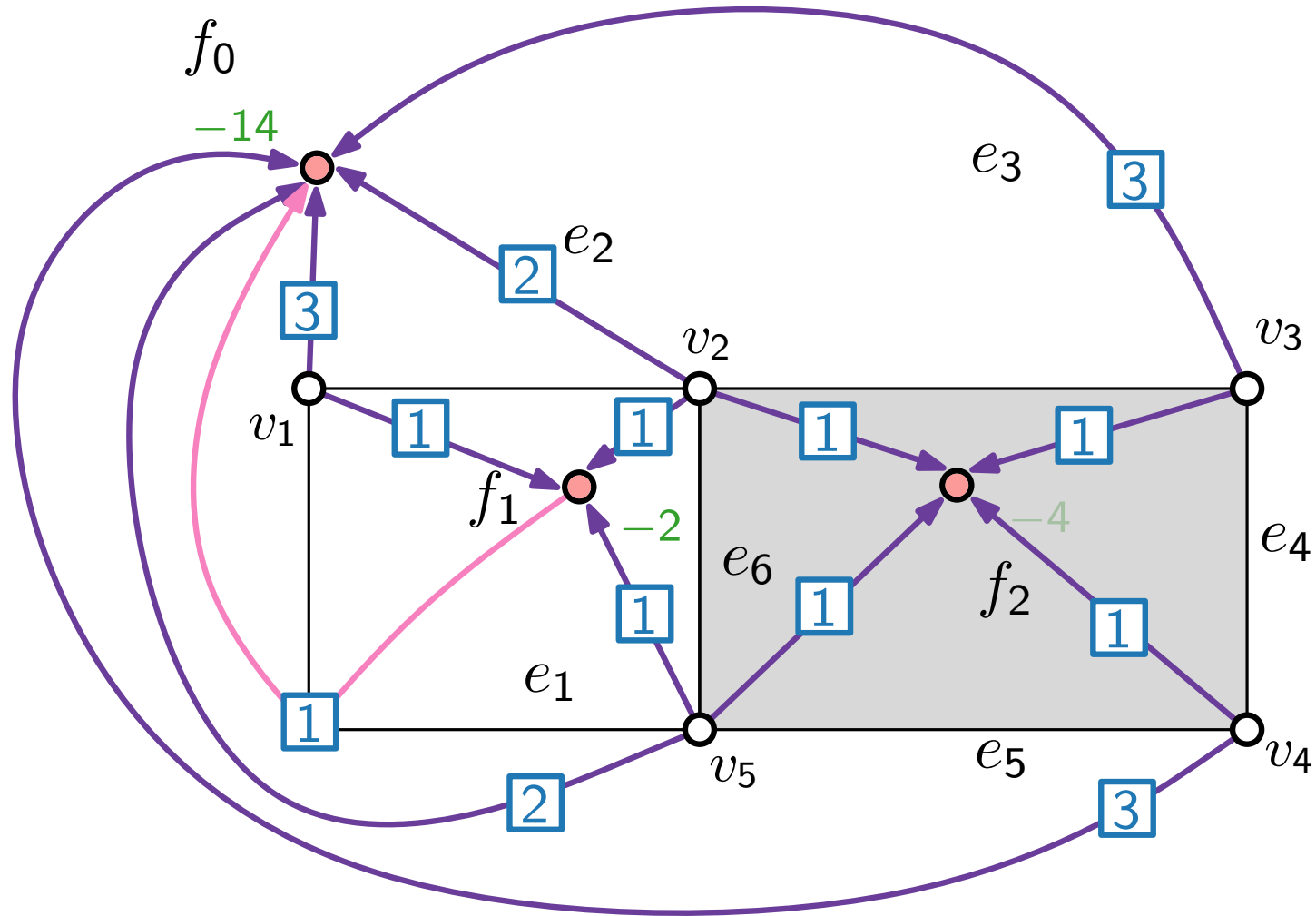
$F \times F \supseteq \xrightarrow{0/\infty/1}$

4 = b-value

3 flow

cost = 1  
one bend  
(outward)

# Flow Network Example



Legend

$V(G)$  ○

$F$  ●

$l/u/cost$

$V(G) \times F \supseteq \xrightarrow{1/4/0}$

$F \times F \supseteq \xrightarrow{0/\infty/1}$

4 =  $b$ -value

3 flow

# Bend Minimization – Result

## Theorem.

[Tamassia '87]

A plane graph  $(G, F, f_0)$  has a valid orthogonal representation  $H(G)$  with  $k$  bends.  $\Leftrightarrow$

The flow network  $N(G)$  has a valid flow  $X$  with cost  $k$ .

## Proof.

“ $\Leftarrow$ ”: Given a valid flow  $X$  in  $N(G)$  of cost  $k$ ,  
construct an orthogonal representation  $H(G)$  with  $k$  bends.

- Transform from flow to orthogonal description.

- Show properties (H1)–(H4).

(H1)  $H(G)$  matches  $F, f_0$  ✓

(H2) Bend order inverted and reversed on opposite sides ✓

(H3) Angle sum of  $f = \pm 4$  ✓

(H4) Total angle at each vertex =  $2\pi$  ✓

(H1)  $H(G)$  corresponds to  $F, f_0$ .

(H2) For each **edge**  $\{u, v\}$  shared by faces  $f$  and  $g$ , sequence  $\delta_1$  is reversed and inverted  $\delta_2$ .

(H3) For each **face**  $f$  it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex**  $v$  the sum of incident angles is  $2\pi$ .

→ *Exercise.*

# Bend Minimization – Result

**Theorem.** [Tamassia '87]

A plane graph  $(G, F, f_0)$  has a valid orthogonal representation  $H(G)$  with  $k$  bends.  $\Leftrightarrow$

The flow network  $N(G)$  has a valid flow  $X$  with cost  $k$ .

- $b(v) = 4 \quad \forall v \in V(G)$
- $b(f) = -2 \deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$
- $\ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$   
 $\text{cost}(v, f) = 0$
- $\ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$   
 $\text{cost}(f, g) = 1$

## Proof.

“ $\Rightarrow$ ”: Given an orthogonal representation  $H(G)$  with  $k$  bends, construct a valid flow  $X$  in  $N(G)$  of cost  $k$ .

- Define flow  $X: E' \rightarrow \mathbb{R}_0^+$ .
- Show that  $X$  is a valid flow and has cost  $k$ .
  - (N1)  $X(vf) = 1/2/3/4$  ✓
  - (N2)  $X((fg)_e) = |\delta|_0$ , where  $(e, \delta, x)$  describes edge  $e$  in  $H(f)$  ✓
  - (N3) capacities, deficit/demand coverage ✓
  - (N4)  $\text{cost} = k$  ✓

# Bend Minimization – Remarks

- The theorem implies that the combinatorial orthogonal bend minimization problem for plane graphs can be solved using an algorithm for min-cost flow.

## Theorem.

[Garg & Tamassia 1996]

The min-cost flow problem for planar graphs with bounded costs and vertex degrees can be solved in  $O(n^{7/4} \sqrt{\log n})$  time.

## Theorem.

[van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]

The minimum-cost flow problem with integral vertex demands, edge capacities & costs can be solved in  $O(m^{1+o(1)} \log U \log C)$  time, where  $U$  is max. capacity and  $C$  are max. costs.

$m \in O(n)$  for planar graphs     $C \in \{0, 1\}$     Further,  $\log n = n^{\log_n \log n} = n^{\log \log n / \log n} \in n^{o(1)}$  since  $\lim_{n \rightarrow \infty} \frac{\log \log n}{\log n} = 0$      $U \in O(n)$  because  $2n + 4$  bends in total are always sufficient [Storer 1984]

## Corollary.

The combinatorial orthogonal bend minimization problem can be solved in  $O(n^{1+o(1)})$  time.

