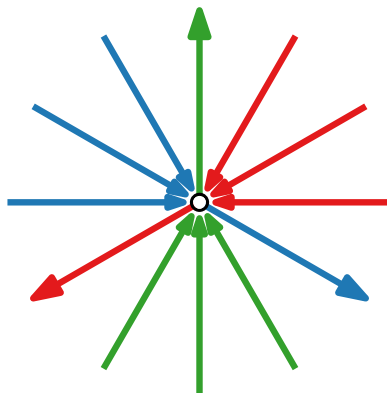
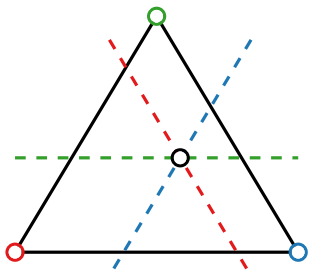


Visualization of Graphs

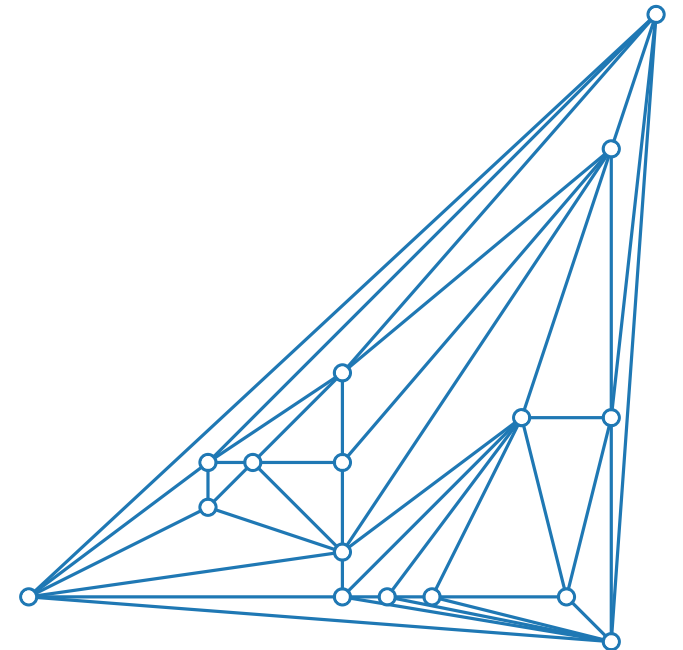
Lecture 4:

Straight-Line Drawings of Planar Graphs II: Schnyder Woods



Alexander Wolff

Summer term 2026



Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$.

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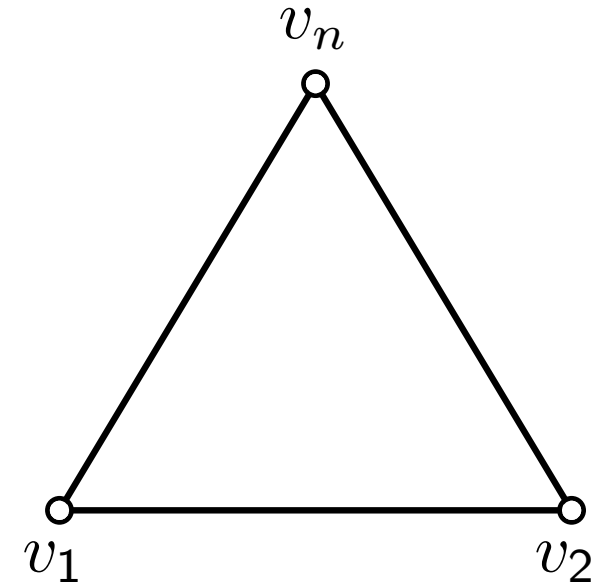
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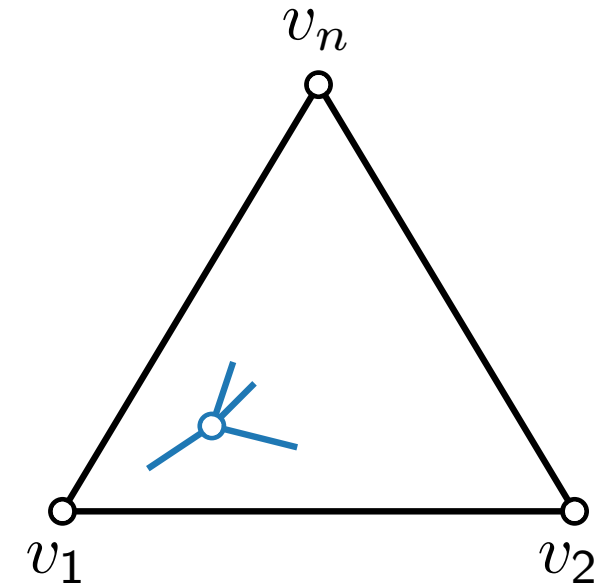
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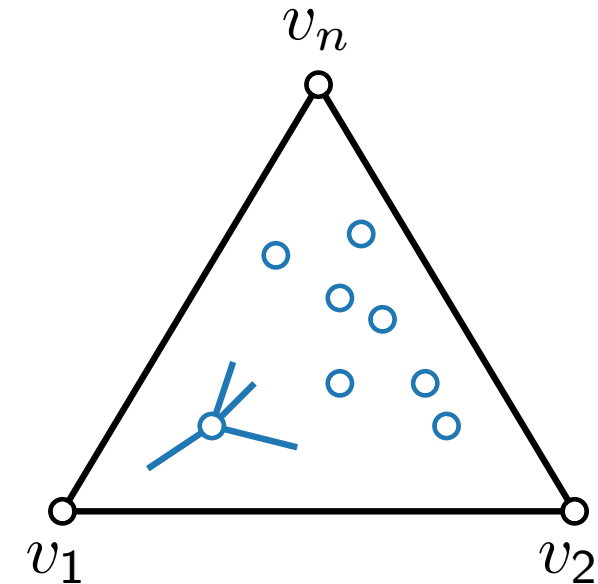
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- Fix outer triangle.
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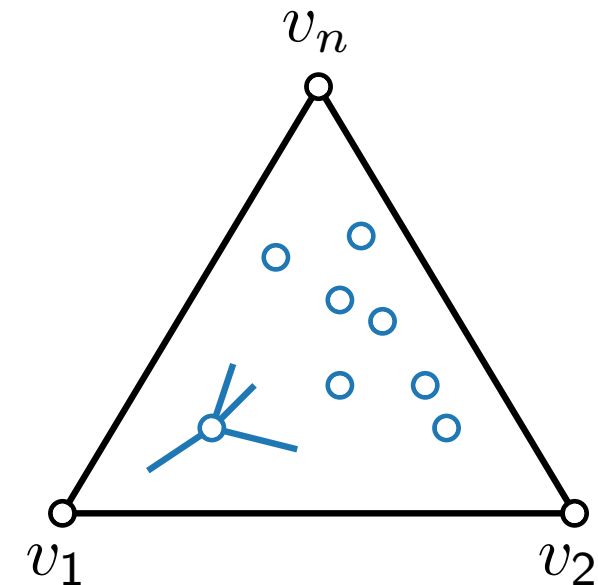
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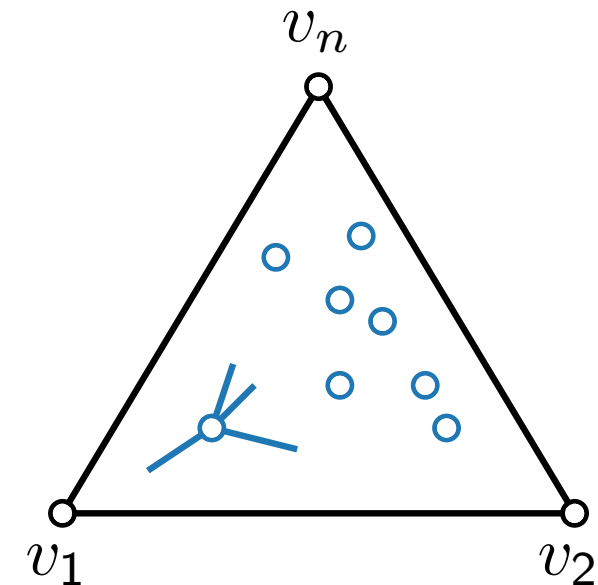
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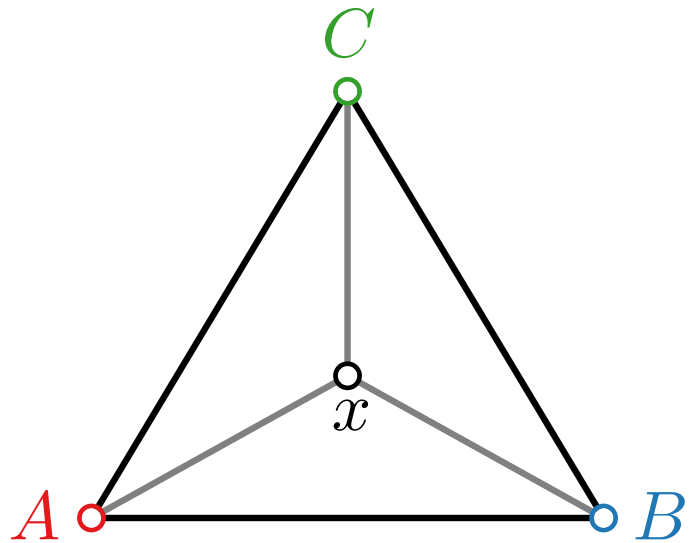
Idea. (easier to show)

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Barycentric Coordinates

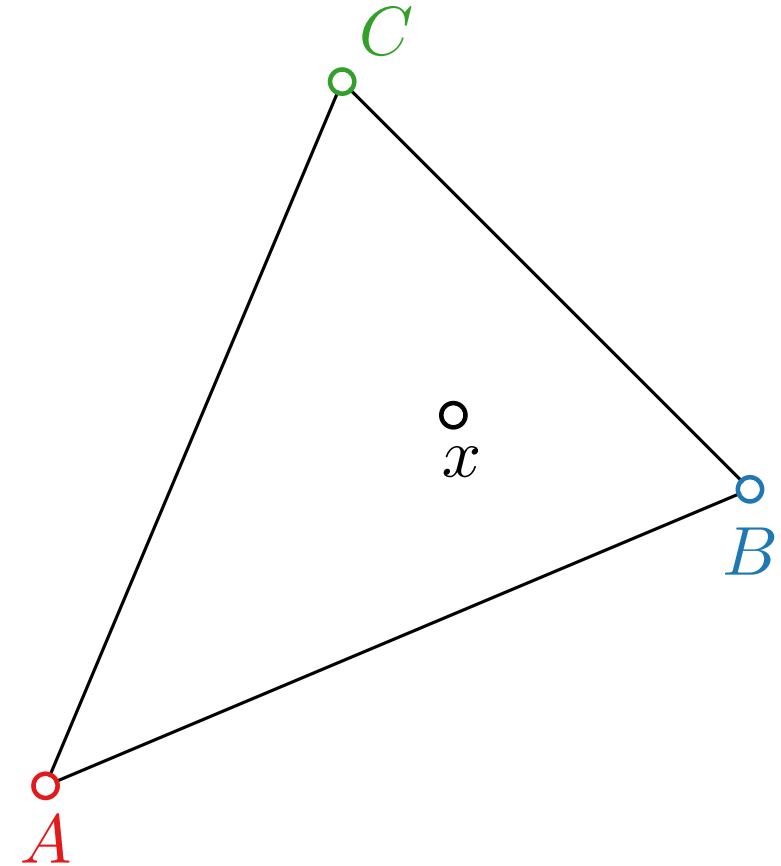
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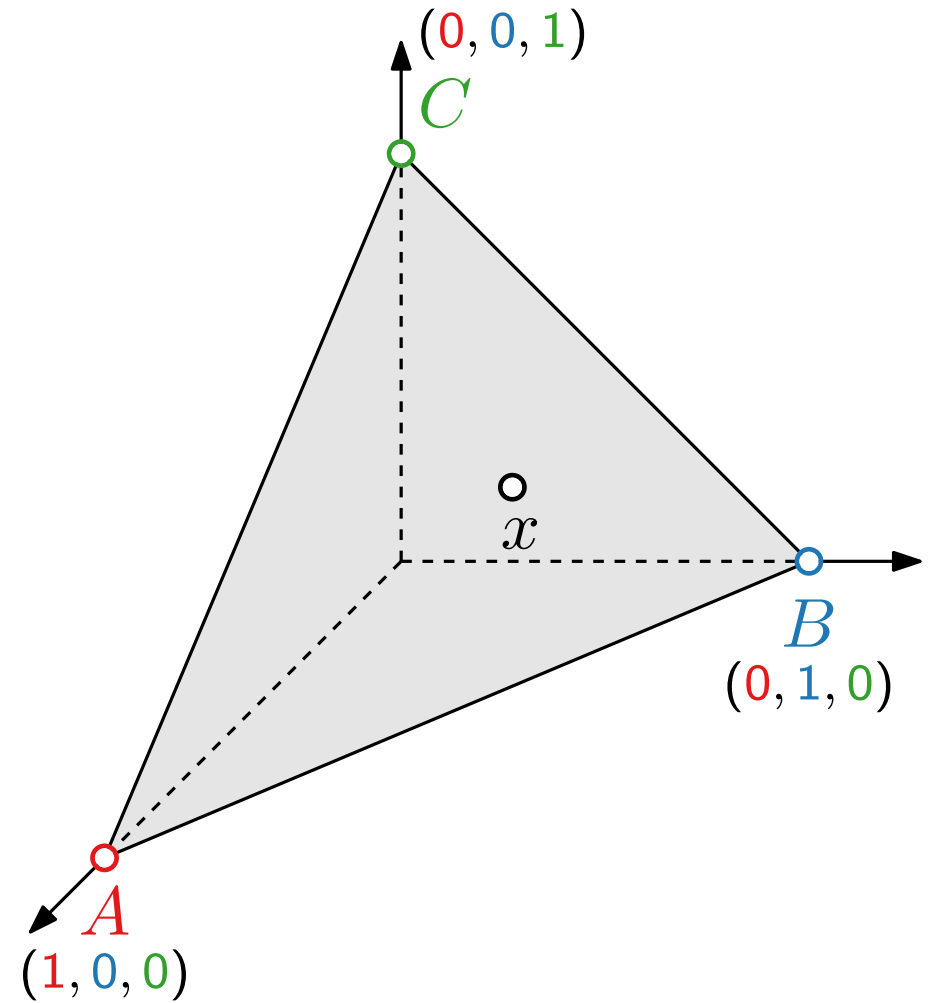
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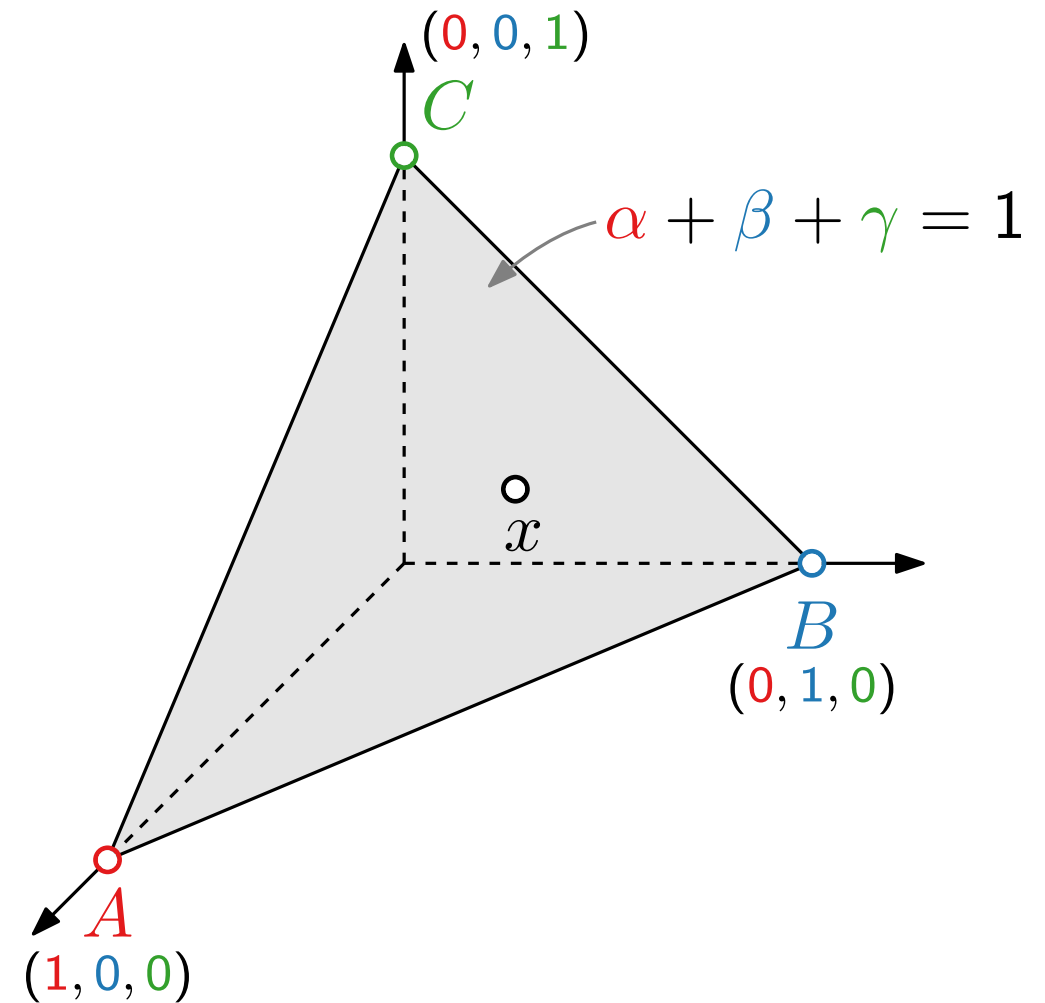
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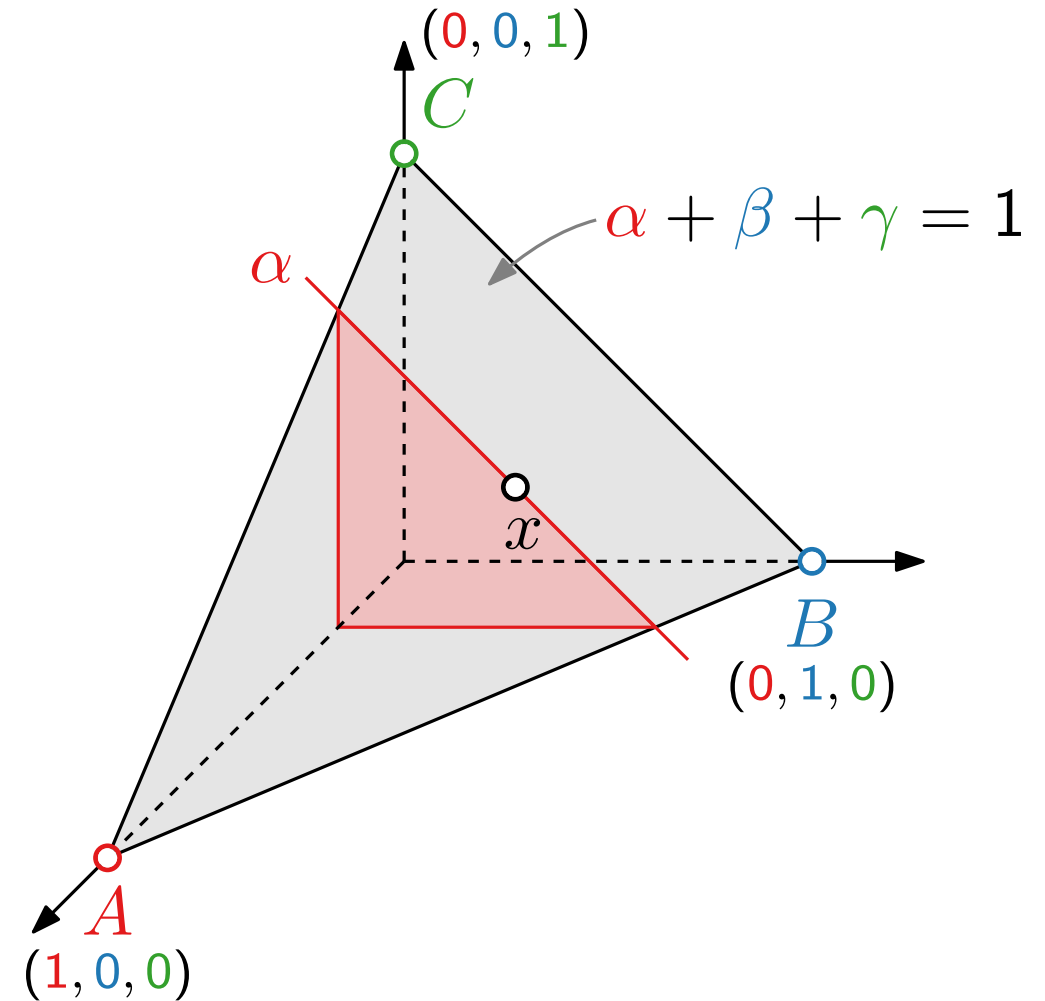
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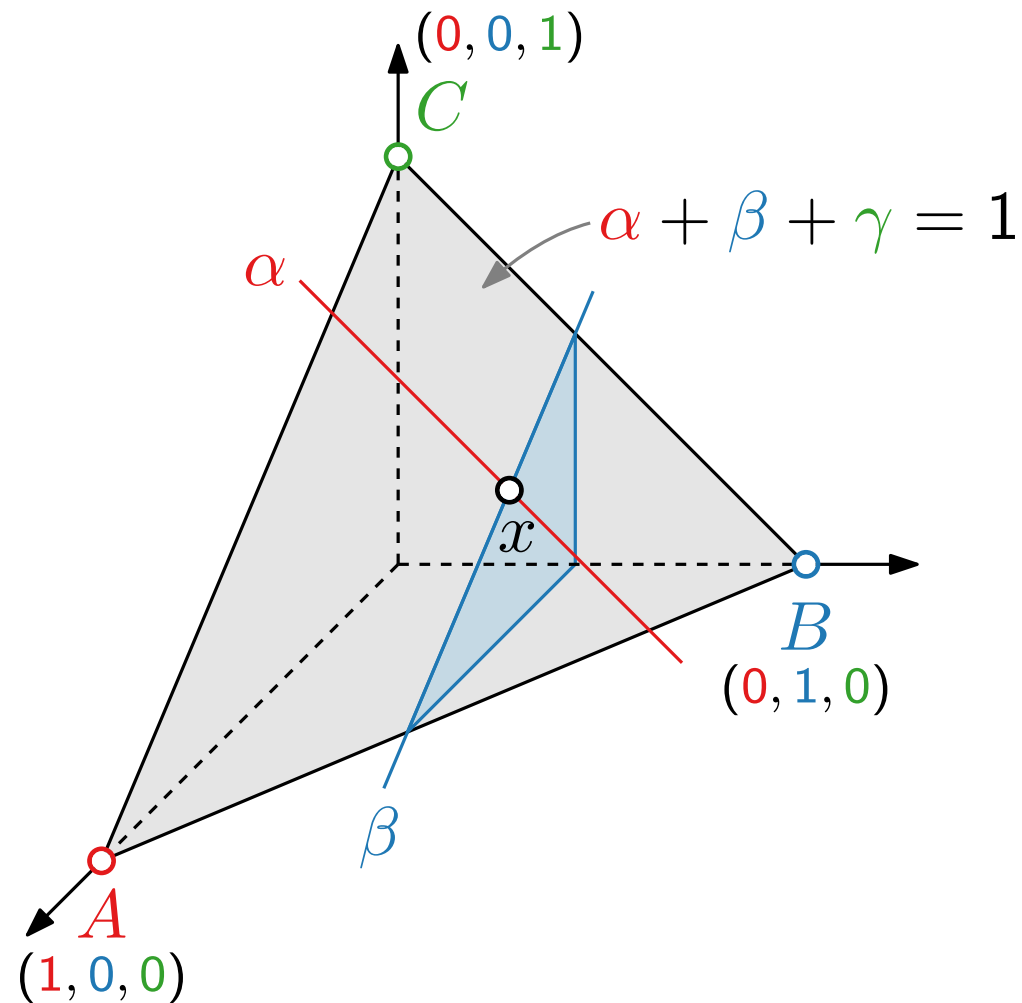
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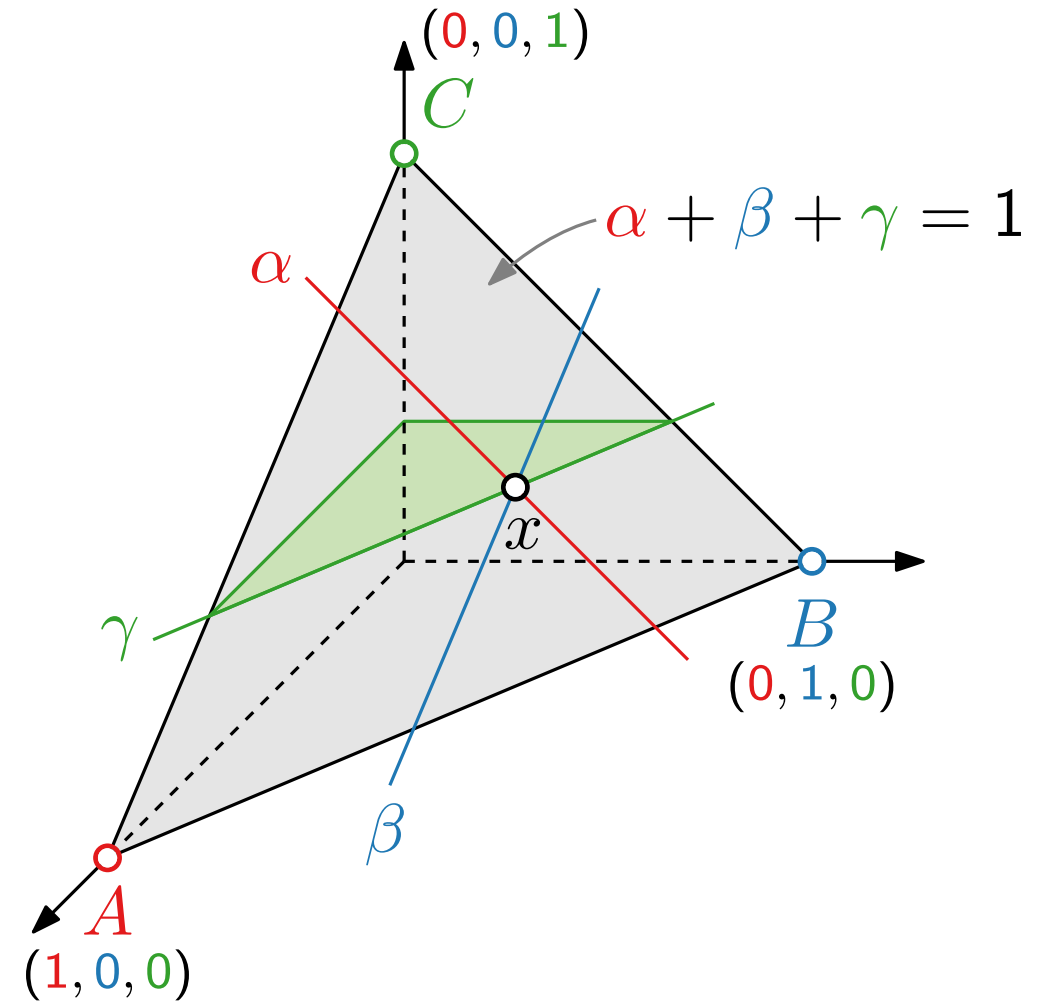
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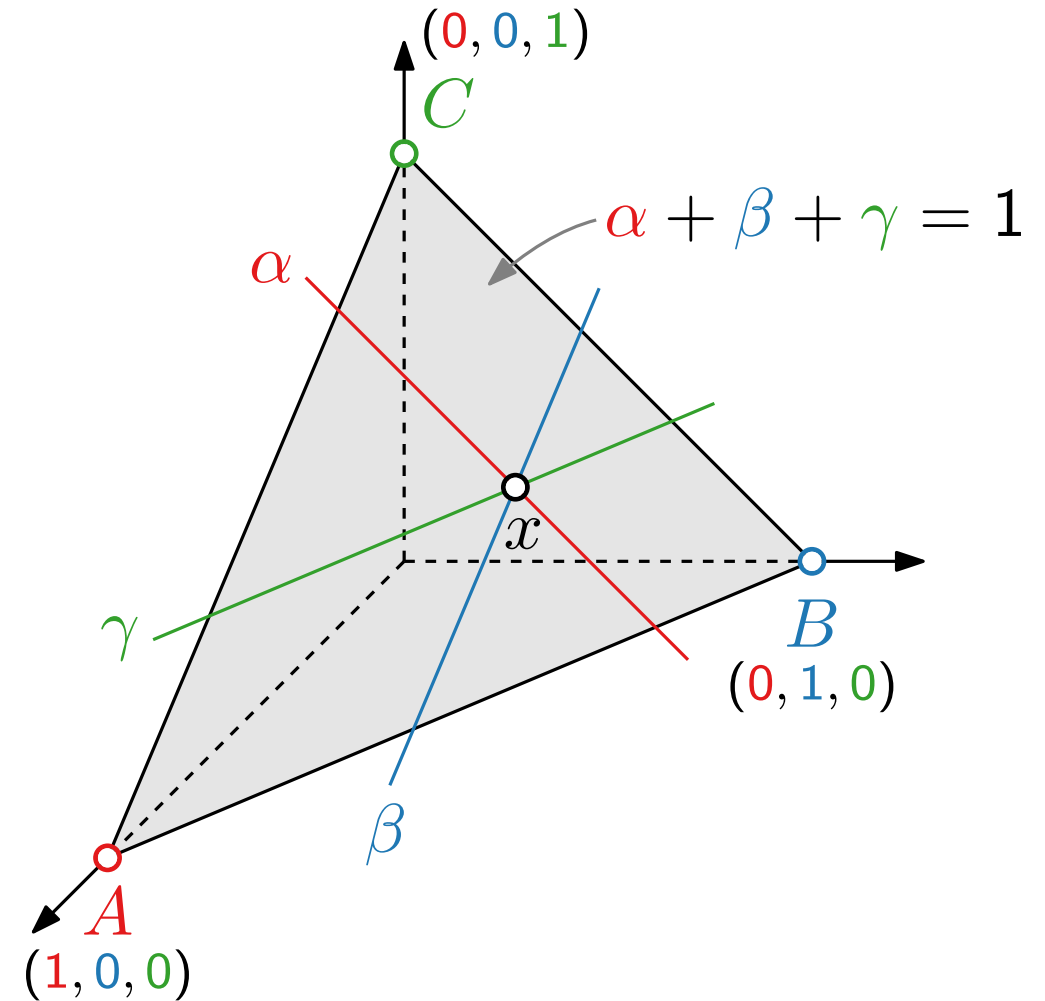
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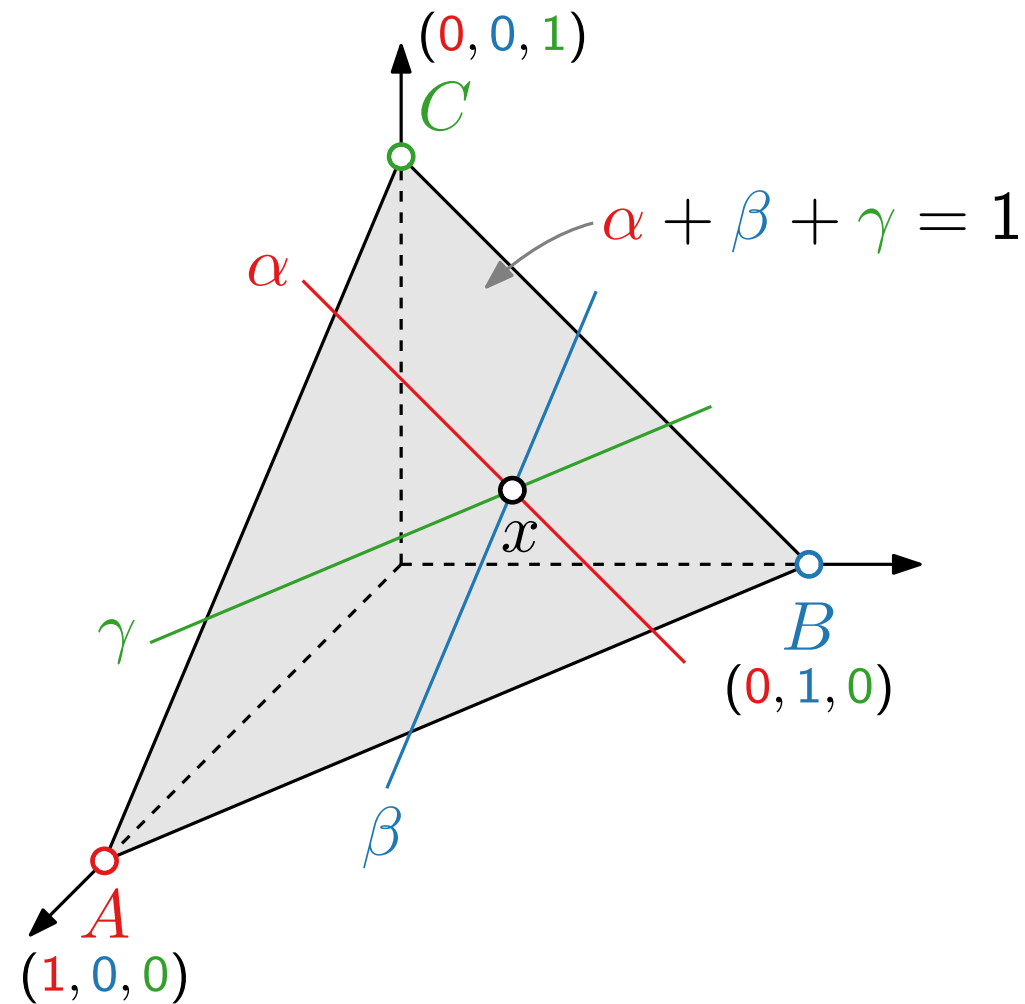
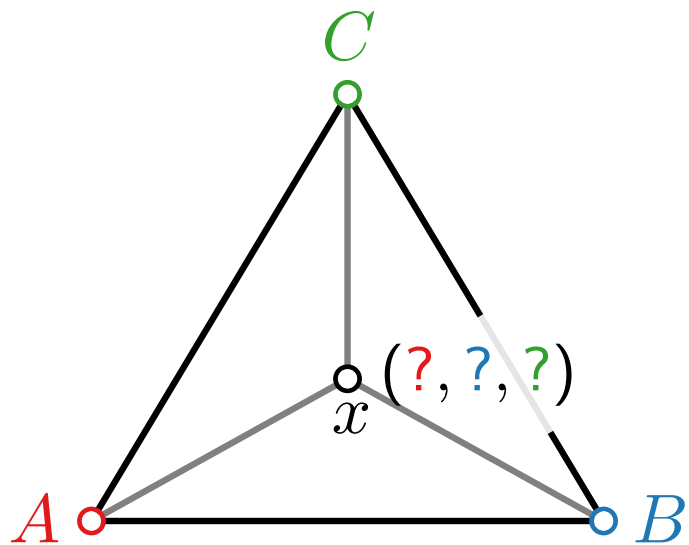
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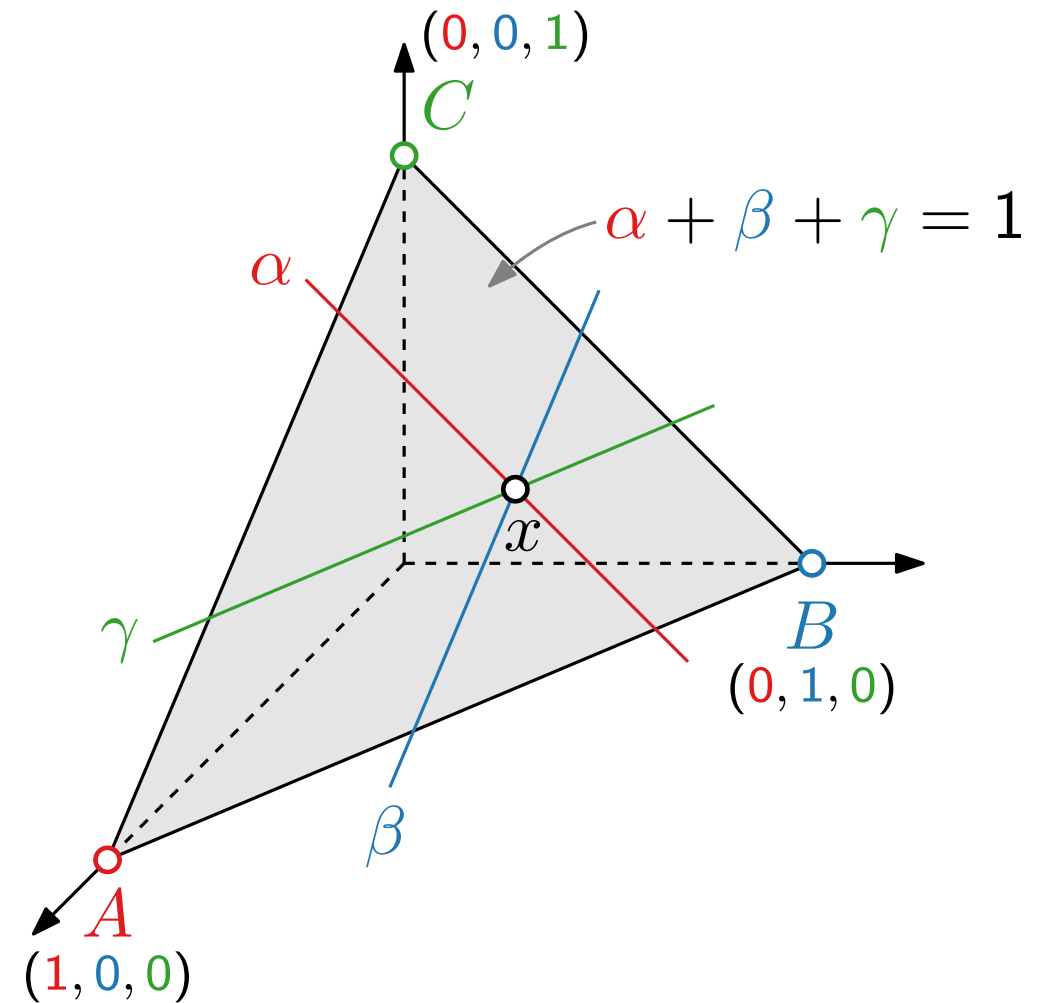
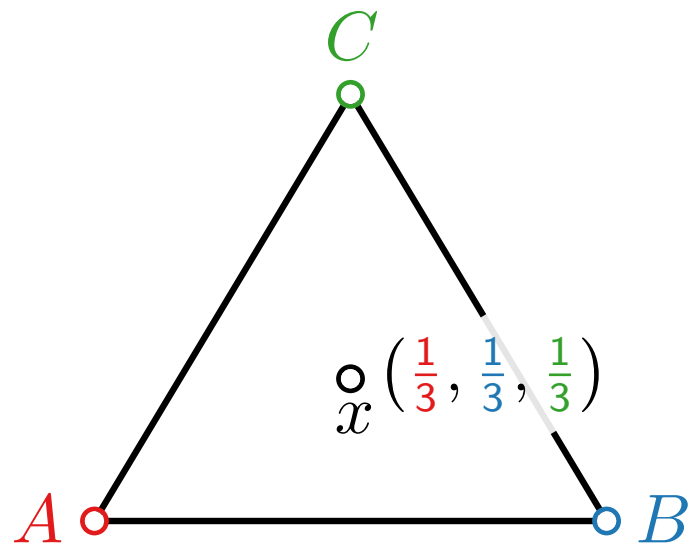
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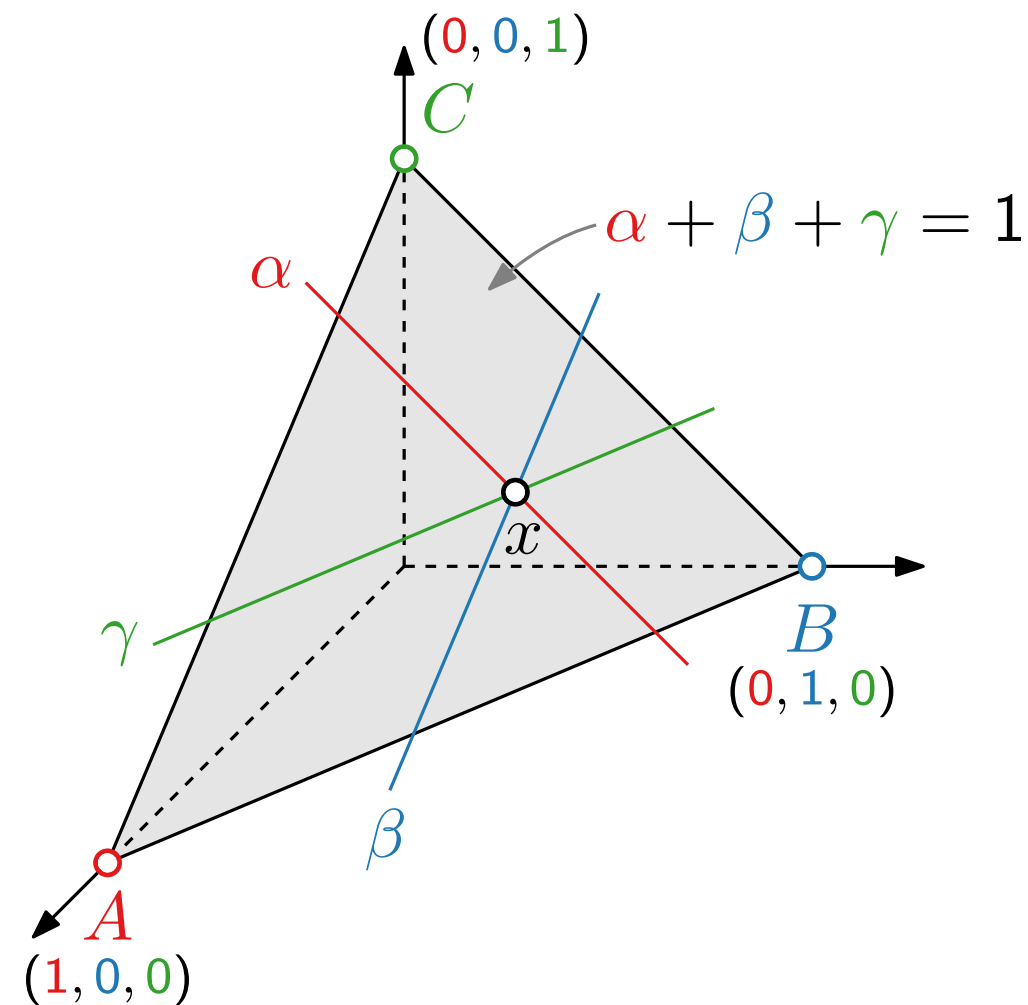
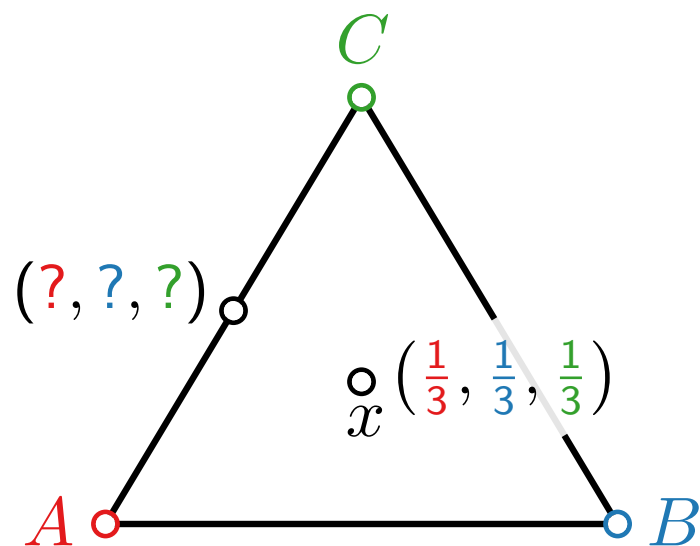
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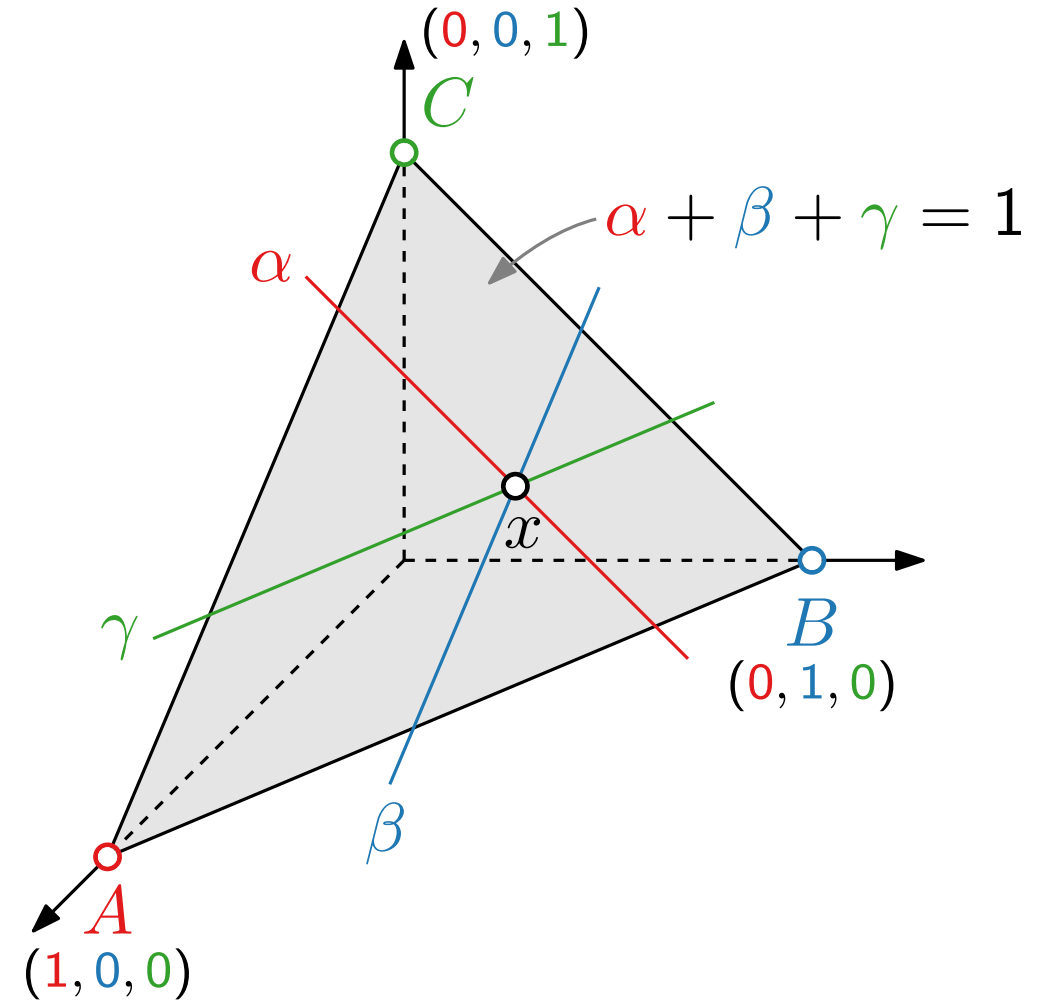
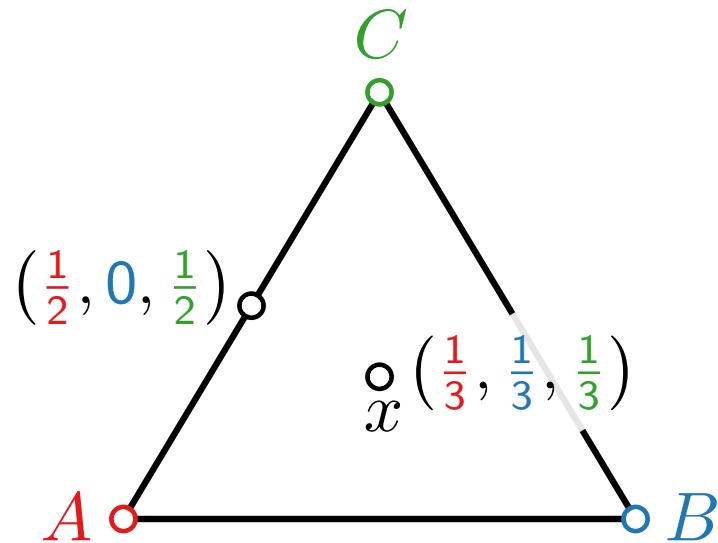


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A **barycentric representation** of a graph G is an assignment of barycentric coordinates to the vertices of G :

$$f: V(G) \rightarrow \mathbb{R}_{\geq 0}^3, \quad v \mapsto (v_1, v_2, v_3)$$

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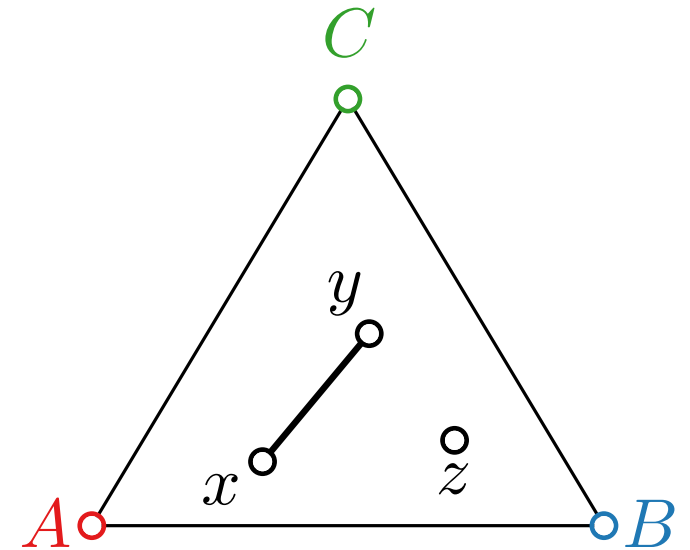
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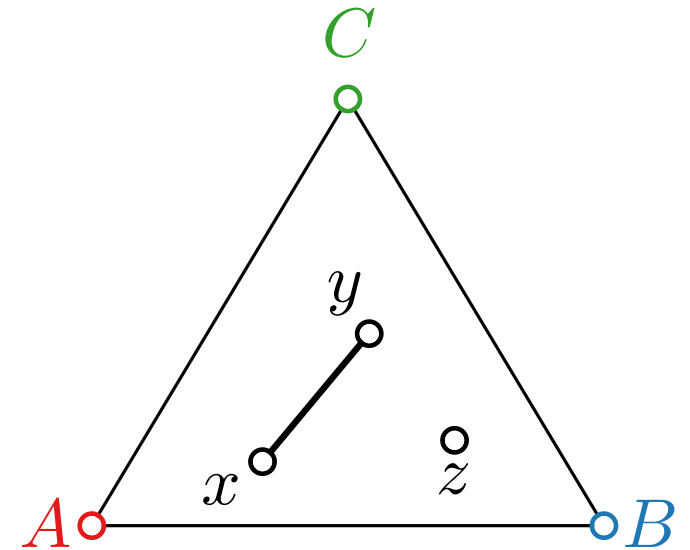
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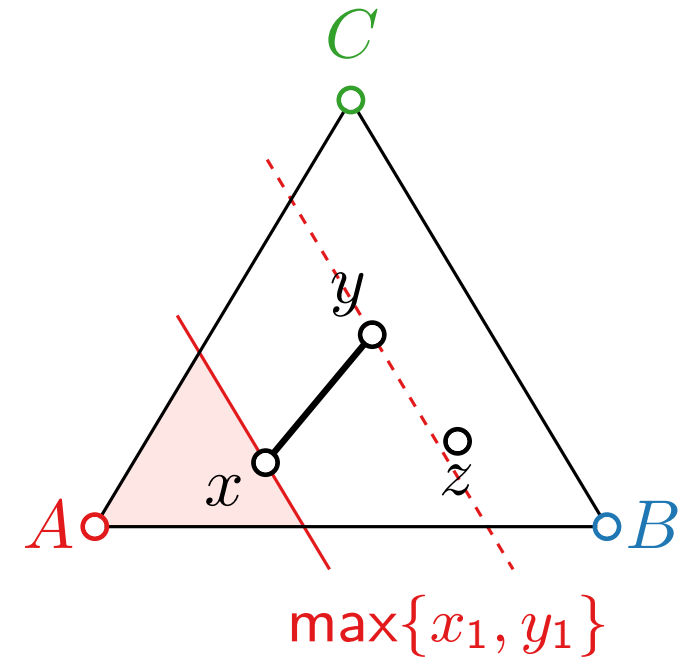
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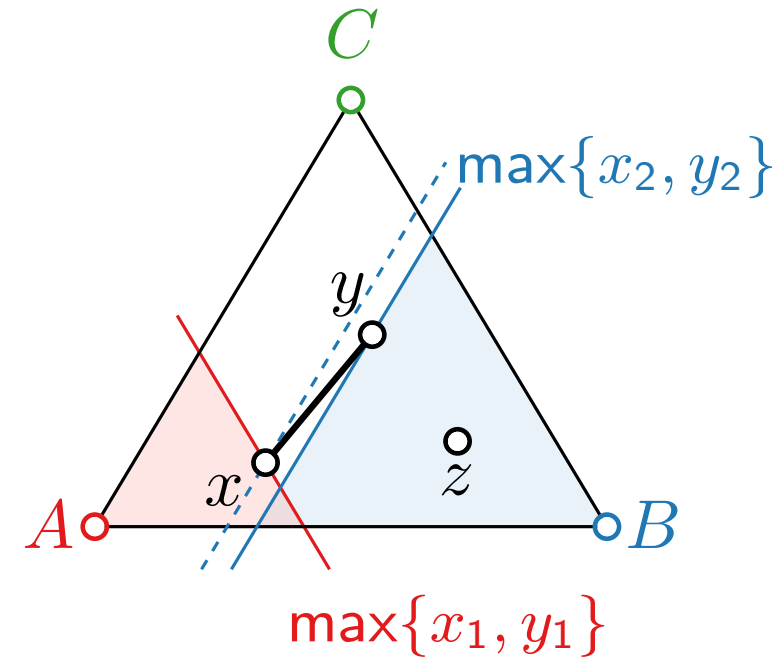
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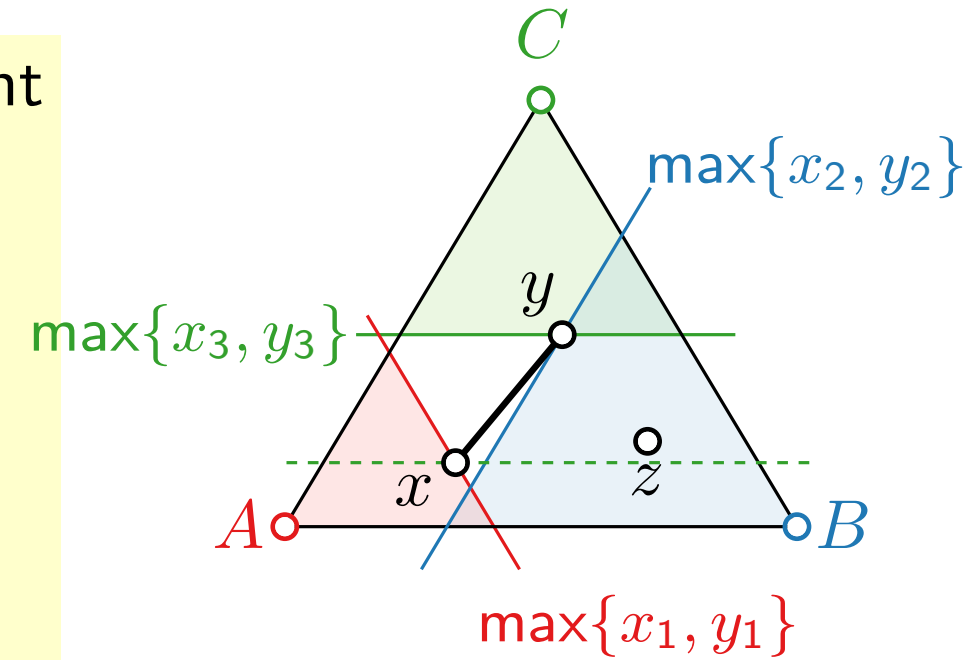
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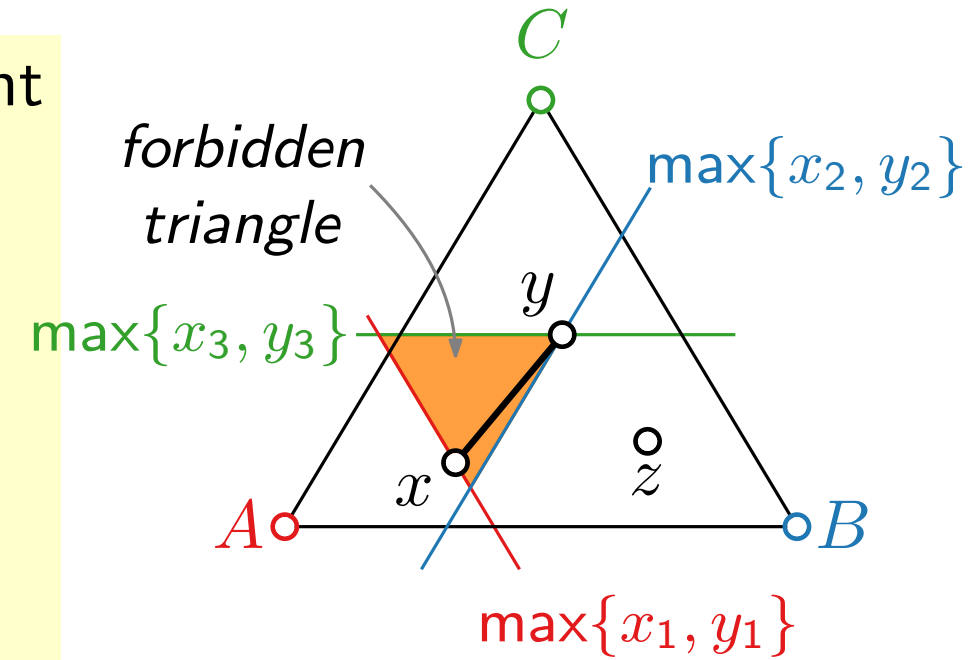
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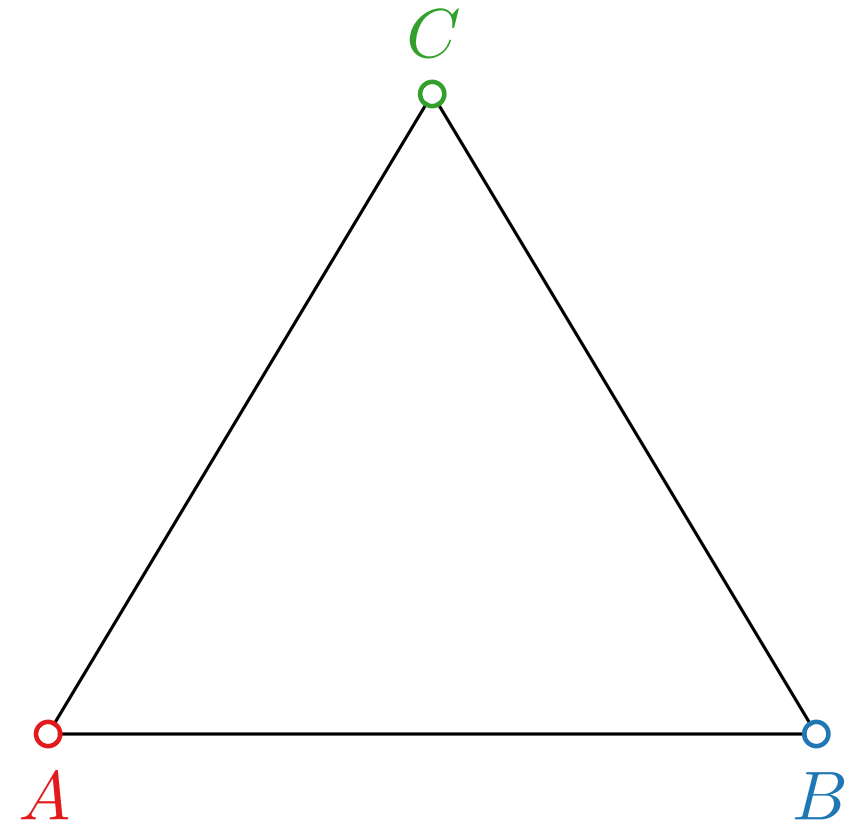
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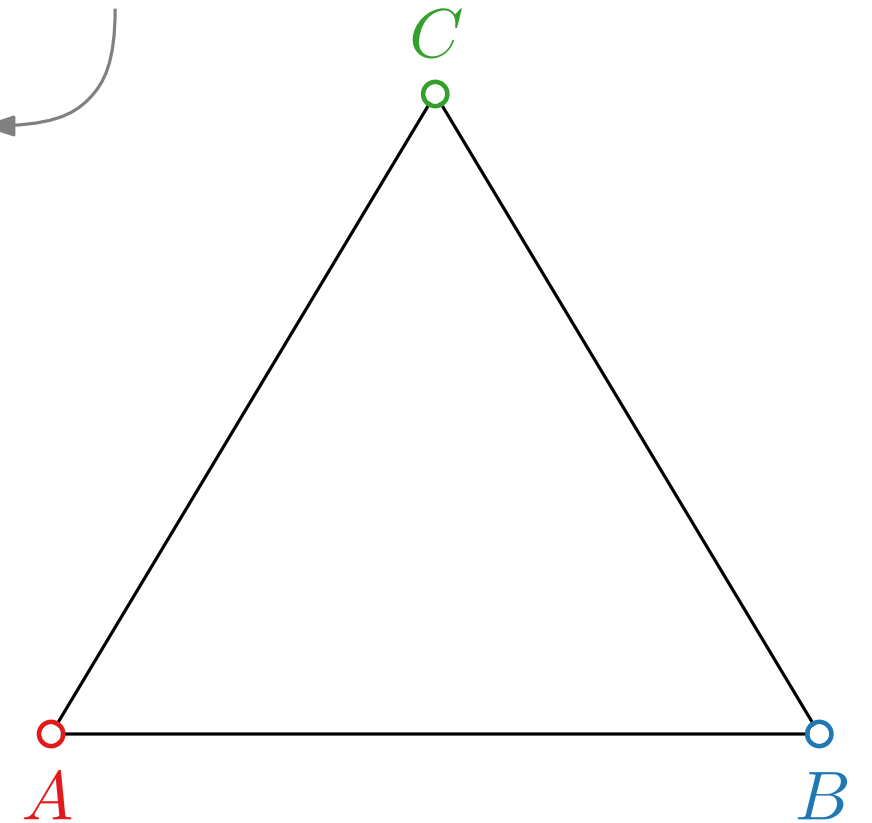


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no three points
on a line



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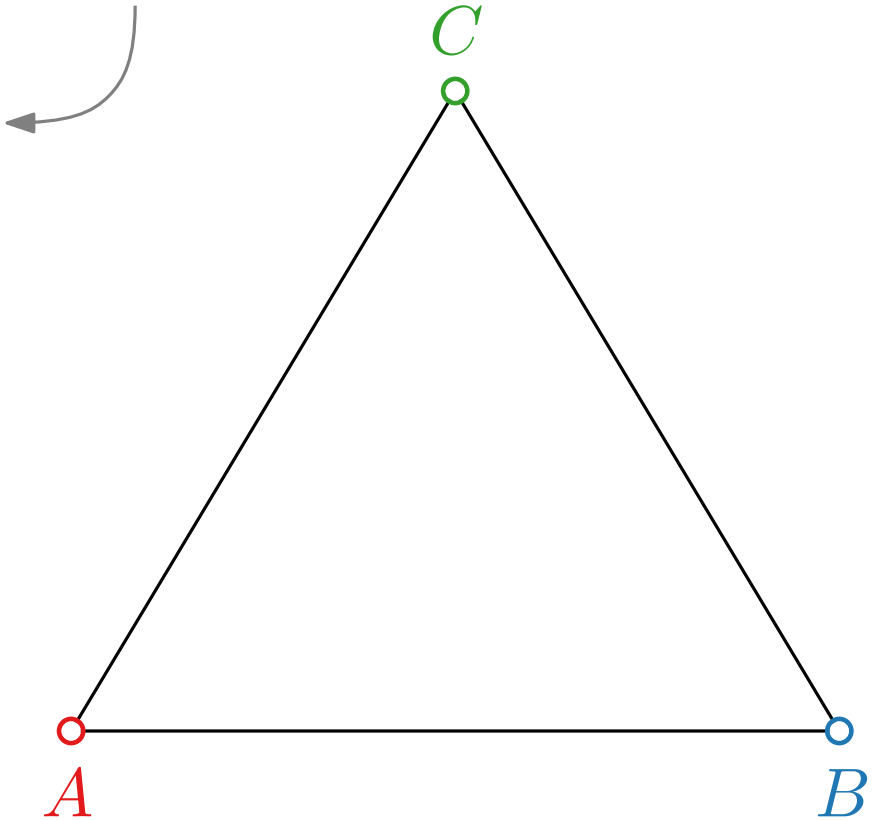
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$$\phi: v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar straight-line** drawing of G inside $\triangle ABC$.

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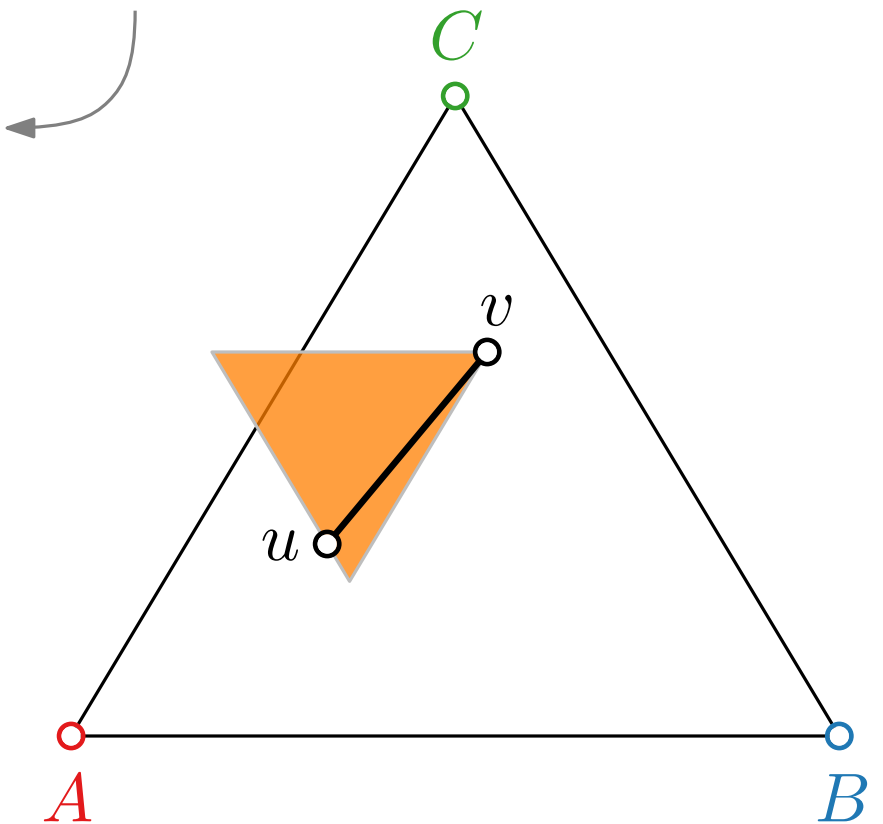
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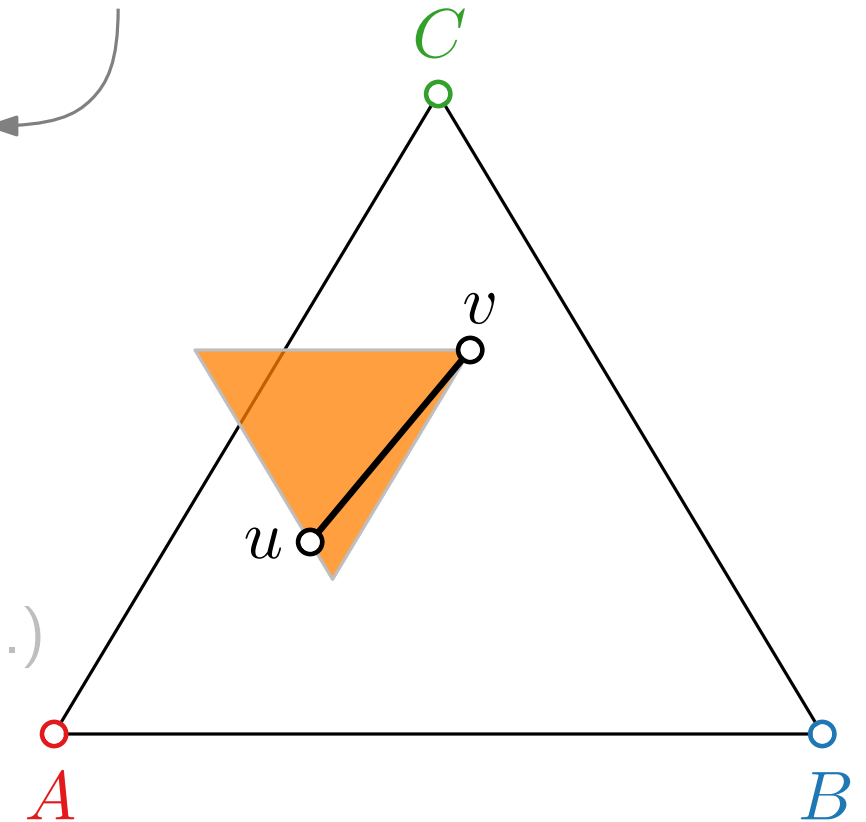
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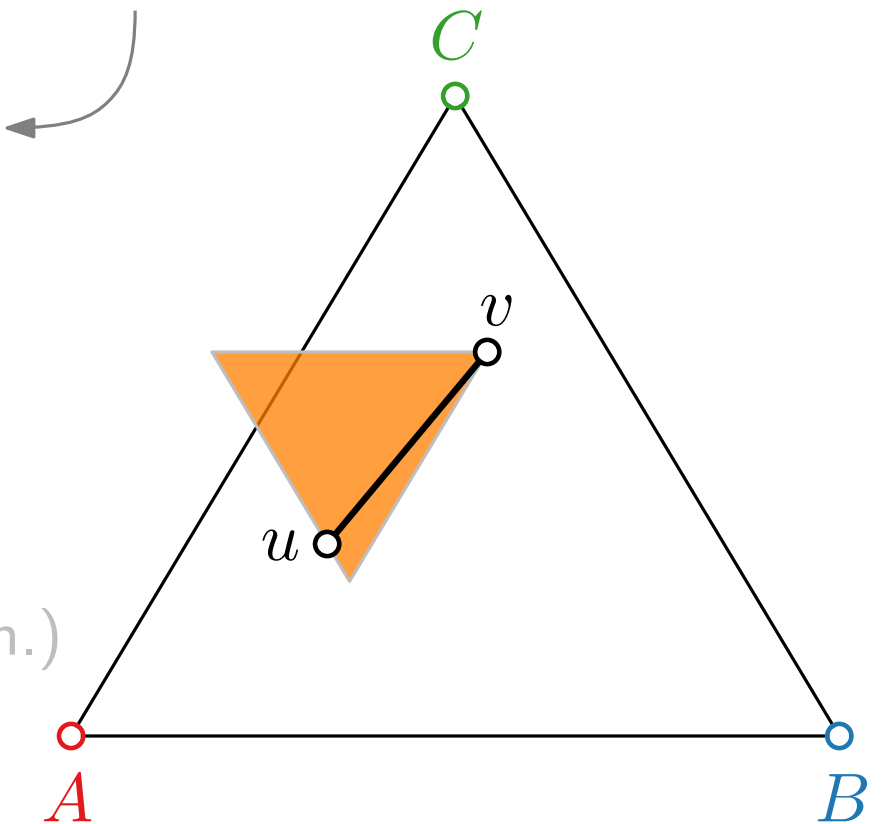
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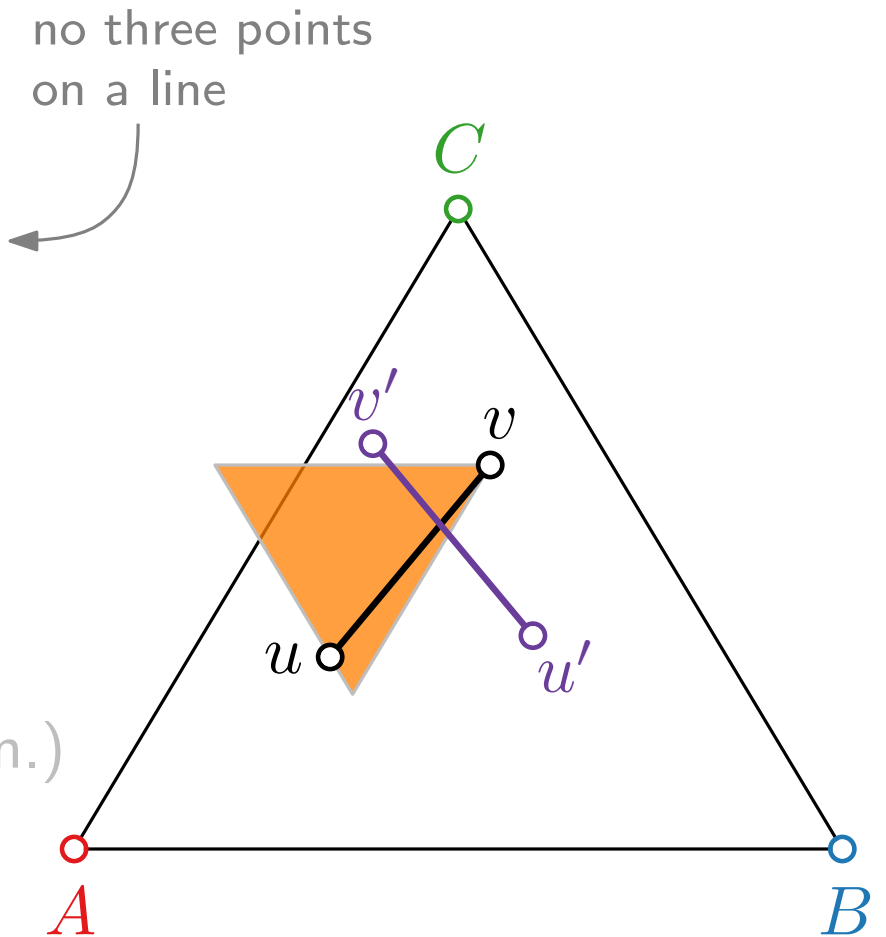
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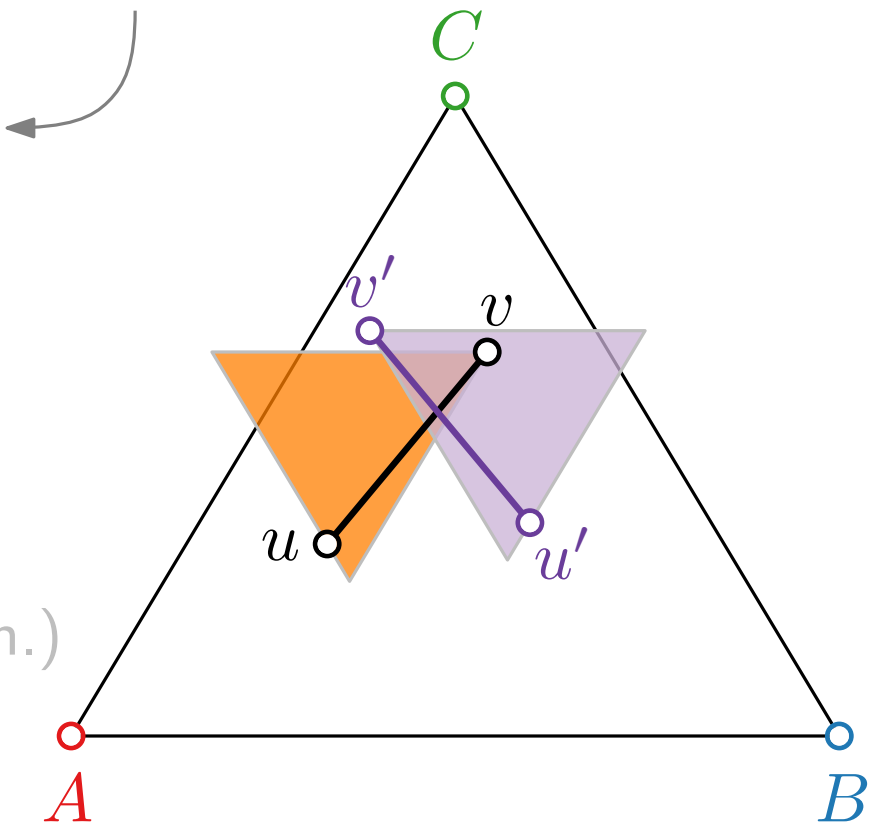
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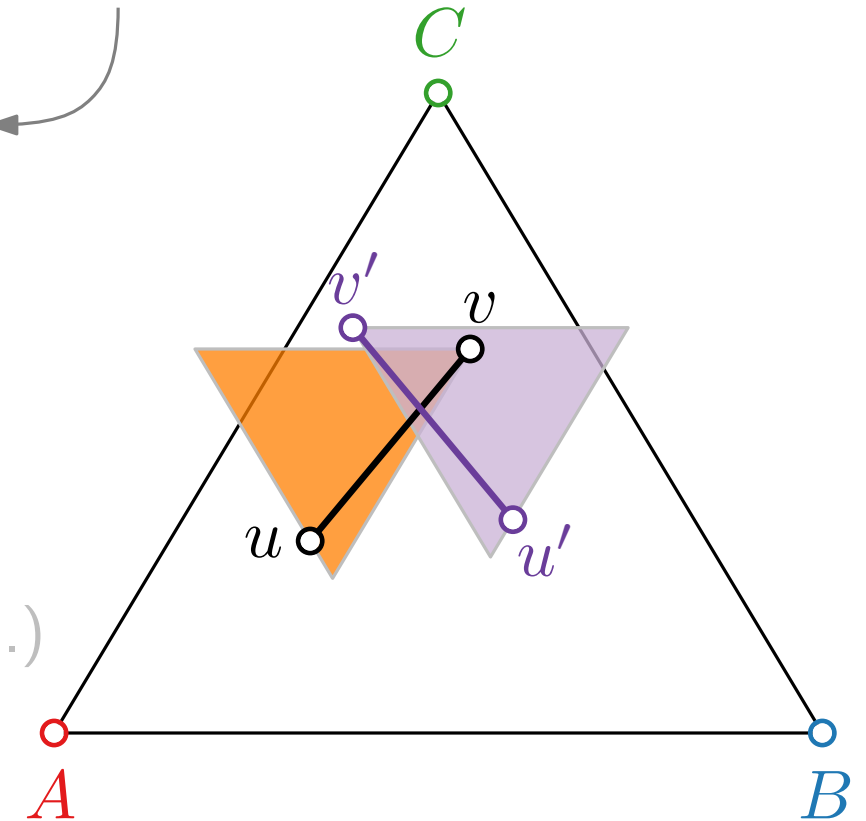
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$$u'_i > u_i, v_i, \quad v'_j > u_j, v_j, \quad u_k > u'_k, v'_k, \quad v_l > u'_l, v'_l$$

no three points
on a line



Barycentric Representations of Planar Graphs

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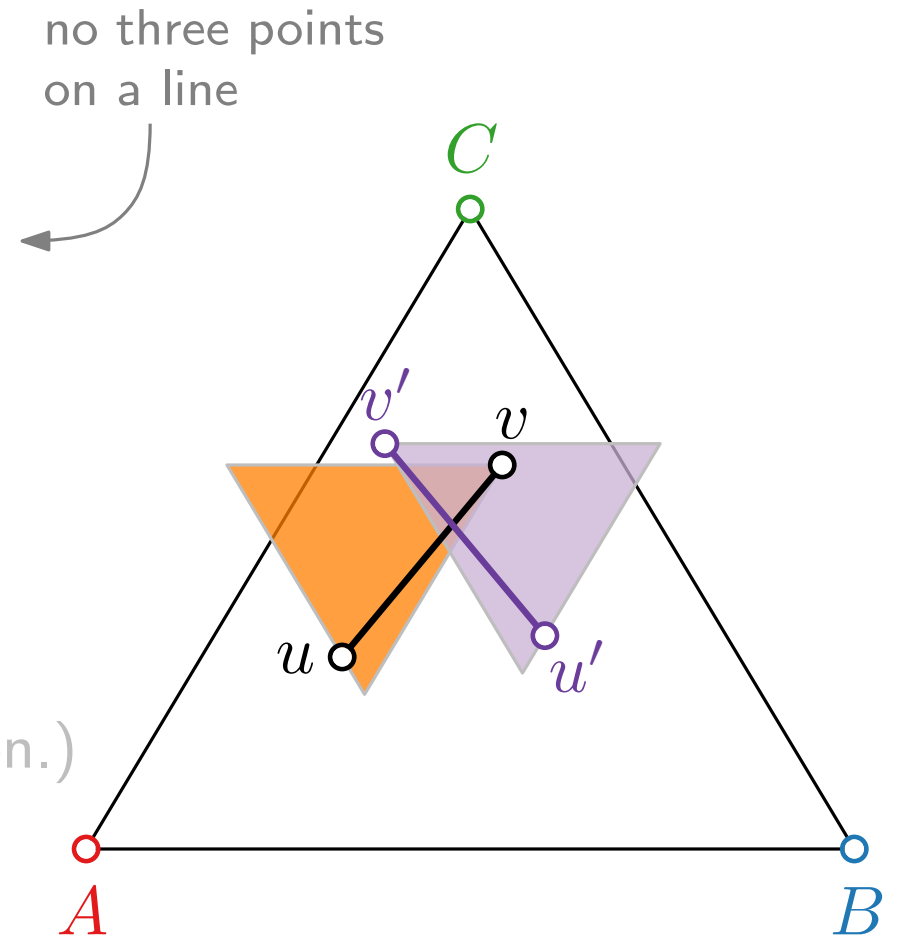
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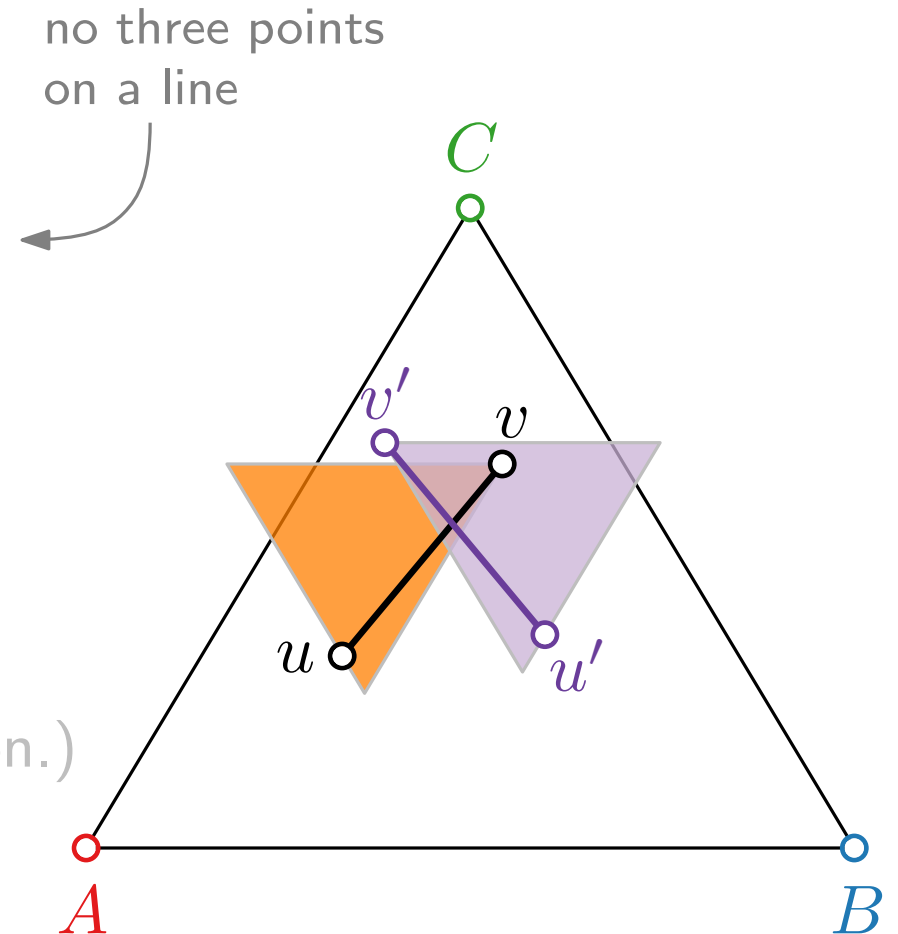
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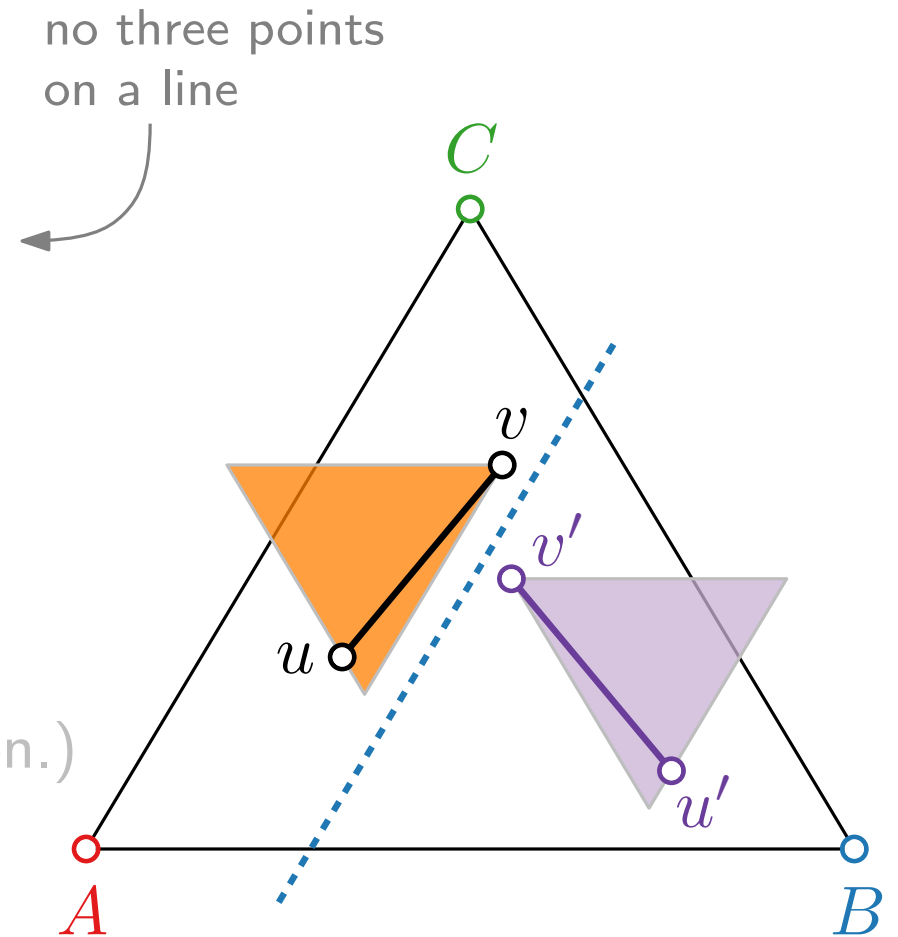
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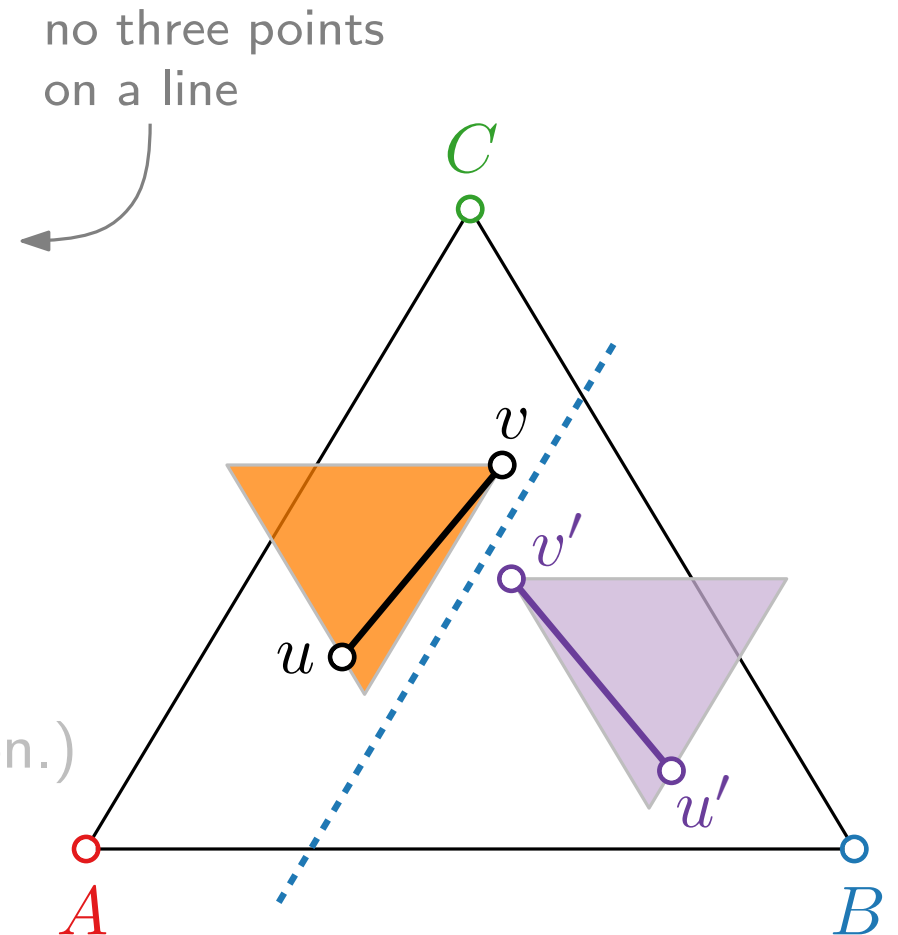
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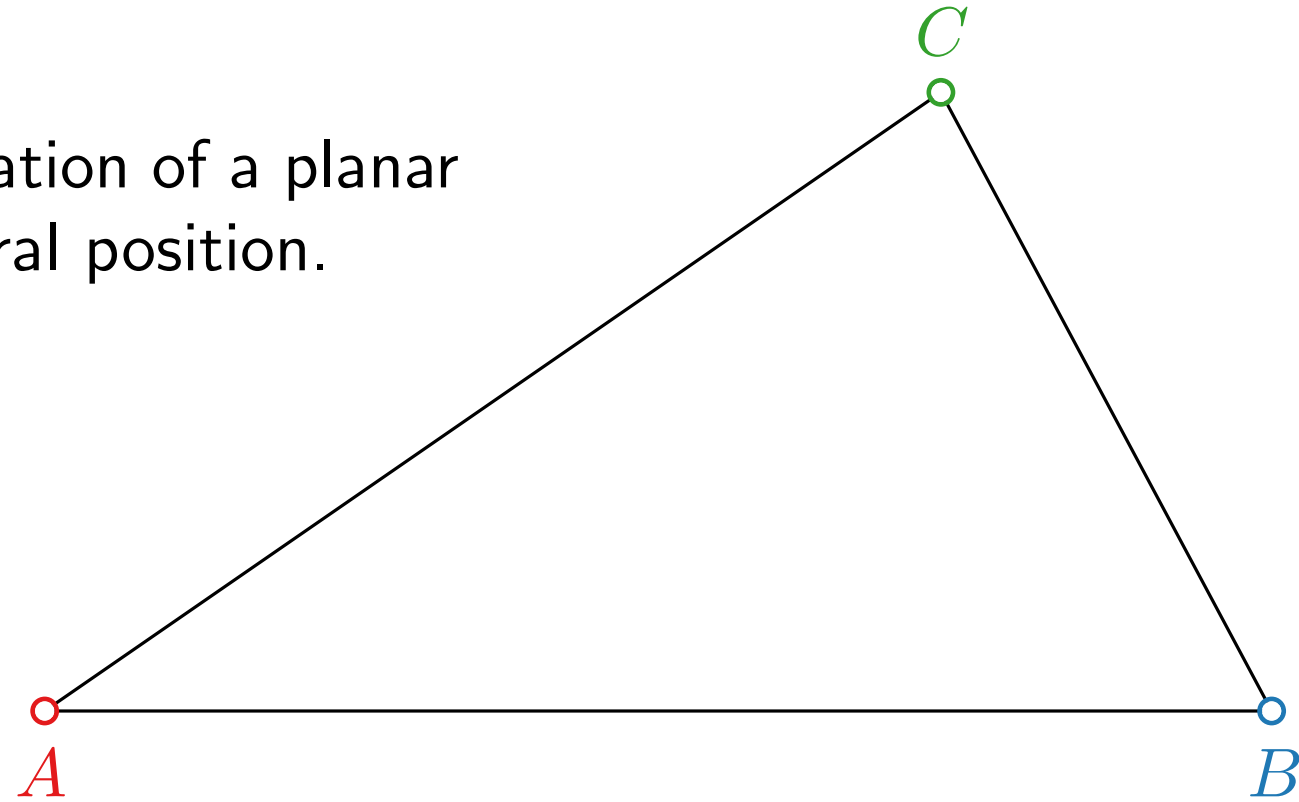
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How to find a barycentric representation?

