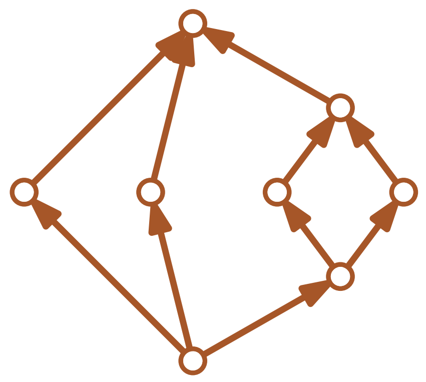
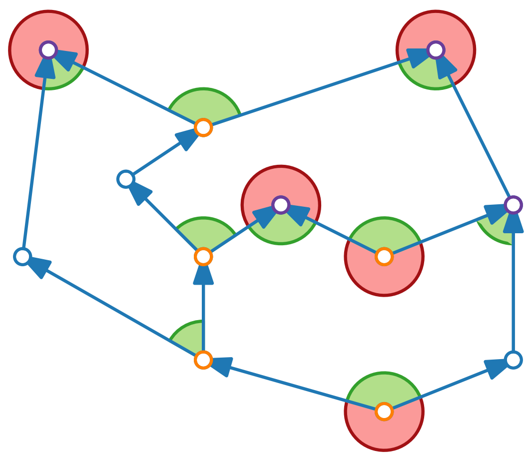
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

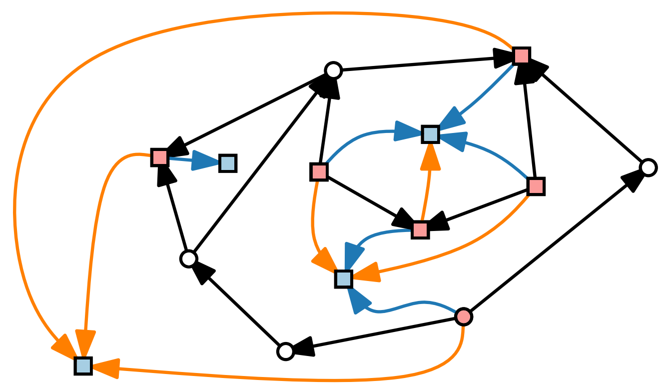
Visualization of Graphs



Lecture 5:
Upward Planar Drawings



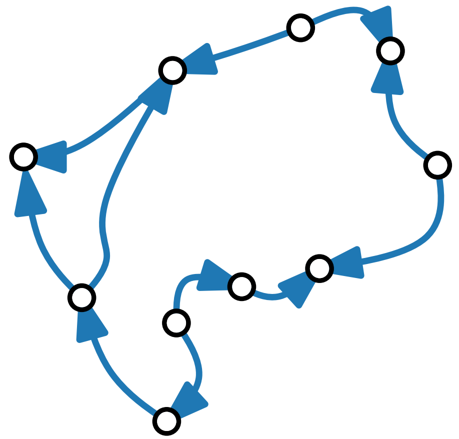
Part I:
Recognition



Alexander Wolff

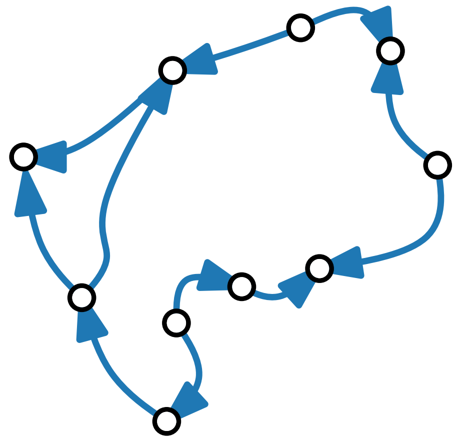
Summer term 2026

Upward Planar Drawings – Motivation

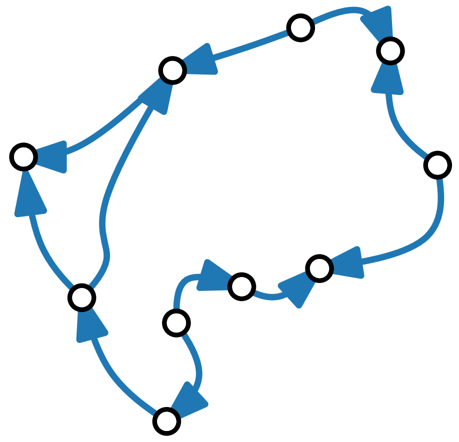


Upward Planar Drawings – Motivation

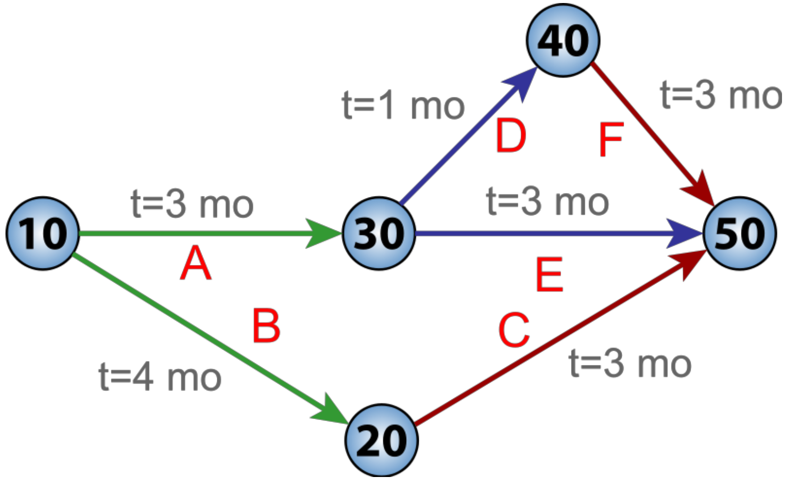
- What may the direction of edges in a directed graph represent?



Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time

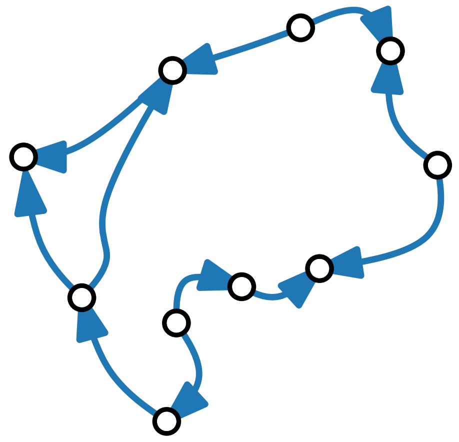


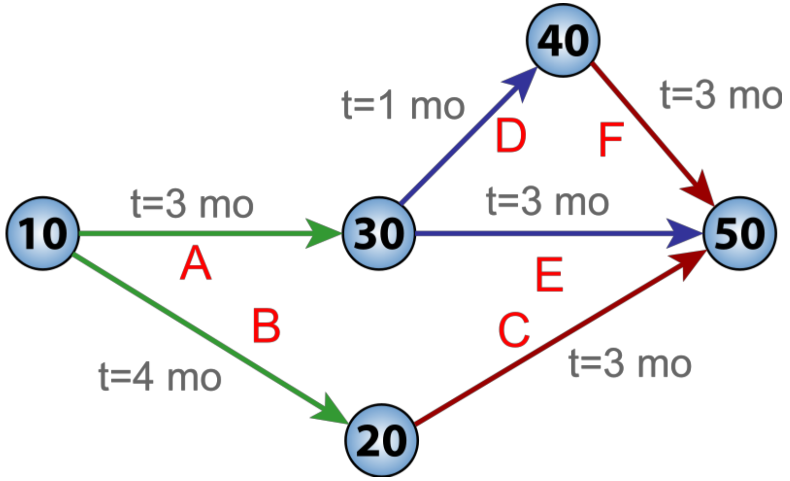
PERT diagram

Program Evaluation and Review Technique
(Project management)

Upward Planar Drawings – Motivation

- What may the direction of edges in a directed graph represent?
  - Time
  - Flow





PERT diagram

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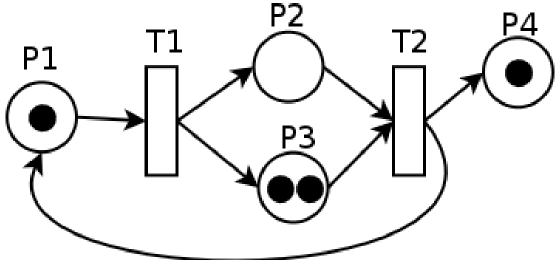
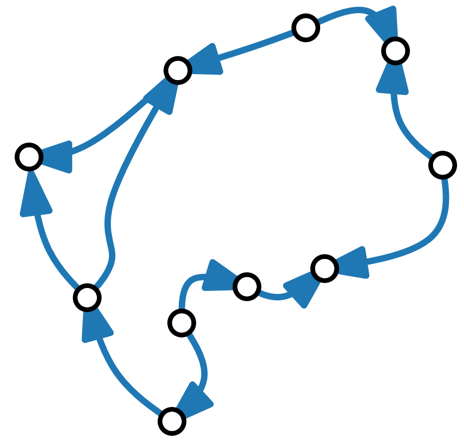


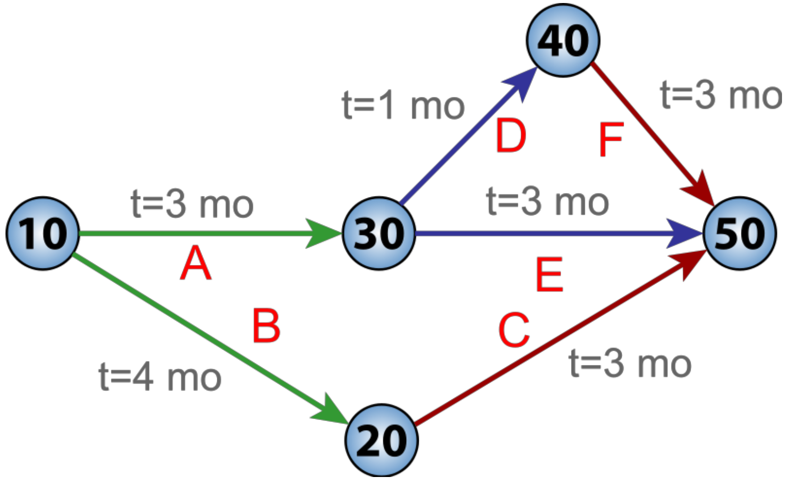
Petri net

Place/Transition net
(Modeling languages for distributed systems)

Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy



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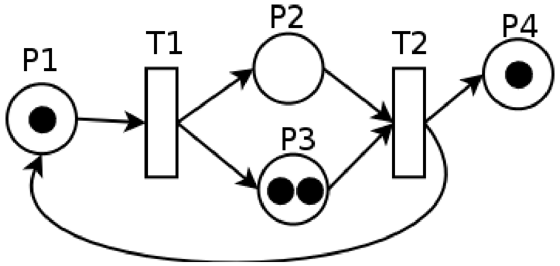

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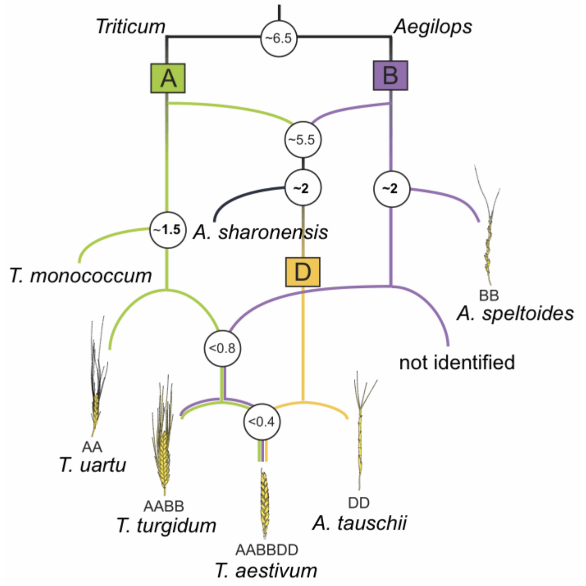
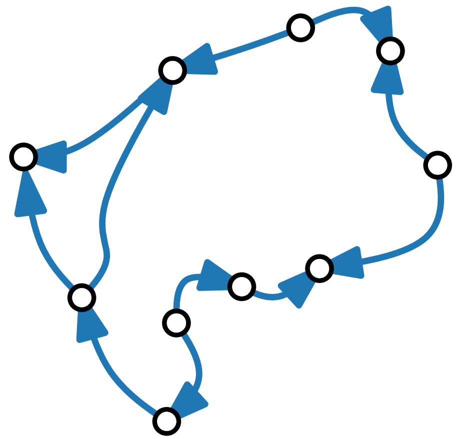


Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
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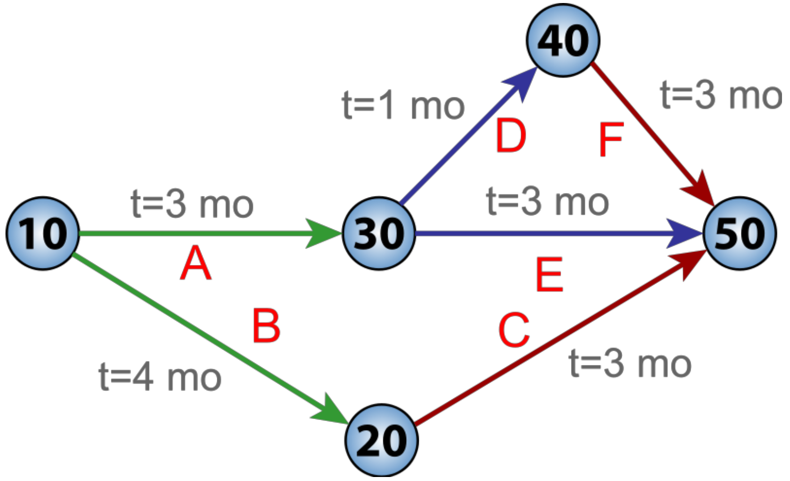
  - Hierarchy
  - ...



PERT diagram

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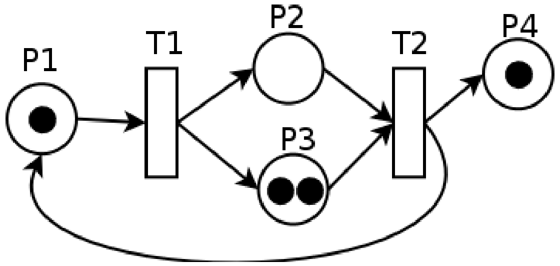

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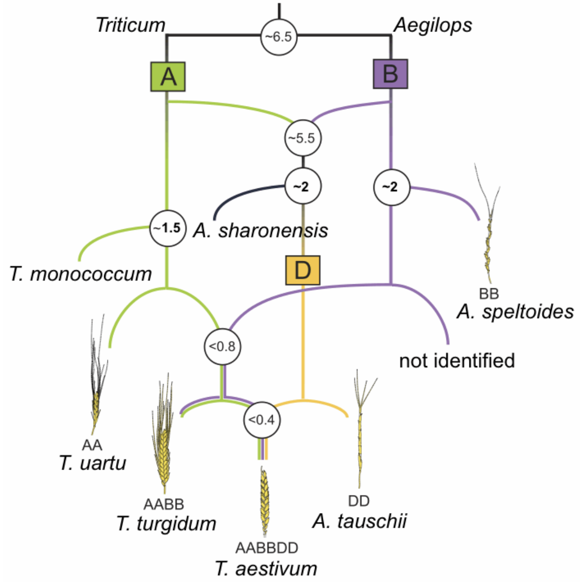

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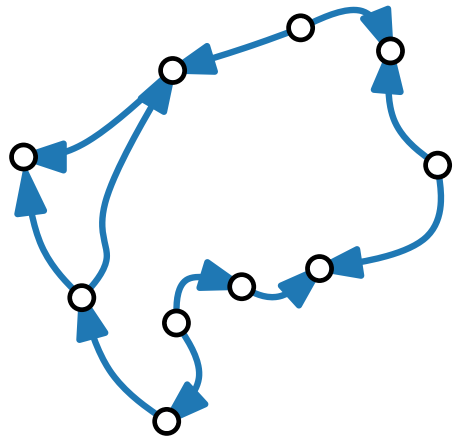
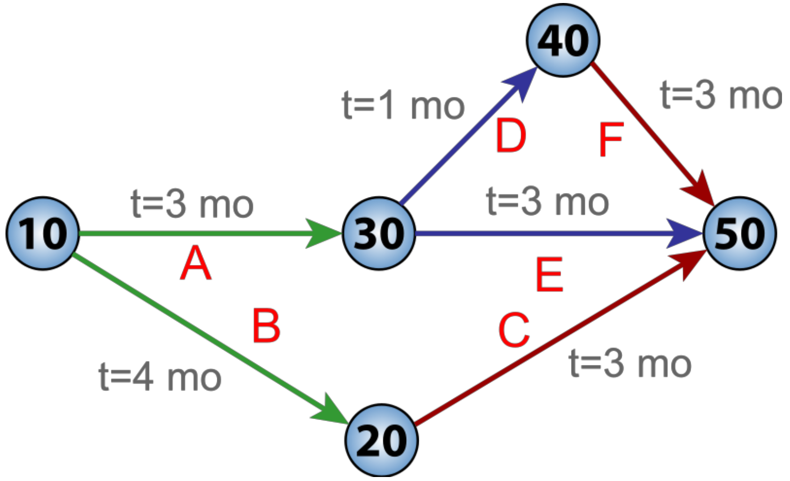
Upward Planar Drawings – Motivation



- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - ...
- We aim for drawings where the general direction is preserved.



PERT diagram

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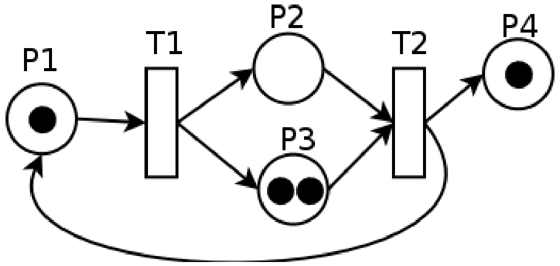

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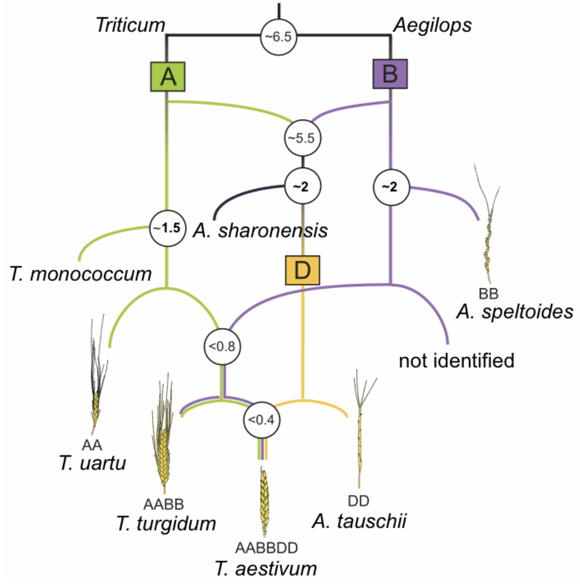

Phylogenetic network

Ancestral trees / networks
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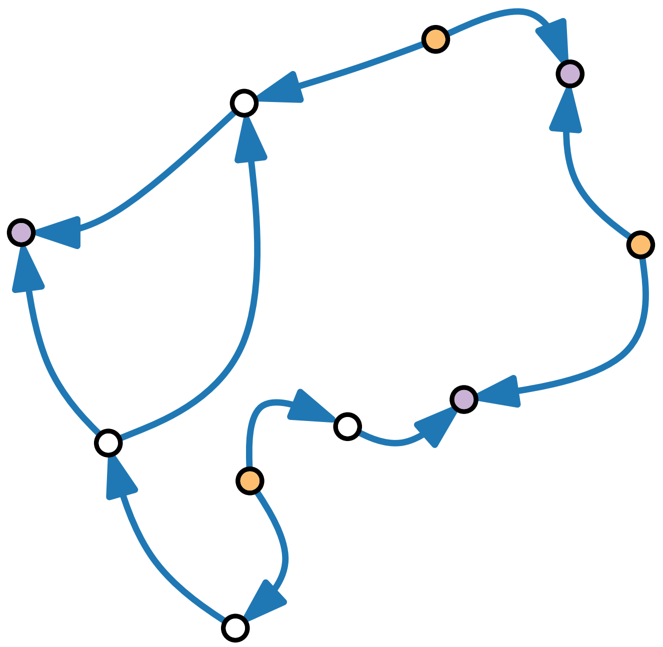
Upward Planar Drawings – Definition

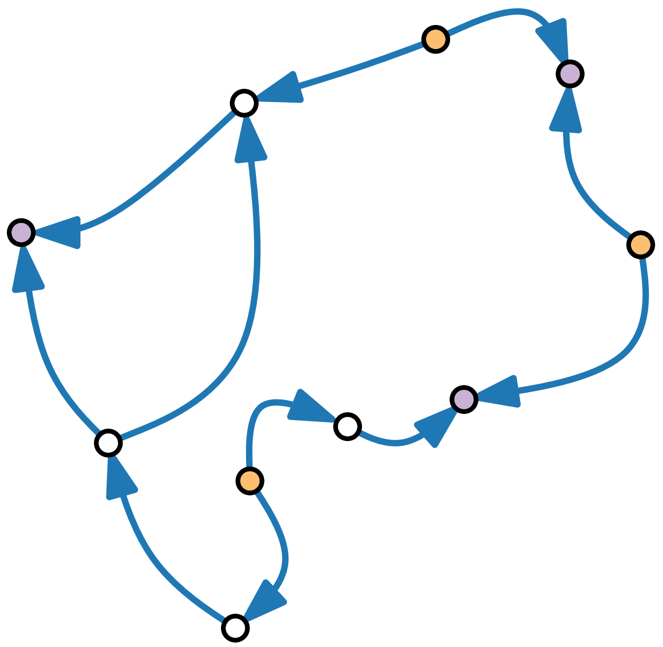
A directed graph (*digraph*) is **upward planar** when it admits a drawing



Upward Planar Drawings – Definition

A directed graph (*digraph*) is **upward planar** when it admits a drawing

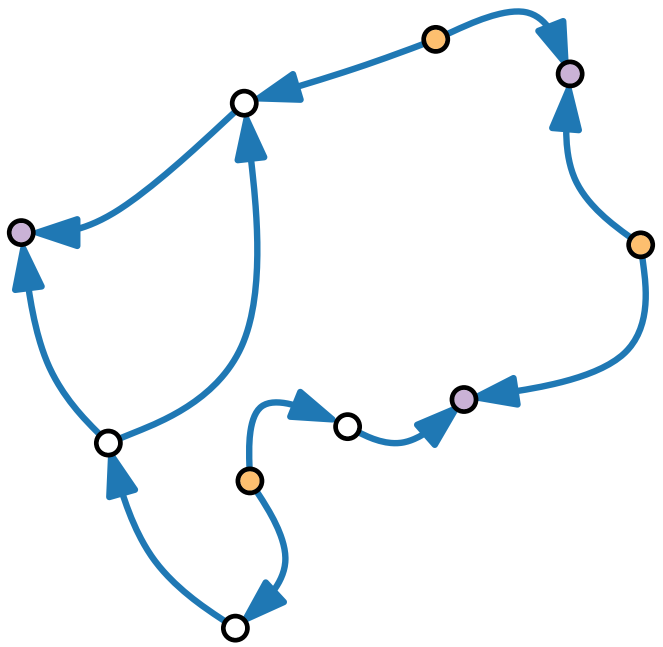
- that is planar



Upward Planar Drawings – Definition

A directed graph (*digraph*) is **upward planar** when it admits a drawing

- that is planar and
- where each edge is drawn as an upward y-monotone curve.



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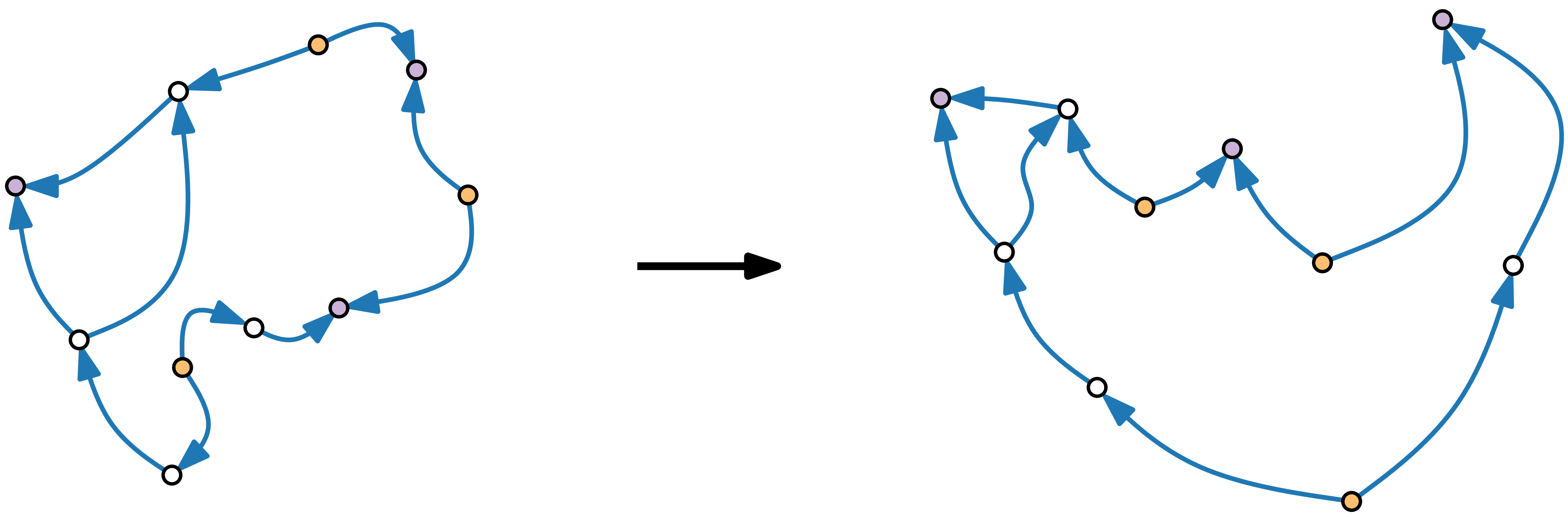
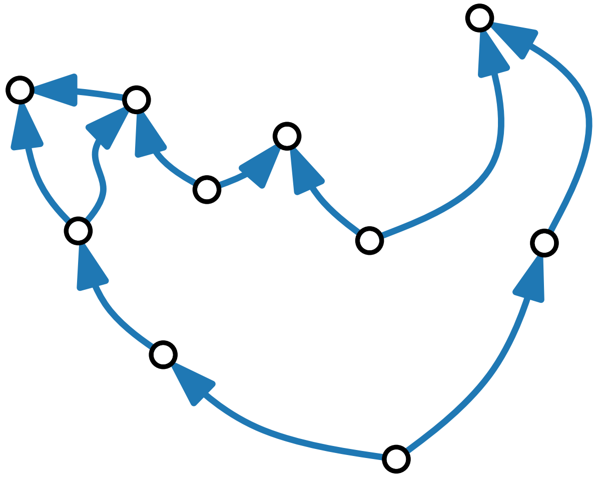


Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .

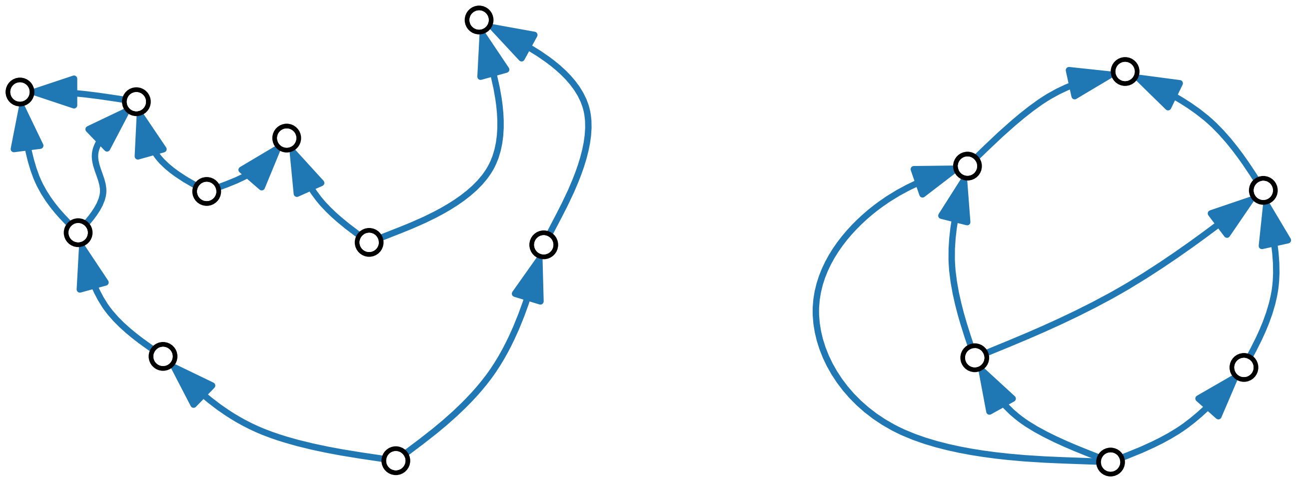
Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar



Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic



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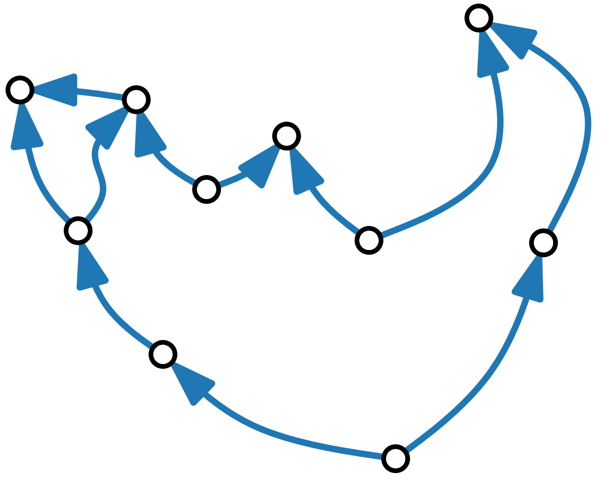

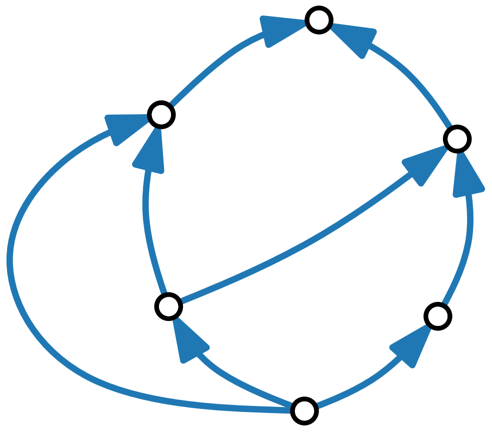

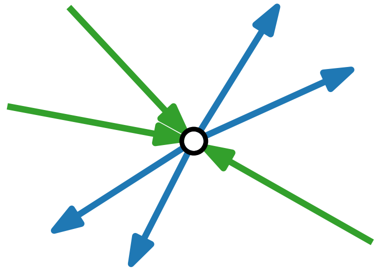

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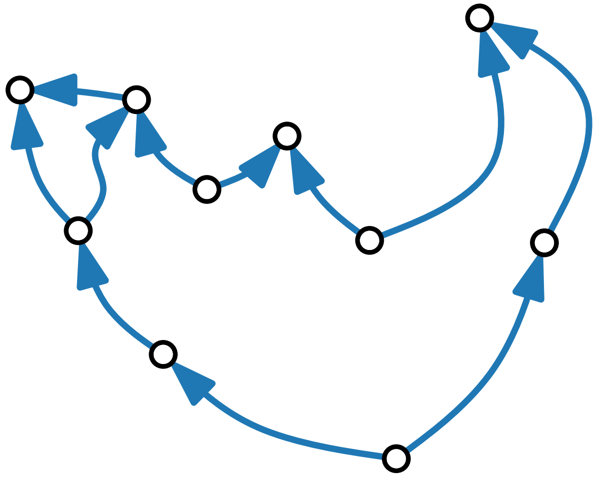

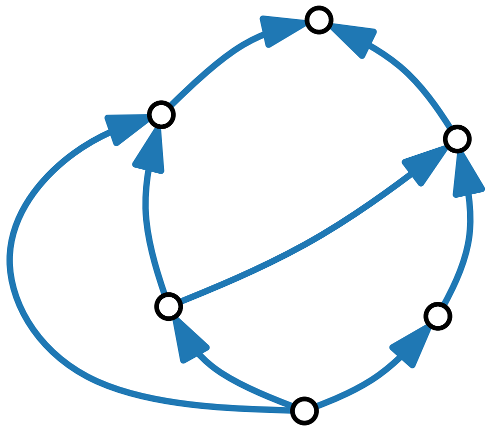

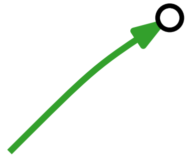

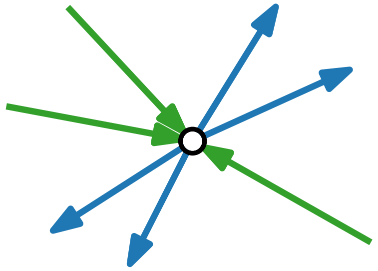

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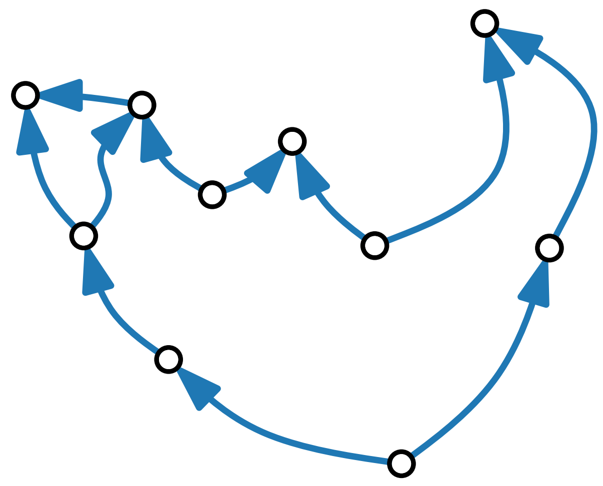

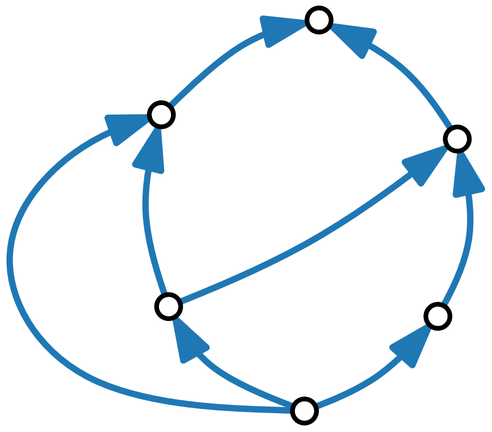

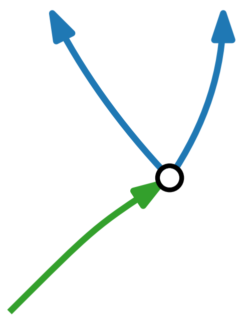

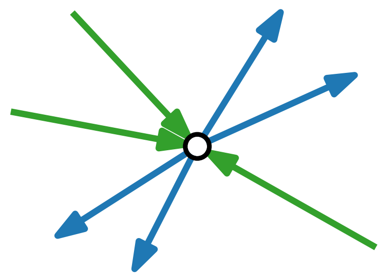

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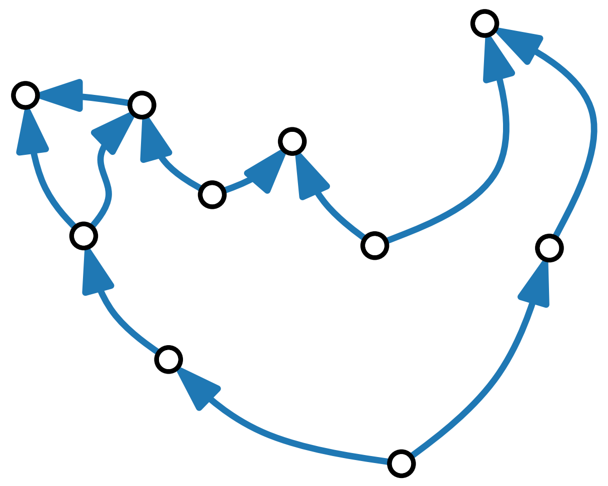

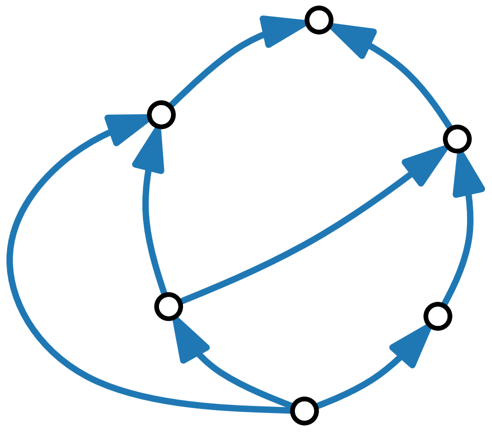

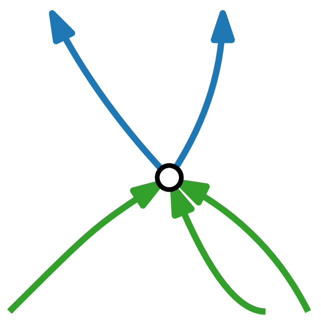

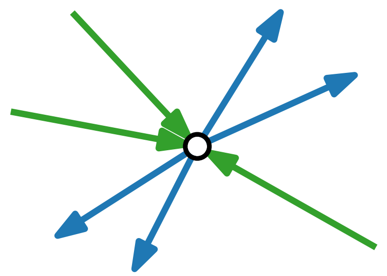

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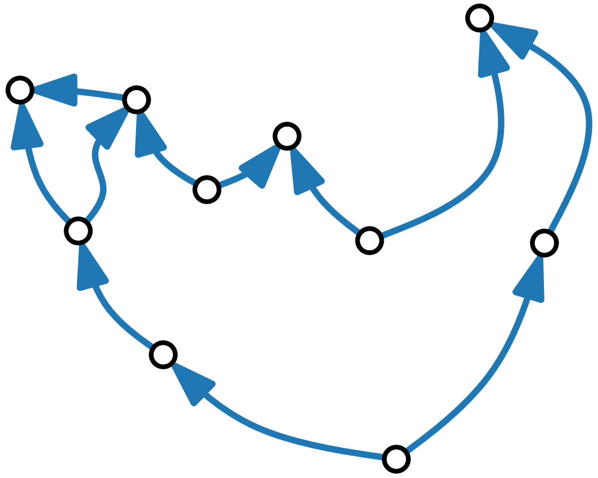

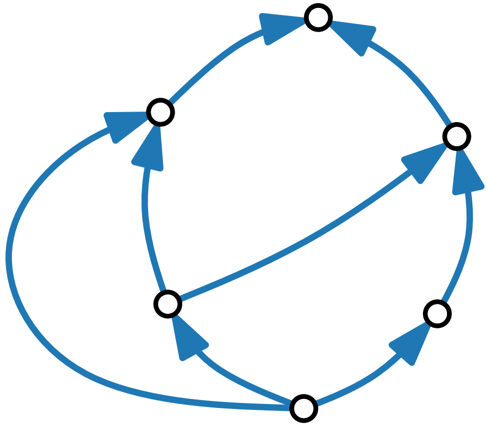

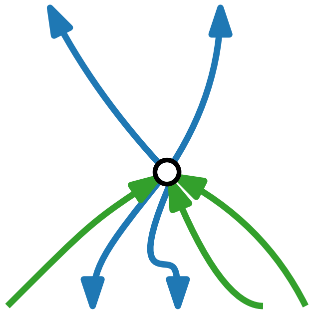

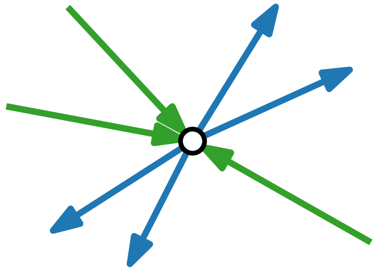

Upward Planarity – Necessary Conditions

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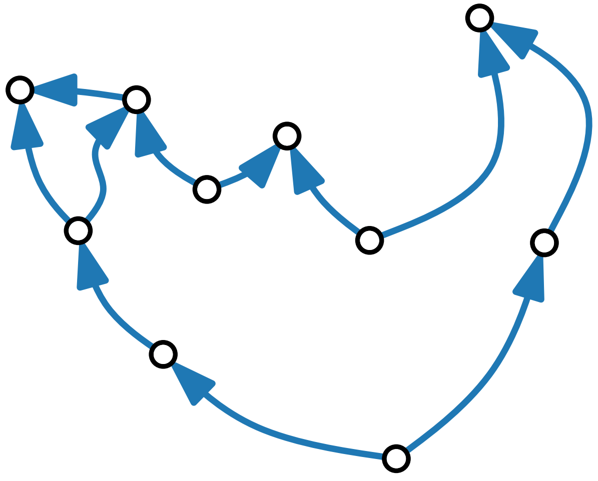

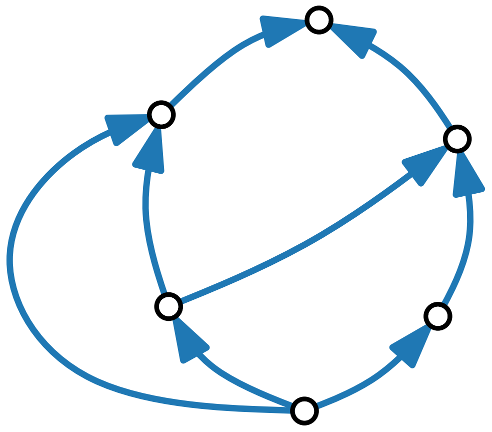

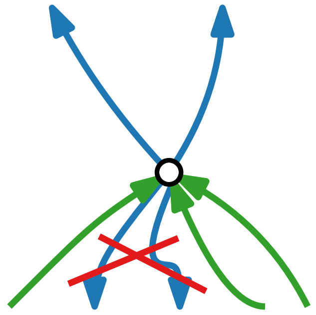

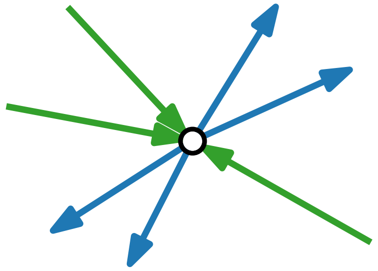

Upward Planarity – Necessary Conditions

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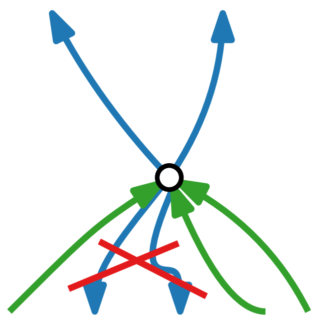
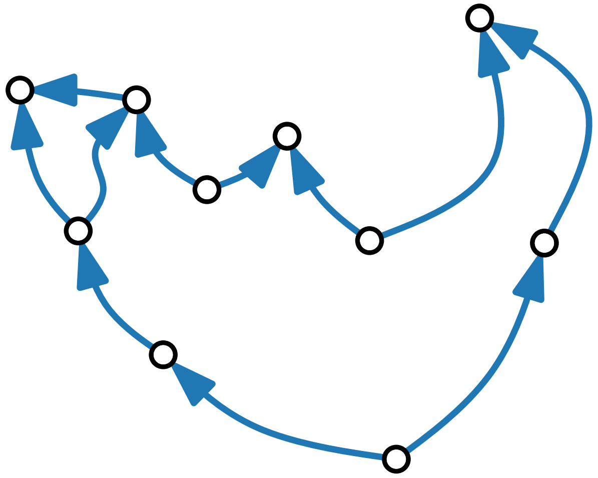
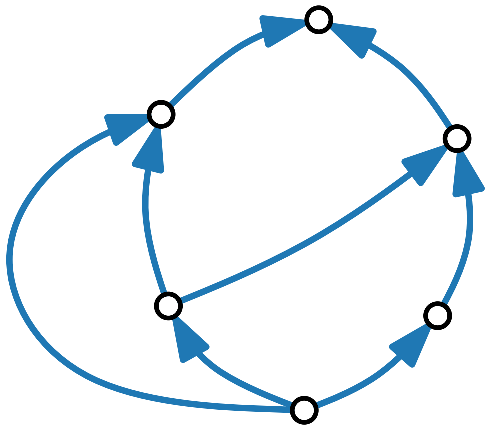
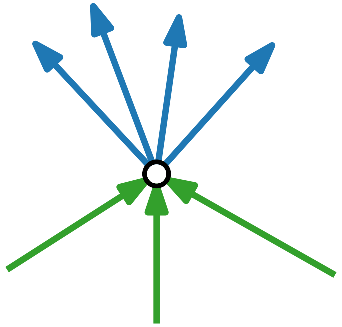
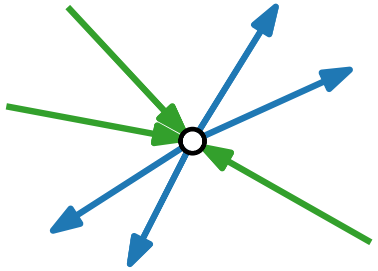




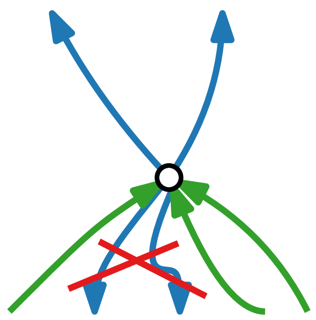


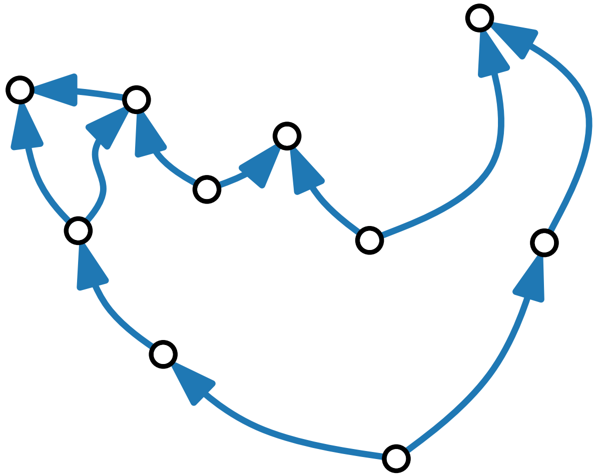


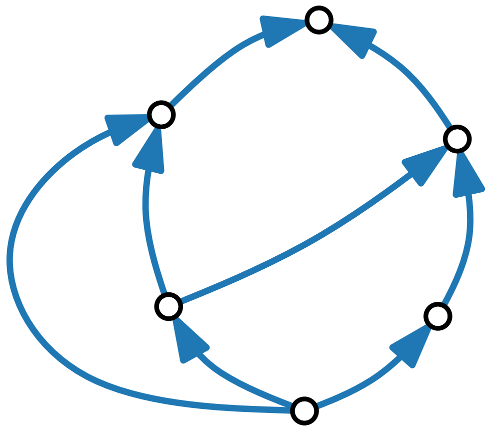


Upward Planarity – Necessary Conditions

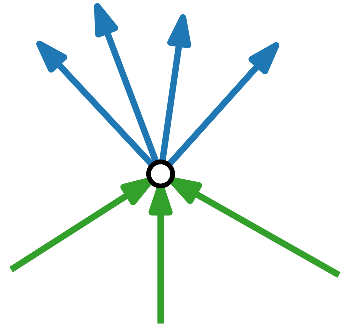
- For an (embedded) digraph to be upward planar, it needs to ...
  - be planar
  - be acyclic



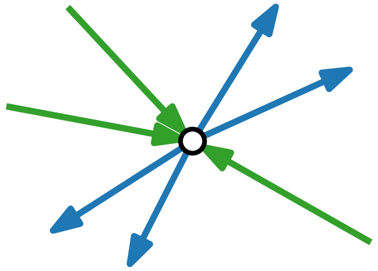




bimodal vertex

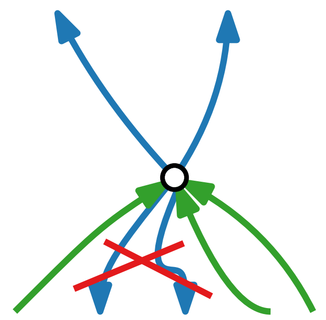


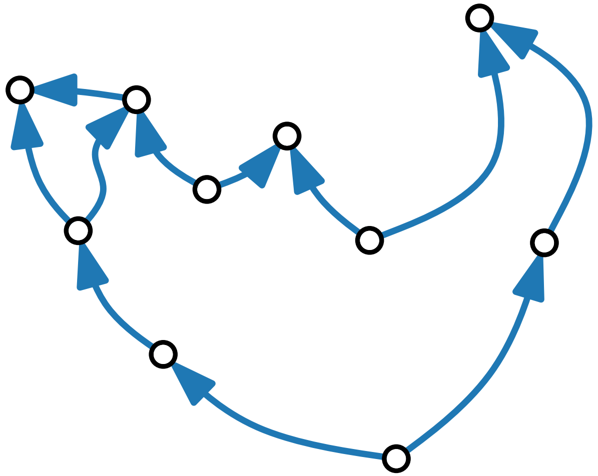
not bimodal

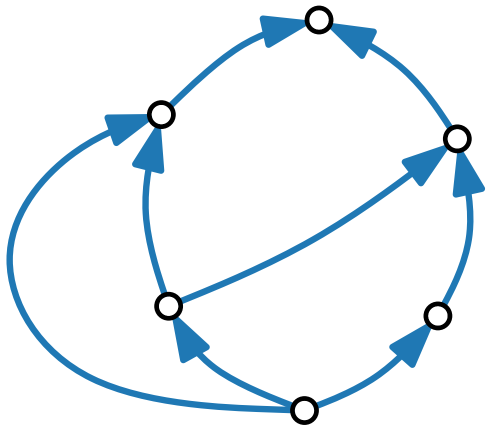


Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to ...
  - be planar
  - be acyclic
  - have a bimodal embedding







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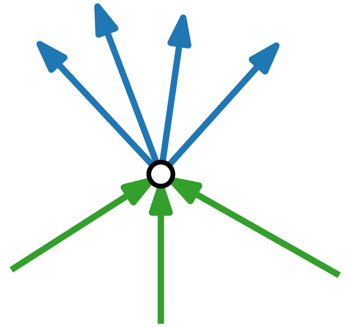

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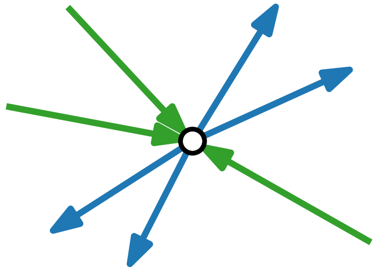
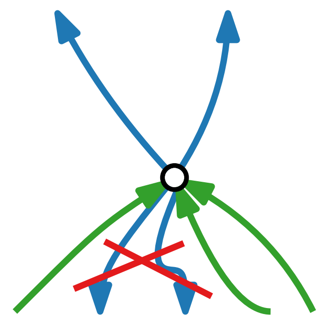
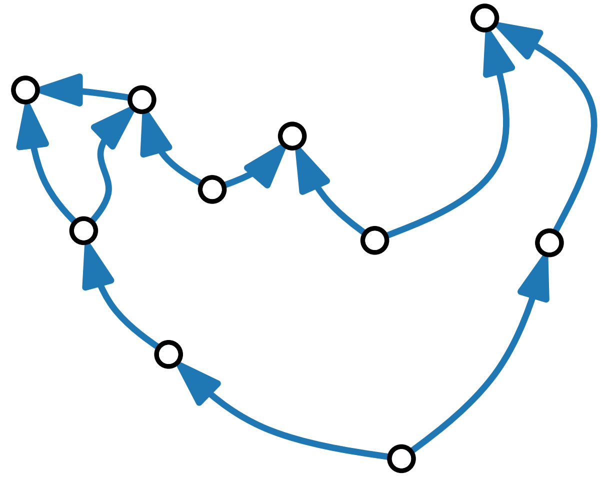
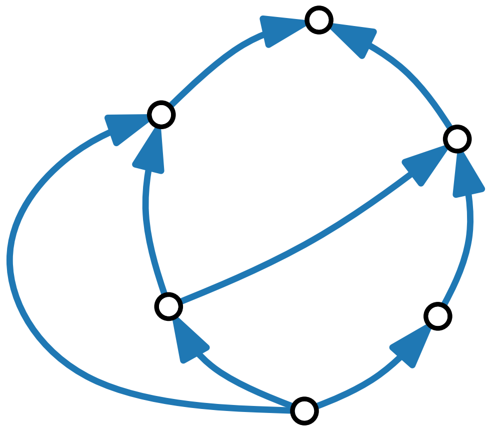


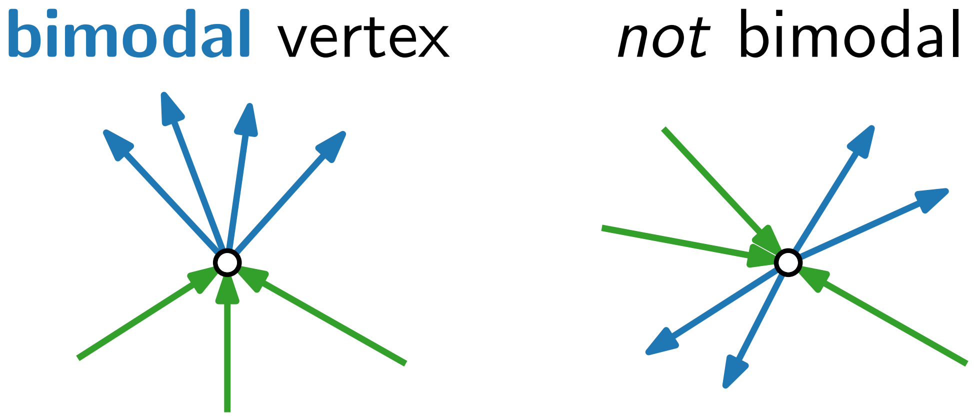
Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic
  - have a bimodal embedding
- . . . but these conditions are *not sufficient*.









Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to ...
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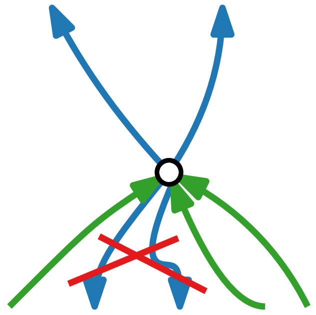
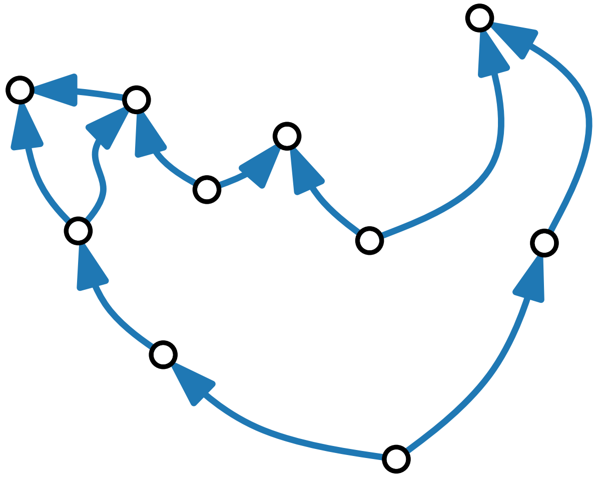
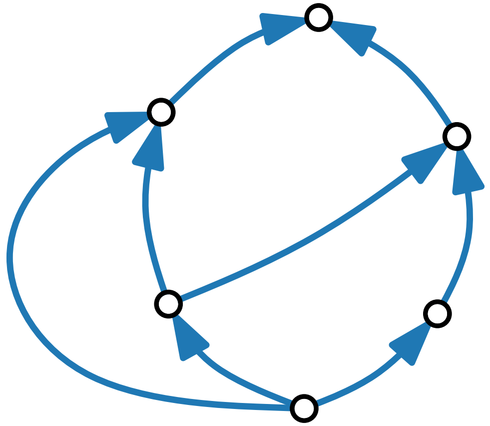
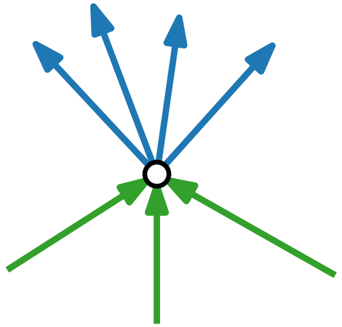


- ... but these conditions are *not sufficient*. $\rightarrow$ **Exercise**





bimodal vertex



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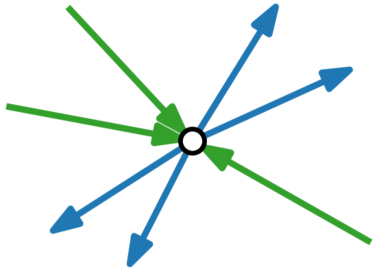

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acyclic digraph with
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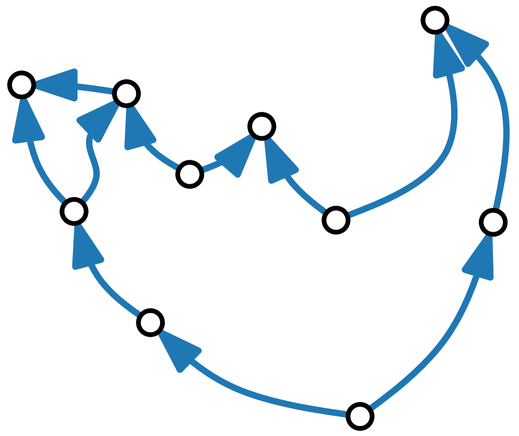

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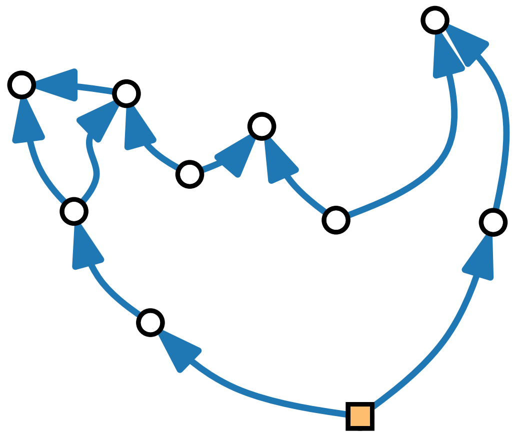

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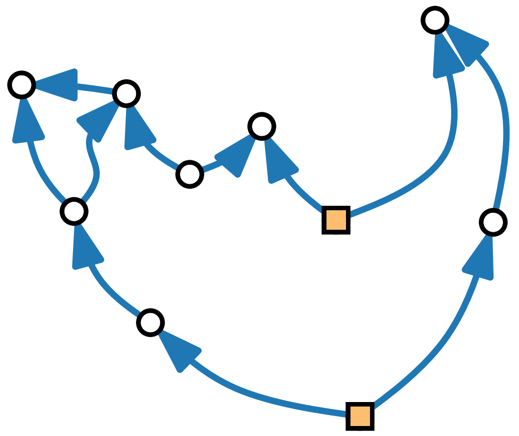

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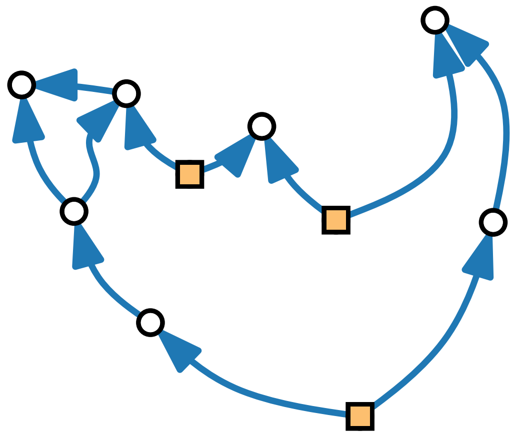

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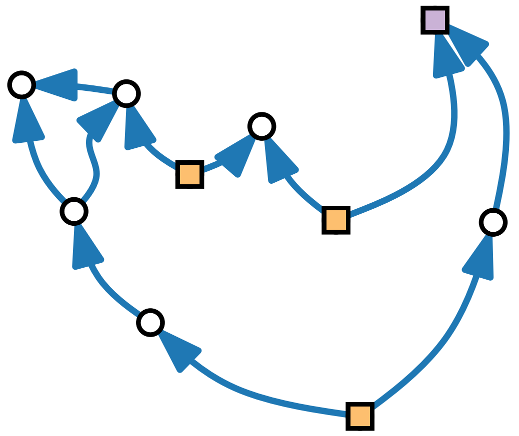

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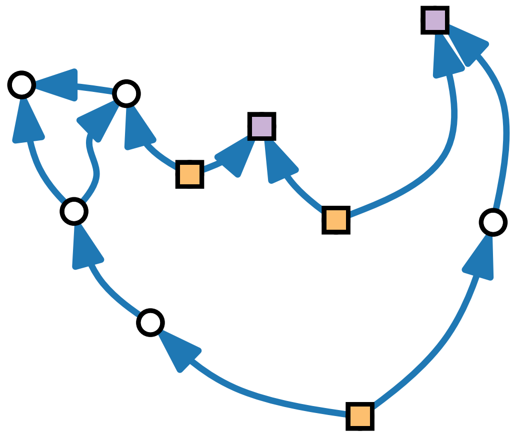

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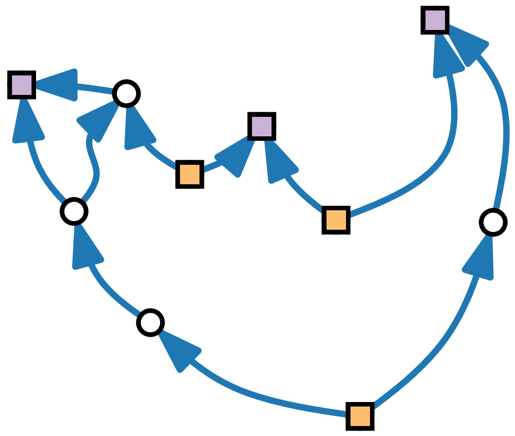

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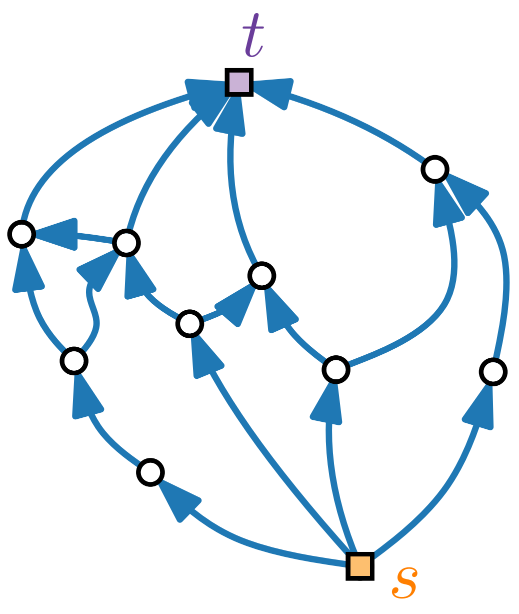

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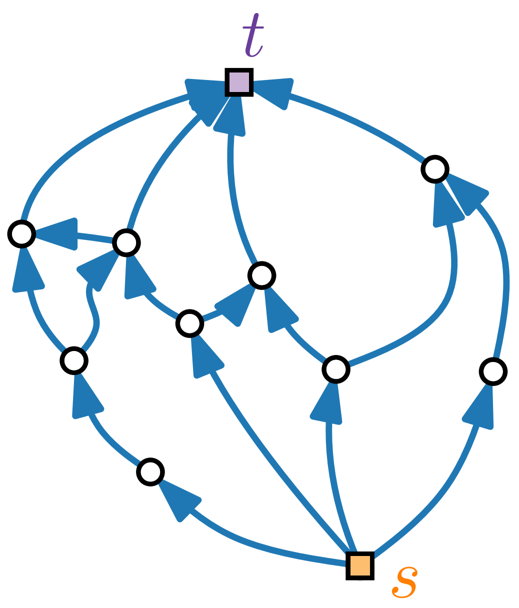

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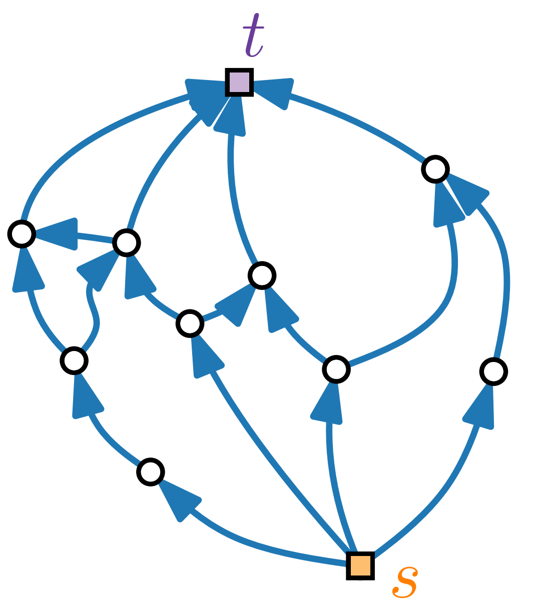

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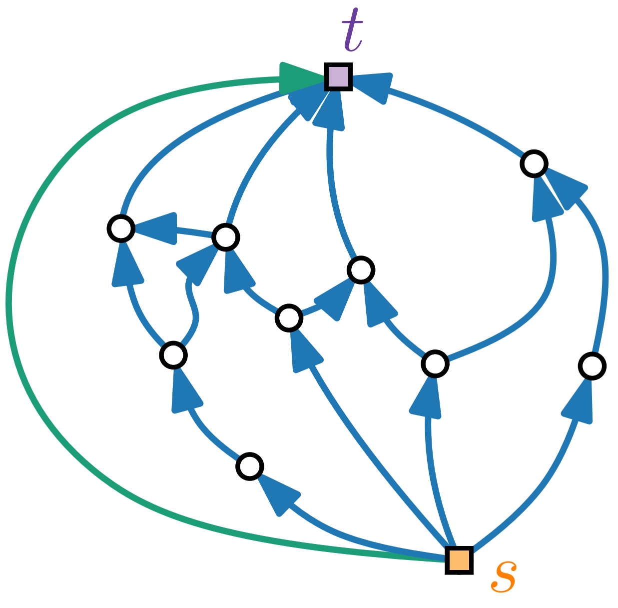

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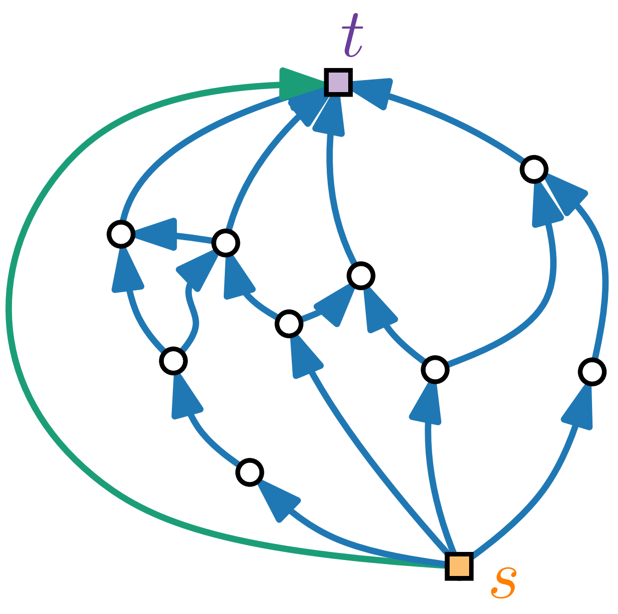

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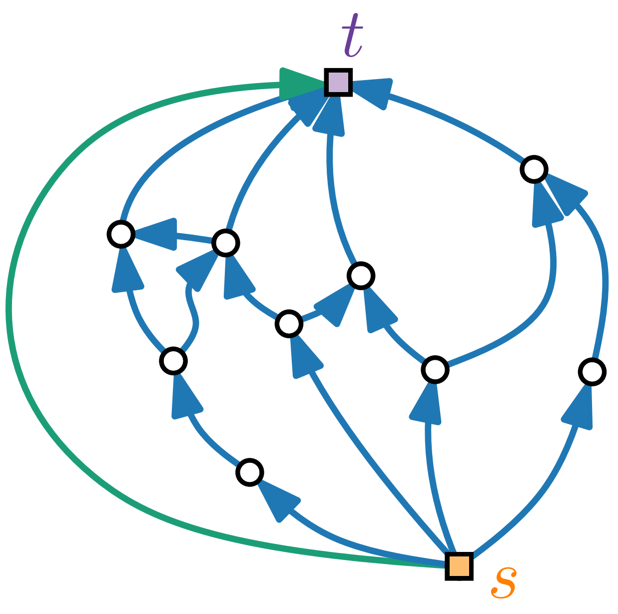

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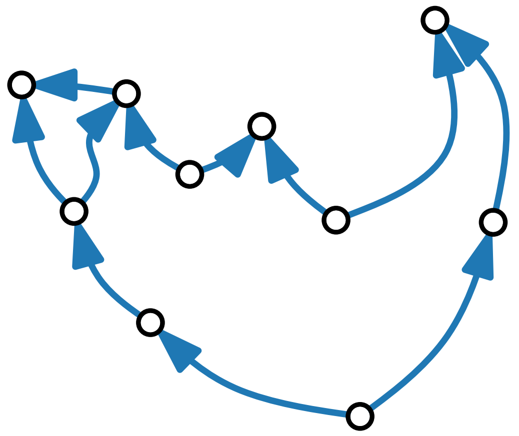

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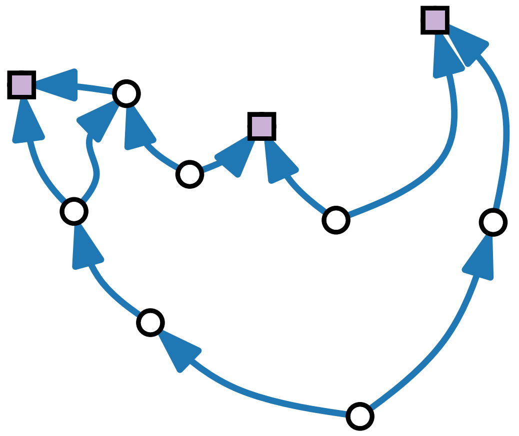

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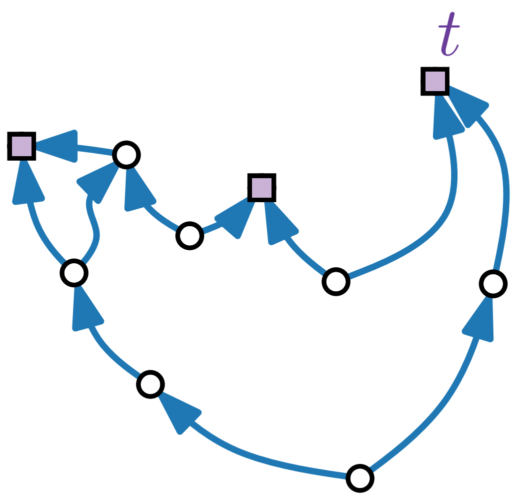

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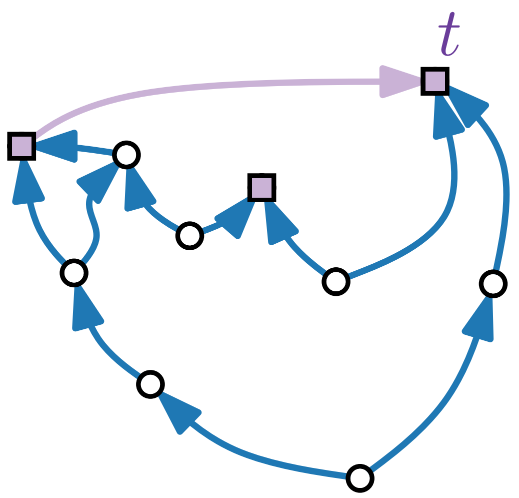

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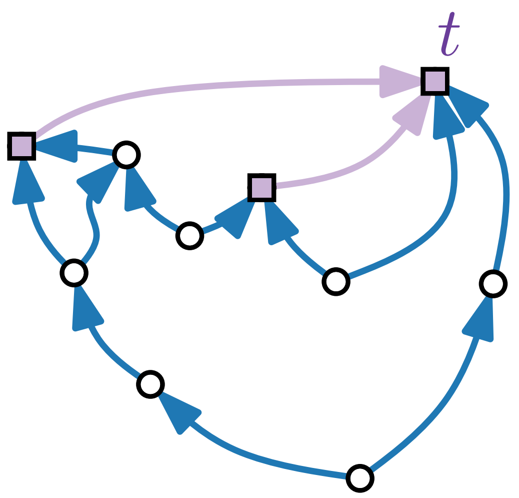

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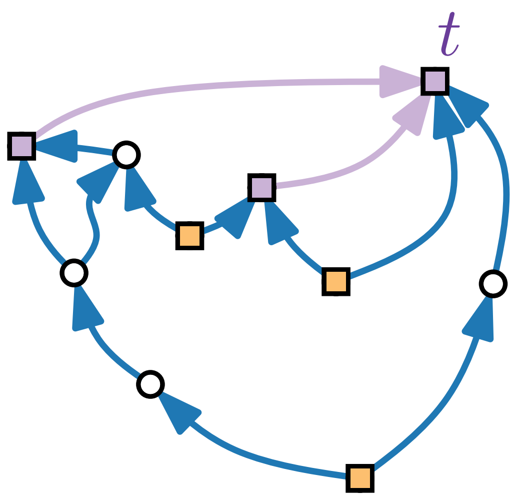

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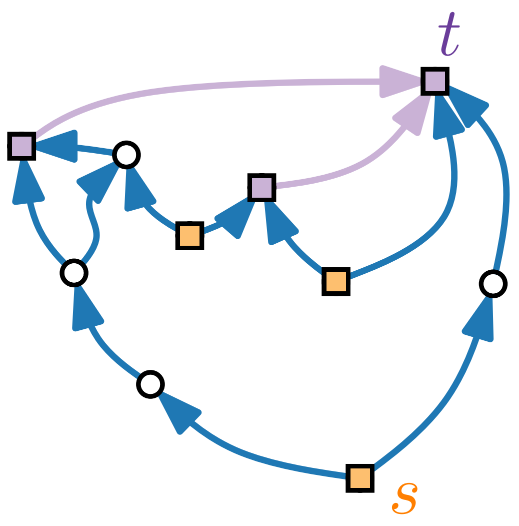

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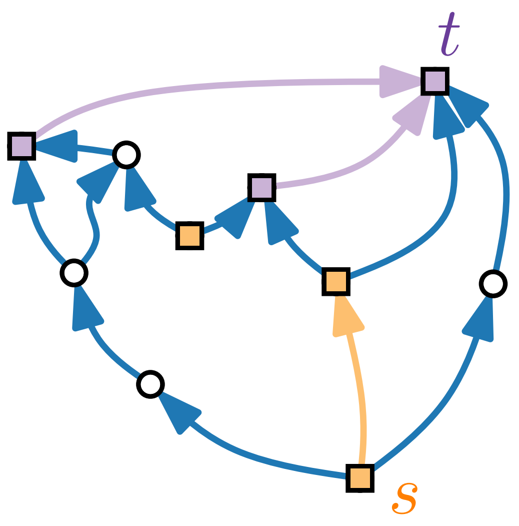

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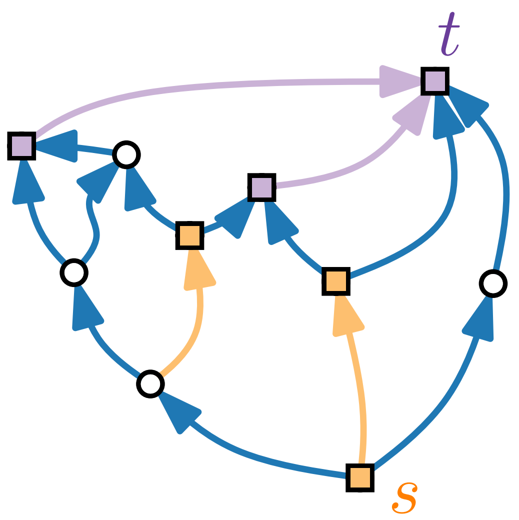

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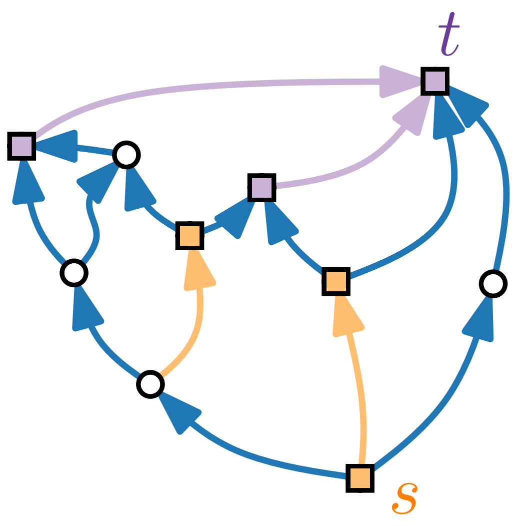

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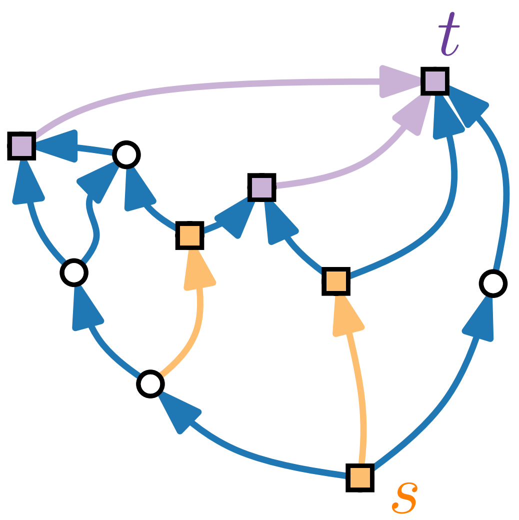

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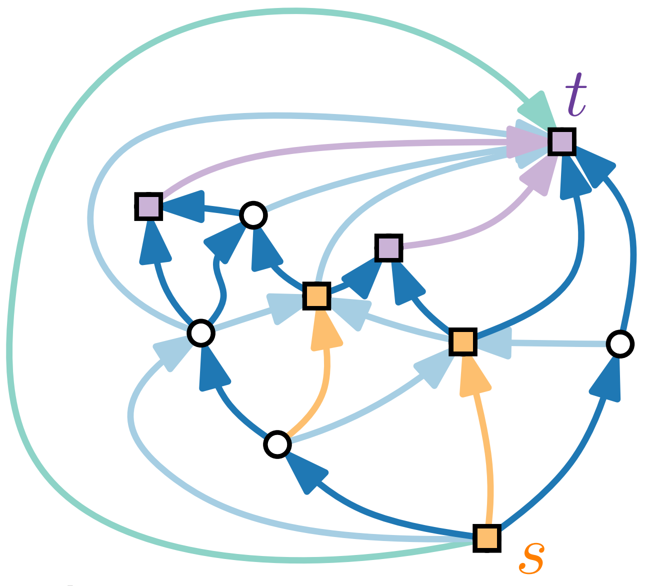

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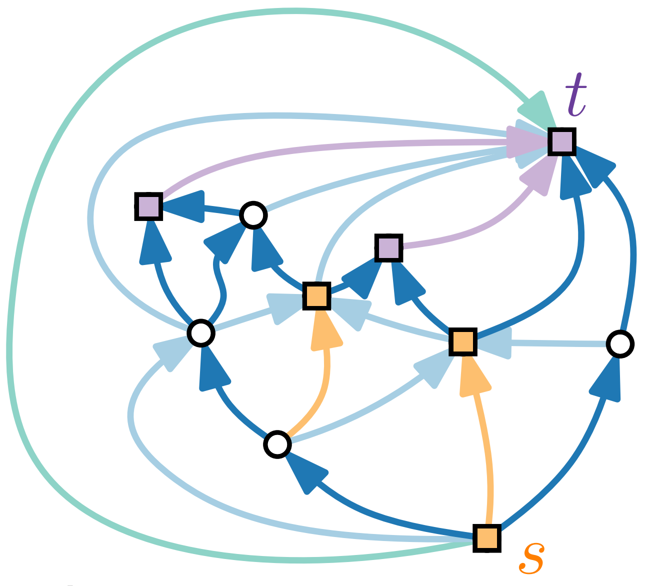

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Claim.
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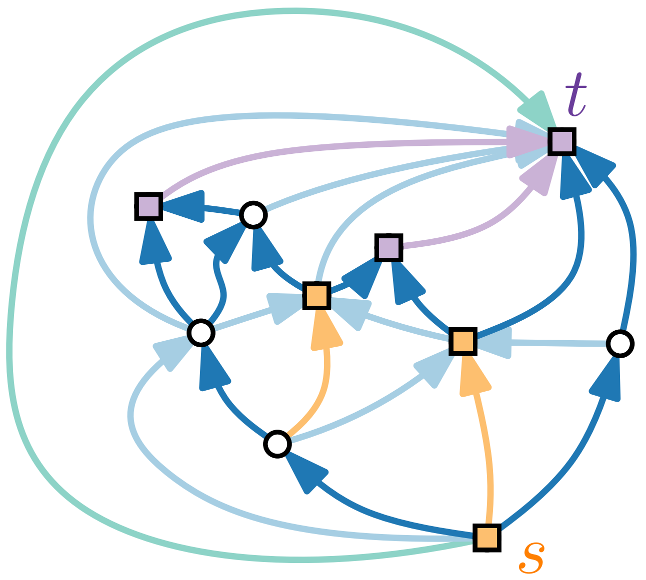

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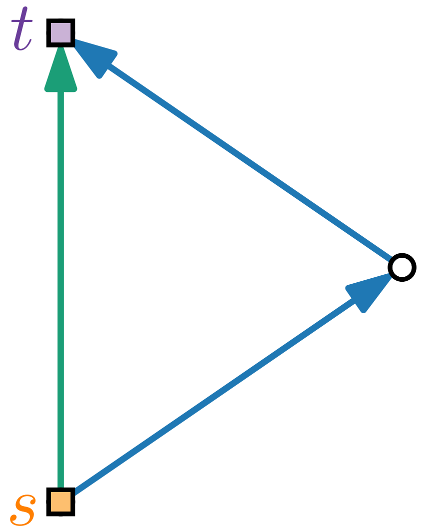

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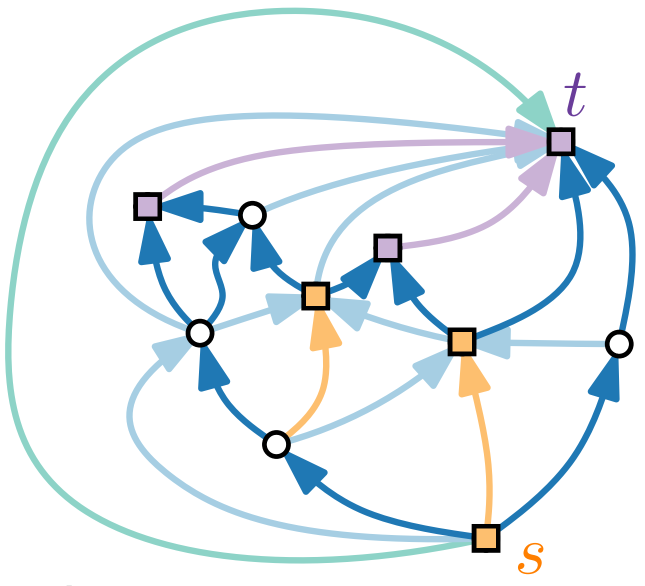

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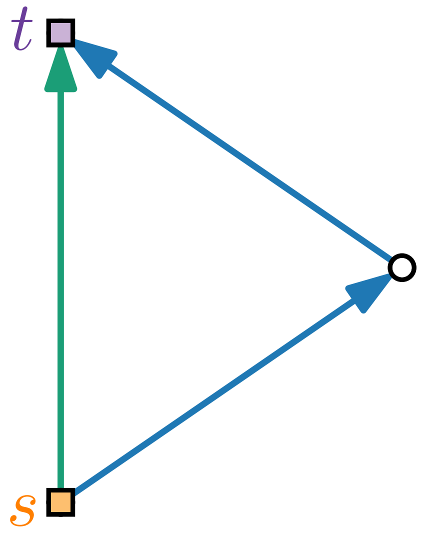
Induction on the number of vertices n.