## Theorem.

[Garg & Tamassia 2001]

Bend minimization without given combinatorial embedding is NP-hard.

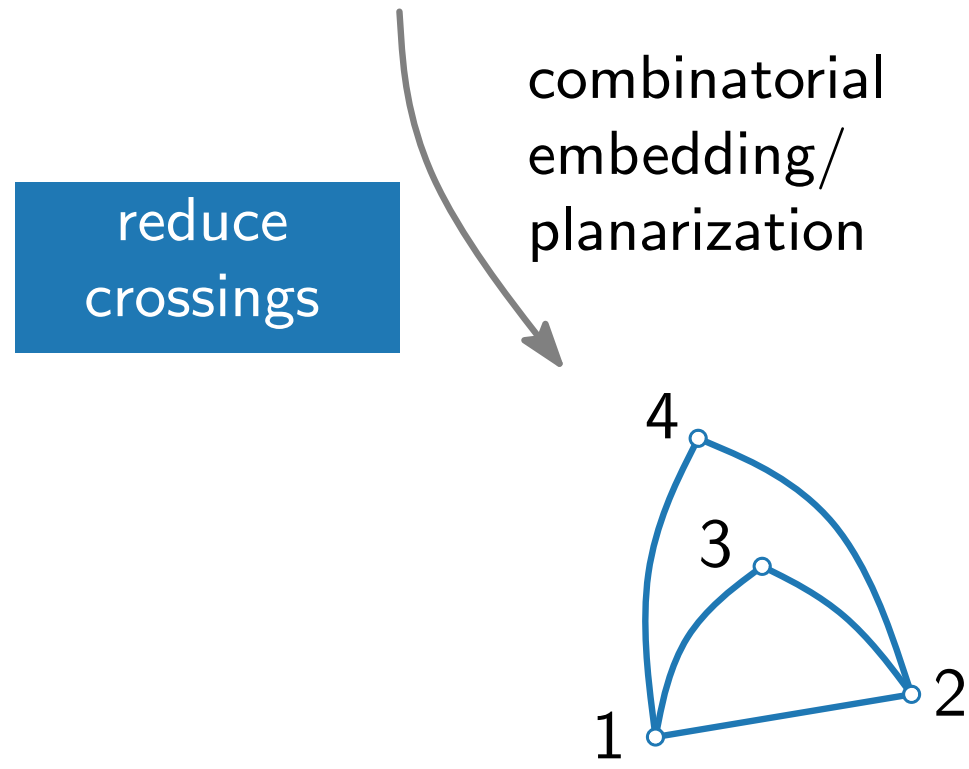
# Topology – Shape – Metrics

Three-step approach:

[Tamassia 1987]

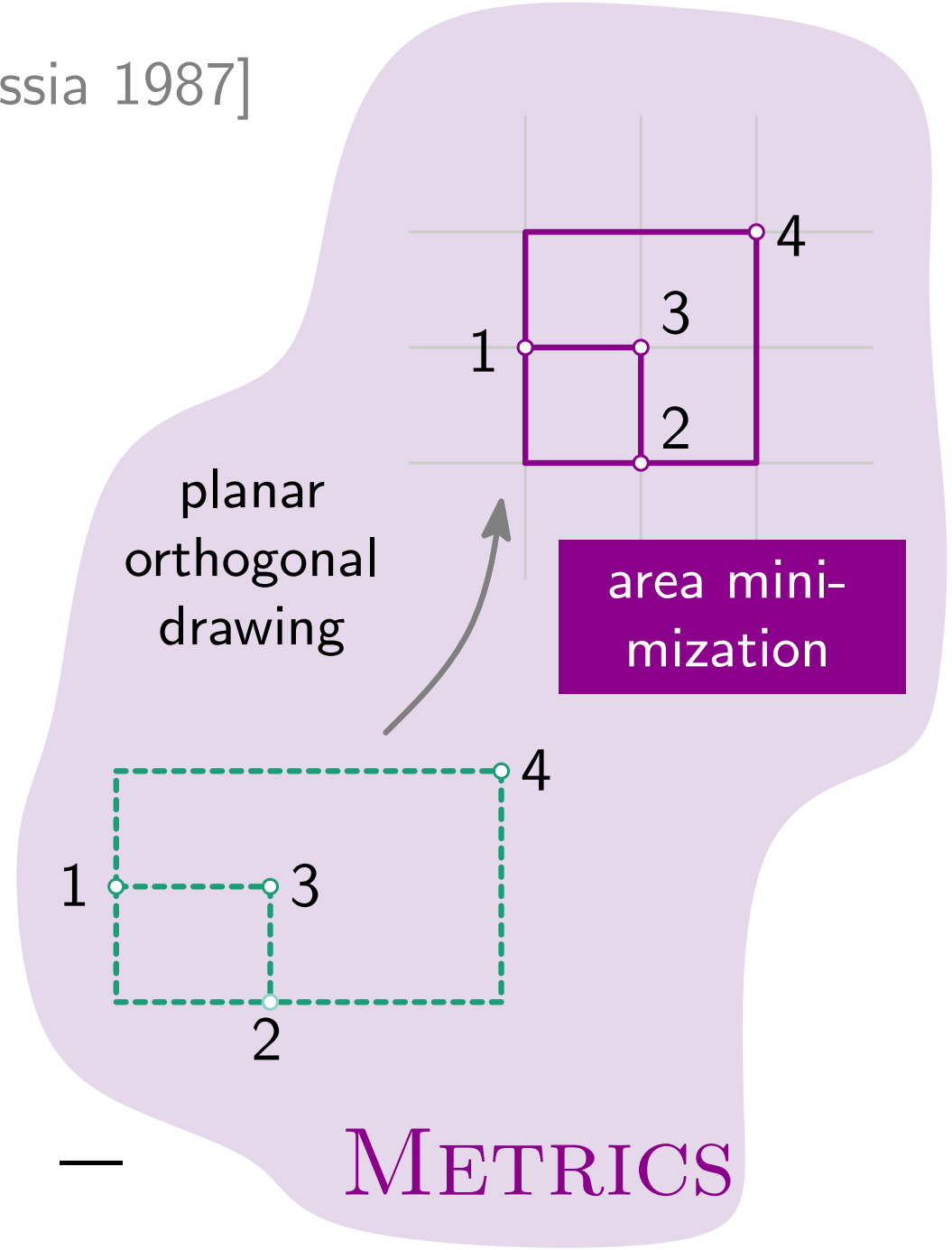
$$V(G) = \{v_1, v_2, v_3, v_4\}$$

$$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$



bend minimization

orthogonal representation



TOPOLOGY

—

SHAPE

—

METRICS

# Compaction

## Compaction problem.

- Given:
- Plane graph  $G$  with maximum degree 4
  - Orthogonal representation  $H(G)$

Find: Compact orthogonal layout of  $G$  that realizes  $H(G)$

## Special case.

All faces are rectangles.