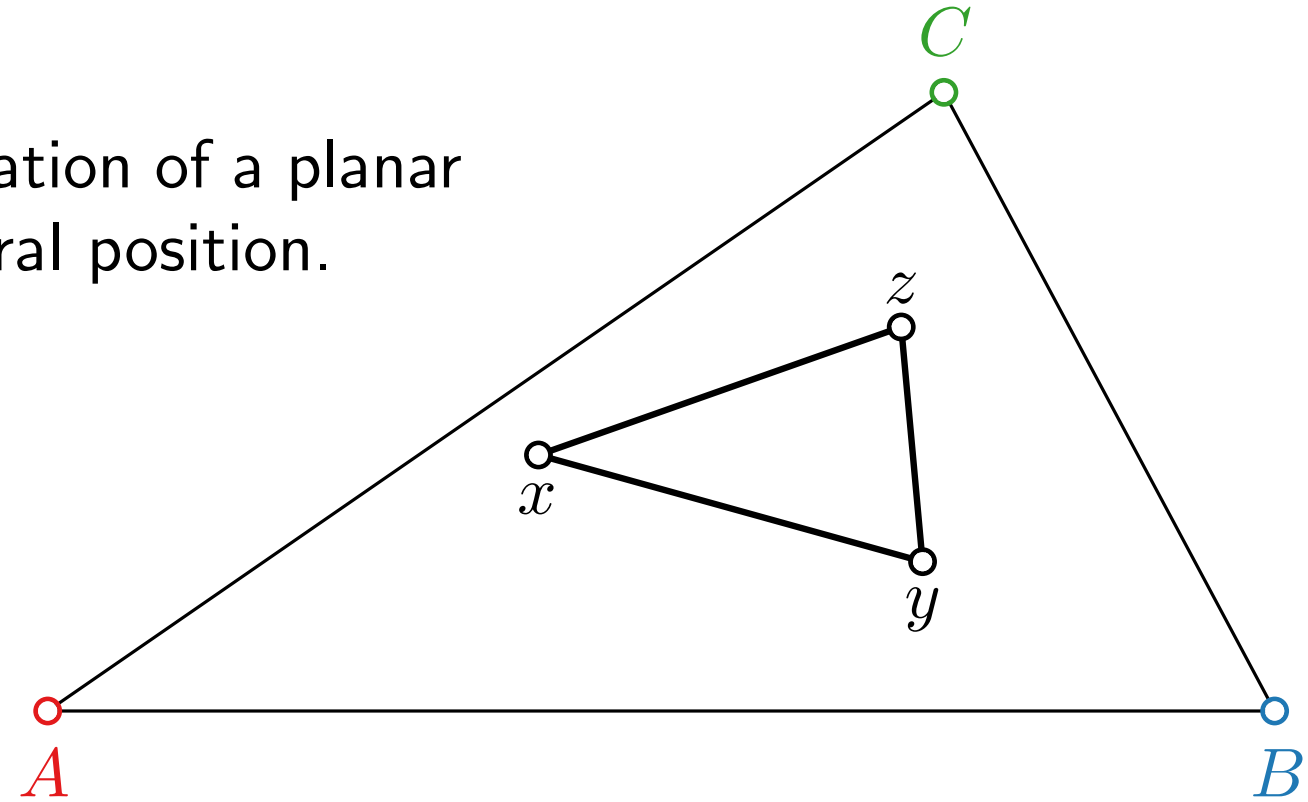
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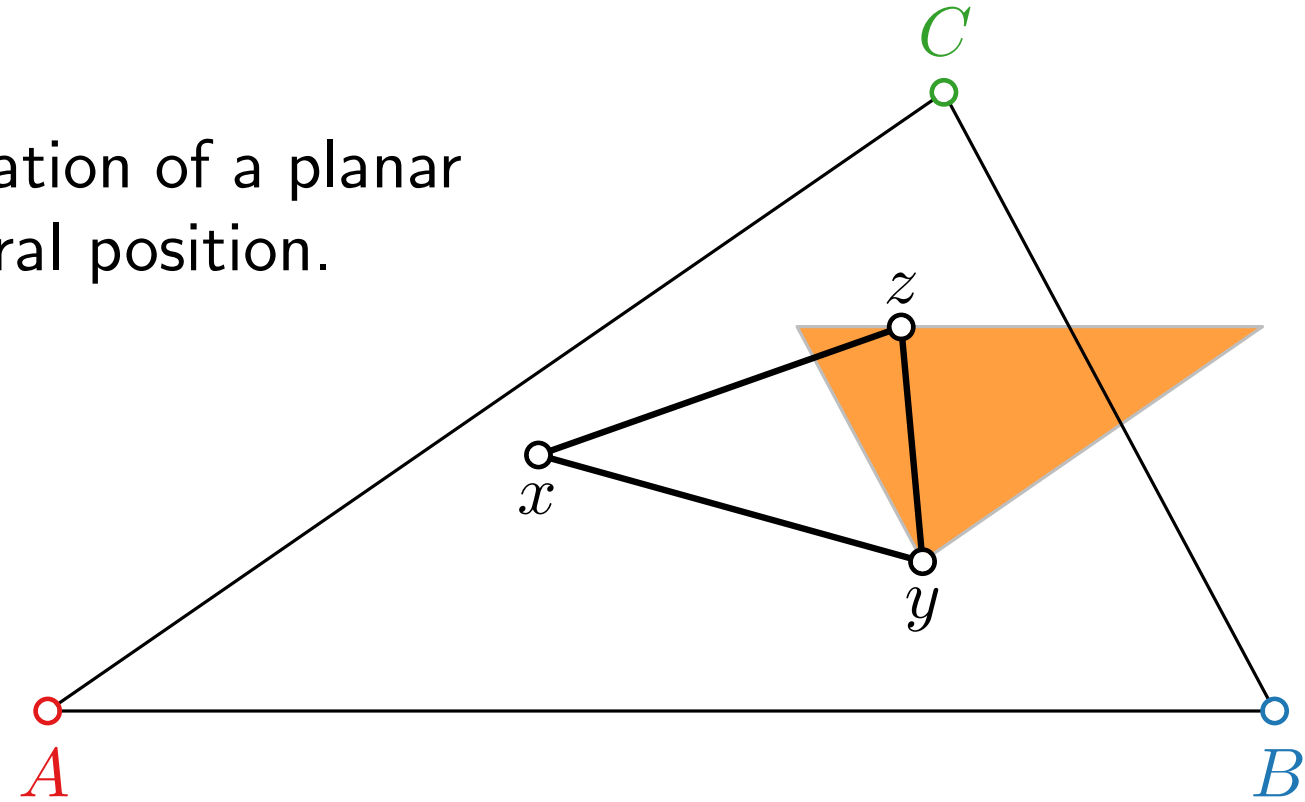
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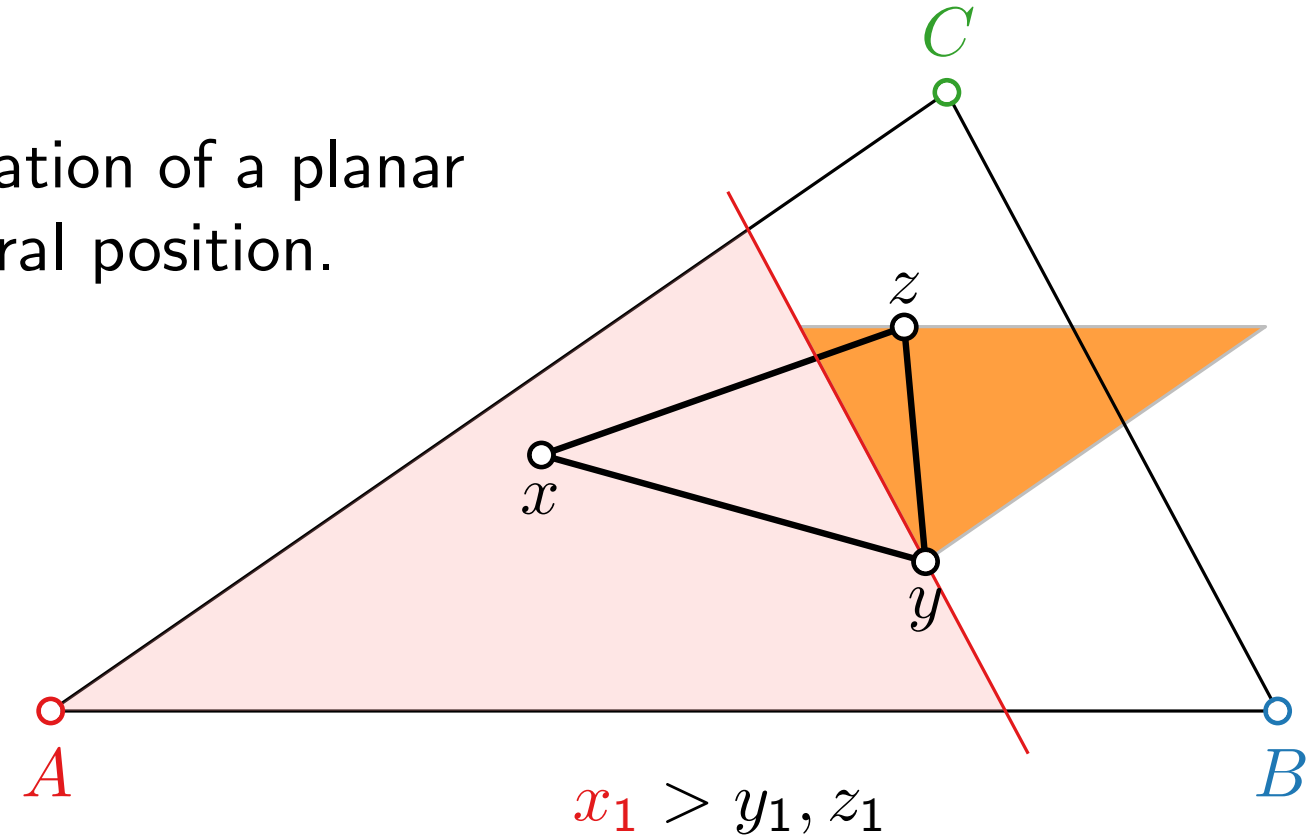
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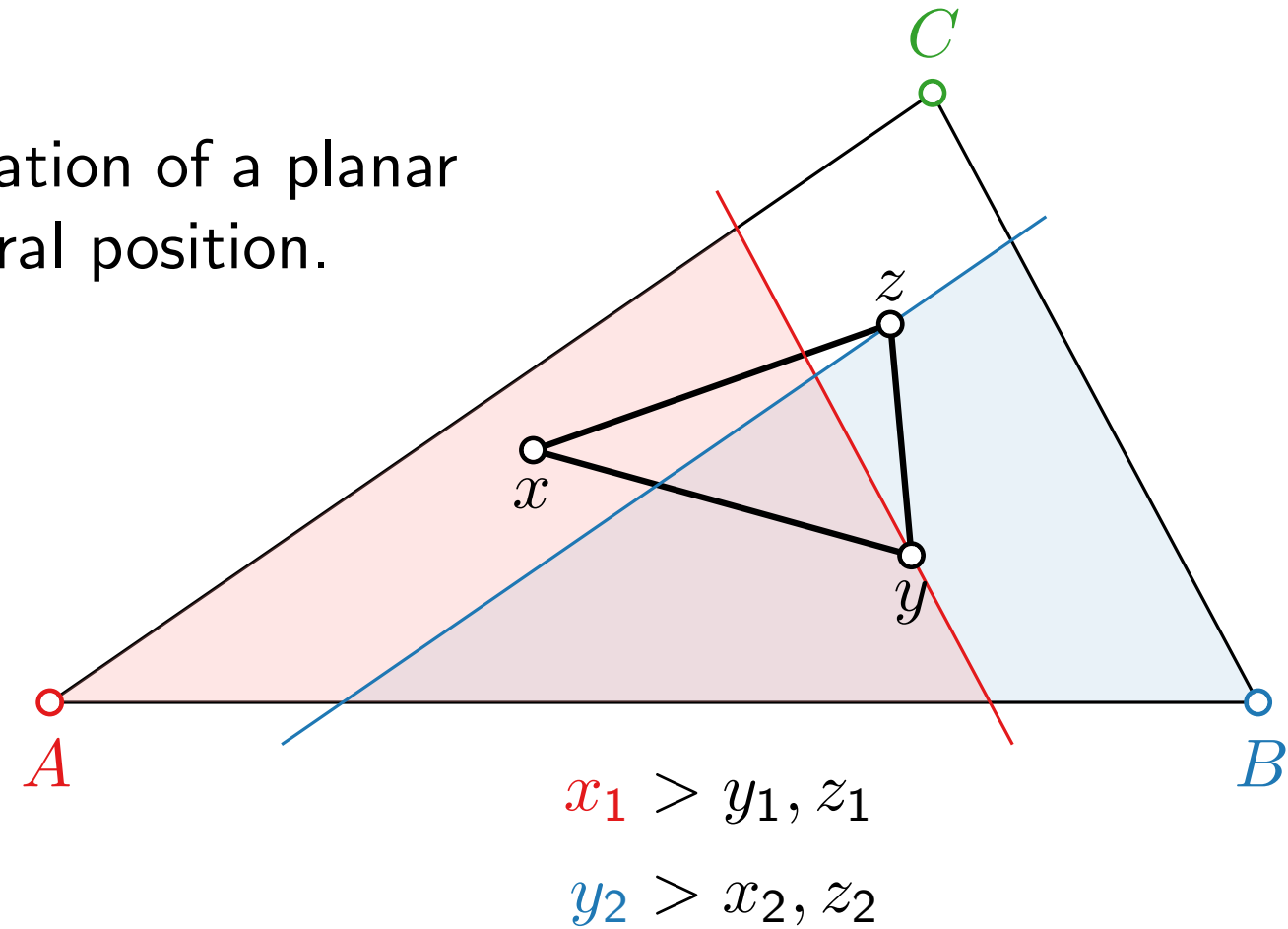
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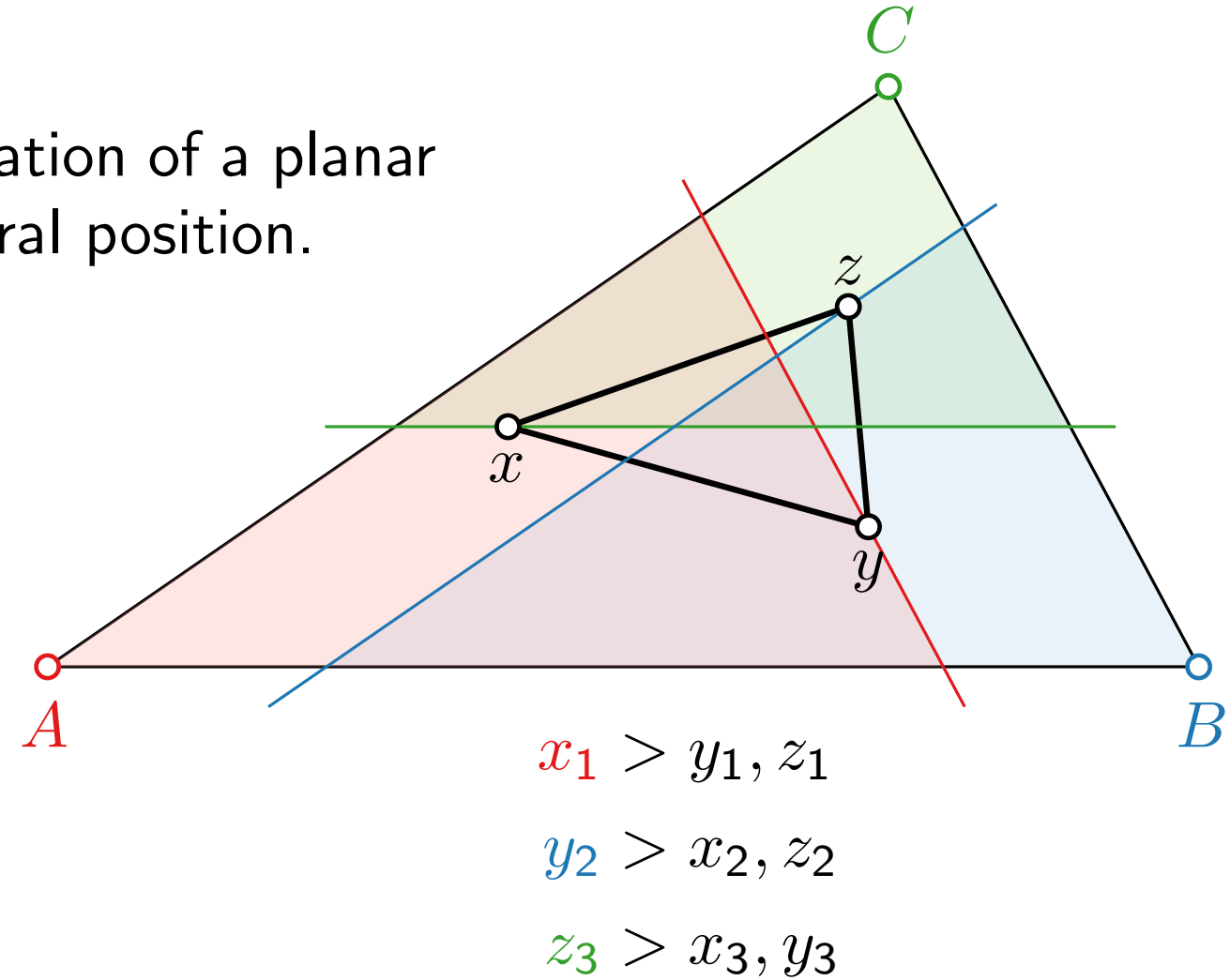
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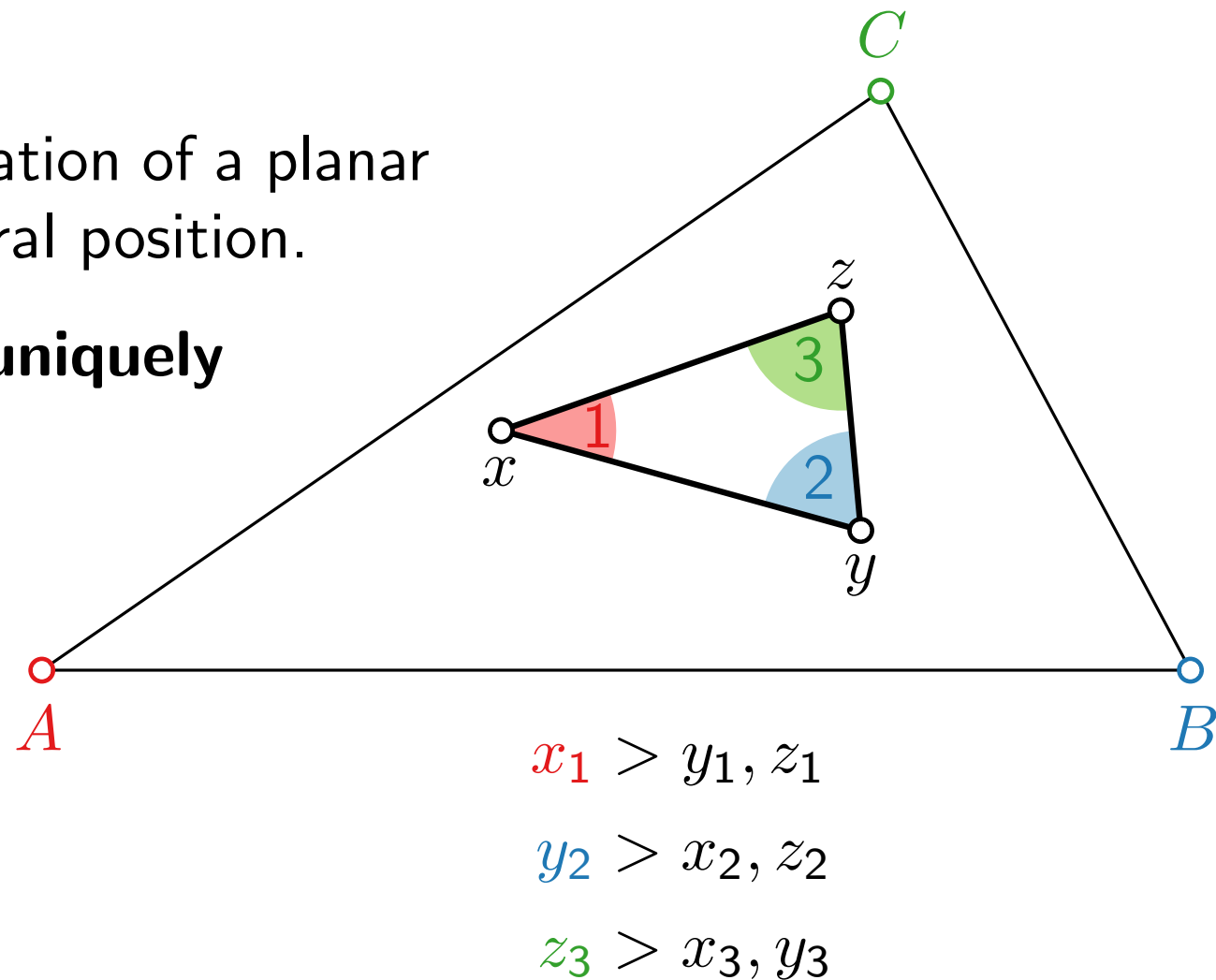
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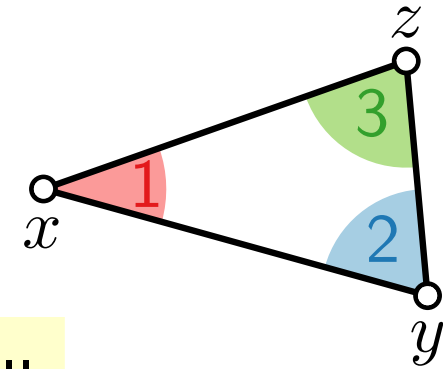
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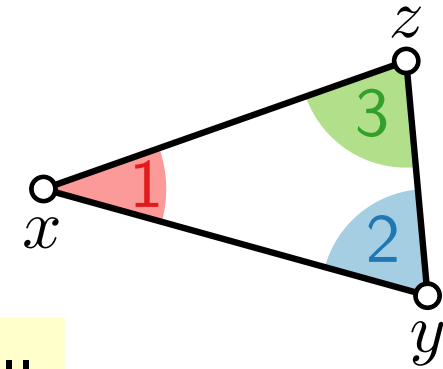


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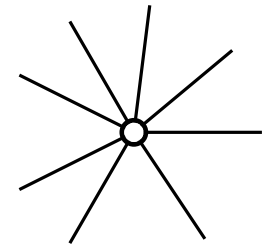
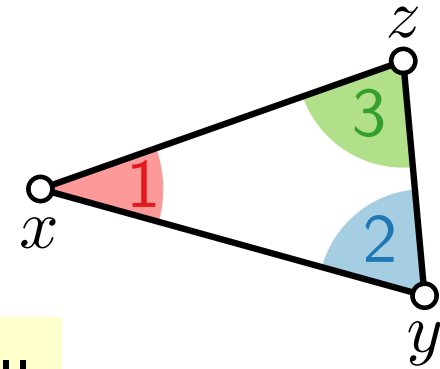
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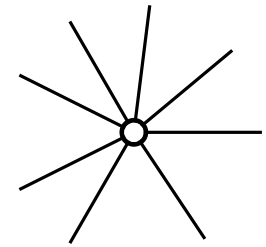
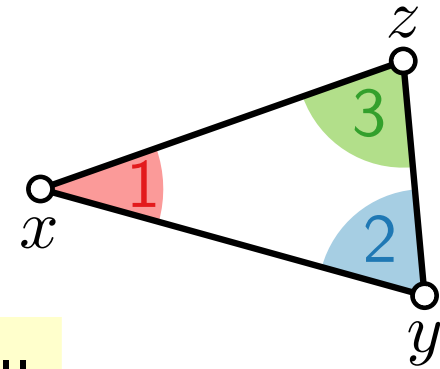
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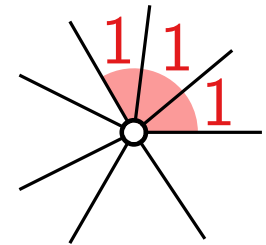
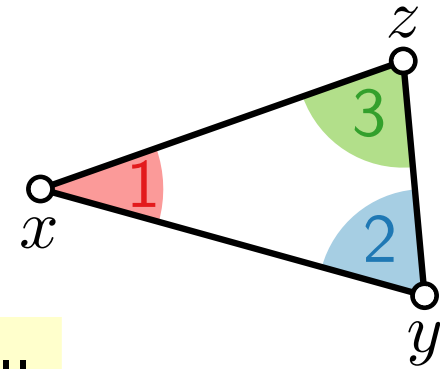
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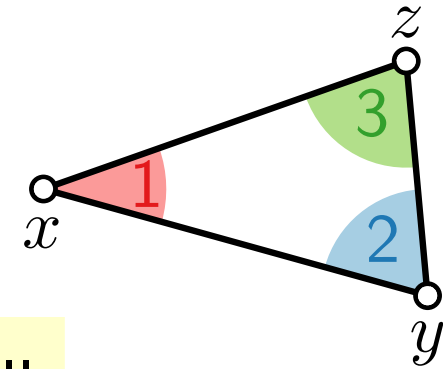
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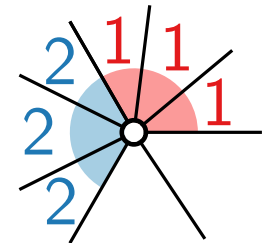


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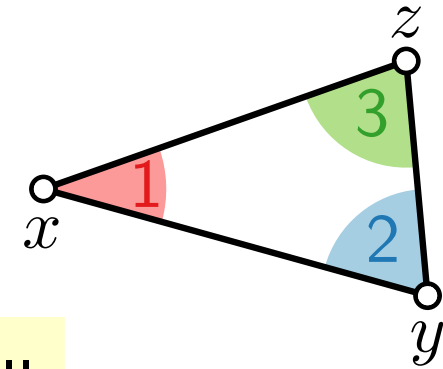
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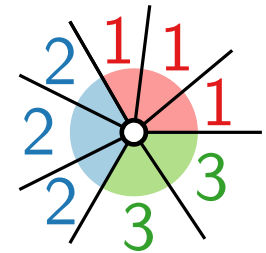


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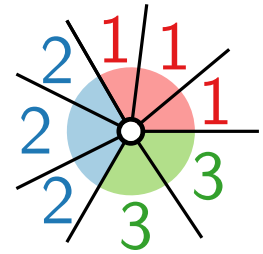
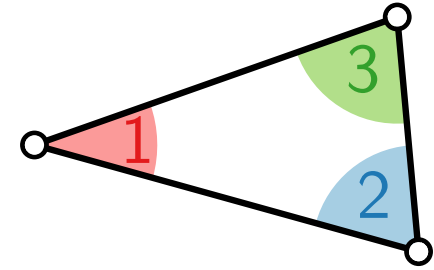
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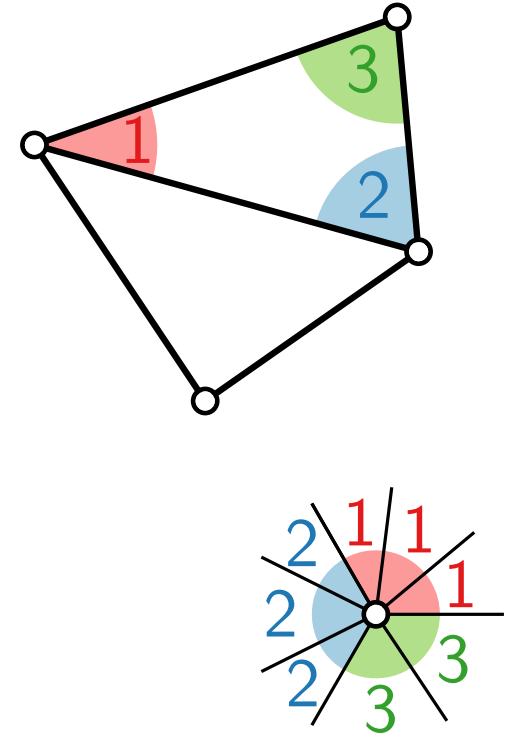
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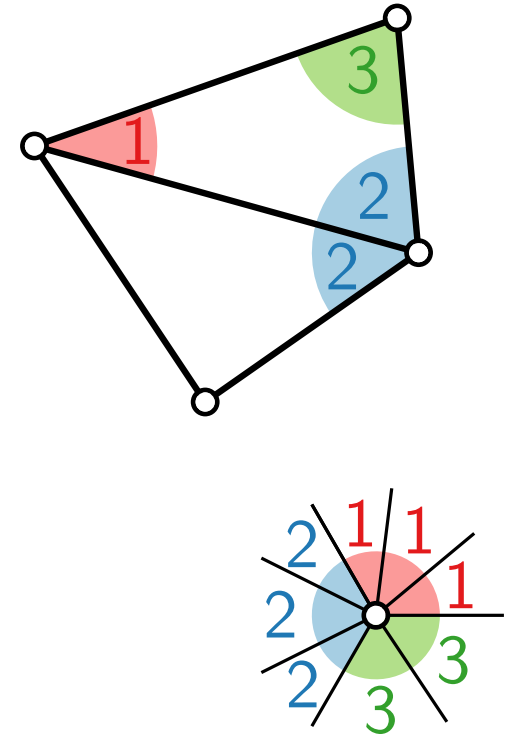
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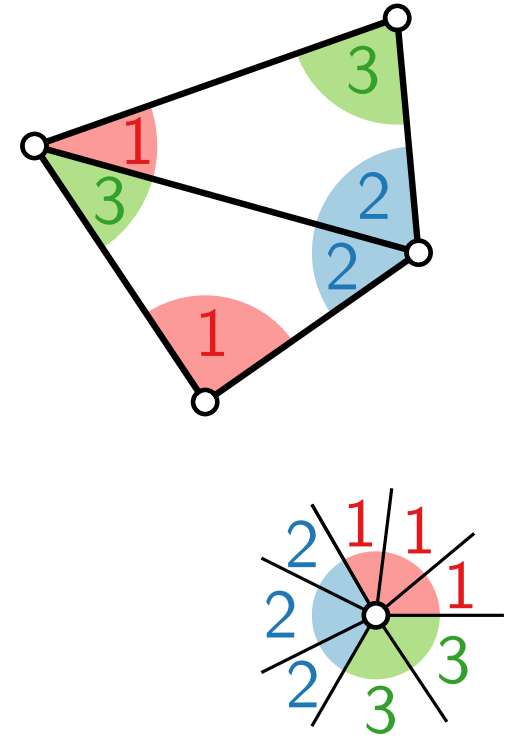
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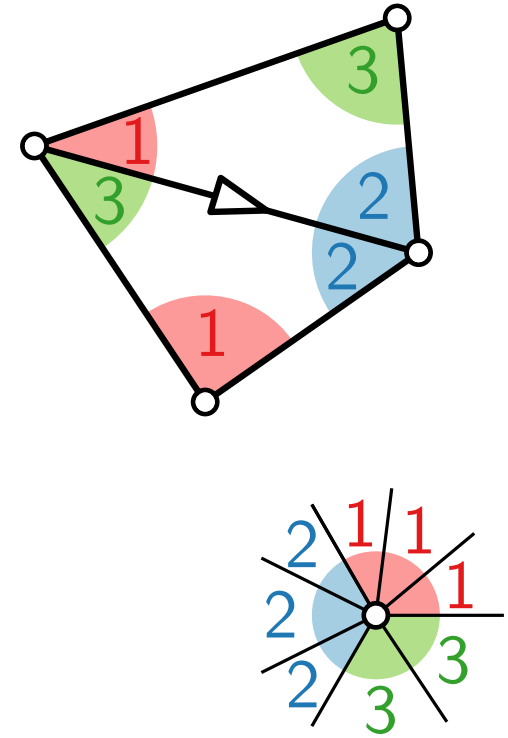
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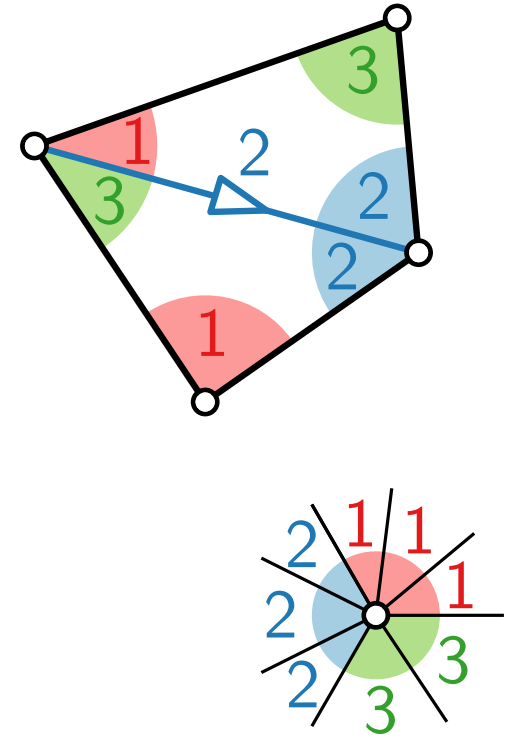
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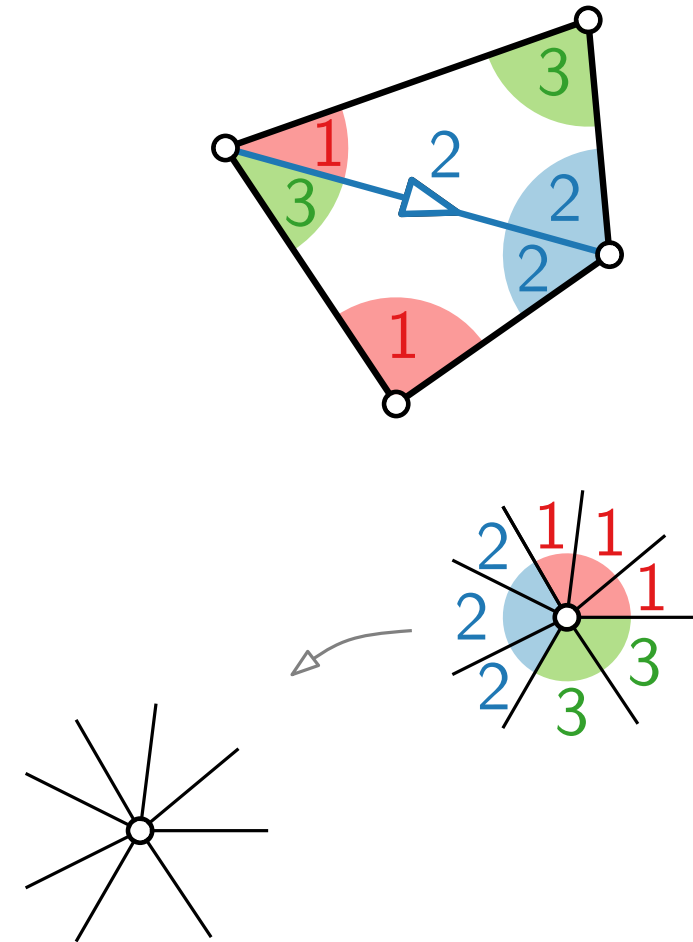
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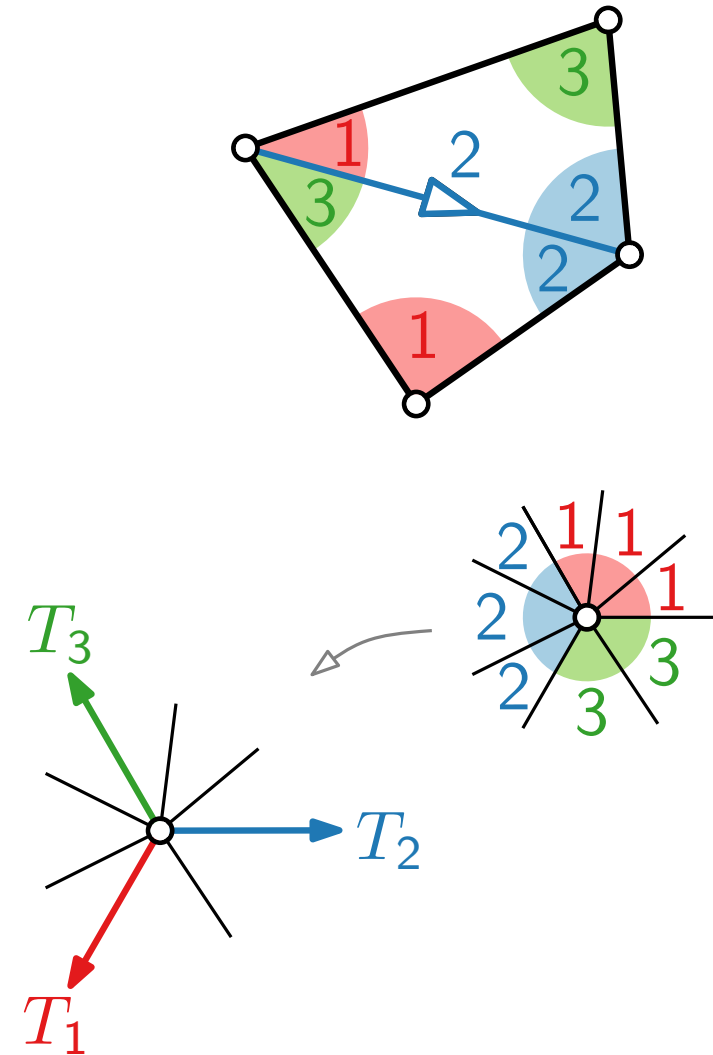


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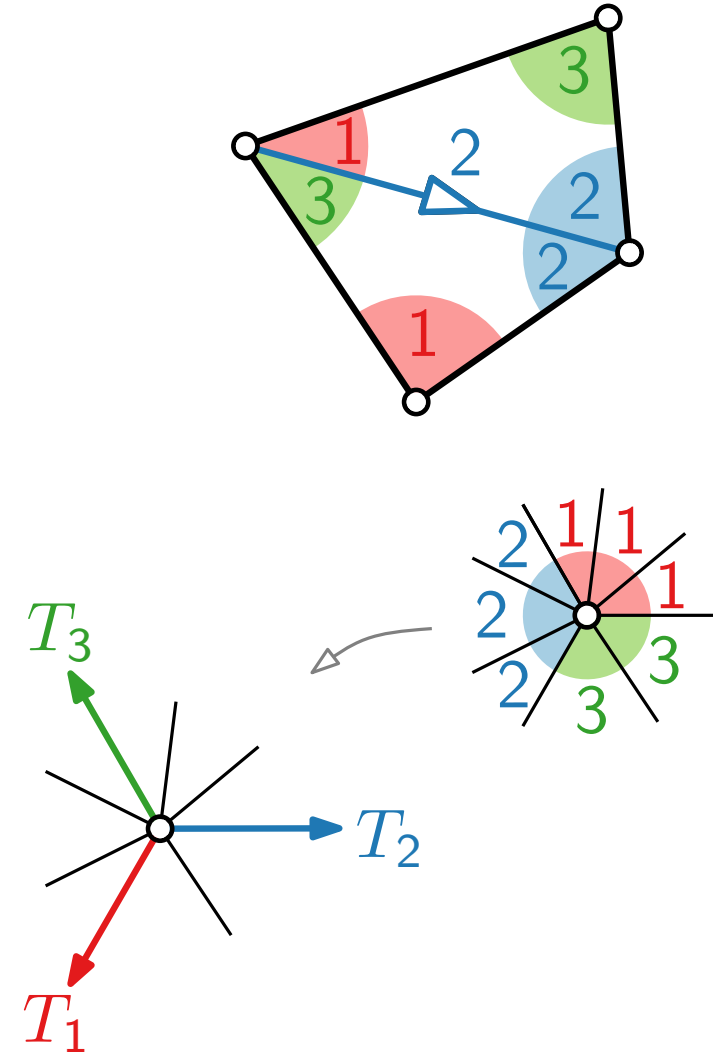


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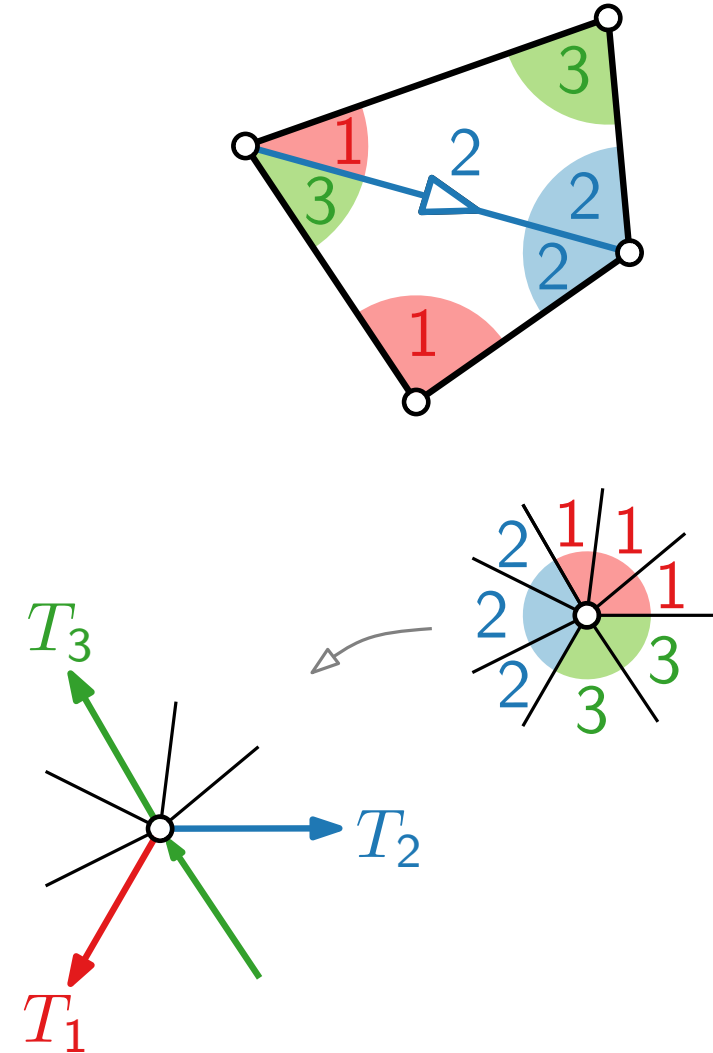


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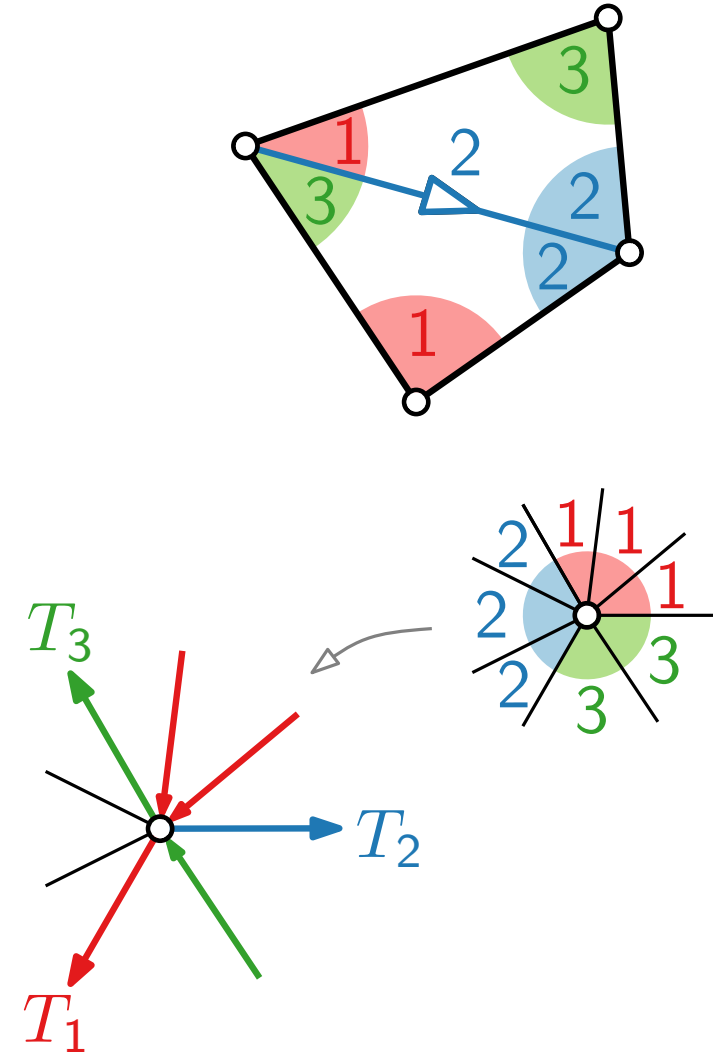


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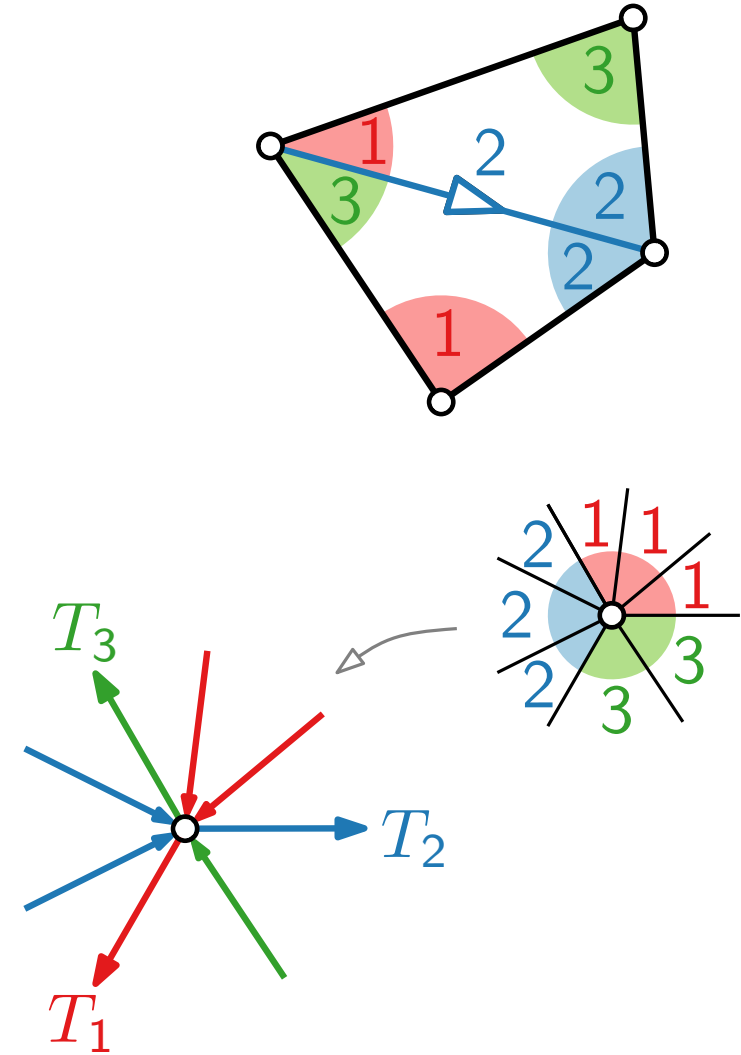


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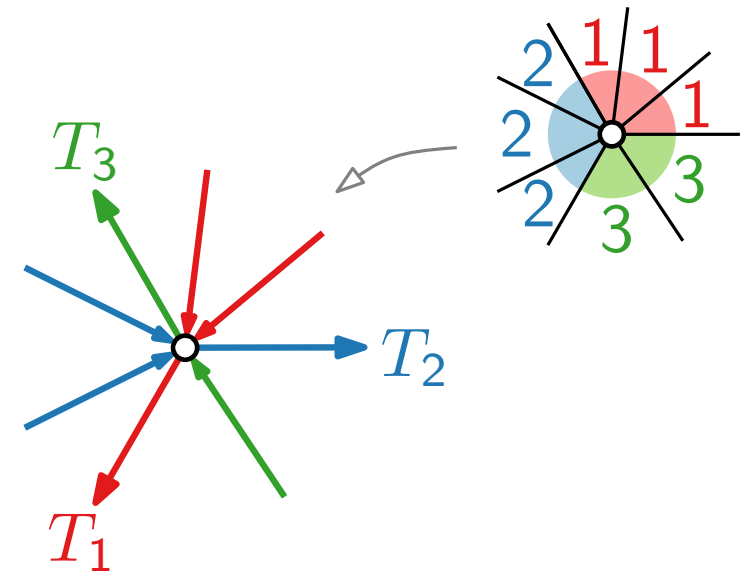
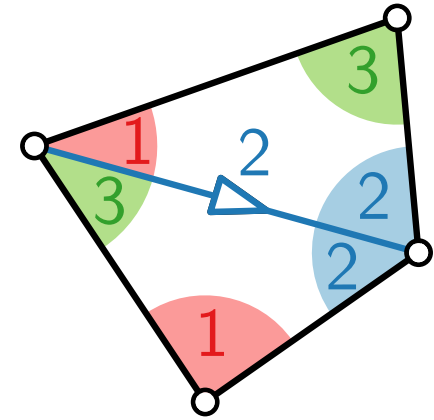
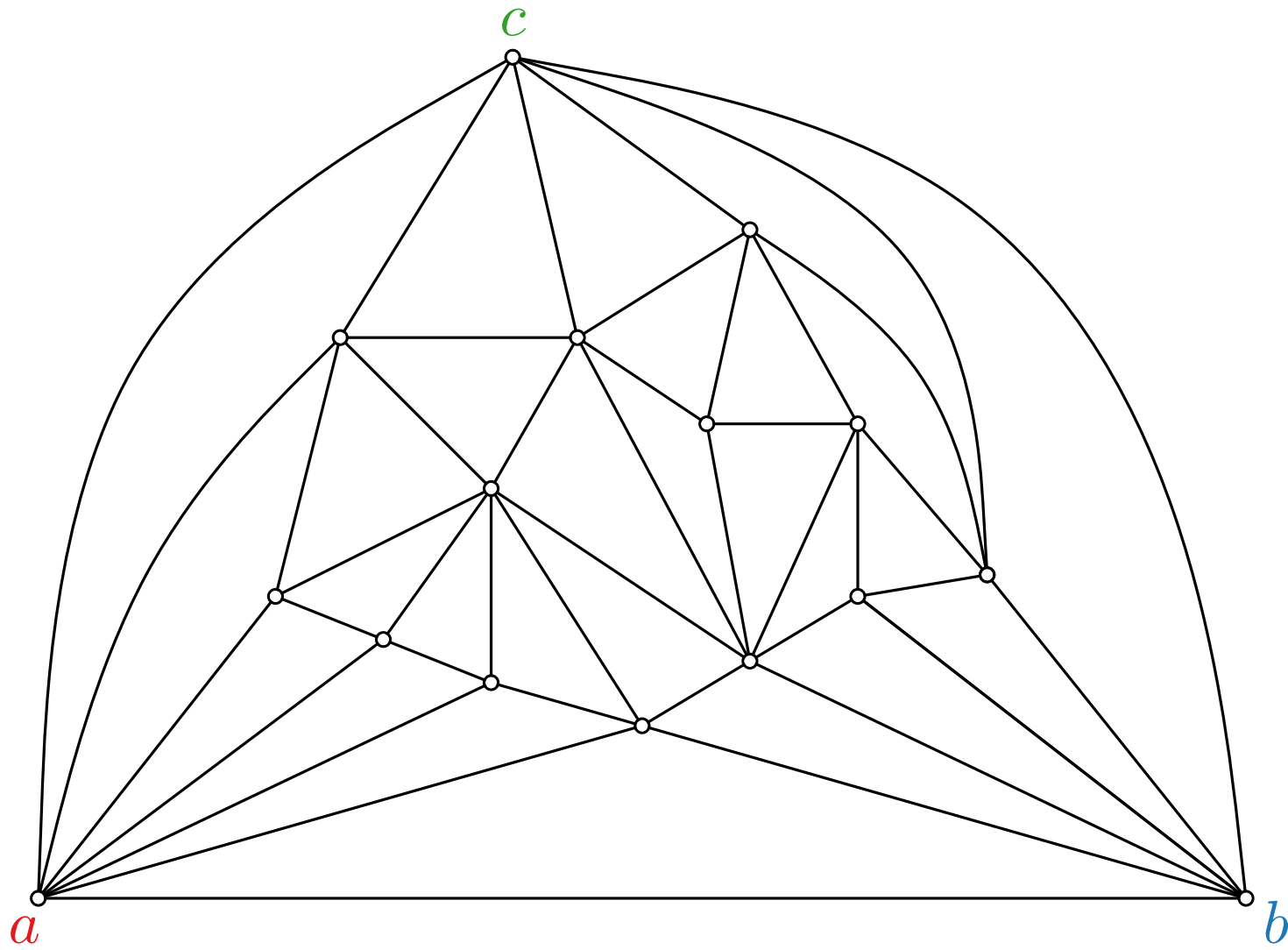
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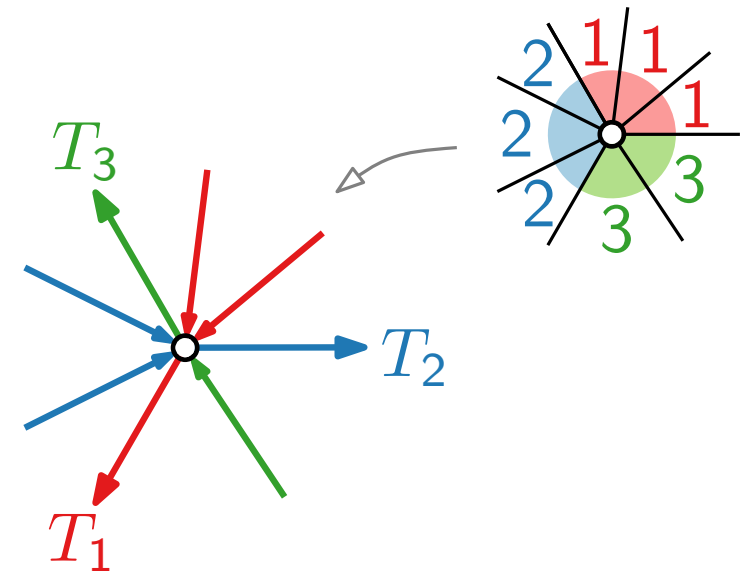
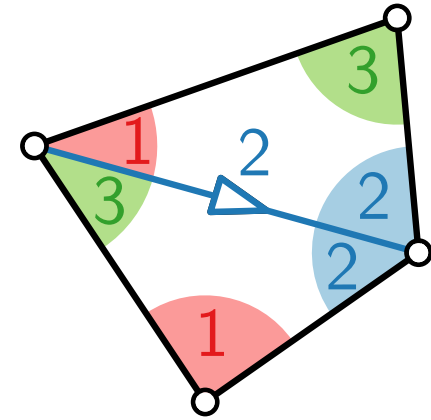
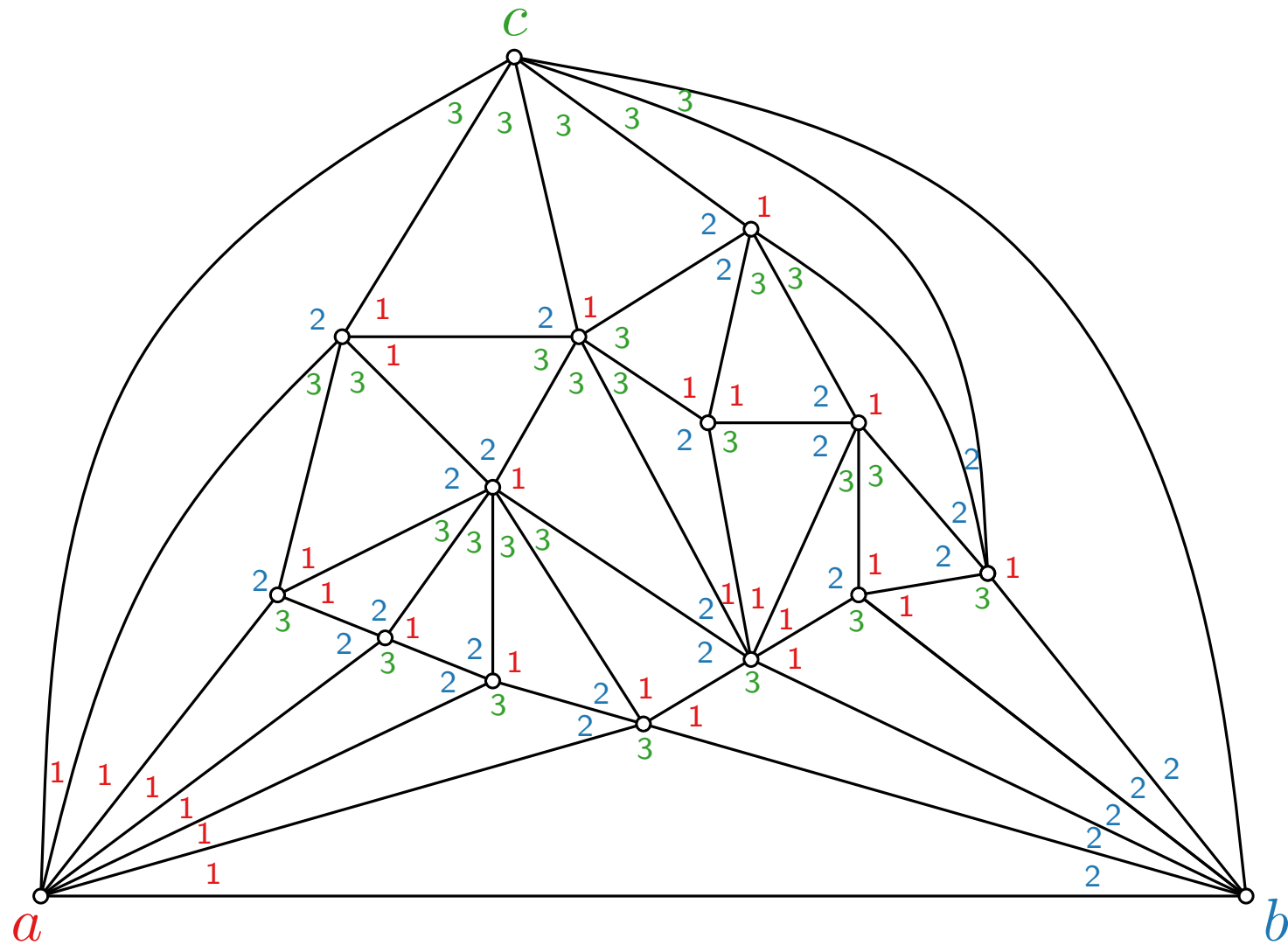
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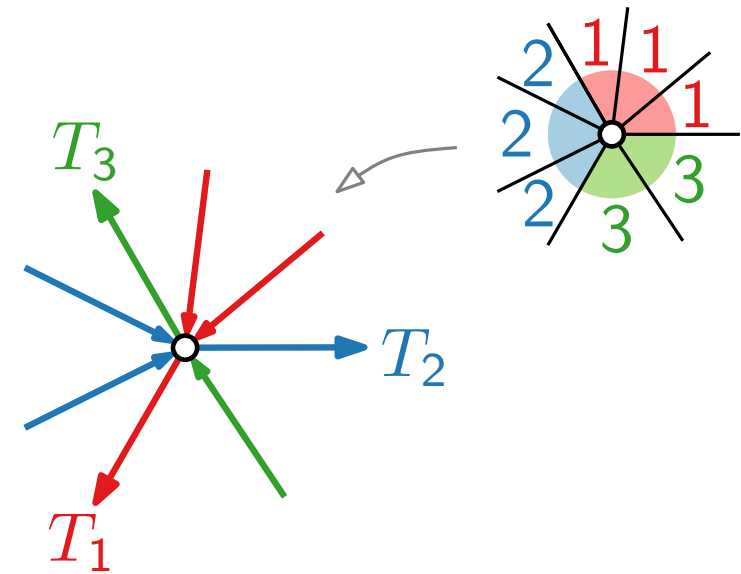
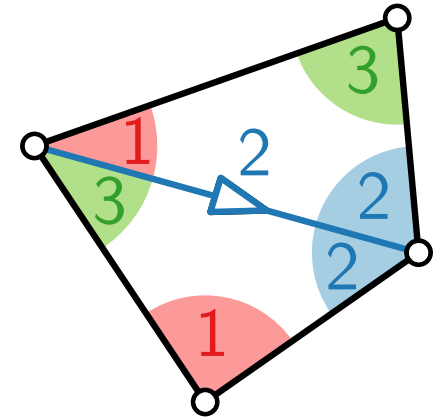
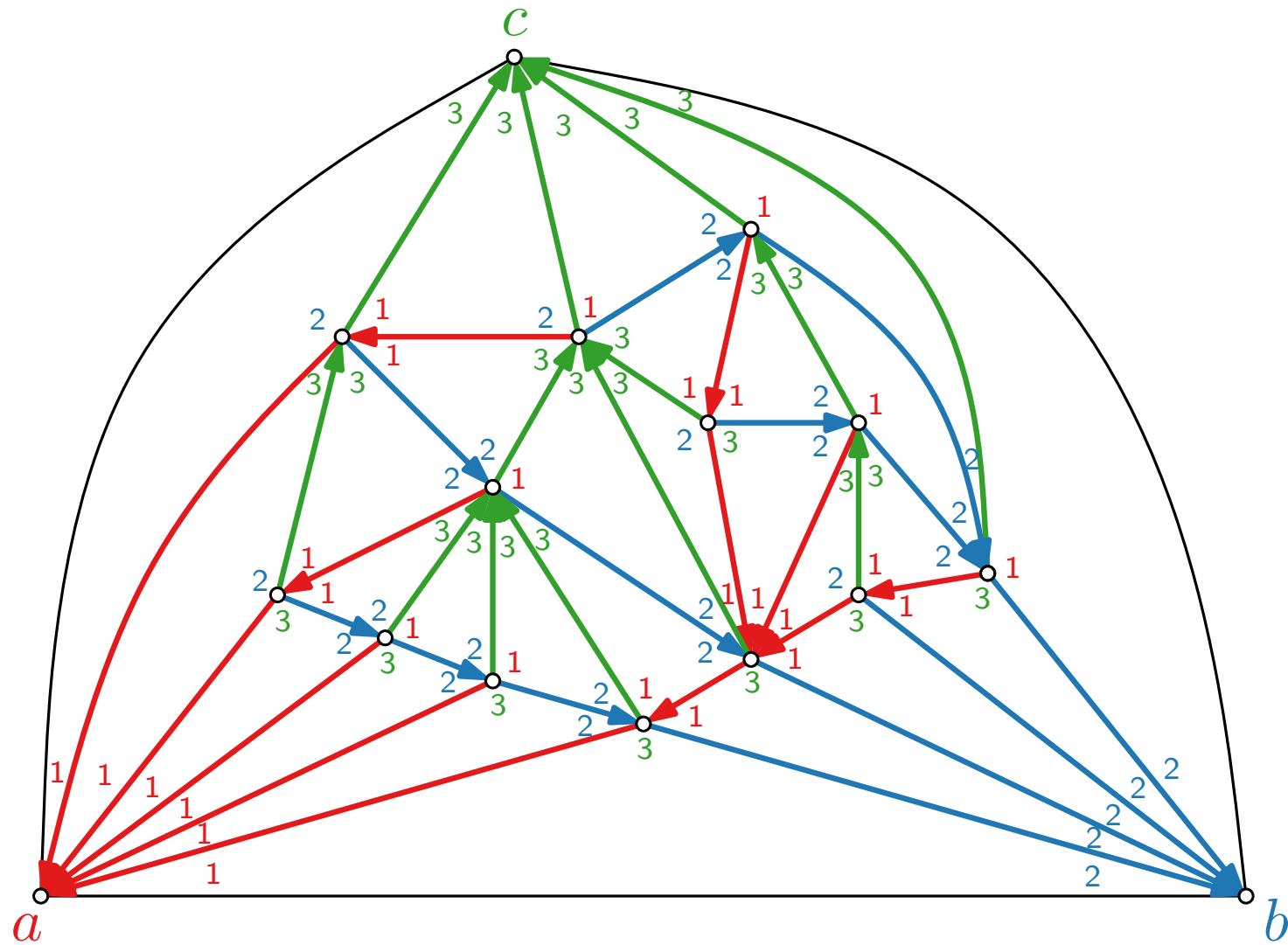
Schnyder Wood – Example and Properties



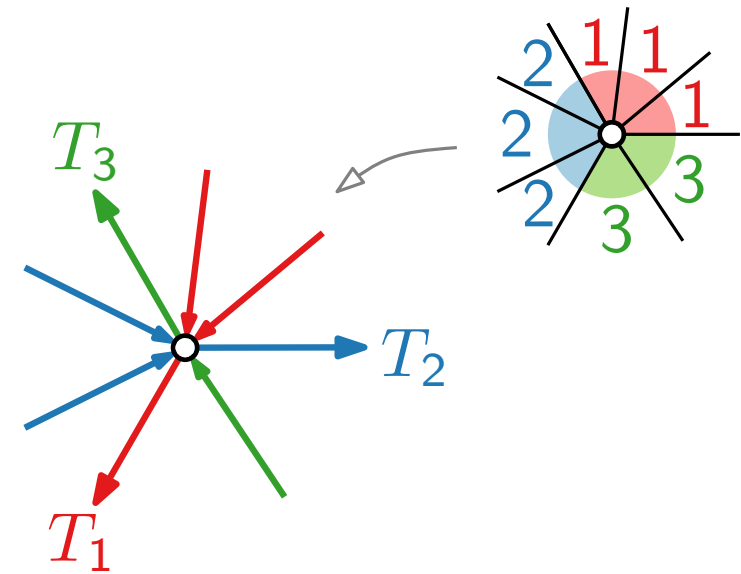
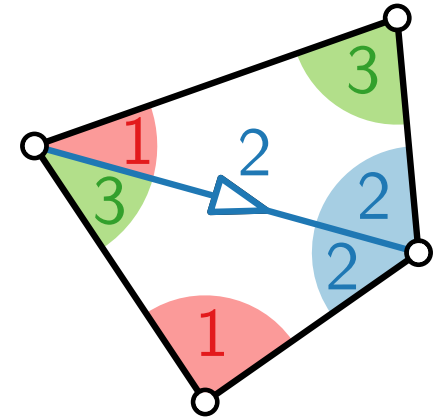
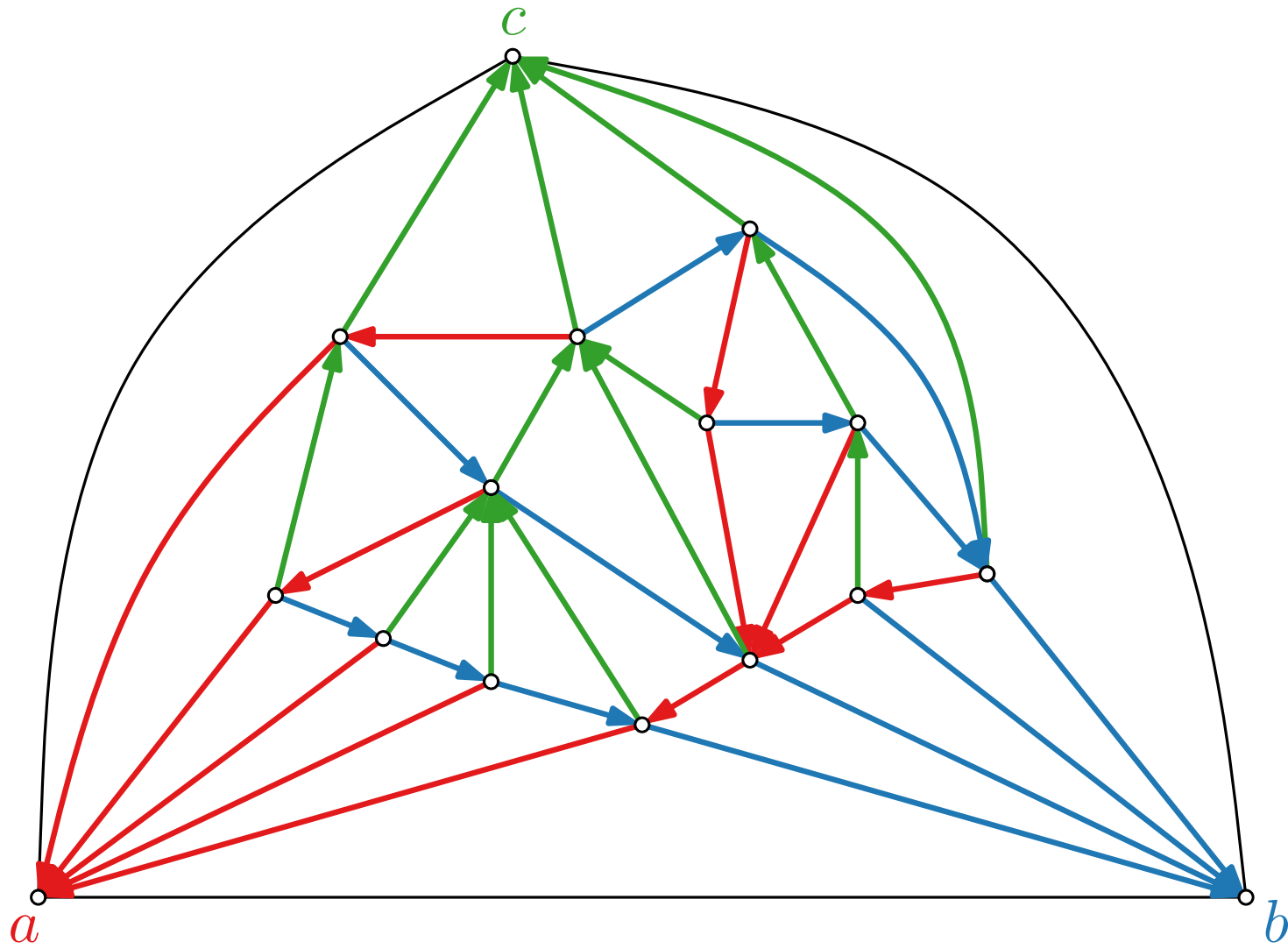
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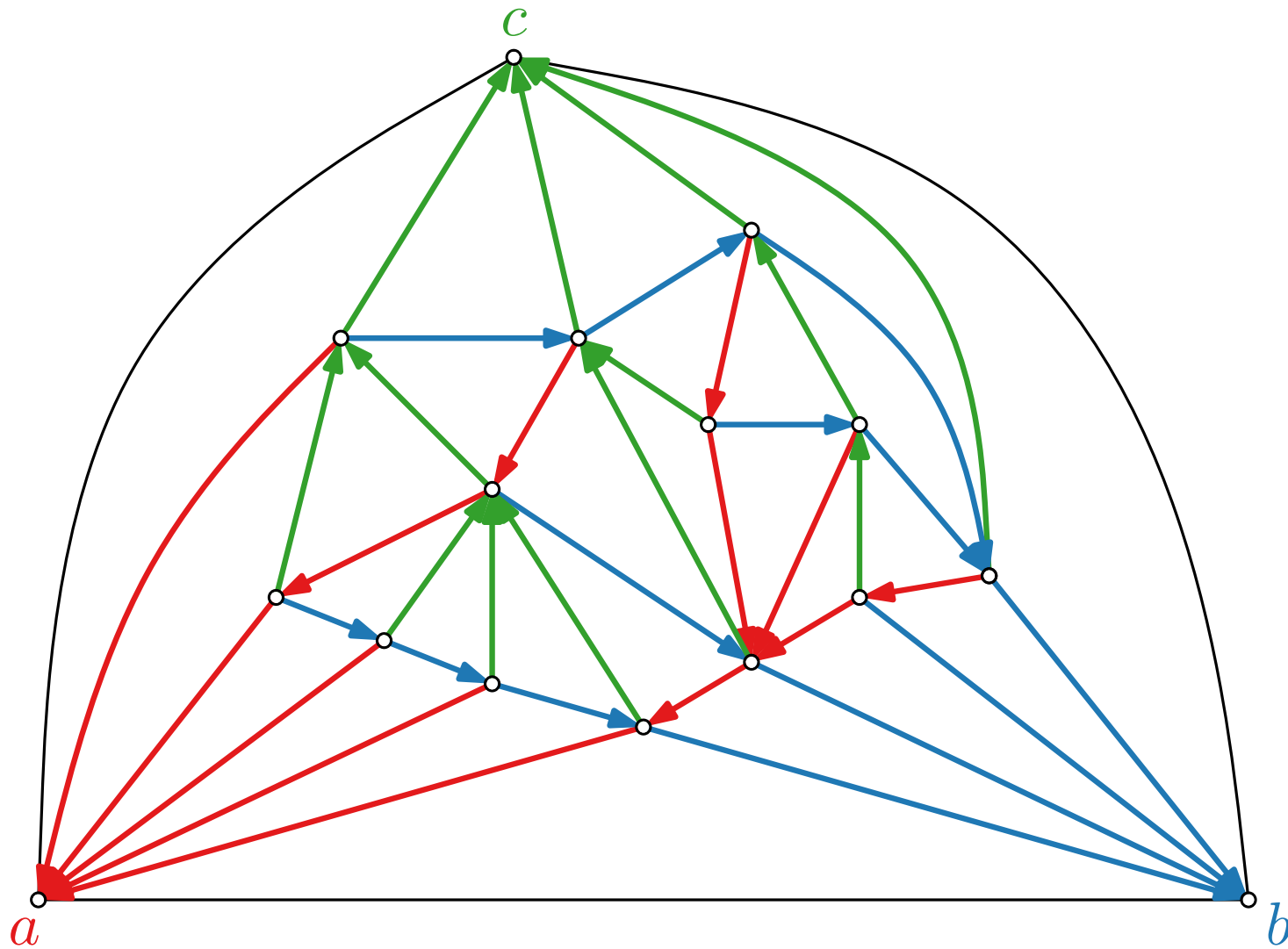
Schnyder Wood – Example and Properties



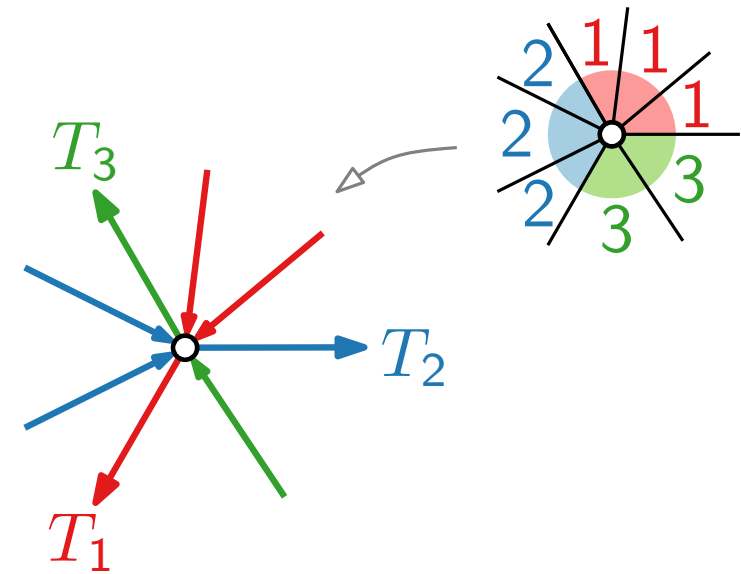
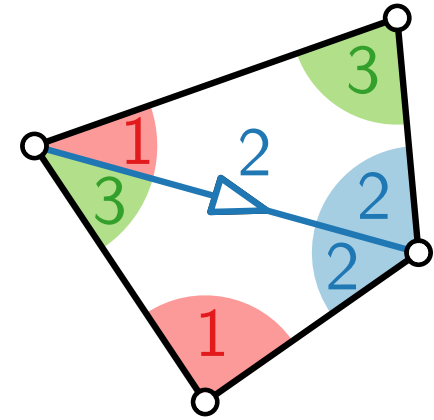
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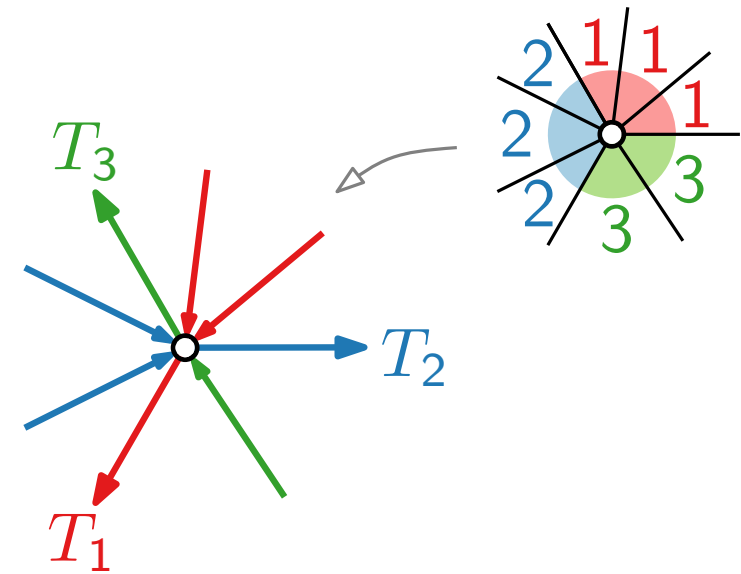
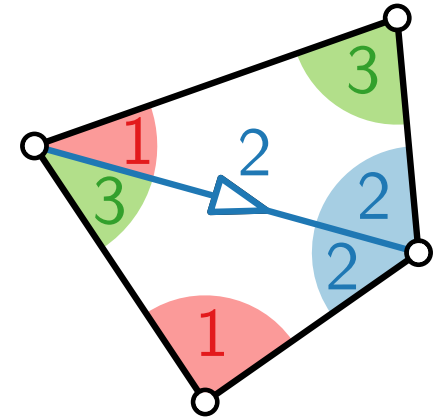
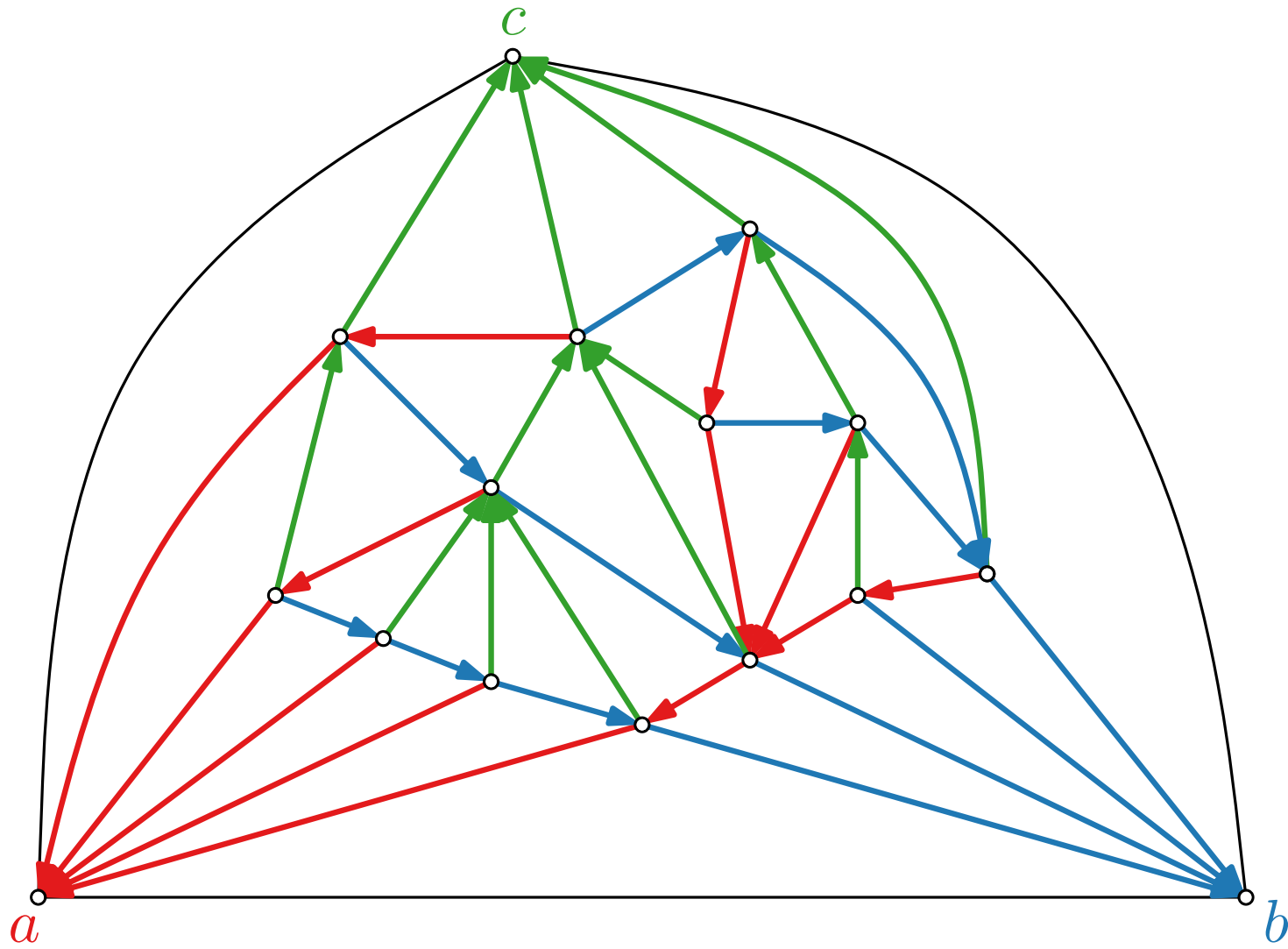
Schnyder Wood – Example and Properties



(a Schnyder labeling is not unique)

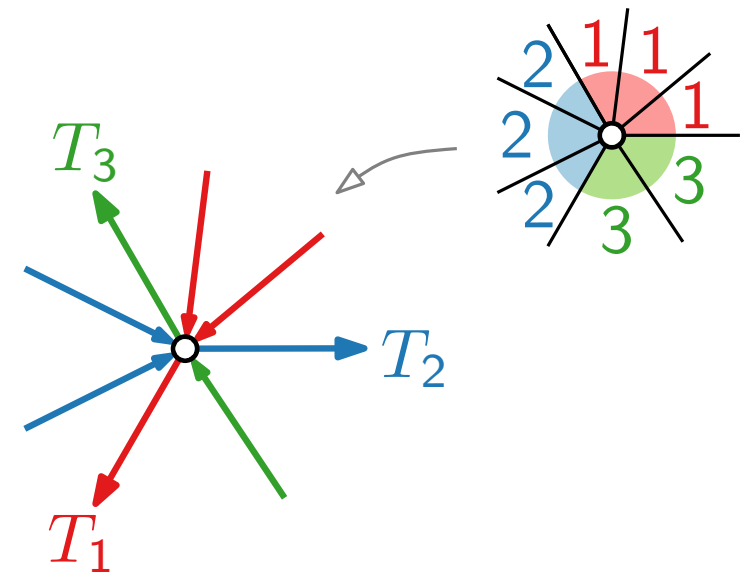
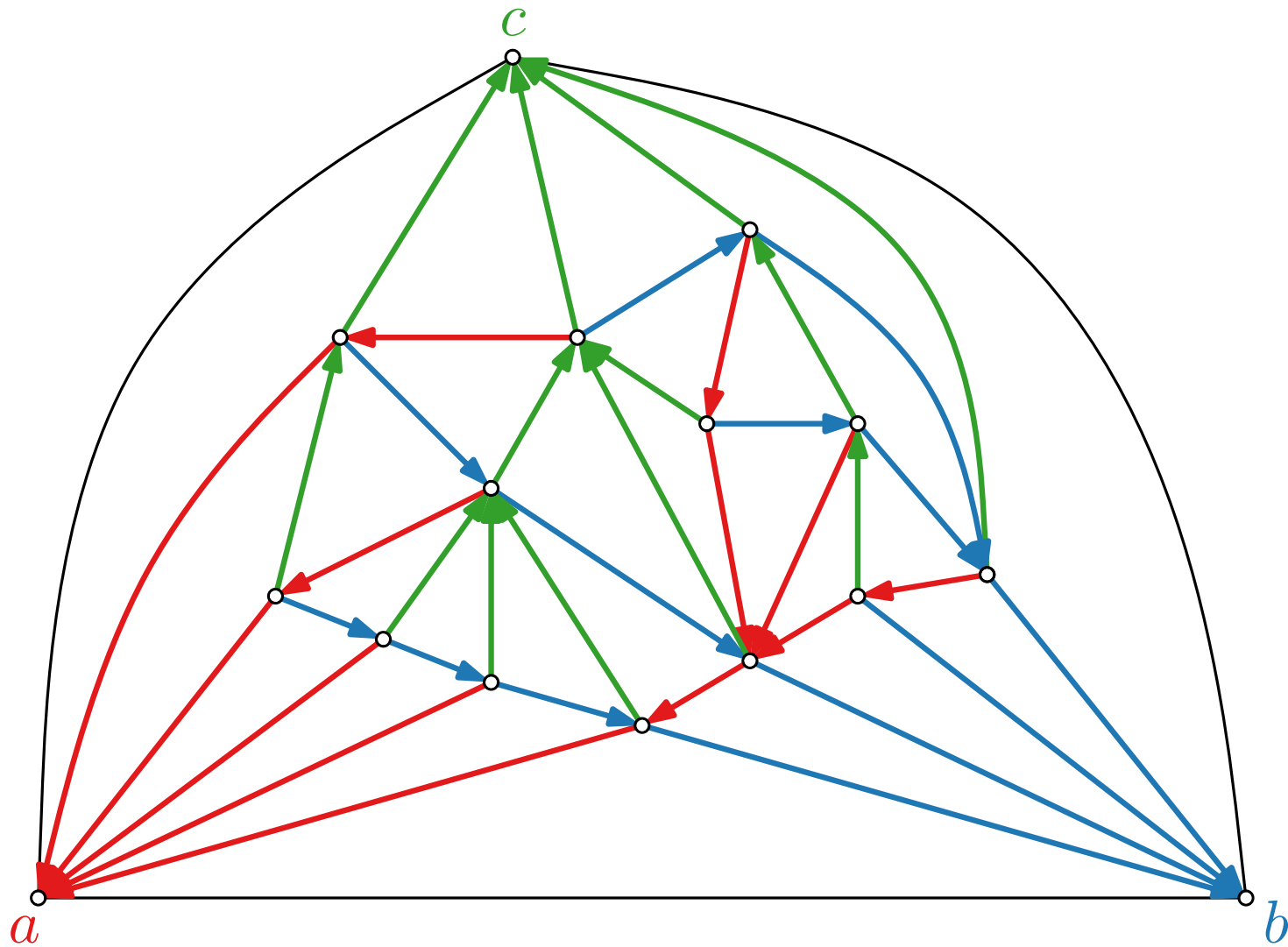


Schnyder Wood – Example and Properties

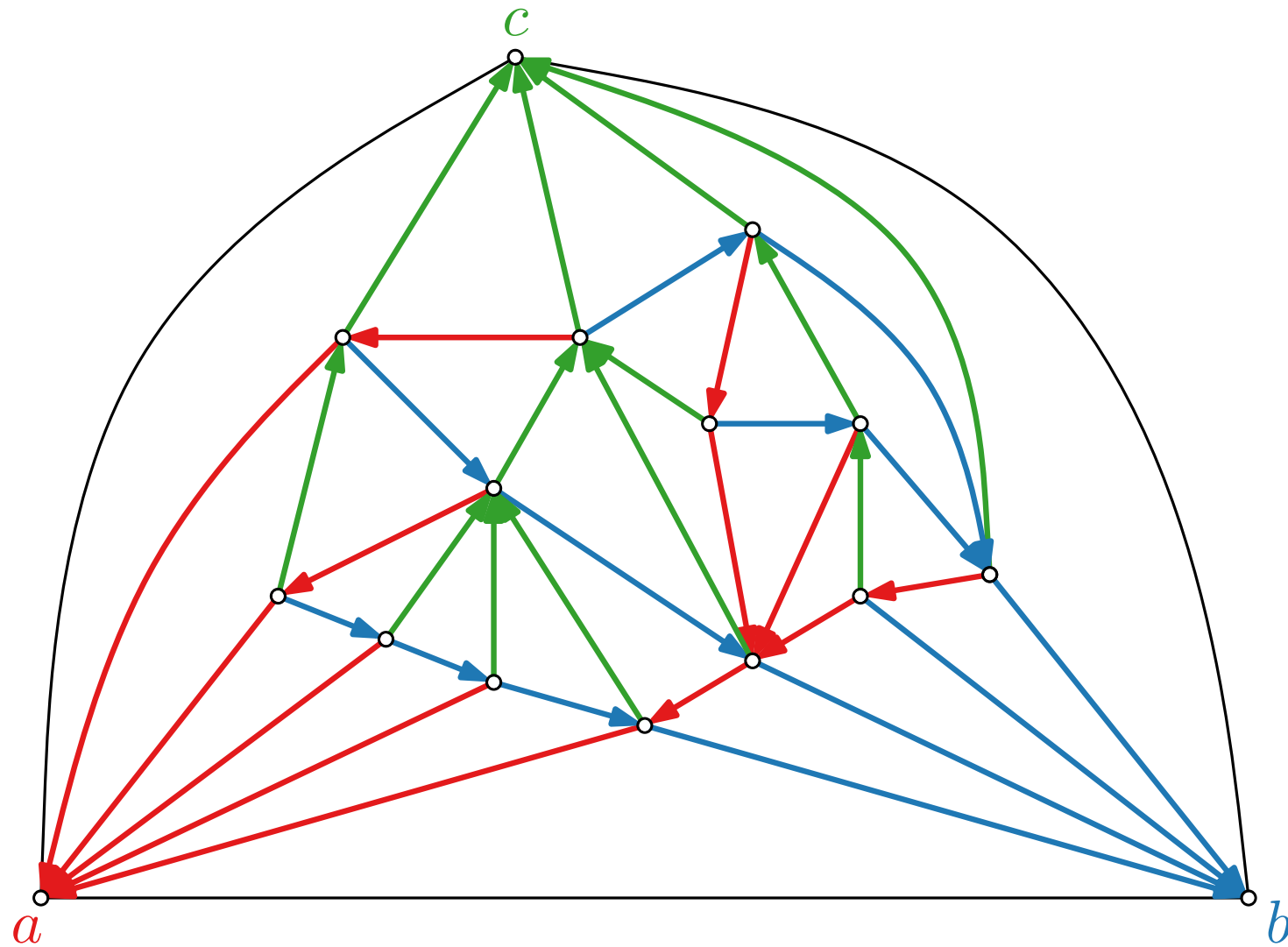


Schnyder Wood – Example and Properties

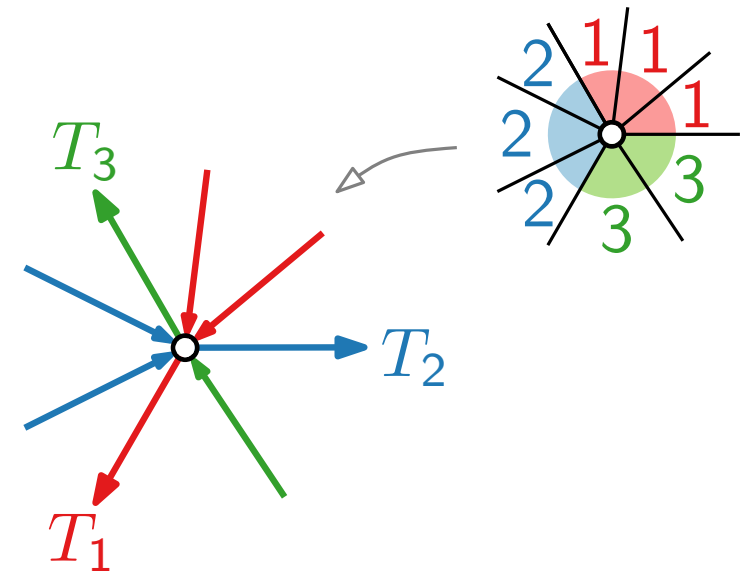
- All inner edges incident to a , b , and c are incoming in the same set (color).



