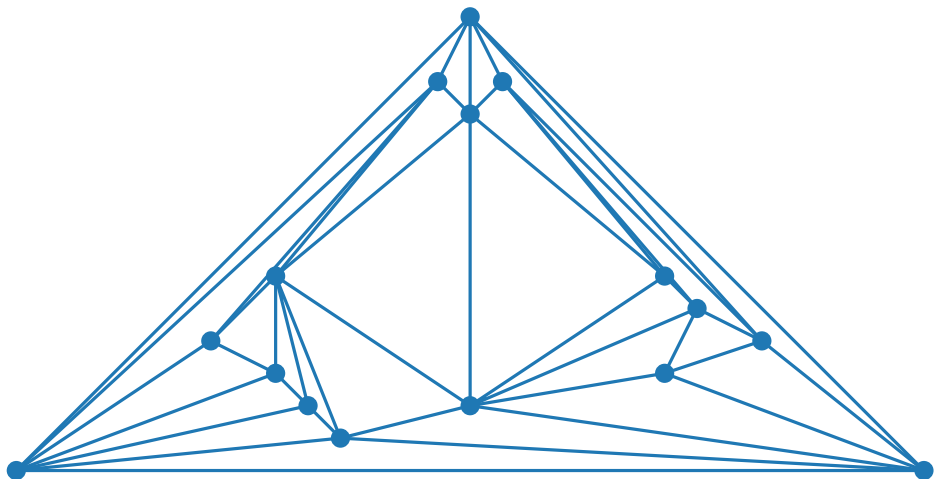


Visualization of Graphs

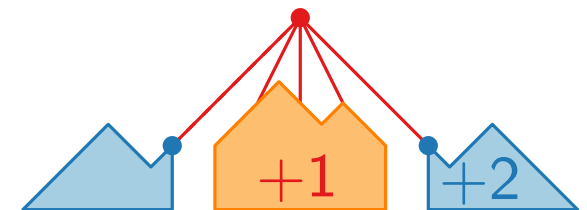
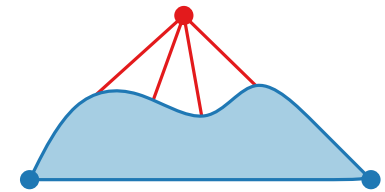
Lecture 3:

Straight-Line Drawings of Planar Graphs I: Canonical Orderings and the Shift Method

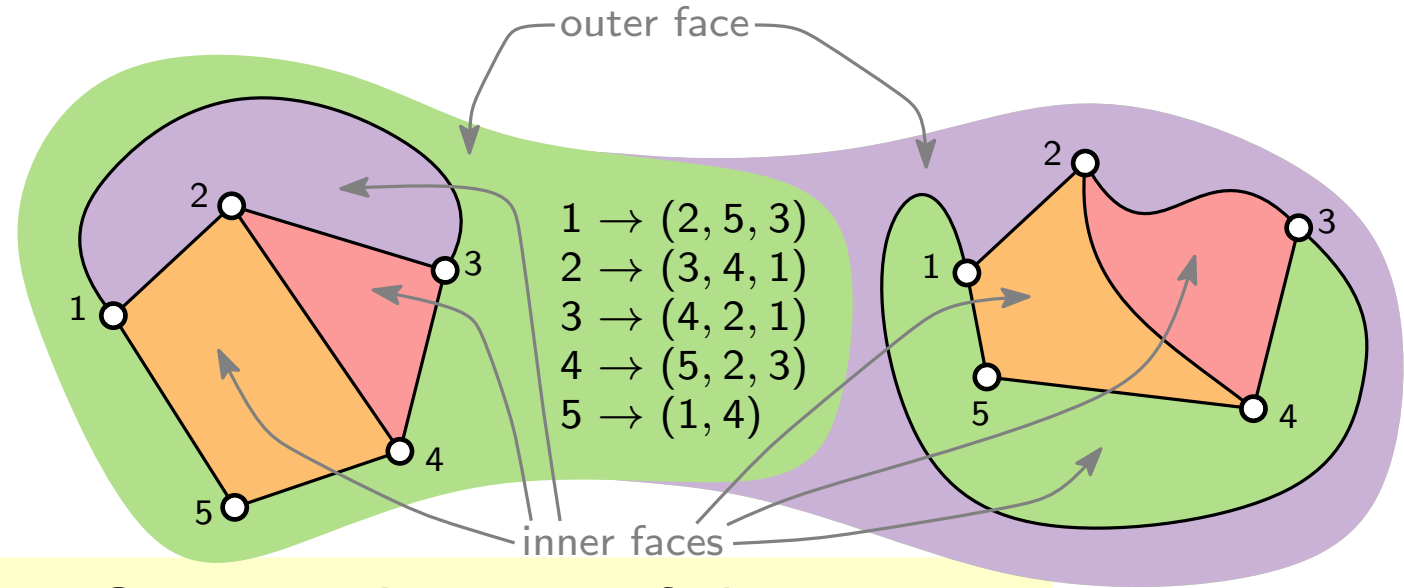
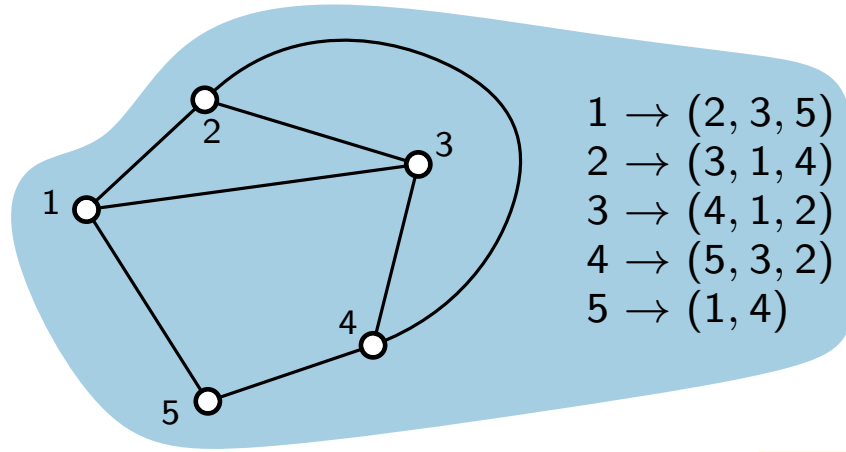
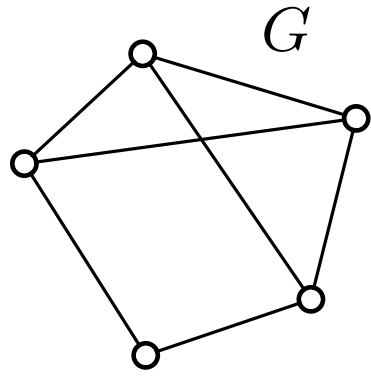


Alexander Wolff

Summer term 2026



Planar Graphs



G is **planar**:

it can be drawn in such a way that no two edges intersect each other.

planar embedding:

clockwise orientation of adjacent vertices around each vertex

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

faces: Connected region of the plane bounded by edges

Euler's polyhedra formula.

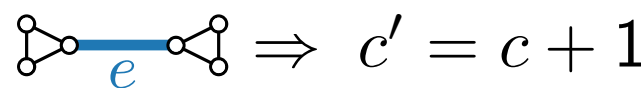
$$\begin{matrix} \# \text{faces} & - & \# \text{edges} & + & \# \text{vertices} & = & \# \text{conn.comp.} & + & 1 \\ f & - & m & + & n & = & c & + & 1 \end{matrix}$$

Proof. By induction on m :

$m = 0 \Rightarrow f = 1$ and $c = n$ ✓

Induction hypothesis in G' :
 $f' - m' + n' = c' + 1$

$m \geq 1 \Rightarrow$ delete some edge $e \Rightarrow m' = m - 1$



Properties of Planar Graphs

Euler's polyhedra formula.

$$\begin{array}{ccccccccc} \# \text{faces} & - & \# \text{edges} & + & \# \text{vertices} & = & \# \text{conn.comp.} & + & 1 \\ f & - & m & + & n & = & c & + & 1 \end{array}$$

Theorem. G simple planar graph with $n \geq 3$ vtc.

1. $m \leq 3n - 6$
2. $f \leq 2n - 4$
3. There is a vertex of degree at most 5.

Proof. 1. Every **edge** incident to ≤ 2 faces
Every **face** incident to ≥ 3 edges

$$\Rightarrow 3f \leq \# \text{ incidences} \leq 2m$$

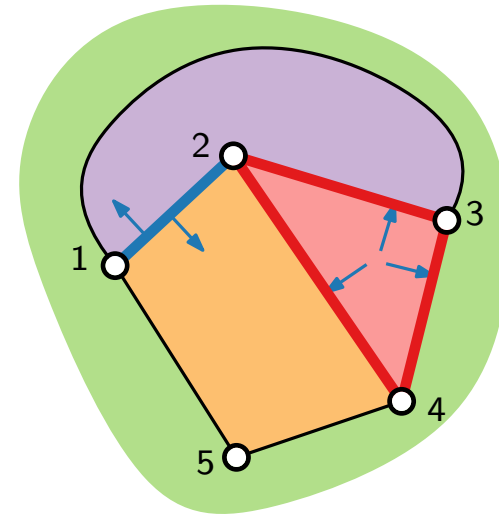
$$\Rightarrow 6 \leq 3c + 3 = 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m$$

$$\Rightarrow m \leq 3n - 6$$

$$2. \quad 3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$$

$$3. \quad \sum_{v \in V(G)} \deg(v) = 2m \leq 6n - 12$$

$$\Rightarrow \min_{v \in V(G)} \deg(v) \leq \text{average degree}(G) = \frac{1}{n} \sum_{v \in V(G)} \deg(v) \leq \frac{6n-12}{n} < 6$$



idea: count
edge-face
incidences

Handshaking lemma.

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

Triangulations

planar graph given with a planar embedding

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.

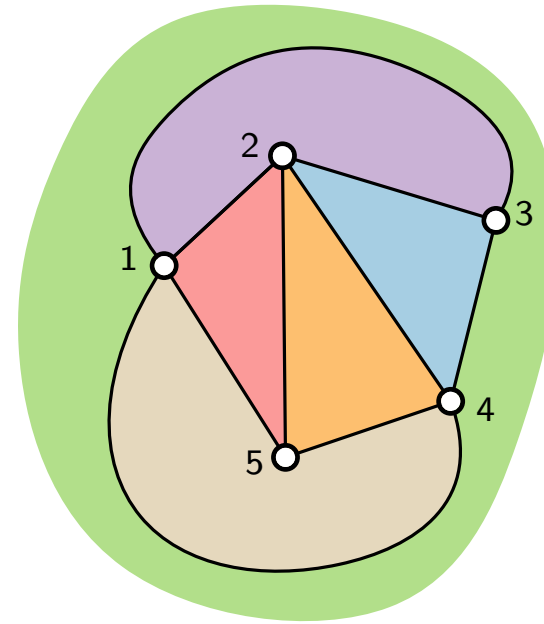
A **maximal planar graph** is a planar graph where adding any edge would violate planarity.

Observation.

Any maximal plane graph is a plane triangulation (and vice versa).

Lemma.

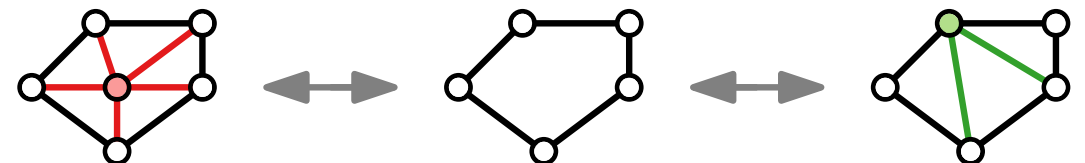
Any plane triangulation is 3-connected and thus has a unique planar embedding (up to mirroring).



We focus on plane triangulations:

Lemma.

Every plane graph is a subgraph of a plane triangulation.



Motivation

- Why planar and straight-line?

[Bennett, Ryall, Spaltzholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

Drawing conventions

- No crossings \Rightarrow planar
- No bends \Rightarrow straight-line

Drawing aesthetics to optimize

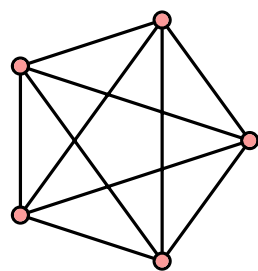
- Area

Towards Straight-Line Drawings

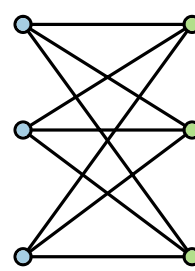
Theorem. [Kuratowski 1930]
 G planar \Leftrightarrow
 neither K_5 nor $K_{3,3}$ minor of G



Kazimierz Kuratowski (1896–1980)



K_5



$K_{3,3}$

Characterization

Theorem. [Hopcroft & Tarjan 1974]
 Let G be a graph with n vertices. There is an
 $\mathcal{O}(n)$ -time algorithm to test whether G is planar.

Also computes a planar embedding in $\mathcal{O}(n)$ time.



John Edward Hopcroft (1939–)
en.wikipedia.org/wiki/User:Shakespeare



Robert Endre Tarjan (1948–)
 Renatokeshet, GFDL via Wikimedia

Recognition

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
 Every planar graph has a planar drawing
 where the edges are straight-line segments.



Klaus Wagner (1910–2000)
 Autor: Konrad Jacobs, wikipedia

Drawing

The algorithms implied by these theorems produce drawings
 whose area is **not** bounded by any polynomial in n .

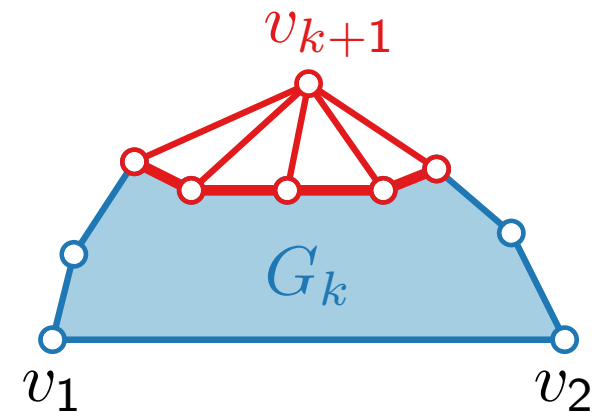
Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$.

Idea.

- Find a *canonical order* (v_1, \dots, v_n) of the vertices of a triangulation.
- Start with the single edge (v_1, v_2) . Let this be the graph G_2 .
- Let $k \in \{3, \dots, n\}$. To obtain G_{k+1} , add v_{k+1} to G_k so that the neighbors of v_{k+1} are on the outer face of G_k .
- The neighbors of v_{k+1} in G_k form a path of length at least two.



Theorem. [Schnyder '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$.

(next lecture)

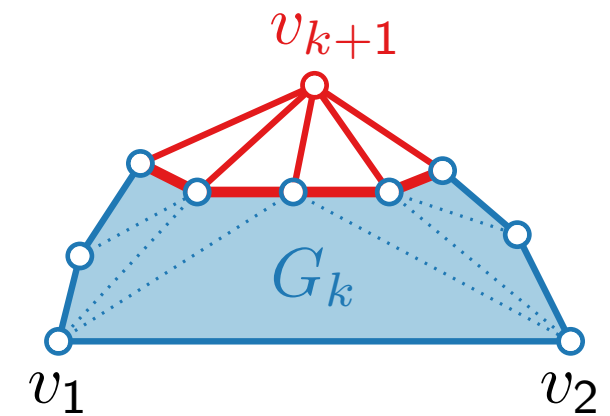
Canonical Order – Definition

Definition.

Let G be a plane triangulation on $n \geq 3$ vertices.

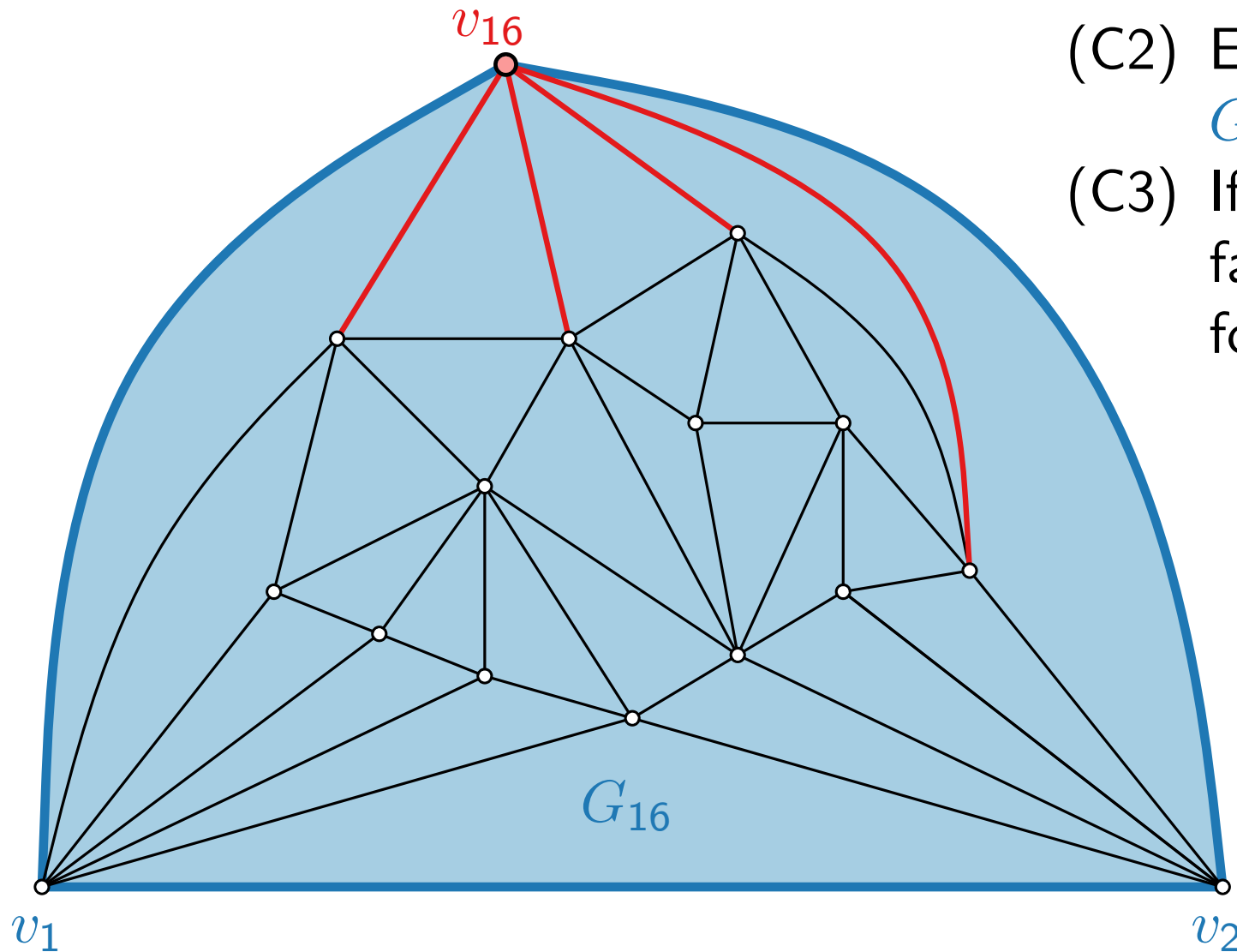
An ordering $\pi = (v_1, v_2, \dots, v_n)$ of $V(G)$ is a **canonical order** if the following conditions hold for each $k \in \{3, 4, \dots, n\}$:

- (C1) Vertices $\{v_1, \dots, v_k\}$ induce a biconnected inner triangulation; call it G_k .
- (C2) Edge (v_1, v_2) belongs to the outer face of G_k .
- (C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k , and the neighbors of v_{k+1} form a path on the boundary of G_k .