- This guarantees:
- minimum total edge length
  - minimum area

## Properties.

- bends only on the outer face
- opposite sides of a face have the same length

## Idea.

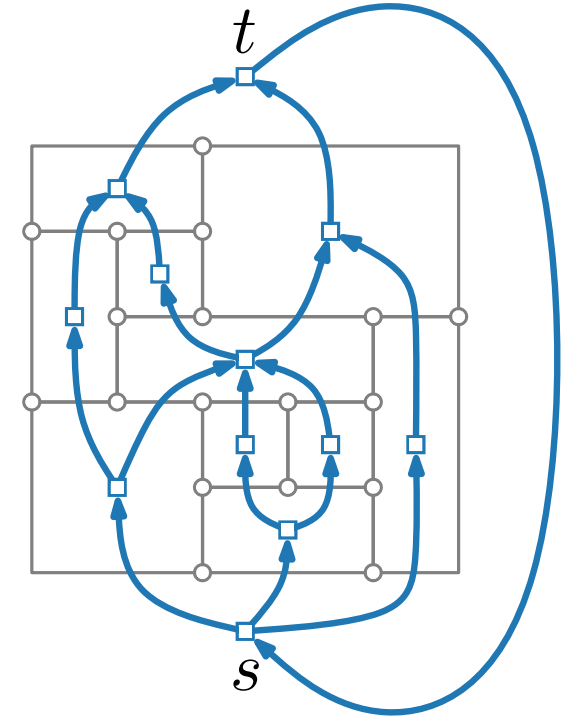
- Formulate flow network for horizontal/vertical compaction

# Flow Network for Edge-Length Assignment

## Definition.

Flow Network  $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$     □
- $E_{\text{hor}} = \{(f, g) \mid f, g \text{ share a horizontal segment and } f \text{ lies below } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

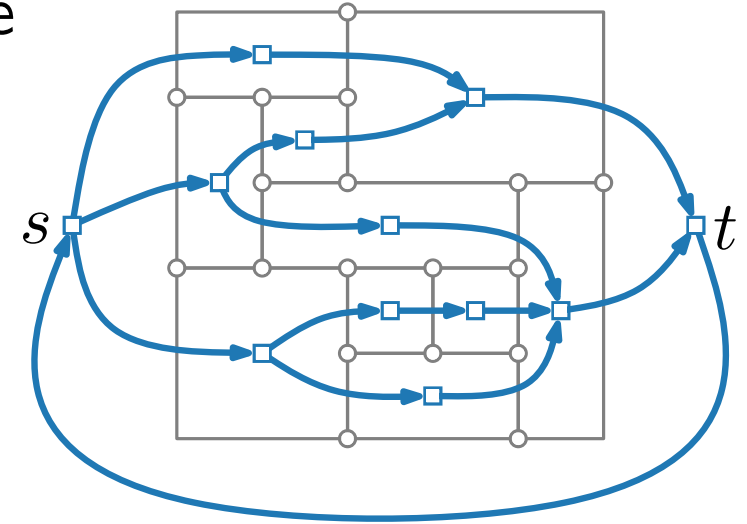


# Flow Network for Edge-Length Assignment

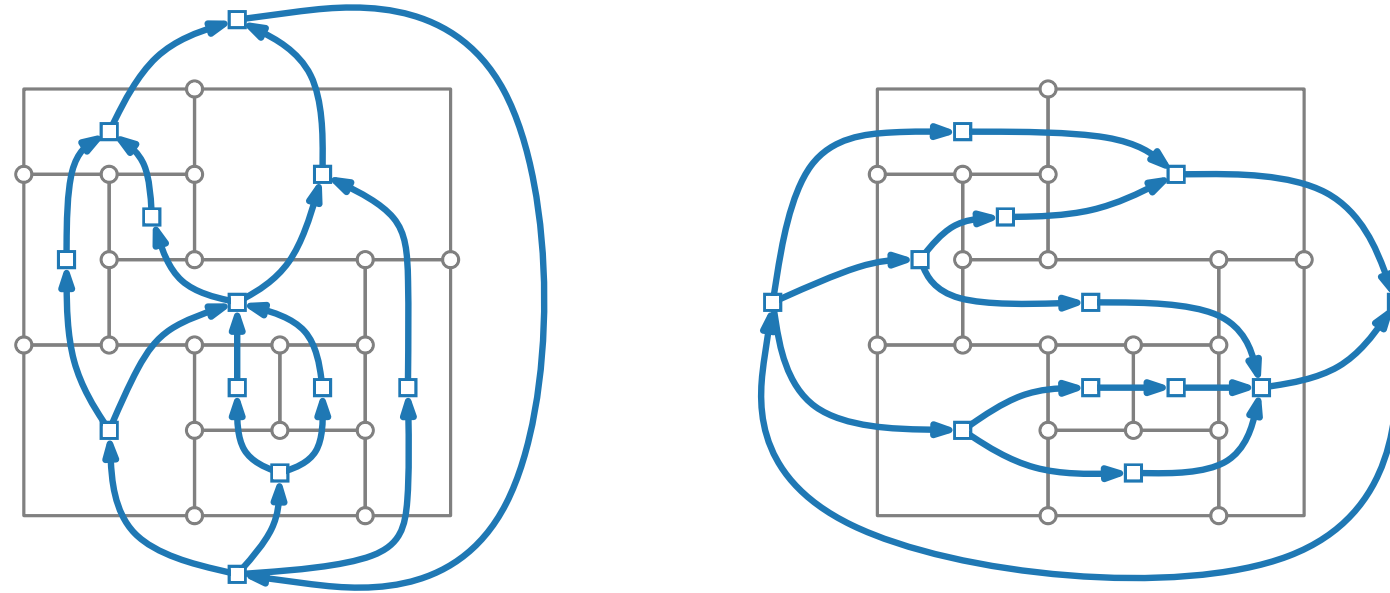
## Definition.

Flow Network  $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$     □
- $E_{\text{ver}} = \{(f, g) \mid f, g \text{ share a } \textit{vertical} \text{ segment and } f \text{ lies to the } \textit{left} \text{ of } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$



# Compaction – Result



What if not all faces are rectangular?

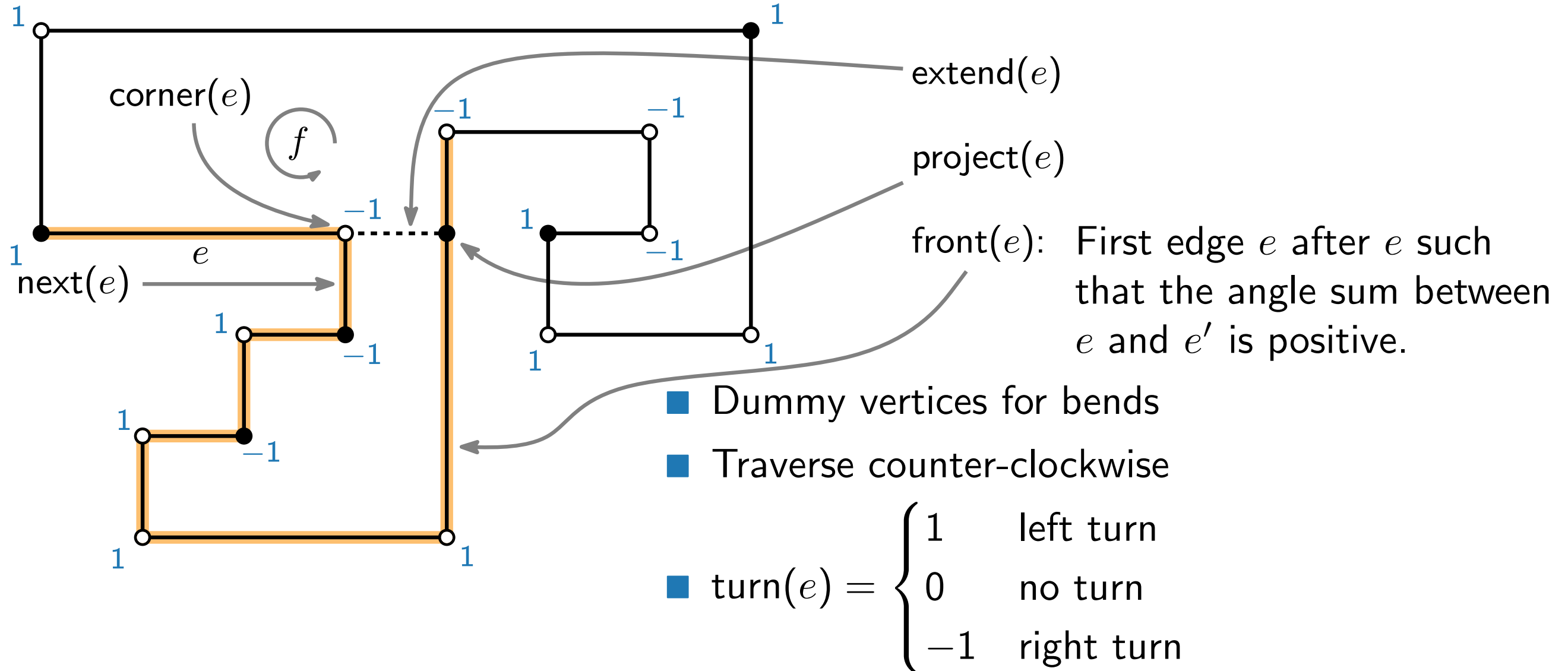
## Theorem.