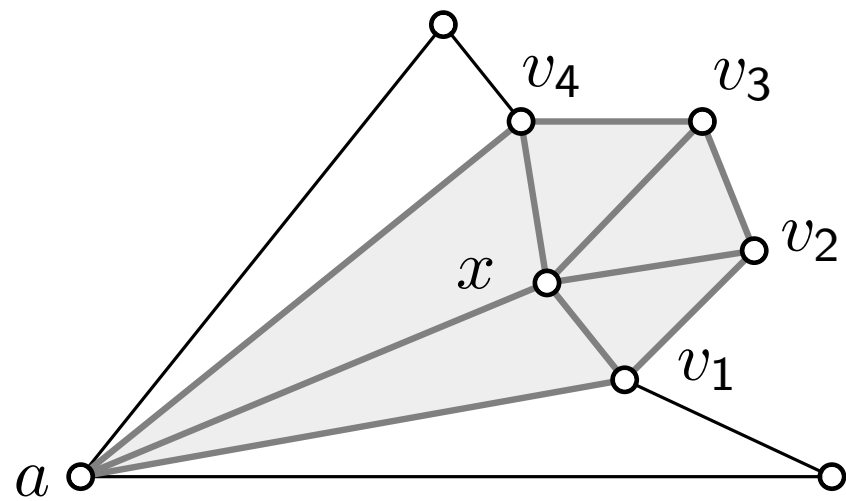
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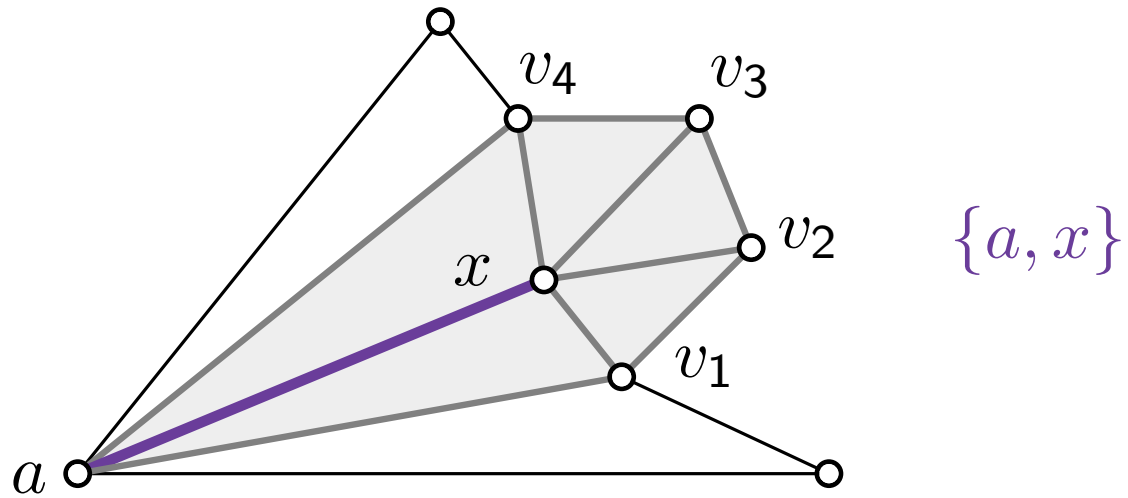
- All inner edges incident to a , b , and c are incoming in the same set (color).
- T_1 , T_2 , and T_3 are trees. Each spans all inner vertices and one outer vertex (its root).



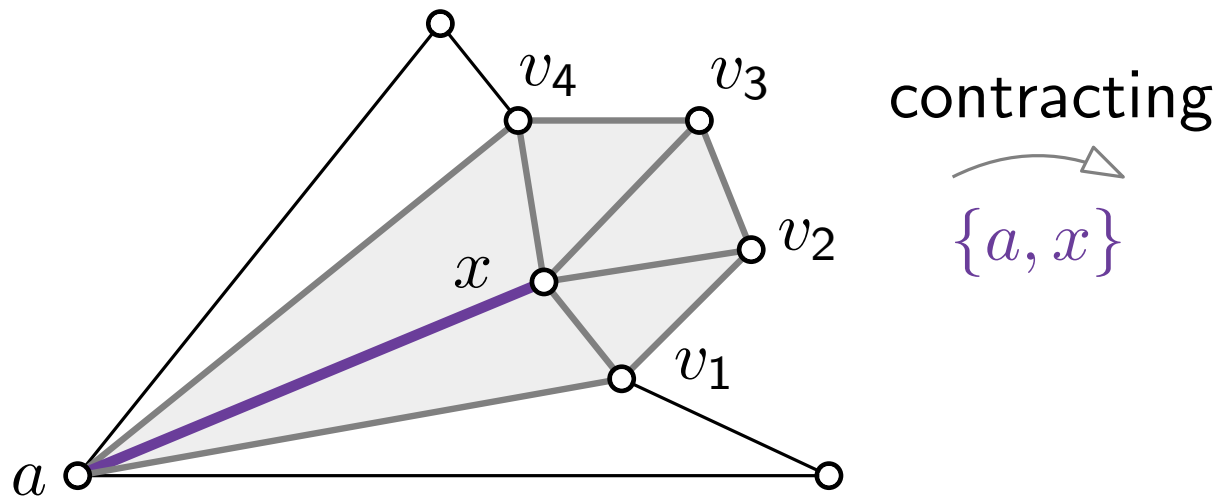
Schnyder Wood – Existence



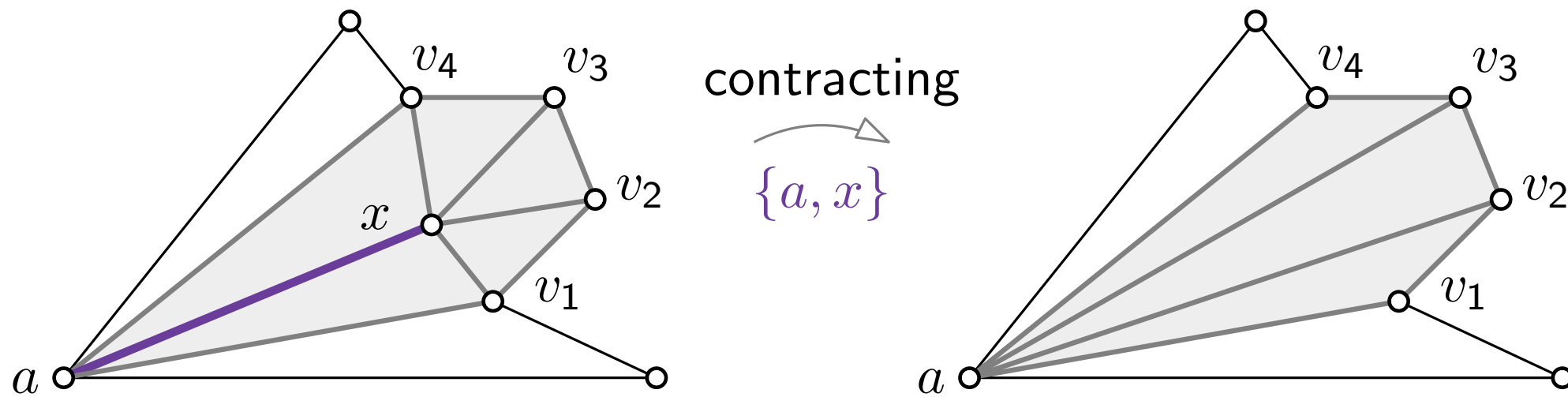
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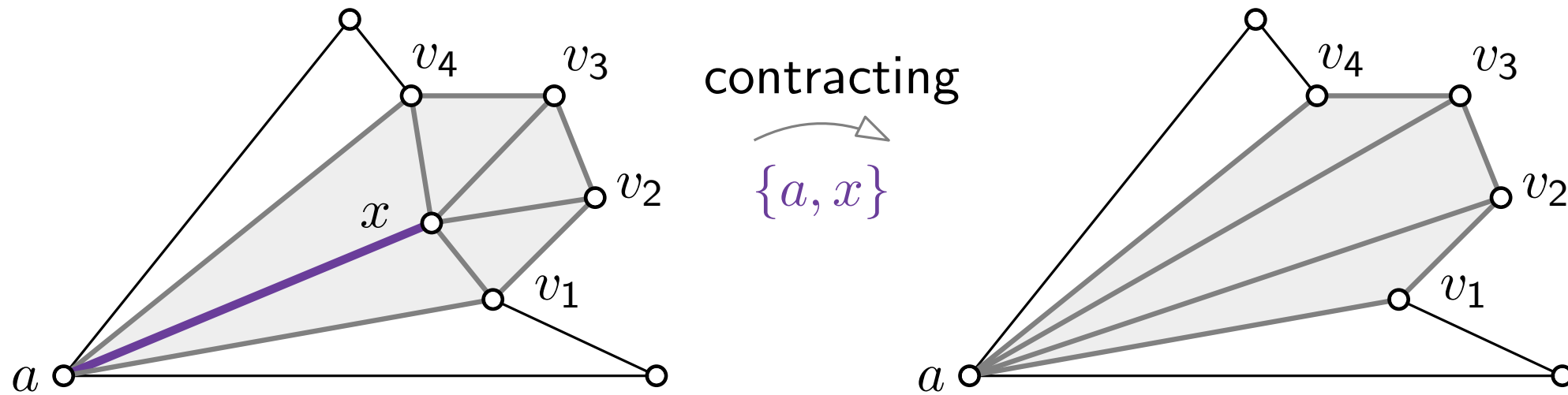
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Schnyder Wood – Existence



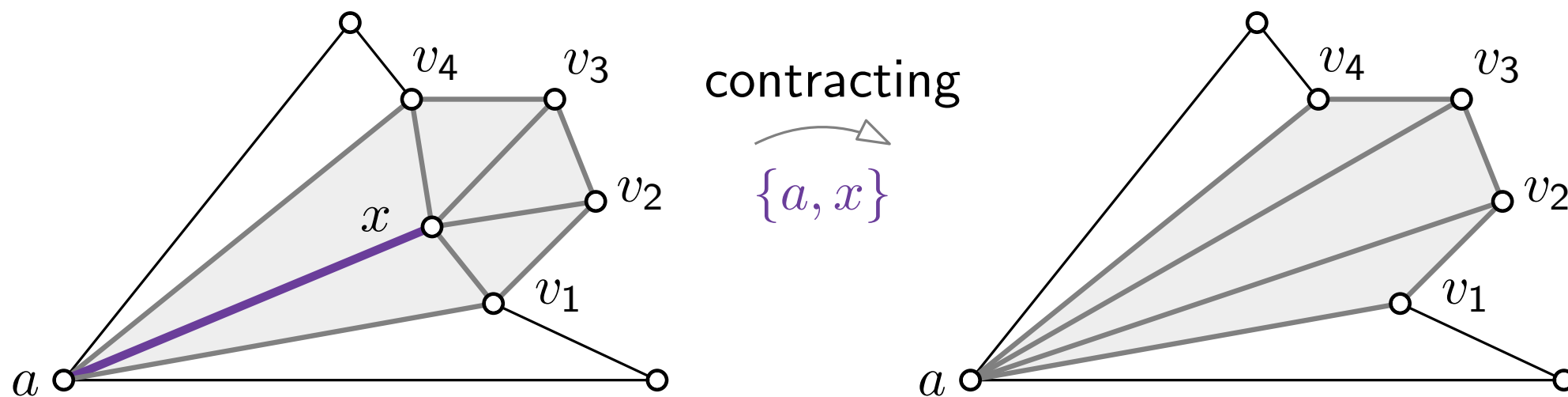
... requires that a and x have exactly two common neighbors.

Schnyder Wood – Existence

Lemma.

[Kampen 1976]

Let G be a plane triangulation with vertices a, b, c on the outer face. Then there exists a **contractible edge** $\{a, x\}$ in G with $x \notin \{b, c\}$.



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Schnyder Wood – Existence

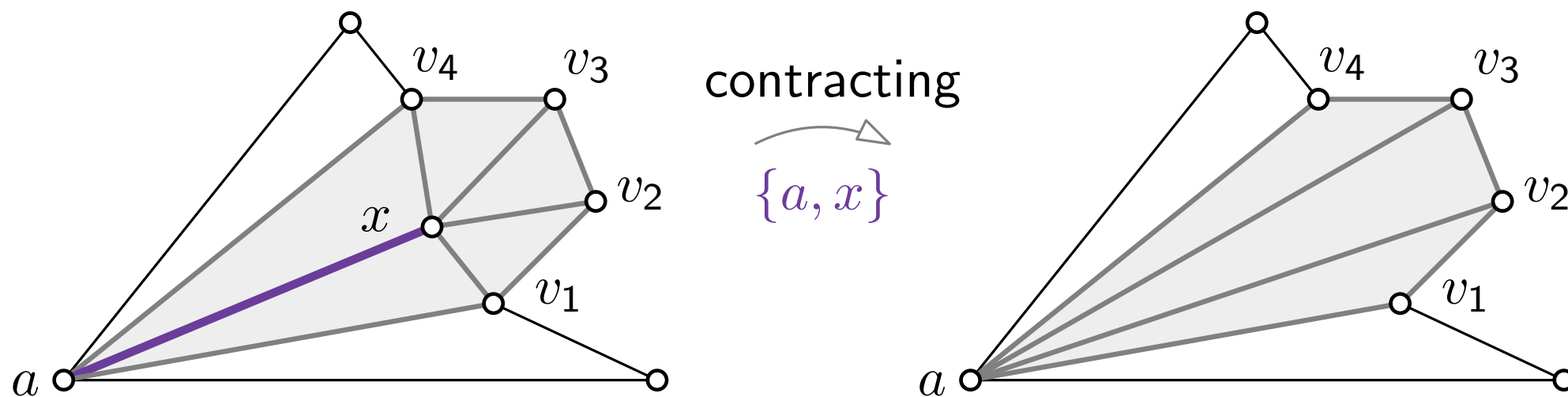
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Theorem.

Every plane triangulation has a Schnyder labeling and a Schnyder wood.



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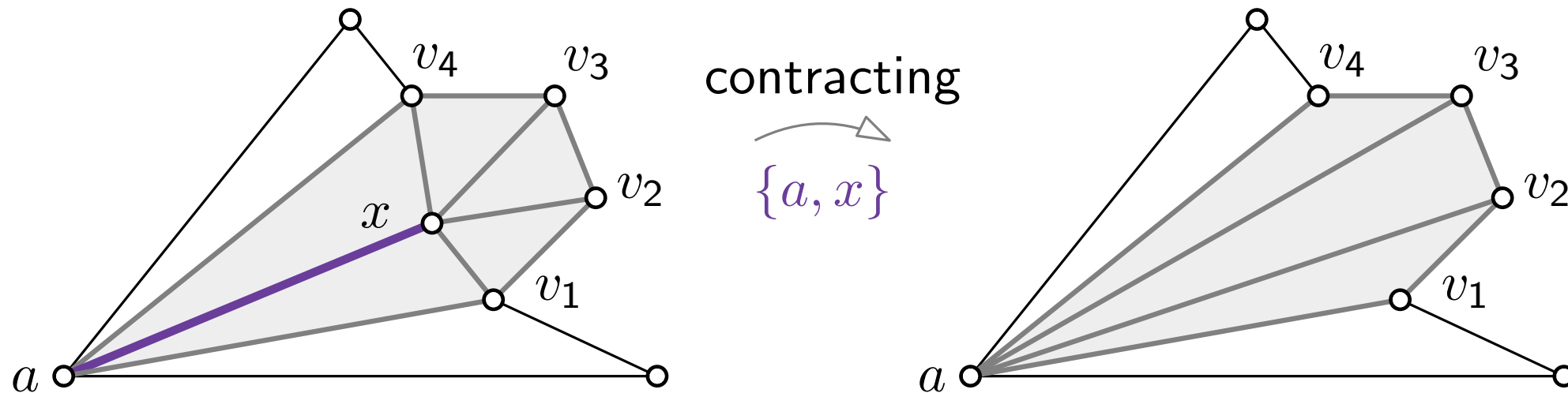
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Proof by induction on $\#$ vertices via edge contractions.



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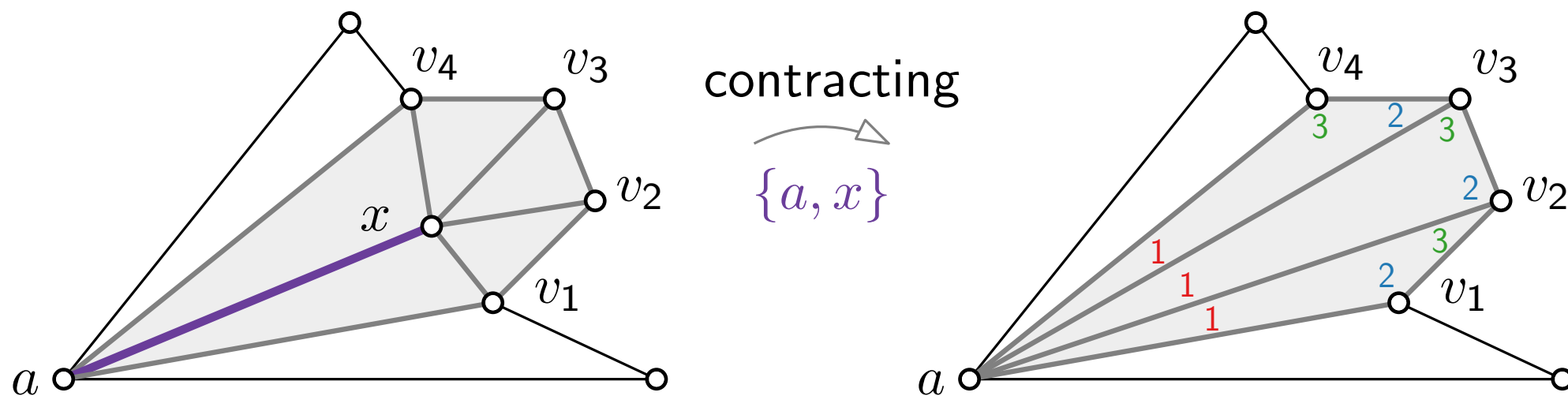
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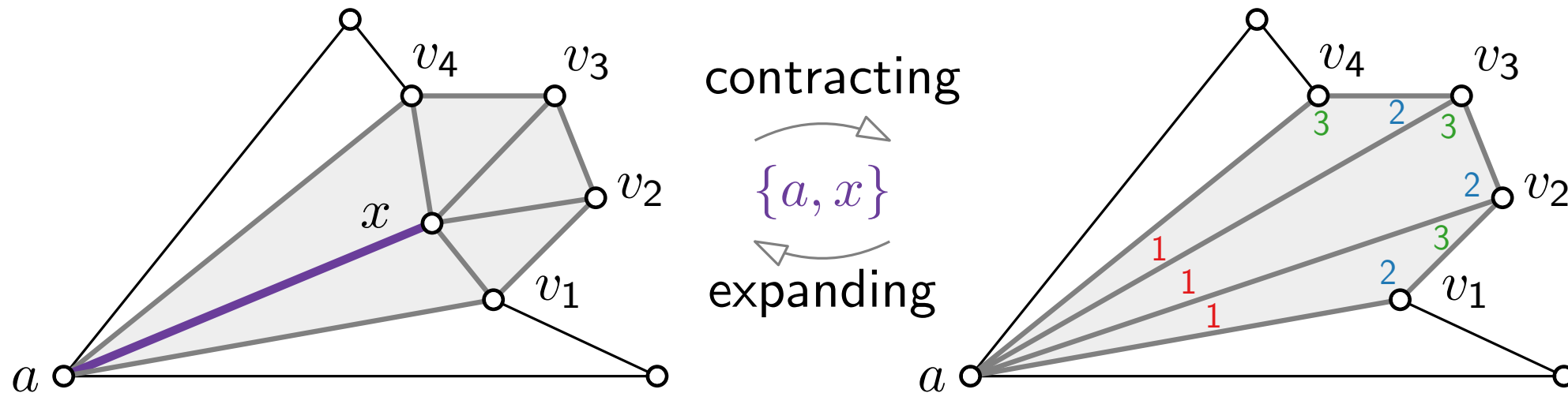
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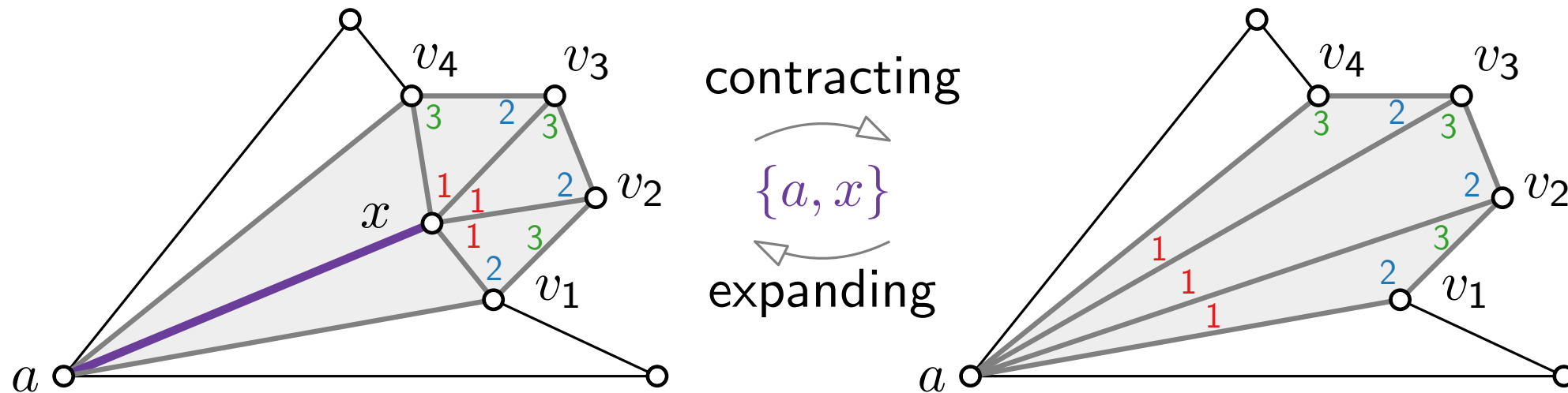
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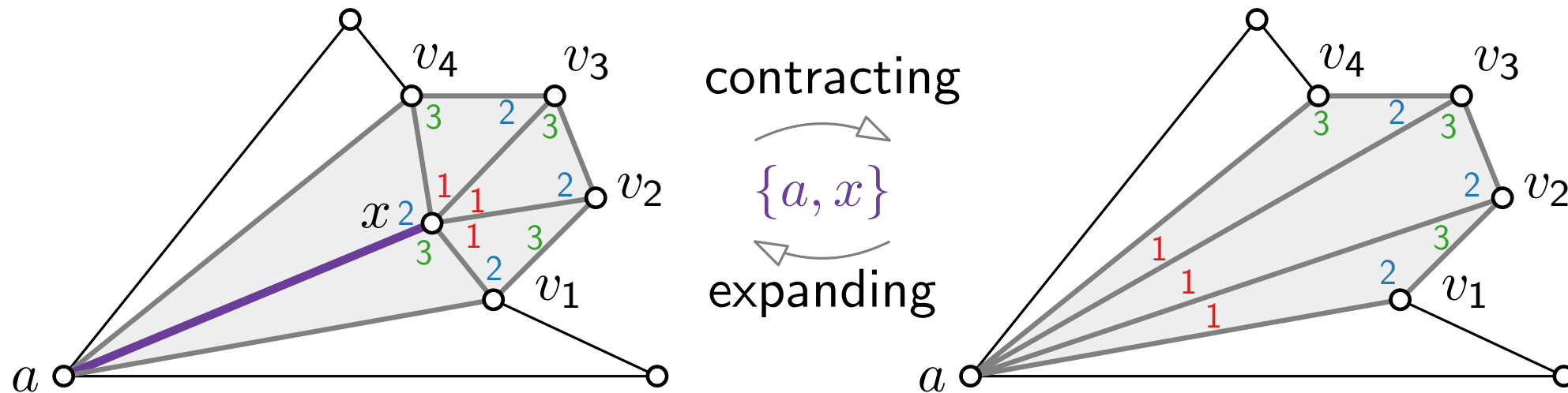
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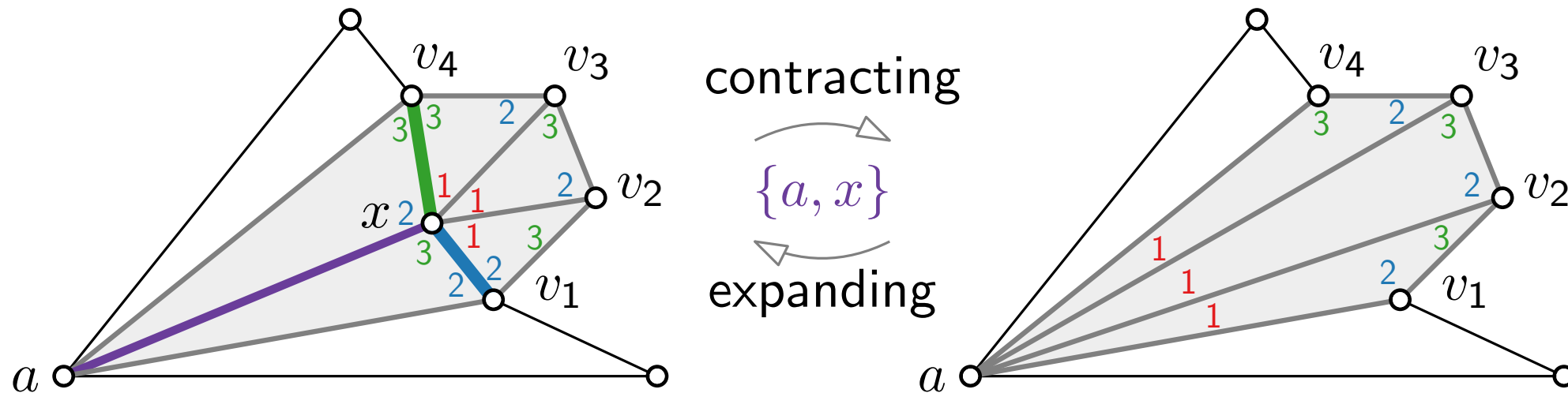
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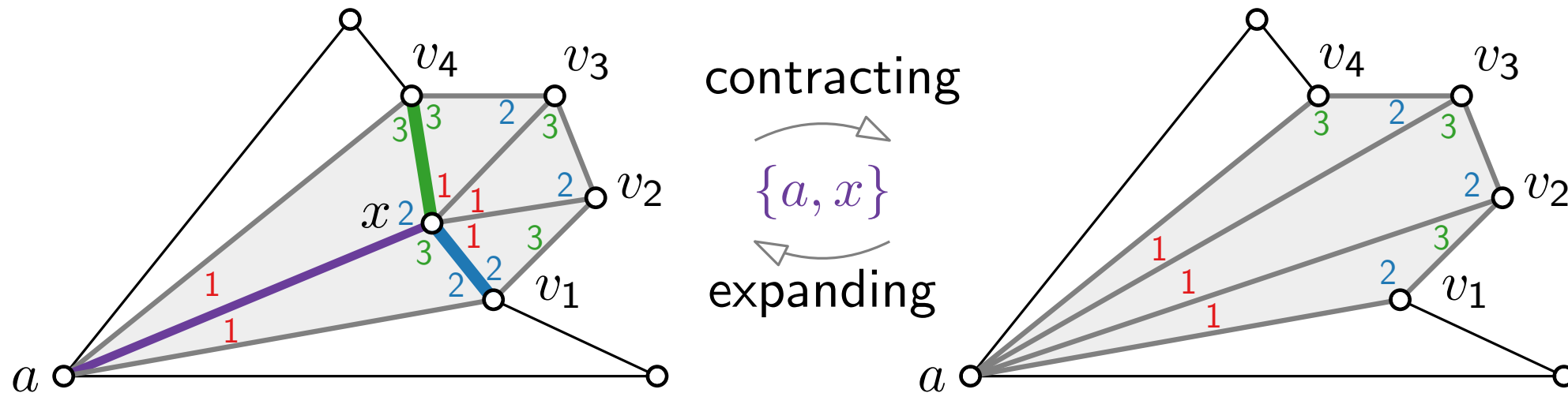
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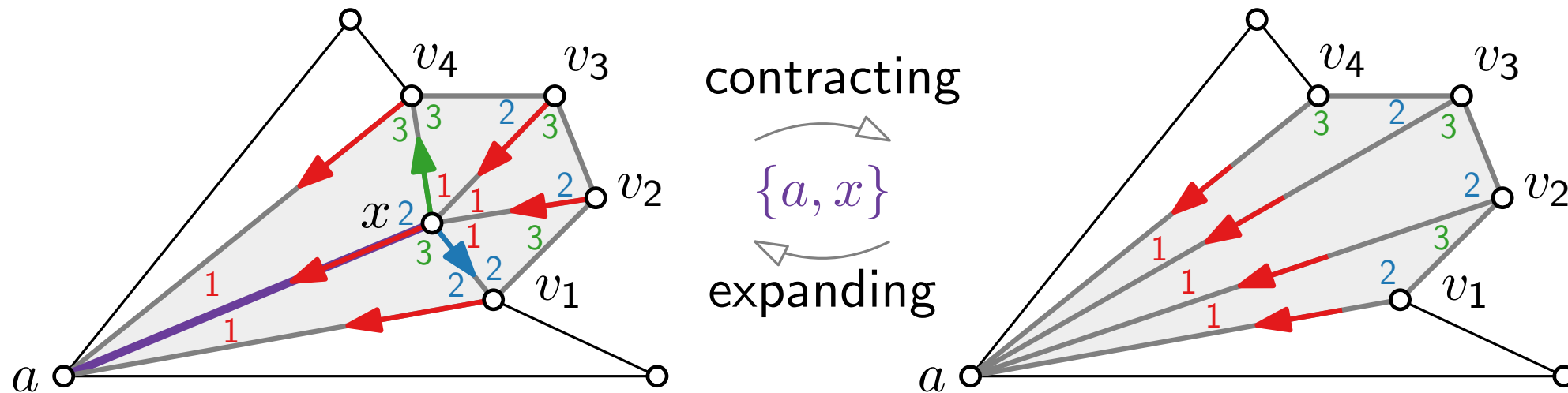
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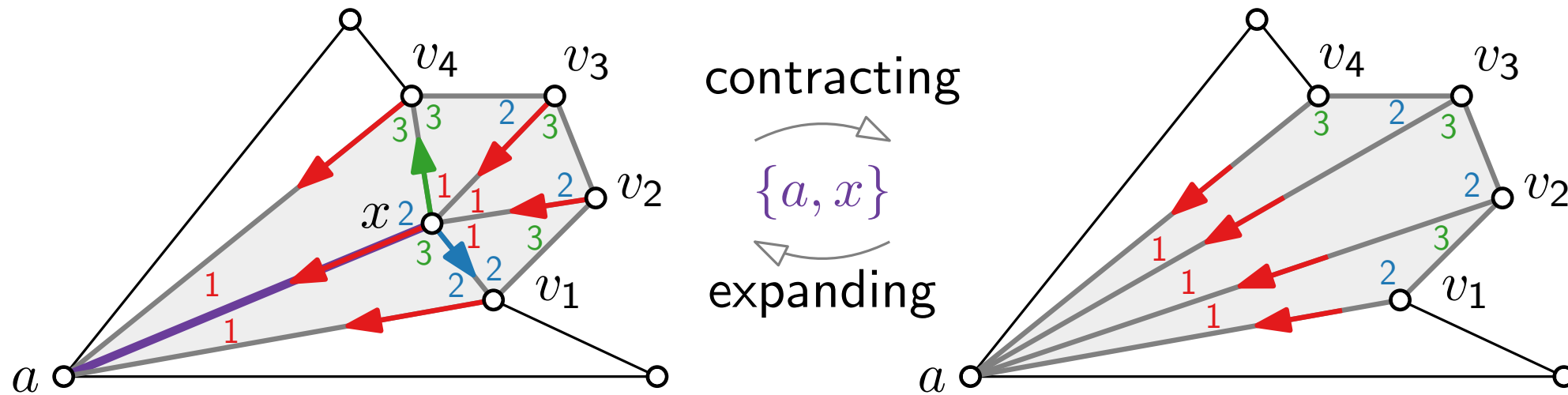
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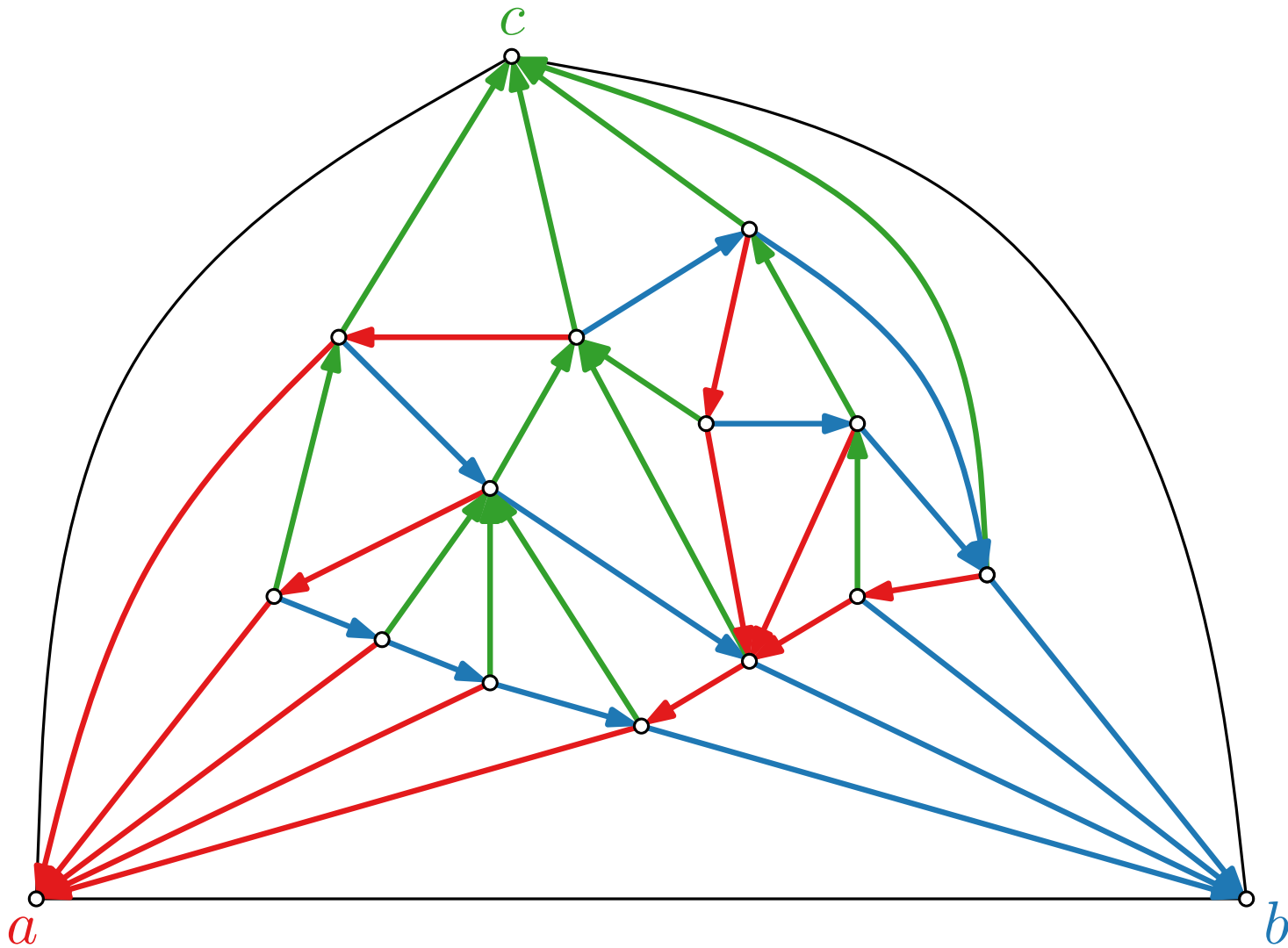


This constructive proof yields an algorithm for computing a Schnyder labeling. It can be implemented to run in $\mathcal{O}(n)$ time.

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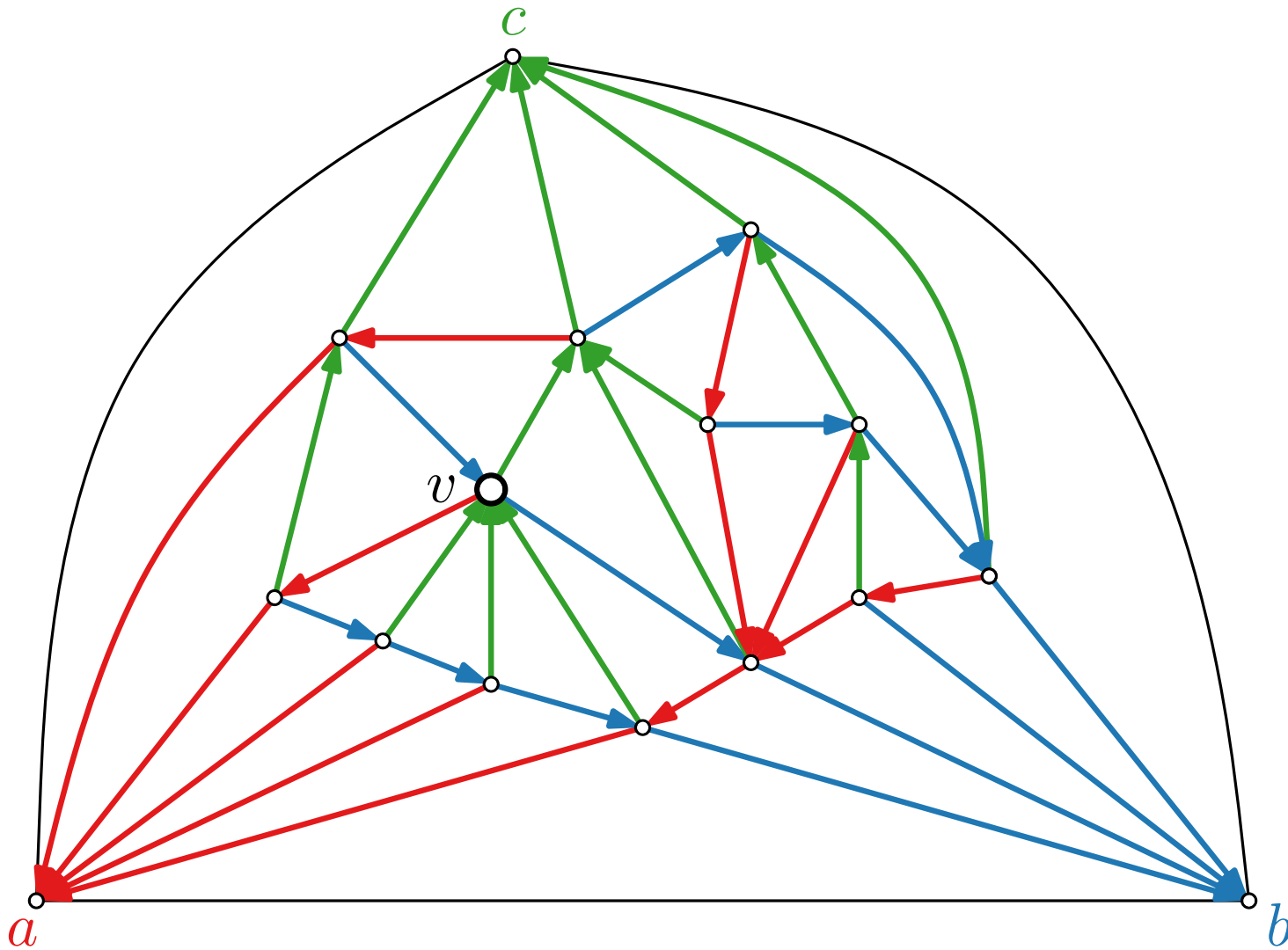
→ *Exercise* 😊

Schnyder Wood – More Properties



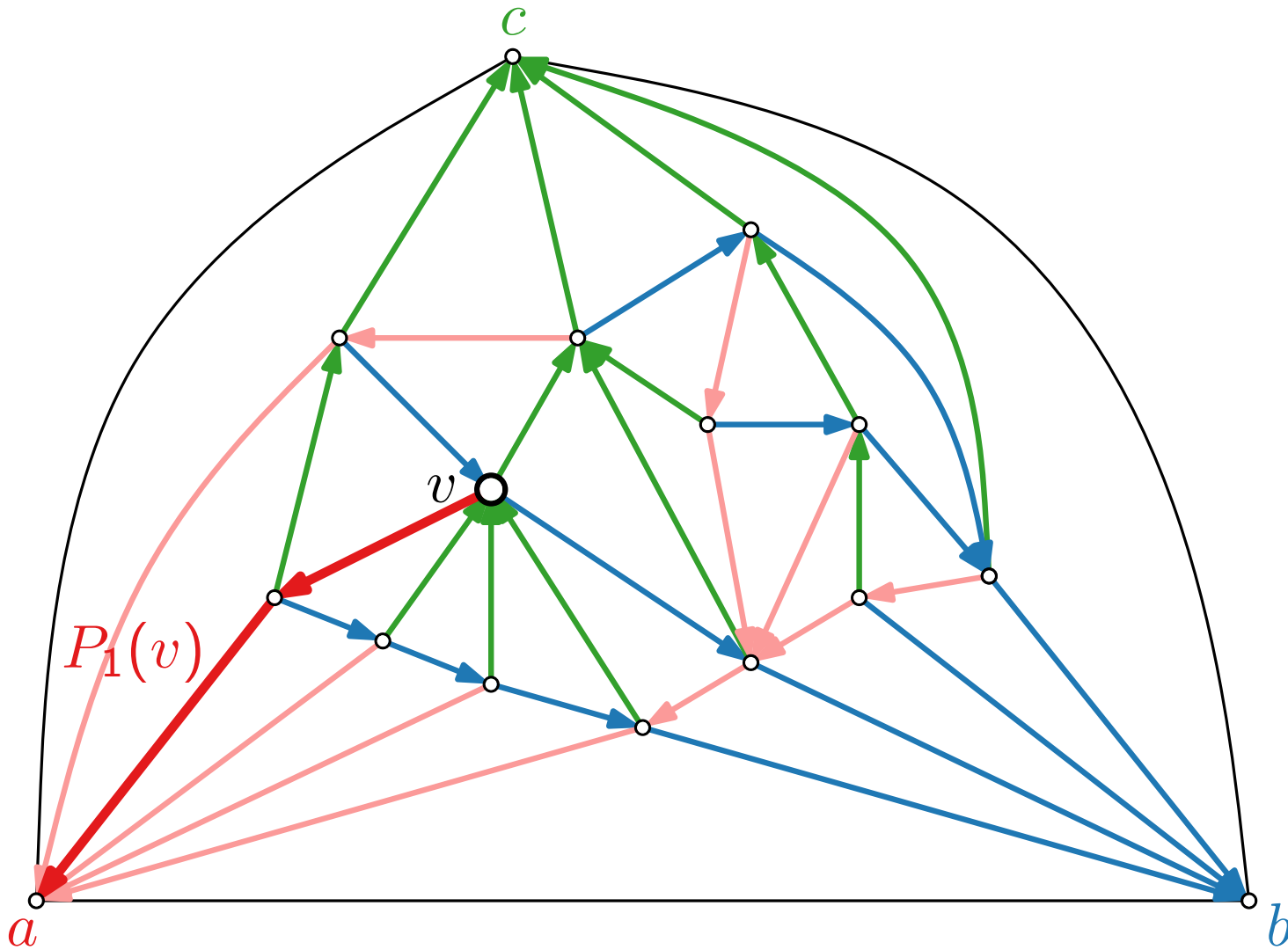
Schnyder Wood – More Properties

- From each vertex v there exists a unique



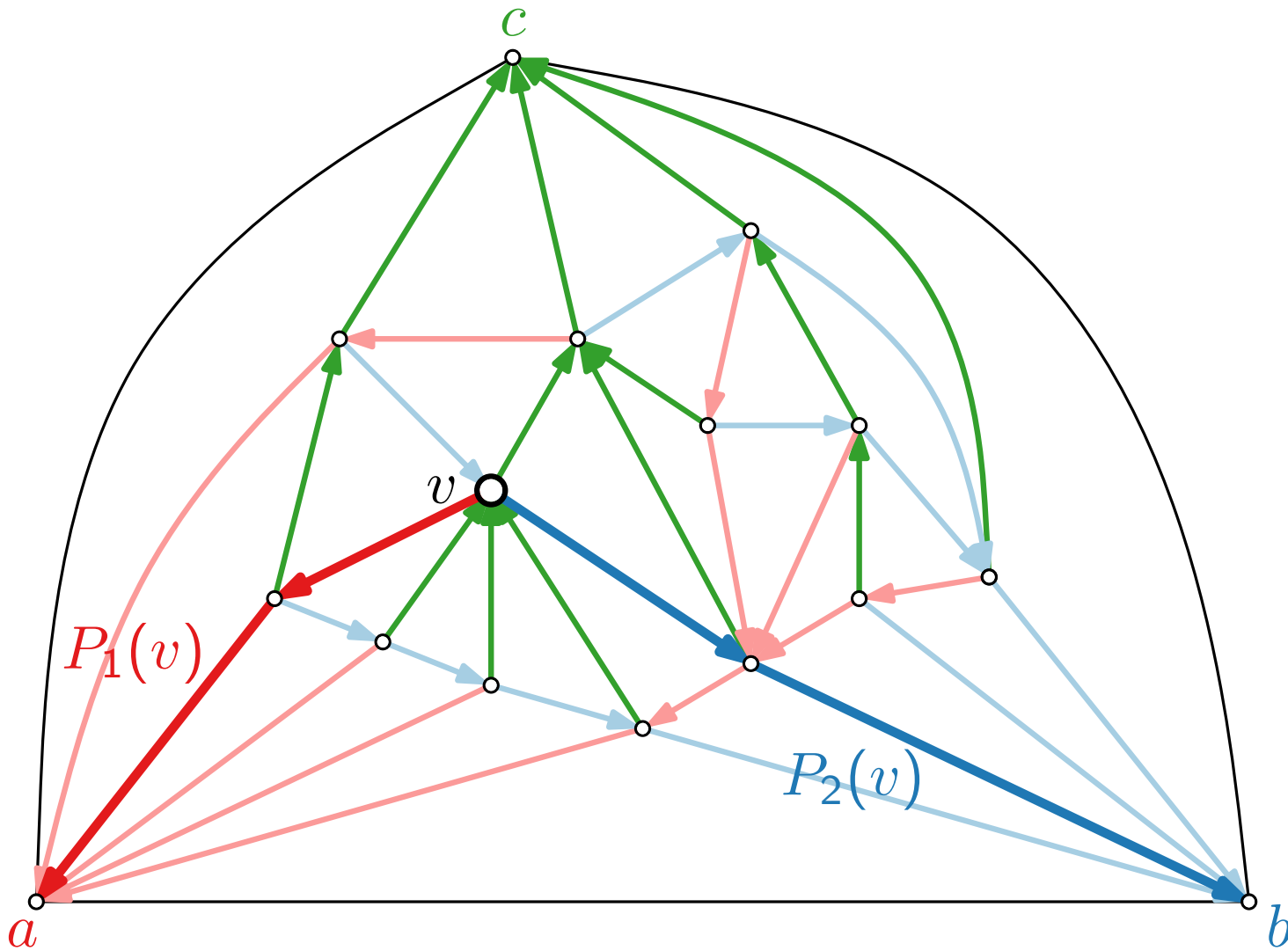
Schnyder Wood – More Properties

- From each vertex v there exists a unique – directed red path $P_1(v)$ to a ,

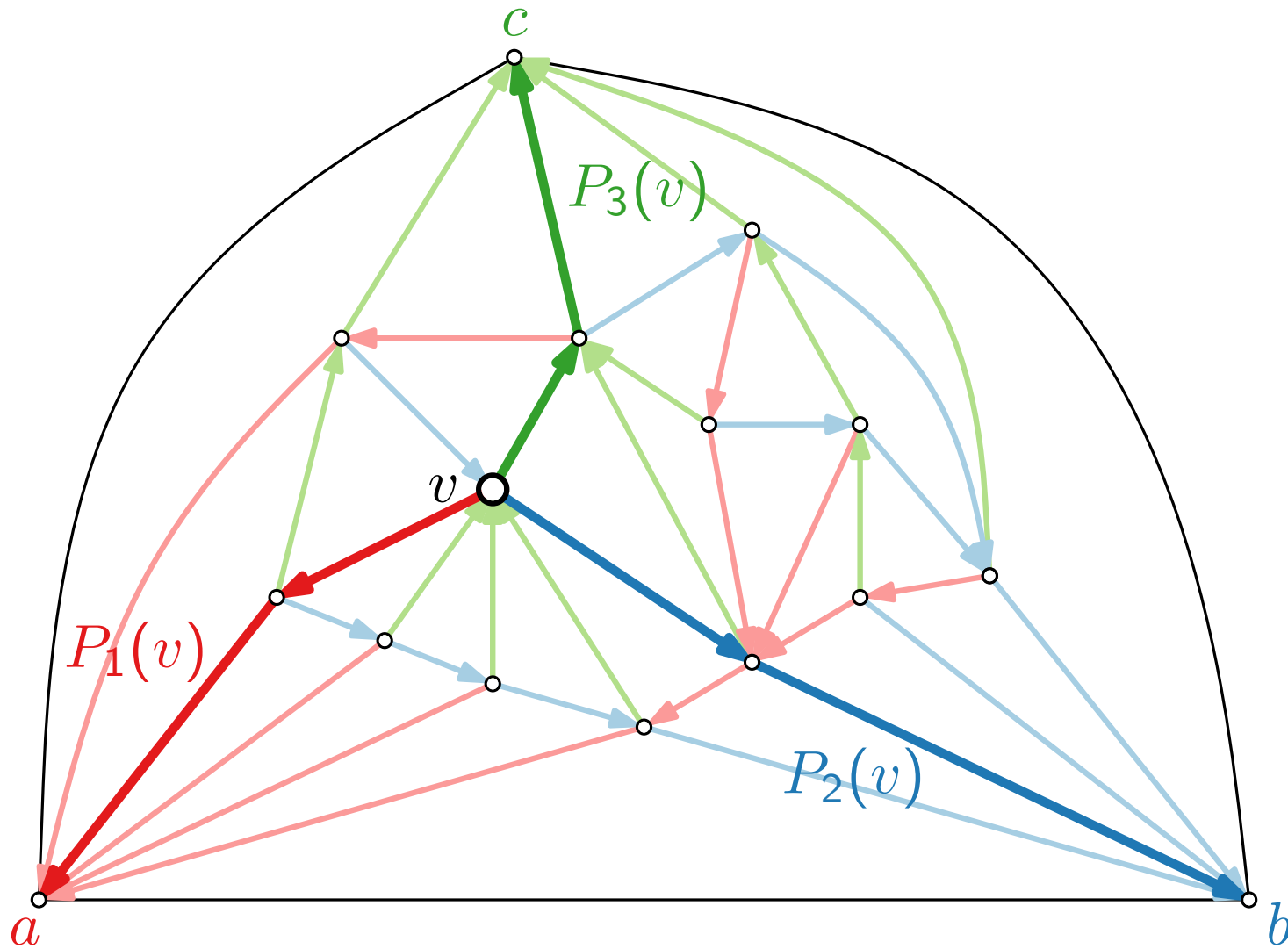


Schnyder Wood – More Properties

- From each vertex v there exists a unique
 - directed **red** path $P_1(v)$ to a ,
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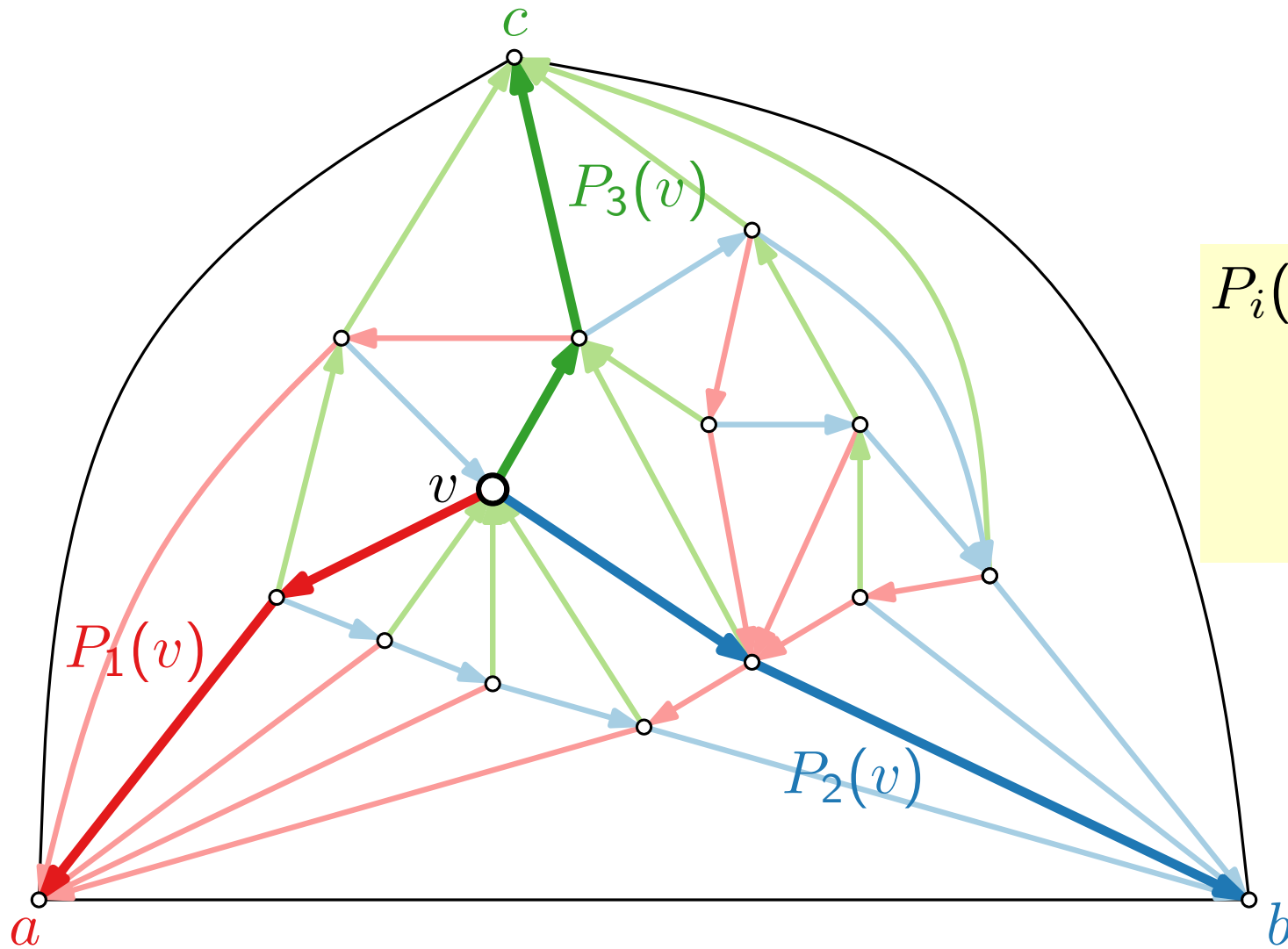
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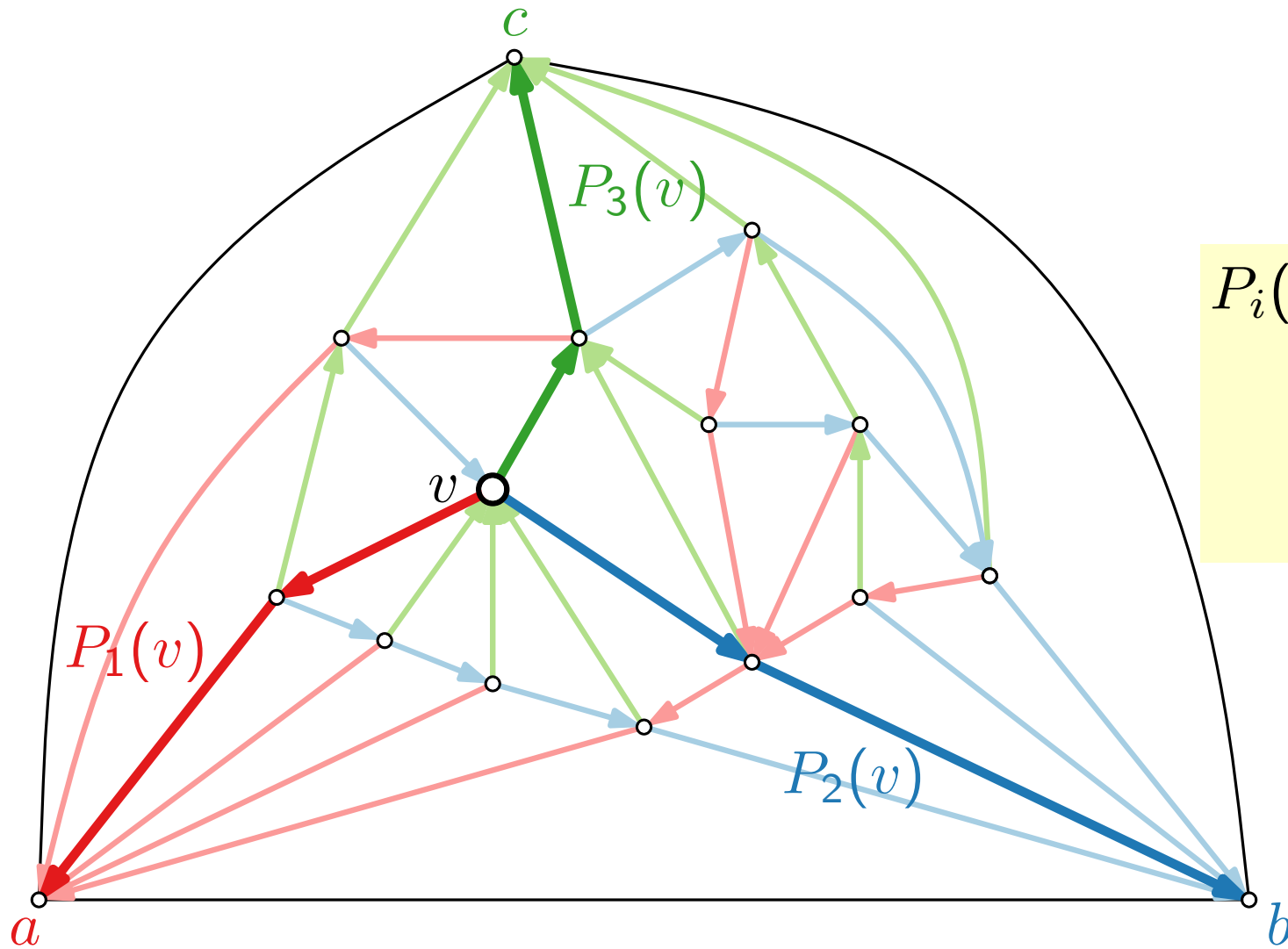
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$P_i(v)$: unique path from v to root of T_i

Schnyder Wood – More Properties



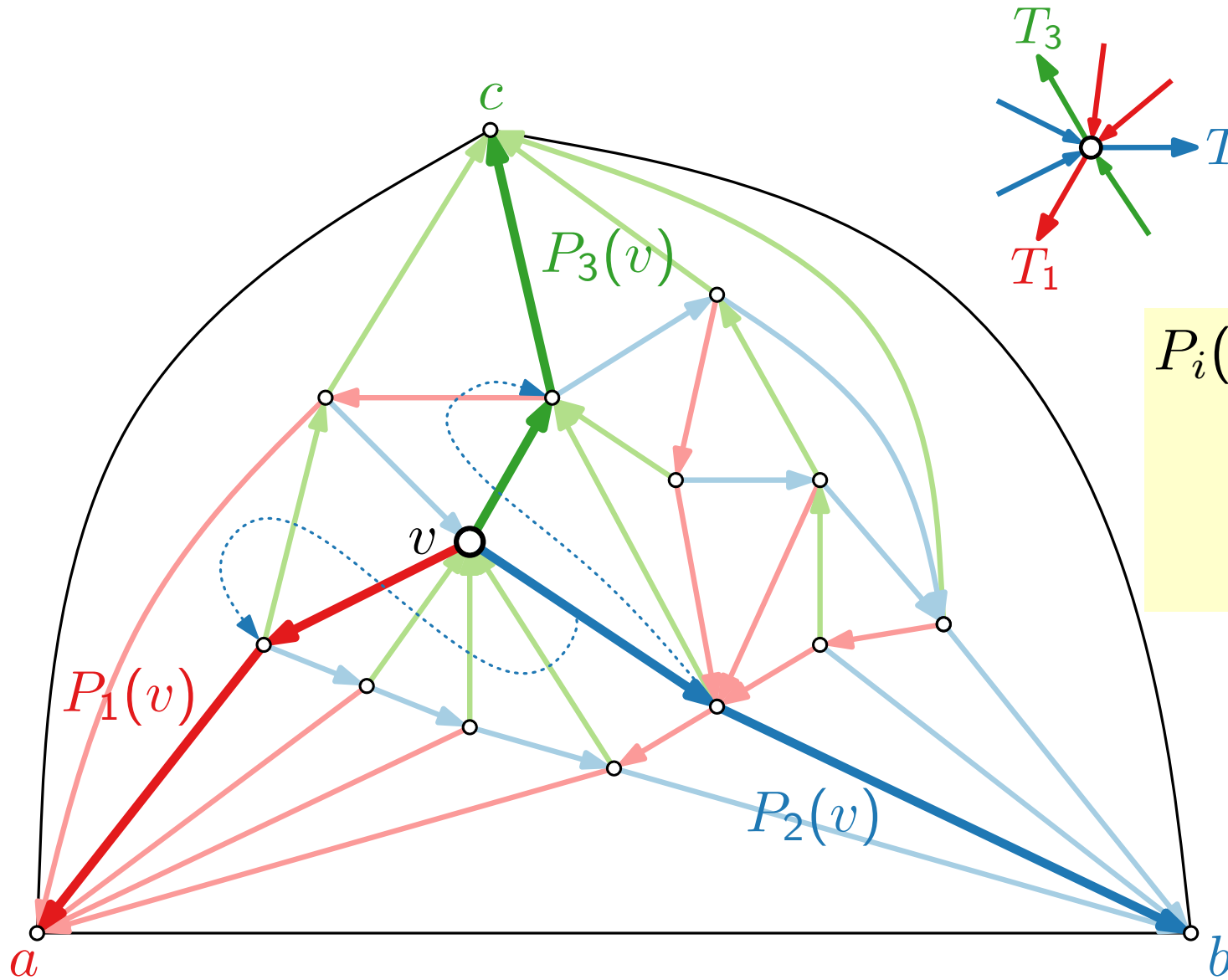
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Lemma.

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Schnyder Wood – More Properties



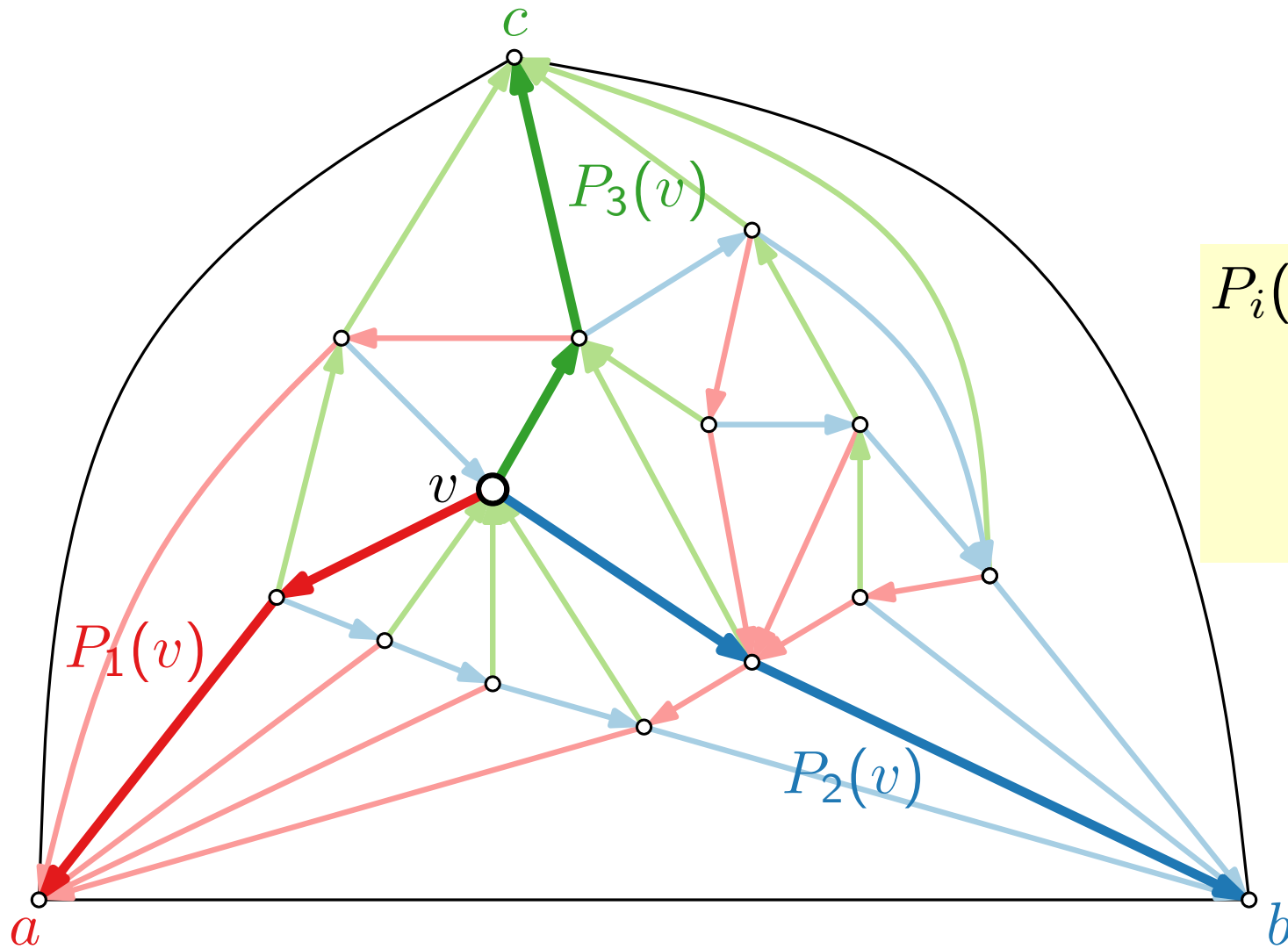
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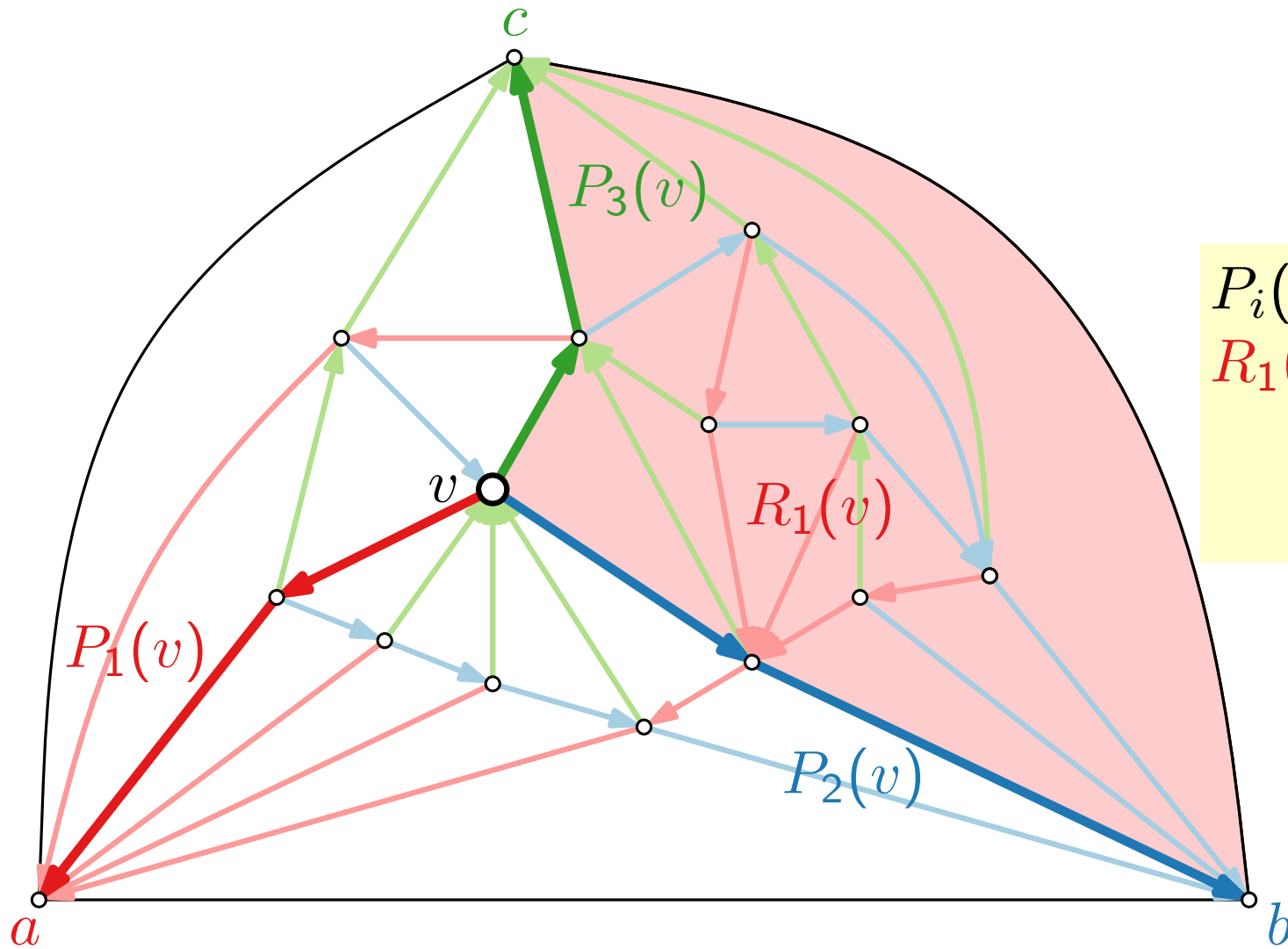
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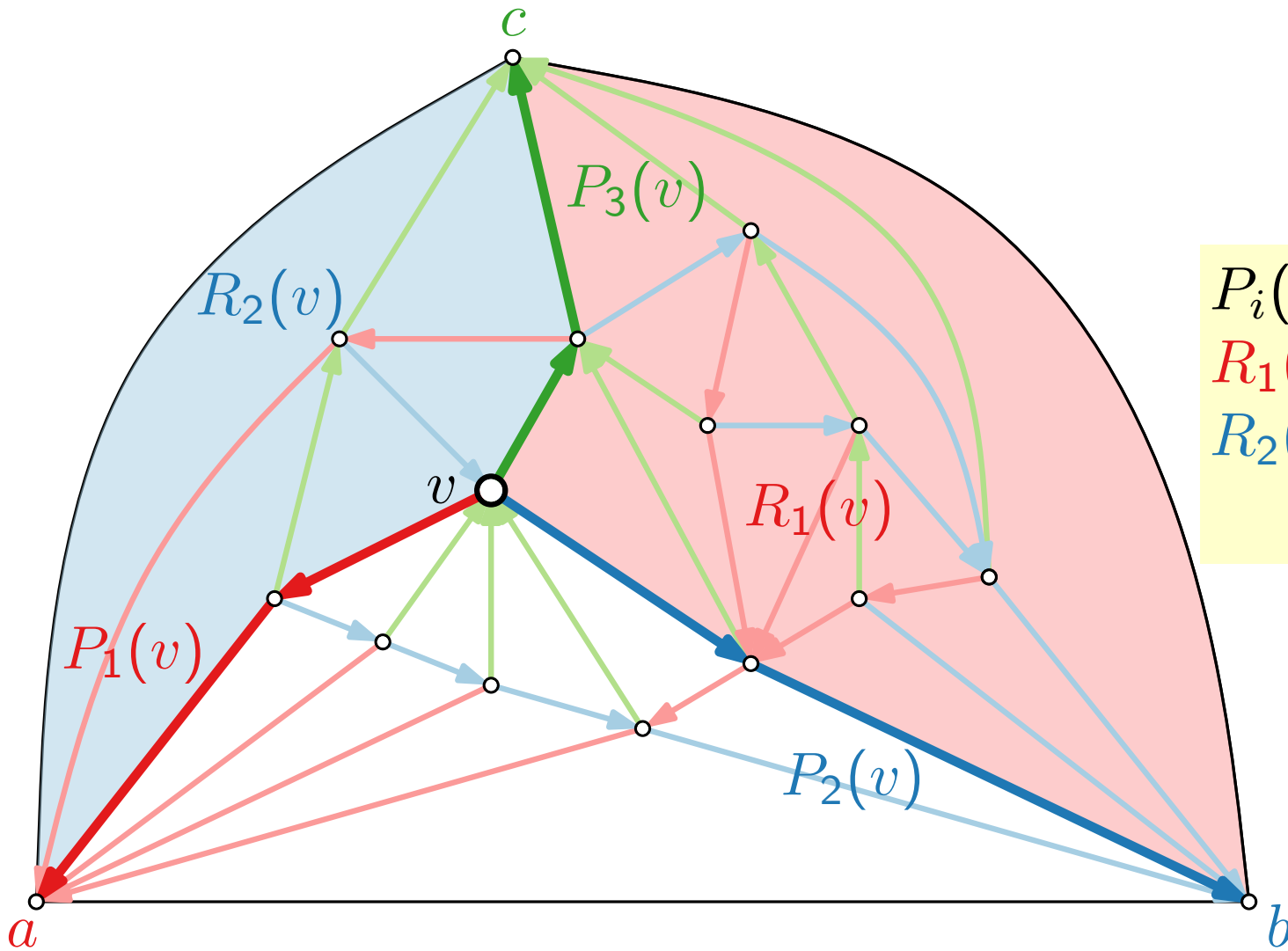
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$R_1(v)$: set of faces bounded by $\langle P_2(v), bc, P_3(v) \rangle$

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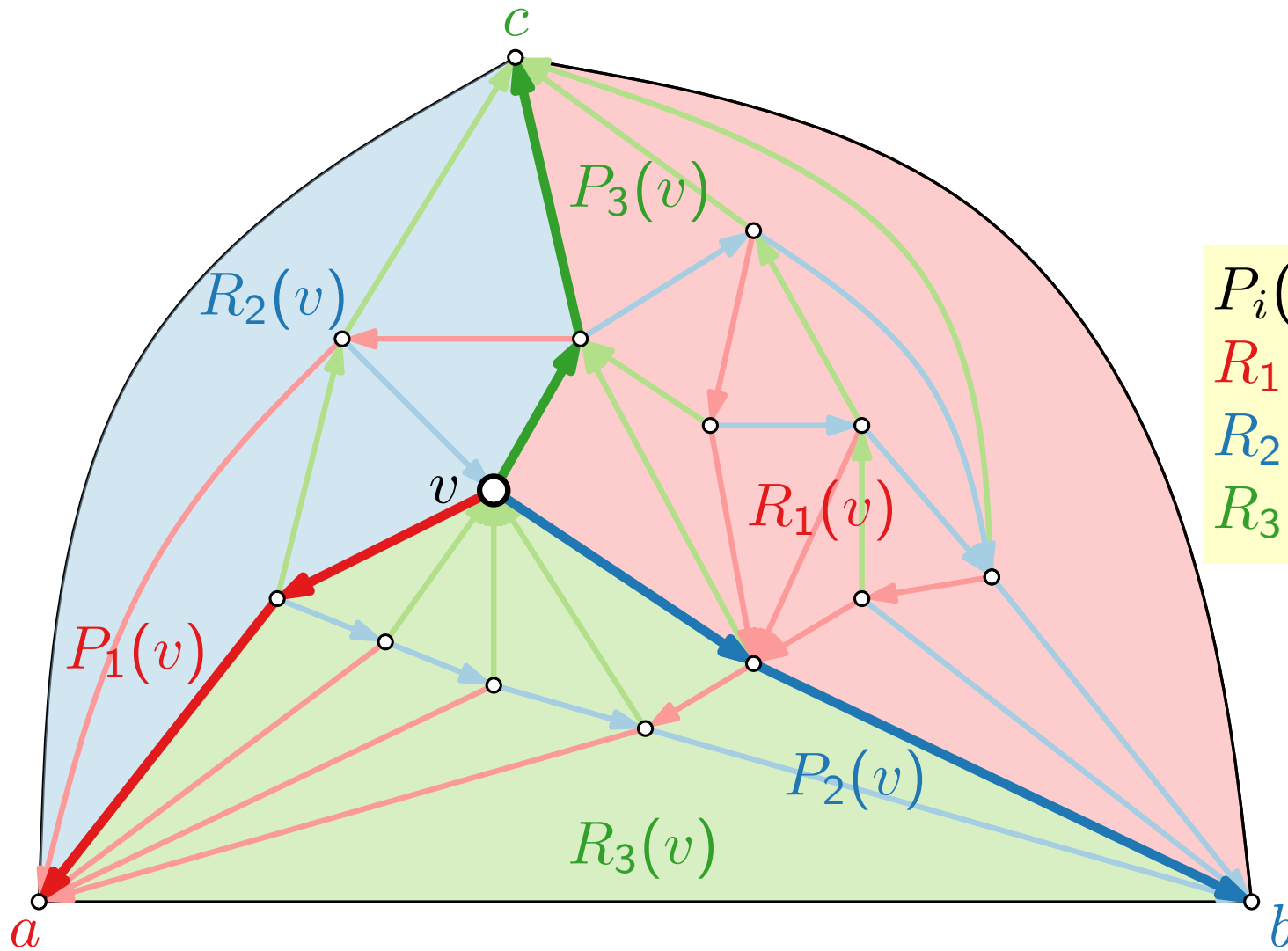
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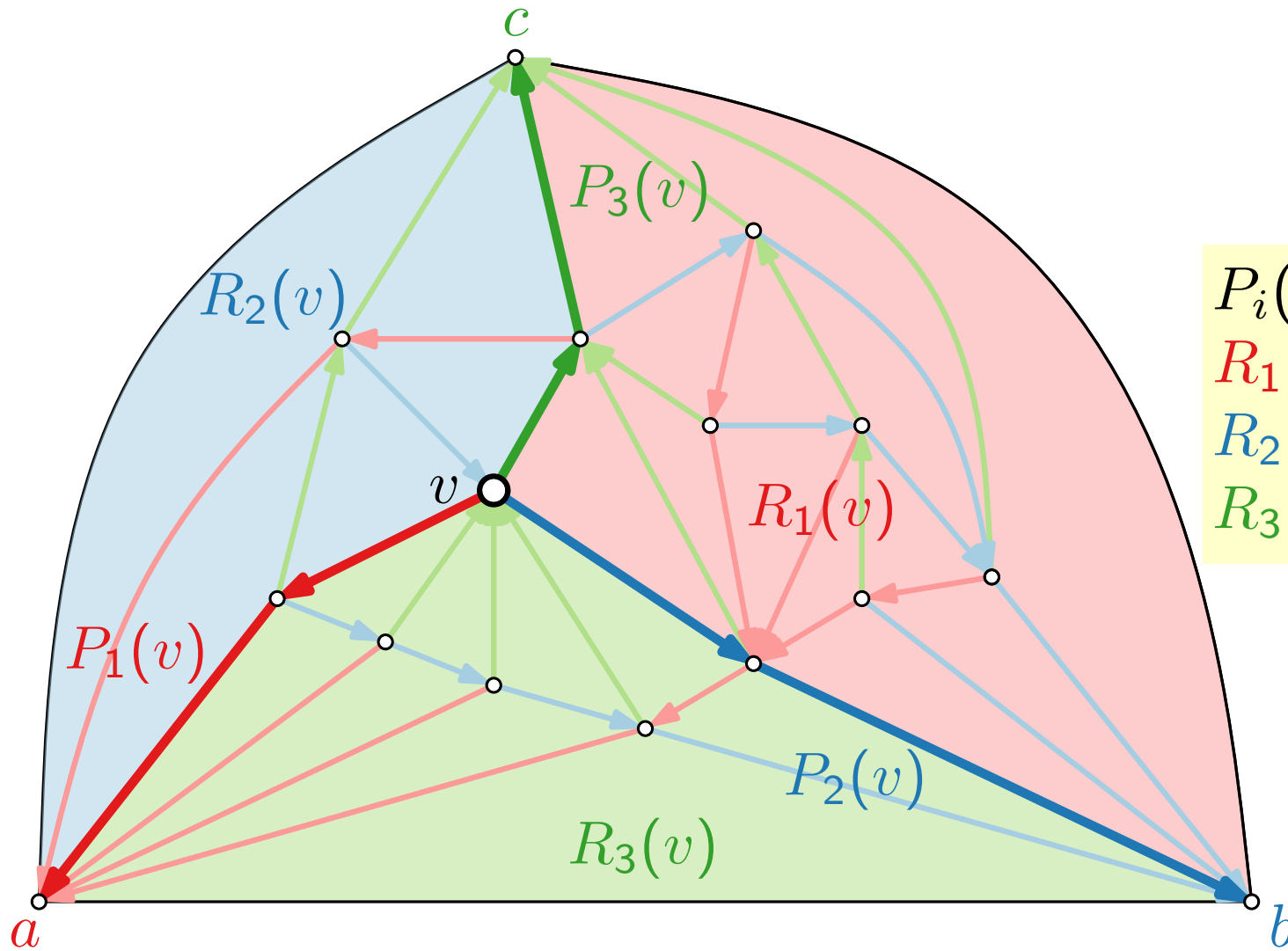
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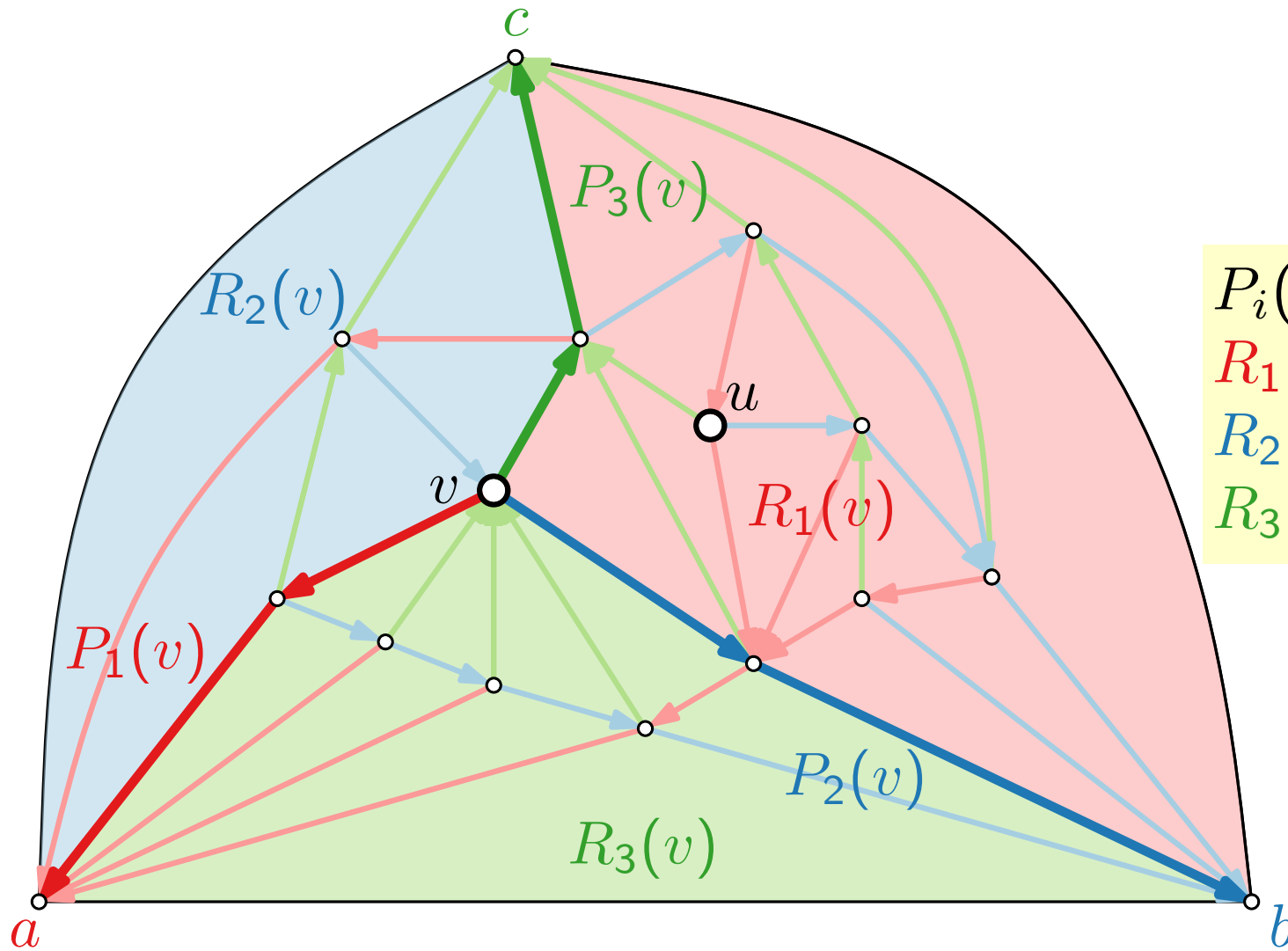
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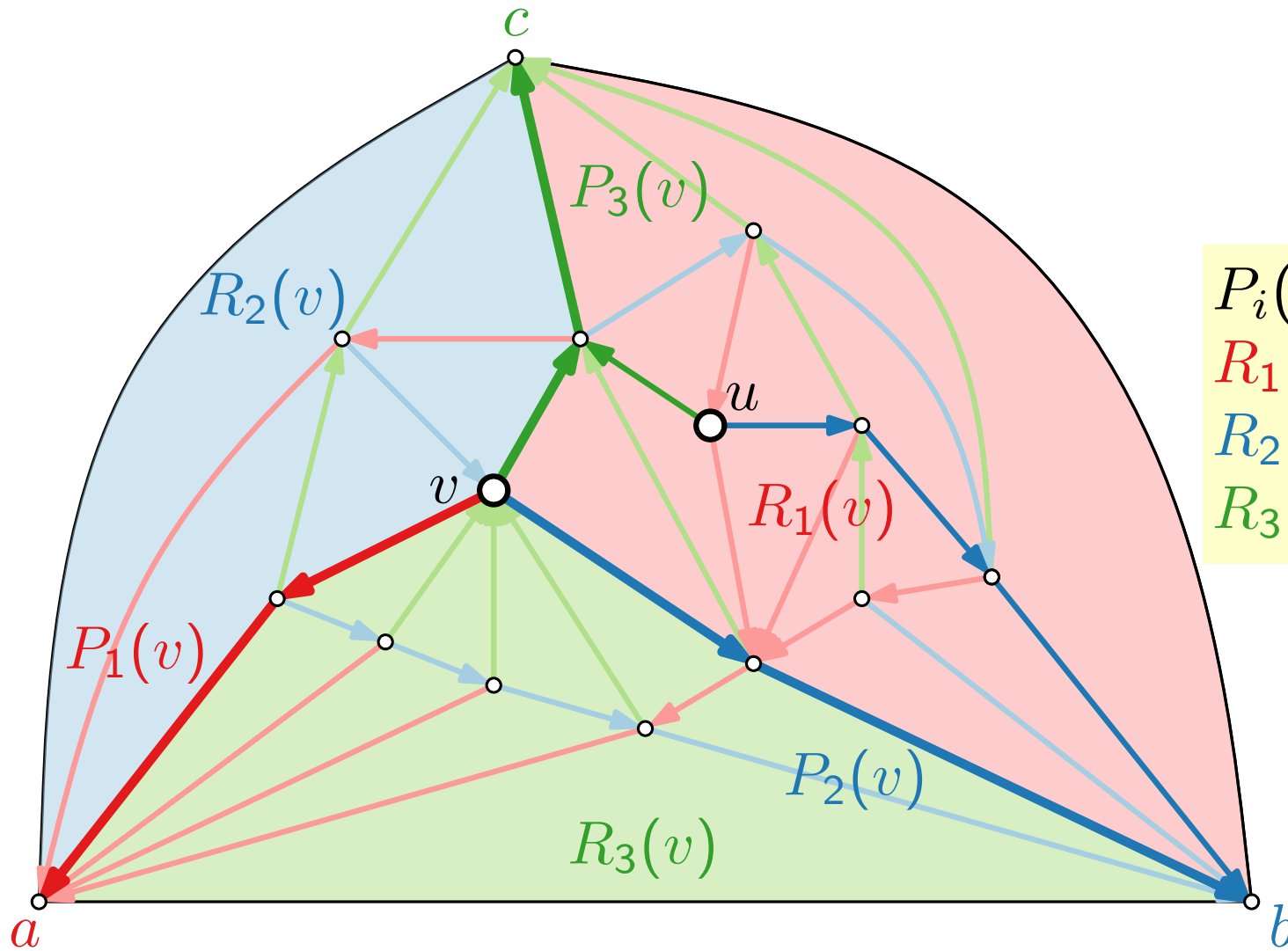
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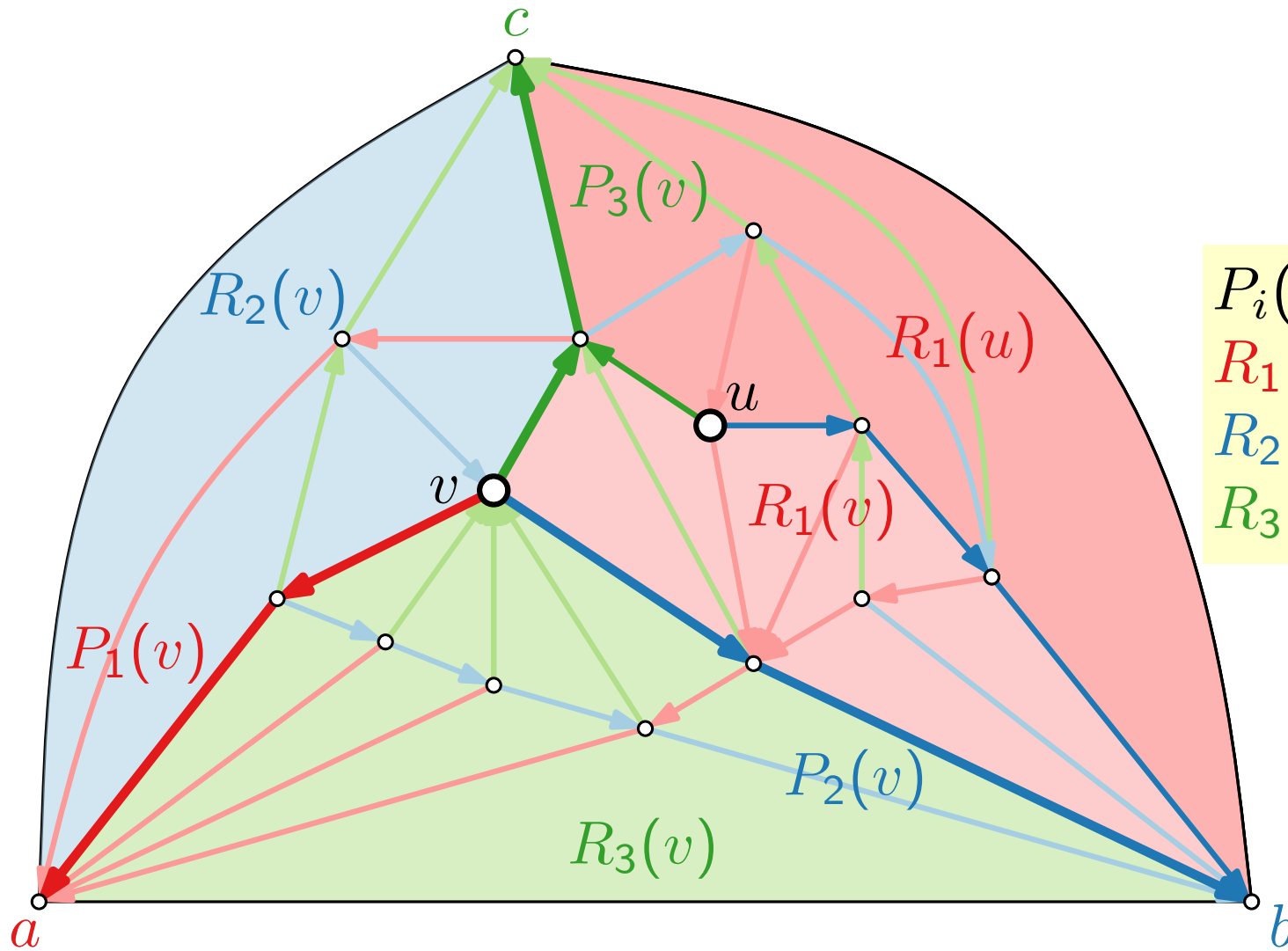
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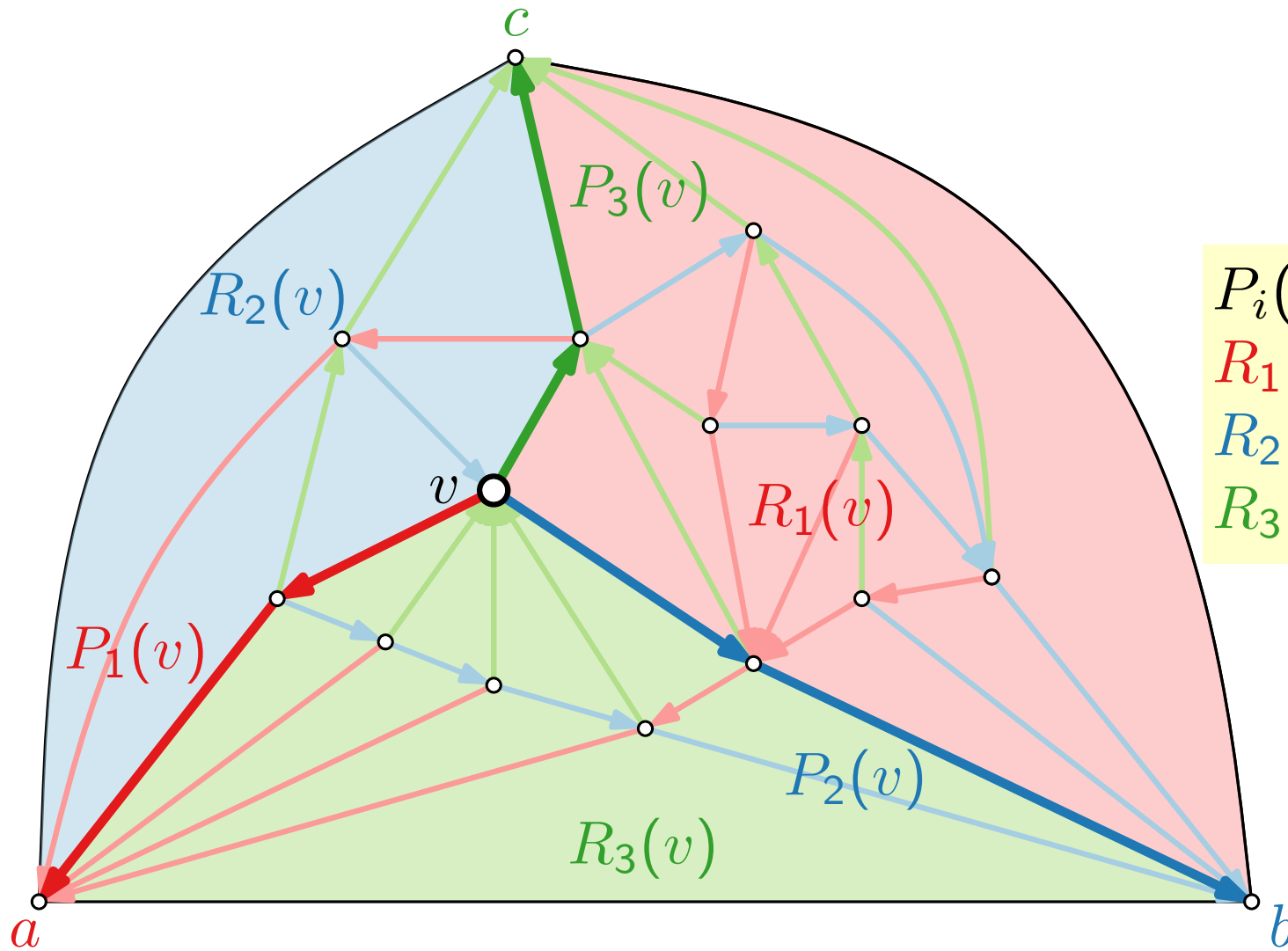
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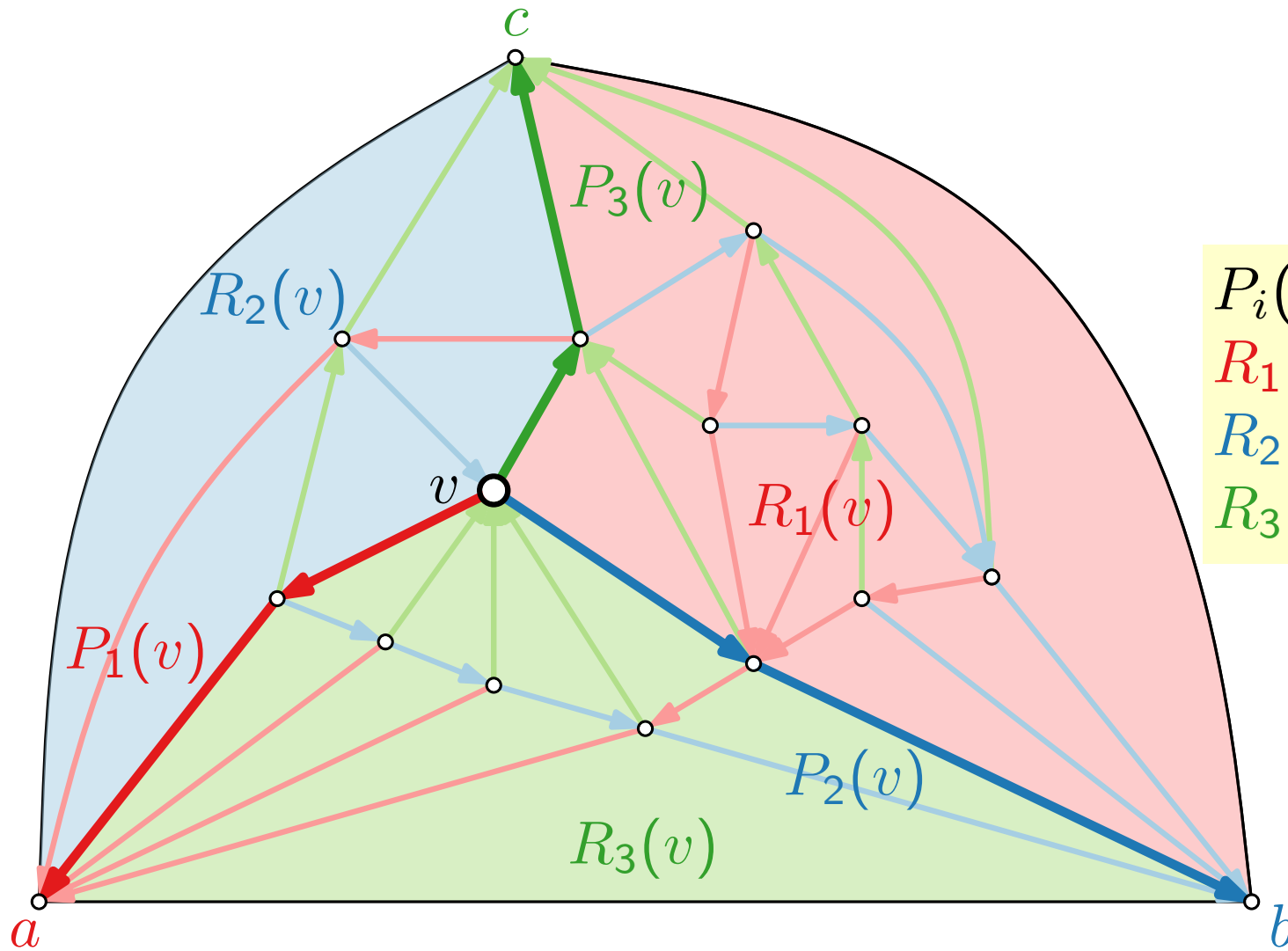
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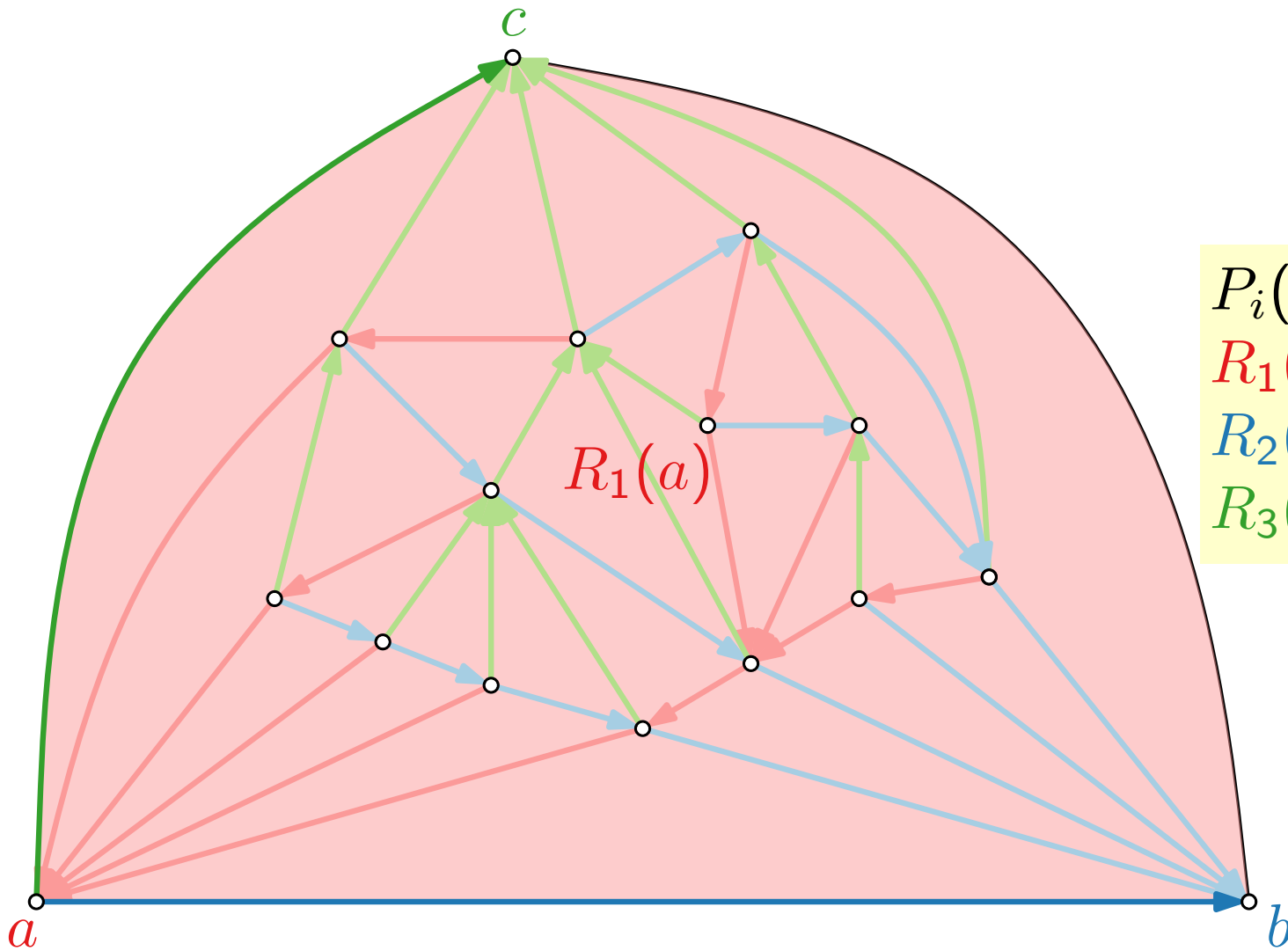
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- $P_1(v), P_2(v), P_3(v)$ cross only at v .
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.
- $|R_1(v)| + |R_2(v)| + |R_3(v)| = 2n - 5$

Schnyder Wood – More Properties



- From each vertex v there exists a unique
 - directed red path $P_1(v)$ to a ,
 - directed blue path $P_2(v)$ to b , and
 - directed green path $P_3(v)$ to c .

$P_i(v)$: unique path from v to root of T_i

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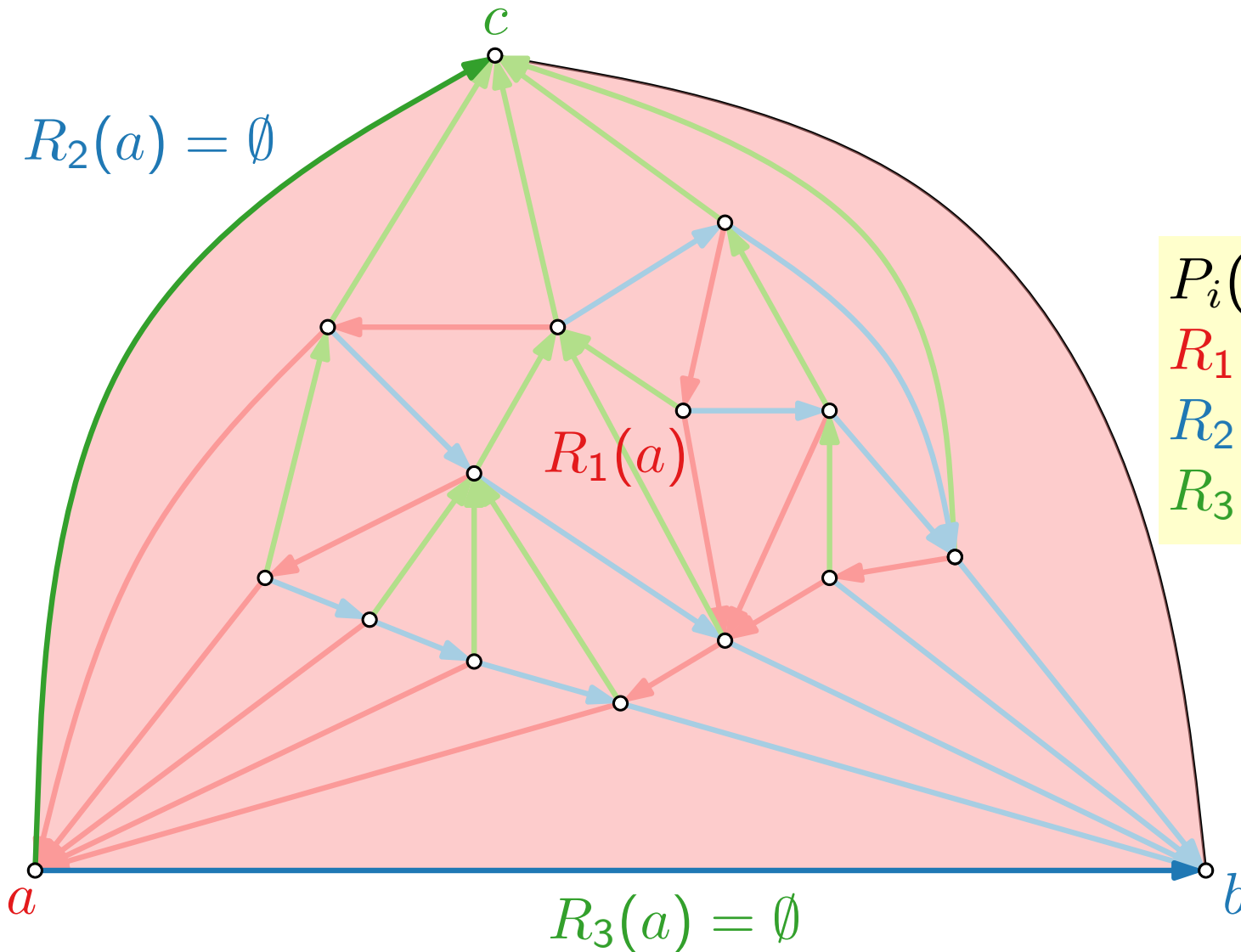
$R_2(v)$: set of faces bounded by $\langle P_3(v), ca, P_1(v) \rangle$

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Schnyder Drawing

Theorem.

[Schnyder '90]

For a plane triangulation G , the mapping

$$f: v \mapsto (v_1, v_2, v_3) = \frac{1}{2n - 5} (|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G and, thus, yields a planar straight-line drawing of G

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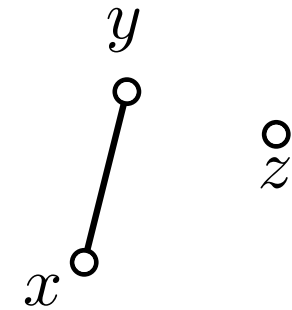
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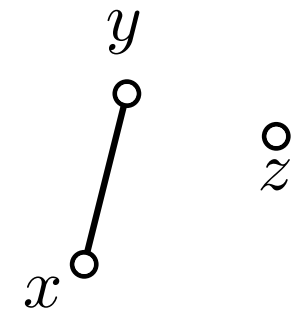
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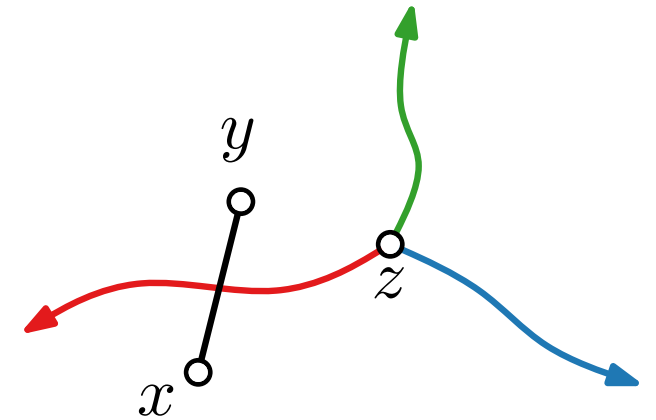
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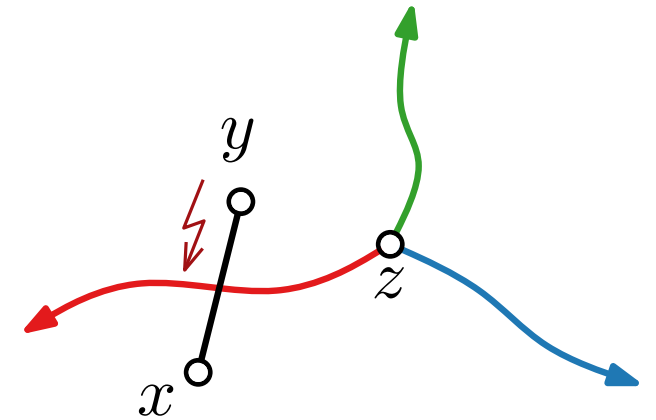
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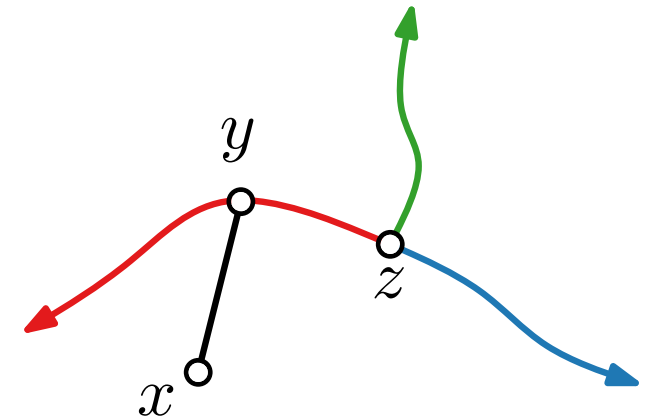
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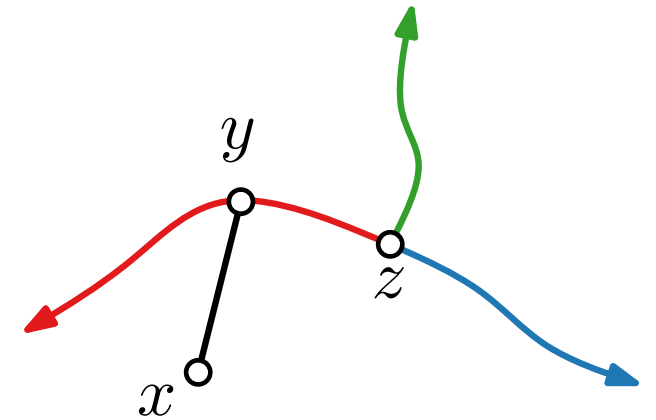
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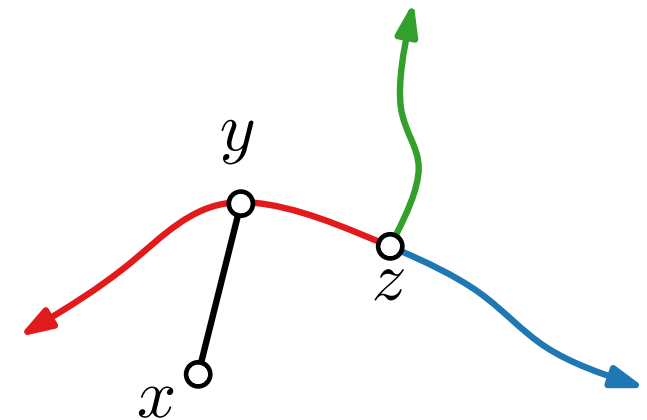
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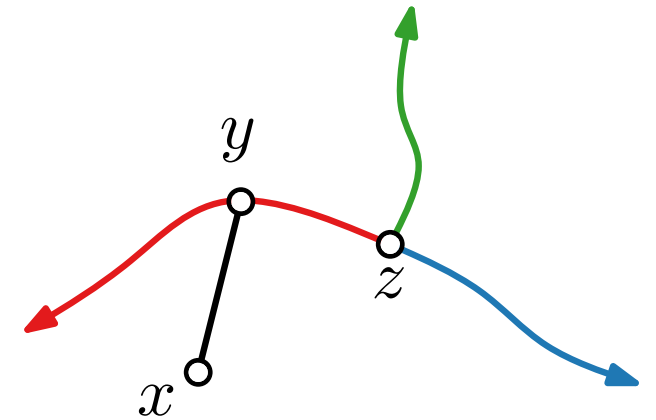
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Schnyder Drawing

Set $A = (0, 0)$, $B = (2n - 5, 0)$, and $C = (0, 2n - 5)$.

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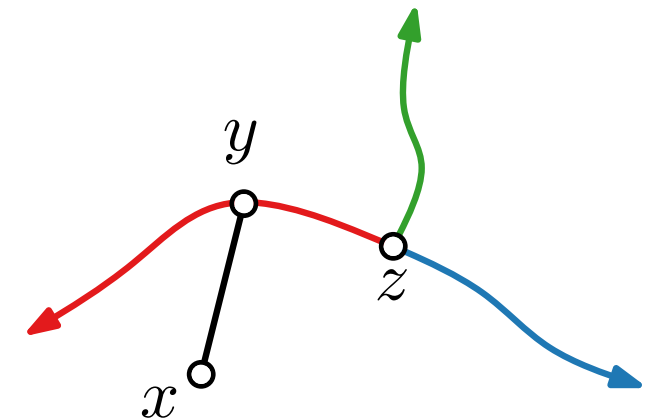
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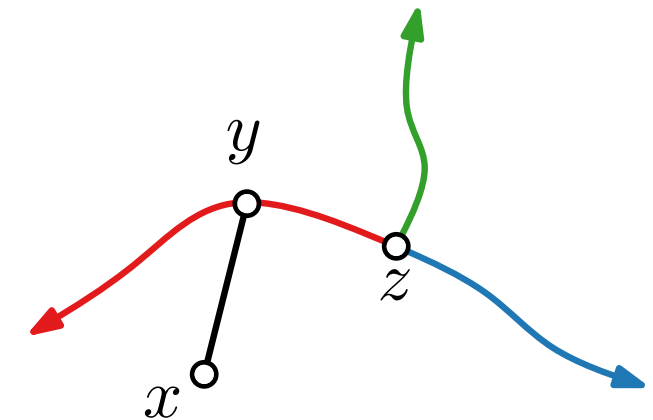
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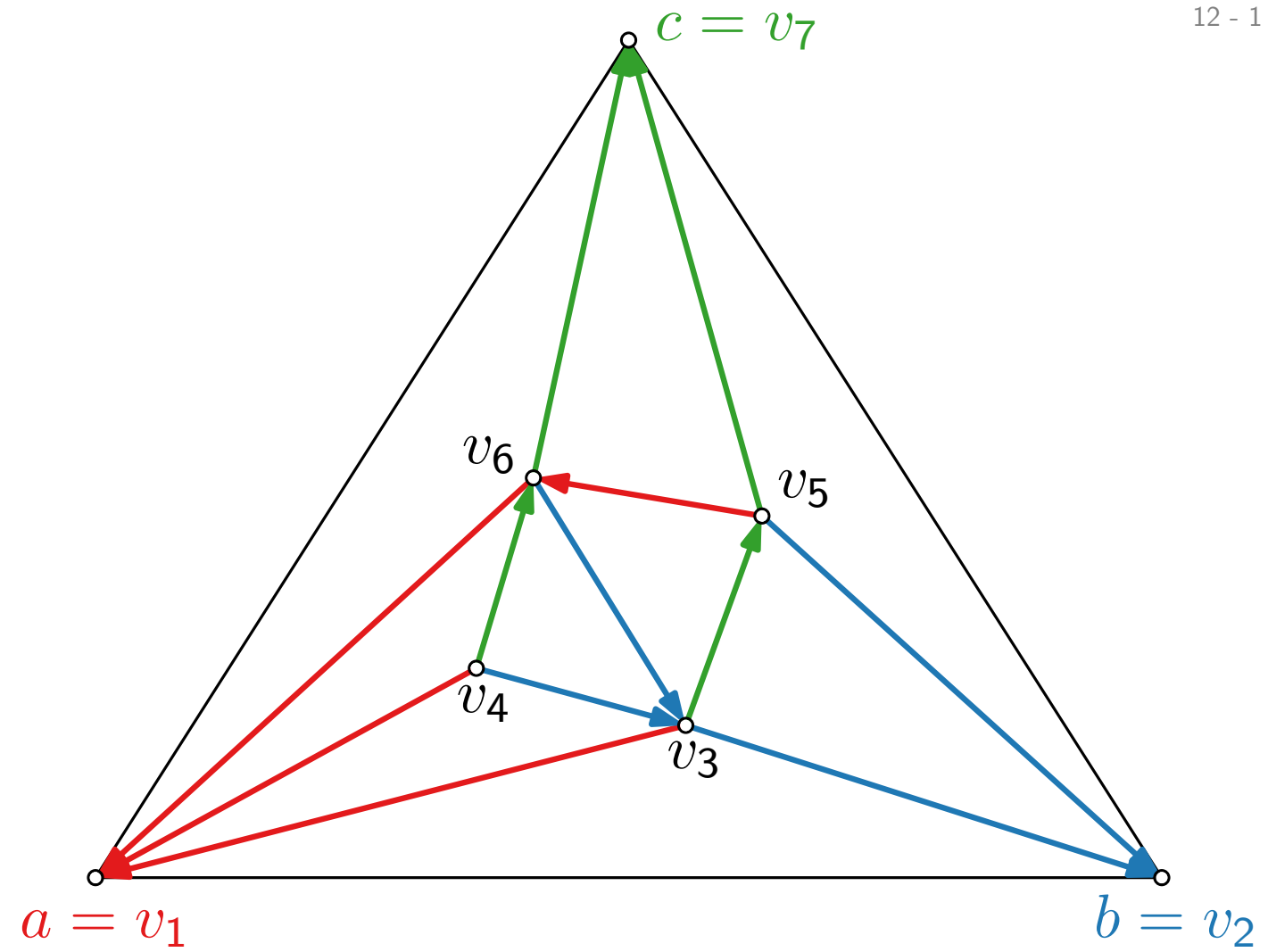
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is a barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(2n - 5) \times (2n - 5)$ grid.

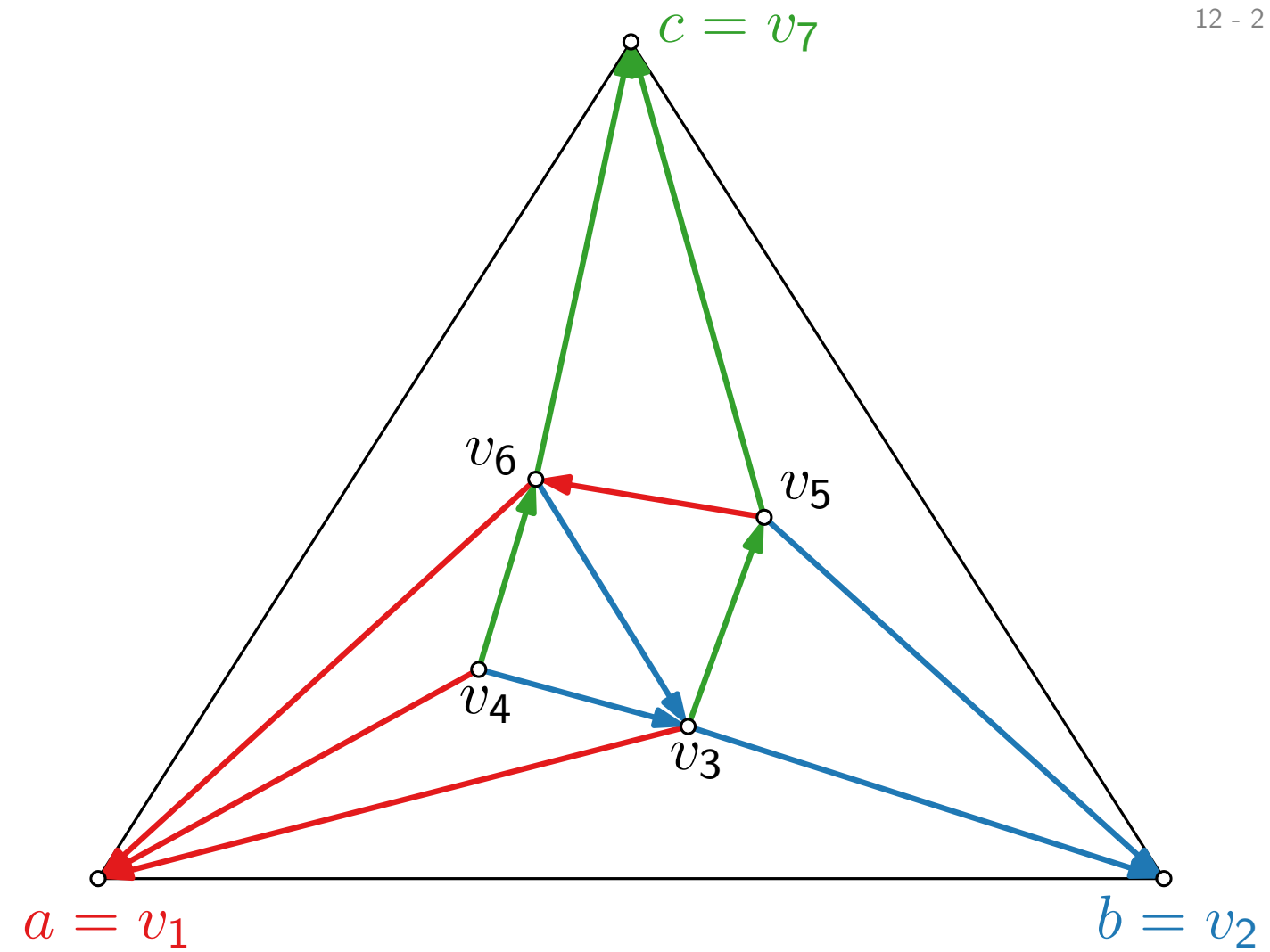
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Schnyder Drawing – Example

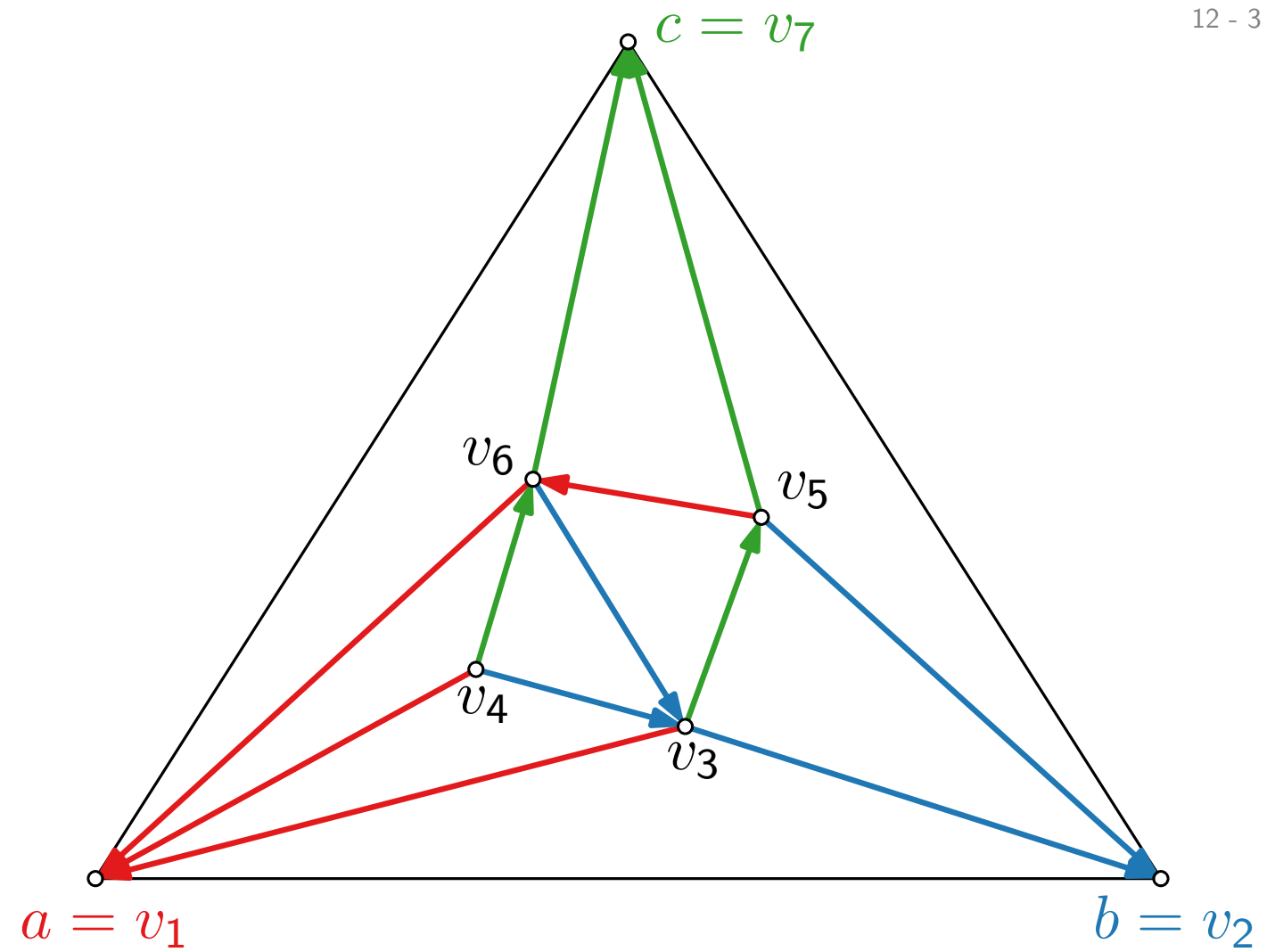


Schnyder Drawing – Example



$$n = 7; \quad 2n - 5 = 9$$

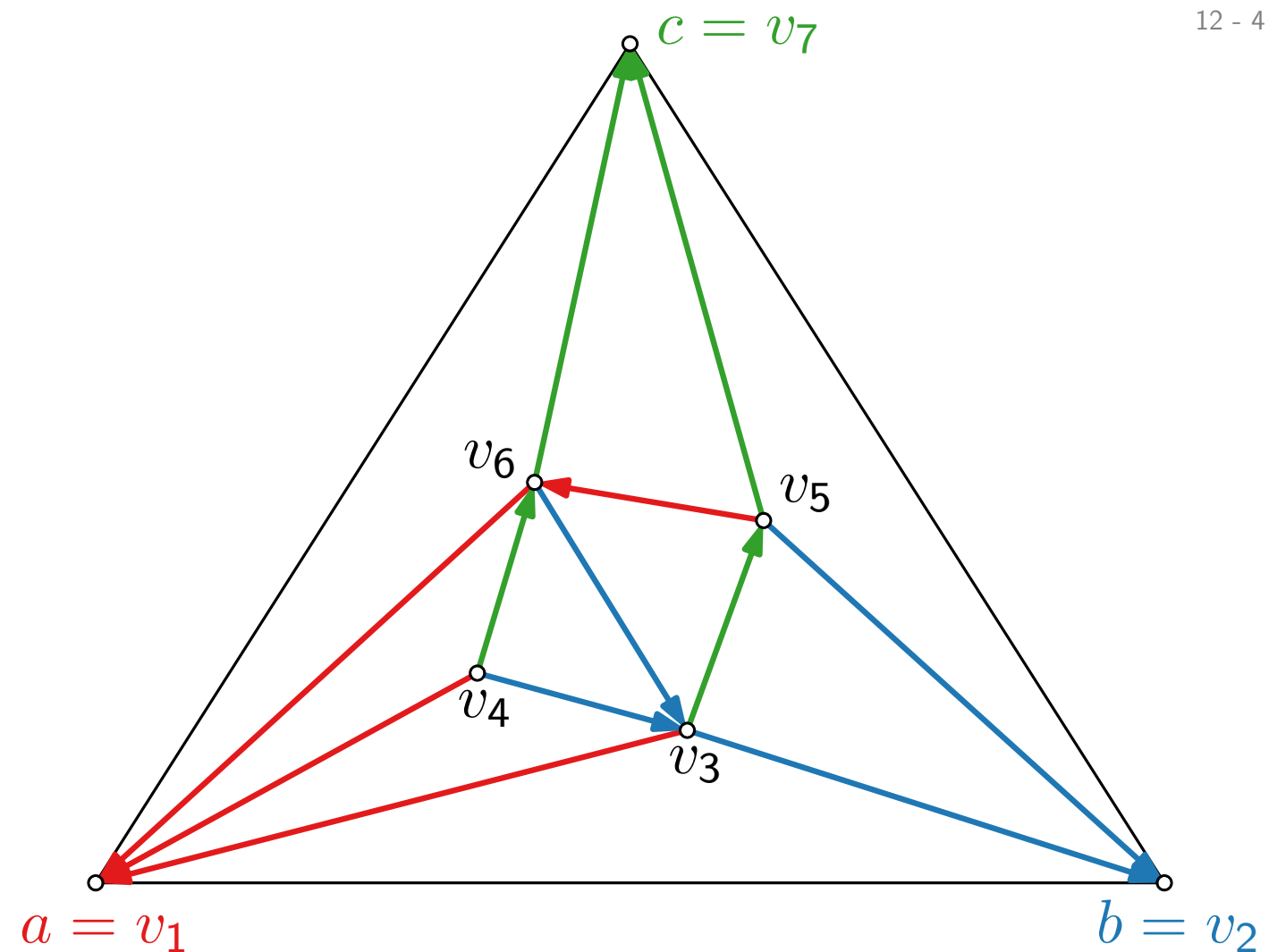
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$$n = 7; \quad 2n - 5 = 9$$

$$f(v_1) = (9, 0, 0)$$

Schnyder Drawing – Example

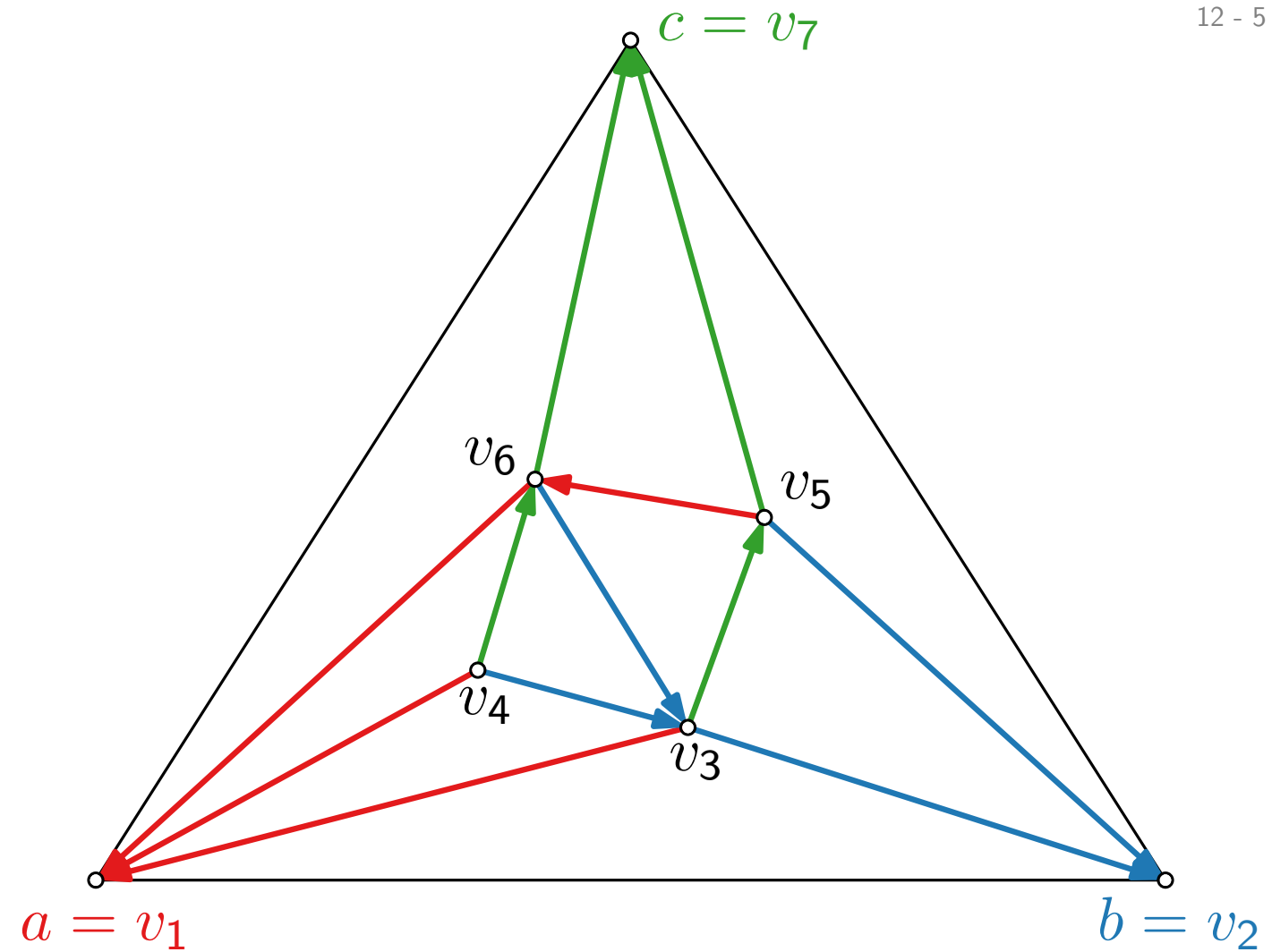


$$n = 7; \quad 2n - 5 = 9$$

$$f(v_1) = (9, 0, 0)$$

$$f(v_2) = (0, 9, 0)$$

Schnyder Drawing – Example



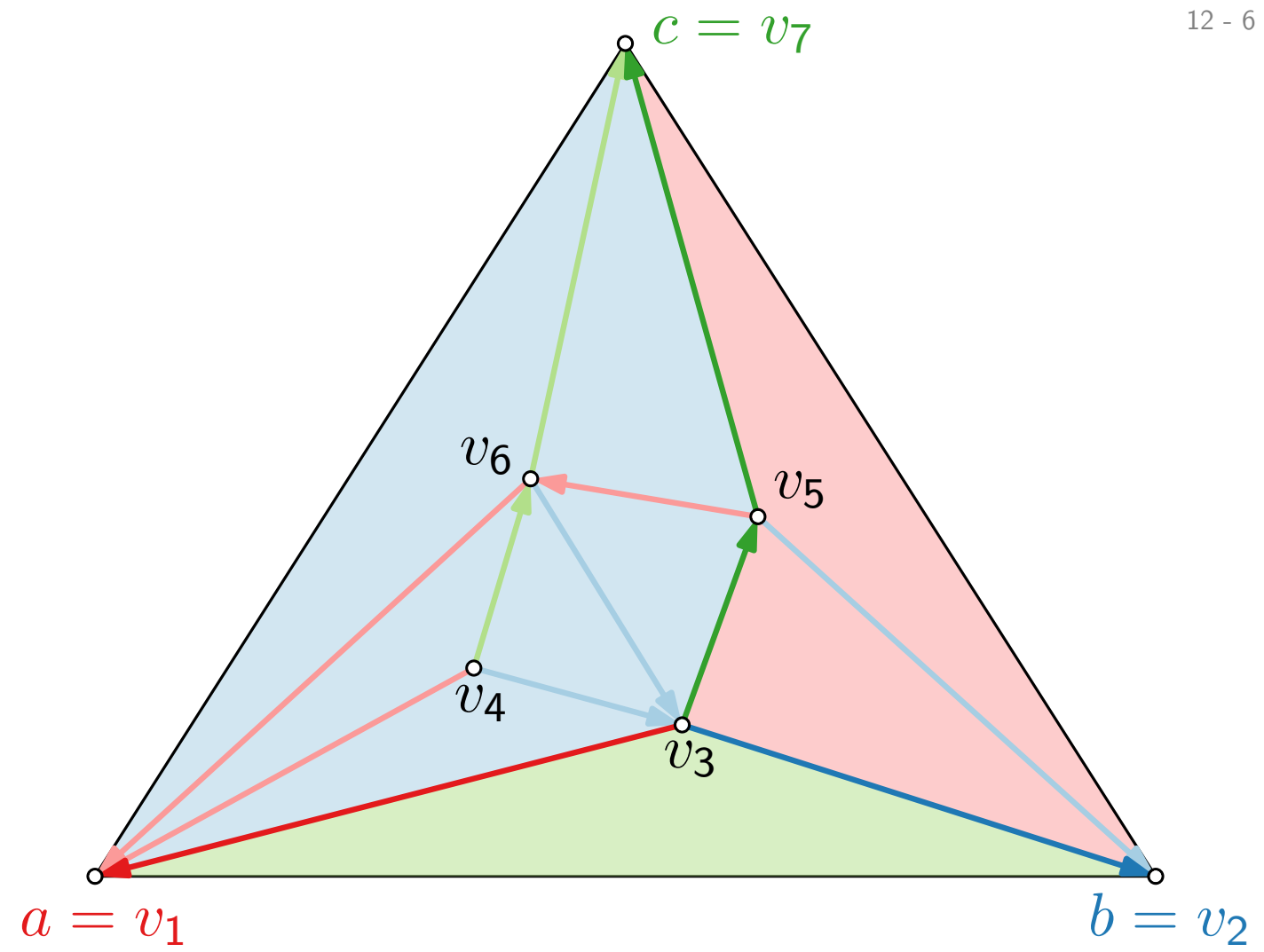
$$n = 7; \quad 2n - 5 = 9 \quad f(v_4) =$$

$$f(v_1) = (9, 0, 0) \quad f(v_5) =$$

$$f(v_2) = (0, 9, 0) \quad f(v_6) =$$

$$f(v_3) = \quad f(v_7) = (0, 0, 9)$$

Schnyder Drawing – Example



$a = v_1$

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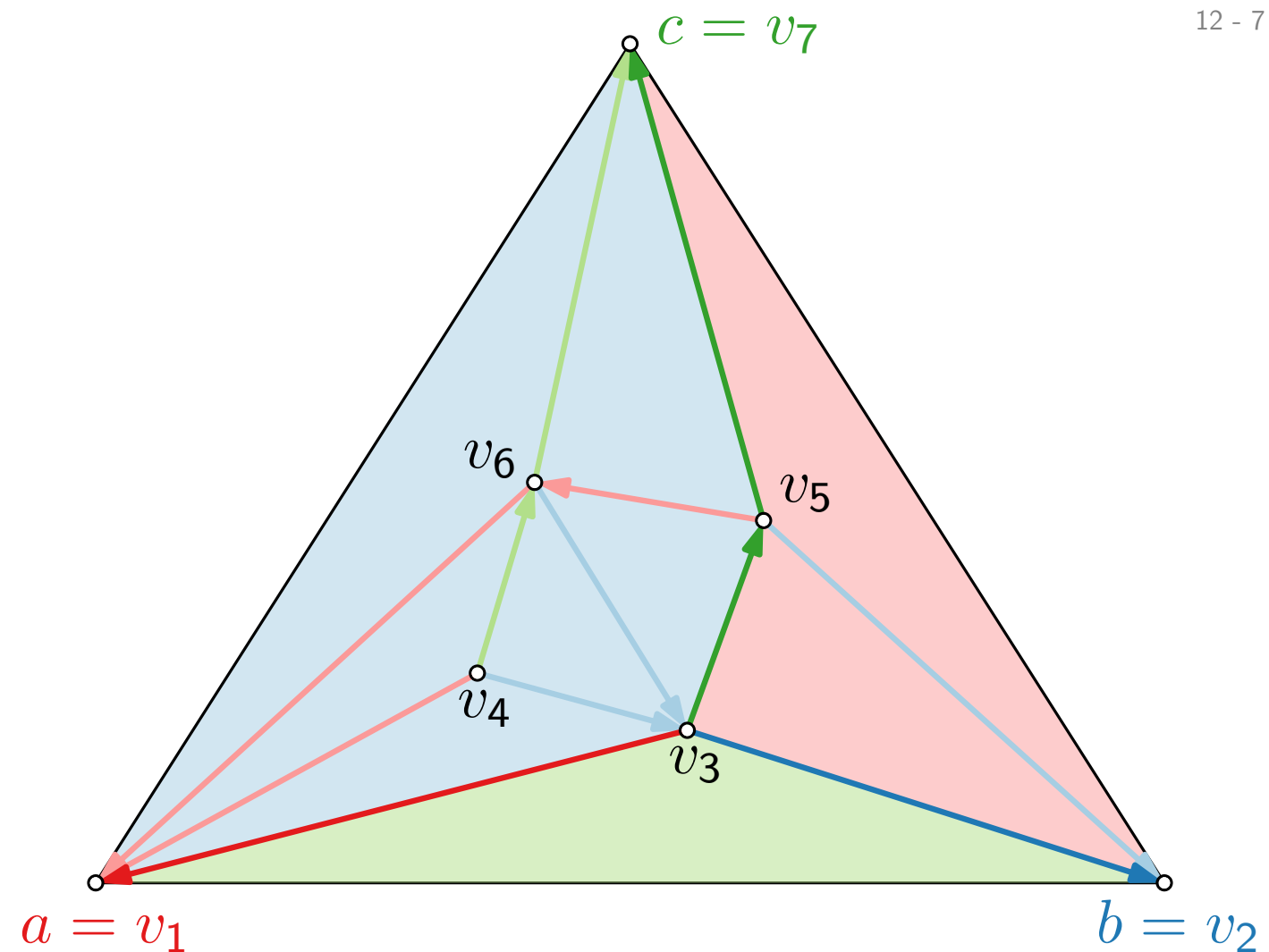
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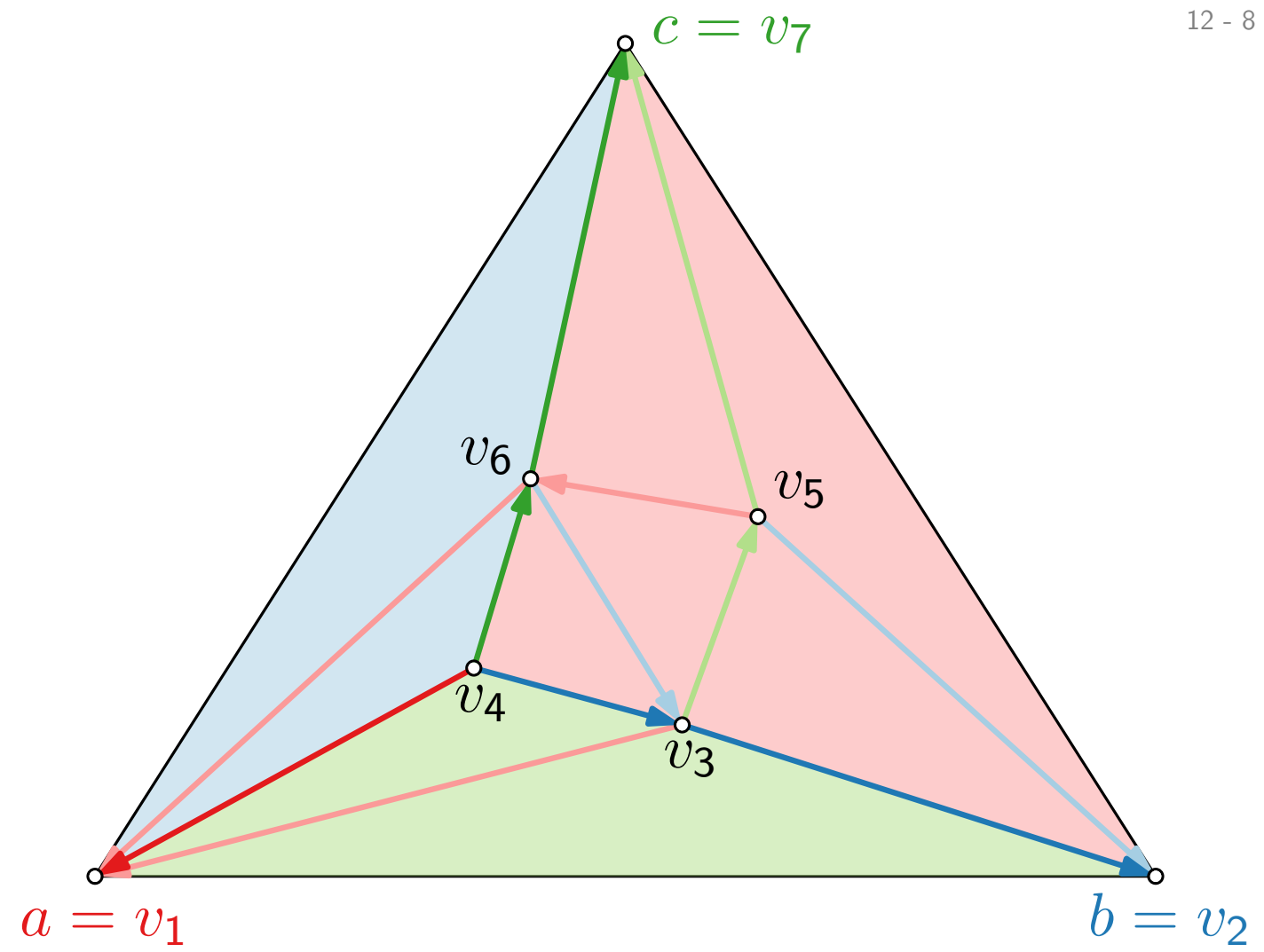
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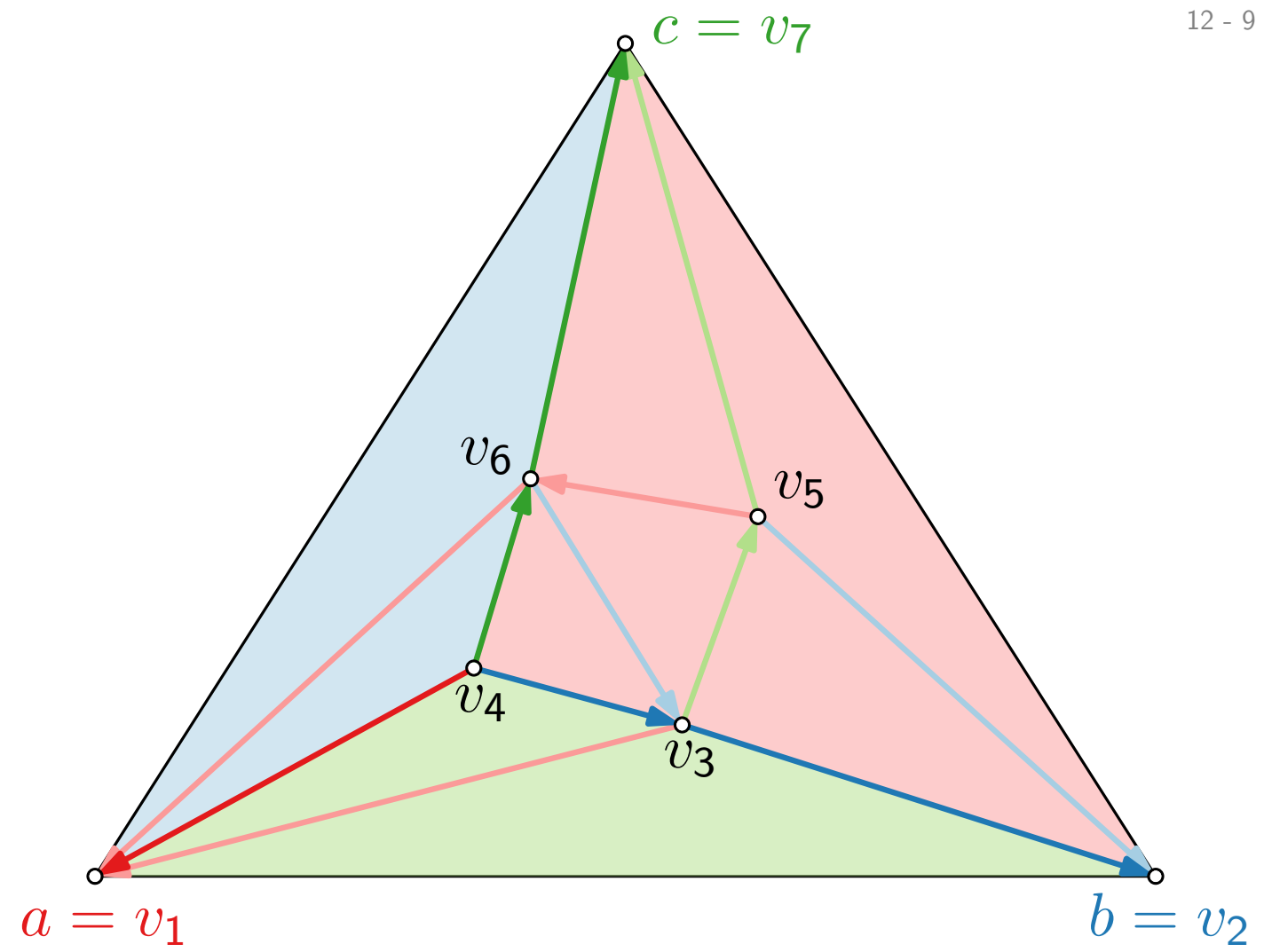
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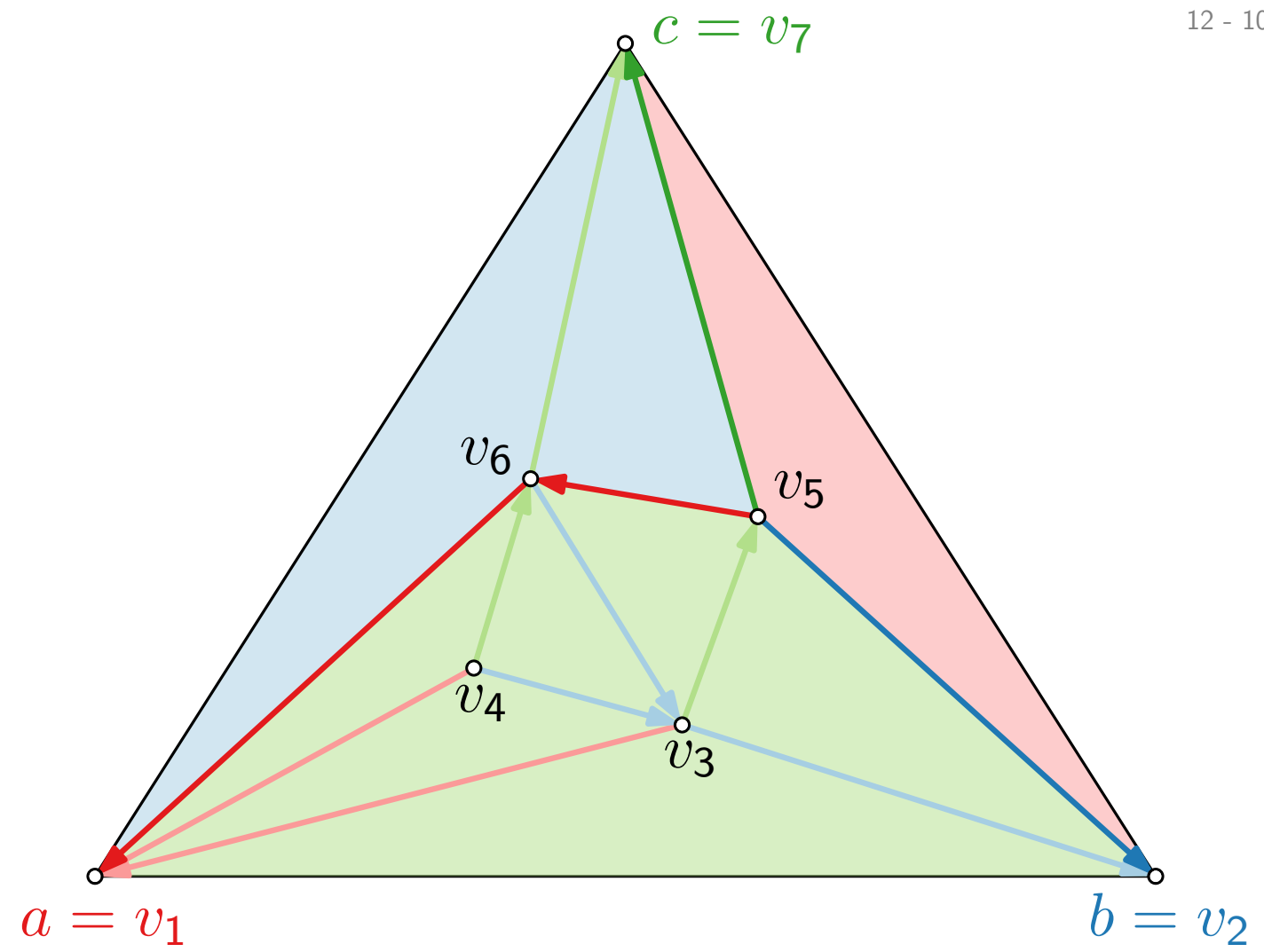
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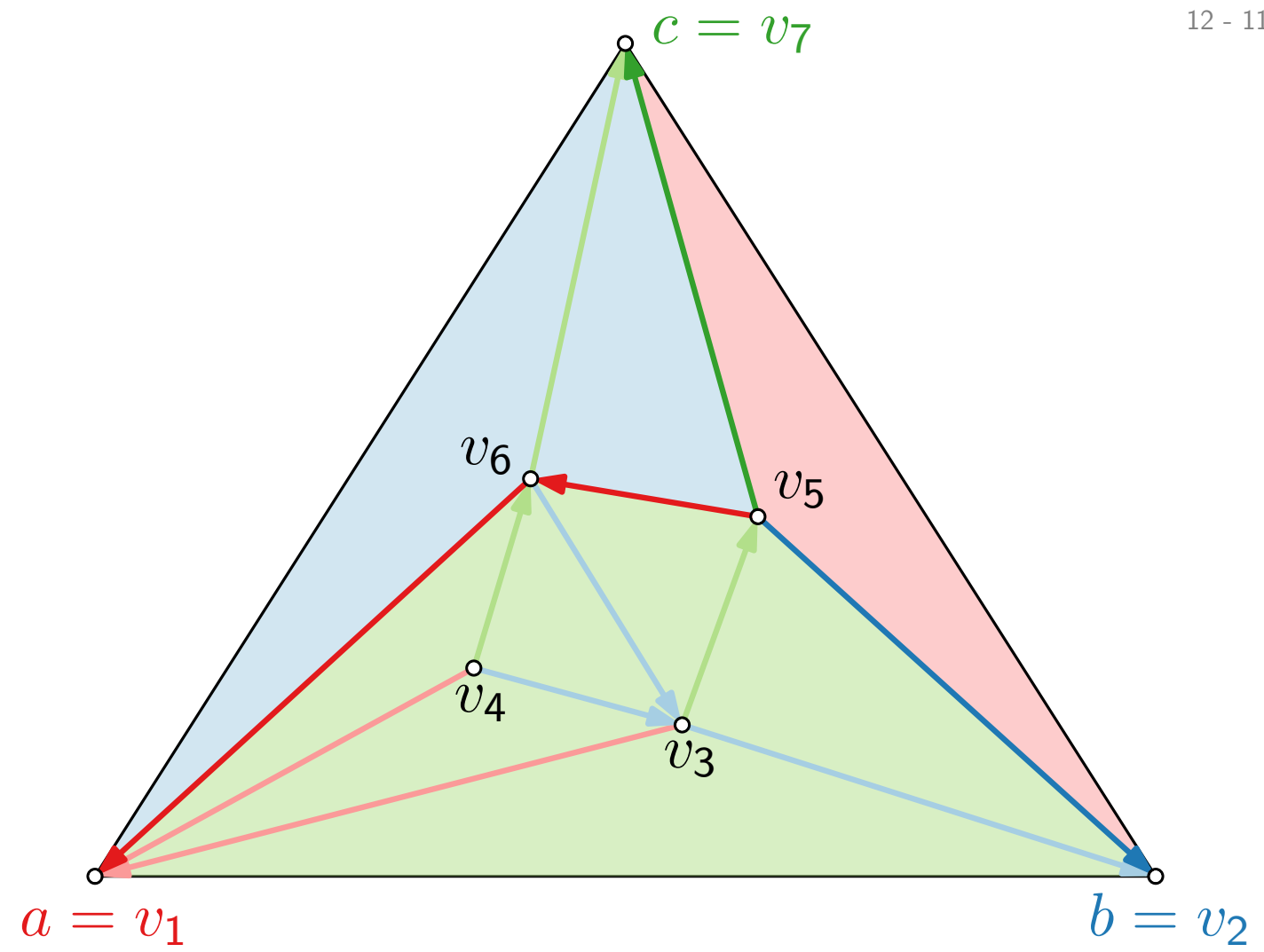
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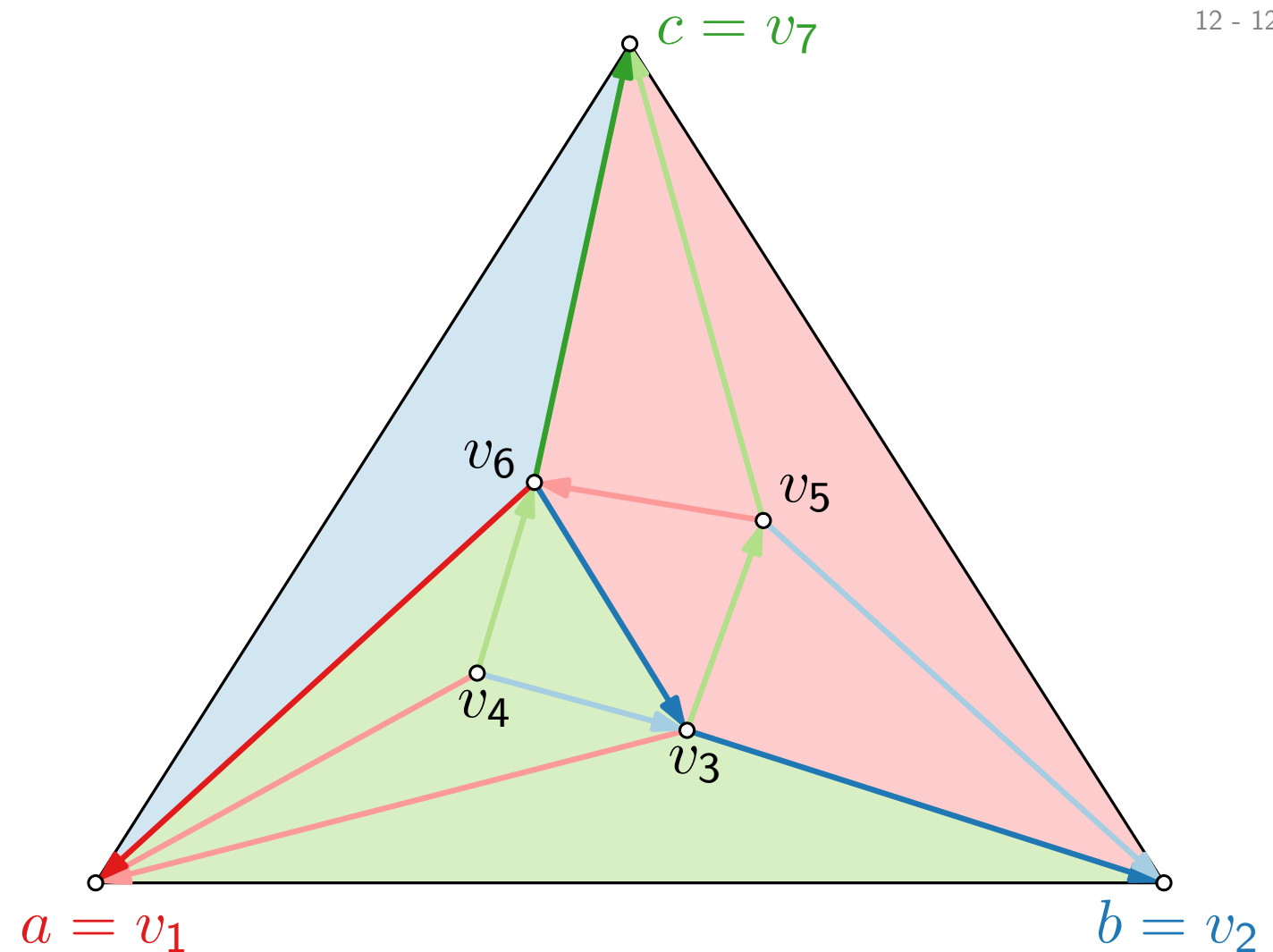
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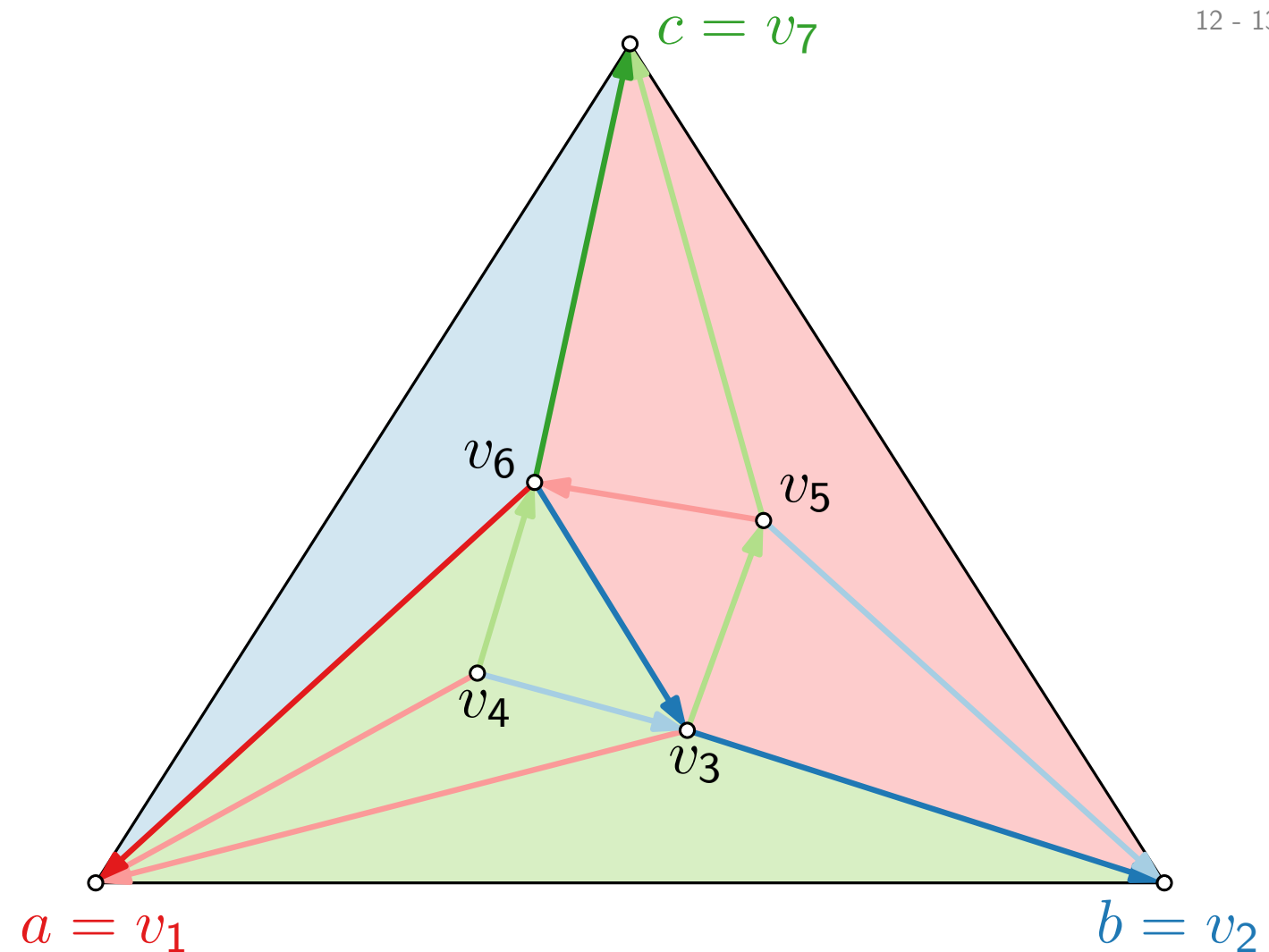
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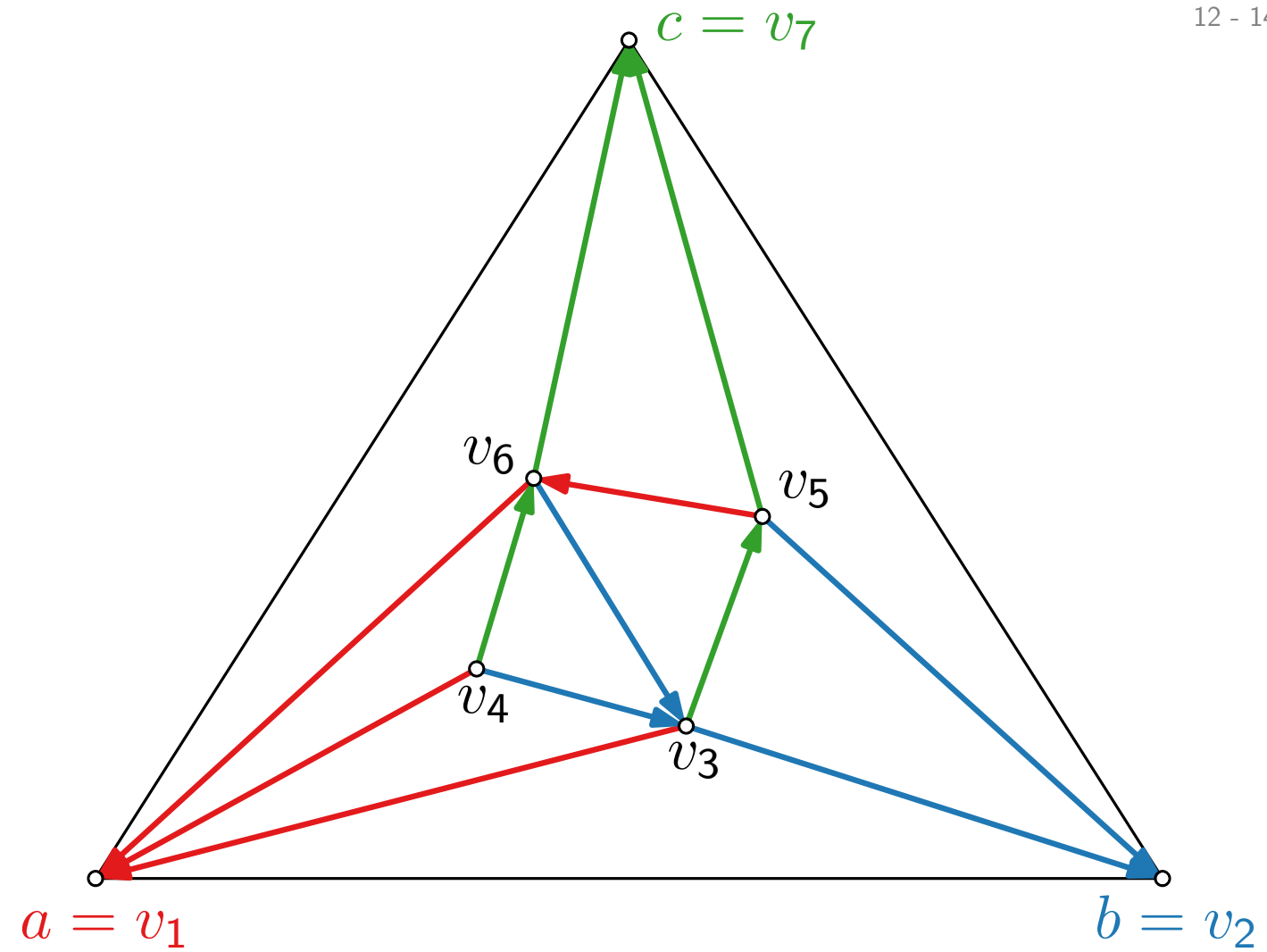
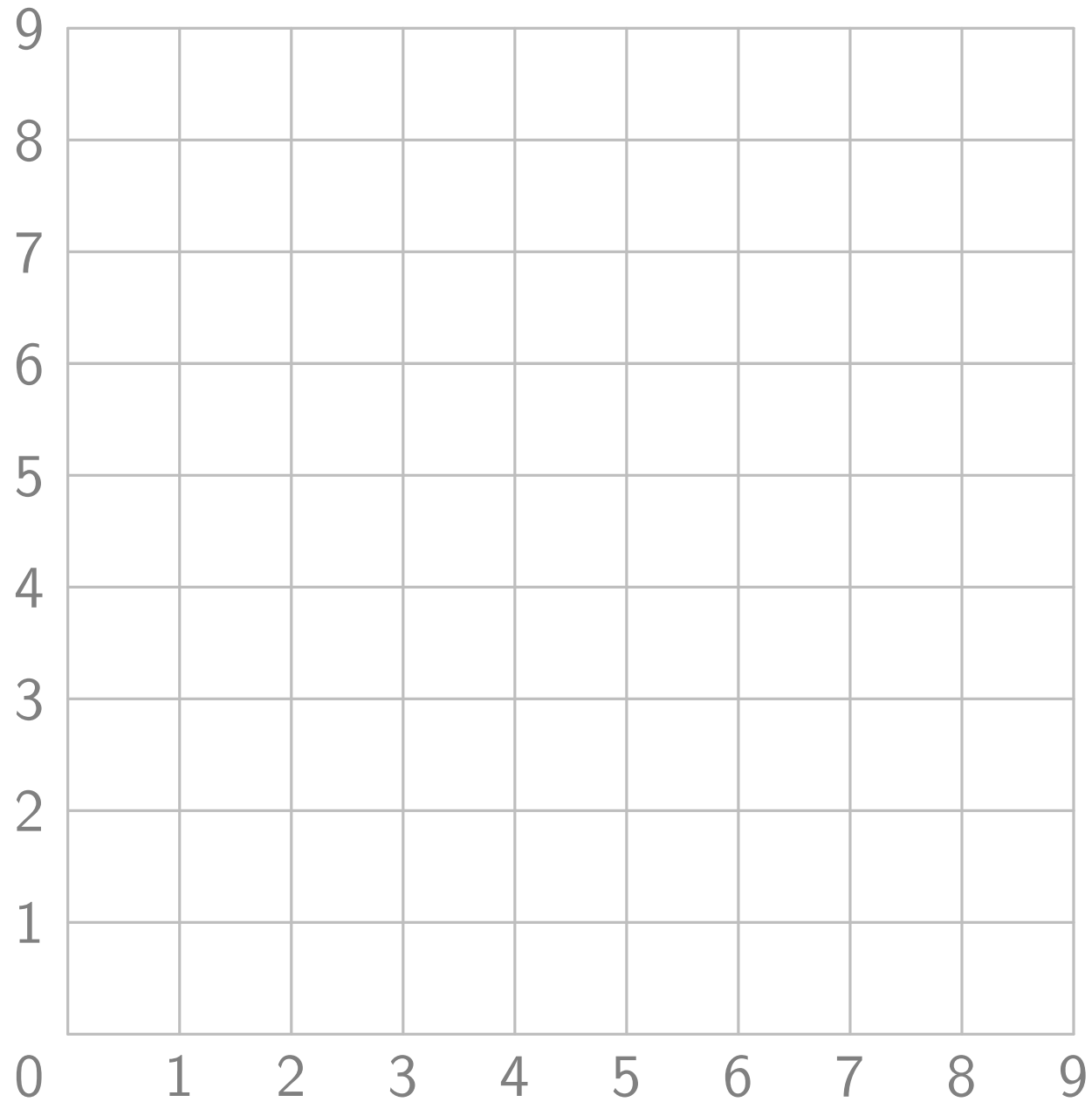
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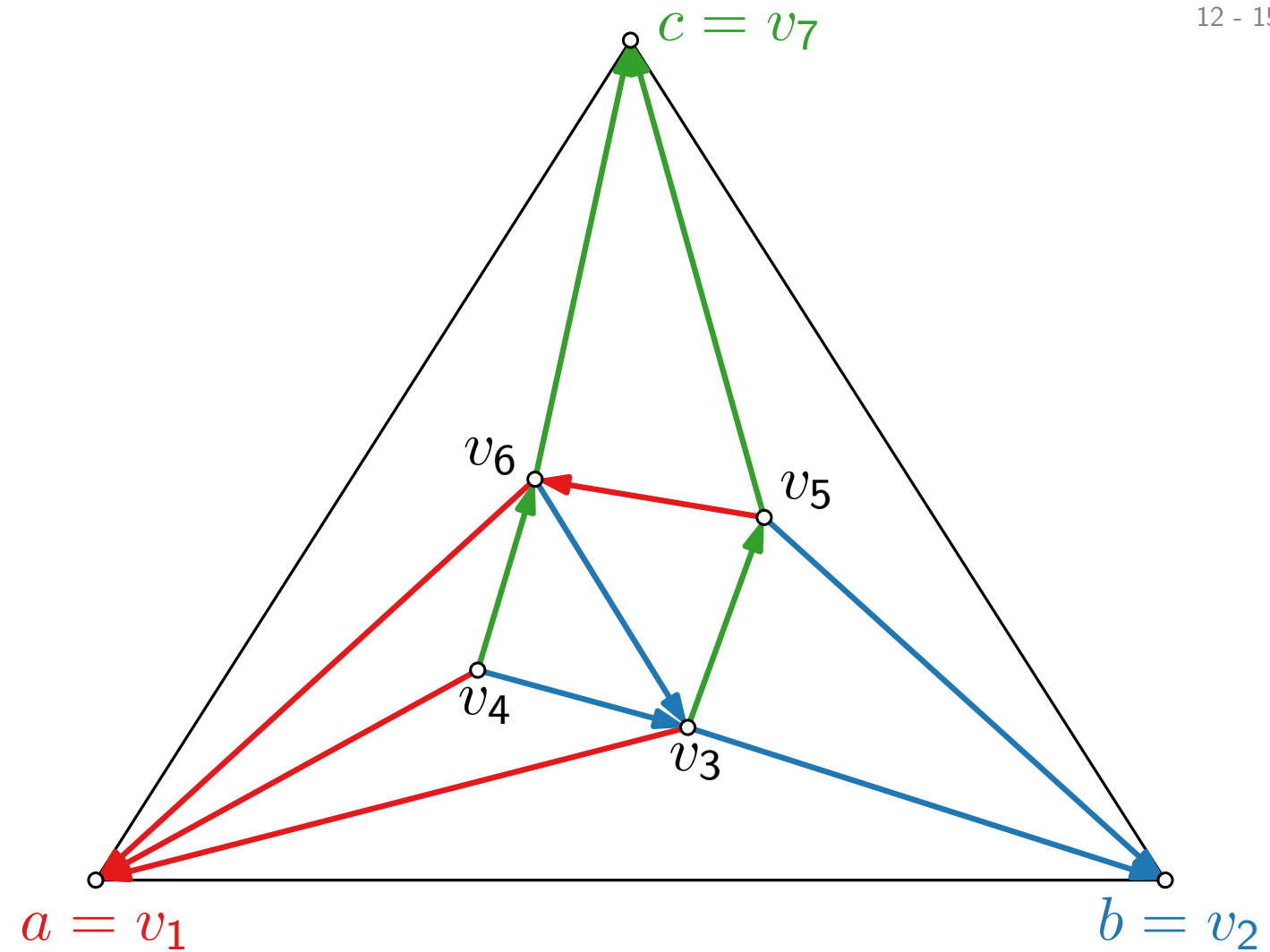
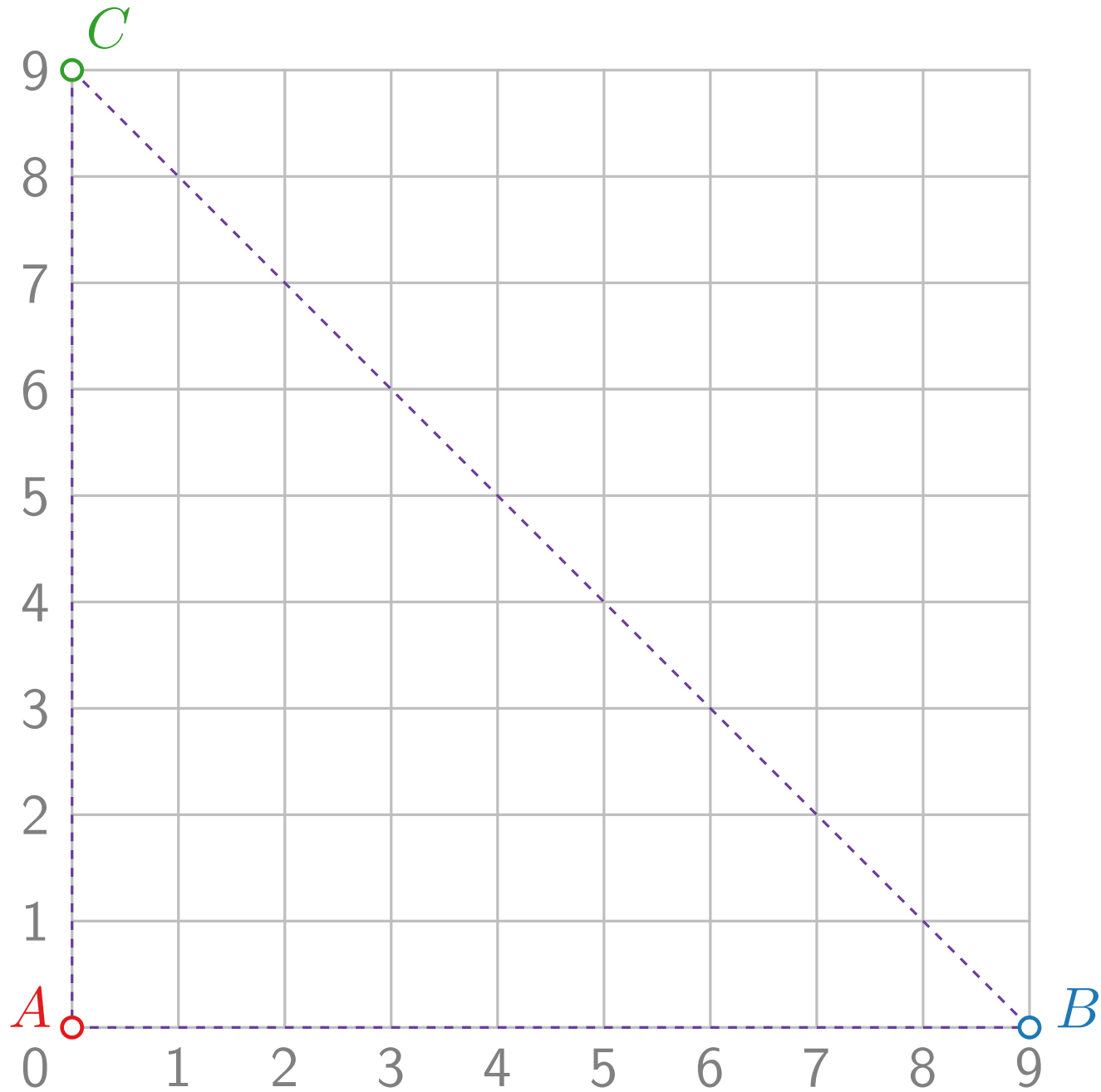
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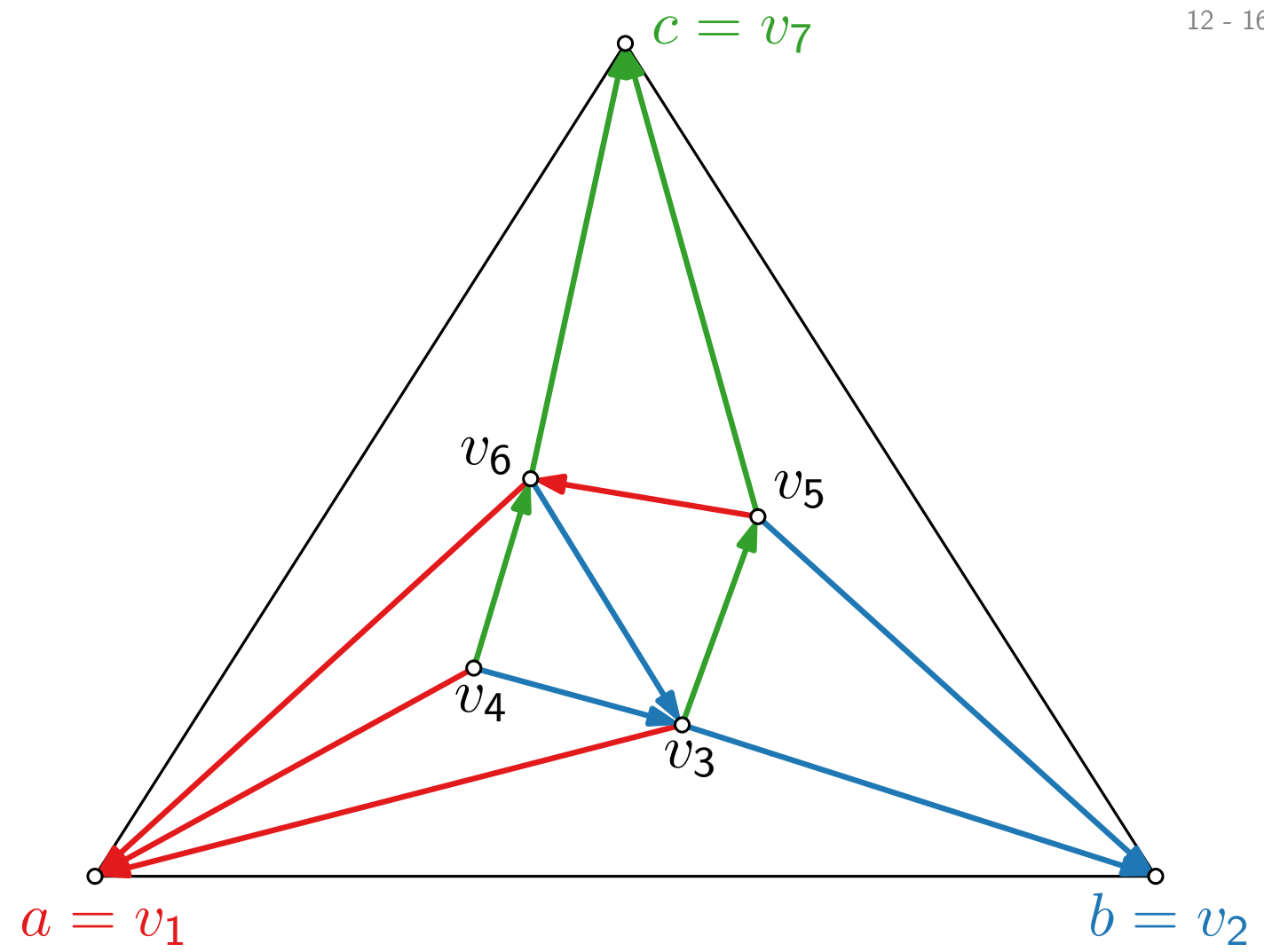
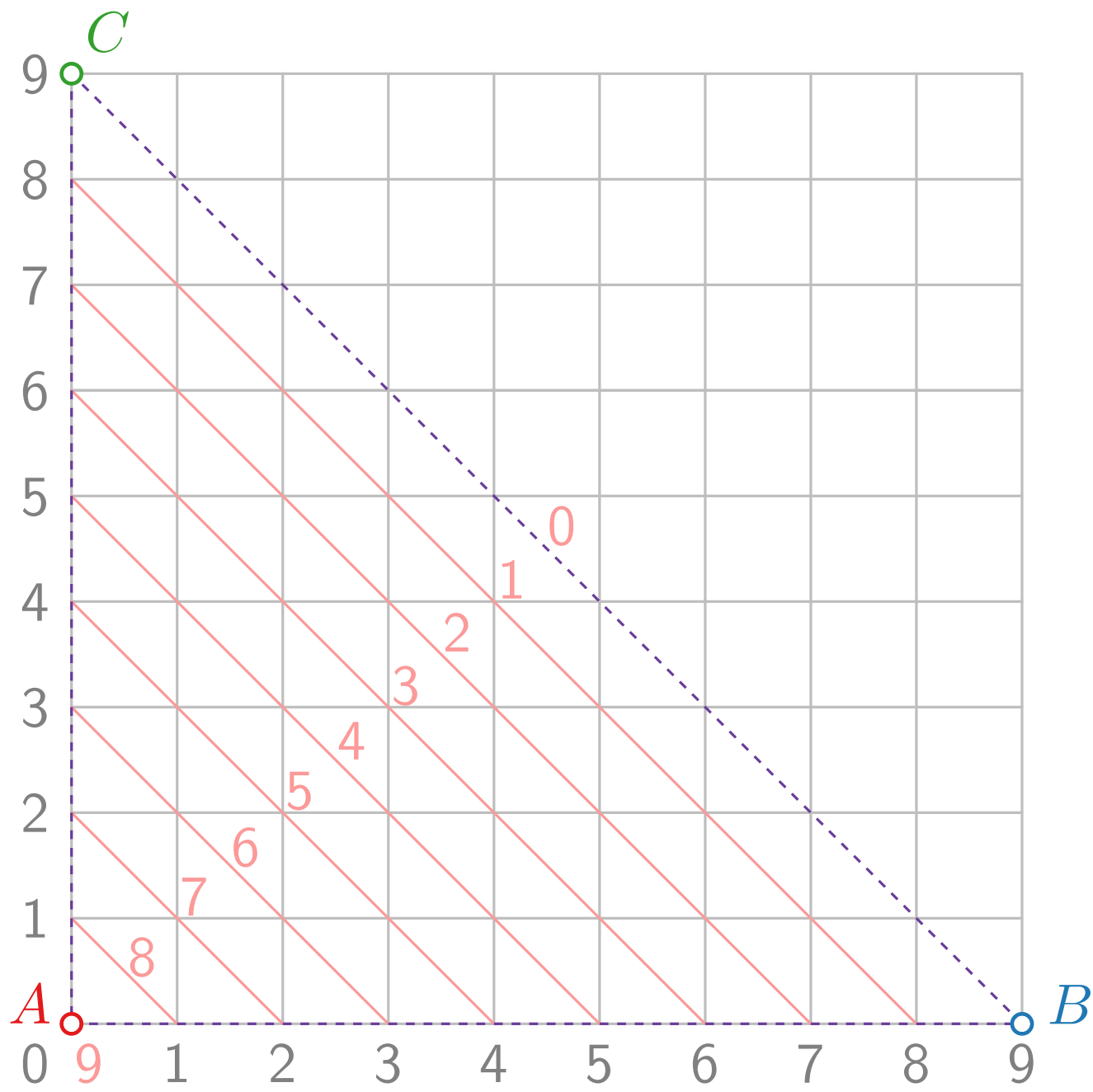
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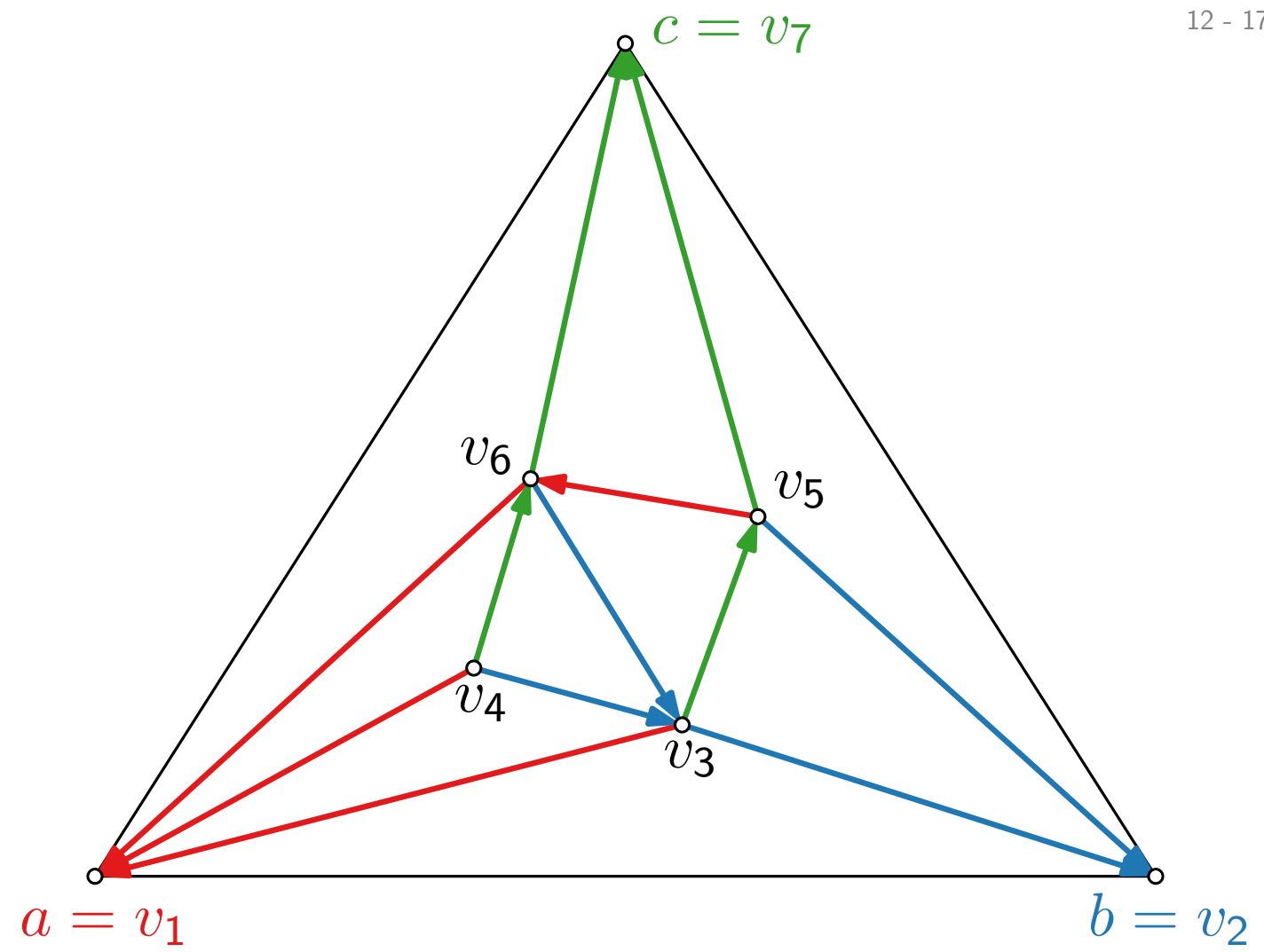
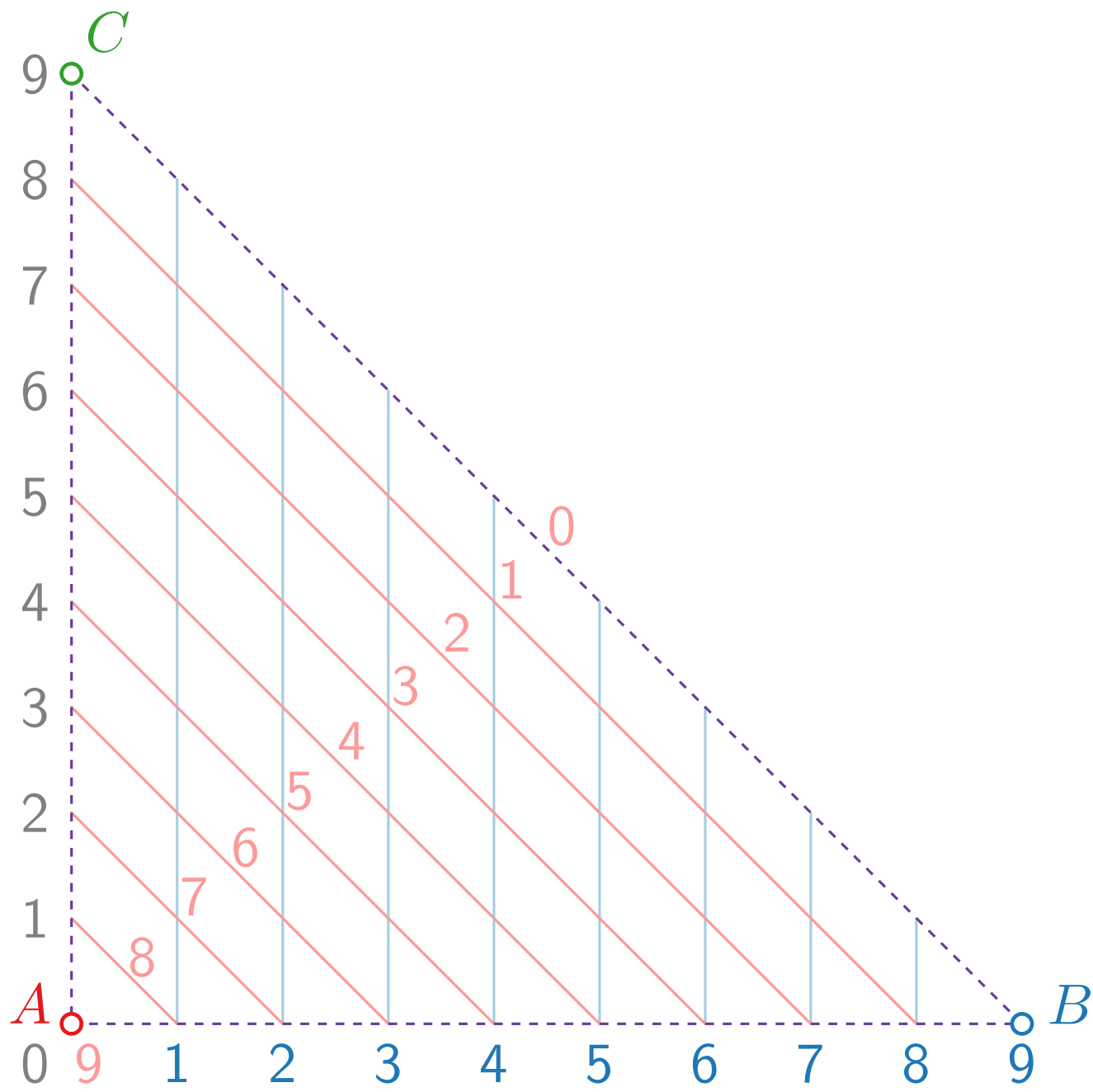
$$f(v_3) = (2, 6, 1) \quad f(v_7) = (0, 0, 9)$$

Schnyder Drawing – Example



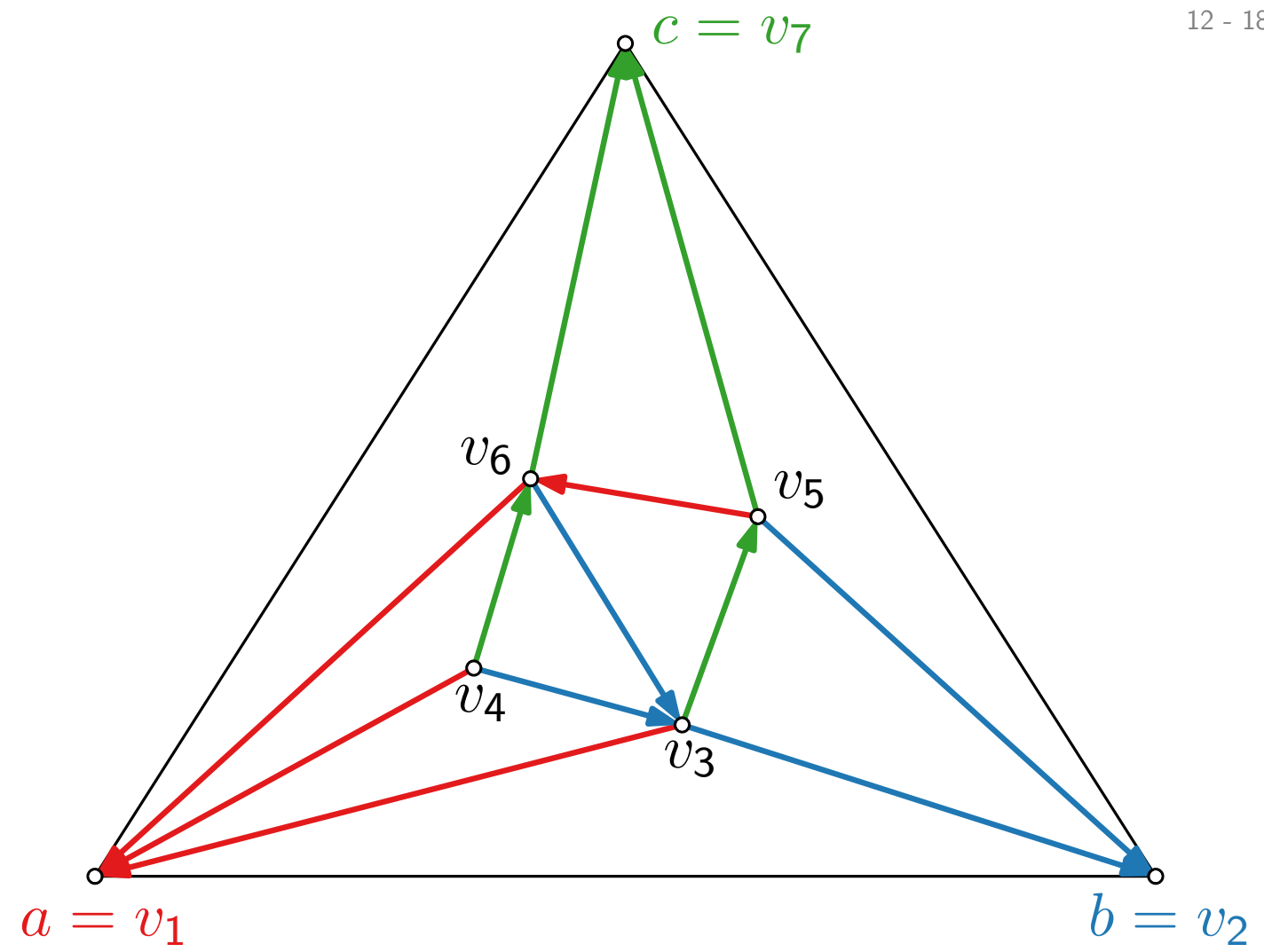
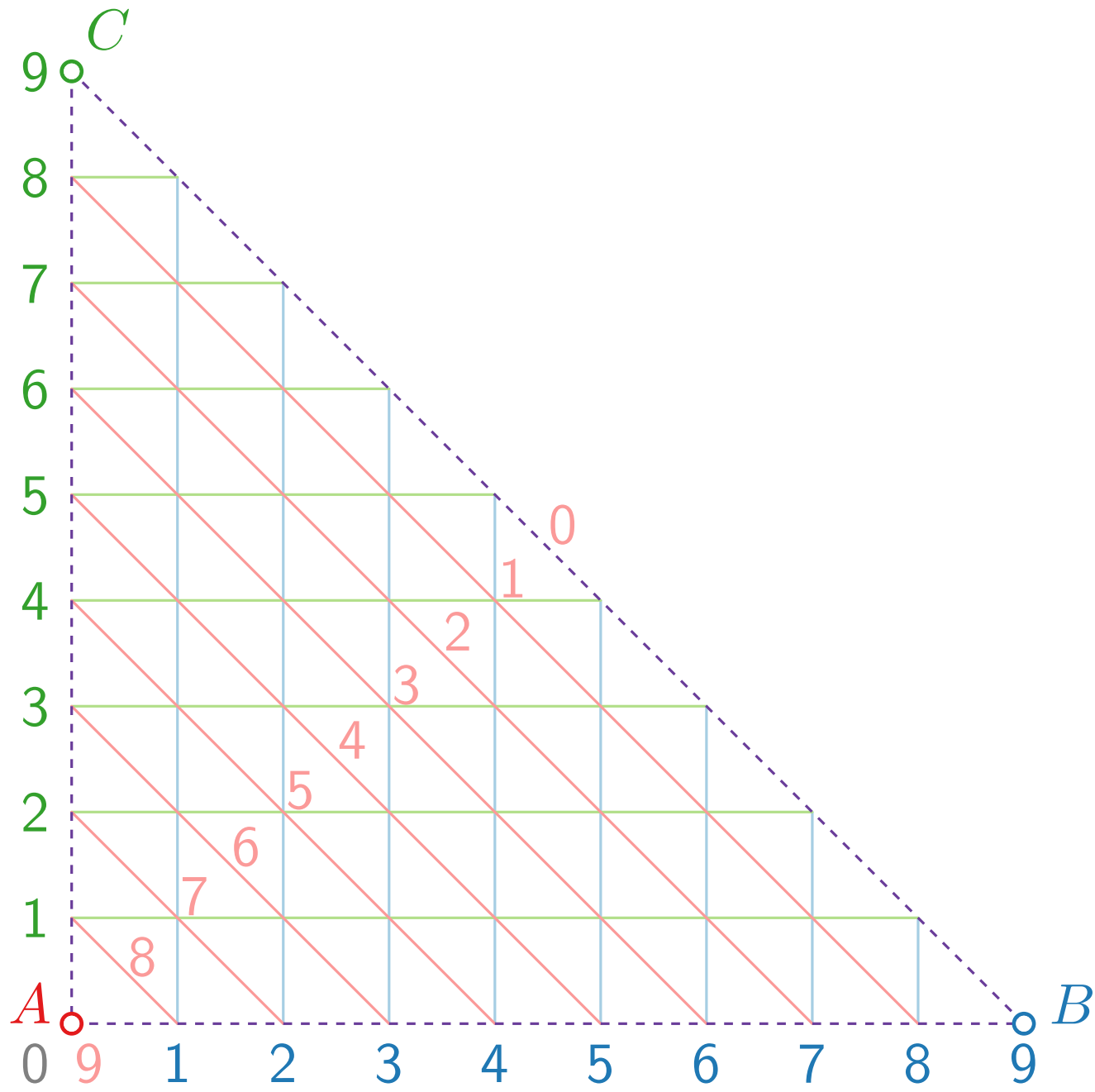
- $n = 7; \quad 2n - 5 = 9$
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- $f(v_5) = (1, 2, 6)$
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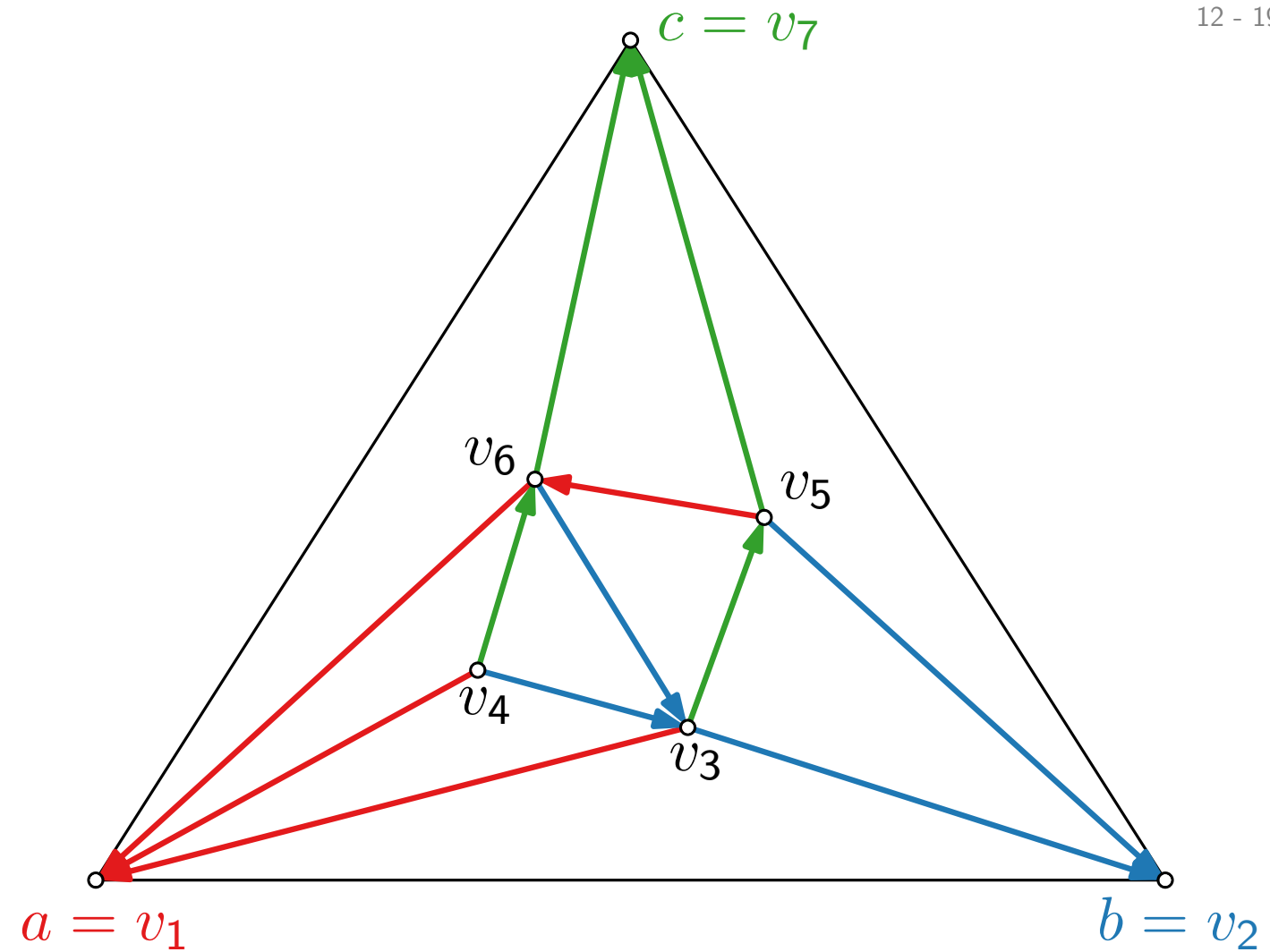
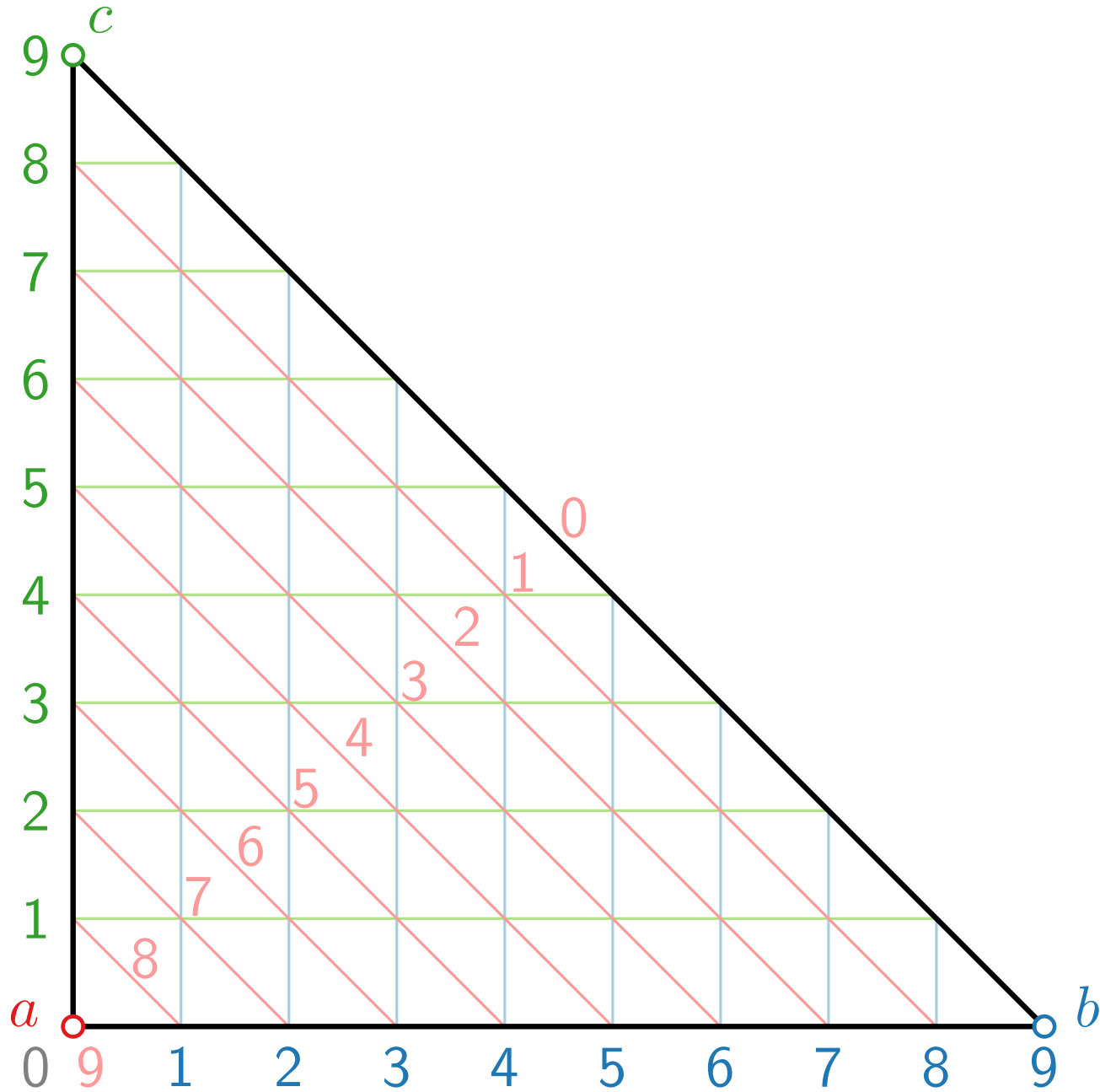
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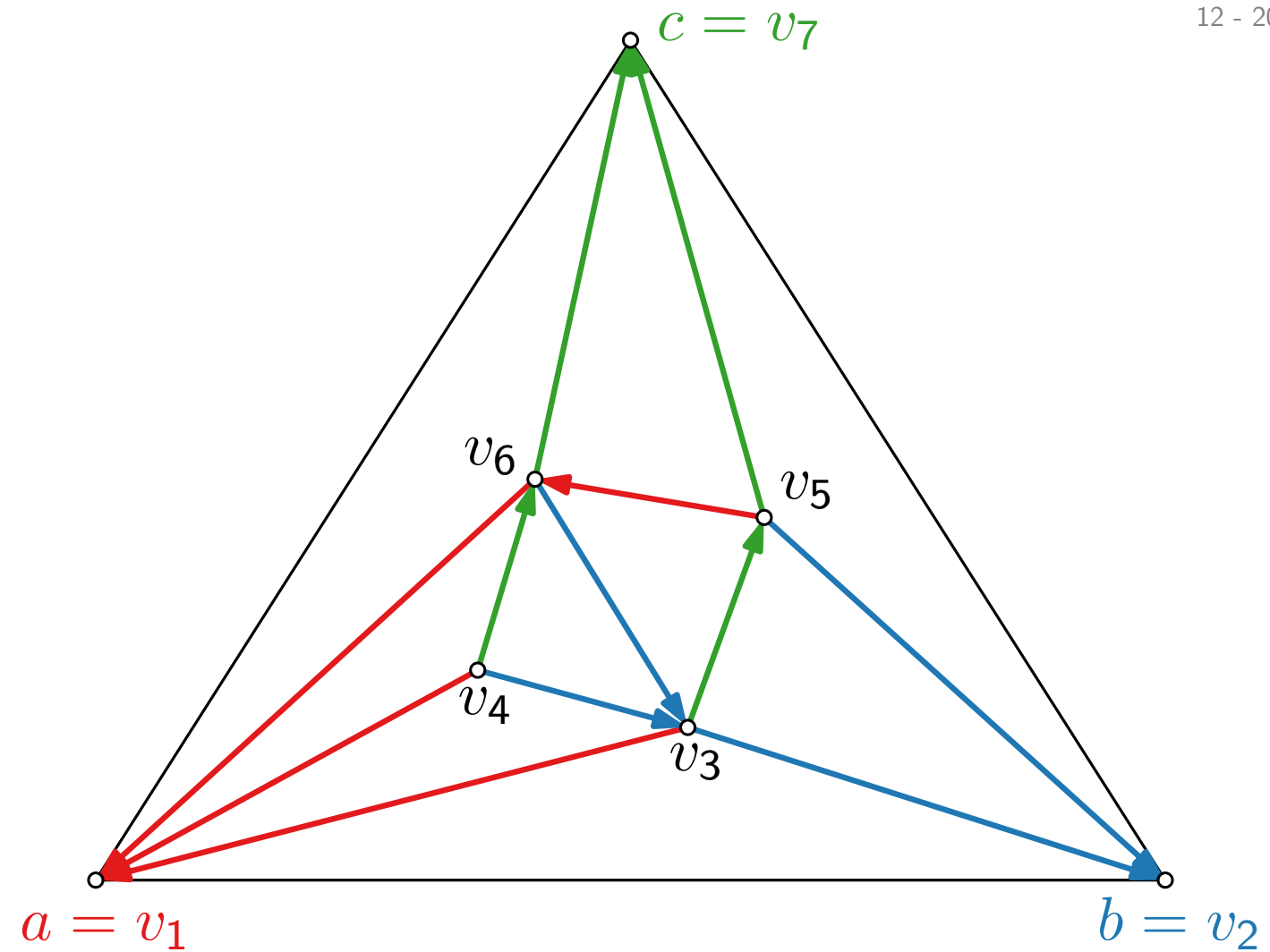
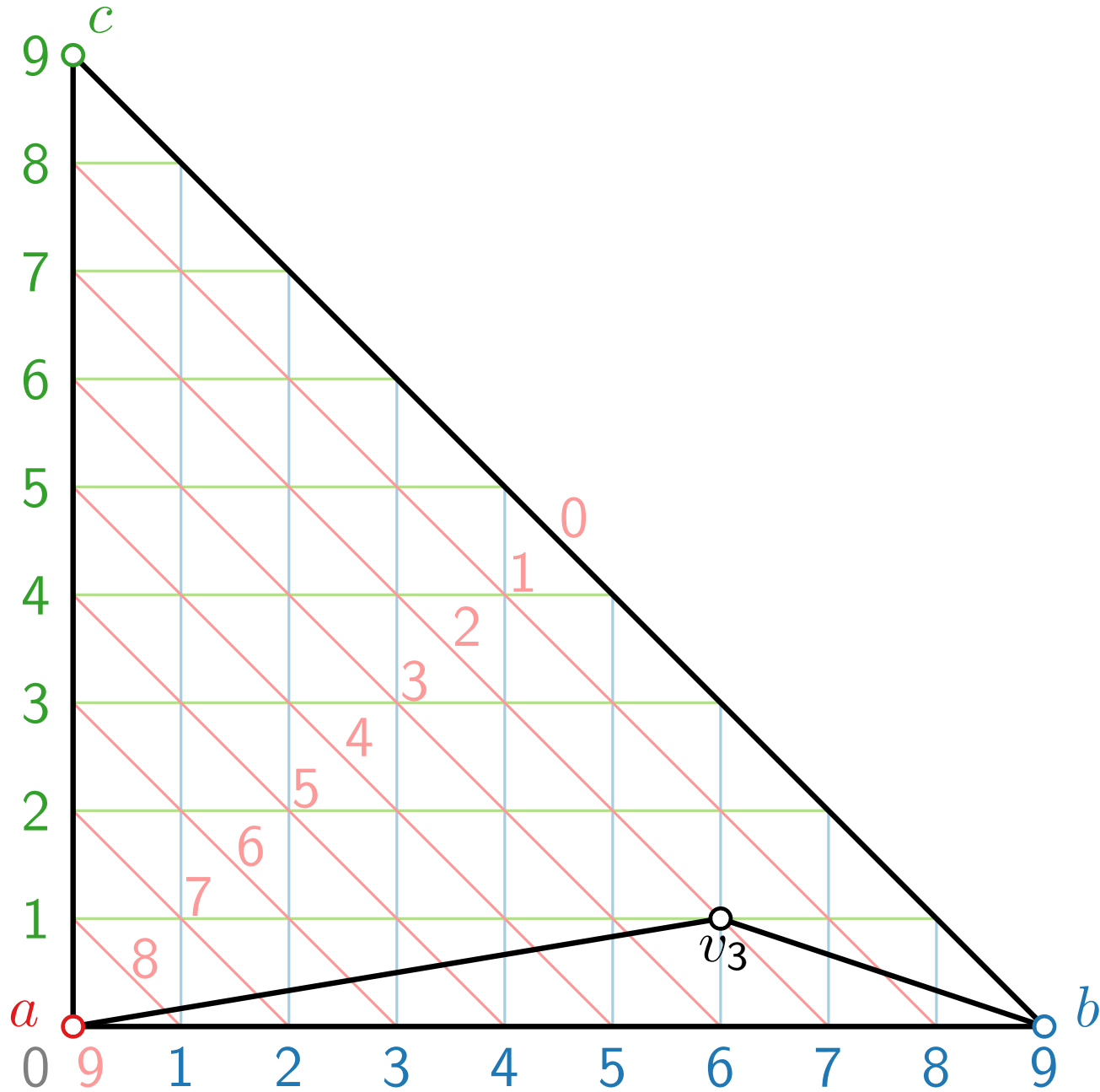
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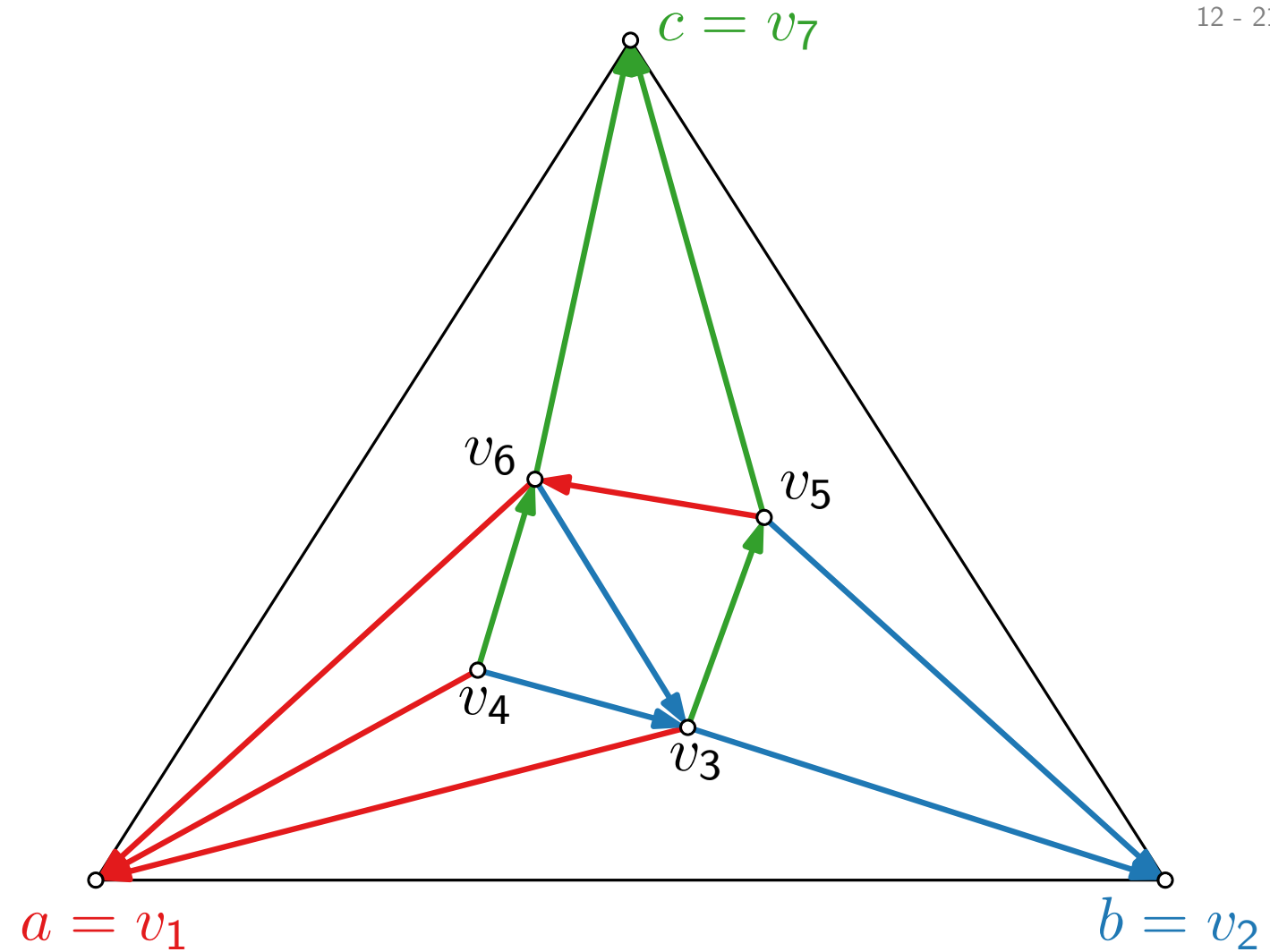
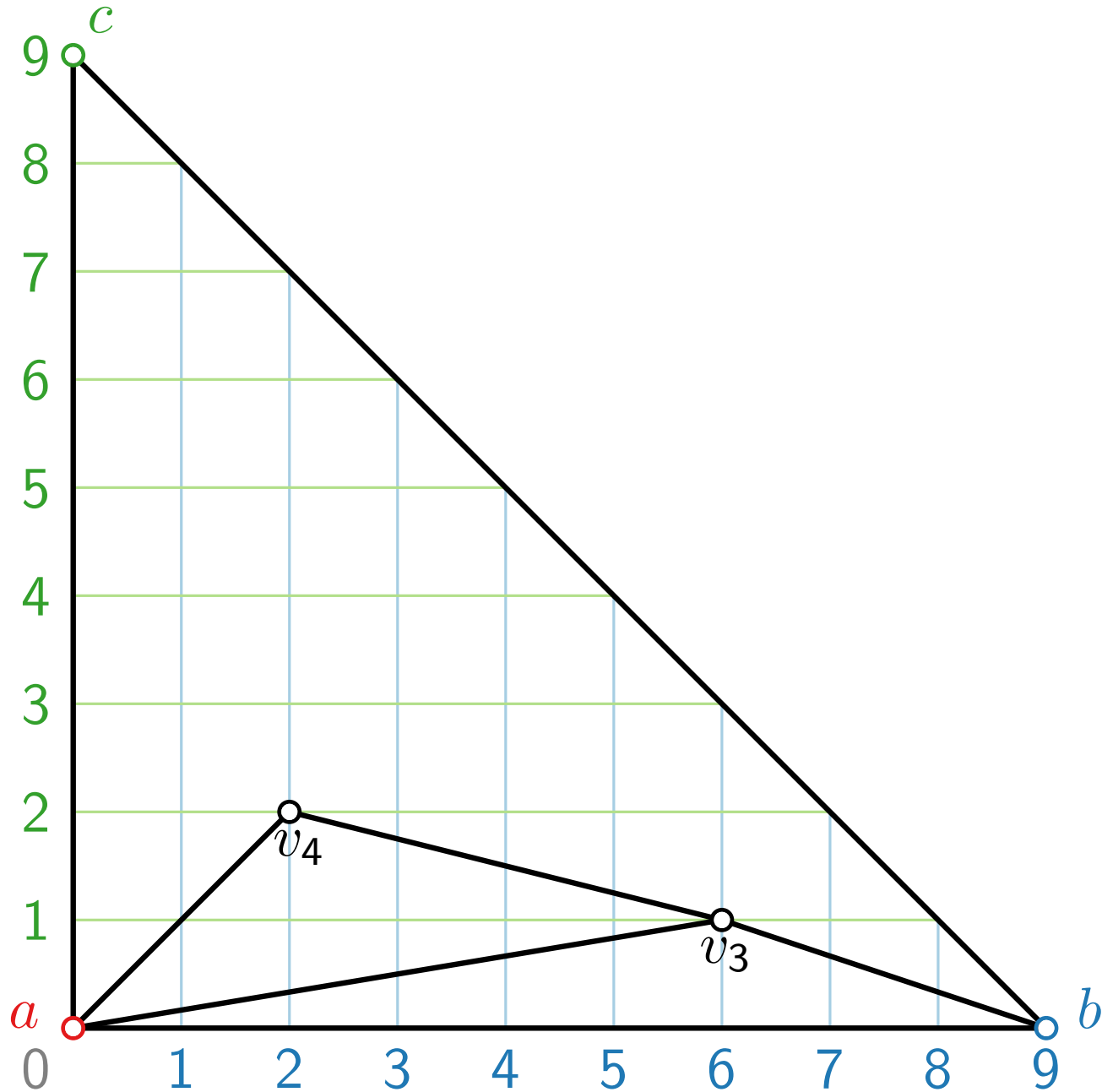
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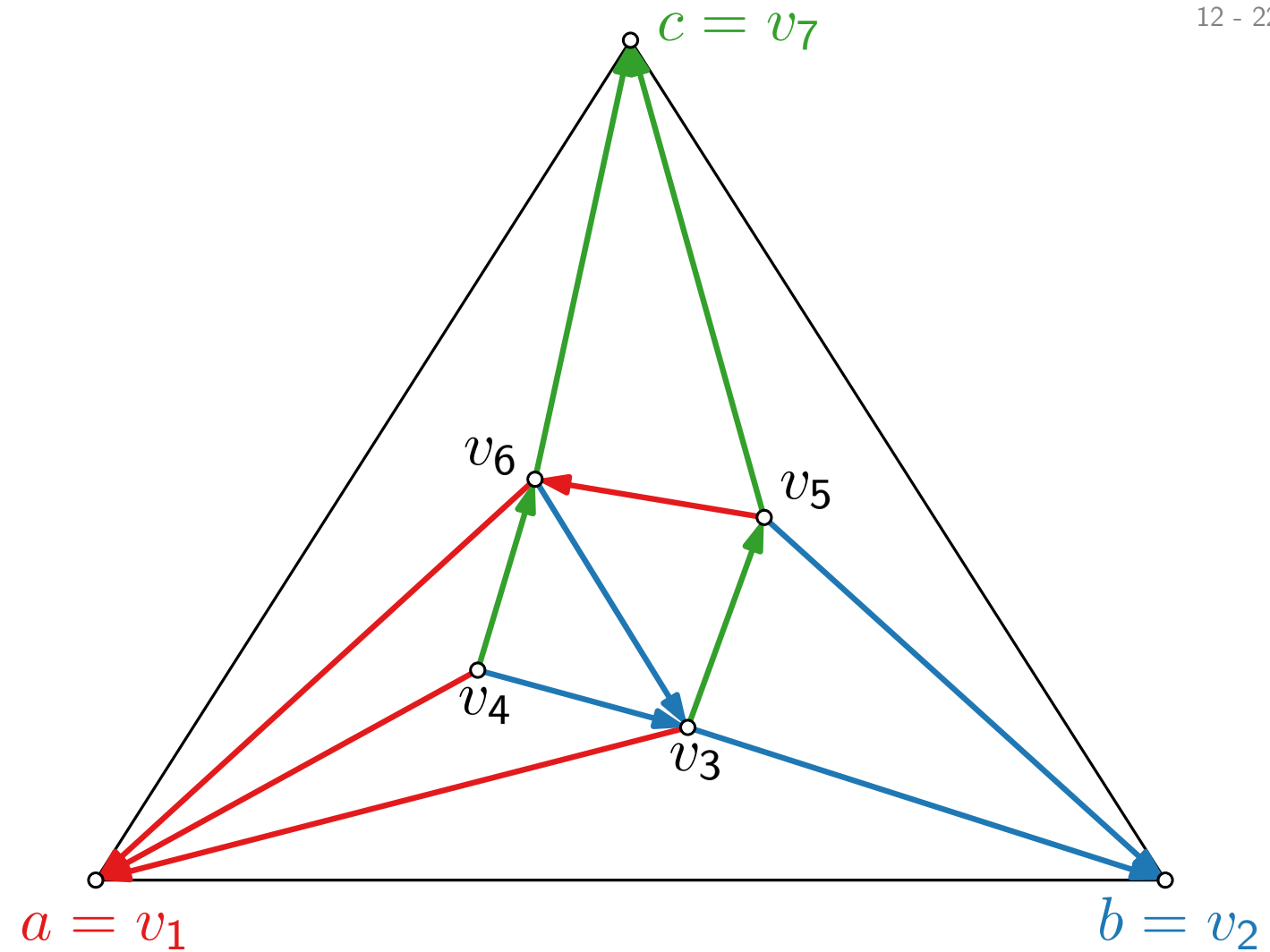
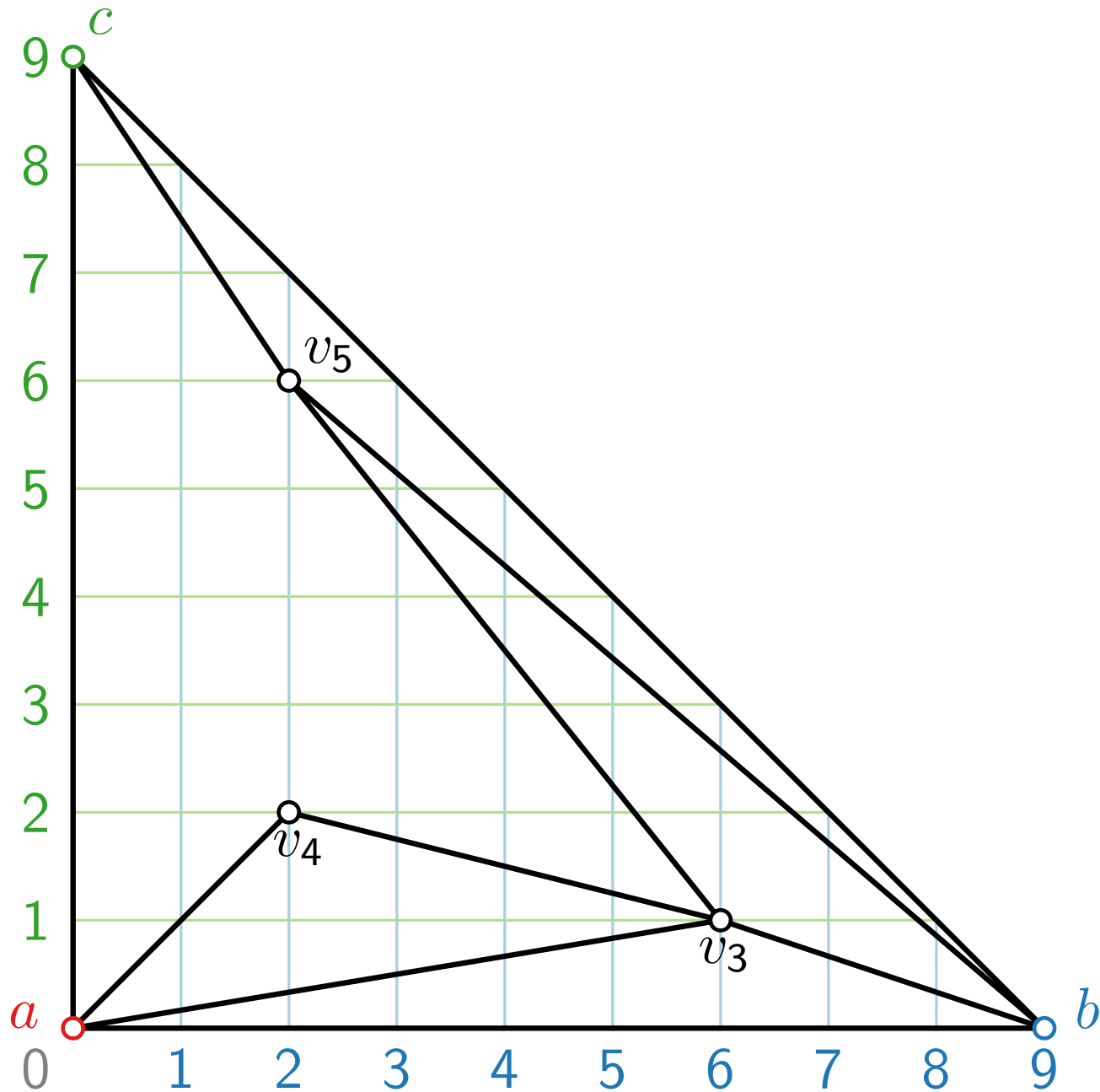
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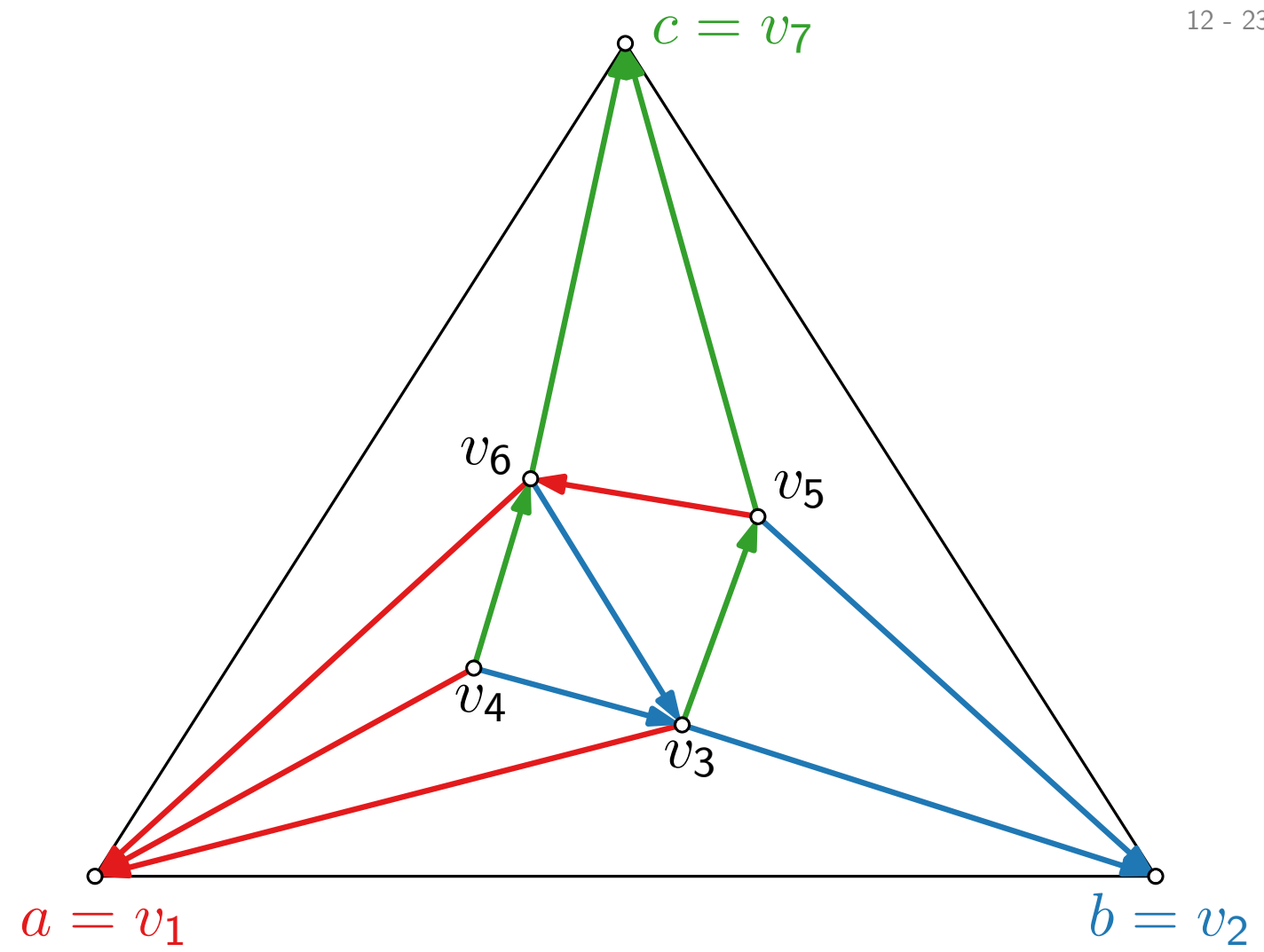
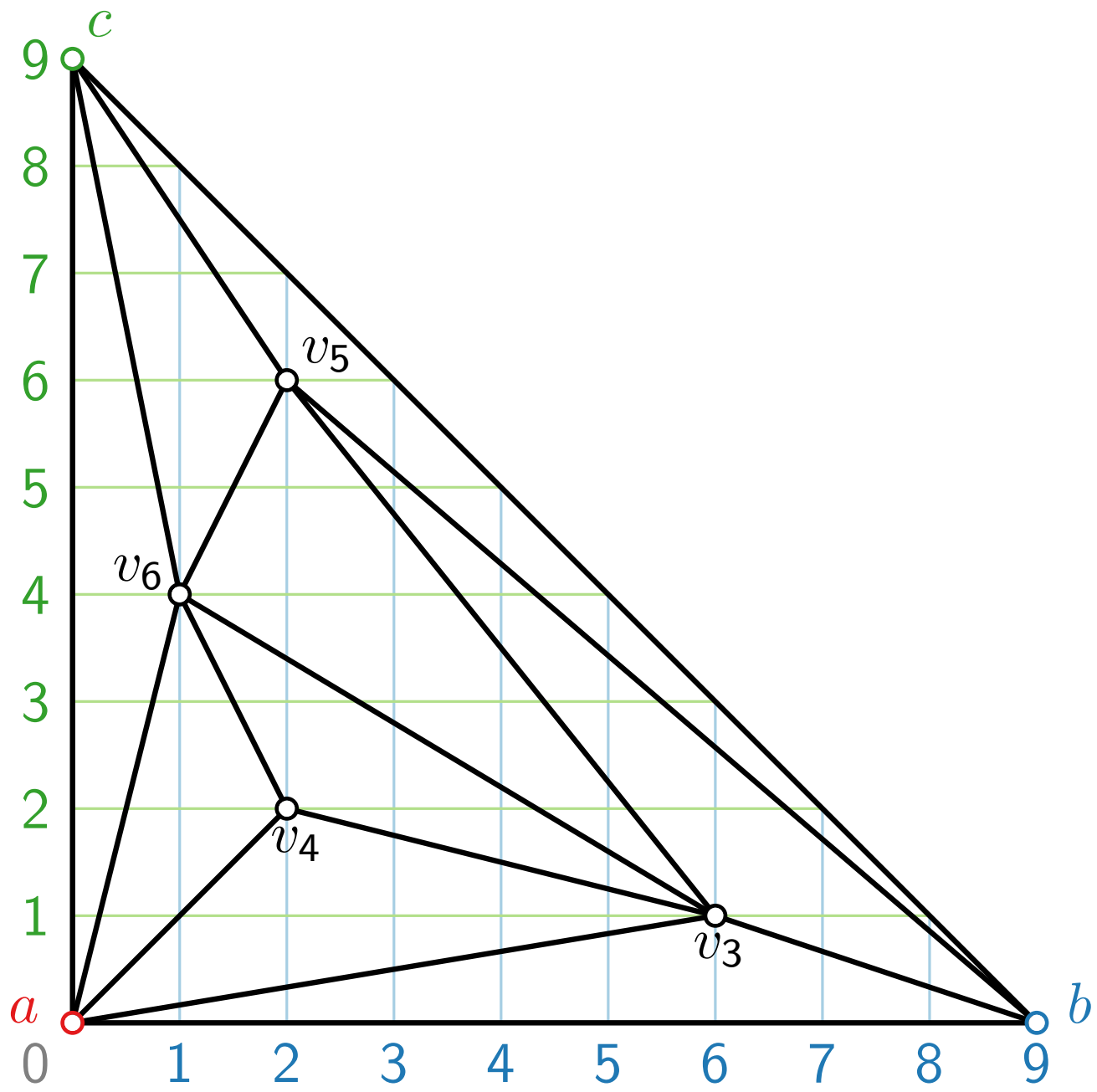
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Weak Barycentric Representation

A **weak barycentric representation** of a graph G is an assignment of barycentric coordinates to $V(G)$:

$$f: V(G) \rightarrow \mathbb{R}_{\geq 0}^3, \quad v \mapsto (v_1, v_2, v_3)$$

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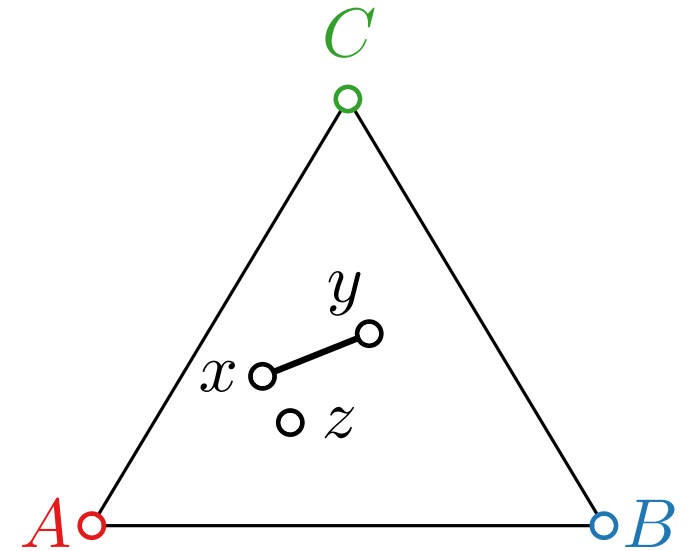
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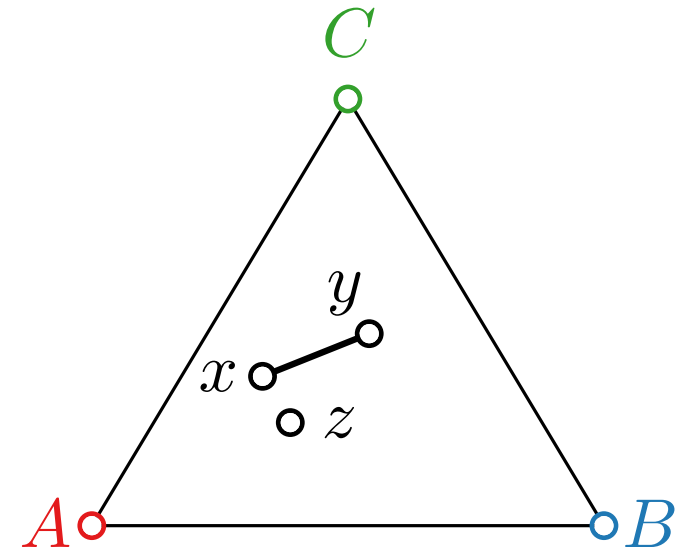
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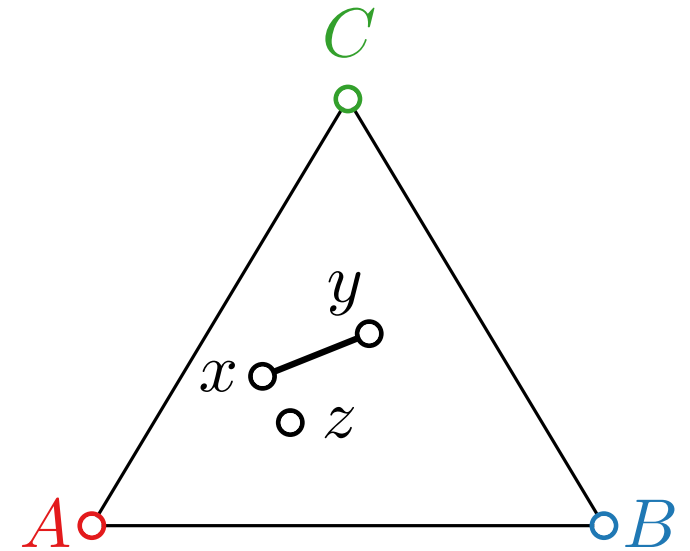
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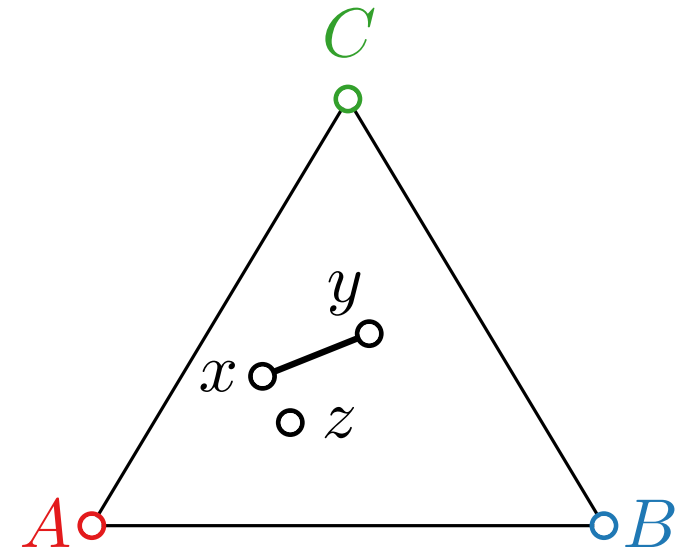
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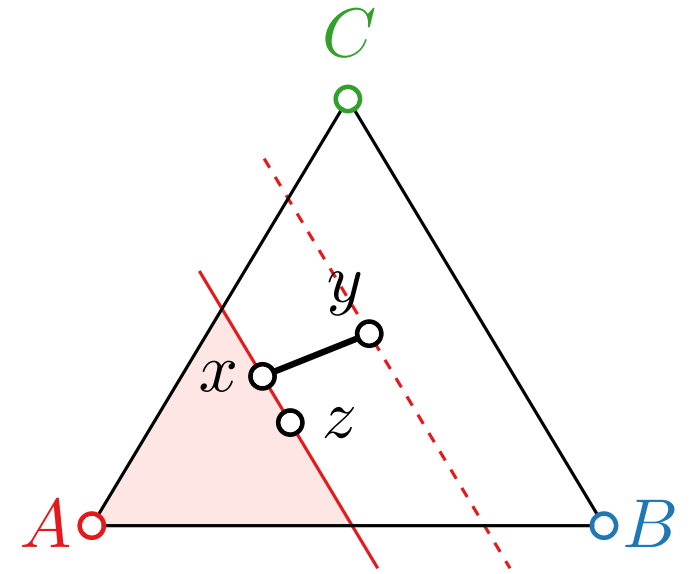
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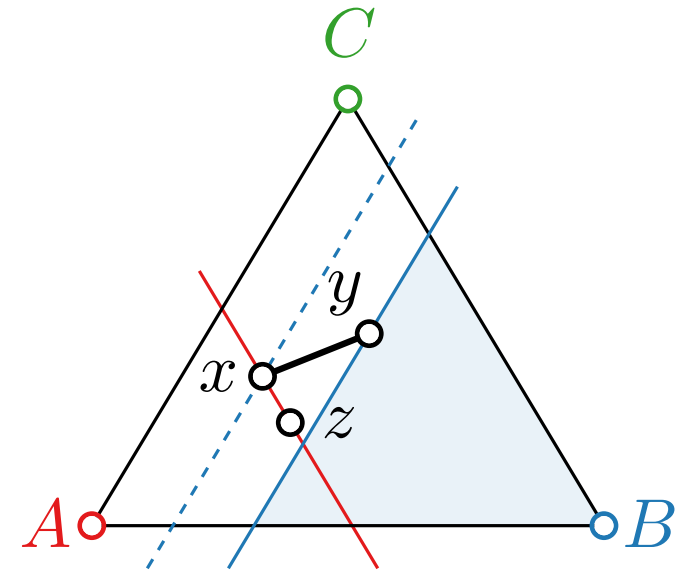
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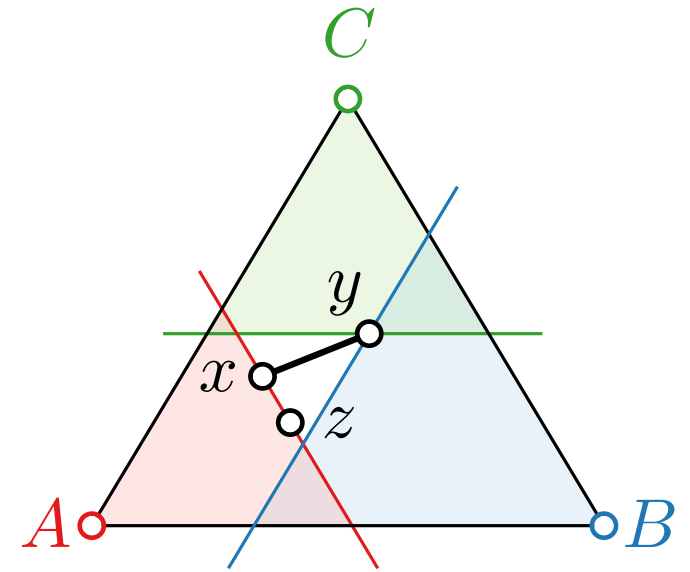
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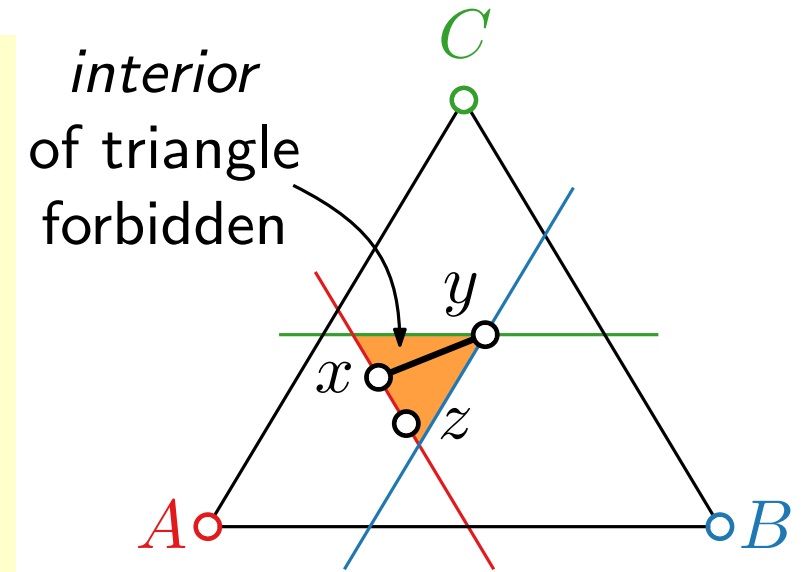
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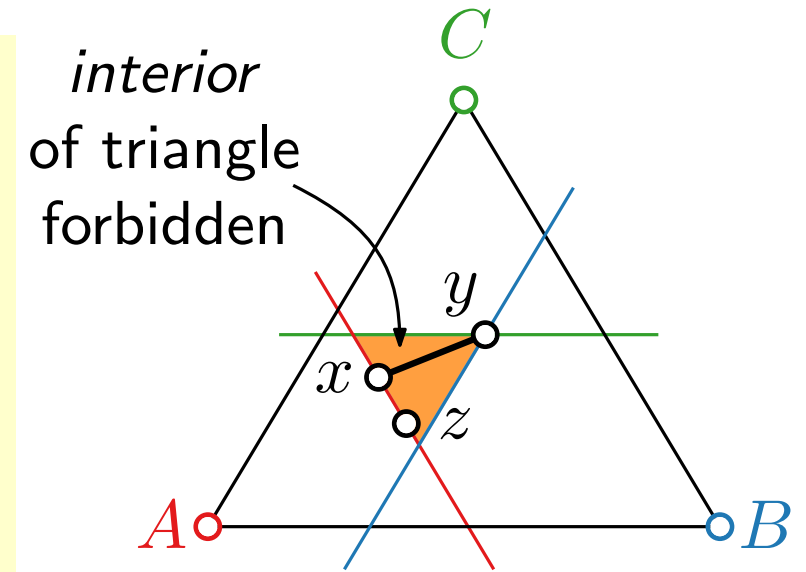
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Lemma.

For a weak barycentric representation $f: v \mapsto (v_1, v_2, v_3)$ and a triangle $\triangle ABC$, the mapping $\phi: V(G) \rightarrow \mathbb{R}^3$ with

$$v \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.



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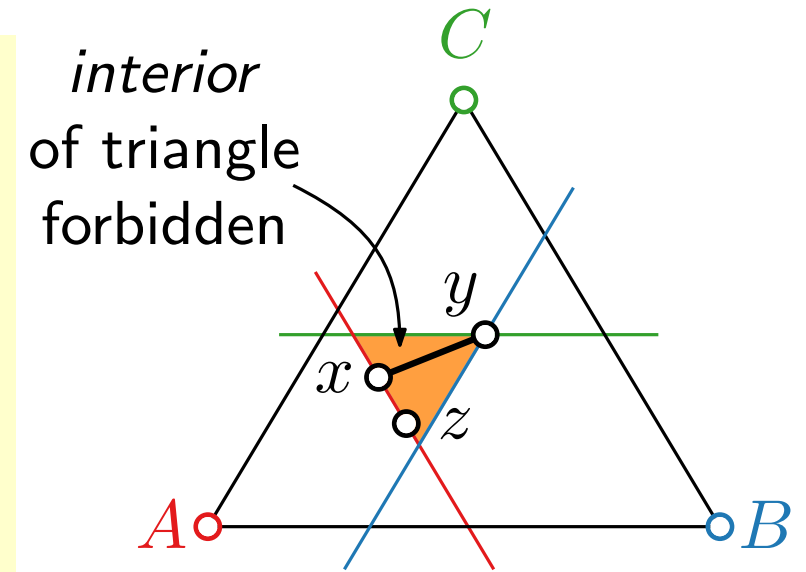
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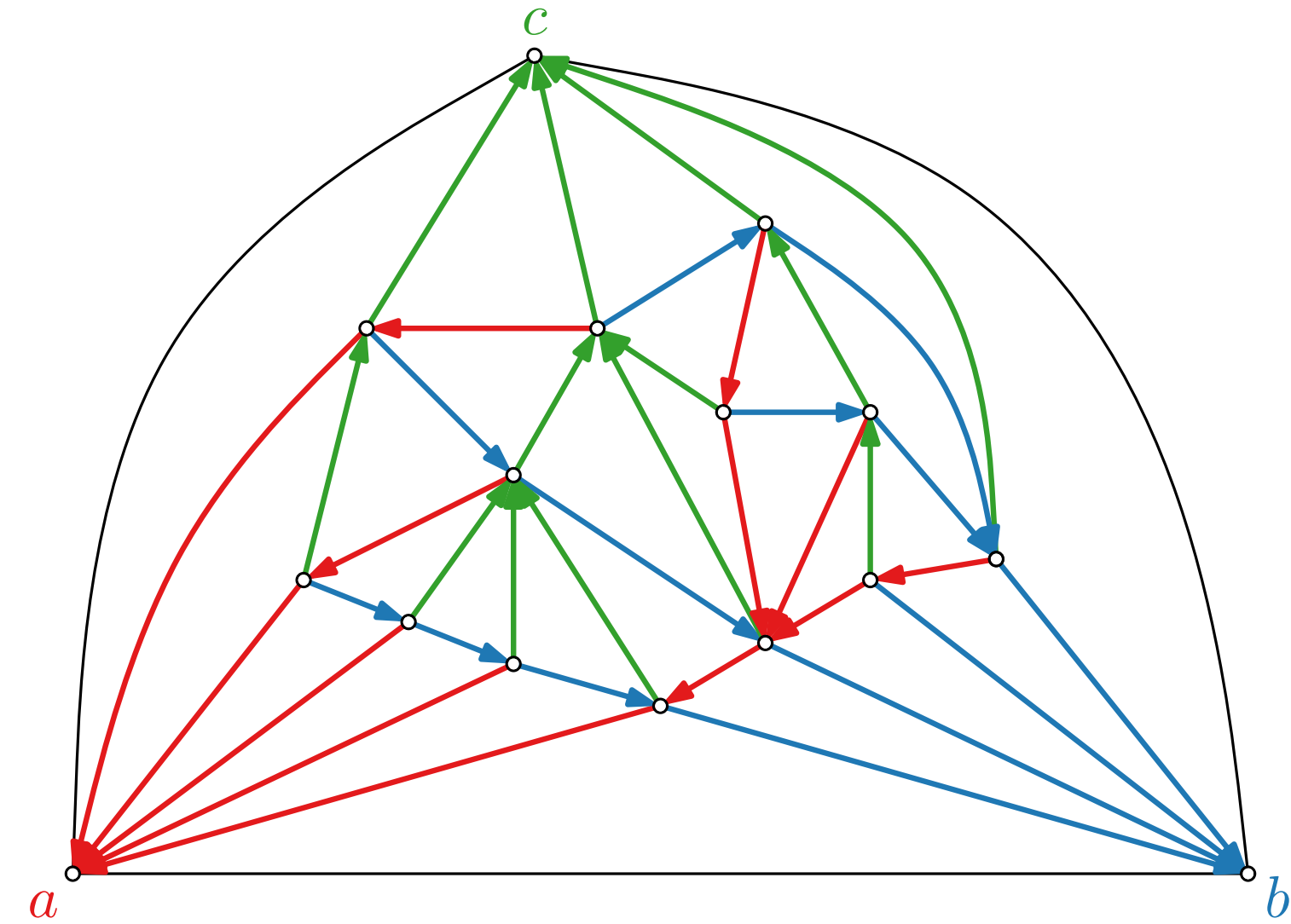


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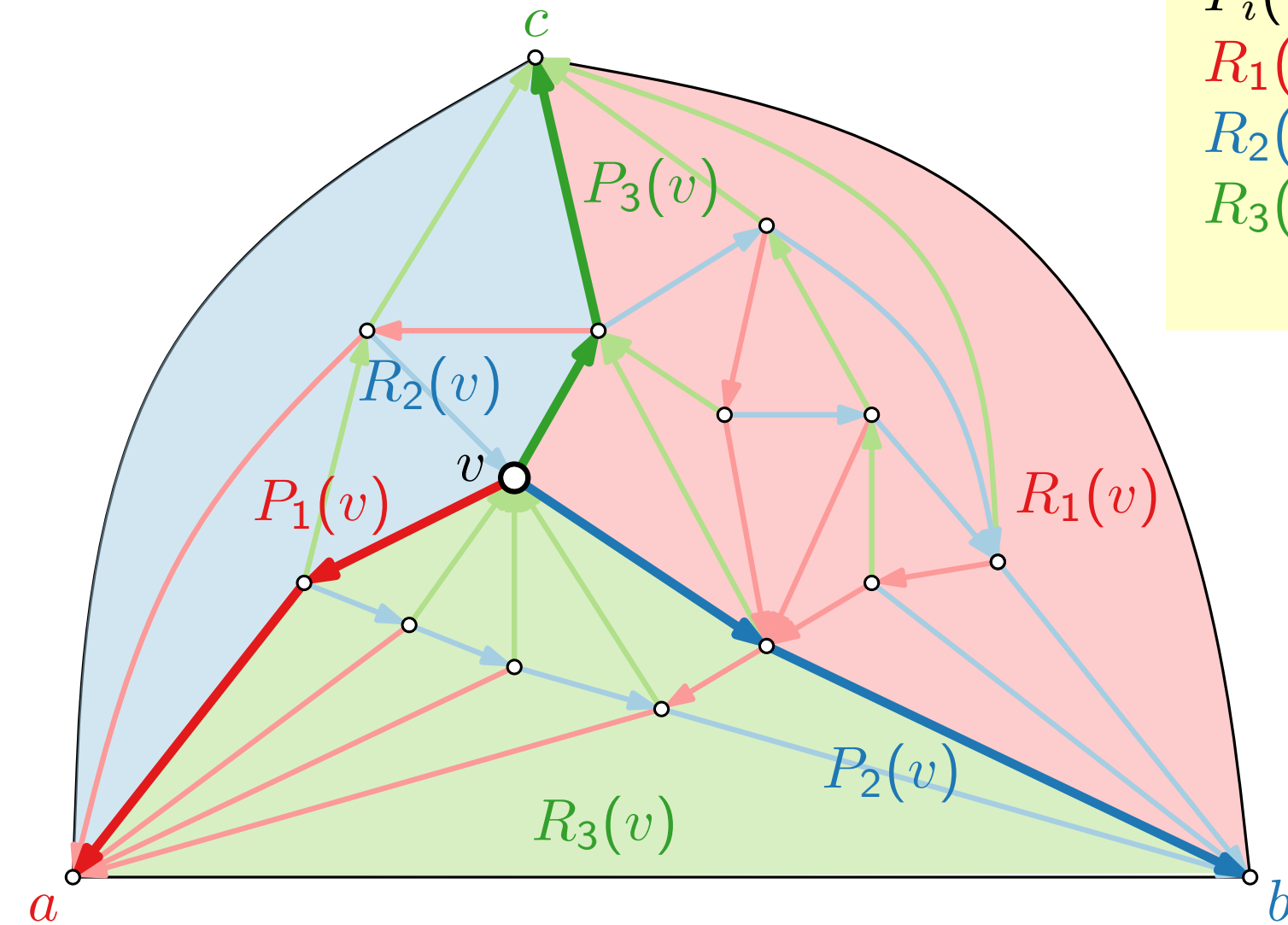
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Proof. \rightarrow *Exercise!*

Counting Vertices



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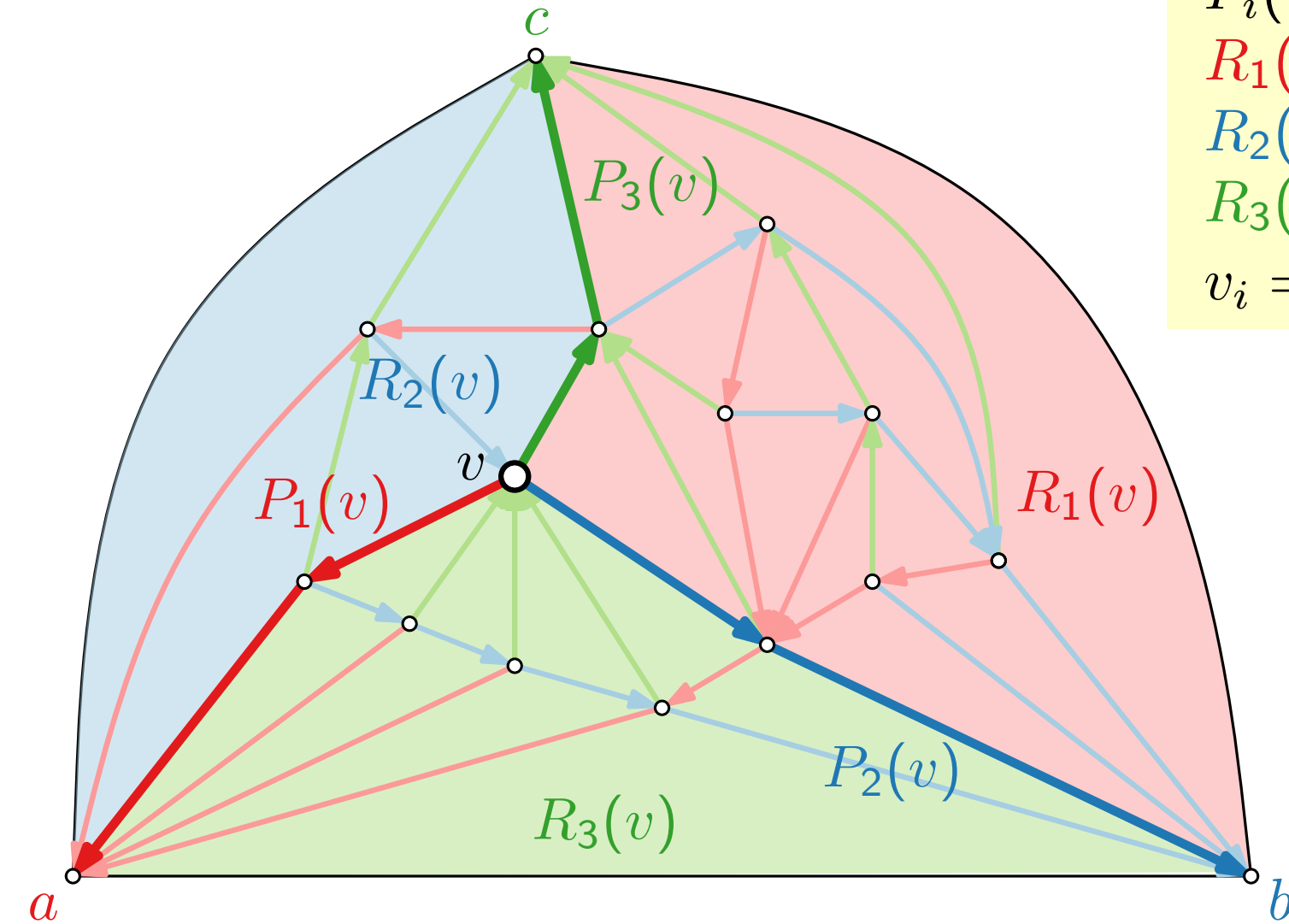
$P_i(v)$: unique path from v to root of T_i

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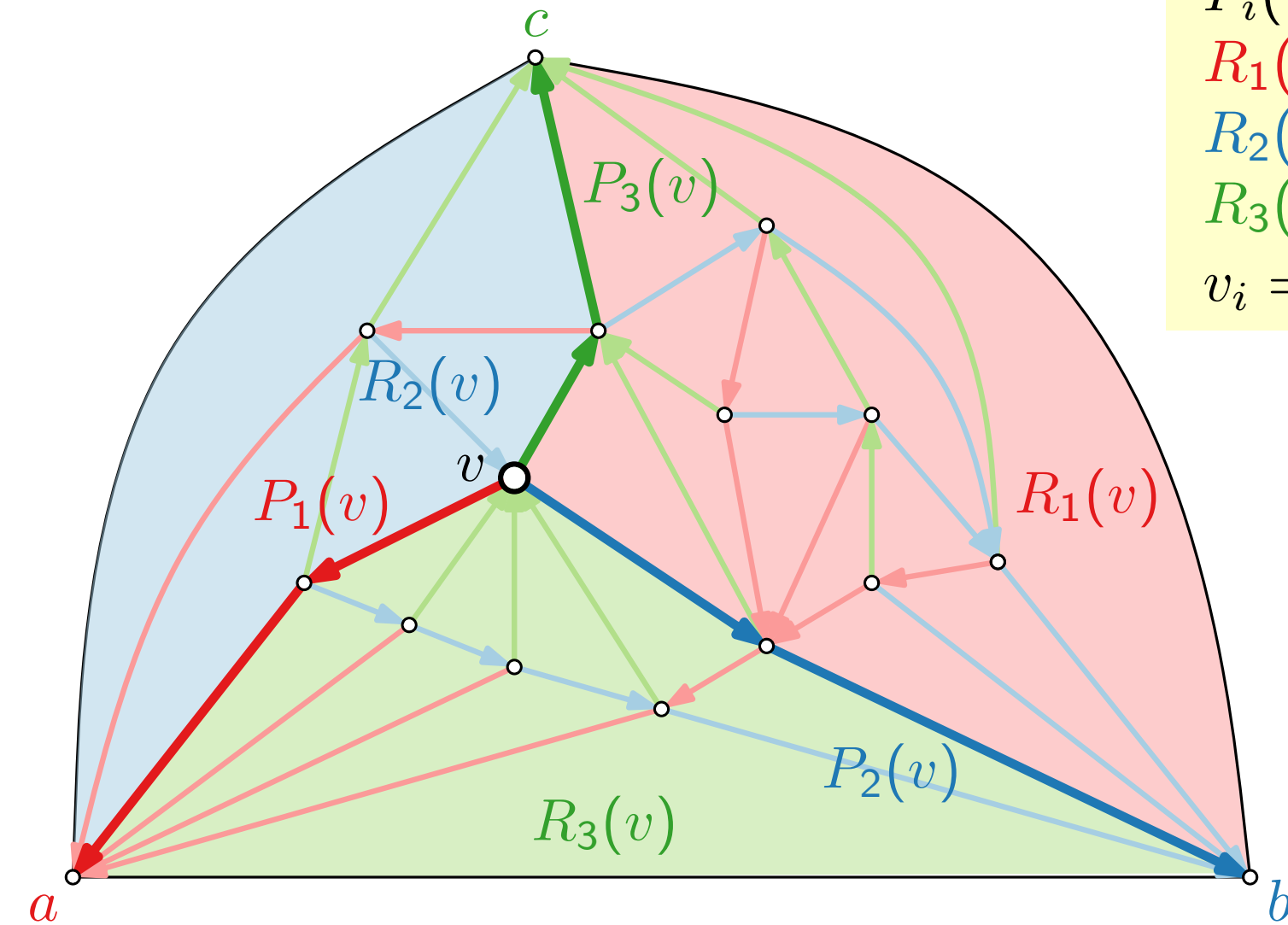
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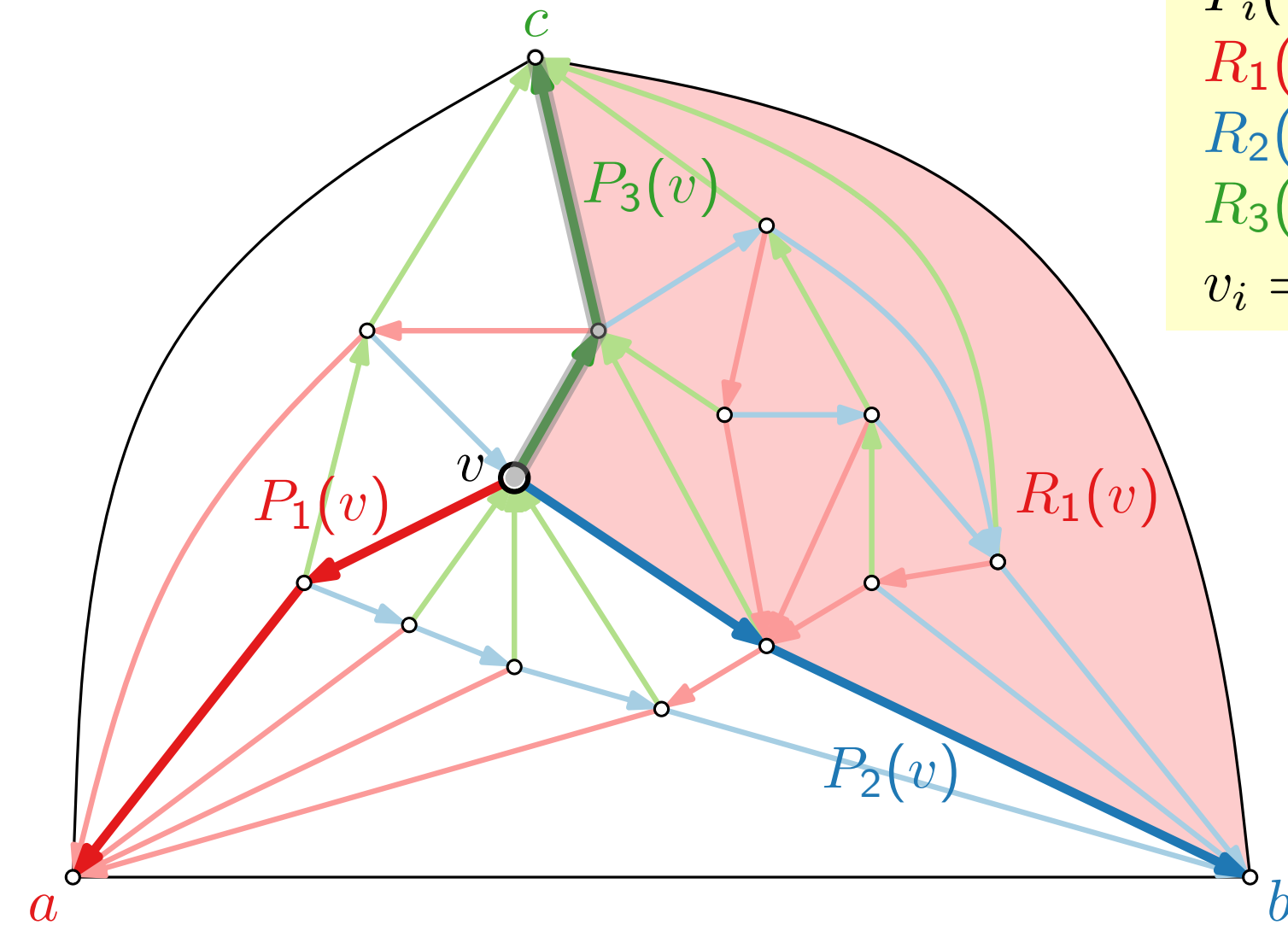
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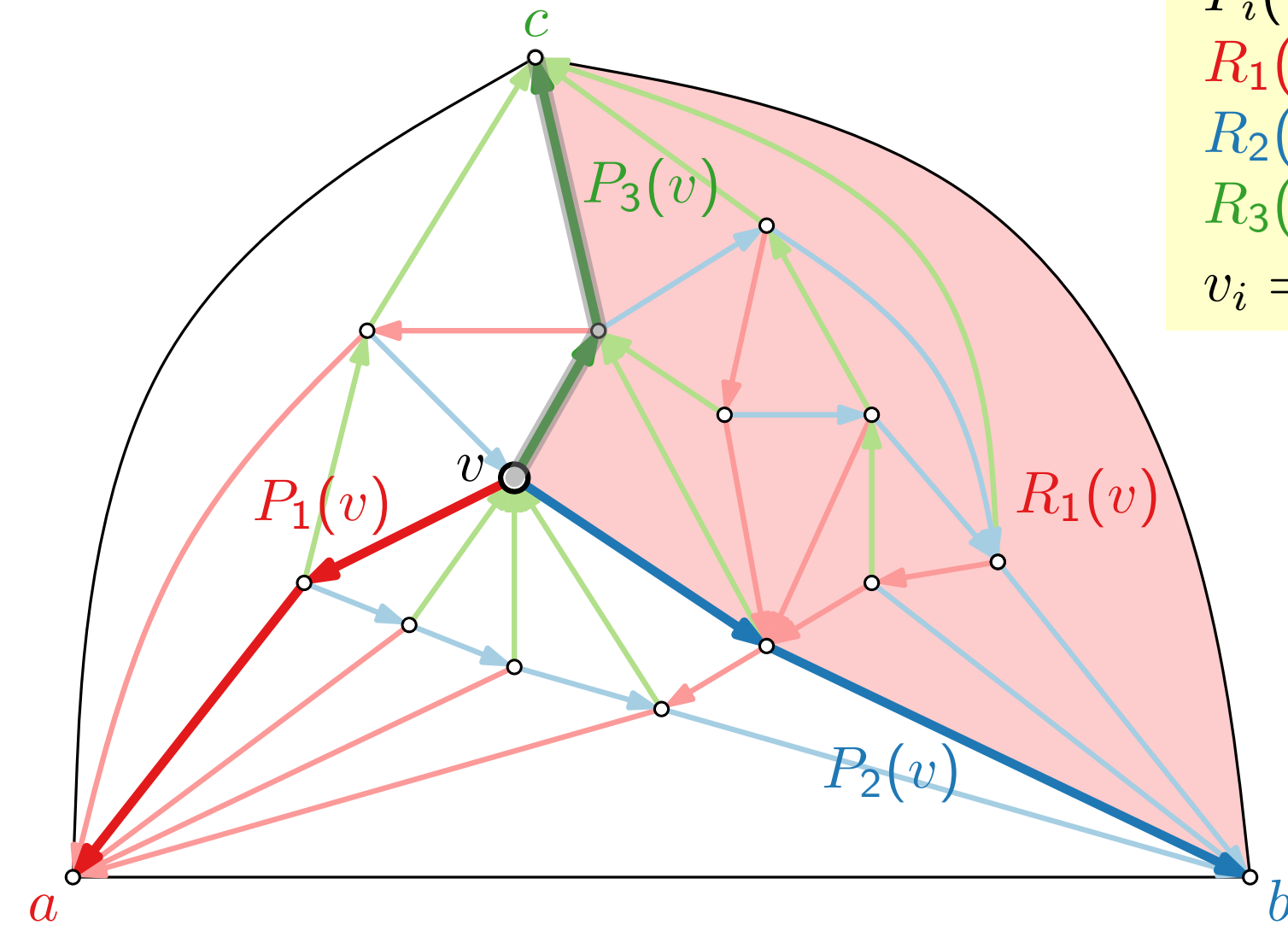
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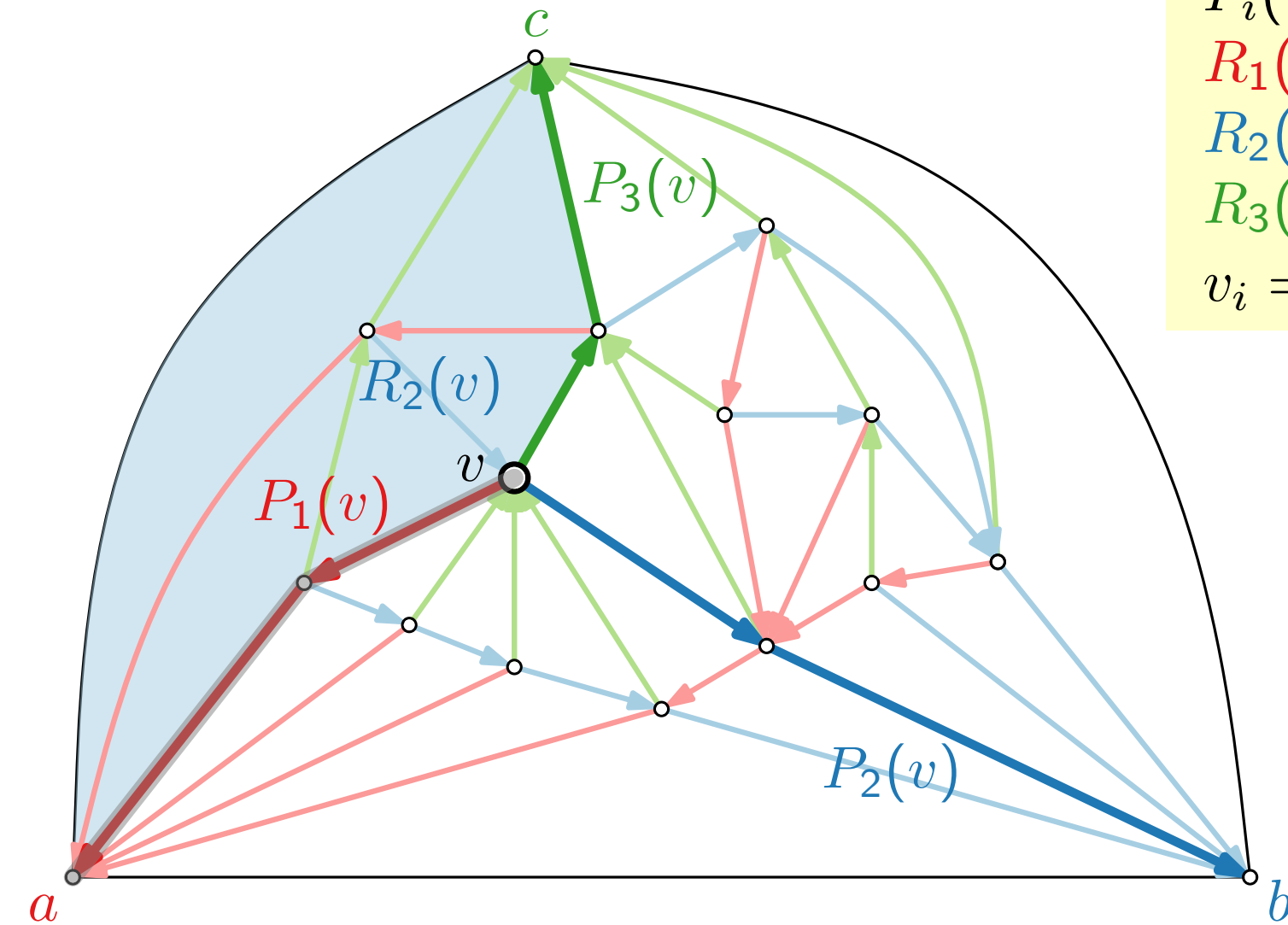
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$$v_1 = 10 - 3 = 7$$

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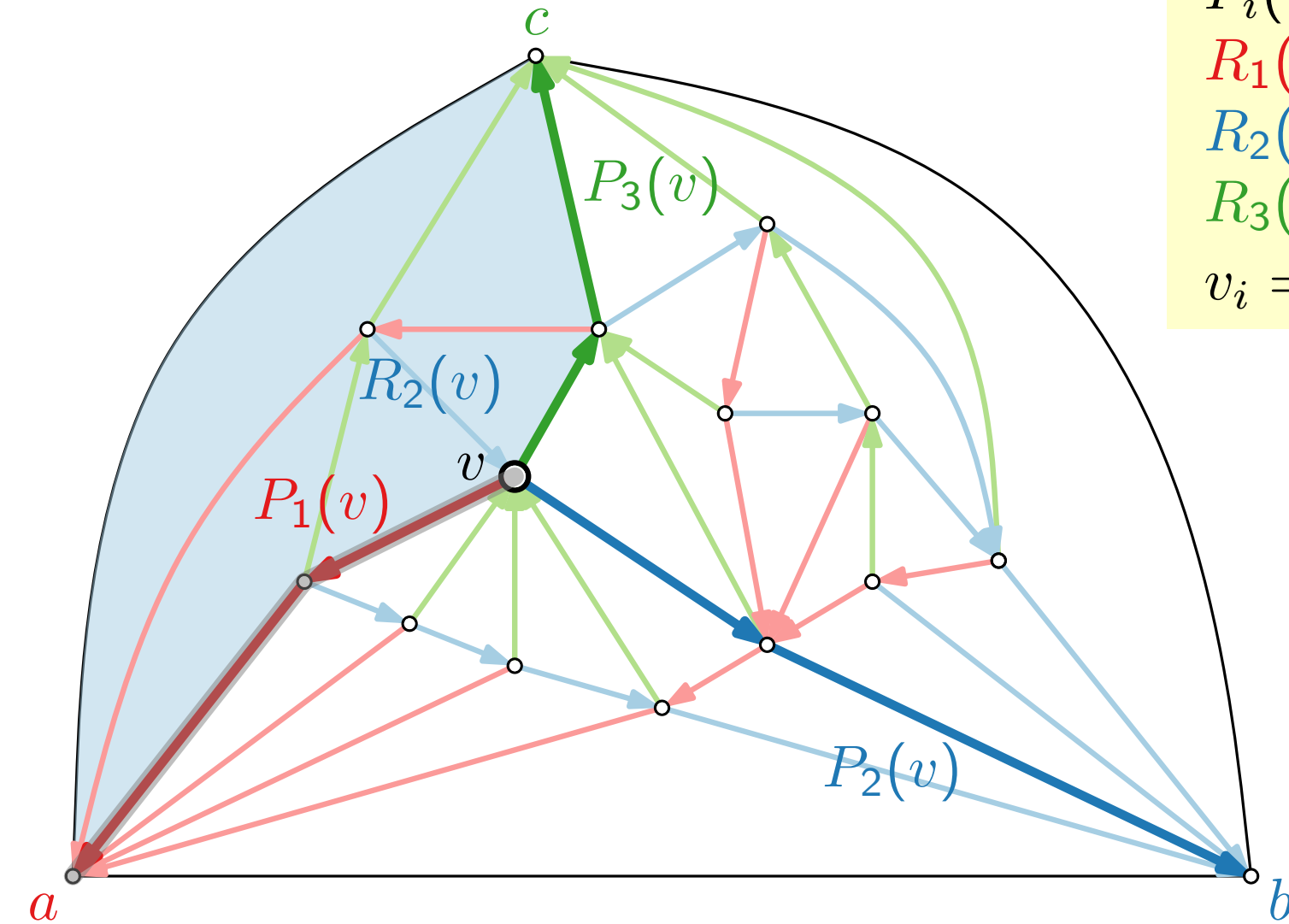
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$$v_2 =$$

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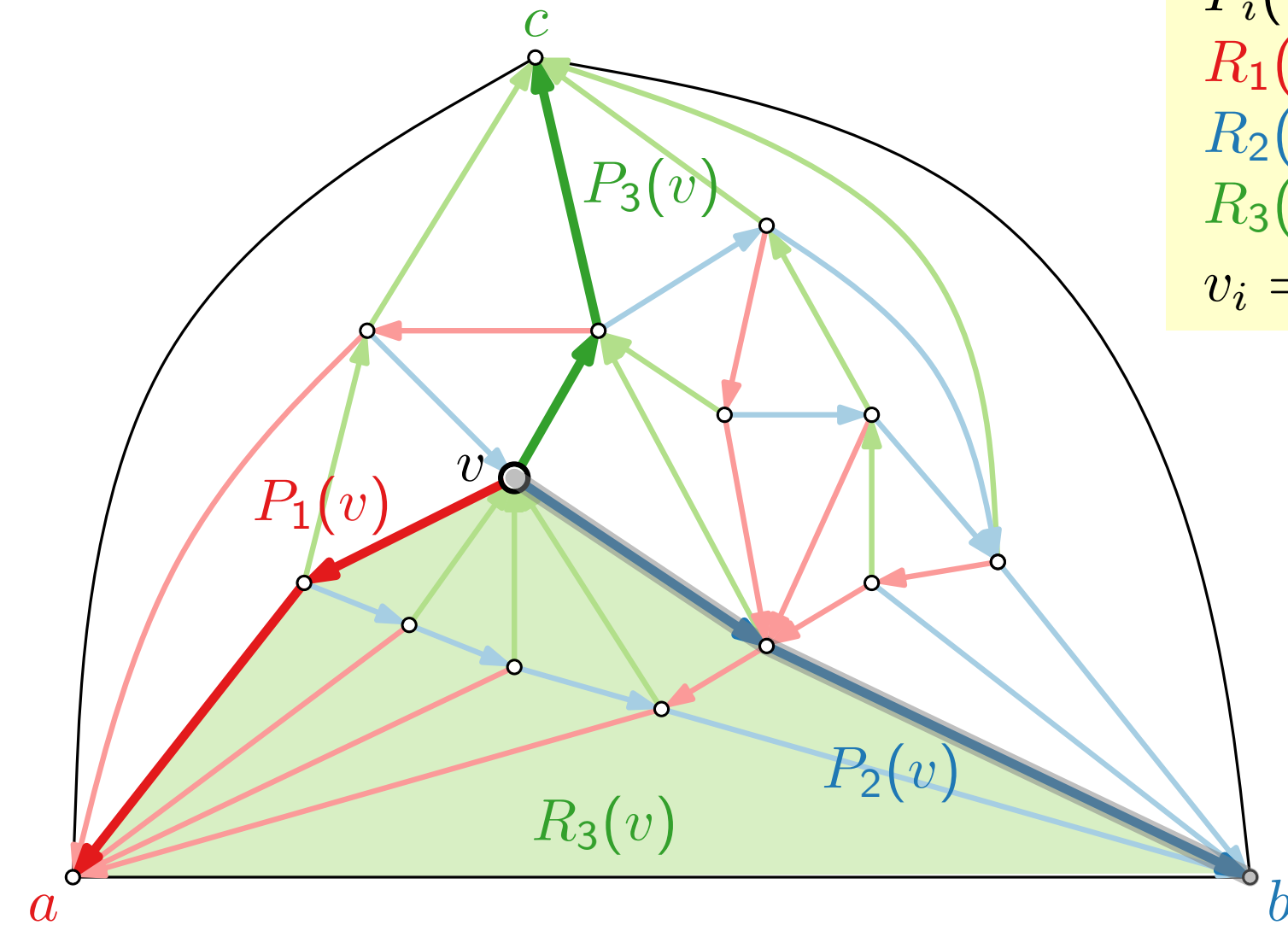
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$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v) \rangle$

$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

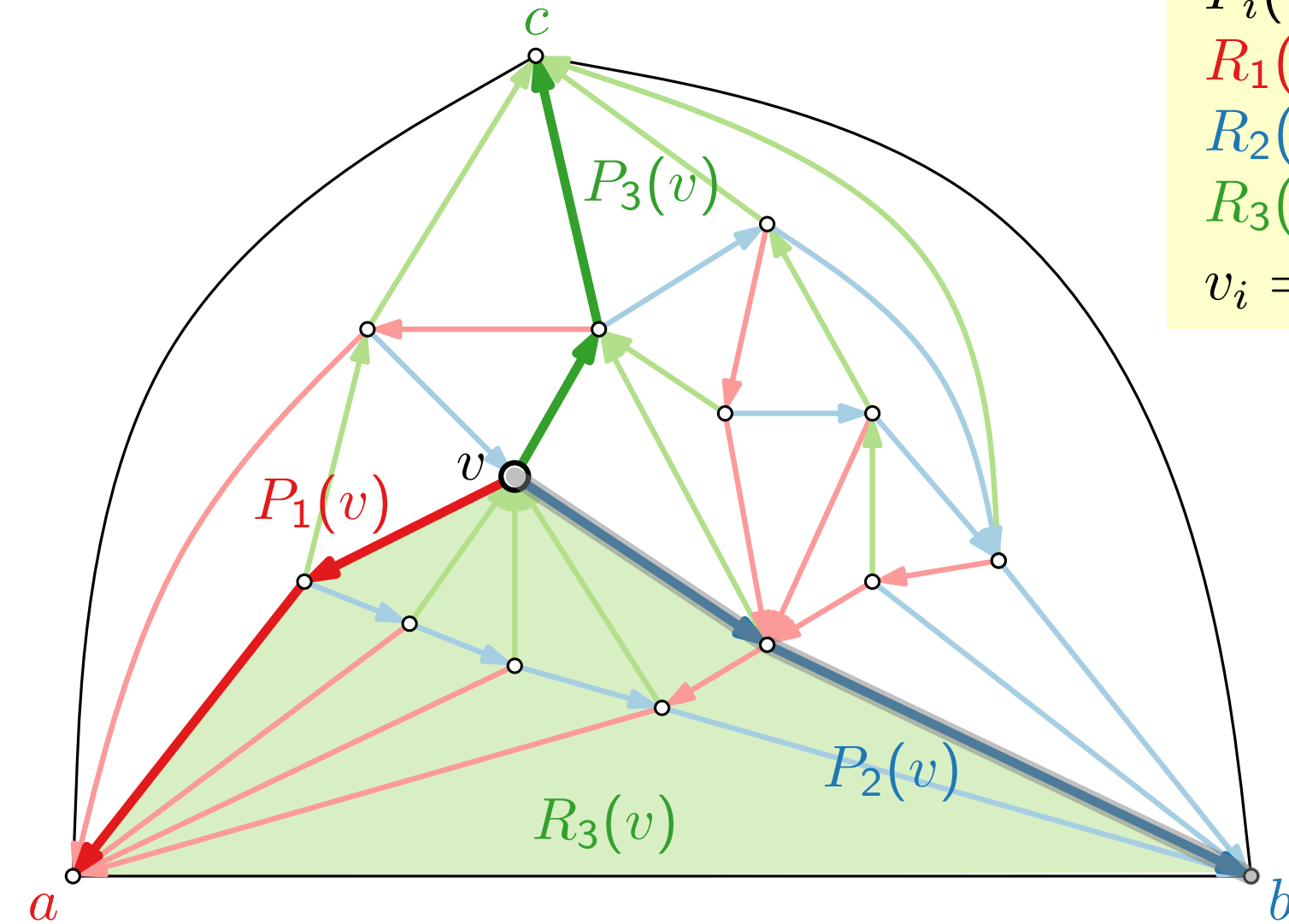
$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

$$v_3 =$$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

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$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

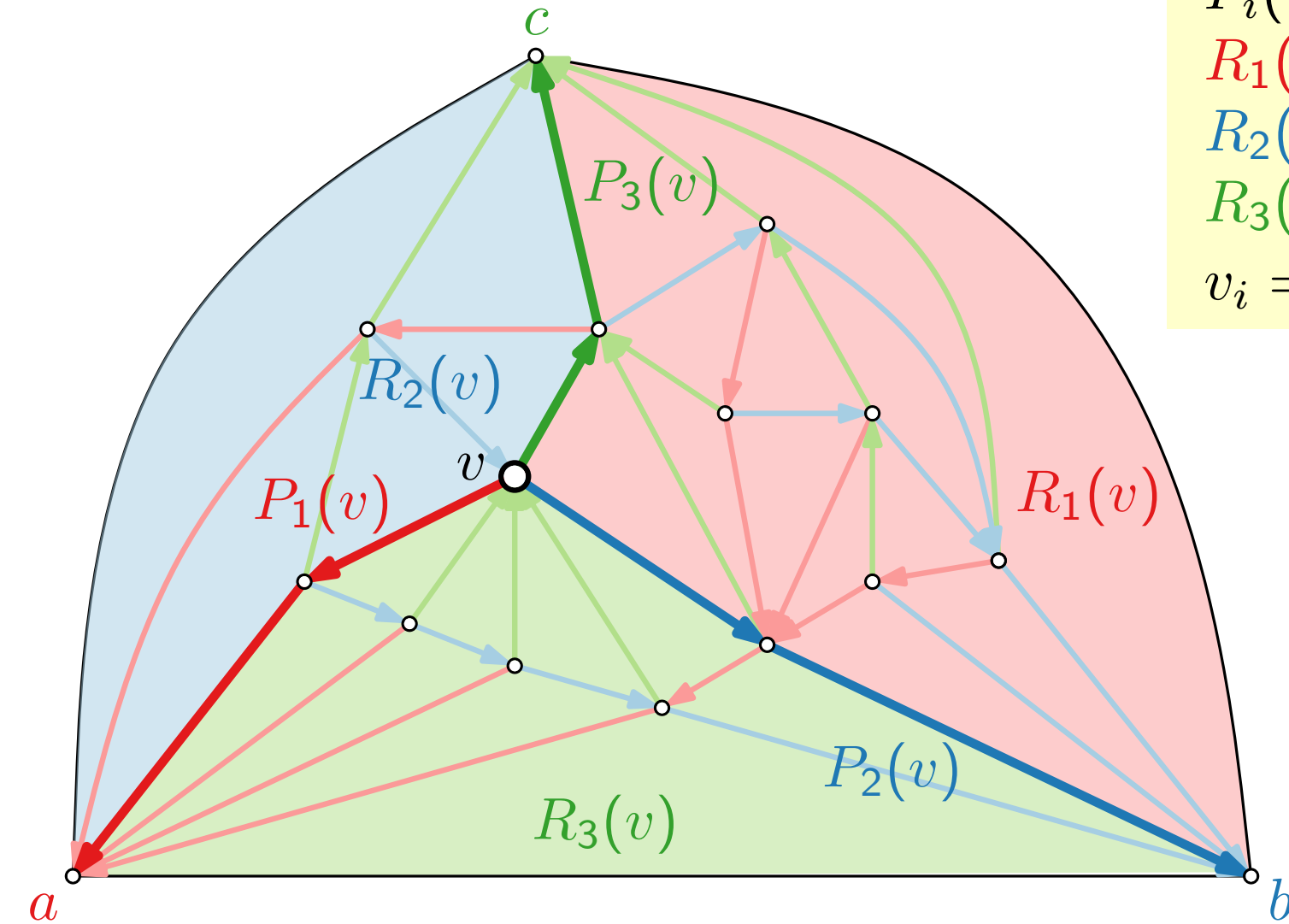
$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v) \rangle$

$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

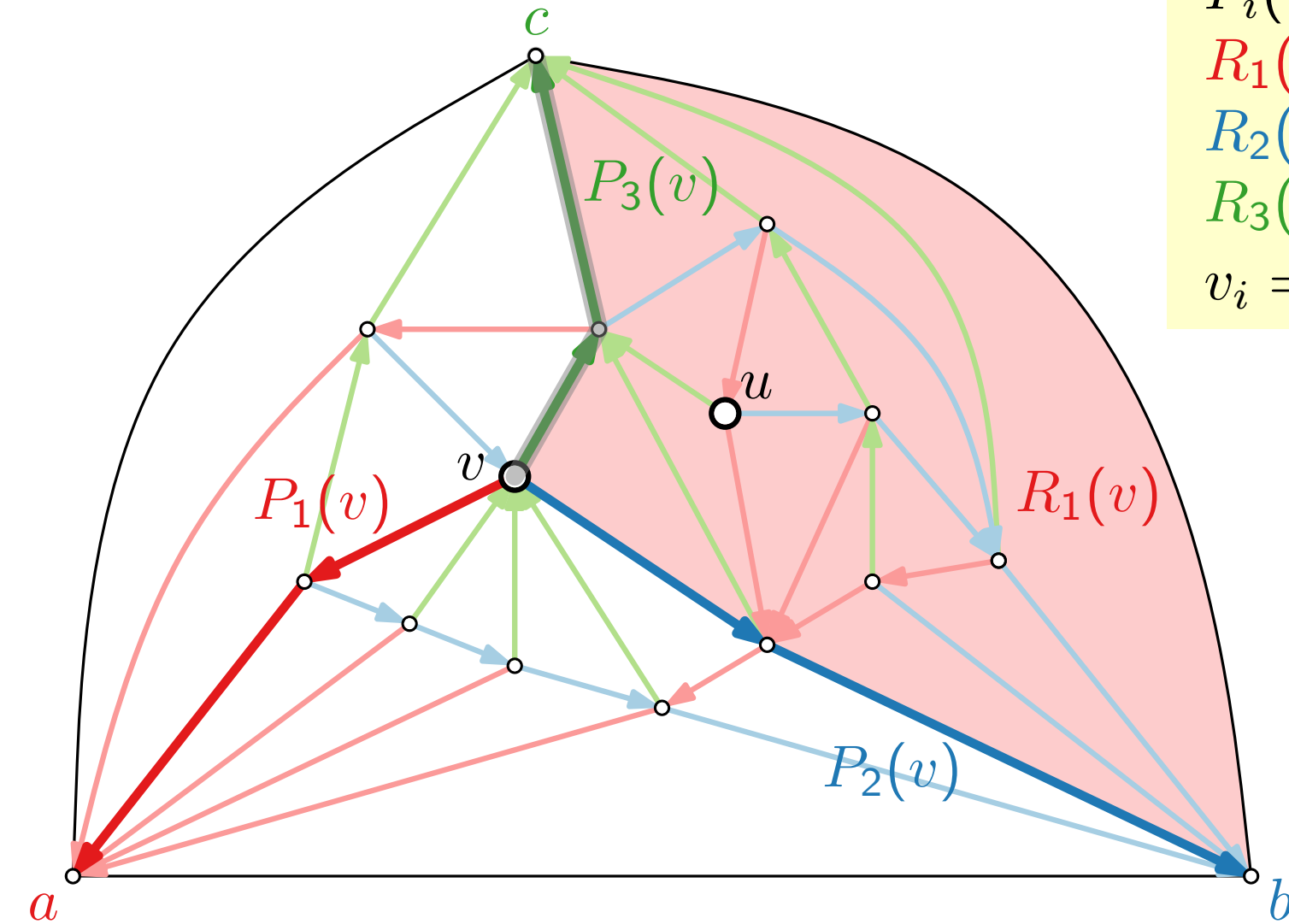
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



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$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v) \rangle$

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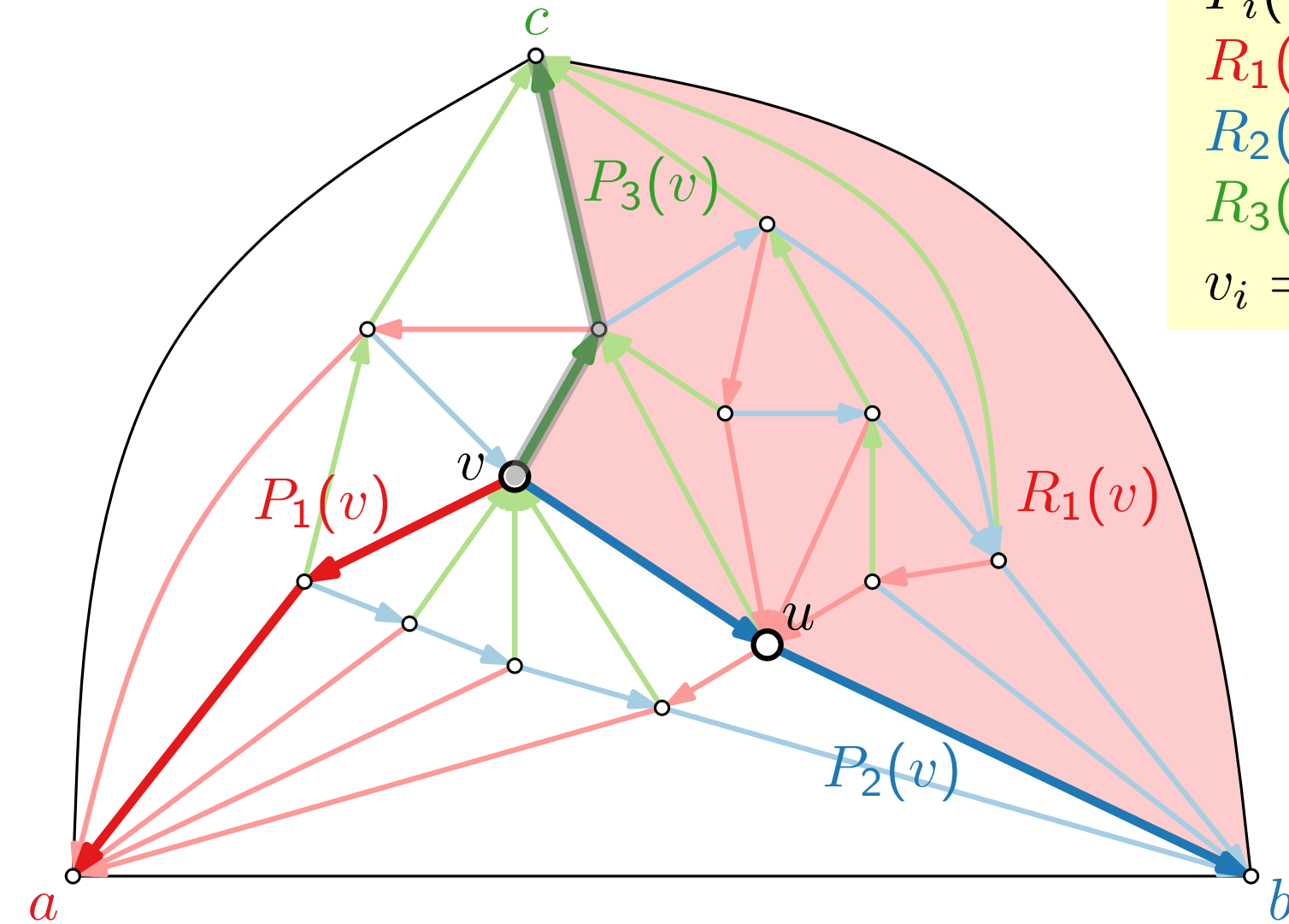
$$v_2 = 6 - 3 = 3$$

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Lemma.

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Counting Vertices



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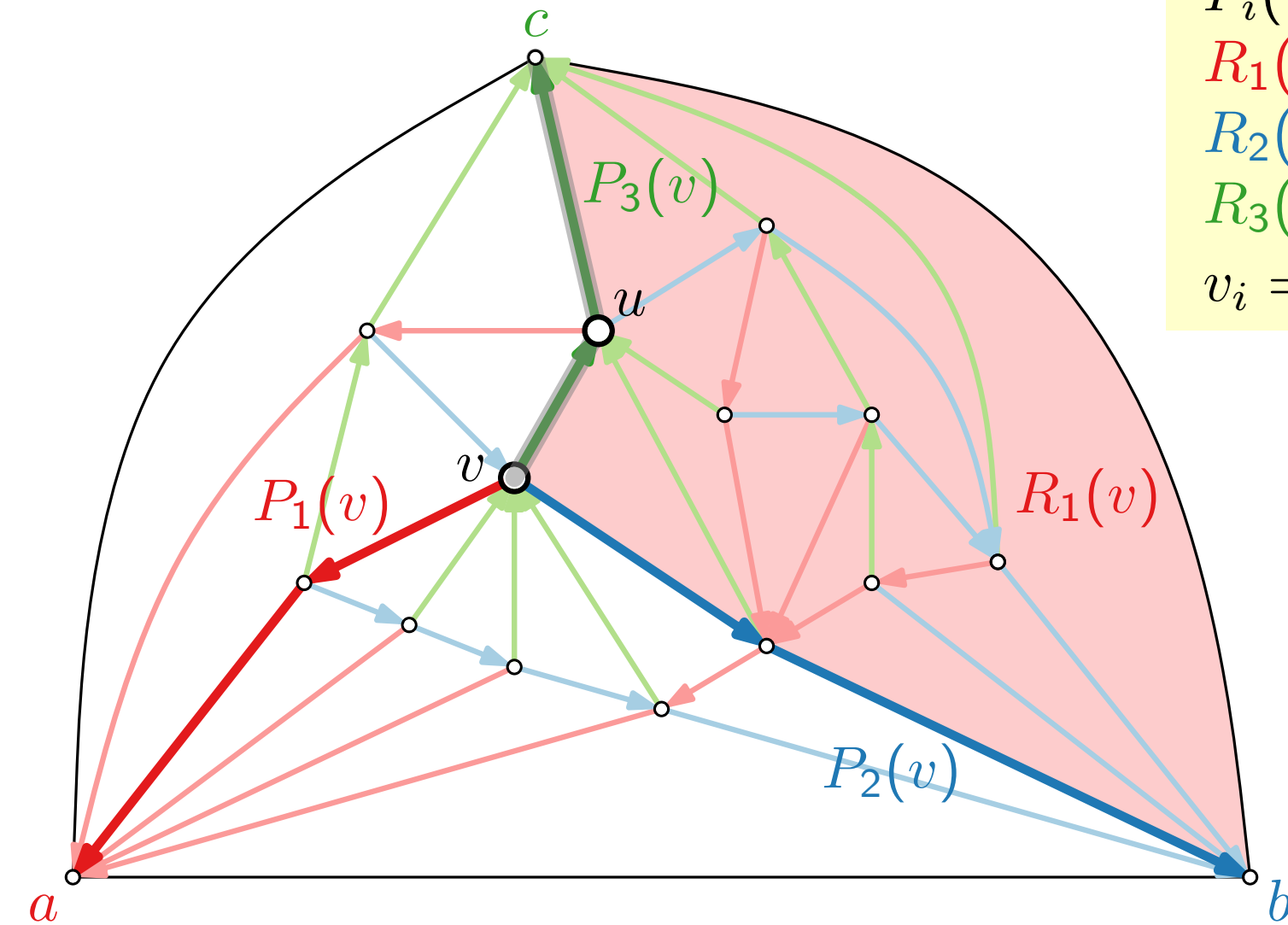
$$v_2 = 6 - 3 = 3$$

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Lemma.

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Counting Vertices



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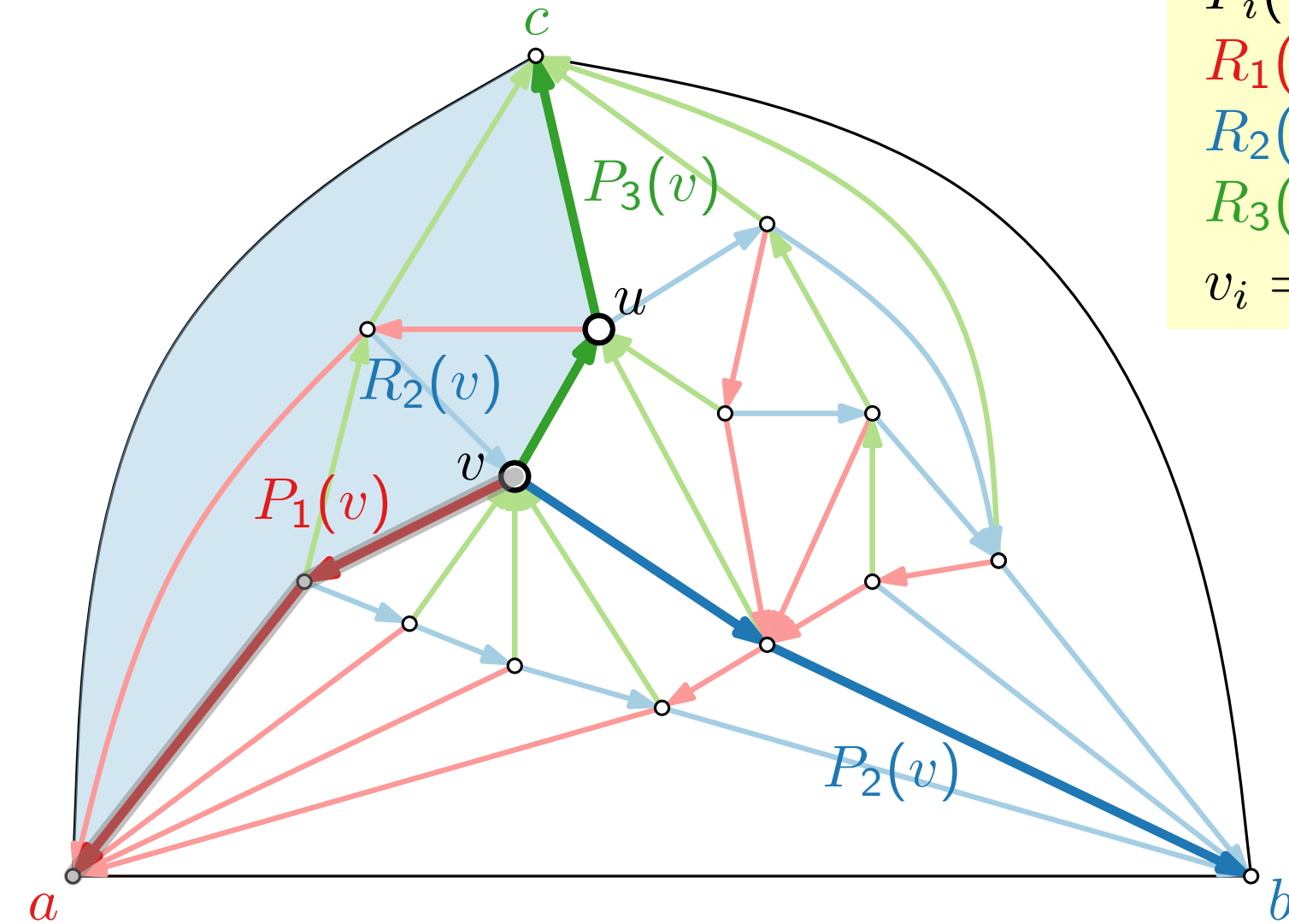
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



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$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

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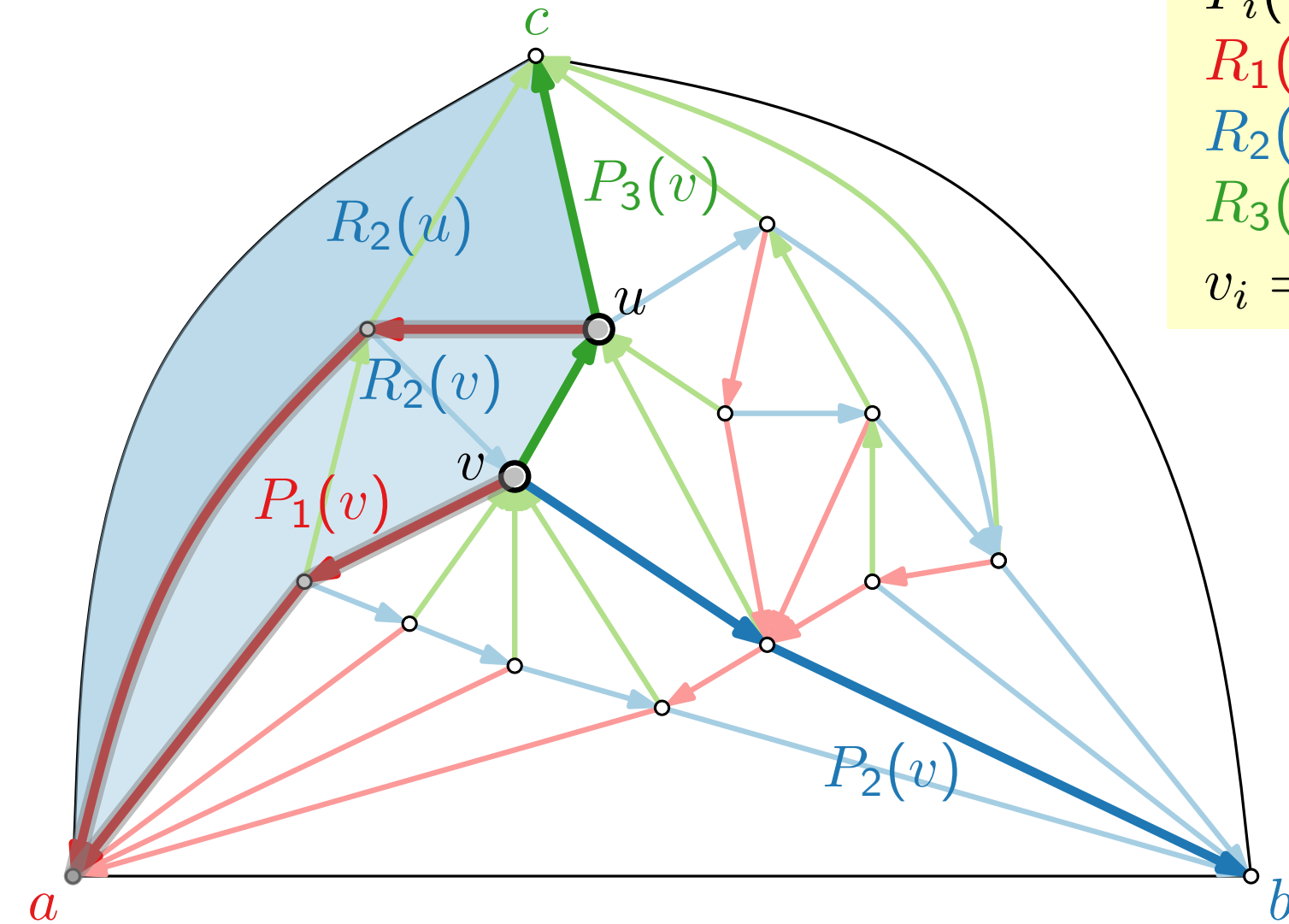
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



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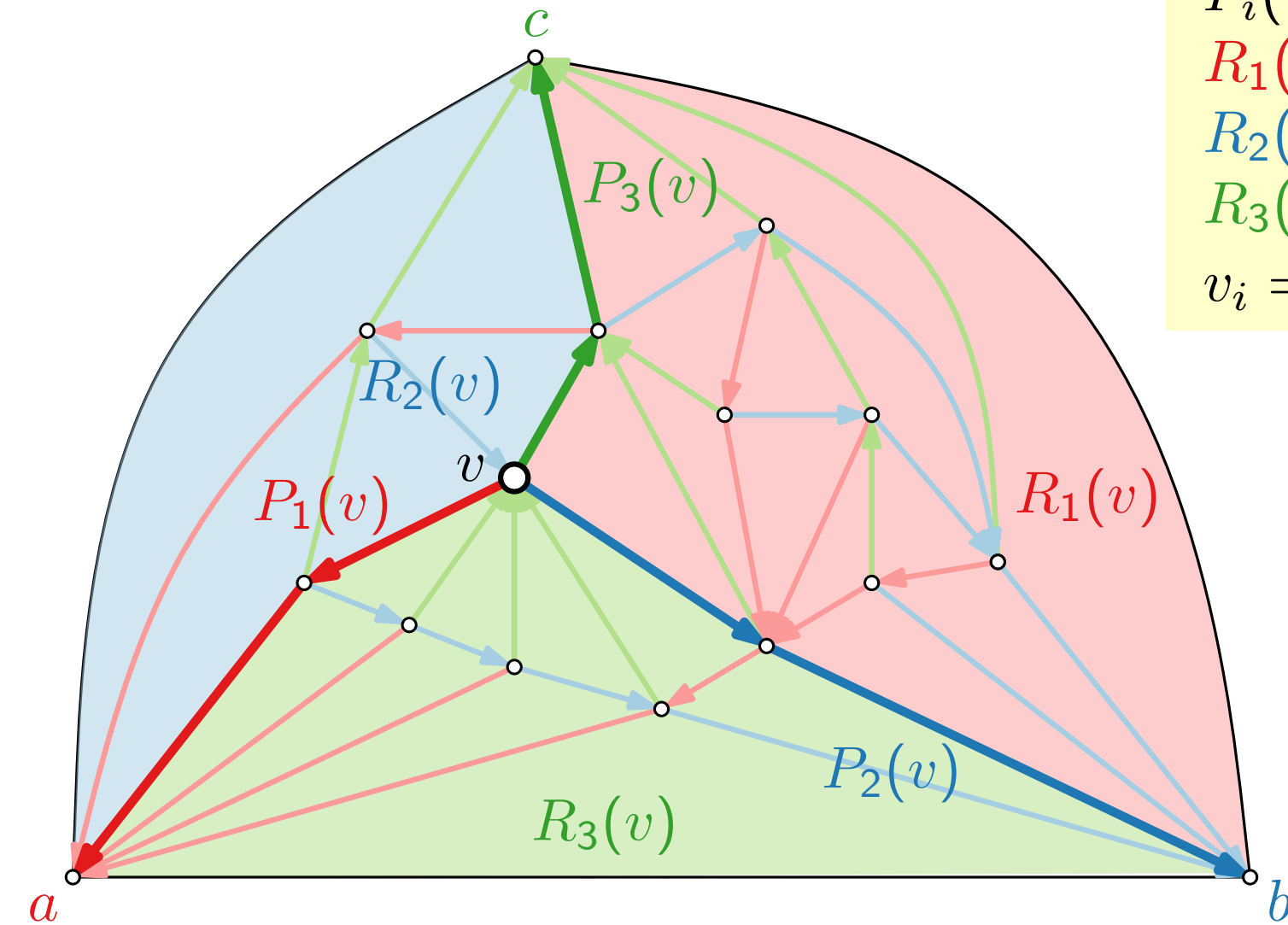
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



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$$v_1 = 10 - 3 = 7$$

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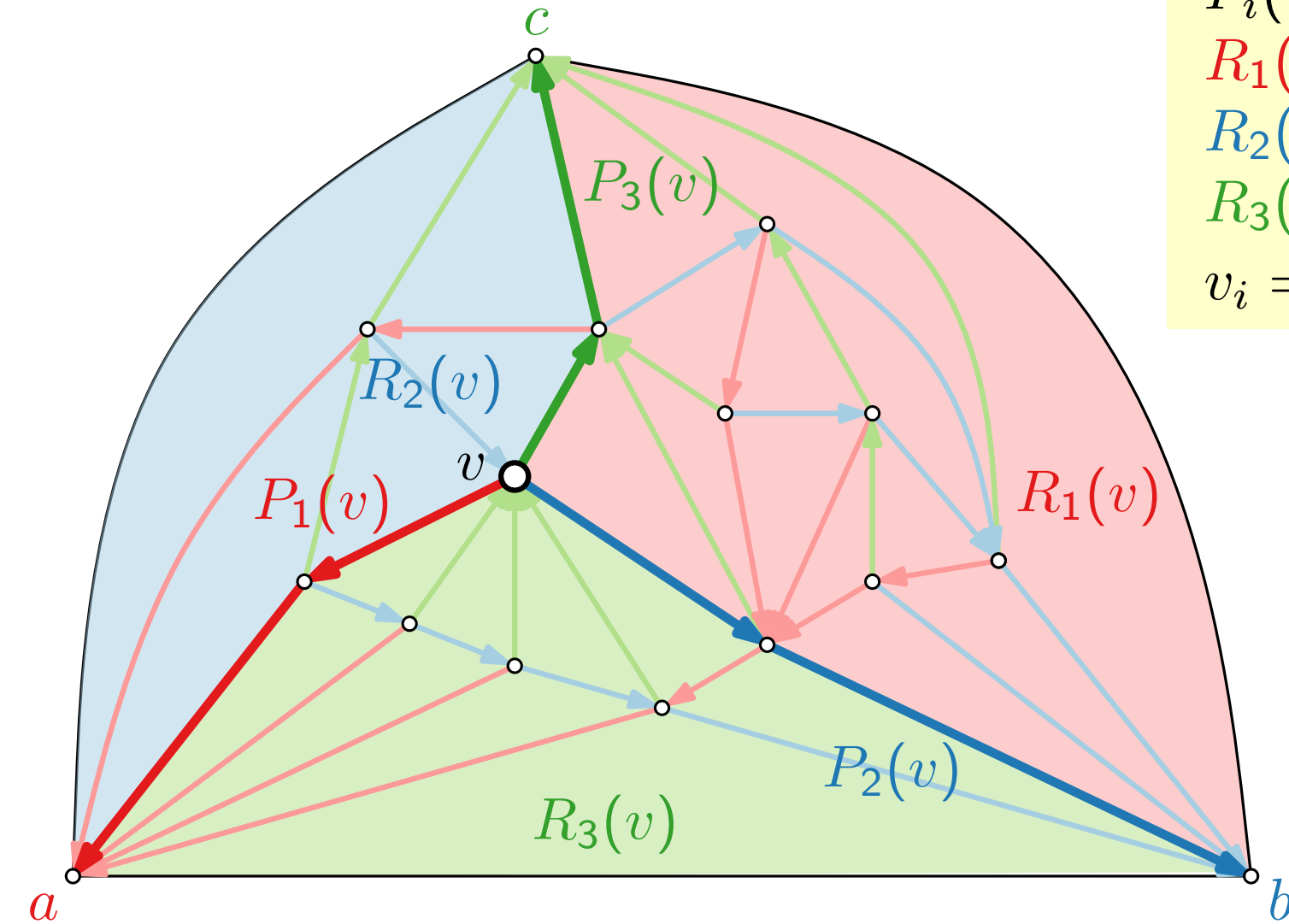
$$v_3 = 8 - 3 = 5$$

Lemma.

■ For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

■ $v_1 + v_2 + v_3 =$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v) \rangle$

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$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

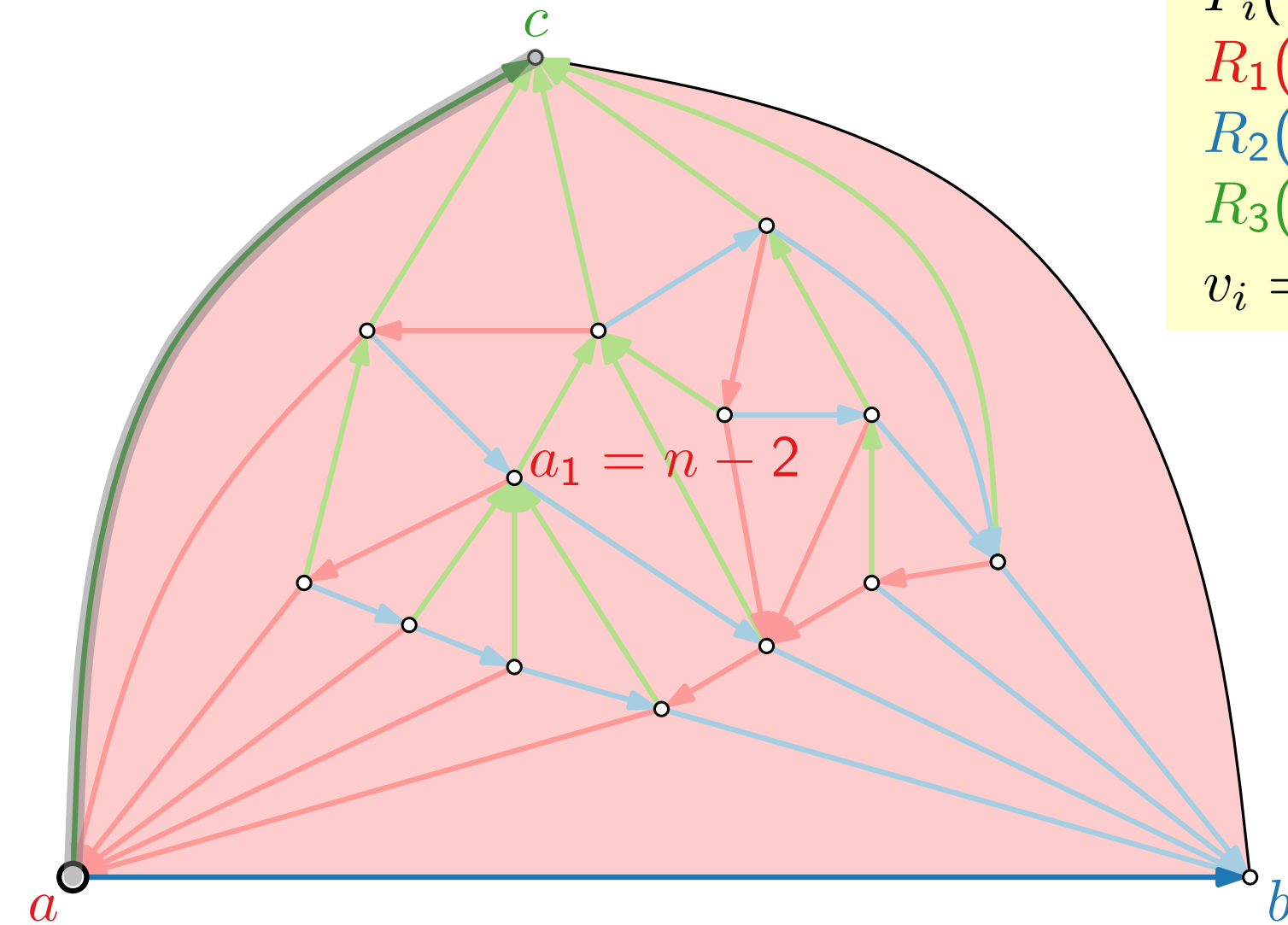
$$v_3 = 8 - 3 = 5$$

Lemma.

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■ $v_1 + v_2 + v_3 = n - 1$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

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$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

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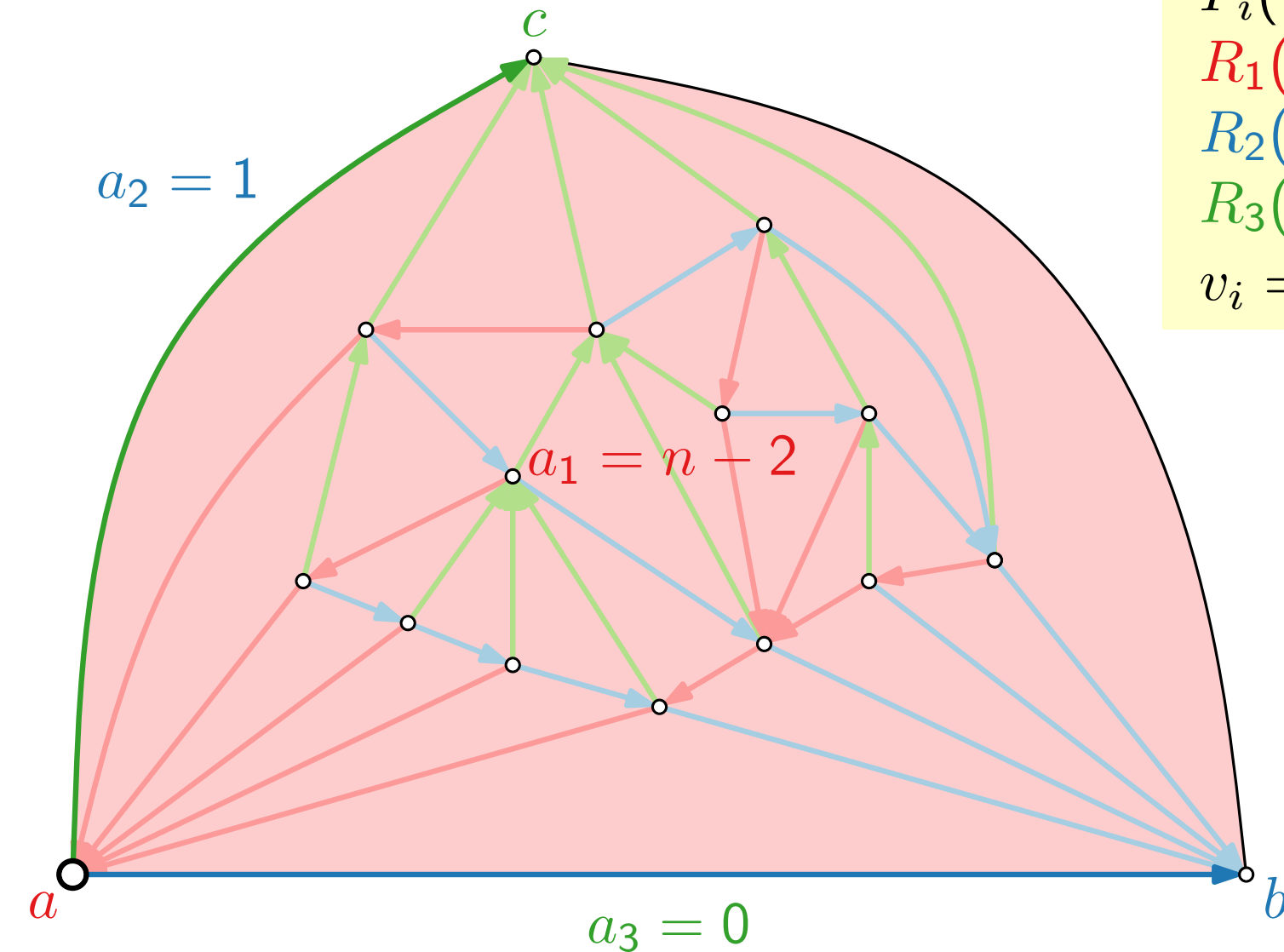
$$v_3 = 8 - 3 = 5$$

Lemma.

■ For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

■ $v_1 + v_2 + v_3 = n - 1$

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Lemma.

■ For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

■ $v_1 + v_2 + v_3 = n - 1$

Schnyder Drawing^{*}

Set $A = (0, 0)$, $B = (n - 1, 0)$, and $C = (0, n - 1)$.

Theorem.

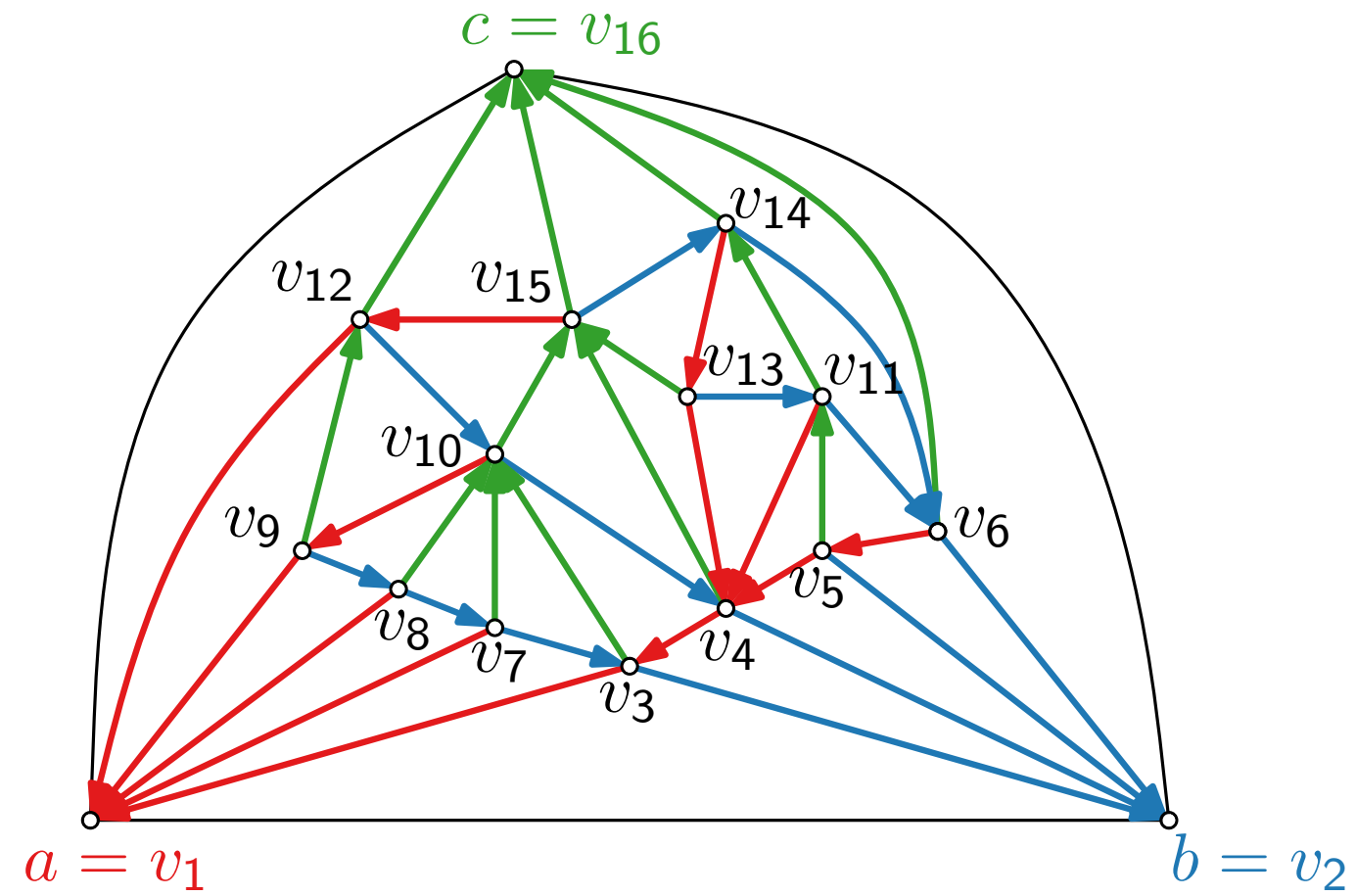
[Schnyder '90]

For a plane triangulation G , the mapping

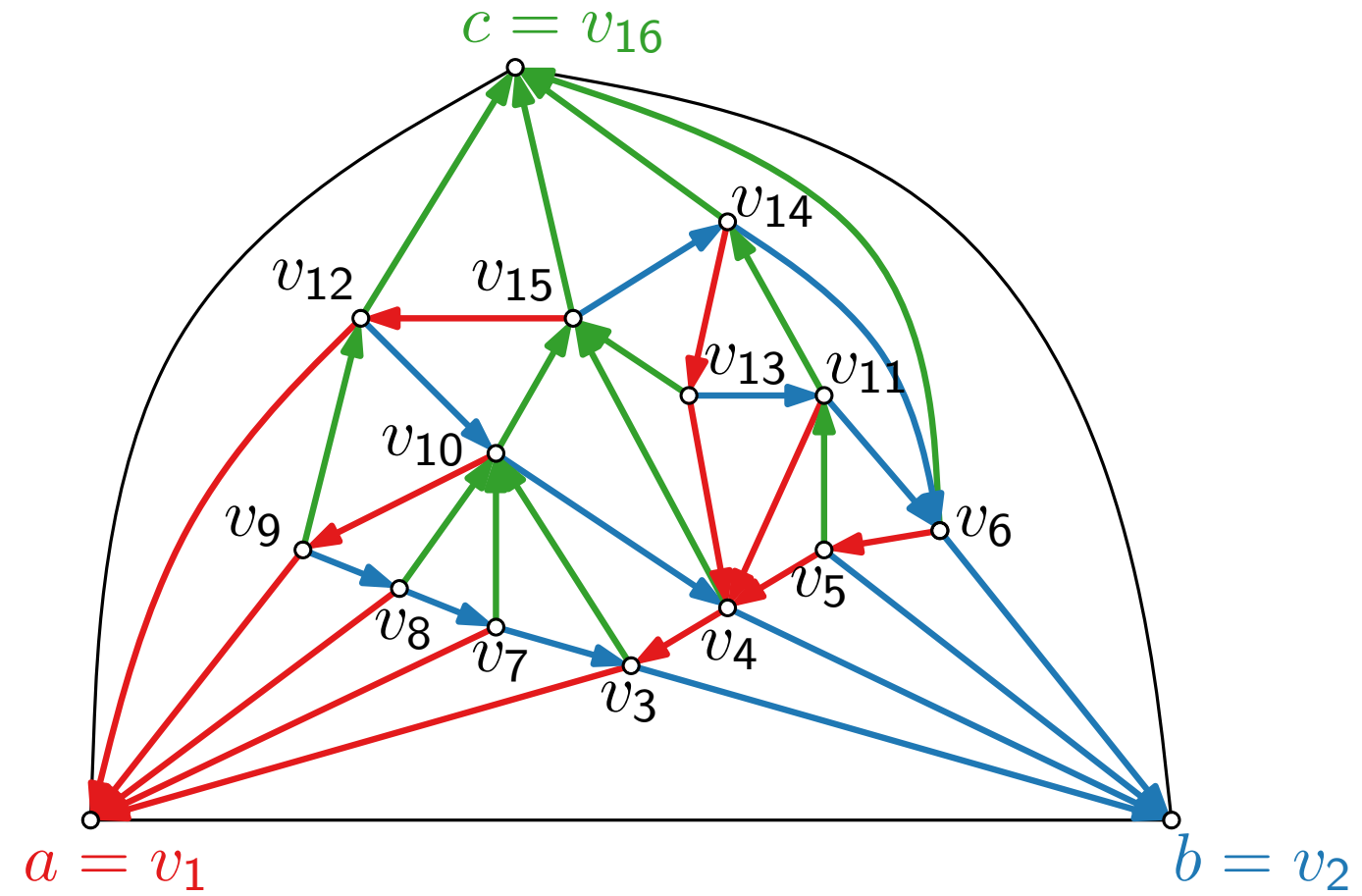
$$f: v \mapsto \frac{1}{n-1} (v_1, v_2, v_3)$$

is a weak barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(n - 2) \times (n - 2)$ grid.