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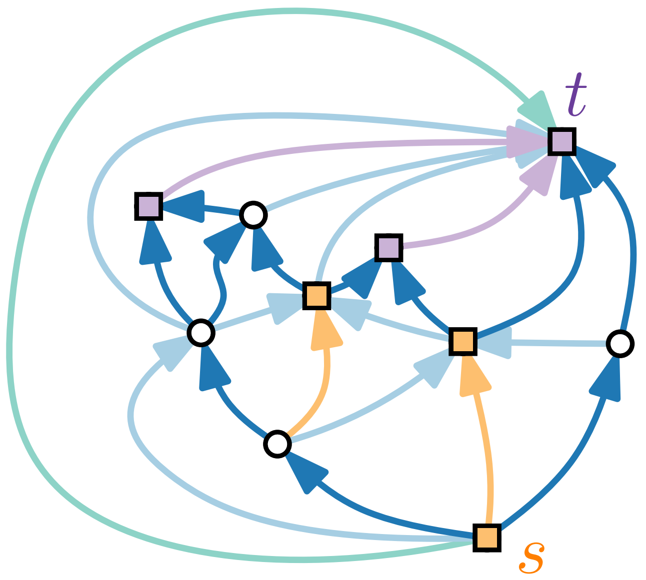

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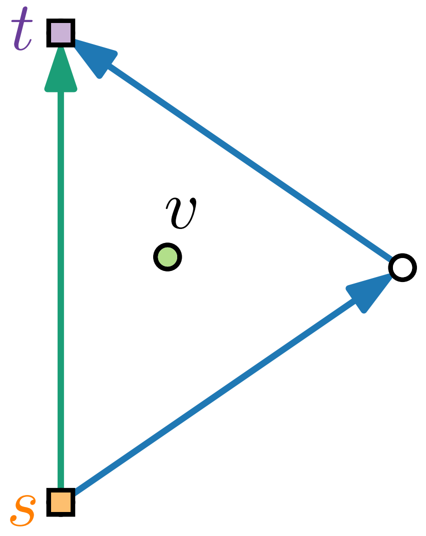
Induction on the number of vertices n.



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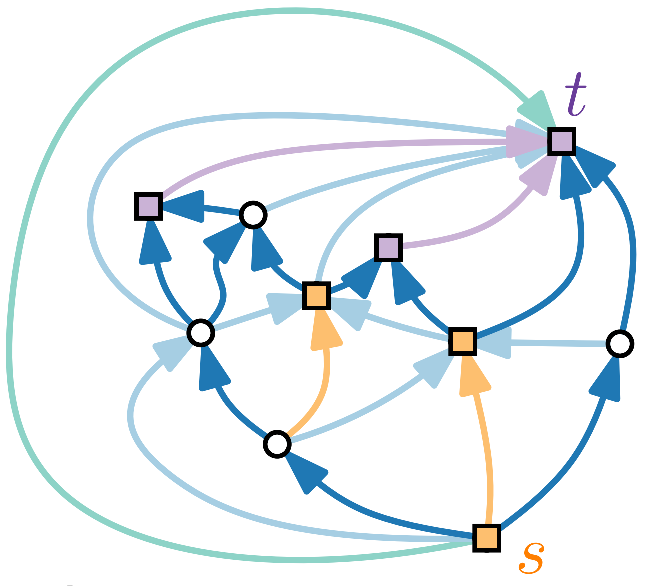

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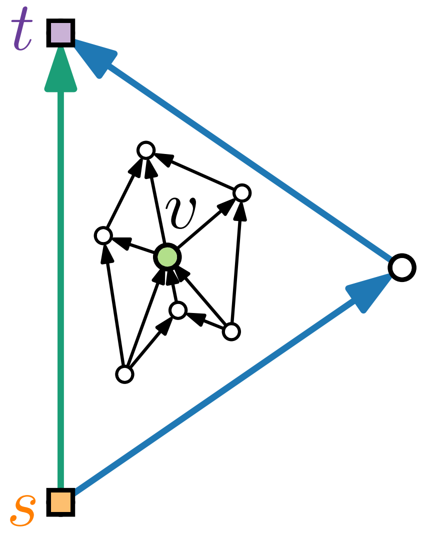

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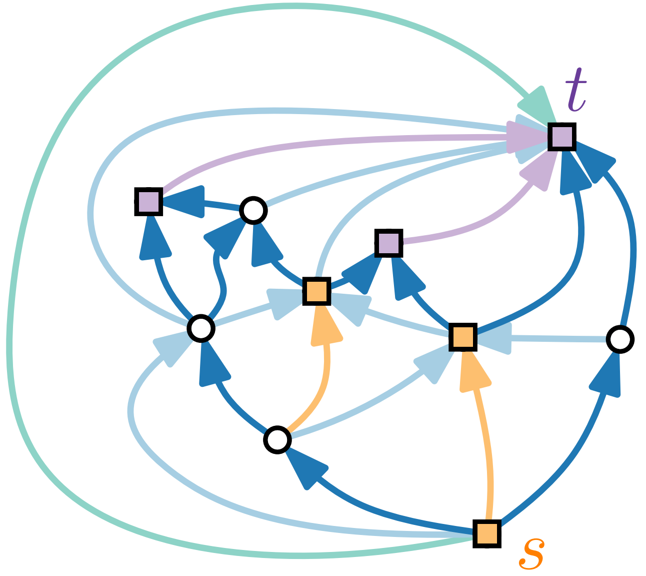

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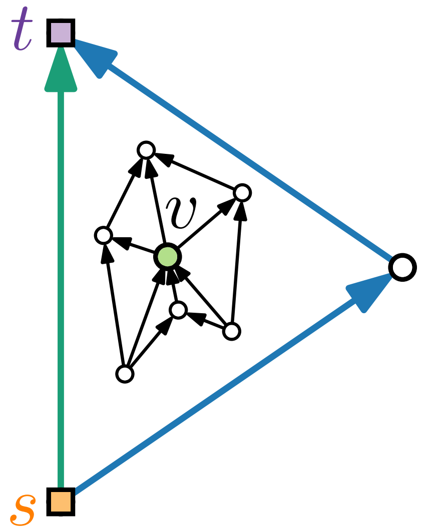

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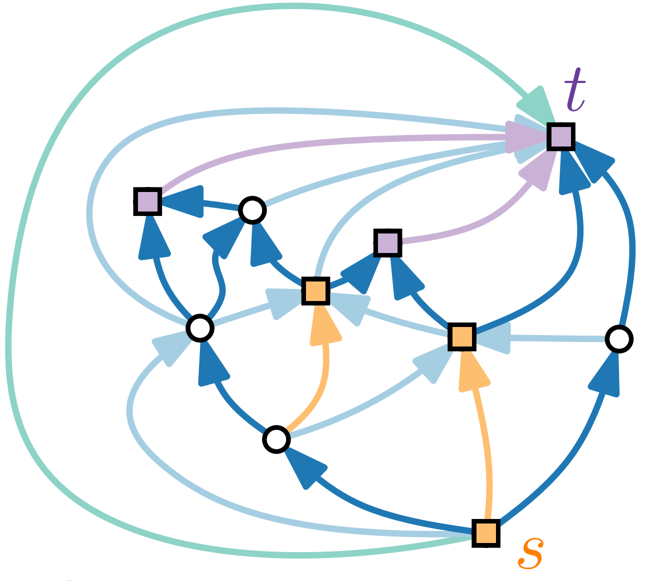

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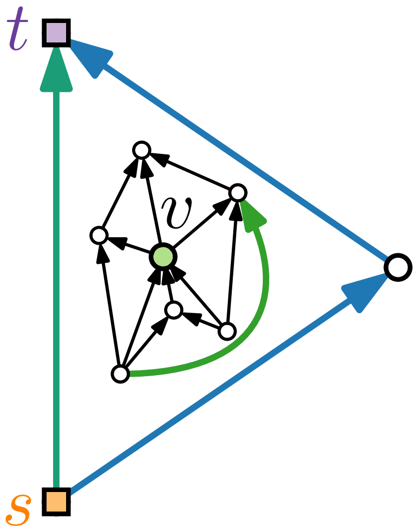

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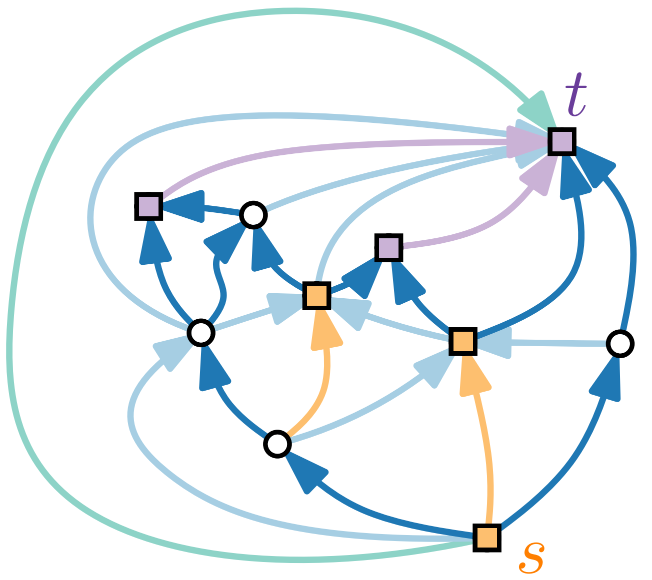

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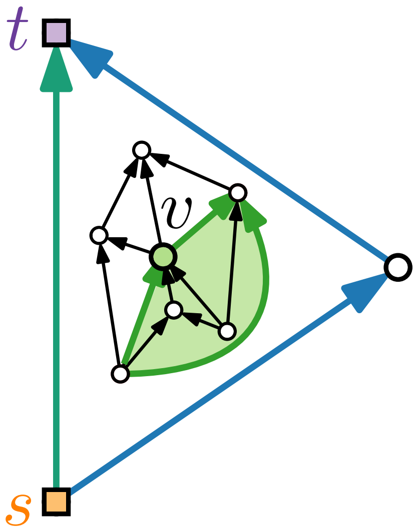

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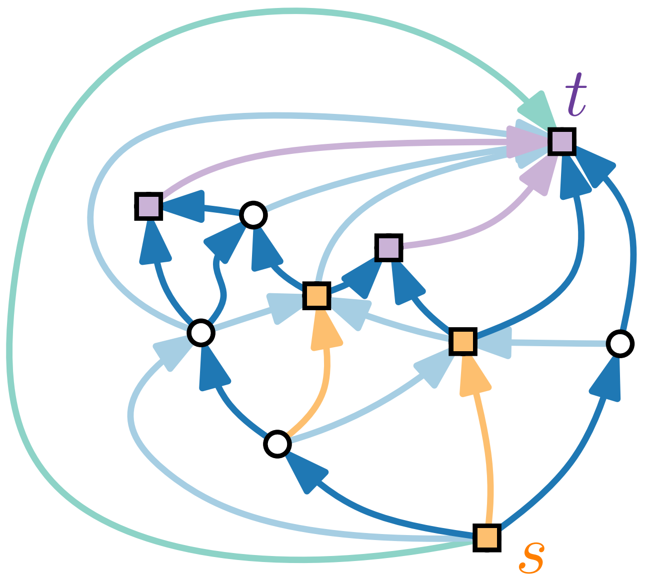

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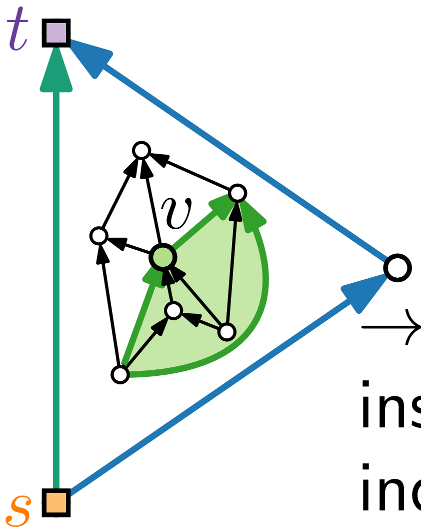

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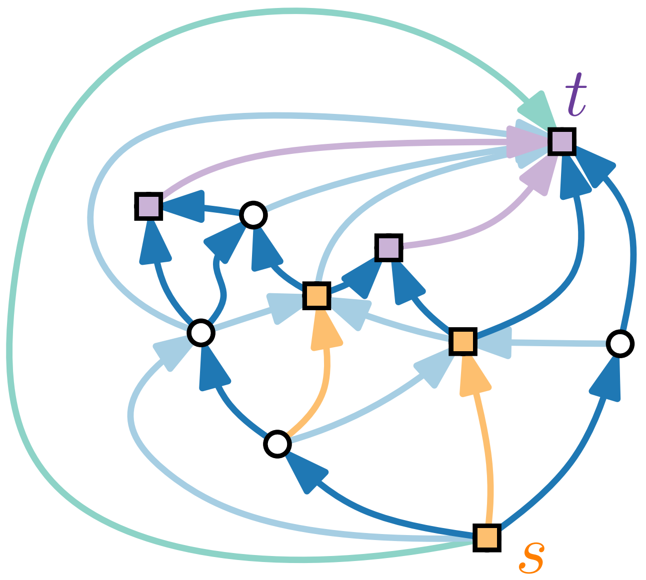

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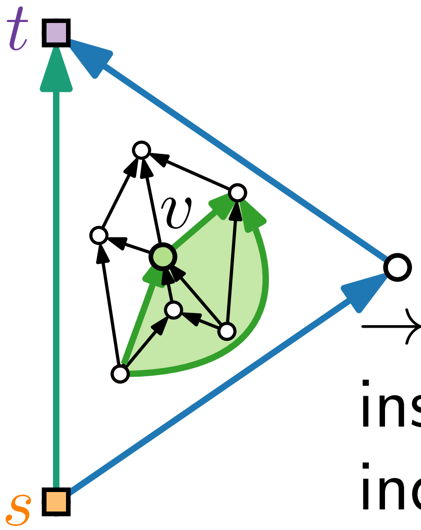

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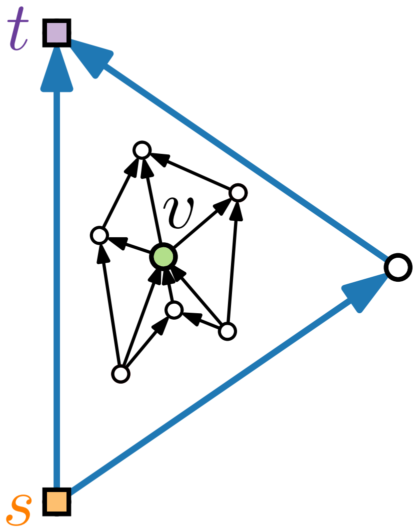

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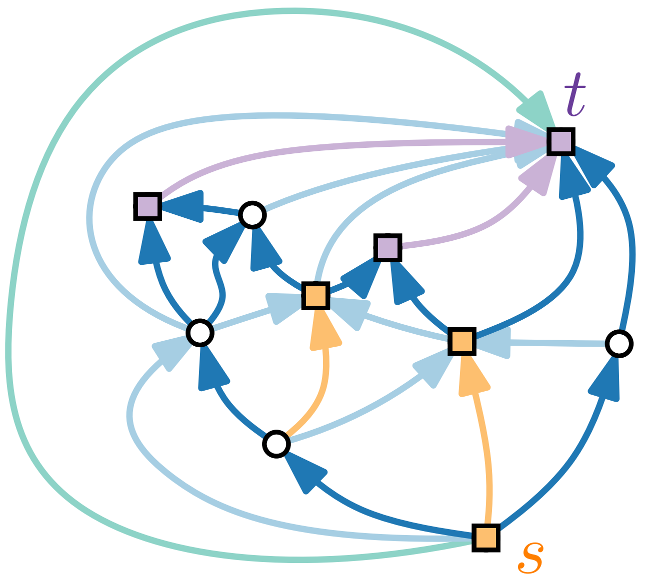

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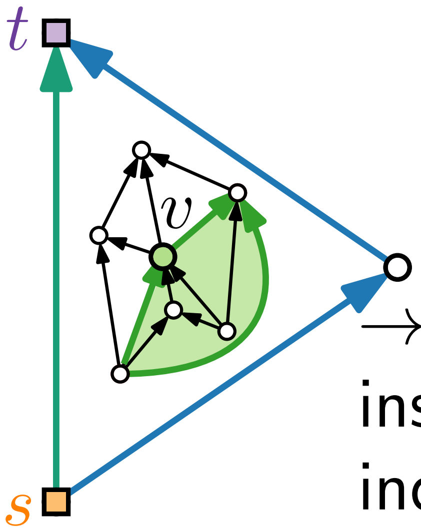

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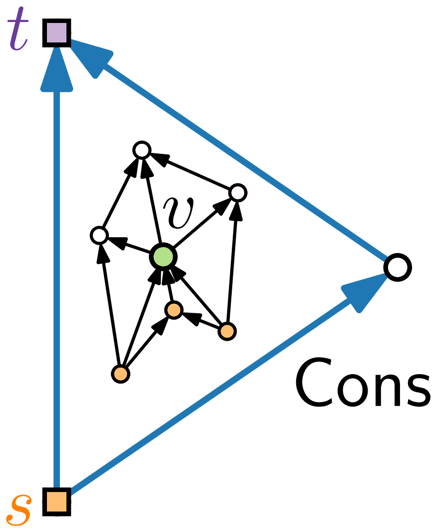

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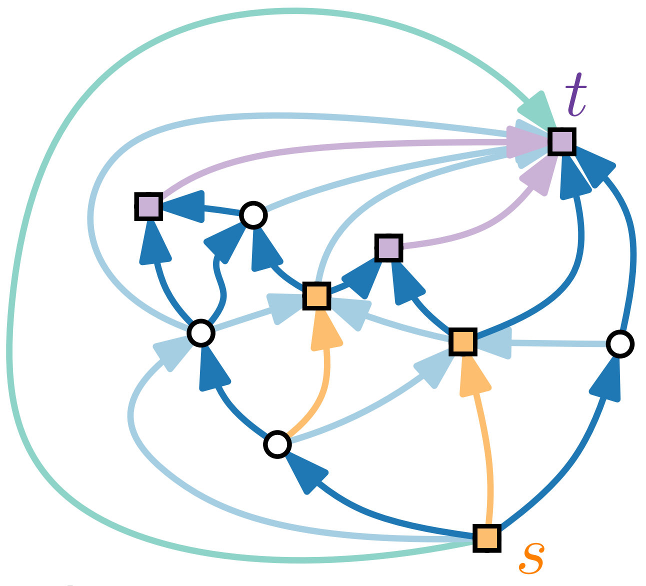

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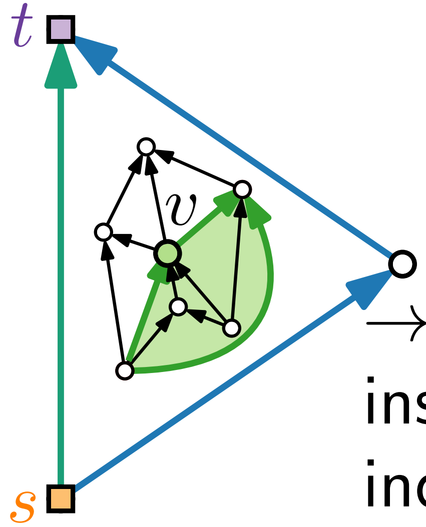

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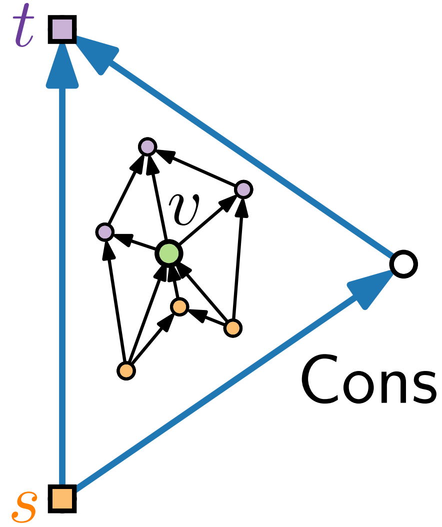

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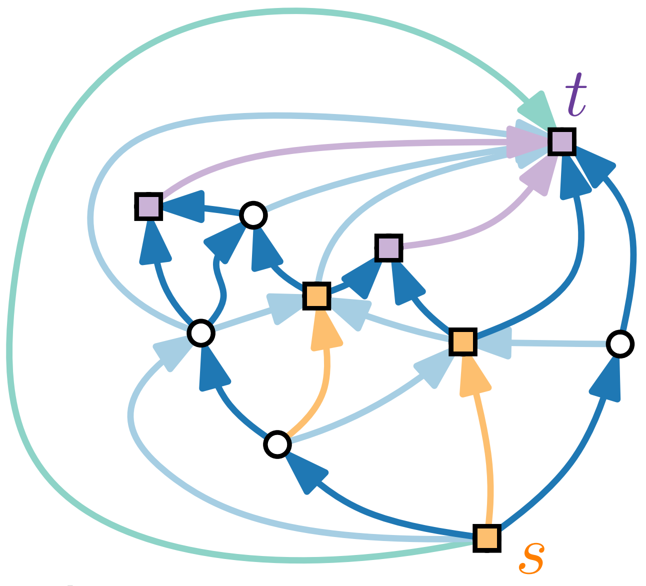

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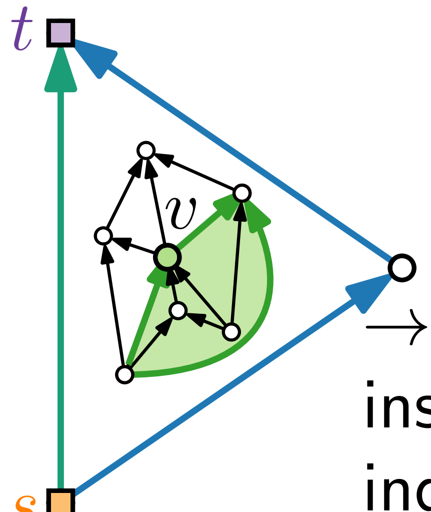

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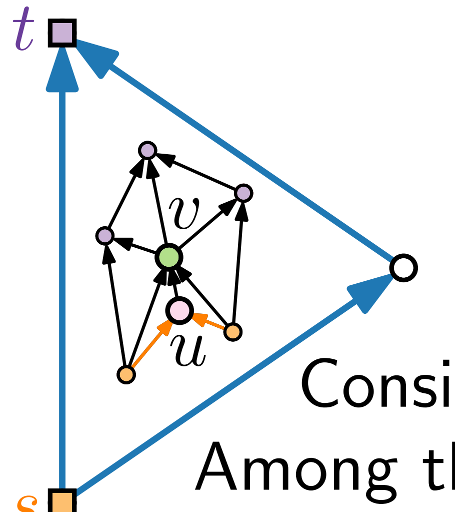

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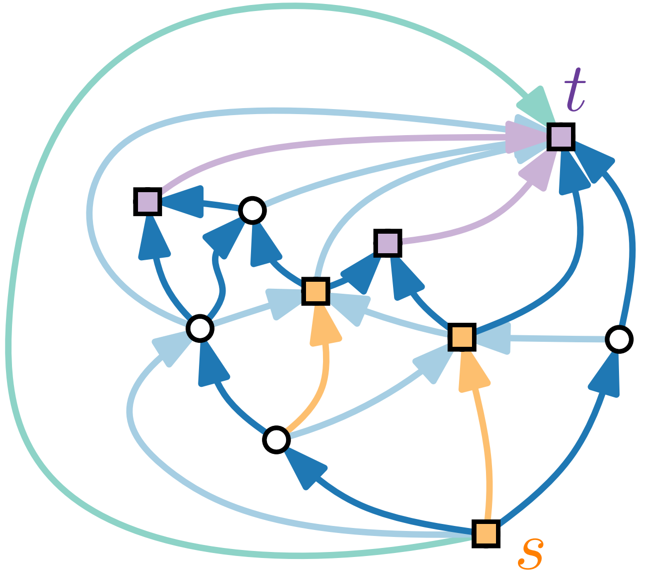

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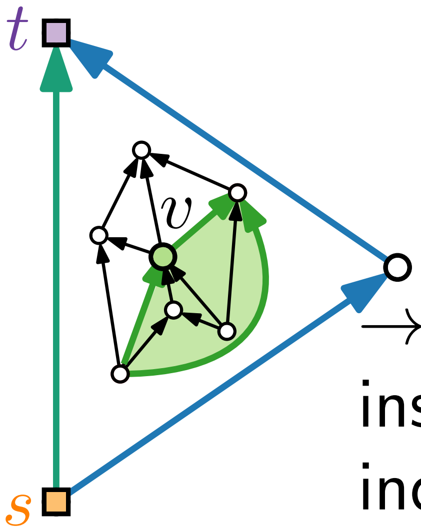

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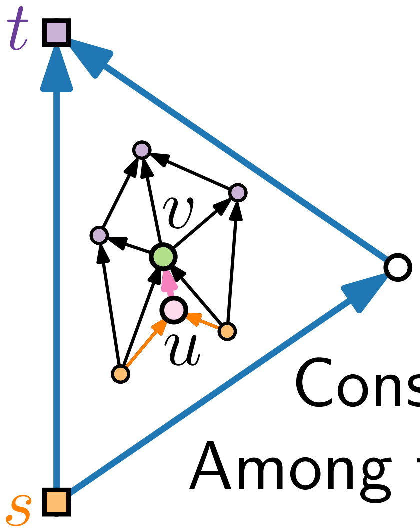

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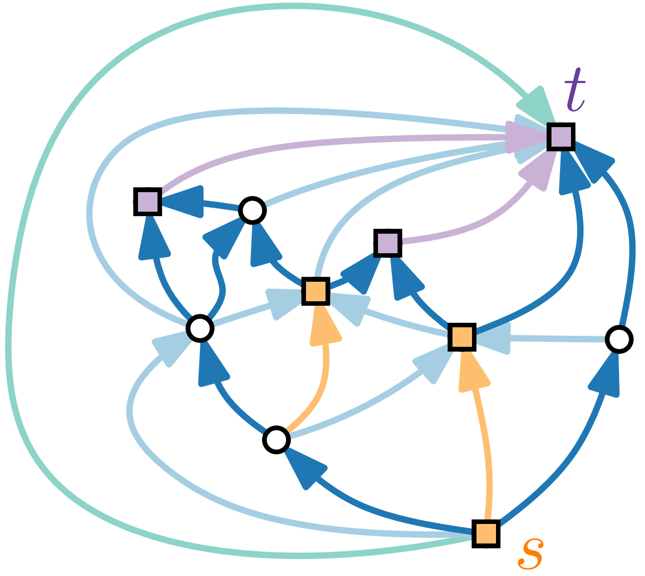

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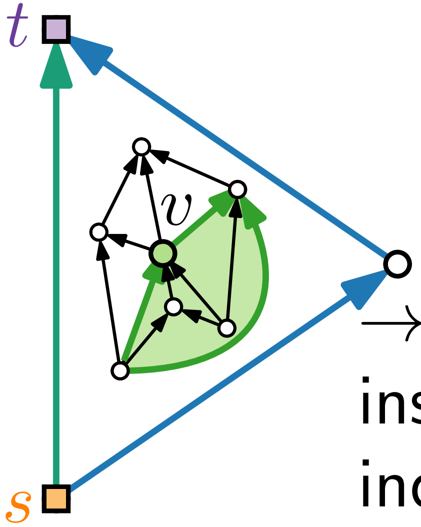

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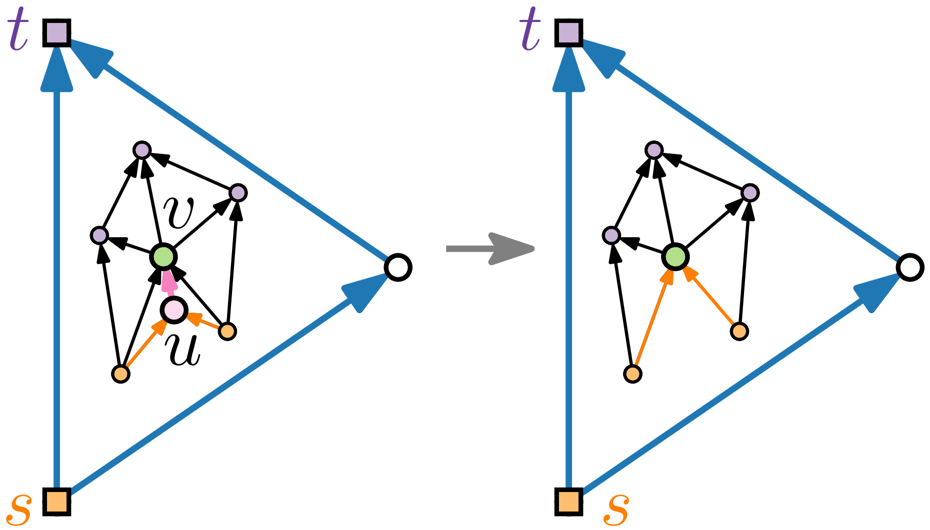

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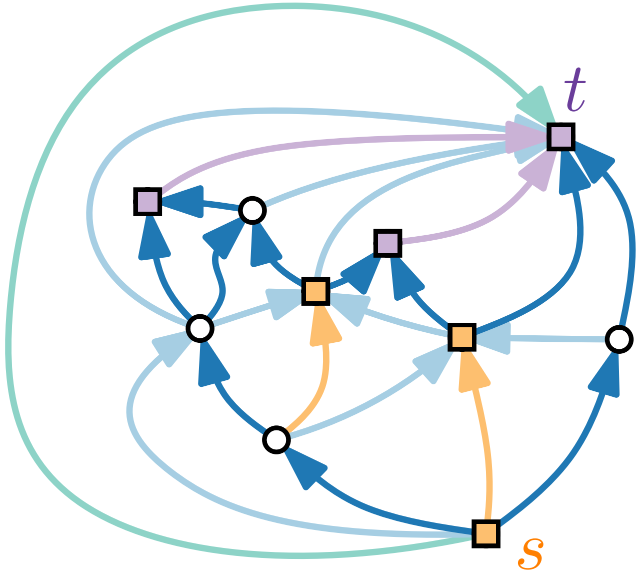

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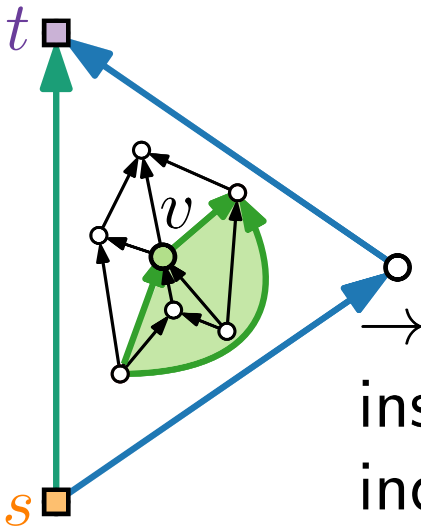

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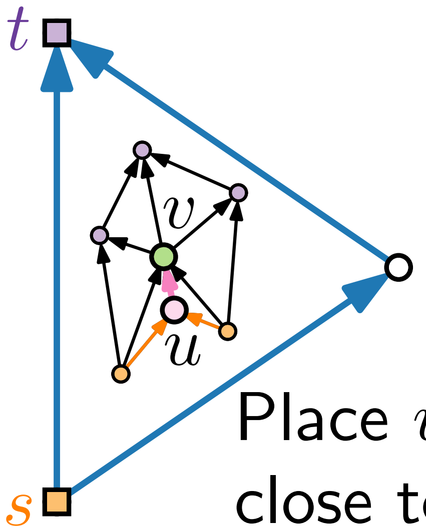
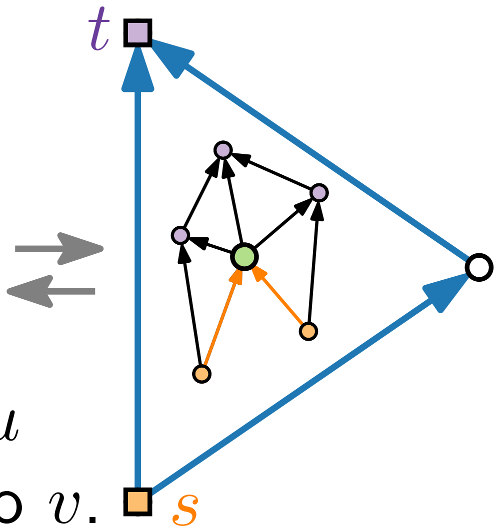


Place u close to v.



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Given a *planar acyclic* digraph G,
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Fixed Embedding Upward Planarity Testing.
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Test whether G is upward planar (w.r.t. to F and f_0).

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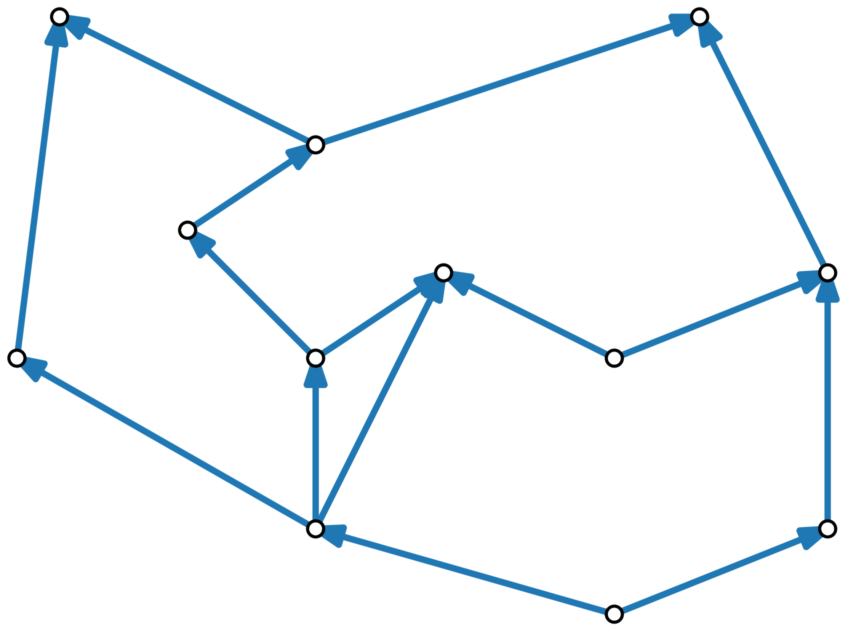
Plan.

- Find a property that any upward planar drawing of G satisfies.
- Formalize this property.
- Specify an algorithm to test this property.

Angles, Local Sources & Sinks

Definitions.



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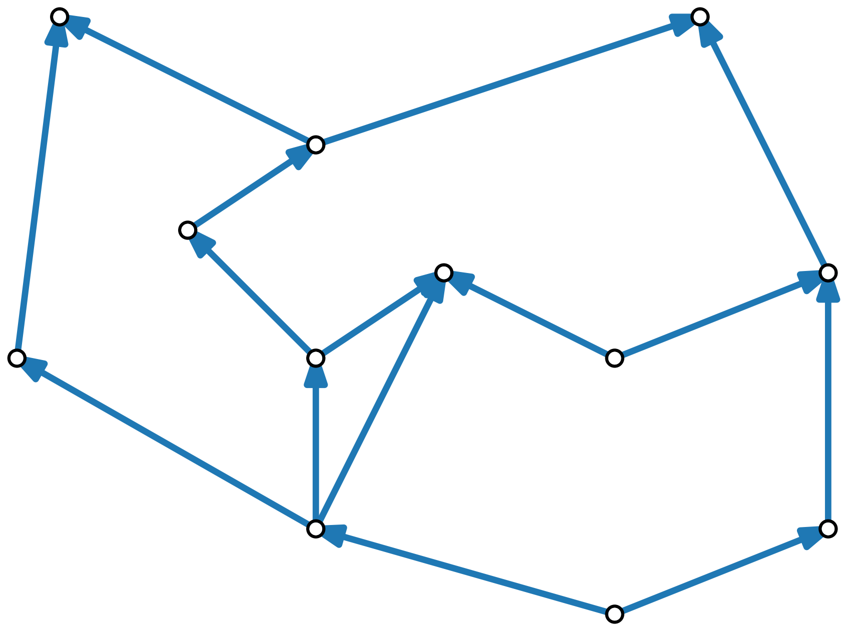

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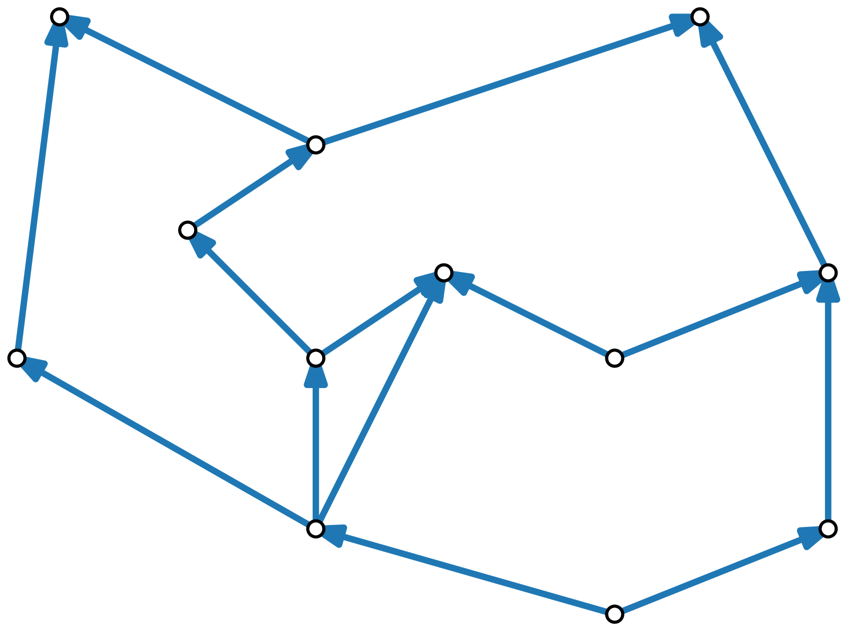

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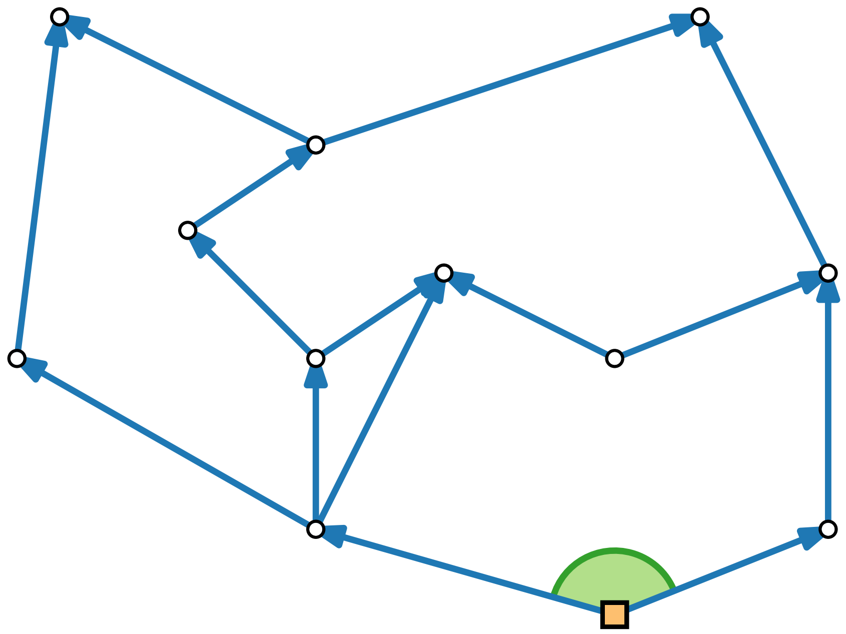

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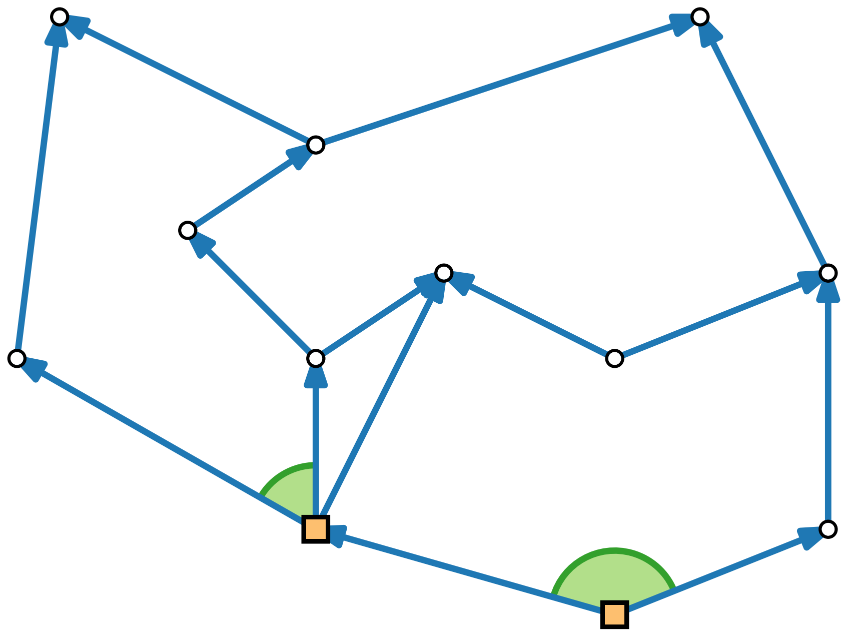

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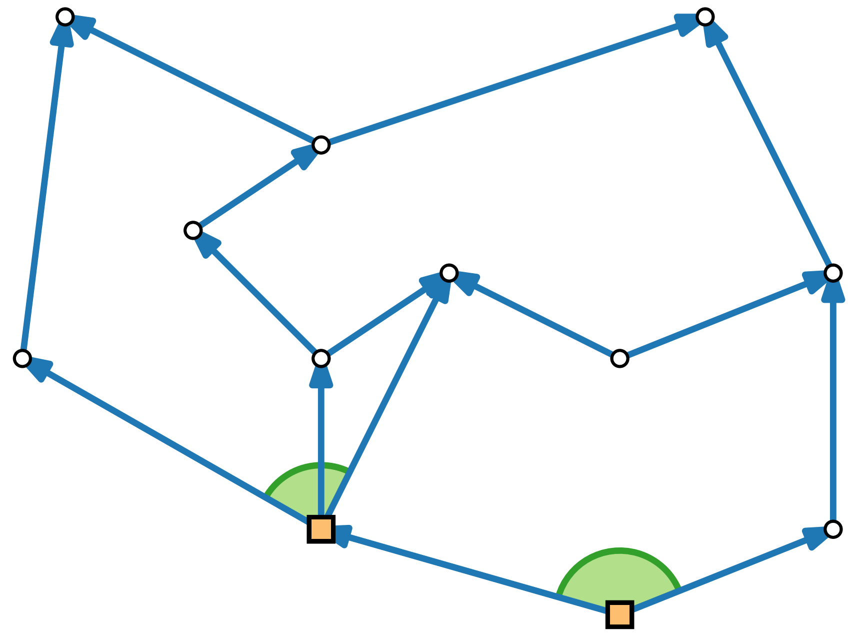

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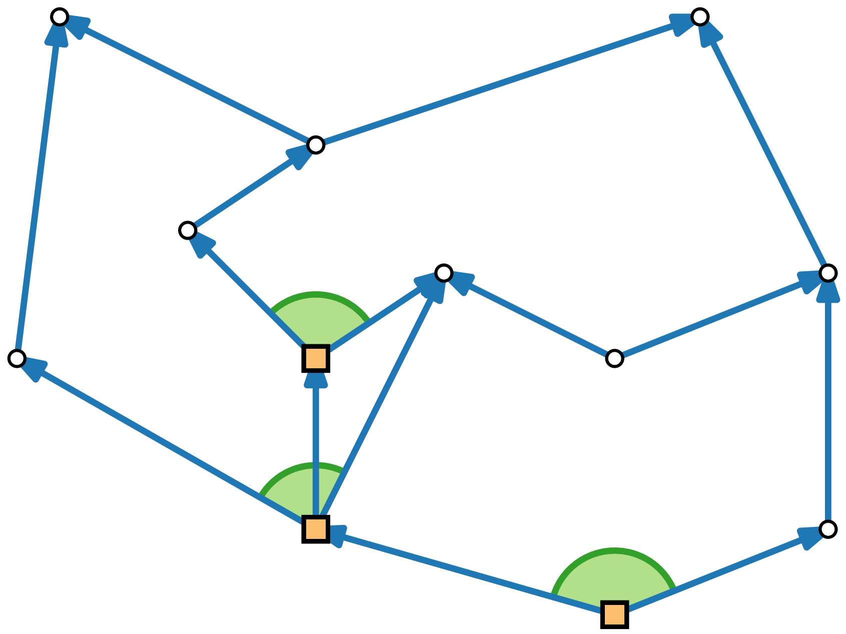

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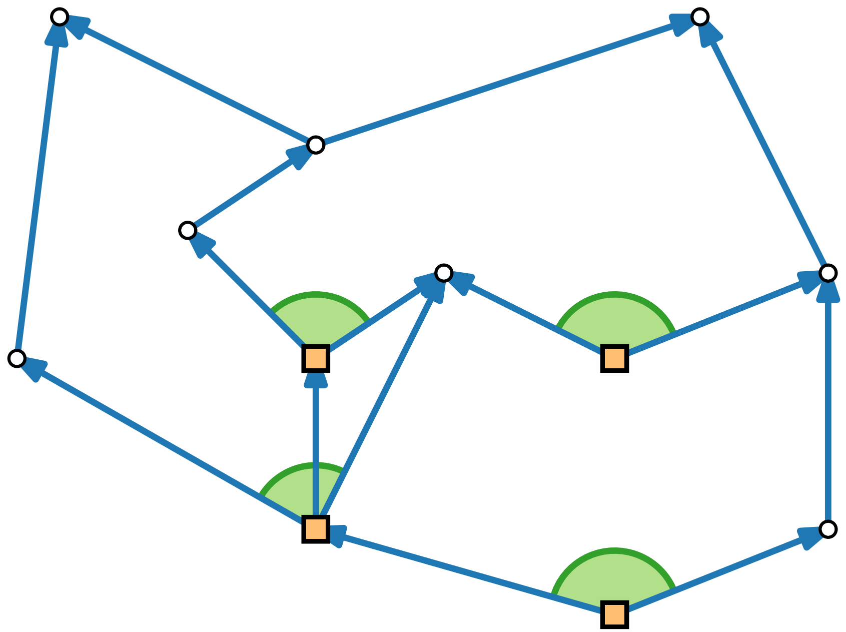

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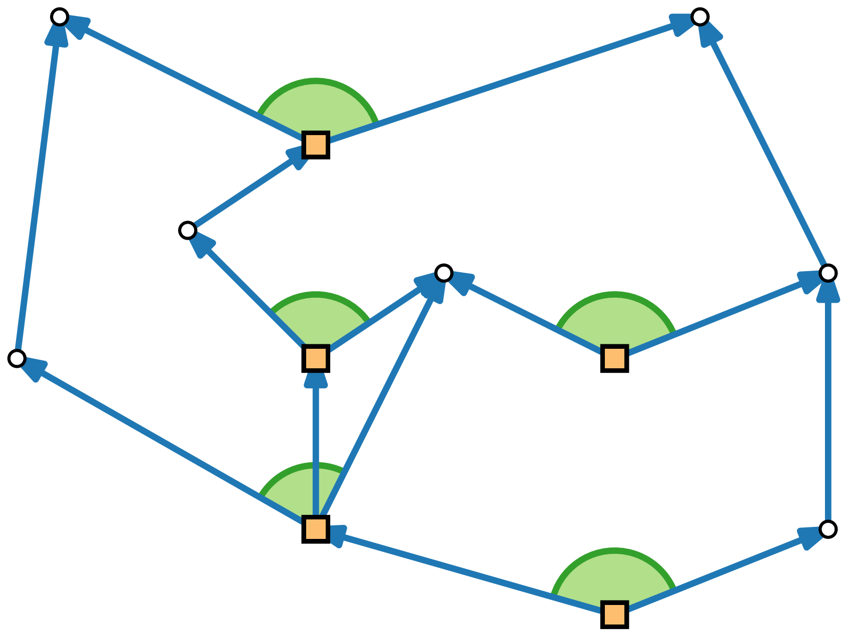

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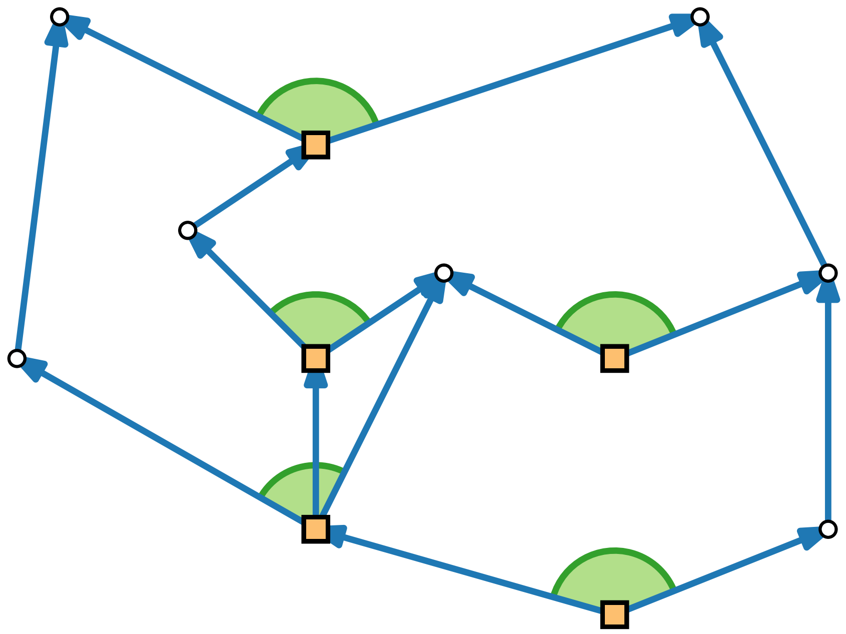

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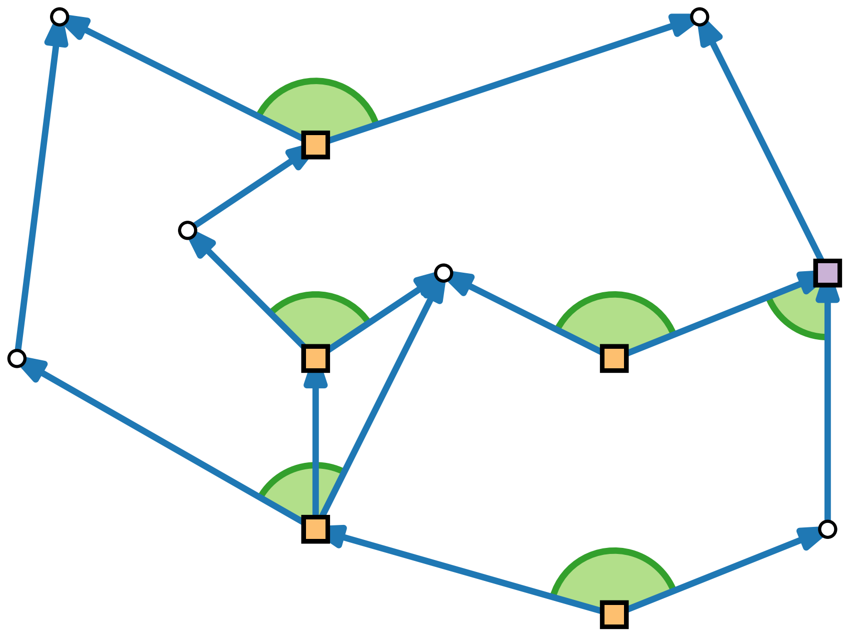

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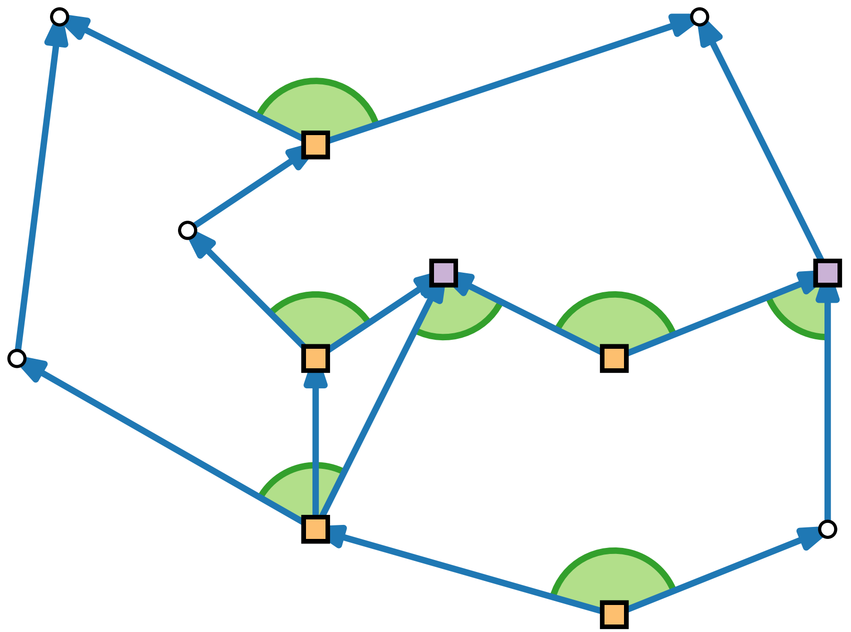

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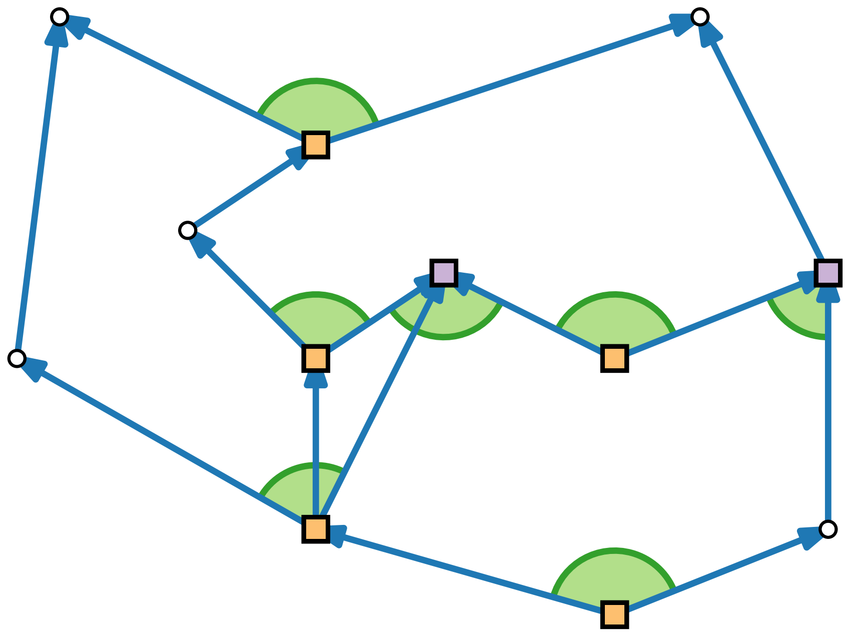

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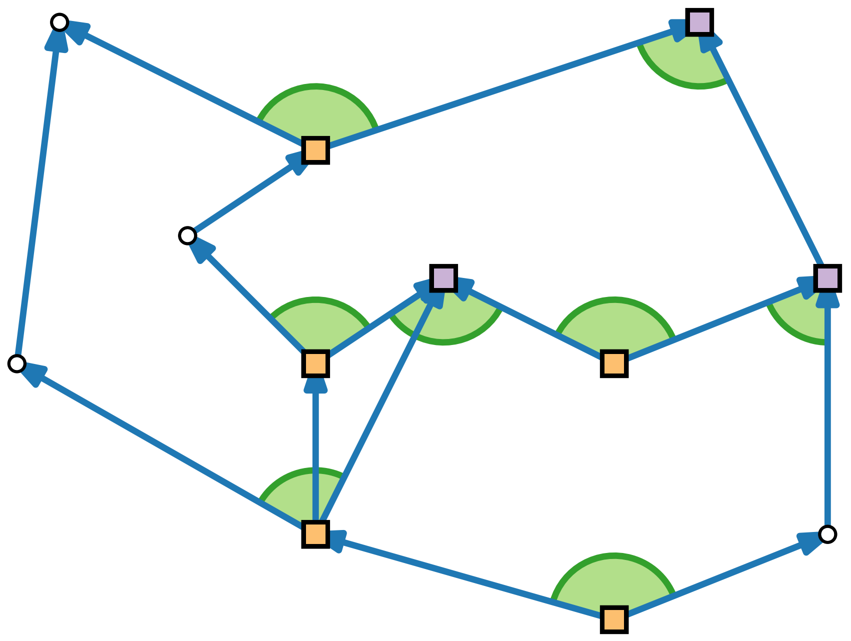

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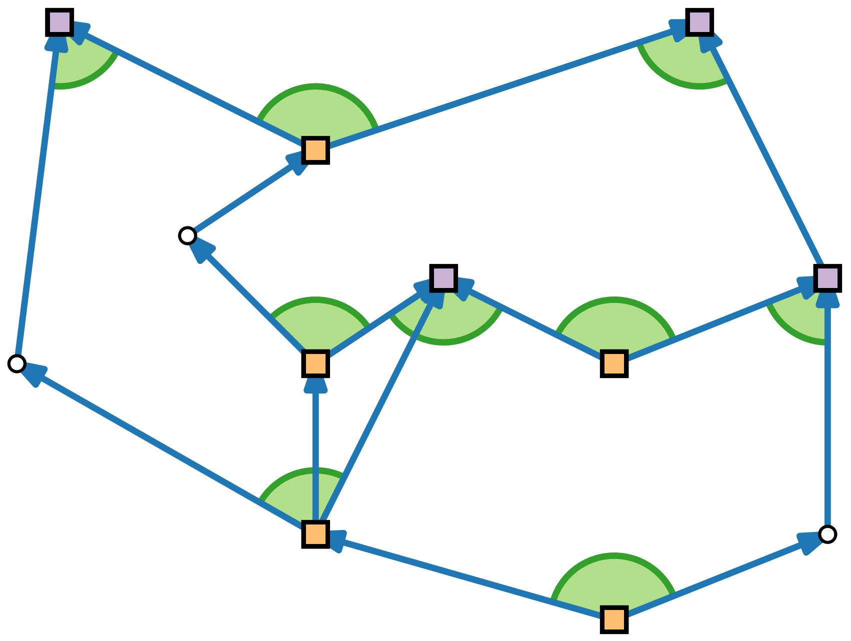

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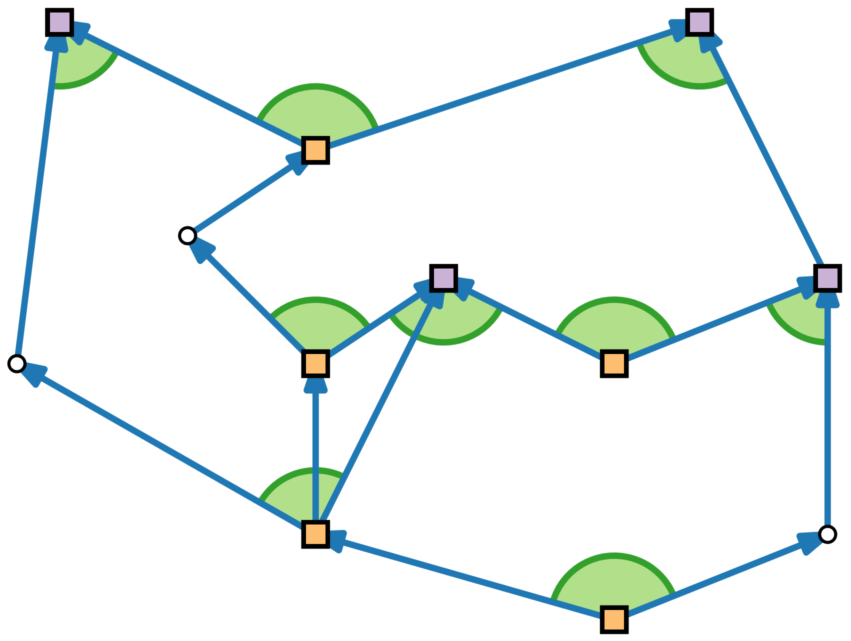

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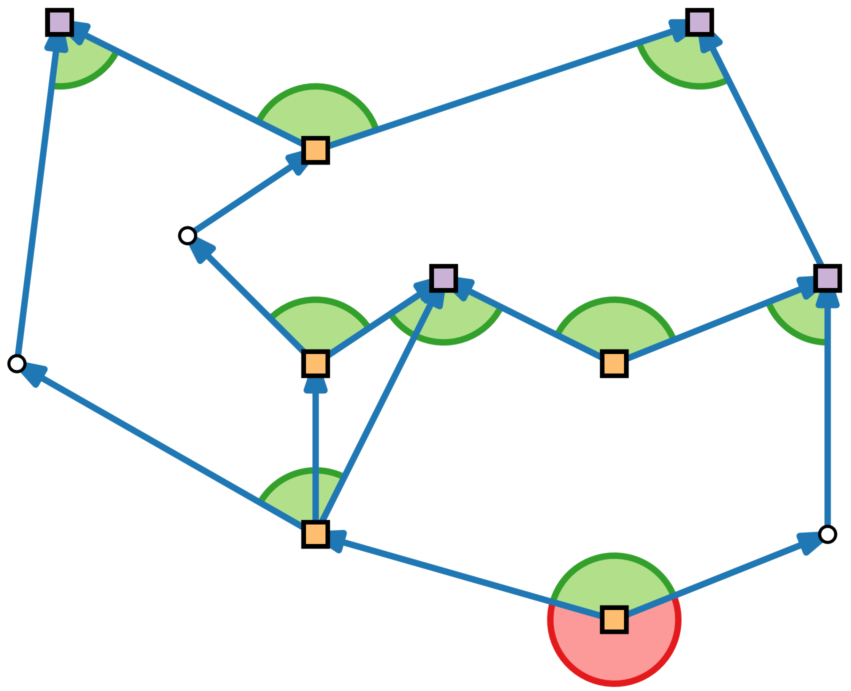

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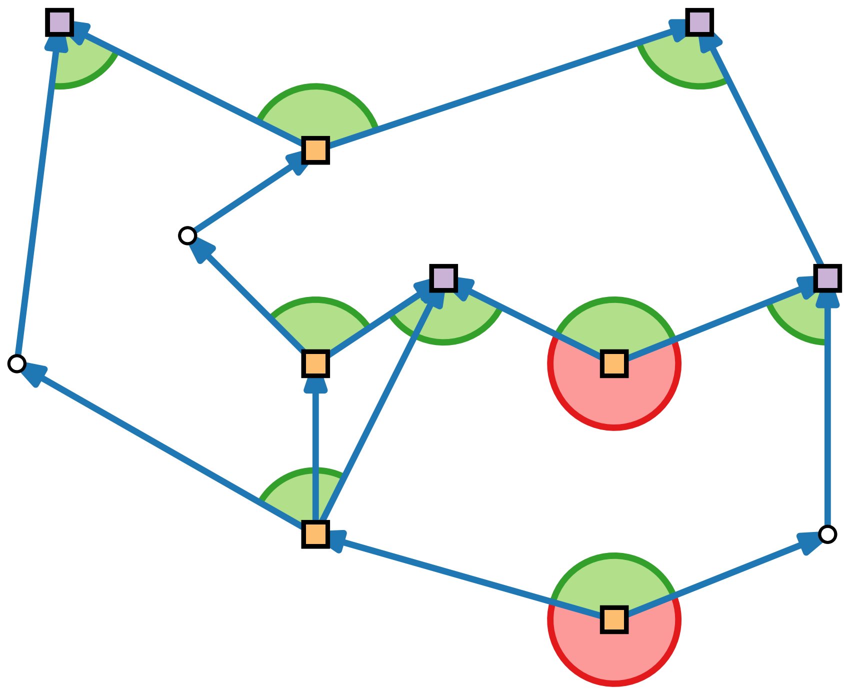

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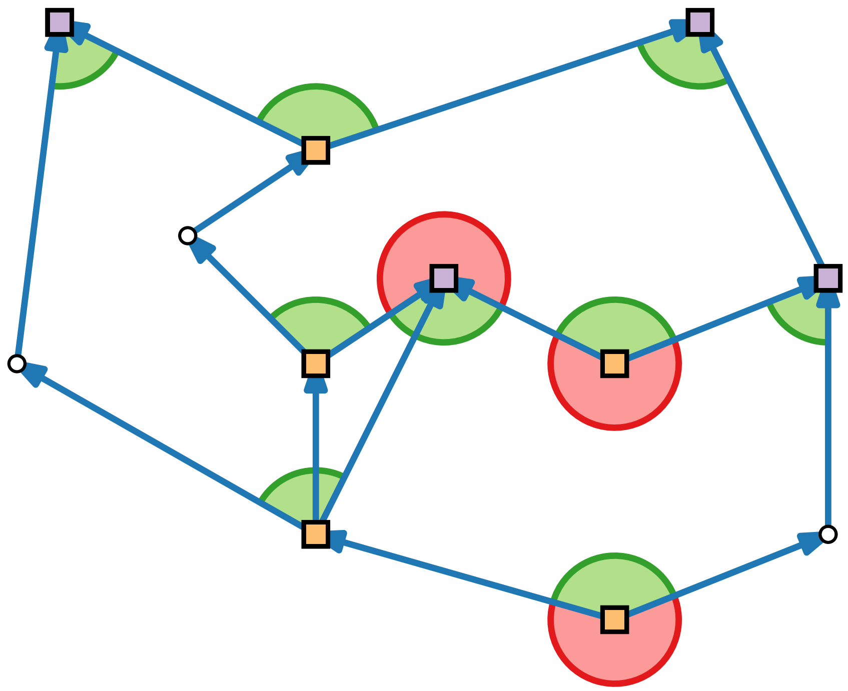

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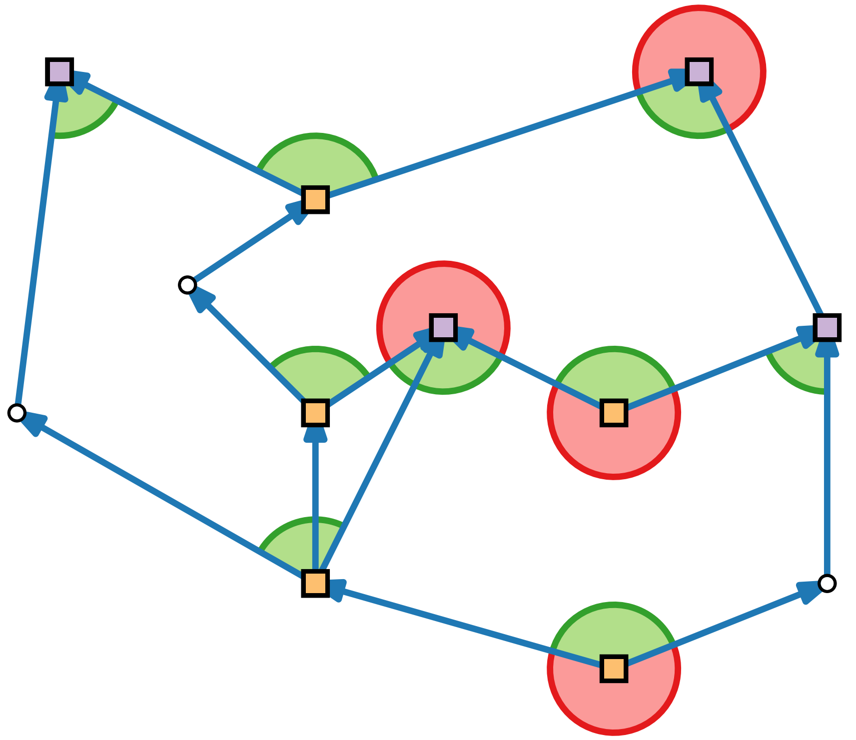