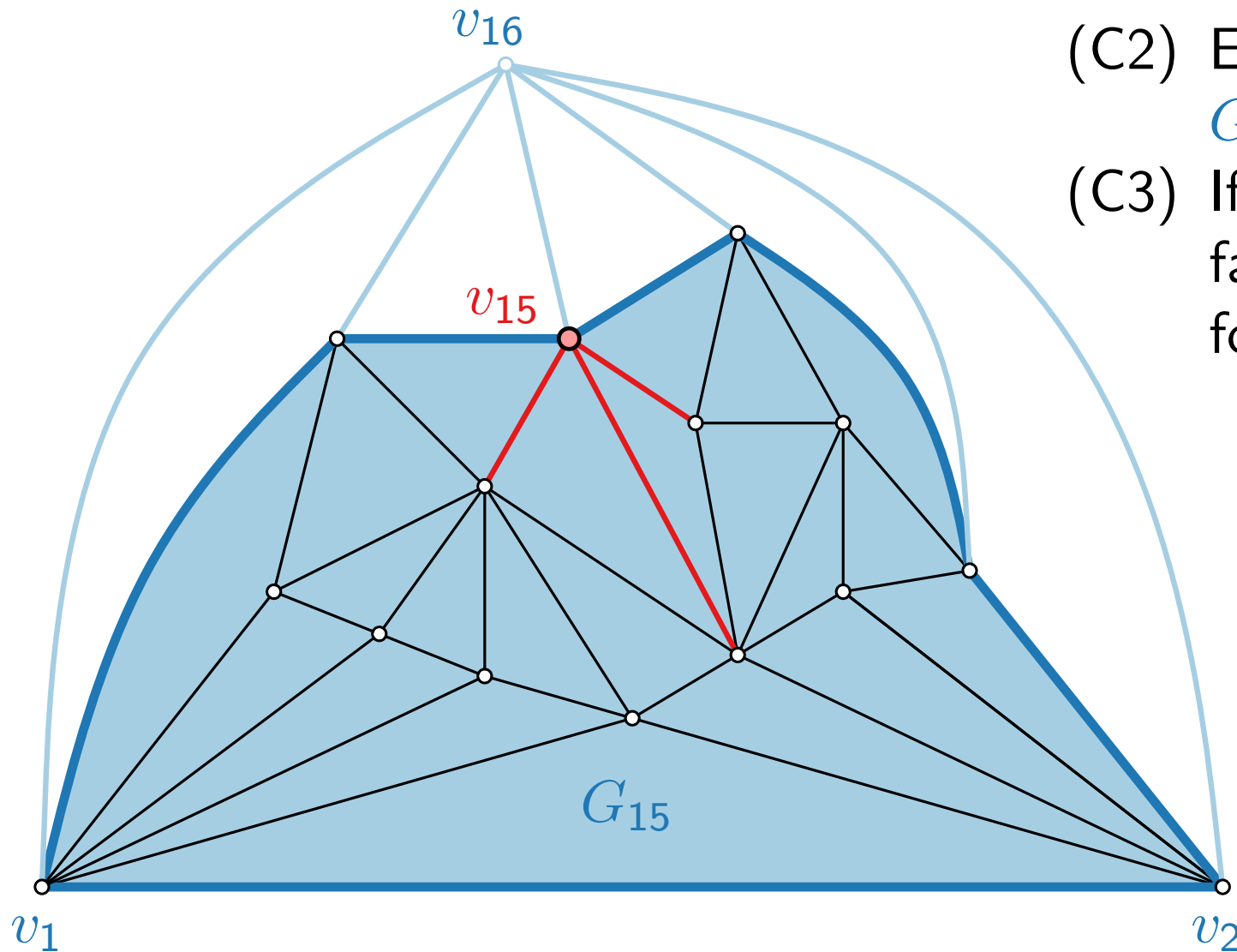
Canonical Order – Example

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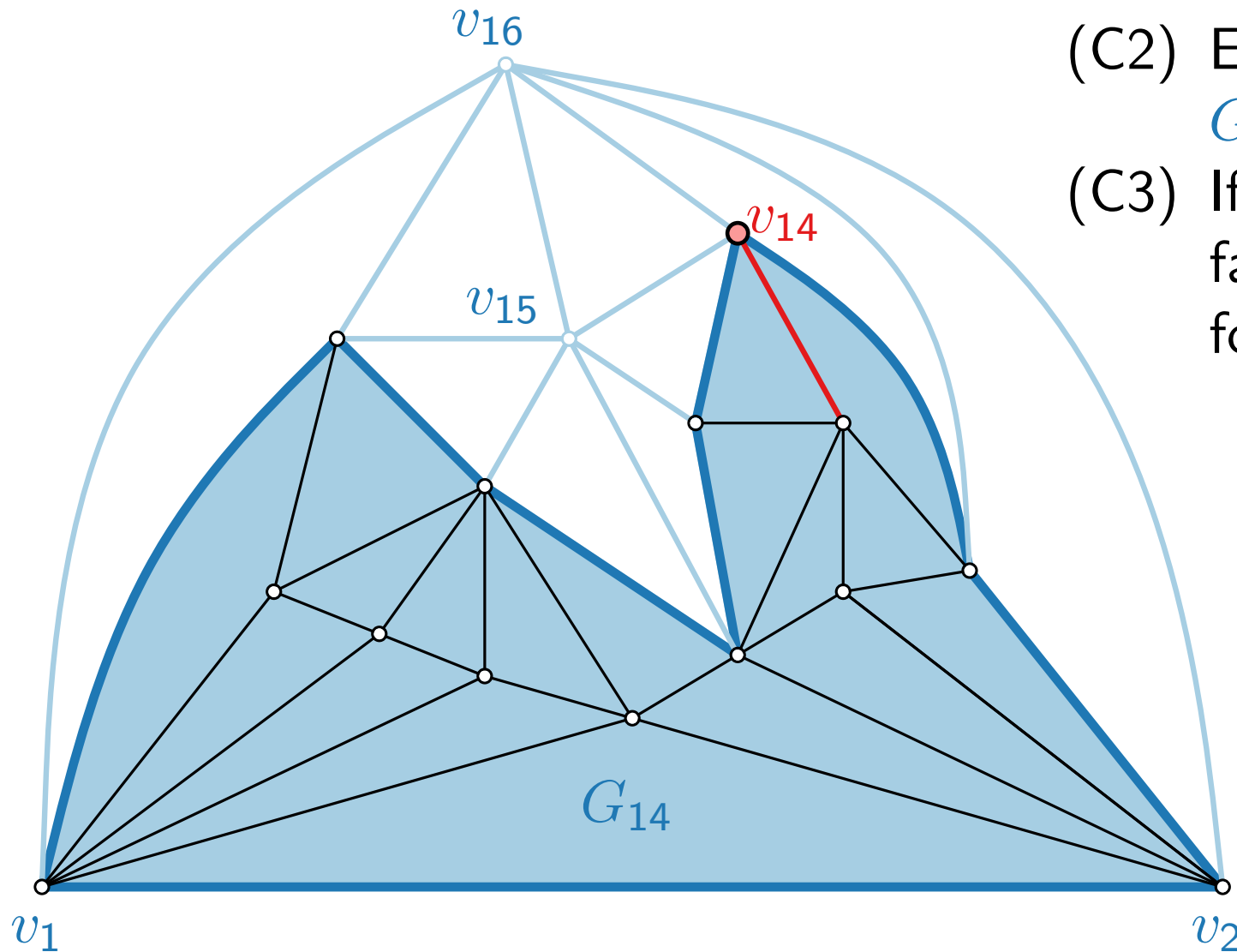
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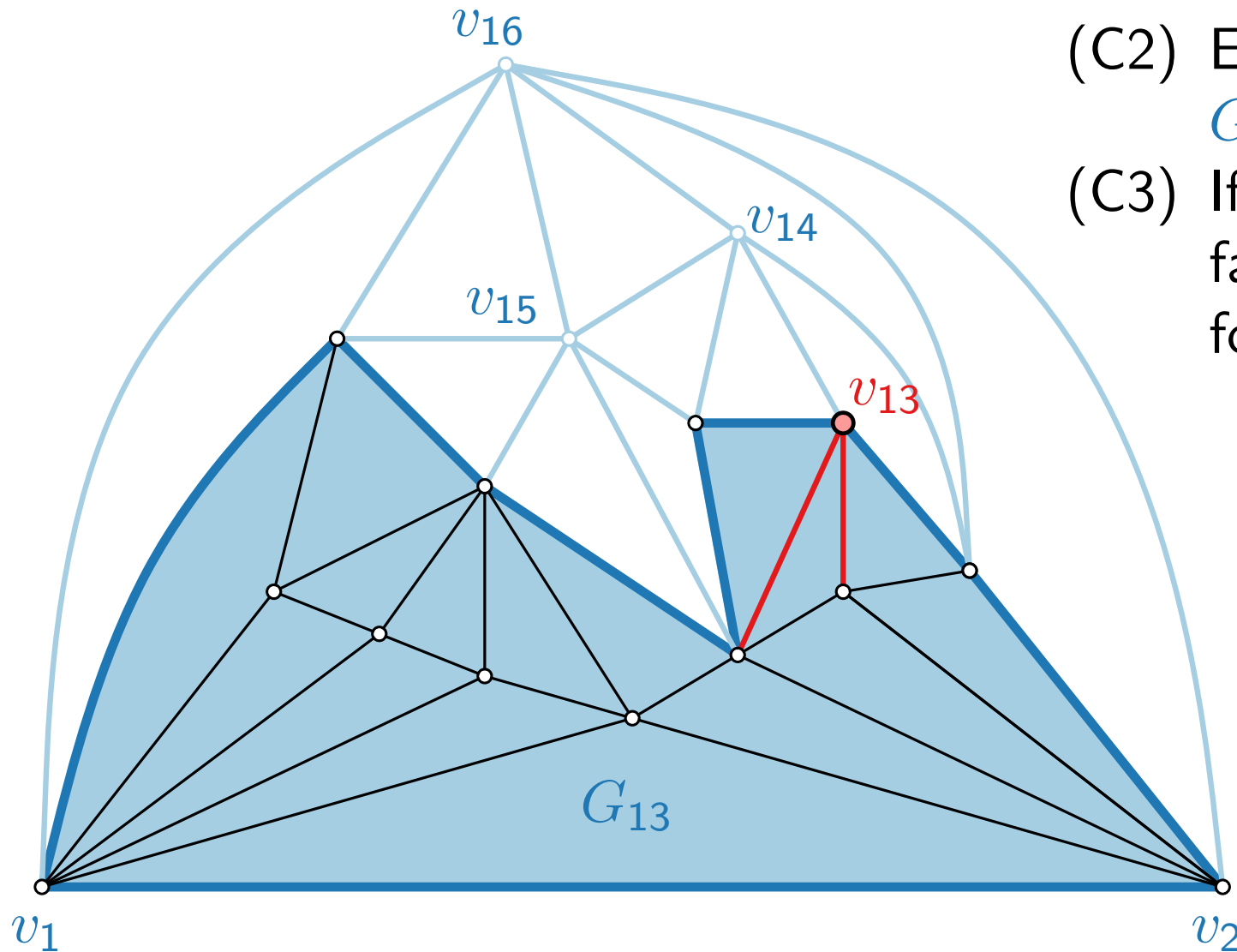
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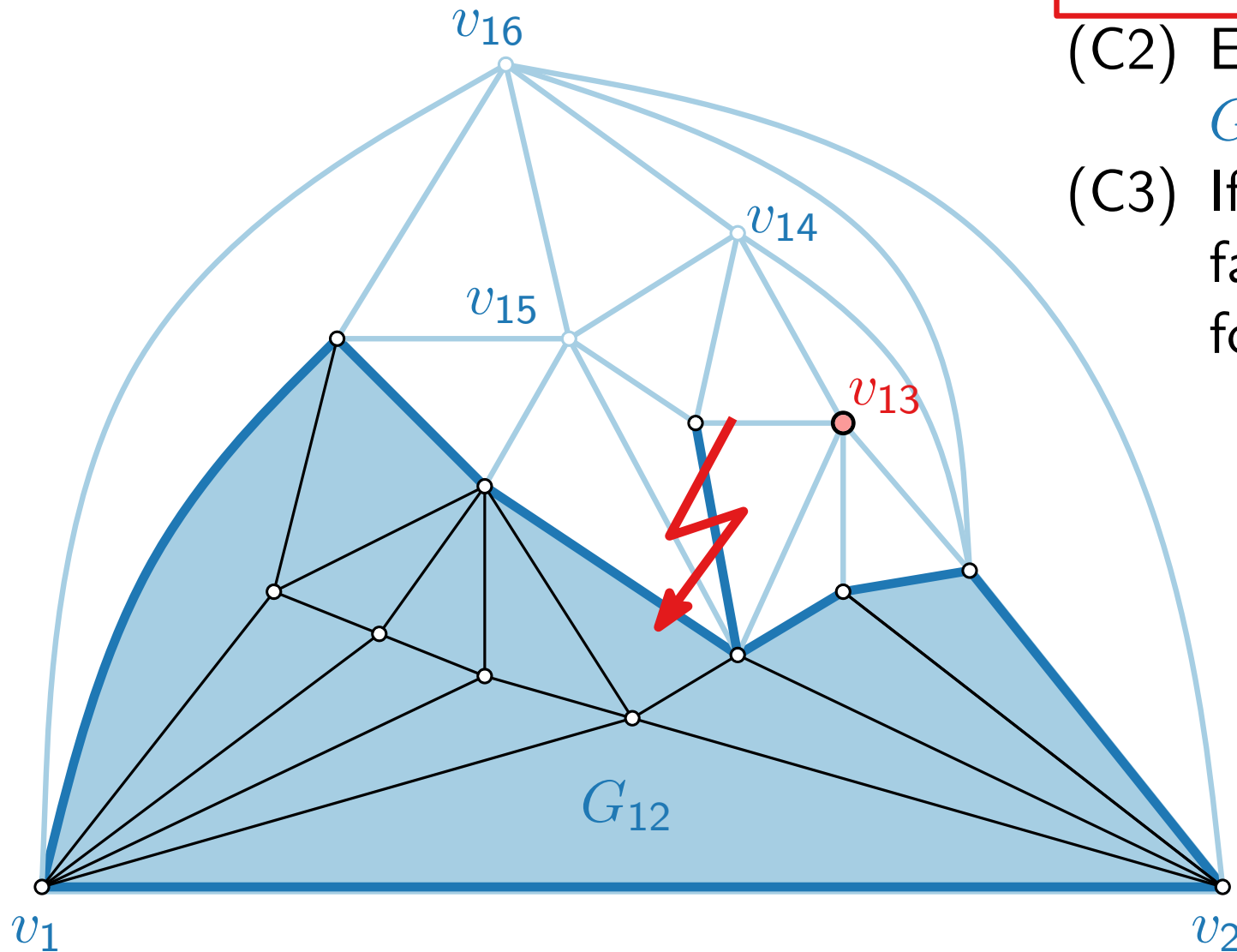
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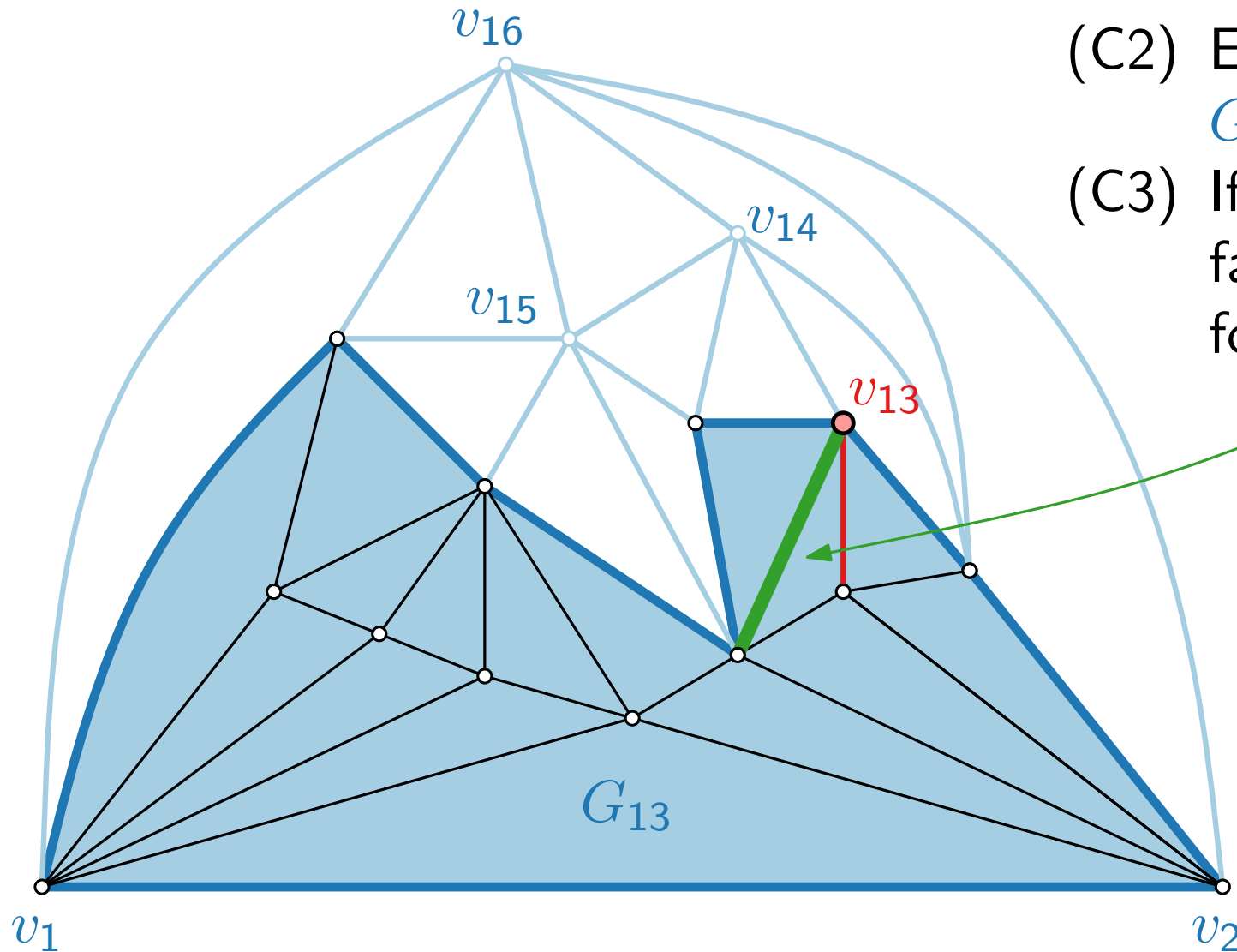
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Canonical Order – Example