Schnyder Drawing* – Example

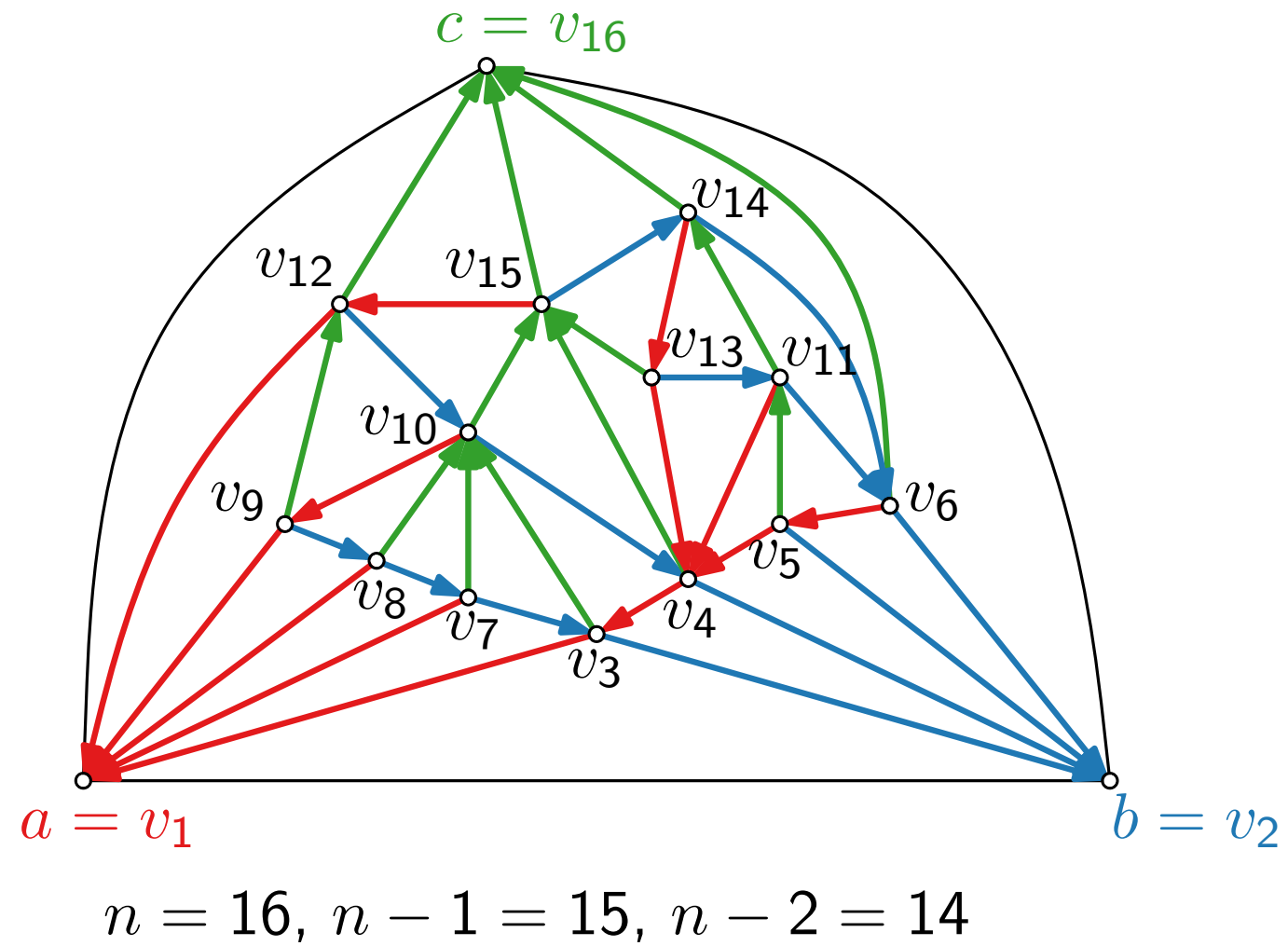
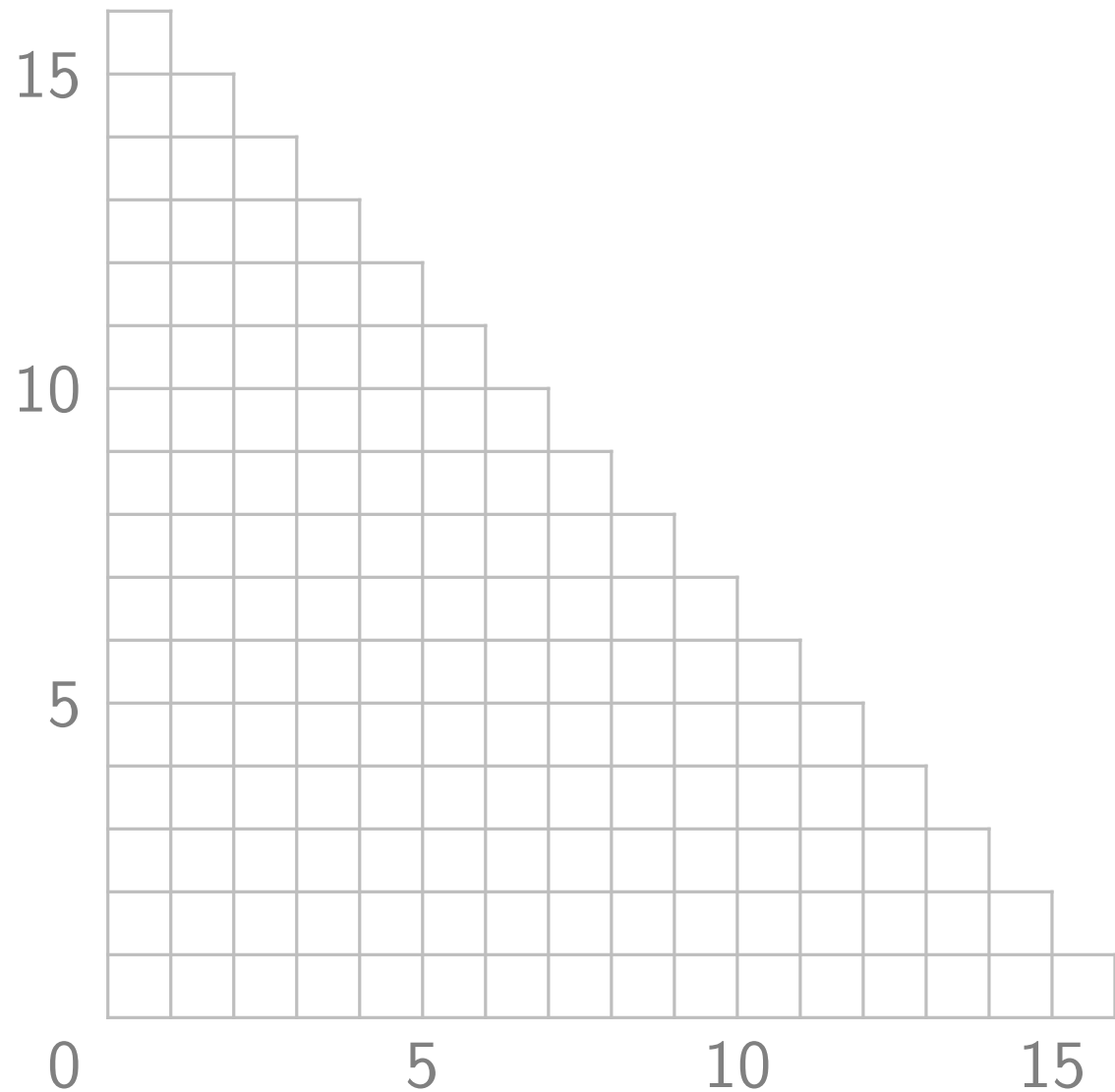


Schnyder Drawing* – Example

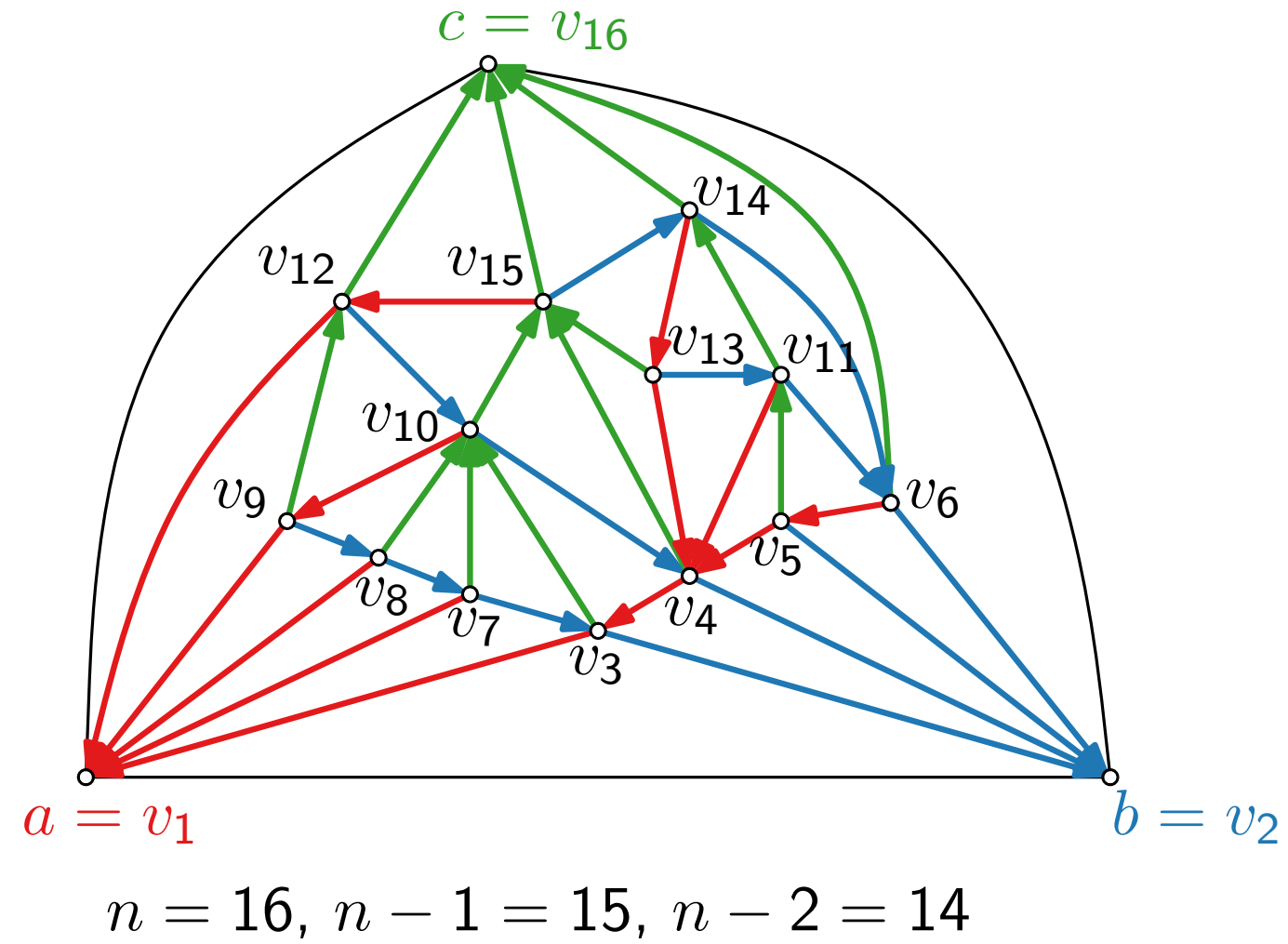
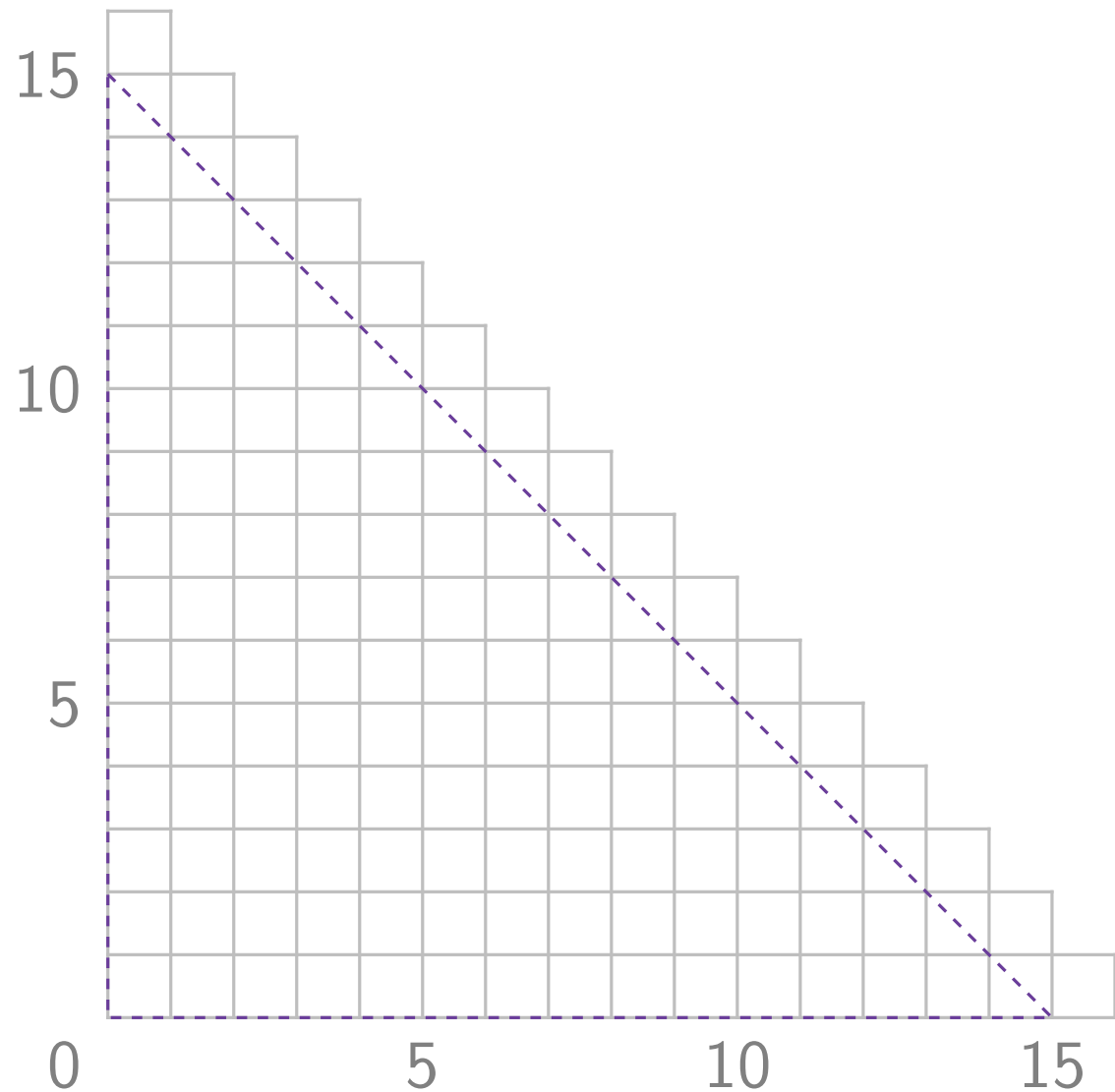


$$n = 16, n - 1 = 15, n - 2 = 14$$

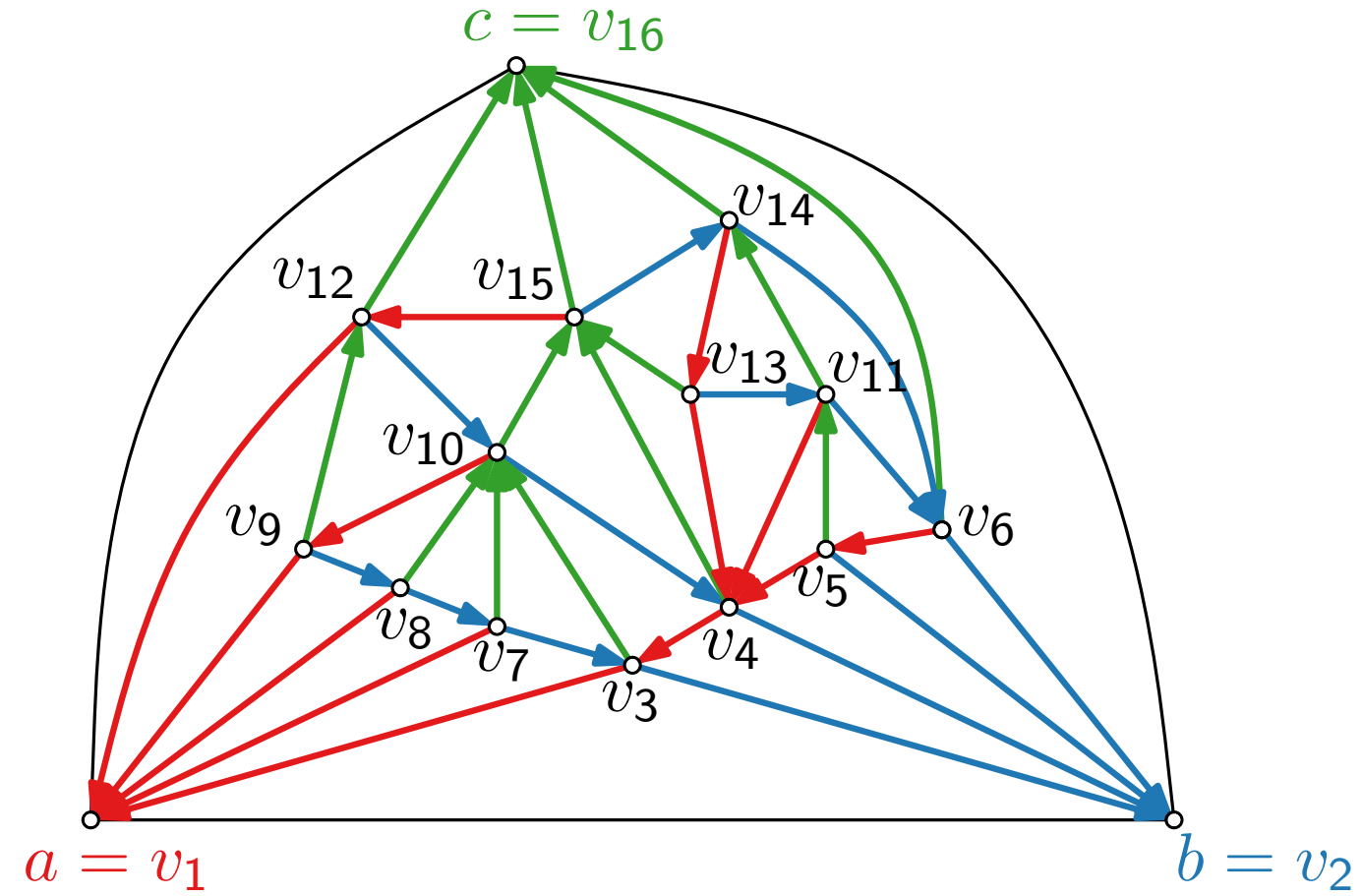
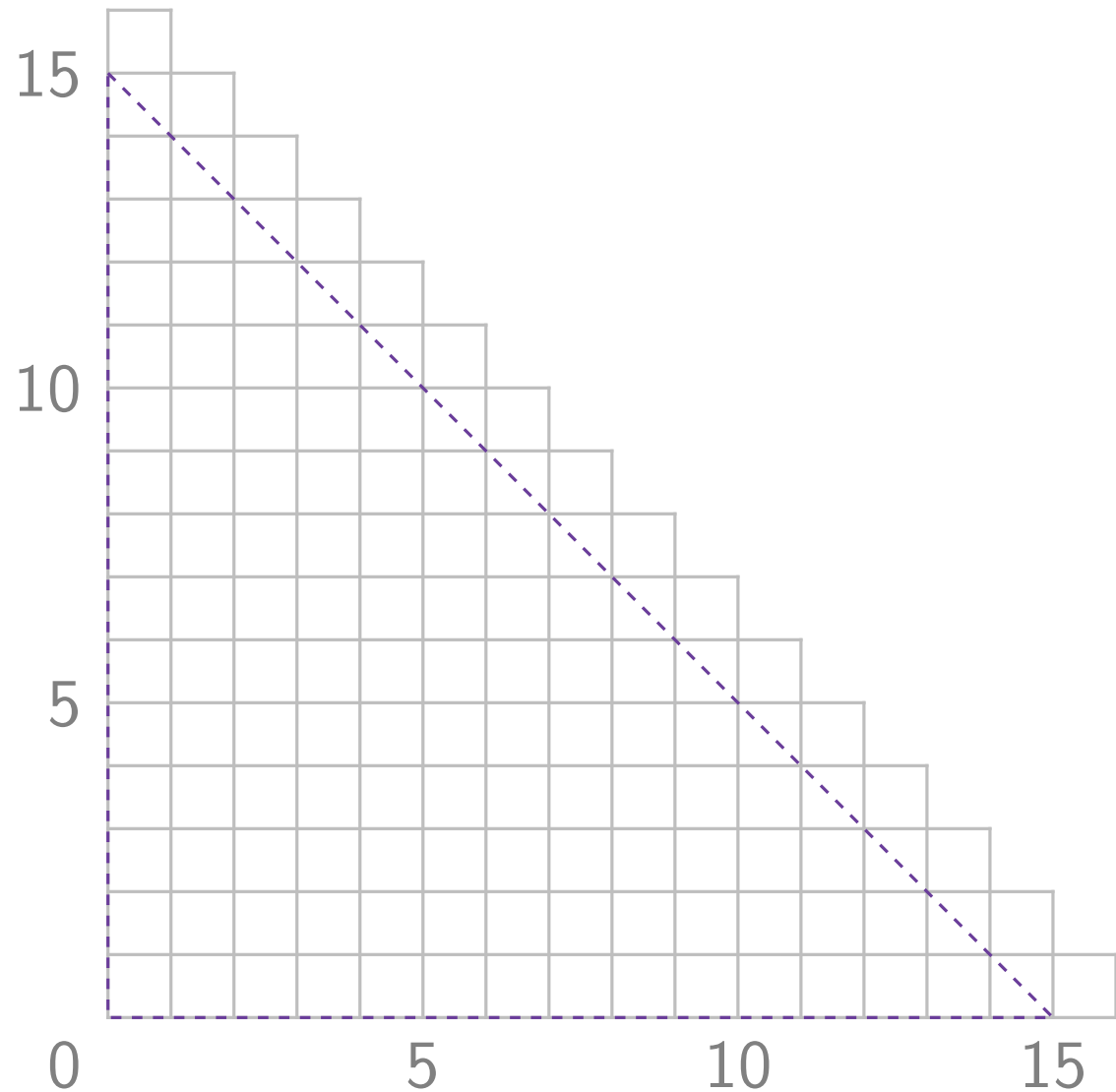
Schnyder Drawing* – Example



Schnyder Drawing* – Example



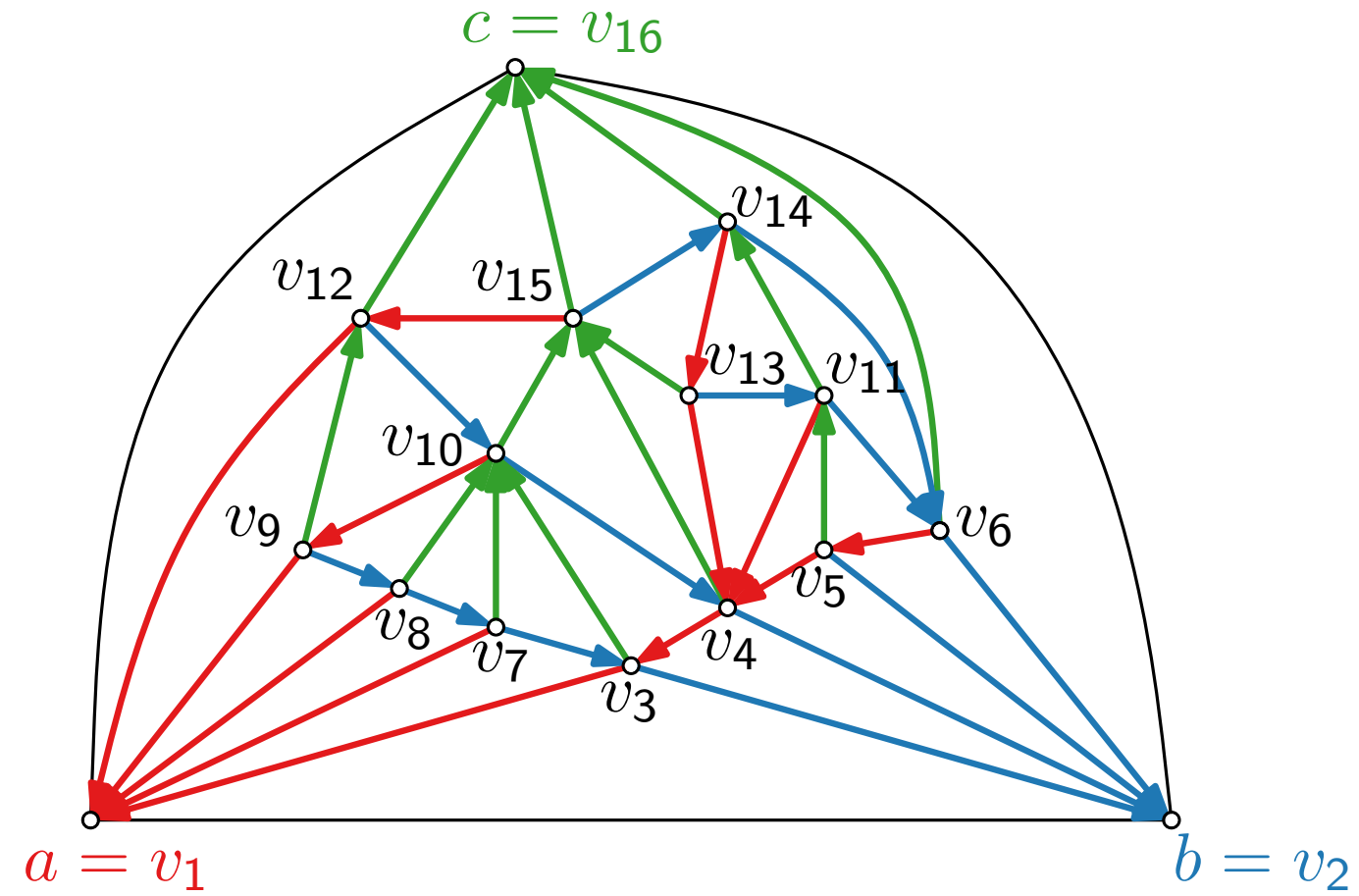
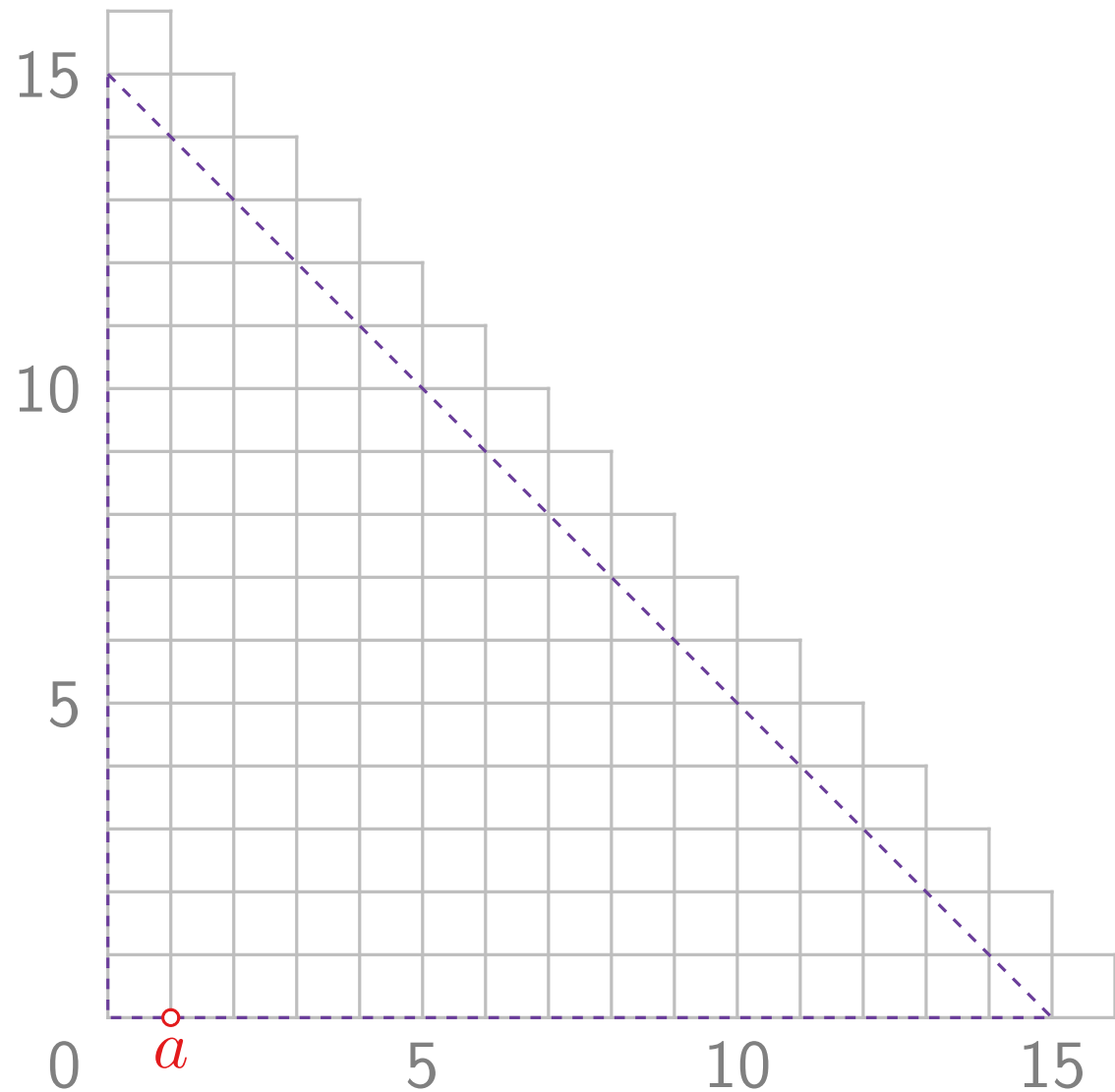
Schnyder Drawing* – Example



$$n = 16, n - 1 = 15, n - 2 = 14$$

$$f(a) =$$

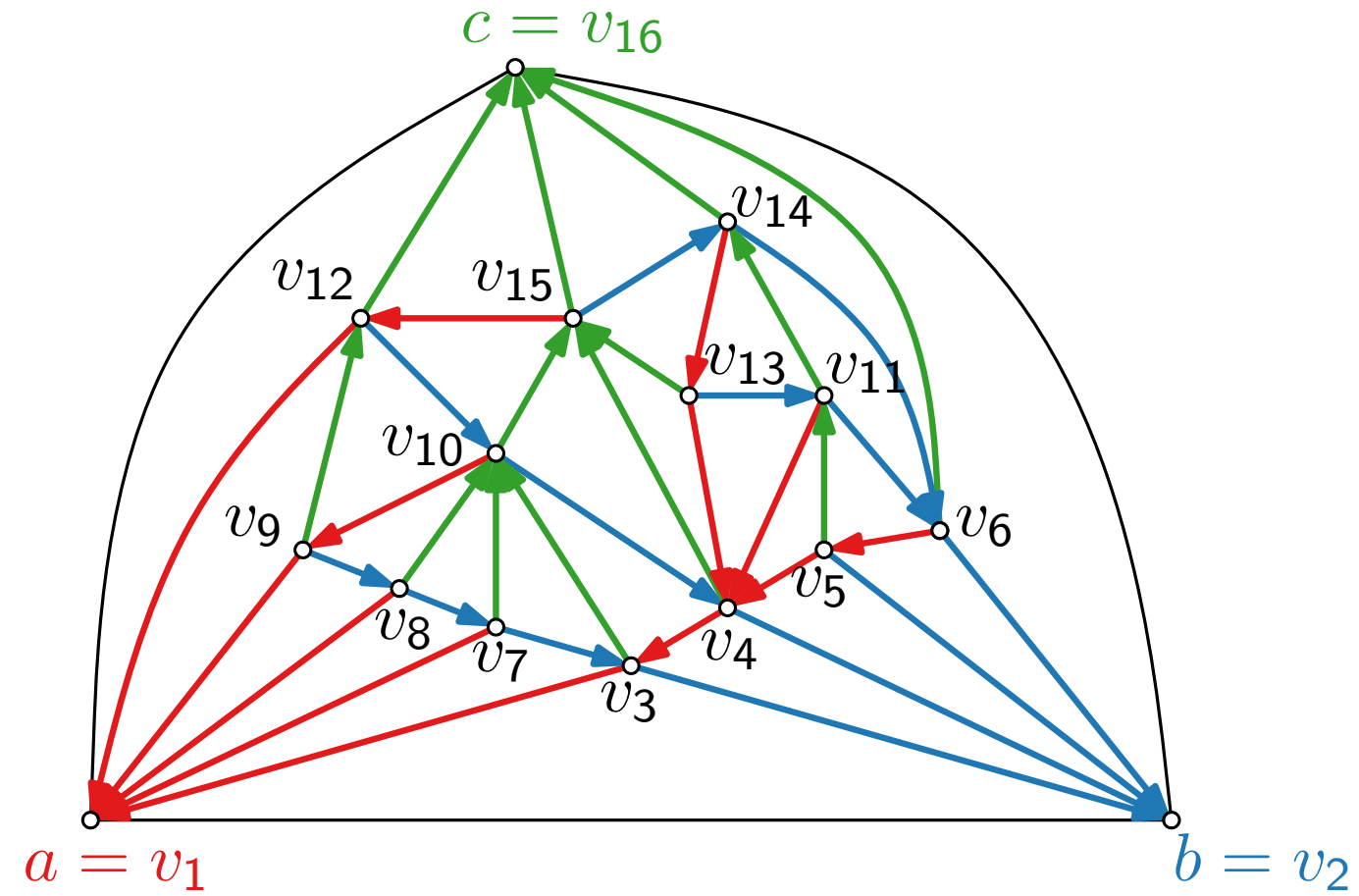
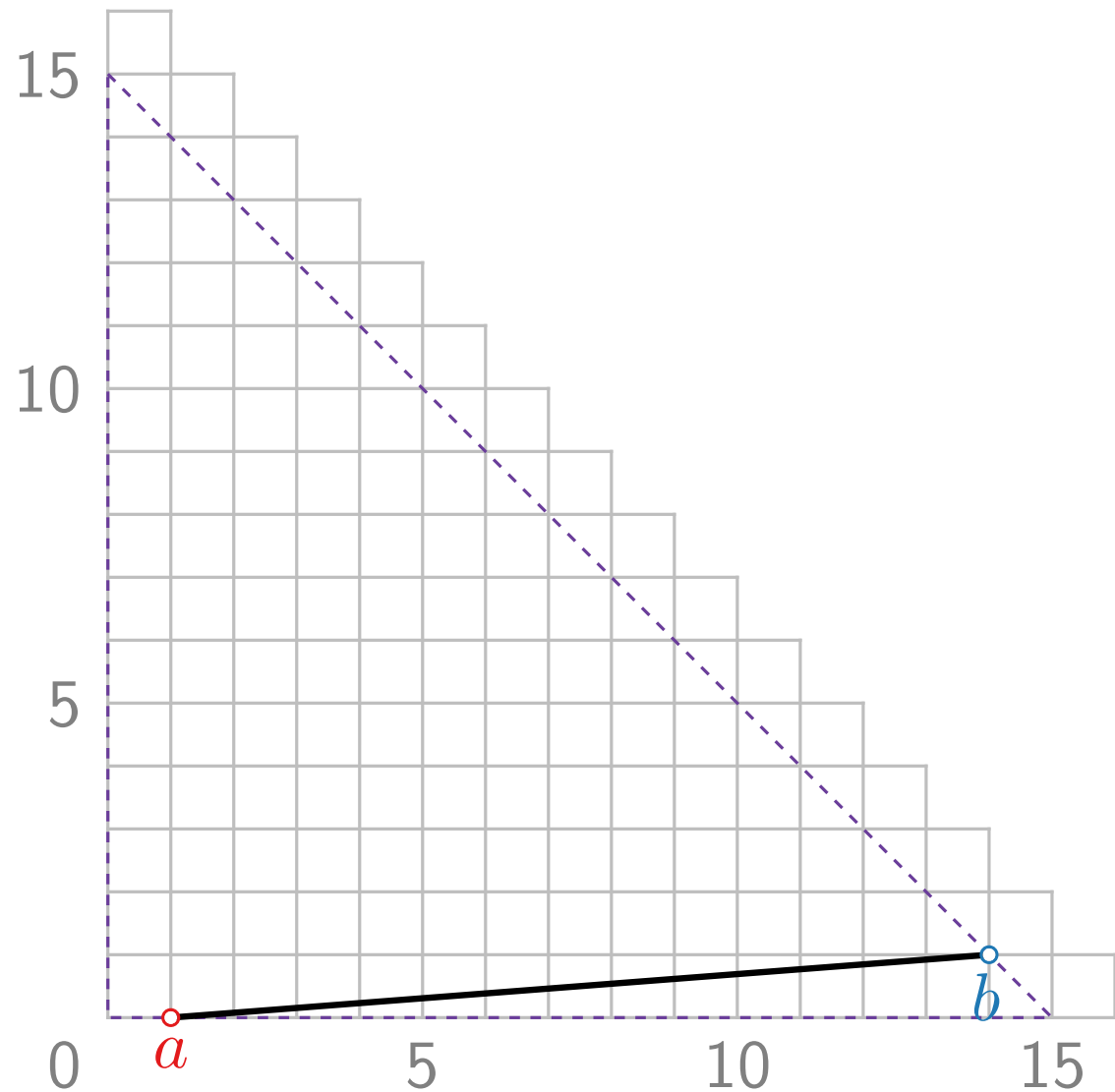
Schnyder Drawing* – Example



$$n = 16, n - 1 = 15, n - 2 = 14$$

$$f(a) = (14, 1, 0)$$

Schnyder Drawing* – Example

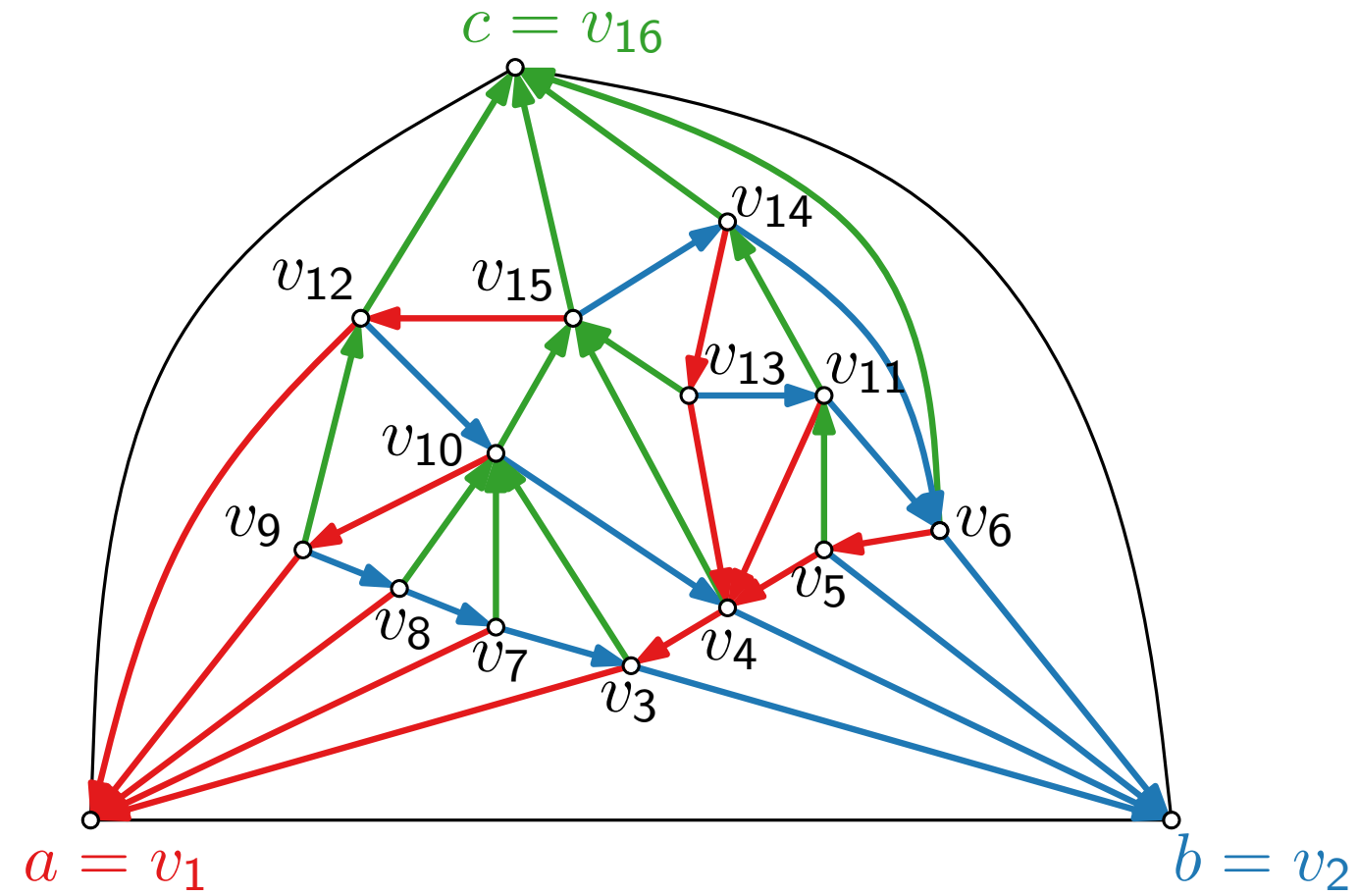
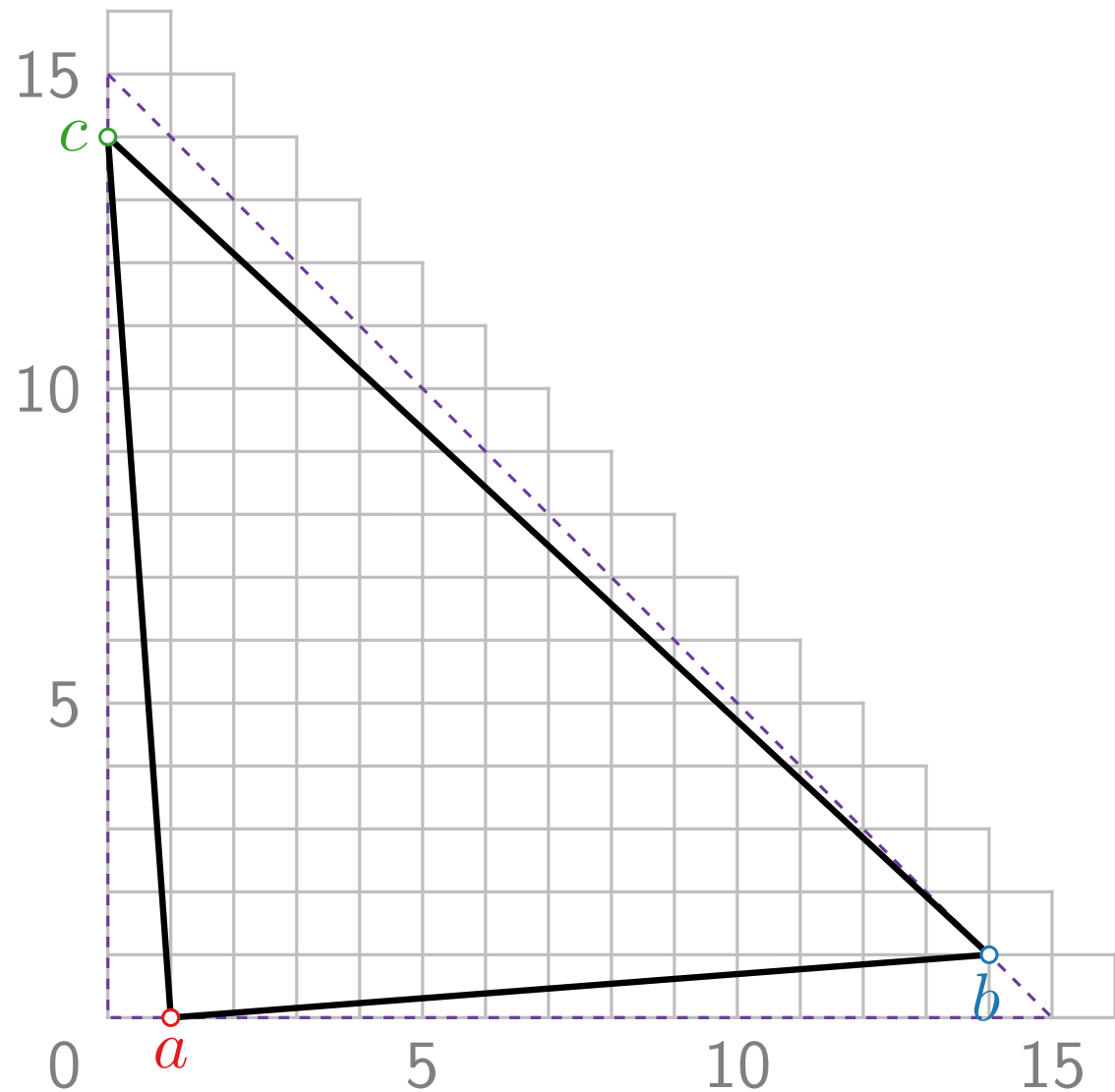


$$n = 16, n - 1 = 15, n - 2 = 14$$

$$f(a) = (14, 1, 0)$$

$$f(b) = (0, 14, 1)$$

Schnyder Drawing* – Example



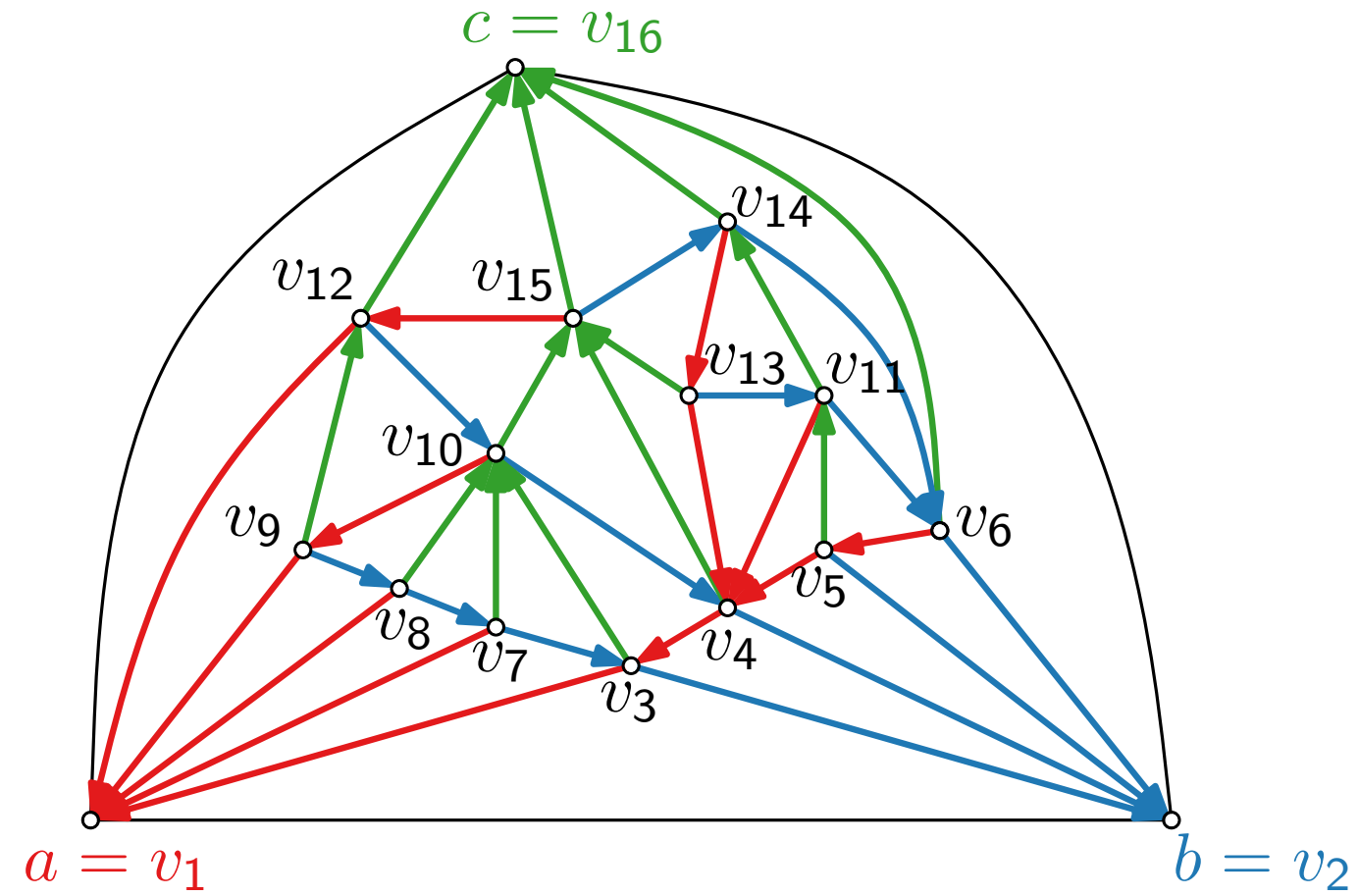
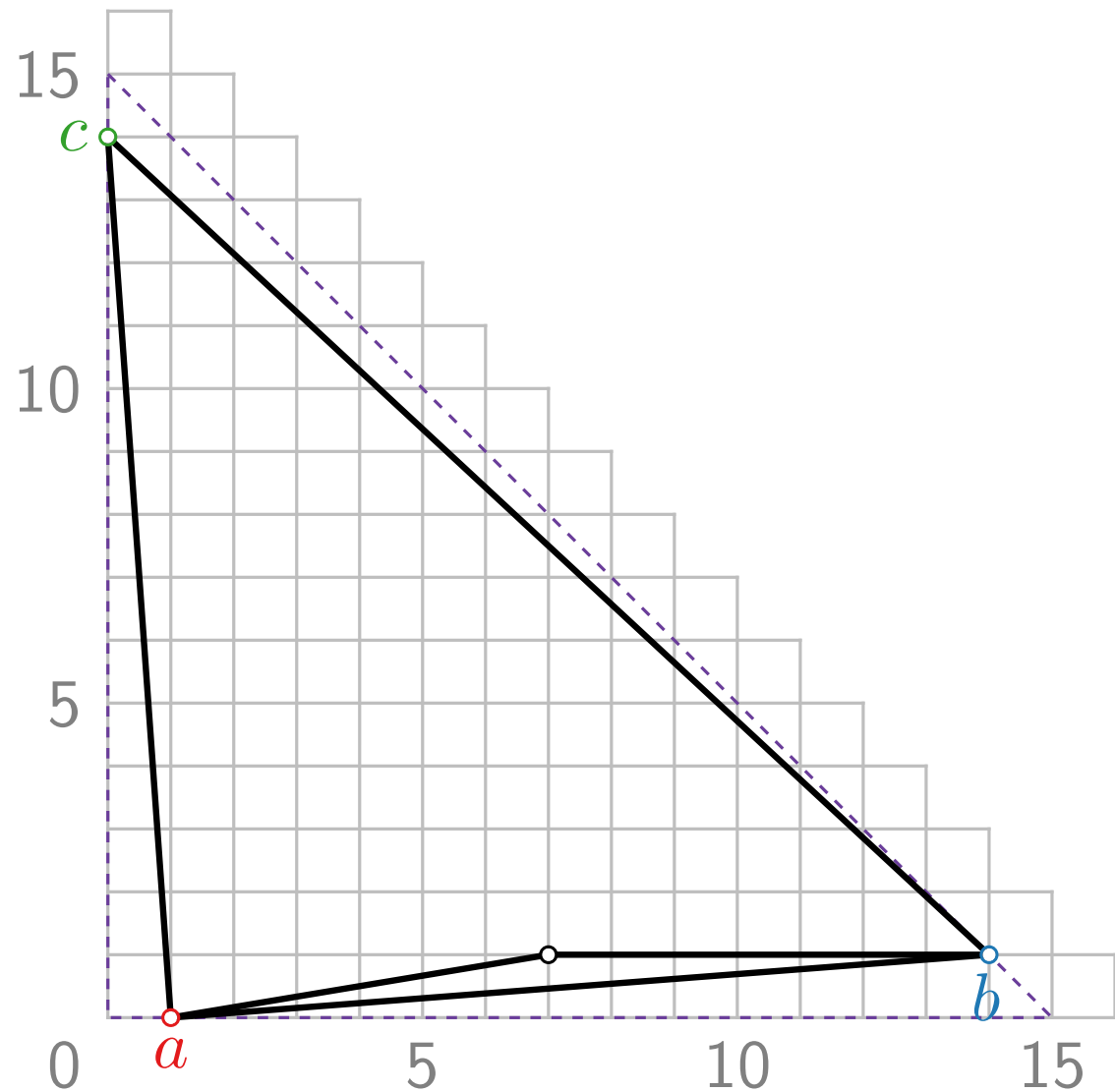
$$n = 16, n - 1 = 15, n - 2 = 14$$

$$f(a) = (14, 1, 0)$$

$$f(b) = (0, 14, 1)$$

$$f(c) = (1, 0, 14)$$

Schnyder Drawing* – Example



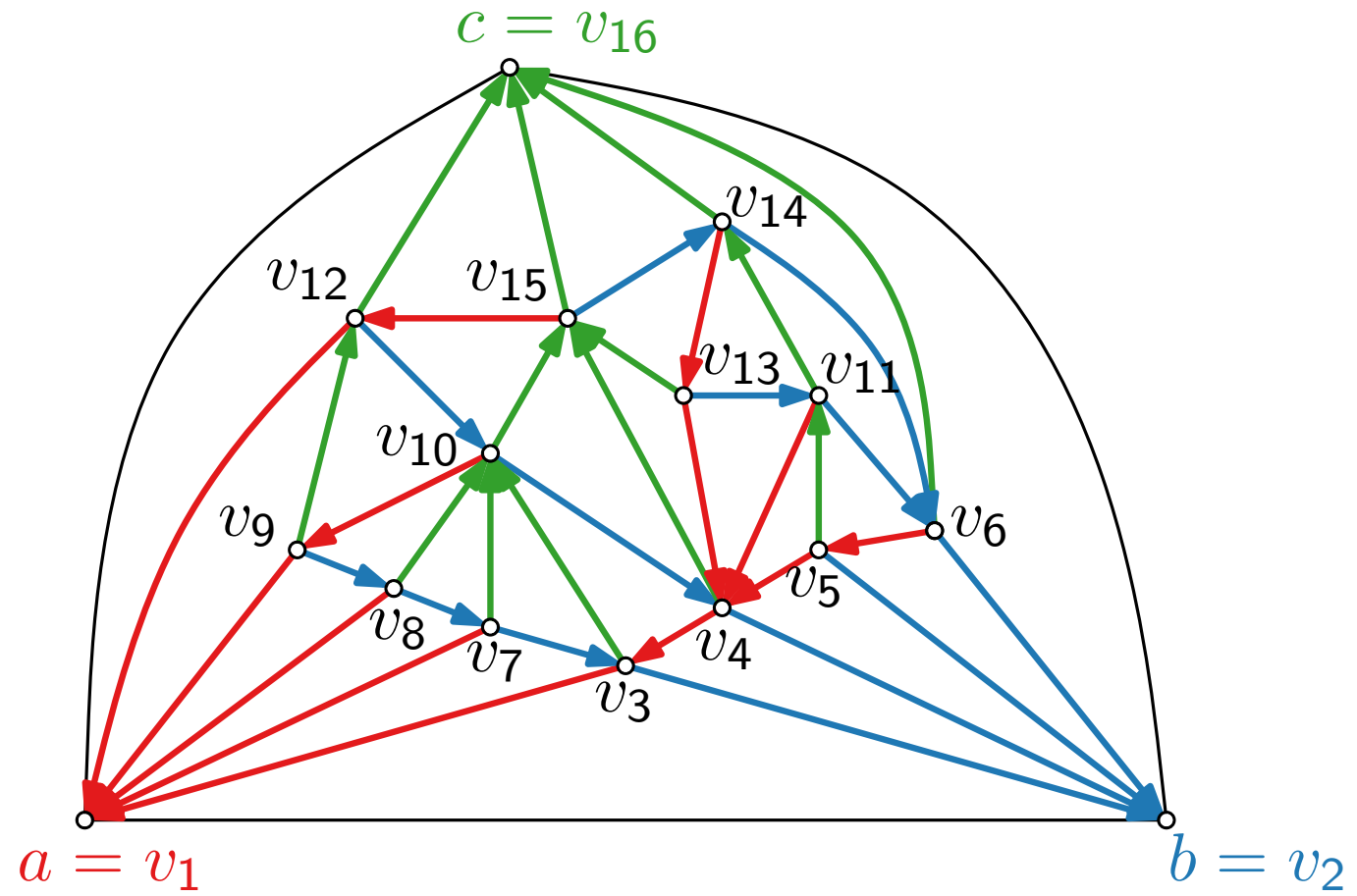
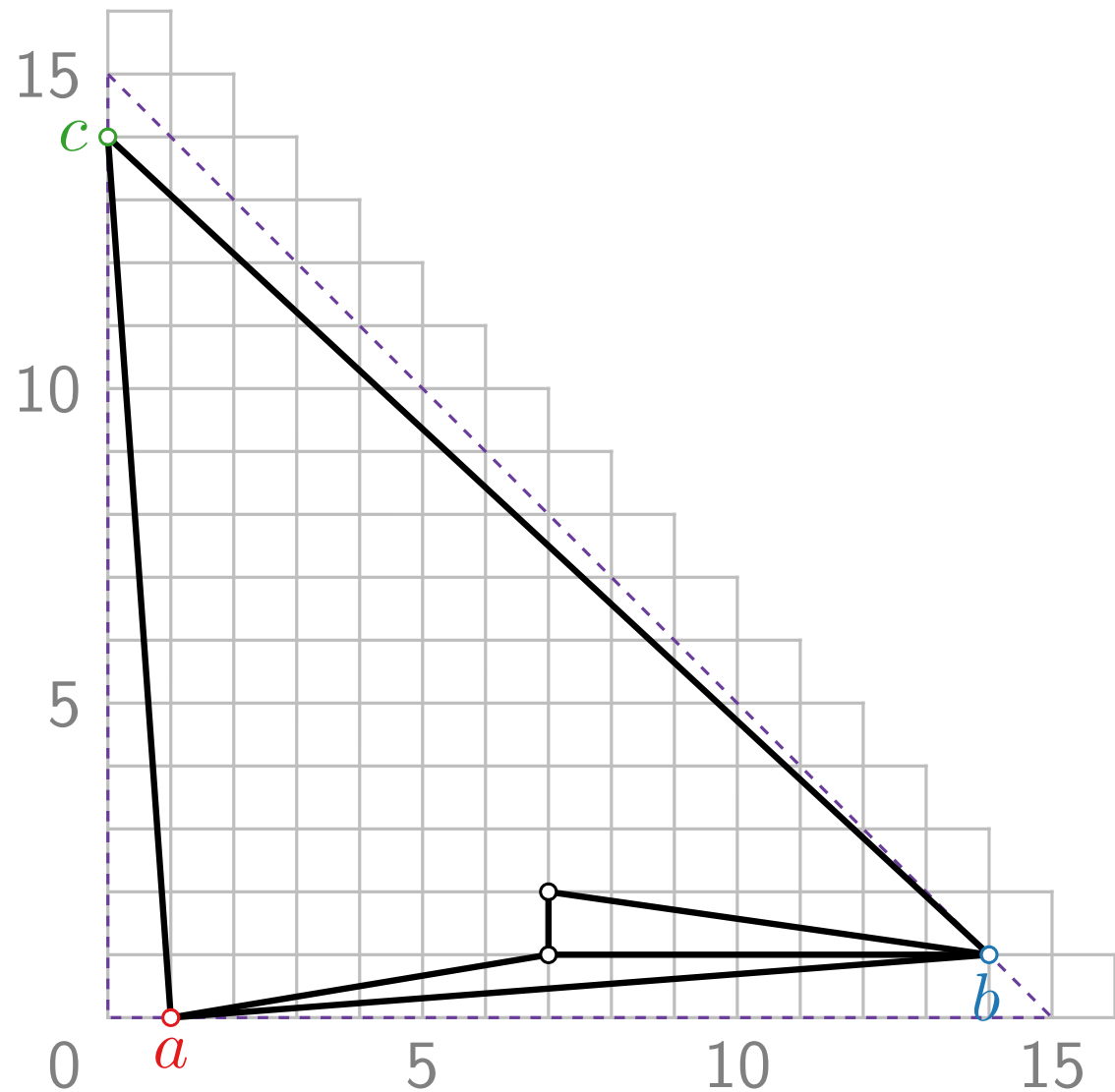
$$n = 16, n - 1 = 15, nf(v_2) = (7, 7, 1)$$

$$f(a) = (14, 1, 0)$$

$$f(b) = (0, 14, 1)$$

$$f(c) = (1, 0, 14)$$

Schnyder Drawing* – Example



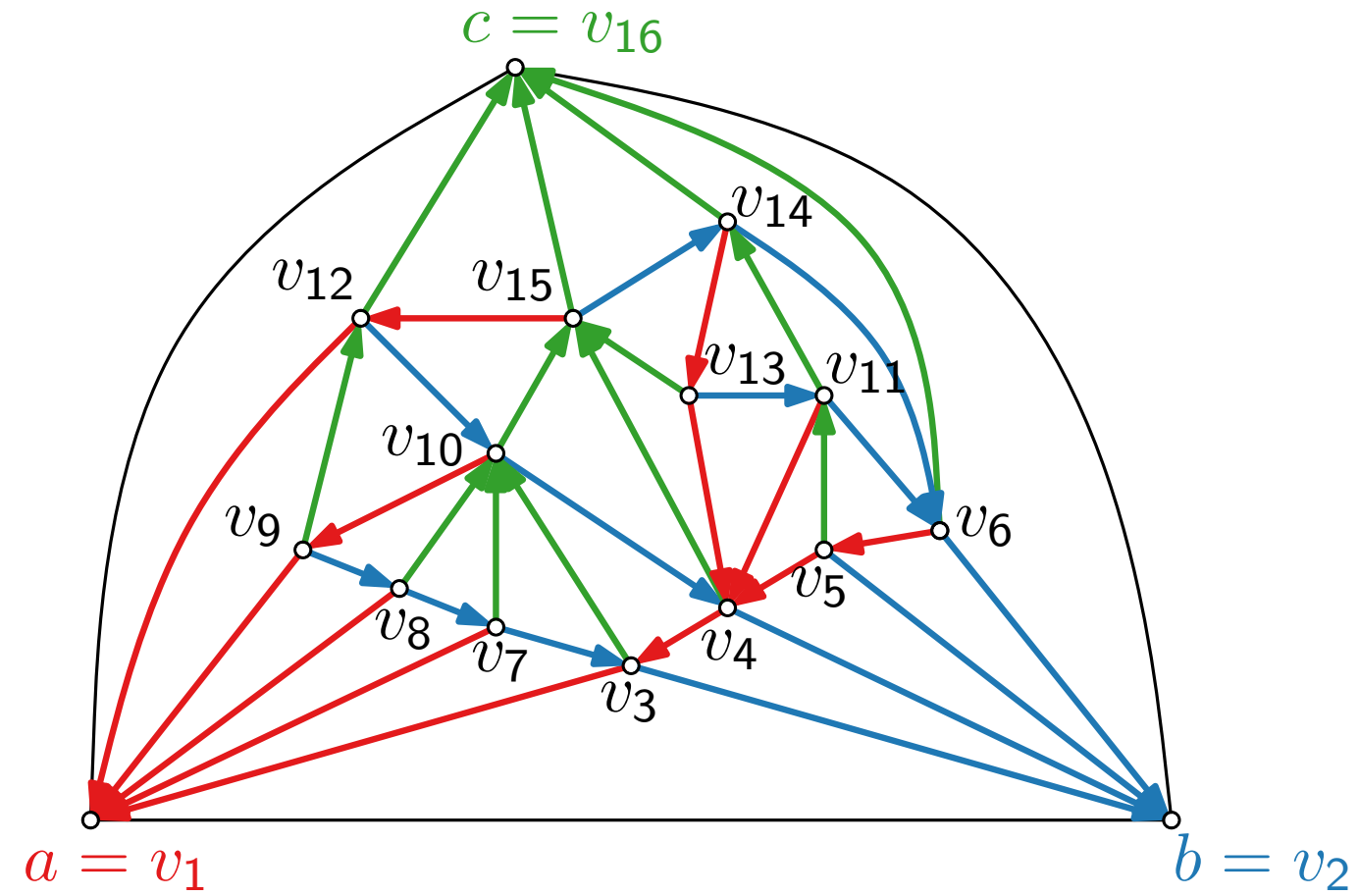
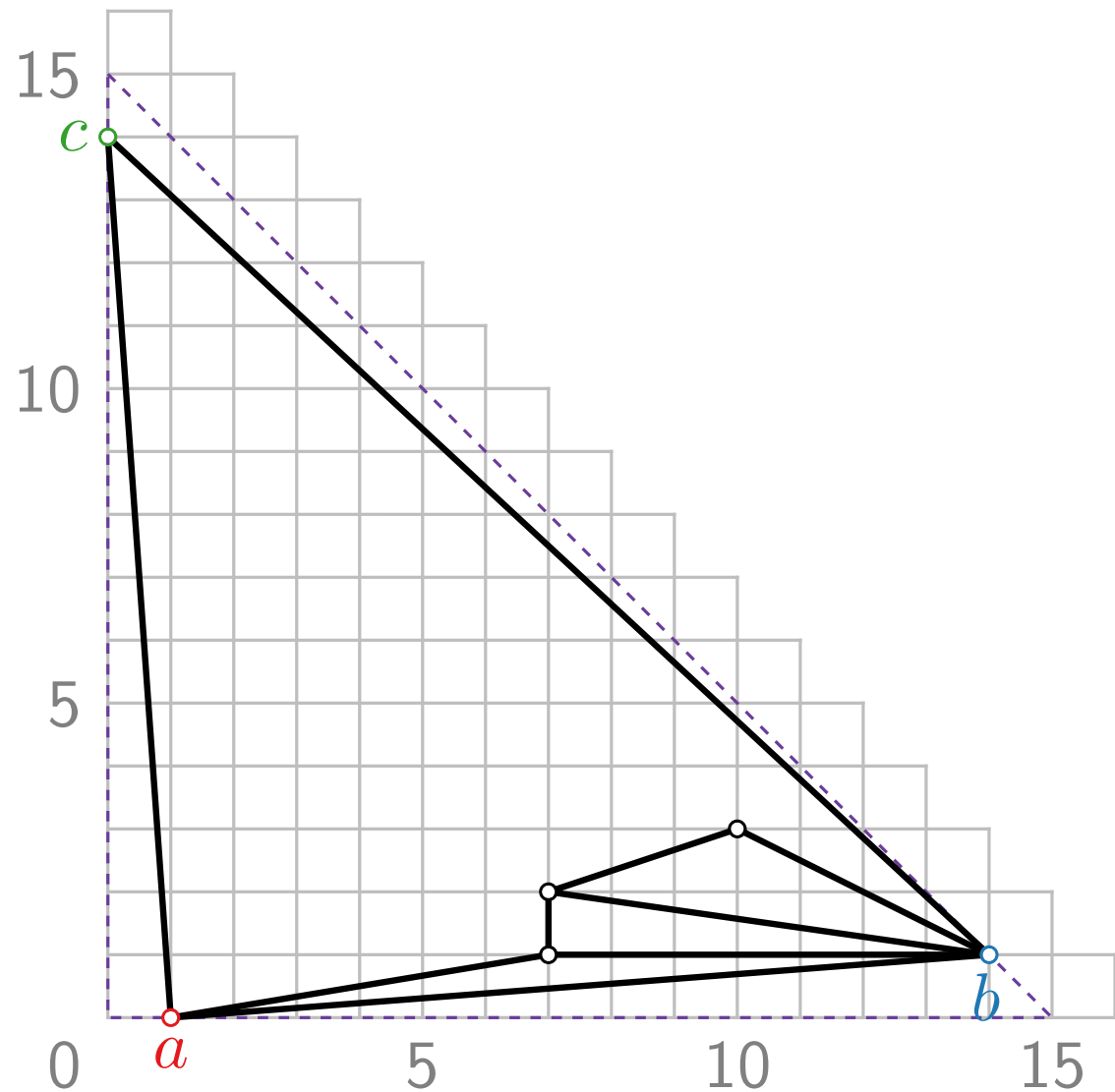
$$n = 16, n - 1 = 15, nf(v_2) = (7, 7, 1)$$

$$f(a) = (14, 1, 0) \quad f(v_4) = (6, 7, 2)$$

$$f(b) = (0, 14, 1)$$

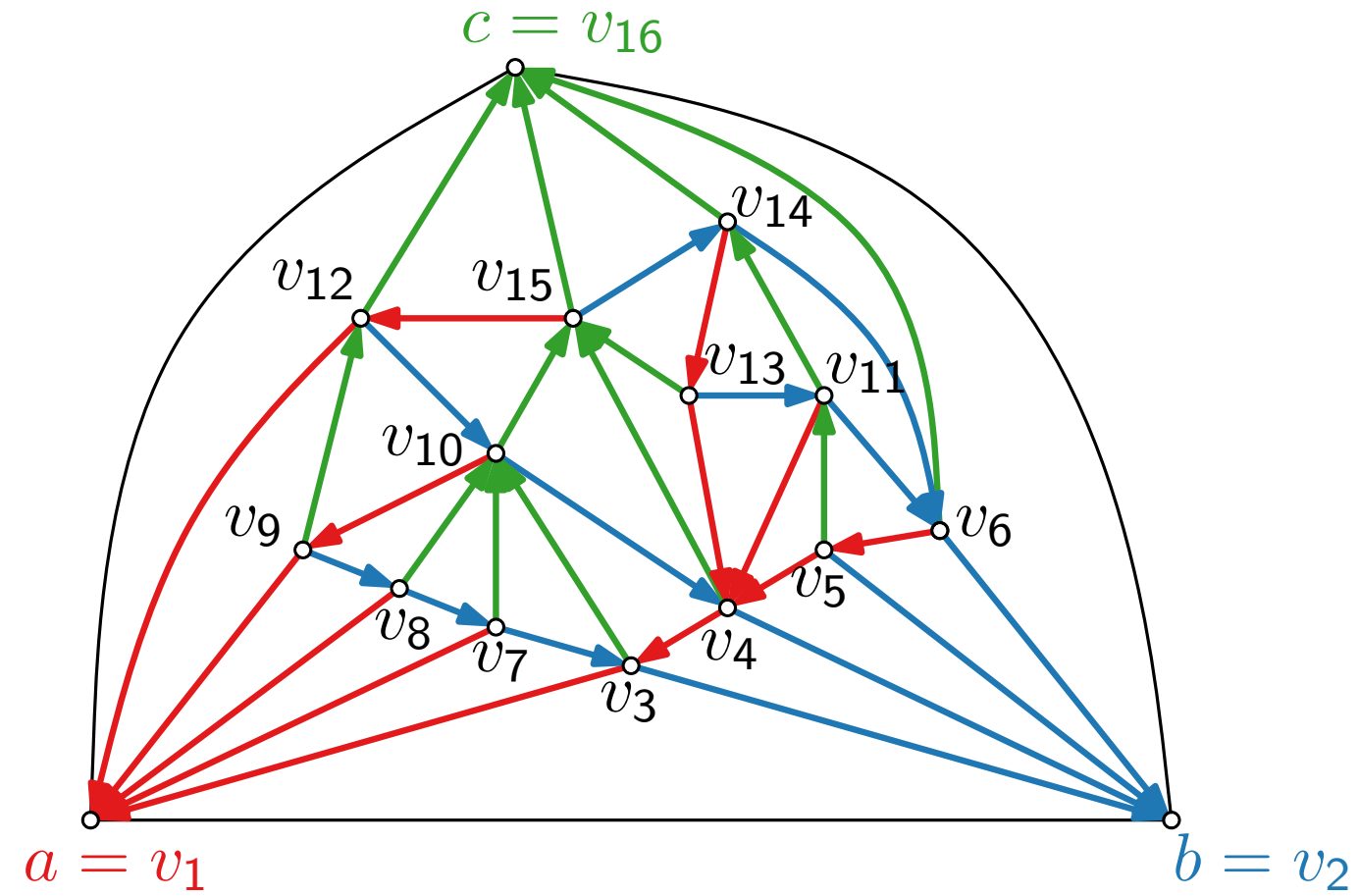
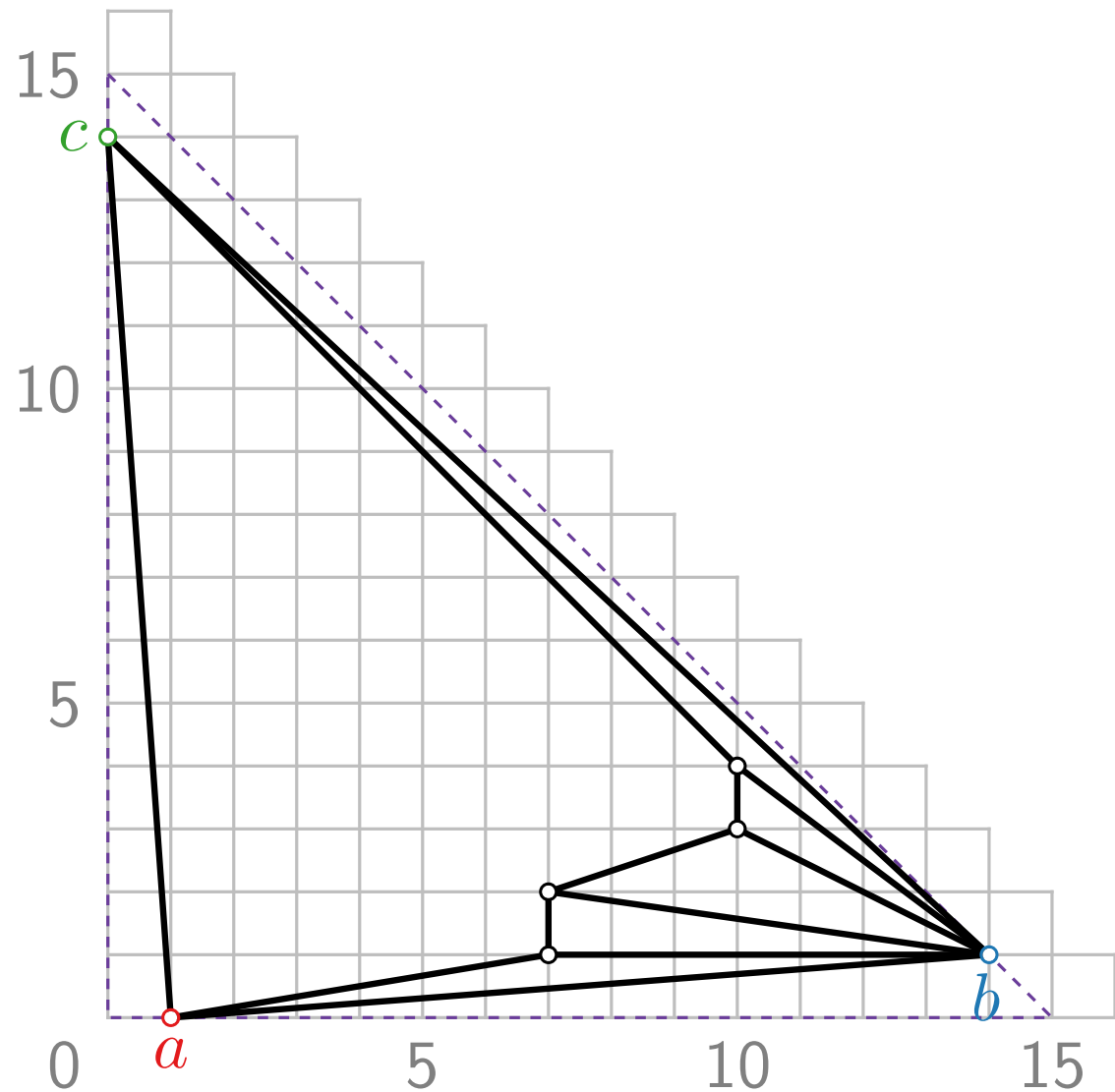
$$f(c) = (1, 0, 14)$$

Schnyder Drawing* – Example



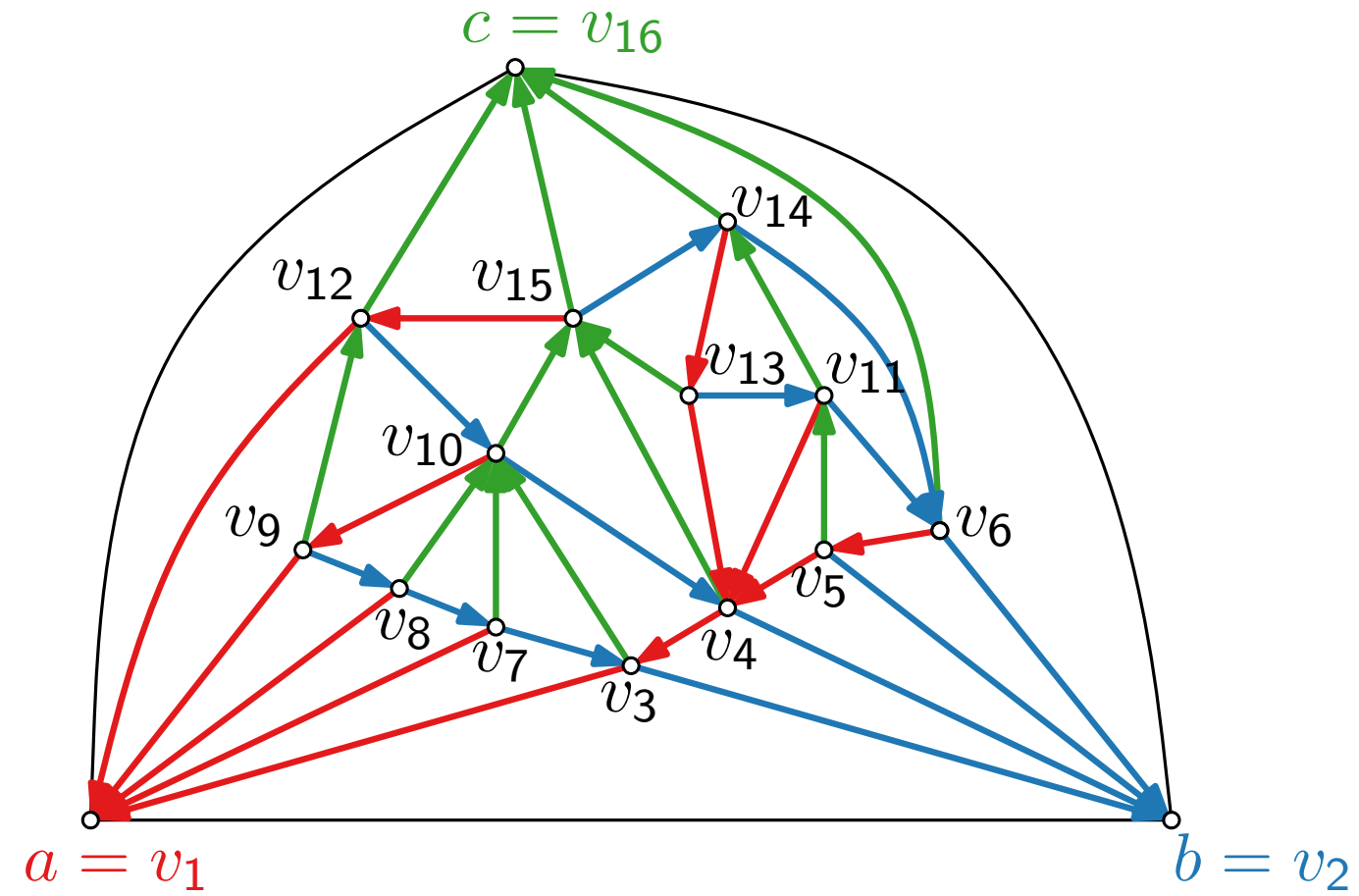
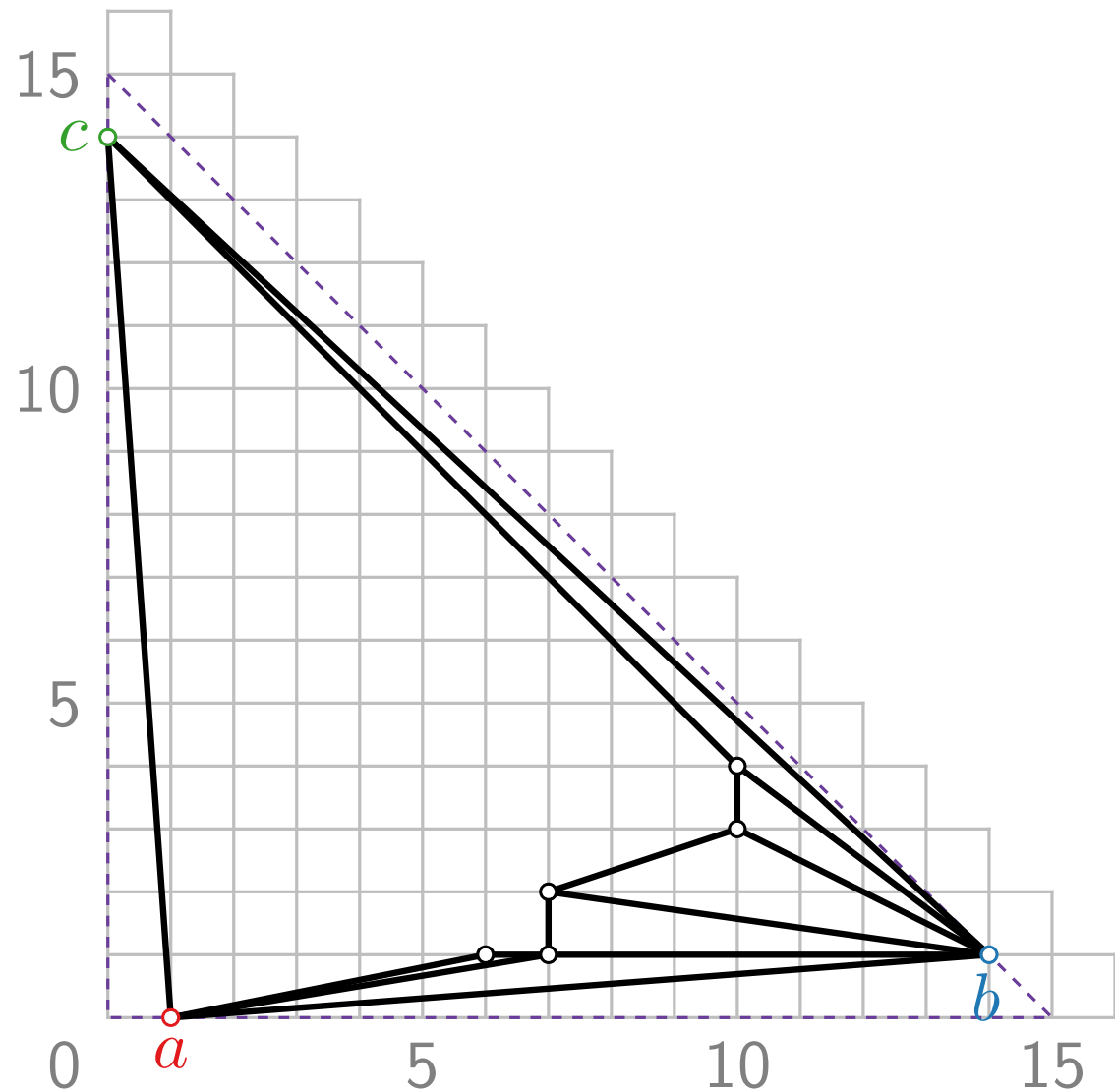
$$\begin{aligned}
 n &= 16, \quad n - 1 = 15, \quad nf(v_2) = (7, 7, 1) \\
 f(a) &= (14, 1, 0) & f(v_4) &= (6, 7, 2) \\
 f(b) &= (0, 14, 1) & f(v_5) &= (2, 10, 3) \\
 f(c) &= (1, 0, 14)
 \end{aligned}$$

Schnyder Drawing* – Example



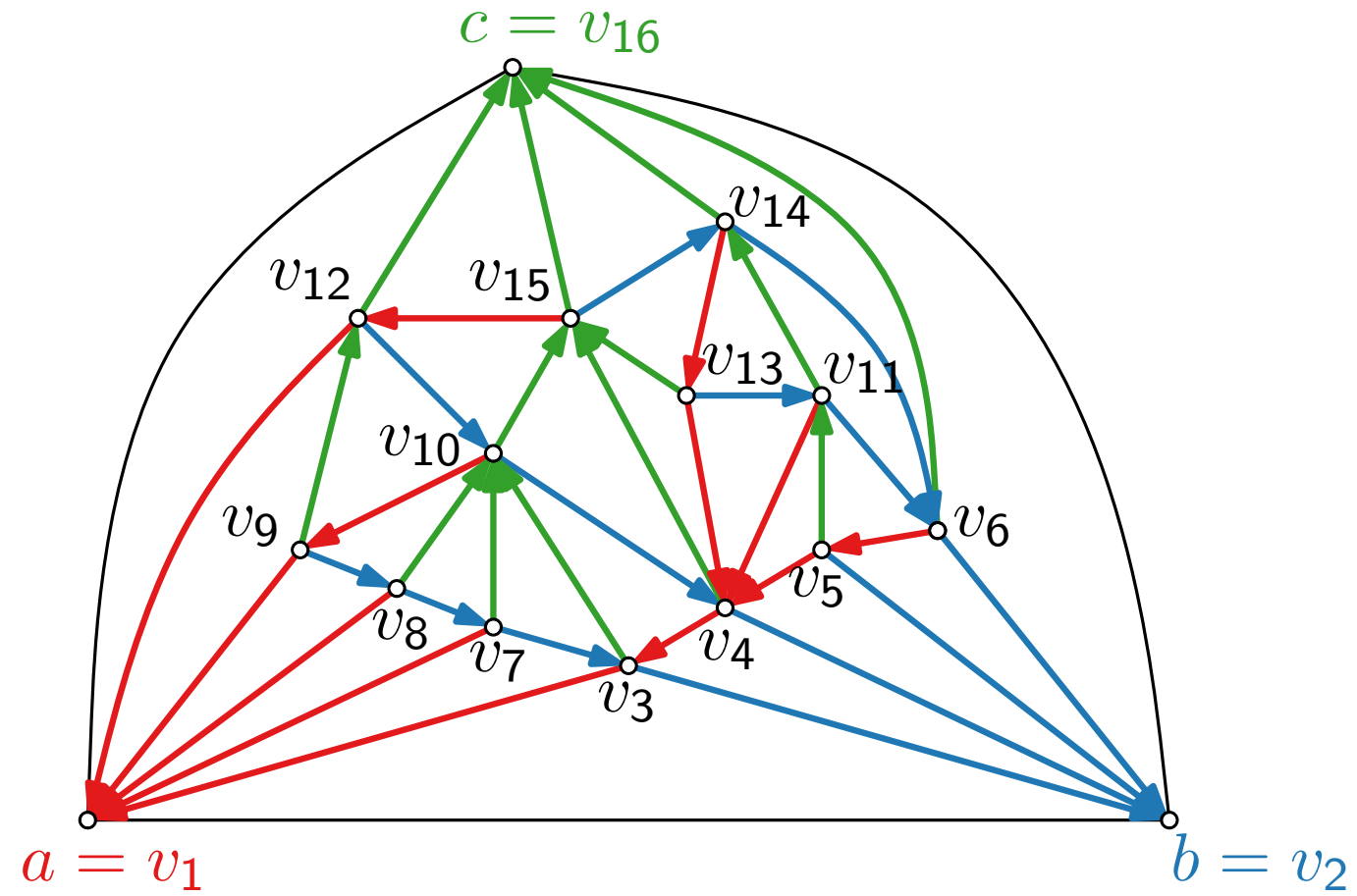
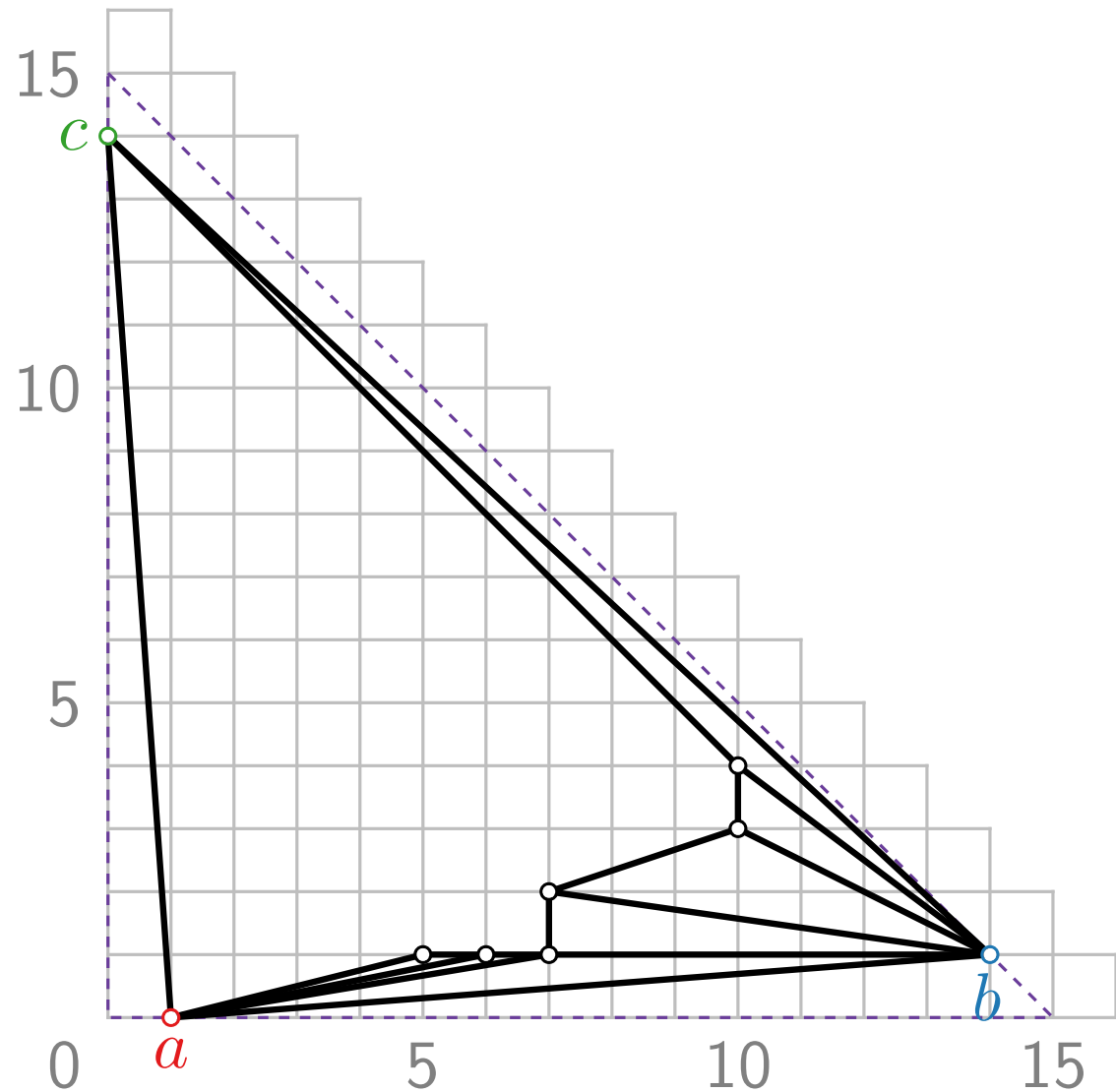
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 \end{aligned}$$