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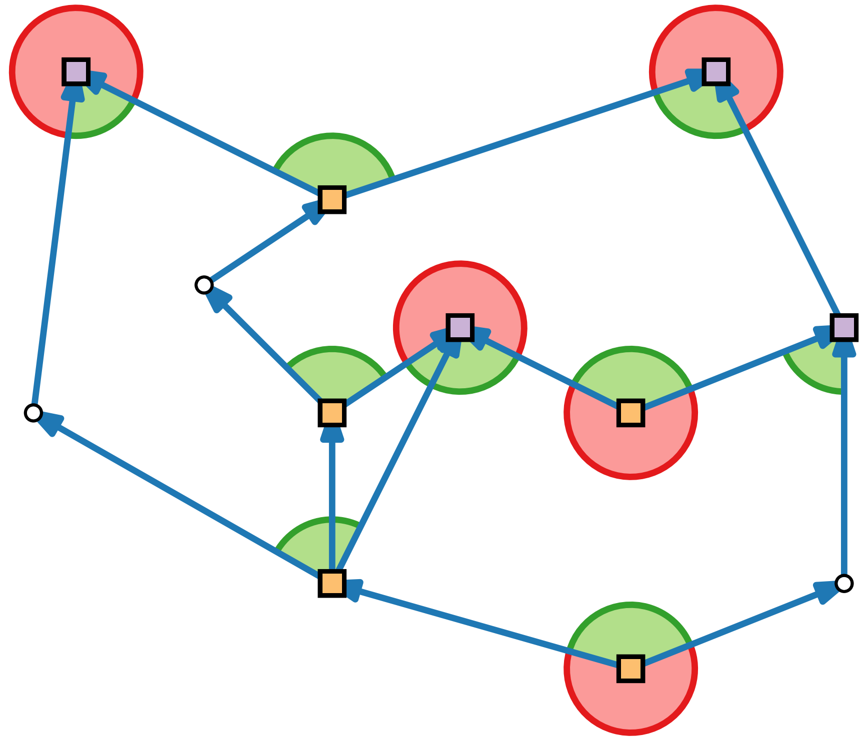

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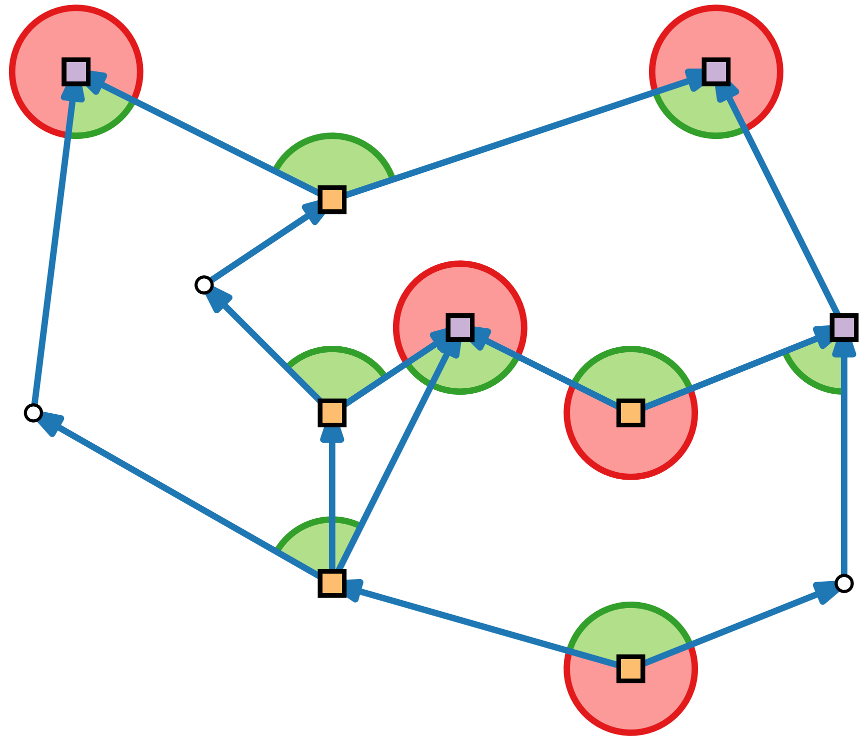

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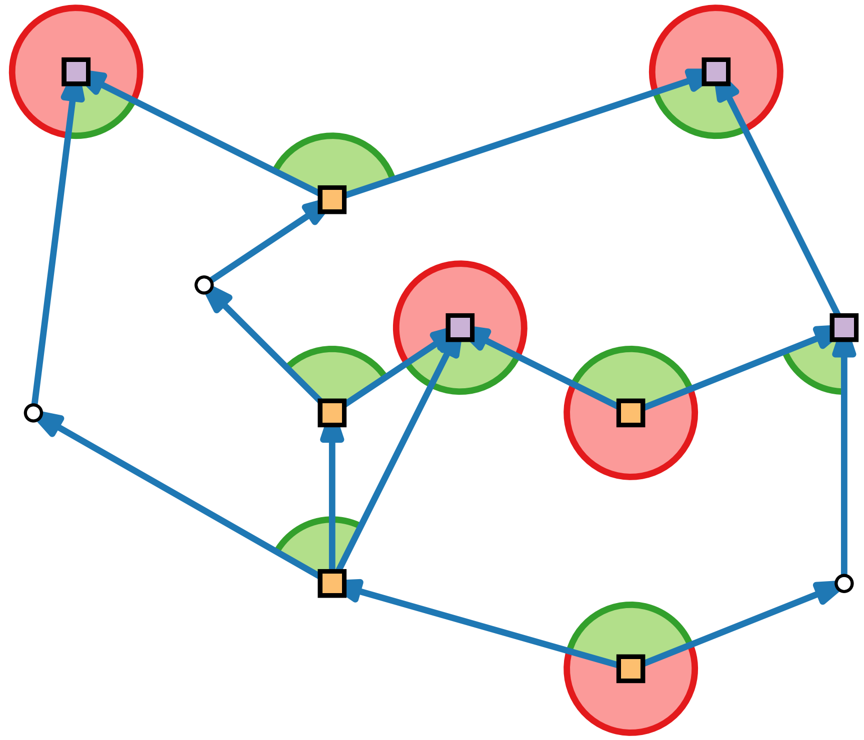

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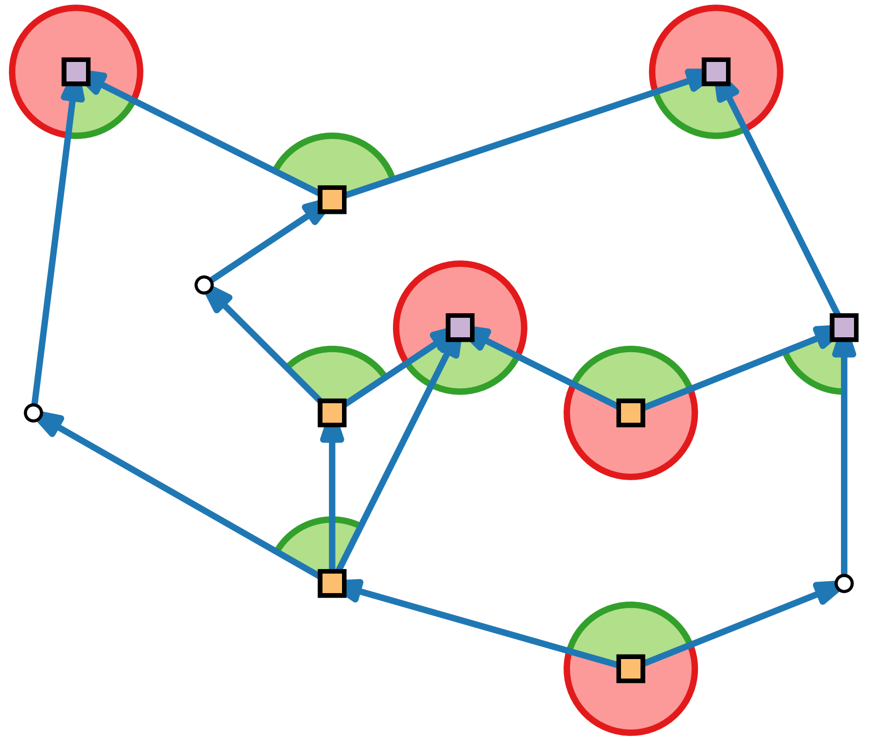

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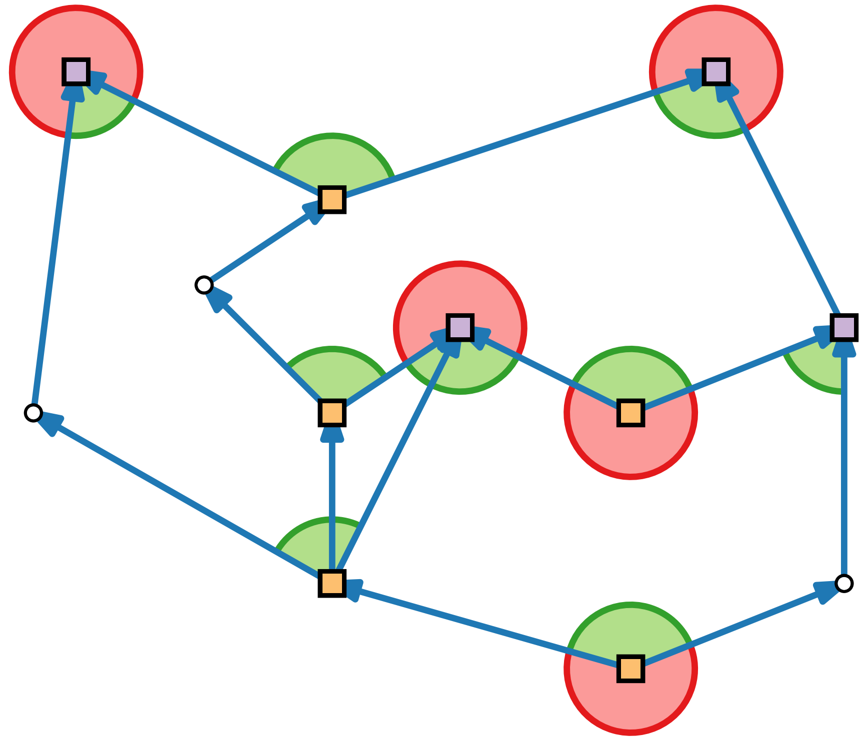

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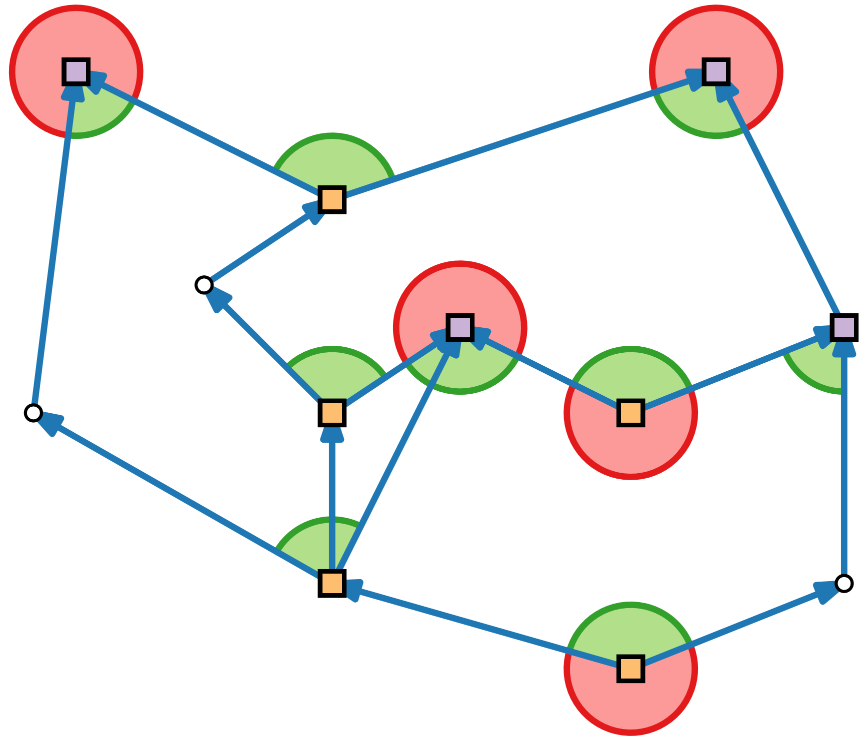

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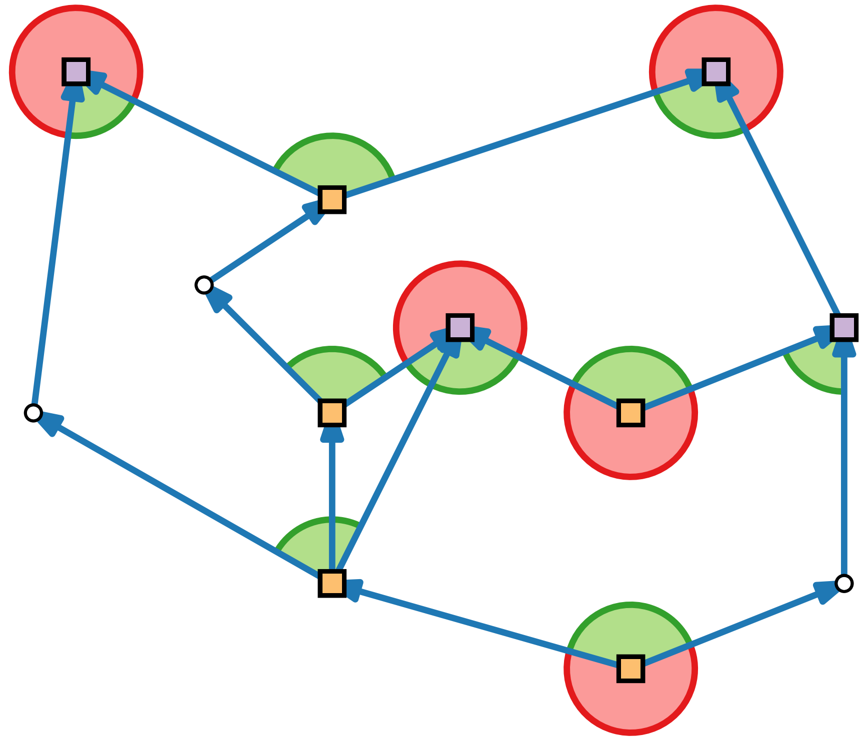

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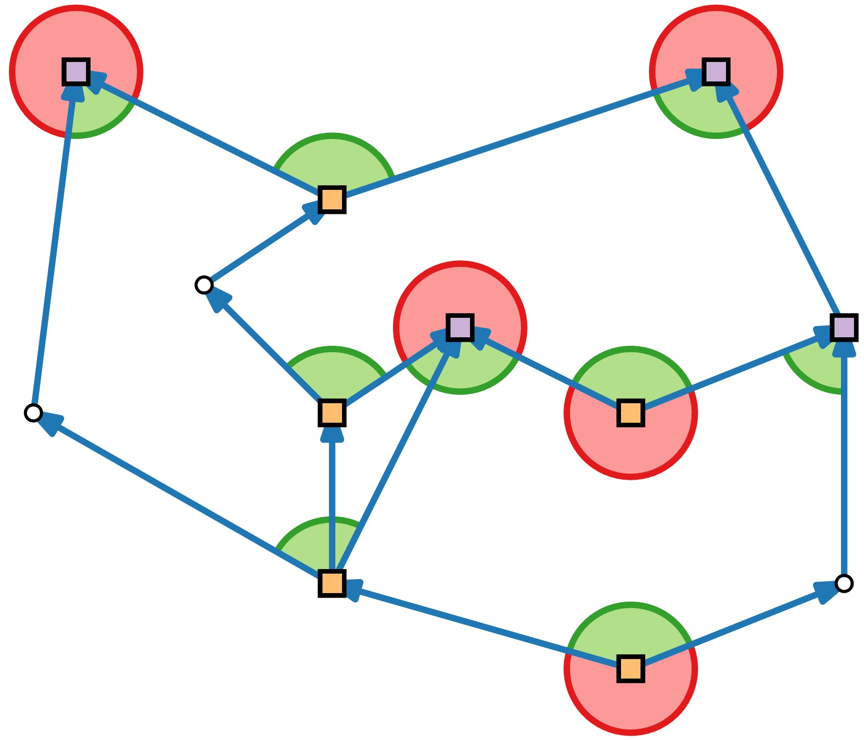

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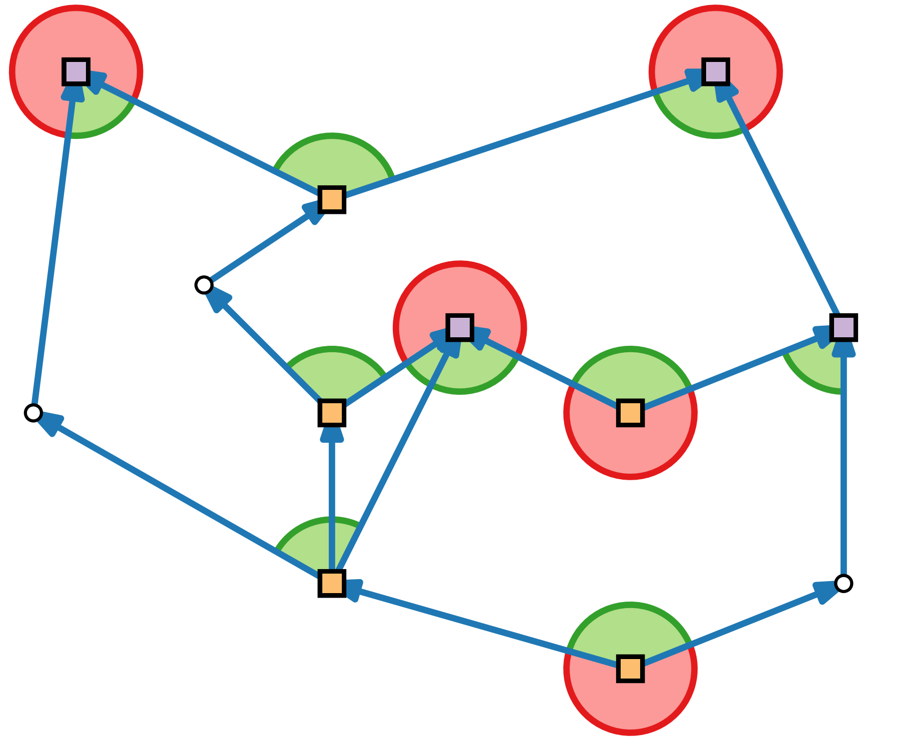

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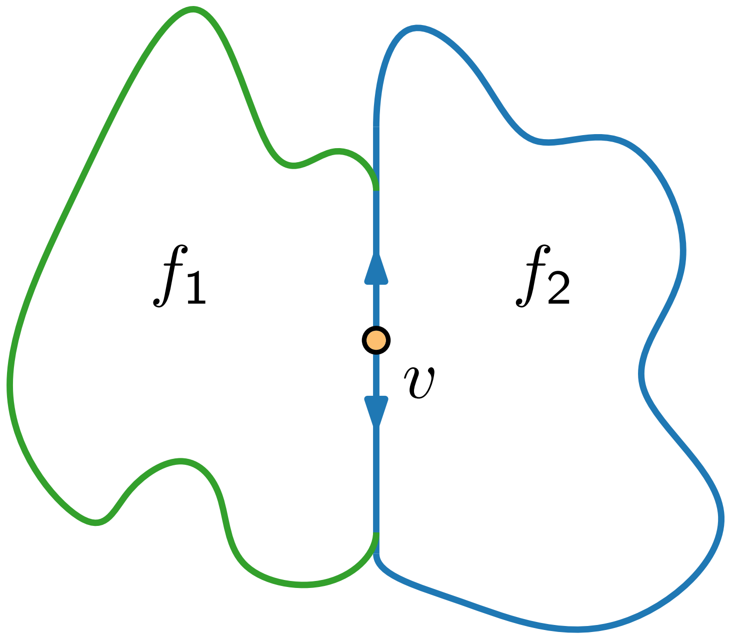

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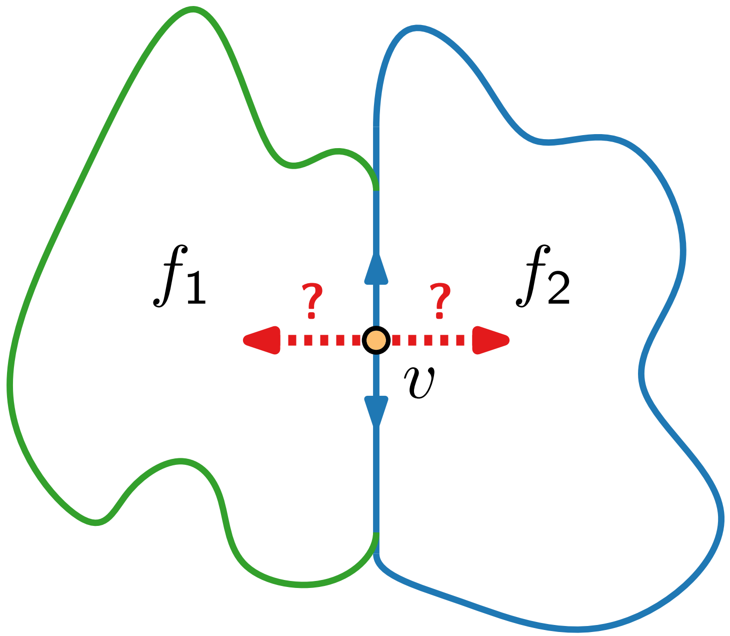

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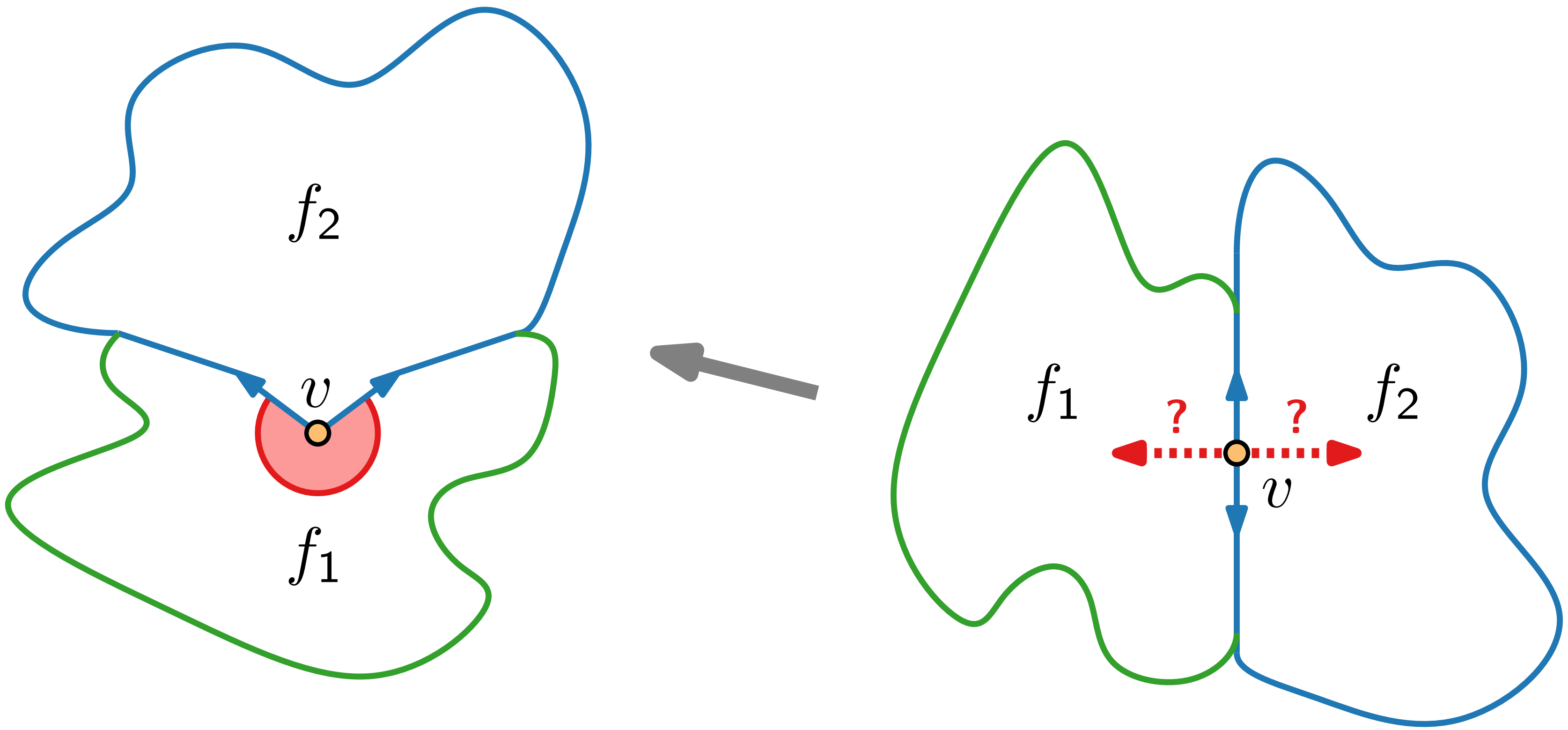

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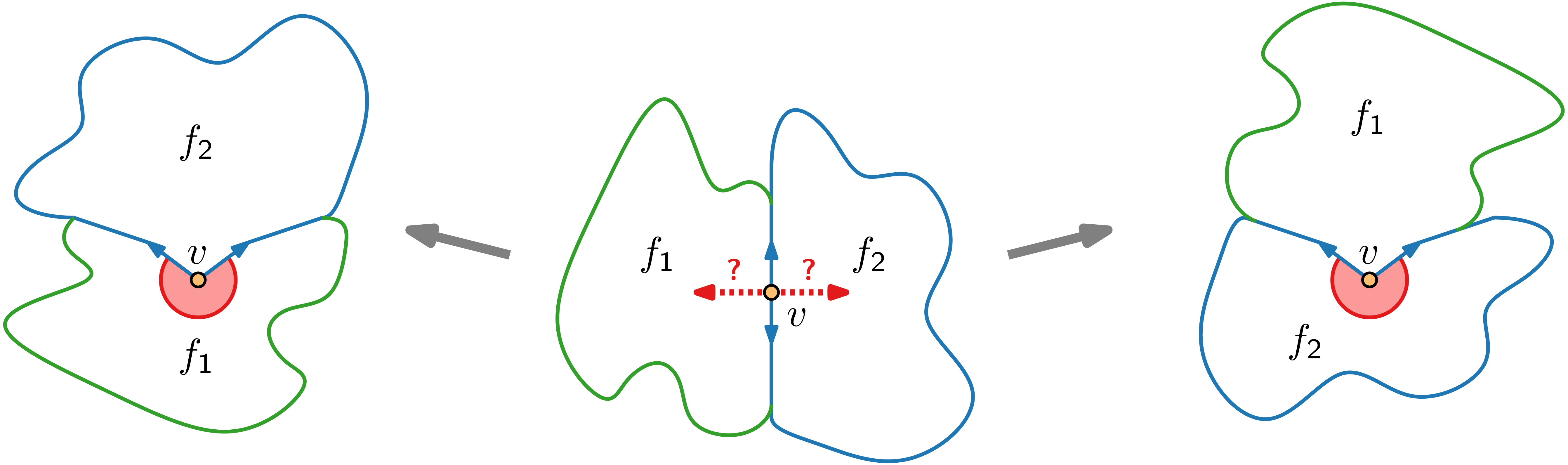

Angle Relations

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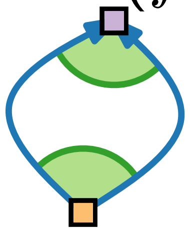

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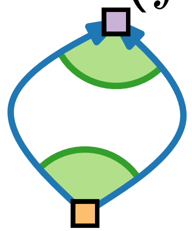
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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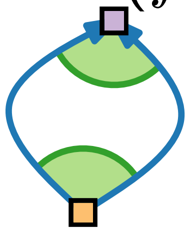
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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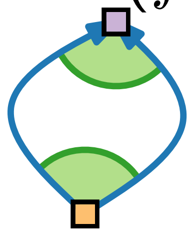
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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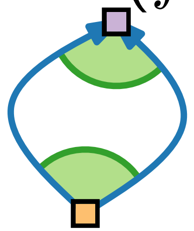
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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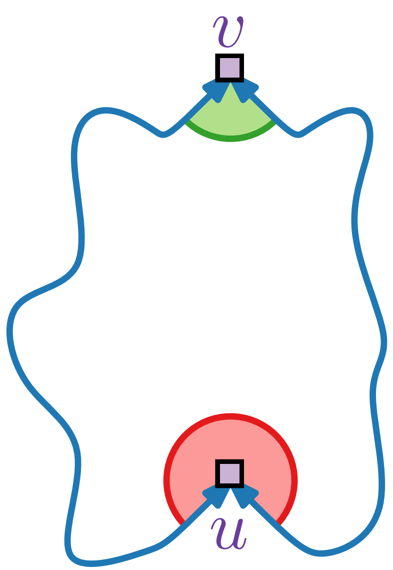

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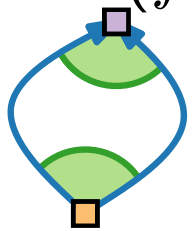
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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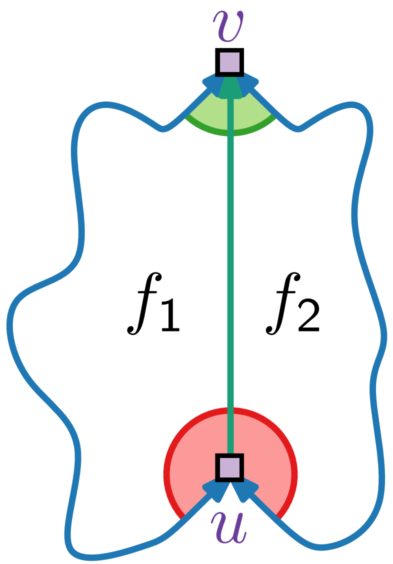

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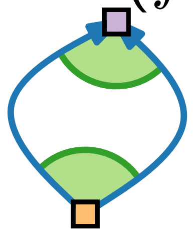
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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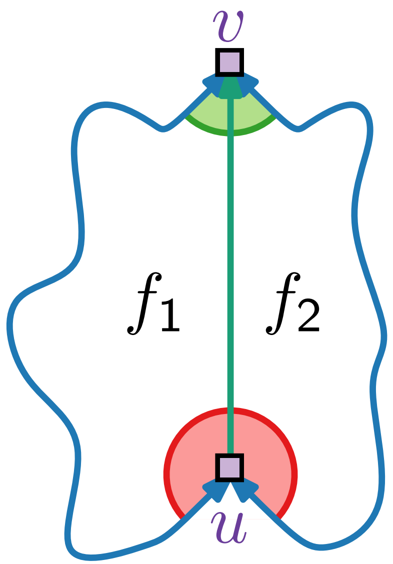

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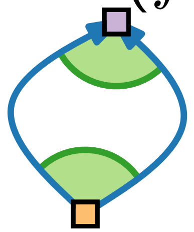
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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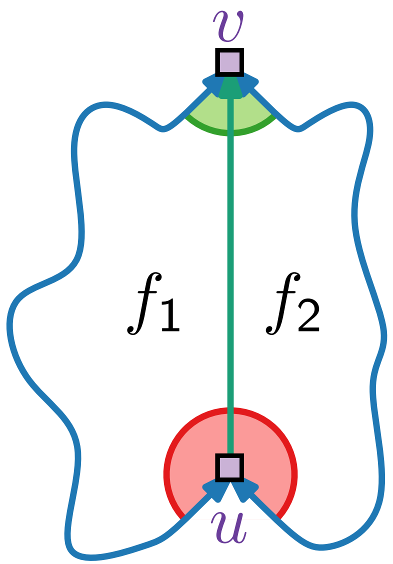

$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

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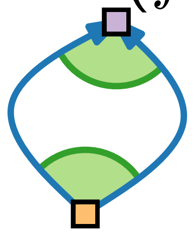
$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$$

Proof by induction on $L(f)$.

- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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Split f with edge from a large angle at a "low" sink u to...

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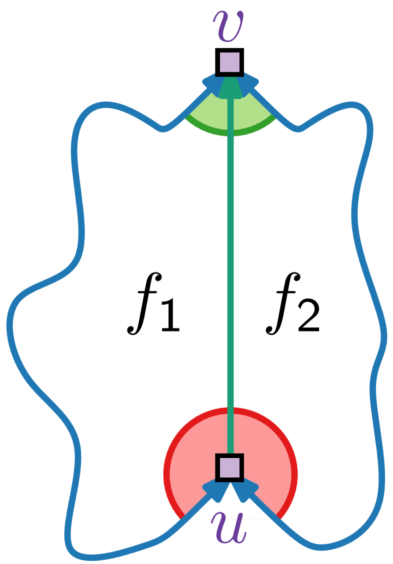

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 \\ - (S(f_1) + S(f_2) - 1)$$

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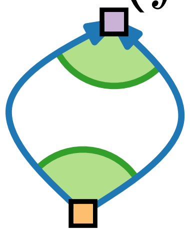
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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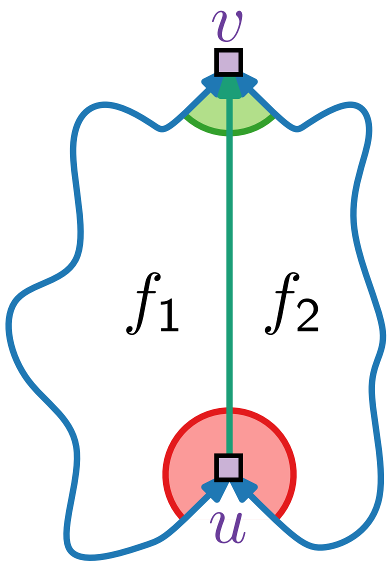

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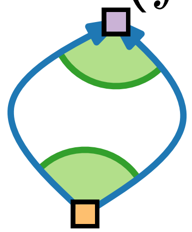
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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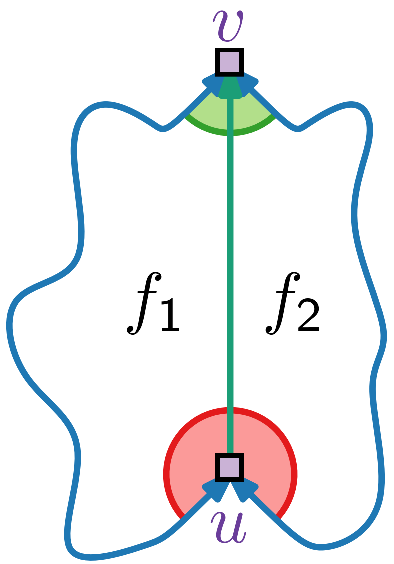

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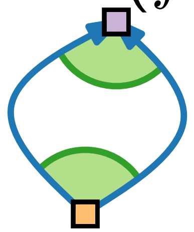
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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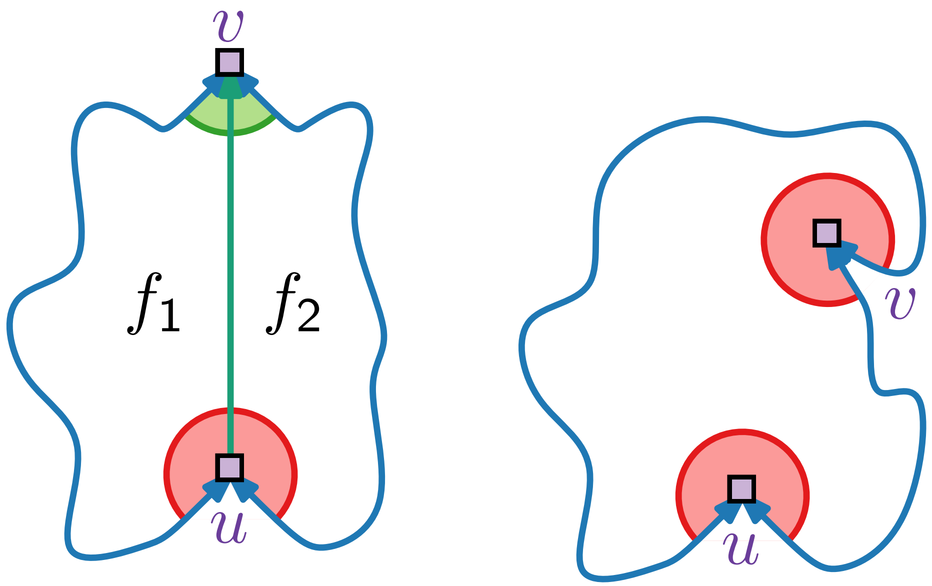

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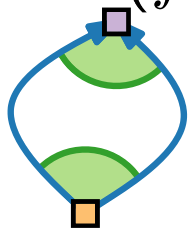

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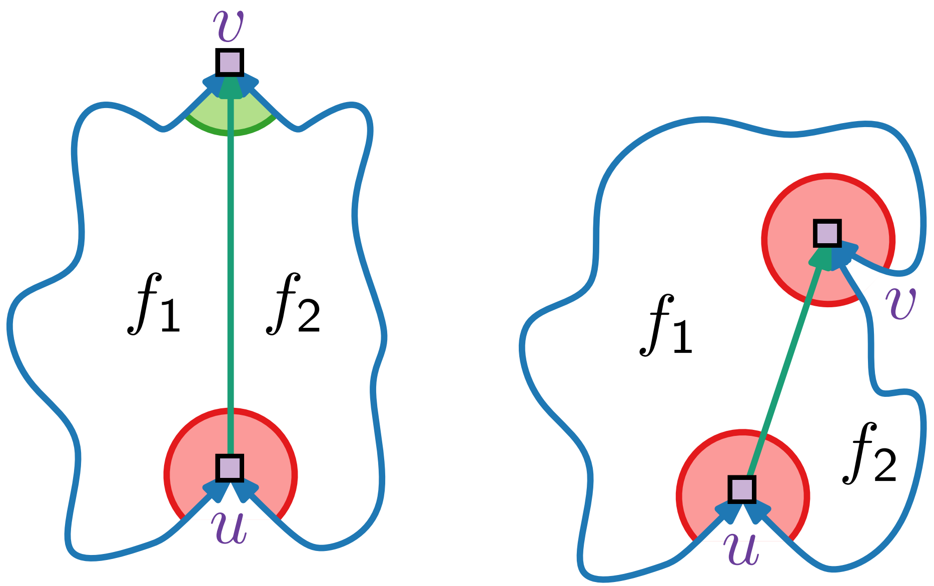

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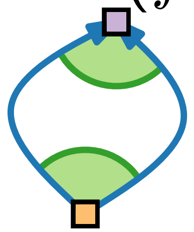
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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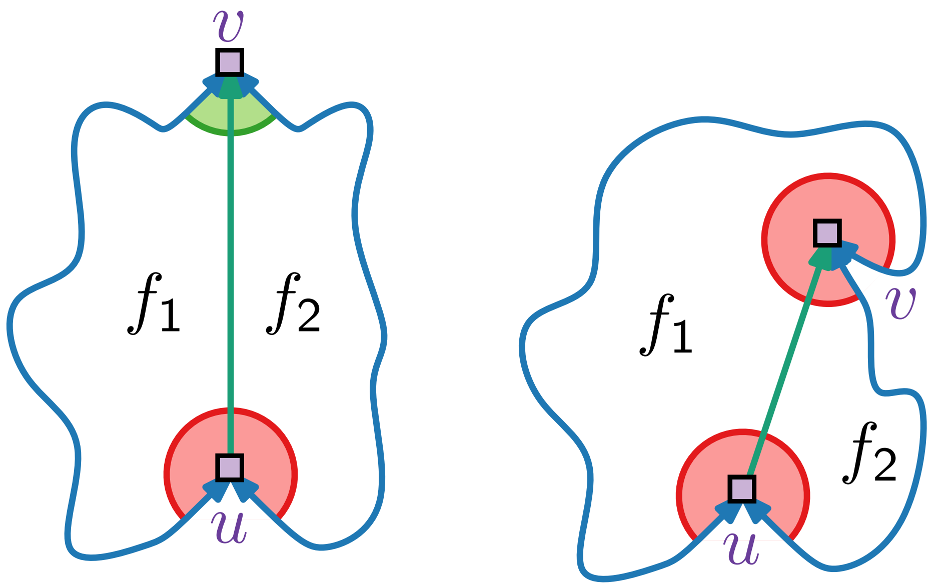

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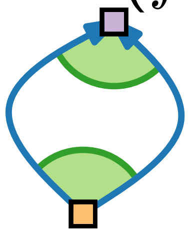
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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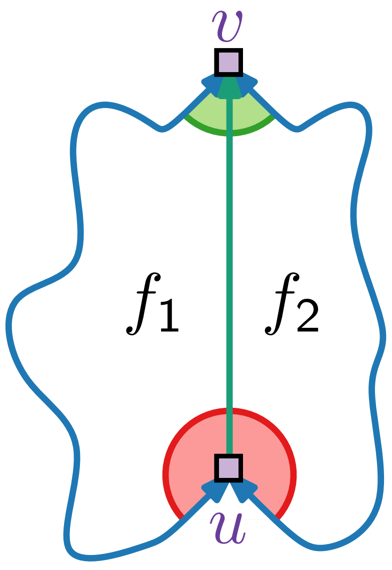

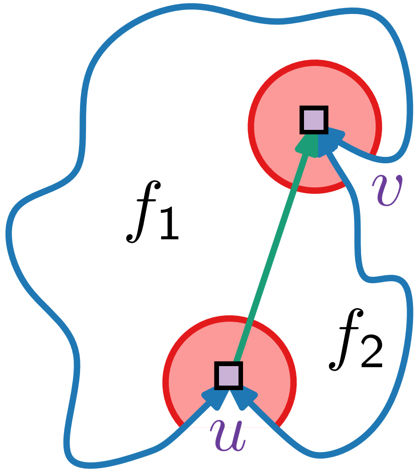

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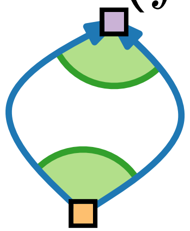
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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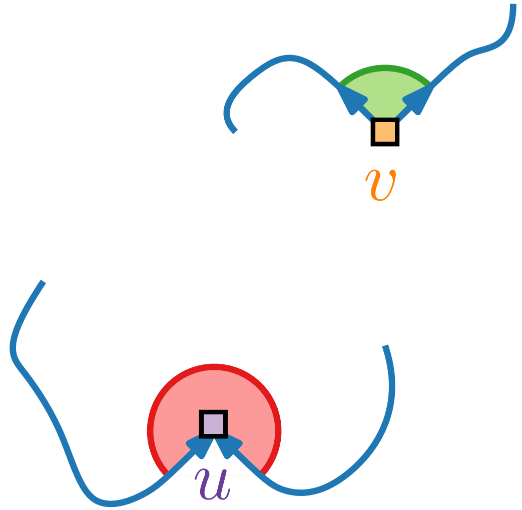

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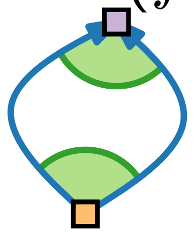
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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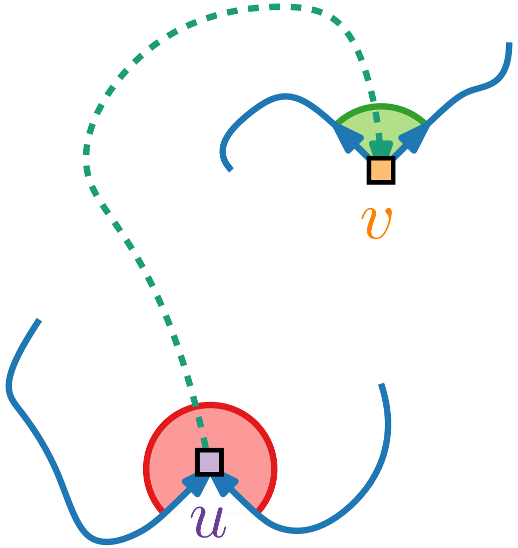

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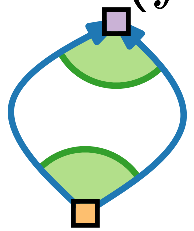
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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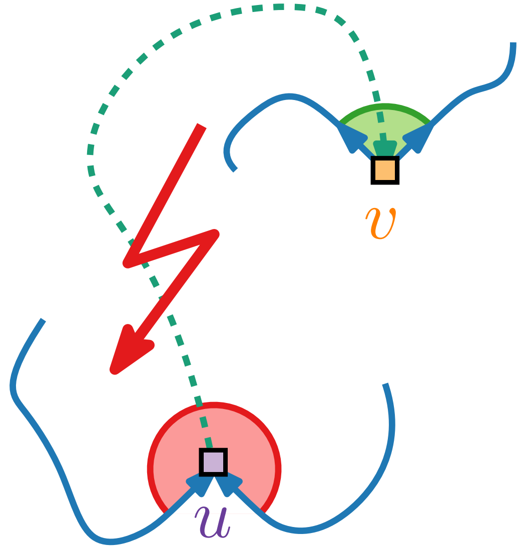

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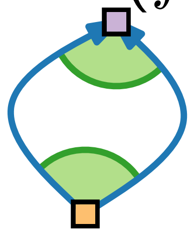
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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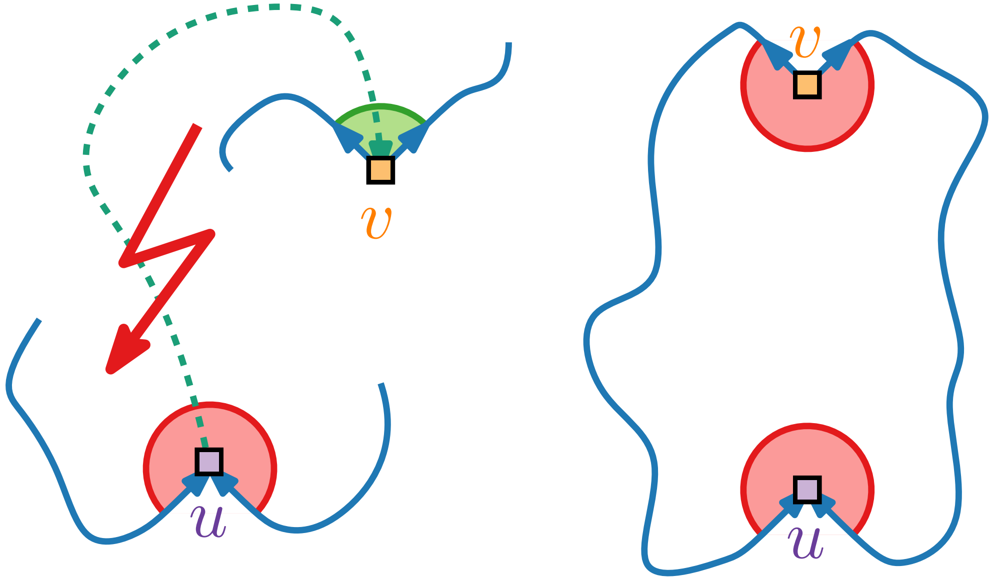

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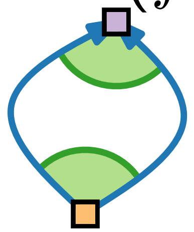
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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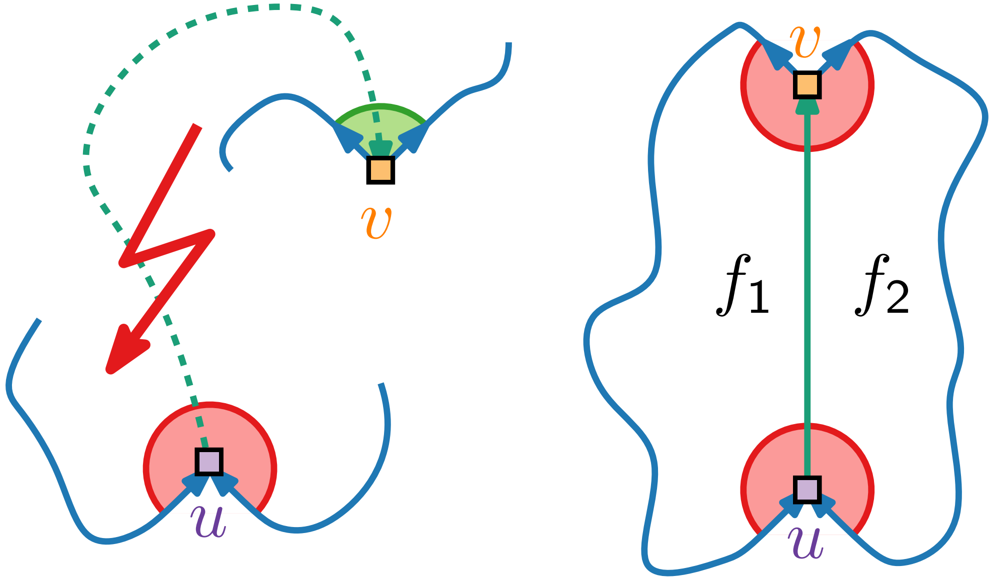

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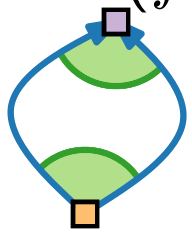
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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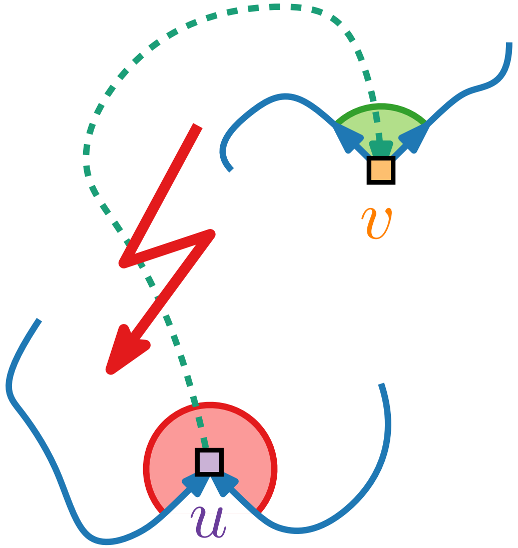

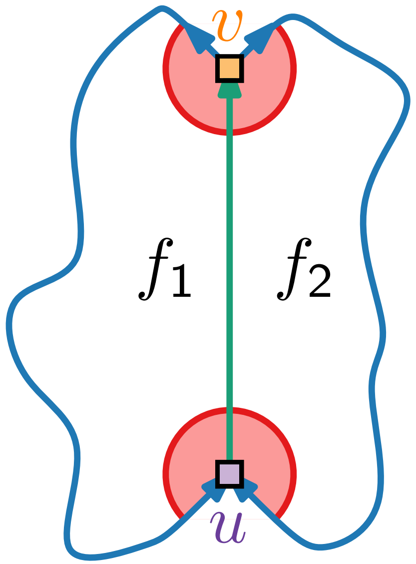

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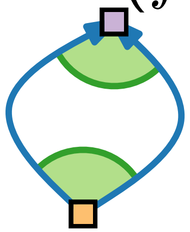

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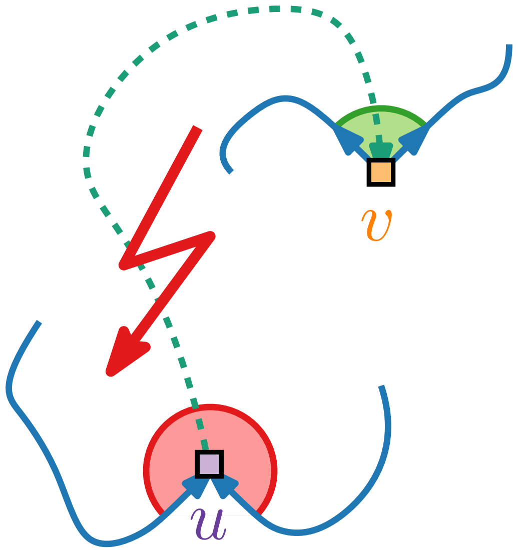

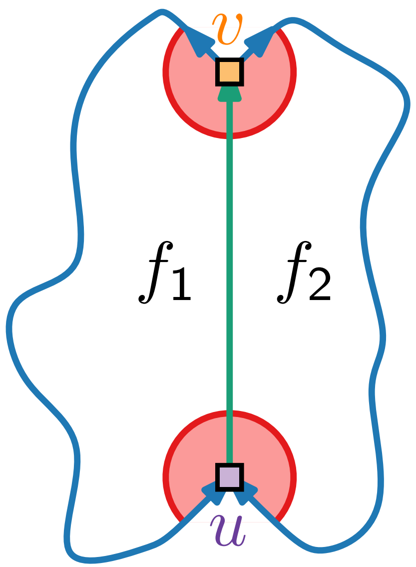

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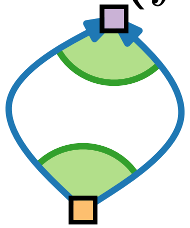

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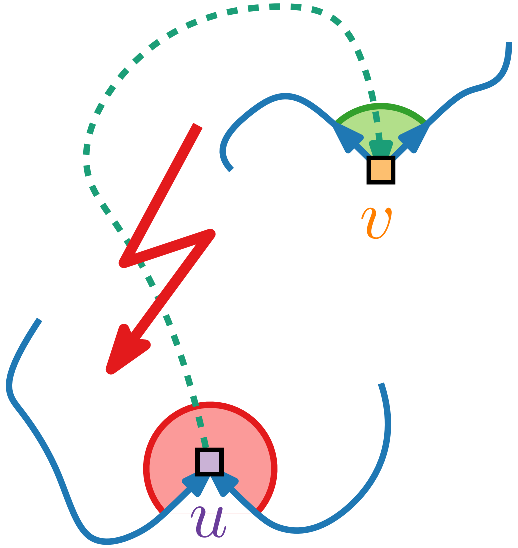

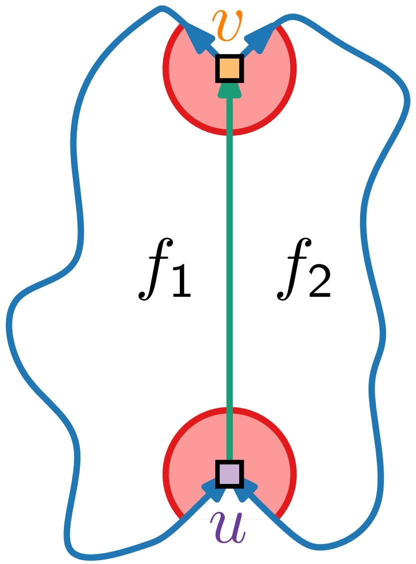

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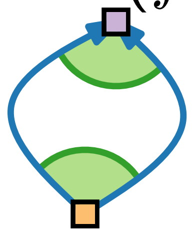

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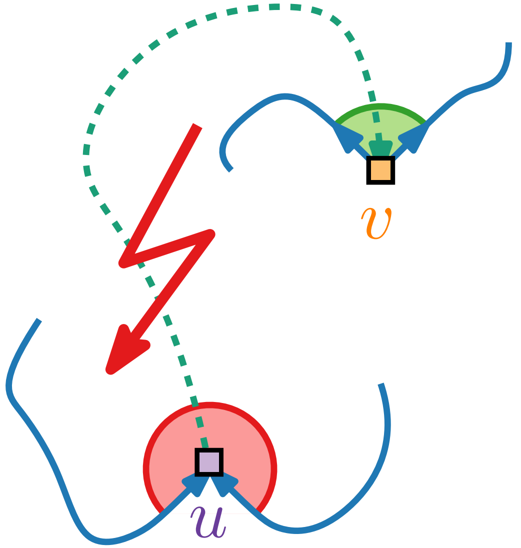

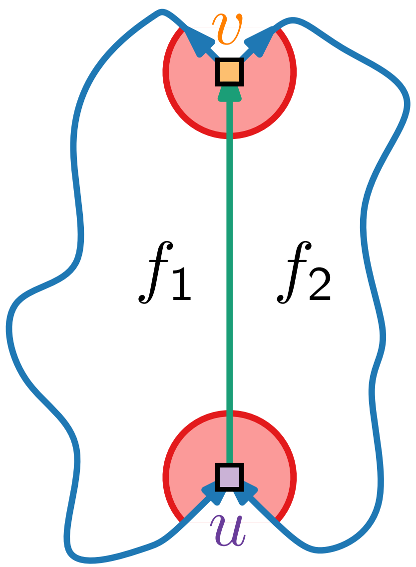

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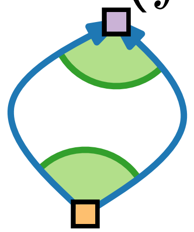
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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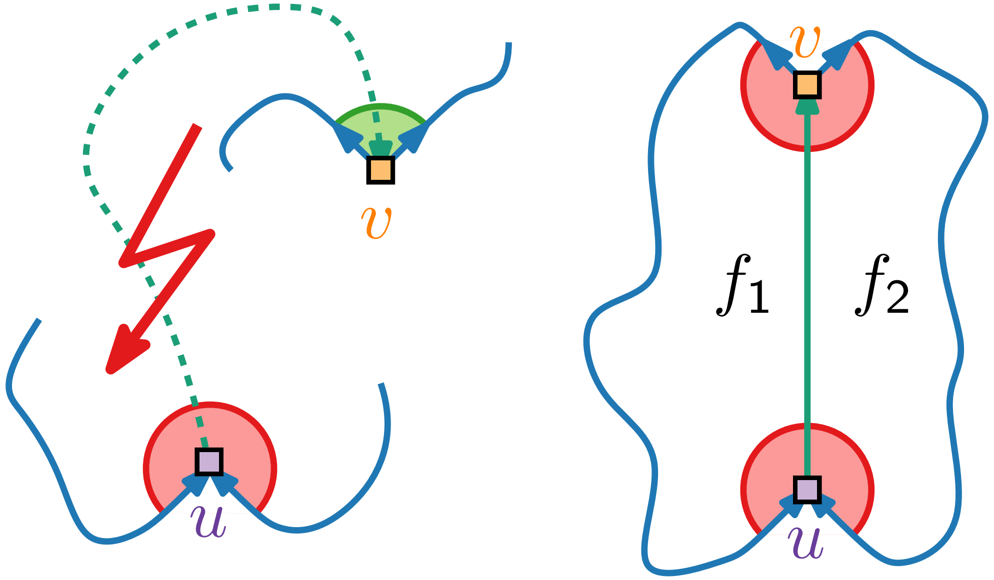

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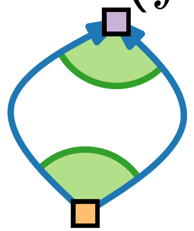
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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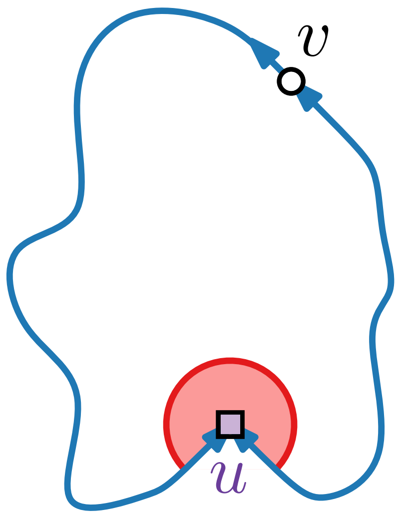

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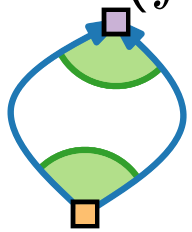
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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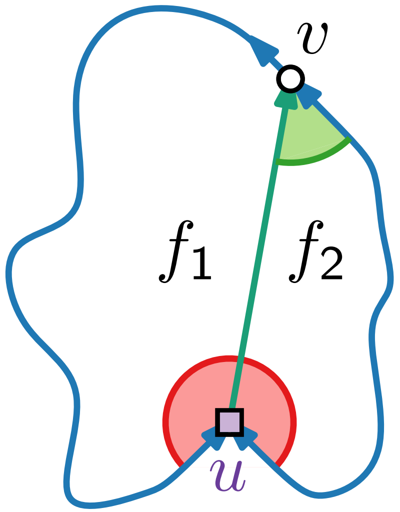

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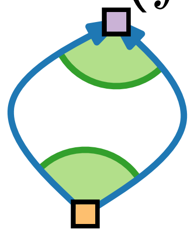

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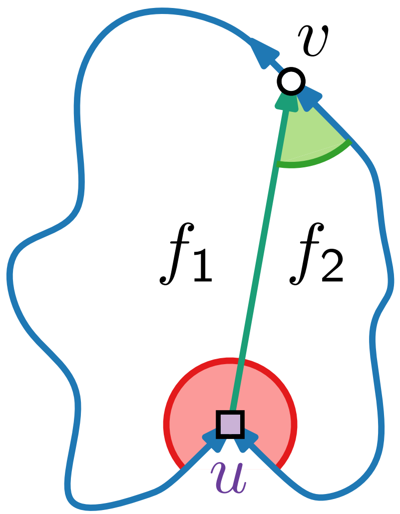

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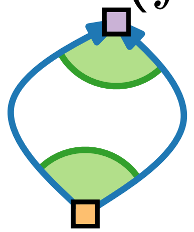
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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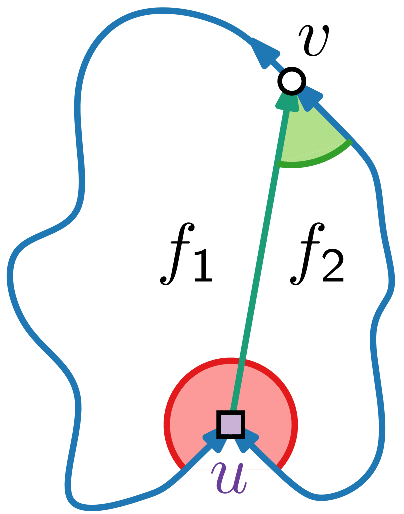

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Angle Relations

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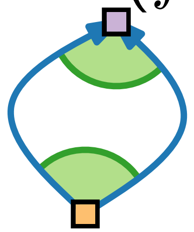
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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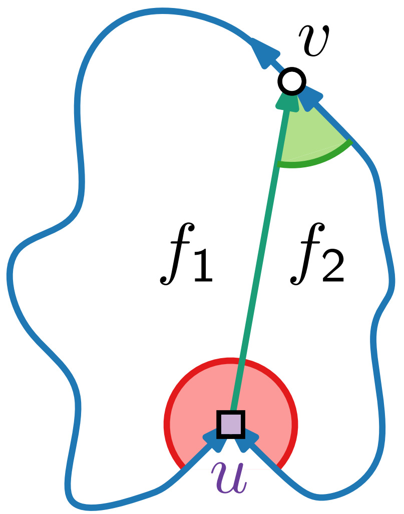

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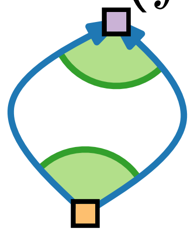
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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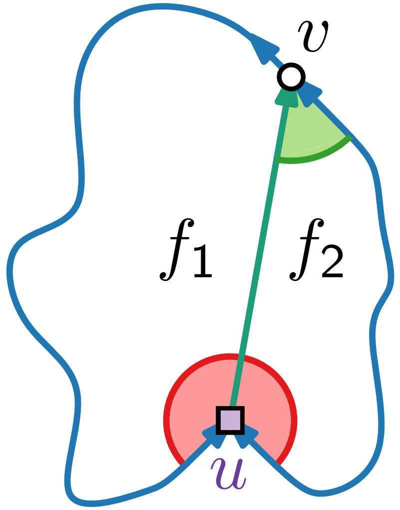

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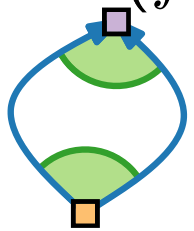
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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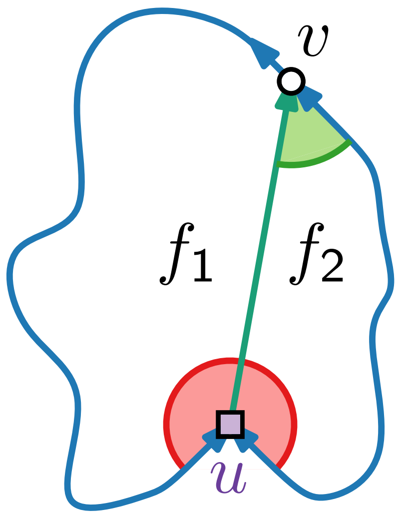

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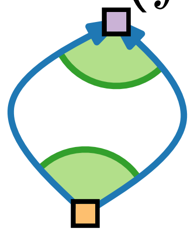
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

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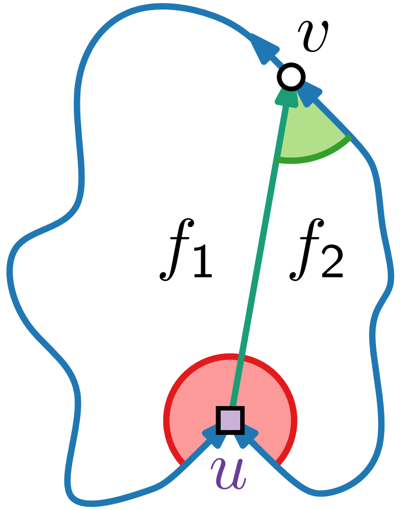

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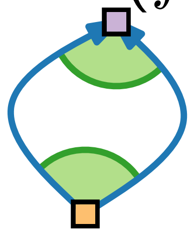

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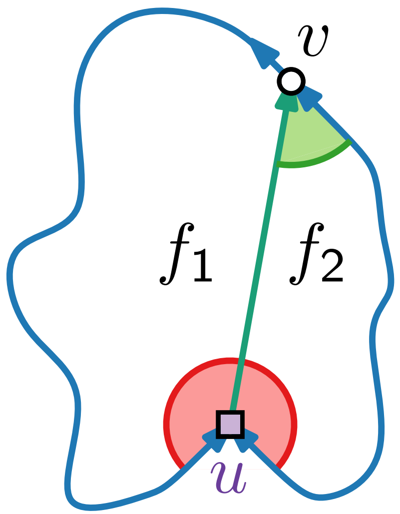

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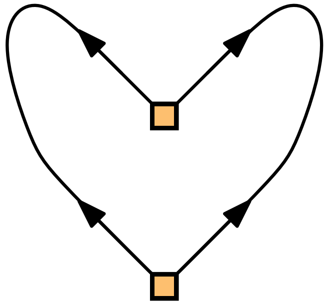

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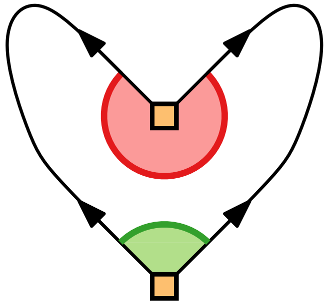

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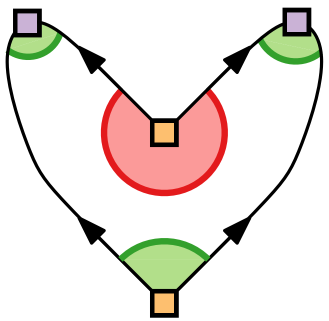

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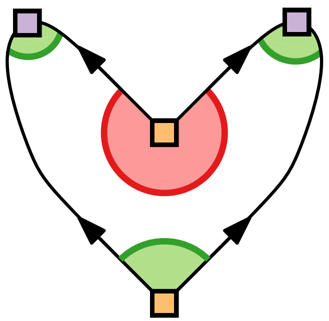

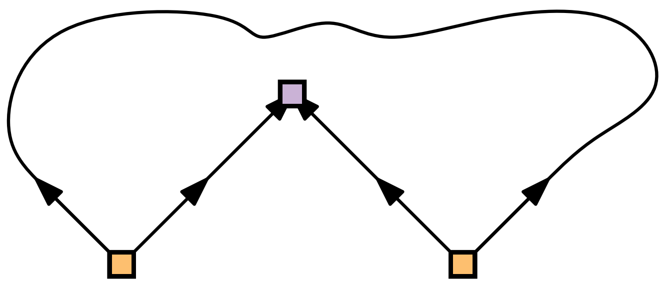

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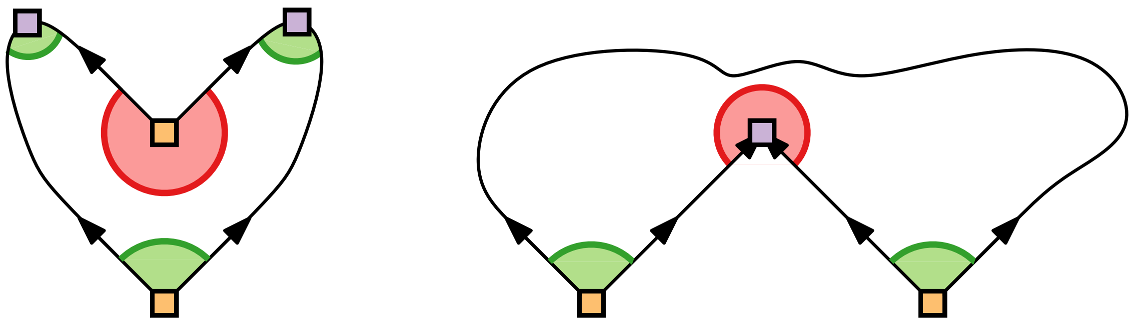

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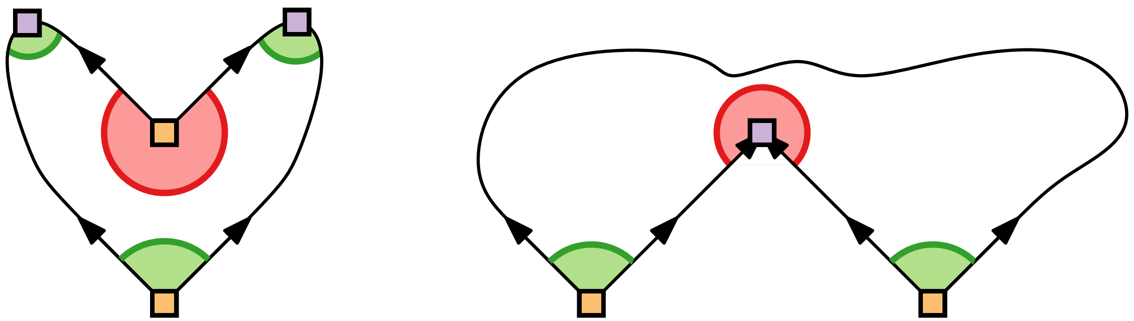

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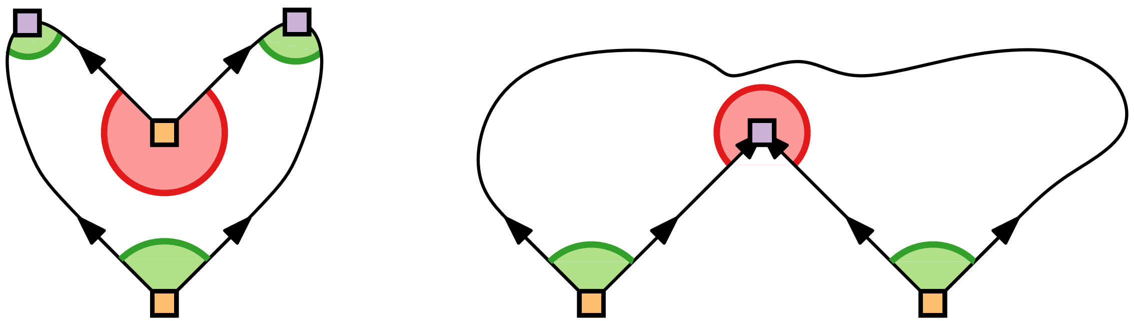

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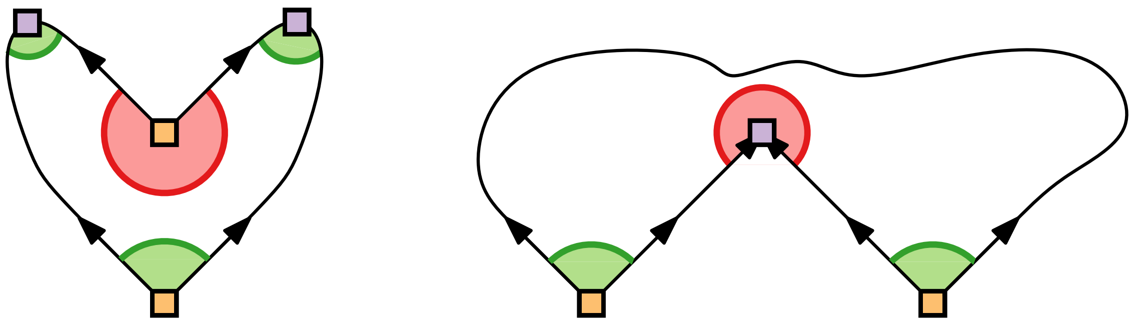

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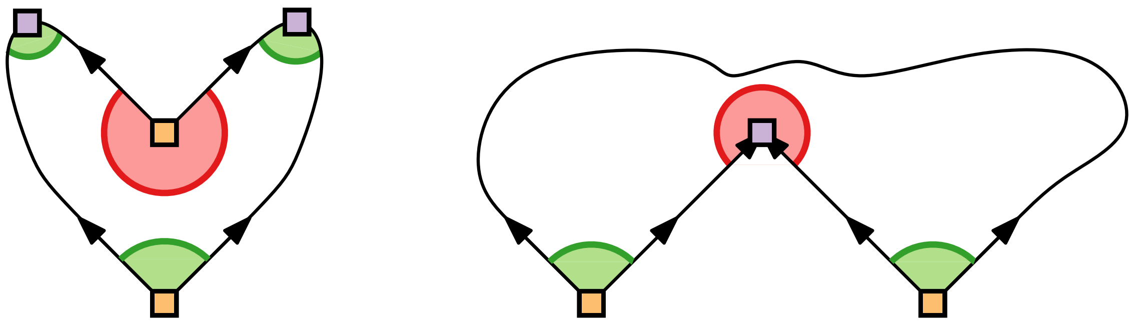

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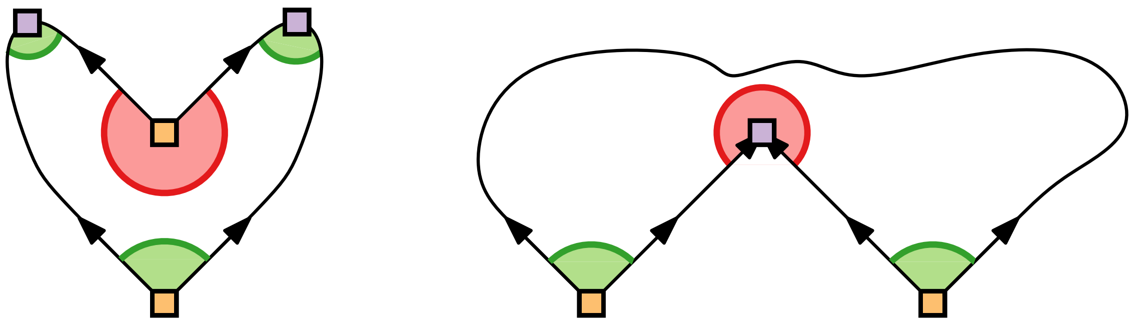