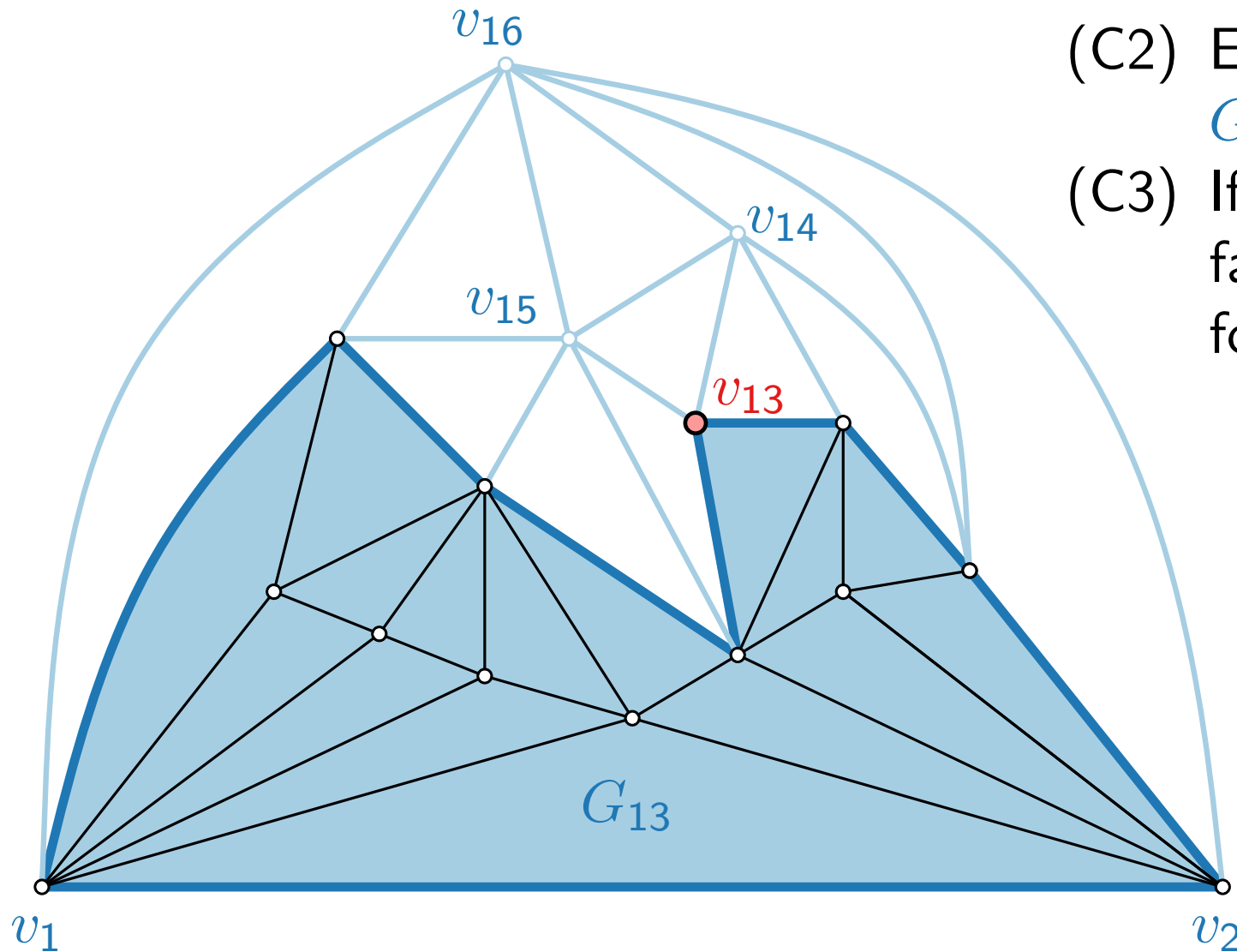
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chord:
edge joining two
non-adjacent
vertices in a cycle

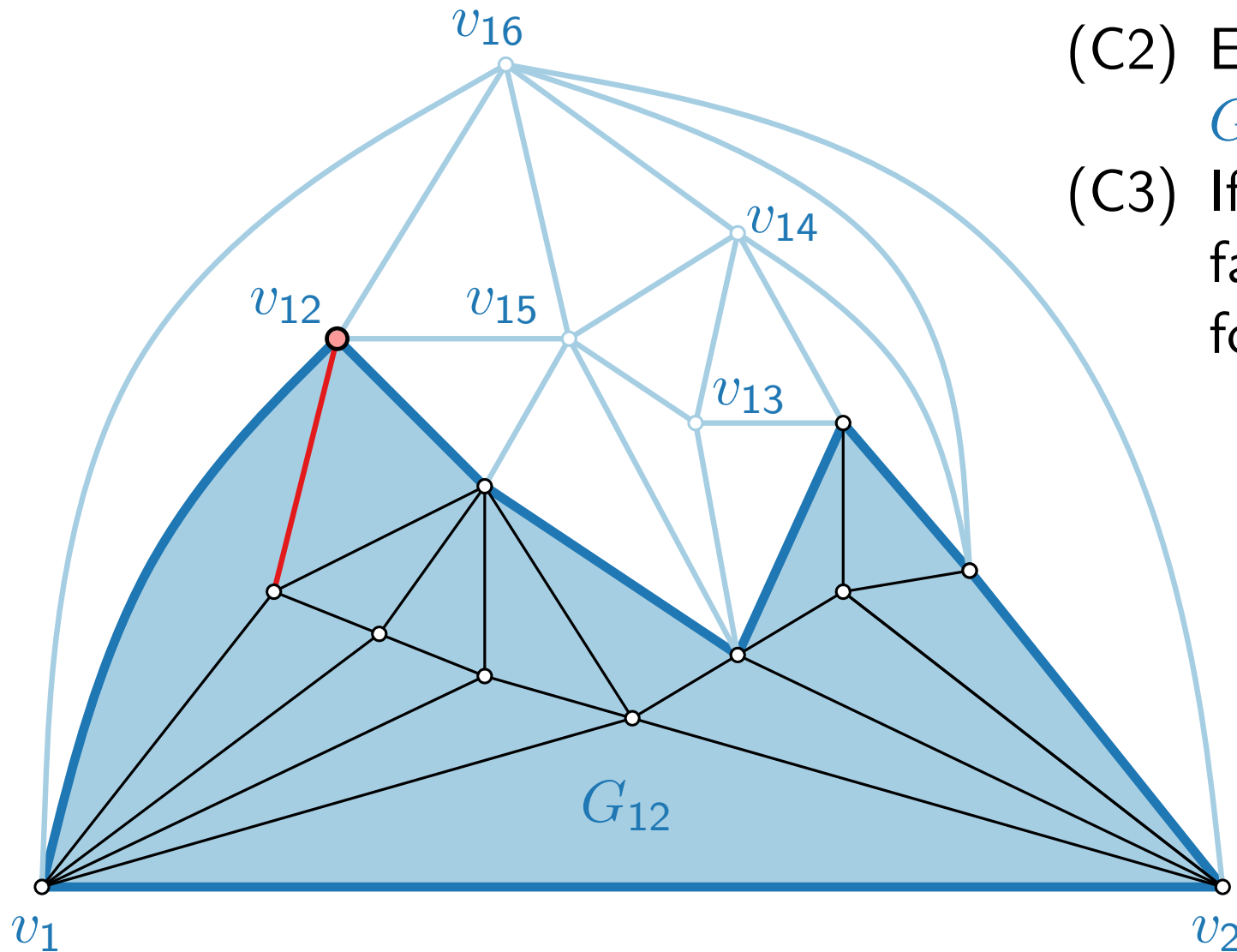
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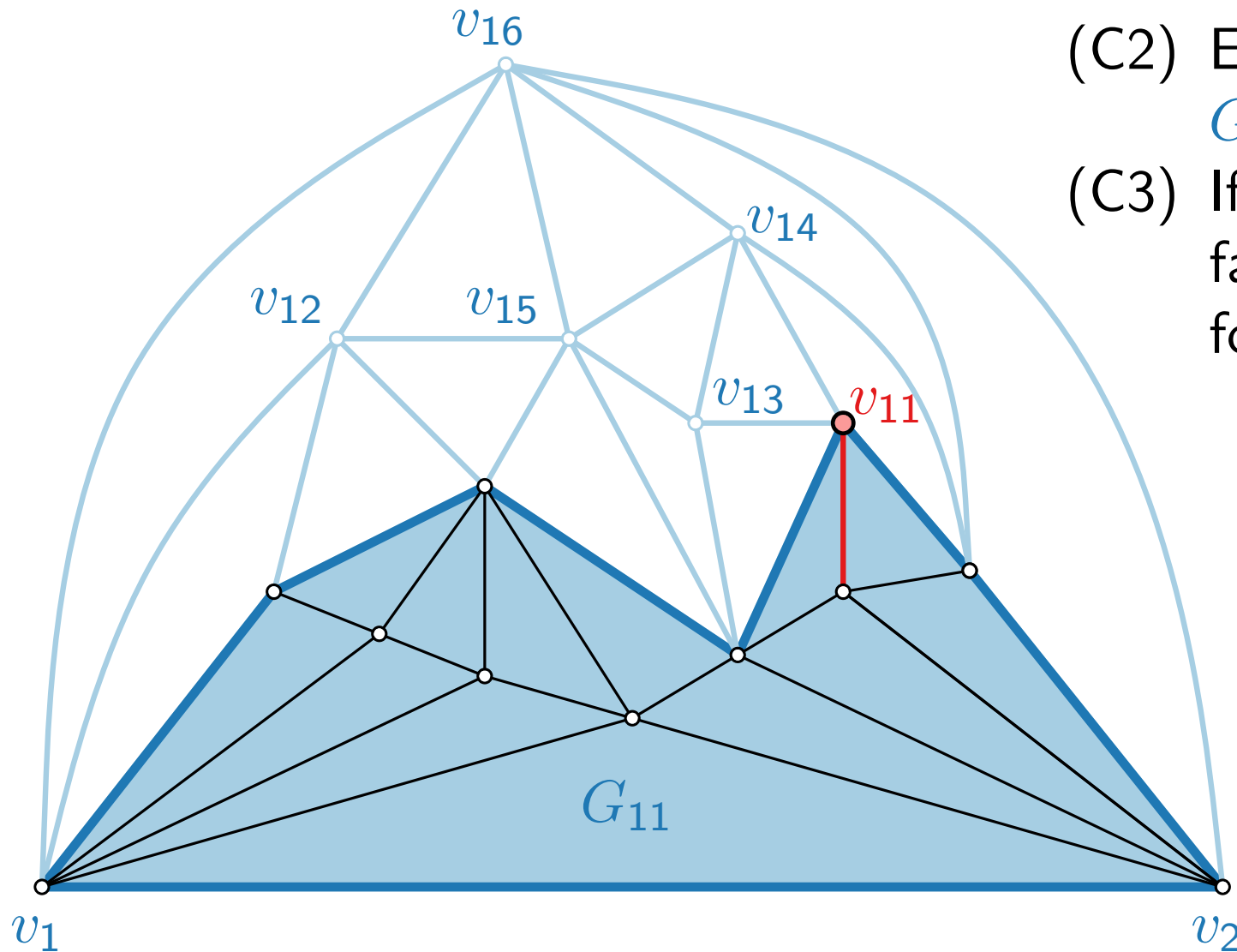
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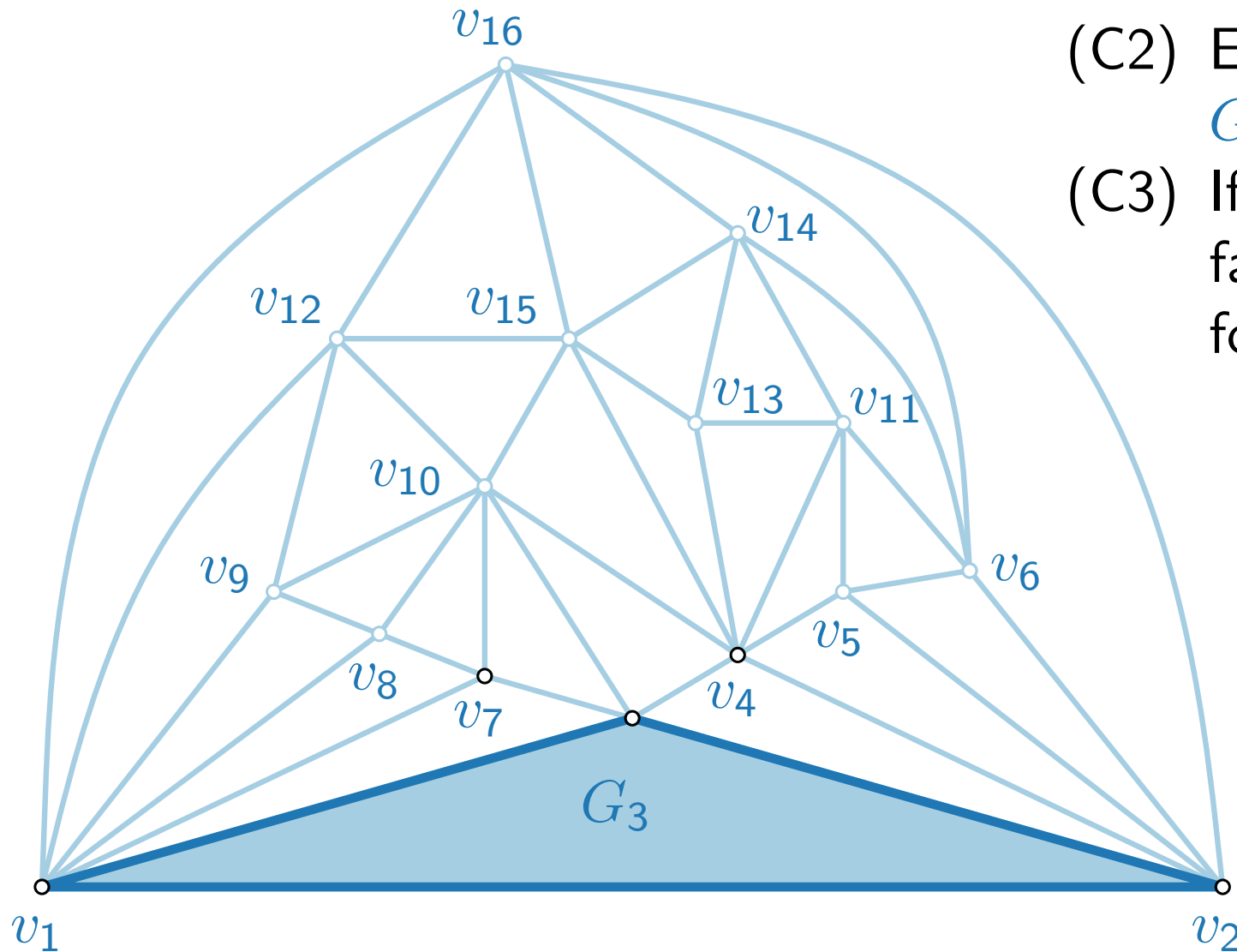
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Canonical Order – Existence

Lemma.

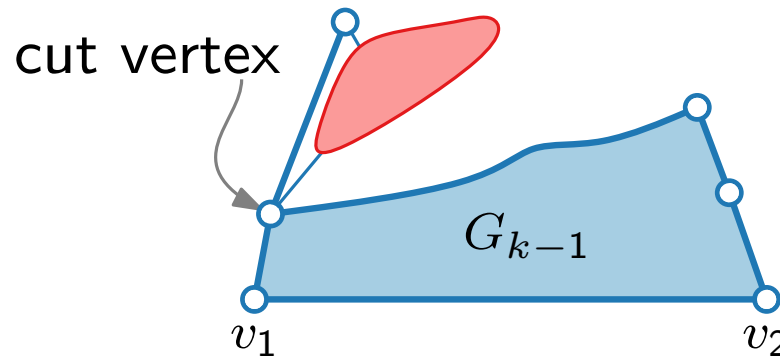
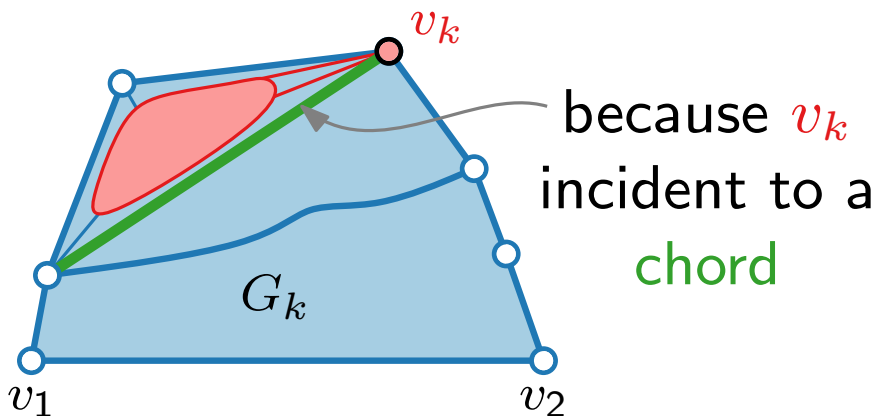
Every plane triangulation has a canonical order.

Consider any n -vertex plane triangulation. We show this statement by induction on k from n down to 3.

Induction base ($k = n$): Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n . Conditions (C1)–(C3) hold.

Induction hypothesis: Vertices v_{n-1}, \dots, v_{k+1} have been chosen such that conditions (C1)–(C3) hold for every $i \in \{k+1, \dots, n\}$.

Induction step: Consider G_k . We search for v_k .



- (C1) G_k biconnected inner triangulation
- (C2) (v_1, v_2) on outer face of G_k
- (C3) $k < n \Rightarrow v_{k+1}$ in outer face of G_k , neighbors of v_{k+1} form path on boundary of G_k

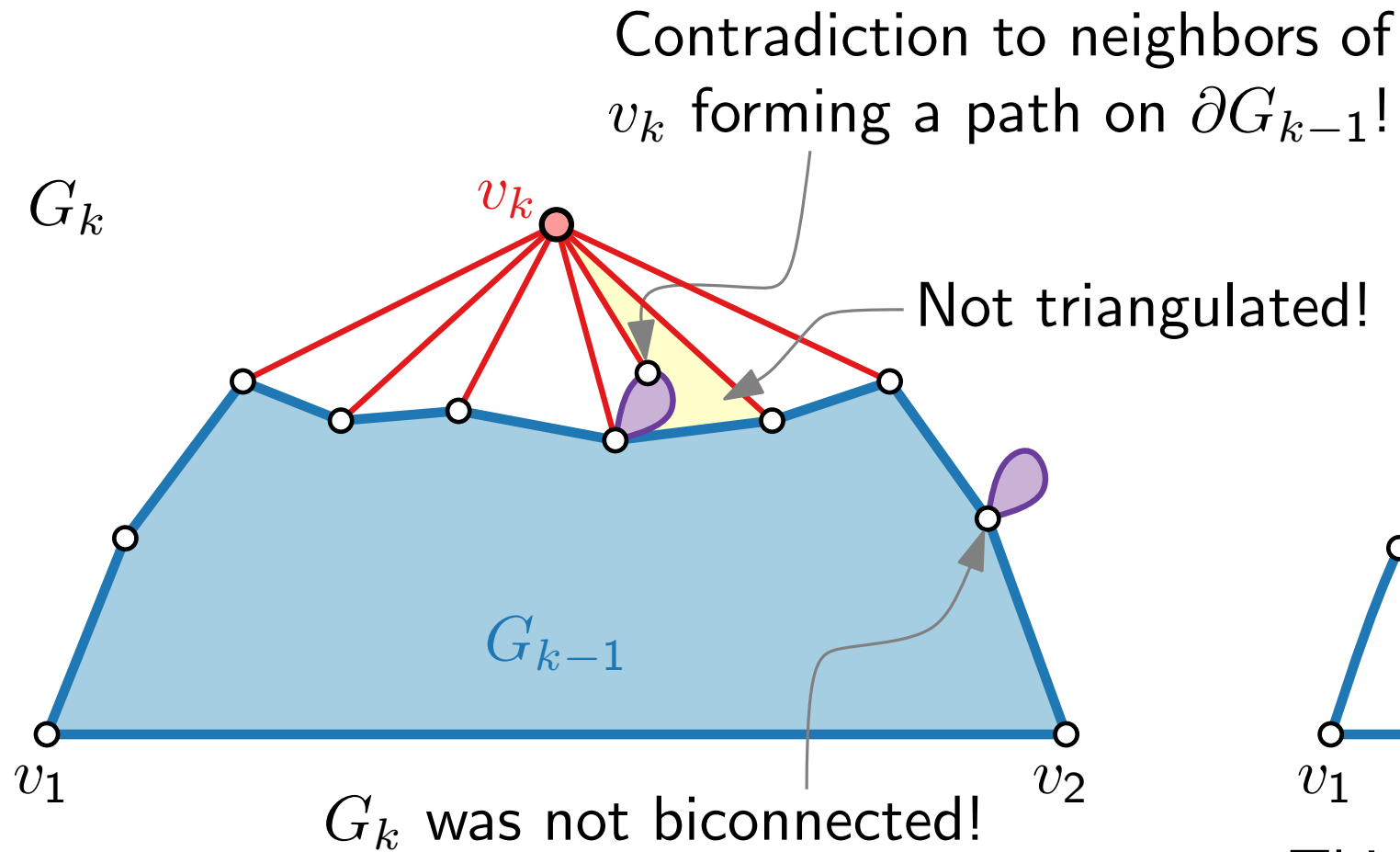
We need to show:

1. v_k not incident to chord is sufficient.
2. Such v_k exists.