A valid flow for  $N_{\text{hor}}$  and  $N_{\text{ver}}$  exists  $\Leftrightarrow$   
 corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

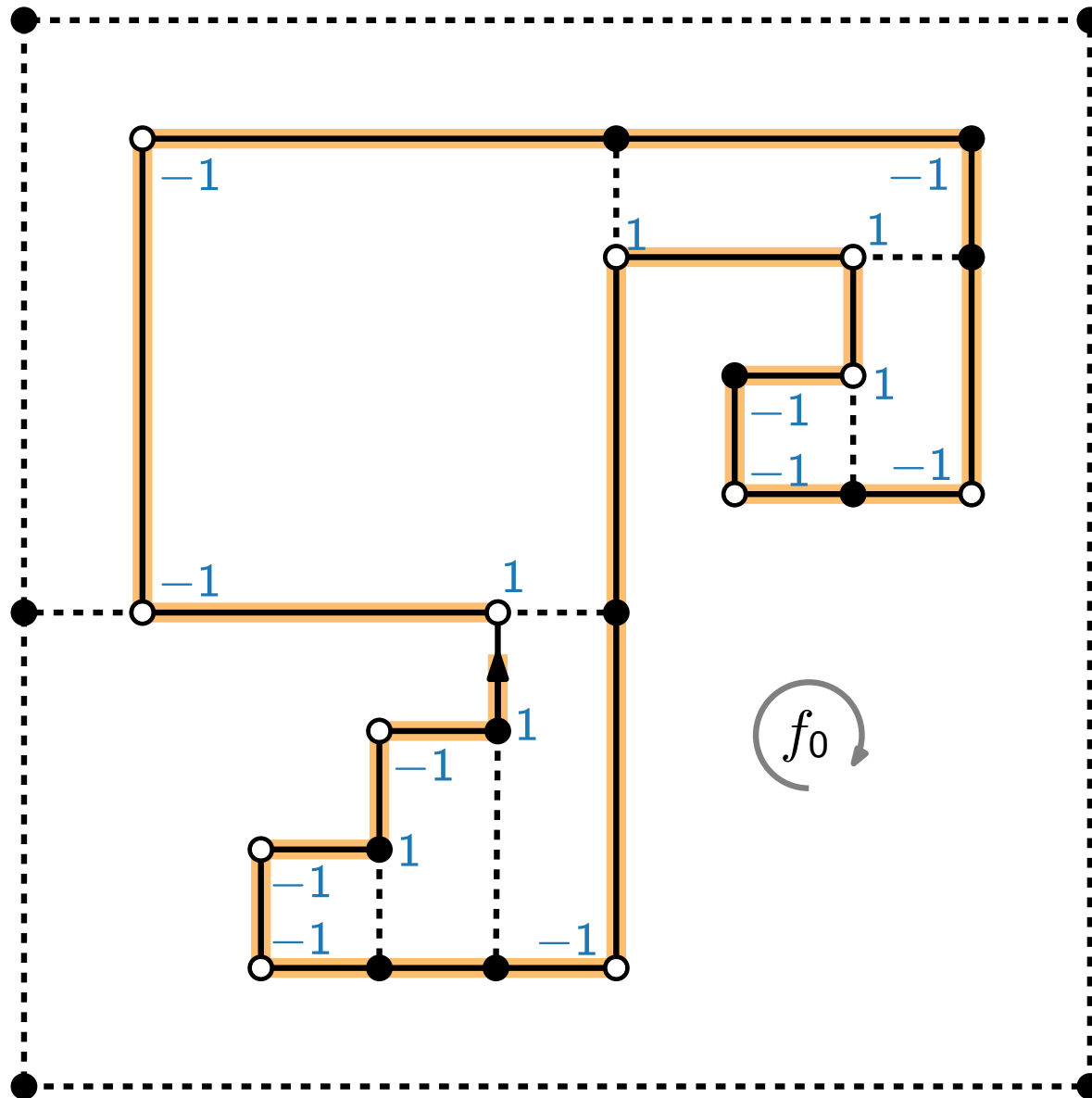
- $|X_{\text{hor}}(t, s)|$  and  $|X_{\text{ver}}(t, s)|$ ? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$  total edge length