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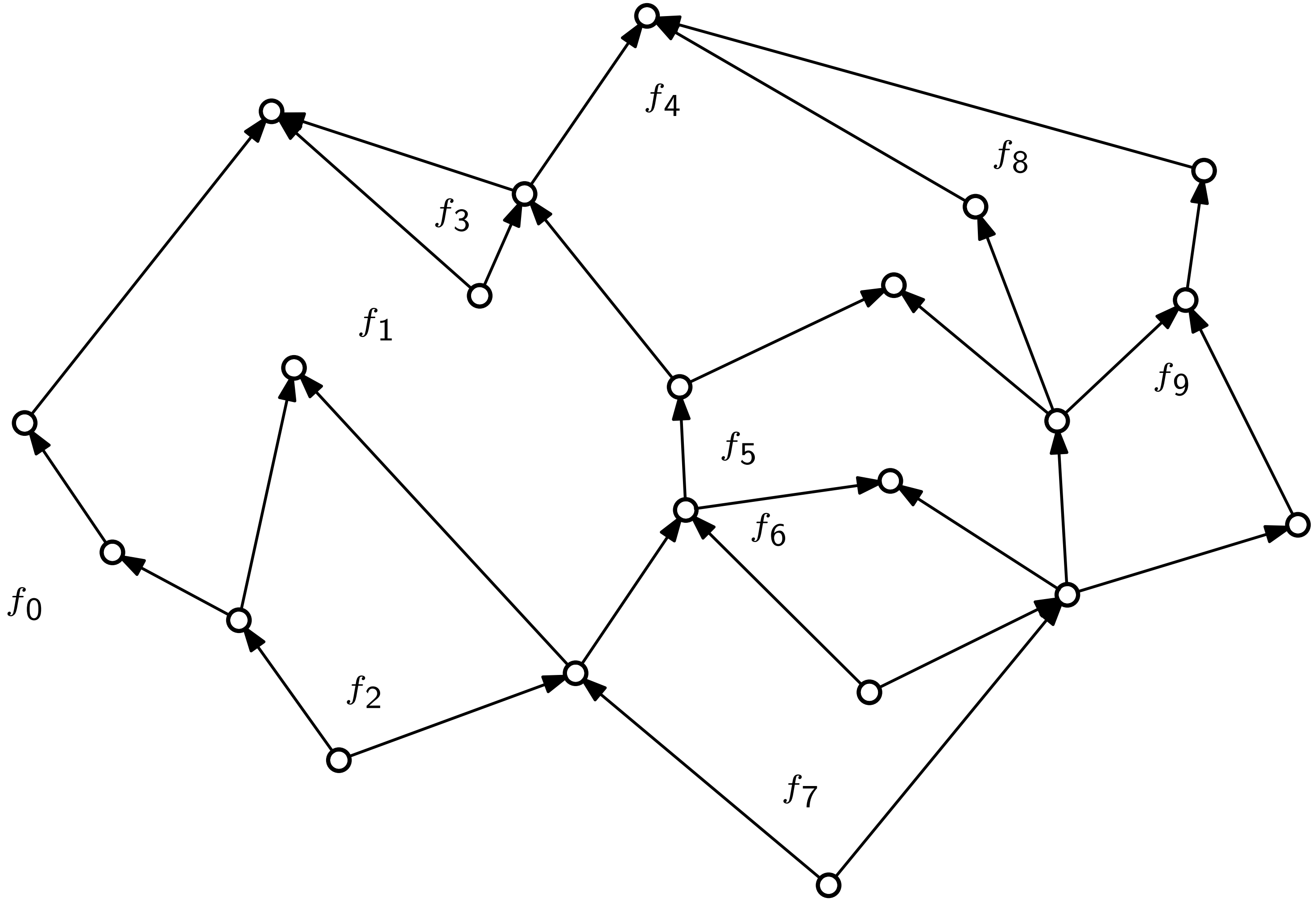

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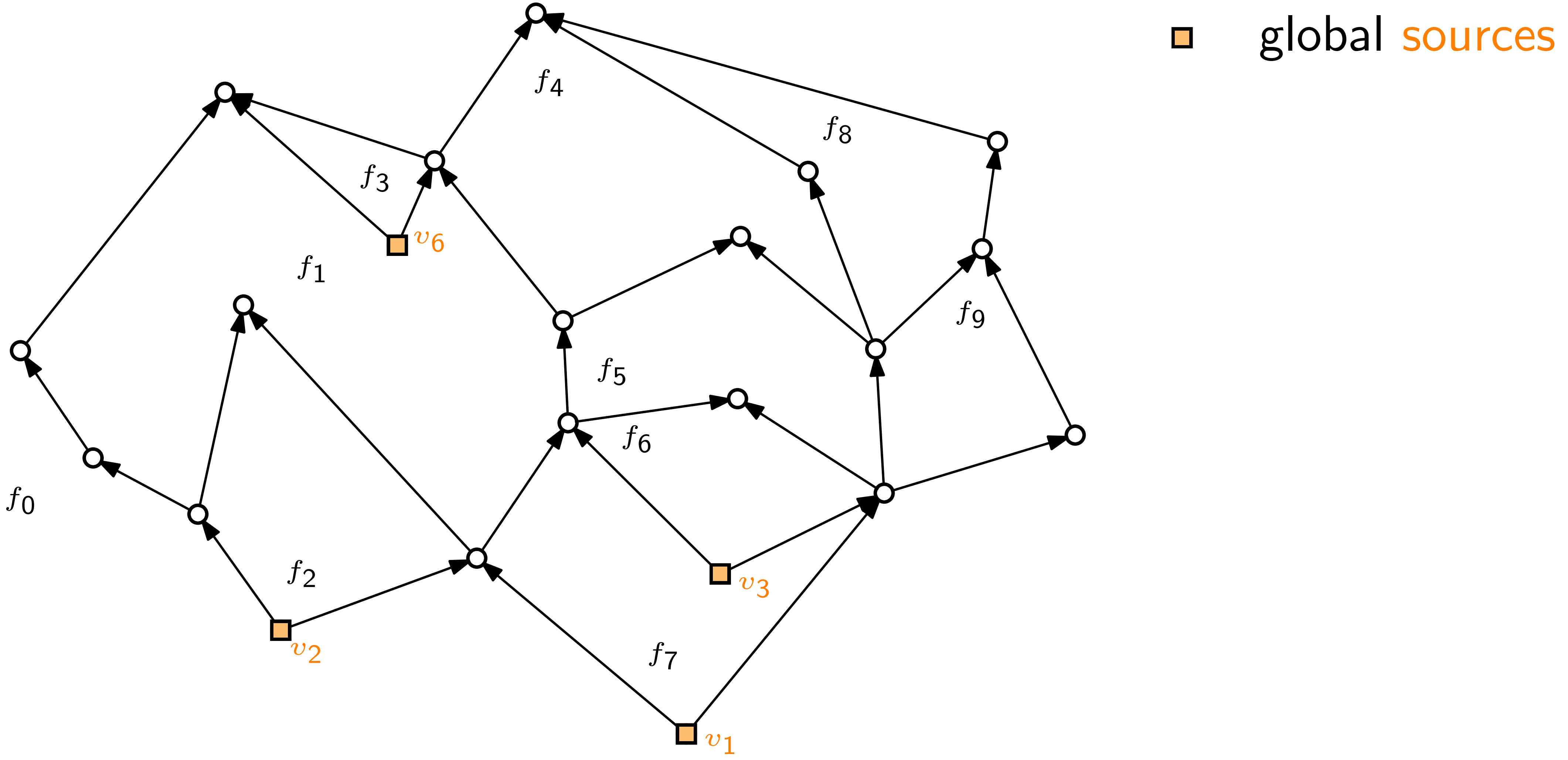

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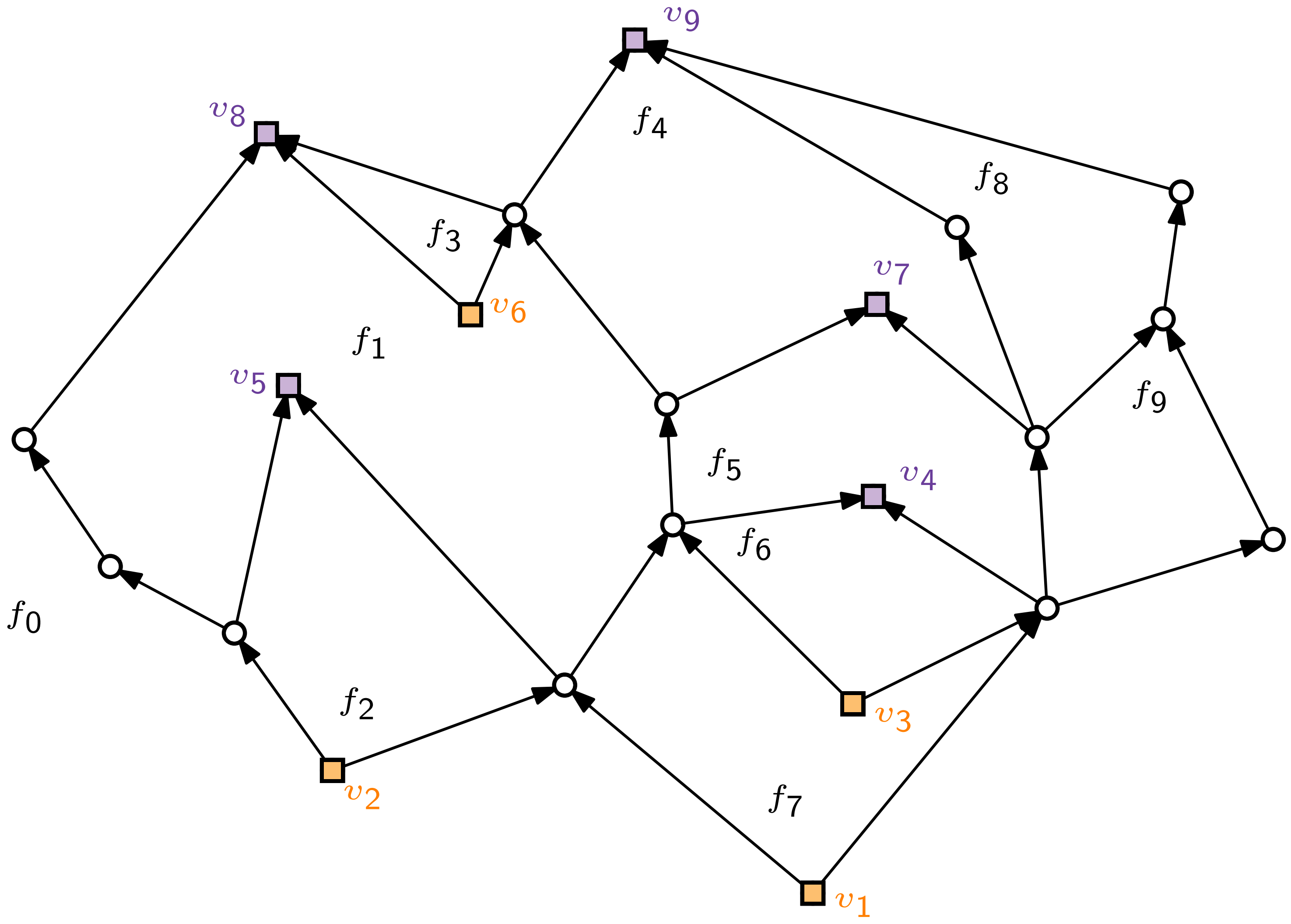

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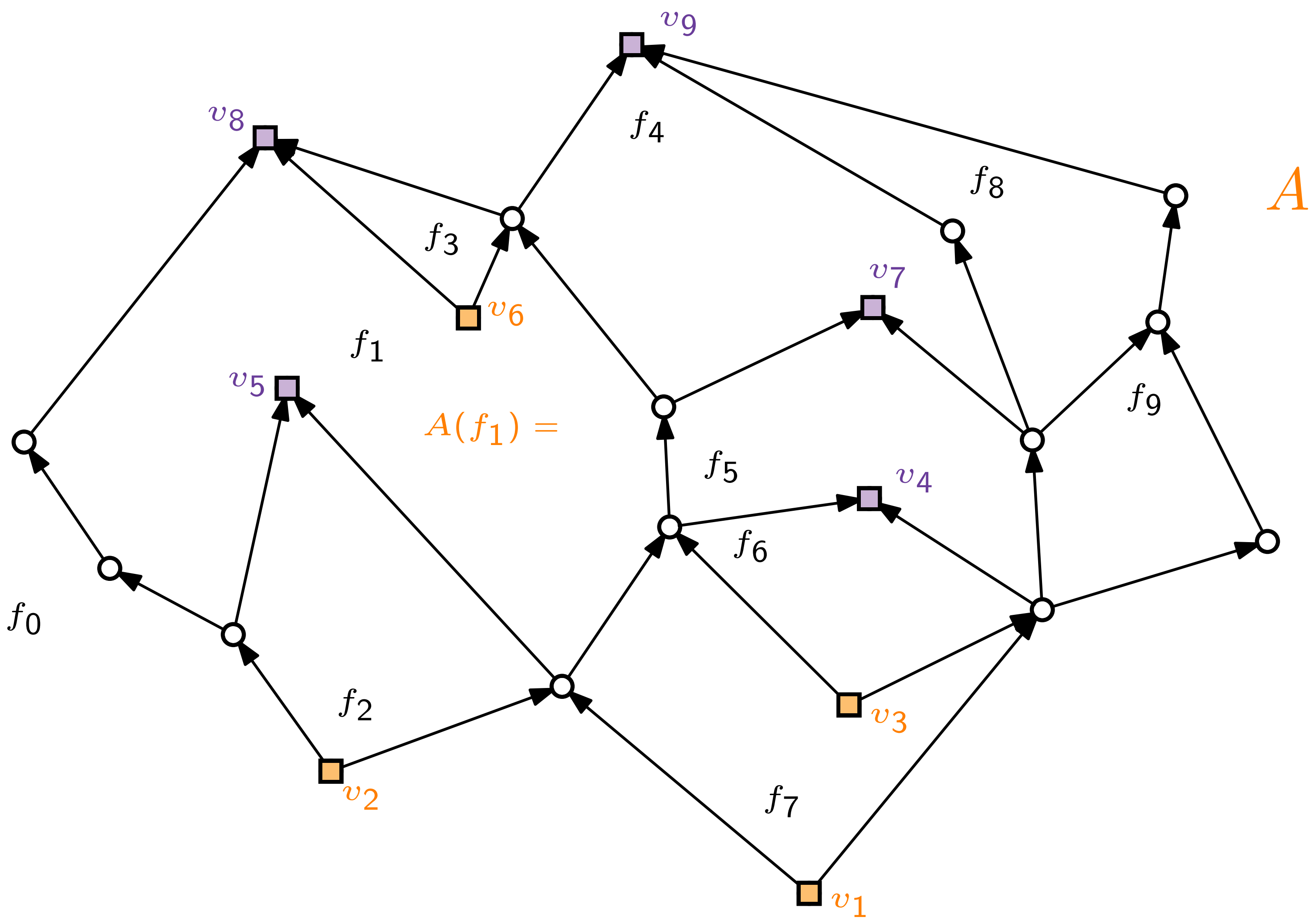

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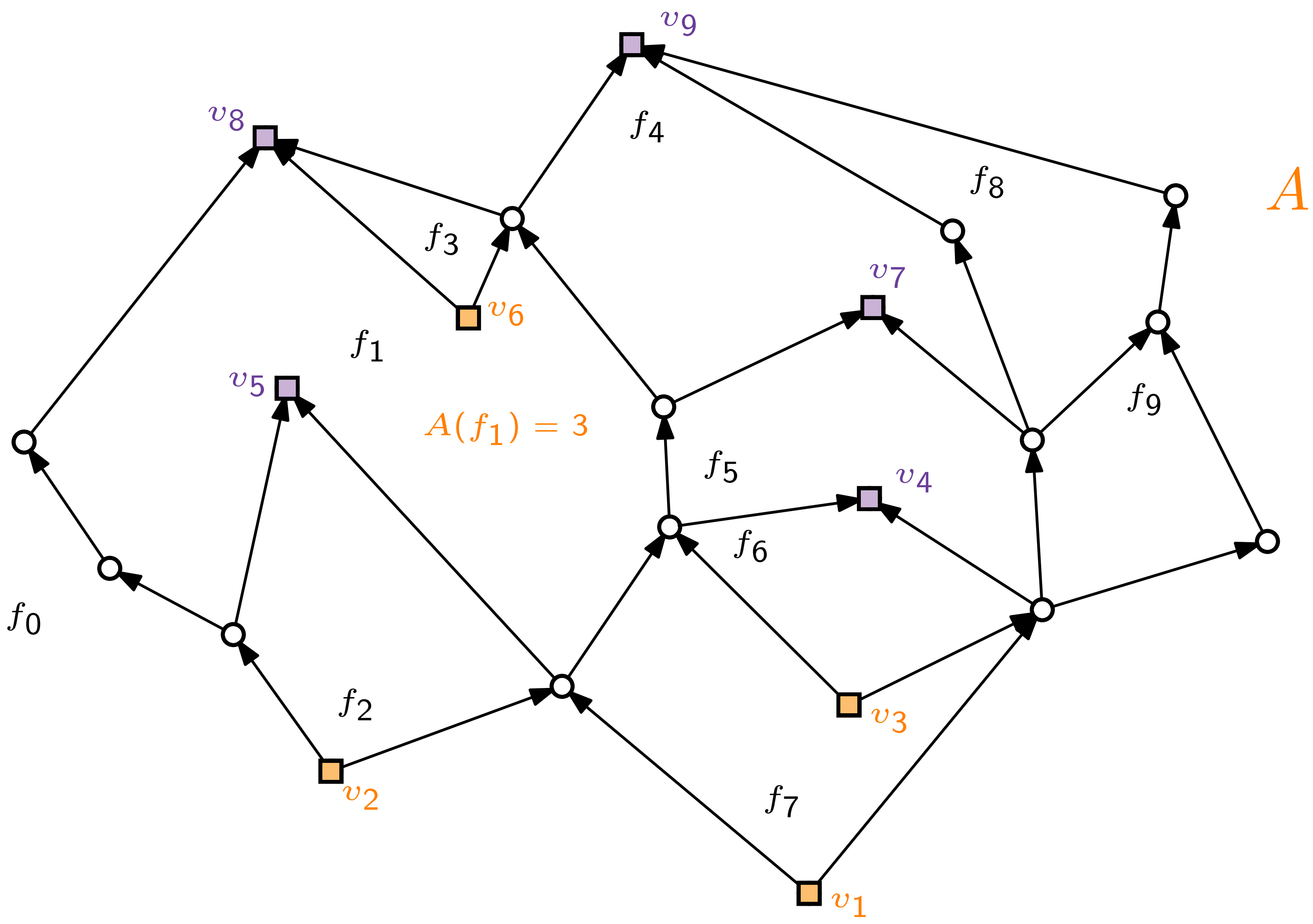

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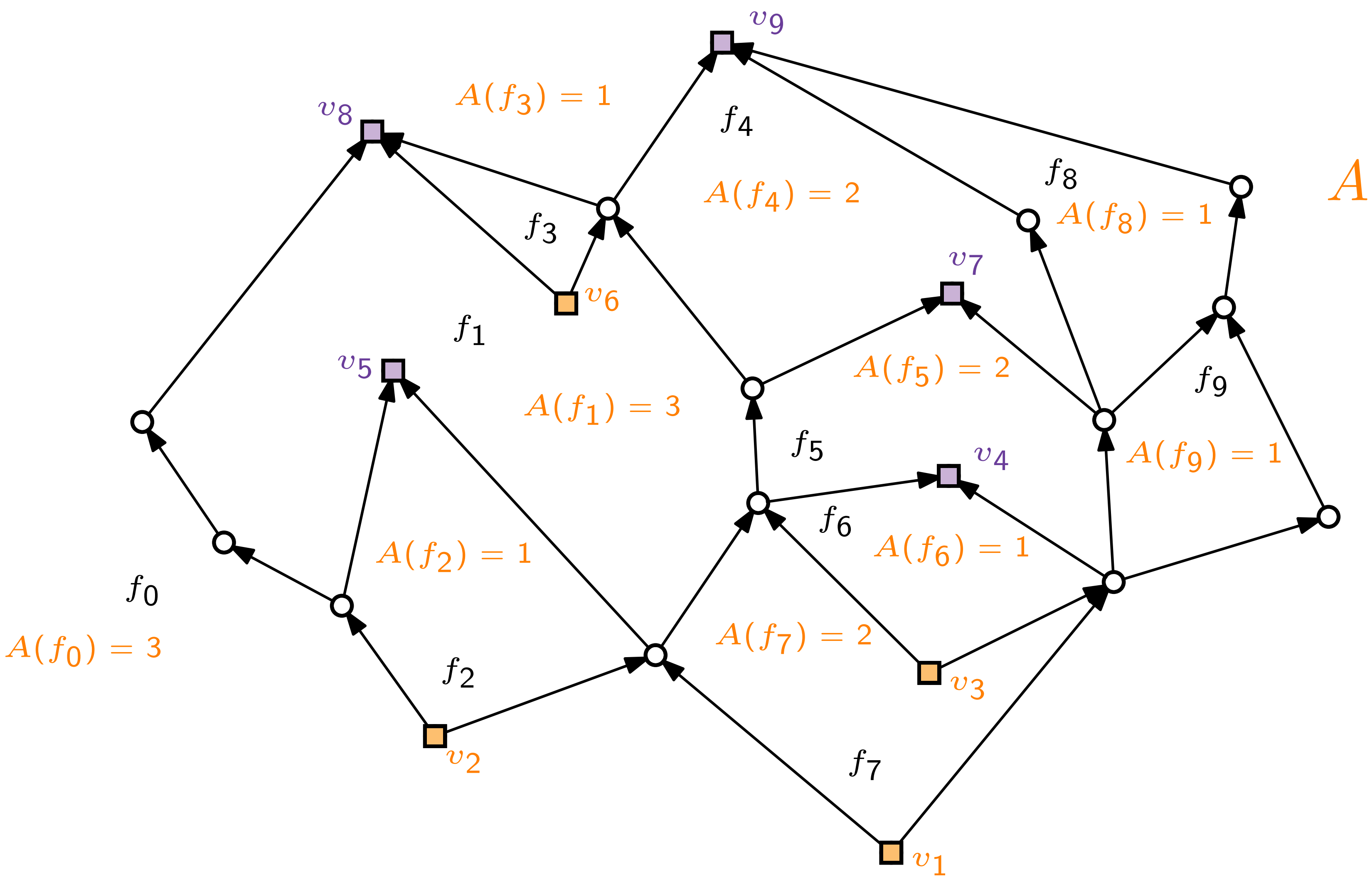

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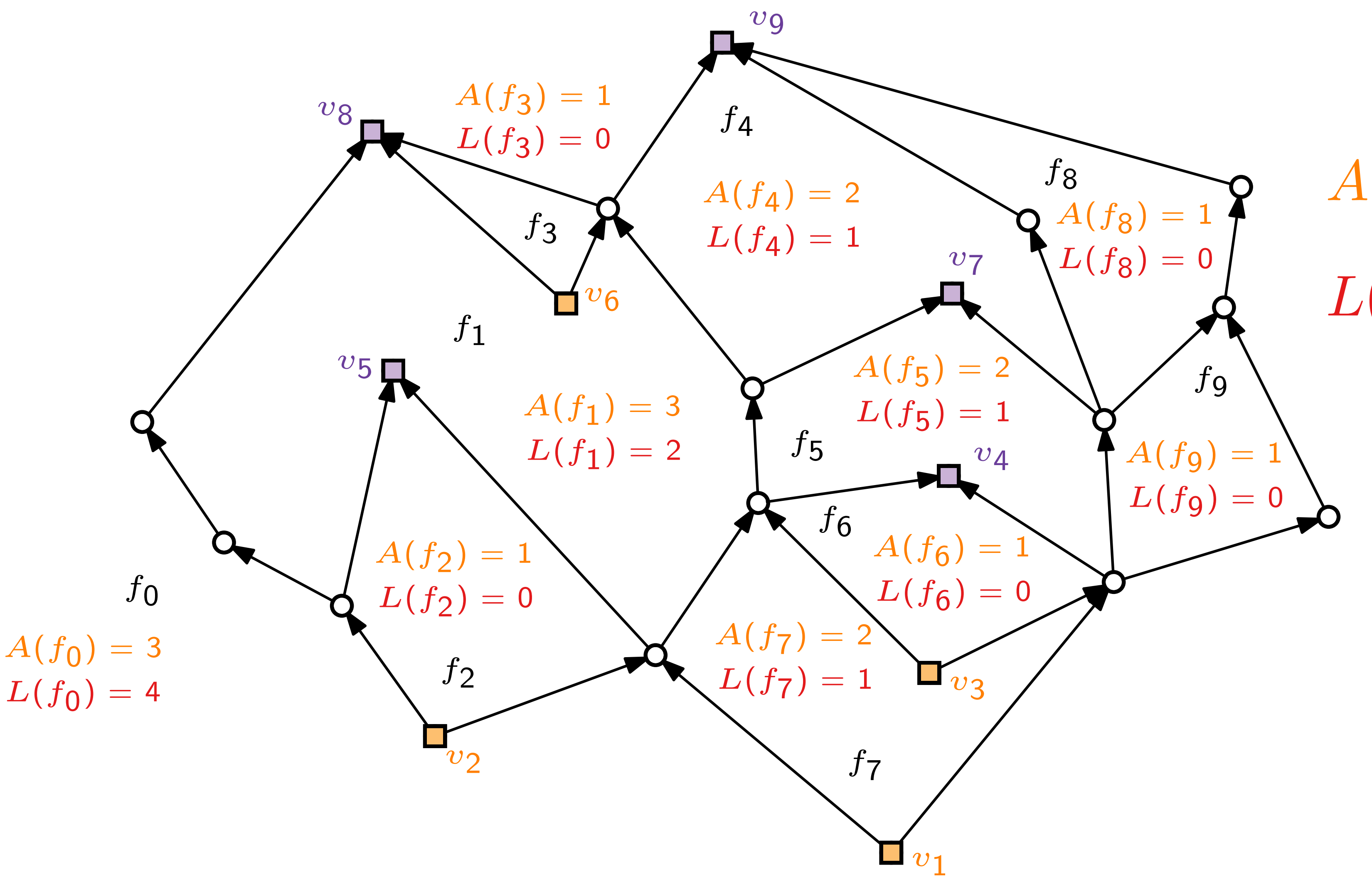

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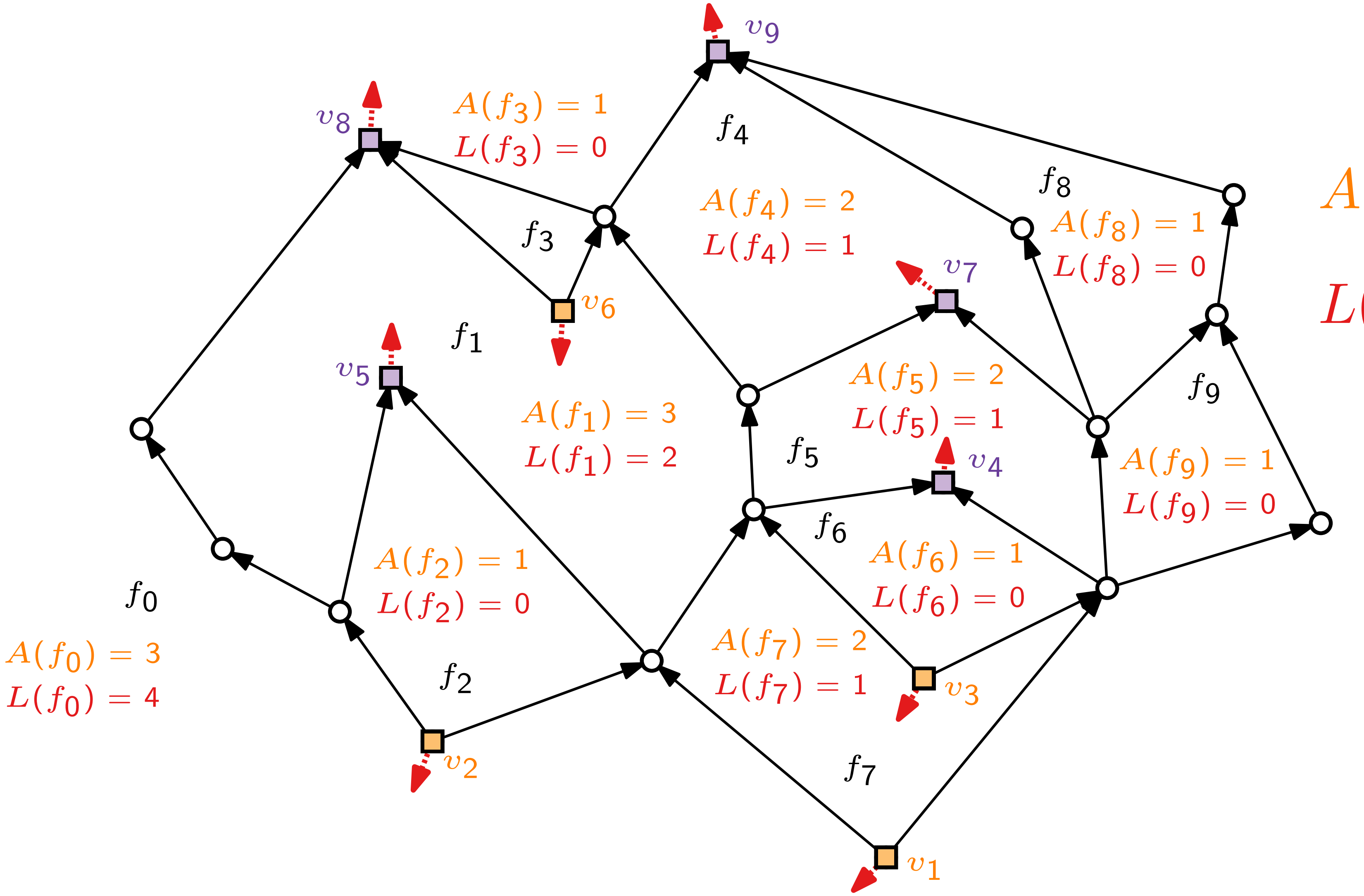

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G is upward planar $\Leftrightarrow$ G is a spanning subgraph of a planar st-digraph.
$\Leftrightarrow$ G admits a straight-line upward planar drawing.

Result Characterization

Theorem 3.
Let G be an acyclic plane digraph with embedding given by F and f_0.

Then G is upward planar (respecting F and f_0)
$\Leftrightarrow$ G is bimodal and there exists a consistent assignment Φ.

Proof.
$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

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(Note: Proof was constructive!)

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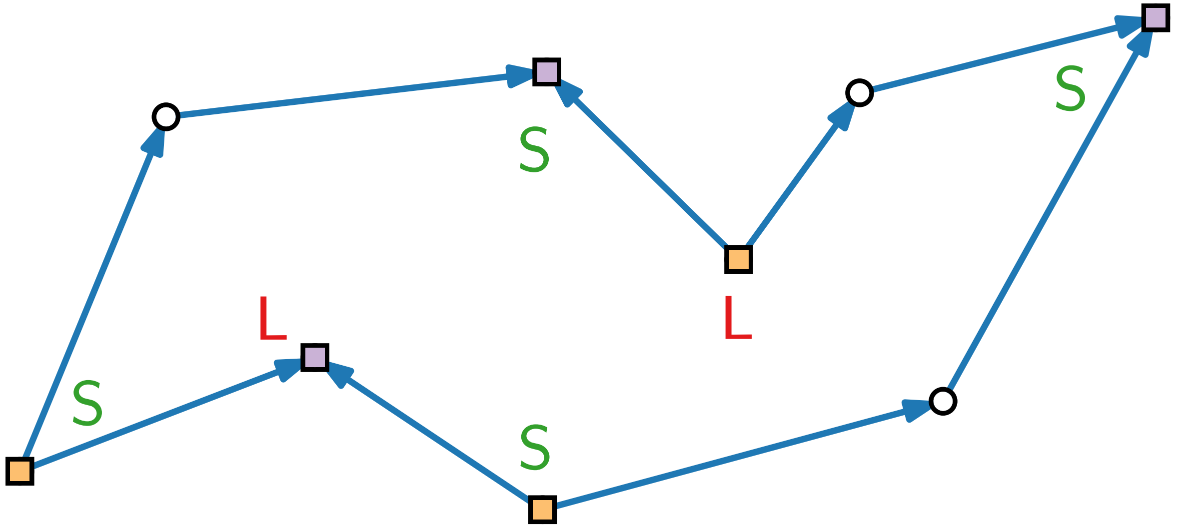

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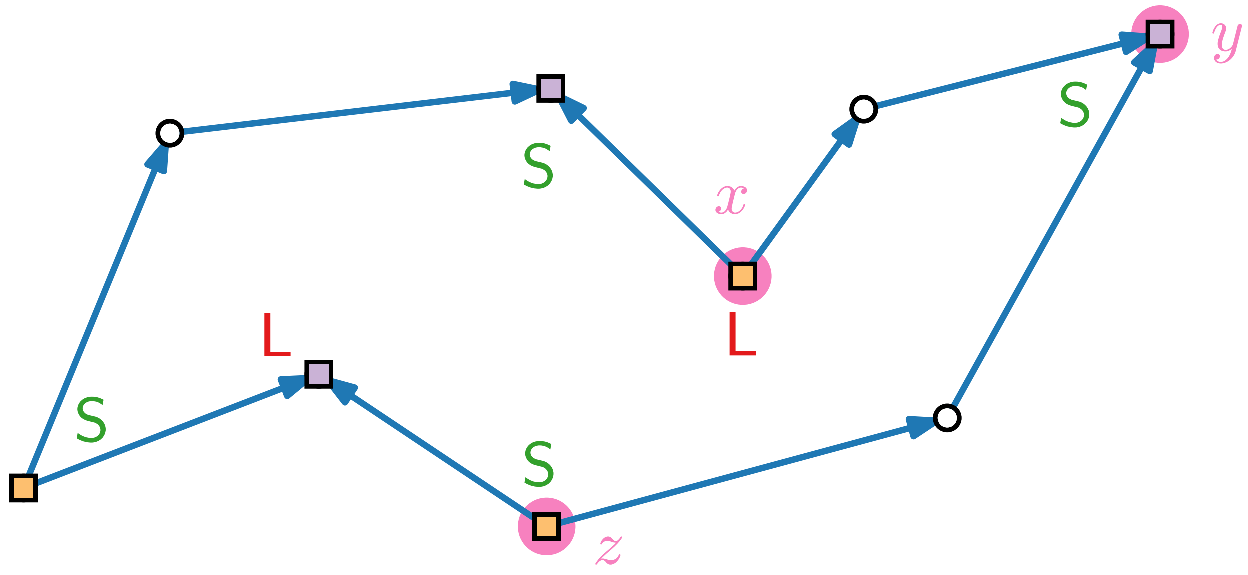

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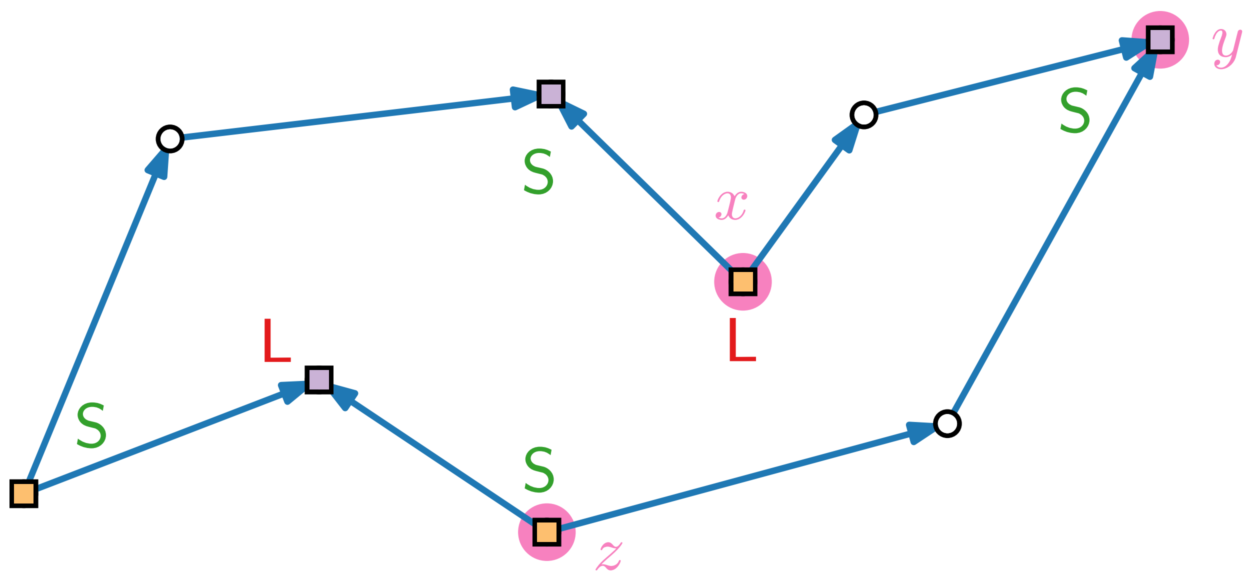

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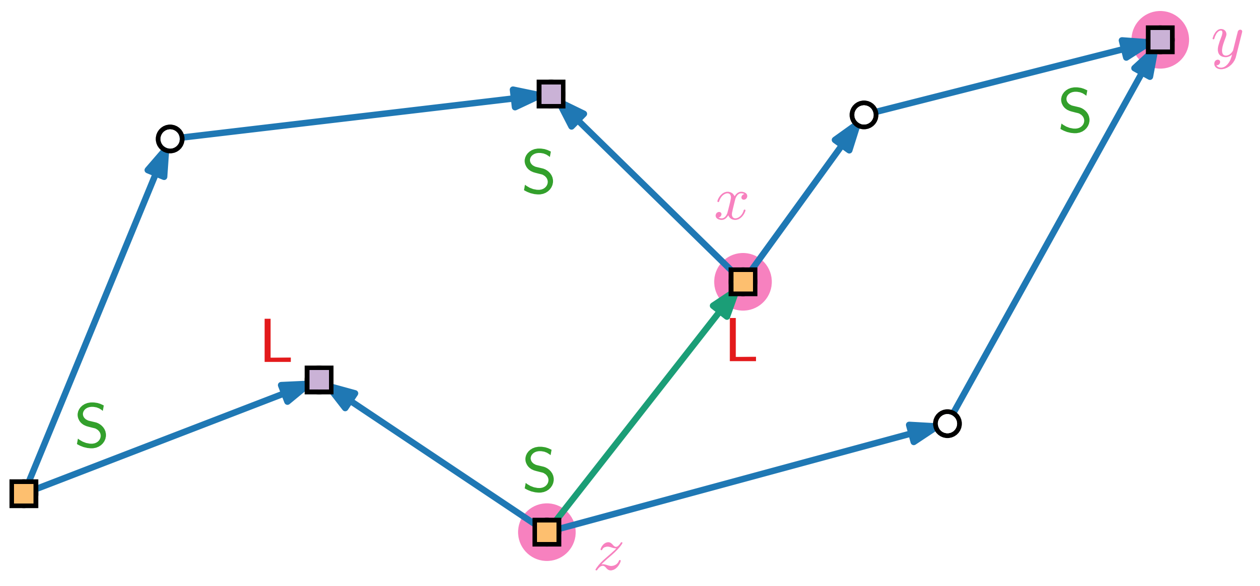

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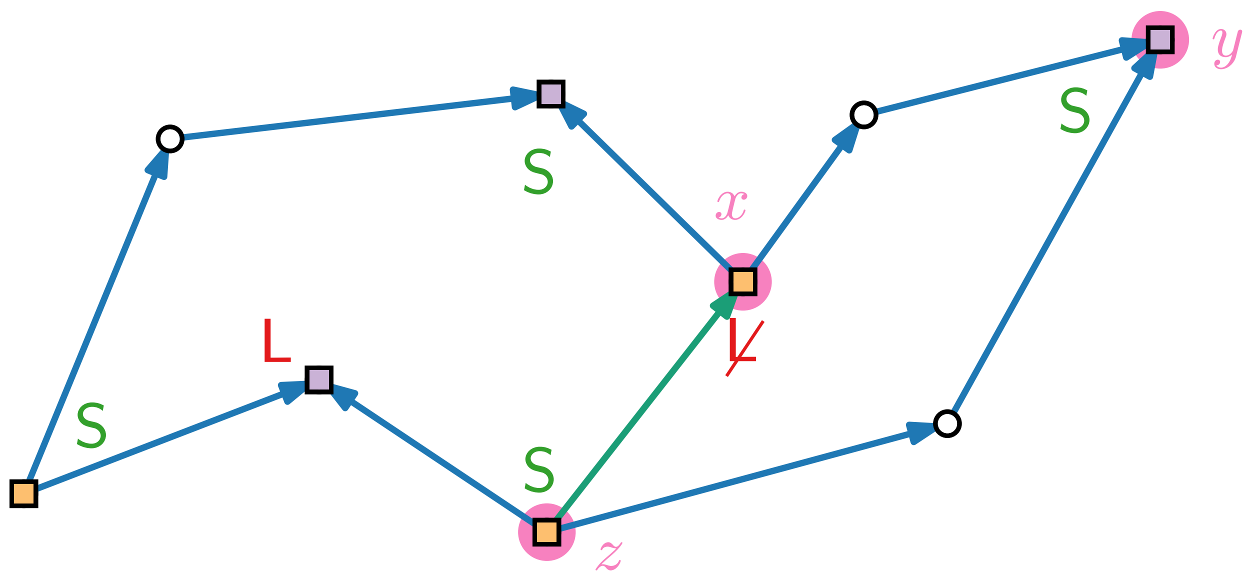

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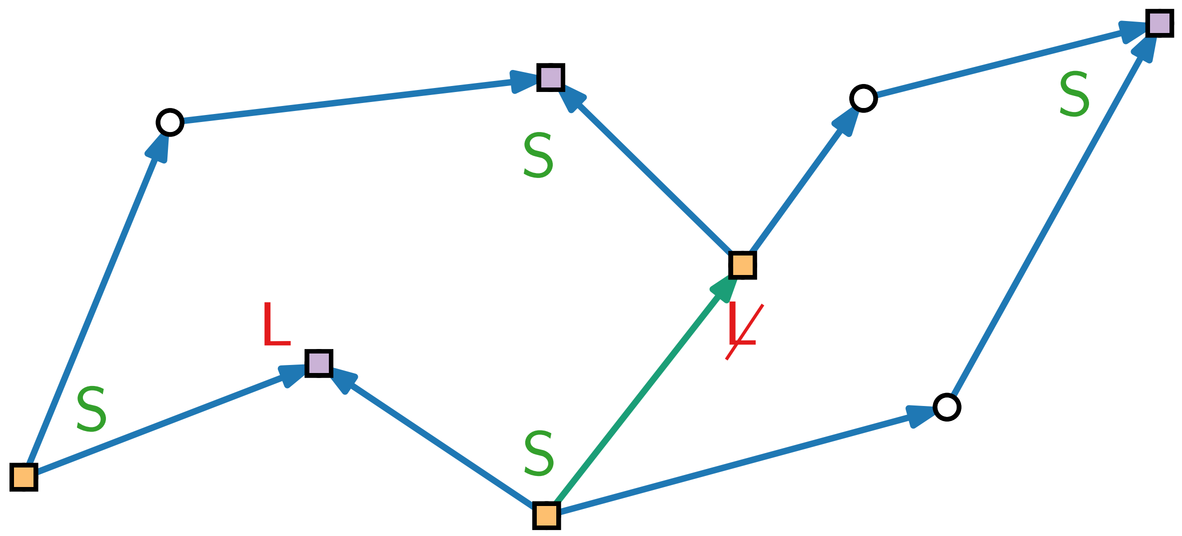

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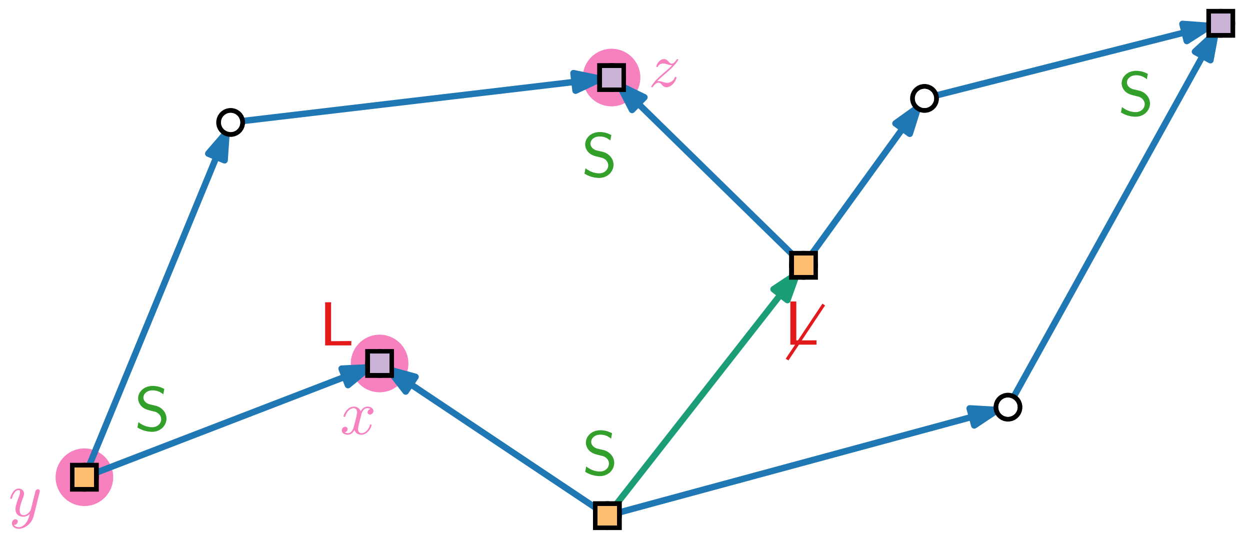

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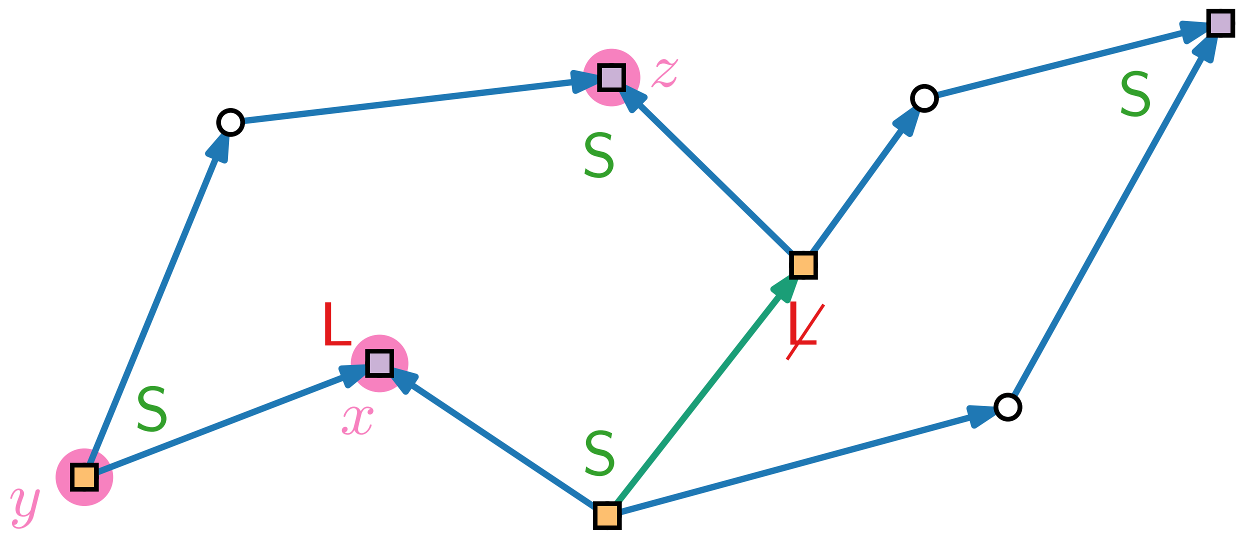

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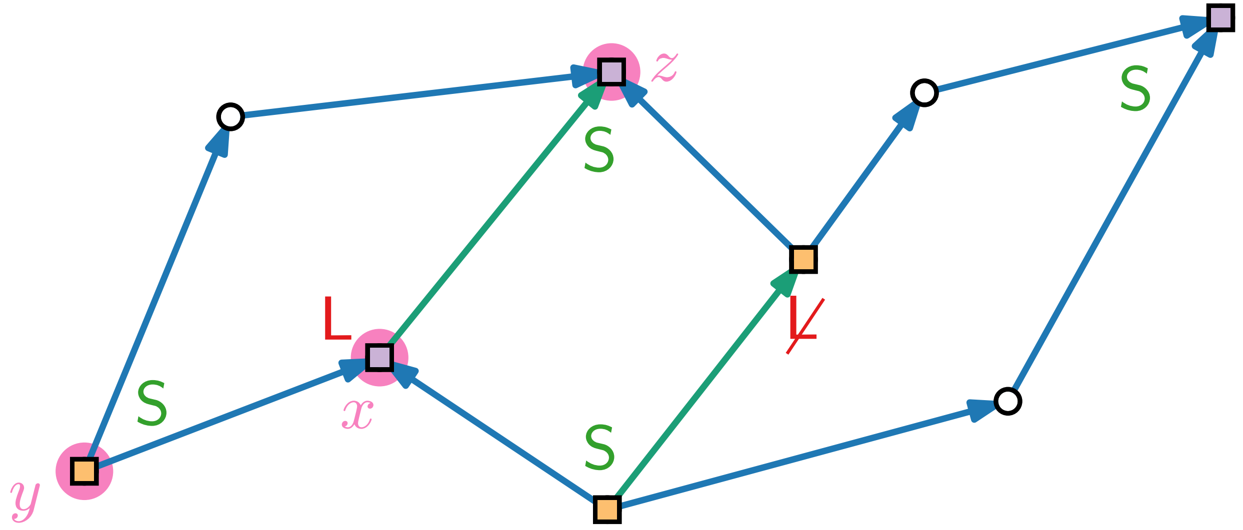

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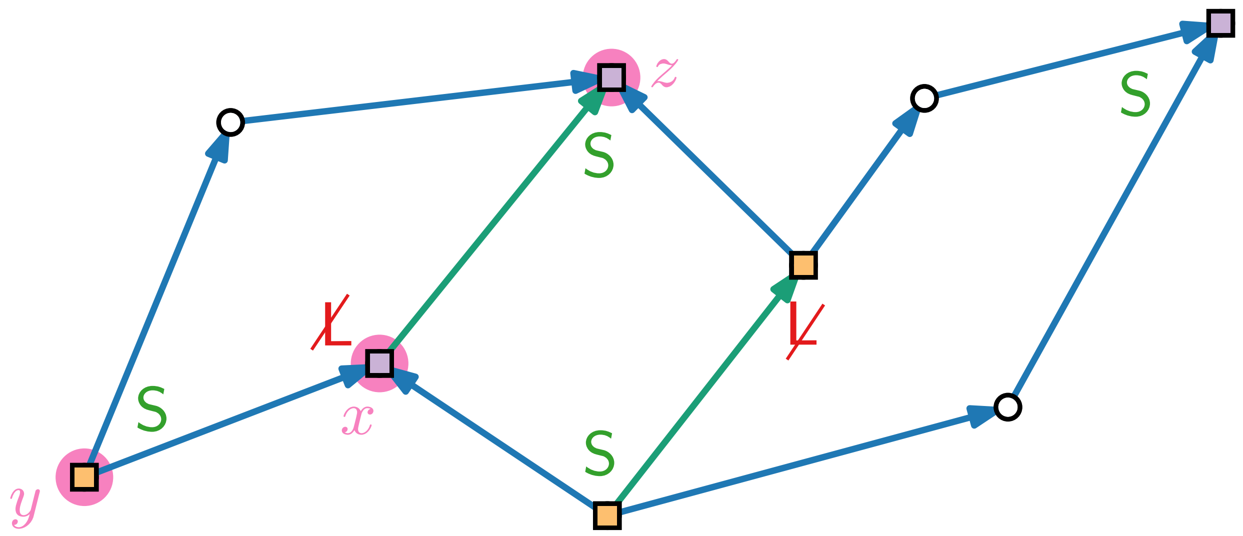

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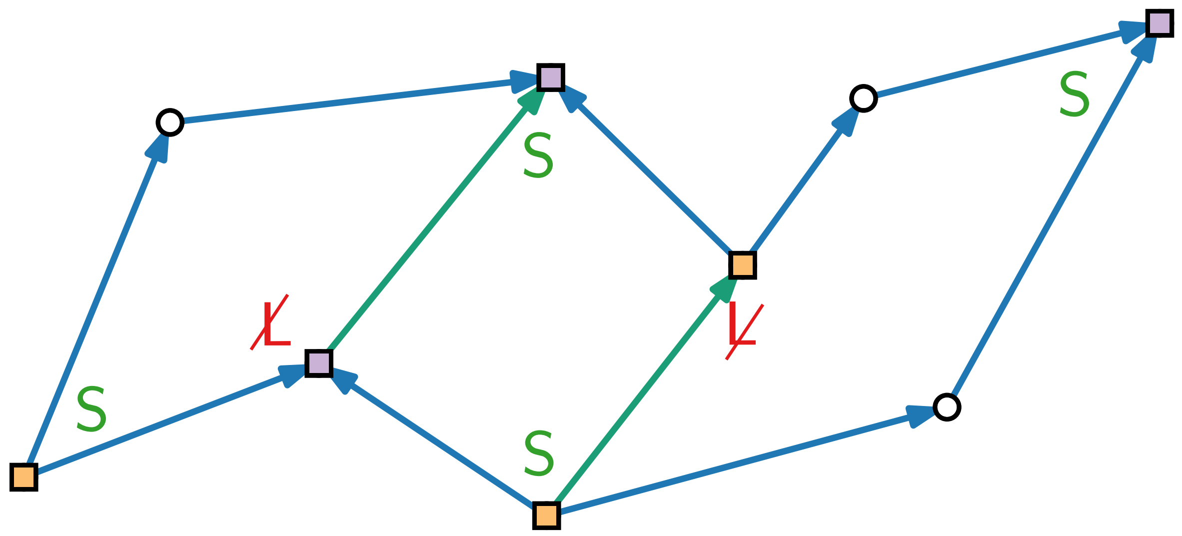

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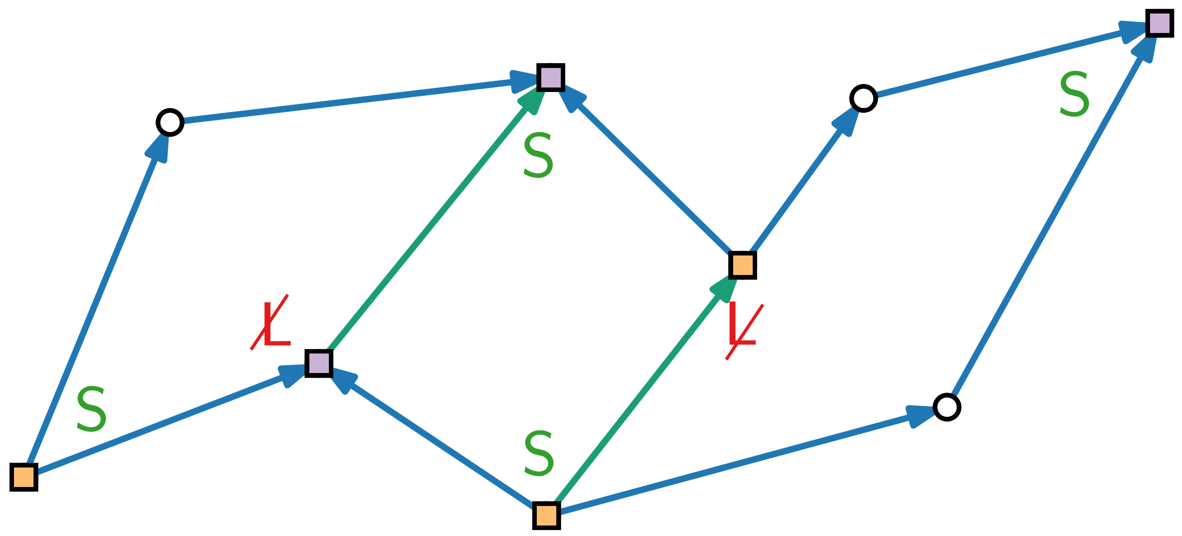

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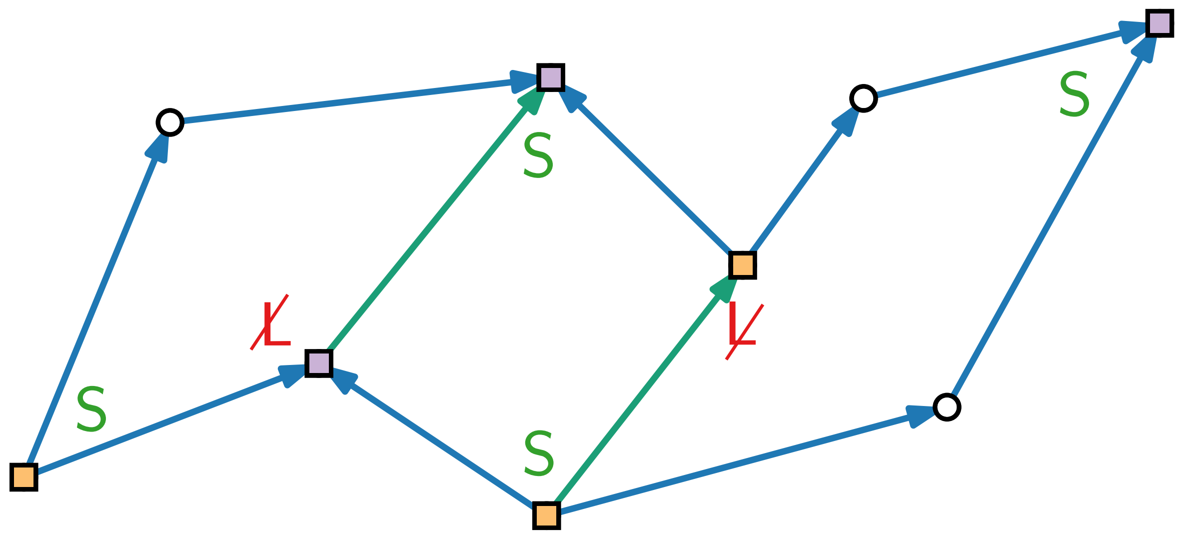

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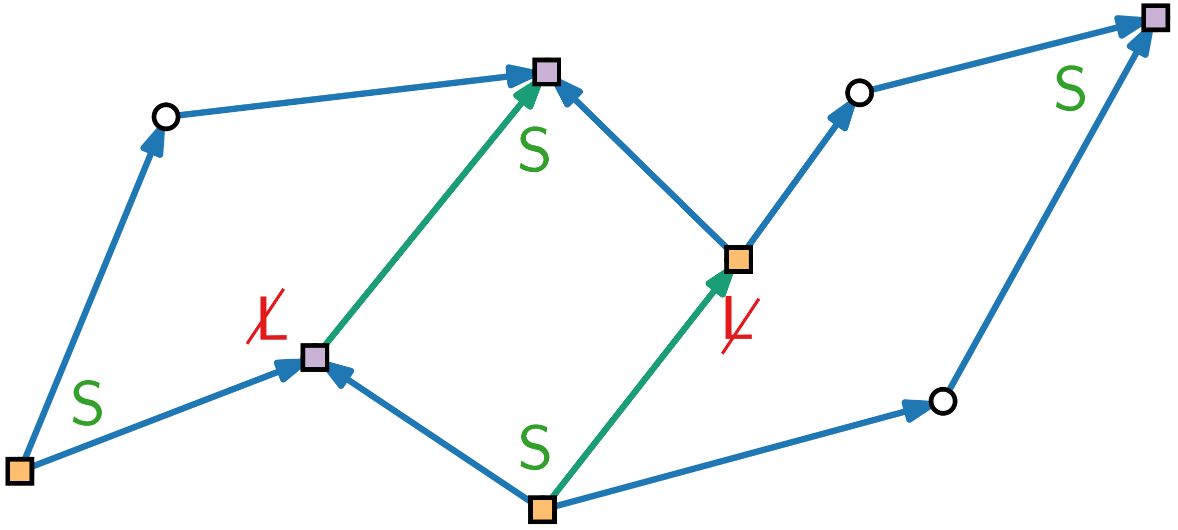

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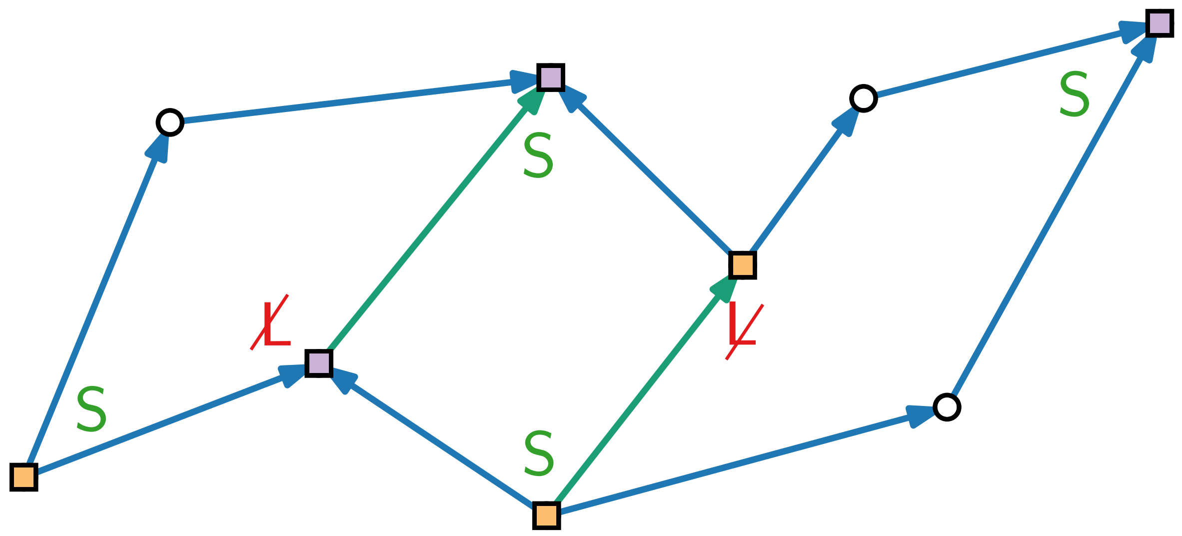
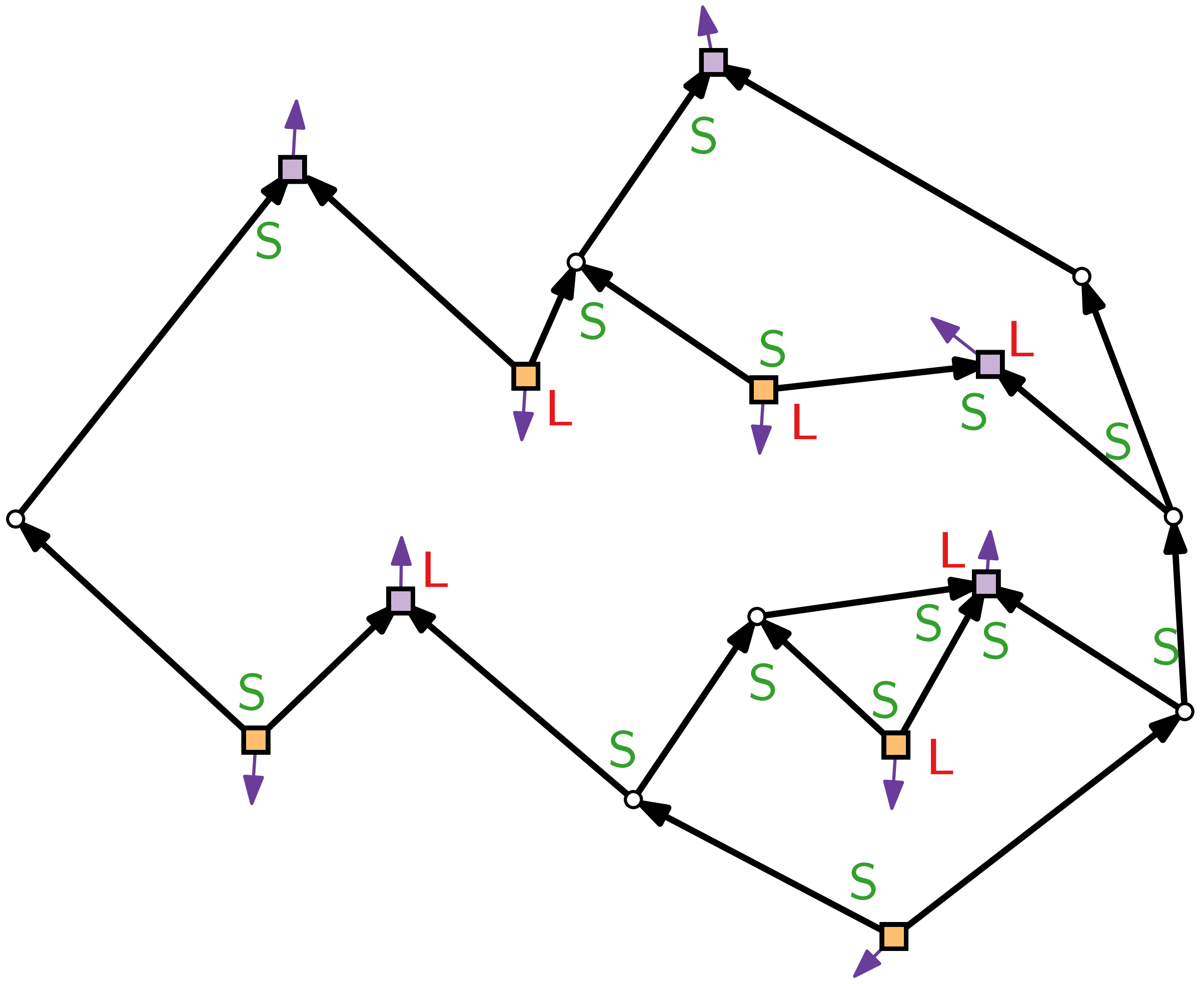


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

Refinement Example



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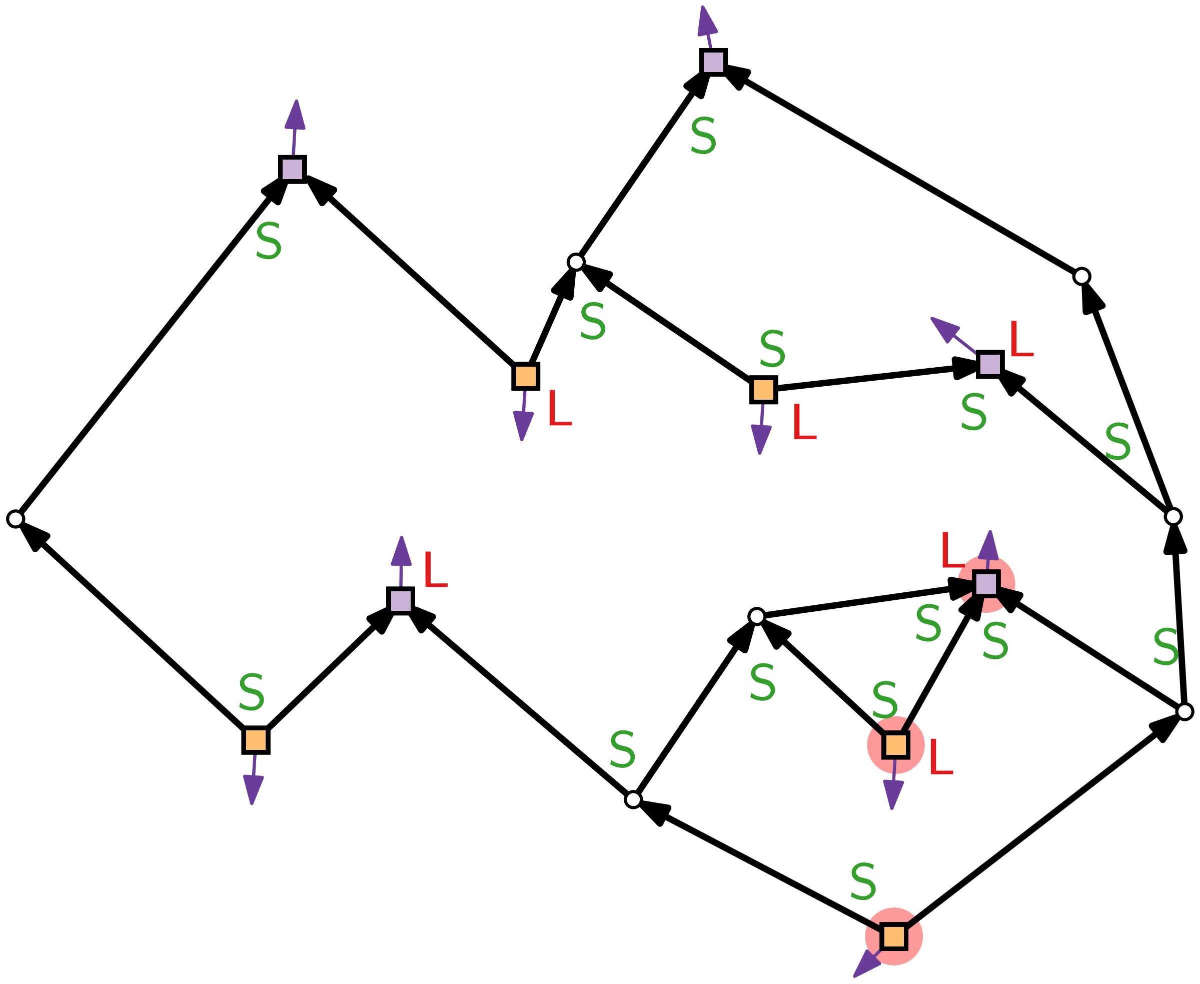

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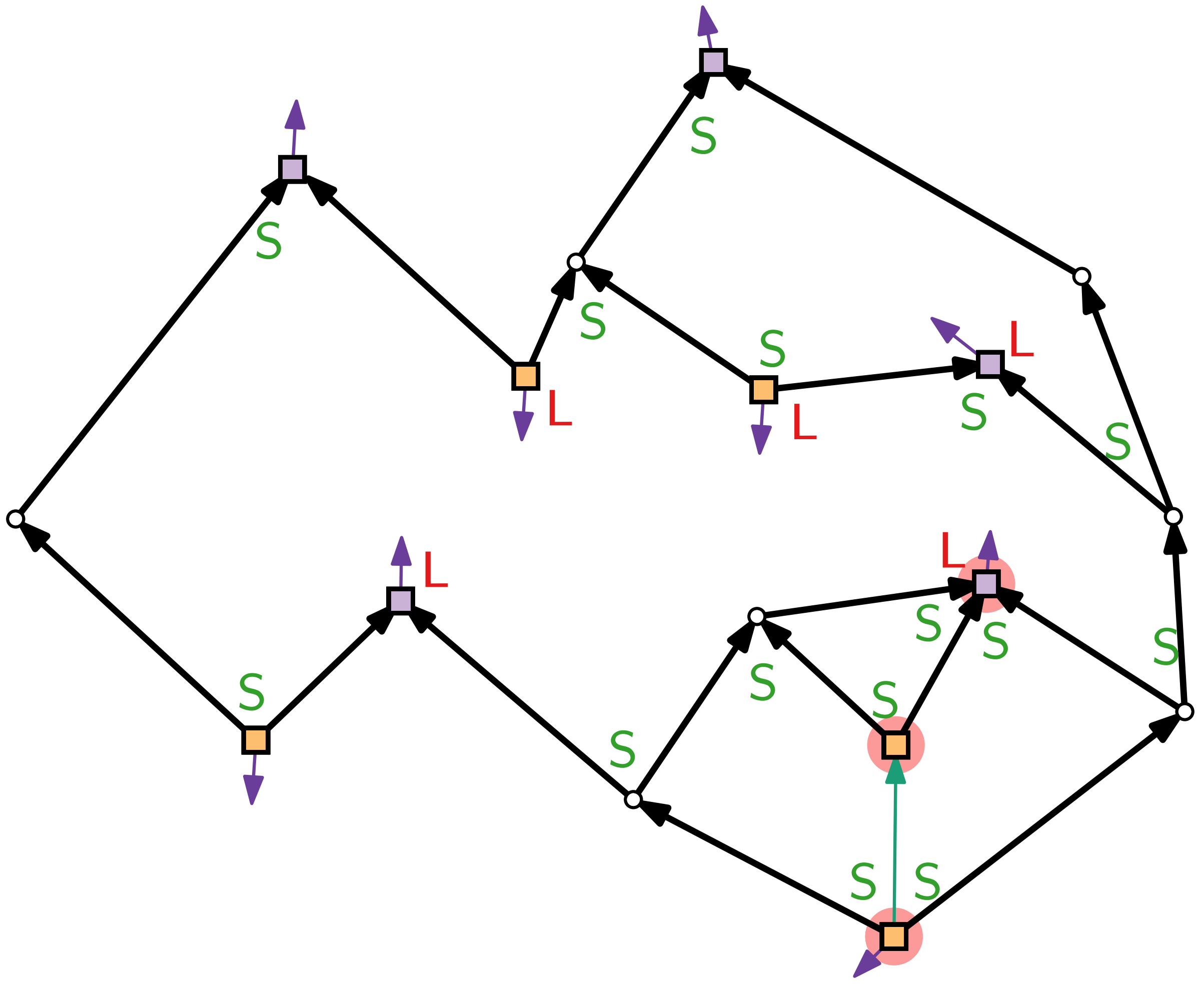