Canonical Order – Existence

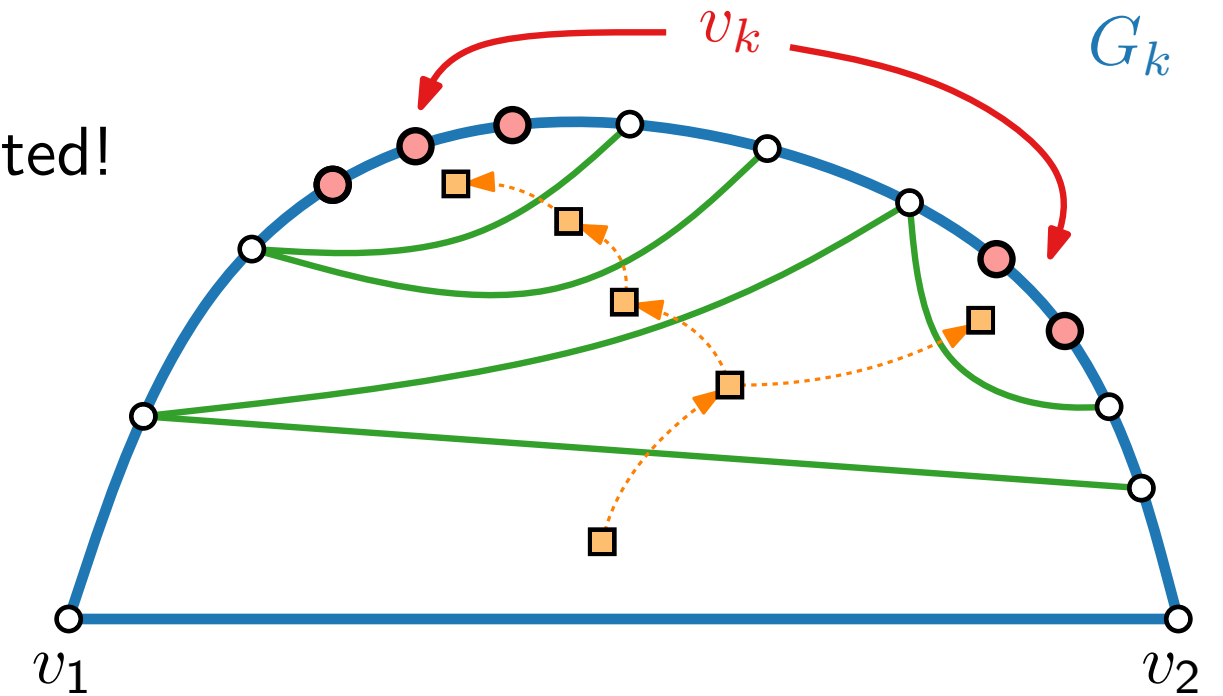
Claim 1.

If v_k is not incident to a chord, then G_{k-1} is biconnected.



Claim 2.

There exists a vertex in G_k that is not incident to a chord as choice for v_k .



This completes the proof of the lemma. \square

Canonical Order – Implementation

outer face

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**

└ chords(v) \leftarrow 0; out(v) \leftarrow false; mark(v) \leftarrow false

out(v_1), out(v_2), out(v_n) \leftarrow true

for $k = n$ **downto** 3 **do** // use list of candidates

choose $v \in V(G) \setminus \{v_1, v_2\}$ such that: \leftarrow

mark(v) = false, out(v) = true, chords(v) = 0

$v_k \leftarrow v$; mark(v_k) \leftarrow true; out(v_k) \leftarrow false

let w_p, \dots, w_q be the ordered unmarked neighbors of v_k

for $i = p + 1$ **to** $q - 1$ **do** // $O(n)$ time in total

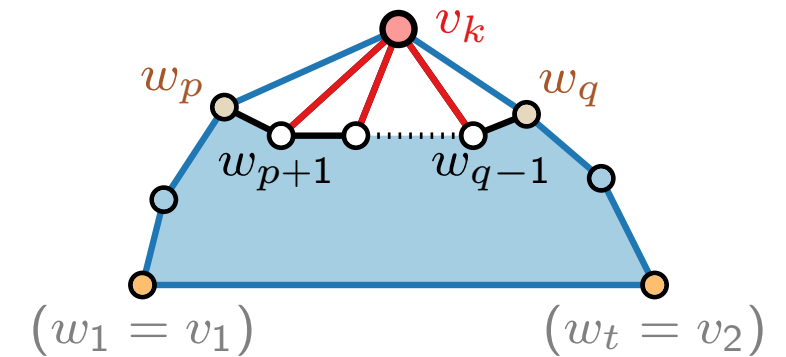
└ out(w_i) \leftarrow true // $O(m) = O(n)$ in total

└ **foreach** $u \in \text{Adj}[w_i] \setminus \{w_{i-1}, w_{i+1}\}$ **do** \leftarrow

└└ **if** out(u) **then** chords(w_i)++, chords(u)++

if $p + 1 = q$ **then** chords(w_p)--, chords(w_q)--

- chord(v) =
chords incident to v
- out(v) = true iff v on boundary of current outer face
- mark(v) = true iff v has received a number $\geq k$



Lemma.

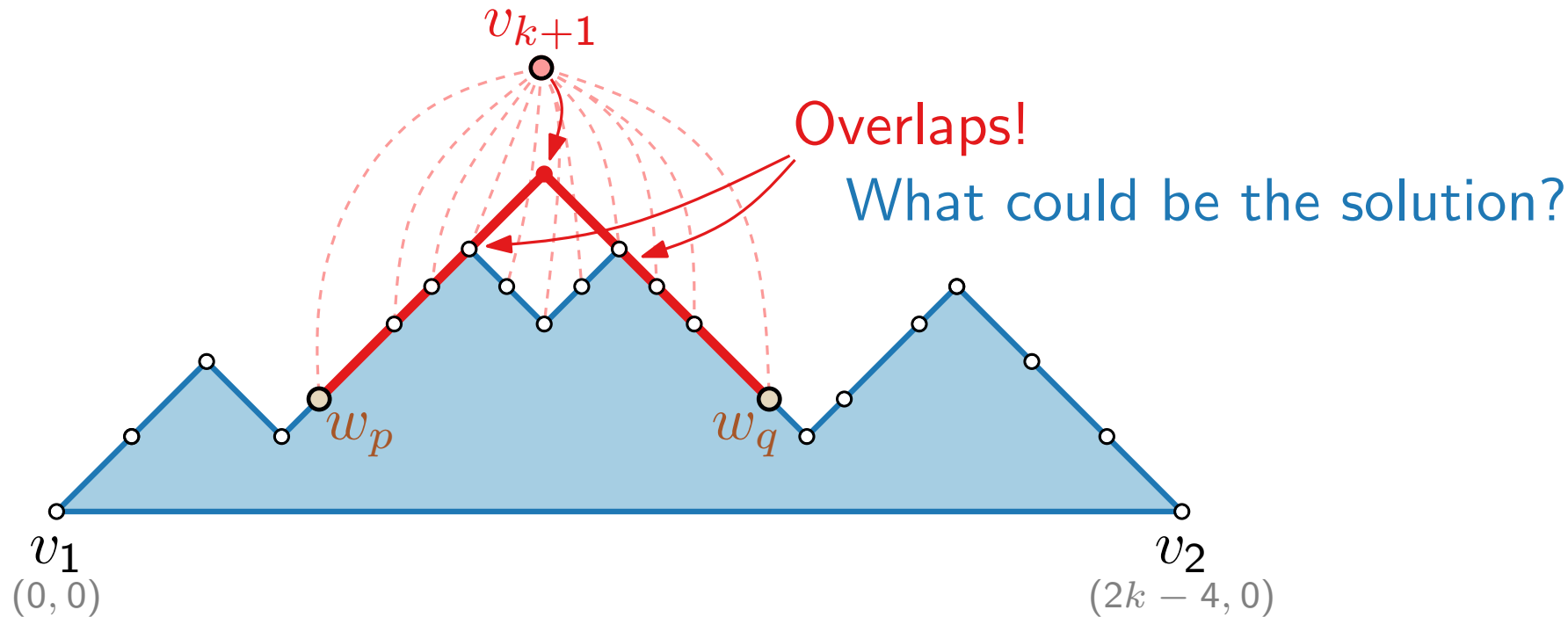
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

Shift Method – Idea

Drawing invariants:

G_k is drawn such that

- v_1 is at $(0, 0)$, v_2 is at $(2k - 4, 0)$,
- boundary of G_k (minus edge $\{v_1, v_2\}$) is drawn x-monotone,
- each edge on the boundary of G_k (except $\{v_1, v_2\}$) is drawn with slopes ± 1 .

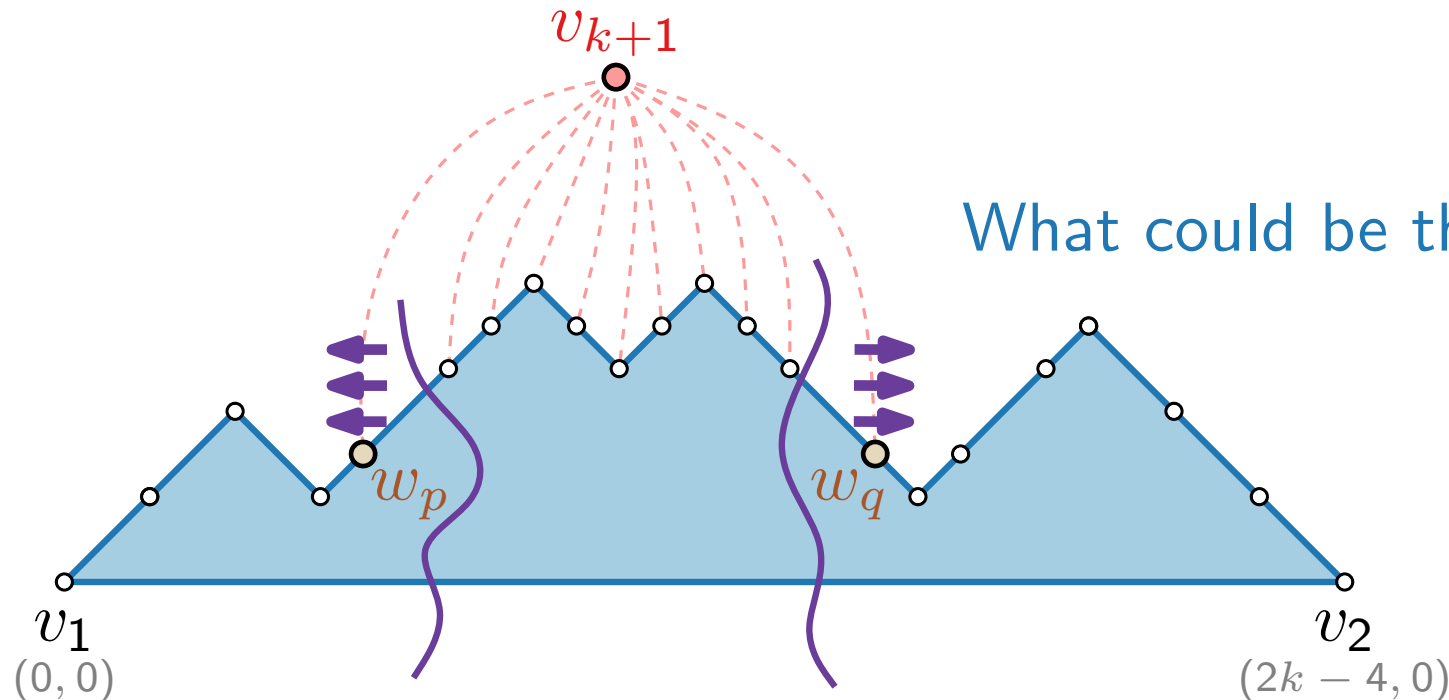


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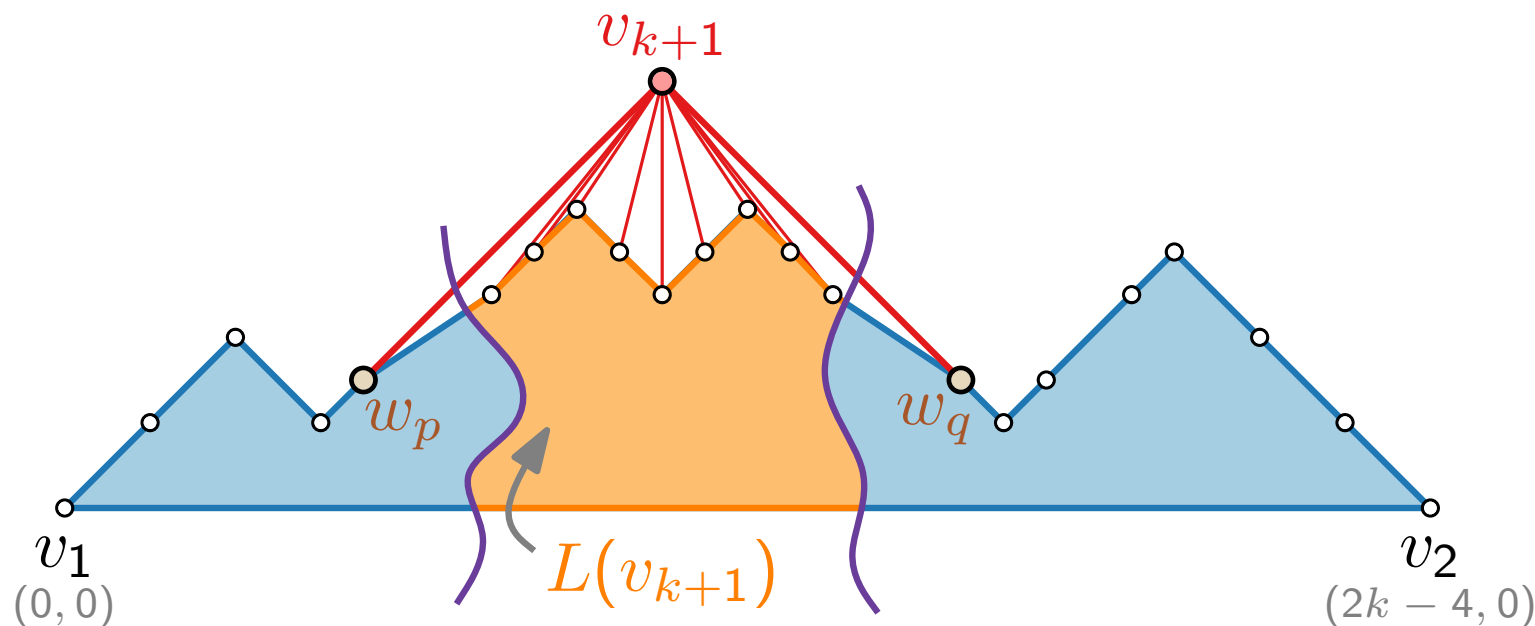