# Refinement of $G$ and $H(G)$ – Inner Face



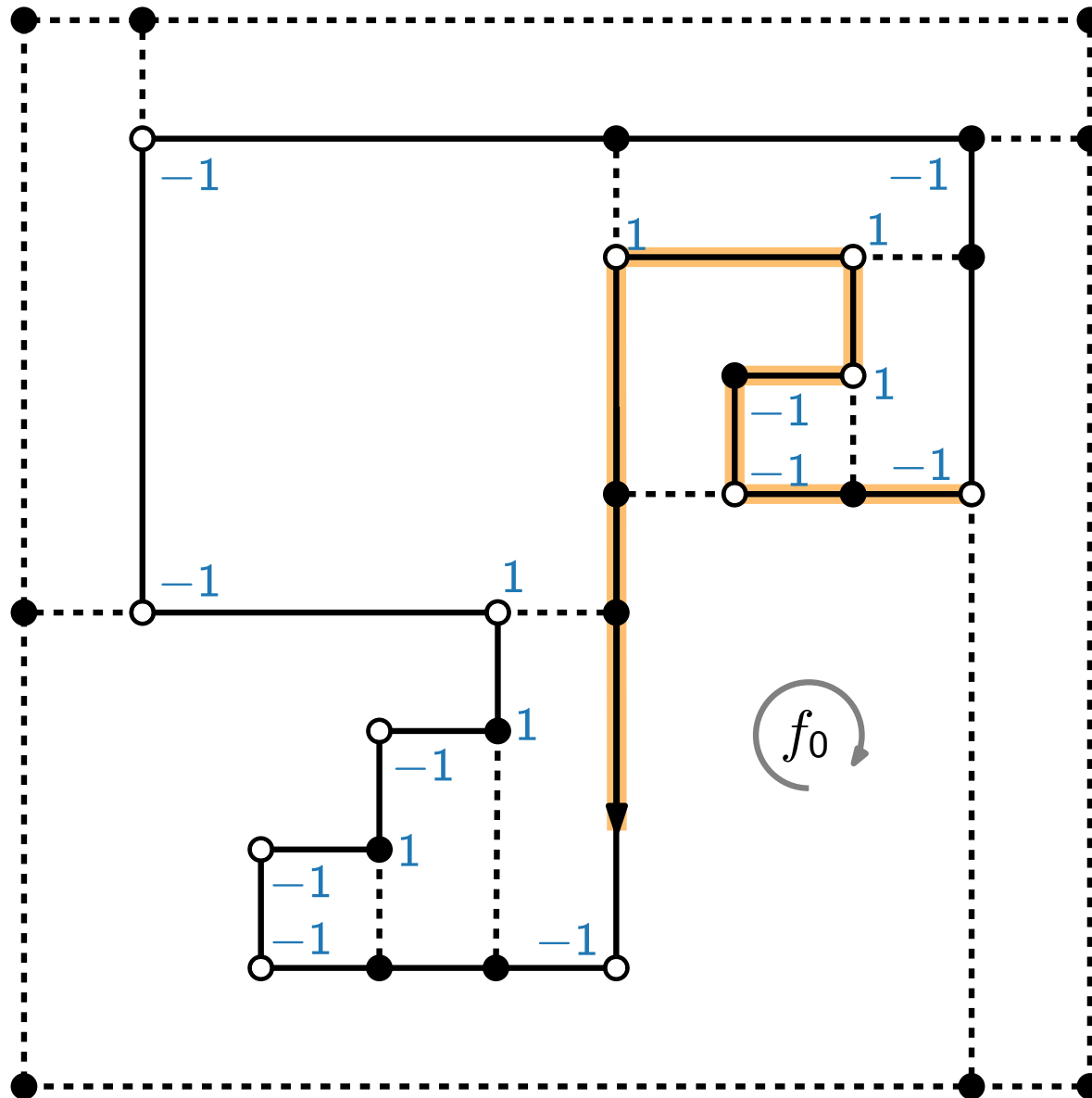


# Refinement of $G$ and $H(G)$ – Outer Face



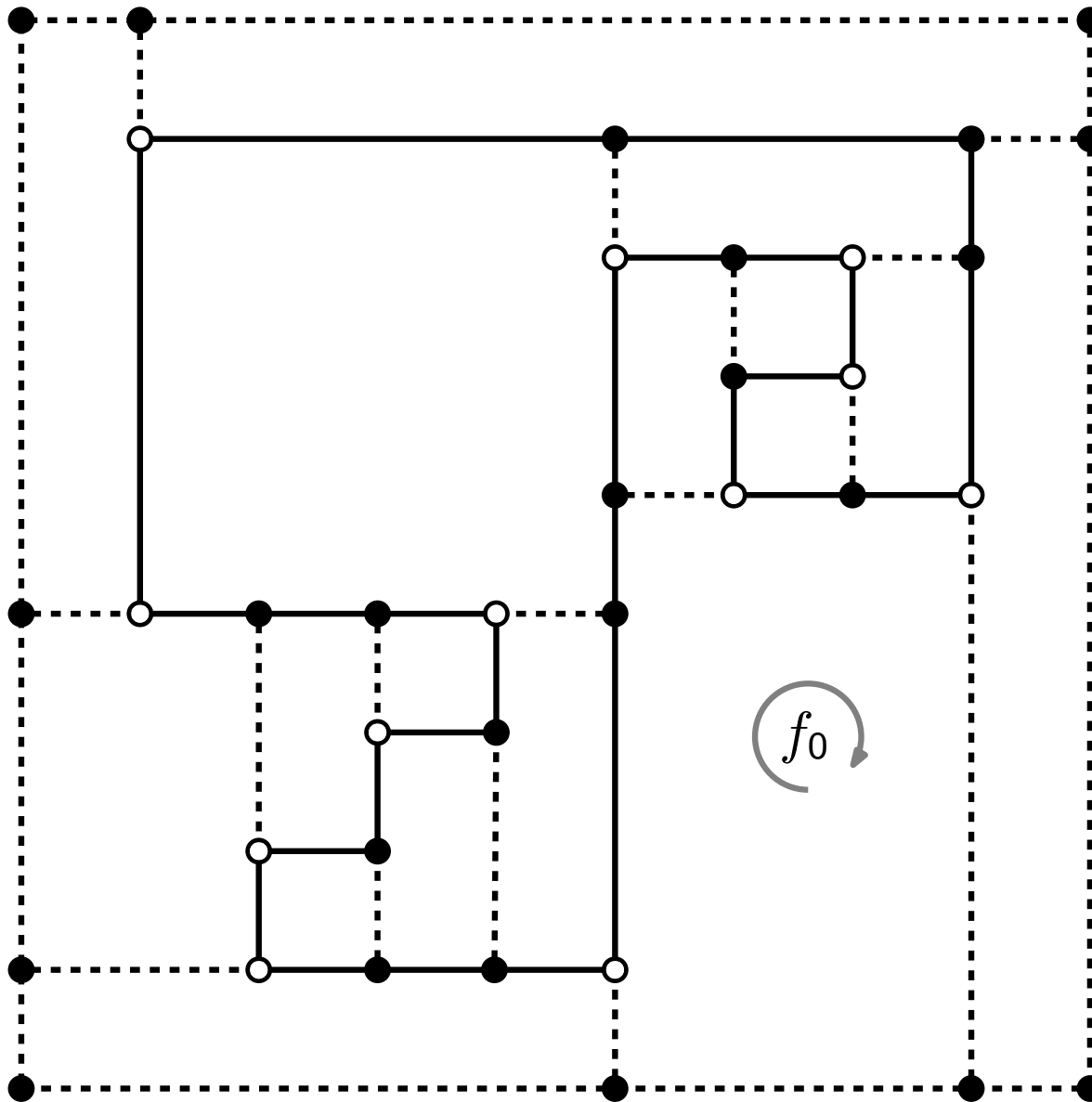
- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face



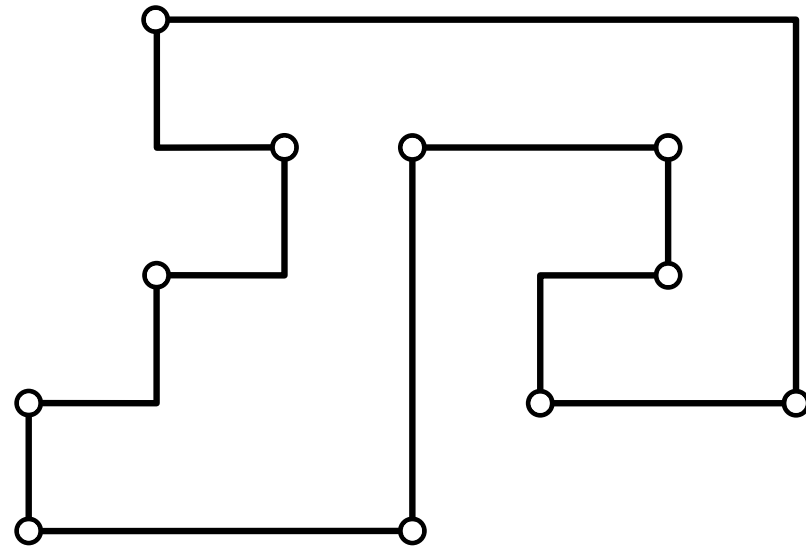
- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face



- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face



Area minimized? **No!**

# vertices

# bends

But we get bound  $O((n + b)^2)$  on the area.

**Theorem.** [Patrignani 2001]

Compaction for a given orthogonal representation is NP-hard in general.

**Theorem.** [EFKSSW 2022]

Compaction is NP-hard even for orthogonal representations of *cycles*.

# Compaction is NP-hard

Polynomial-time reduction from the NP-complete satisfiability problem (SAT).

In an instance of the SAT problem we have:

- set of  $n$  Boolean variables  $X = \{x_1, x_2, \dots, x_n\}$
- $m$  clauses  $C_1, C_2, \dots, C_m$ , where each clause is a disjunction of **literals** from  $X$ ,  
e.g.,  $C_1 = x_1 \vee \neg x_2 \vee x_3$ 

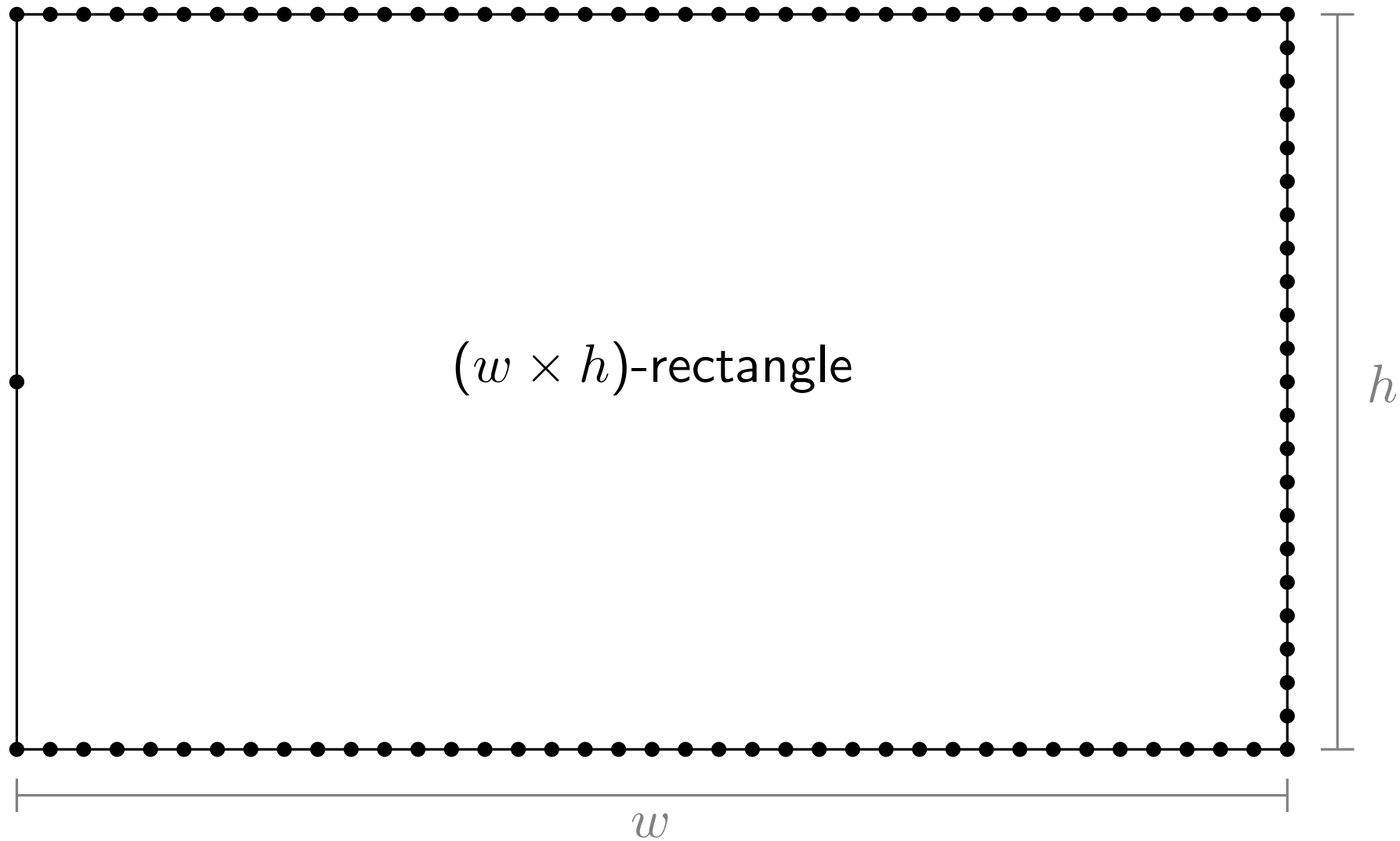
← a literal is a variable  $x$  or a negated variable  $\neg x$
- Boolean formula  $\Phi = C_1 \wedge C_2 \wedge \dots \wedge C_m$

Question: Is there an assignment of truth values to the variables in  $X$  such that  $\Phi$  is true?

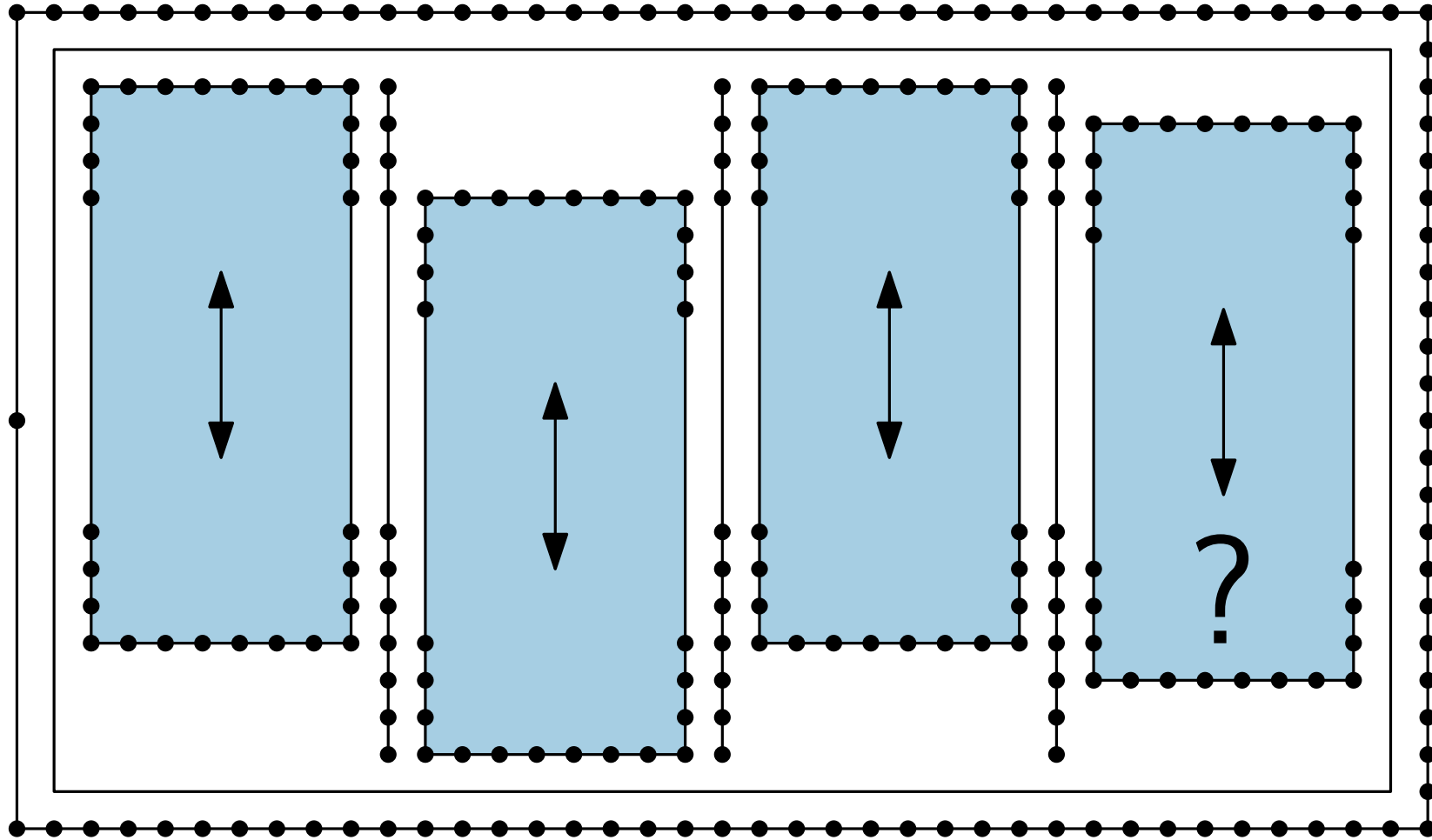
Idea of the reduction:

- Given SAT instance  $\Phi \Rightarrow$  construct a plane graph  $G$  and a orthogonal description  $H(G)$
- $\Phi$  is satisfiable  $\Leftrightarrow G$  can be drawn w.r.t.  $H(G)$  in area  $K$  for some specific number  $K$

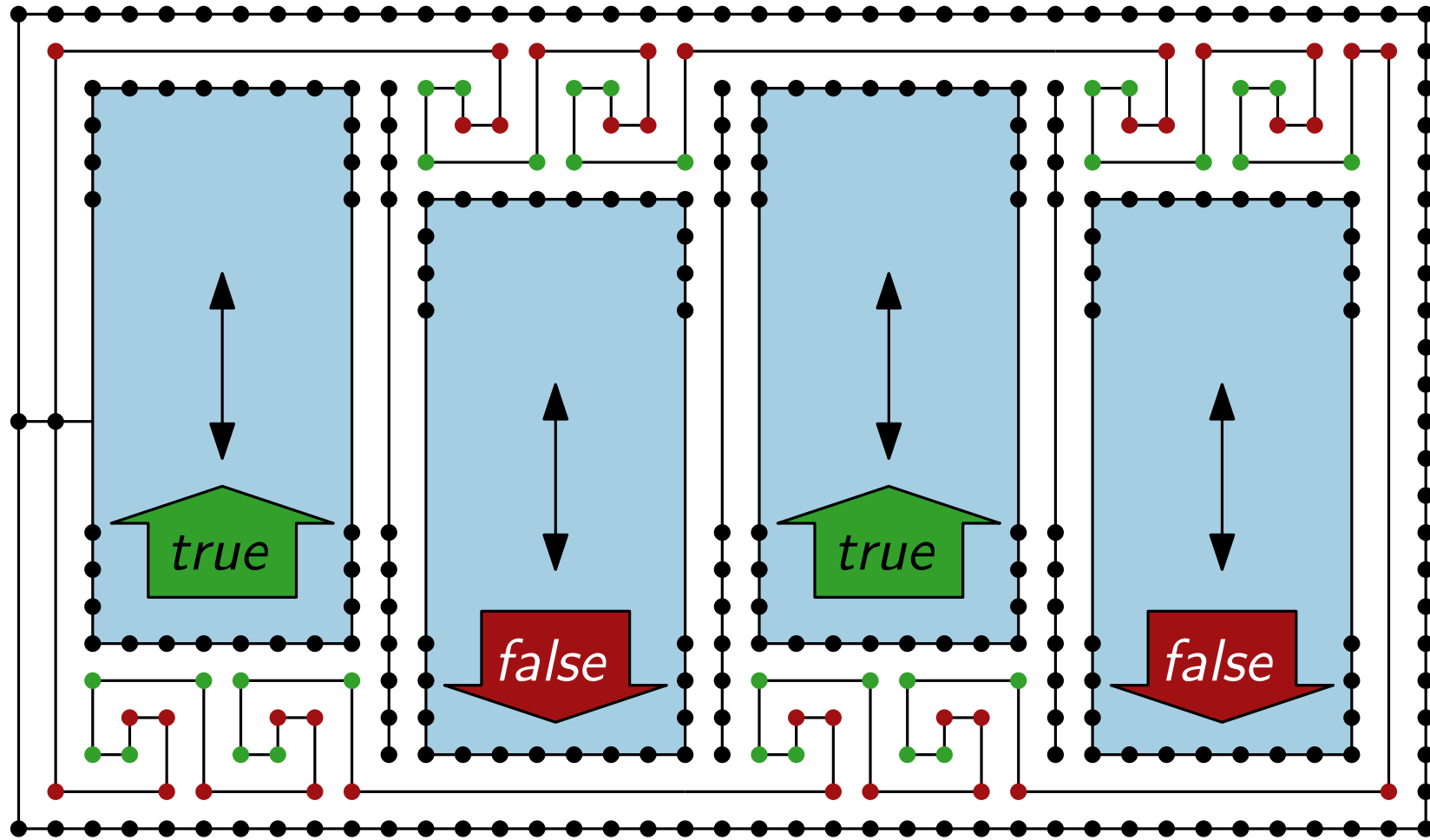
# Boundary, Belt, and “Piston” Gadget



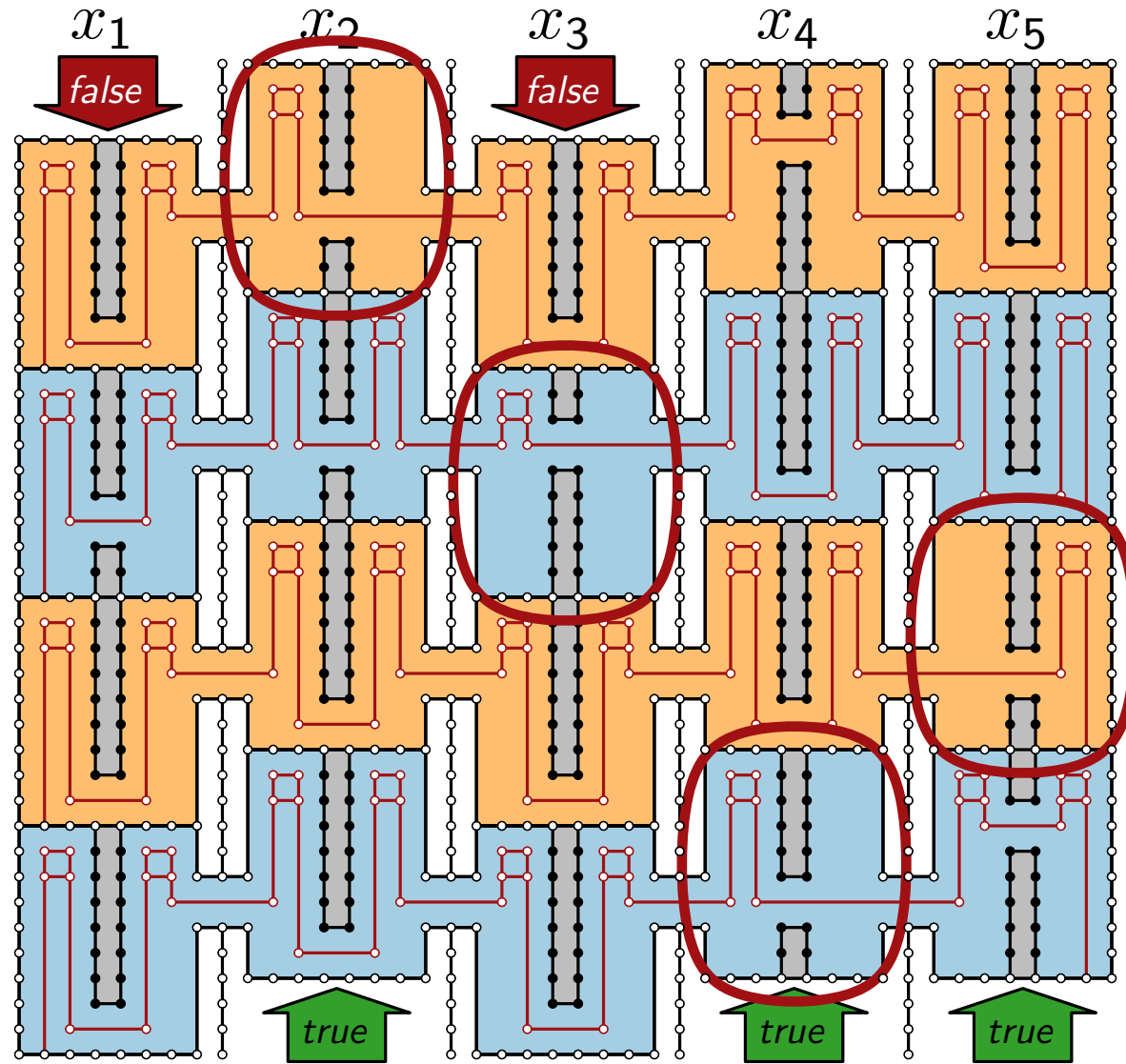
# Boundary, Belt, and "Piston" Gadget



# Boundary, Belt, and "Piston" Gadget



# Clause Gadgets



Example:

$$C_1 = x_2 \vee \neg x_4$$

$$C_2 = x_1 \vee x_2 \vee \neg x_3$$

$$C_3 = x_5$$

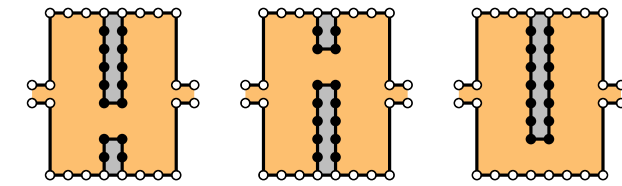
$$C_4 = x_4 \vee \neg x_5$$

$C_1$

$C_2$

$C_3$

$C_4$



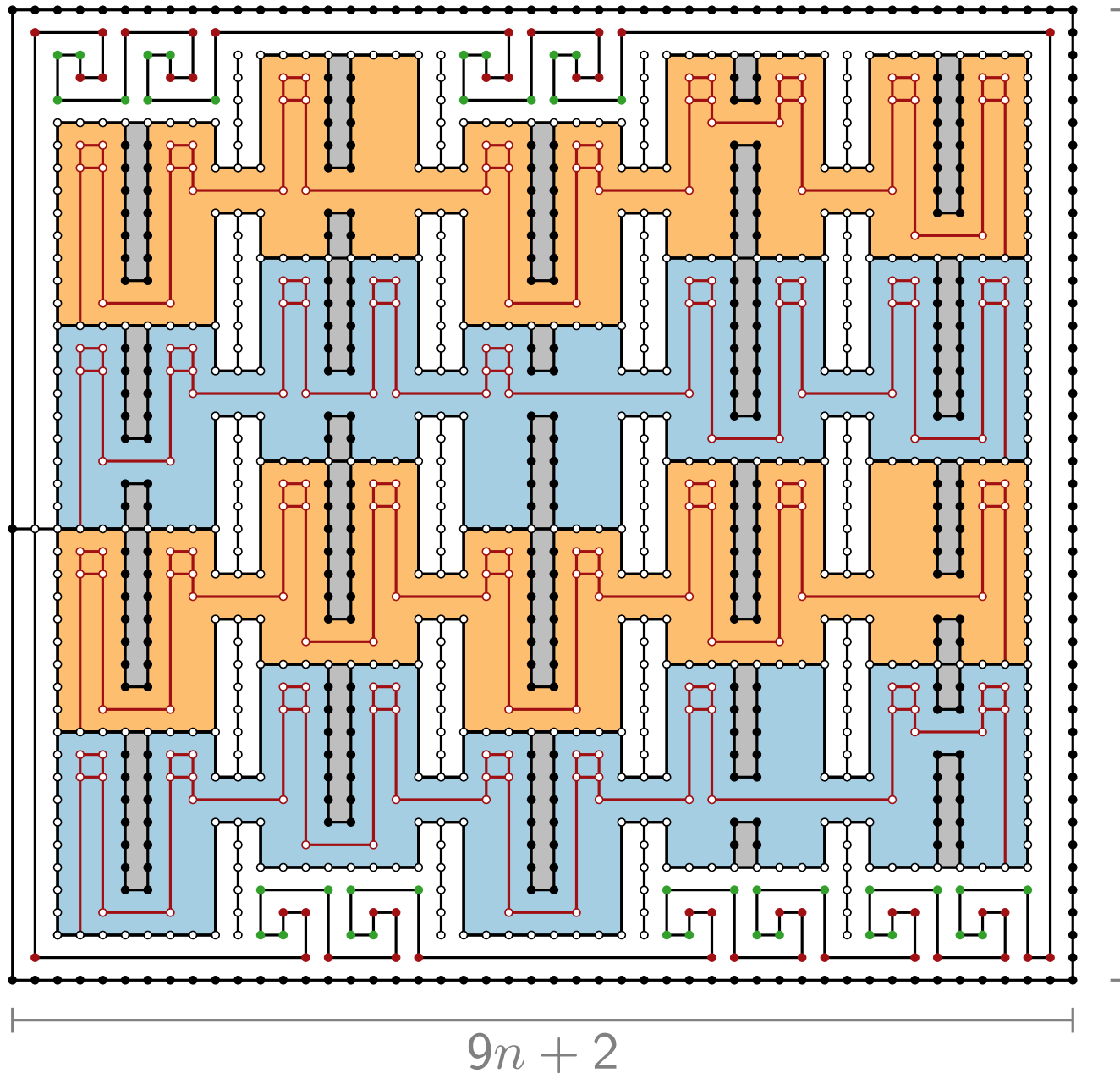
$x$        $\neg x$        $\emptyset$



insert  $(2n - 1)$ -chain  
through each clause

→ for every clause, there needs to be  $\geq 1$  “gap of a literal” to be on the same height as the “tunnel” to the next literal

# Complete Reduction



Pick

$$K = (9n + 2) \times (9m + 7)$$

$$9m + 7$$

Then:

$G$  under  $H(G)$  has an  
orthogonal drawing in area  $K$



$\Phi$  satisfiable



# Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] “On embedding a graph in the grid with the minimum number of bends”  
Original paper on flow for bend minimization.
- [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]  
“A Deterministic Almost-Linear Time Algorithm for Minimum-Cost Flow”  
State-of-the-art algorithm for solving the minimum-cost flow problem  
(published recently in the proceedings of the FOCS 2023 conference).
- [Patrignani 2001] “On the complexity of orthogonal compaction”  
NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.
- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]  
“Minimum rectilinear polygons for given angle sequences”: Compacting cycles is NP-hard.
- [Antić, Liotta, Masařík Ortali, Pfretzschner, Stumpf, Wolff, Zink 2025]  
“Unbent Collections of Orthogonal Drawings”: It is NP-hard to find two drawings such that each edge is straight in one and the total number of bends is minimum.