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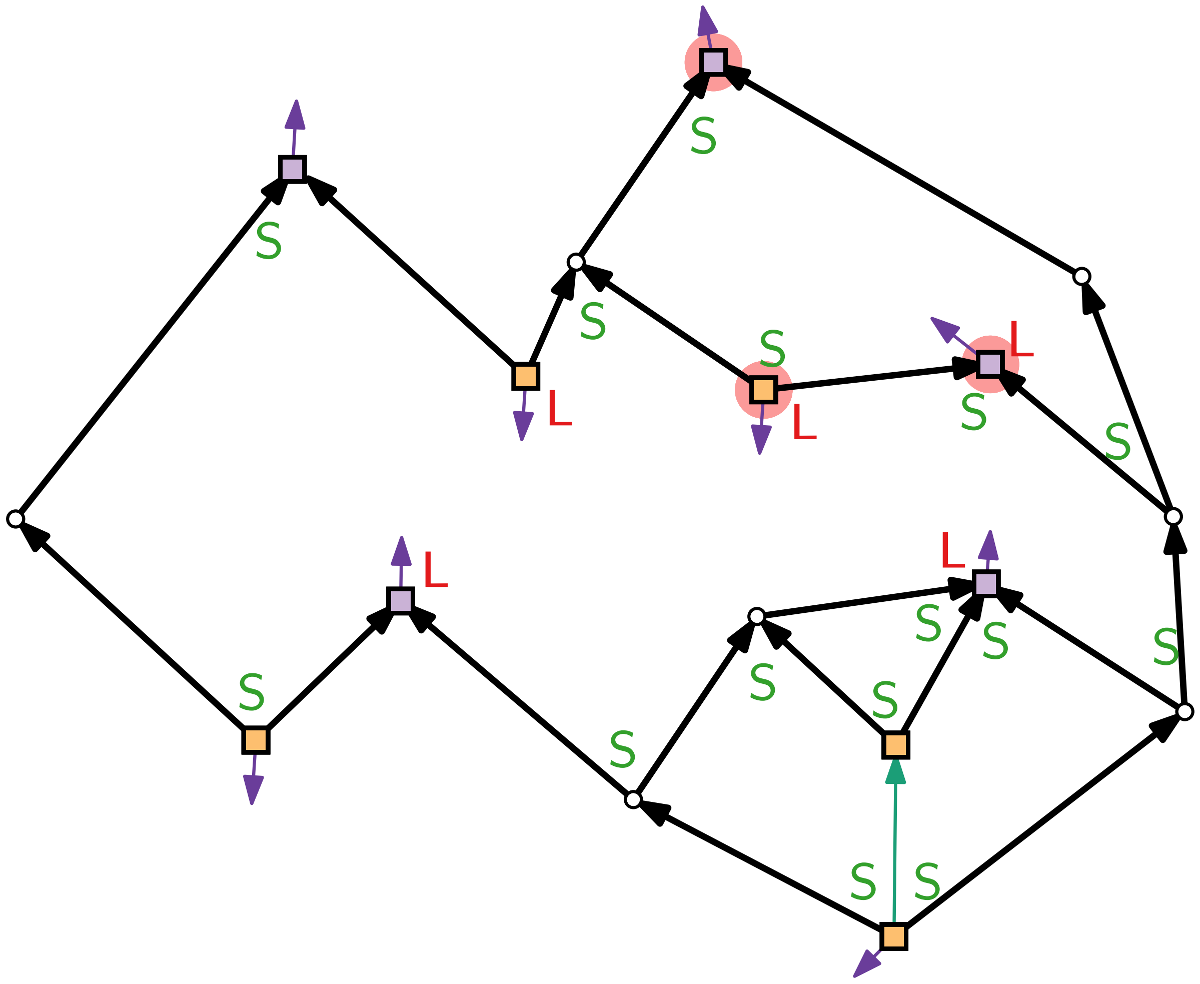

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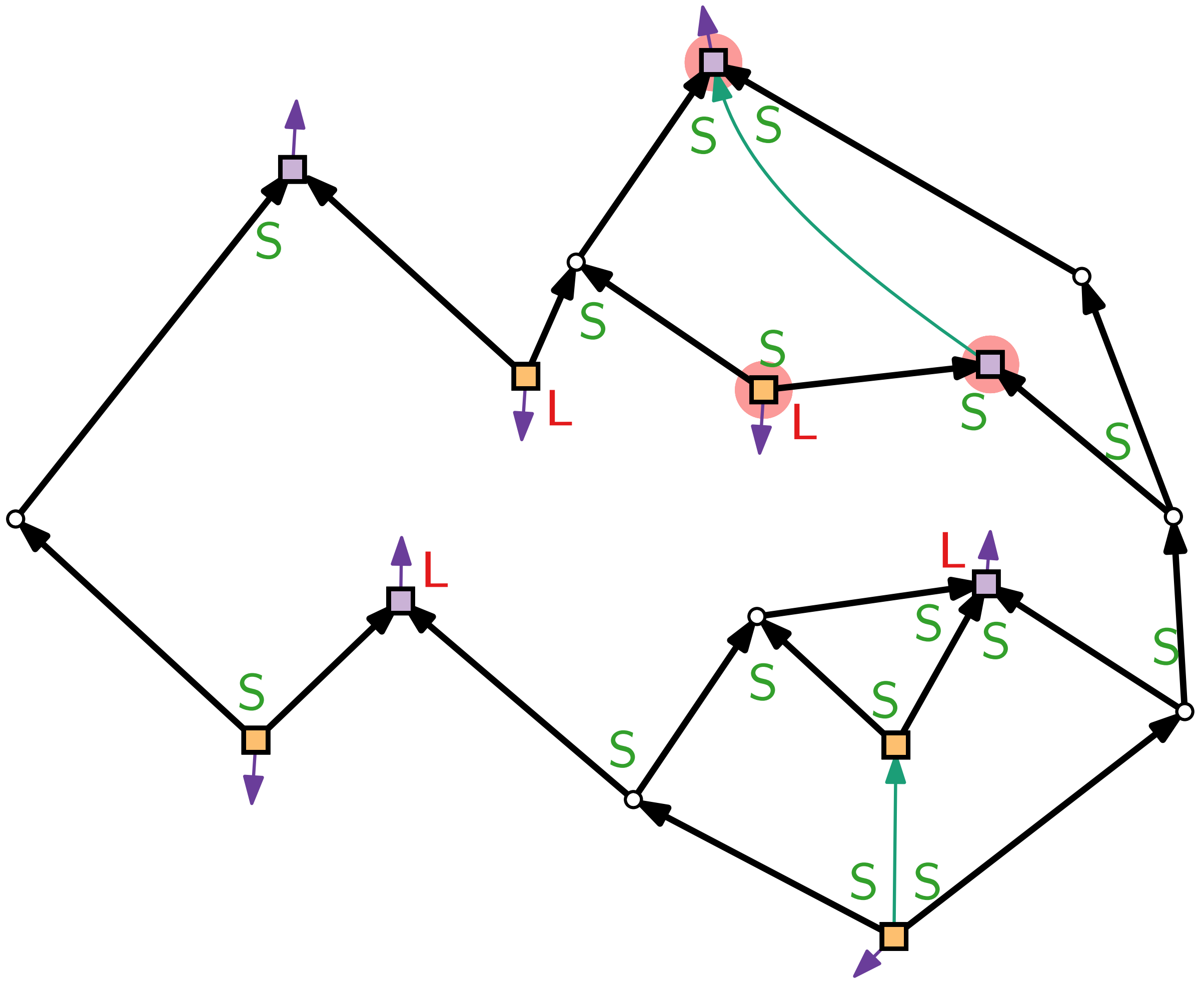

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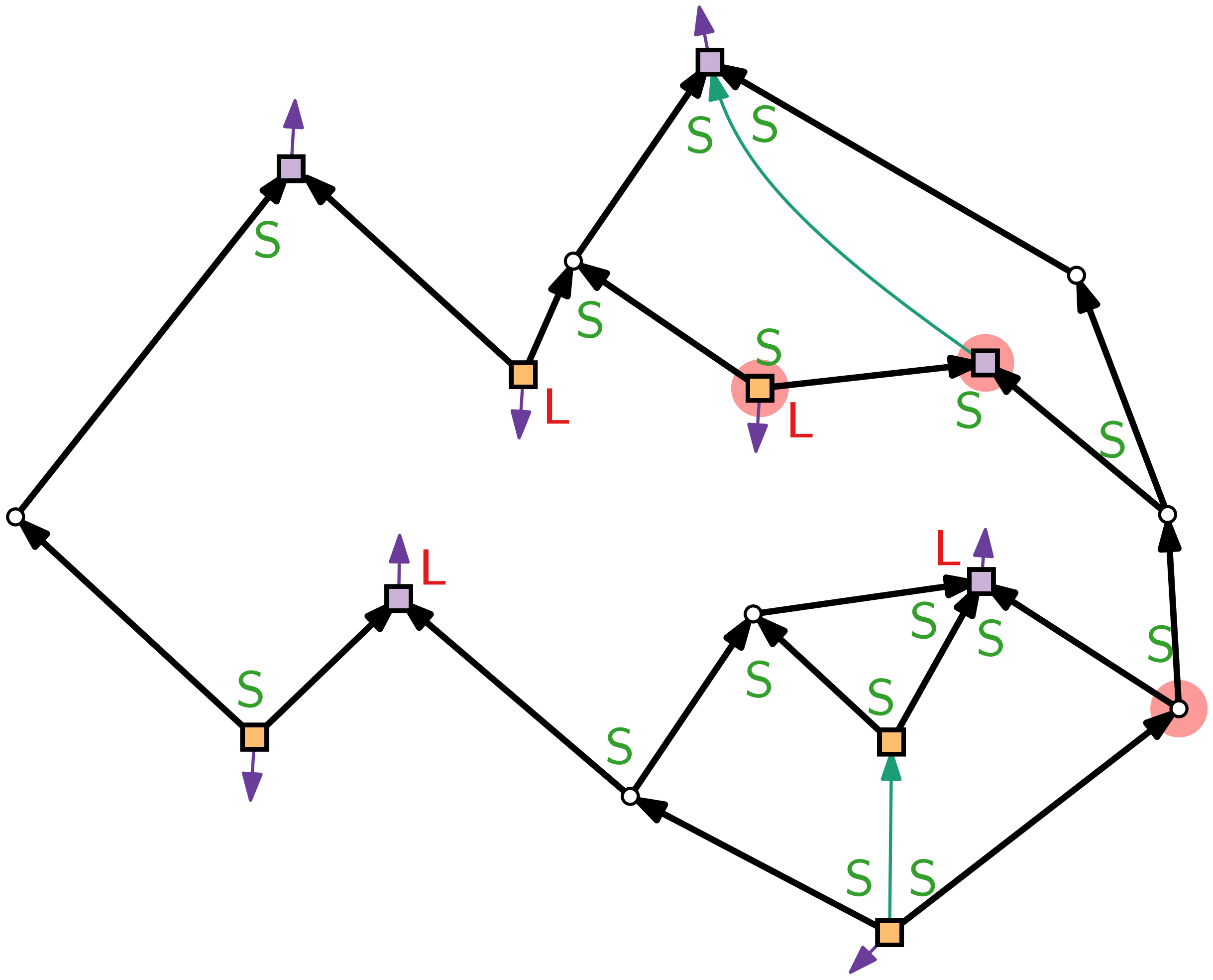

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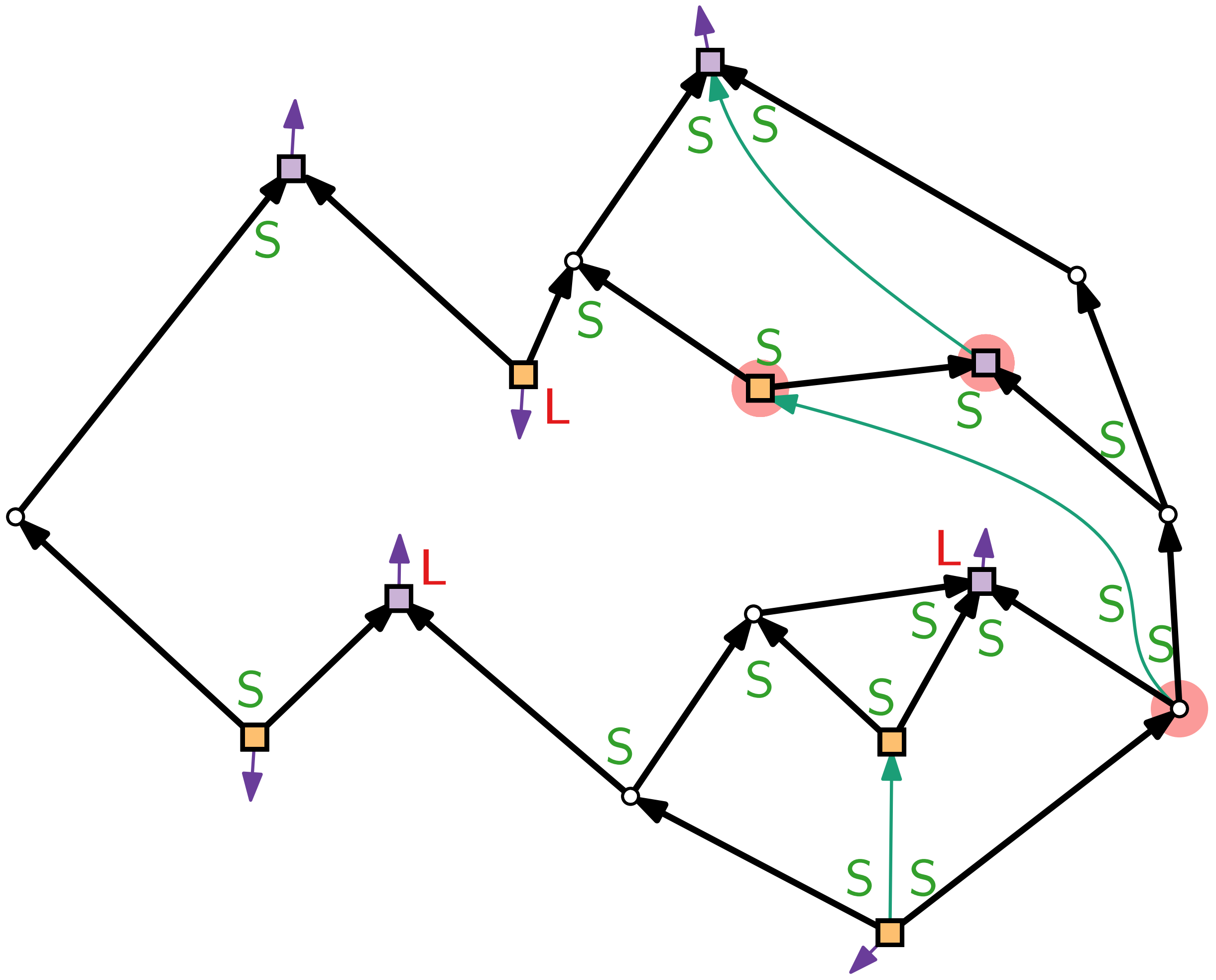

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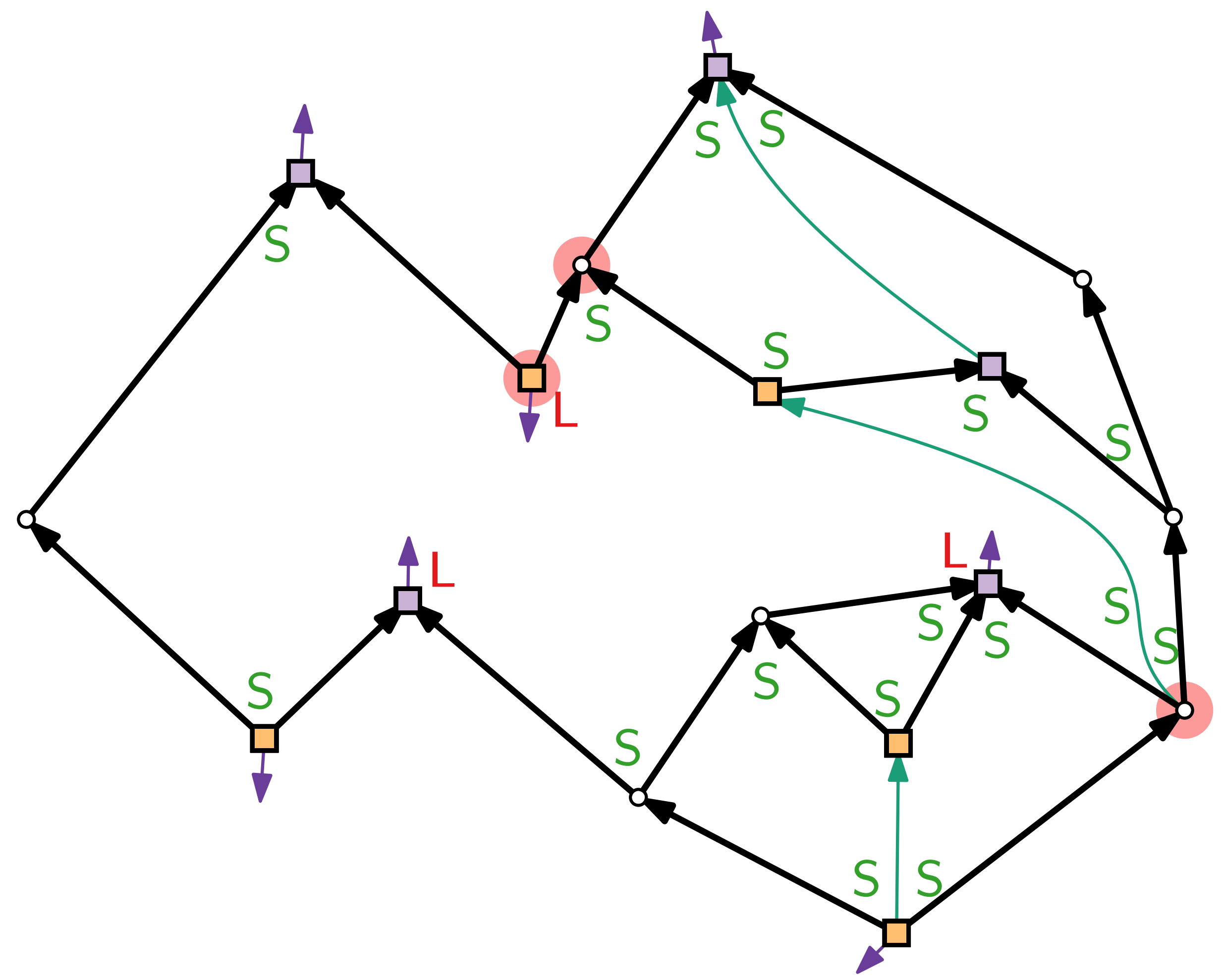

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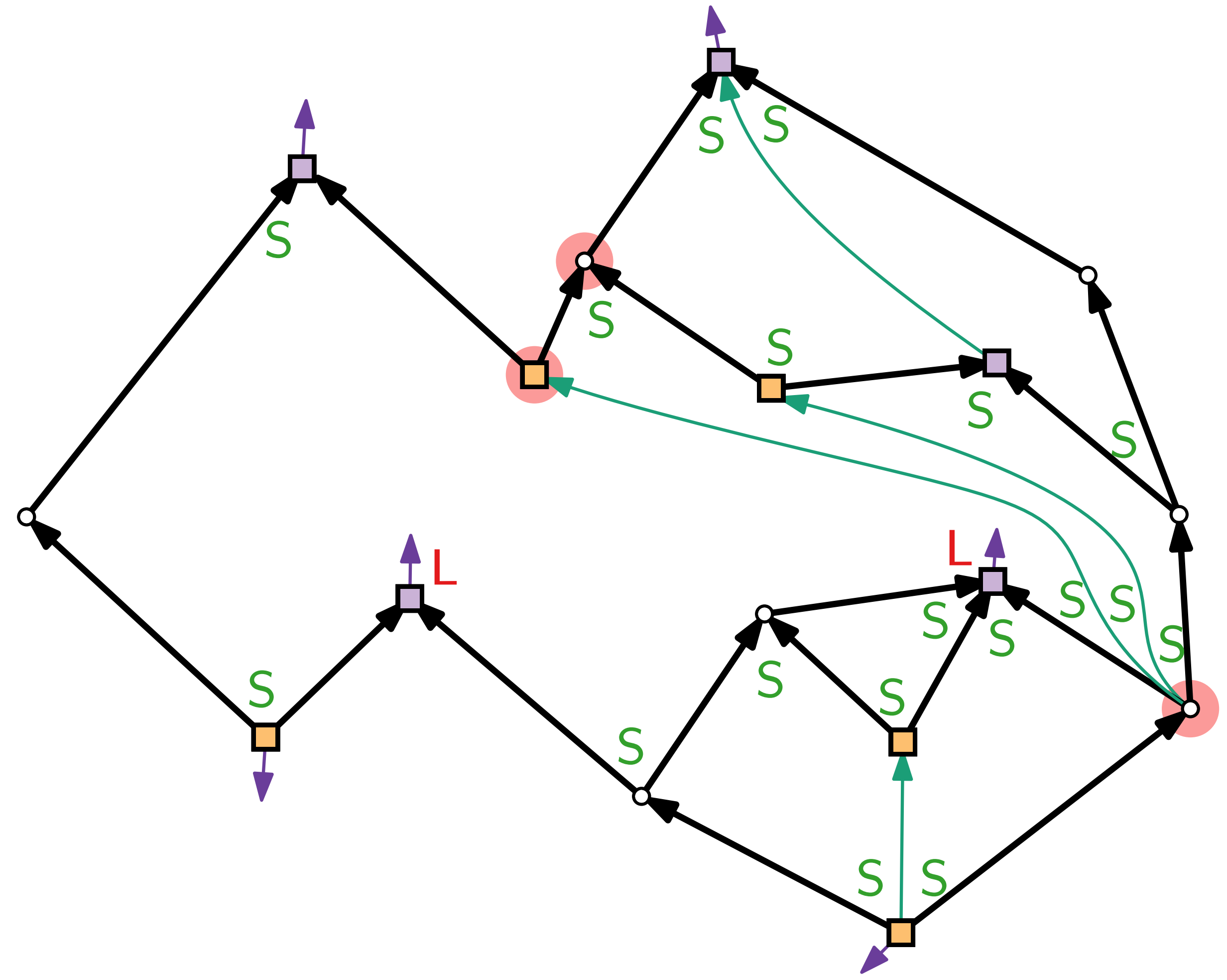

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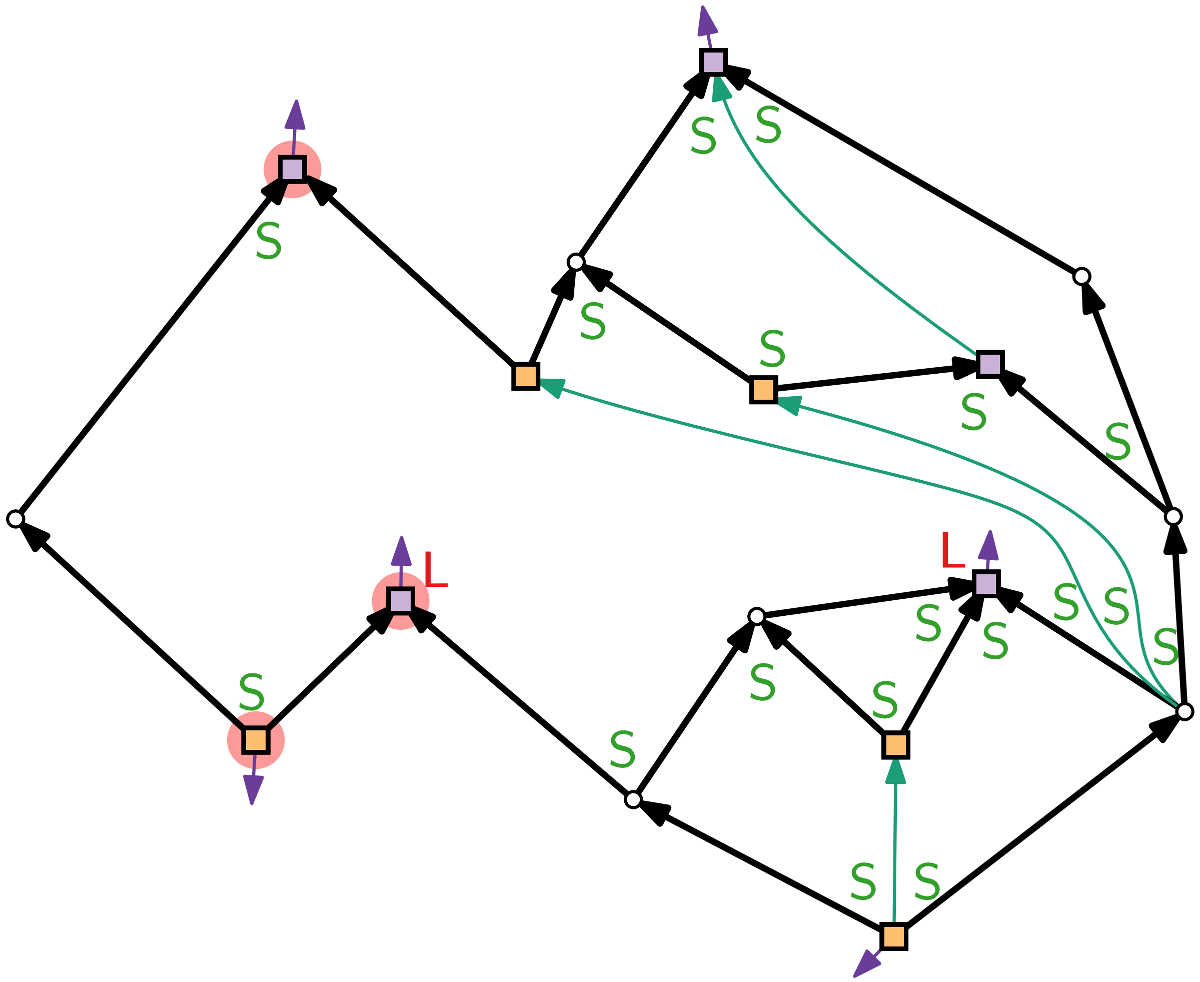

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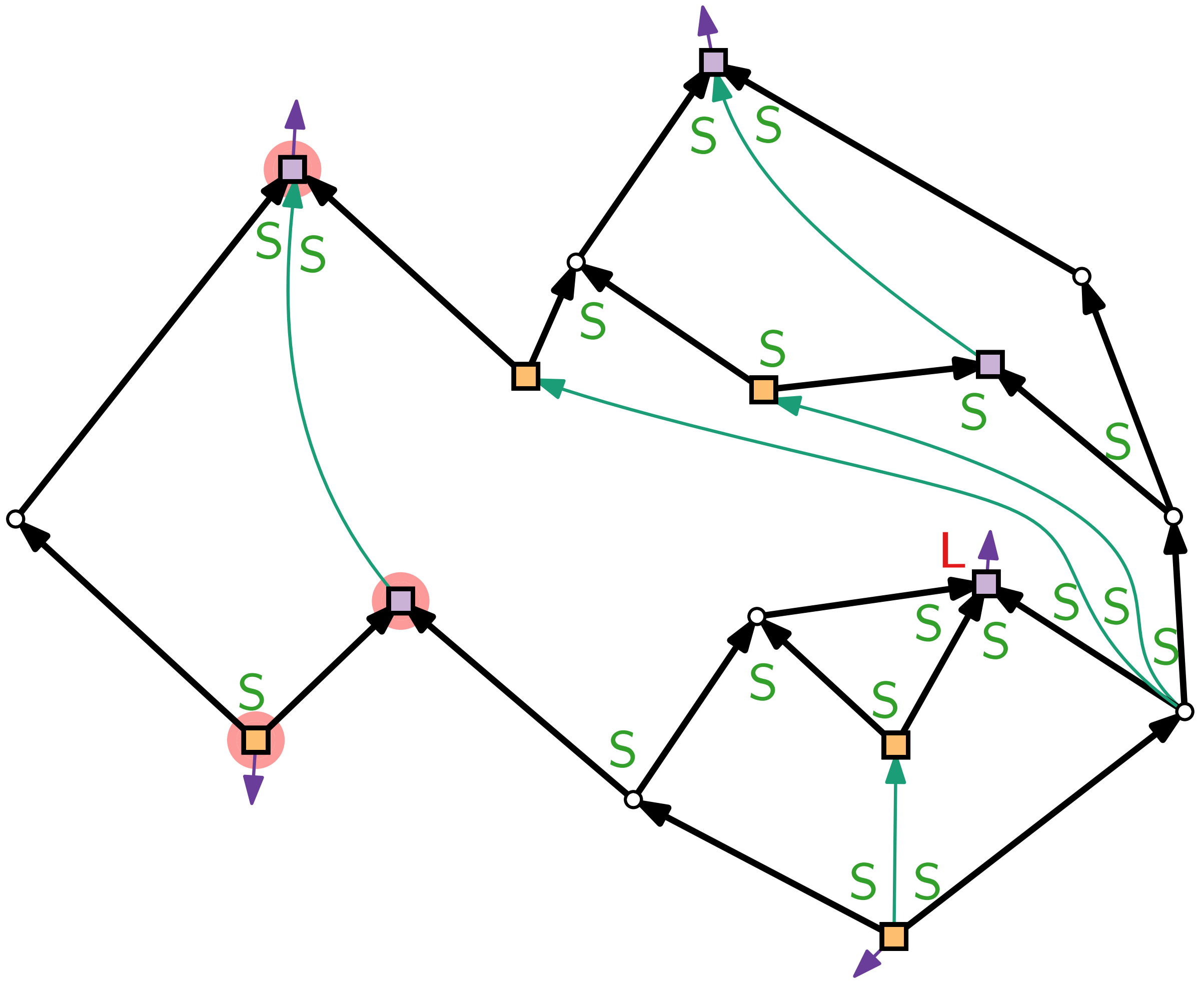

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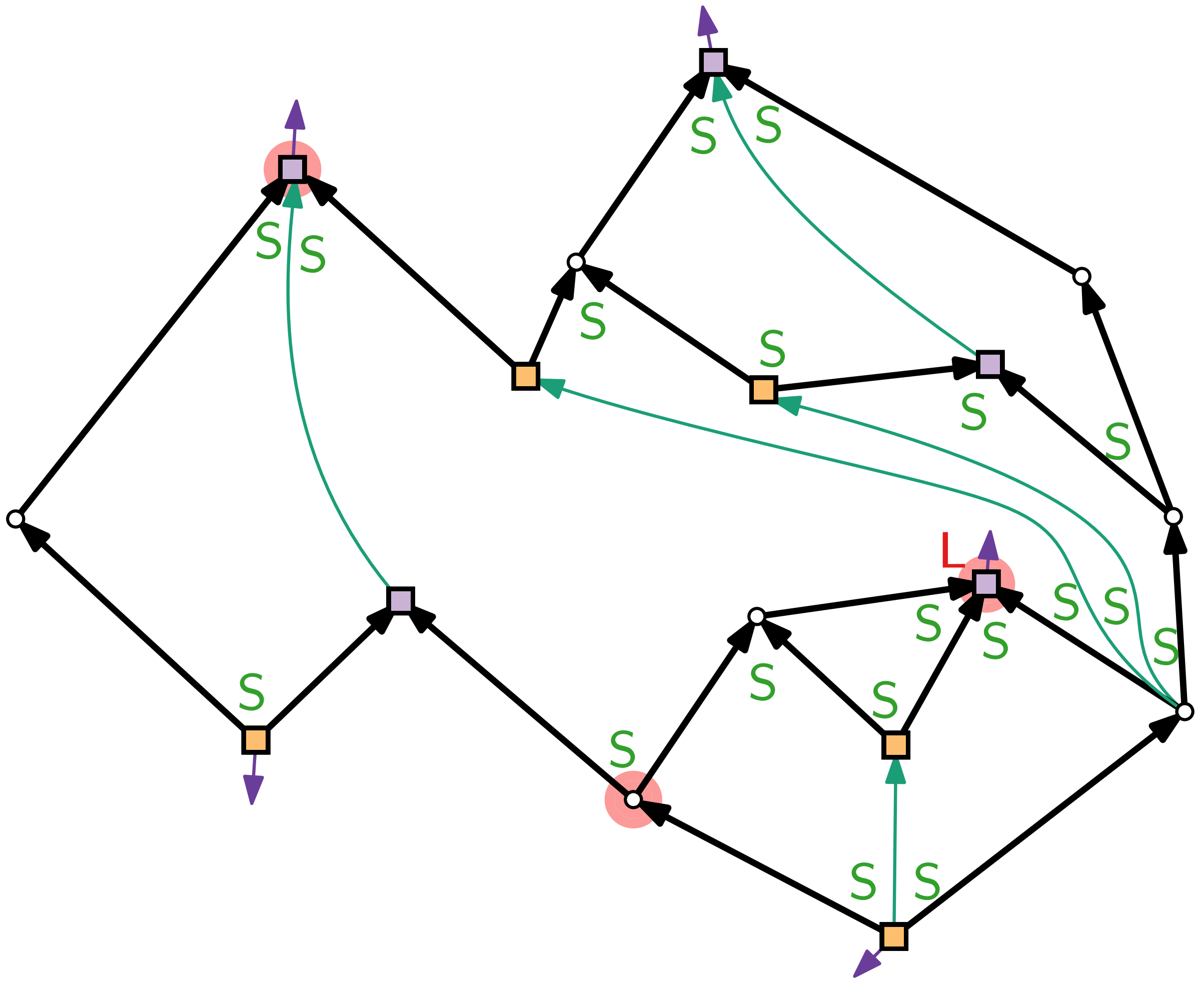

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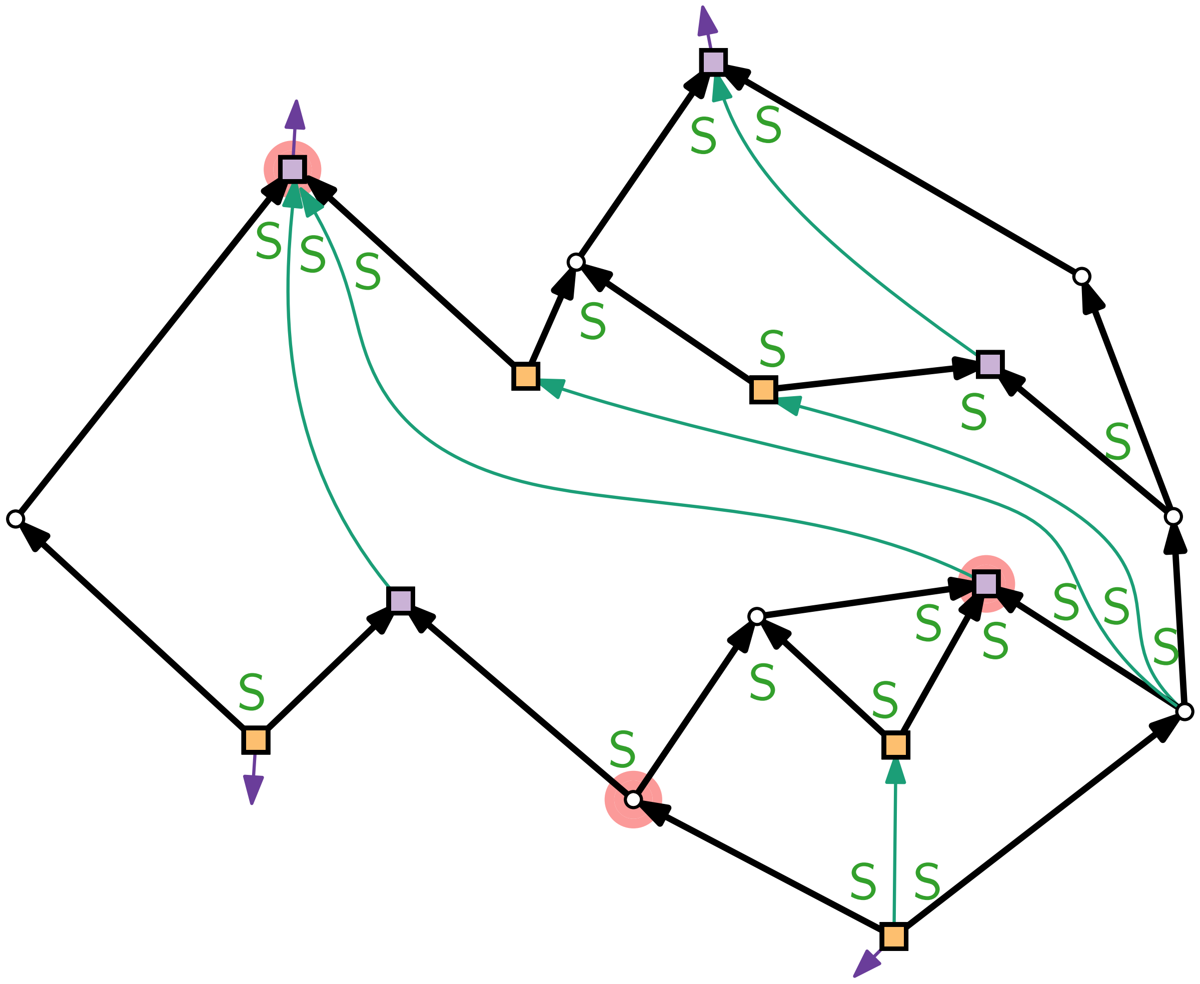

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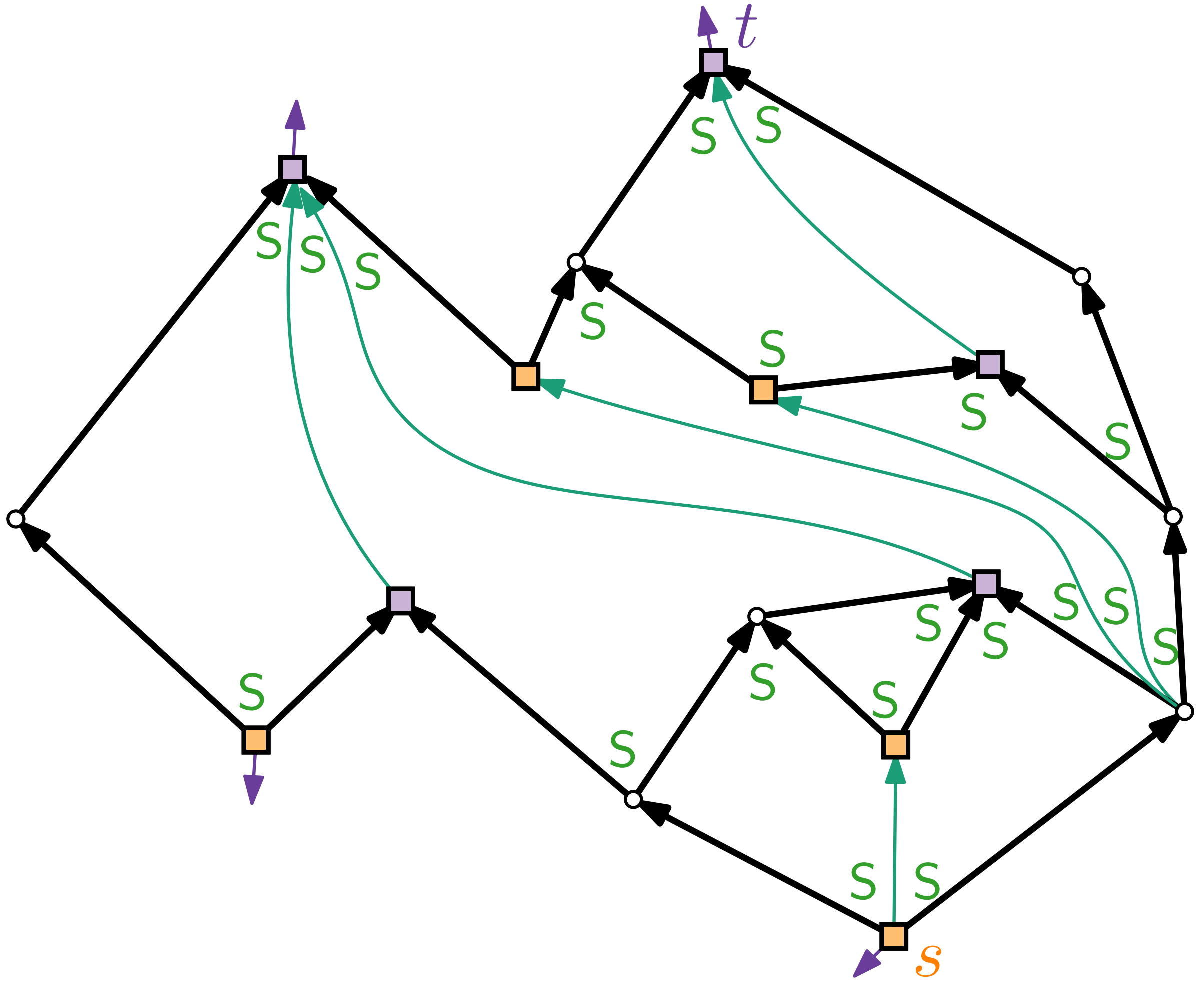

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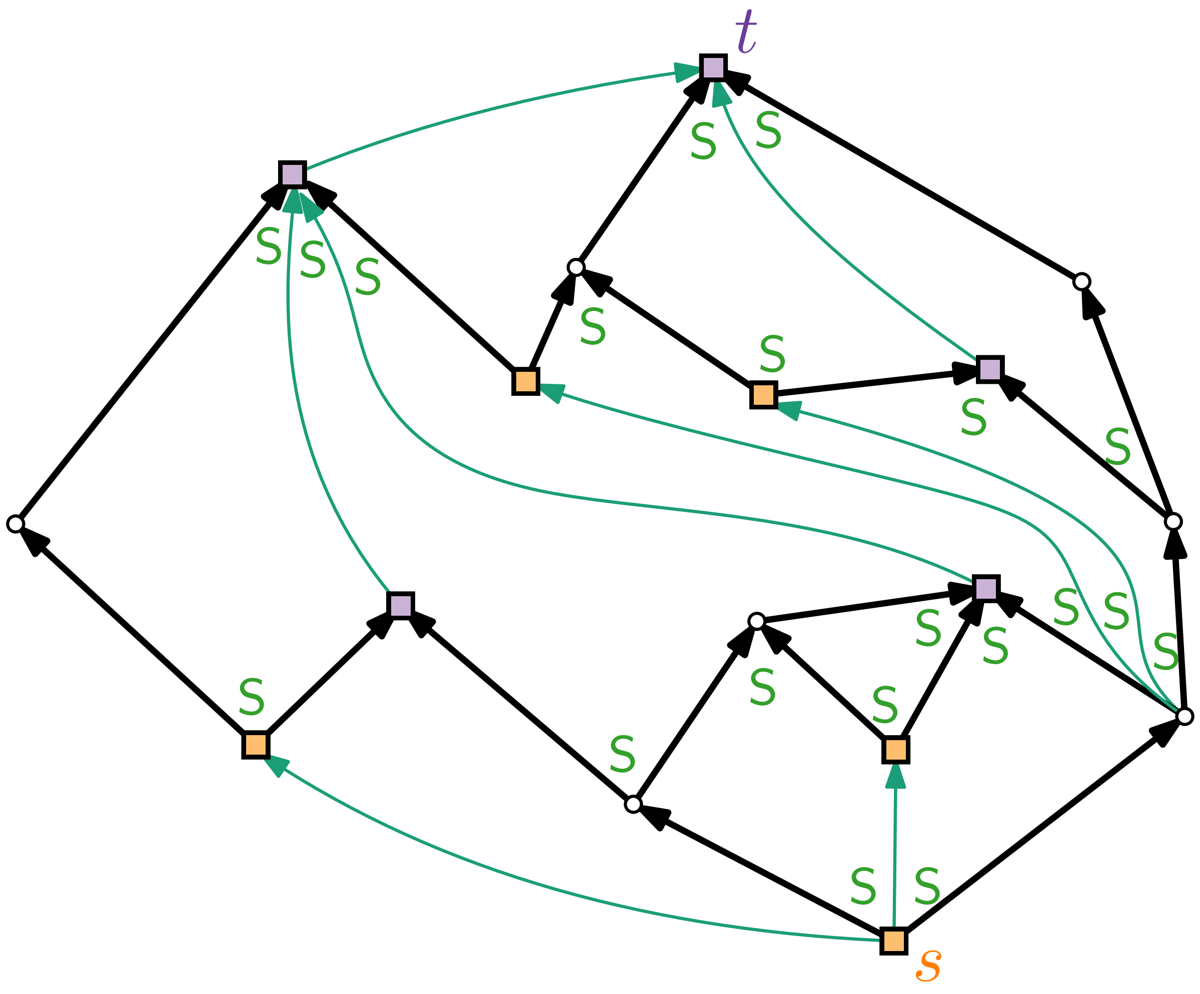

Result Upward Planarity Test

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- Deleted edges added in refinement step.

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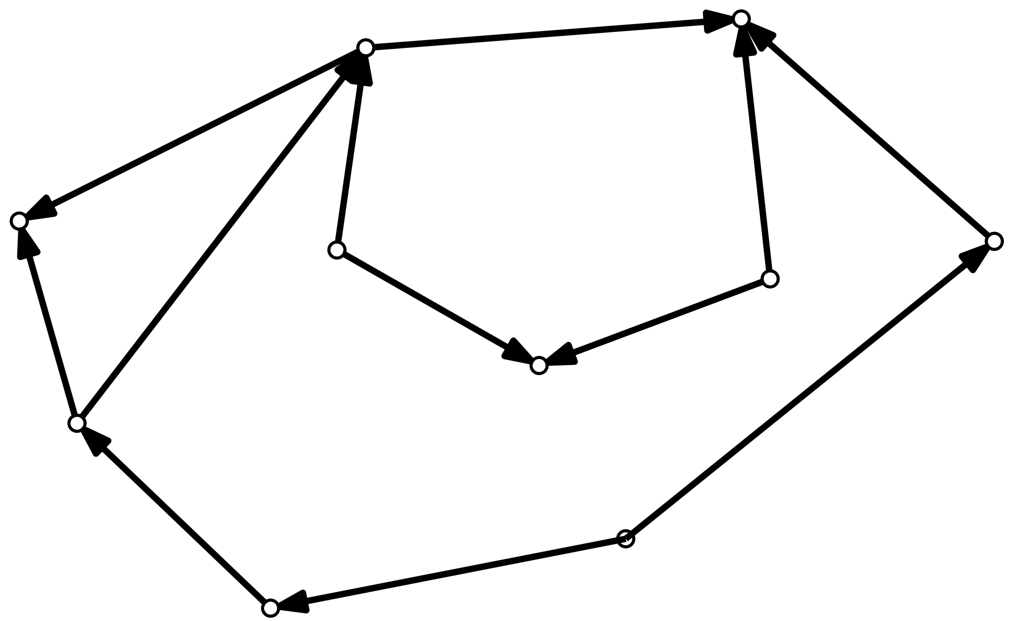

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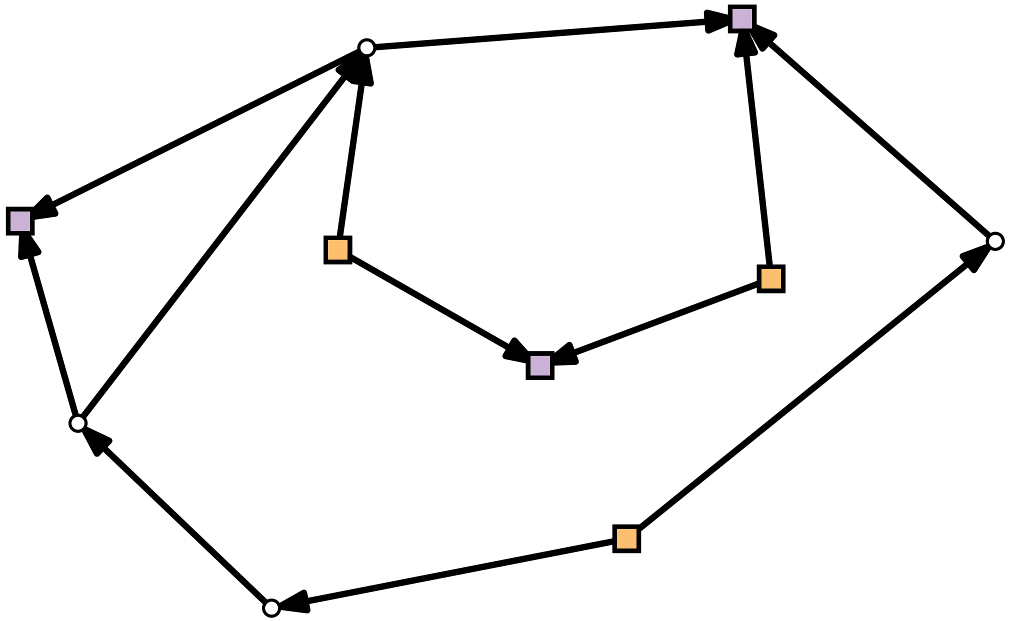

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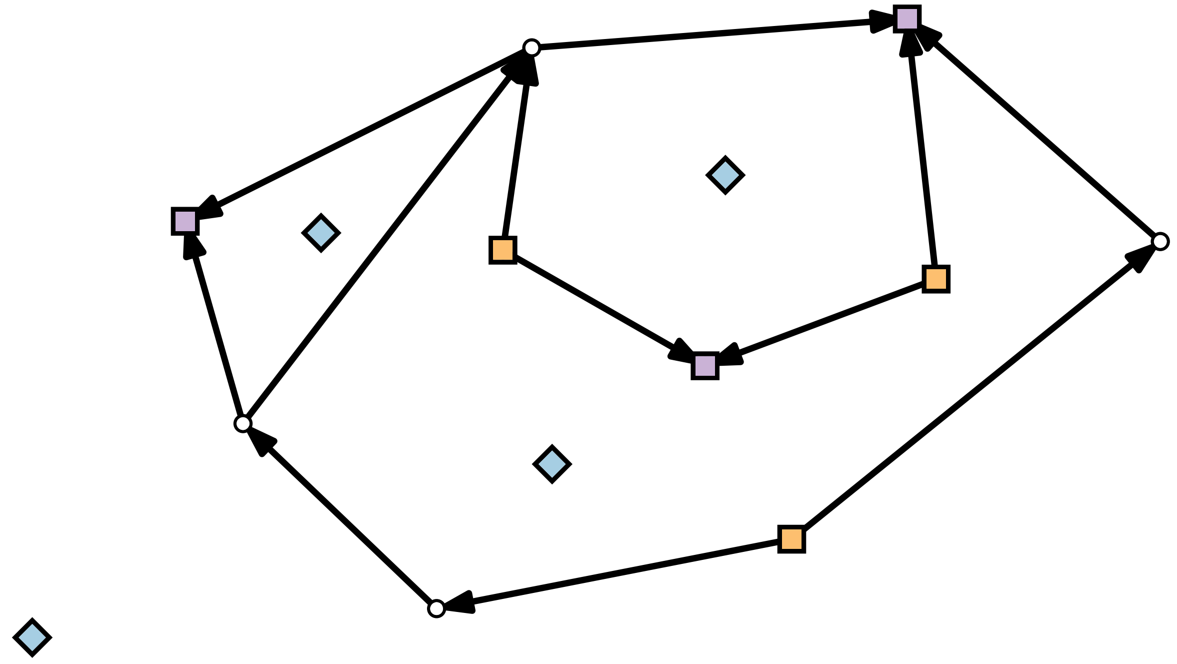

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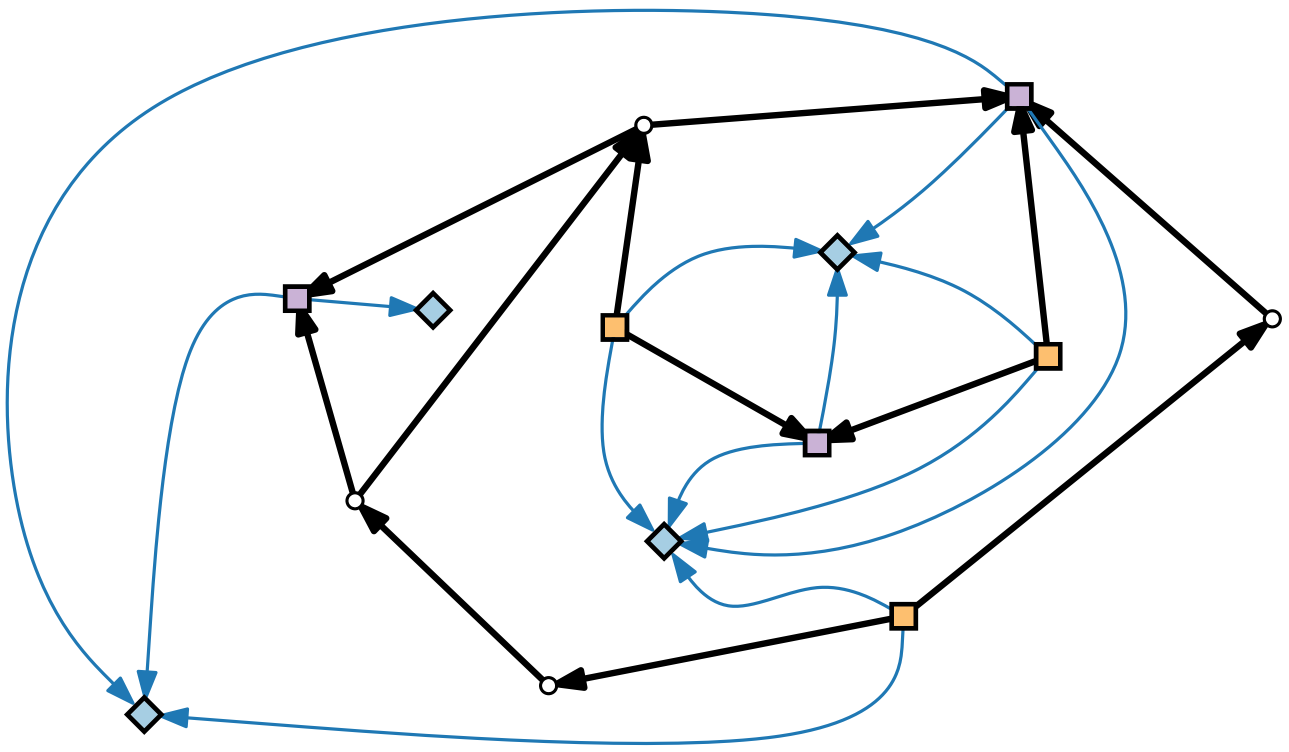

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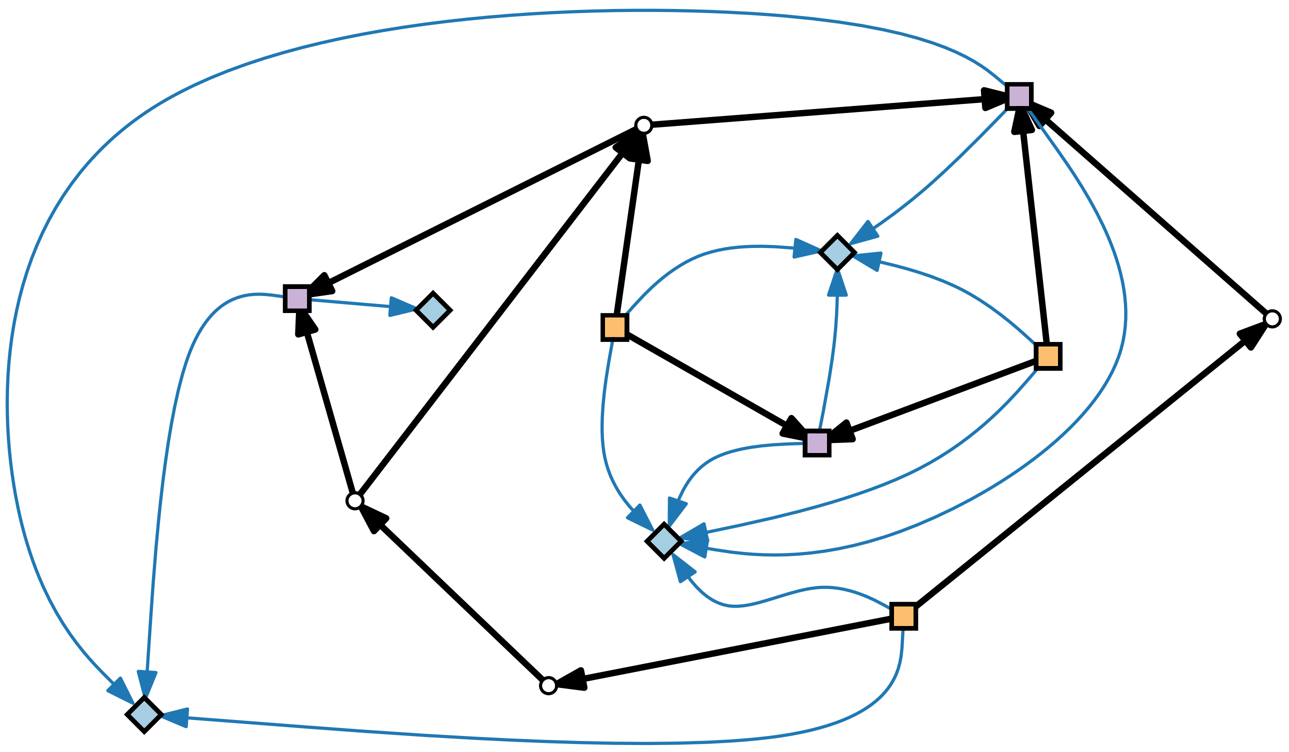

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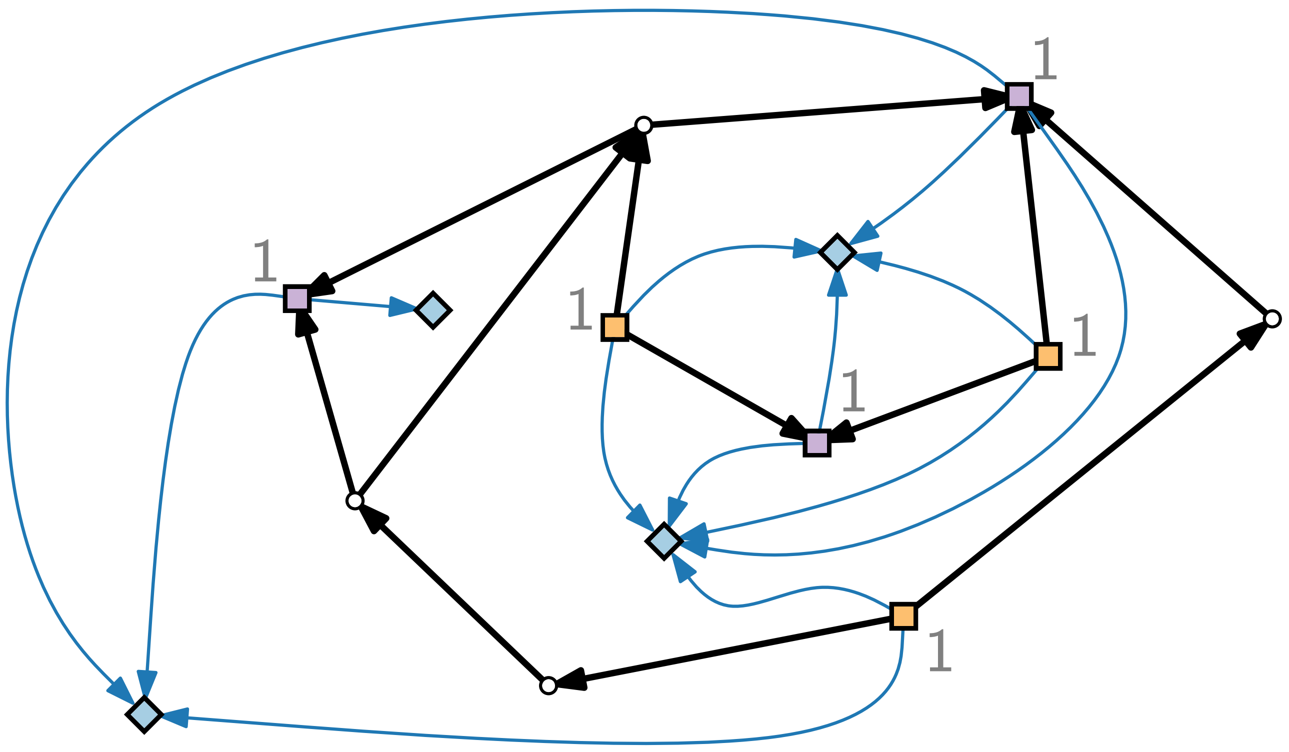

Finding a Consistent Assignment

Idea. Flow $(v, f) = 1$
from global source / sink v to the incident face f its large angle gets assigned to.

nodes of flow network   edges of flow network   supplies/demands of nodes   lower/upper bounds on edge capacities

Flow network.

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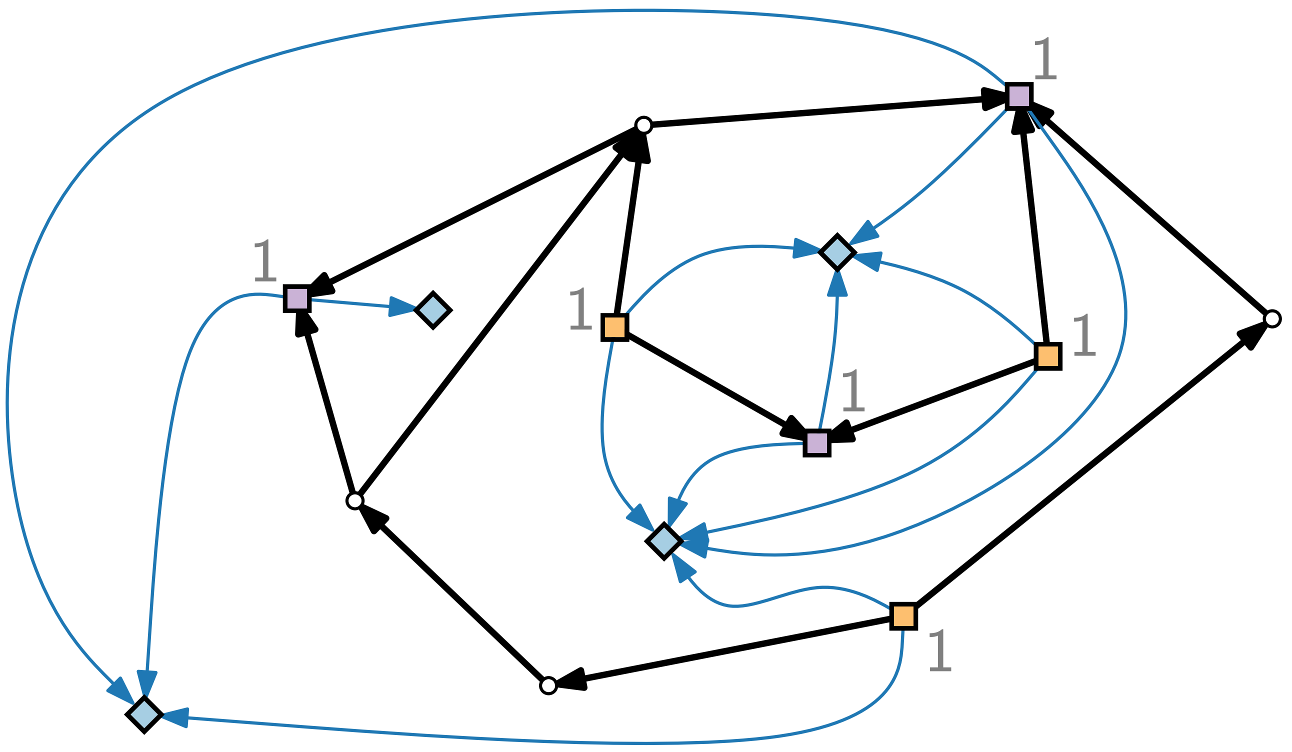

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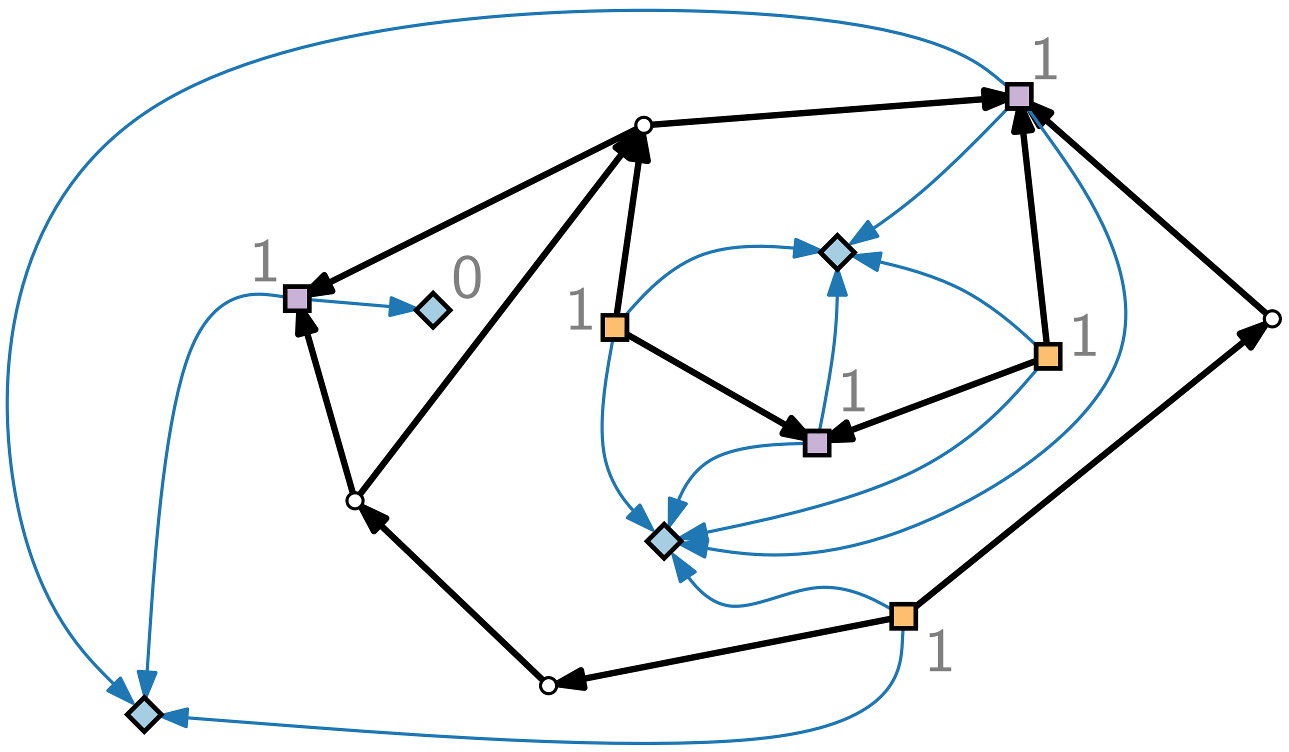

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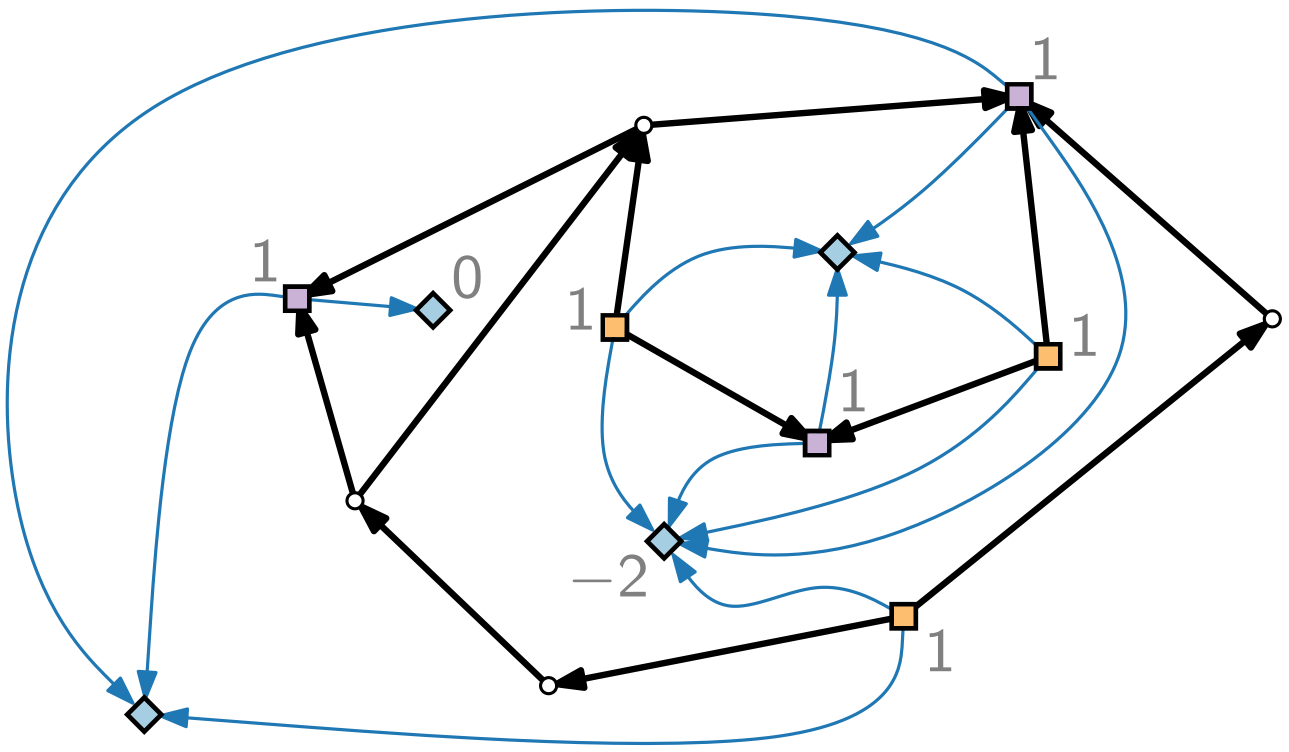

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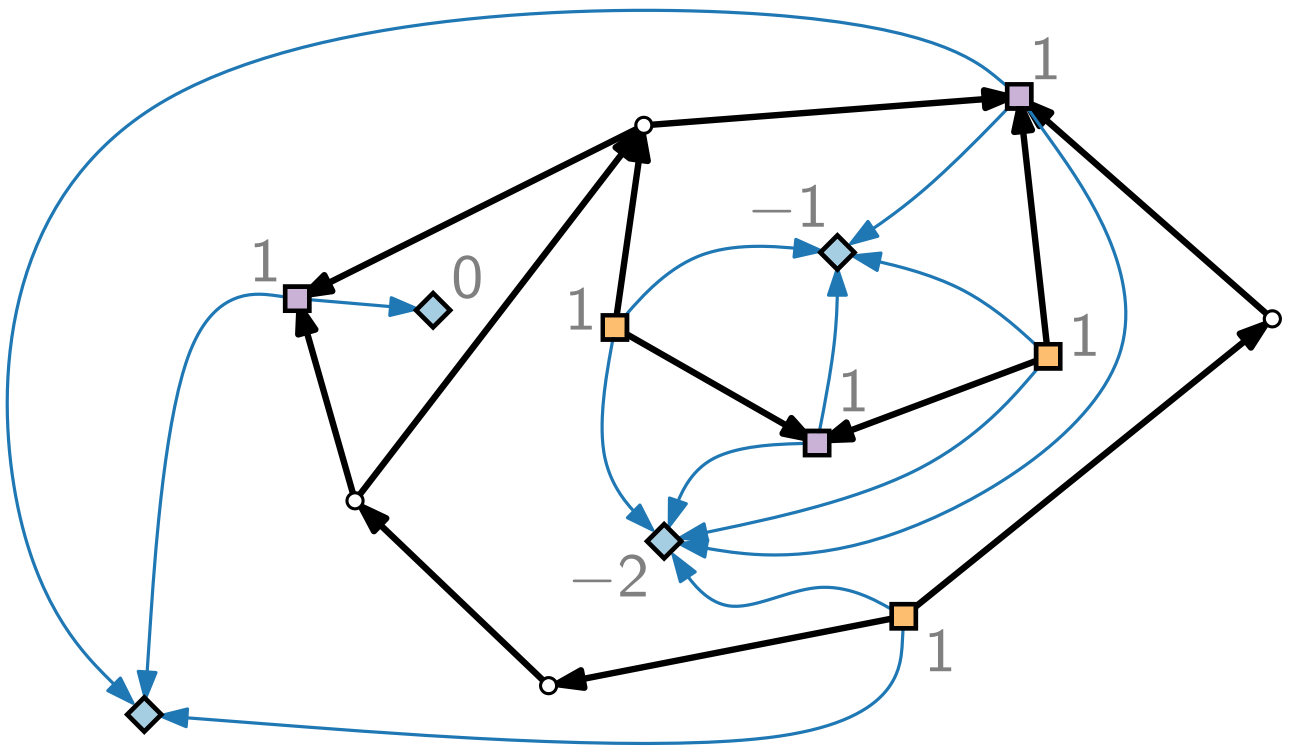

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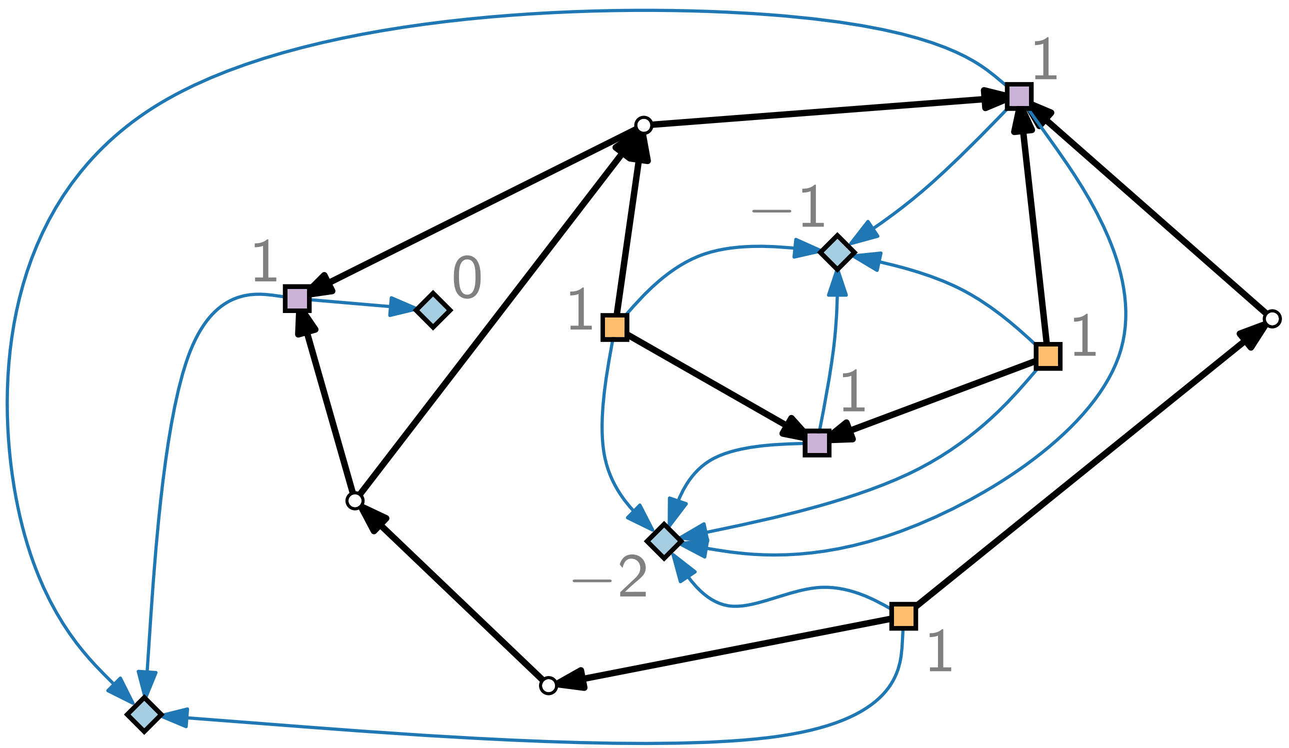