Schnyder Drawing* – Example



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Schnyder Drawing* – Example



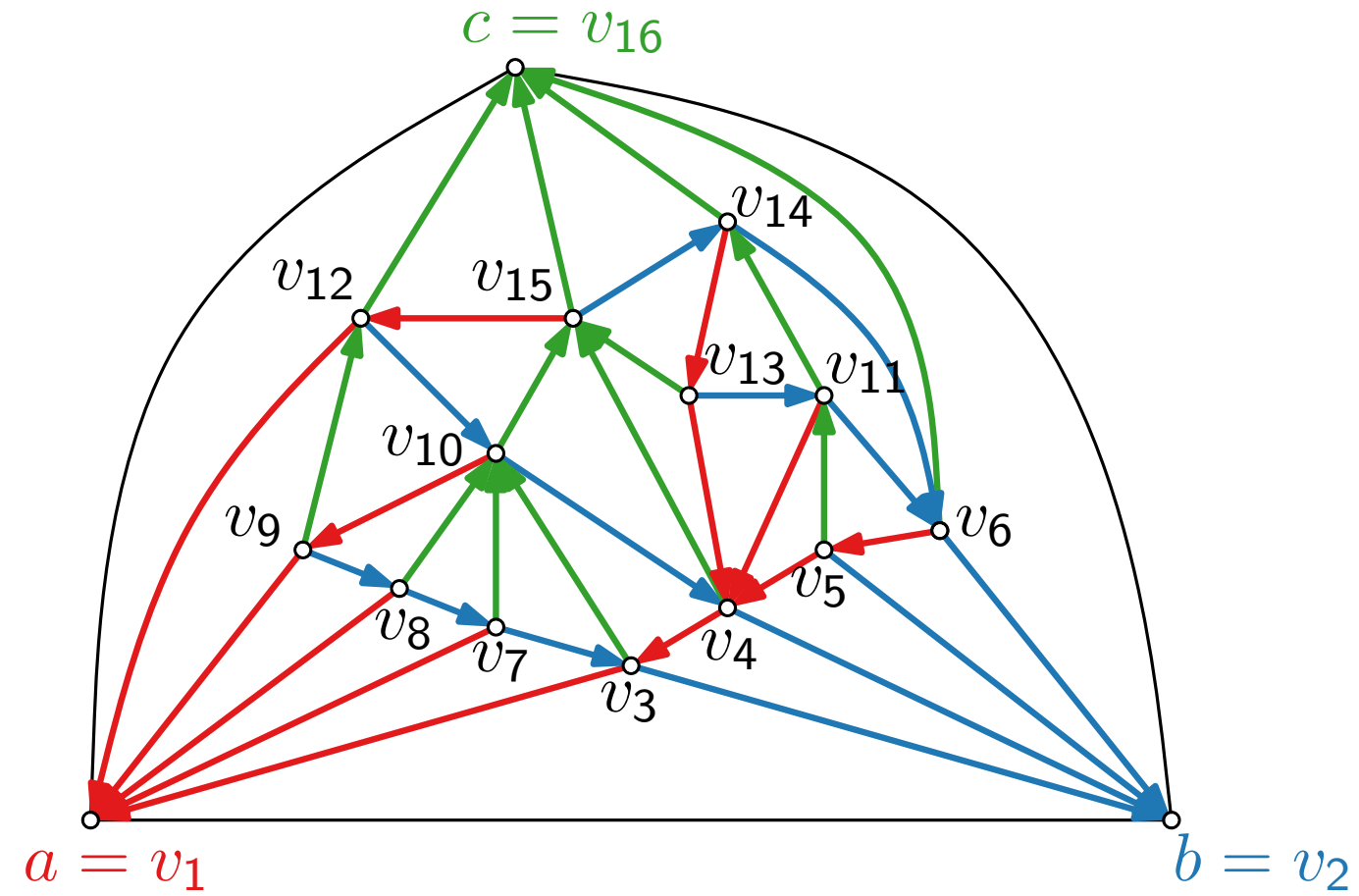
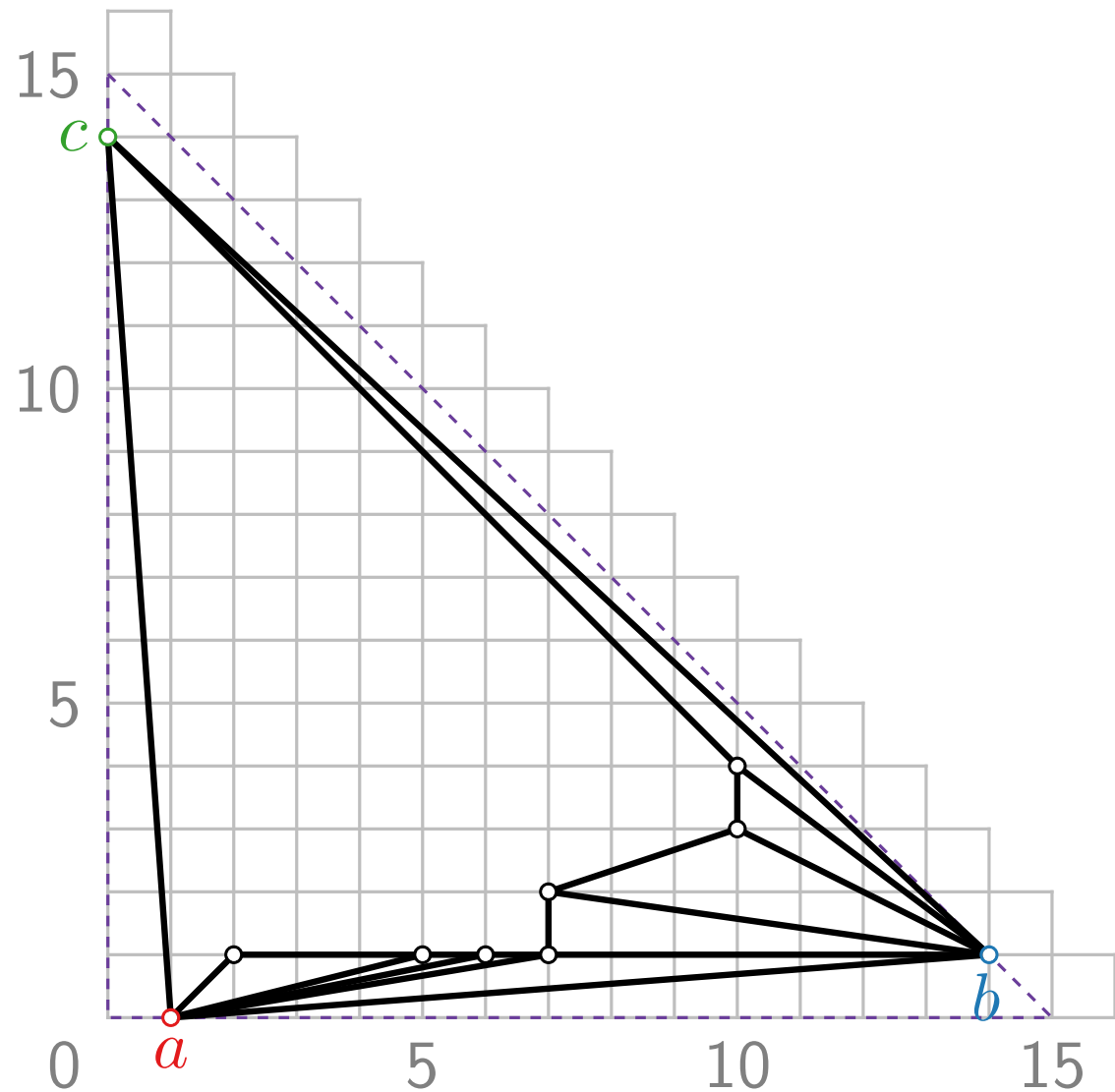
$$n = 16, n - 1 = 15, nf(v_2) = (7, 7, 1)$$

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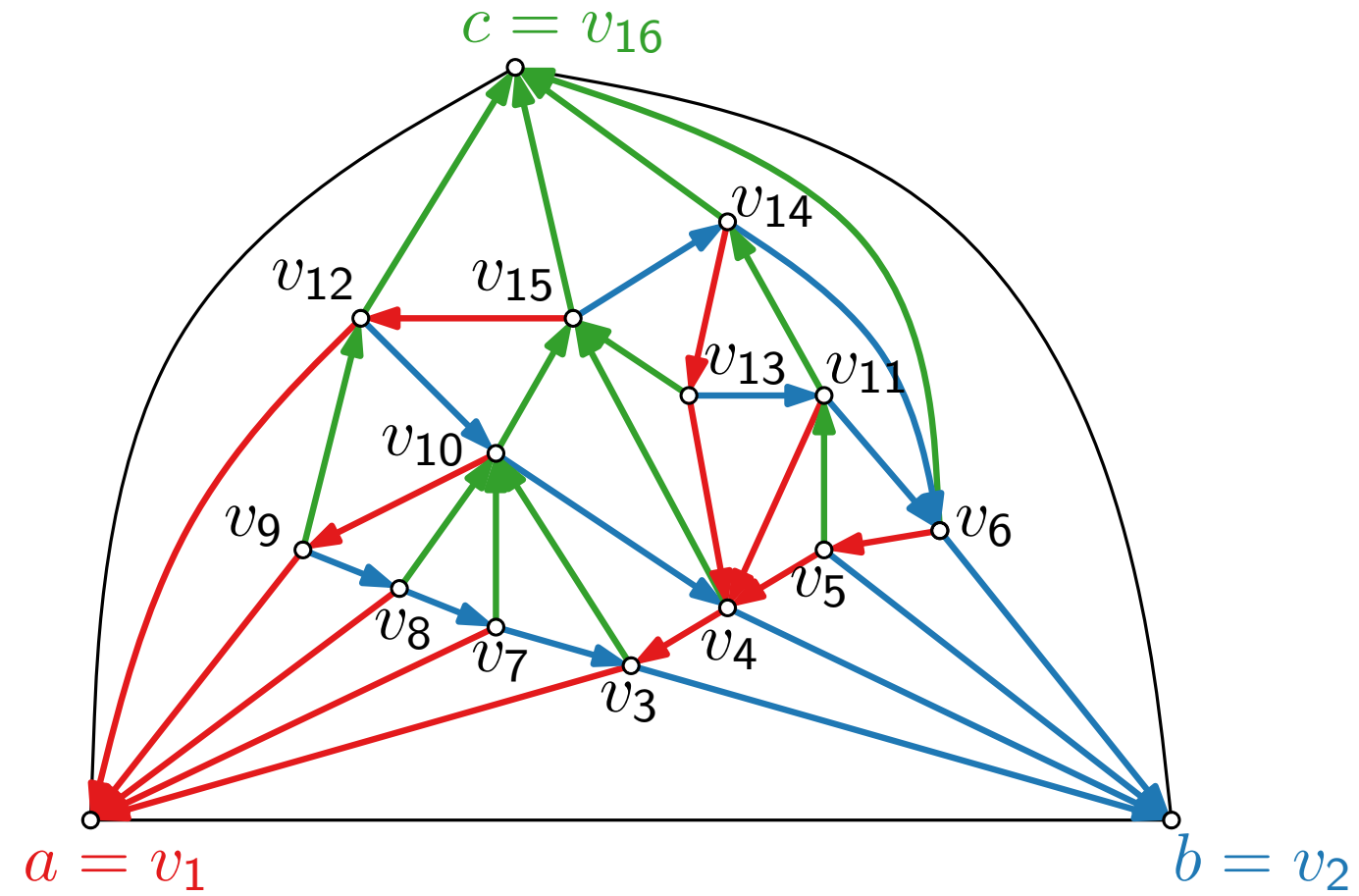
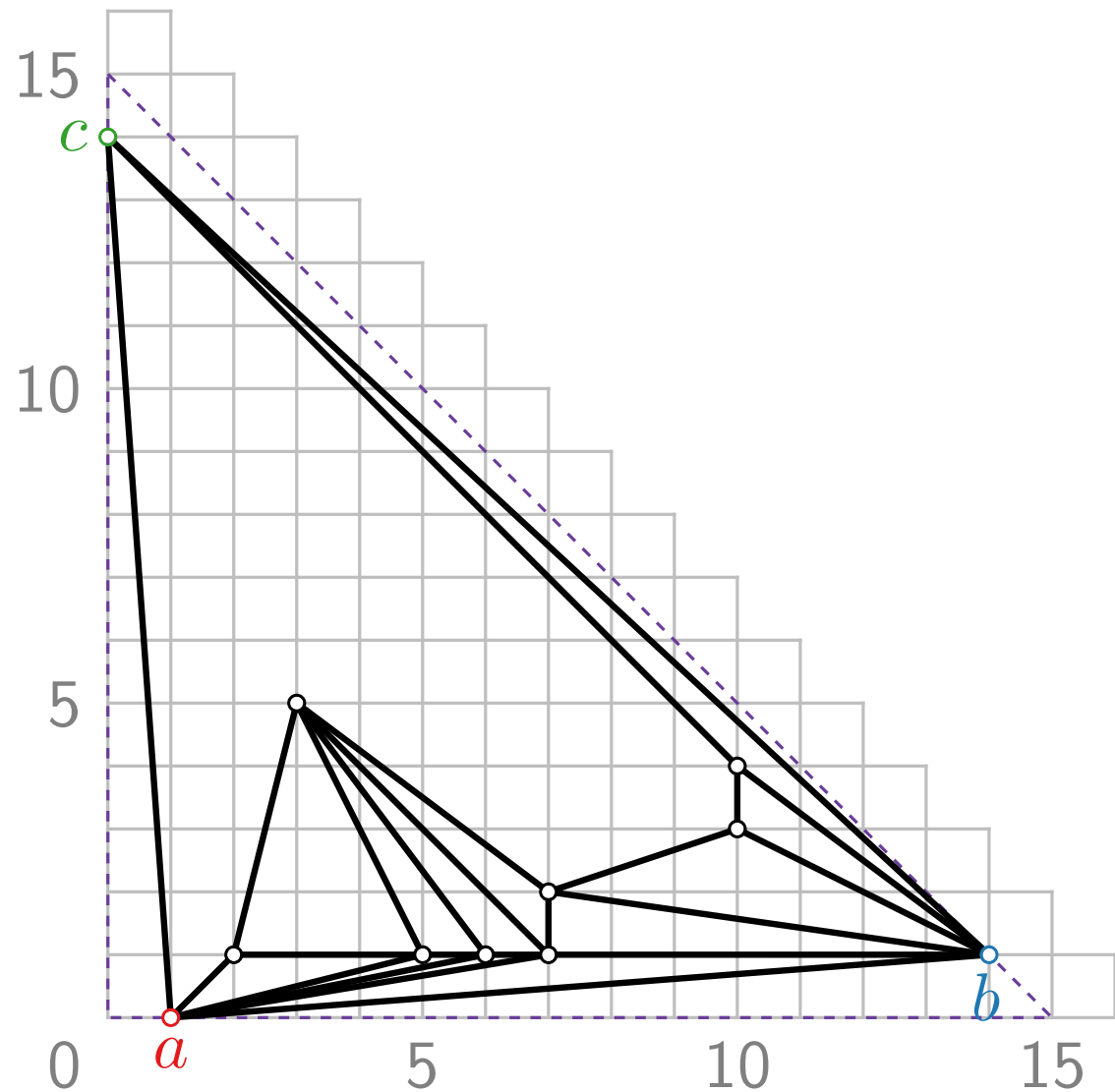
$$f(c) = (1, 0, 14) \quad \vdots$$

Schnyder Drawing* – Example



$$\begin{aligned}
 n &= 16, \quad n - 1 = 15, \quad nf(v_2) = (7, 7, 1) \\
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Schnyder Drawing* – Example



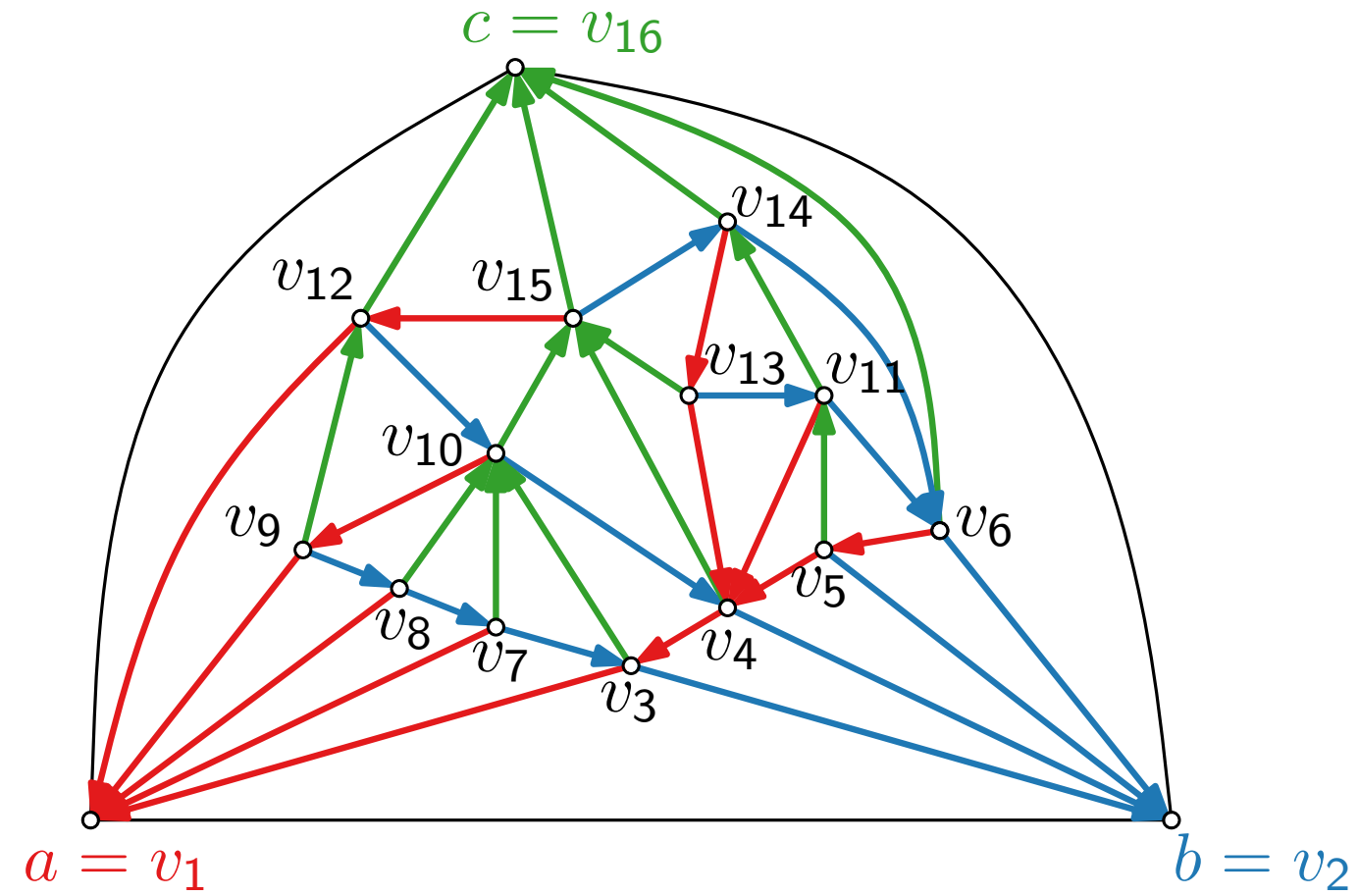
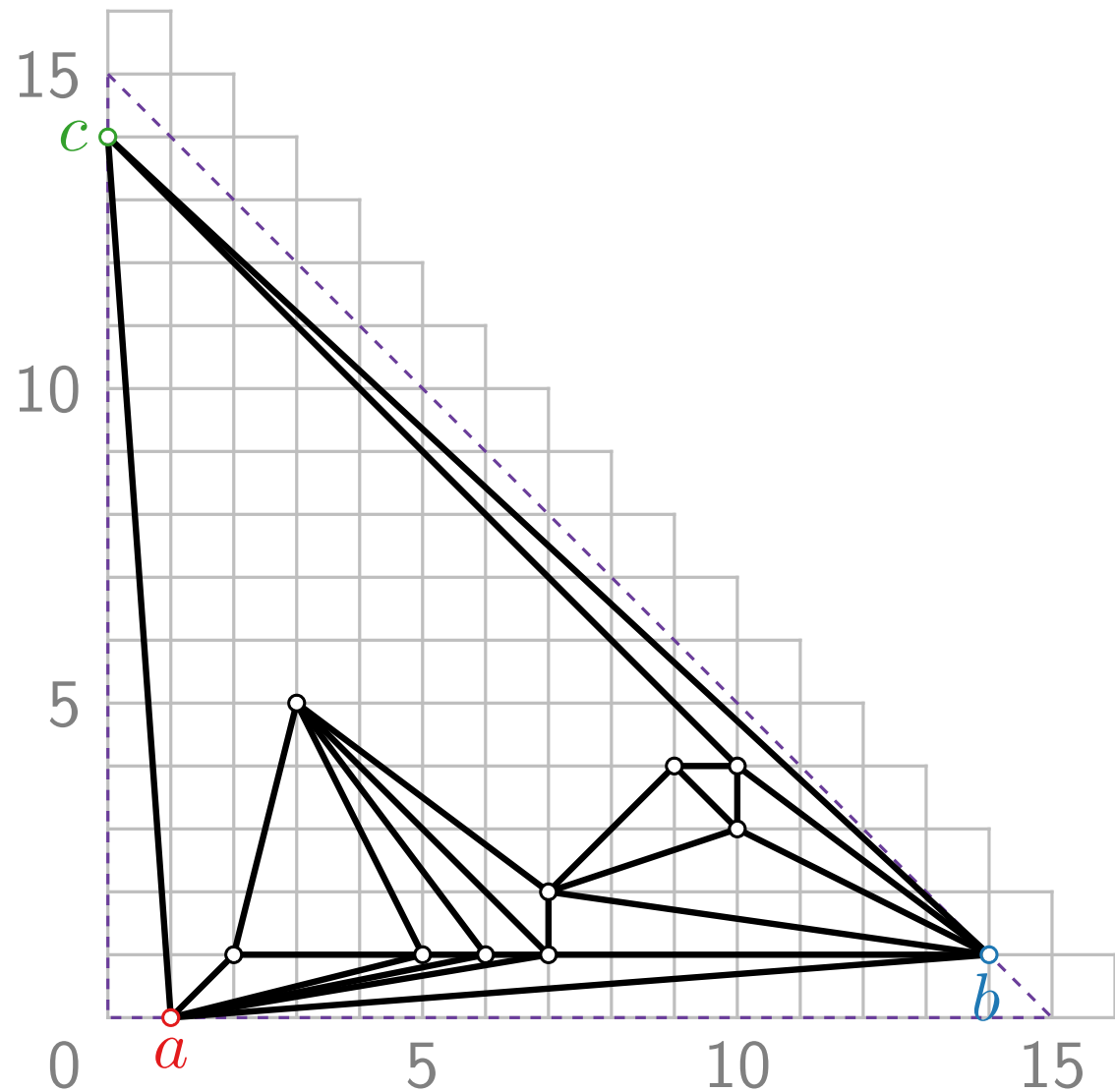
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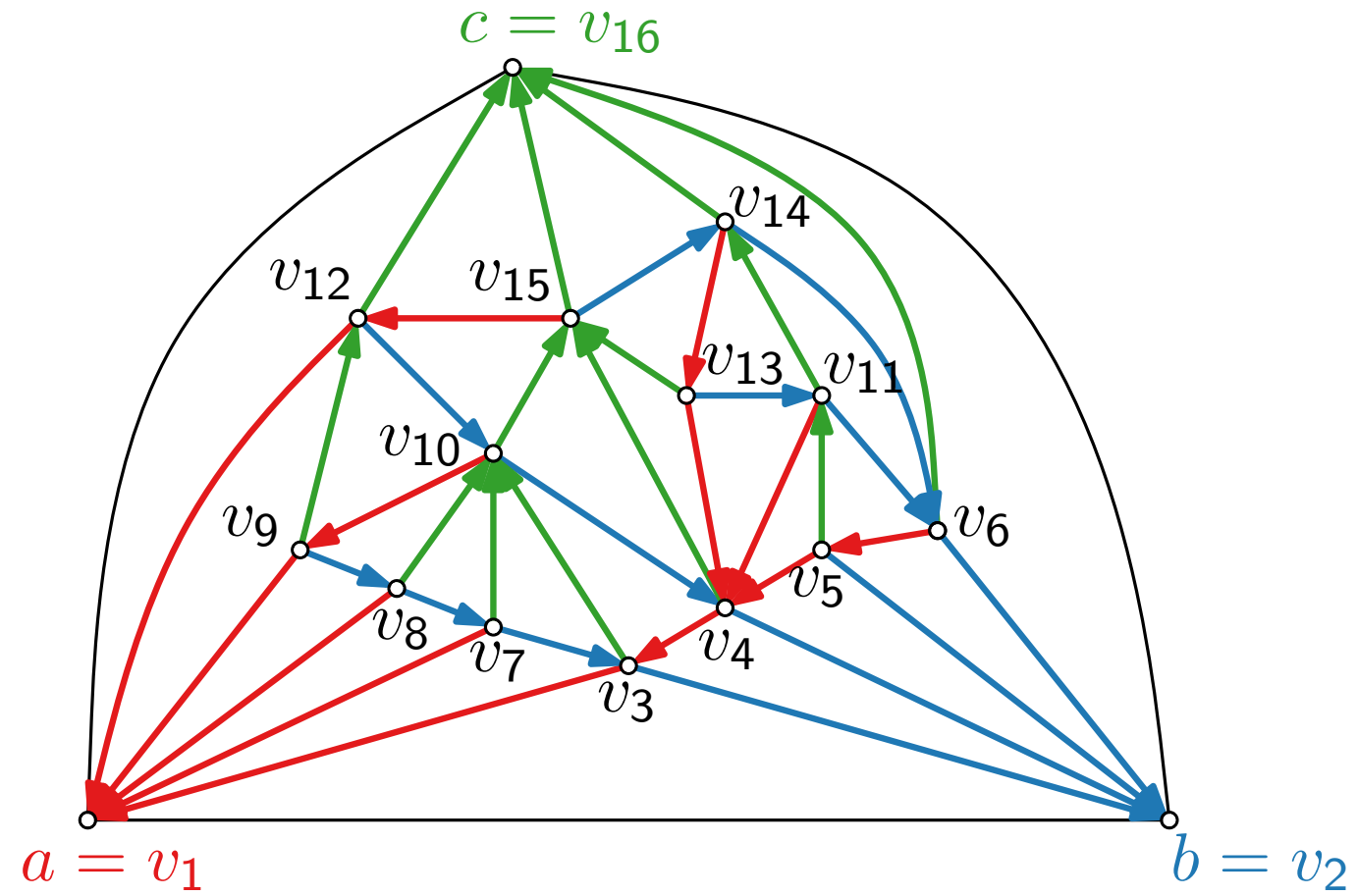
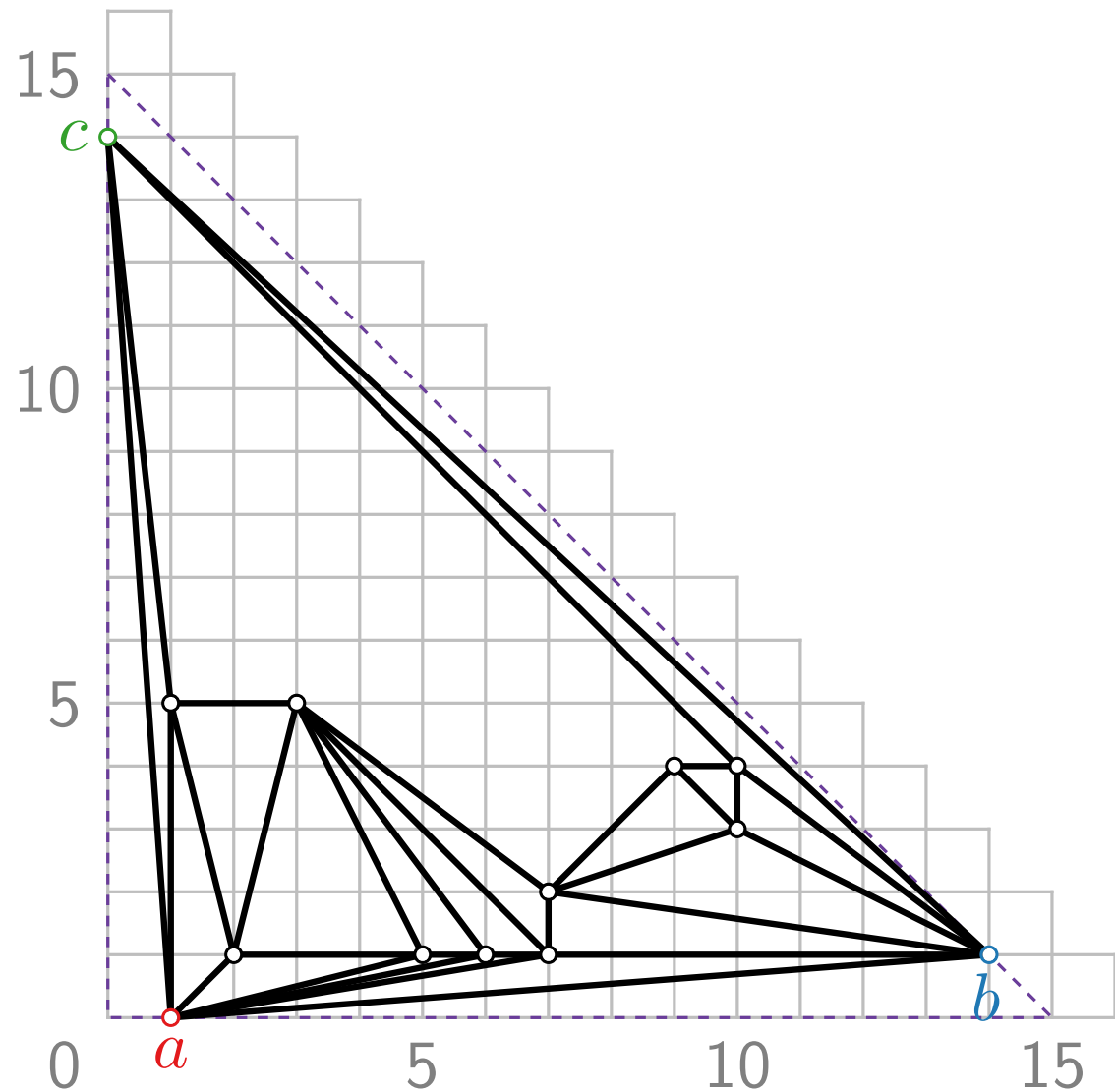
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Schnyder Drawing* – Example



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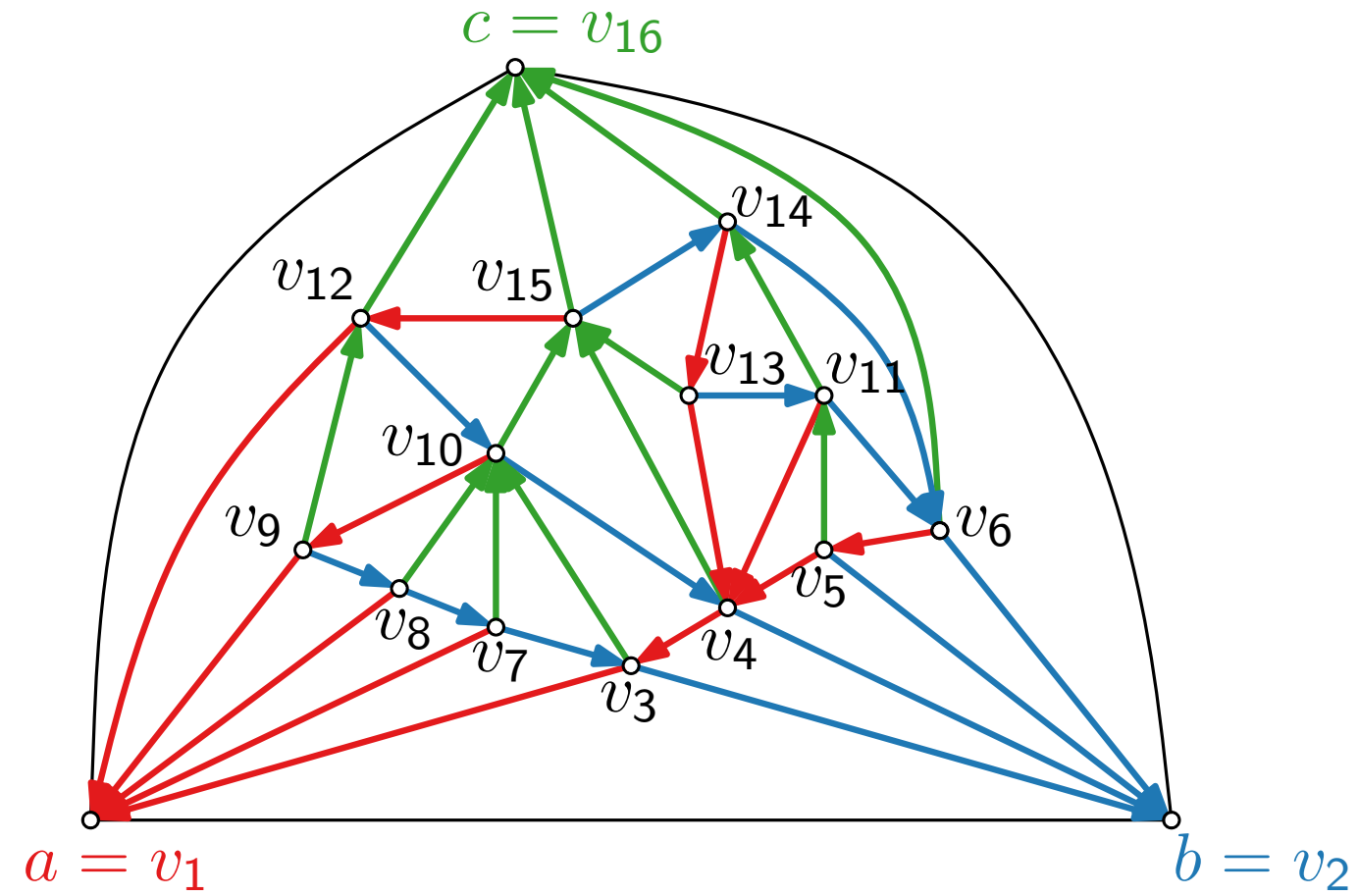
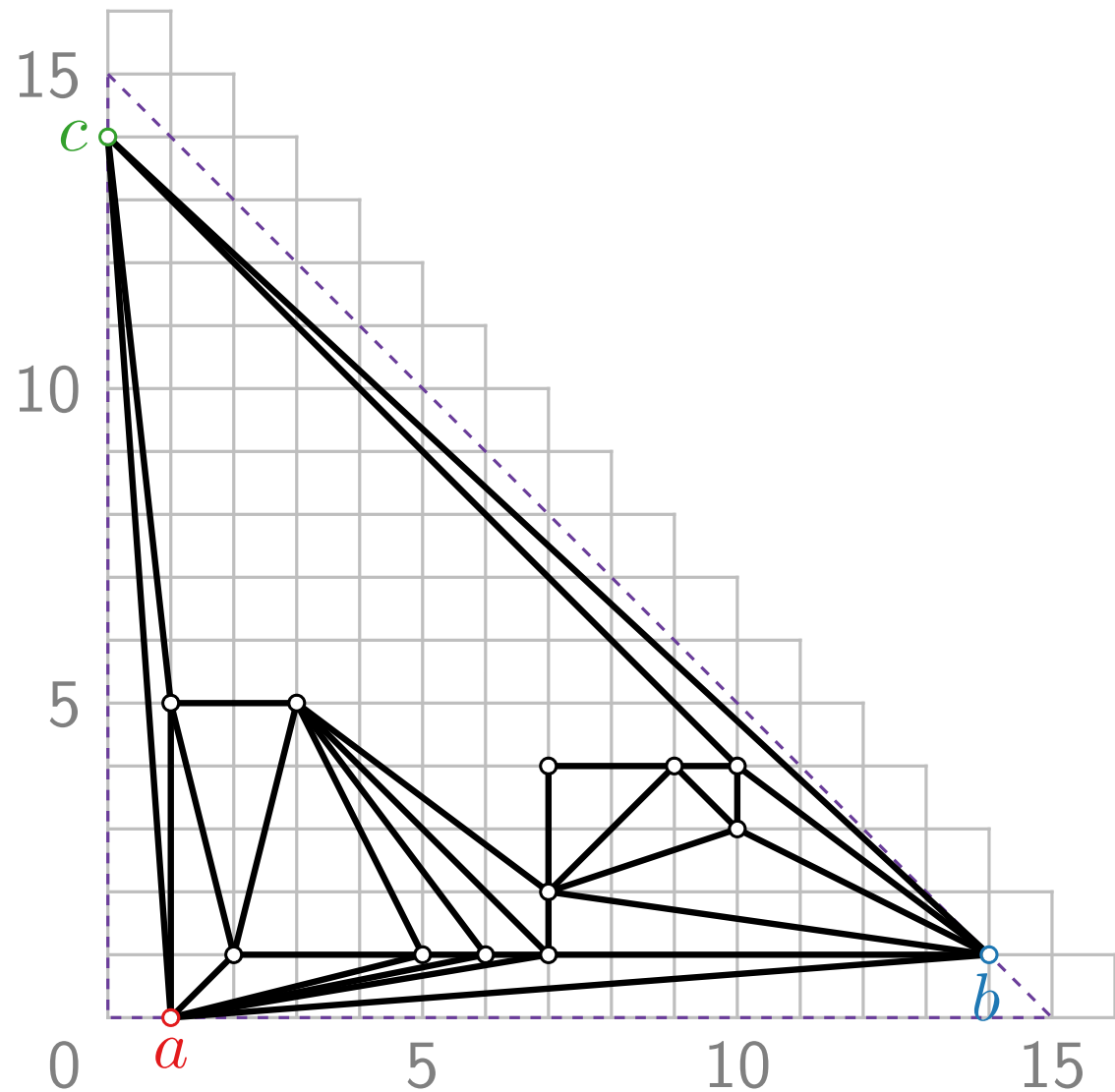
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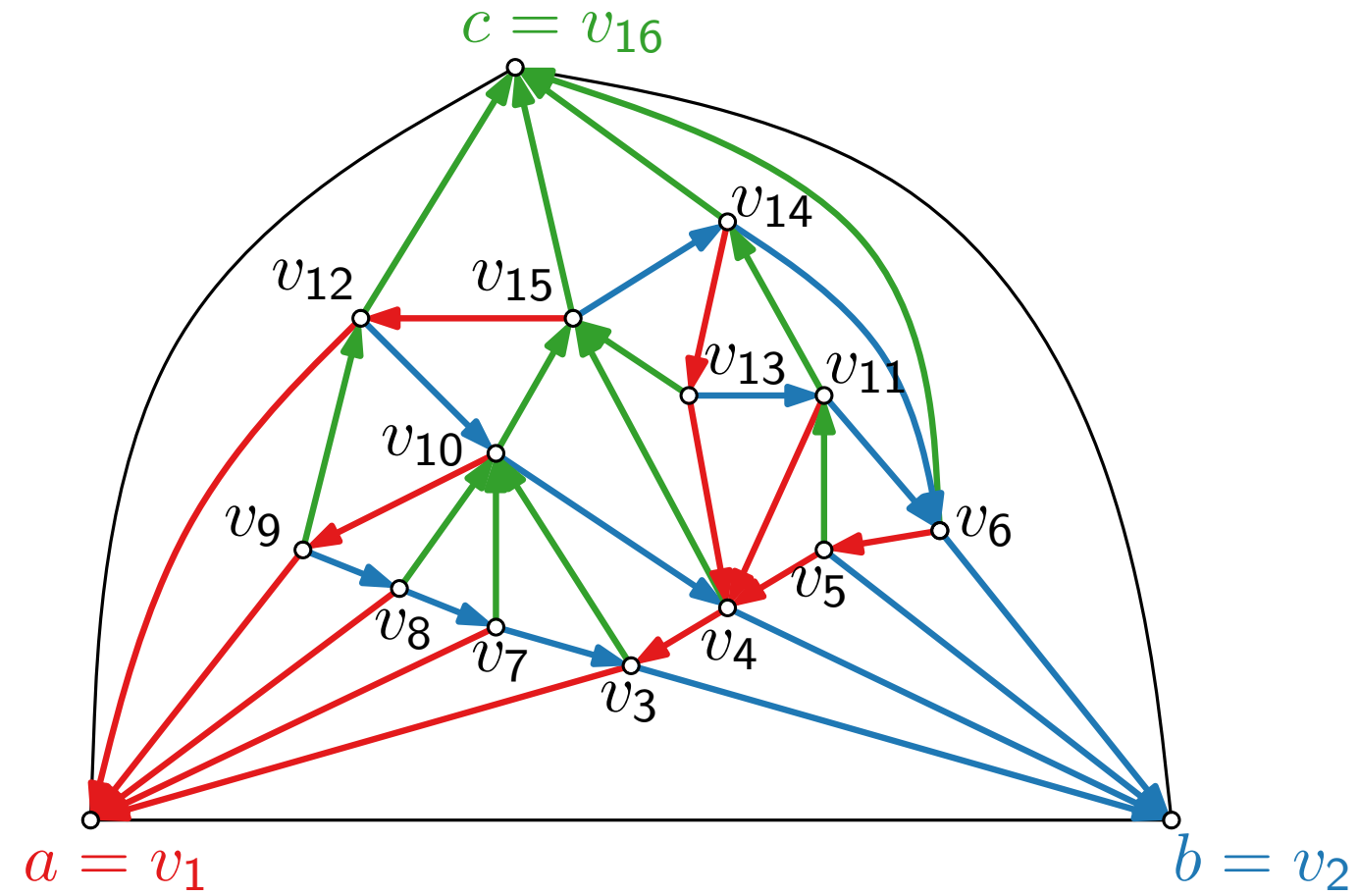
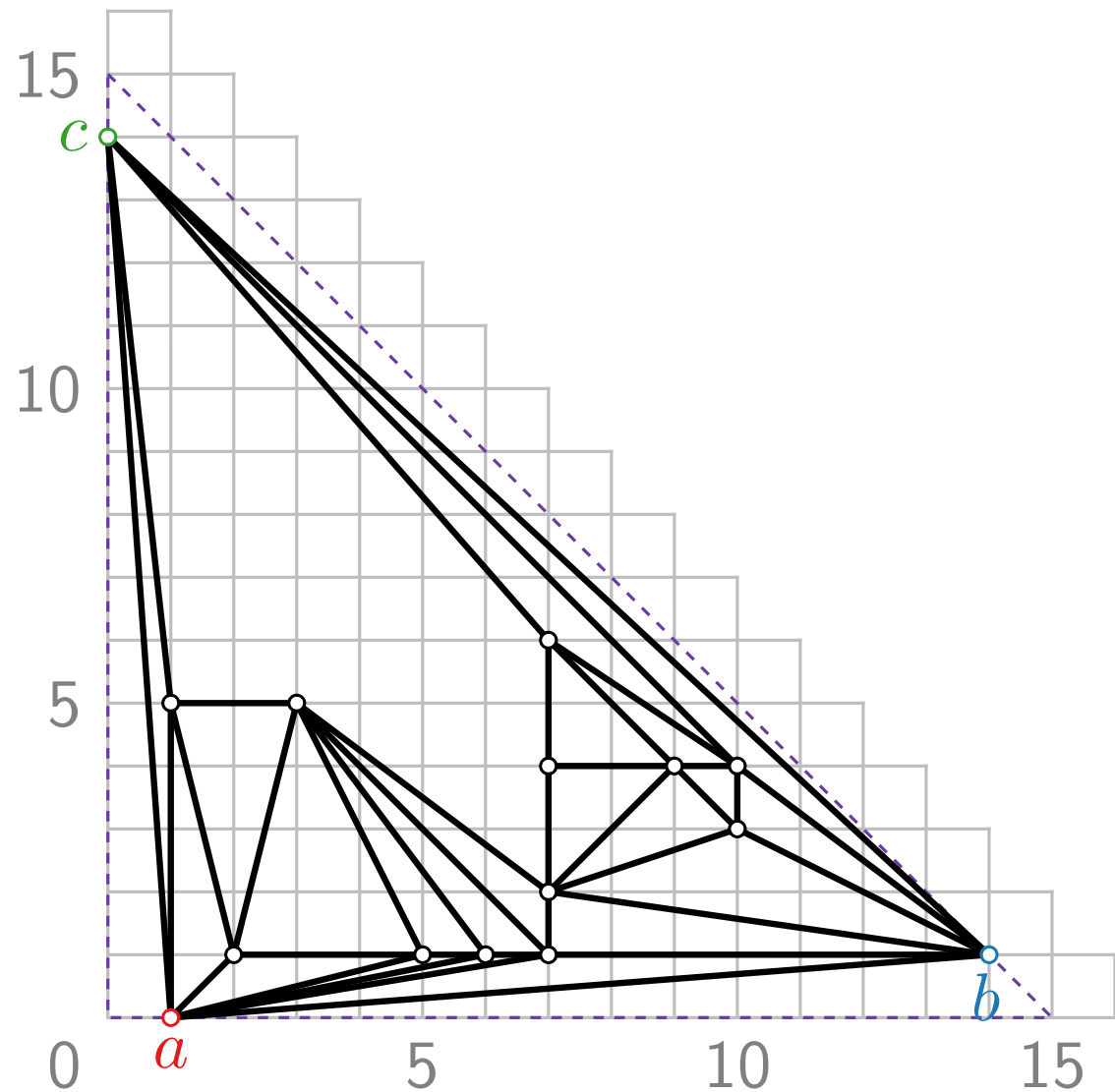
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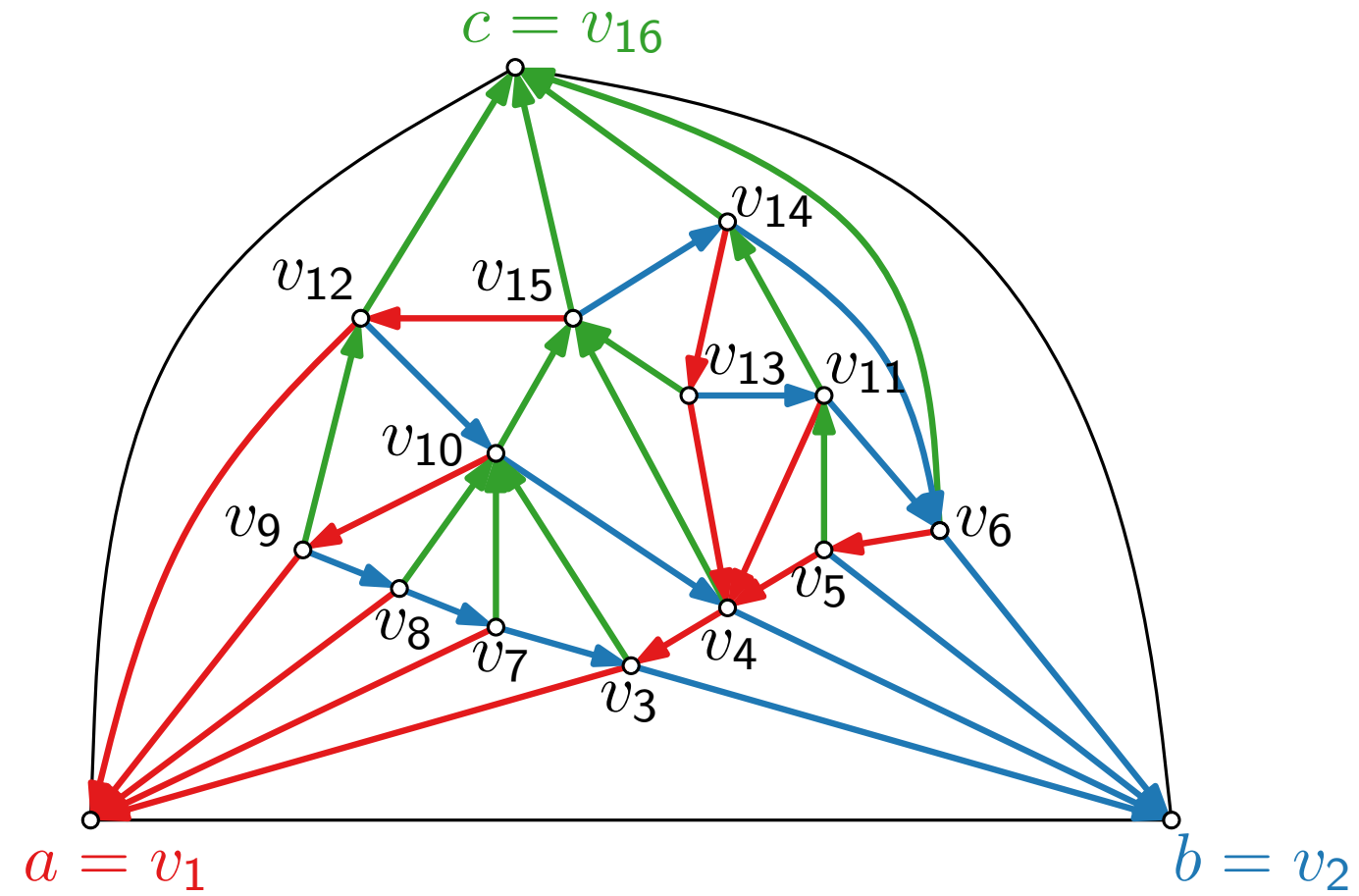
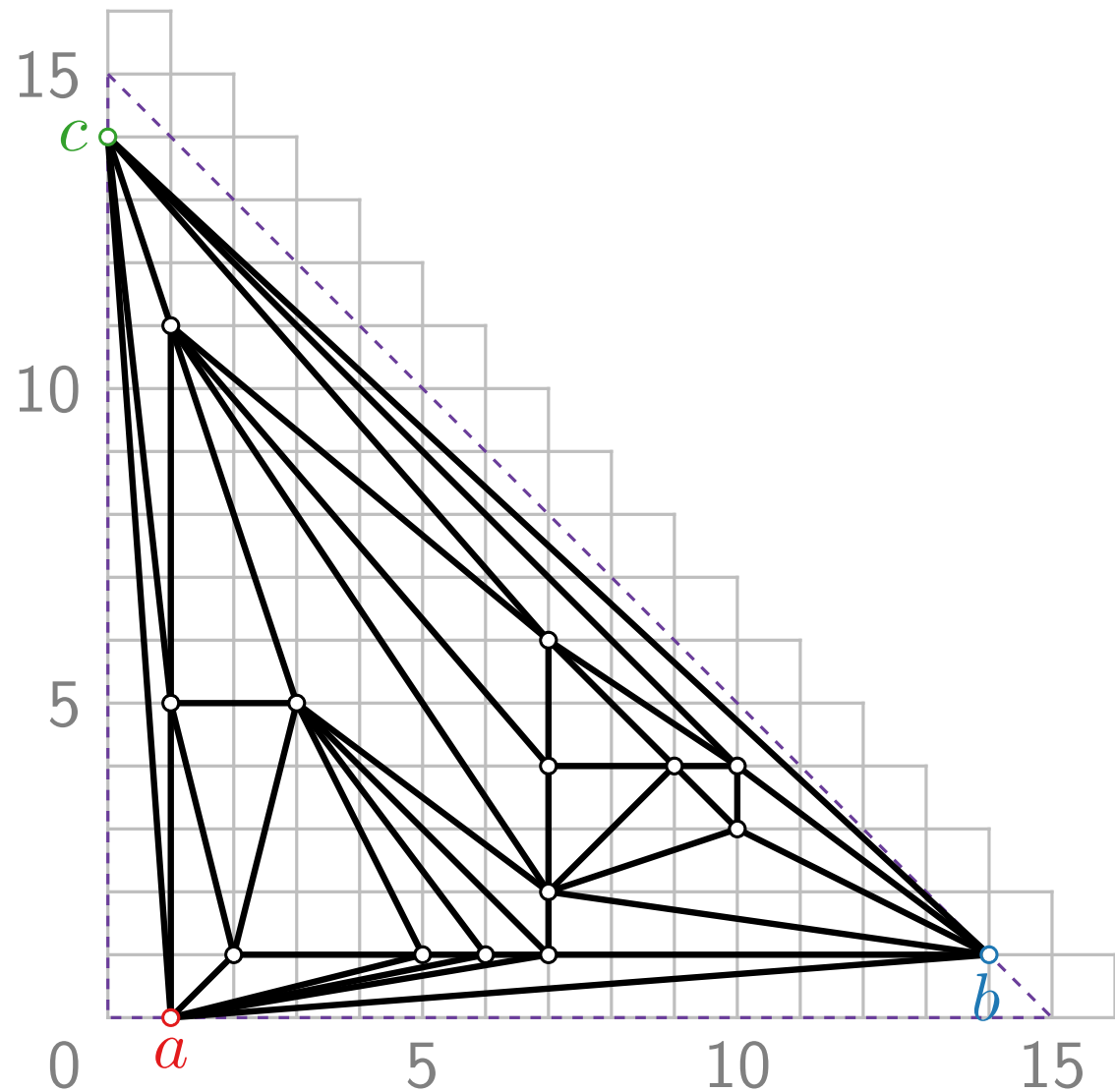
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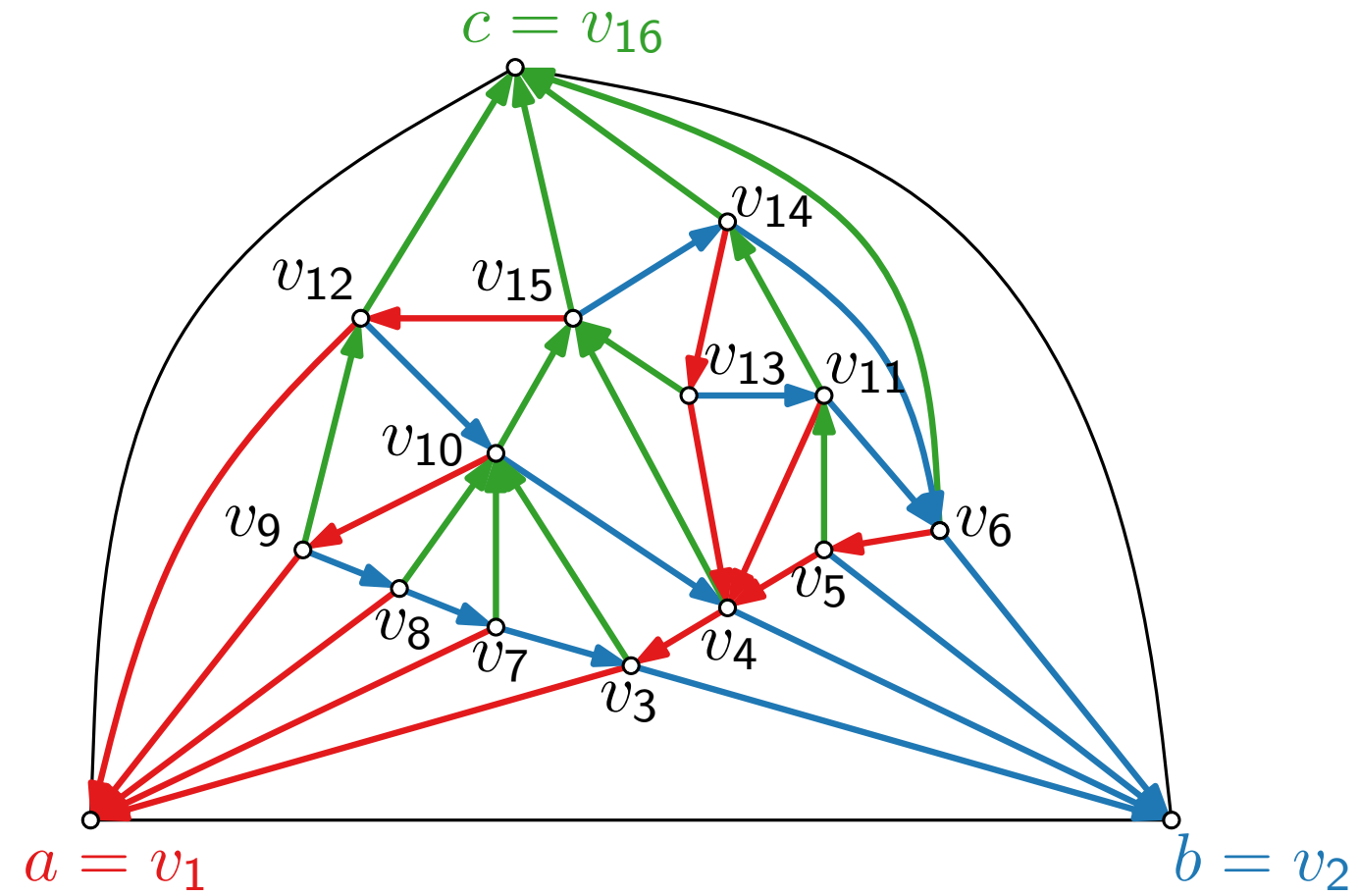
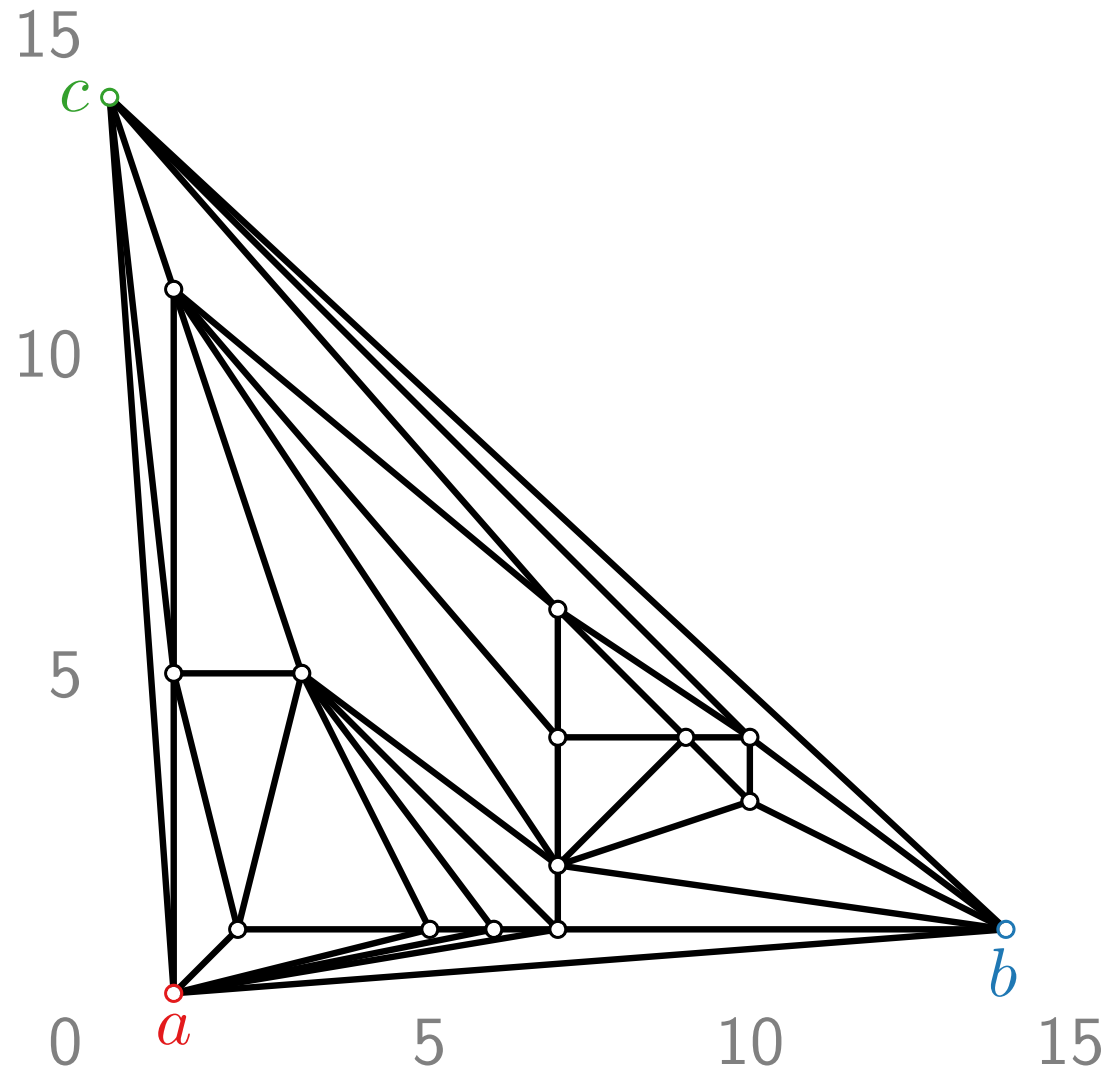
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Results & Variations

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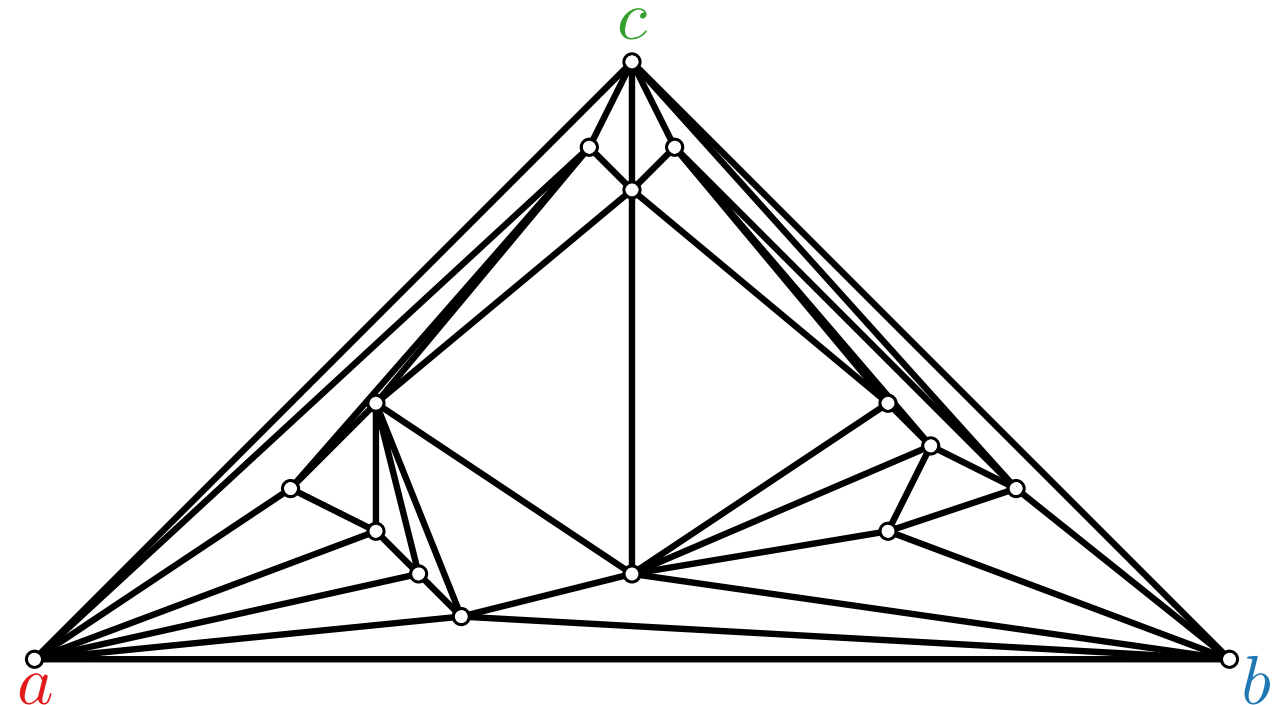
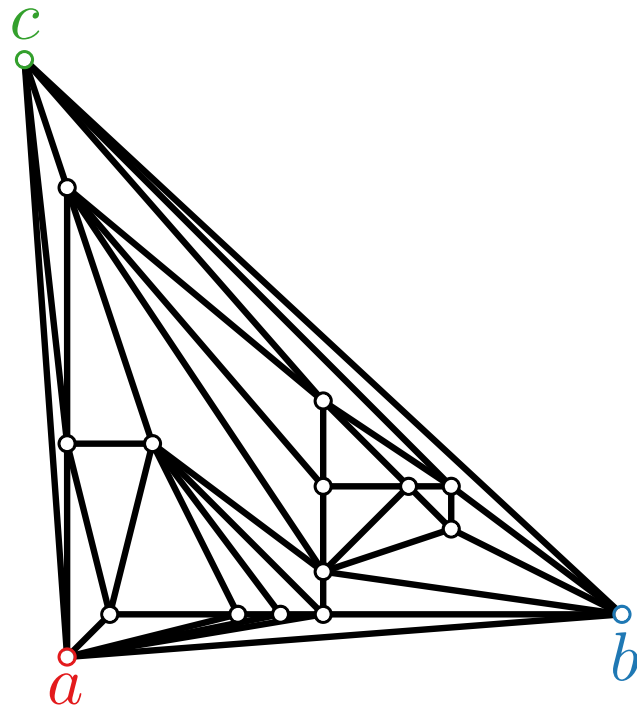
[De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

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Results & Variations

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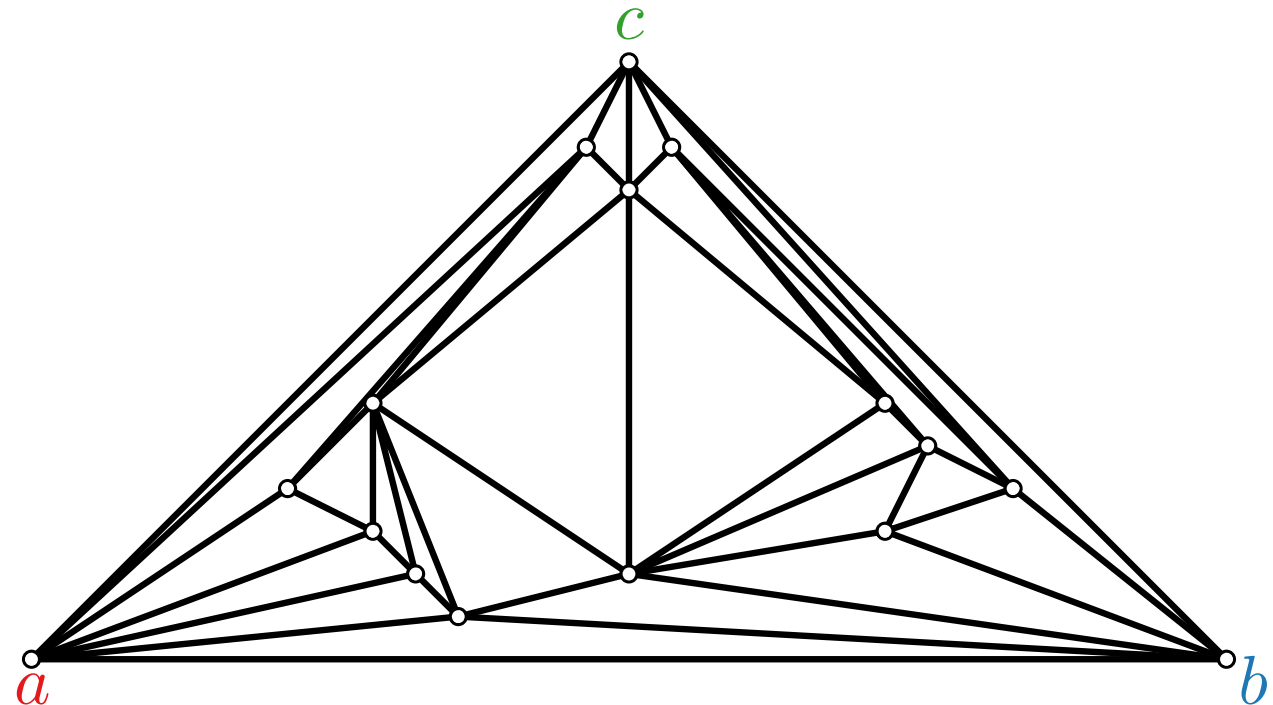
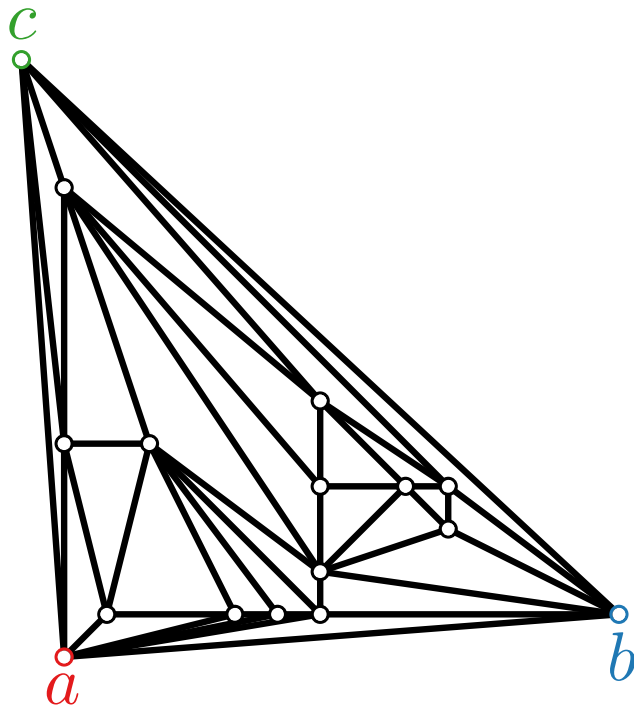
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Results & Variations

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Results & Variations

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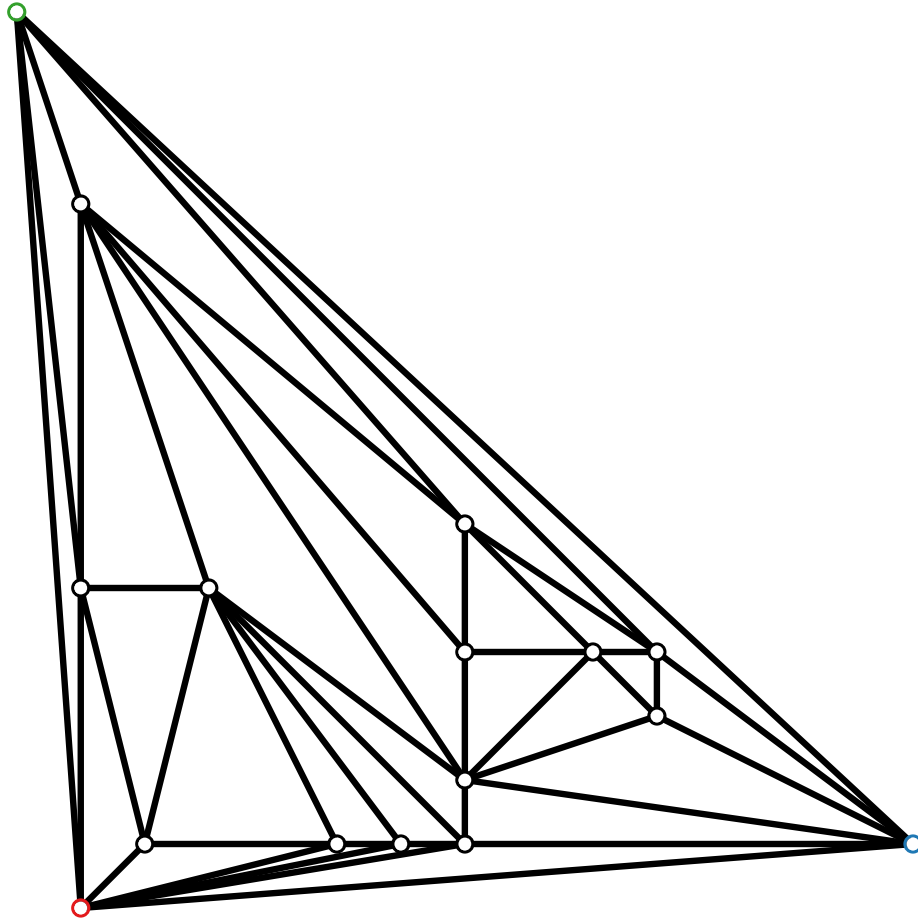
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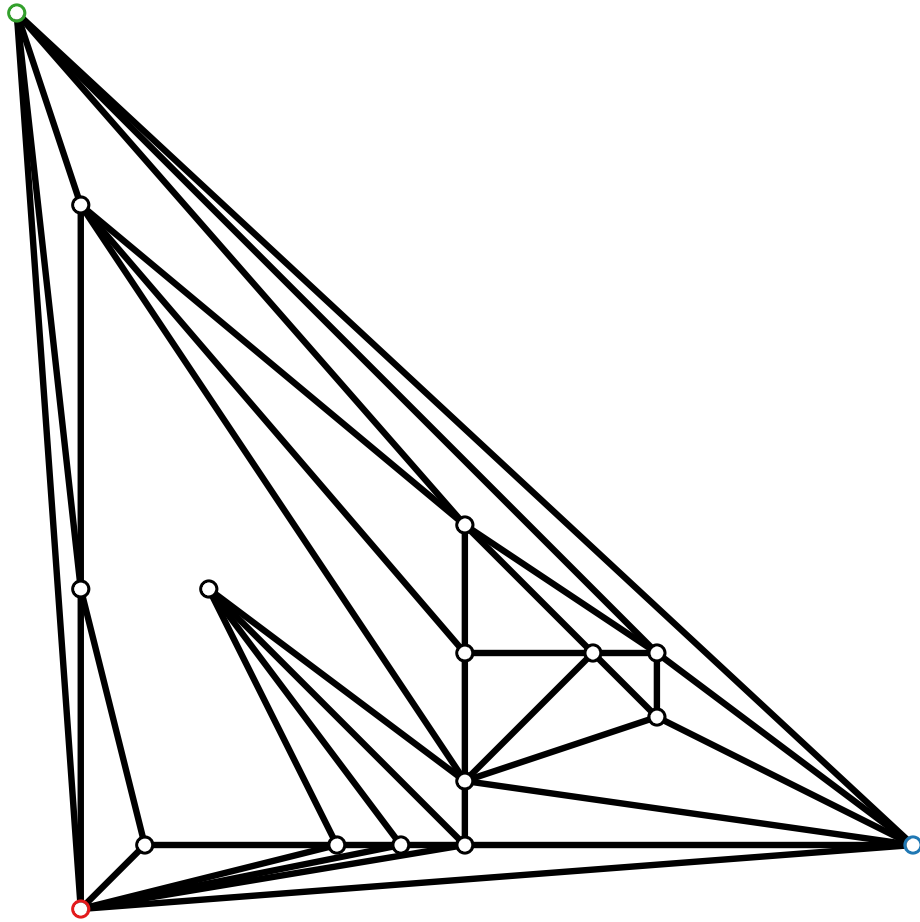
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[Fрати, Patrignani '07] Area at least $n^2/9 + \Omega(n)$ in the variable-embedding setting.

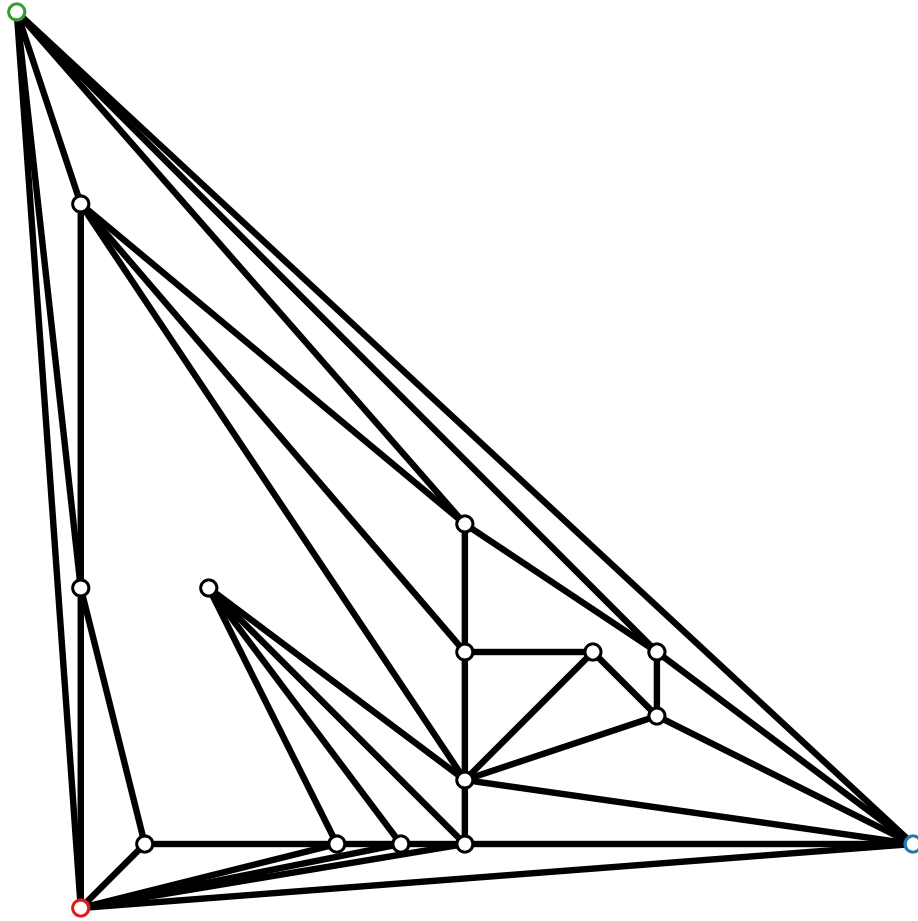
Results & Variations



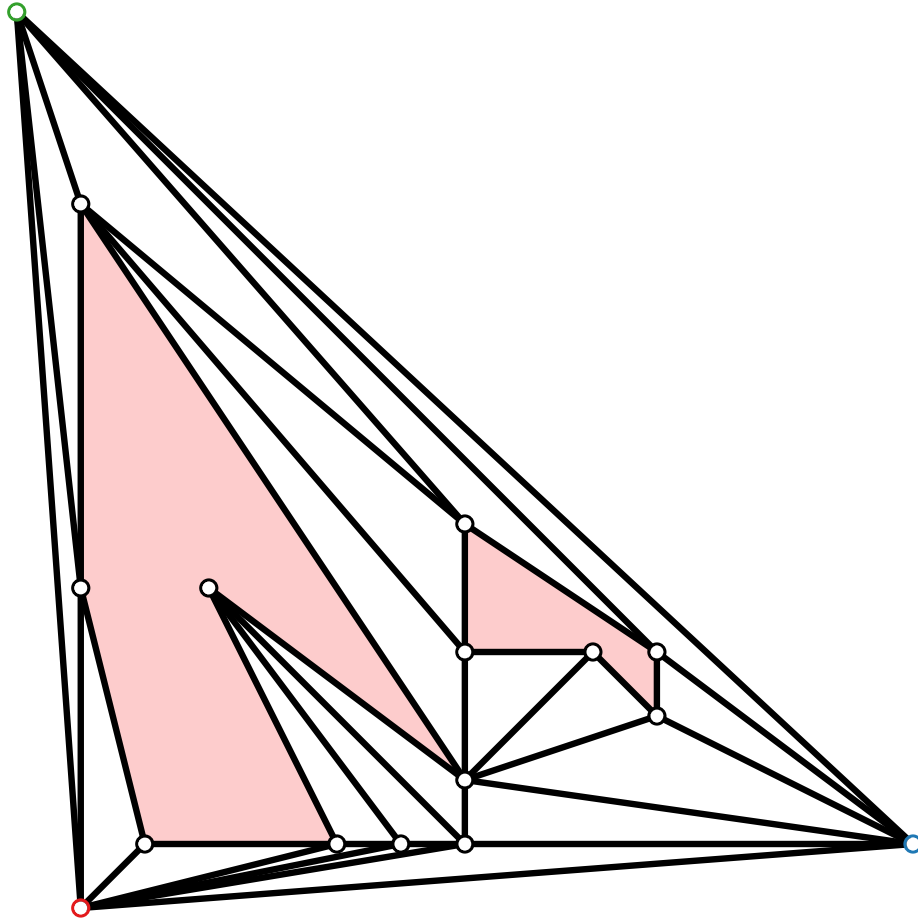
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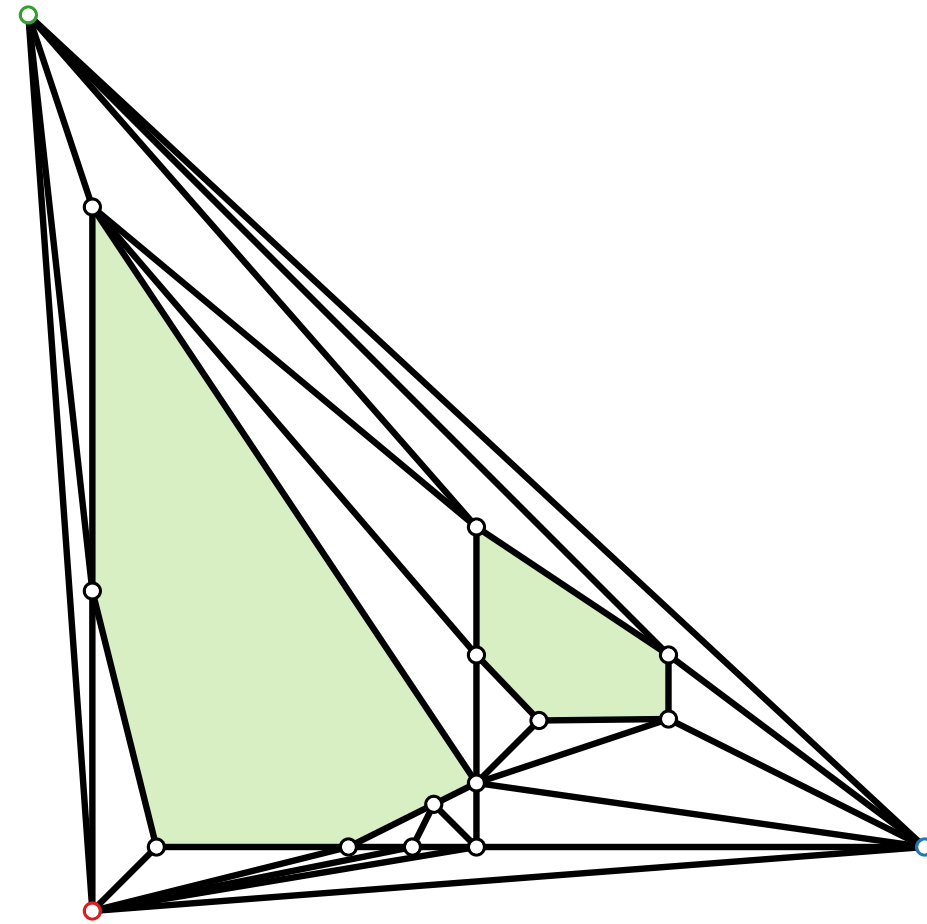
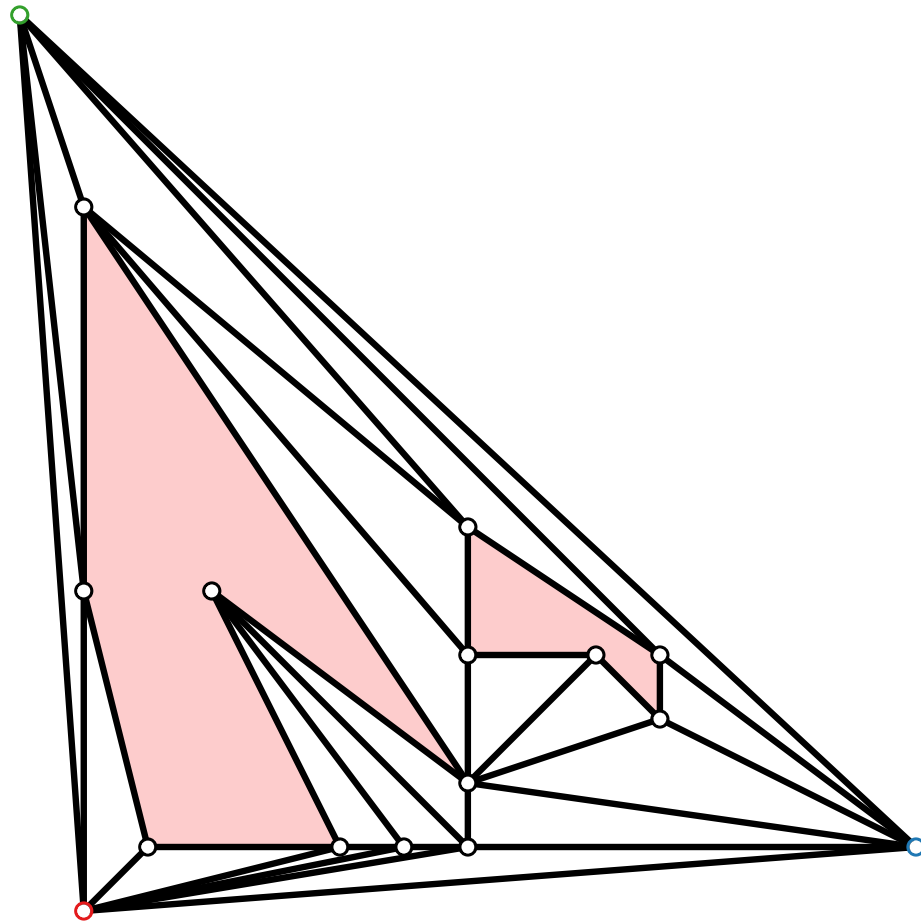
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Results & Variations

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[Felsner '01]

Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f - 1) \times (f - 1)$ where all faces are drawn convex. Such a drawing can be computed in $O(n)$ time.

Literature

- [PGD Ch. 4.3] for detailed explanation of Schnyder woods etc.
- [Sch90] “Embedding planar graphs on the grid”, Walter Schnyder, SoCG 1990 – original paper on Schnyder realizer method.