Shift Method – Idea

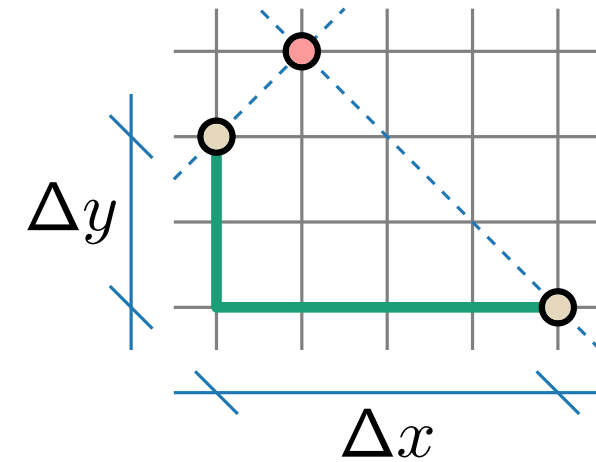
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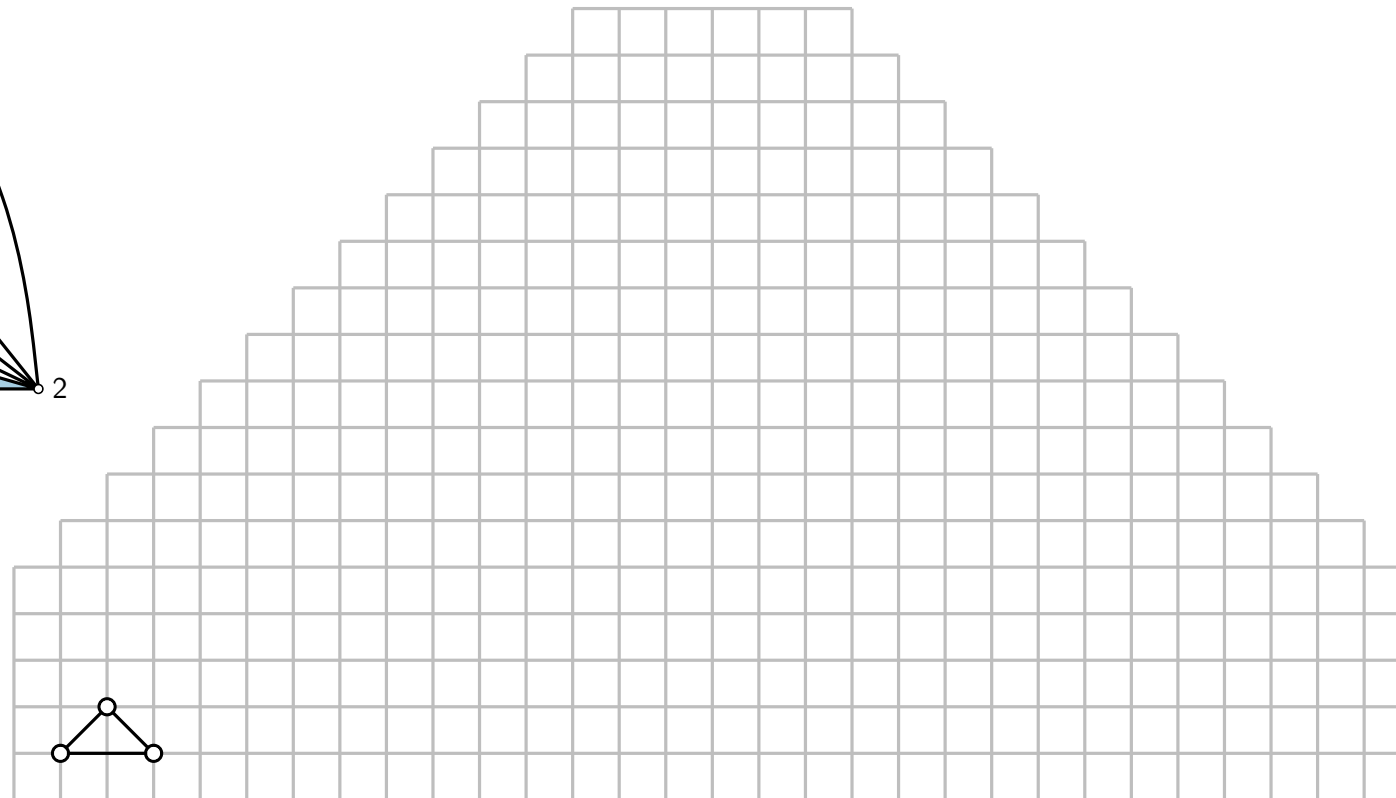
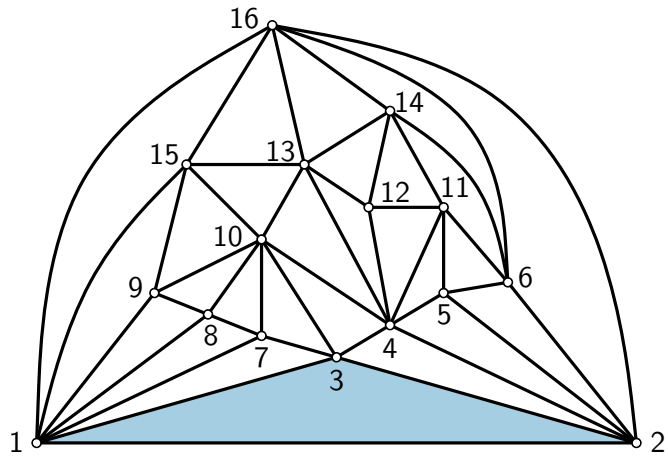
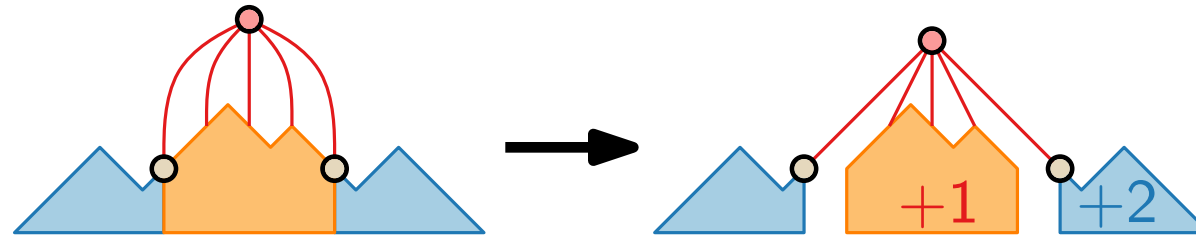
Will v_{k+1} lie on the grid?



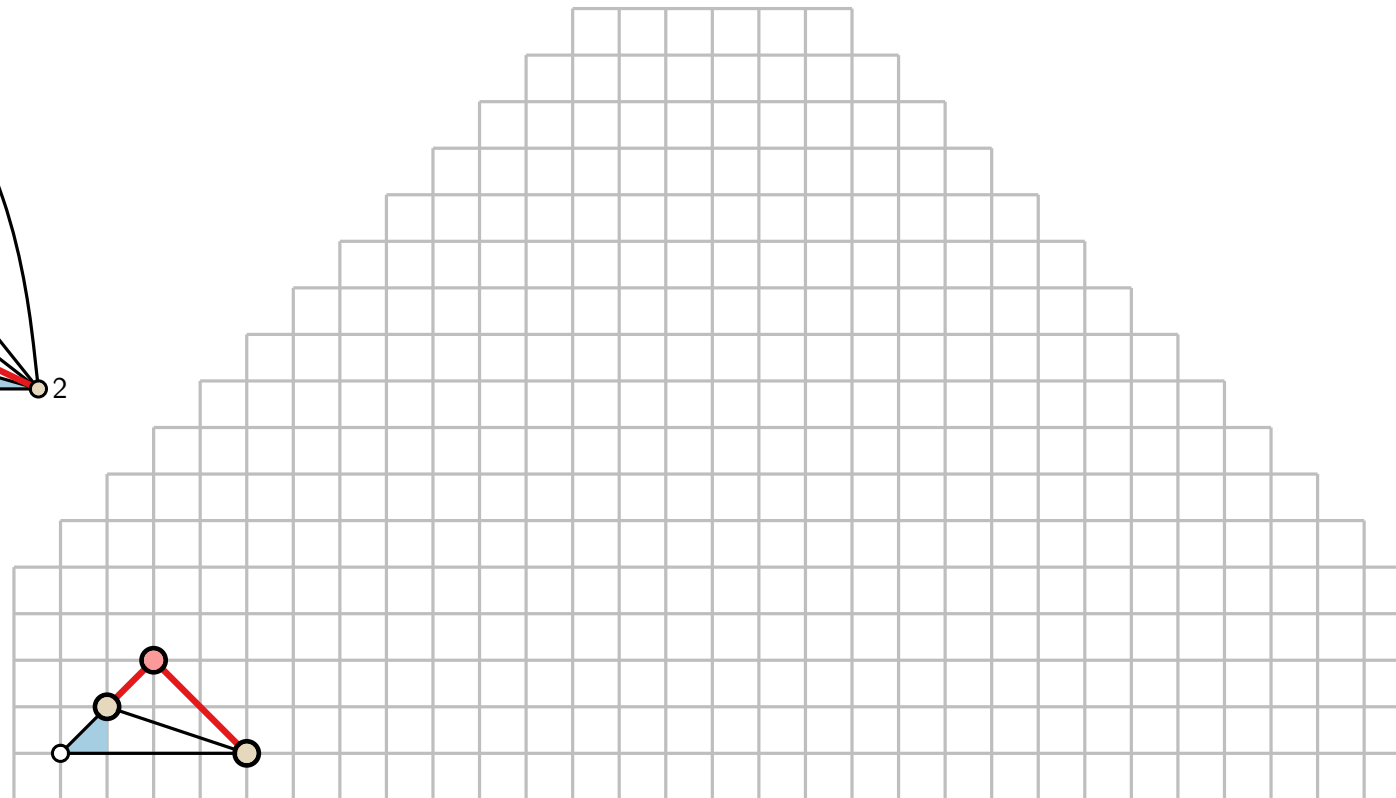
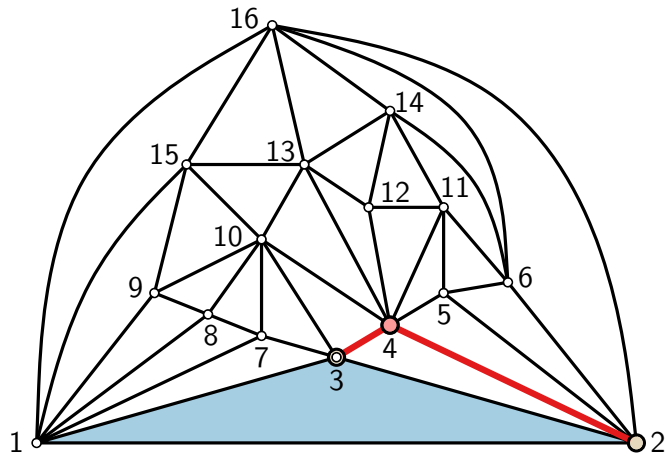
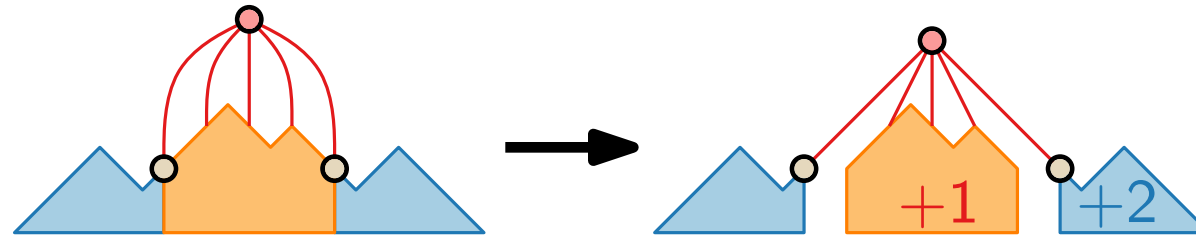
Yes, since w_p and w_q have even Manhattan distance $\Delta x + \Delta y$.

If we follow the boundary of G_k from w_p to w_q , then from one grid point g to the next, the current Manhattan distance from w_p to g changes by 0 or 2.

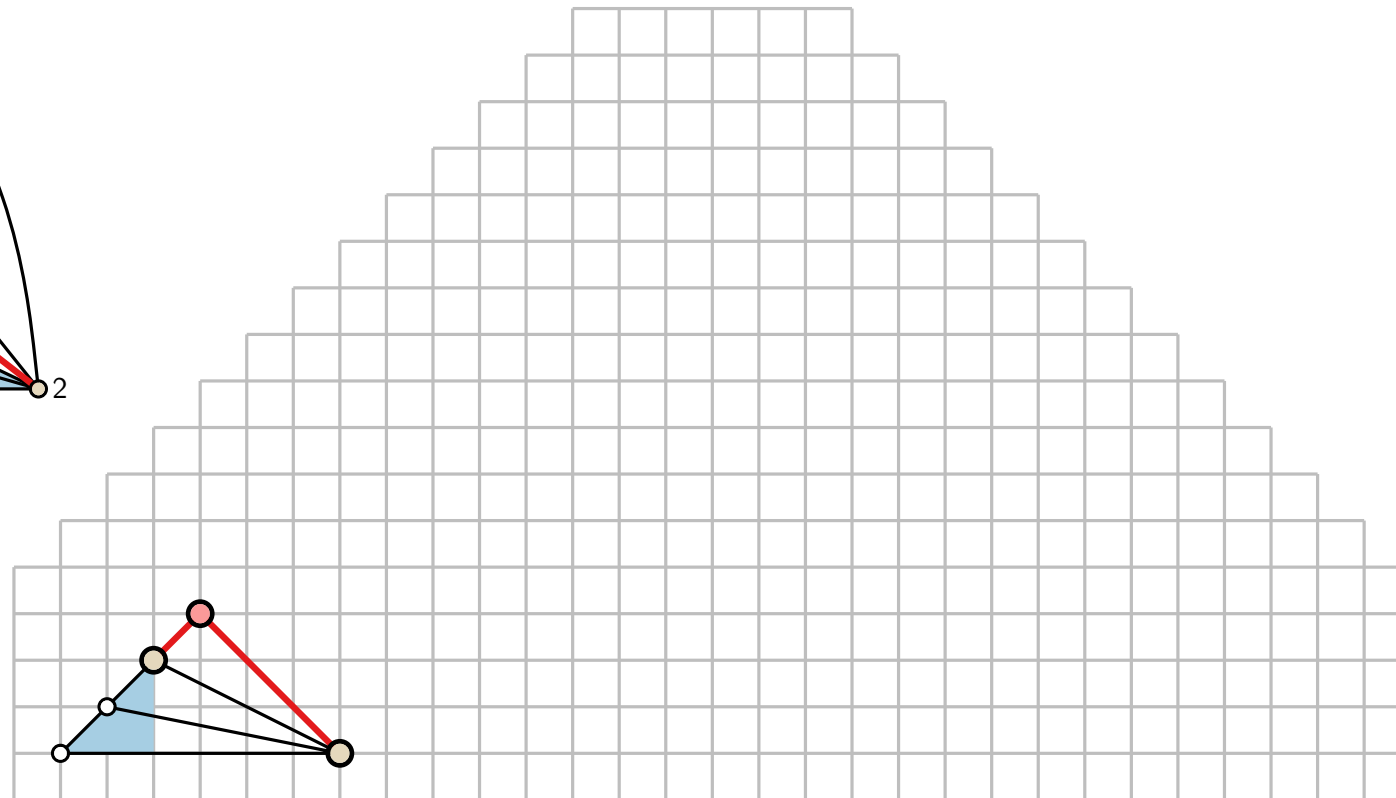
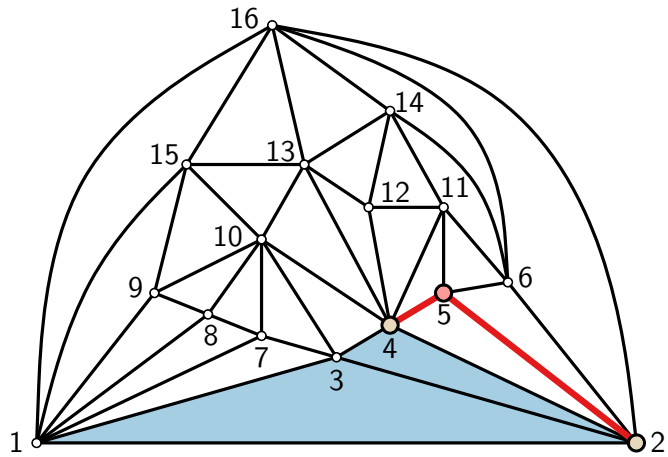
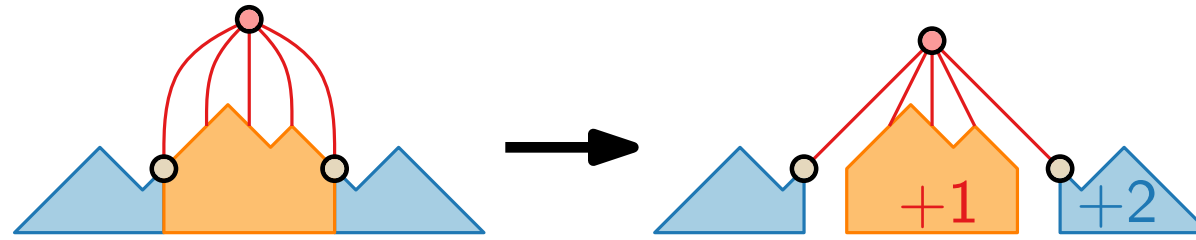
Shift Method – Example



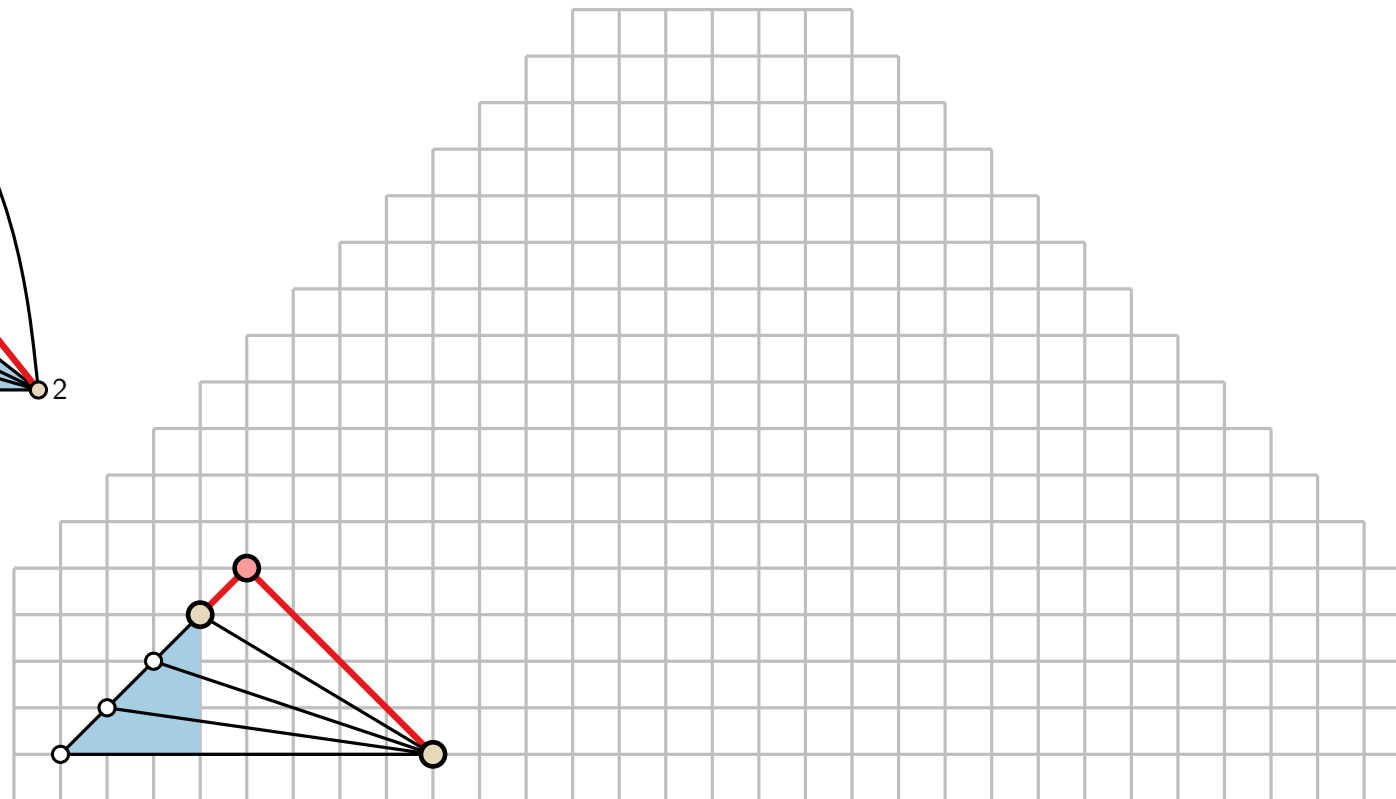
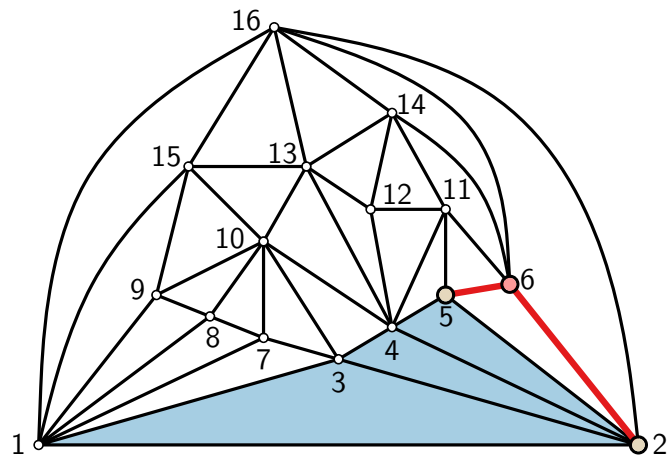
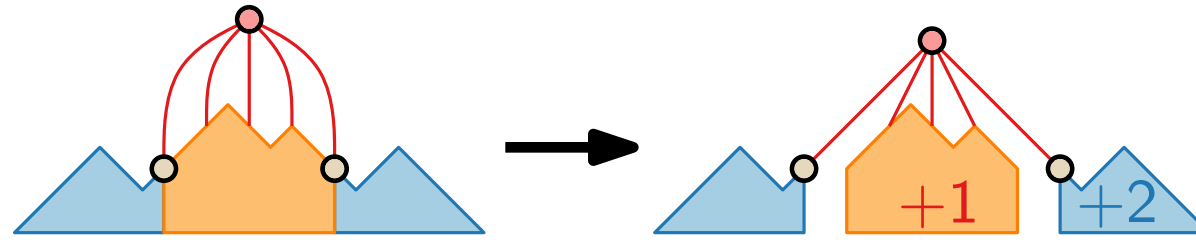
Shift Method – Example