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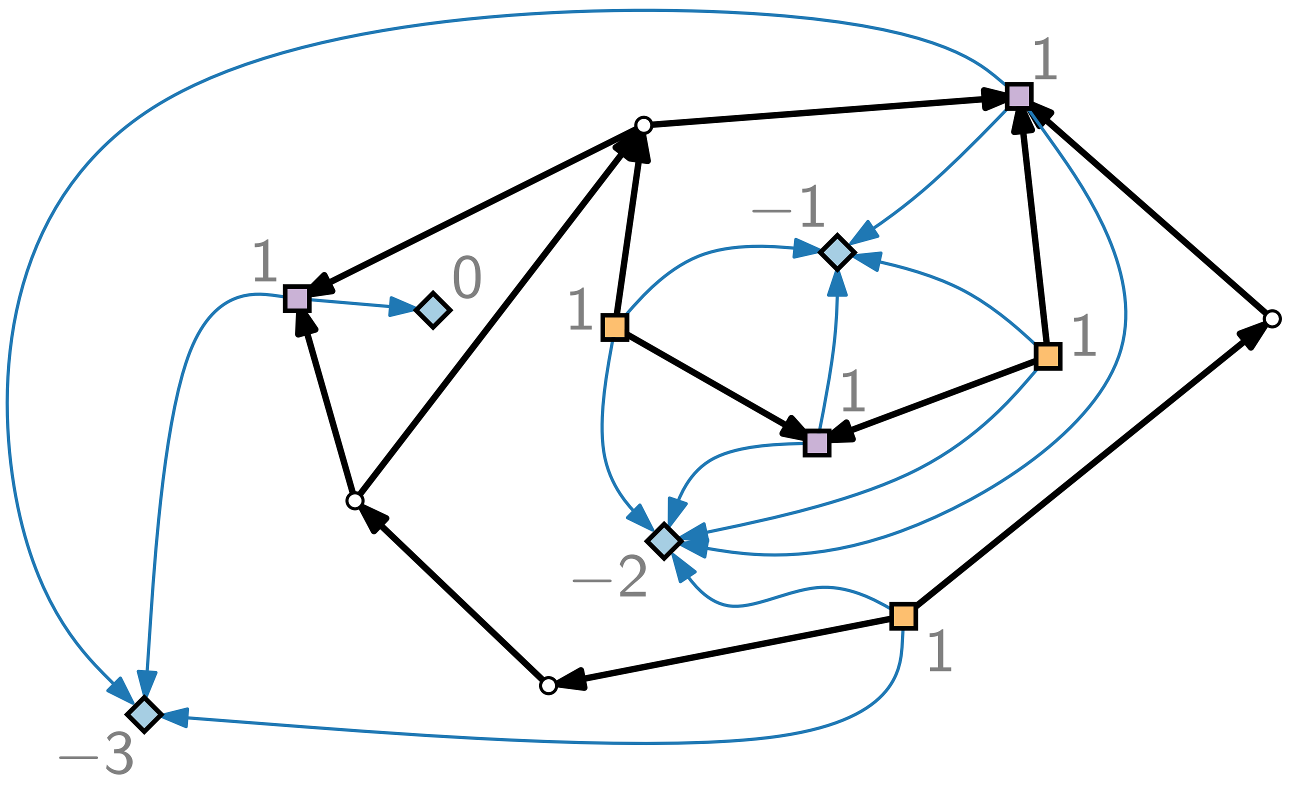

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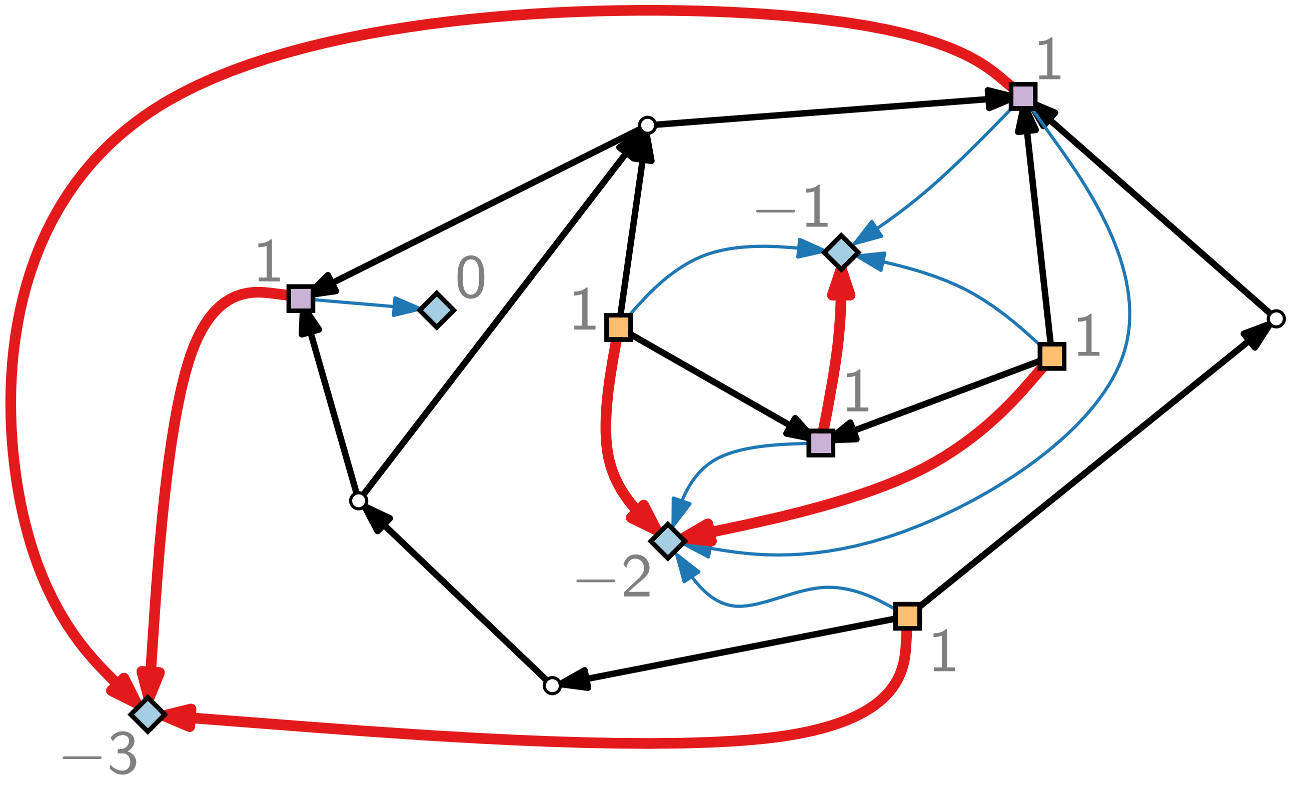
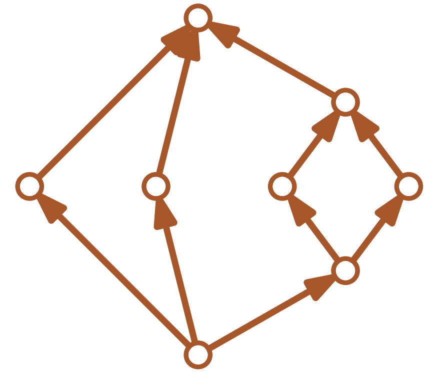
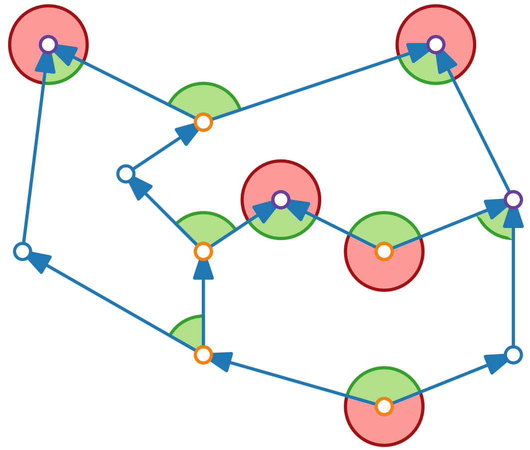


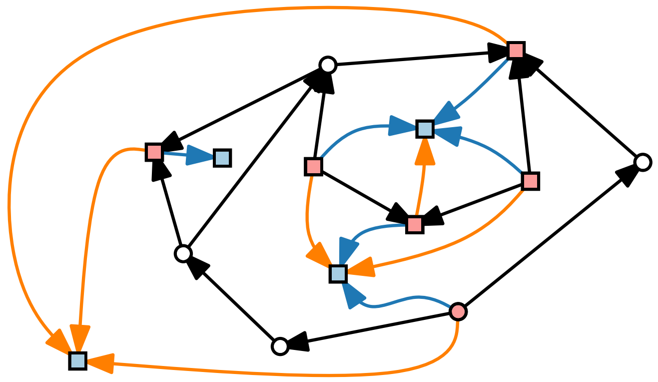
Visualization of Graphs

Lecture 5:
Upward Planar Drawings



Part II:
Series-Parallel Graphs





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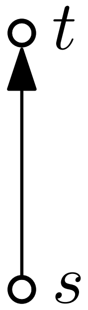

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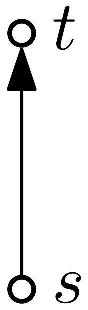

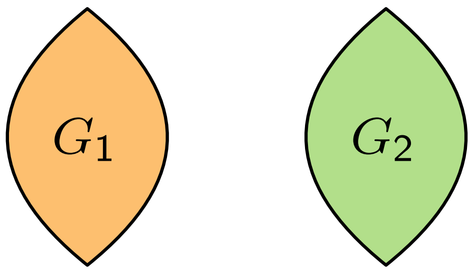

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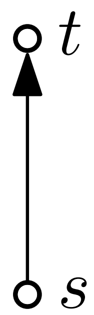

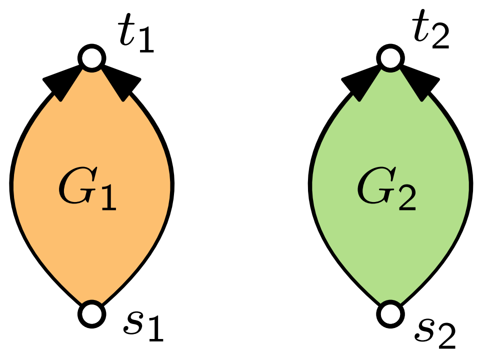

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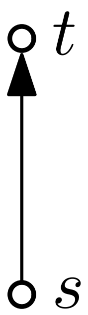

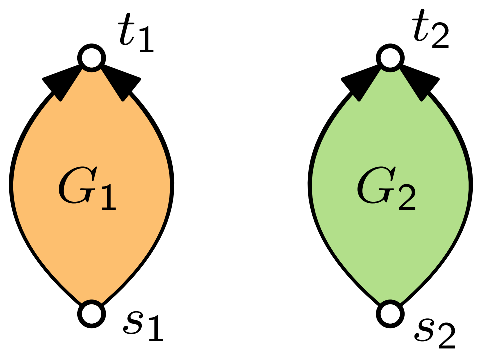

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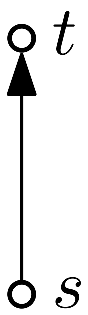
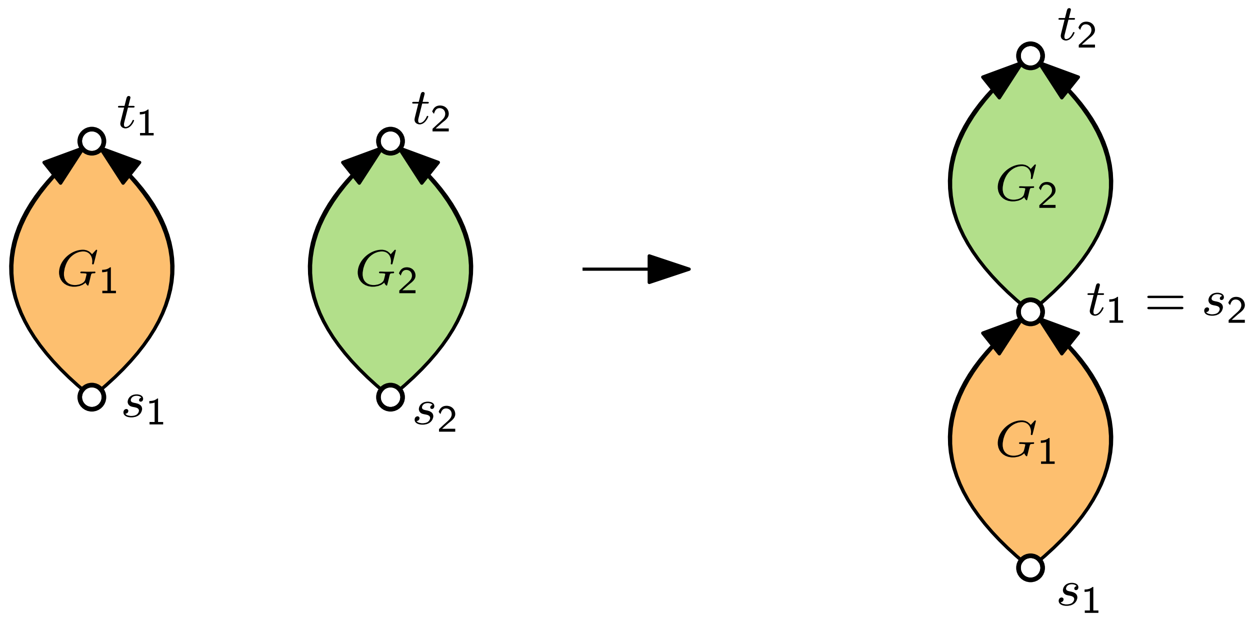


Series composition



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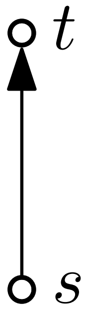

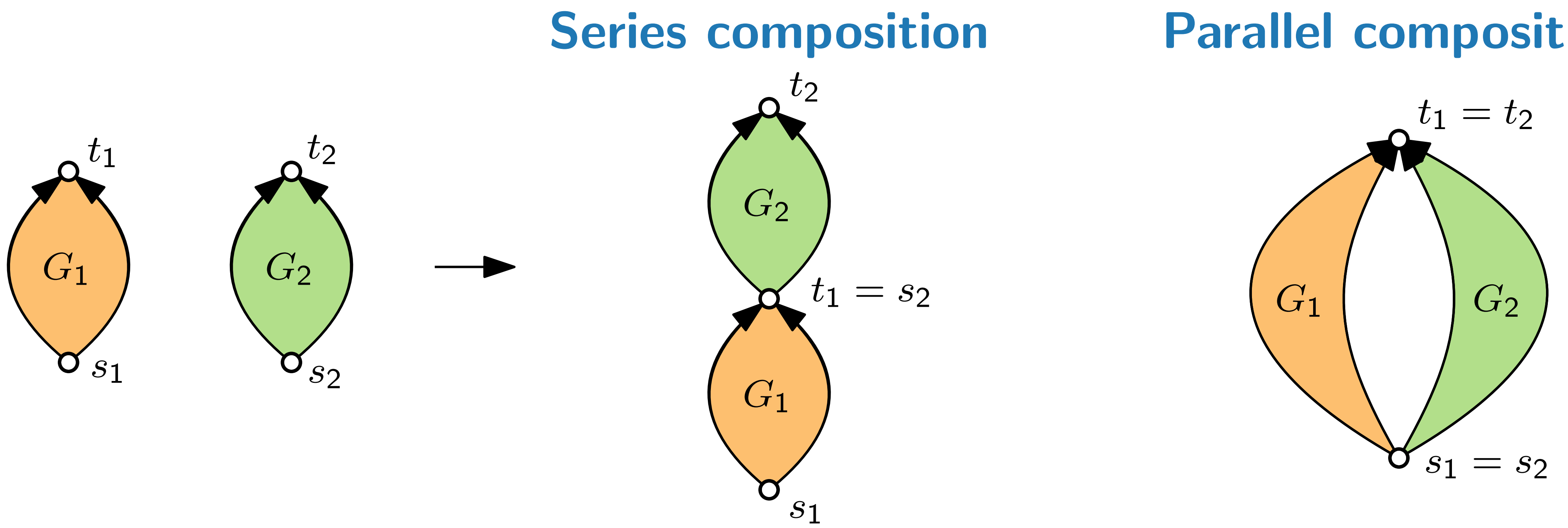

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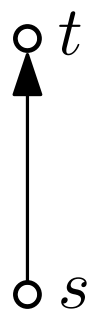
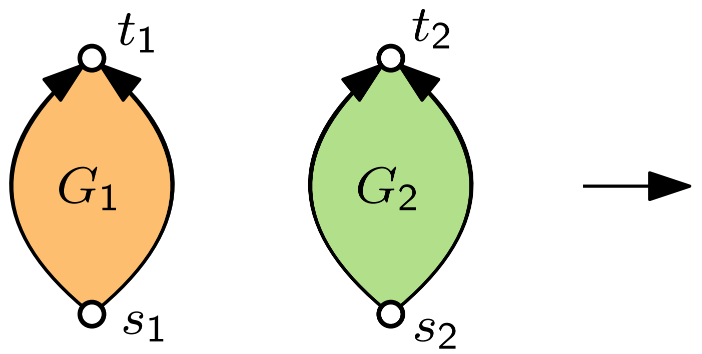


Convince yourself that series-parallel graphs are (upward) planar!



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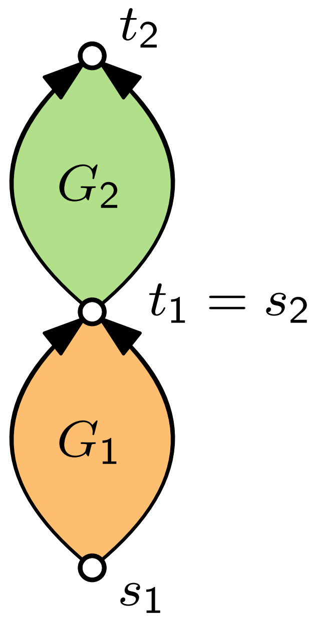
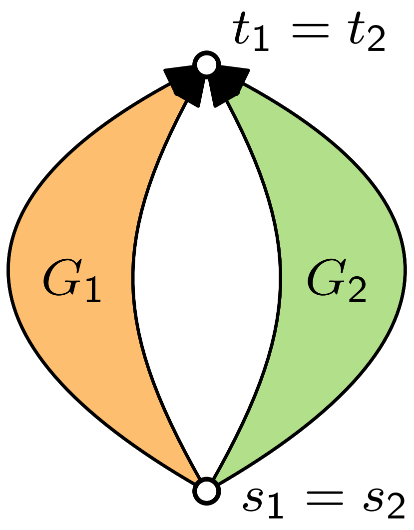


Parallel composition



Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**.

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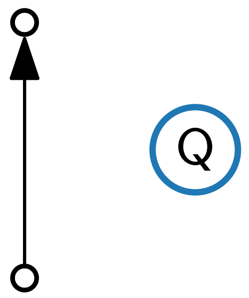

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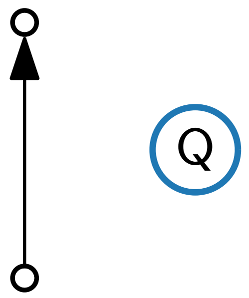

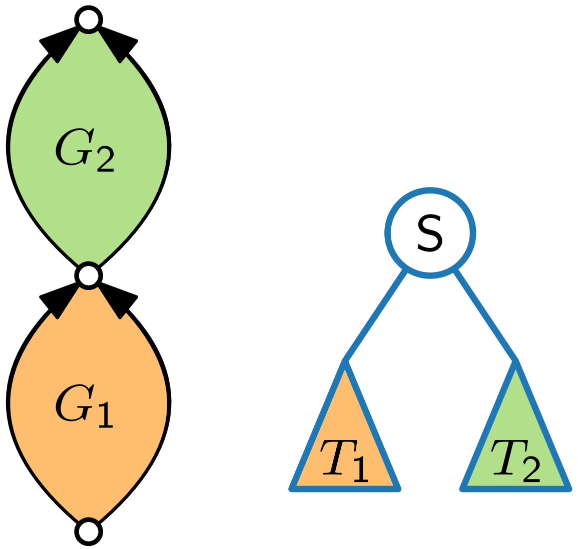

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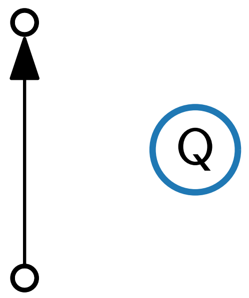
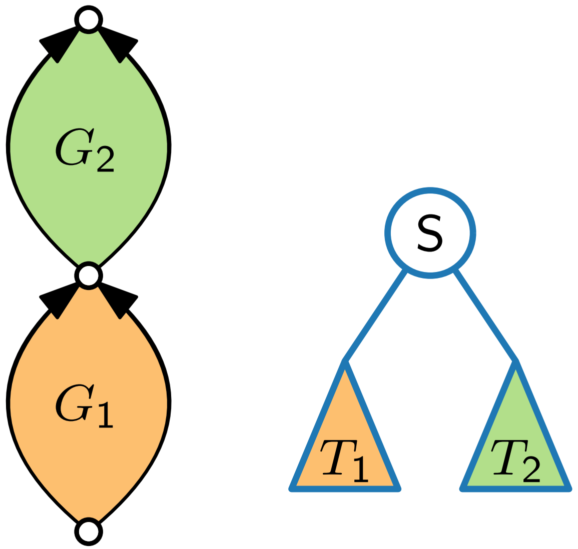
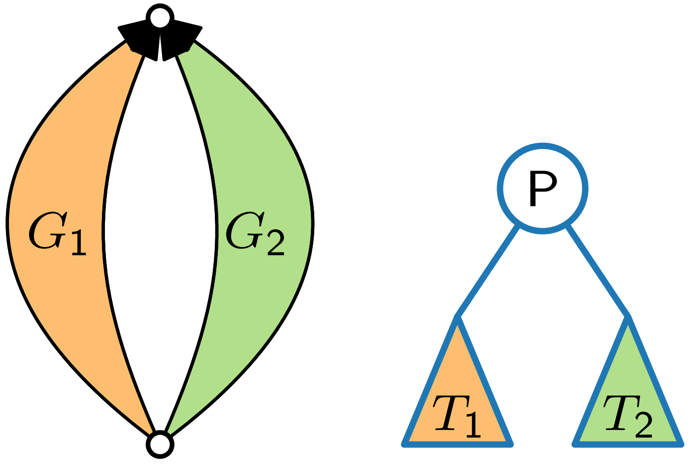
- A **P**-node represents a parallel composition; its children T_1 and T_2 represent G_1 and G_2







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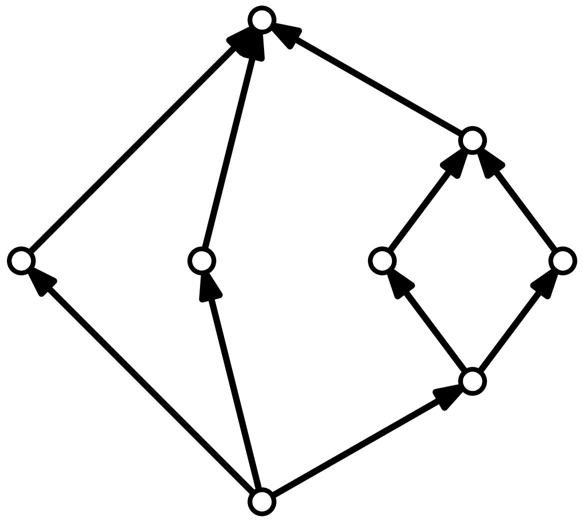

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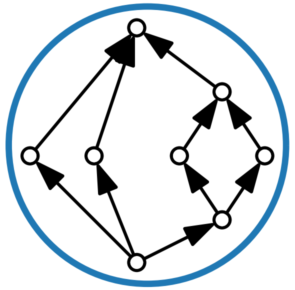

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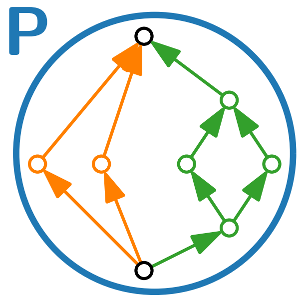

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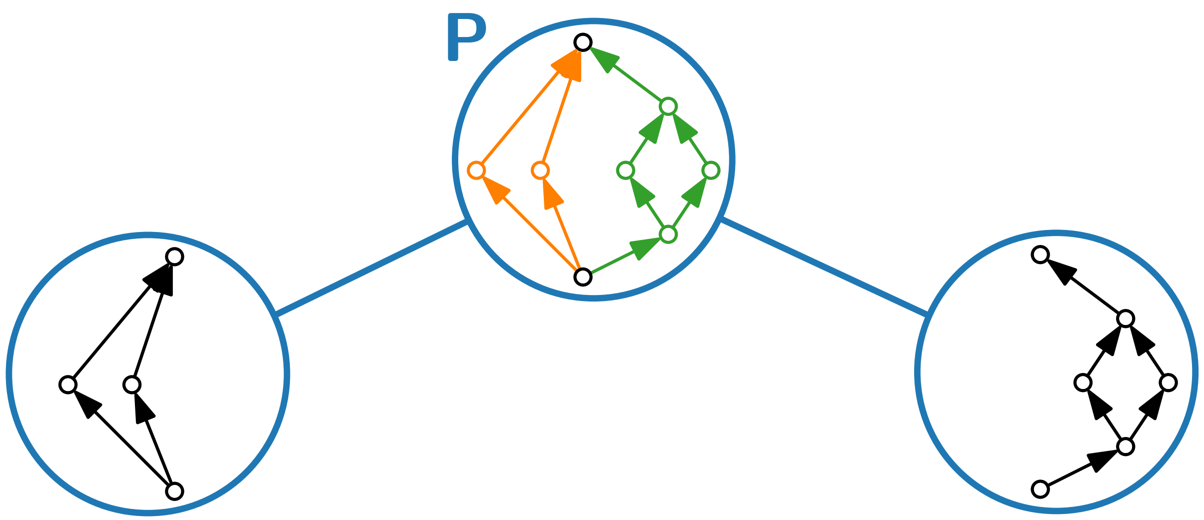

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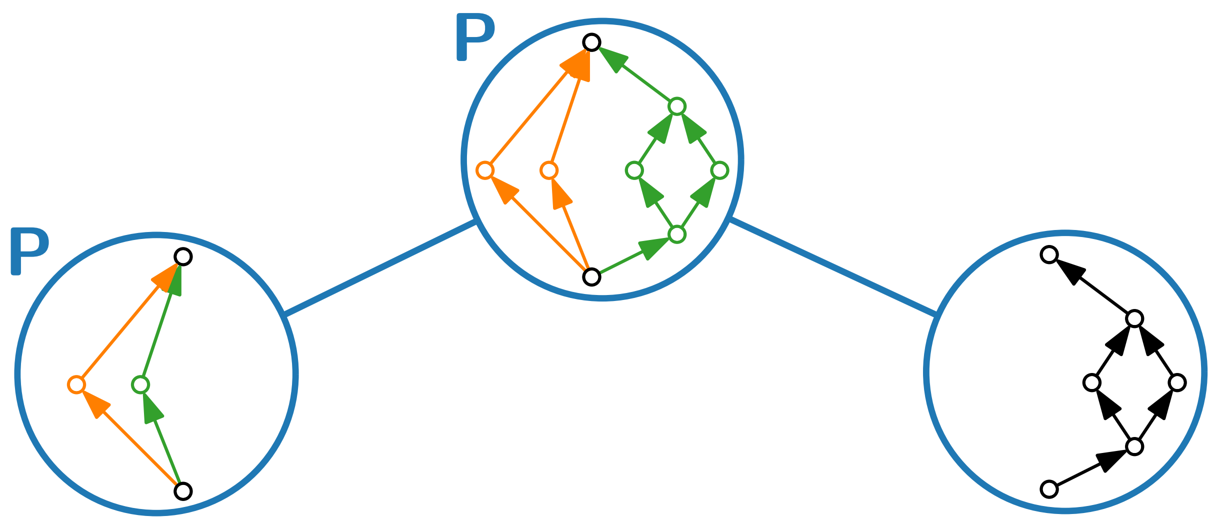

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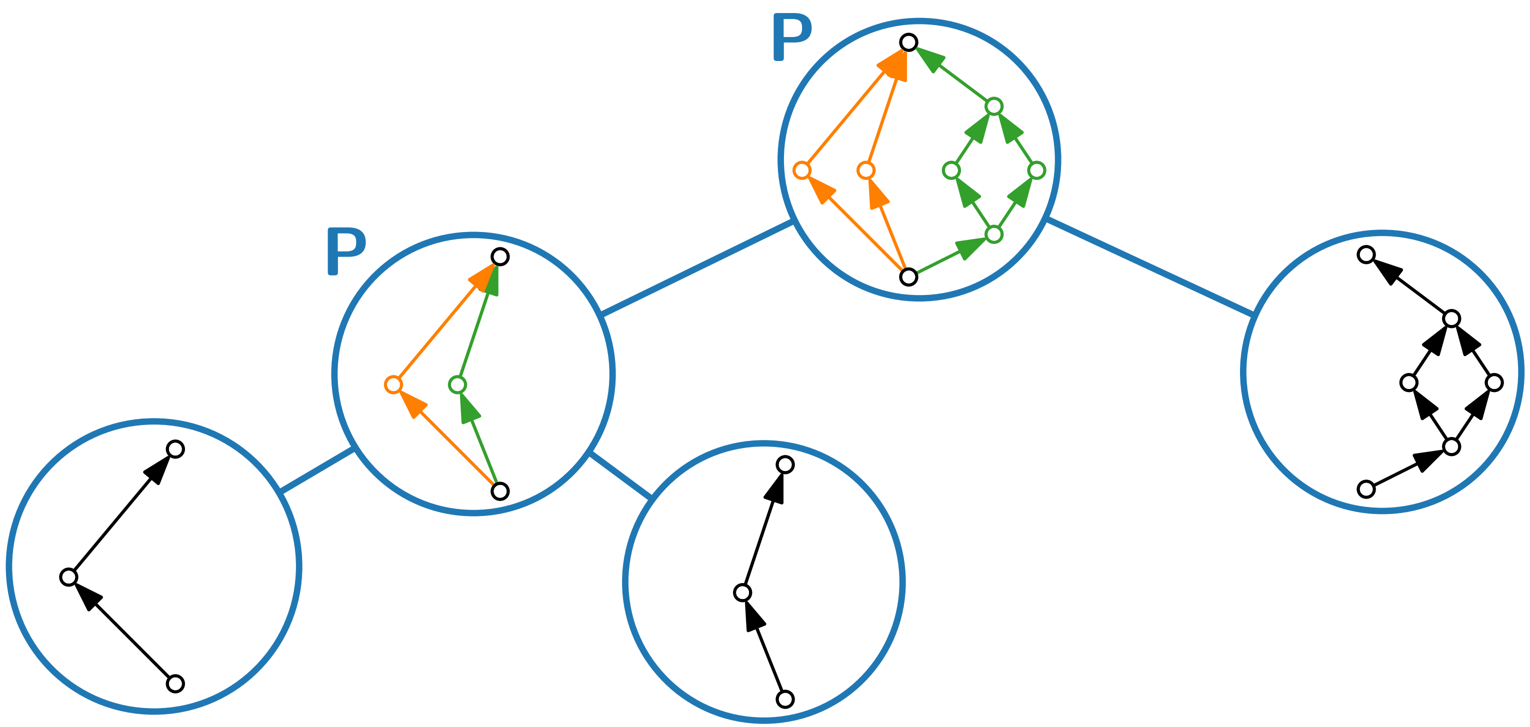

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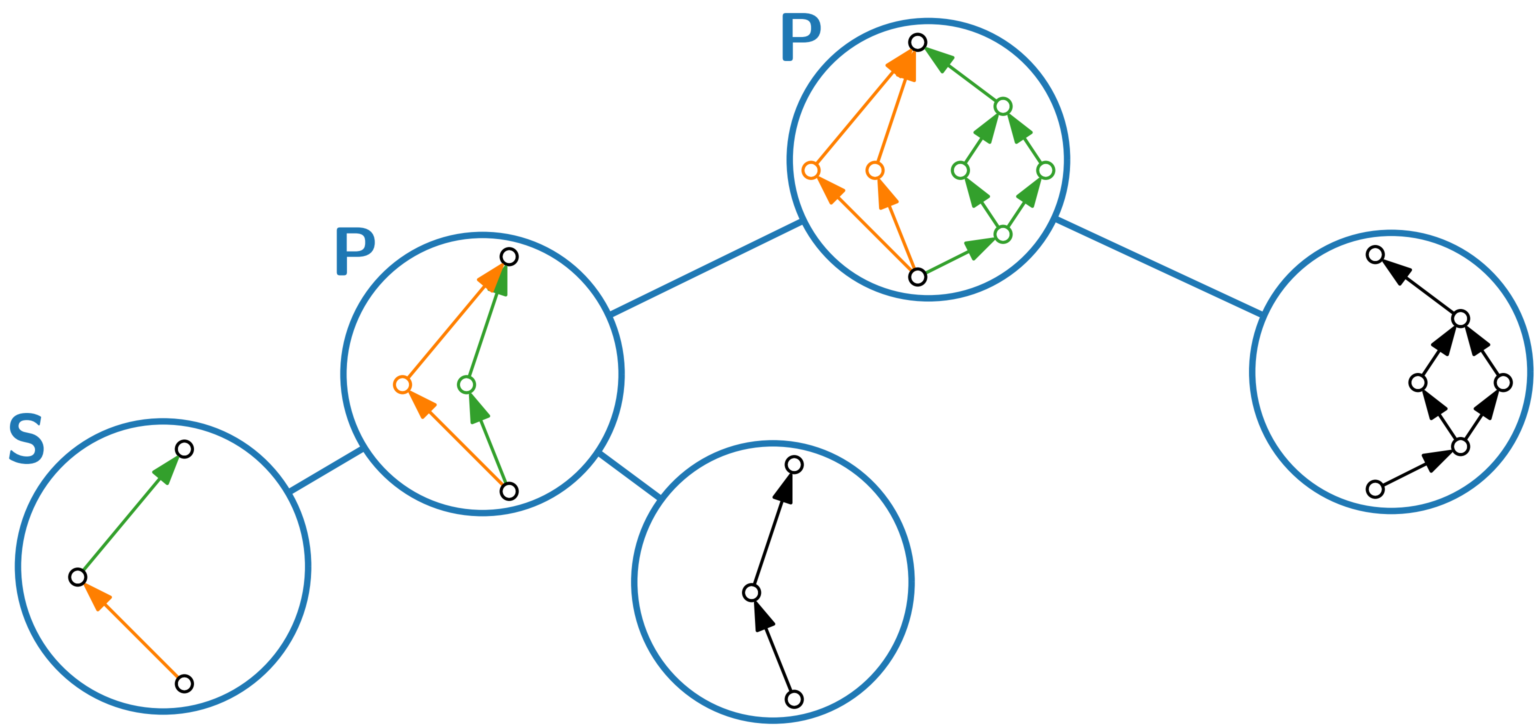

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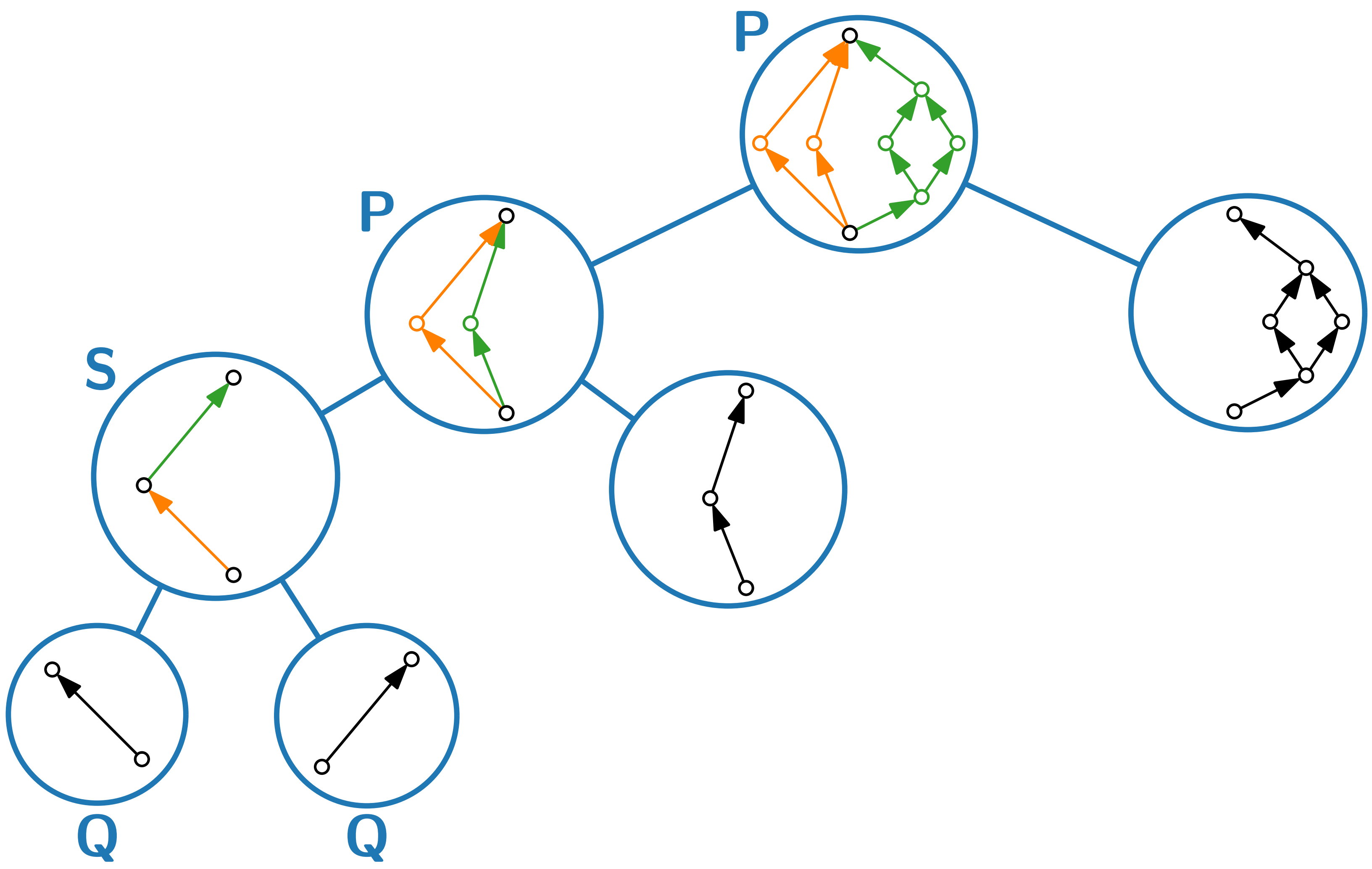

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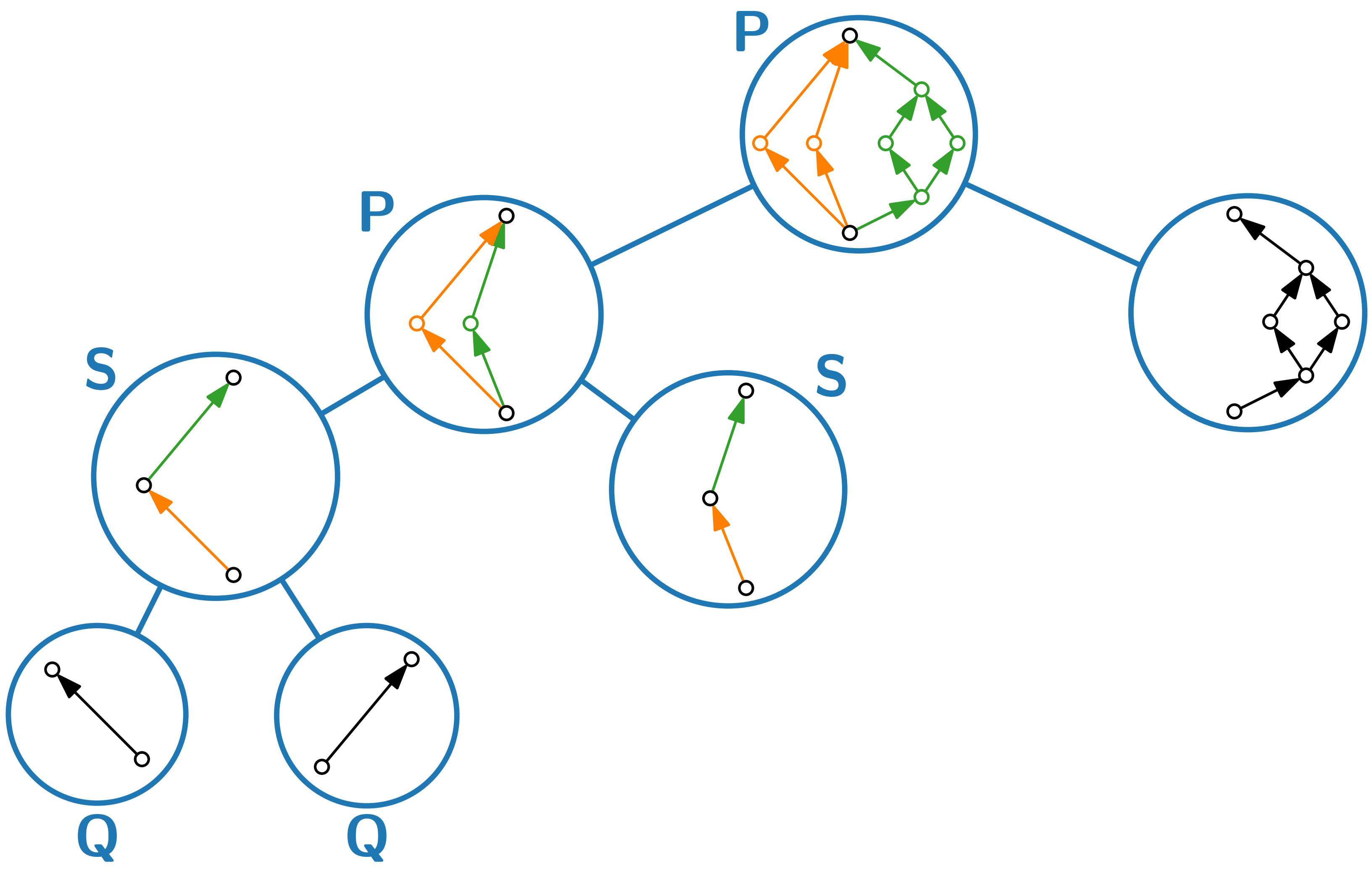

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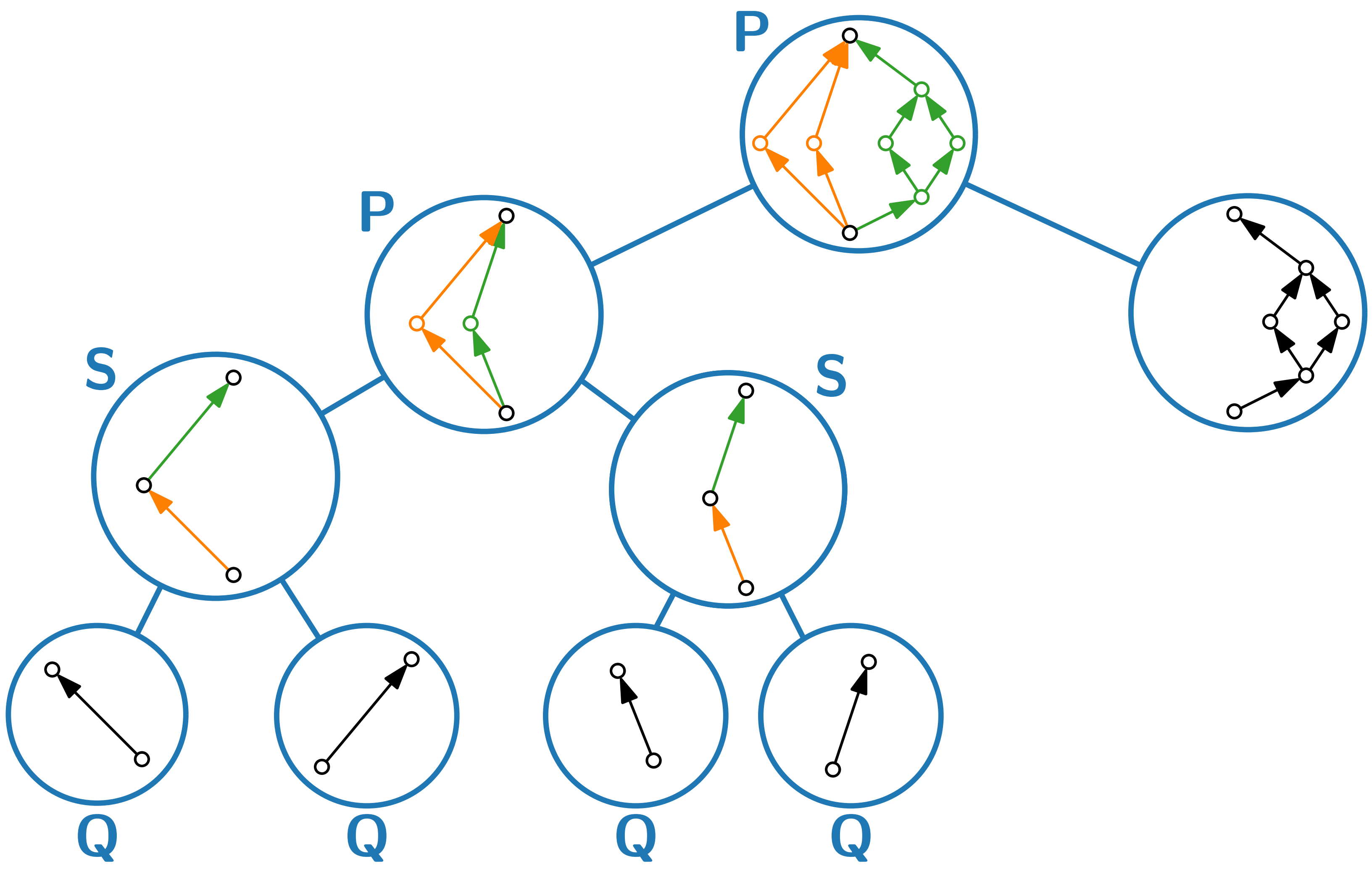

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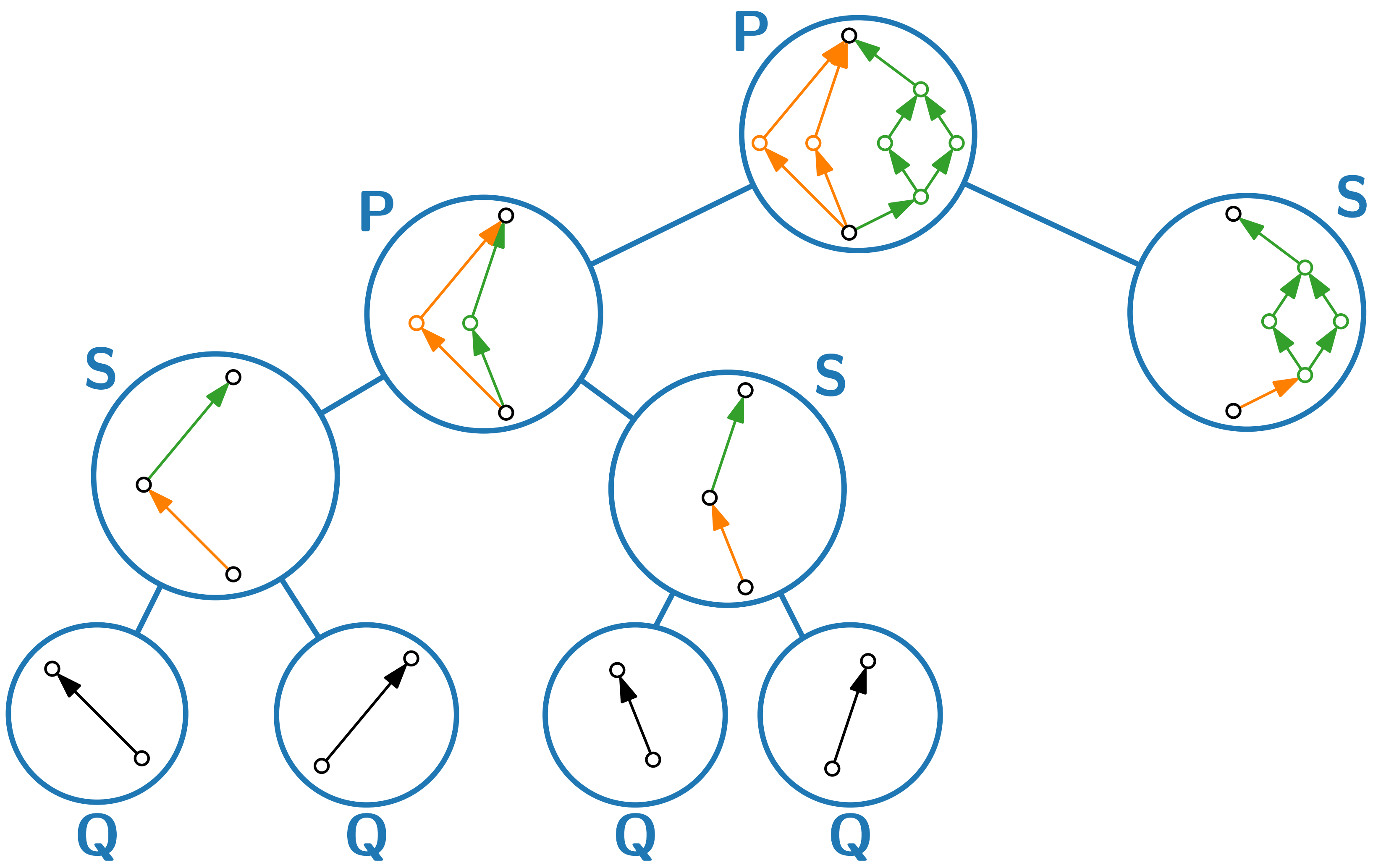

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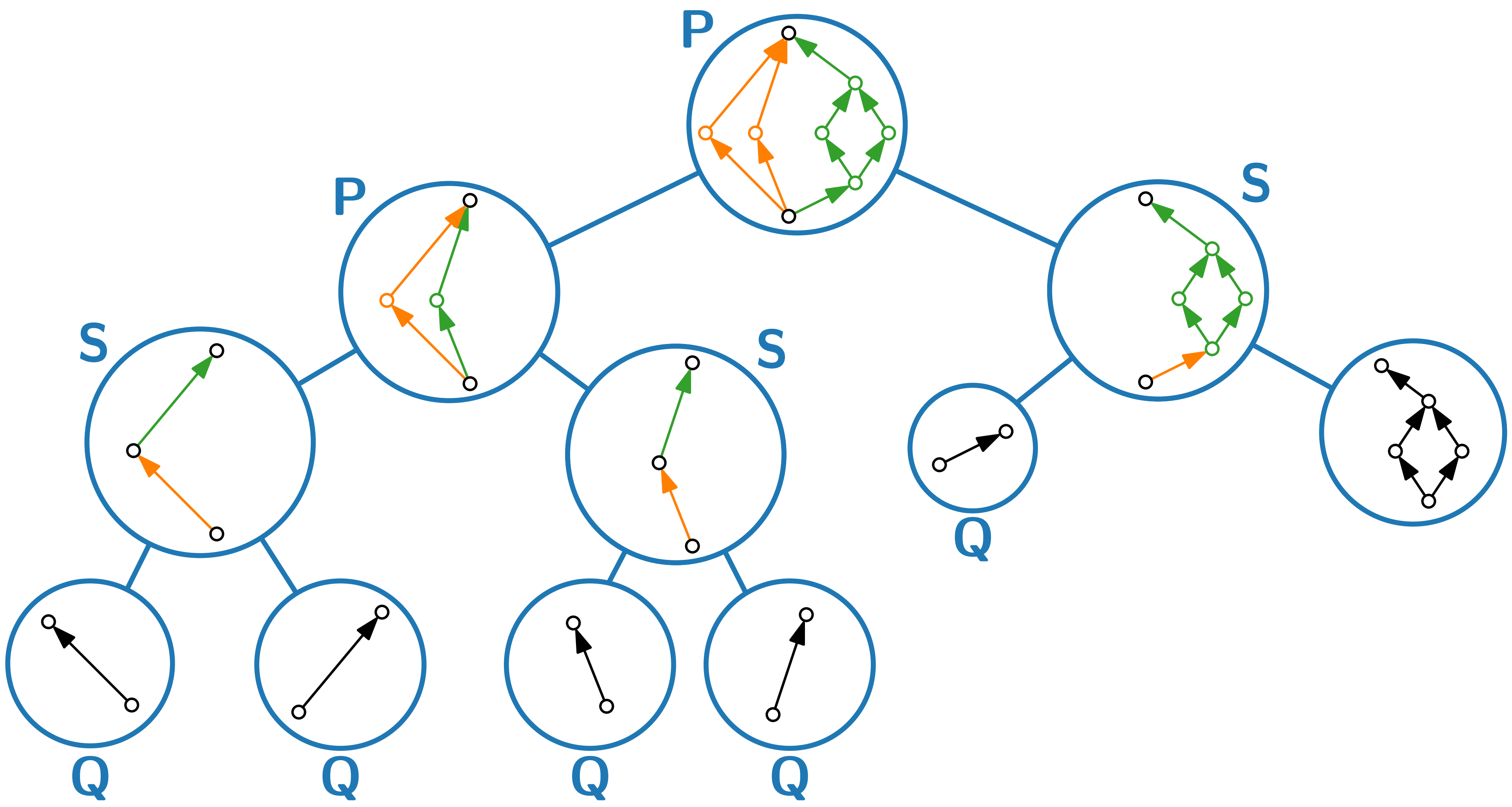

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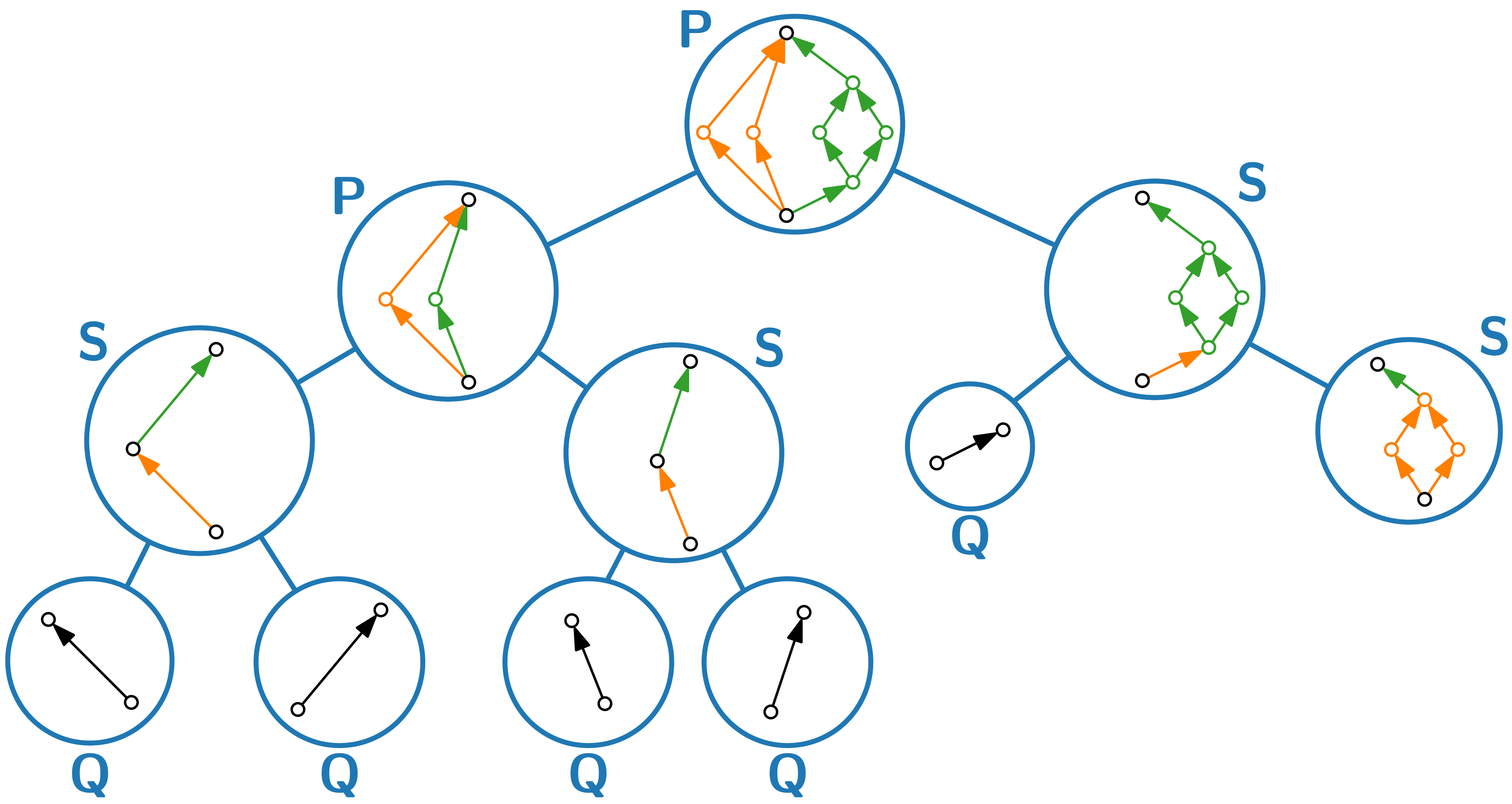

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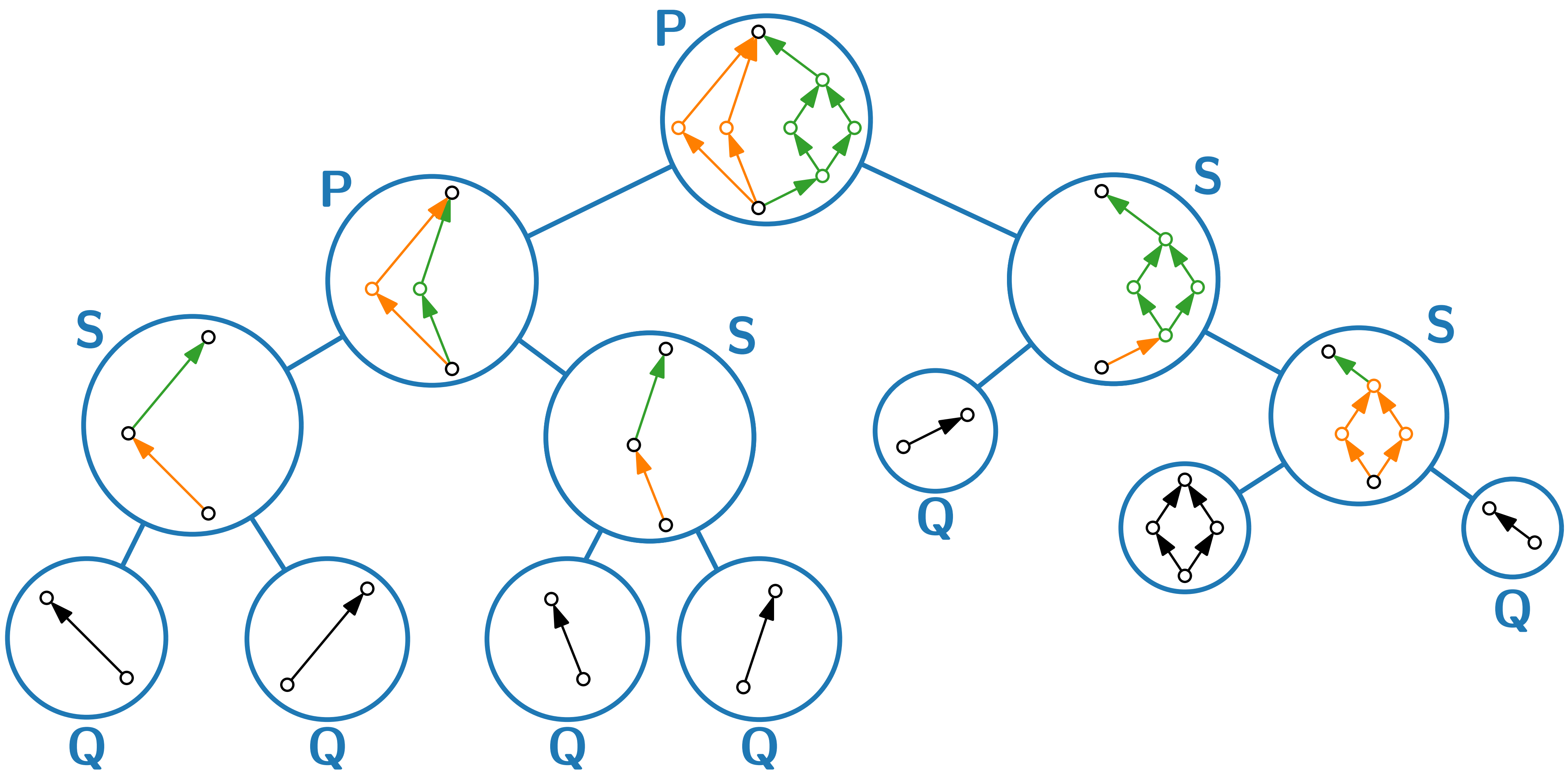

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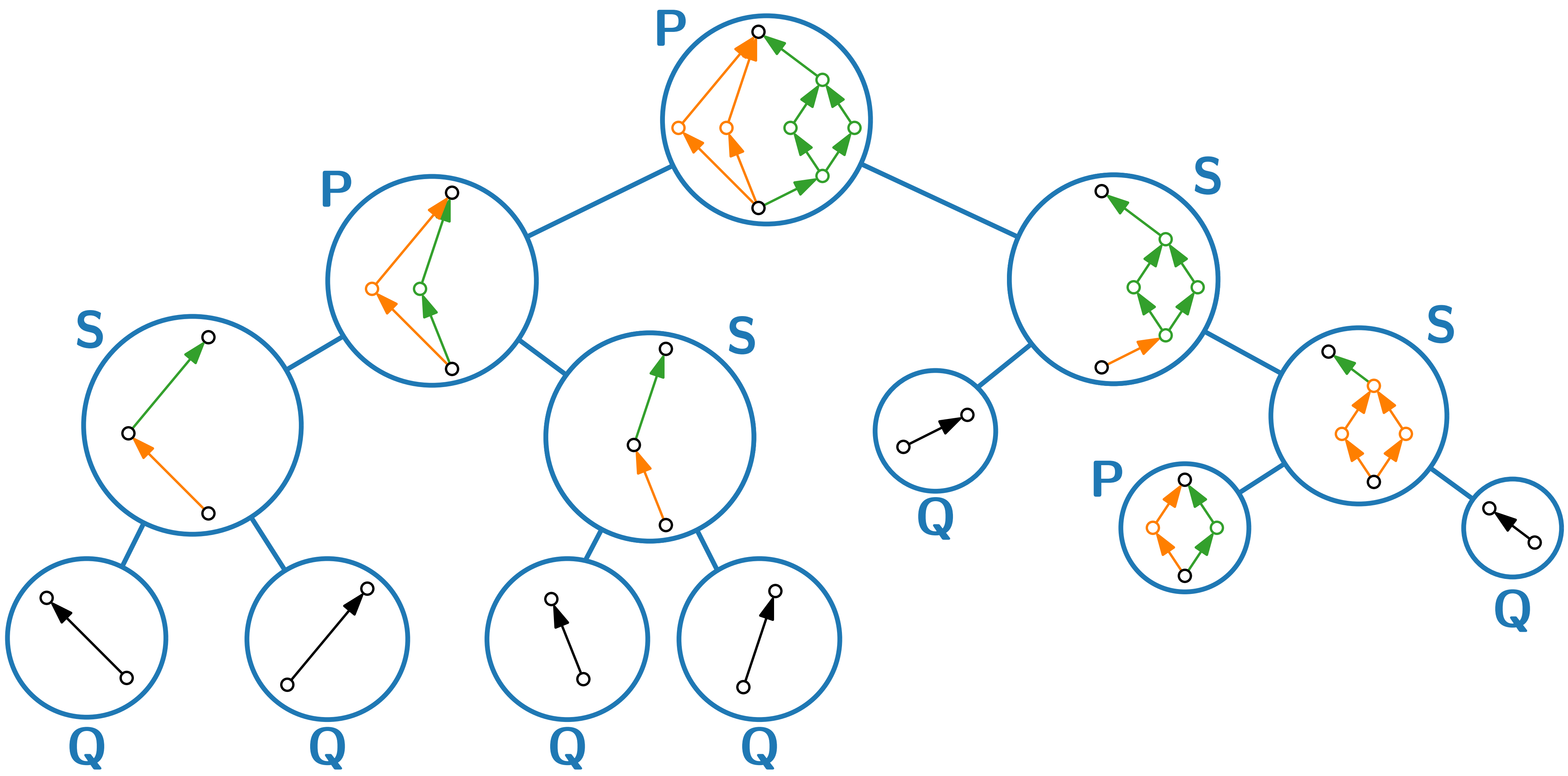

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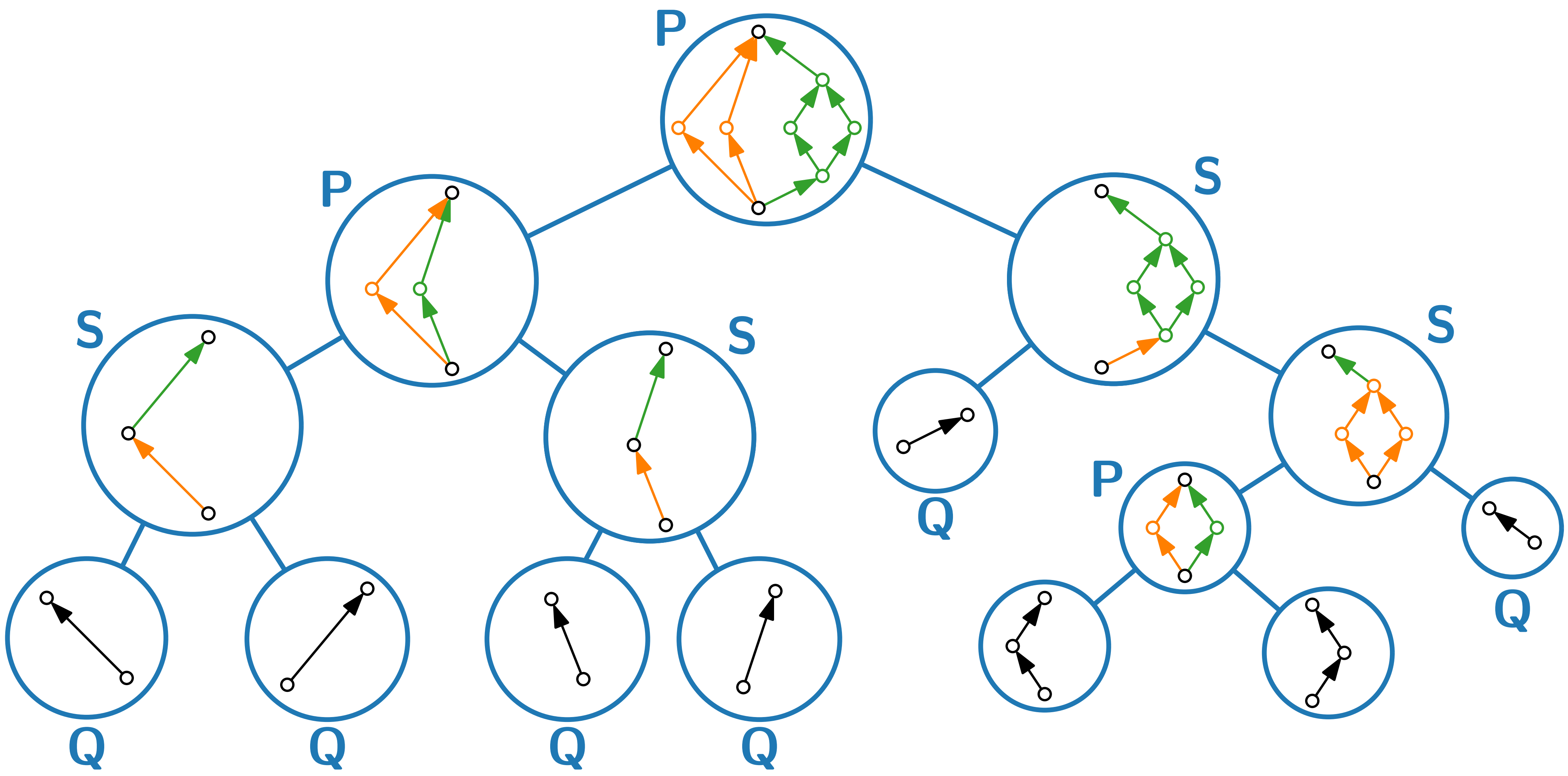

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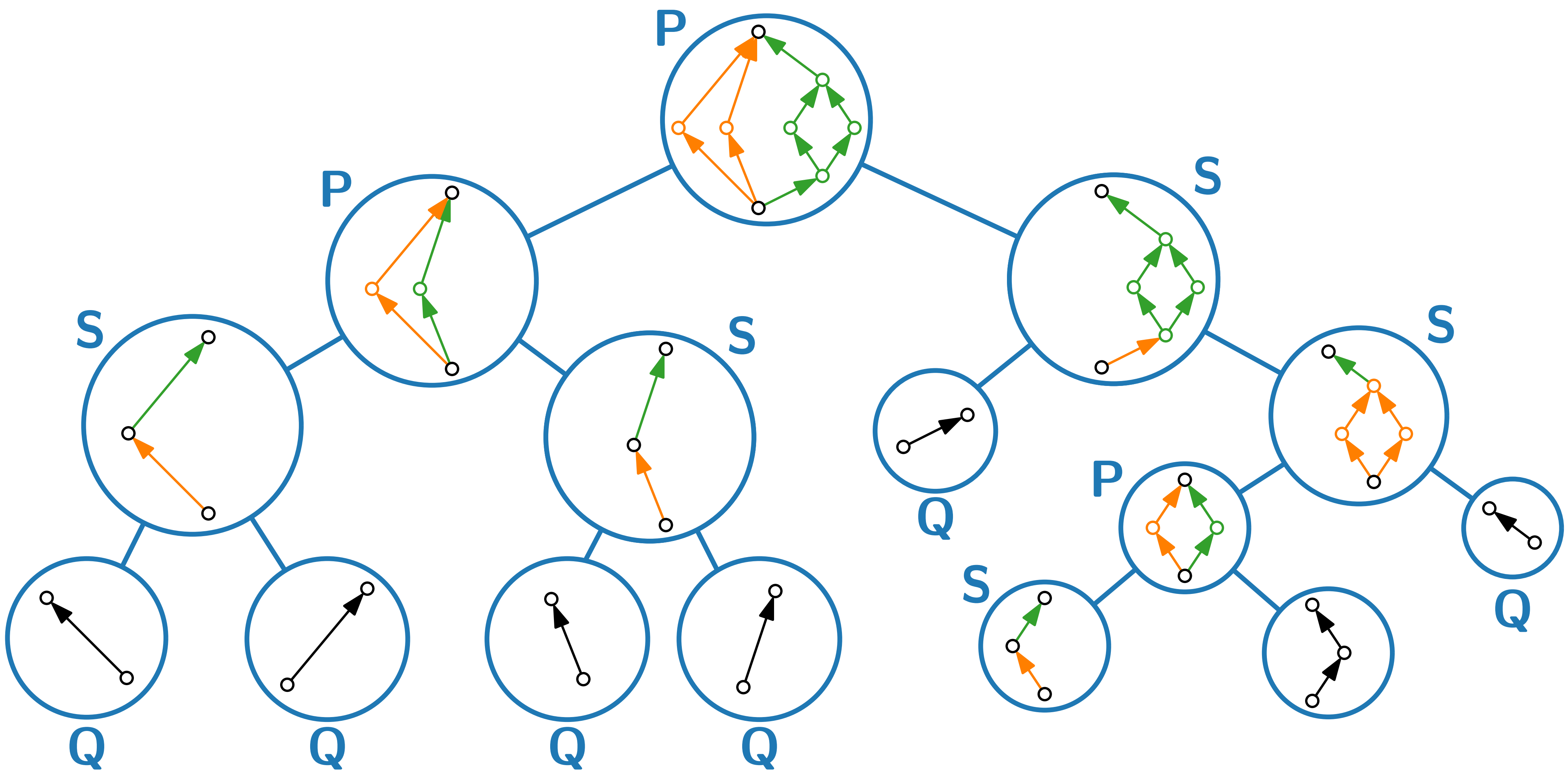

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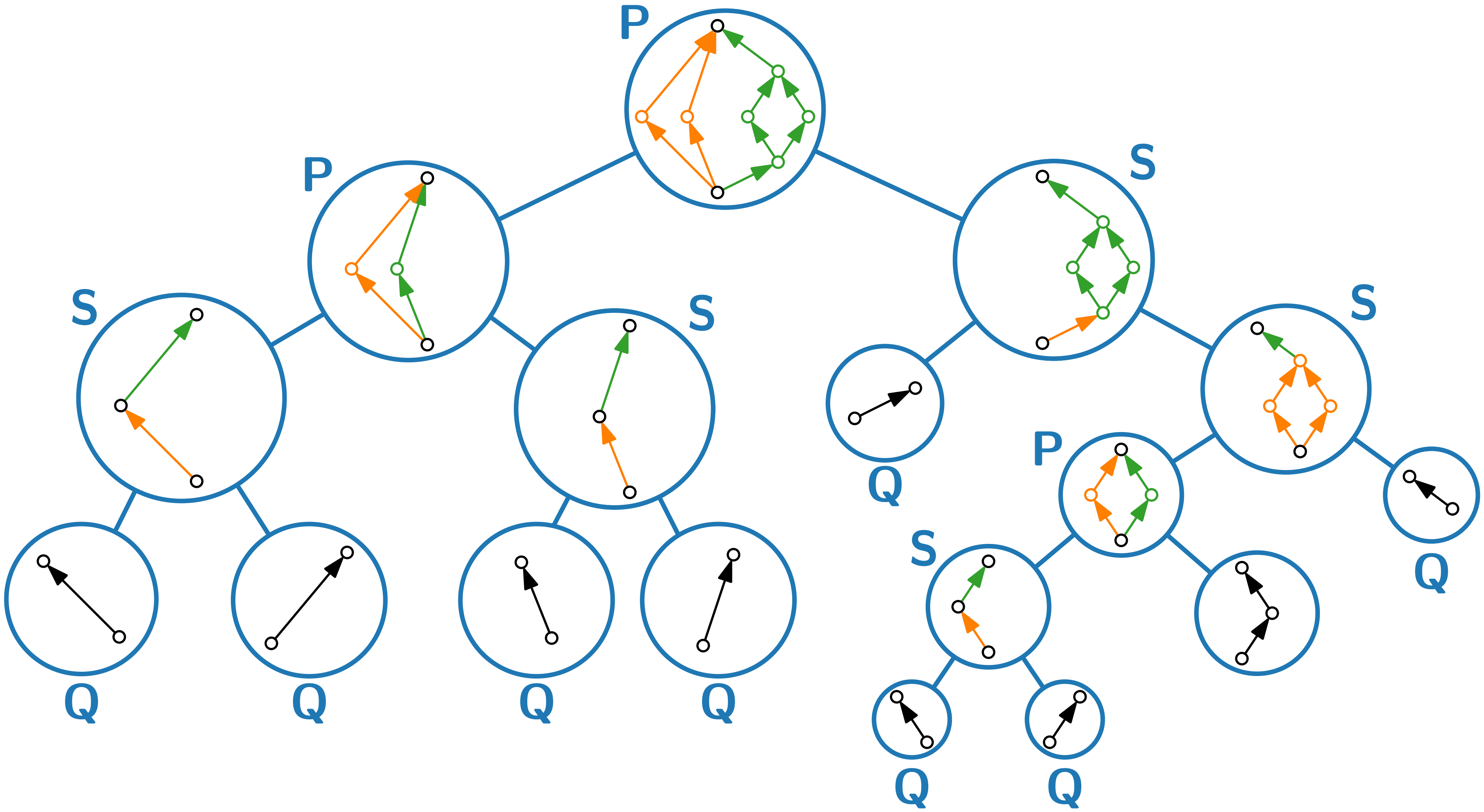
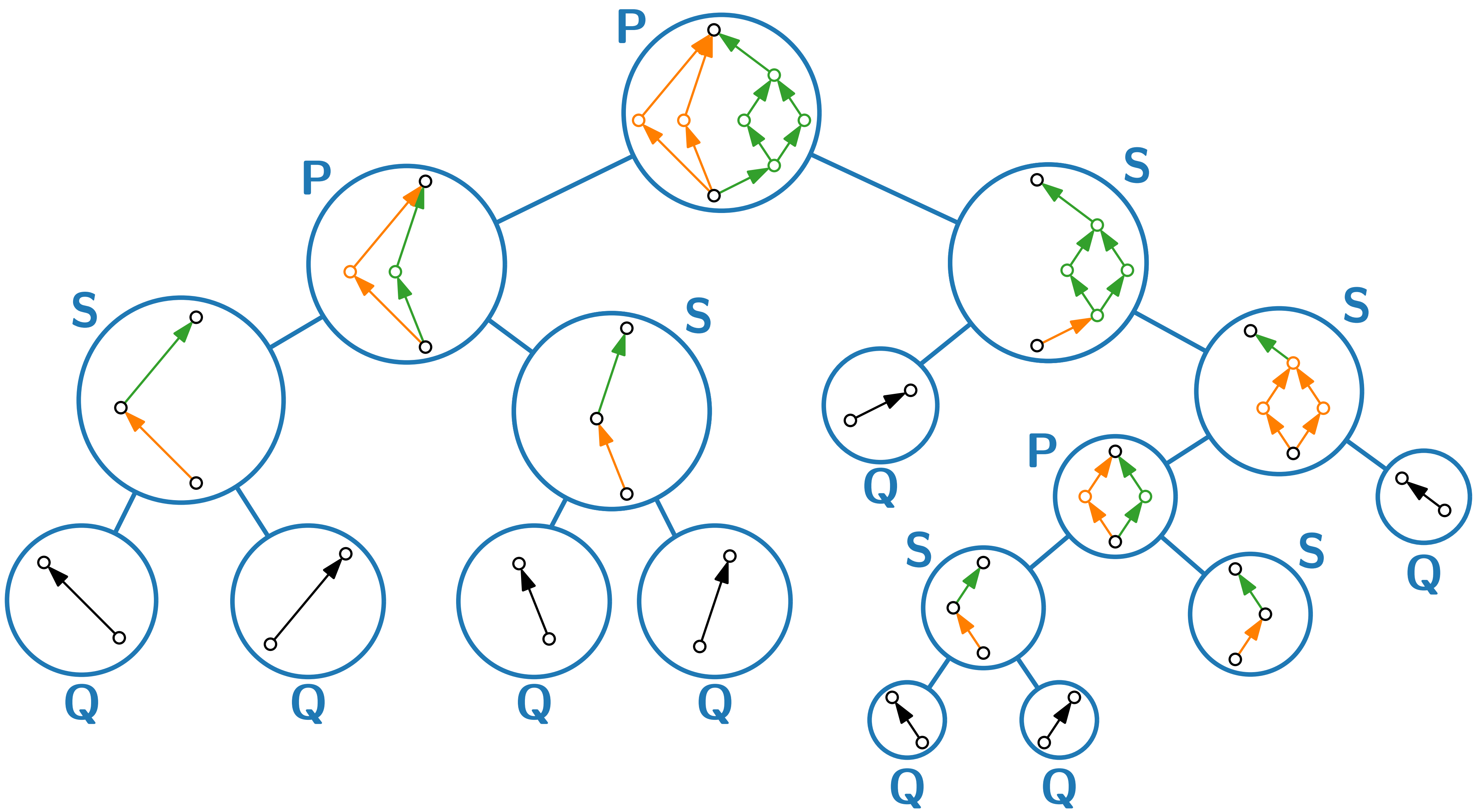
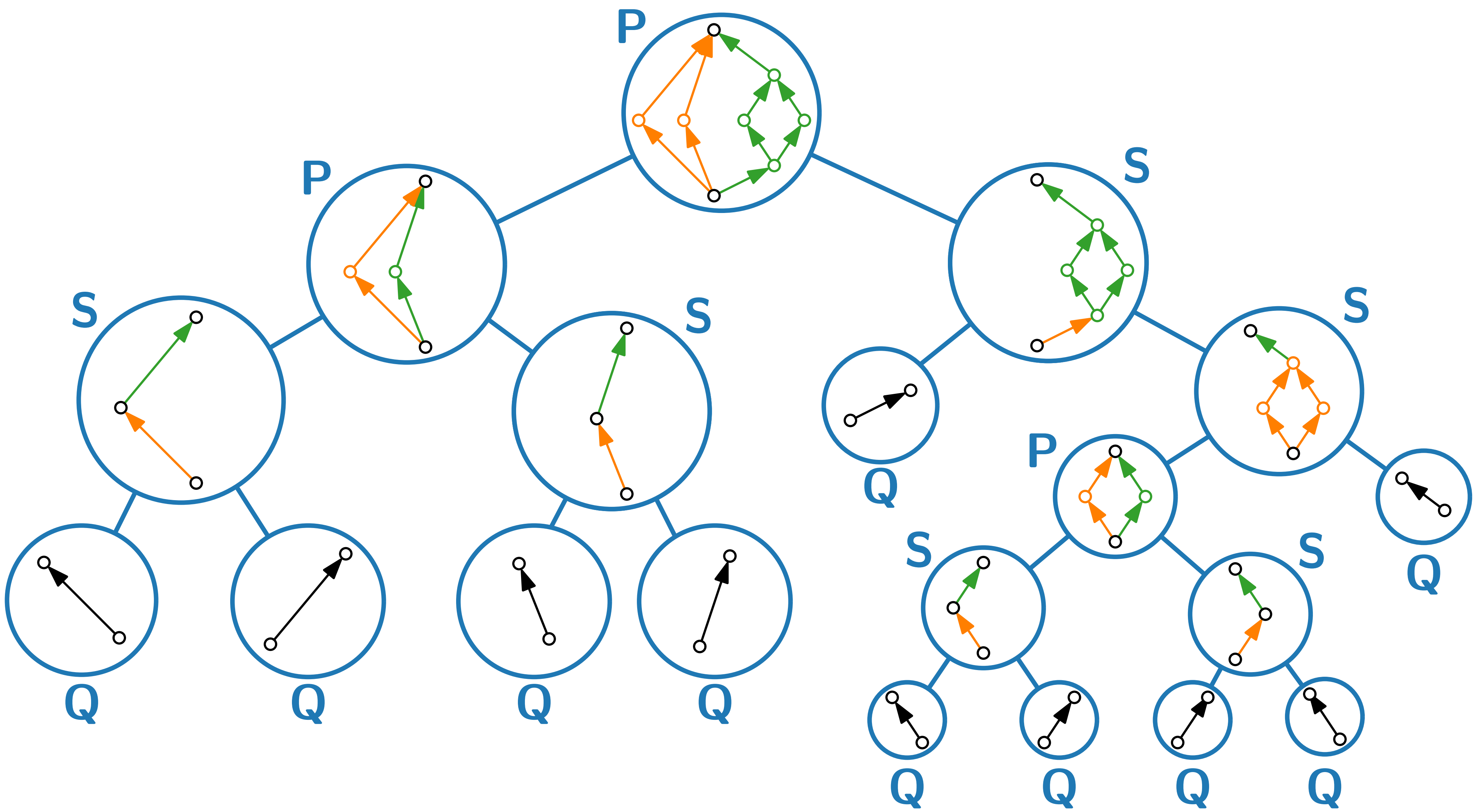


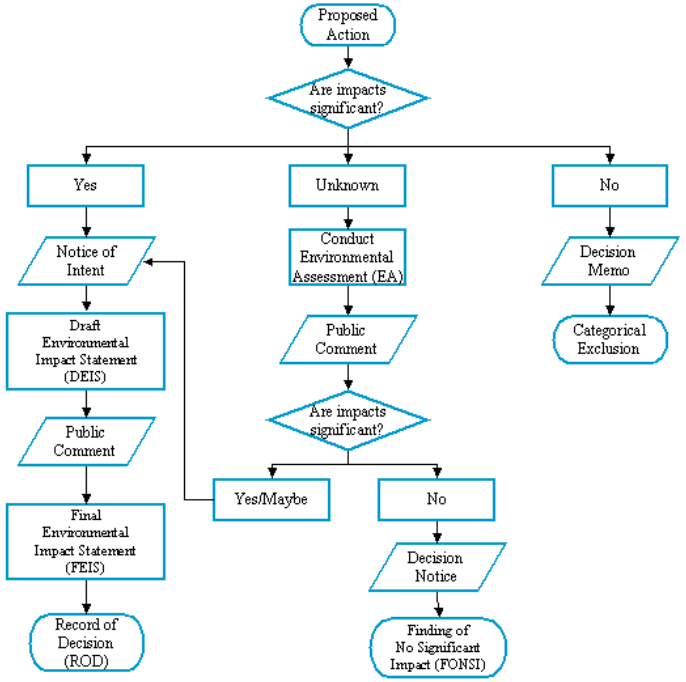
Series-Parallel Graphs – Decomposition Example



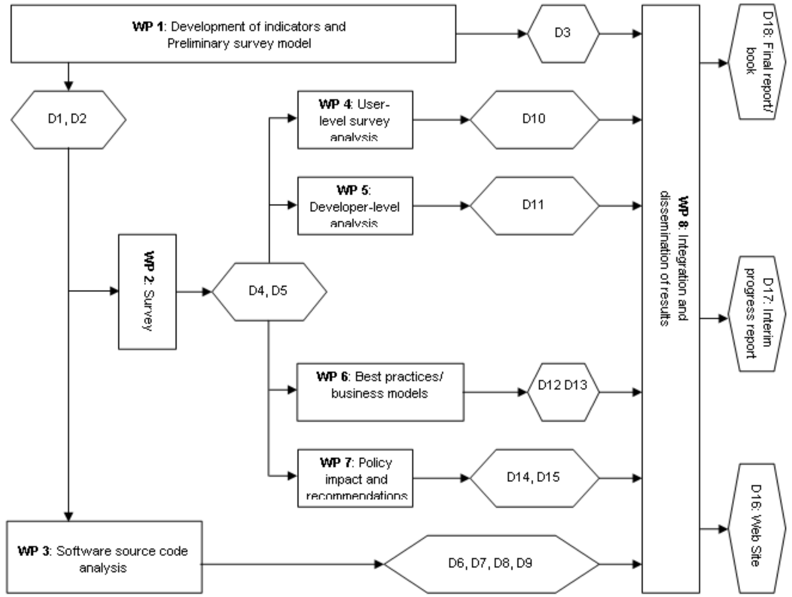
Series-Parallel Graphs – Decomposition Example



Series-Parallel Graphs – Applications



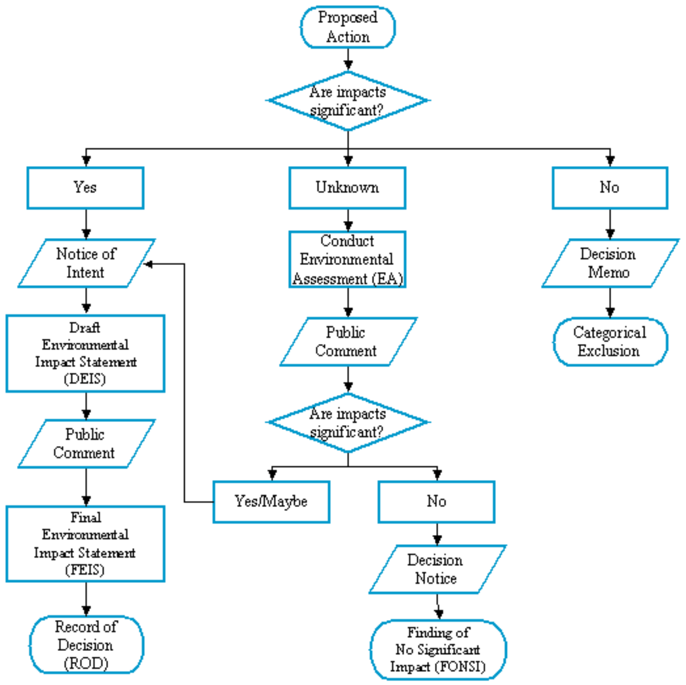
Flowcharts



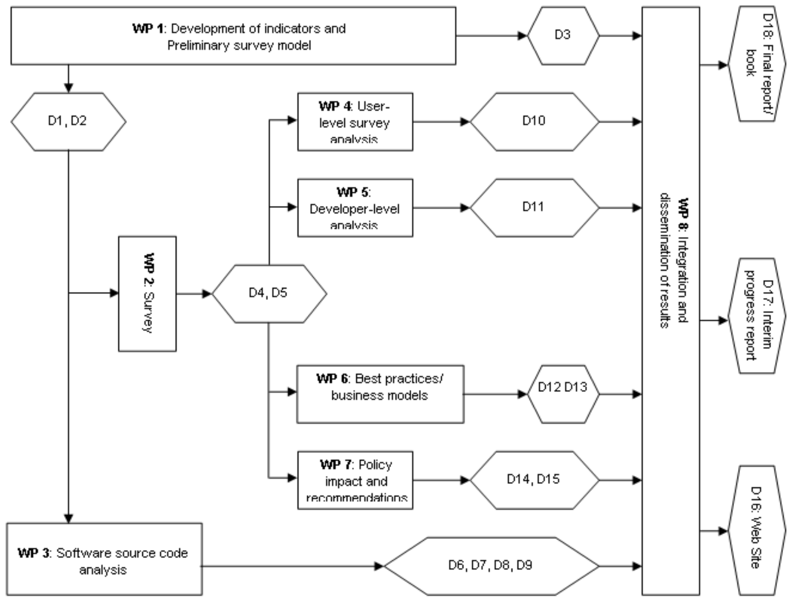
PERT-Diagrams

(Program Evaluation and Review Technique)

Series-Parallel Graphs – Applications



Flowcharts



PERT-Diagrams

(Program Evaluation and Review Technique)

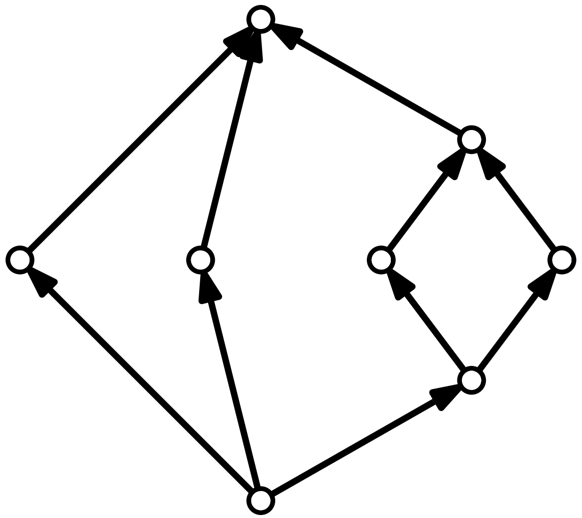
Computational complexity:

Series-parallel graphs often admit linear-time algorithms for problems that are NP-hard in general, e.g., minimum maximal matching, maximum independent set, Hamiltonian completion.

Series-Parallel Graphs – Drawing Style

Drawing conventions

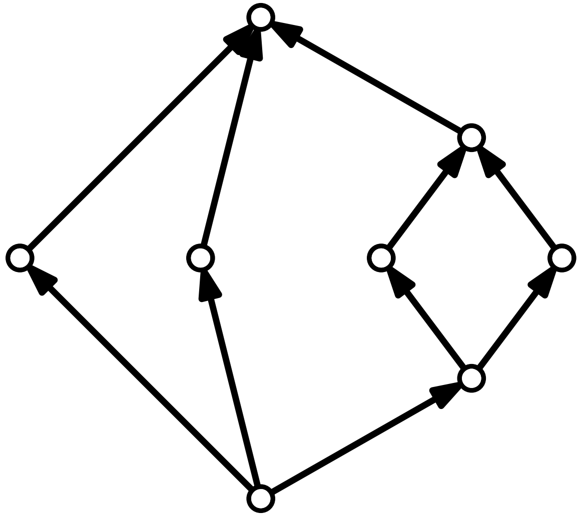
Drawing aesthetics to optimize



Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity

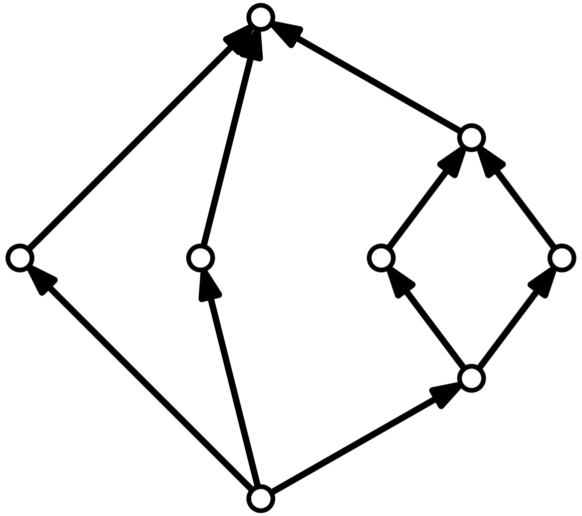


Drawing aesthetics to optimize

Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges

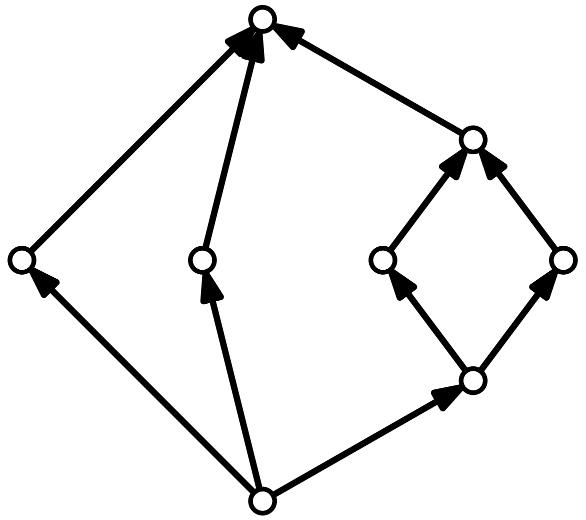


Drawing aesthetics to optimize

Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward



Drawing aesthetics to optimize

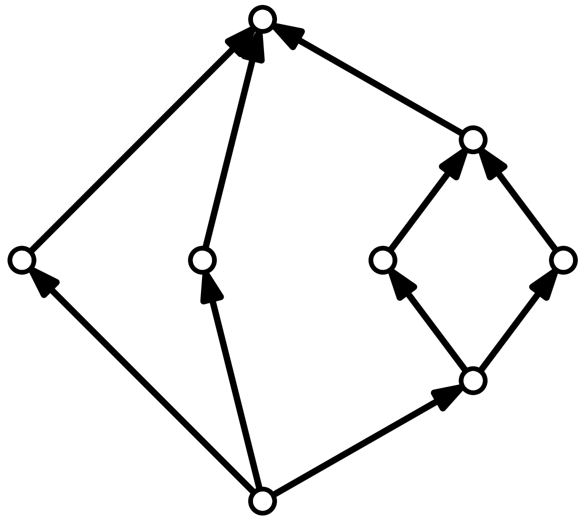
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize

- Area



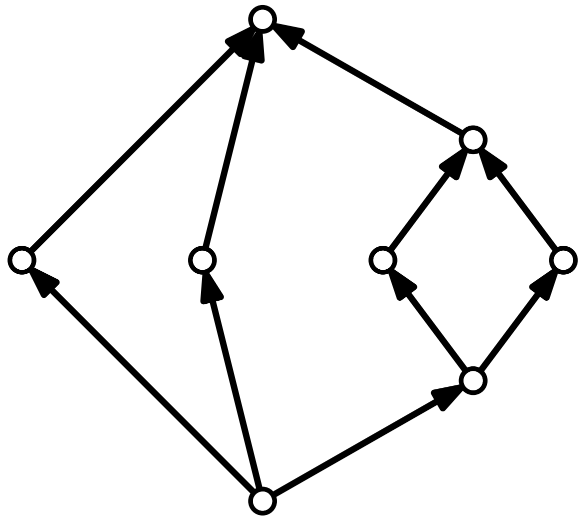
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics to optimize

- Area
- Symmetry



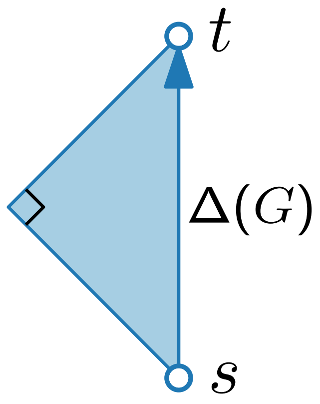
Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

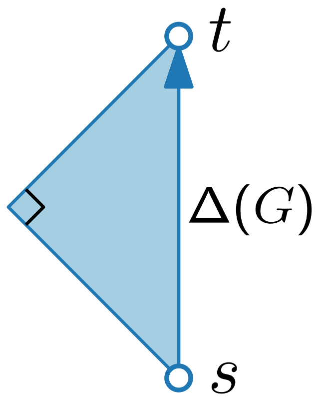
Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top



Series-Parallel Graphs – Straight-Line Drawings



Divide-and-conquer algorithm using the decomposition tree

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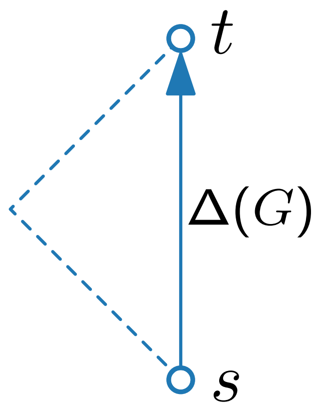
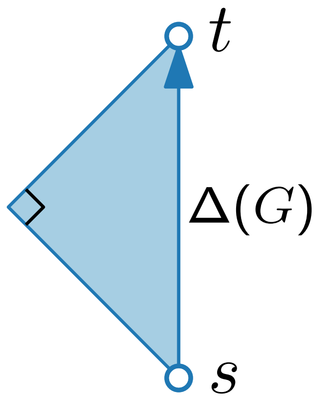
Base case: Q-nodes



Series-Parallel Graphs – Straight-Line Drawings



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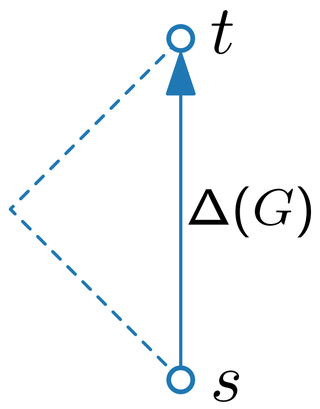

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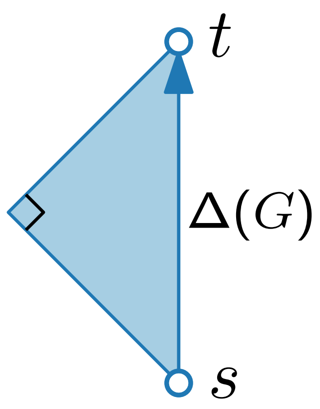
Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree



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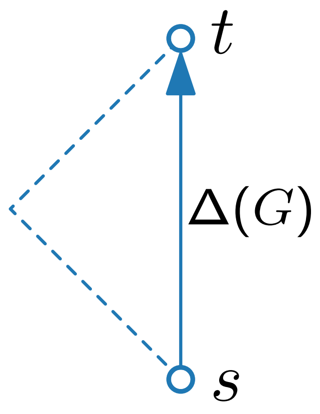

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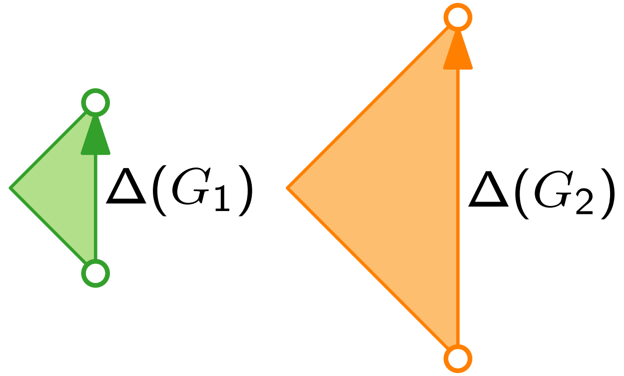
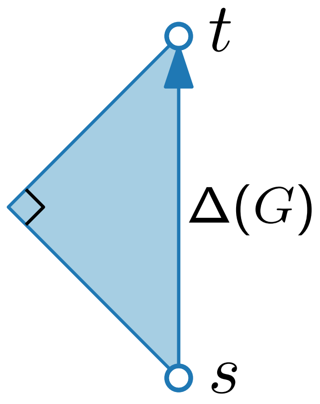


Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree



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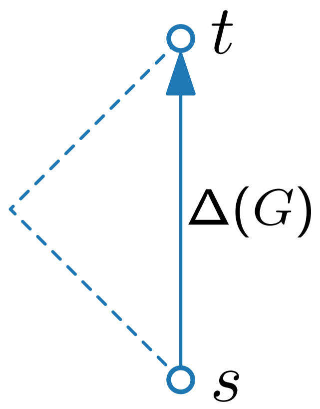

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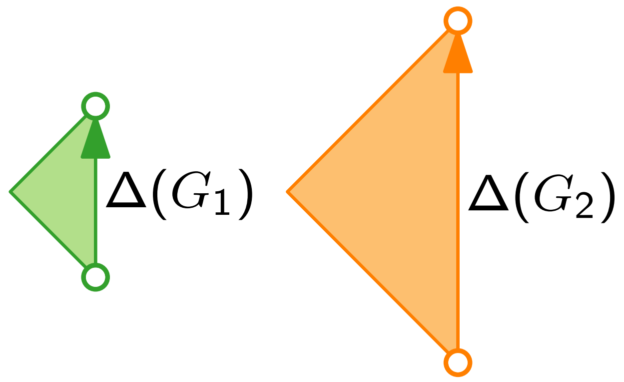
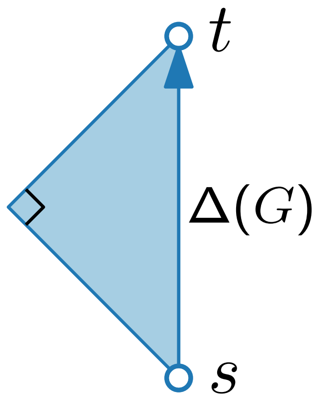


Conquer:

Series-Parallel Graphs – Straight-Line Drawings



Divide-and-conquer algorithm using the decomposition tree

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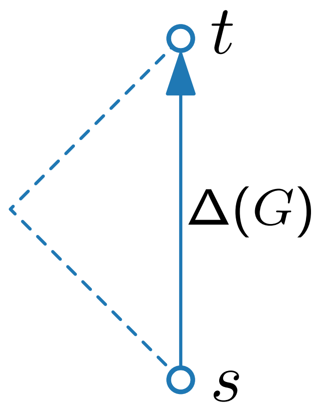

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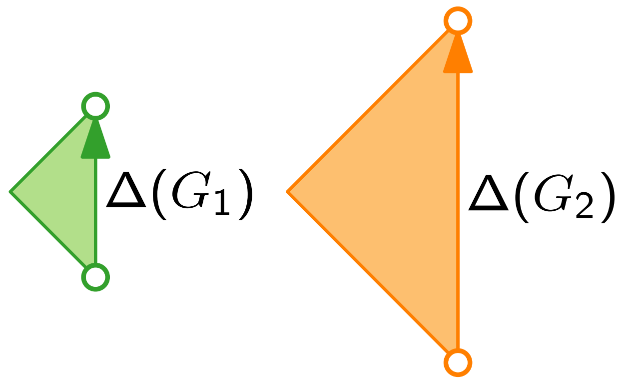

Conquer:

- S-nodes: series compositions

Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

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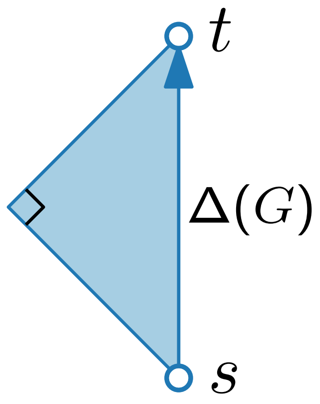

Base case: Q-nodes

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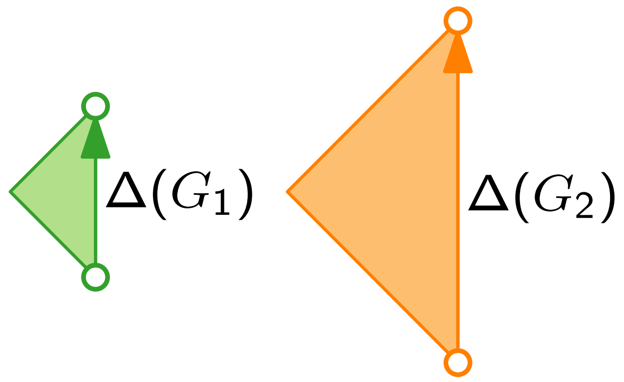

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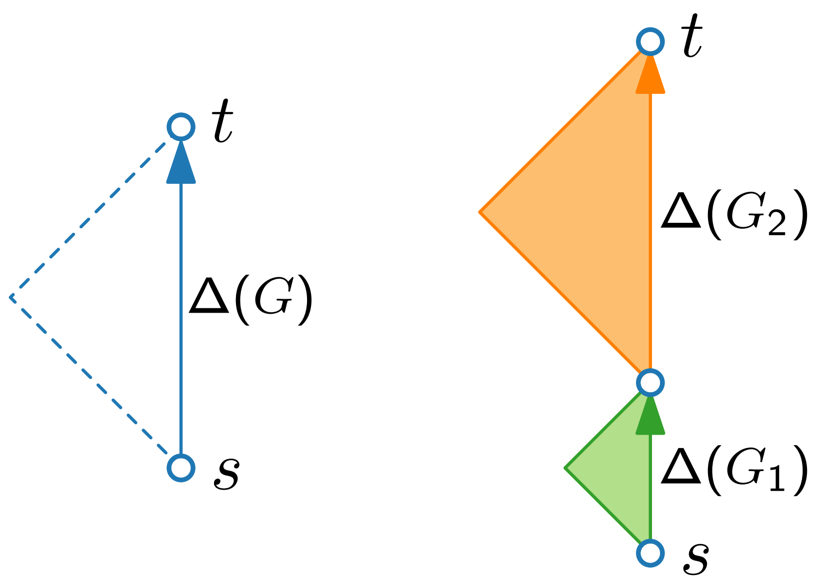
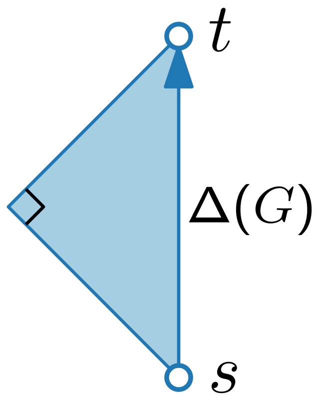


Series-Parallel Graphs – Straight-Line Drawings



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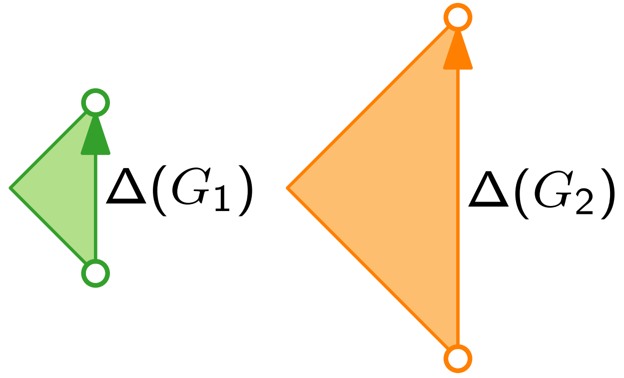

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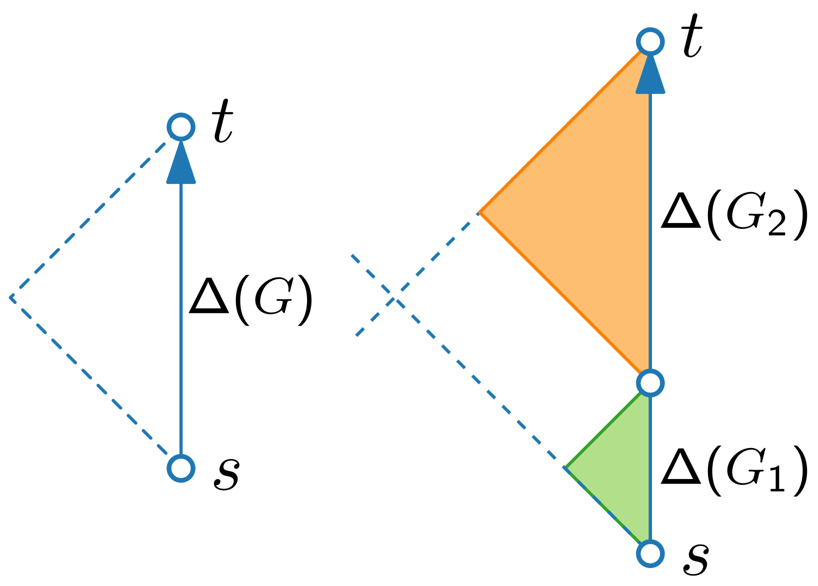
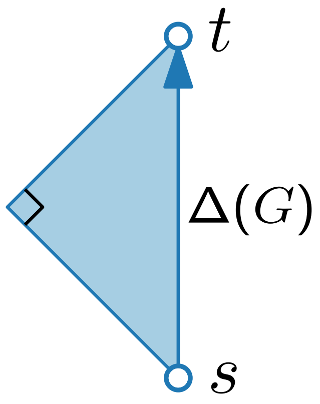


Series-Parallel Graphs – Straight-Line Drawings



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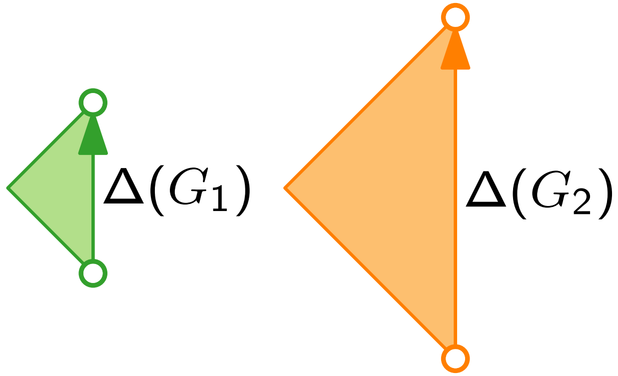
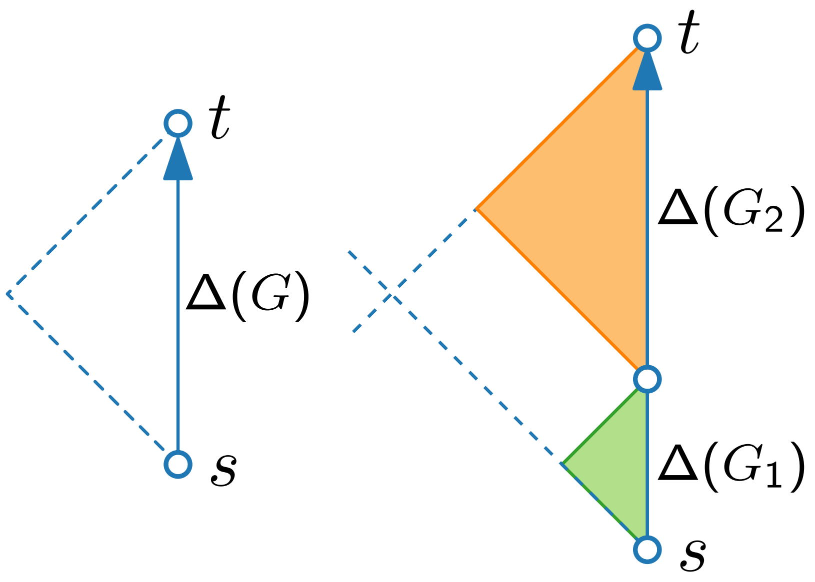
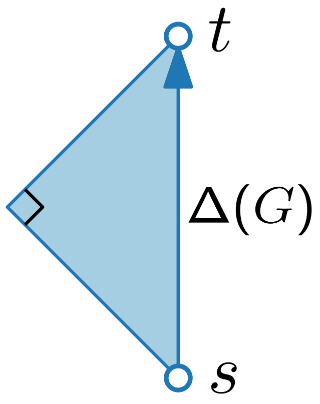


Conquer:

- S-nodes: series compositions
- P-nodes: parallel compositions



Series-Parallel Graphs – Straight-Line Drawings



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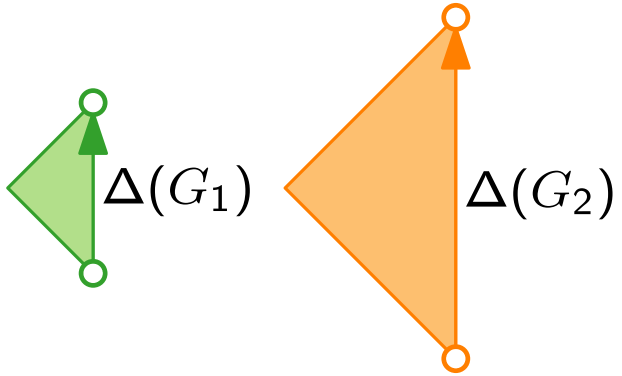

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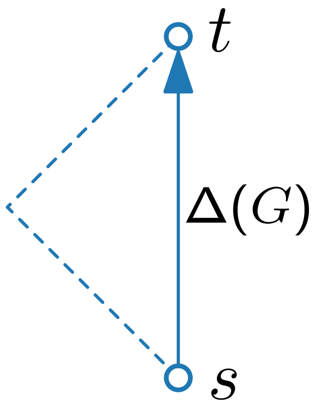

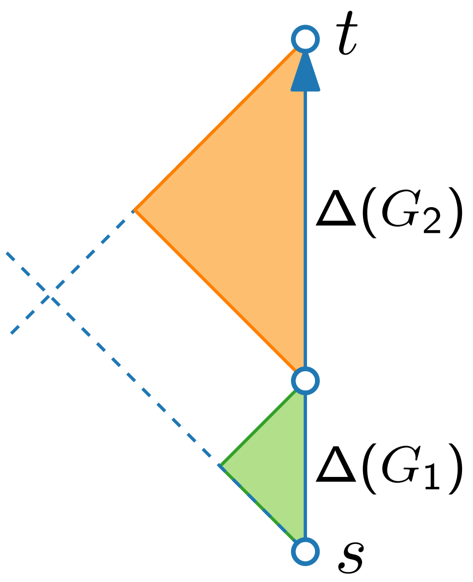

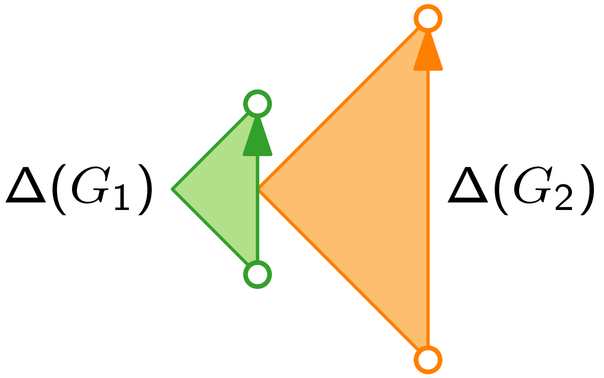

Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

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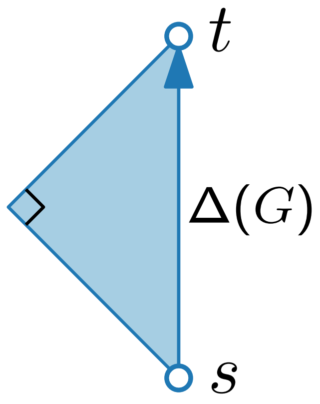

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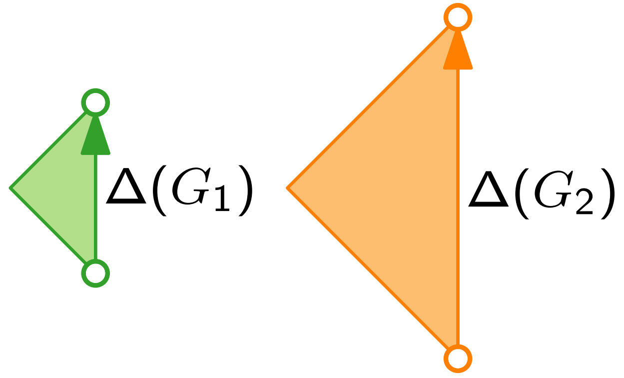

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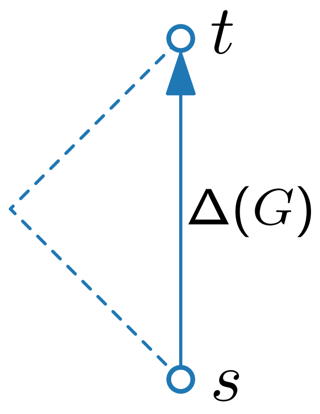

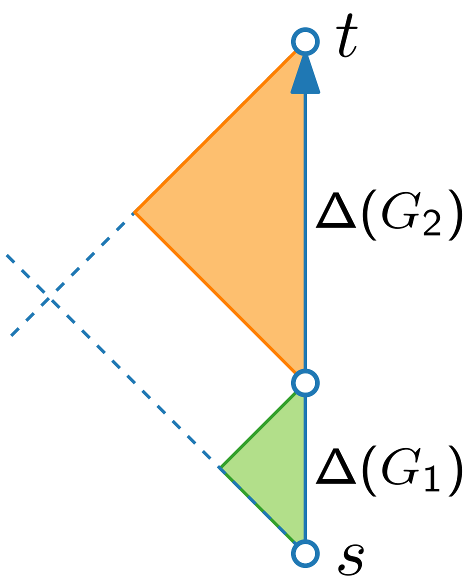

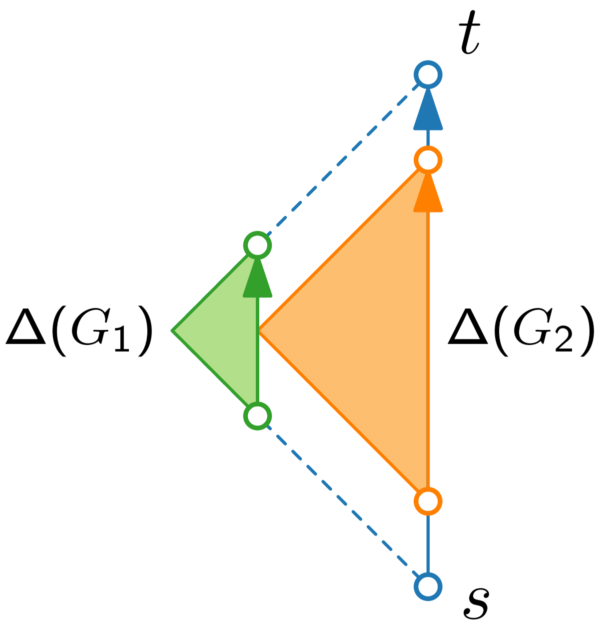

Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

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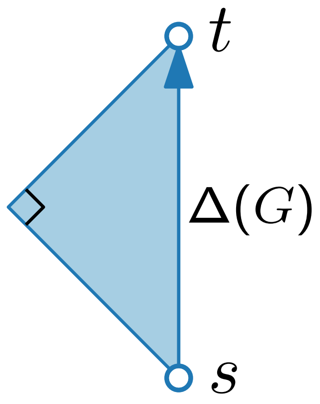

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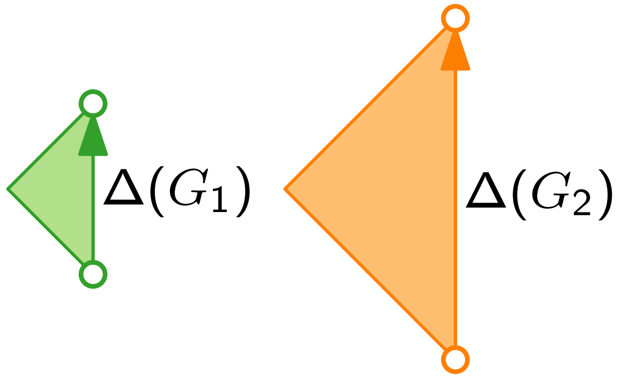

Conquer:

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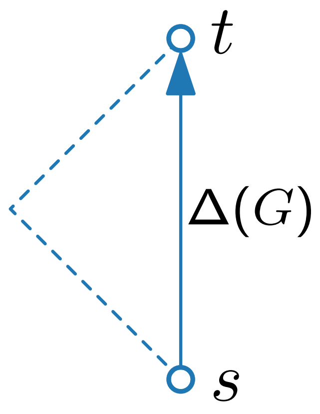

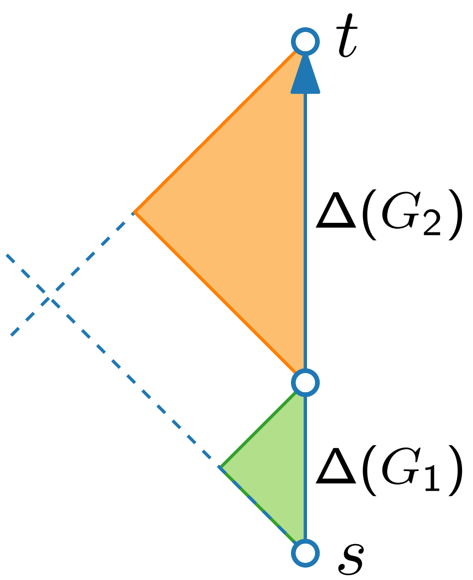

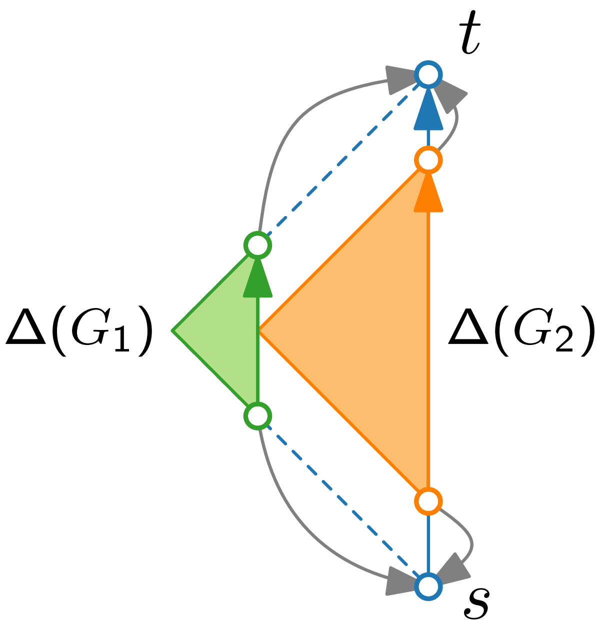

Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree

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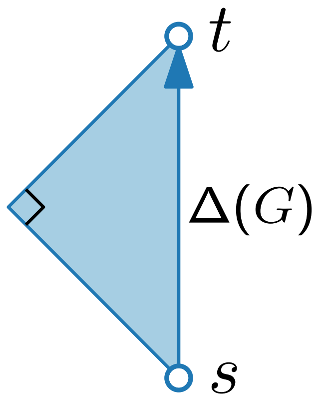

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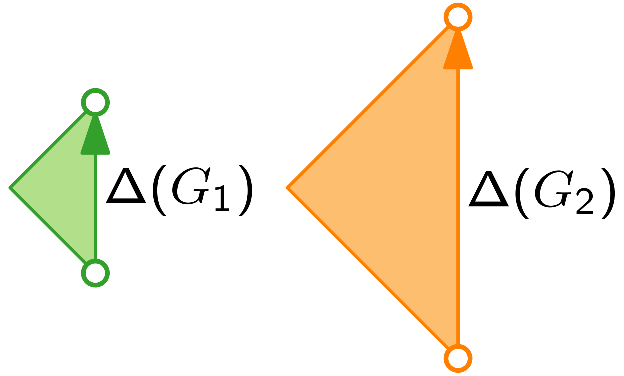

Conquer:

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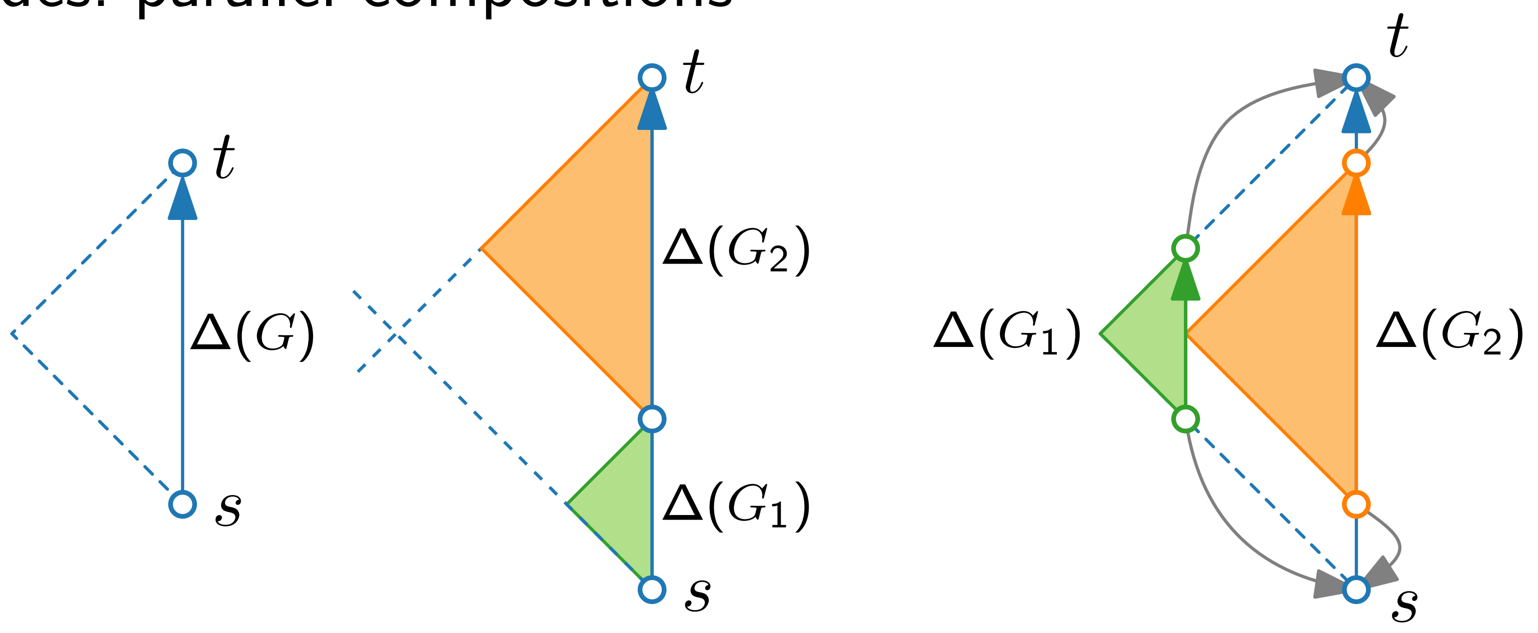

Do you see any problem?

Series-Parallel Graphs – Straight-Line Drawings

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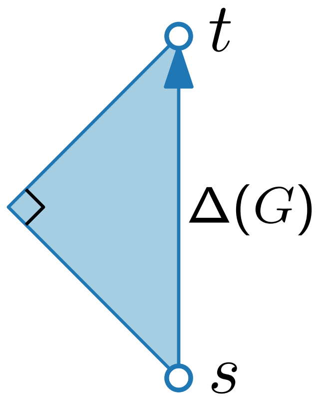

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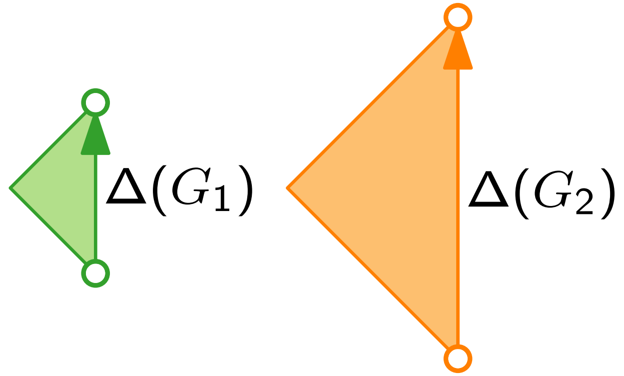

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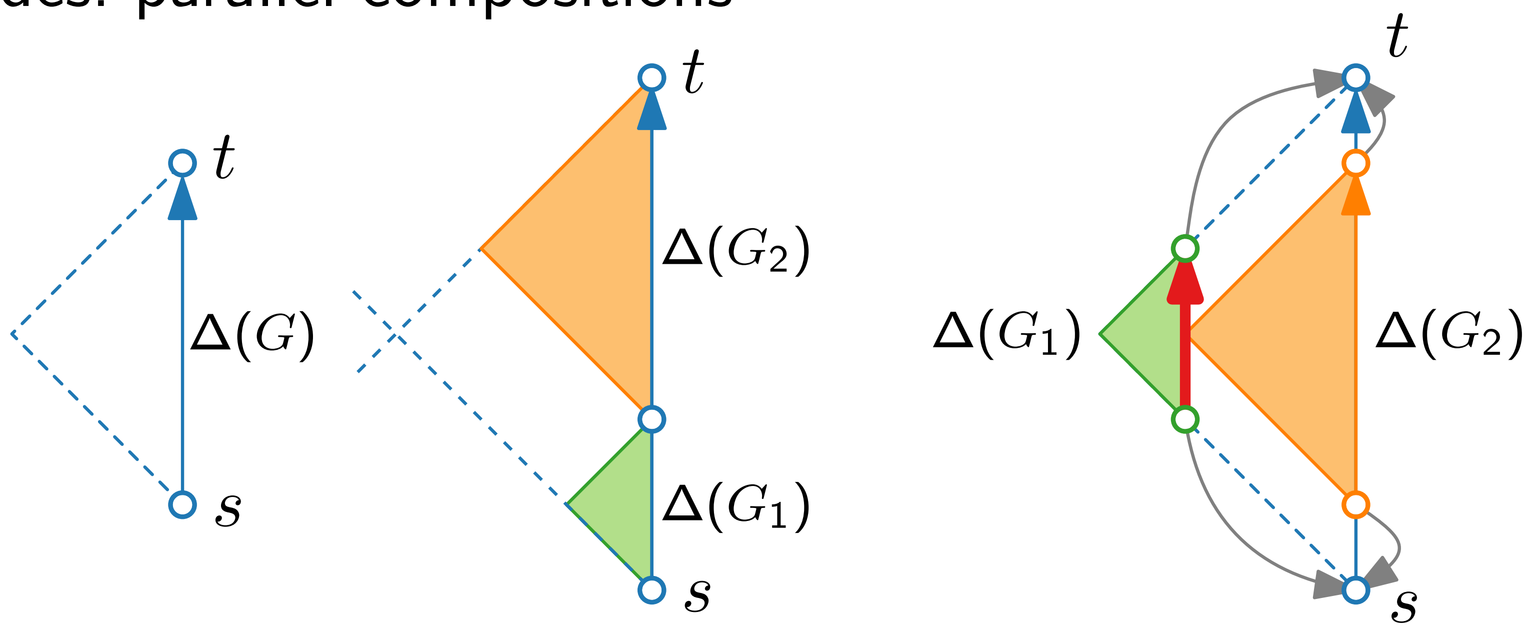
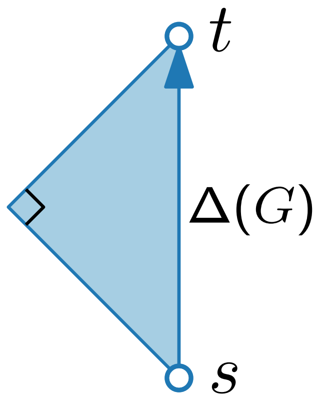


Do you see any problem?

single edge

Series-Parallel Graphs – Straight-Line Drawings

Divide-and-conquer algorithm using the decomposition tree



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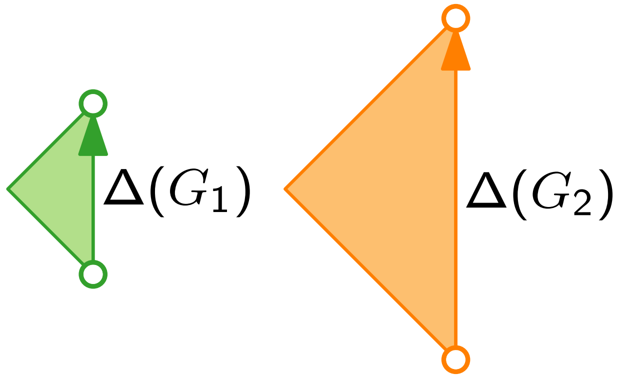

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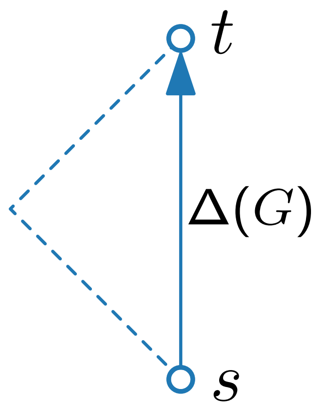
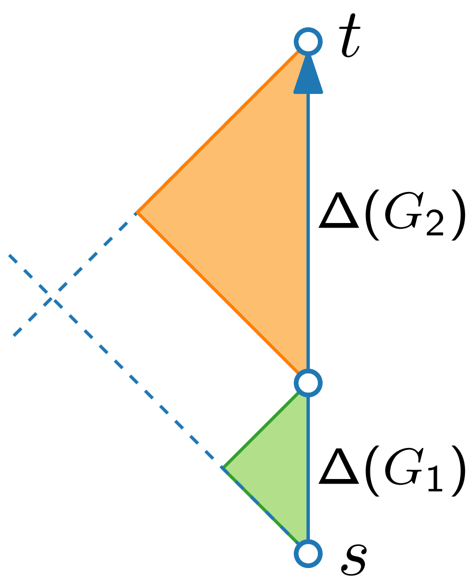
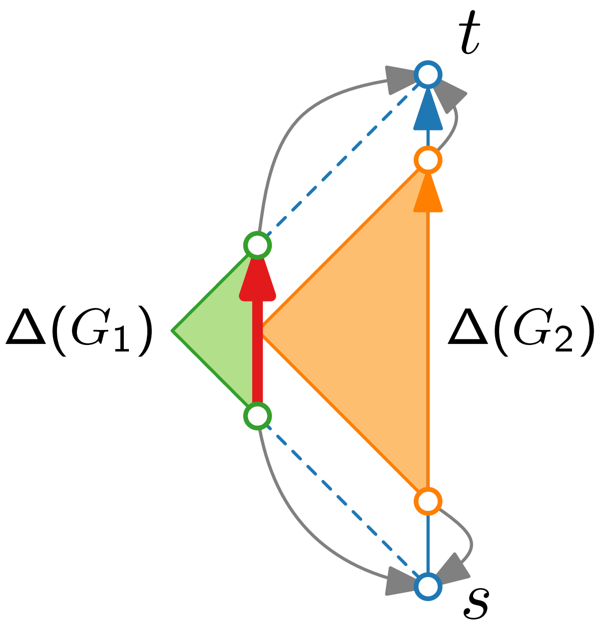




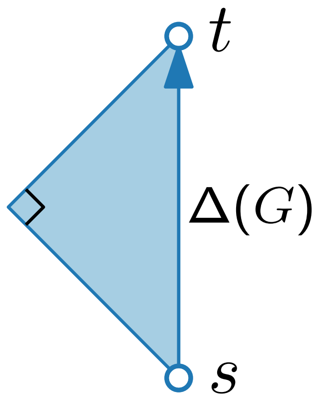


Do you see any problem?

single edge

change embedding!

Series-Parallel Graphs – Straight-Line Drawings



Divide-and-conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top

Base case: Q-nodes

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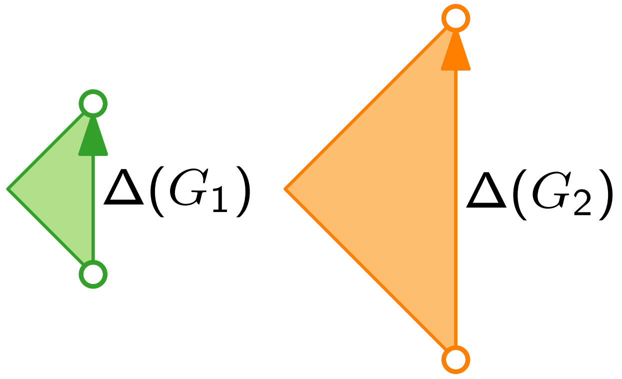

Conquer:

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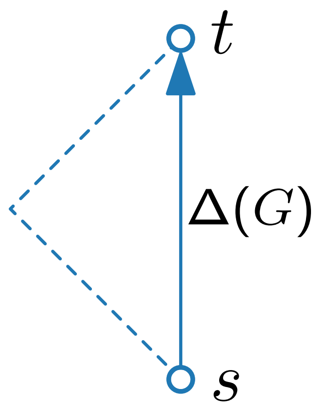
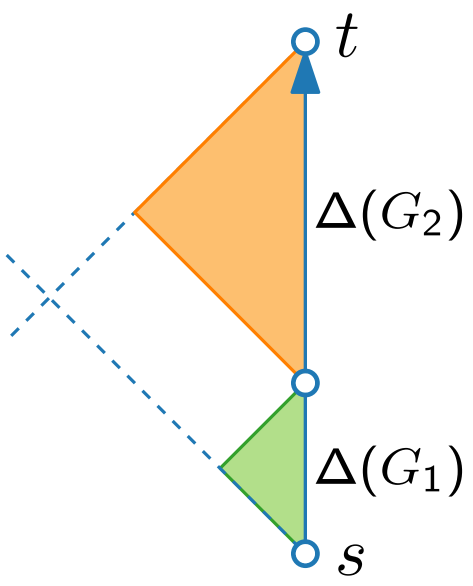
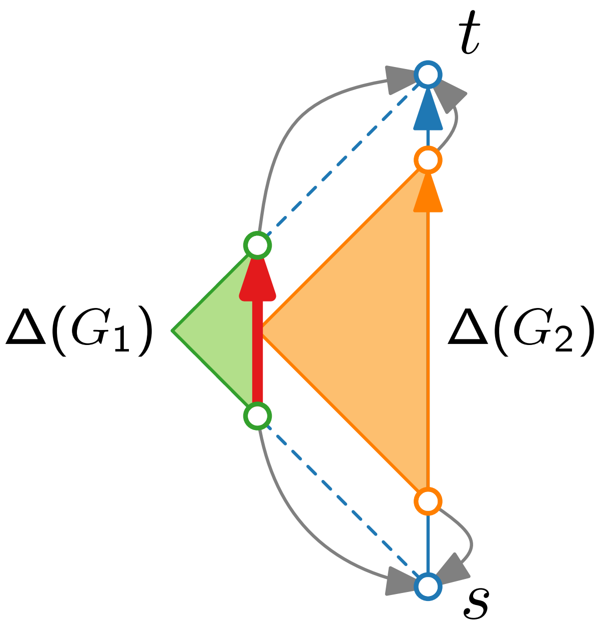
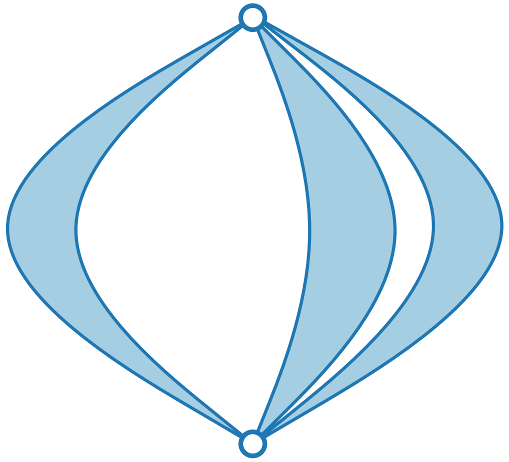
- P-nodes: parallel compositions









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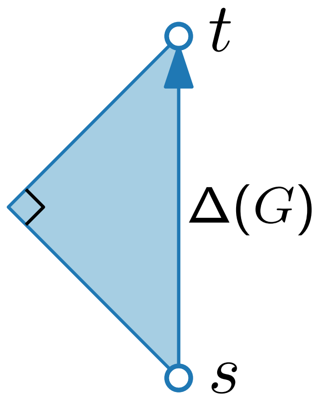

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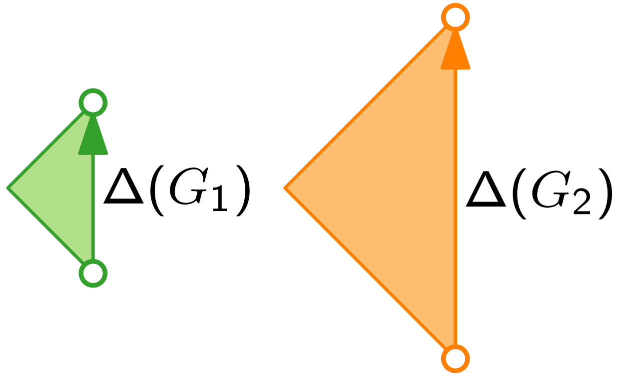

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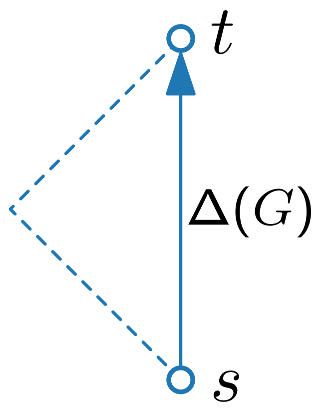

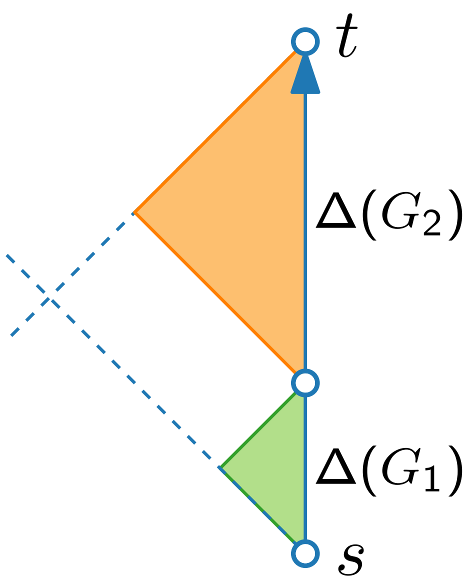

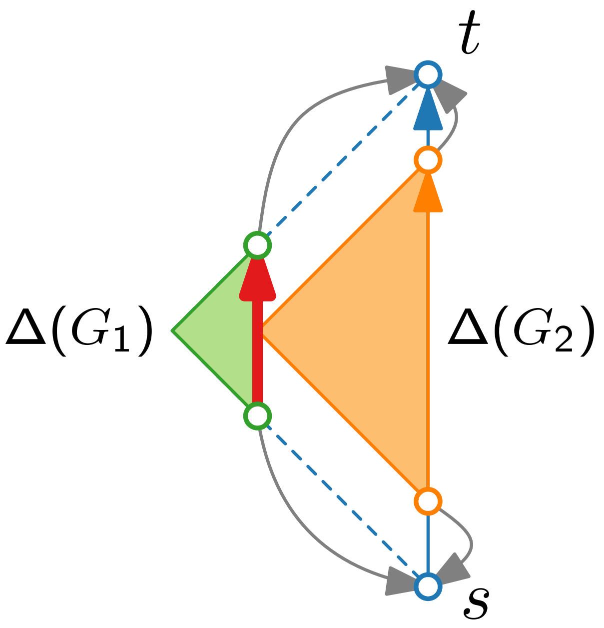

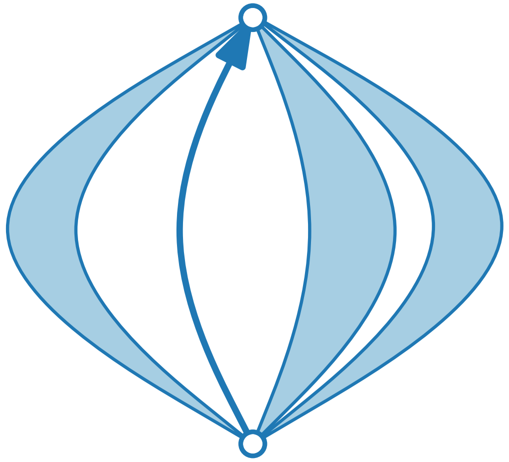

Series-Parallel Graphs – Straight-Line Drawings

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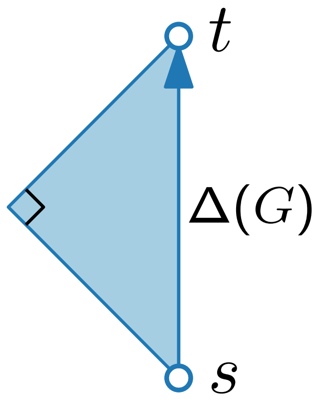

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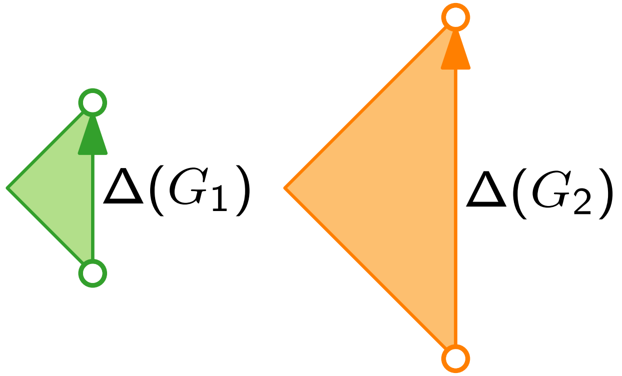

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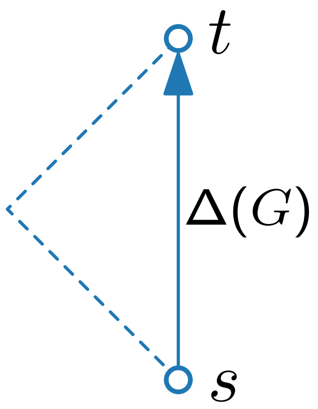

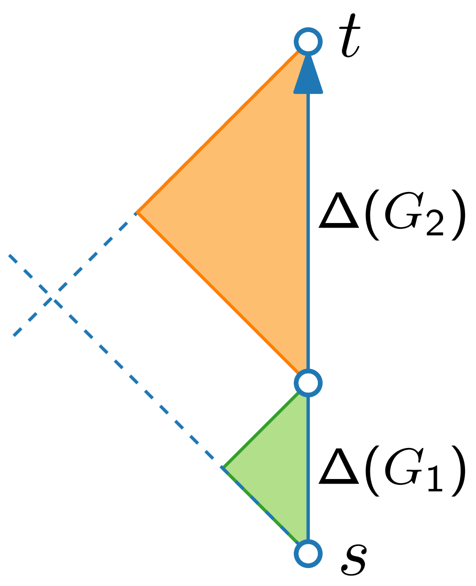

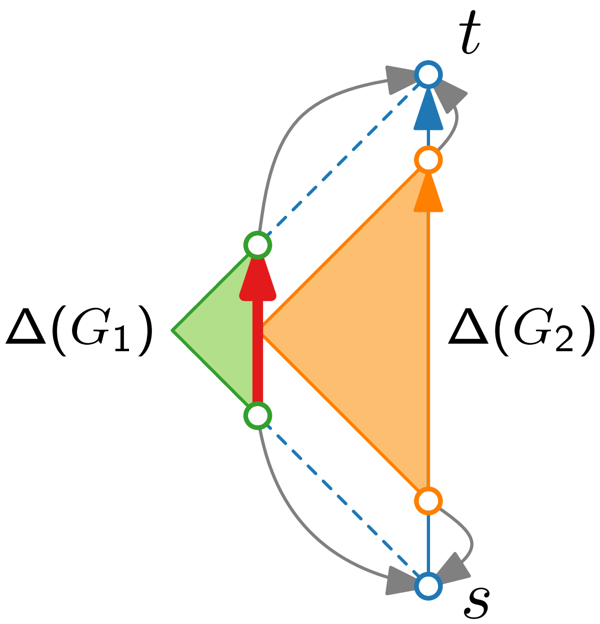

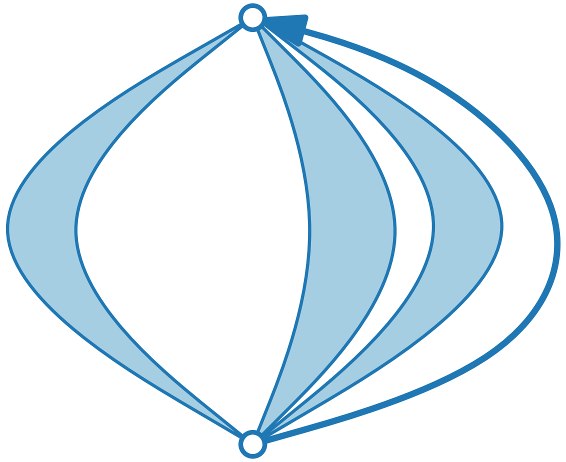

Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?