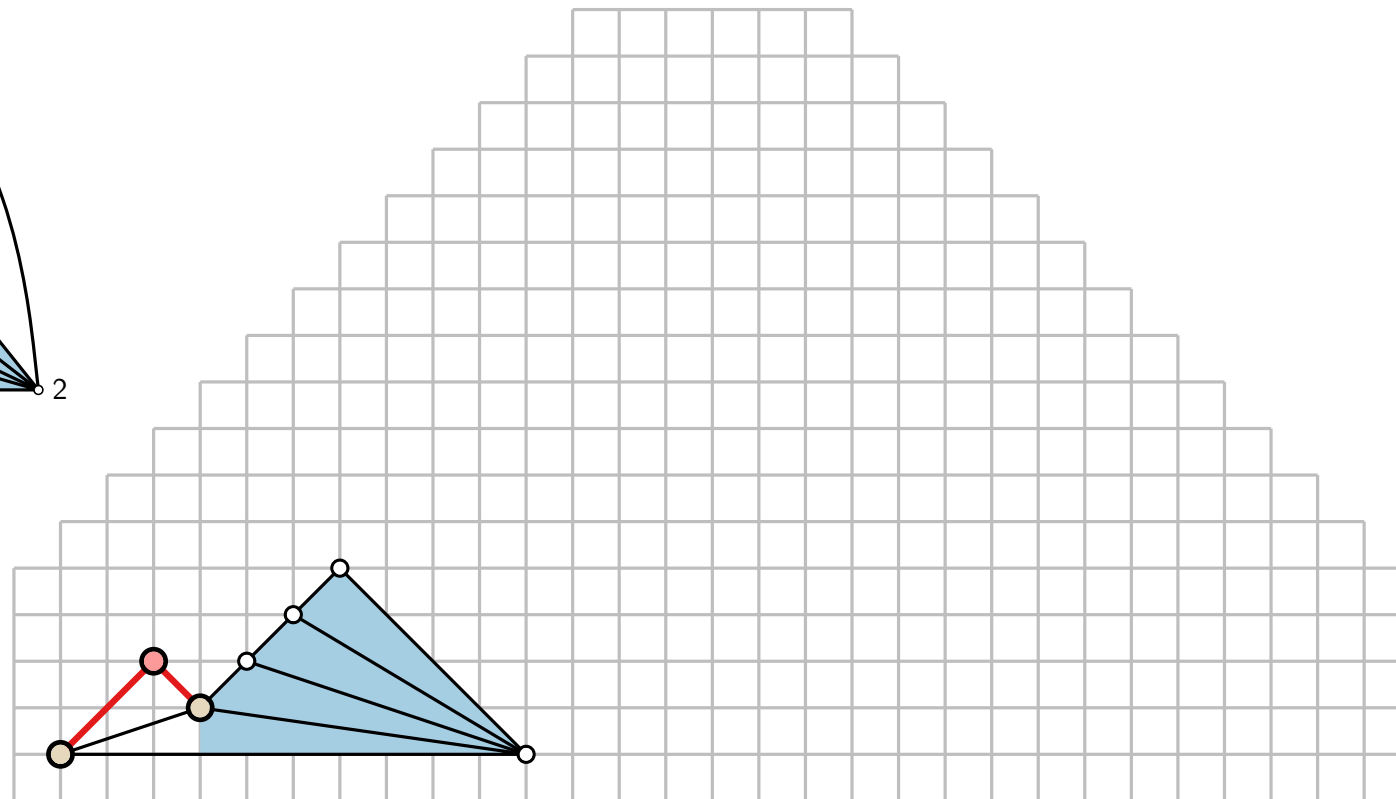
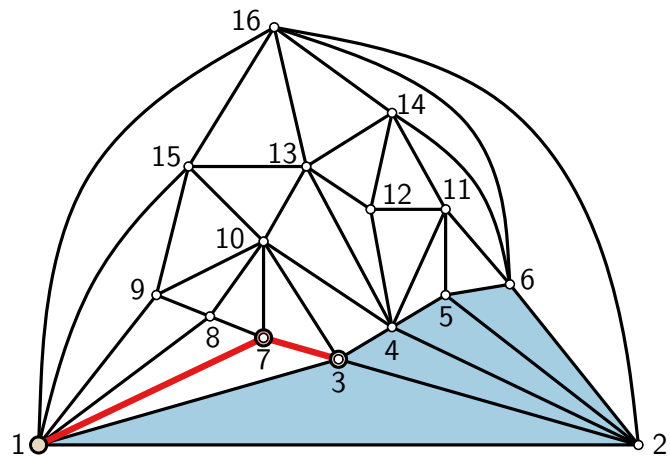
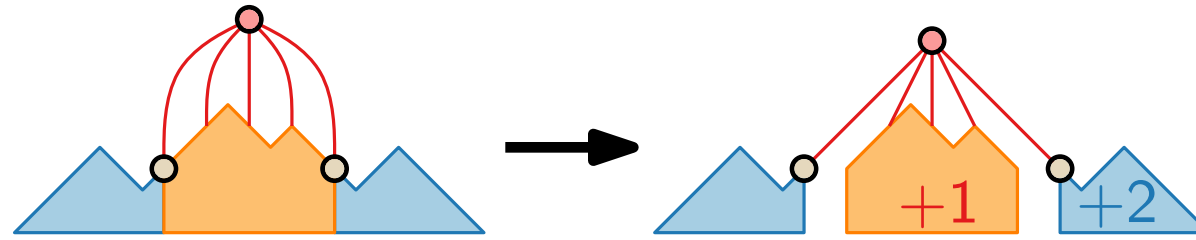
Shift Method – Example



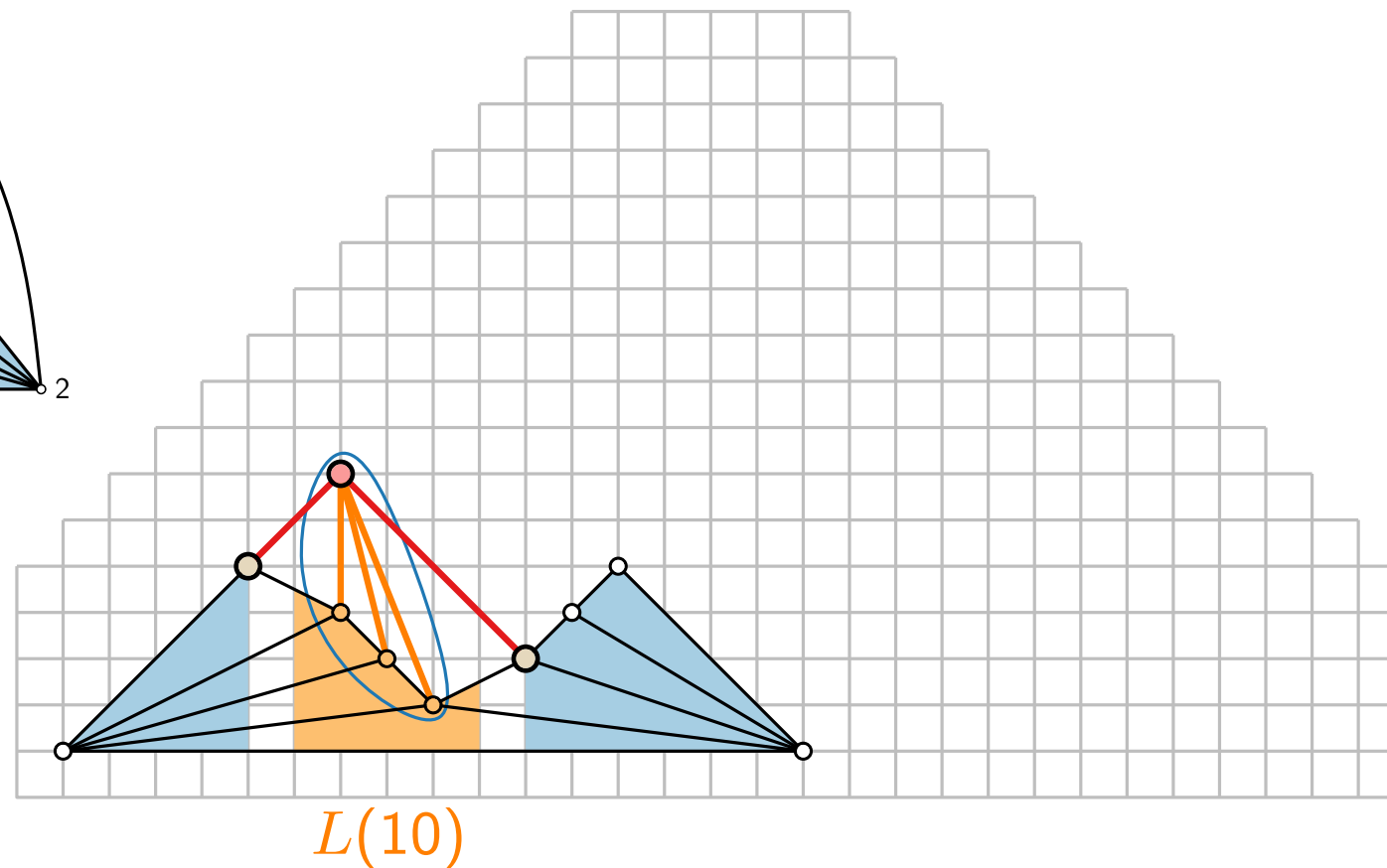
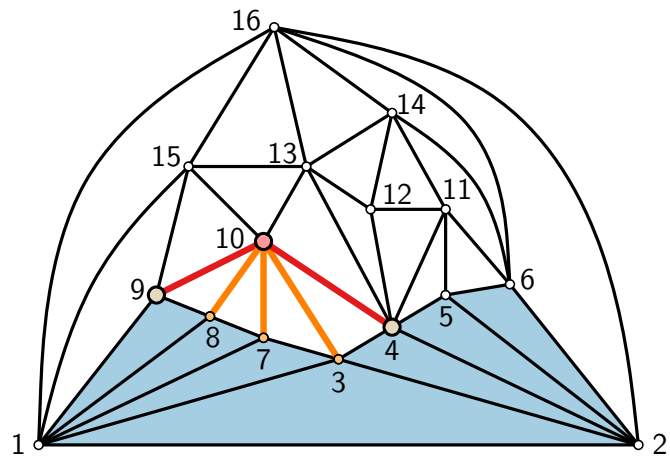
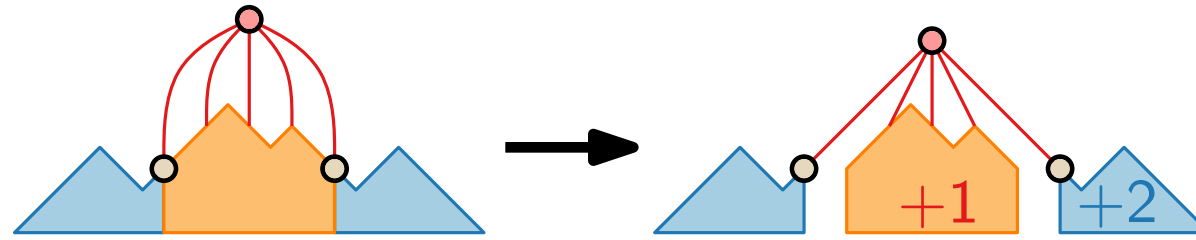
Shift Method – Example



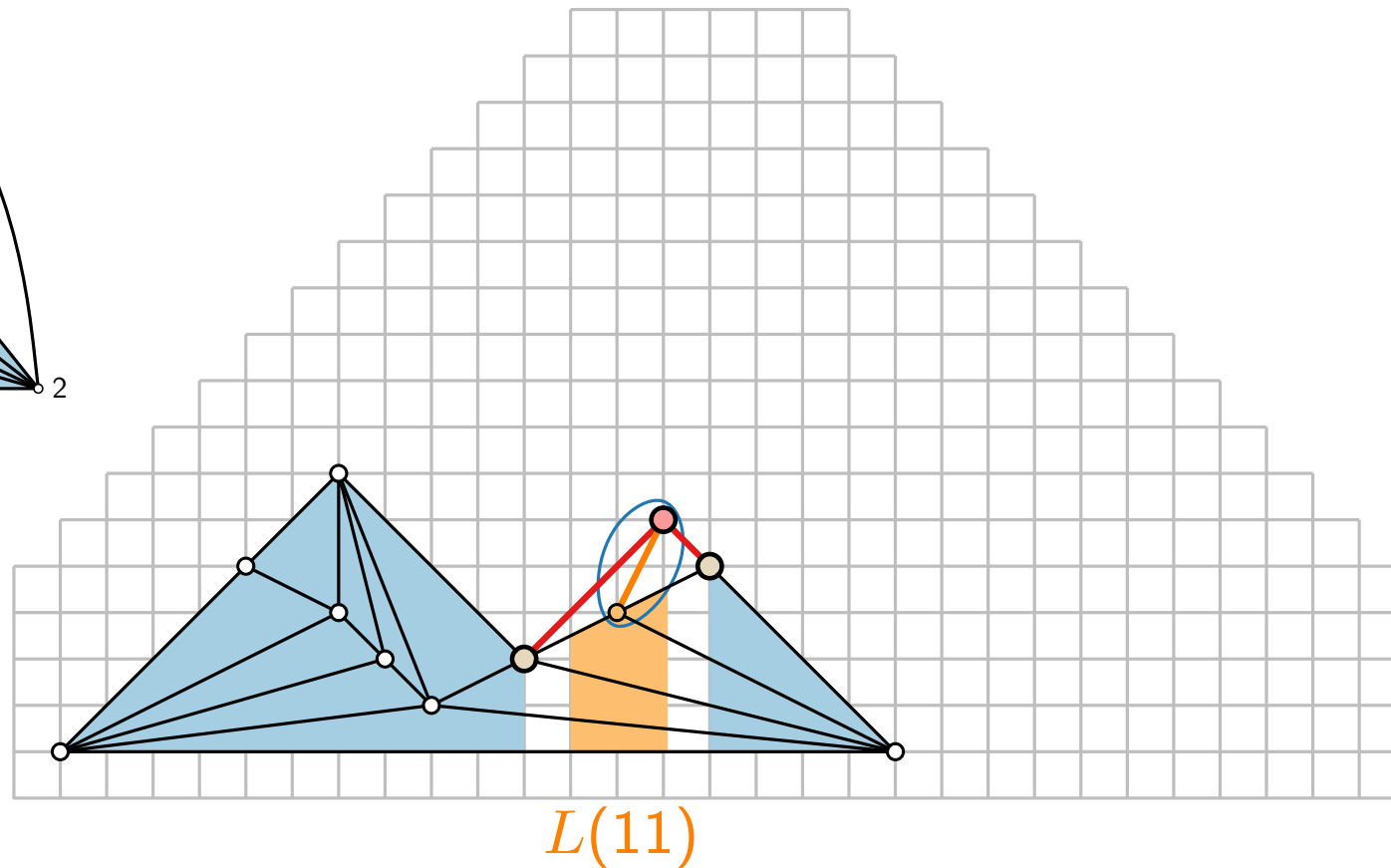
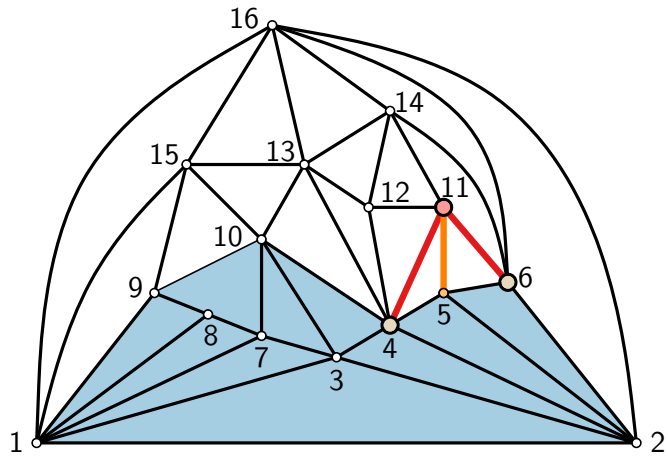
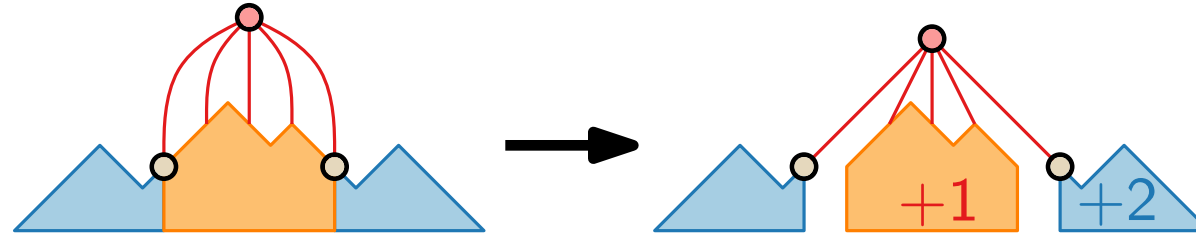
Shift Method – Example



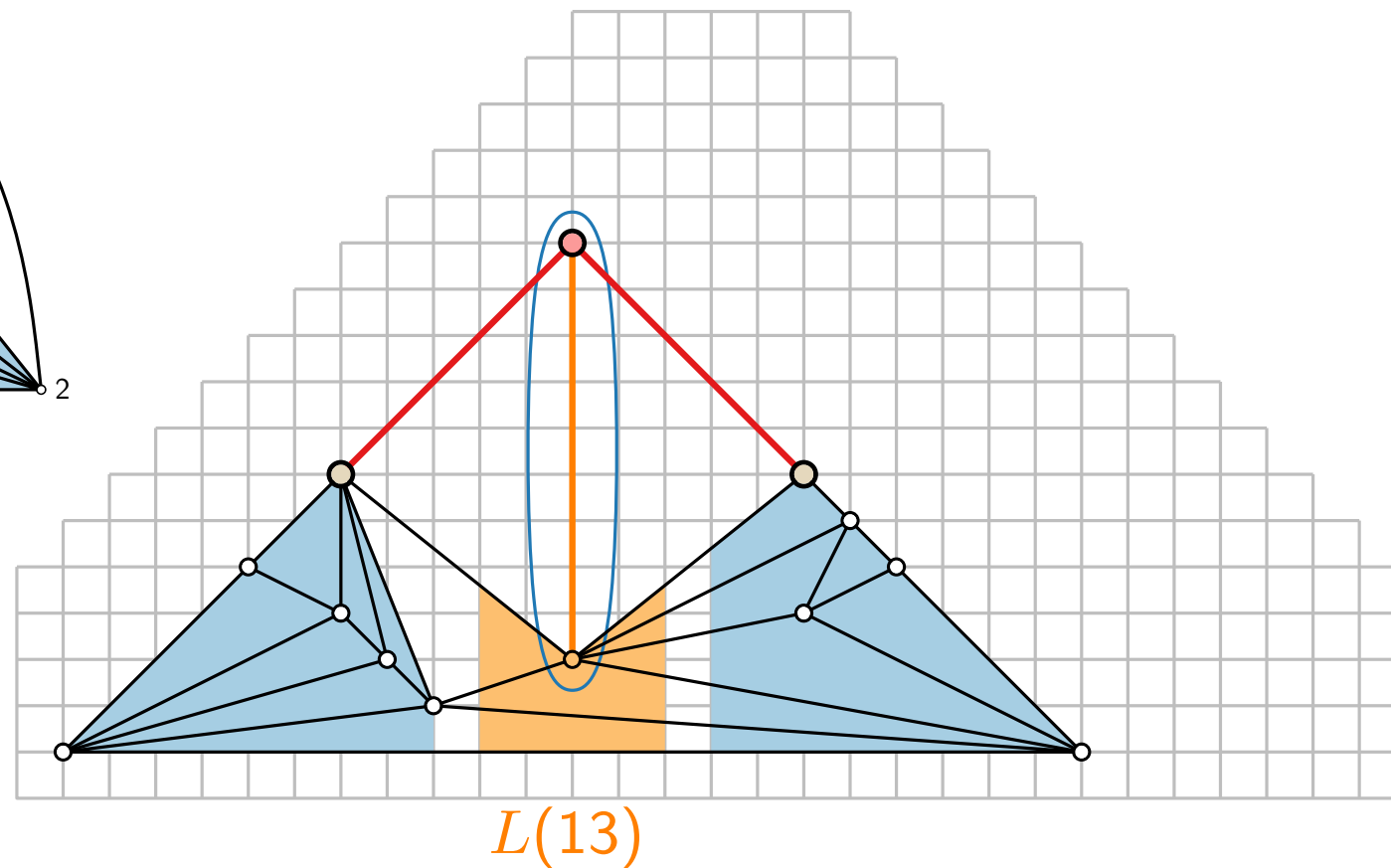
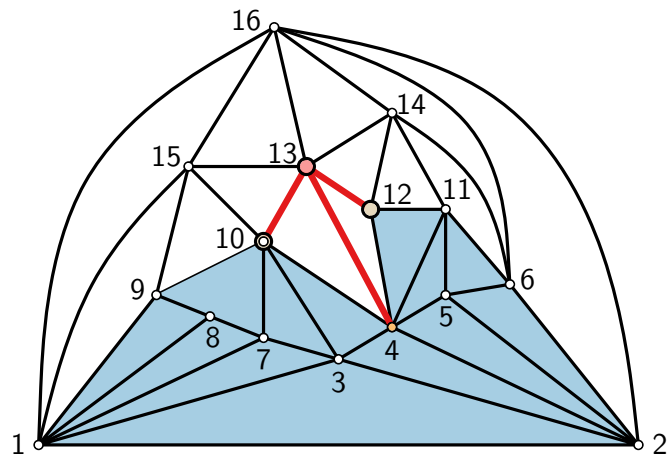
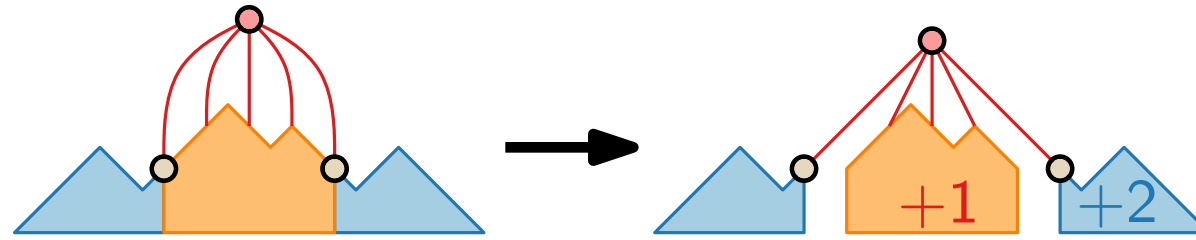
Shift Method – Example



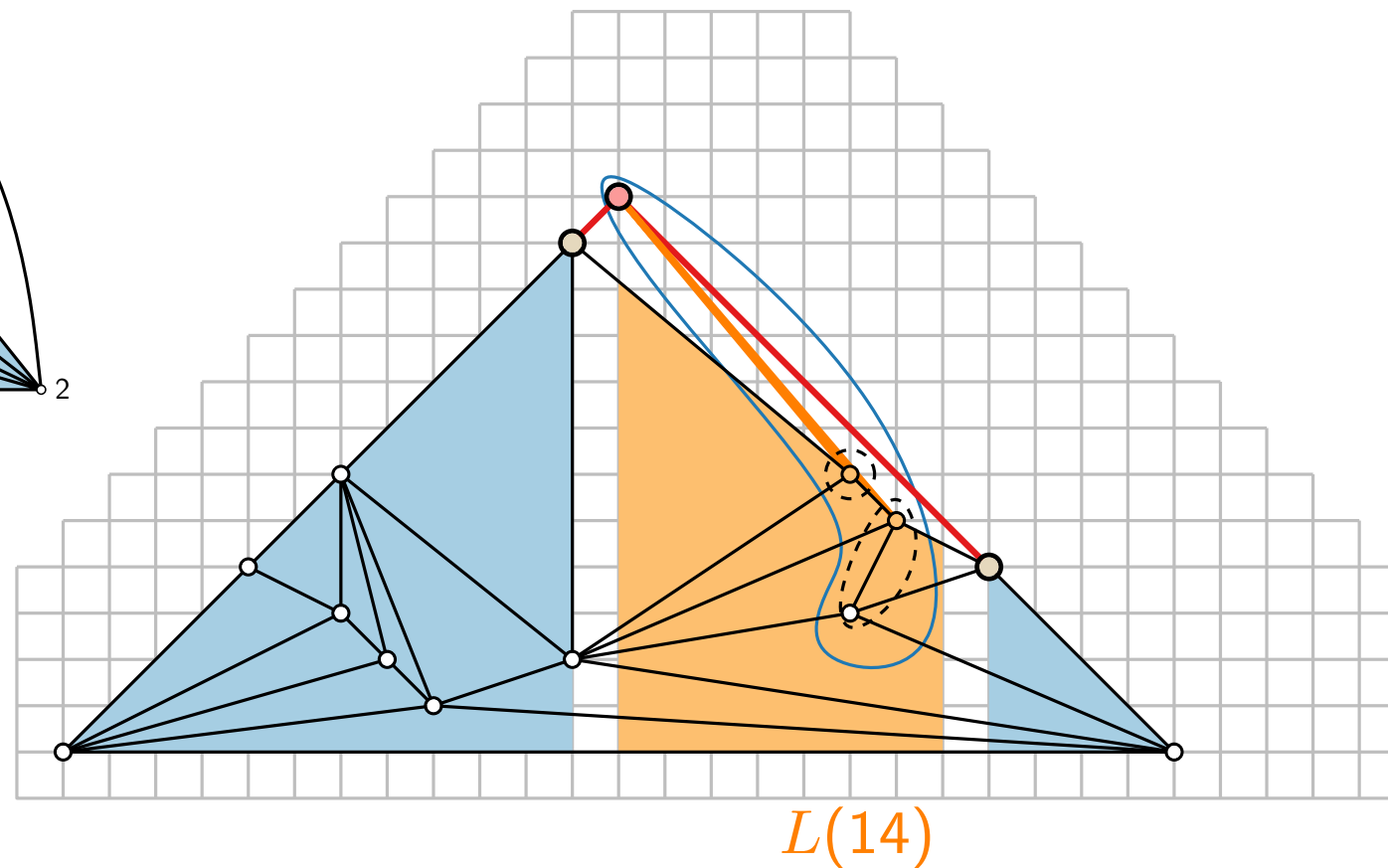
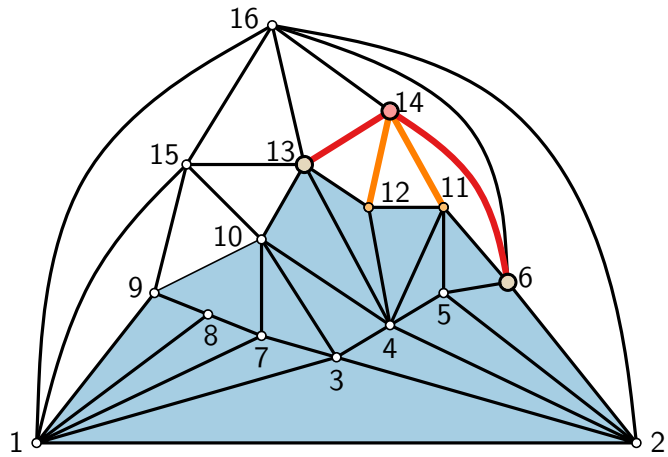
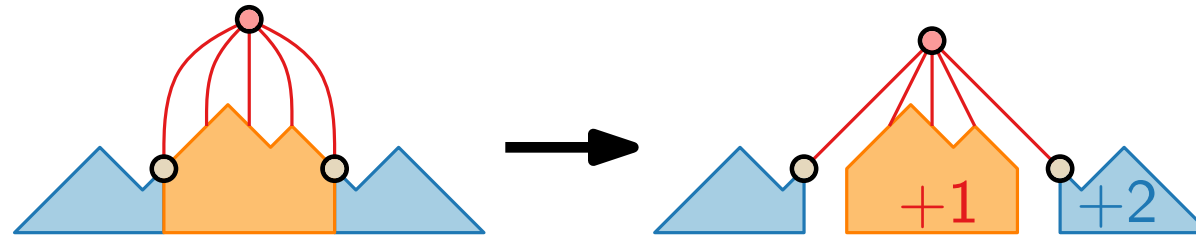
Shift Method – Example



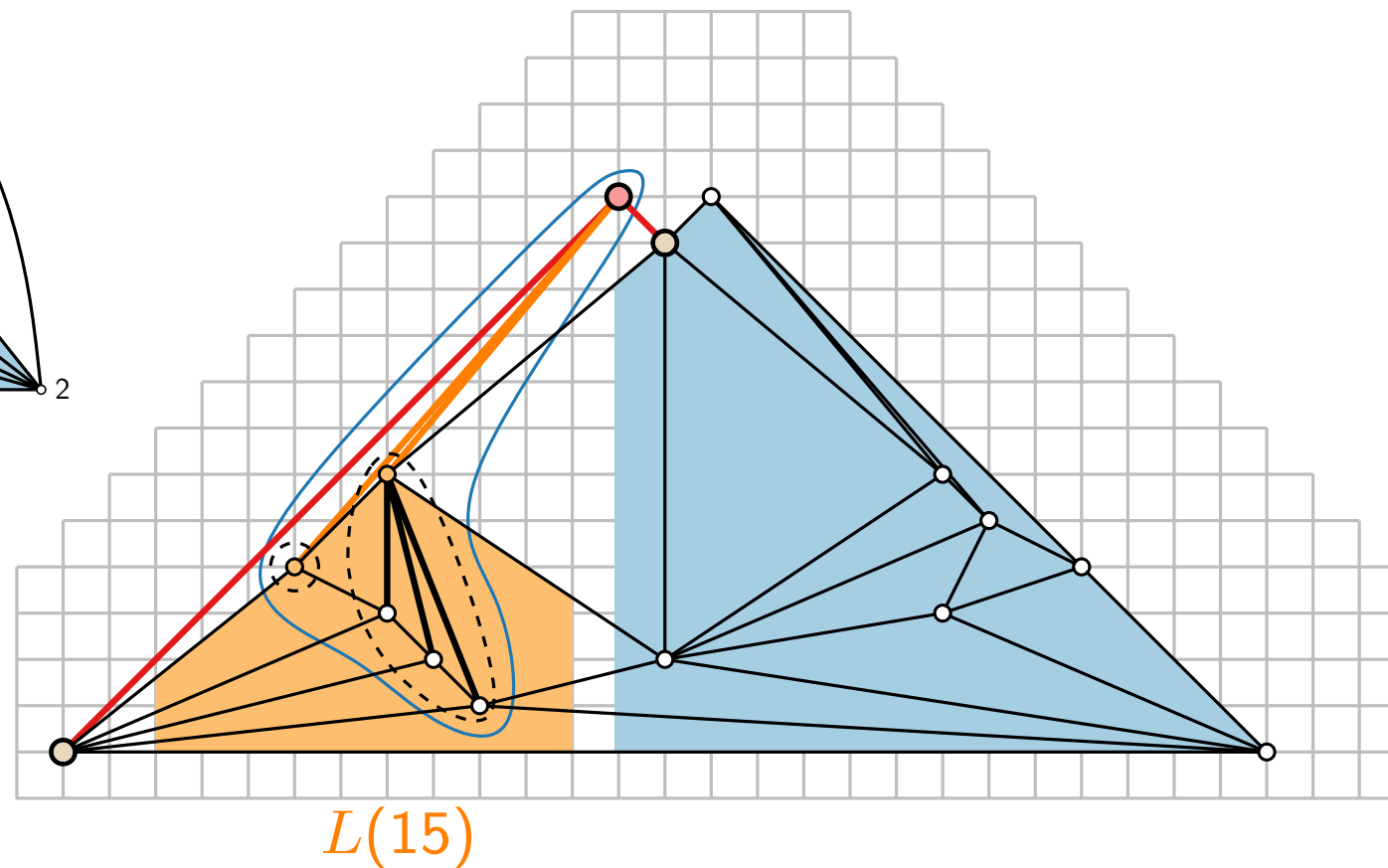
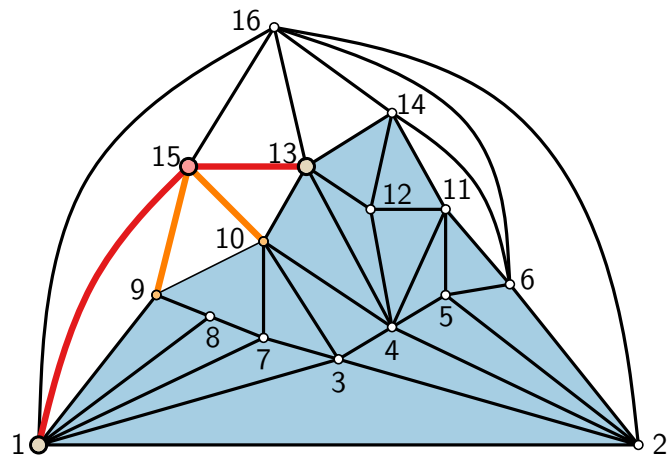
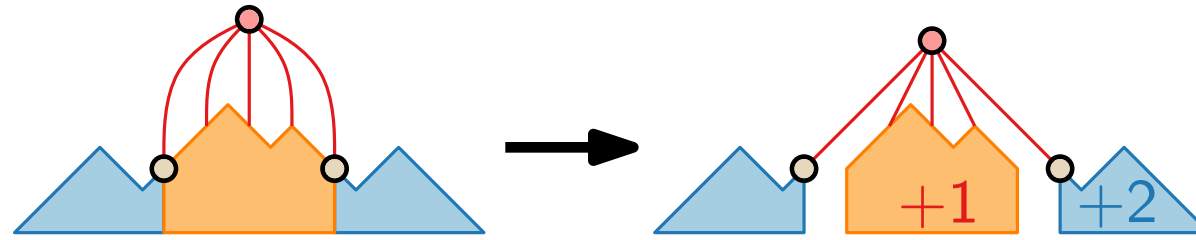
Shift Method – Example



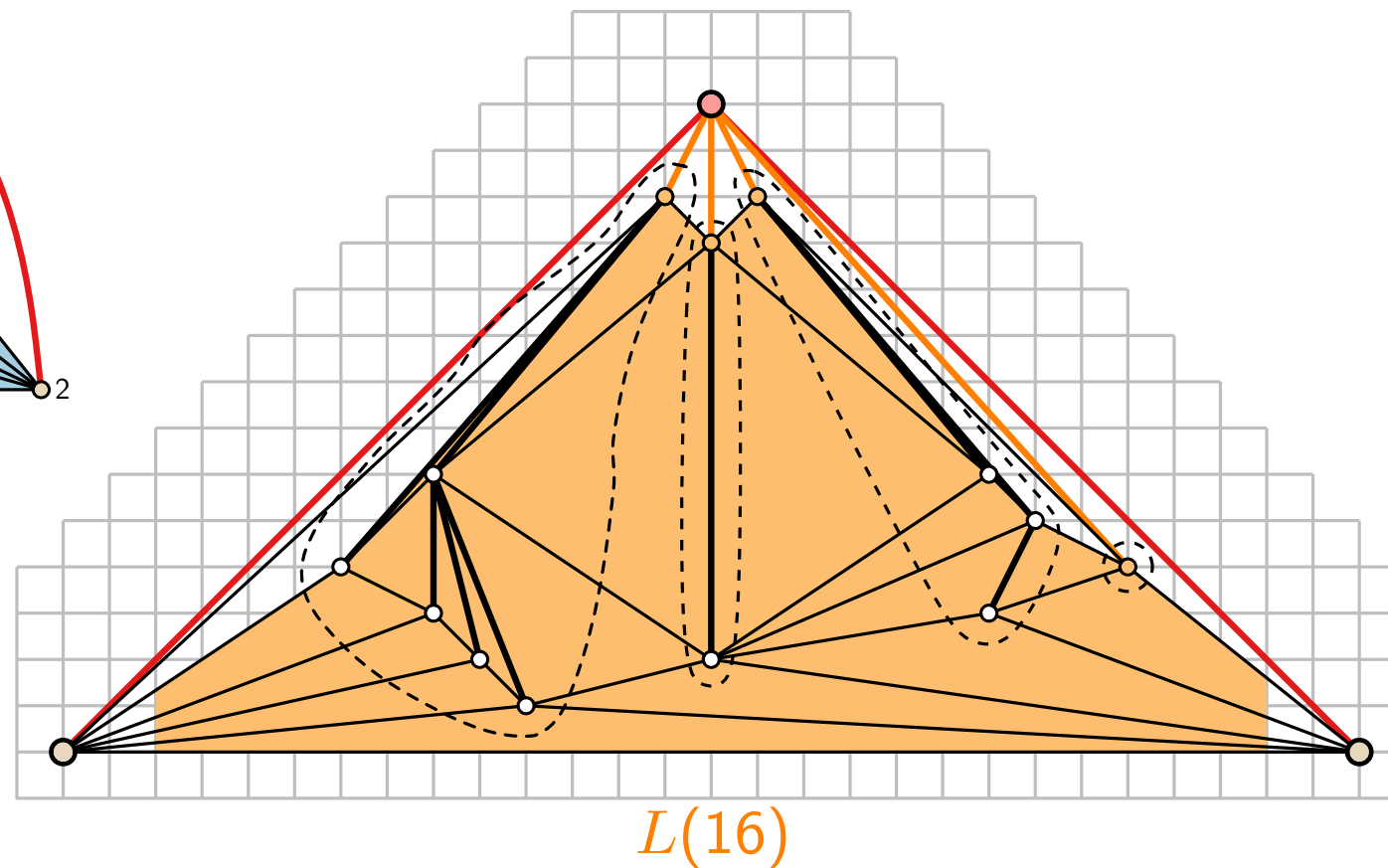
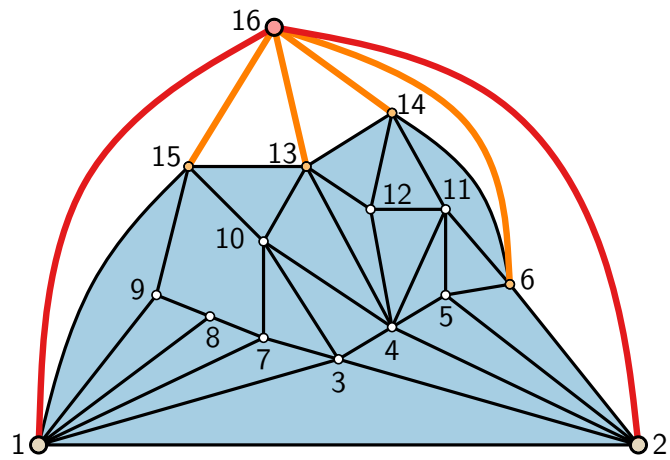
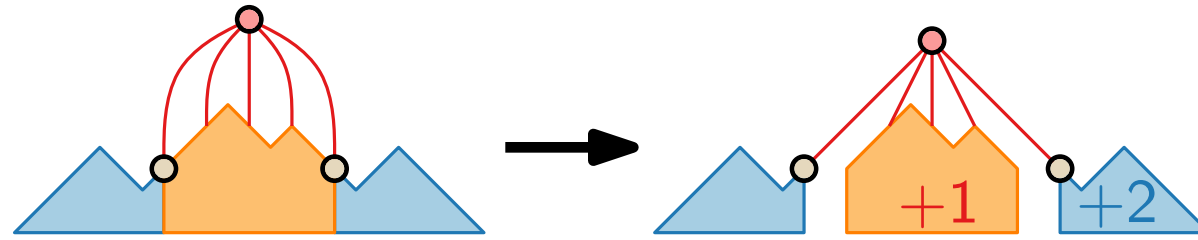
Shift Method – Example



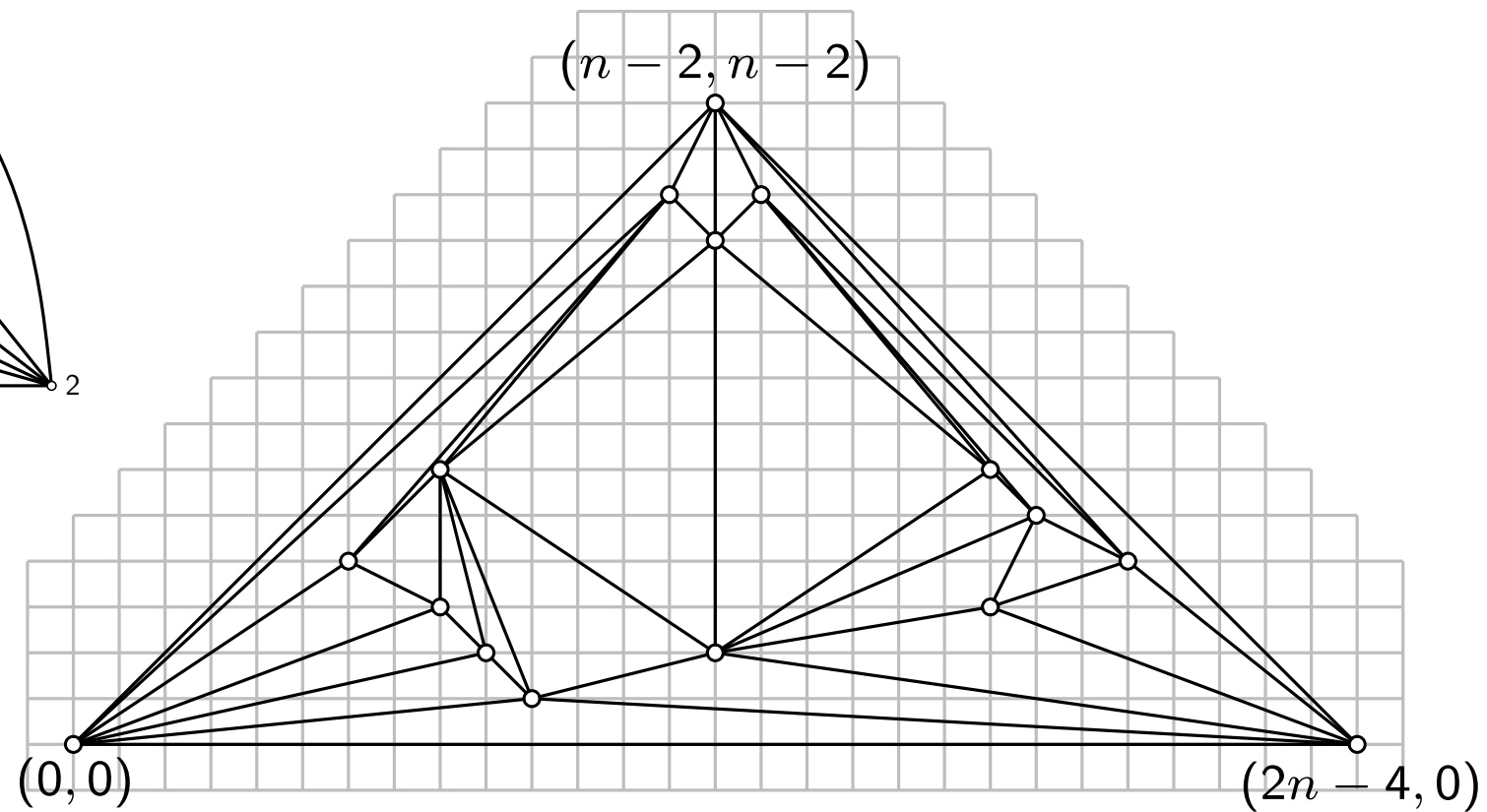
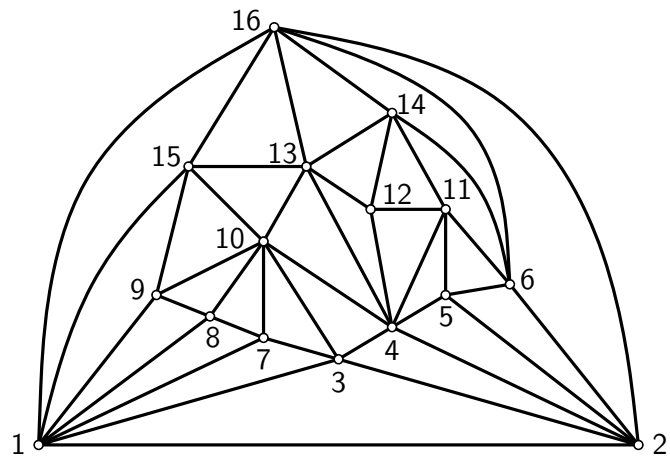
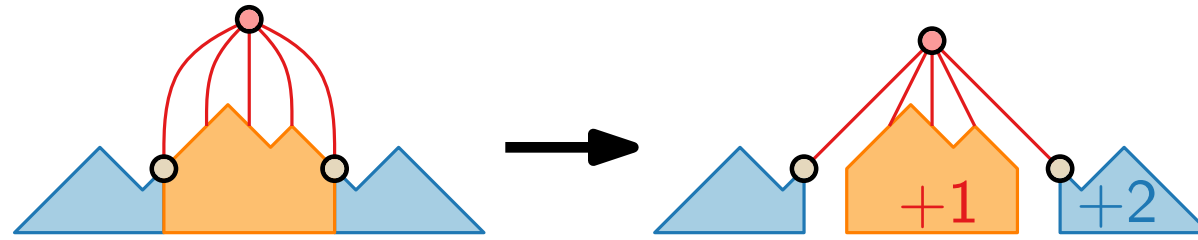
Shift Method – Example



Shift Method – Example



Shift Method – Example



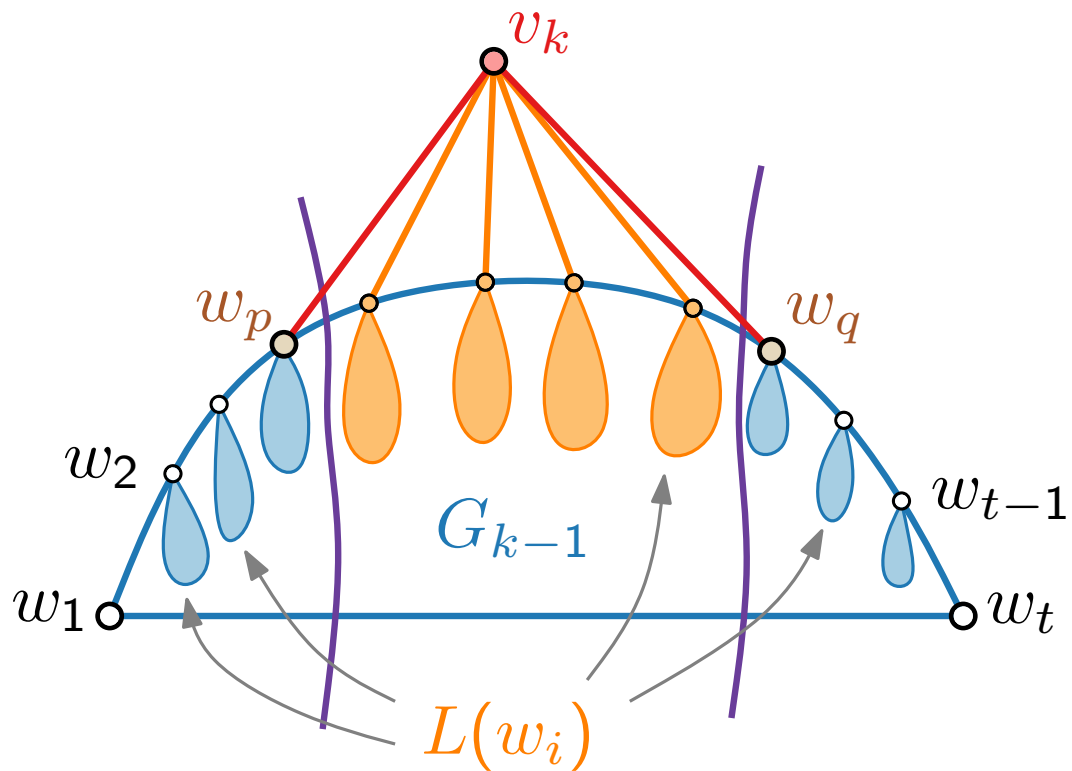
Shift Method – Planarity

Observations.

- Each internal vertex is **covered** exactly once.
- Covering relation defines a tree in G
- and a forest in G_i , $1 \leq i \leq n - 1$.

Lemma.

Let $0 \leq \delta_1 \leq \delta_2 \leq \dots \leq \delta_t \in \mathbb{N}$,
 s.t. $\delta_{p+1} - \delta_p \geq 1$, $\delta_q - \delta_{q-1} \geq 1$,
 $\delta_q - \delta_p \geq 2$ and even. If we shift
 $L(w_i)$ by δ_i to the right, then we
 get a planar straight-line drawing.

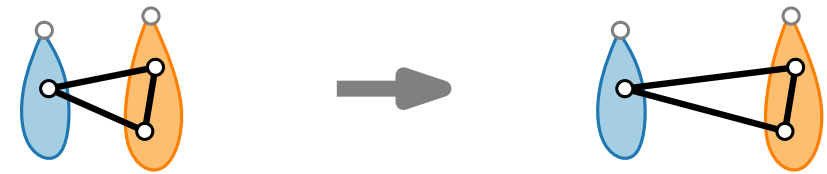


Proof by induction:

If G_{k-1} is drawn planar and straight-line, then so is G_k .

Ideas:

- New edges don't intersect other edges (\rightarrow invariants).
- Edges within each $L(w_i)$ do not change.
- Other edges lie within triangles that only become flatter without causing new intersections.



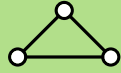
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod($G, (v_1, v_2, \dots, v_n)$)

for $k = 1$ to 3 **do**

$L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0, 0); P(v_2) \leftarrow (2, 0), P(v_3) \leftarrow (1, 1)$ 

for $k = 4$ to n **do**

 Let ∂G_{k-1} be $v_1 = w_1, w_2, \dots, w_{t-1}, w_t = v_2$.

 Let w_p, \dots, w_q be the neighbors of v_k .

foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do** // $\mathcal{O}(n^2)$ in total

$x(v) \leftarrow x(v) + 1$

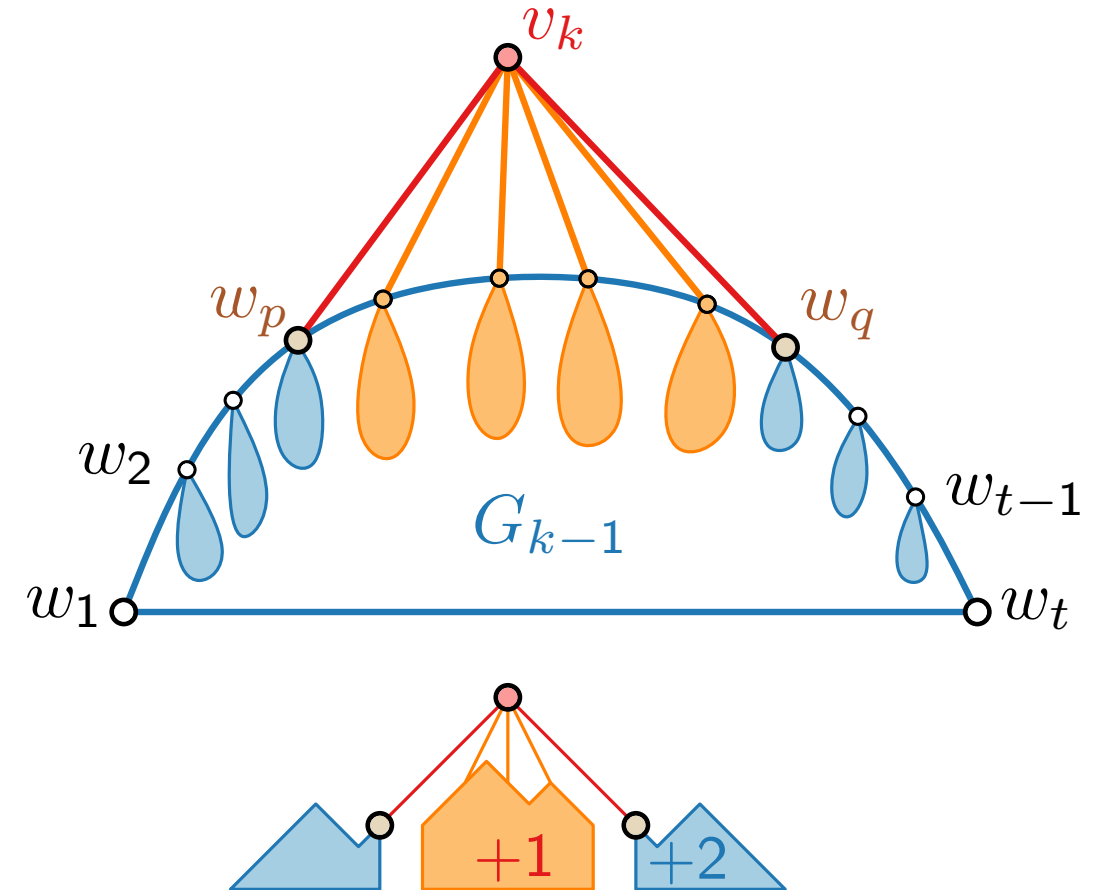
foreach $v \in \bigcup_{i=q}^t L(w_i)$ **do** // $\mathcal{O}(n^2)$ in total

$x(v) \leftarrow x(v) + 2$

$P(v_k) \leftarrow$ intersection of slope- ± 1 diagonals
 through $P(w_p)$ and $P(w_q)$

$L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$

return $P(v_1), \dots, P(v_n)$



Running Time?

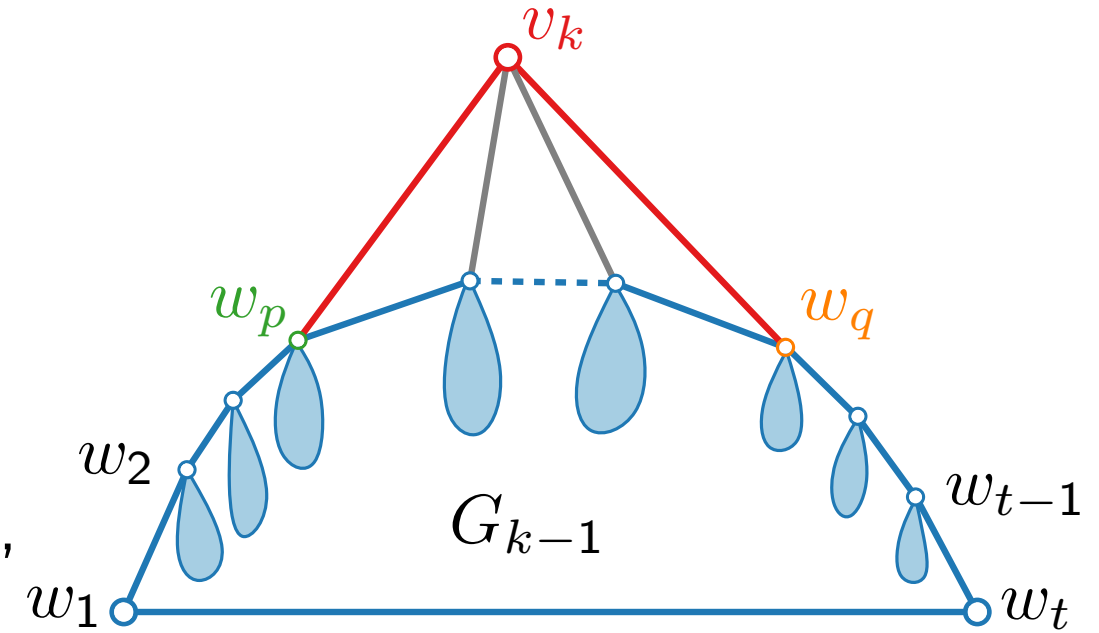
Shift Method – Linear-Time Implementation

Idea 1.

To compute $x(v_k)$ and $y(v_k)$, we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$

Idea 2.

Instead of storing explicit (absolute) x-coordinates, we store, for each vertex within a specific spanning tree, the x-distance to its parent (v_1 is the root).



After all x-distances have been computed, use an additional preorder traversal to compute all x-coordinates.

$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2) \quad y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

$$(3) \quad x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})++$, $\Delta_x(w_q)++$
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \dots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3) ■ $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$



takes $\mathcal{O}(n)$ time in total 😊

$$(1) \quad x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2) \quad y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

$$(3) \quad \underbrace{x(v_k) - x(w_p)}_{\Delta_x(v_k)} = \frac{1}{2}(\underbrace{x(w_q) - x(w_p)}_{\Delta_x(w_p, w_q)} + y(w_q) - y(w_p))$$

Discussion

- The shift method by de Fraysseix, Pach, and Pollack provides an algorithmic tool to efficiently draw a plane graph onto a polynomial-size grid using only straight-line edges.
- The linear-time implementation was later proposed by Chrobak and Payne.
- Although we are guaranteed to get a very small grid, only straight-line edges, and no edge crossings, the resulting drawings are not always visually pleasing: the drawings tend to have very small angles and a big variance in the size of the triangular faces.
- A quite different approach yielding similar results is by Schnyder (\rightarrow next lecture).

Literature

- [PGD Ch. 4.2] for detailed explanation of the shift method
- [de Fraysseix, Pach, Pollack 1990] “How to draw a planar graph on a grid”
 - original paper introducing the shift method
- [Chrobak, Payne 1995] “A linear-time algorithm for drawing a planar graph on a grid”
 - original paper on how to implement the shift method in linear time