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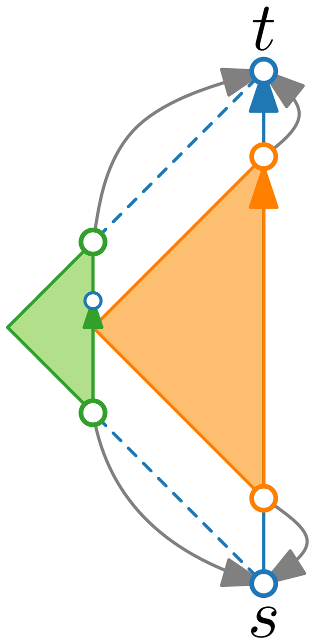

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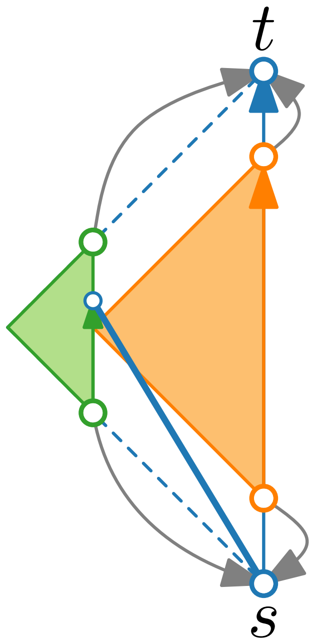

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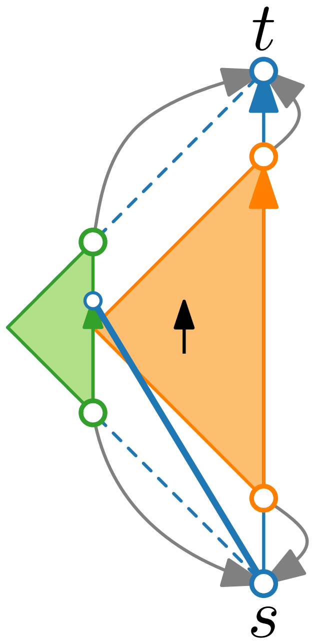

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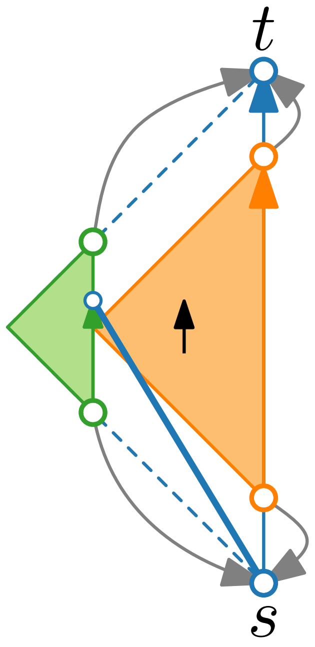

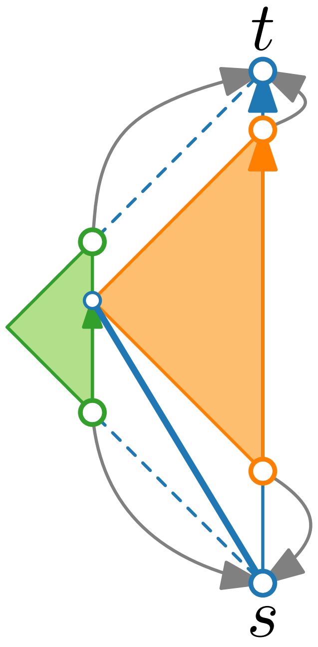

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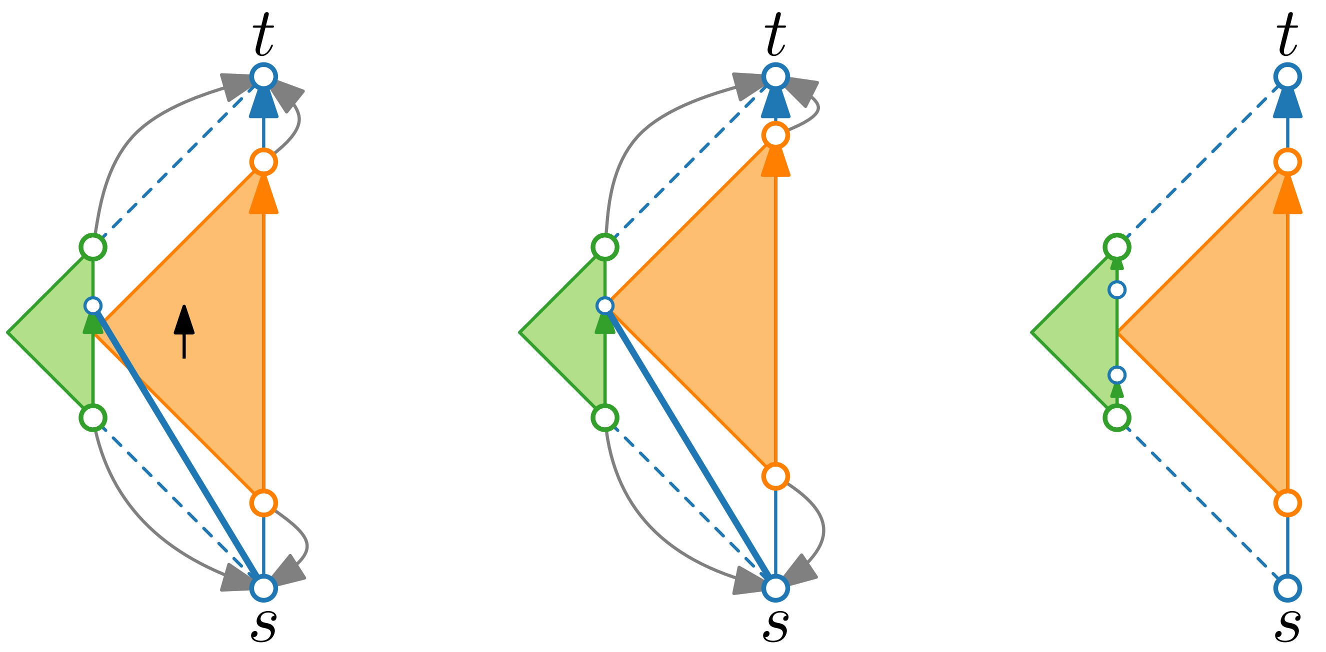

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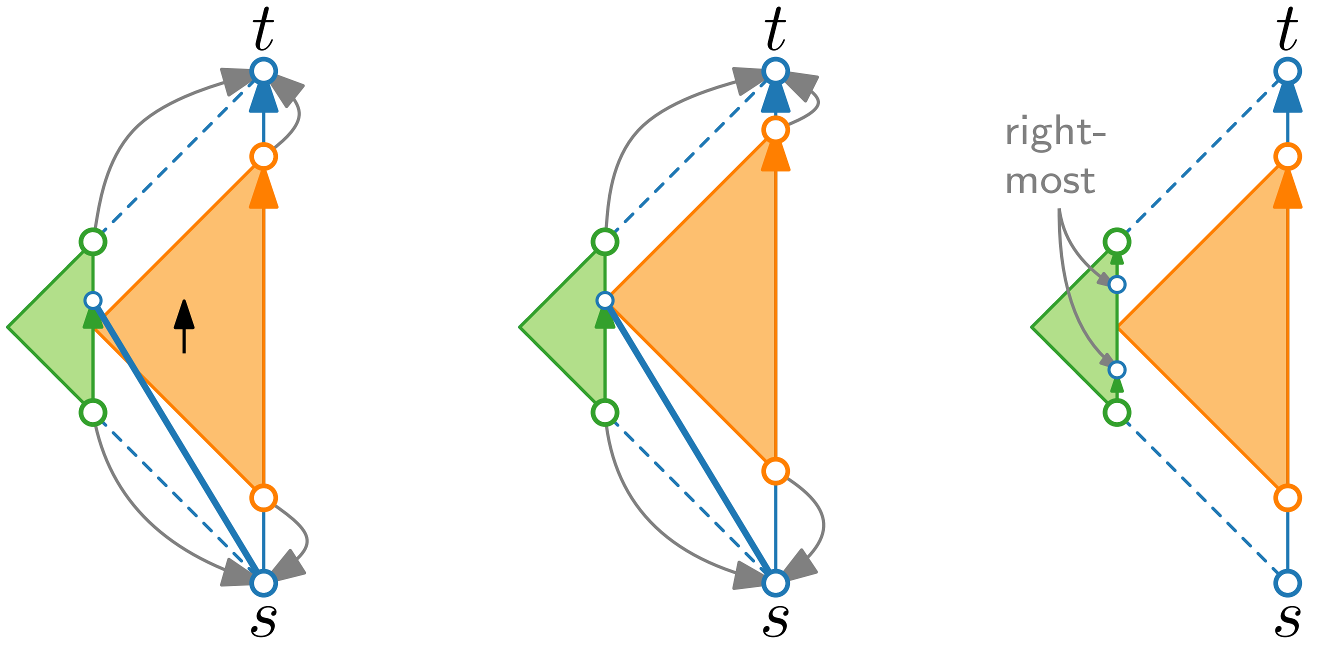

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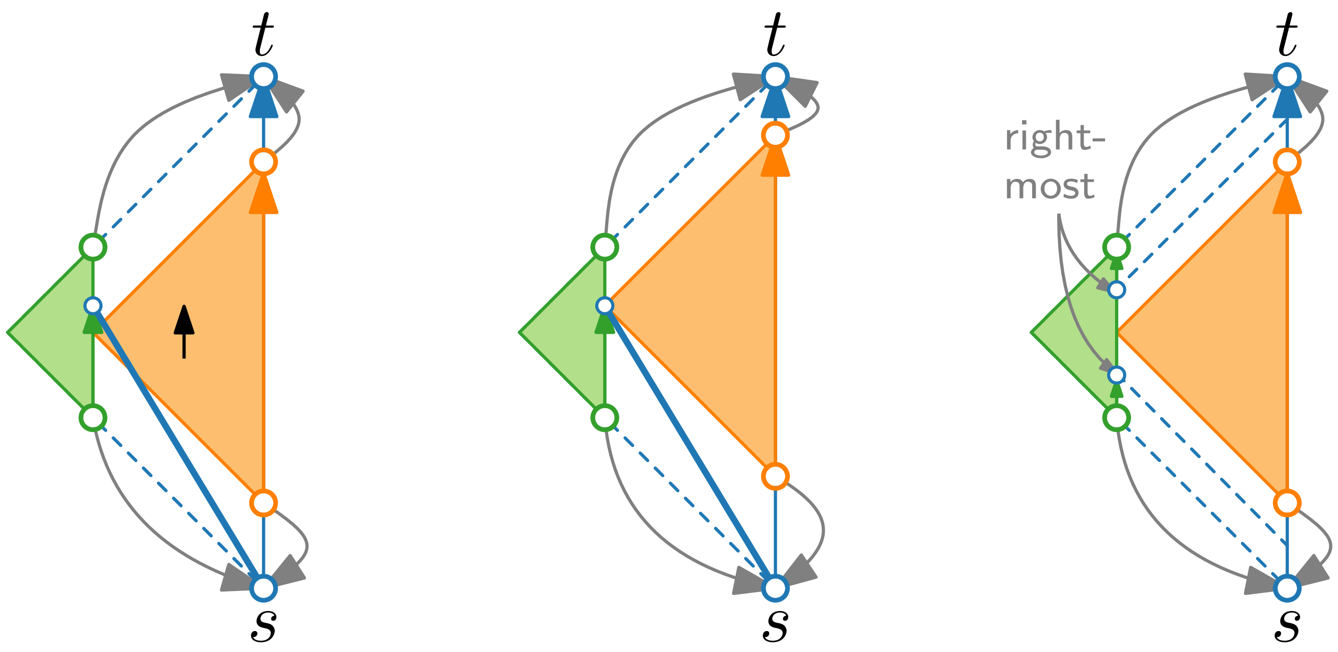

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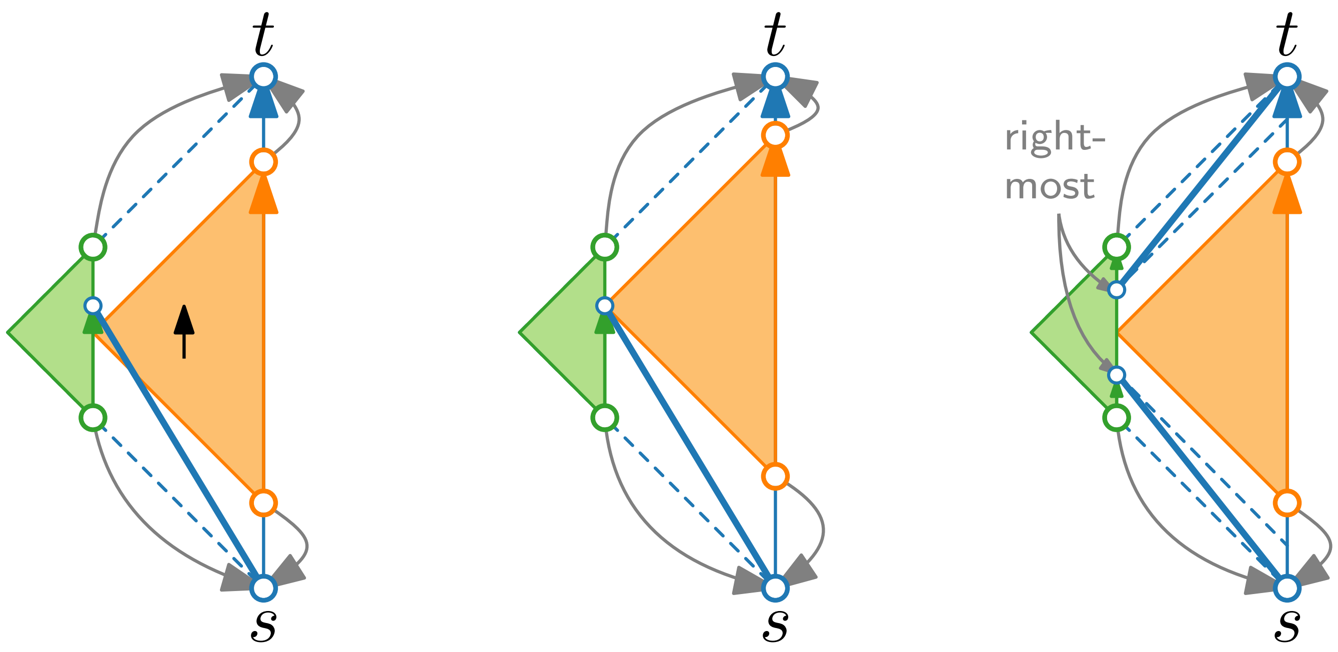

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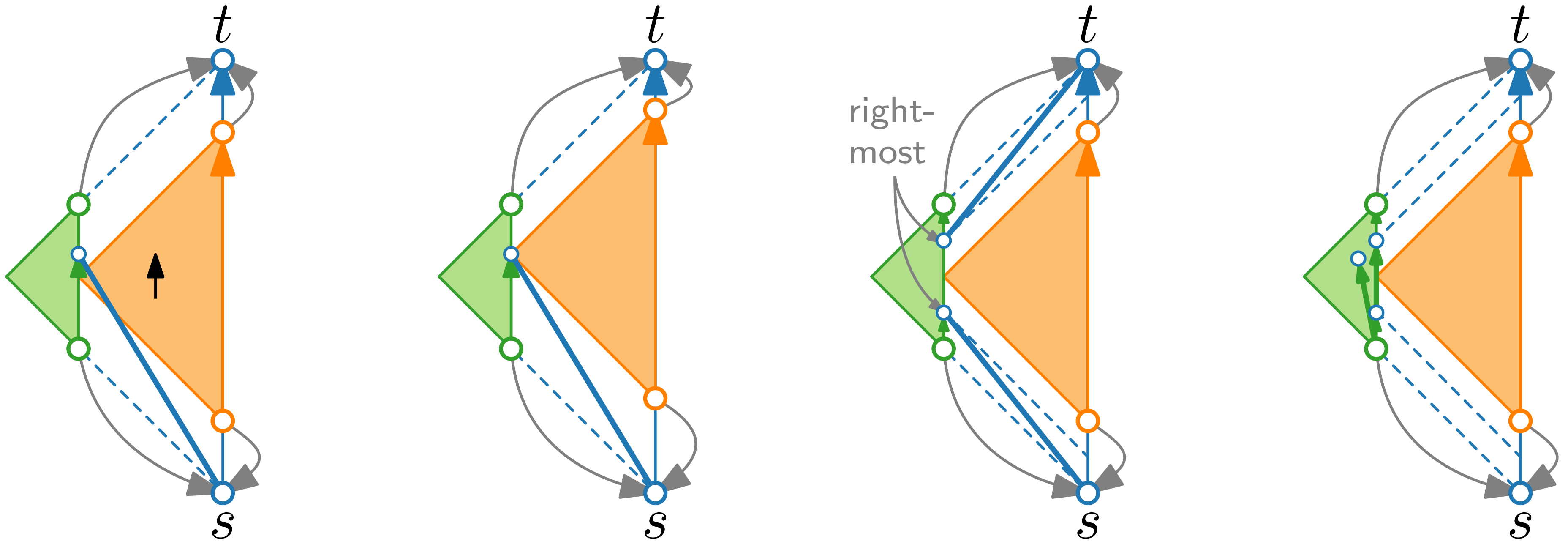

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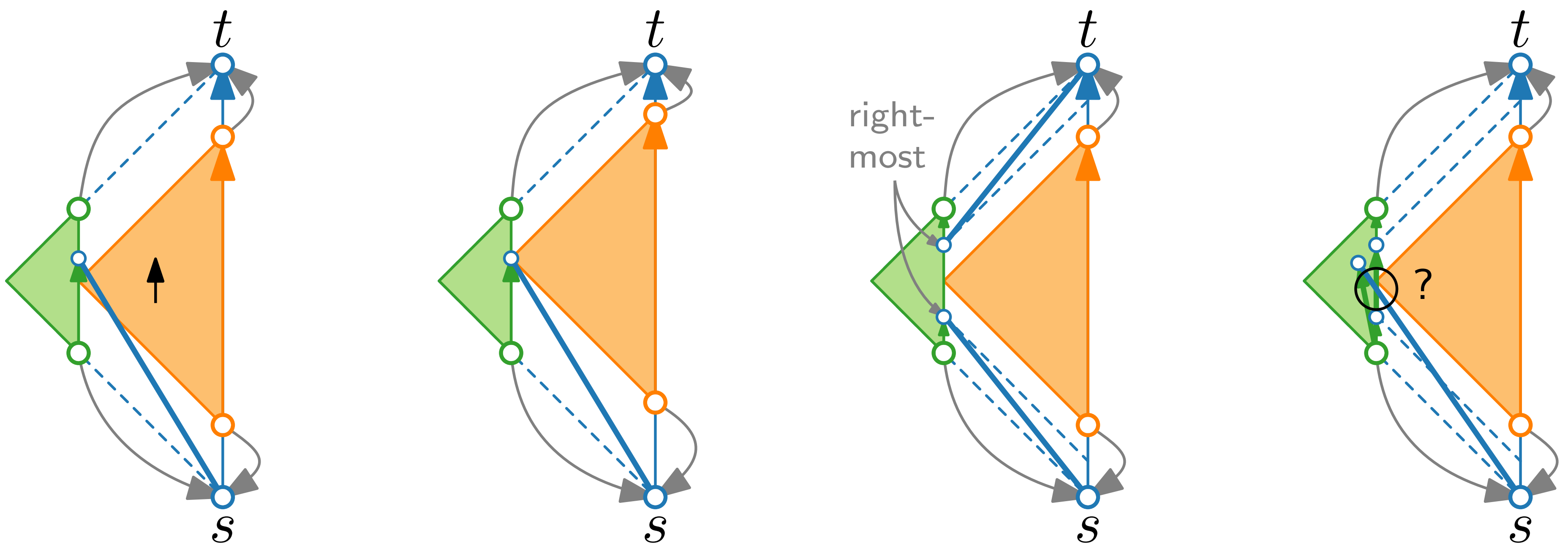

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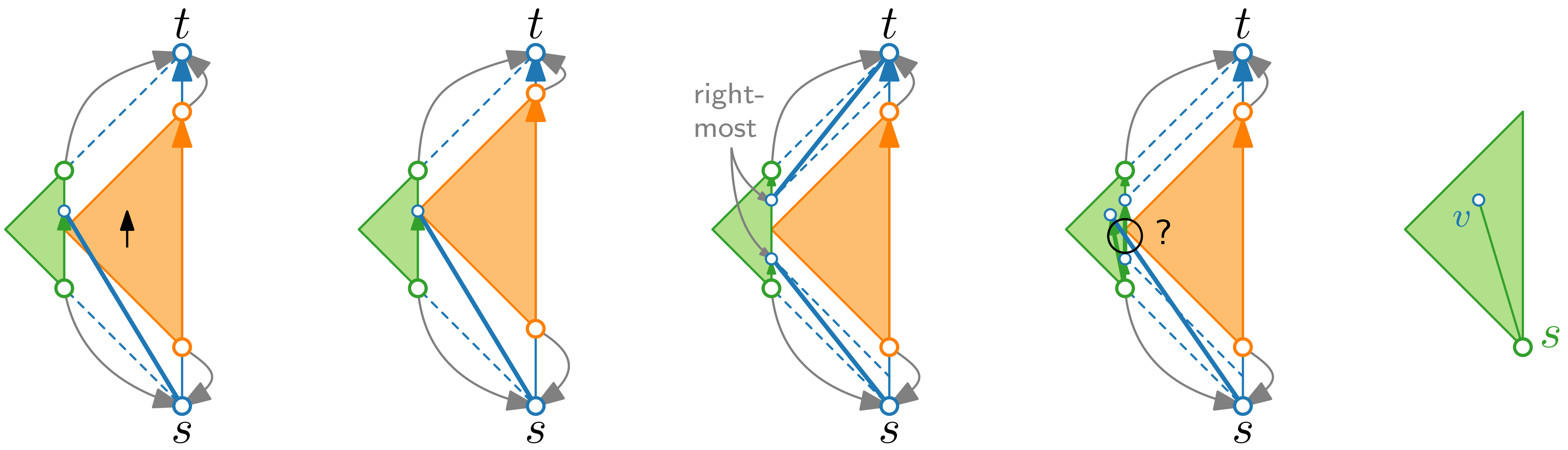

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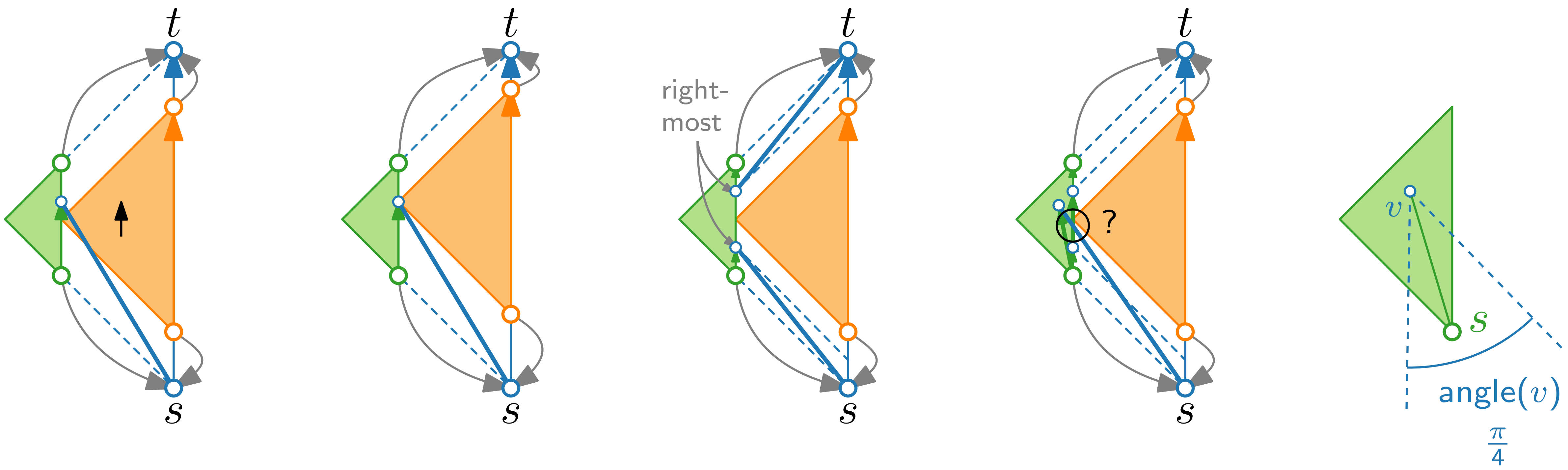

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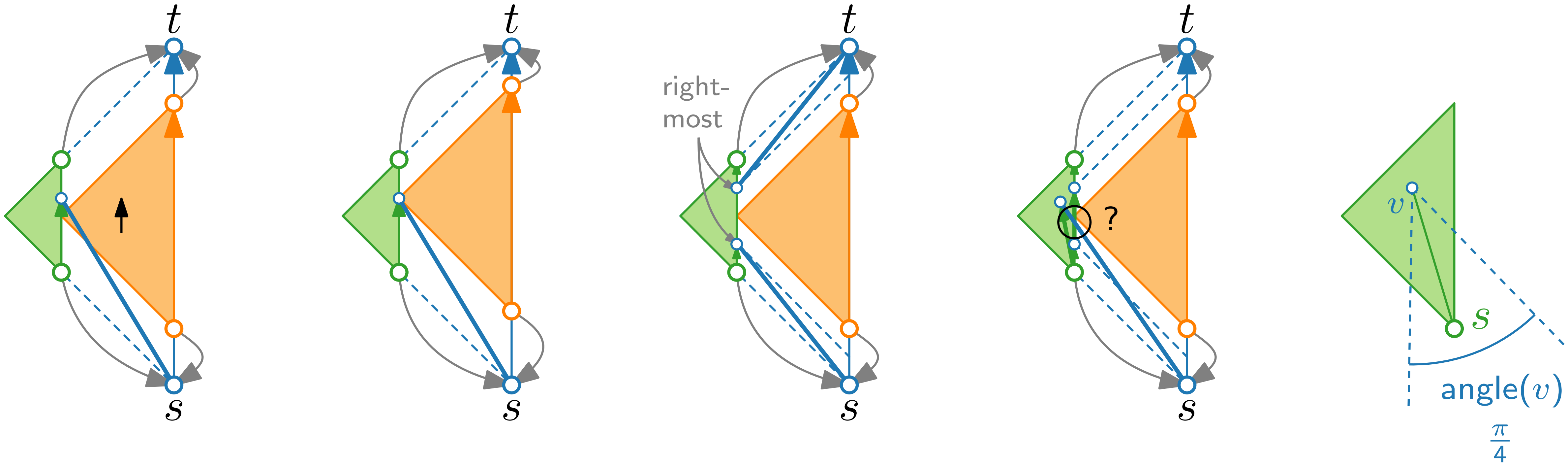

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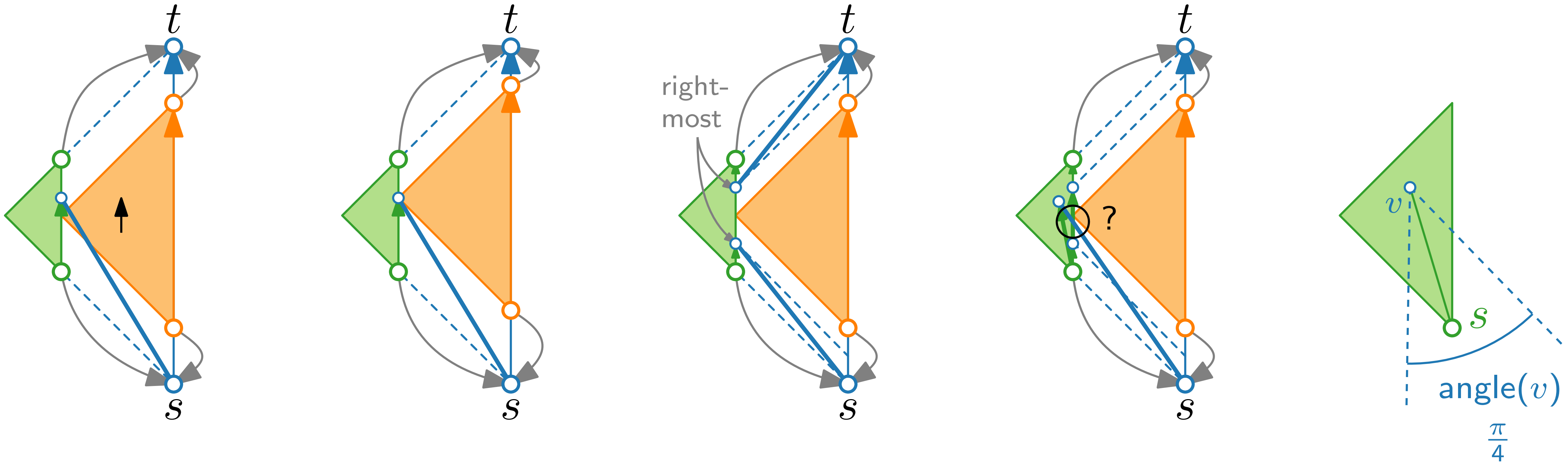

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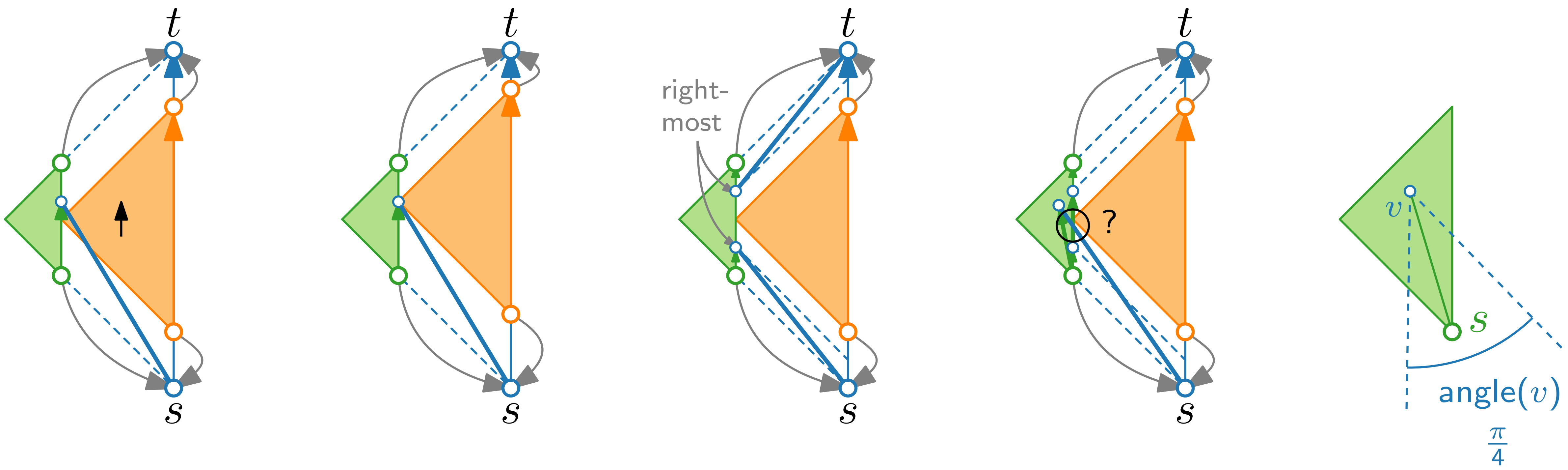

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Lemma.
The drawing produced by the algorithm is planar.

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Theorem.
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Γ can be computed in linear time.

Series-Parallel Graphs – Fixed Embedding

Theorem. [Bertolazzi, Di Battista, Mannino, Tamassia '94]

For any $n \geq 1$, there exists a $2n$-vertex series-parallel graph G_n in an embedding such that any upward planar straight-line drawing of G_n that respects the given embedding requires $\Omega(4^n)$ area.

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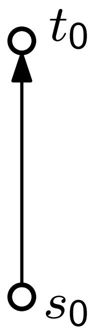

G_1

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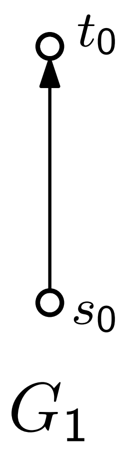

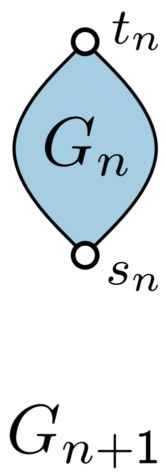

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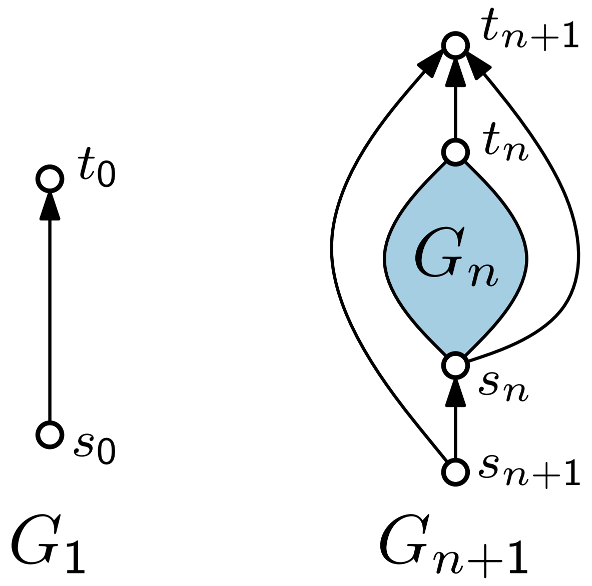

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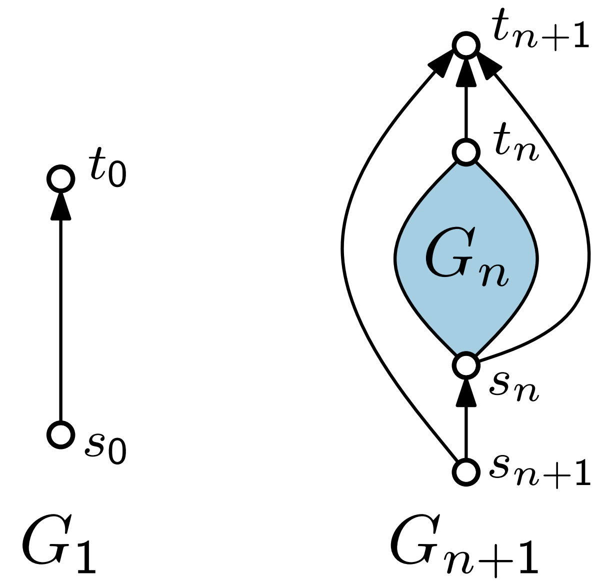

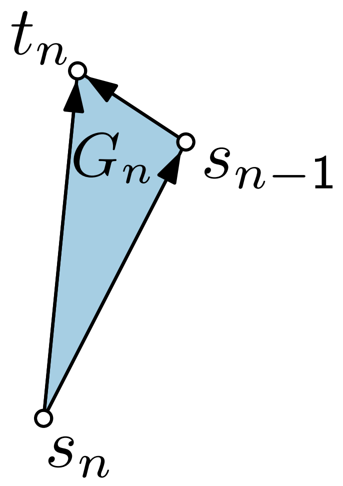

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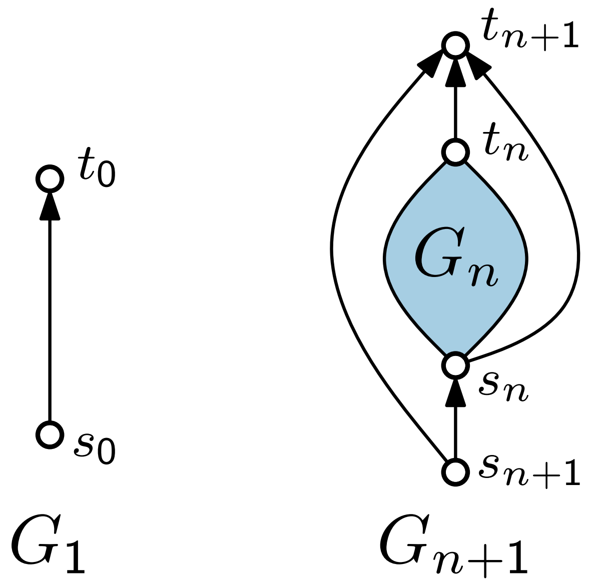

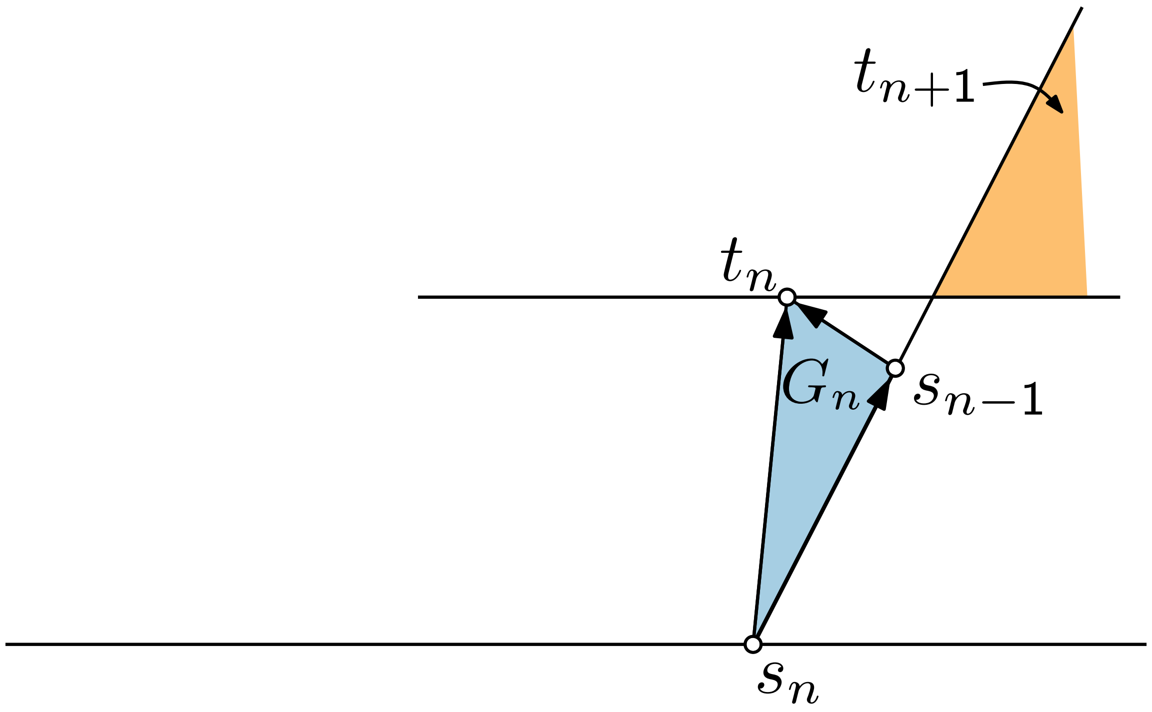

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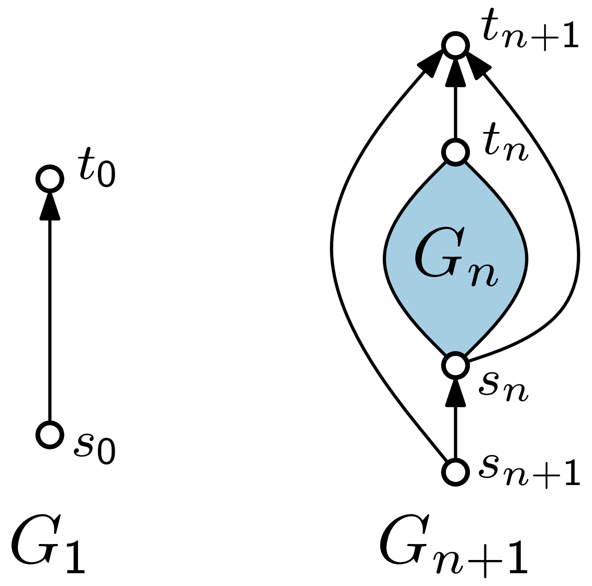

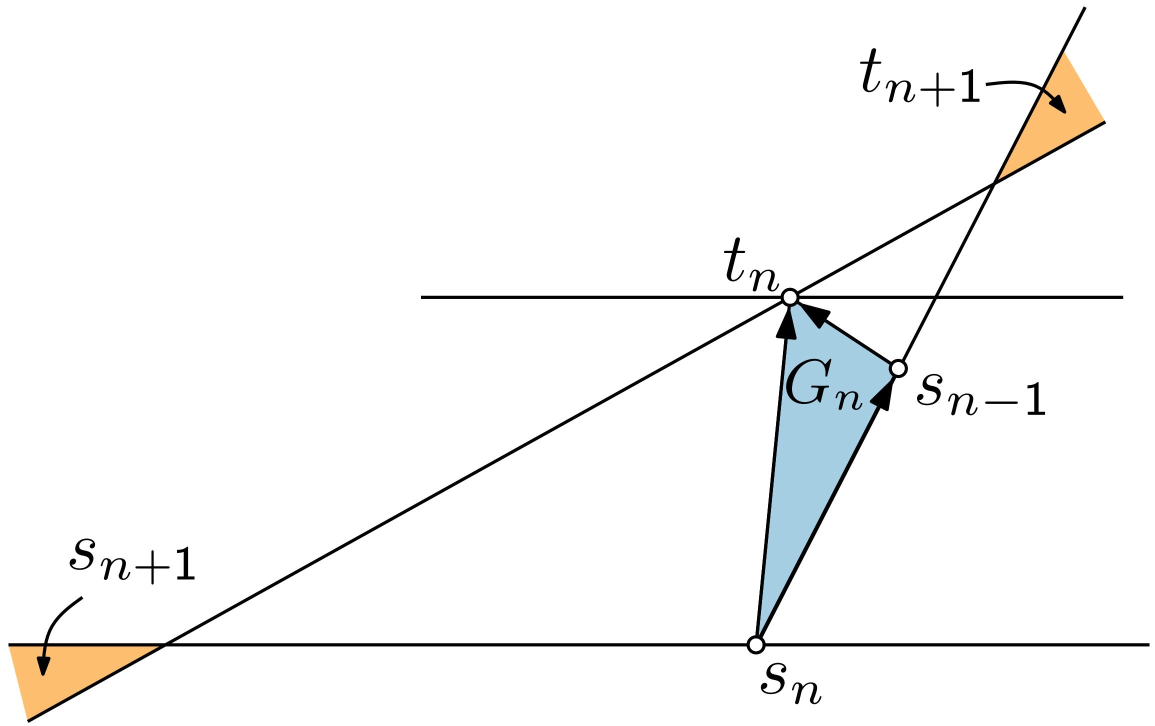

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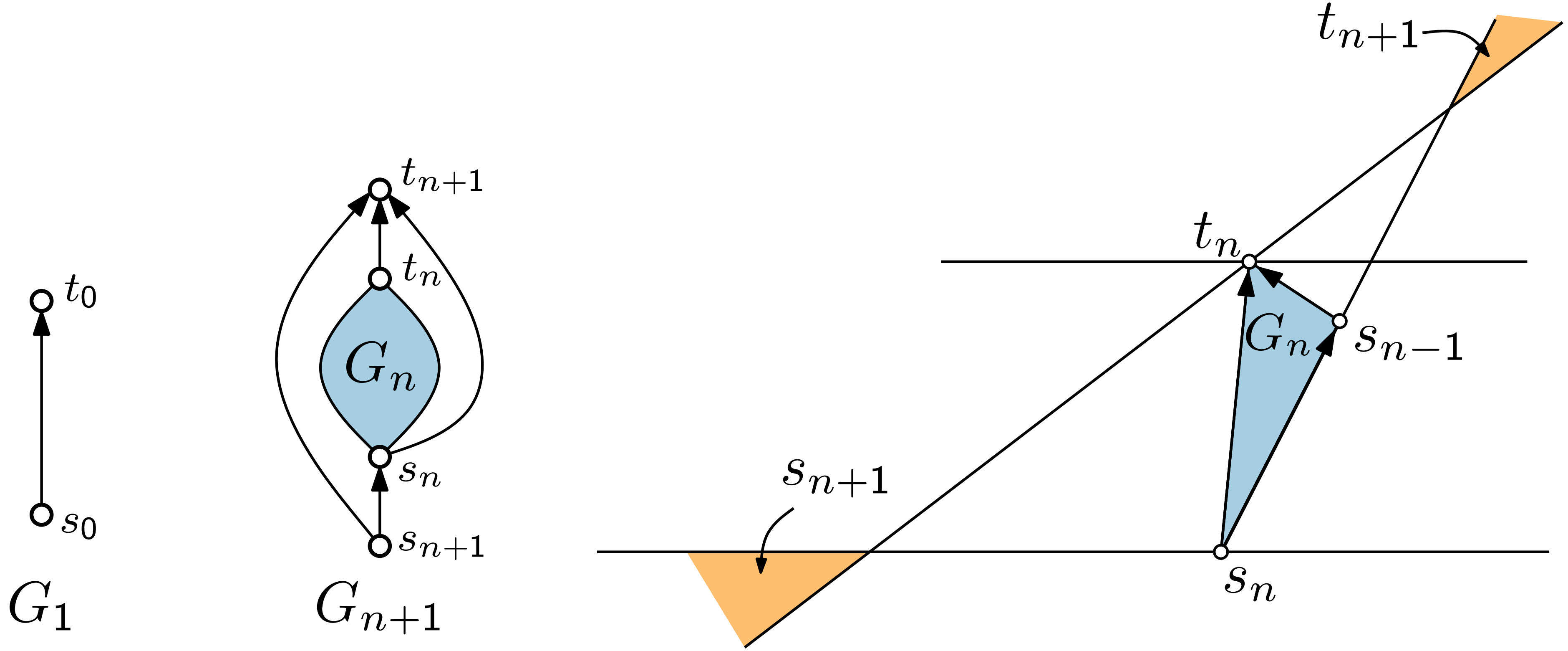

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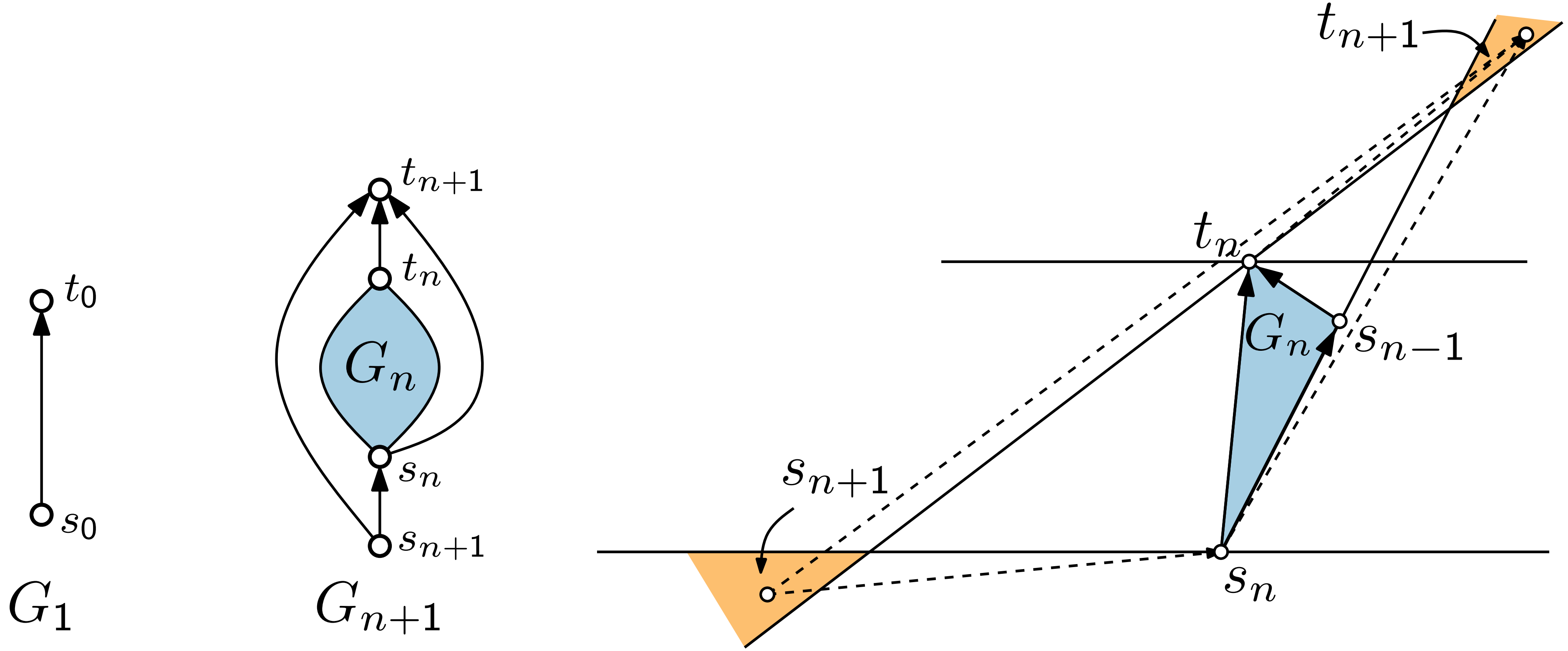

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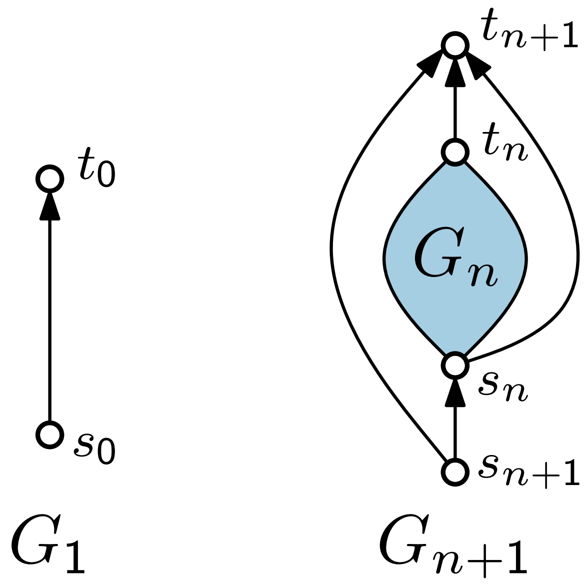

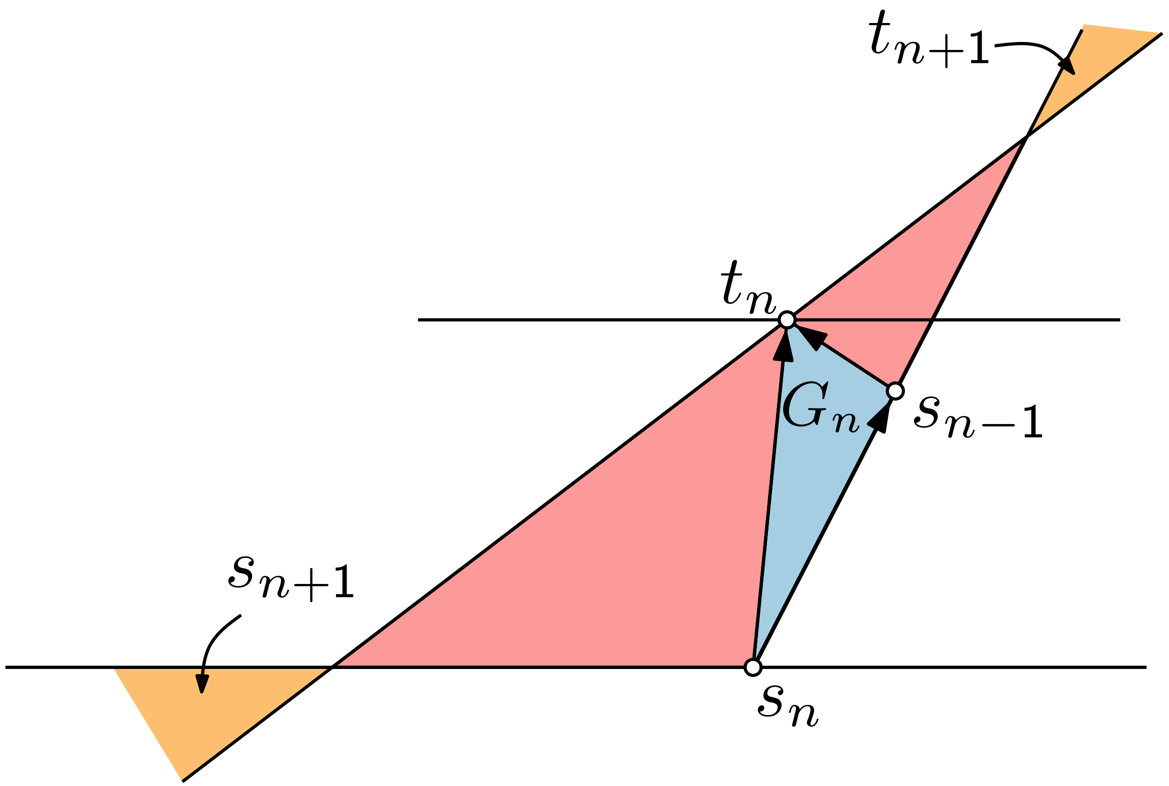

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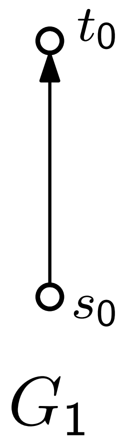

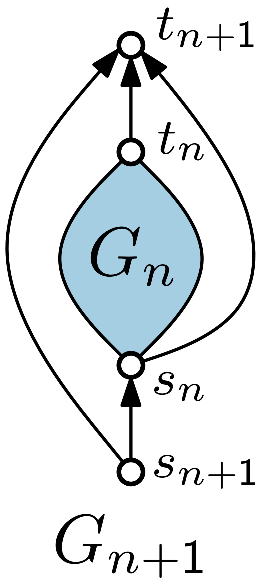

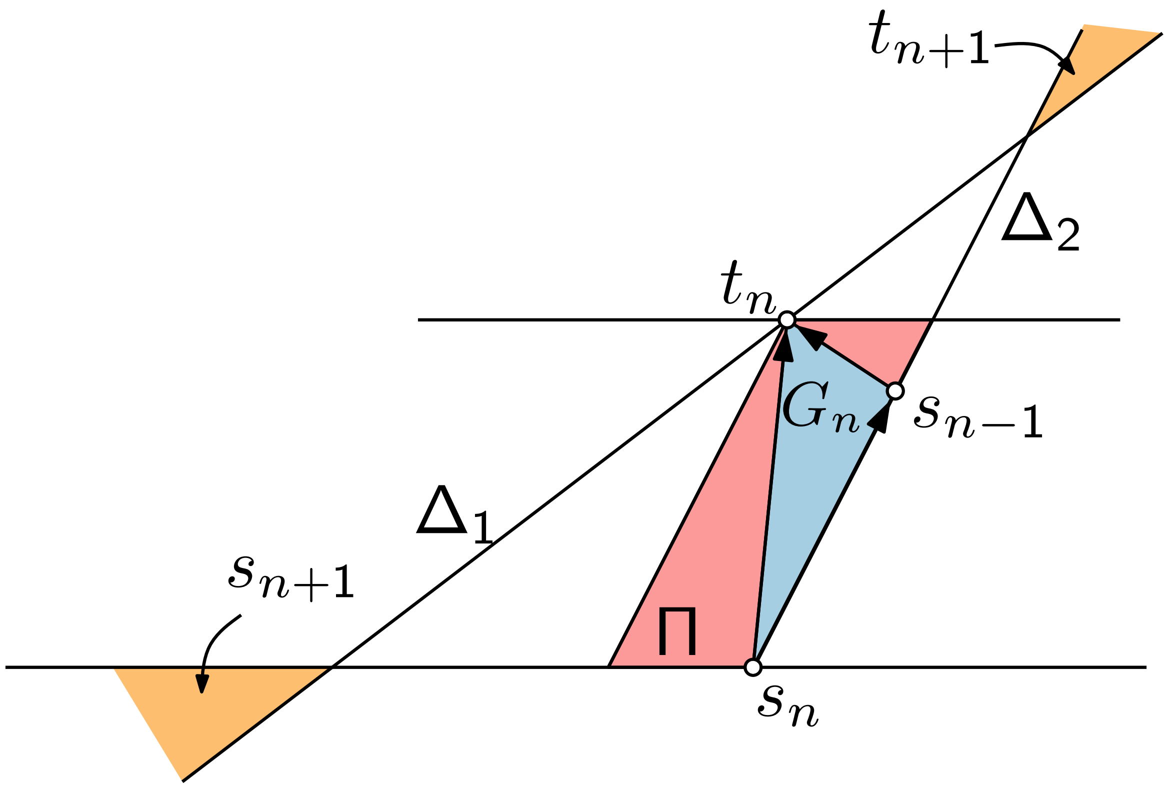

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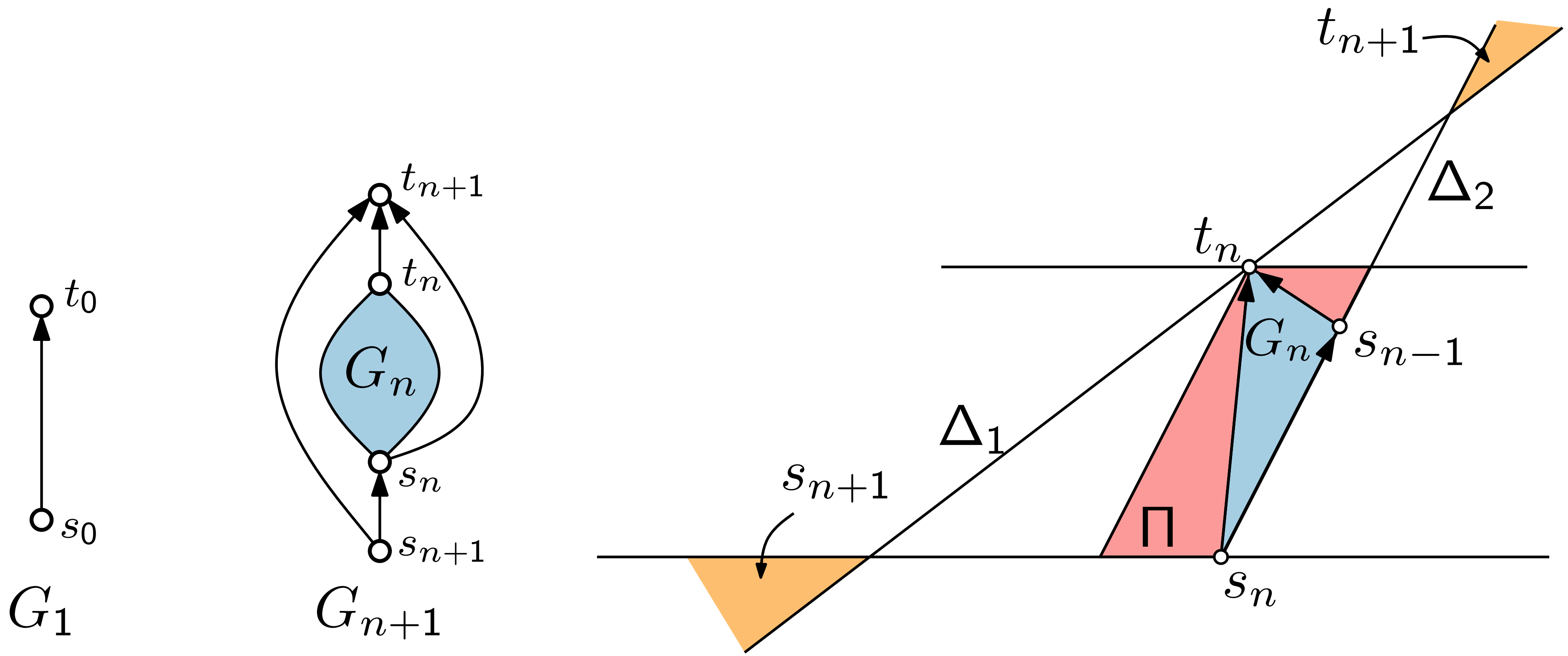

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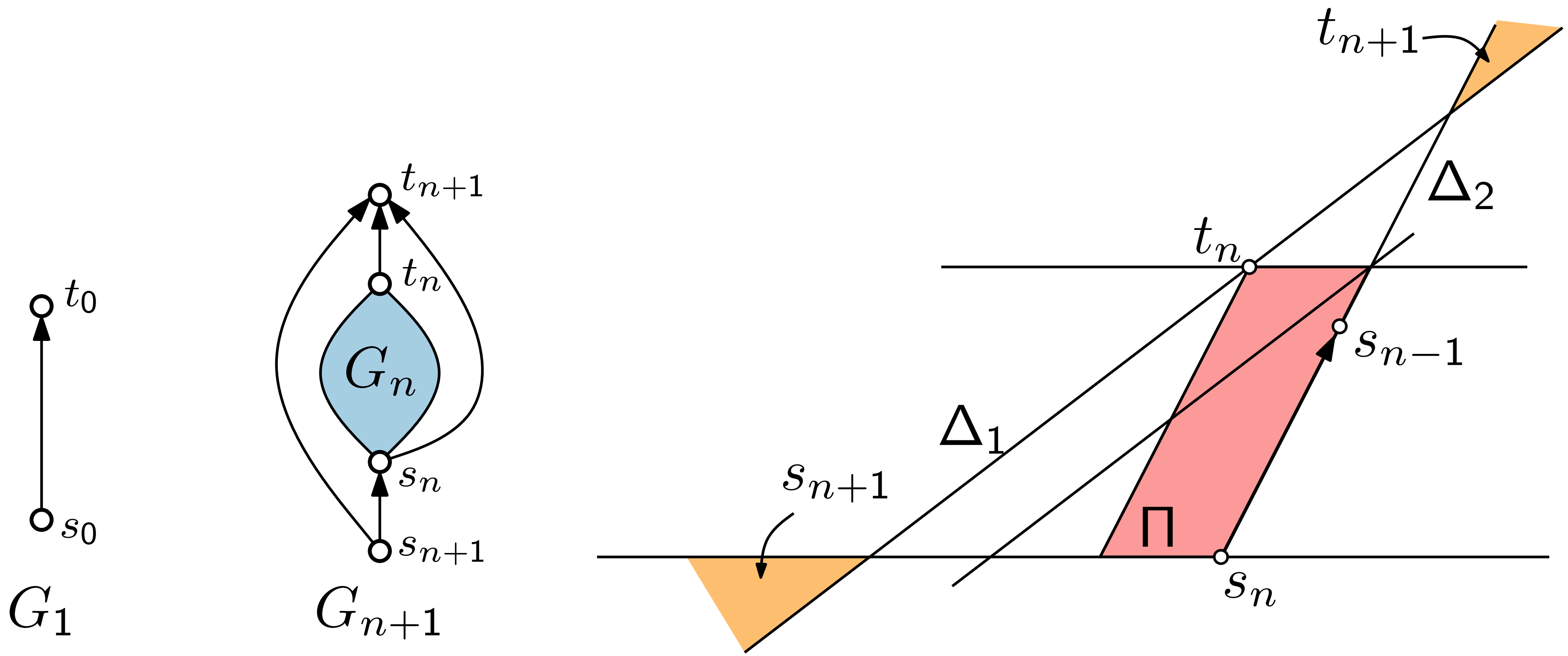

Series-Parallel Graphs – Fixed Embedding

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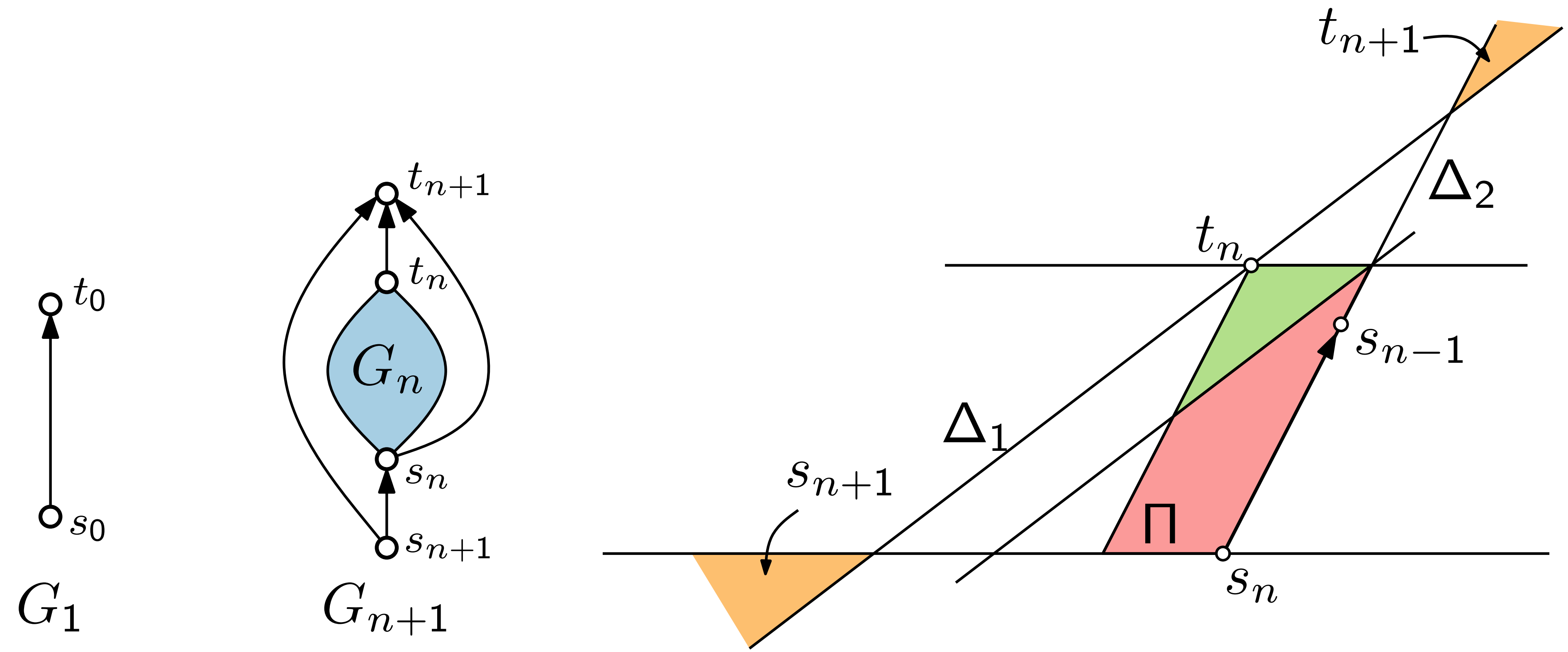

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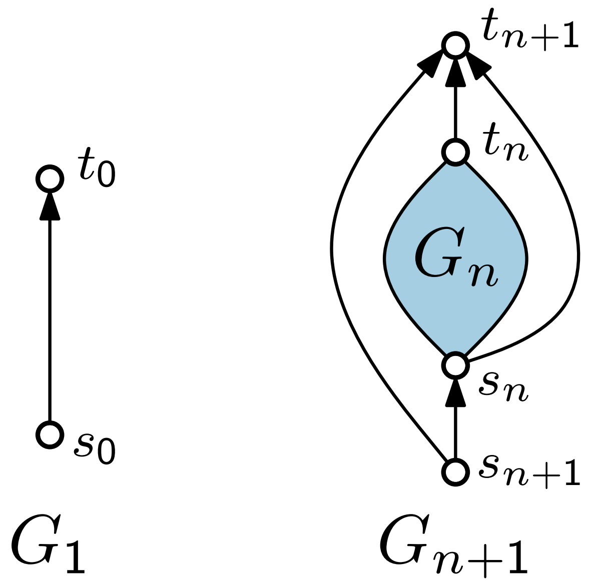

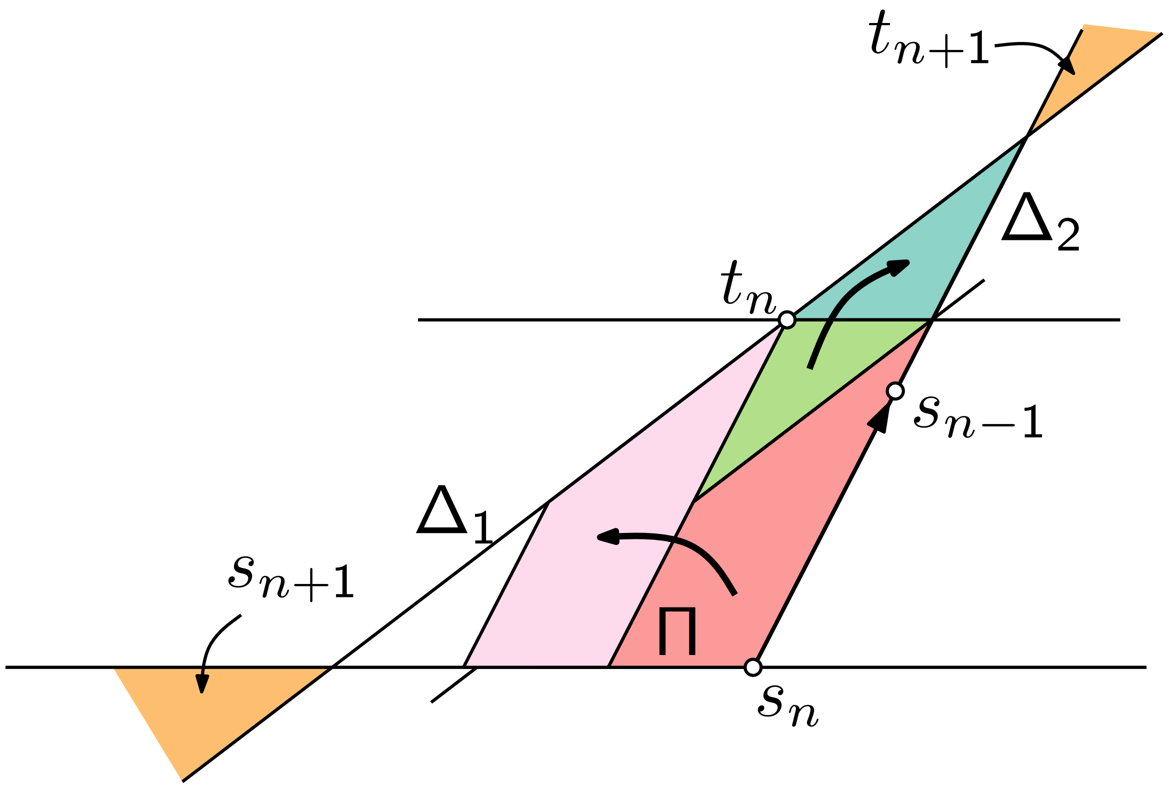

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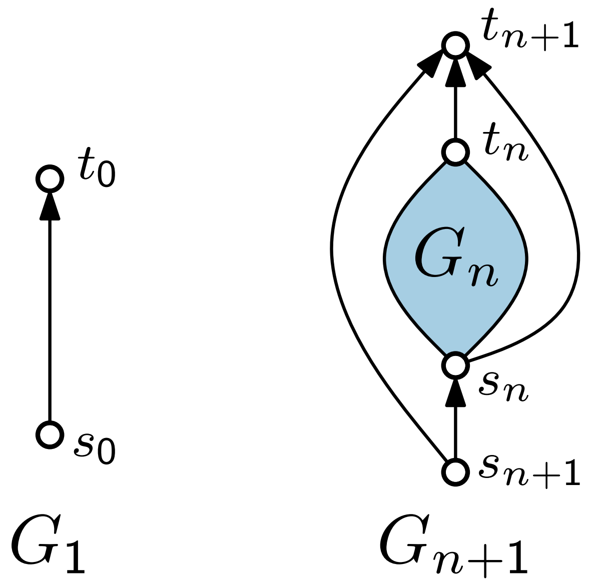

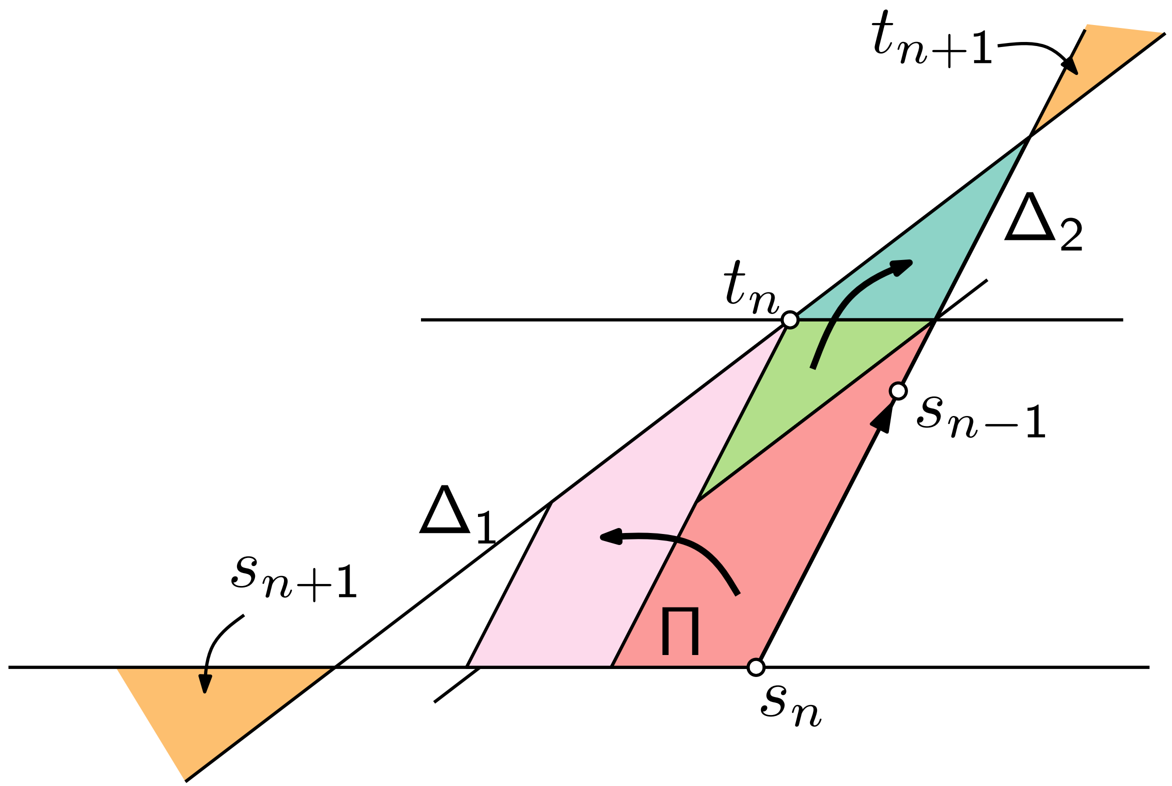

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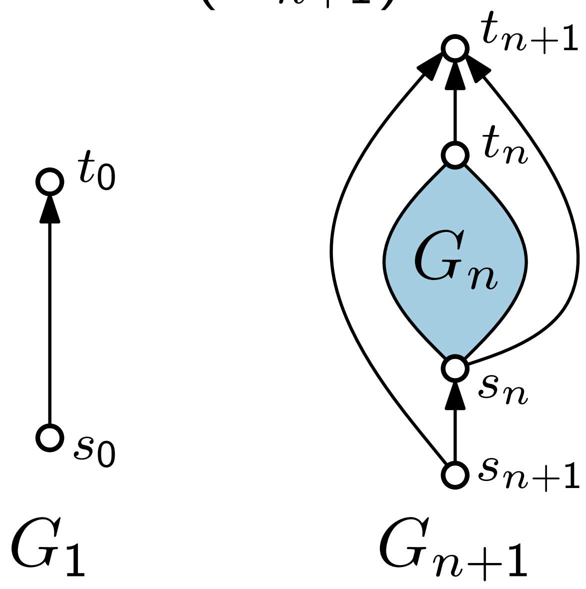
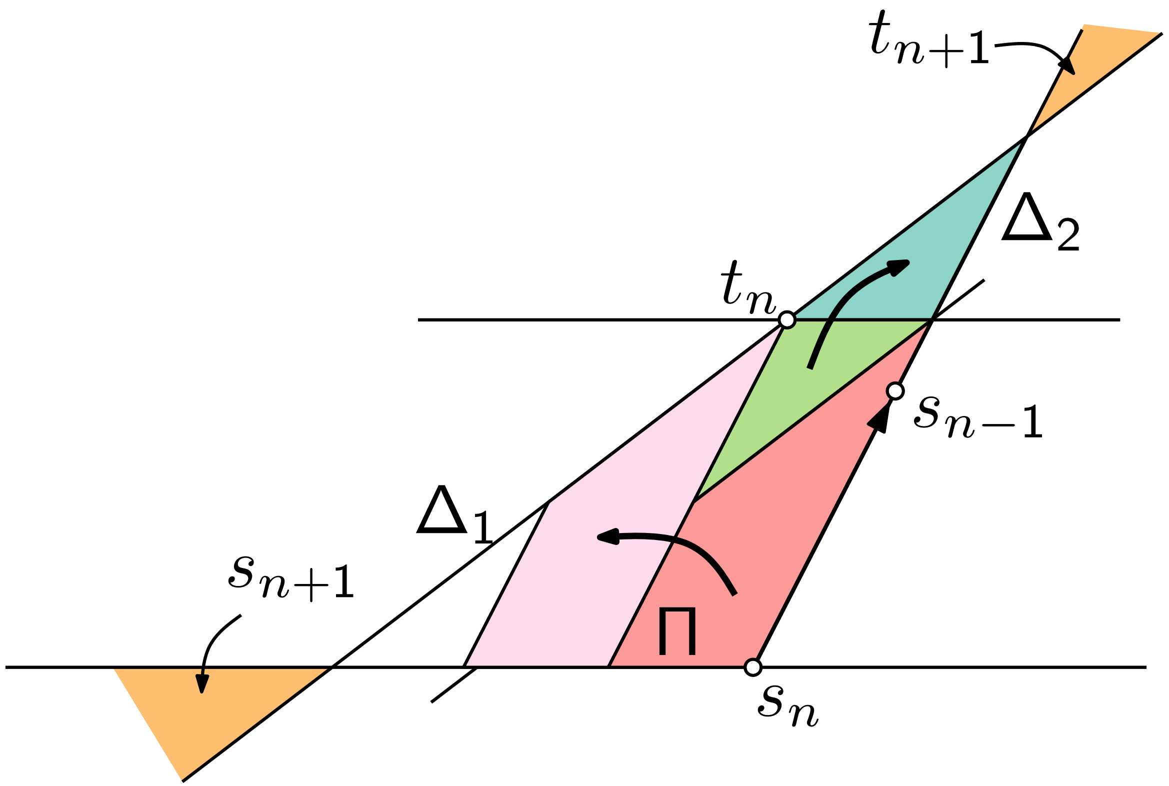
$\Rightarrow 4 \cdot \text{Area}(G_n) < \text{Area}(G_{n+1})$





Discussion

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- Many related concepts have been studied:
  upward drawings of mixed graphs, upward drawings with layers for the vertices, upward planarity on cylinder/torus, upward k-planarity, ...

Literature

- [GD Ch. 6] Detailed explanation on upward planarity.
- [GD Ch. 3] Divide-and-conquer methods for series-parallel graphs.

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giordano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing