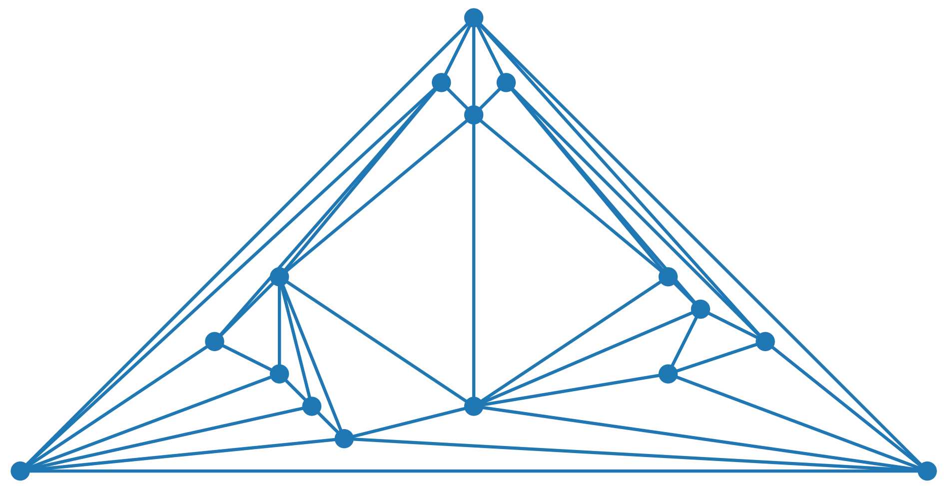
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

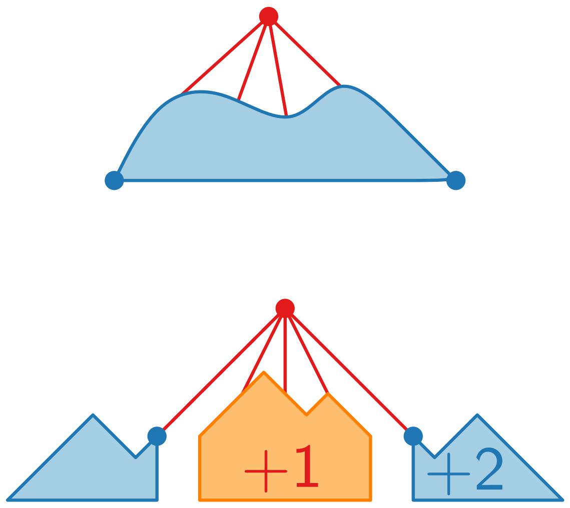
Visualization of Graphs

Lecture 3:
Straight-Line Drawings of Planar Graphs I:
Canonical Orderings and the Shift Method

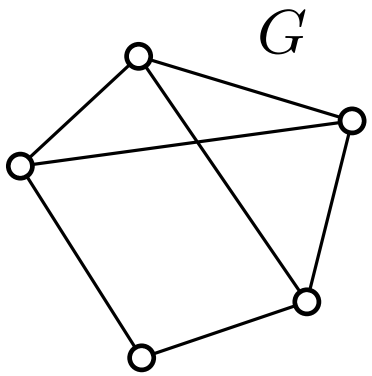


Alexander Wolff

Summer term 2026



Planar Graphs



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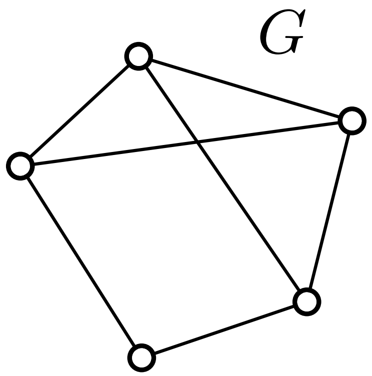

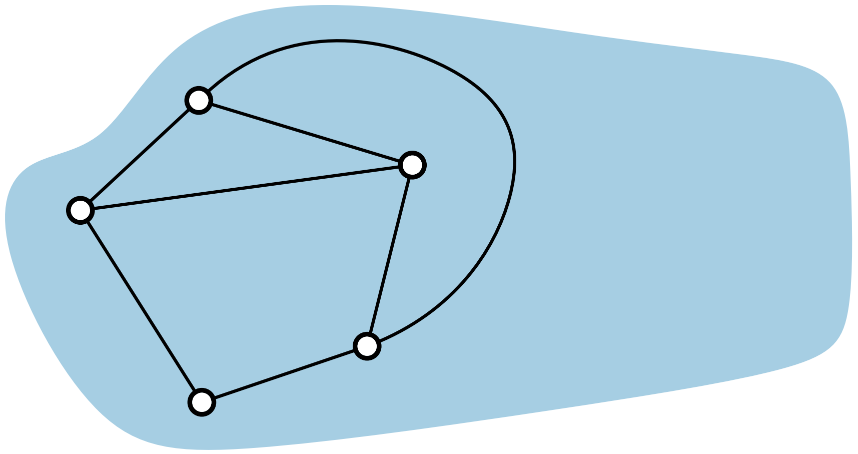

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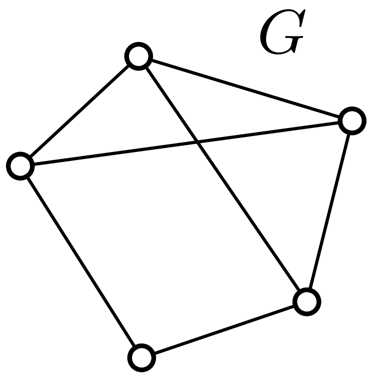

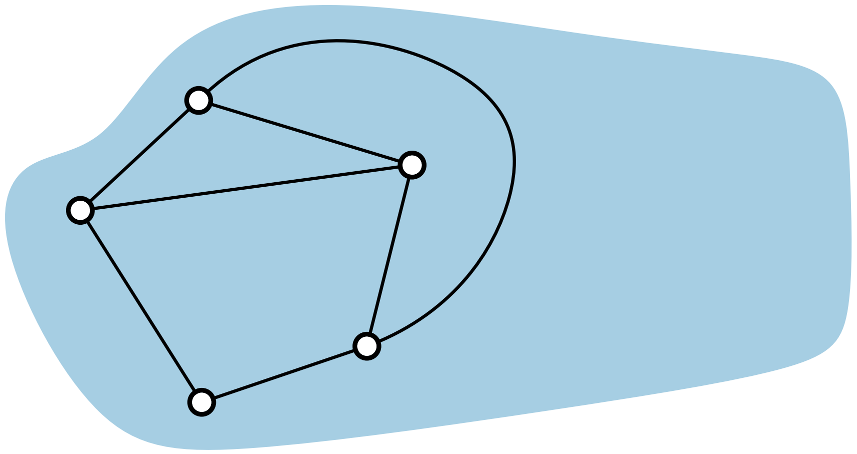

G is **planar**:
it can be drawn in such a way that
no two edges intersect each other.

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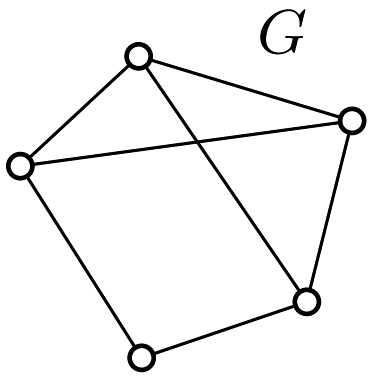

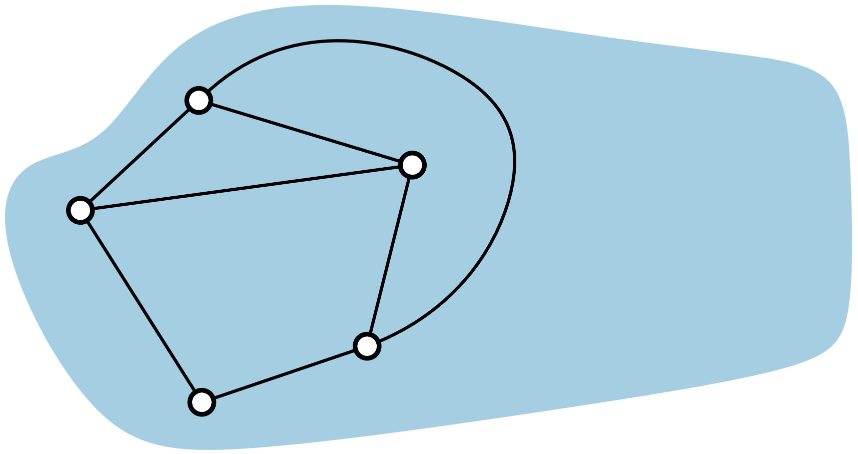

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clockwise orientation of adjacent vertices around each vertex

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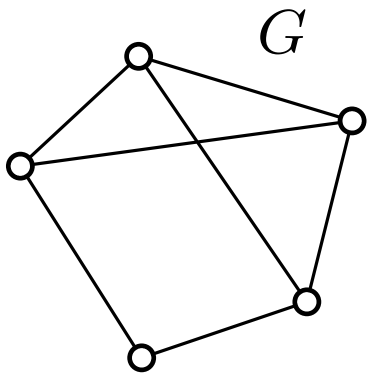

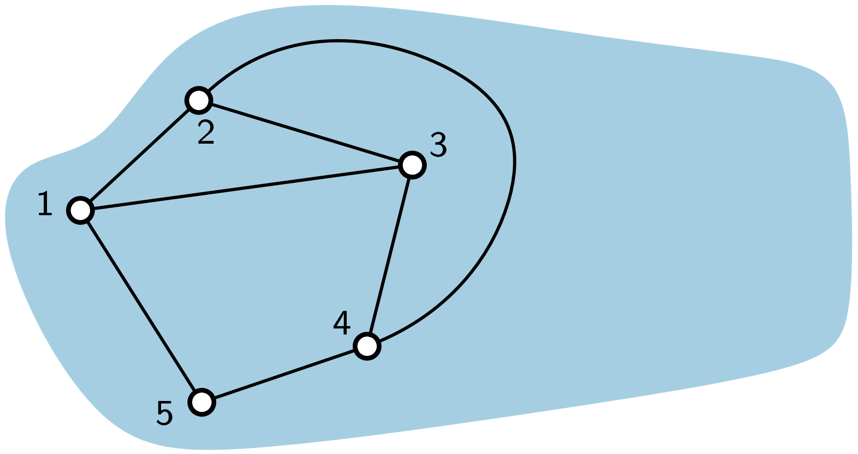

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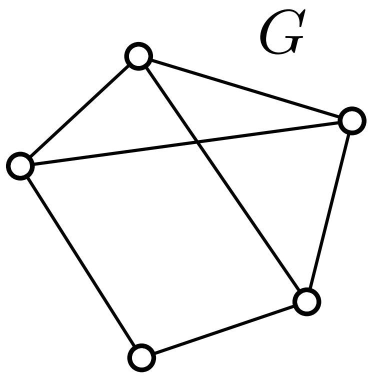

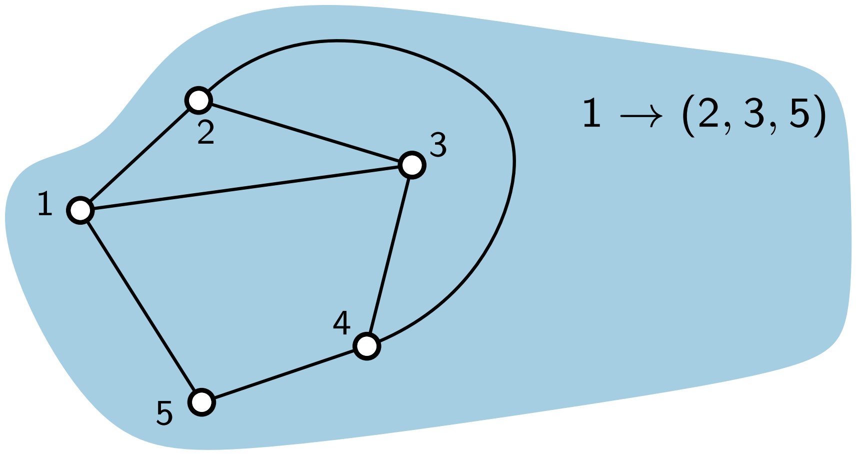

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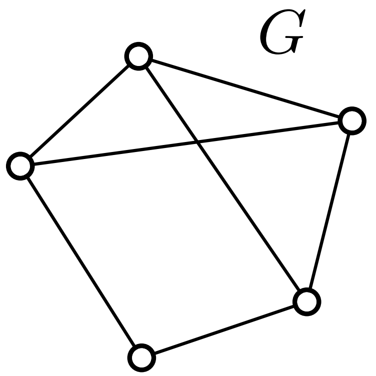

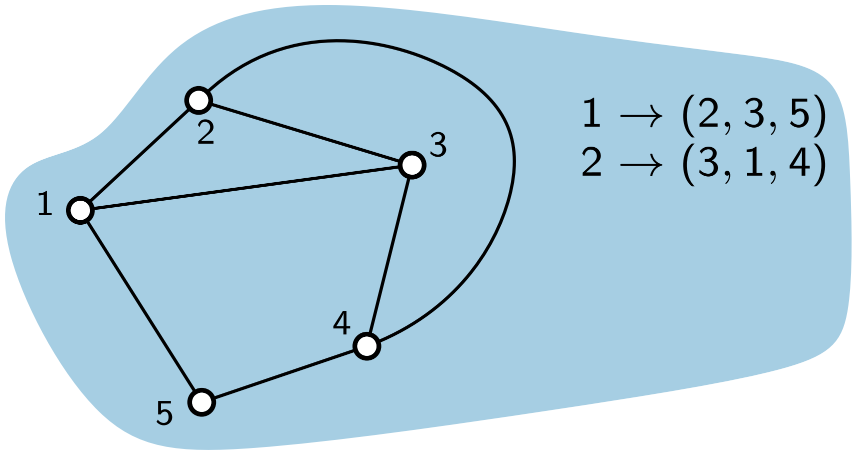

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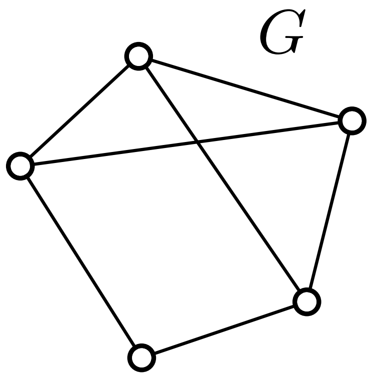

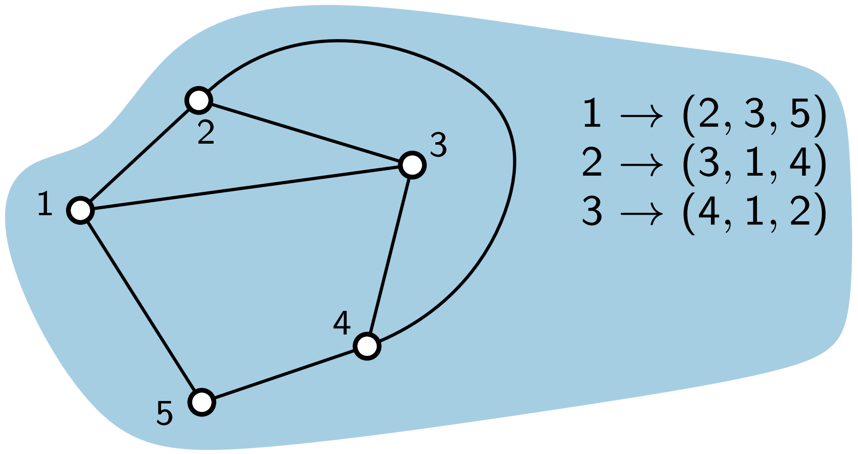

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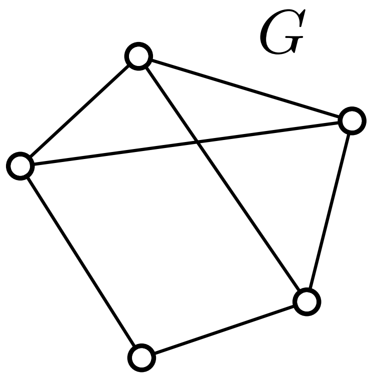

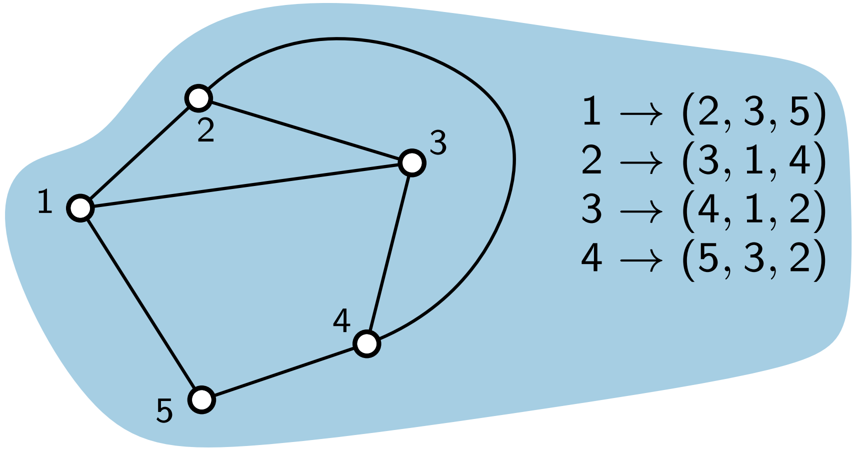

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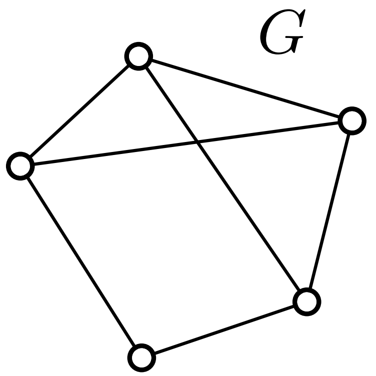

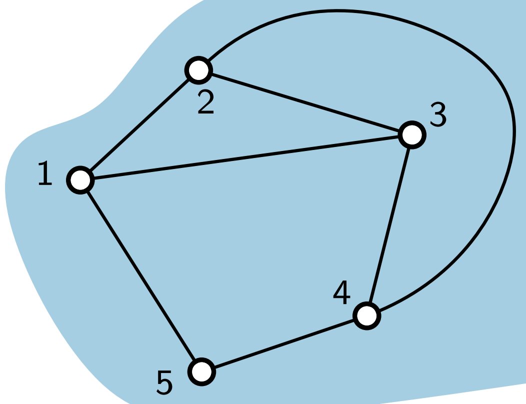

$1 \to (2, 3, 5)$
$2 \to (3, 1, 4)$
$3 \to (4, 1, 2)$
$4 \to (5, 3, 2)$
$5 \to (1, 4)$

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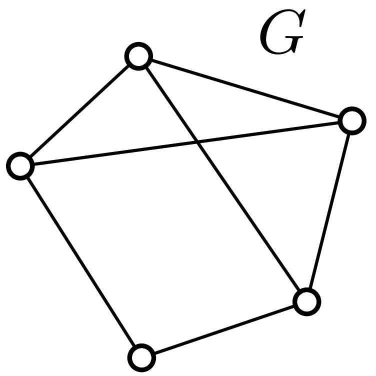

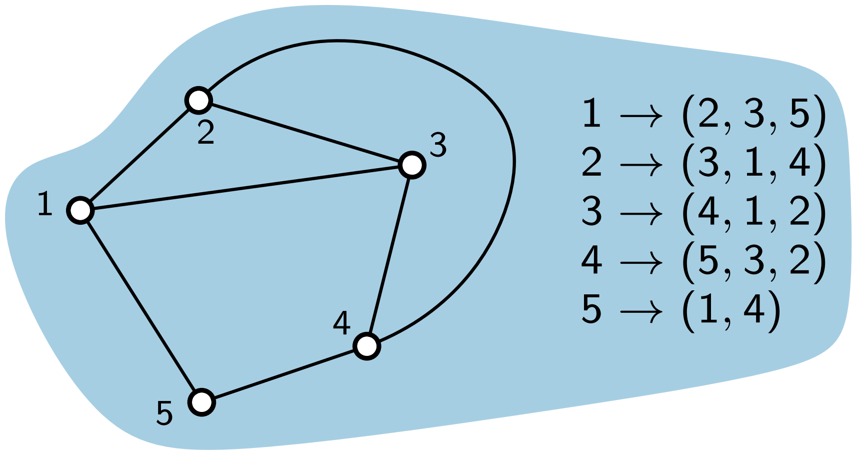

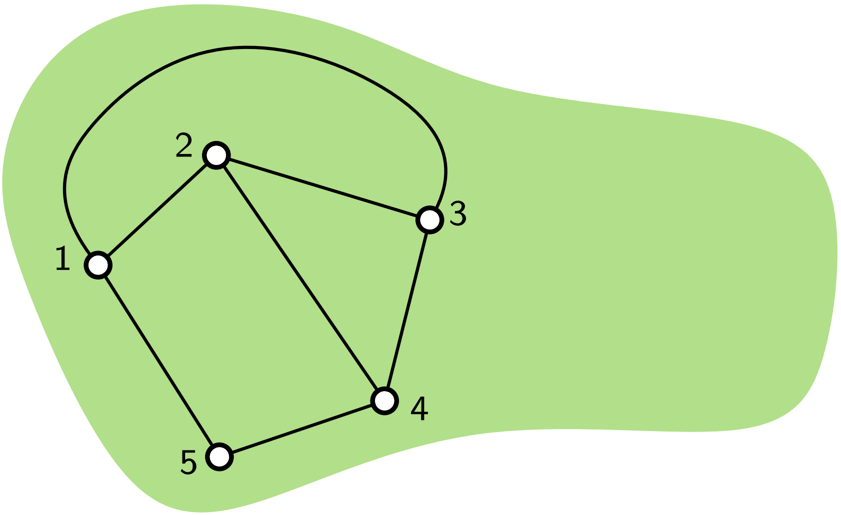

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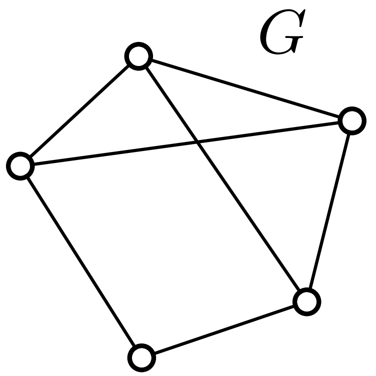

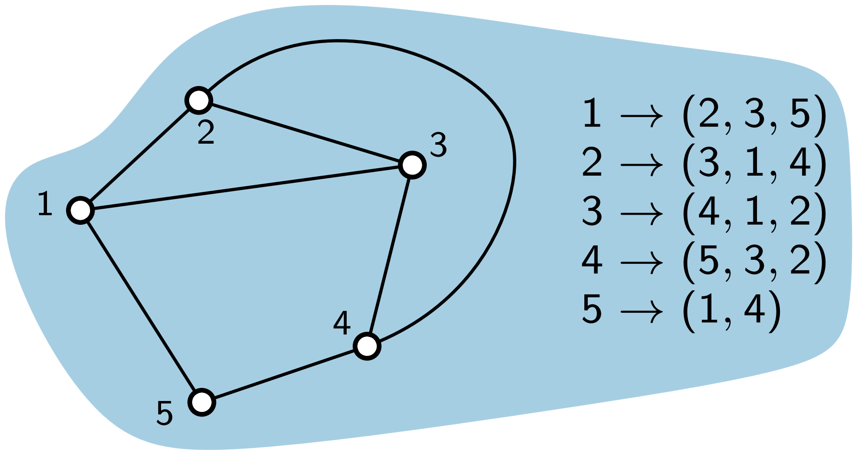

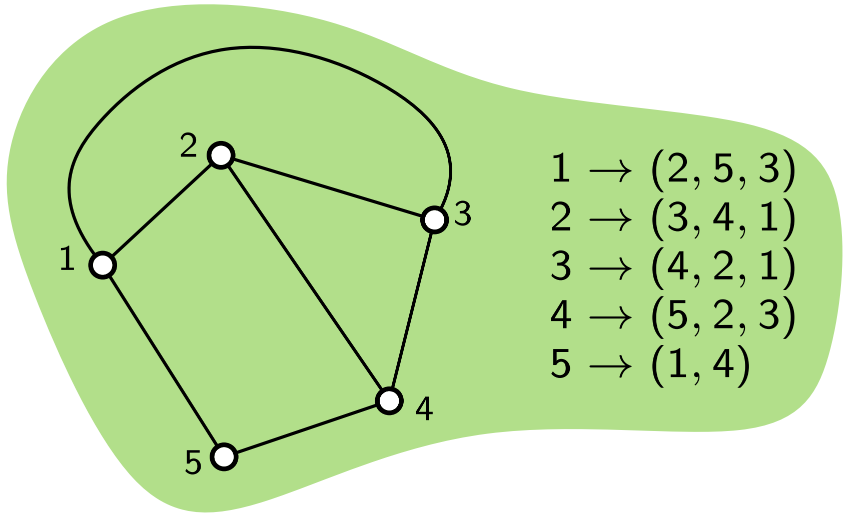

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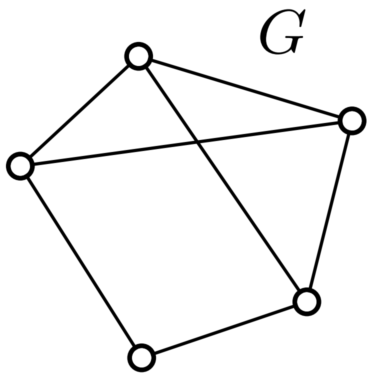

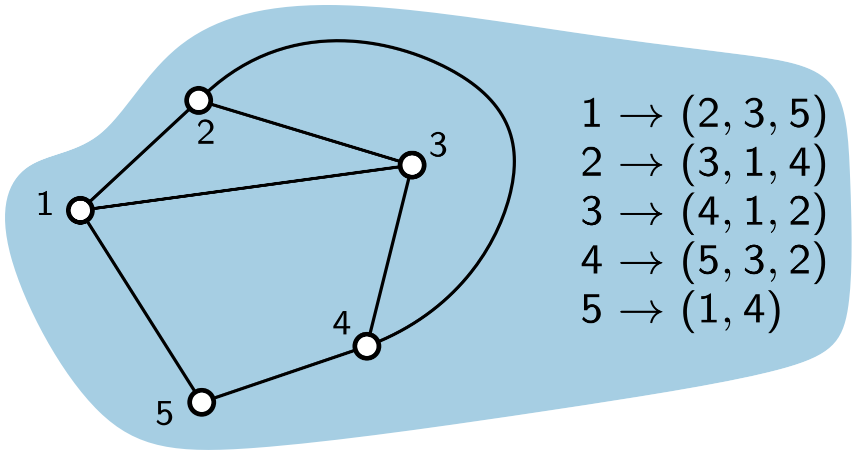

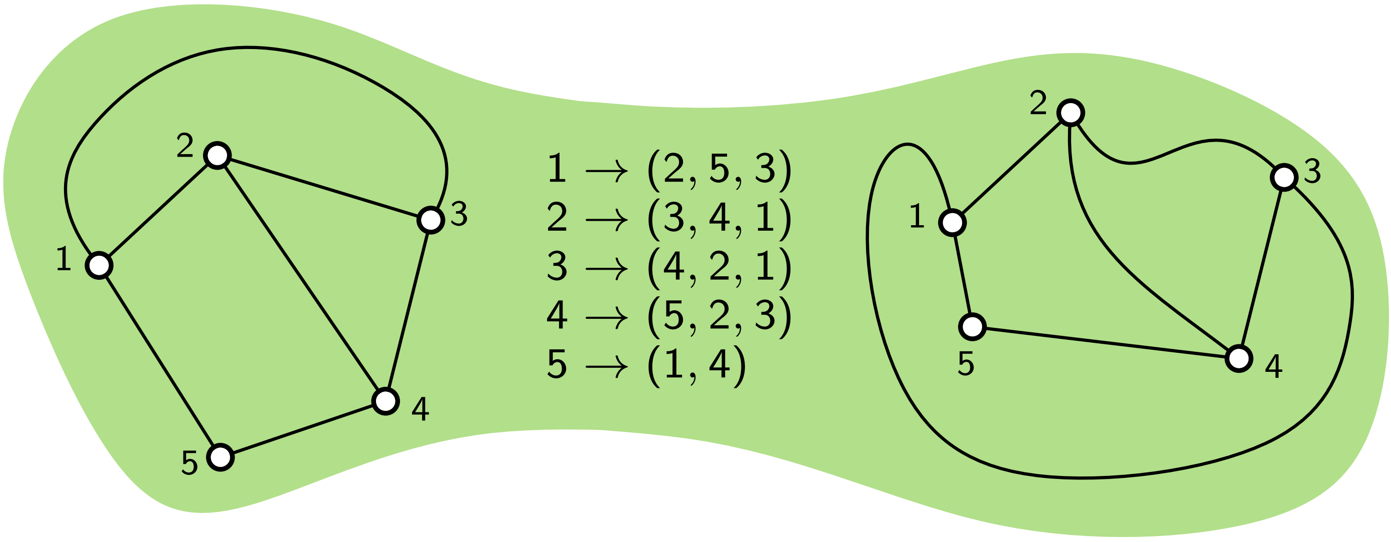

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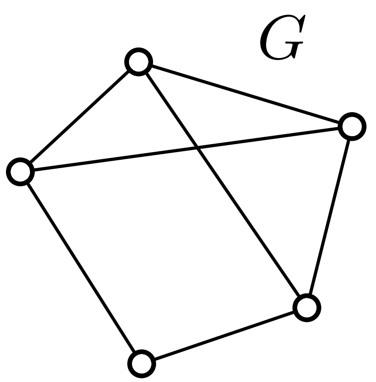

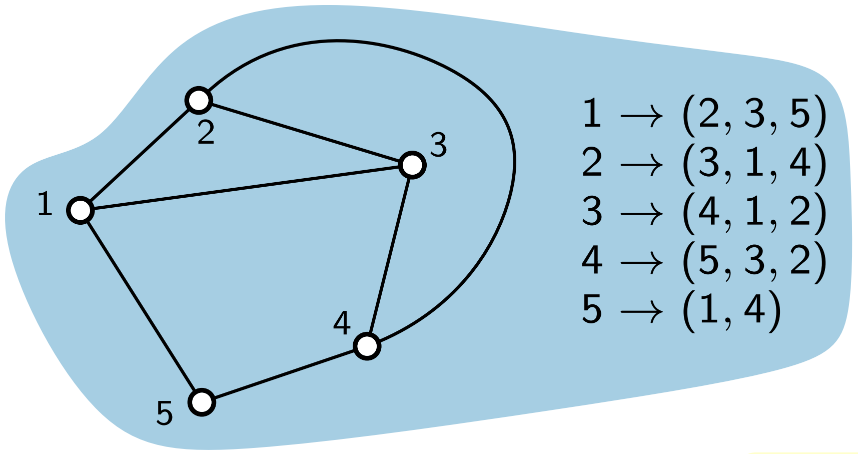

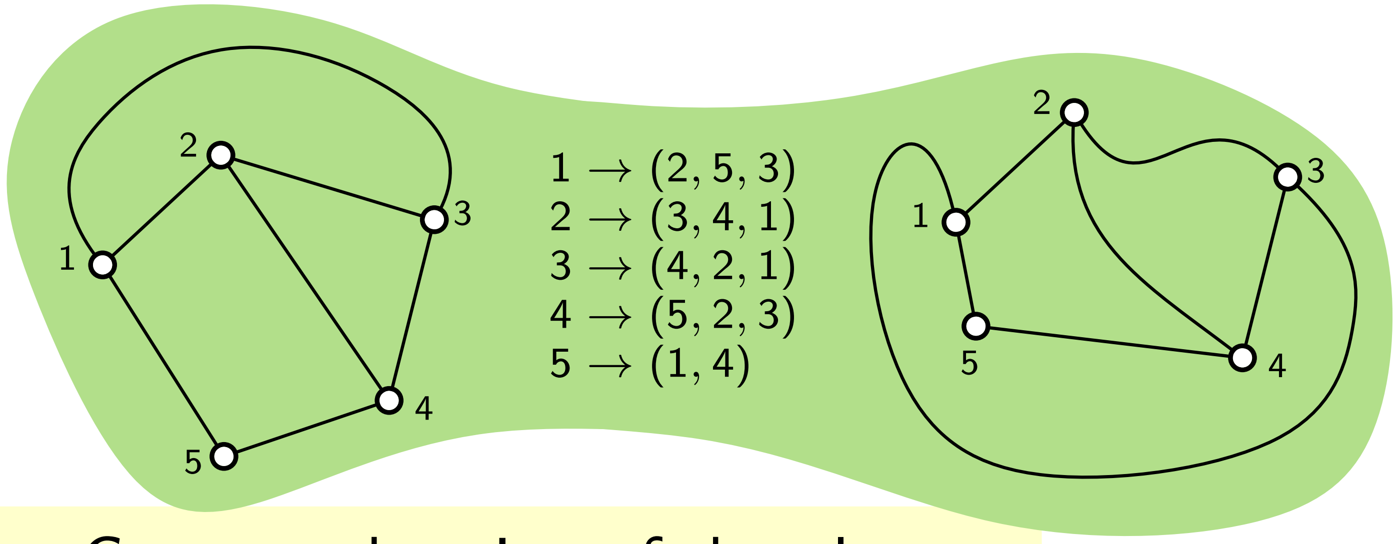

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faces: Connected region of the plane bounded by edges

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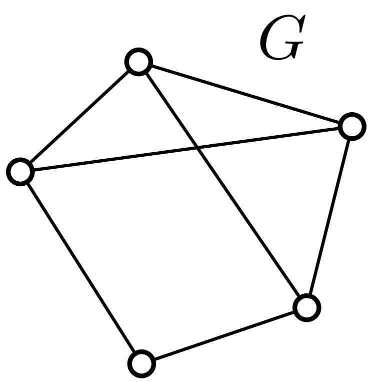

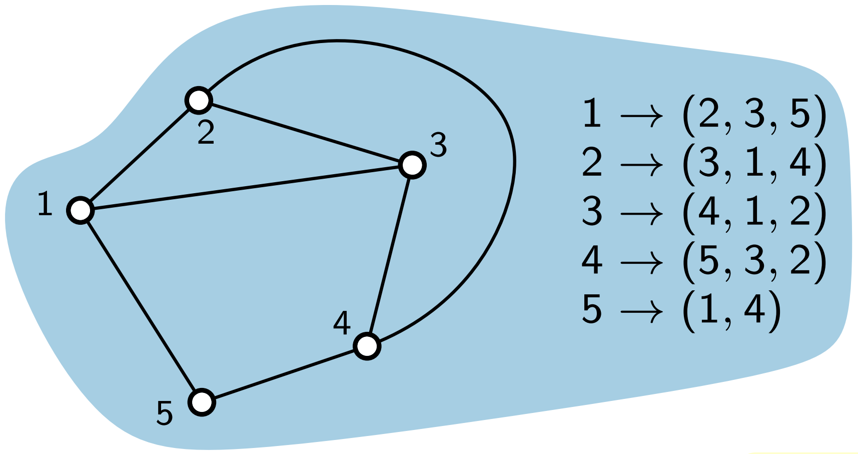

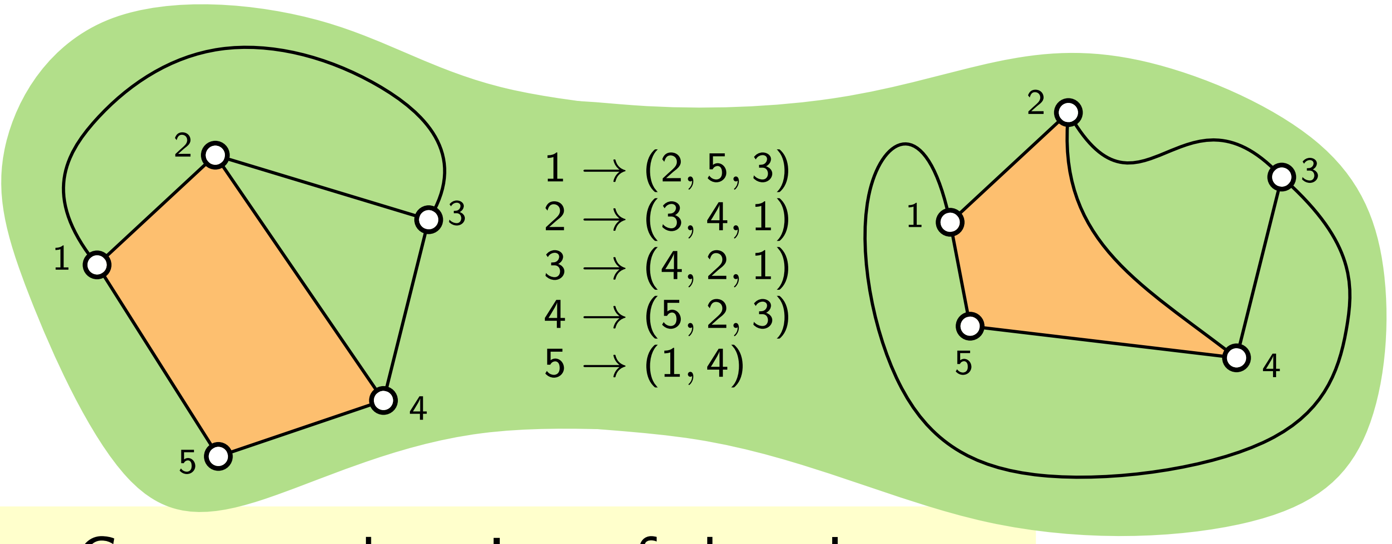

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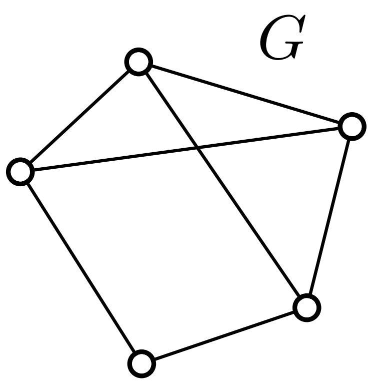

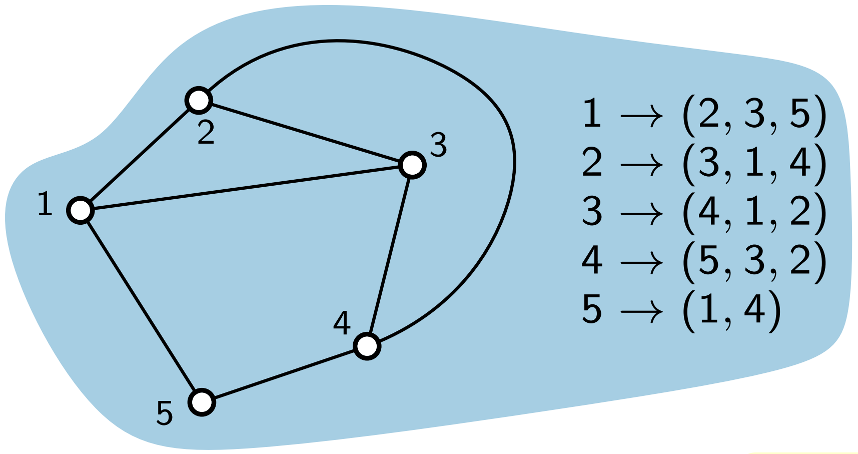

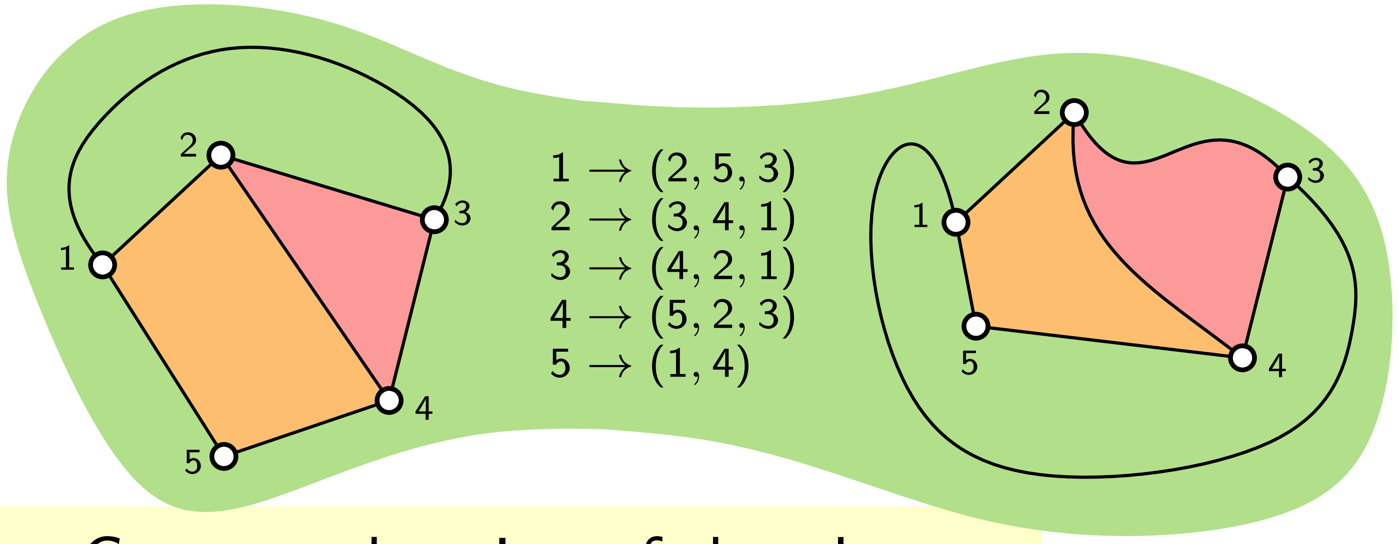

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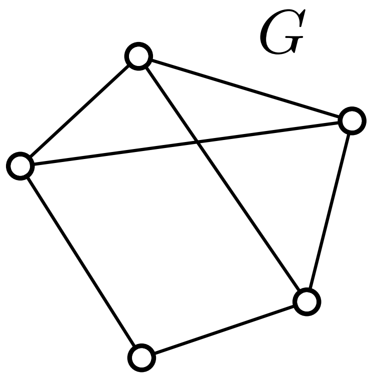

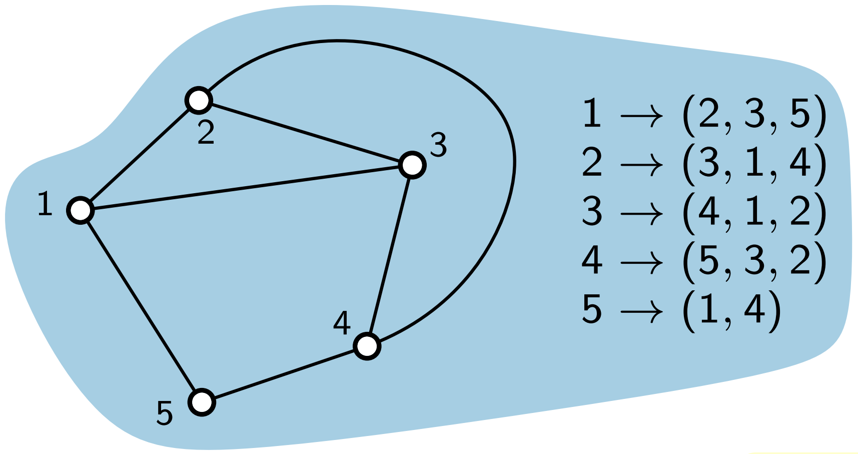

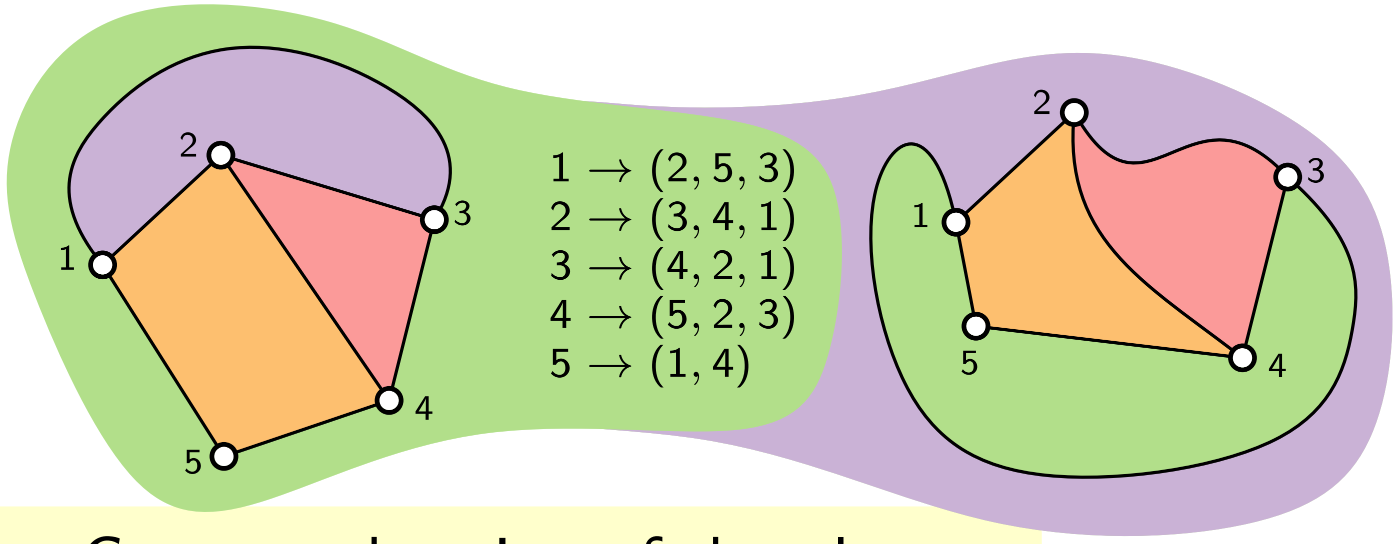

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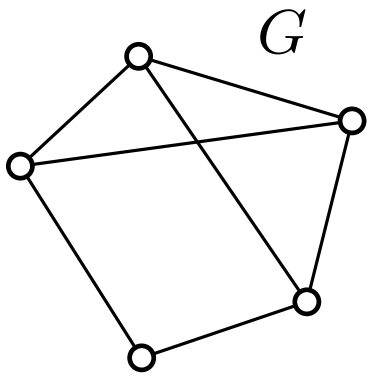

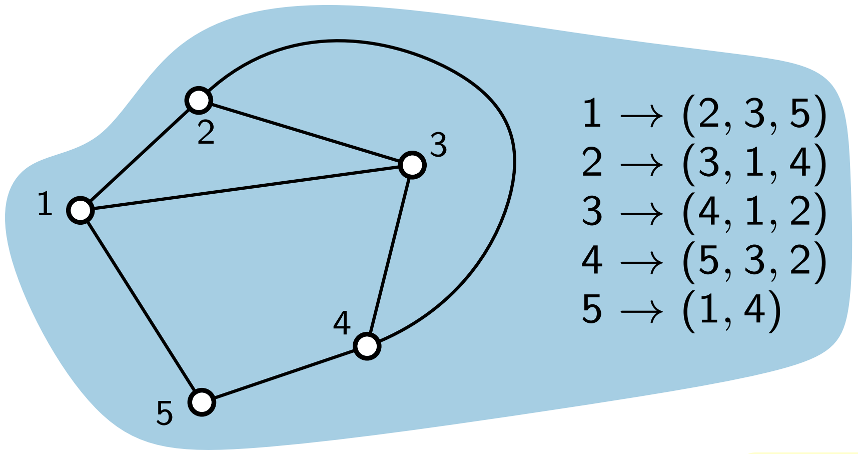

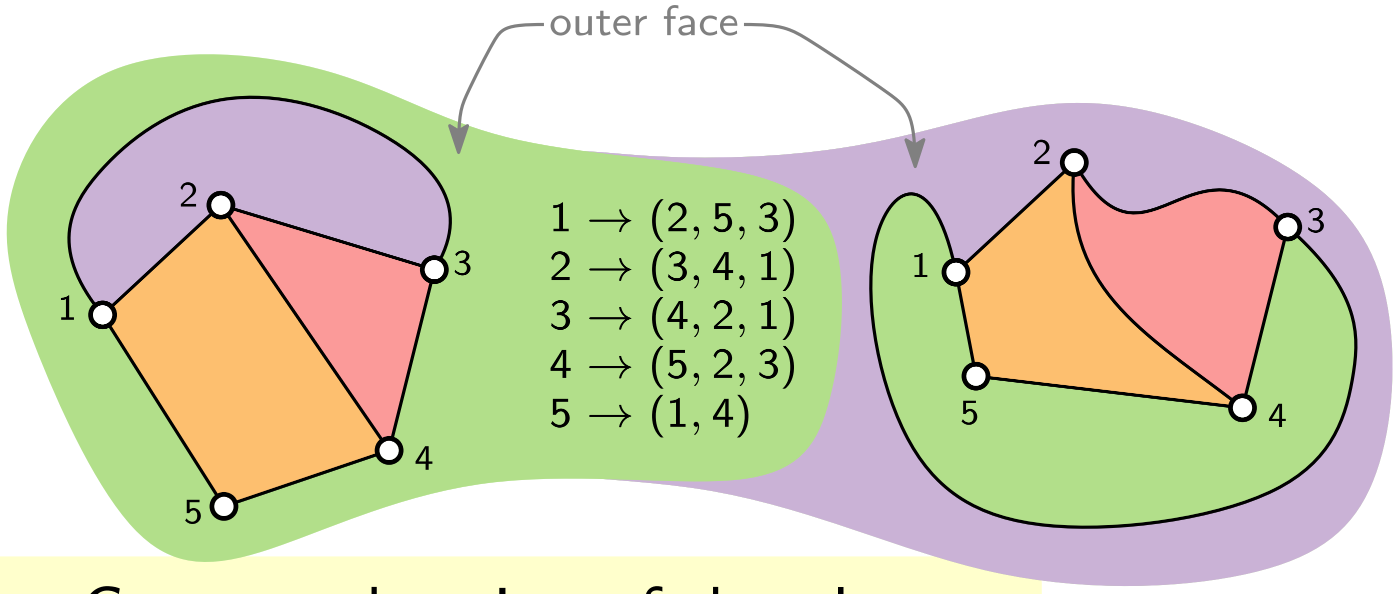

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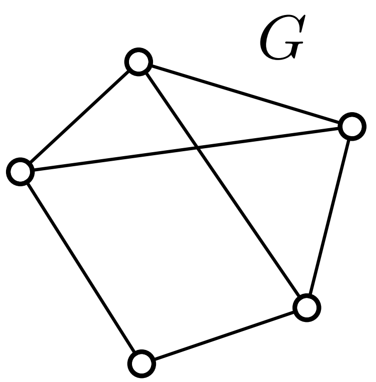

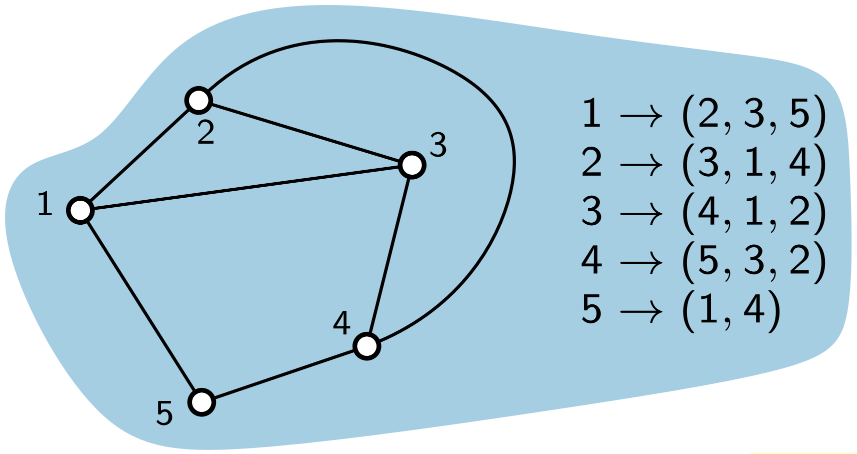

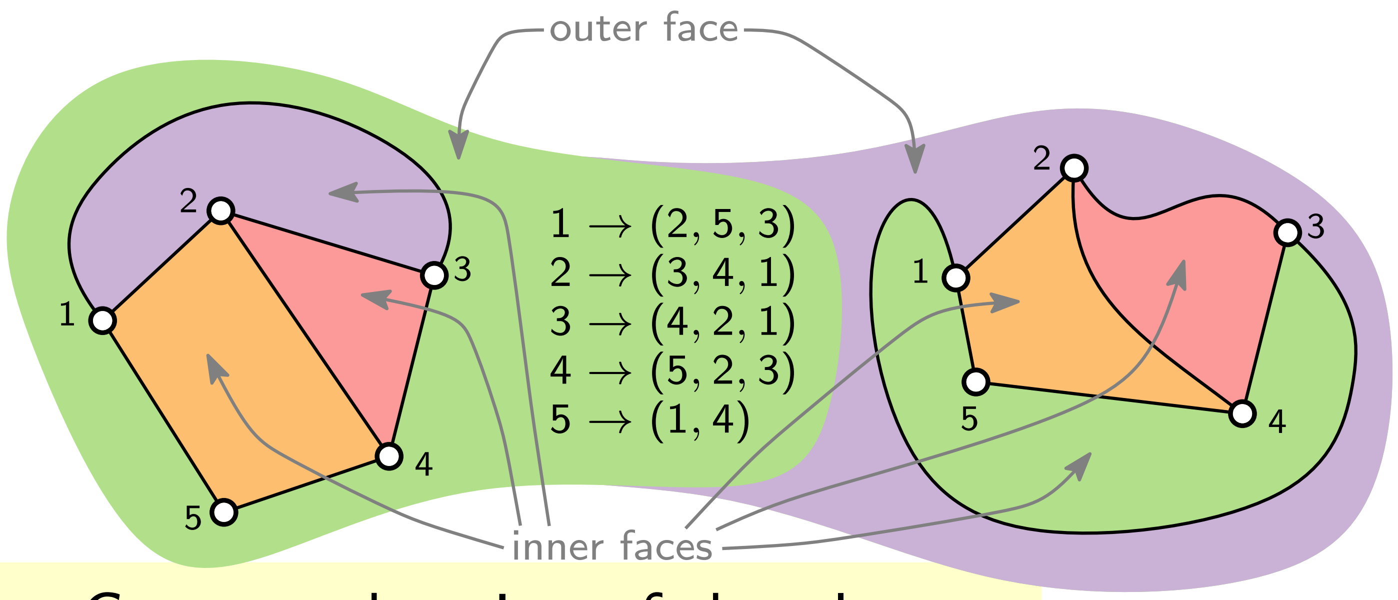

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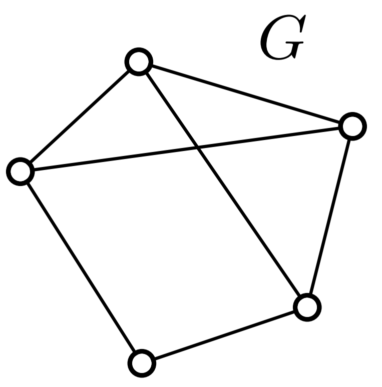

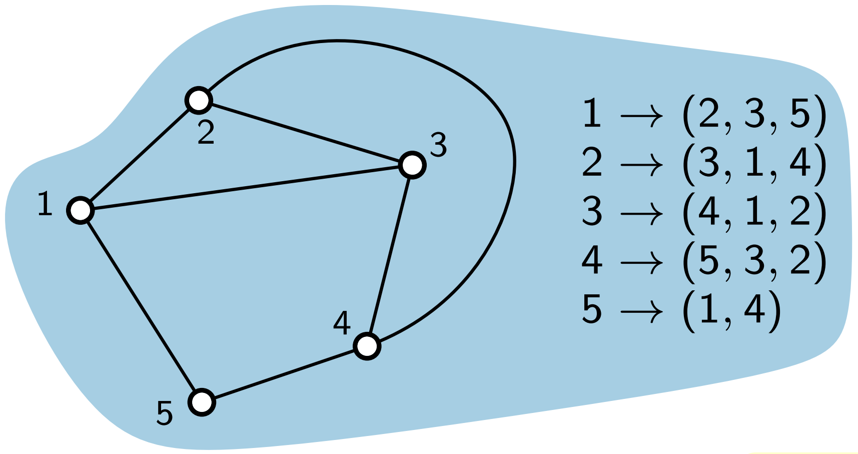

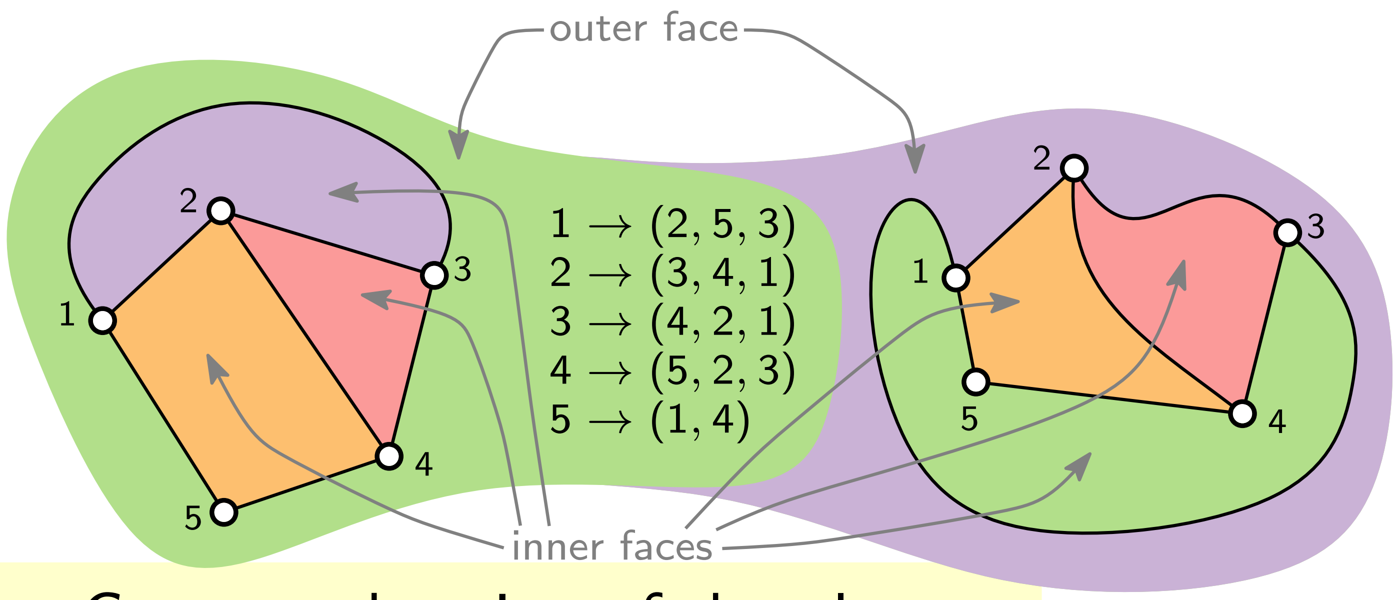

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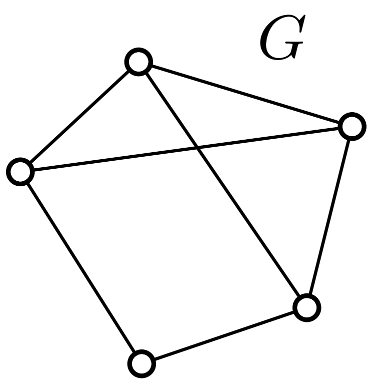
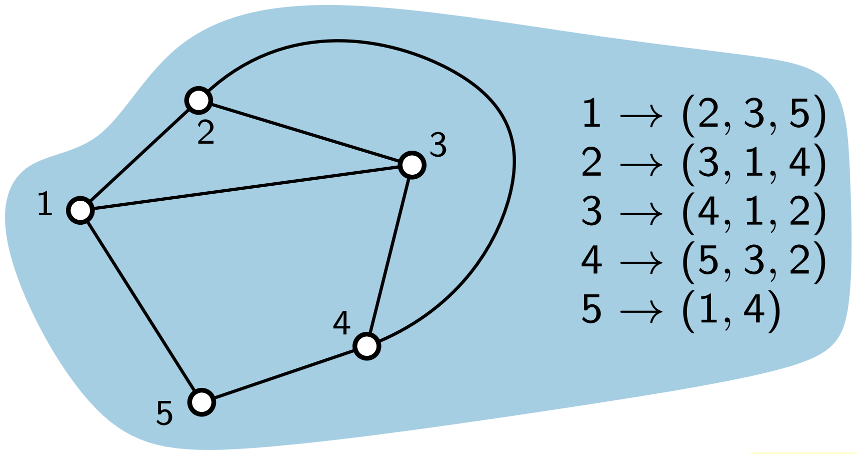
faces: Connected region of the plane bounded by edges

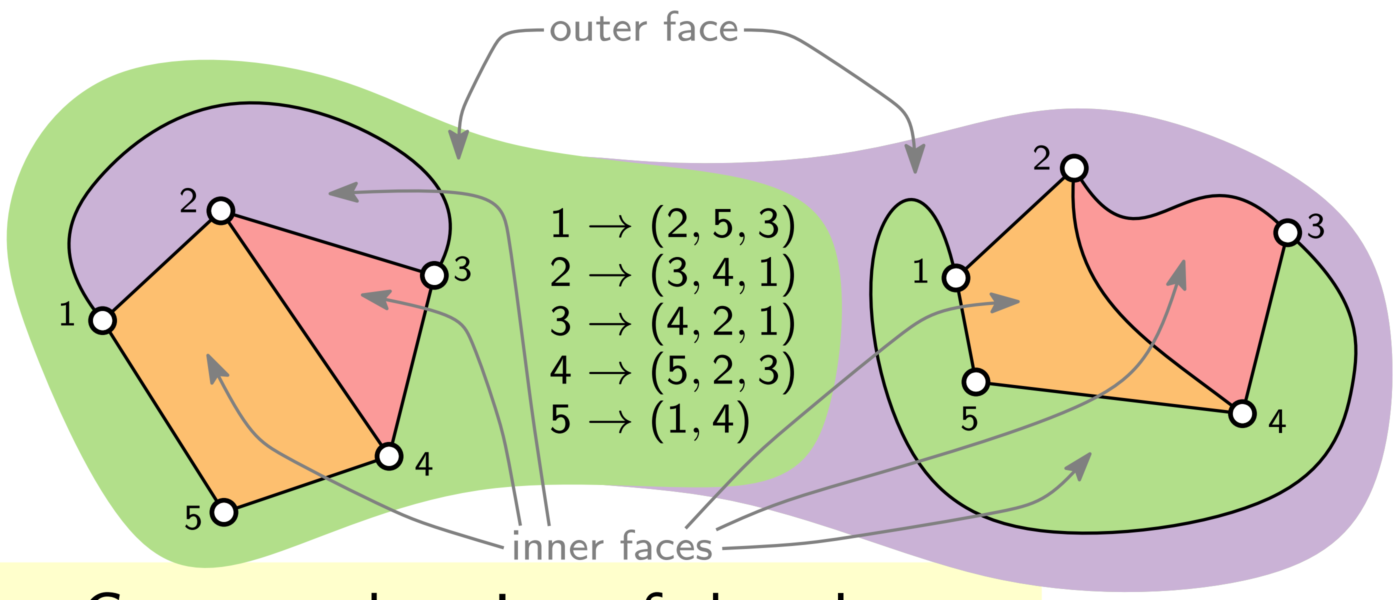
Euler's polyhedra formula.

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Planar Graphs







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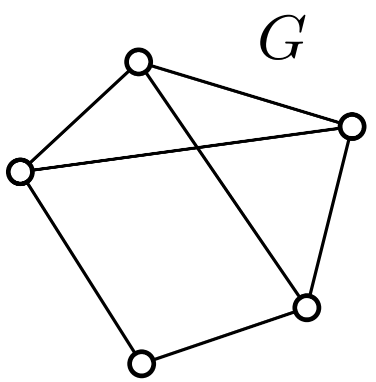
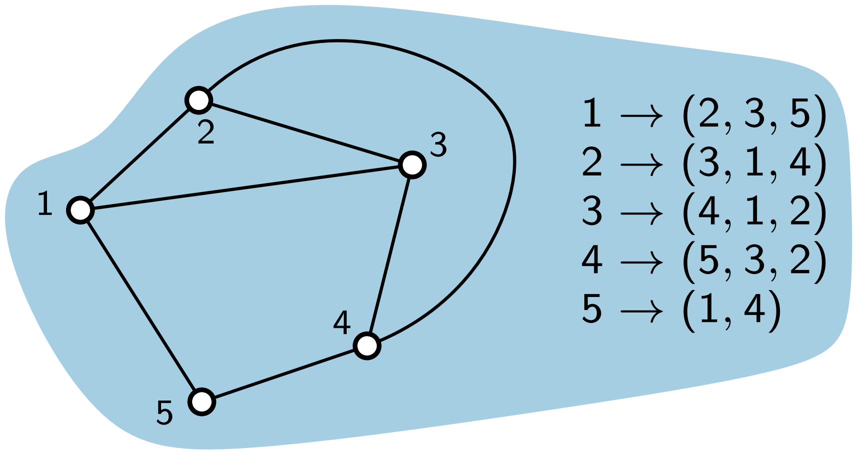
Euler's polyhedra formula.

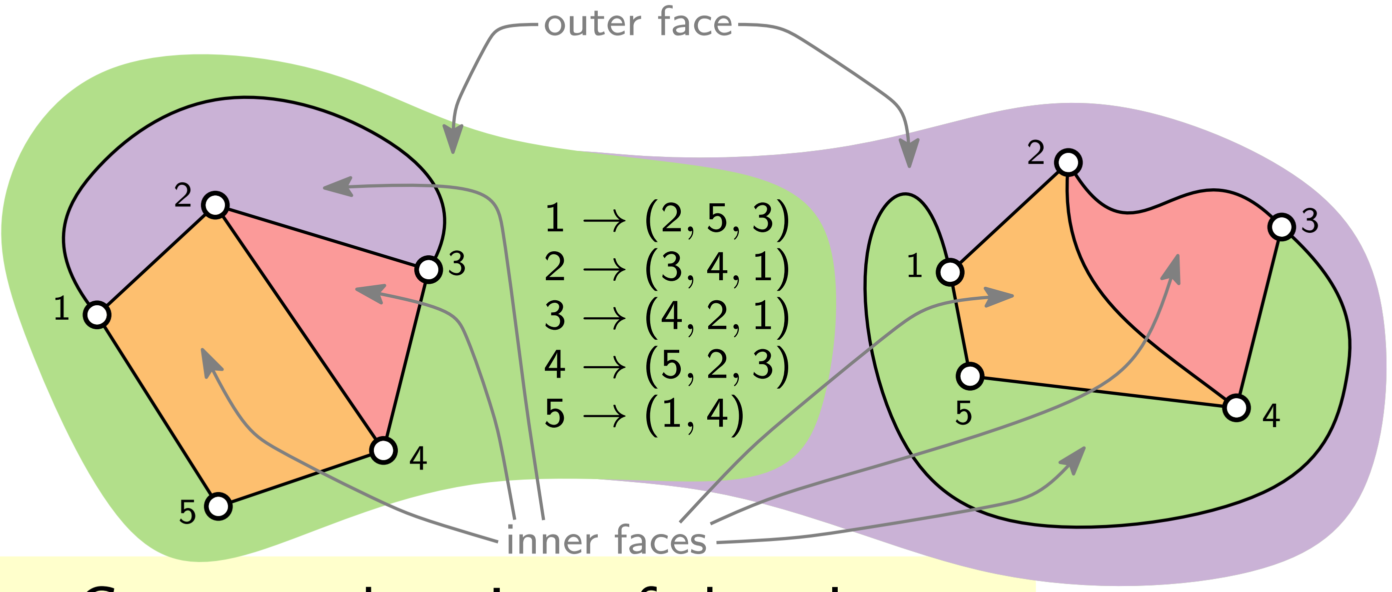
$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof.

Planar Graphs







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Euler's polyhedra formula.

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof. By induction on m:

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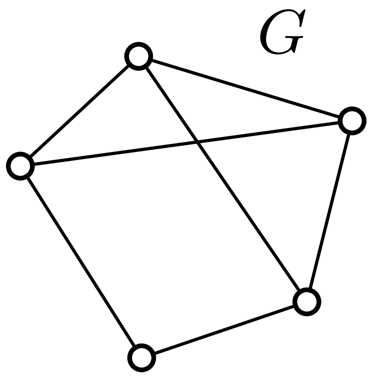

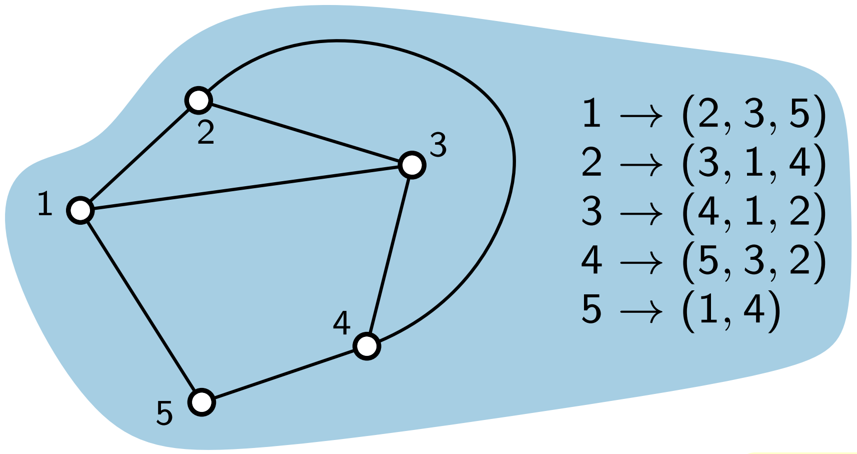

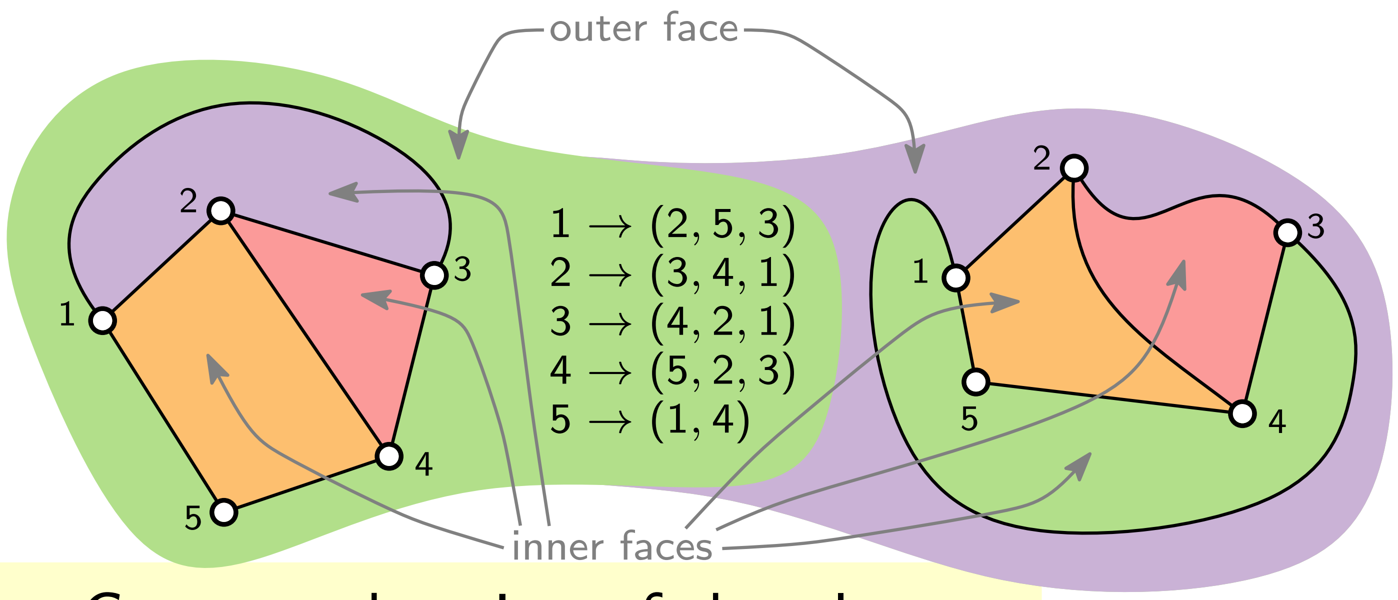

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$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof. By induction on m:

$m = 0 \Rightarrow$

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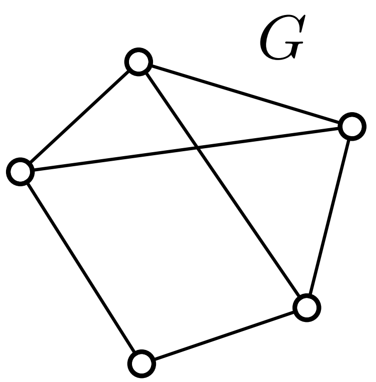

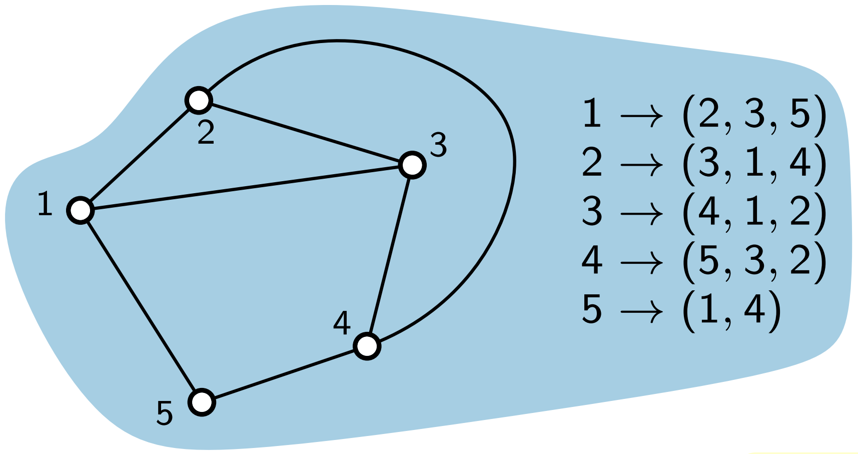

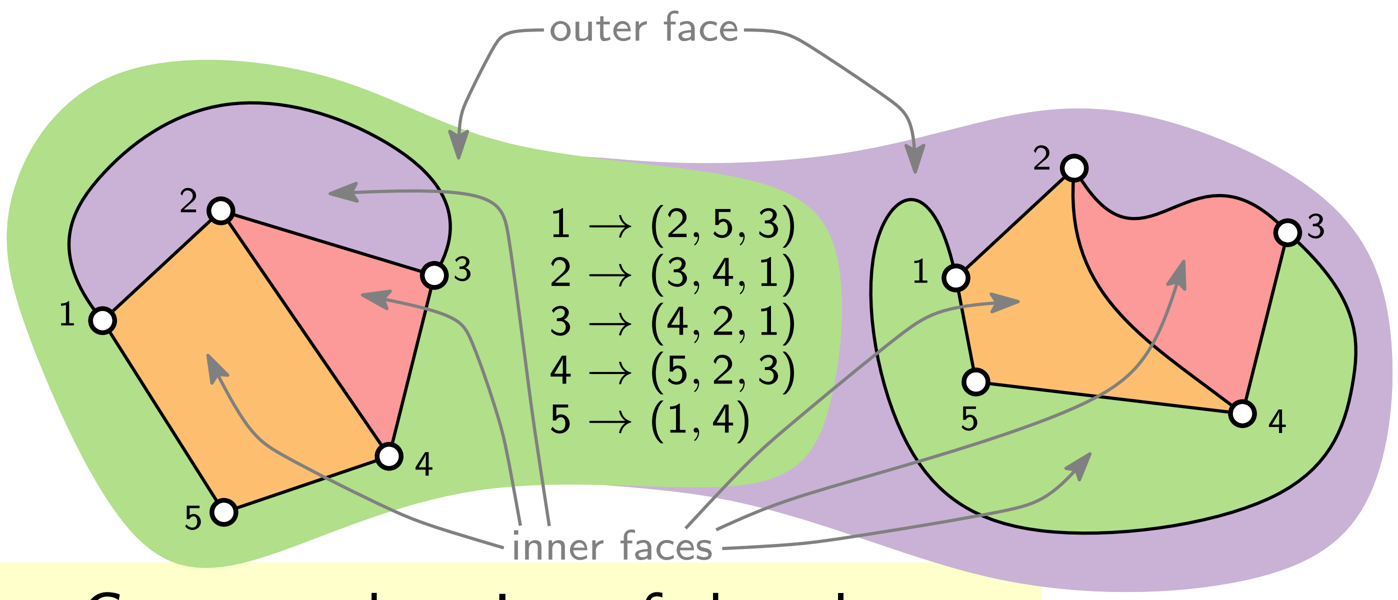

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it can be drawn in such a way that no two edges intersect each other.

planar embedding:
clockwise orientation of adjacent vertices around each vertex

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

faces: Connected region of the plane bounded by edges

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$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof. By induction on m:
$m = 0 \Rightarrow f =$? and $c =$?

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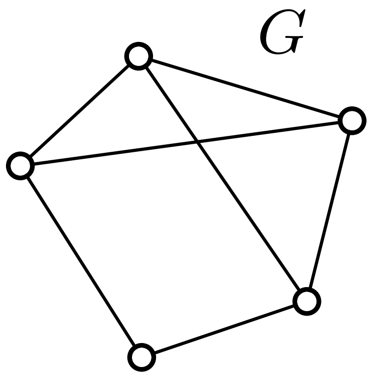

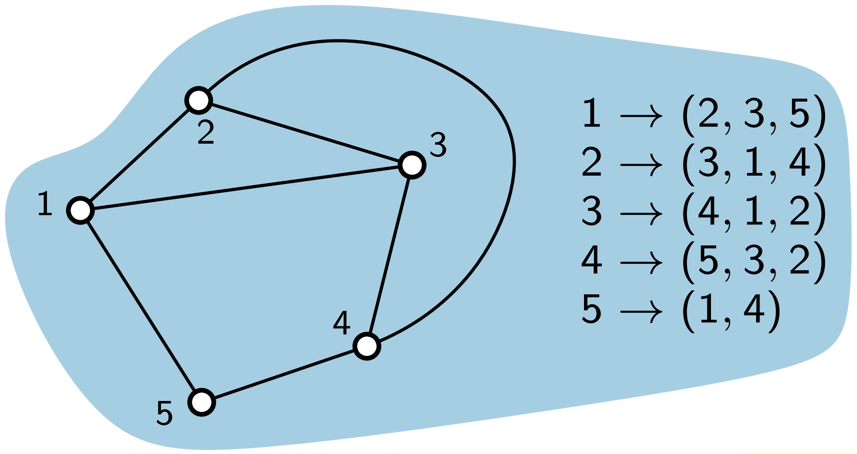

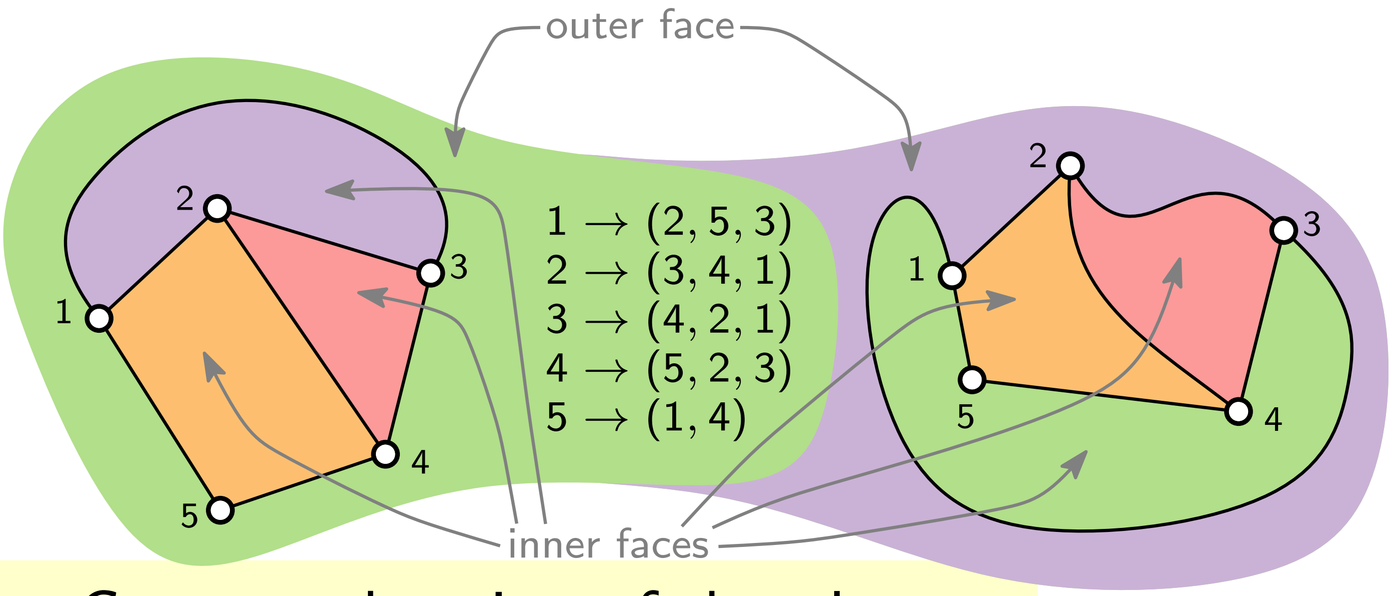

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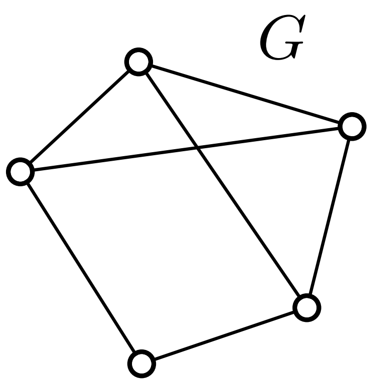

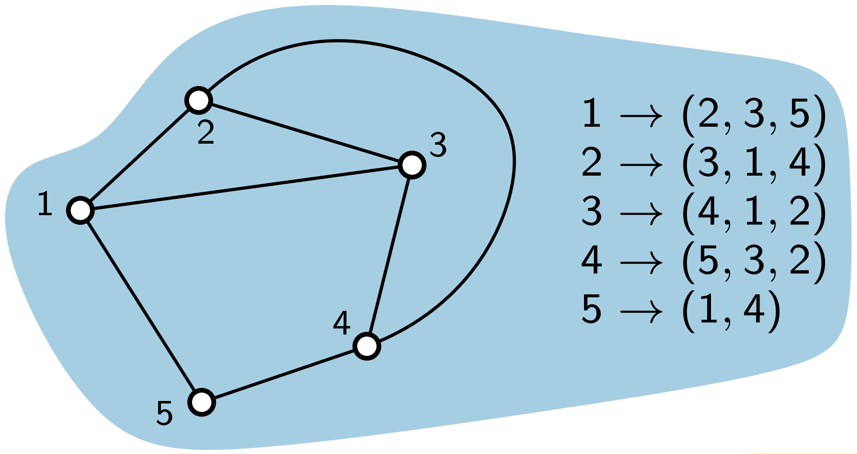

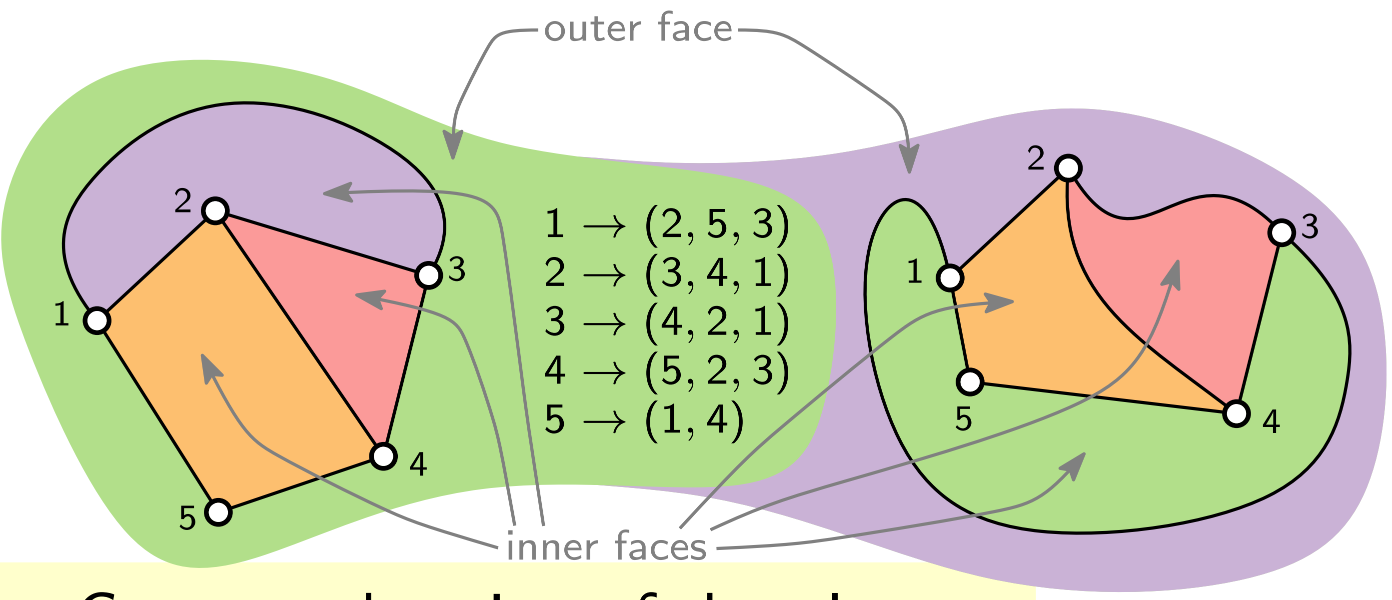

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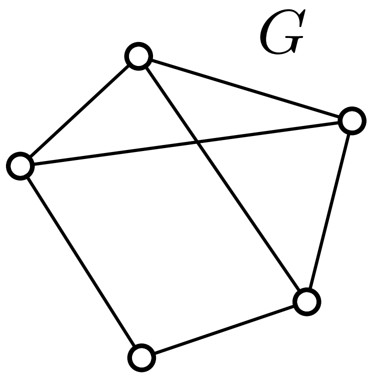

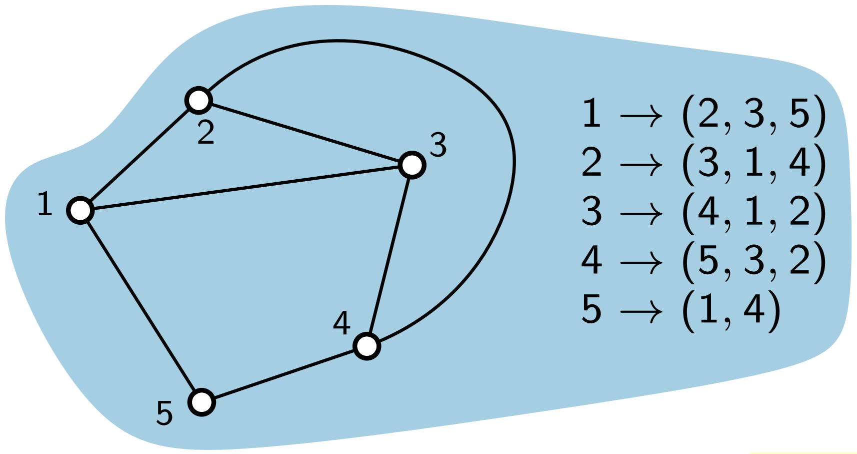

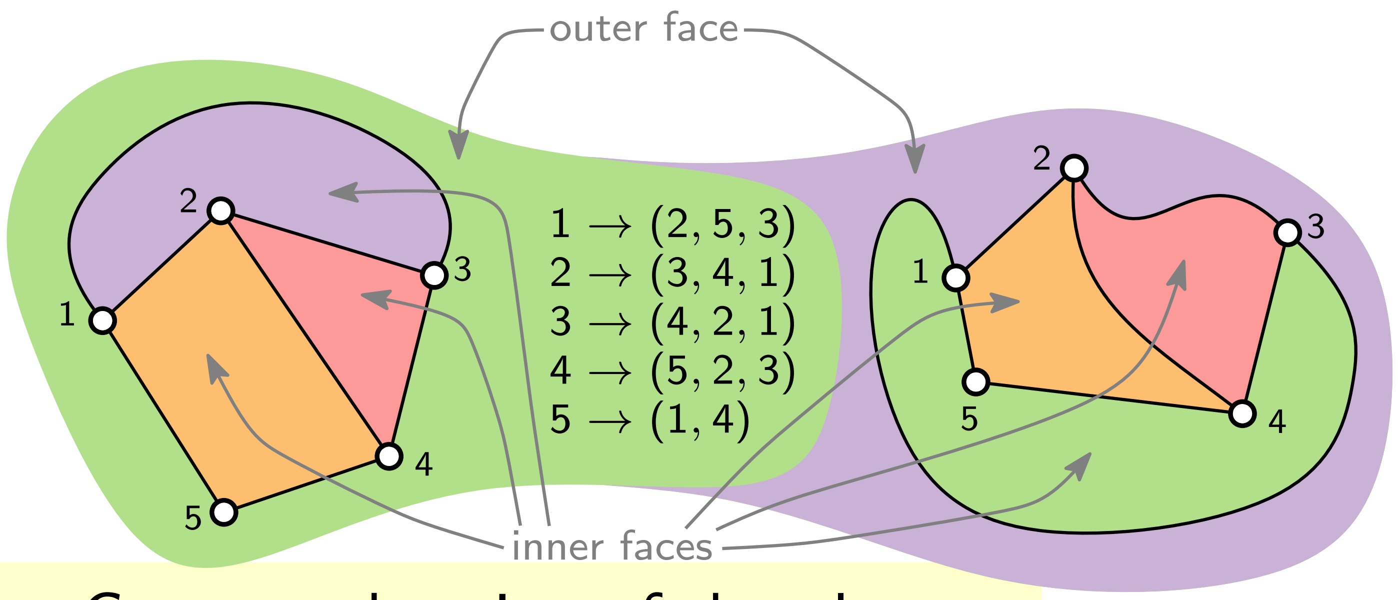

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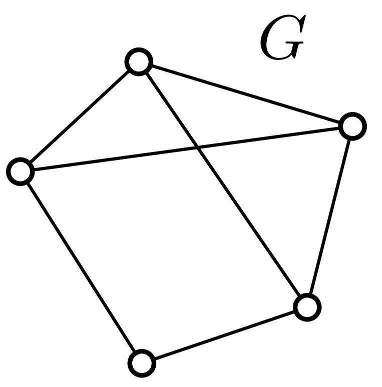

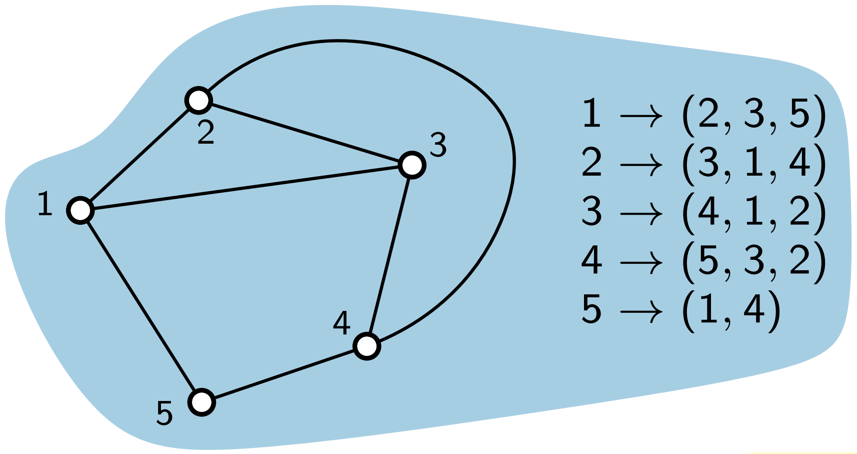

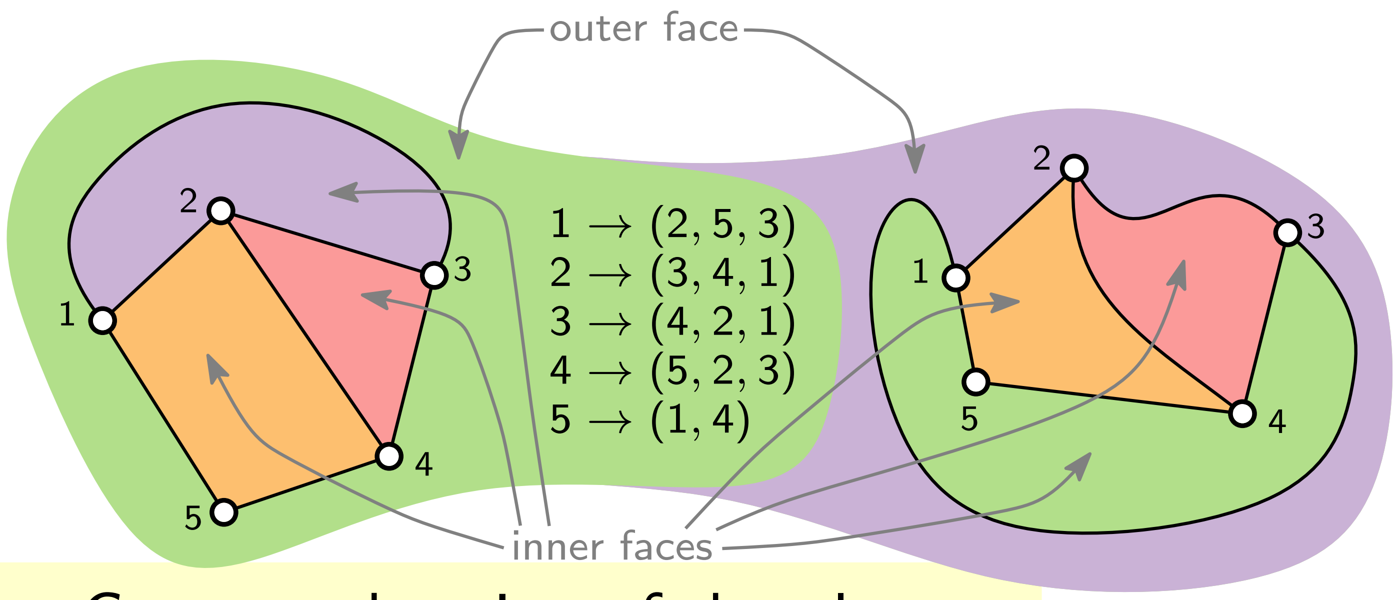

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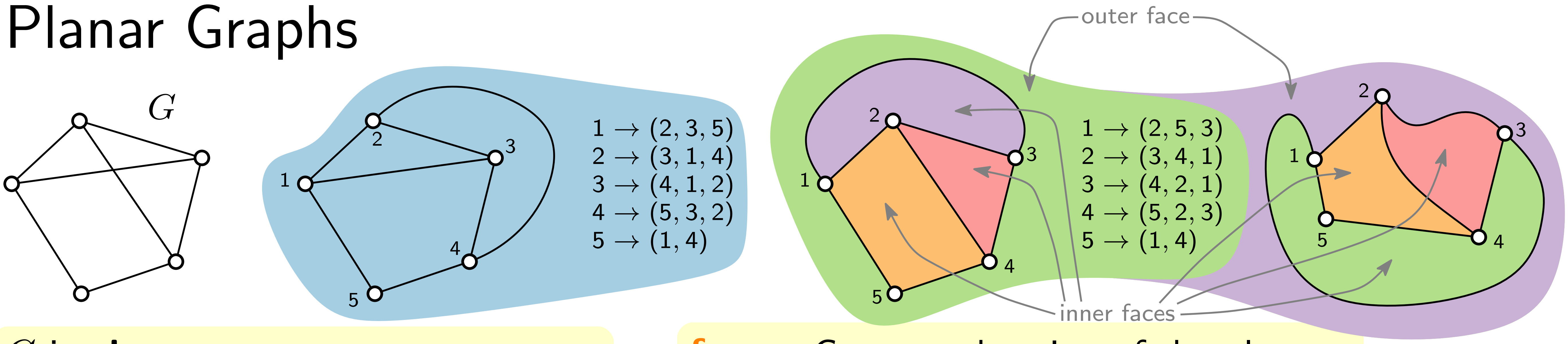

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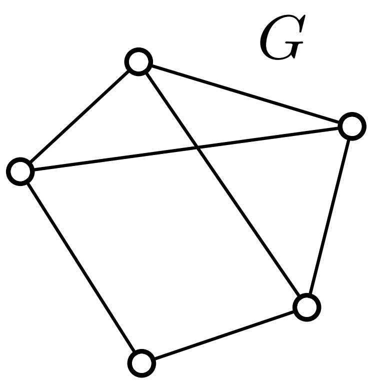

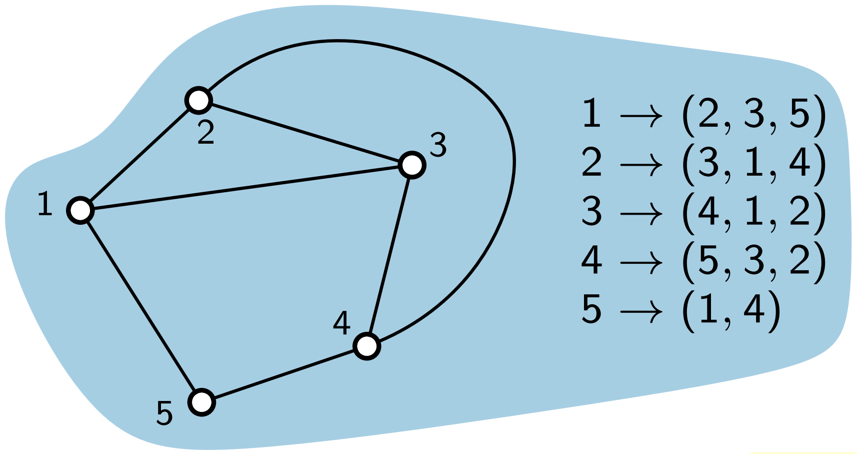

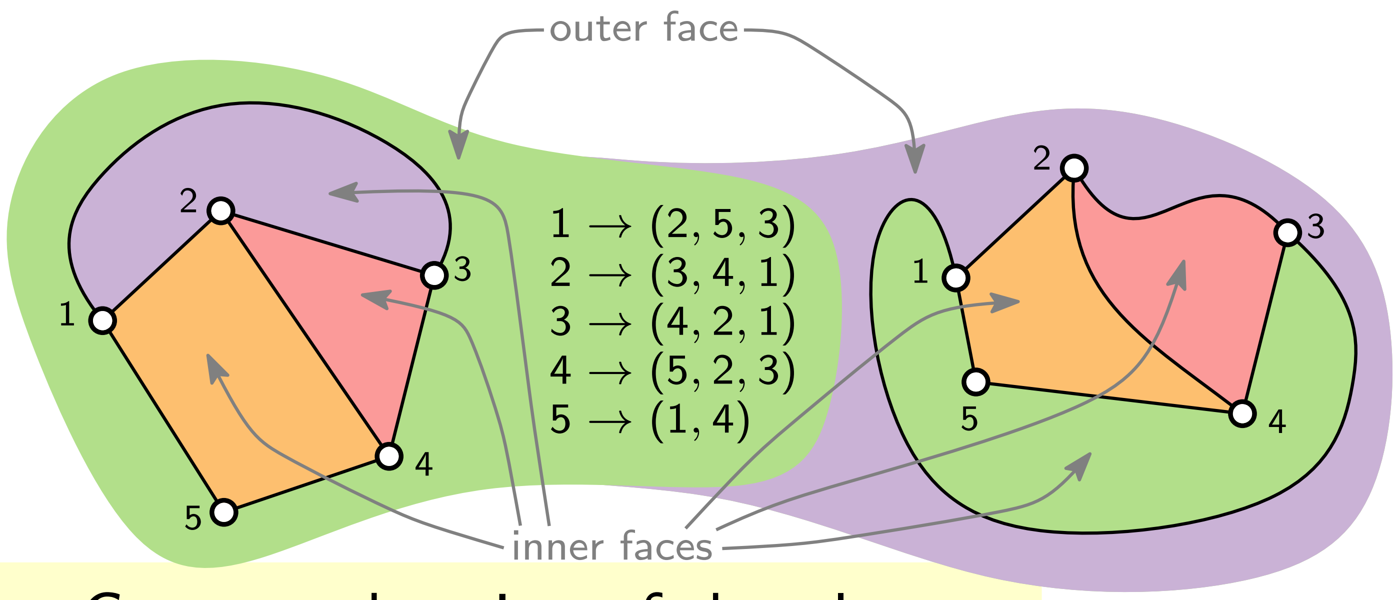

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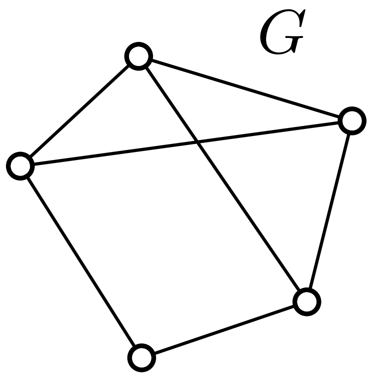

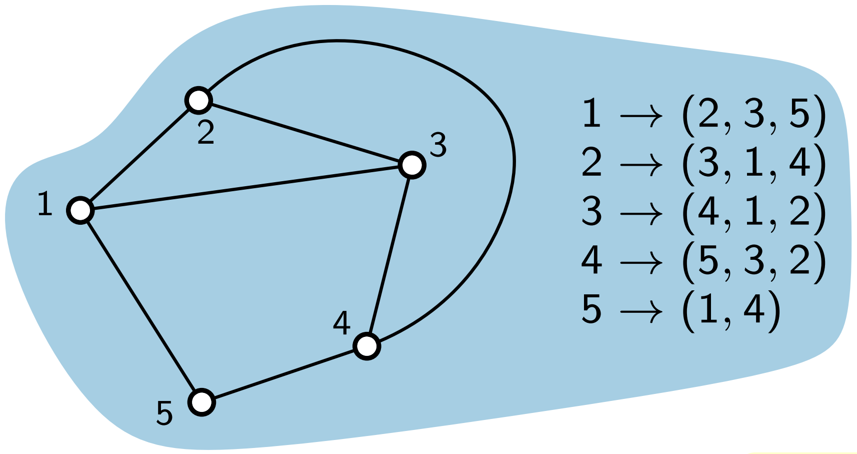

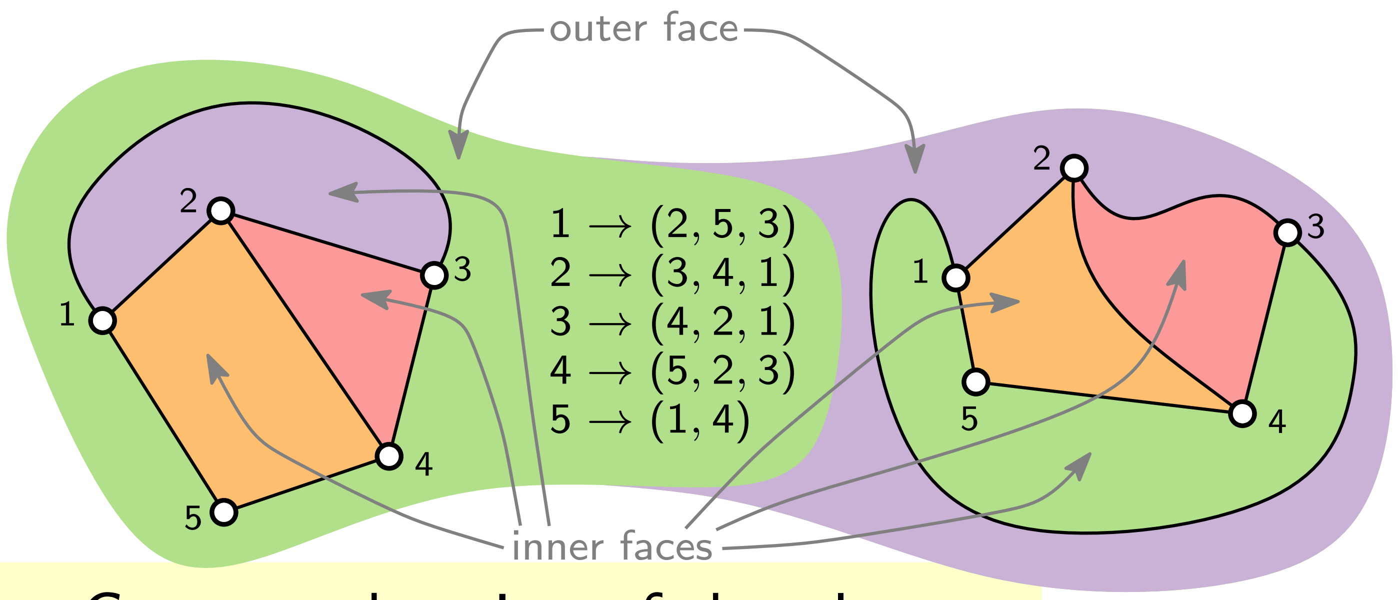

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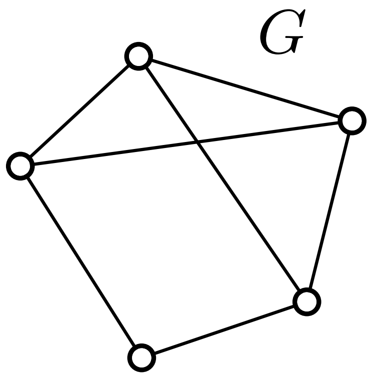

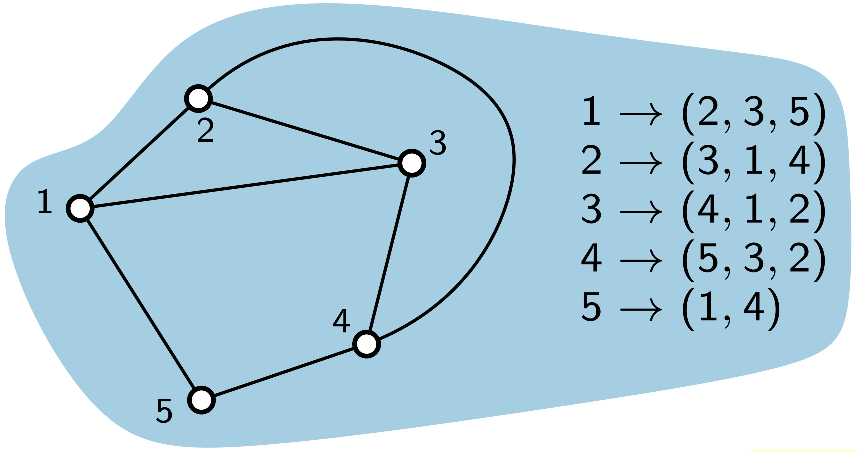

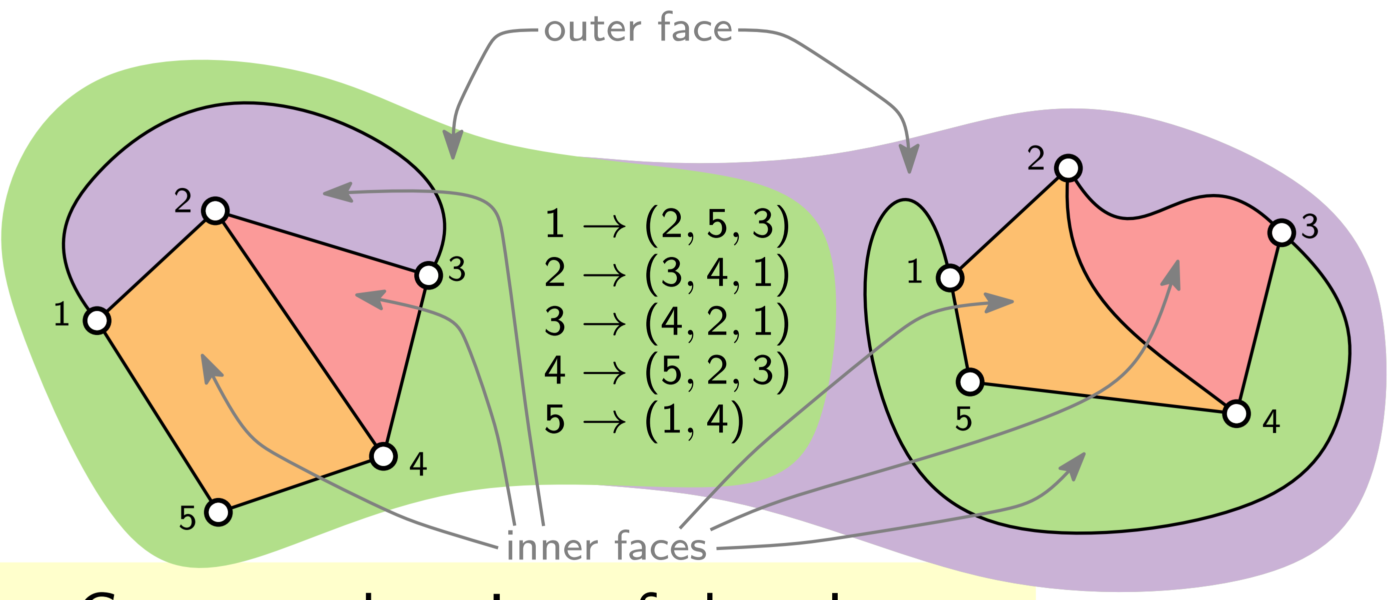

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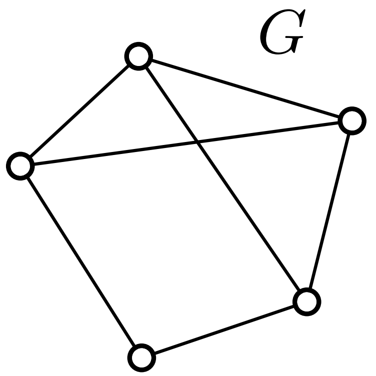

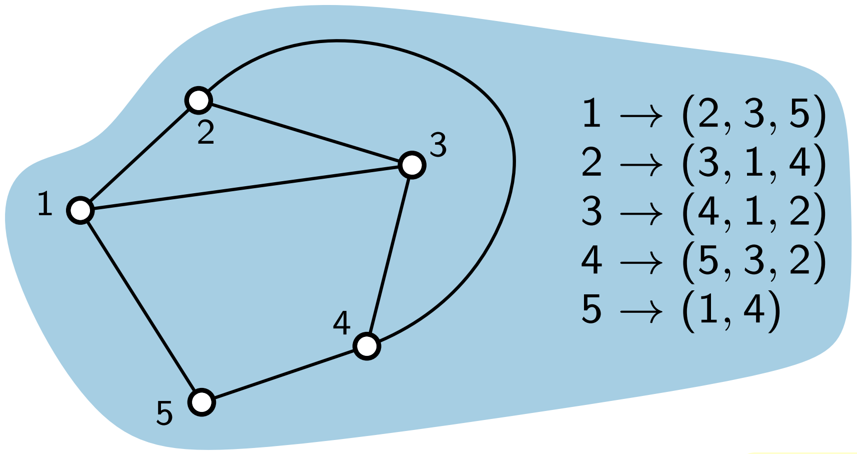

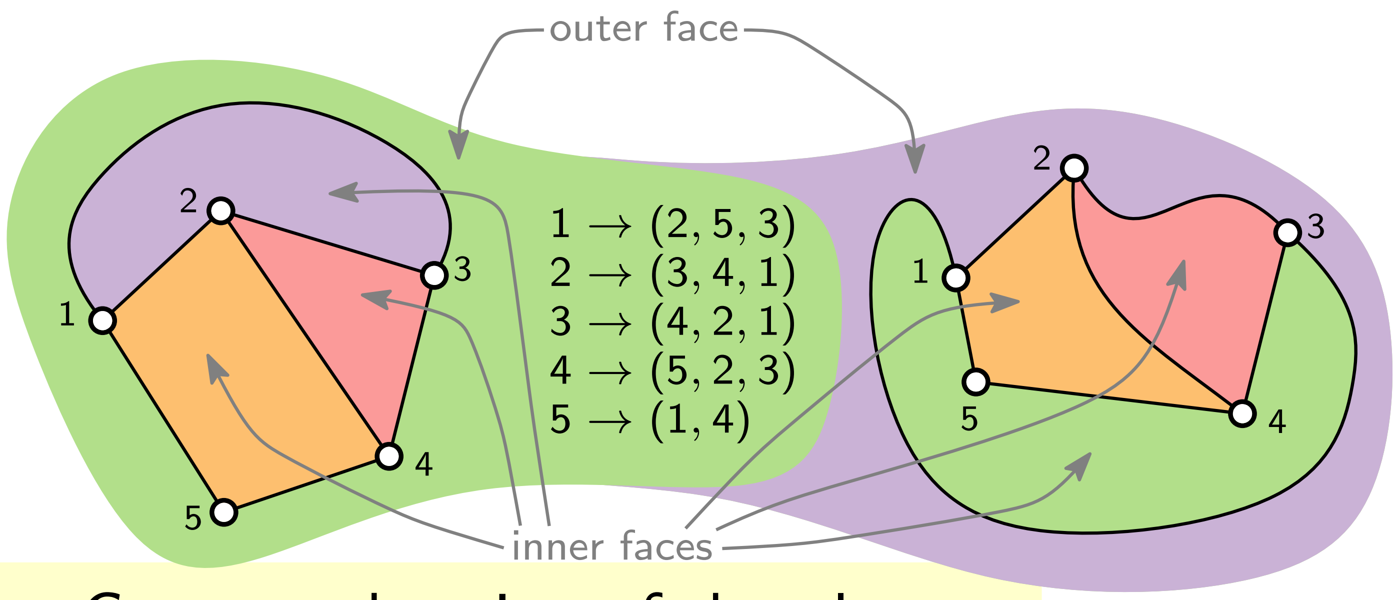

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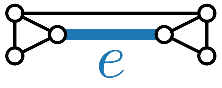

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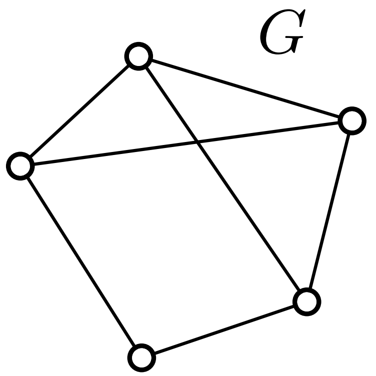

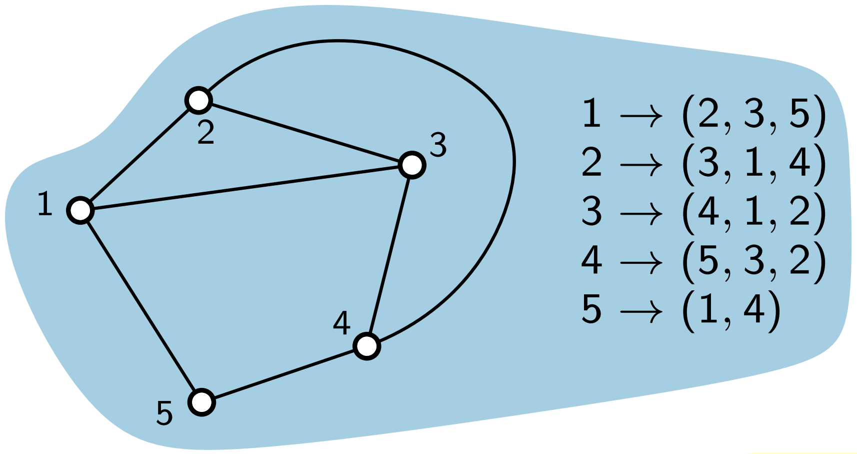

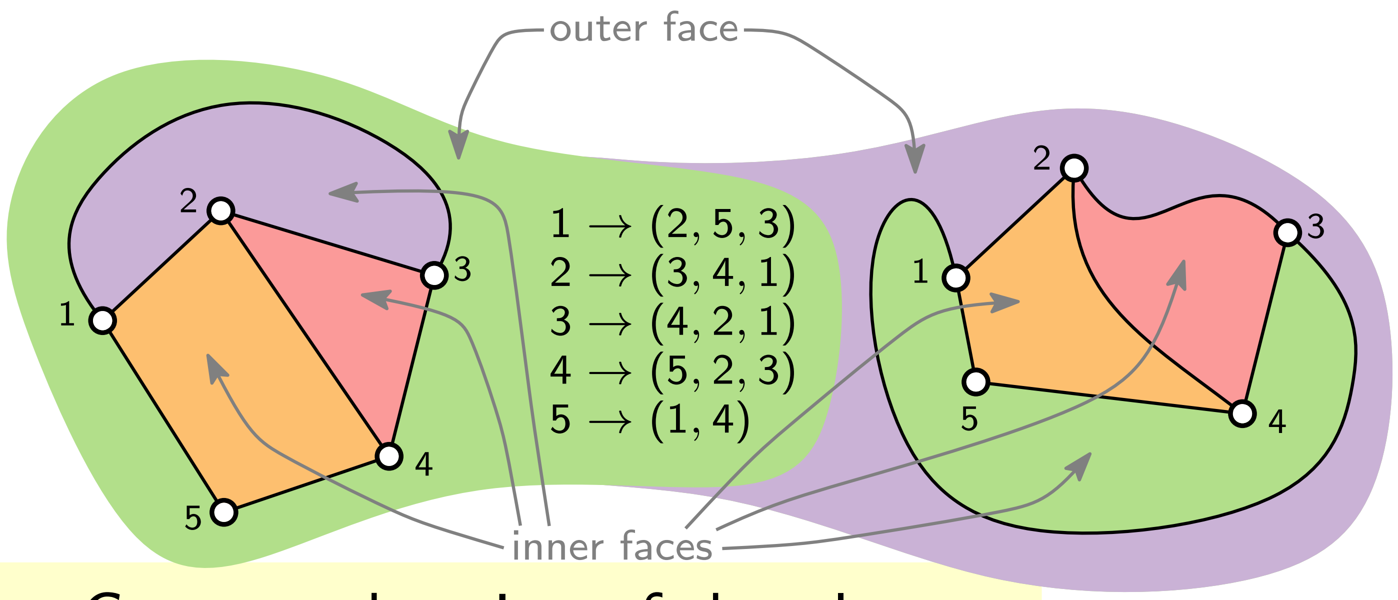

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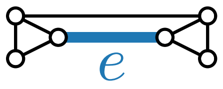

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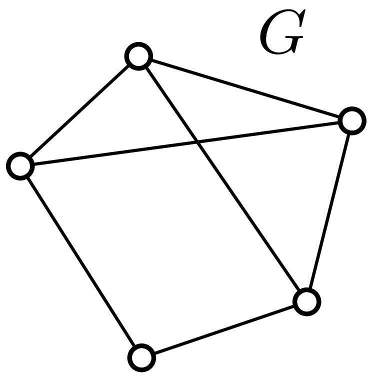

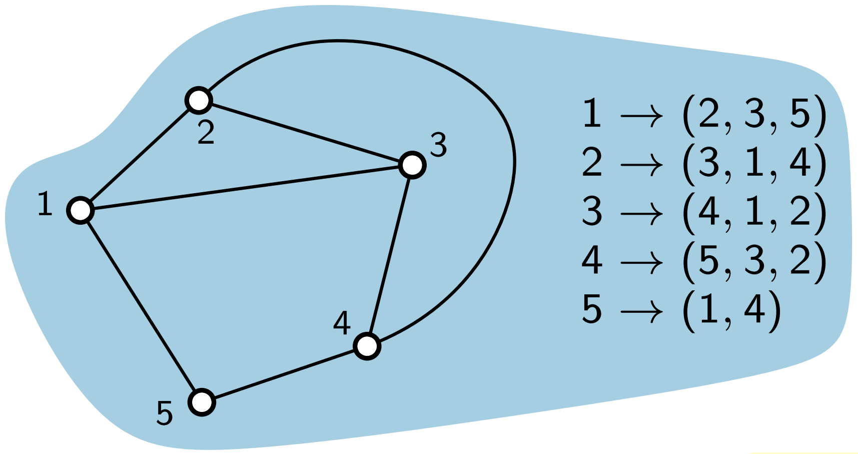

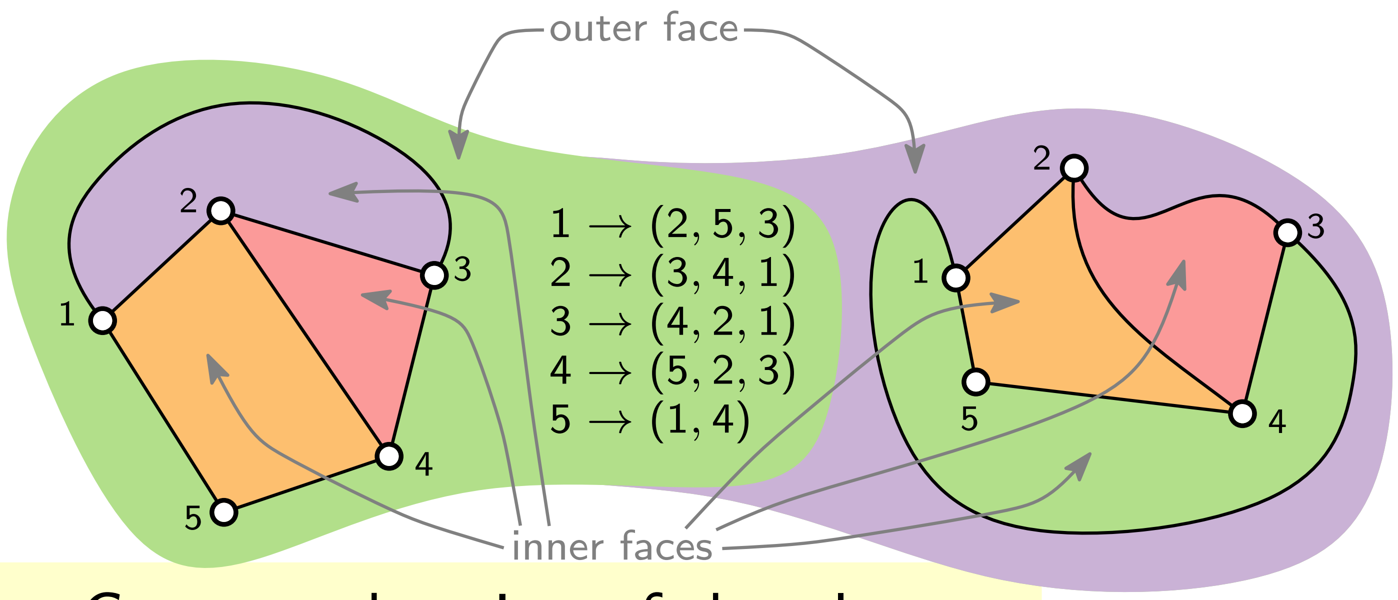

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Theorem. G simple planar graph with $n \geq 3$ vtc.

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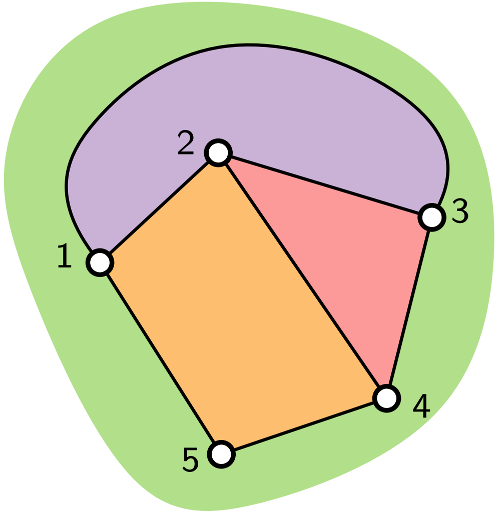

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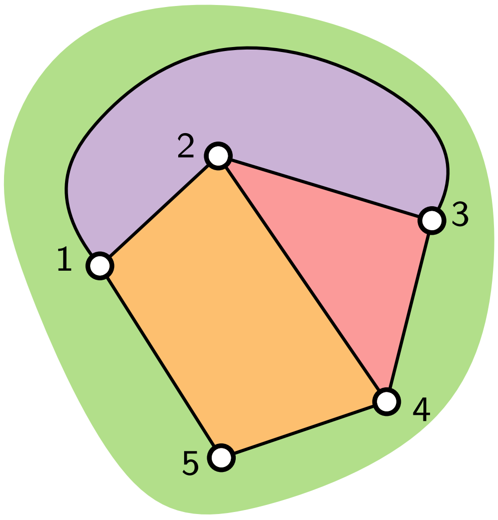
idea: count edge–face incidences



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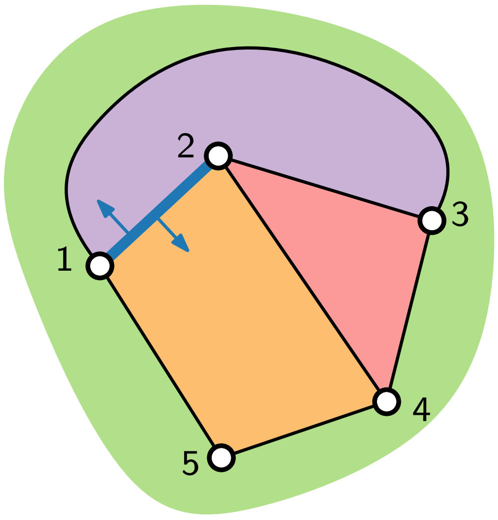

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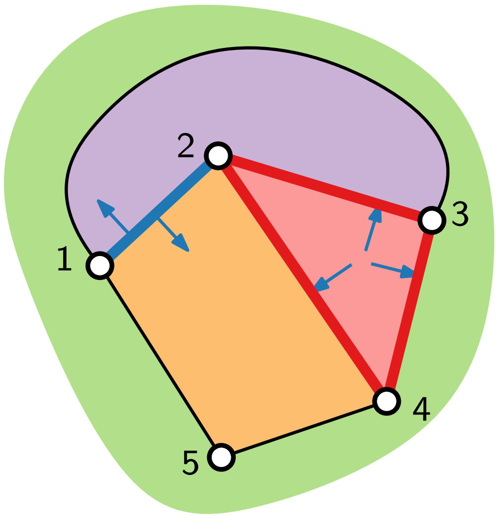

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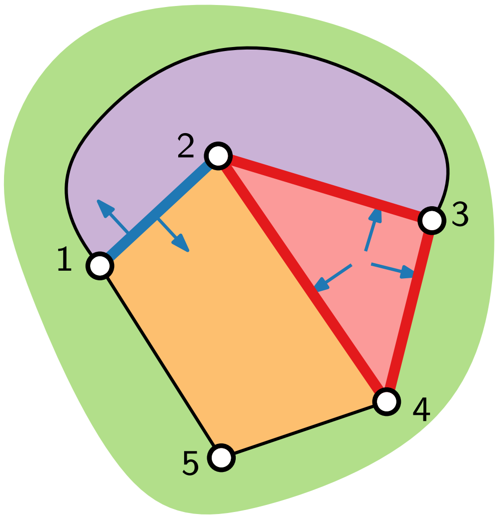

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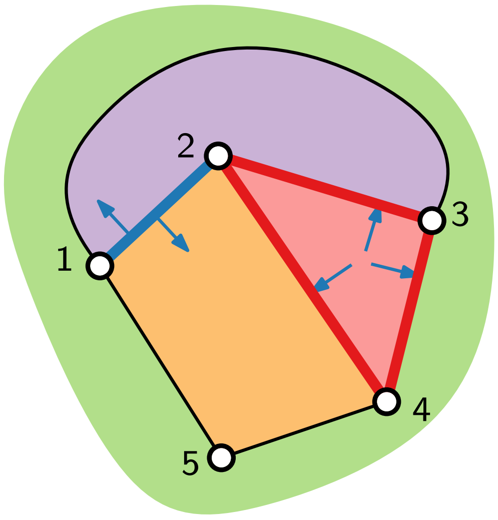

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$\Rightarrow c + 1 = f - m + n$

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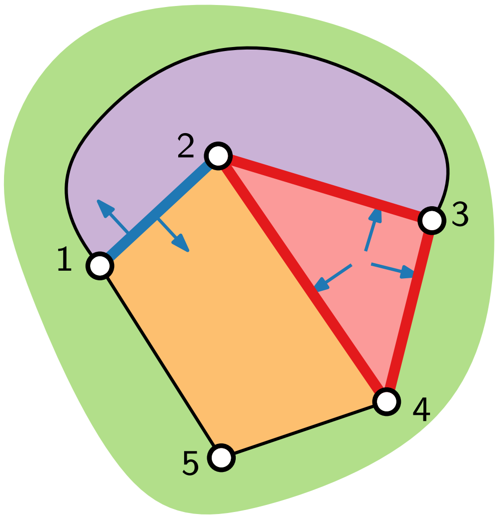

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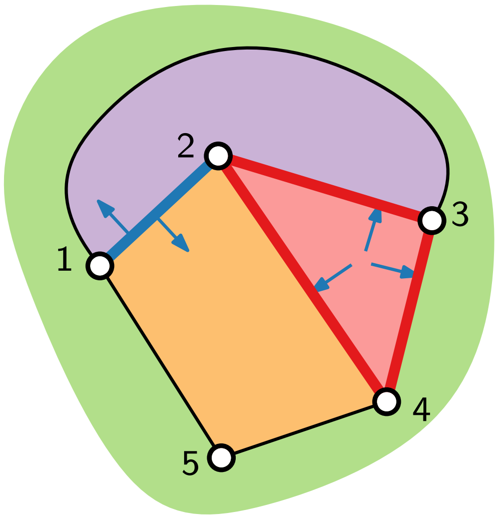

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idea: count edge–face incidences

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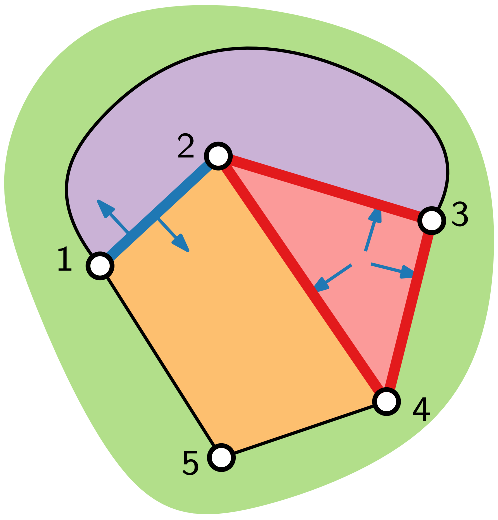

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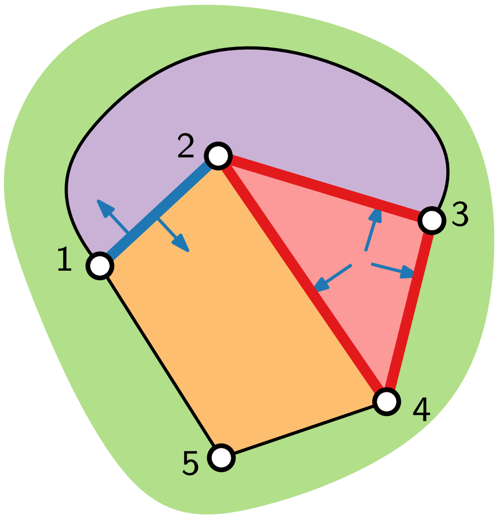

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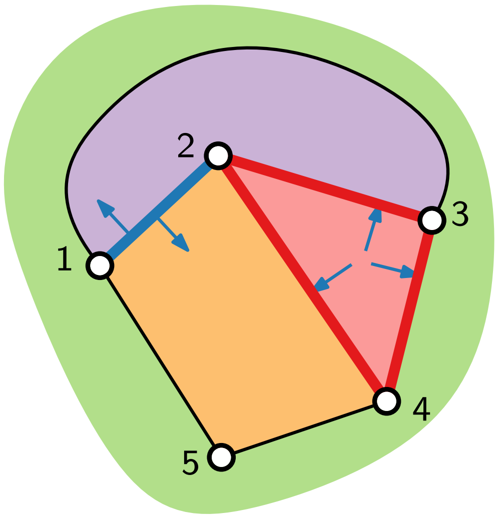

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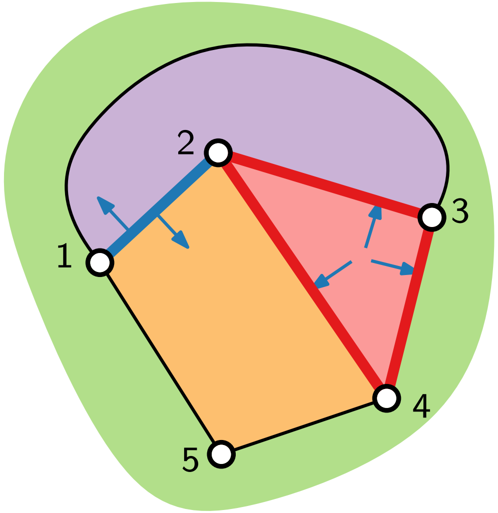

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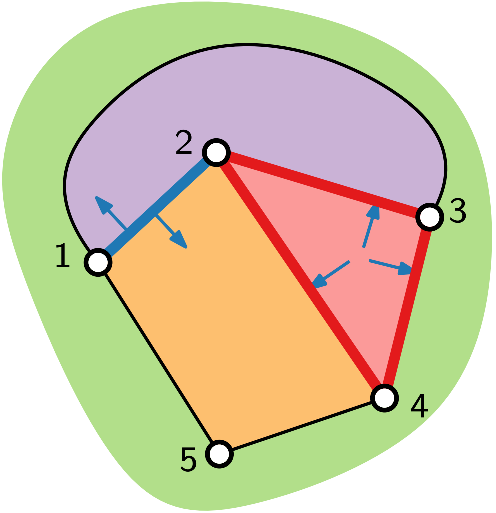

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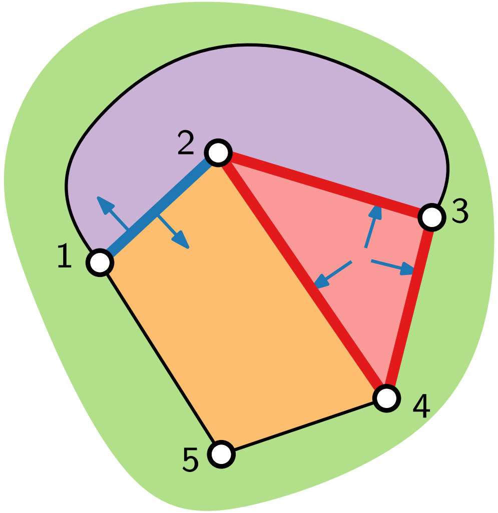

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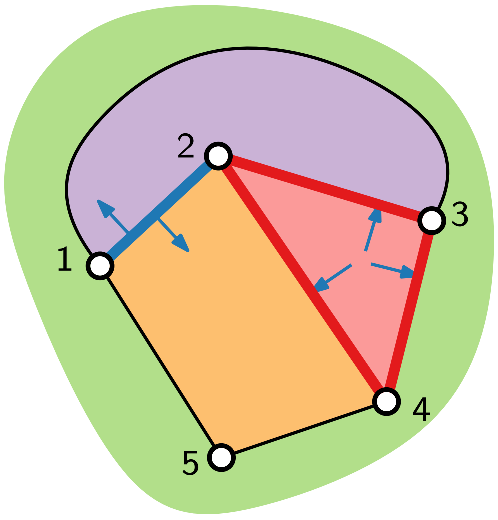

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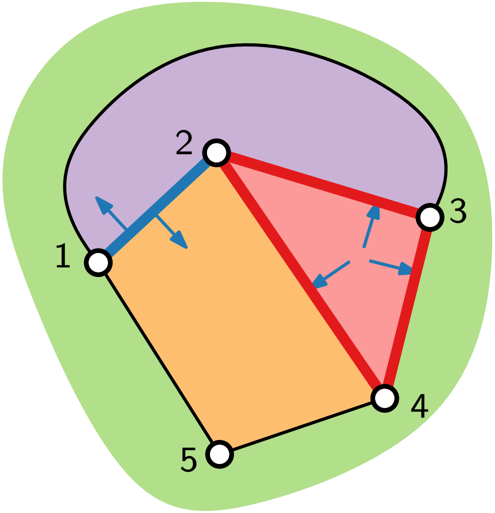

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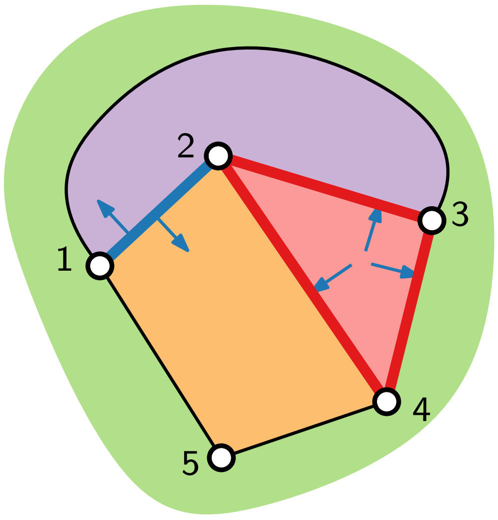

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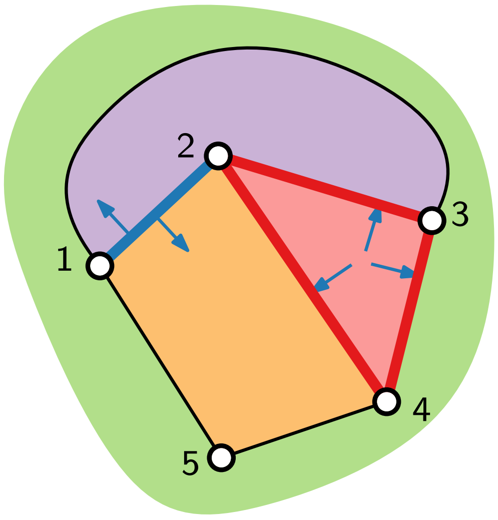

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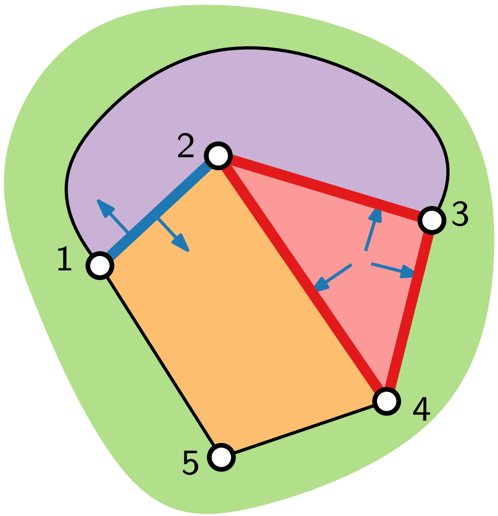

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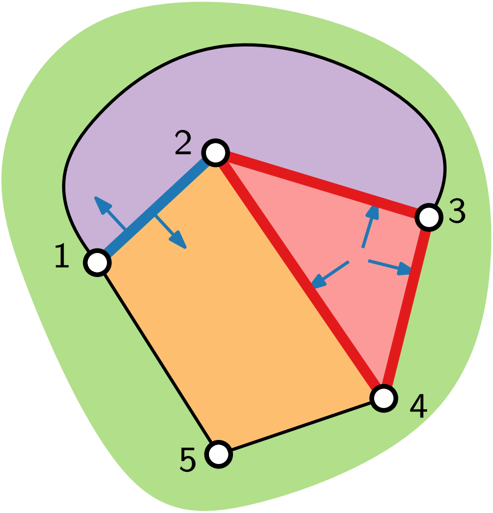

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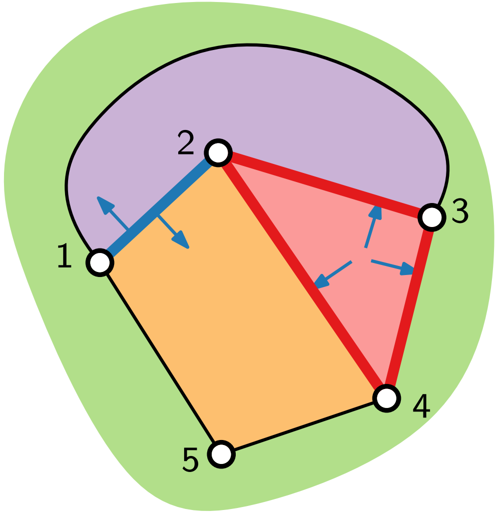

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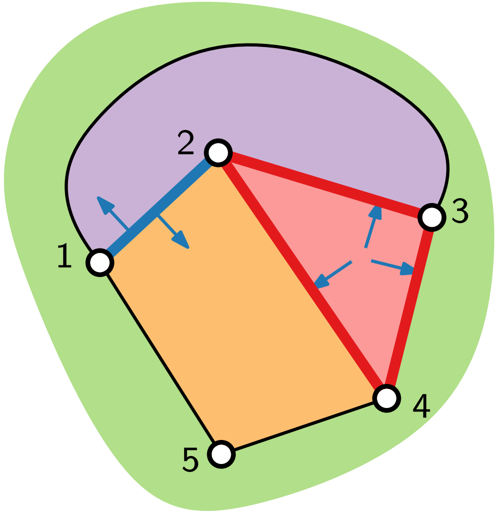

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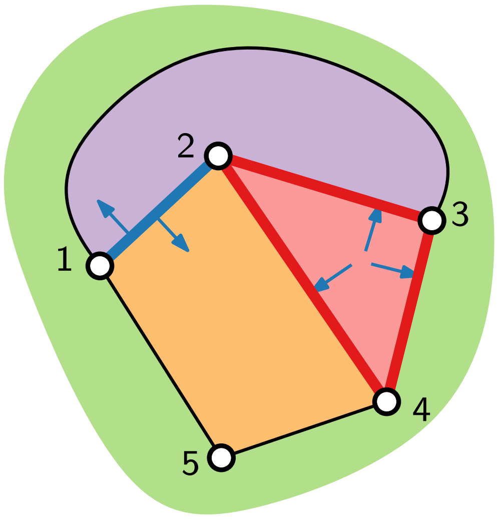

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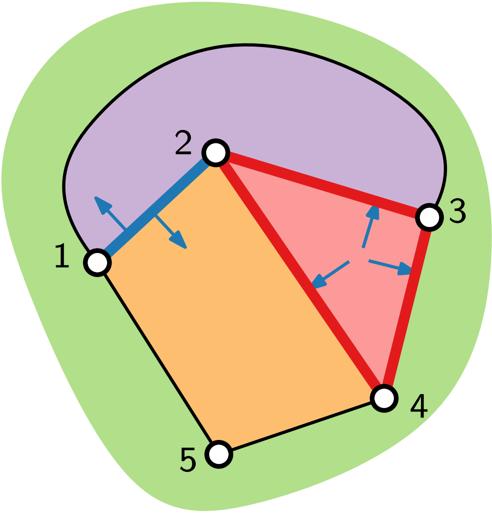

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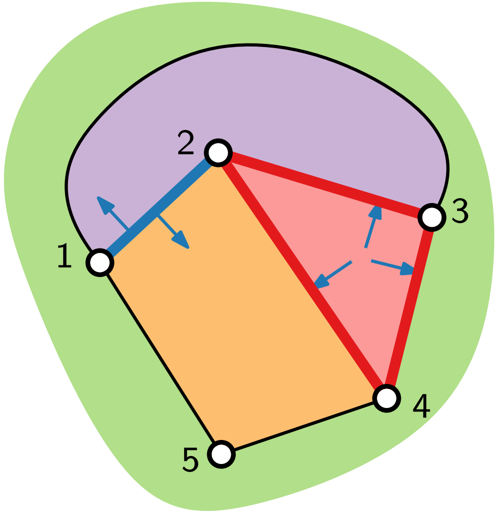

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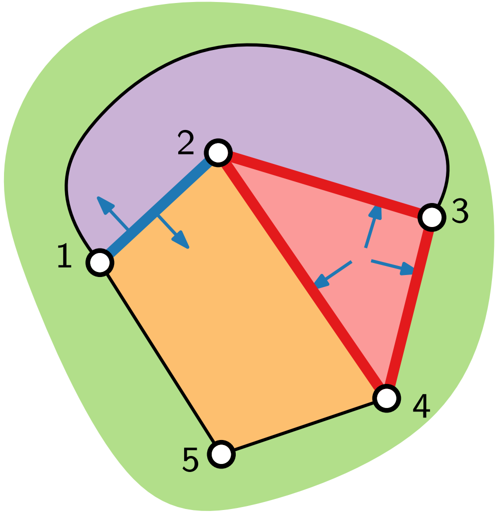

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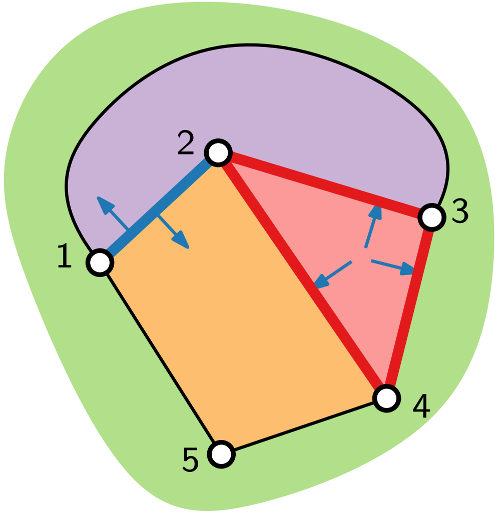

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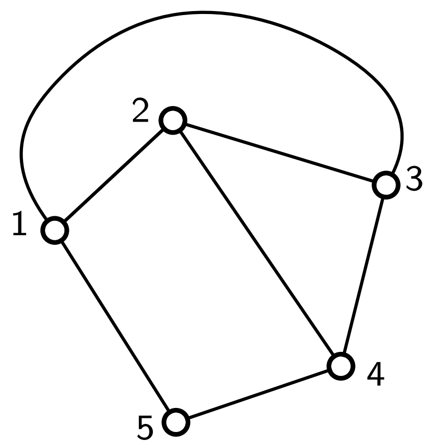
A **plane triangulation** is a plane graph where every face is a triangle.



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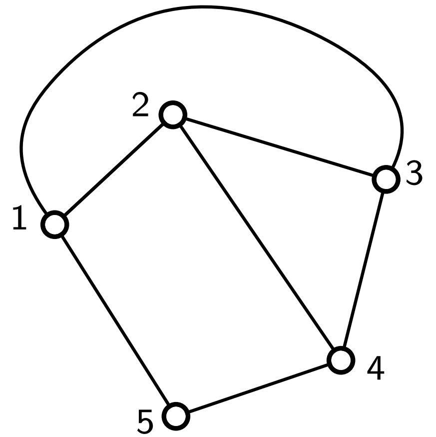

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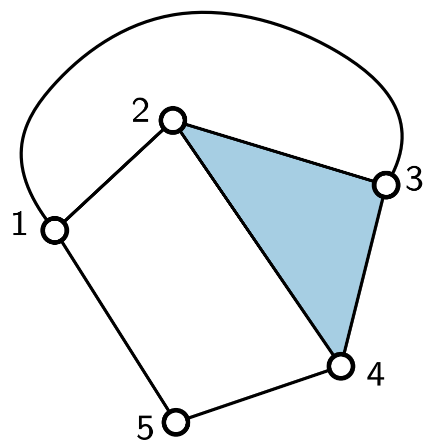

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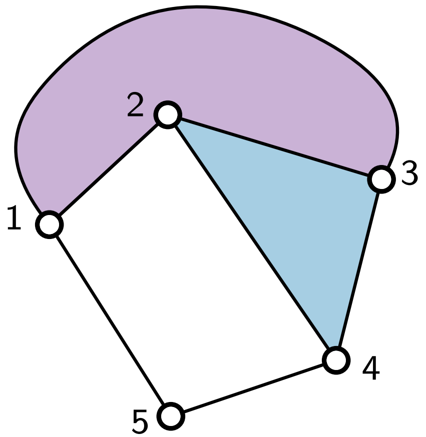

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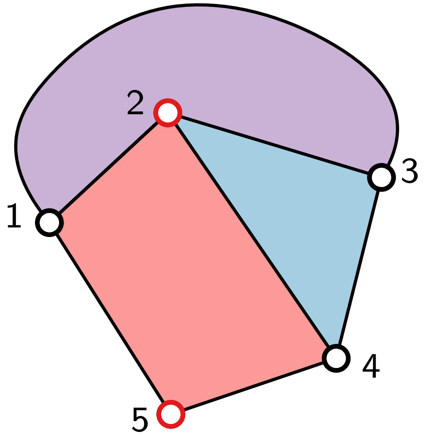

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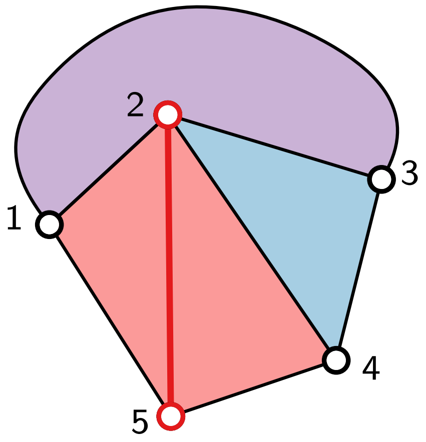

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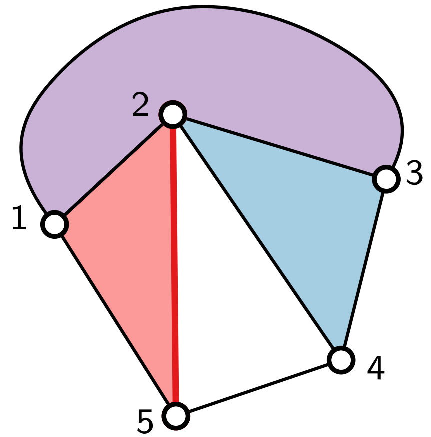

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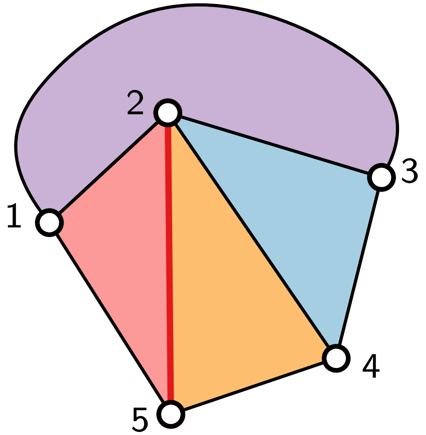

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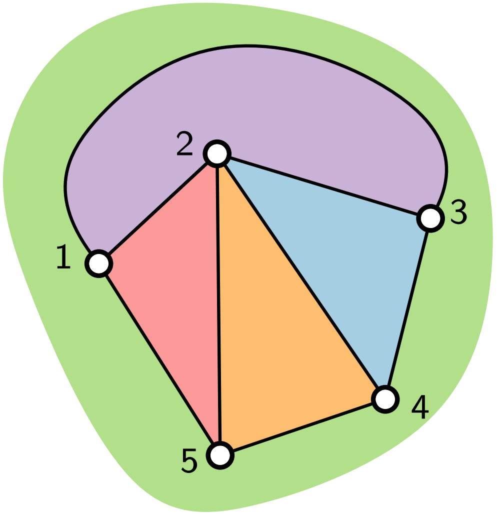

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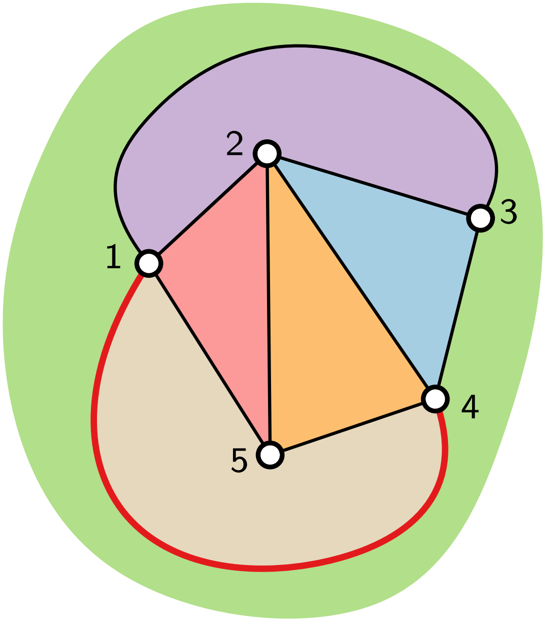

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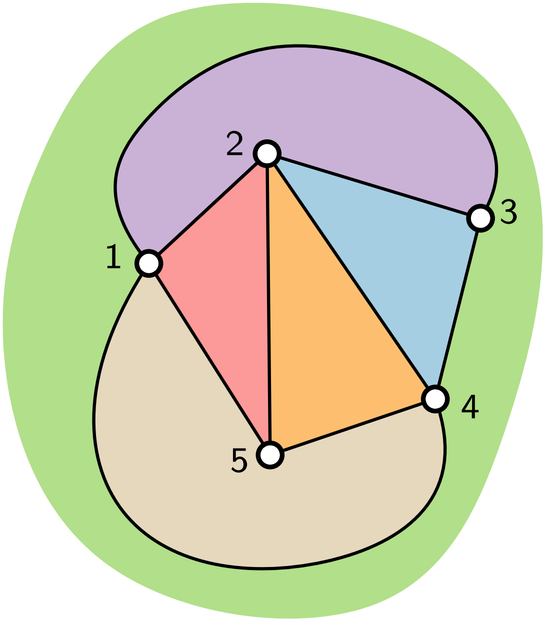

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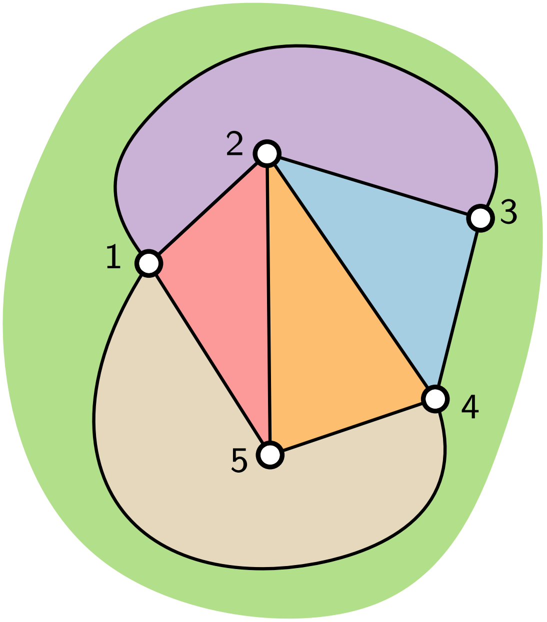

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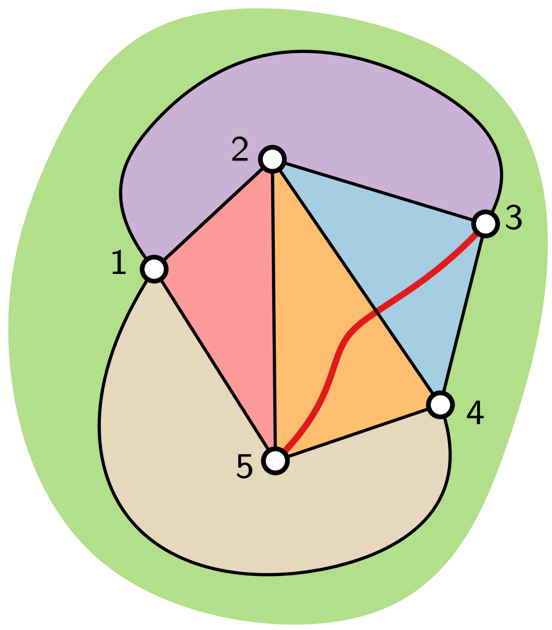

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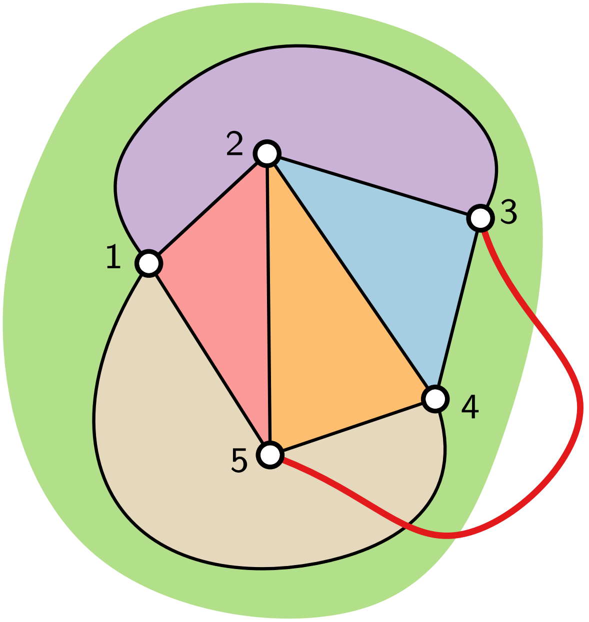

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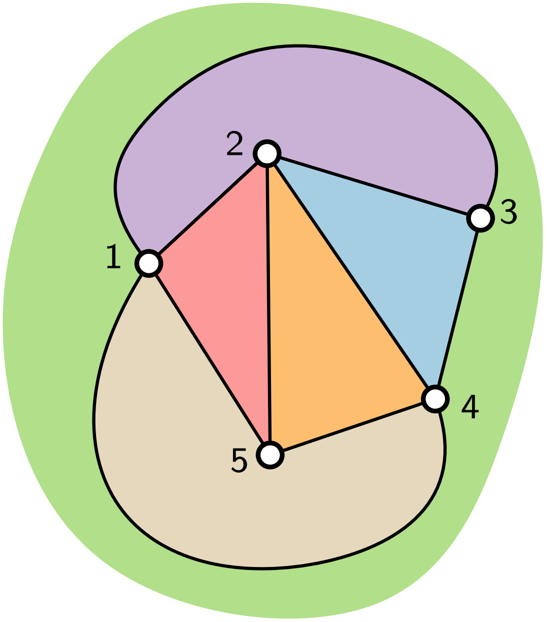

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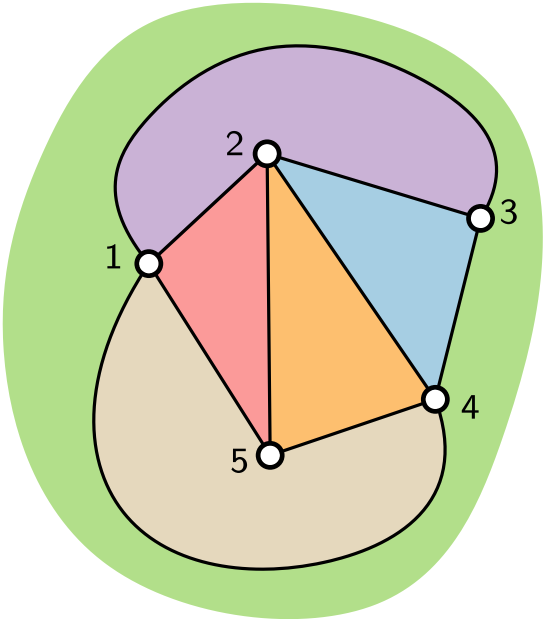

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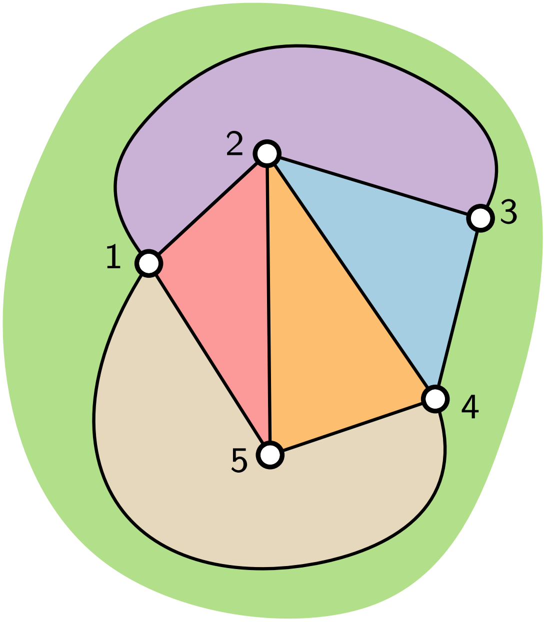

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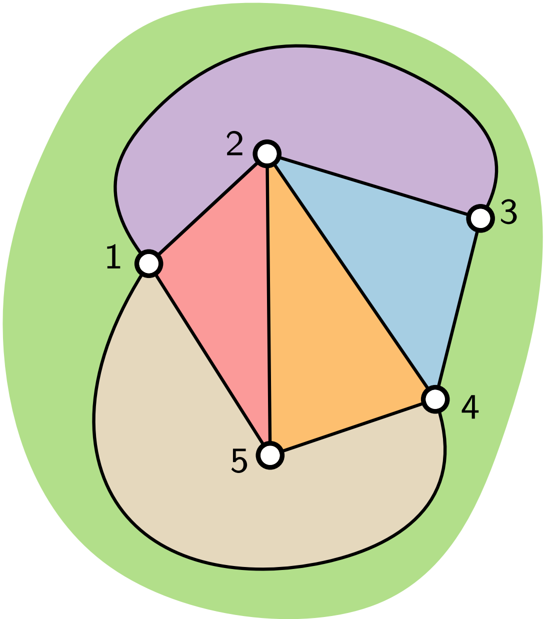

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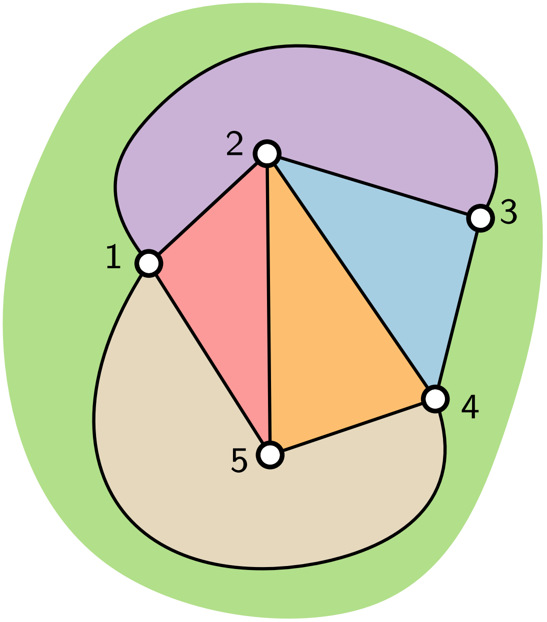

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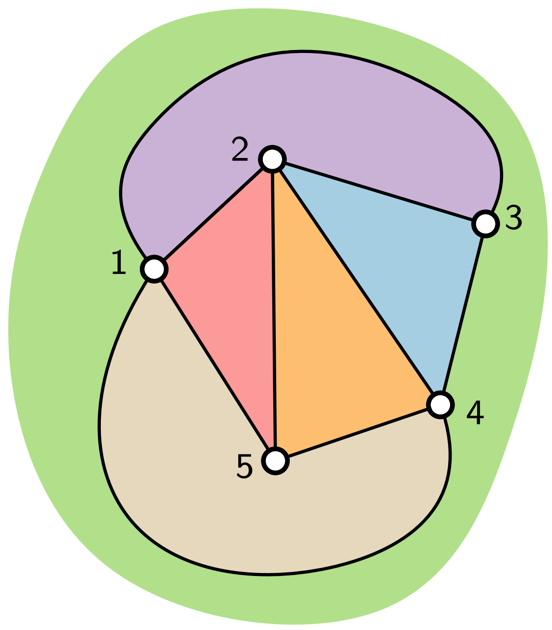

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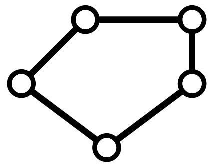

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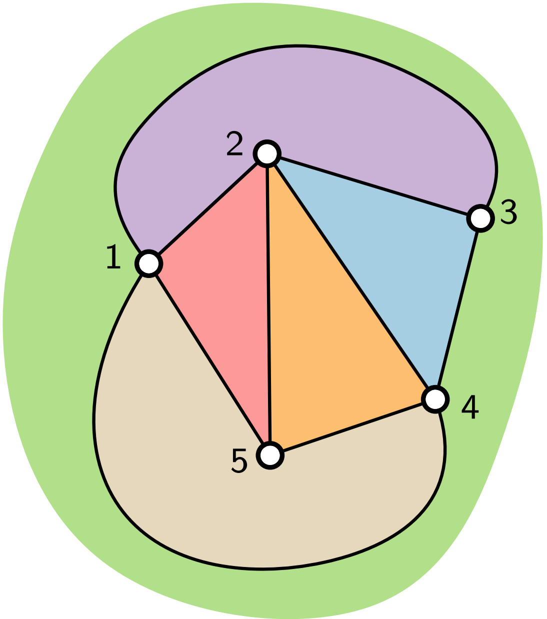

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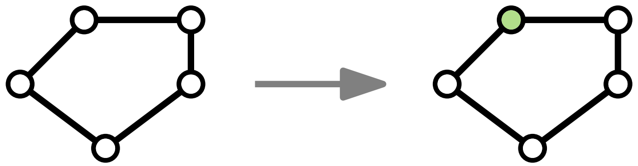

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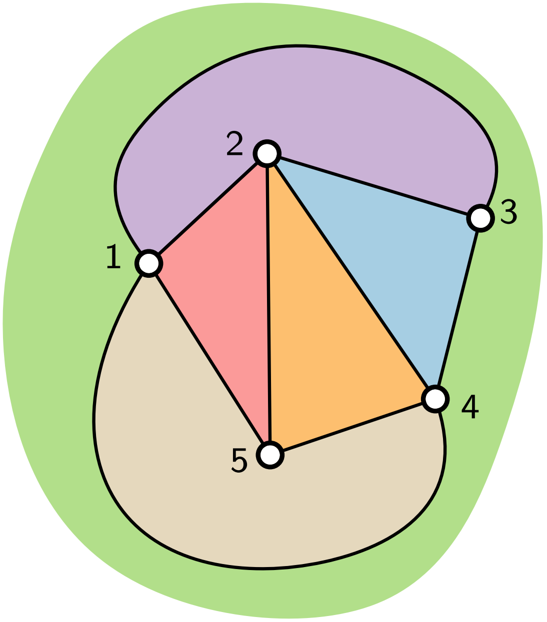

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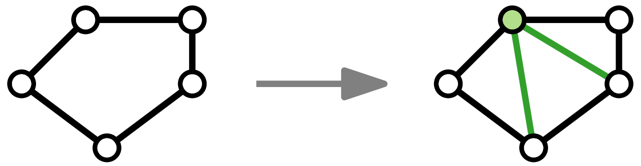

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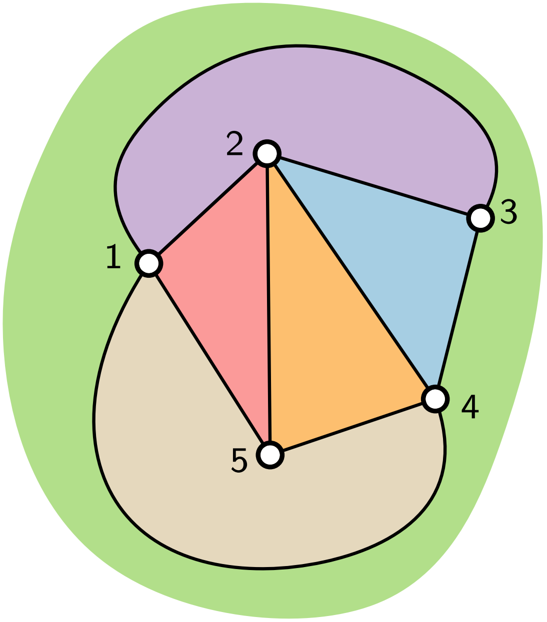

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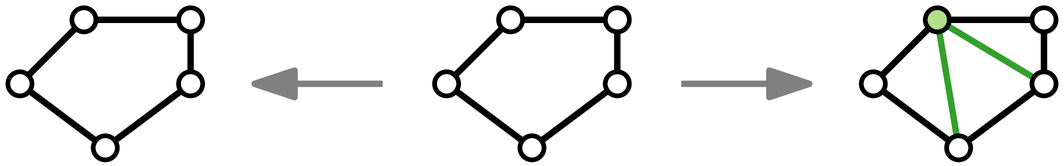

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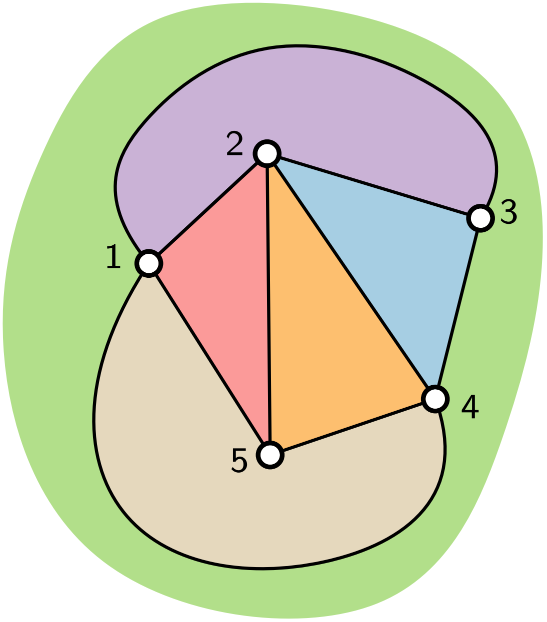

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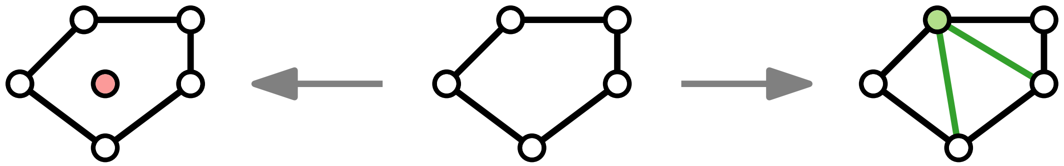

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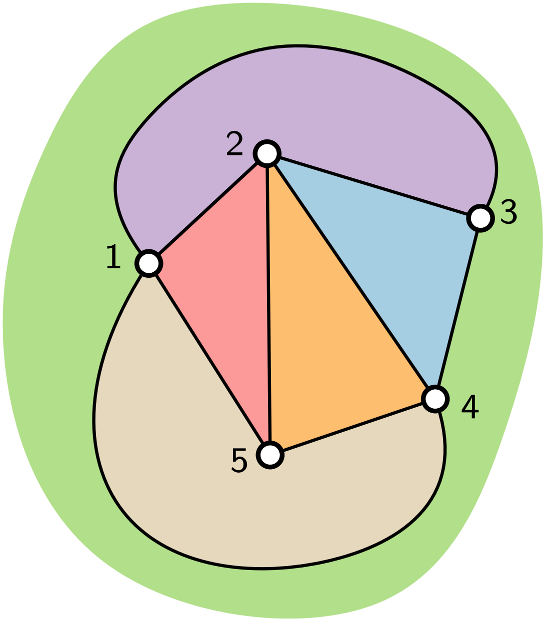

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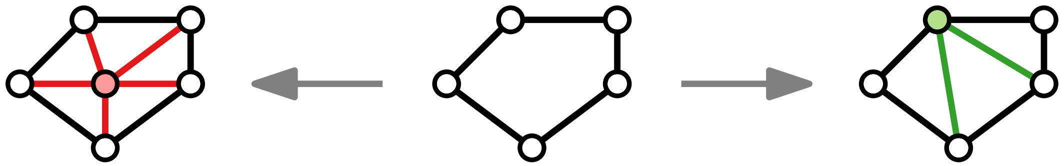

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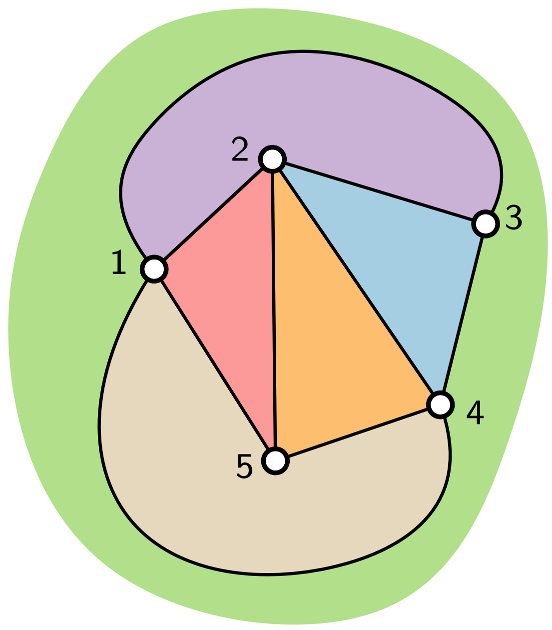

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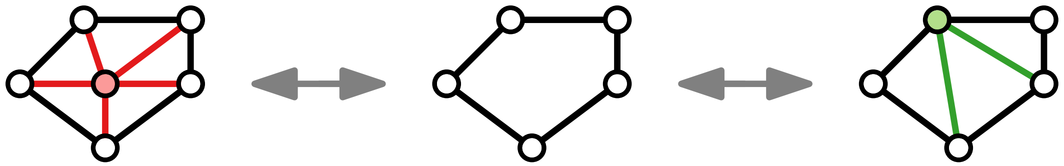

Motivation

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[Bennett, Ryall, Spaltzeholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

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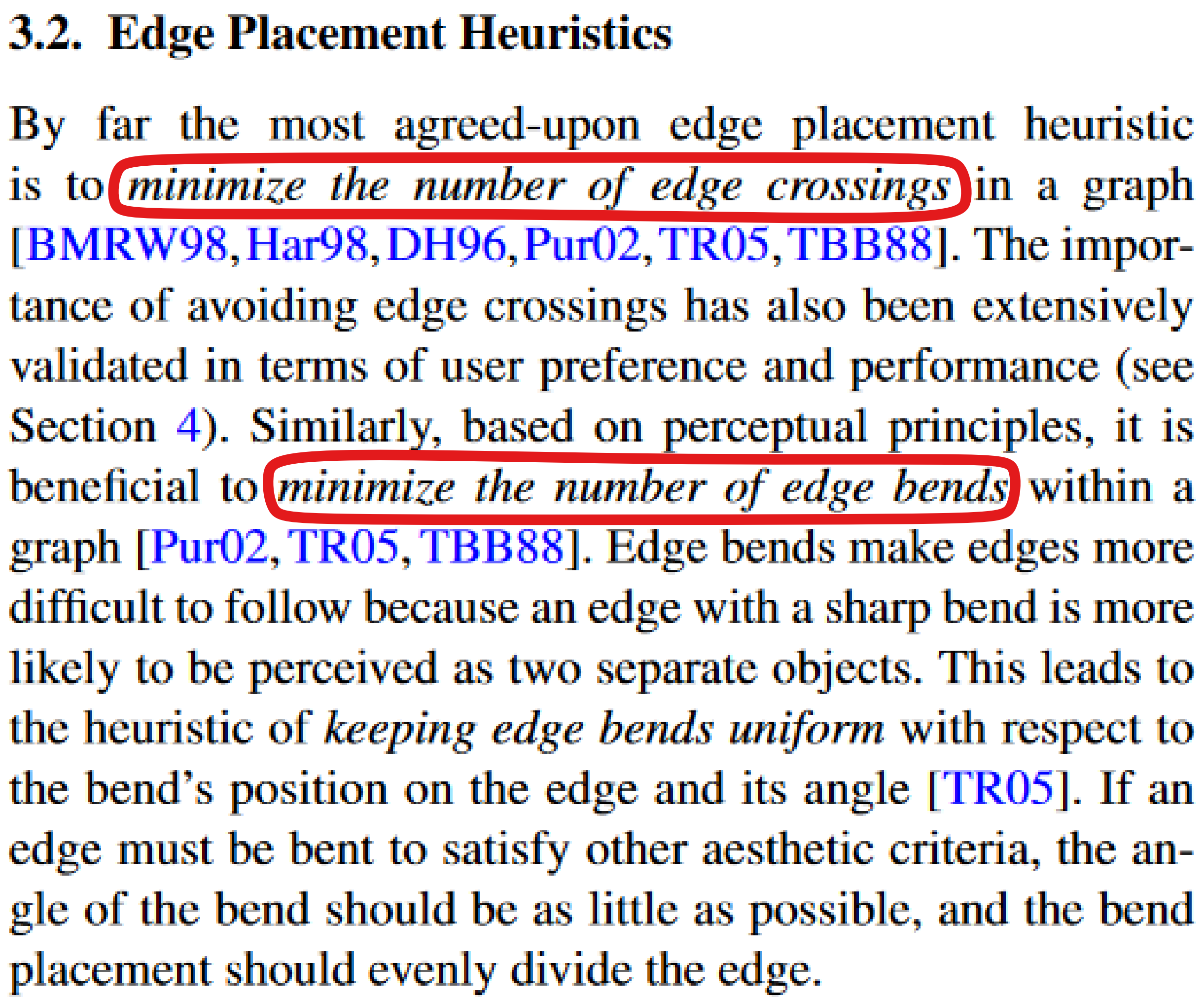
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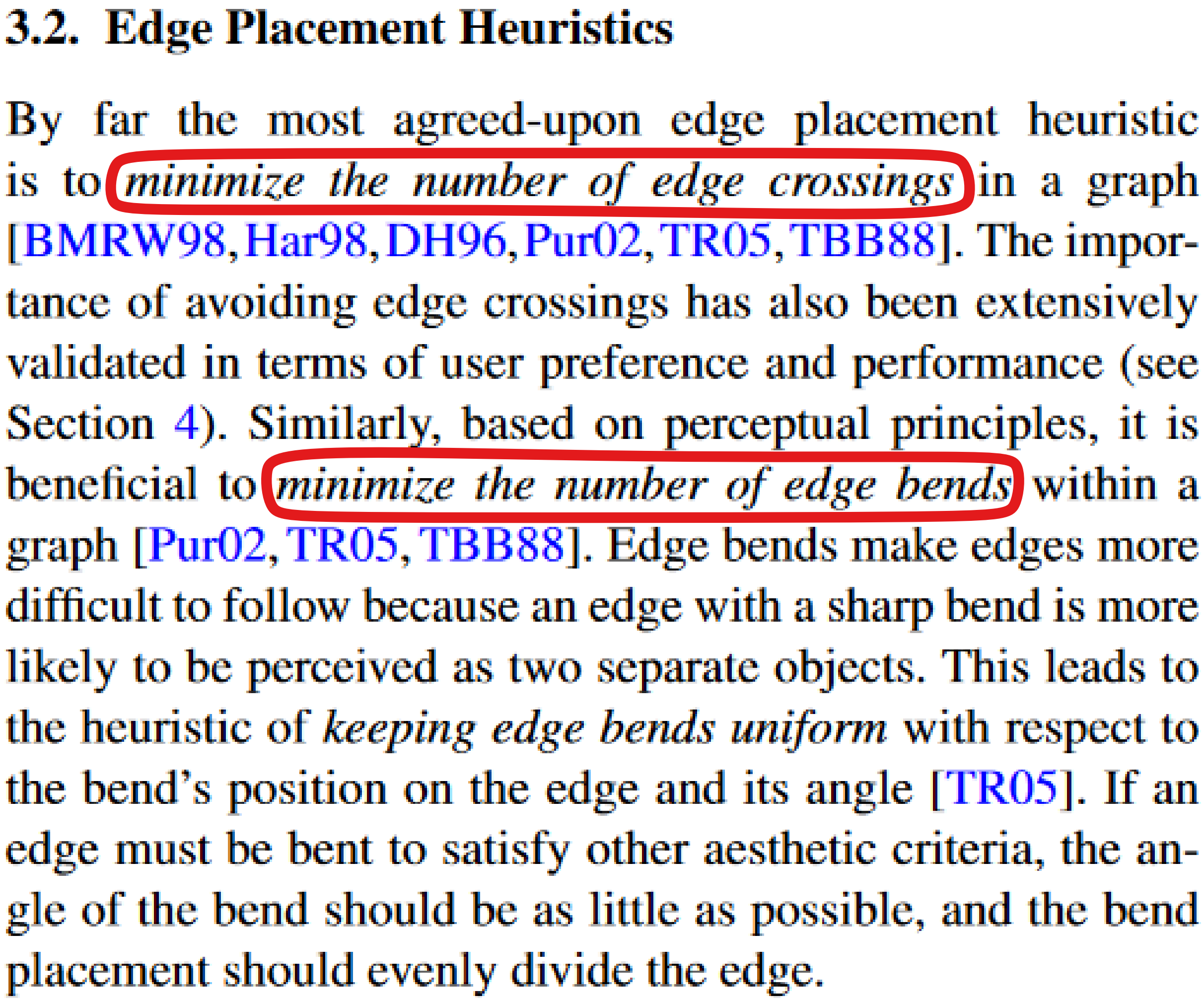

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- Area

Towards Straight-Line Drawings

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Characterization

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Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
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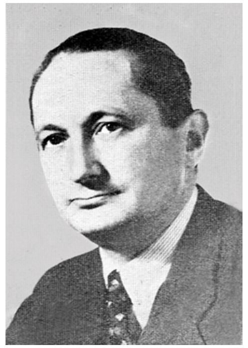
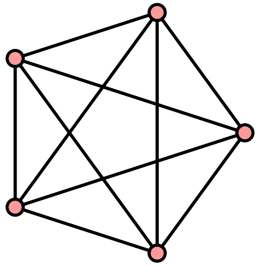


Kazimierz Kuratowski (1896–1980)



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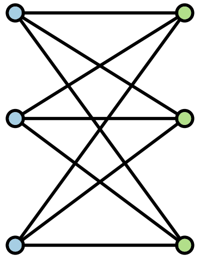

$K_{3,3}$

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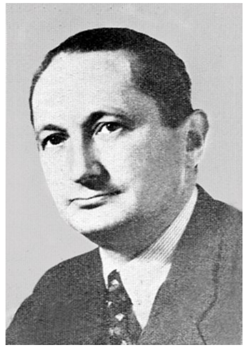

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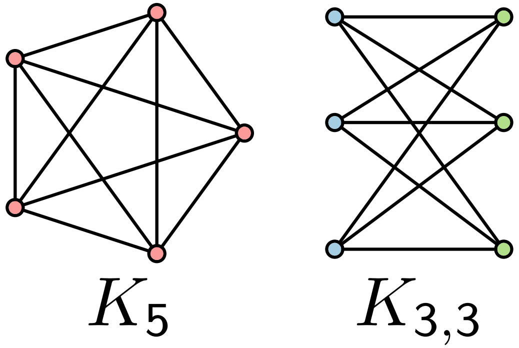

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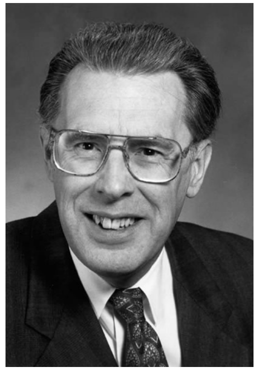

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Renatokeshet, GFDL via Wikimedia

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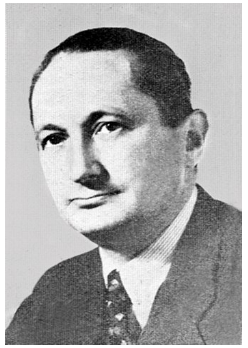

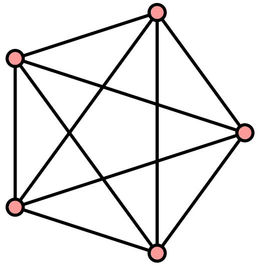

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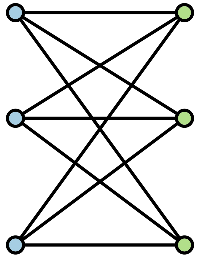

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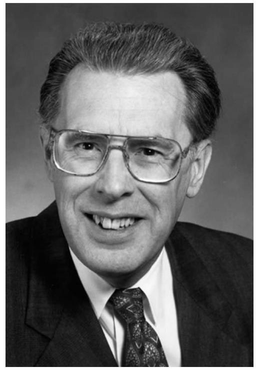

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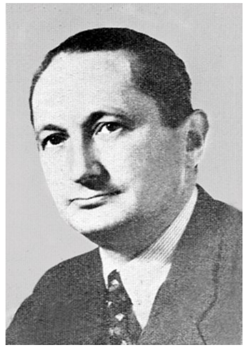

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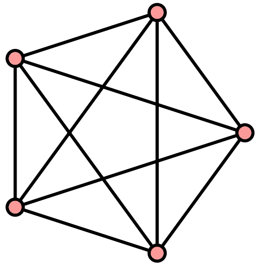

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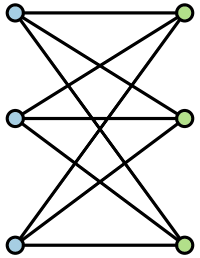

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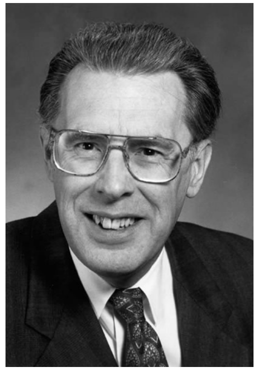

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Recognition

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has a planar drawing where the edges are straight-line segments.

Klaus Wagner (1910–2000)
Autor: Konrad Jacobs, wikipedia

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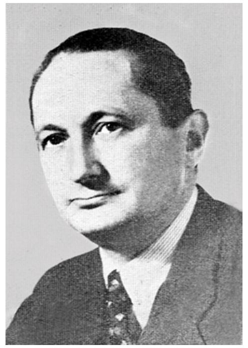
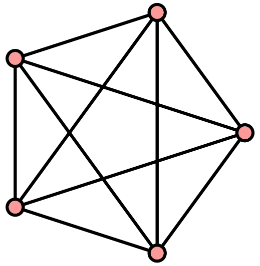
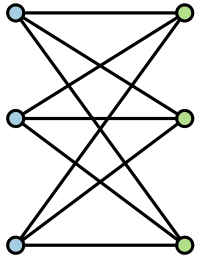

Kazimierz Kuratowski (1896–1980)

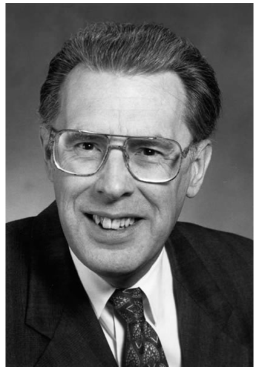

K_5


$K_{3,3}$

Characterization

Theorem. [Hopcroft & Tarjan 1974]
Let G be a graph with n vertices. There is an $\mathcal{O}(n)$-time algorithm to test whether G is planar.

Also computes a planar embedding in $\mathcal{O}(n)$ time.


John Edward Hopcroft (1939–)
en.wikipedia.org/wiki/User:Shakespeare


Robert Endre Tarjan (1948–)
Renatokeshet, GFDL via Wikimedia

Recognition

Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has a planar drawing where the edges are straight-line segments.

The algorithms implied by these theorems produce drawings whose area is **not** bounded by any polynomial in n.


Klaus Wagner (1910–2000)
Autor: Konrad Jacobs, wikipedia

Drawing

Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$.

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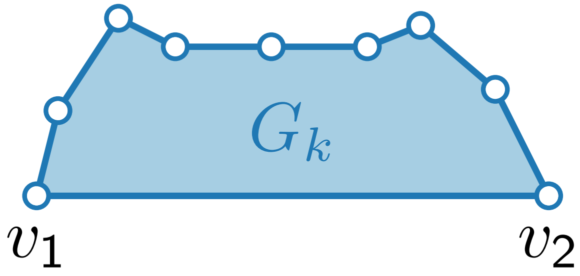

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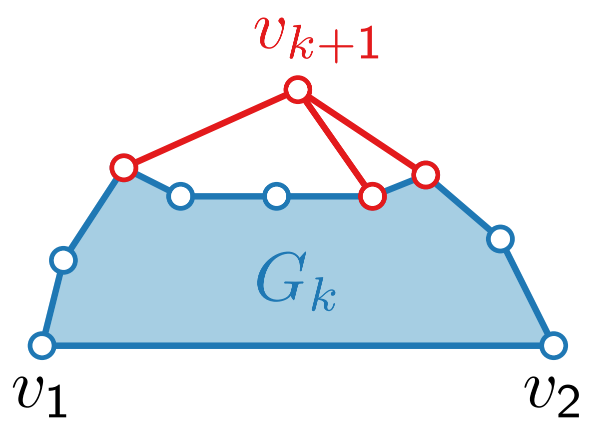

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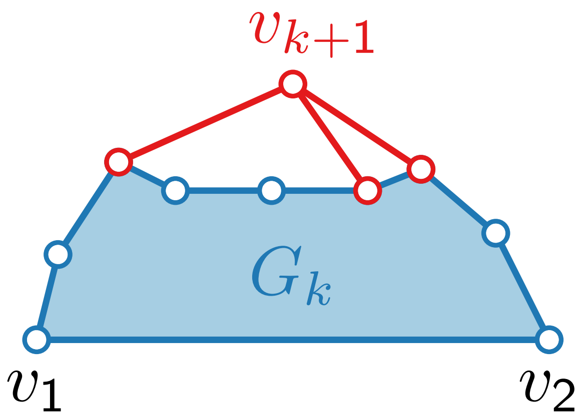

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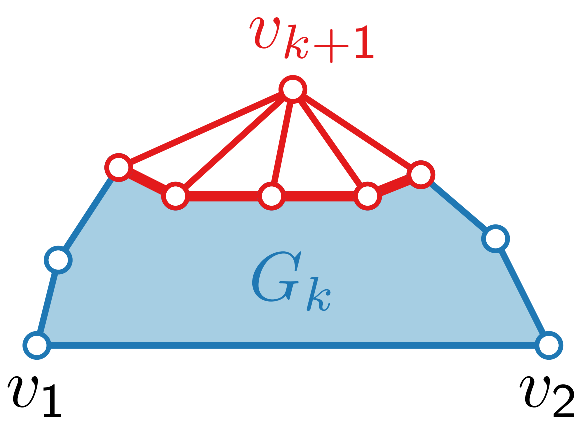

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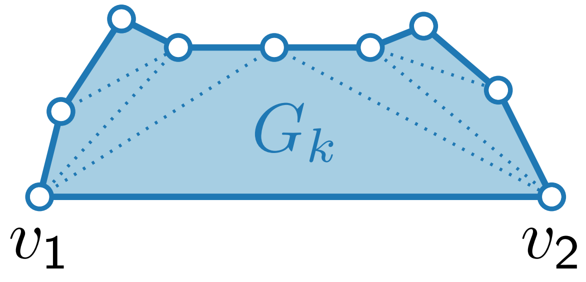

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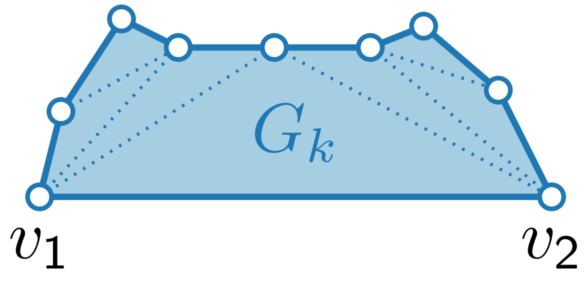

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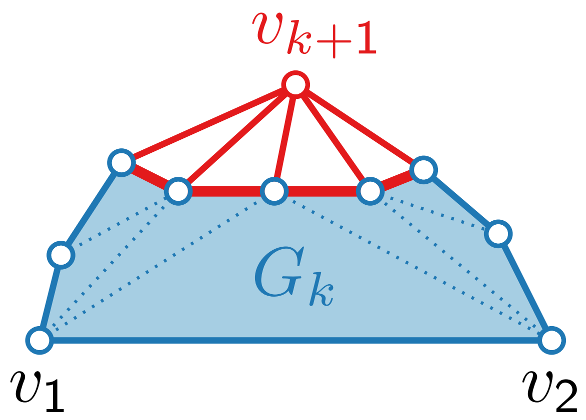

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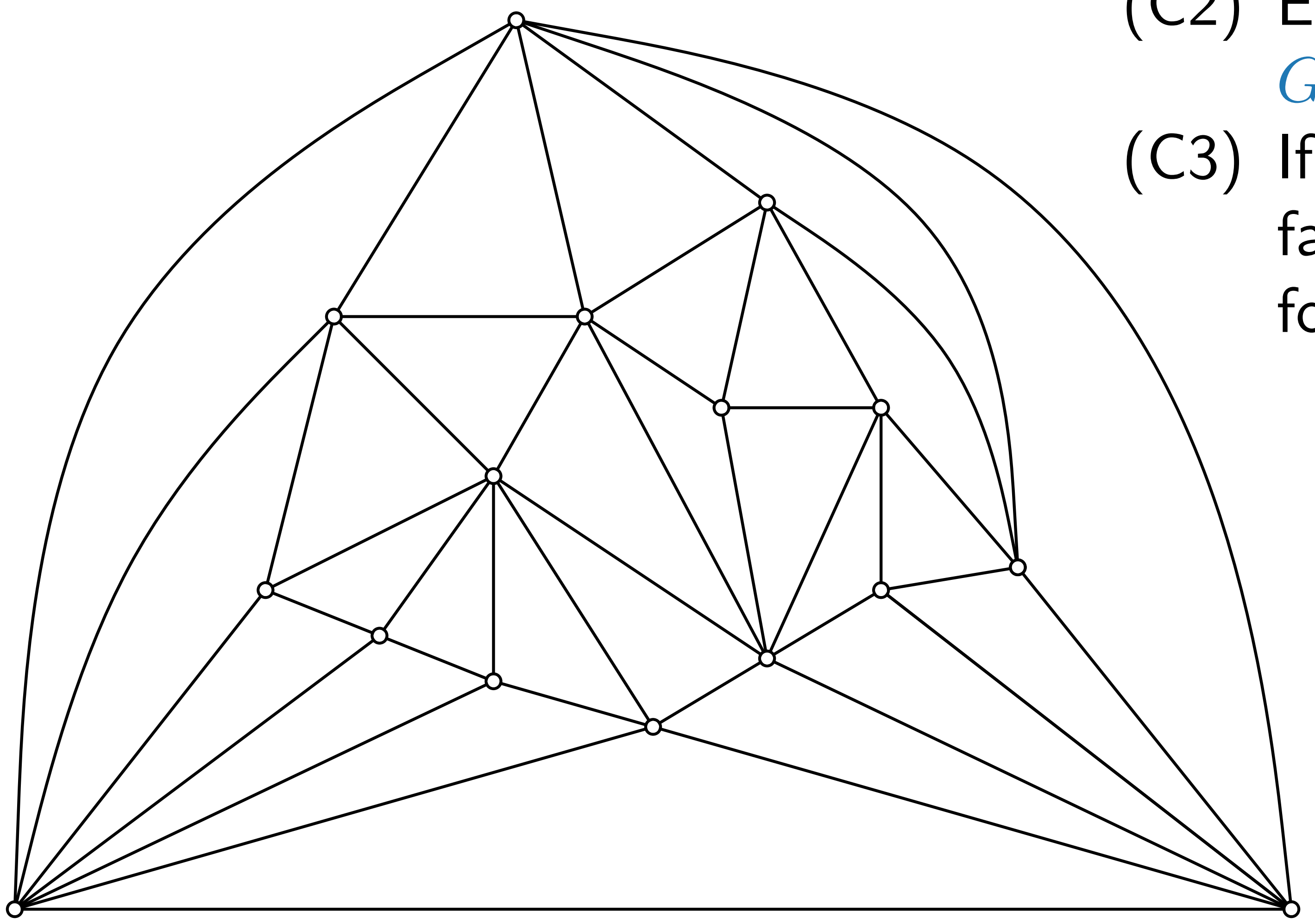

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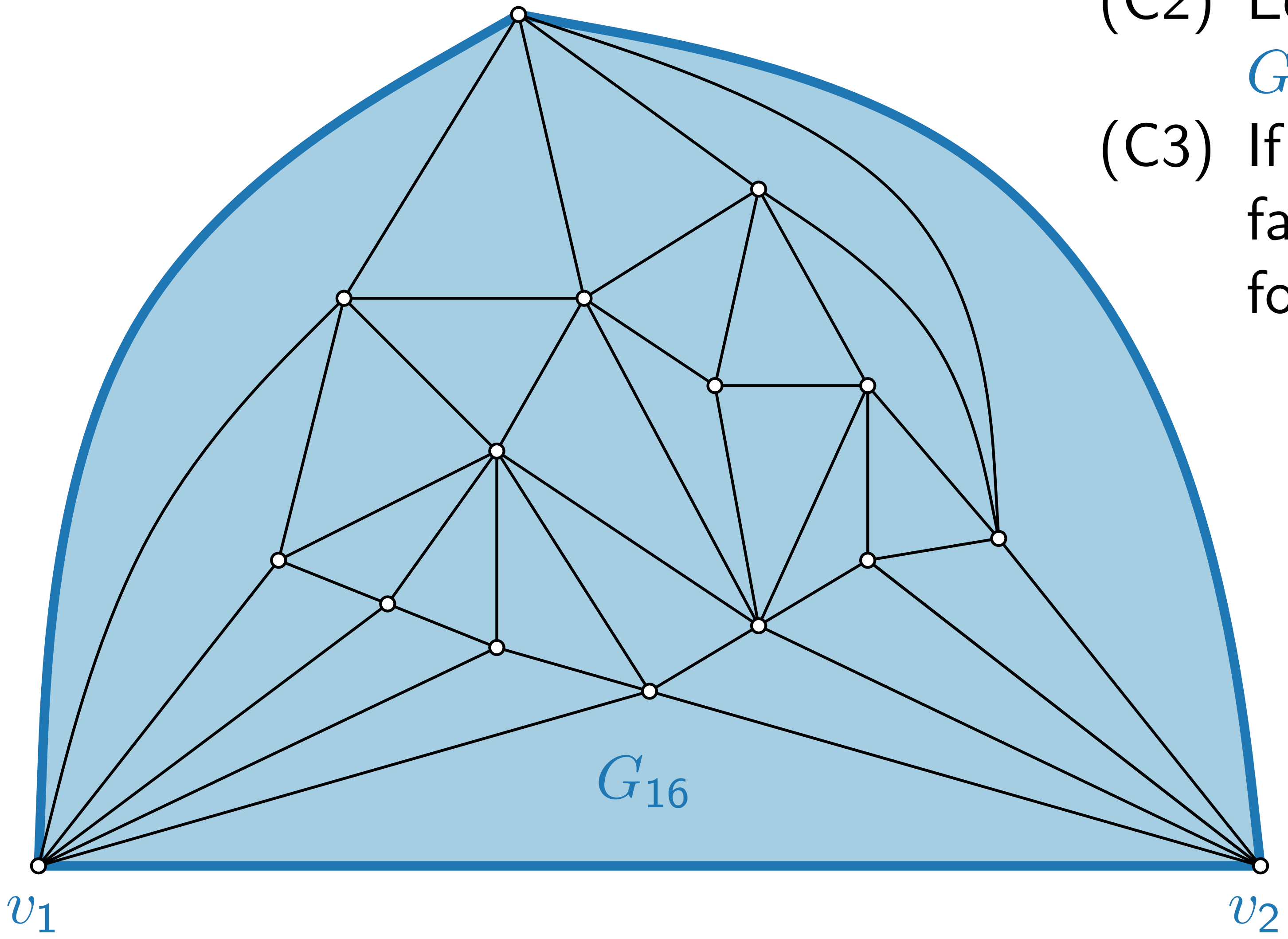

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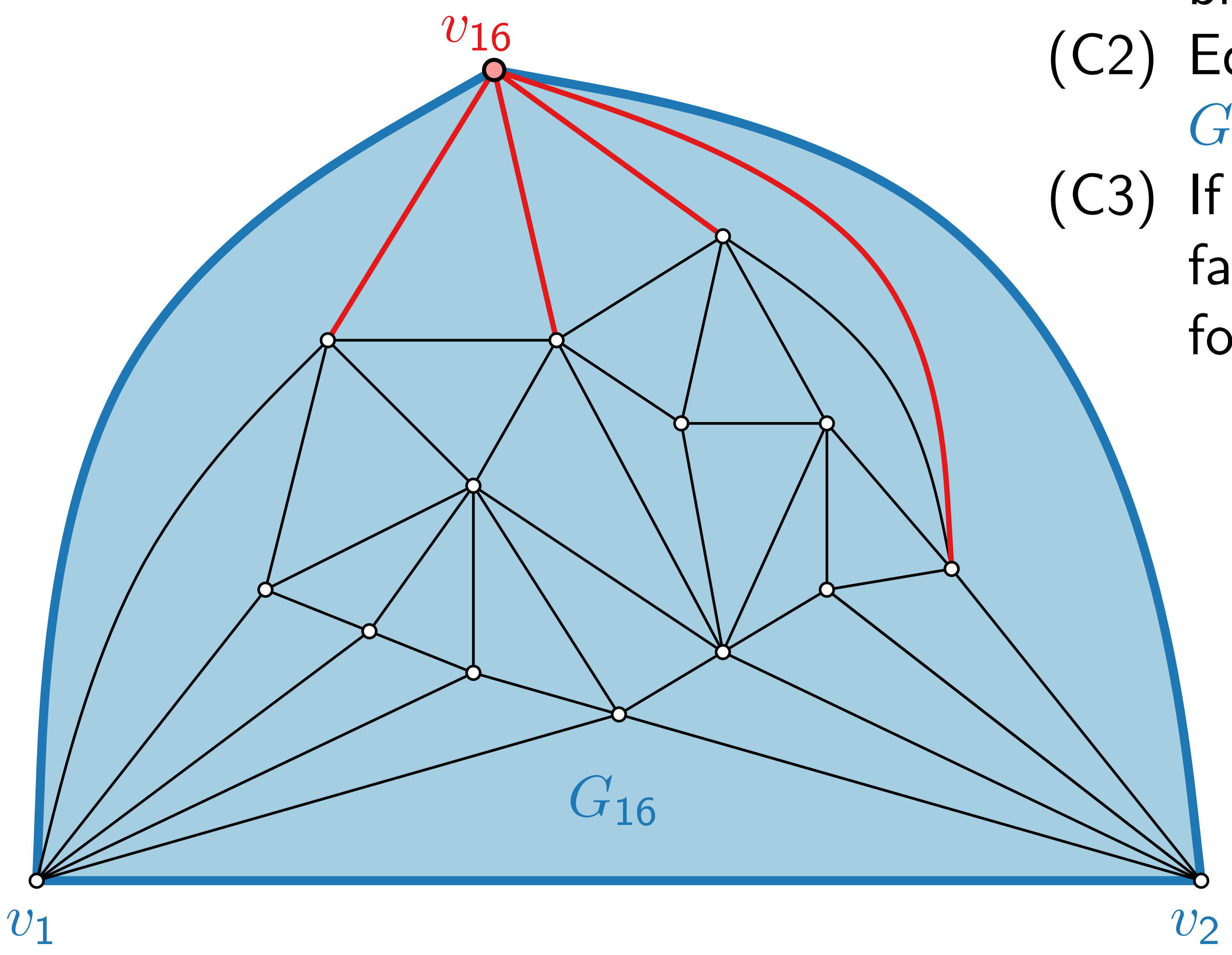

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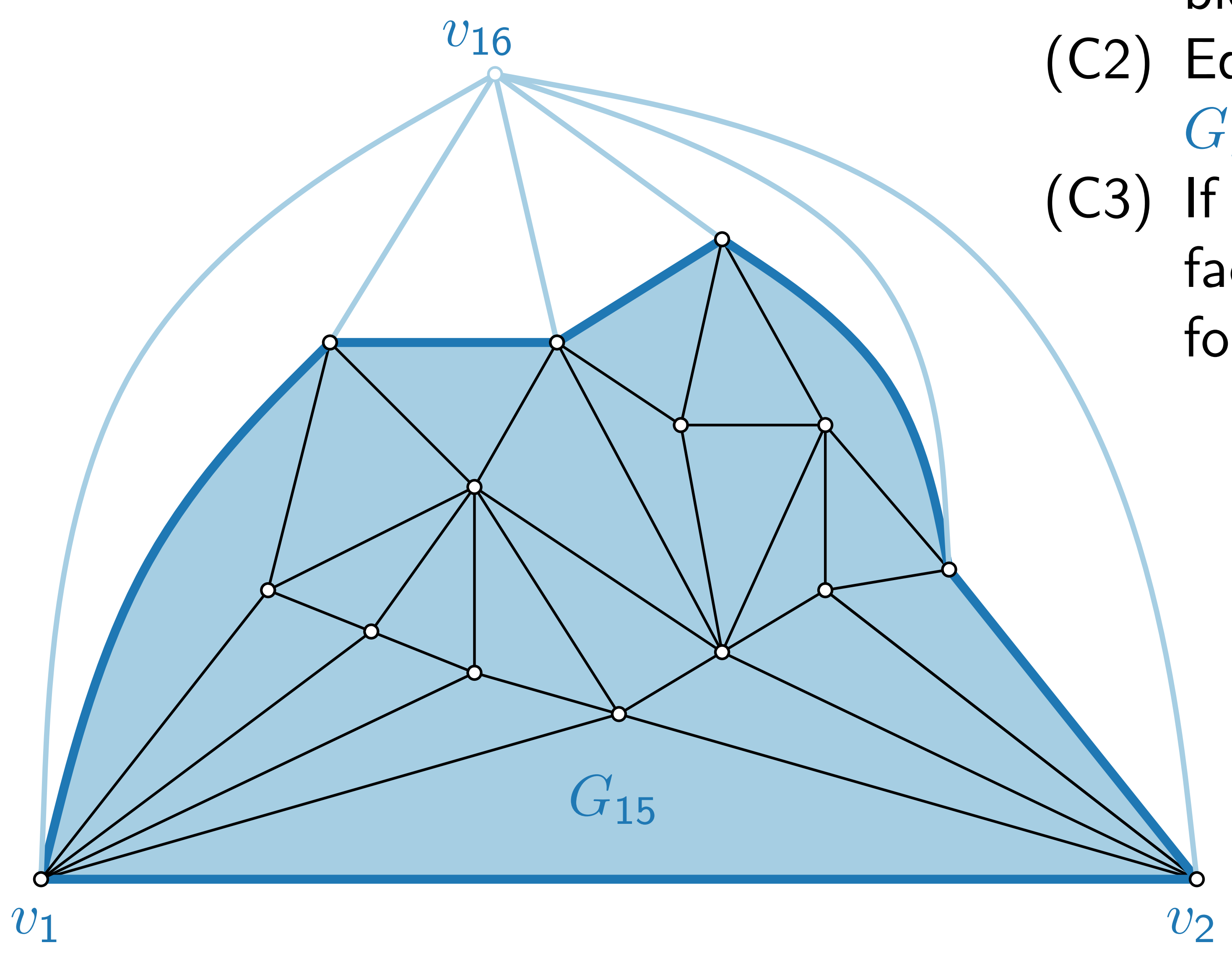

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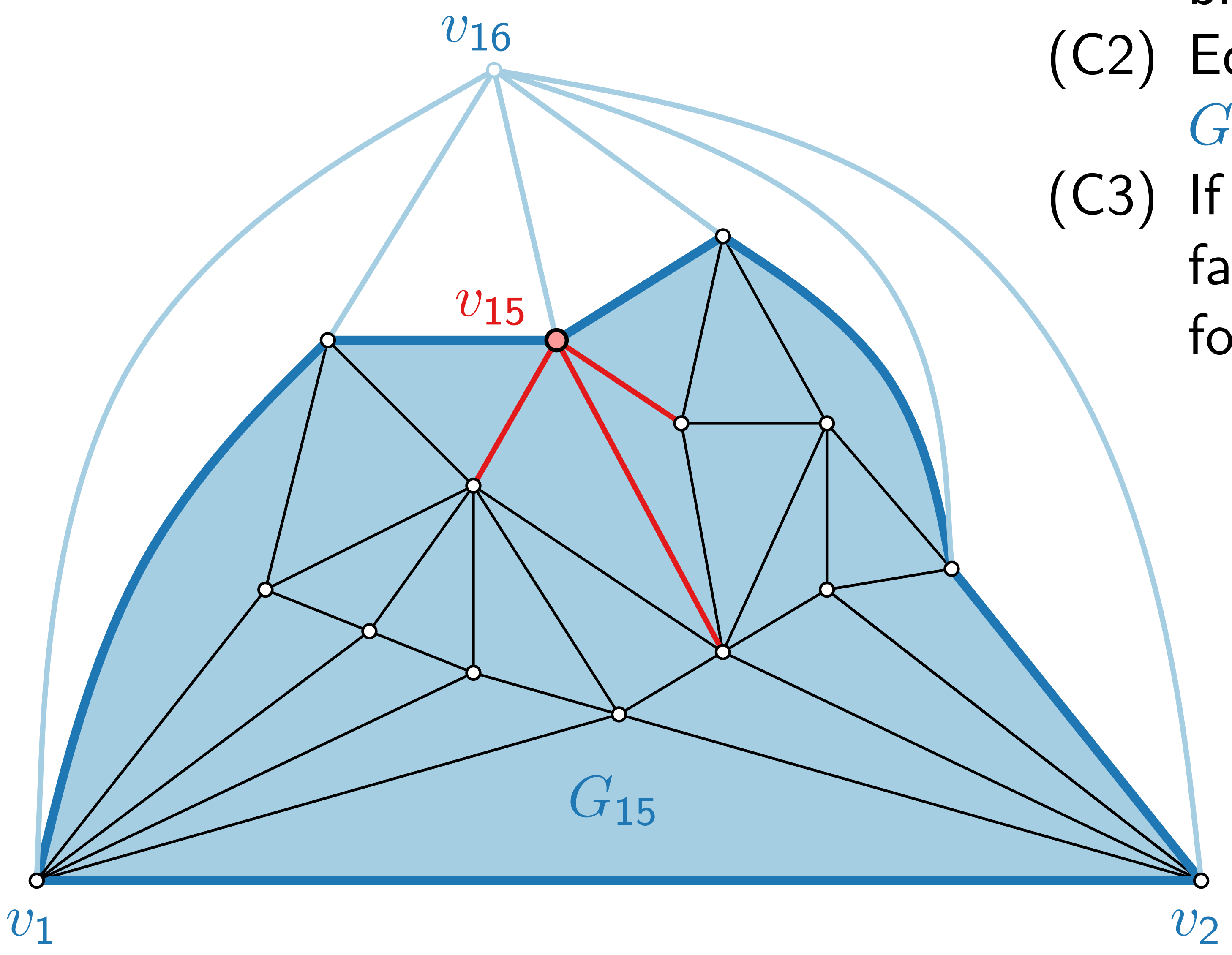

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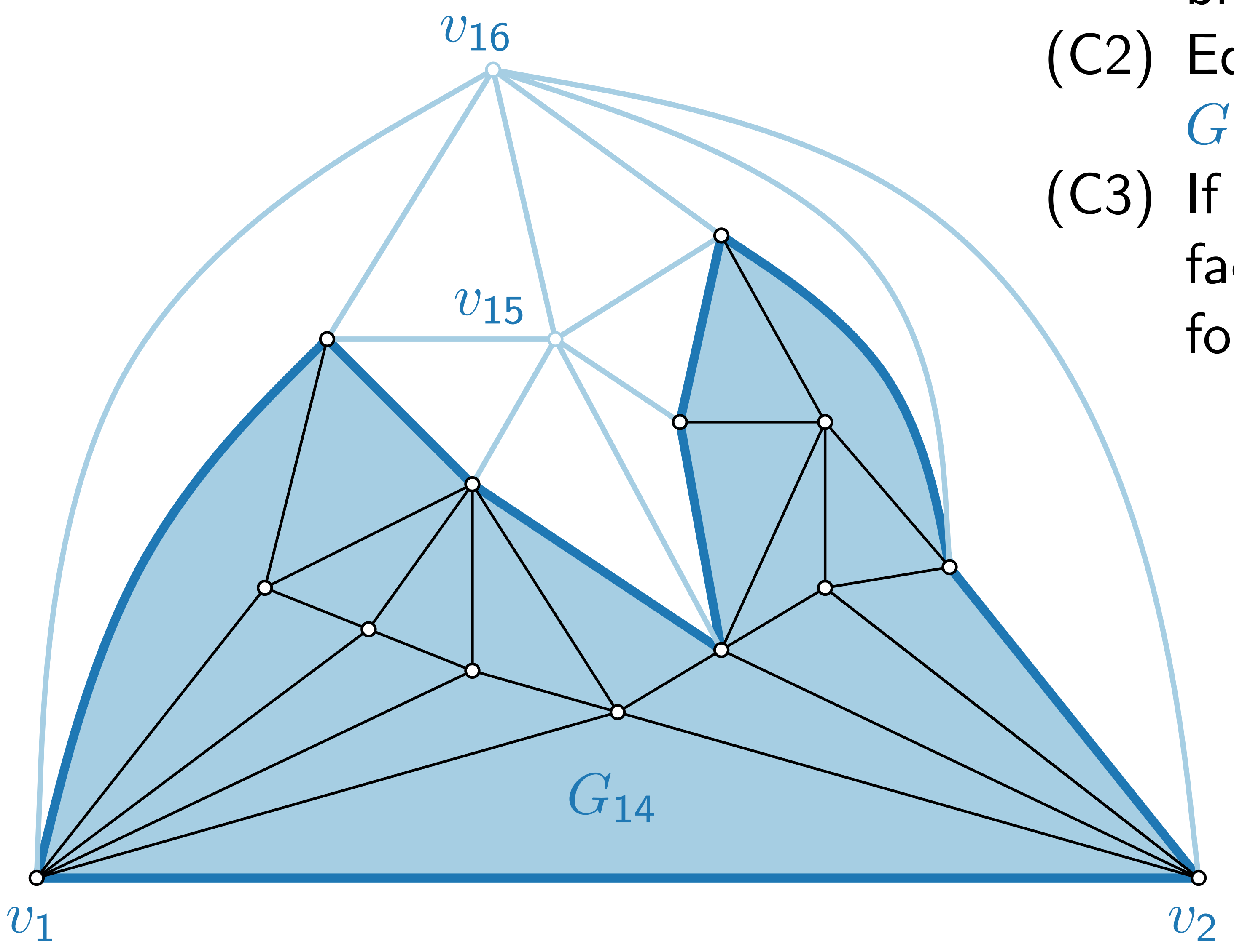

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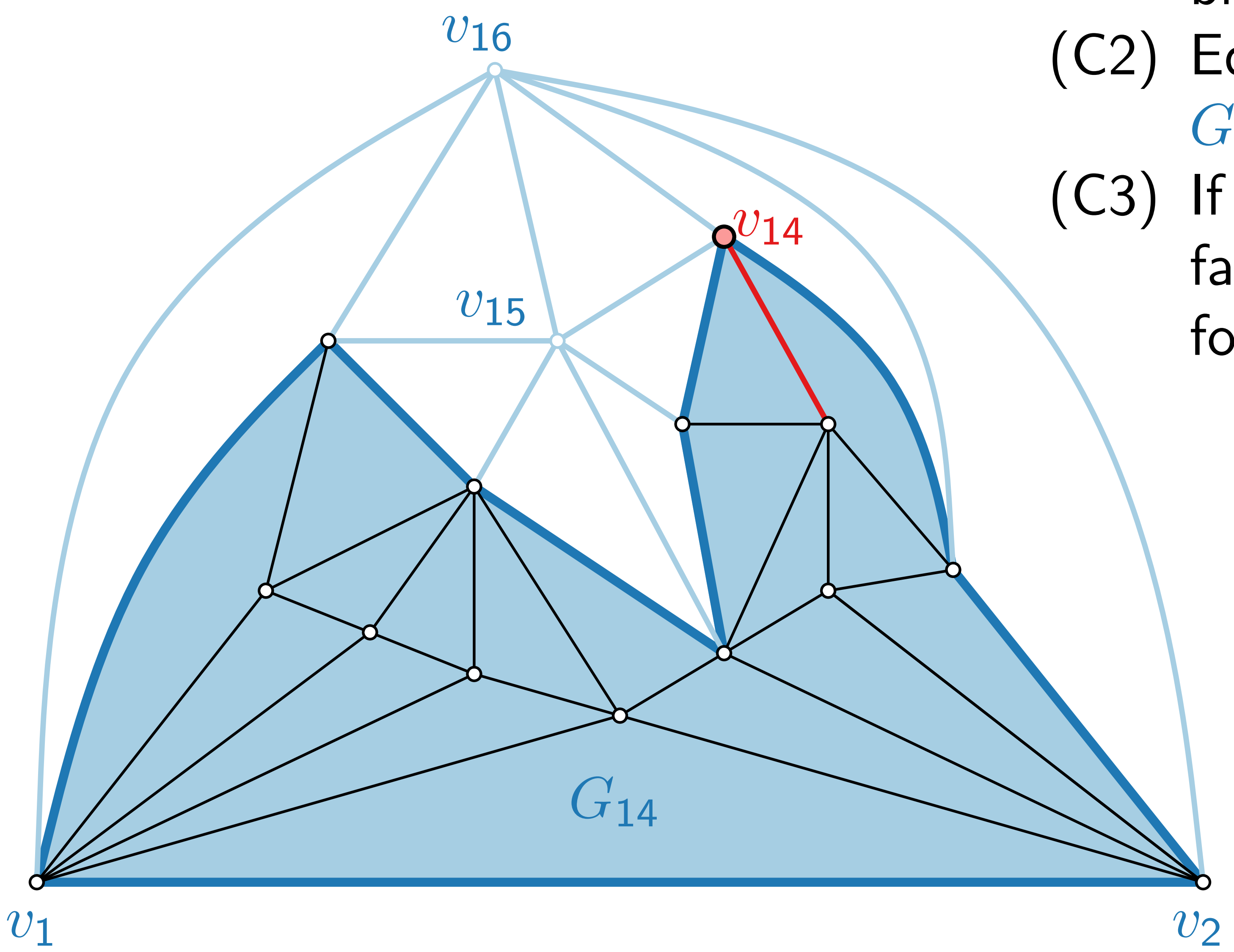

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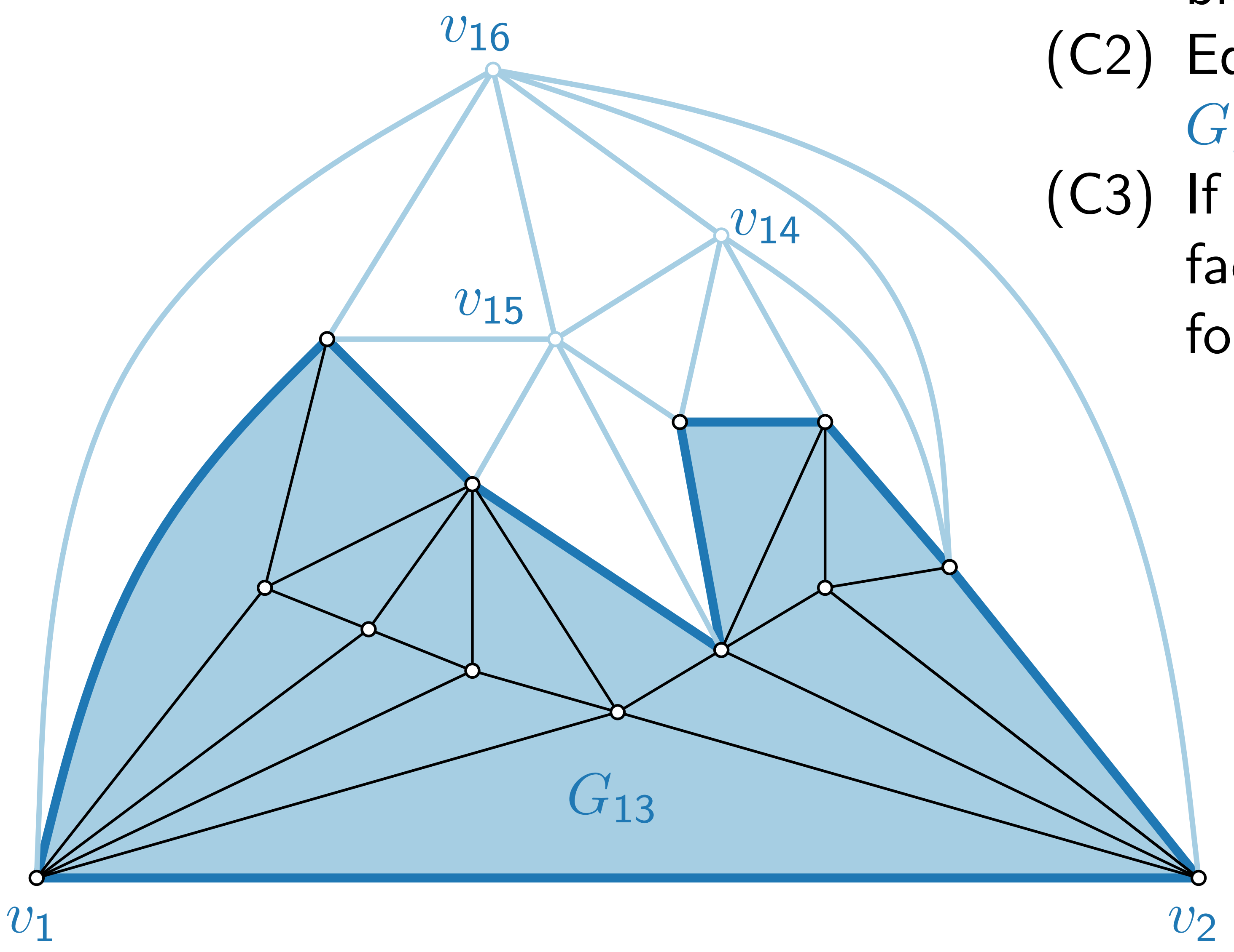

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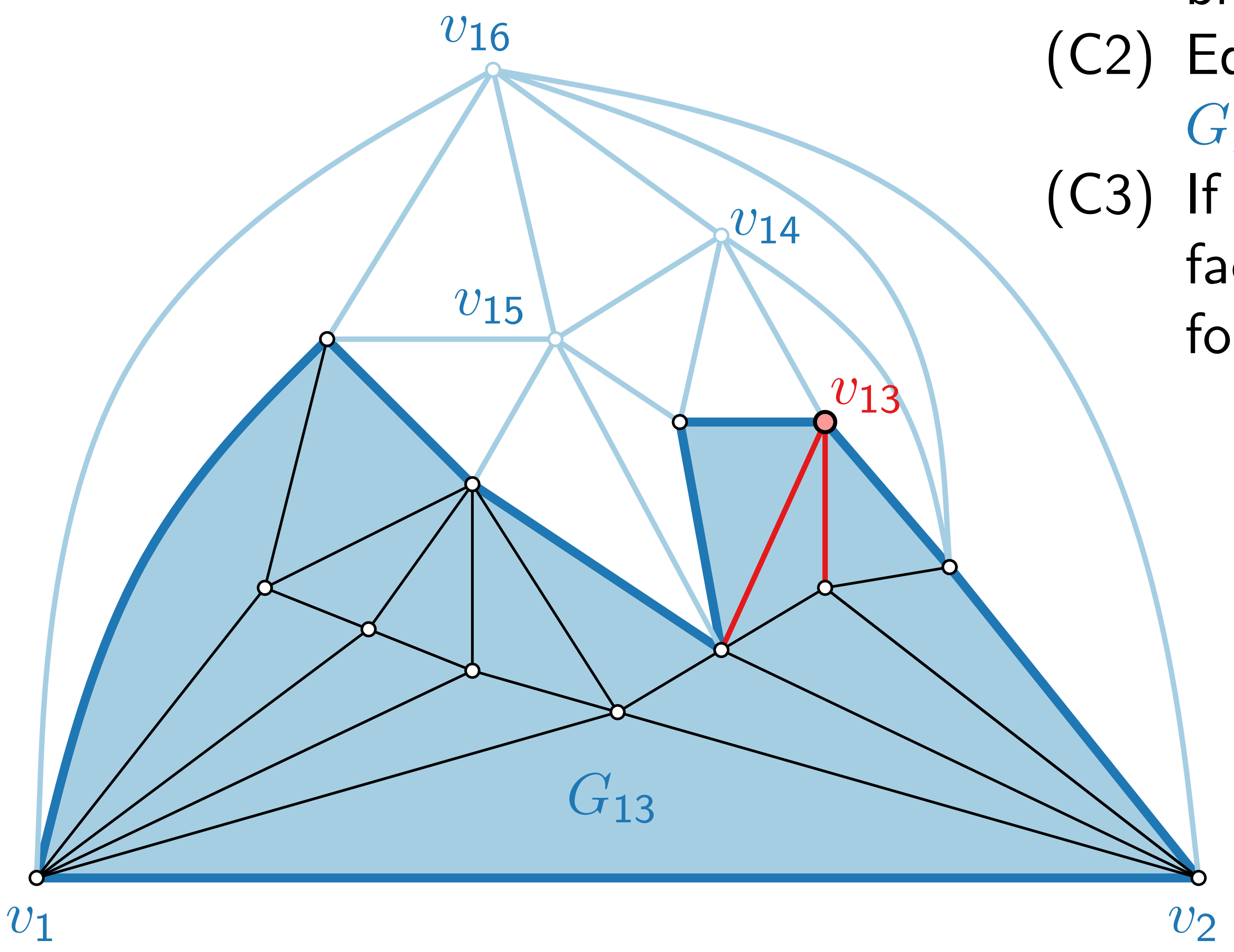

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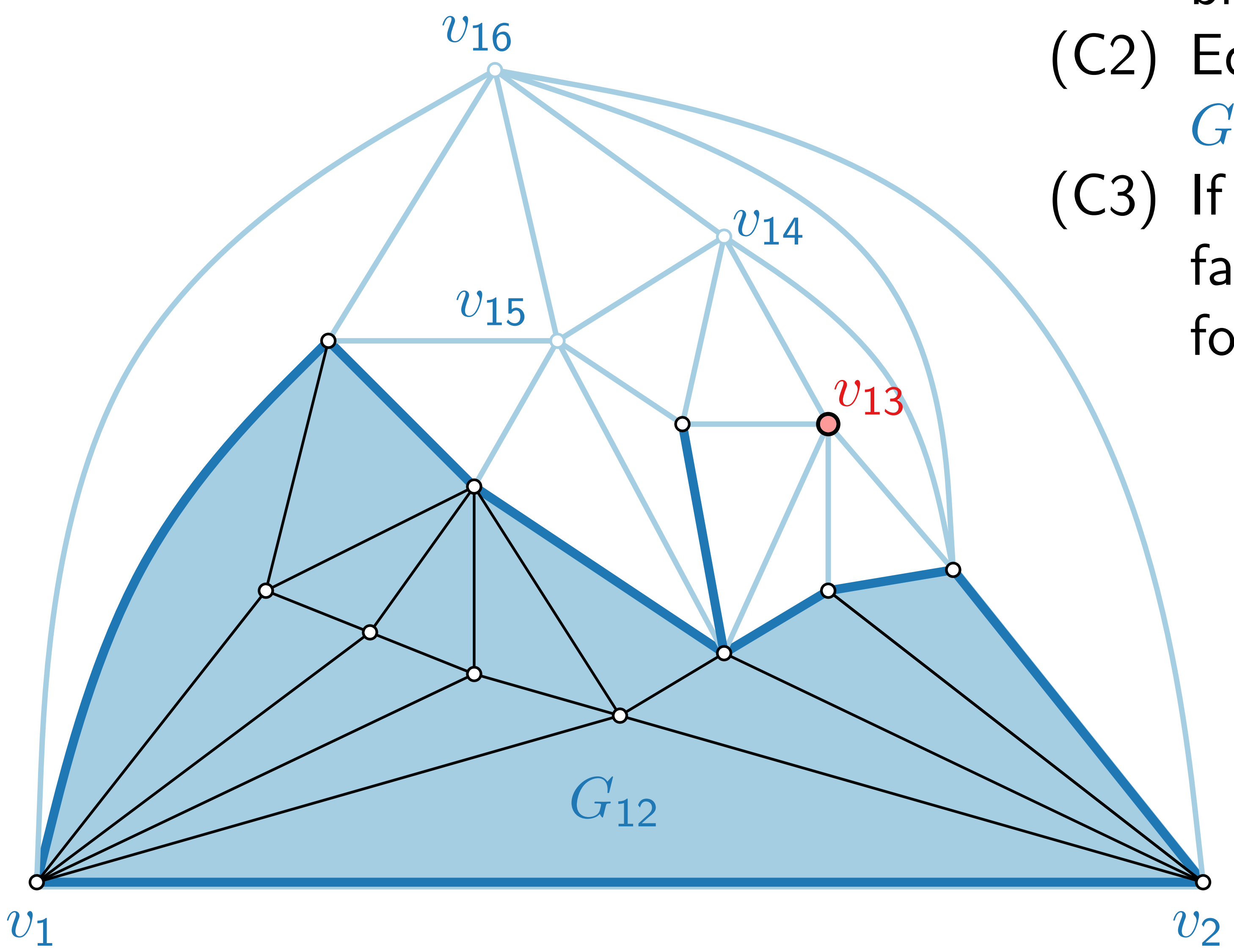

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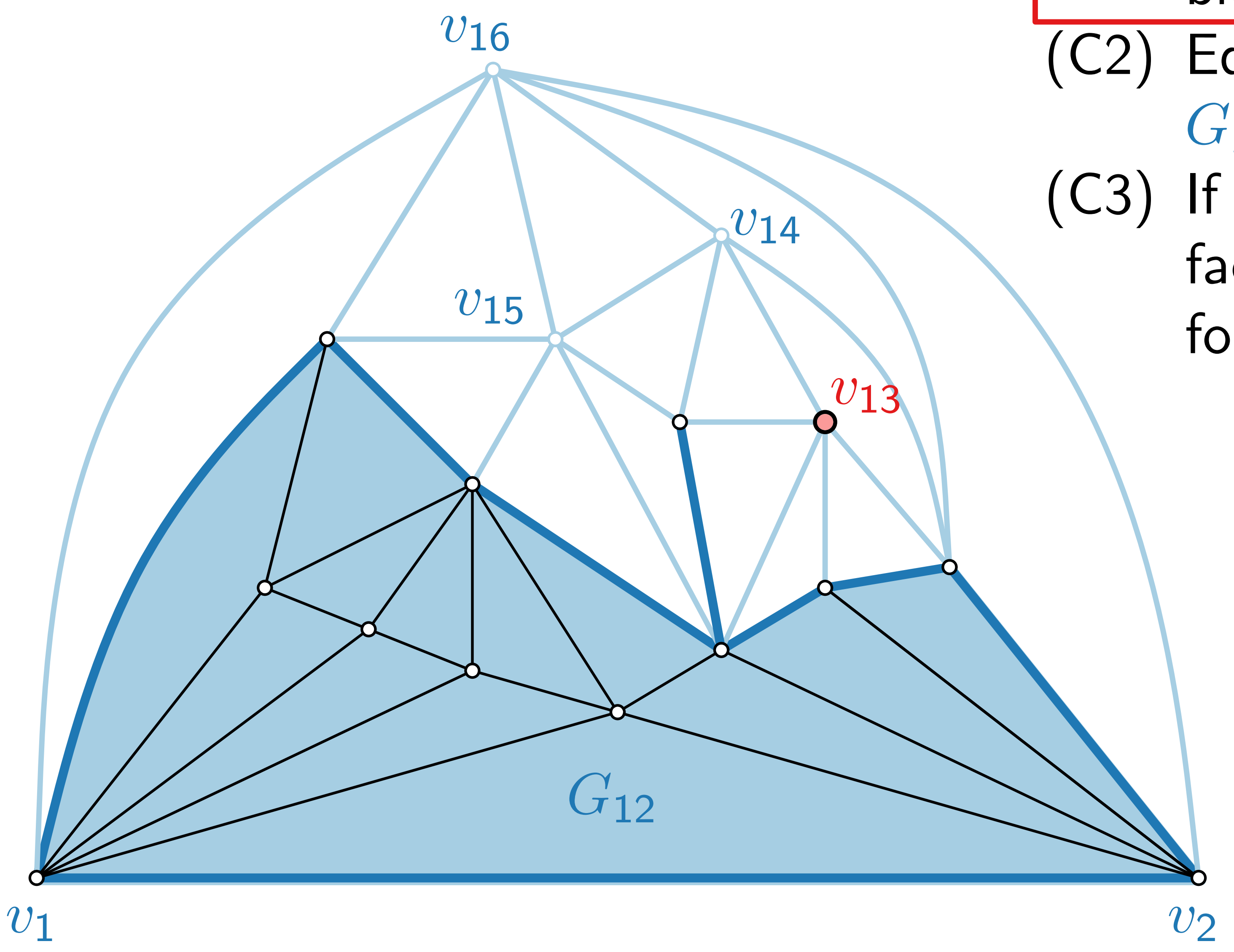

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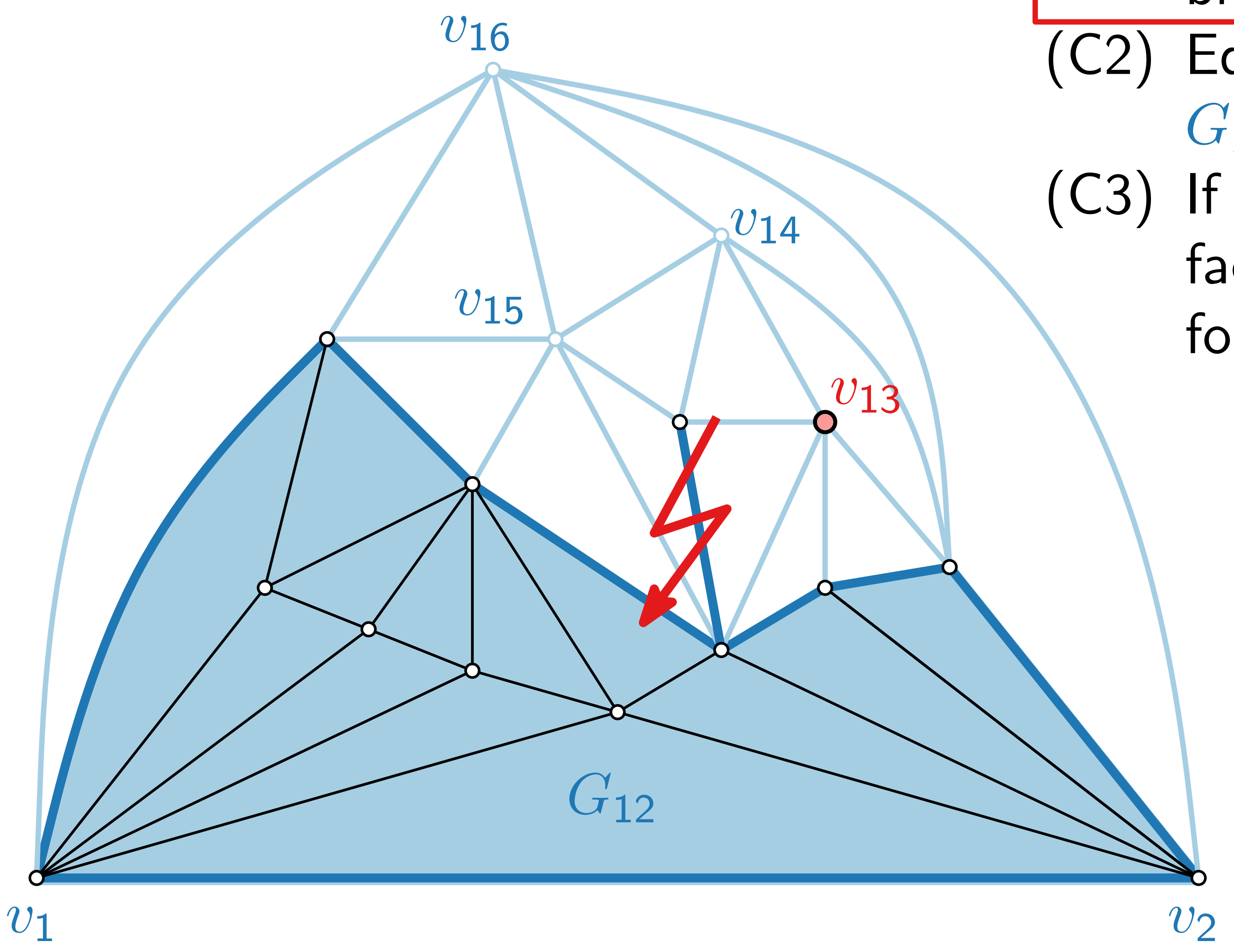

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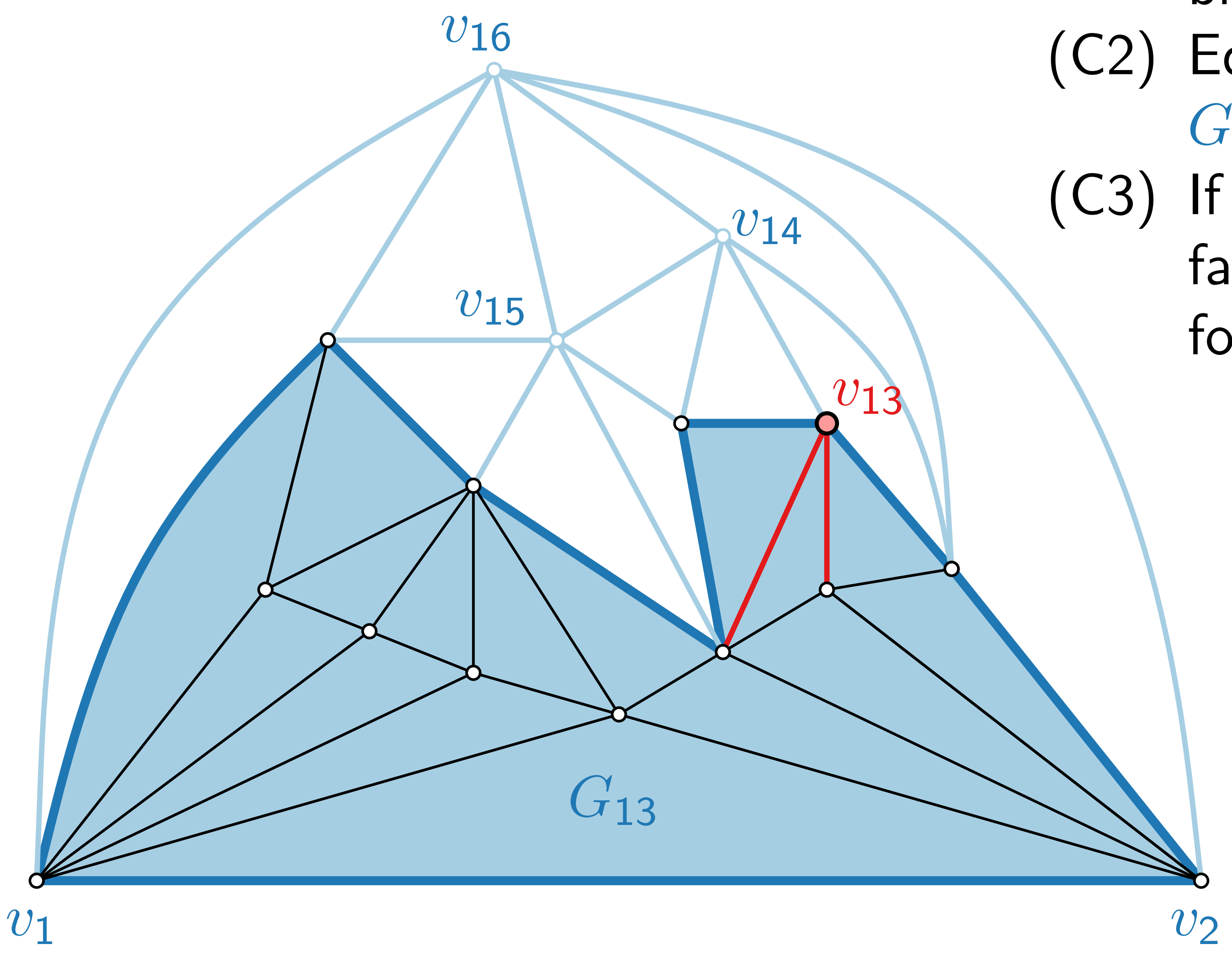

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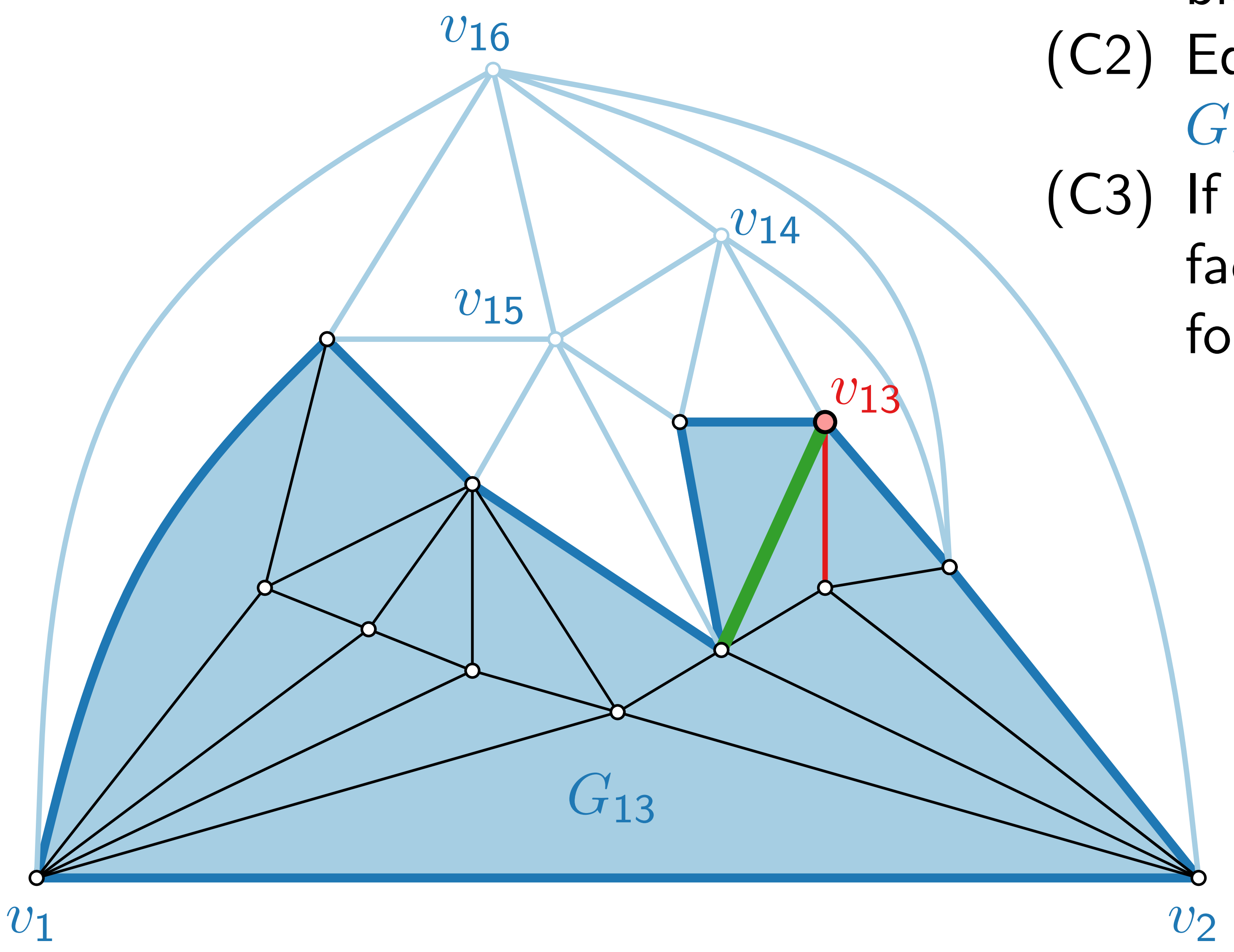

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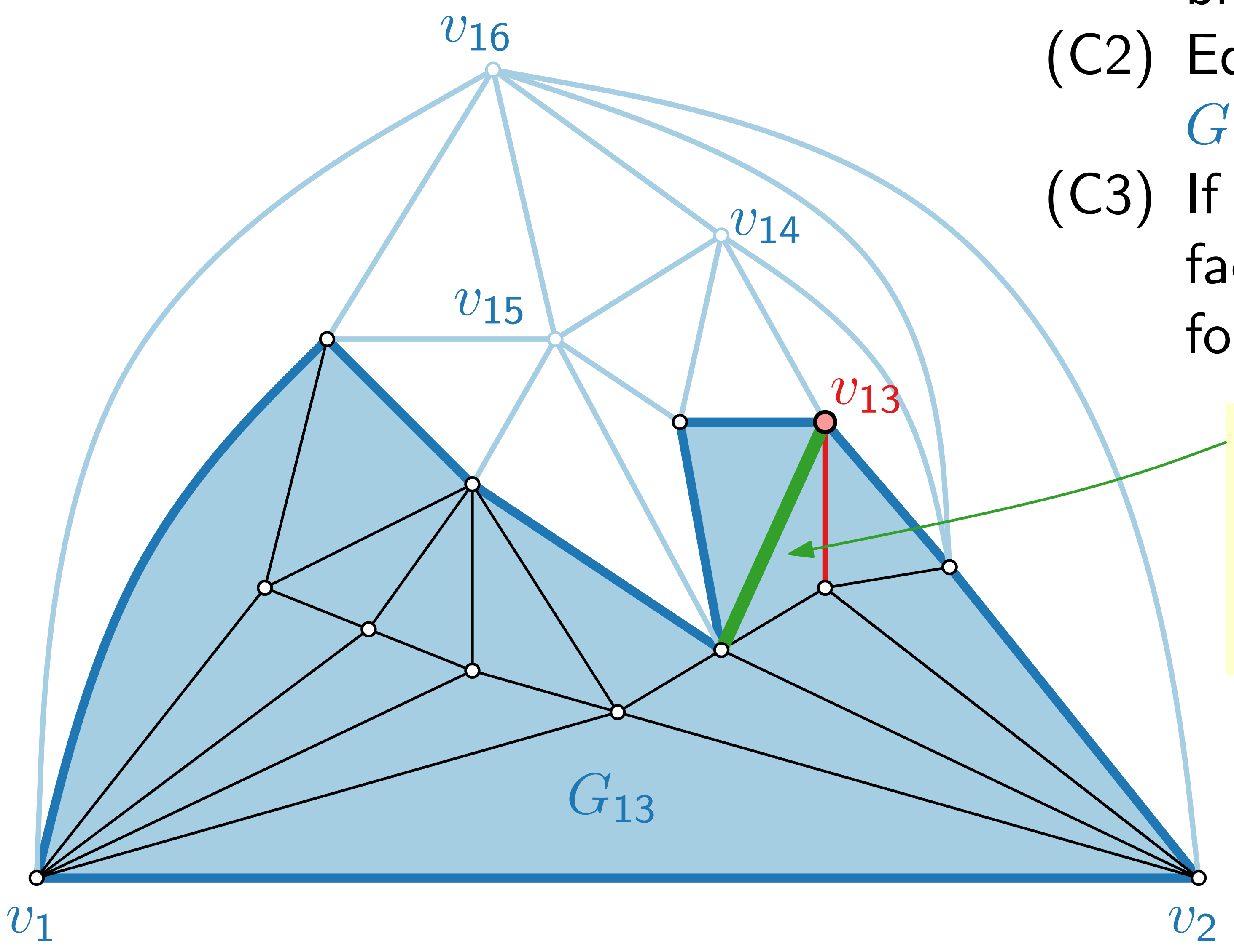

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chord:

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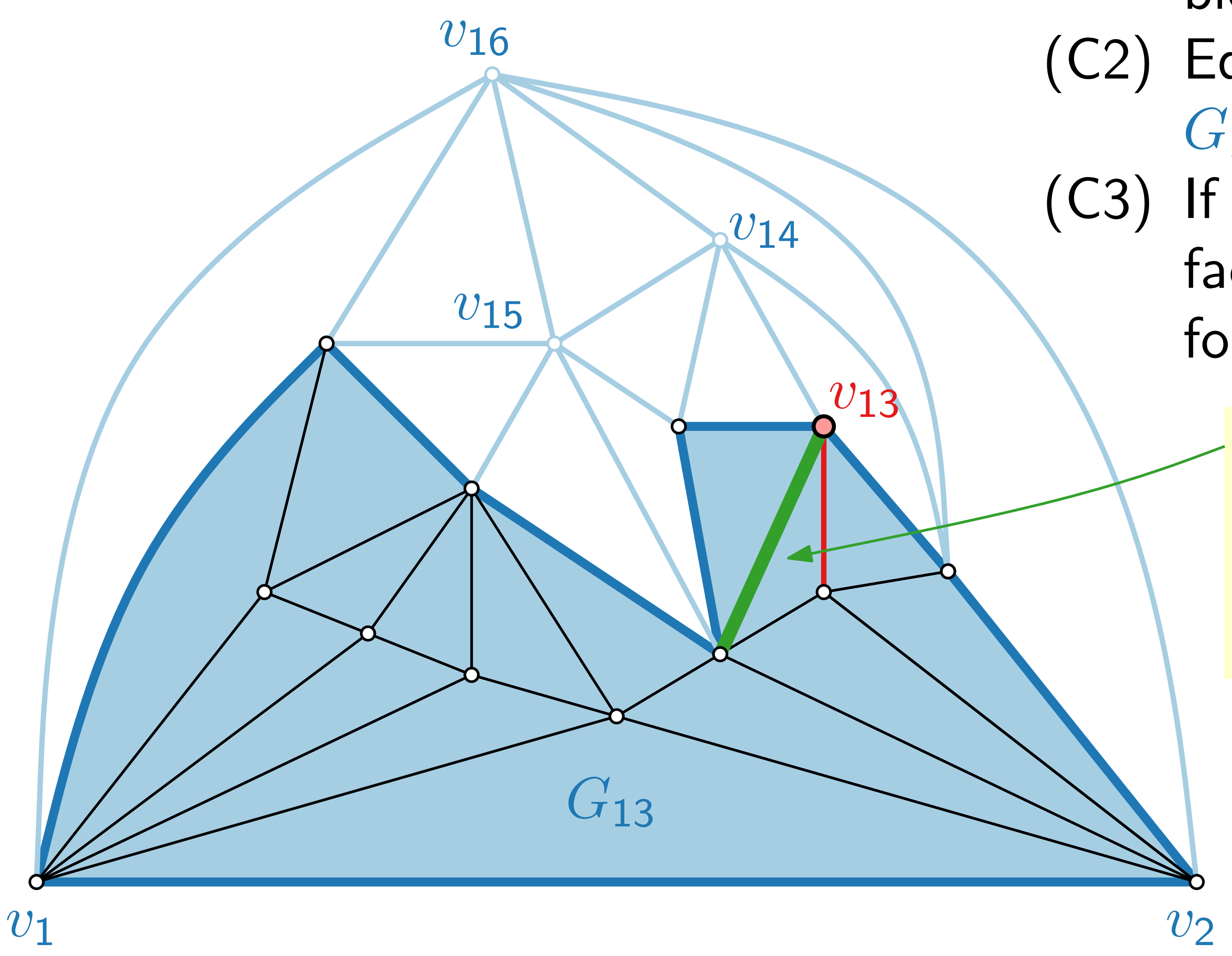

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chord:
edge joining two non-adjacent vertices in a cycle

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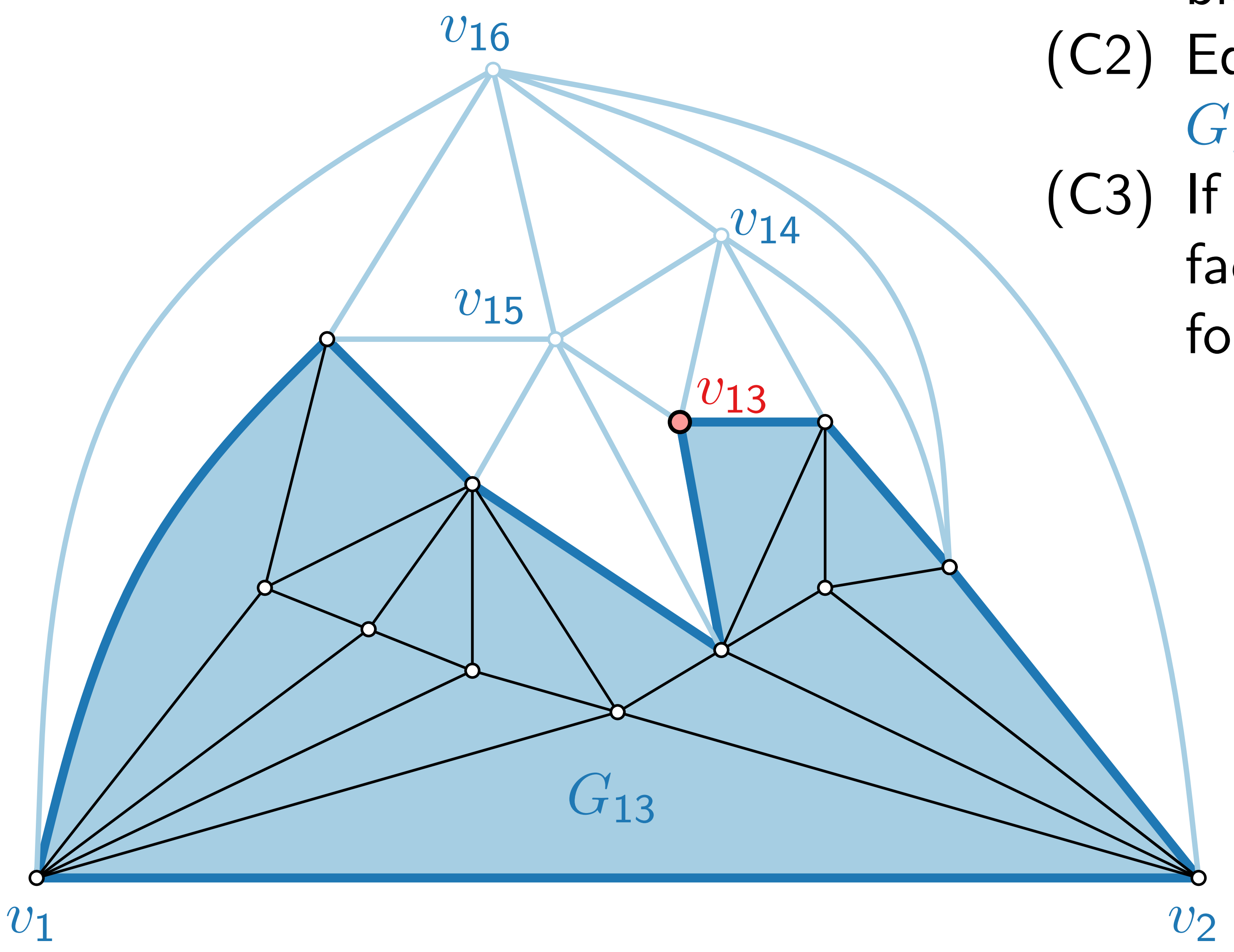

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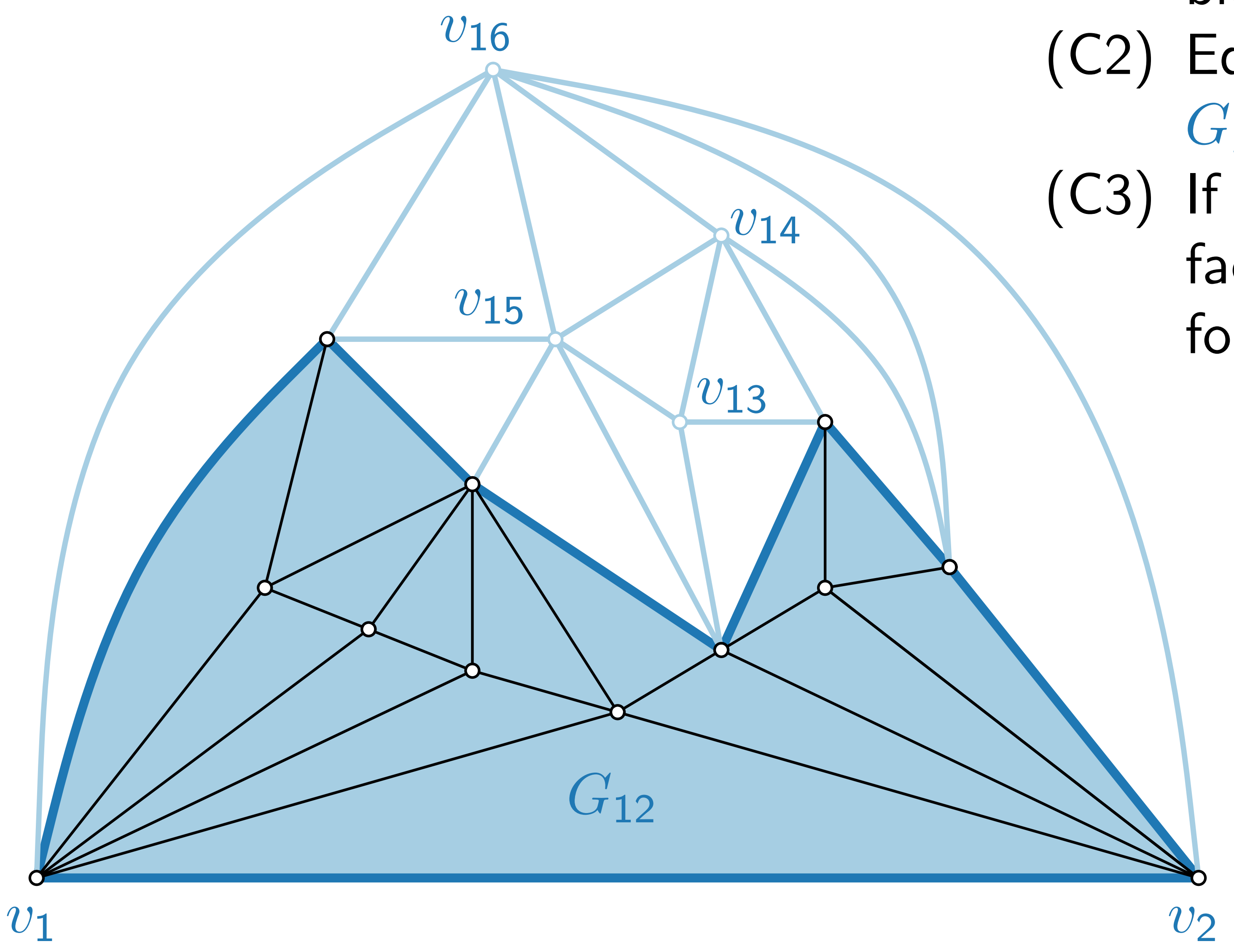

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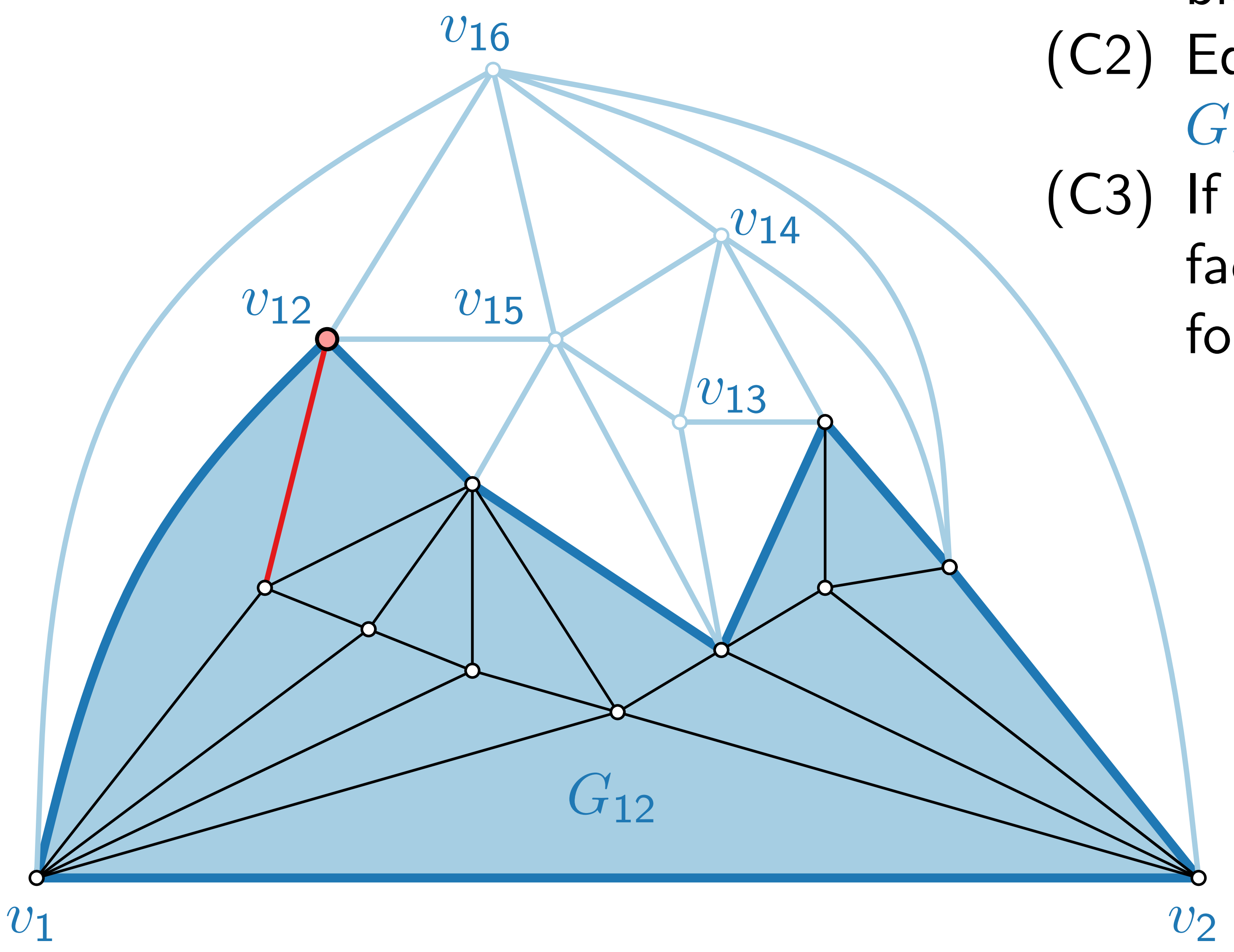

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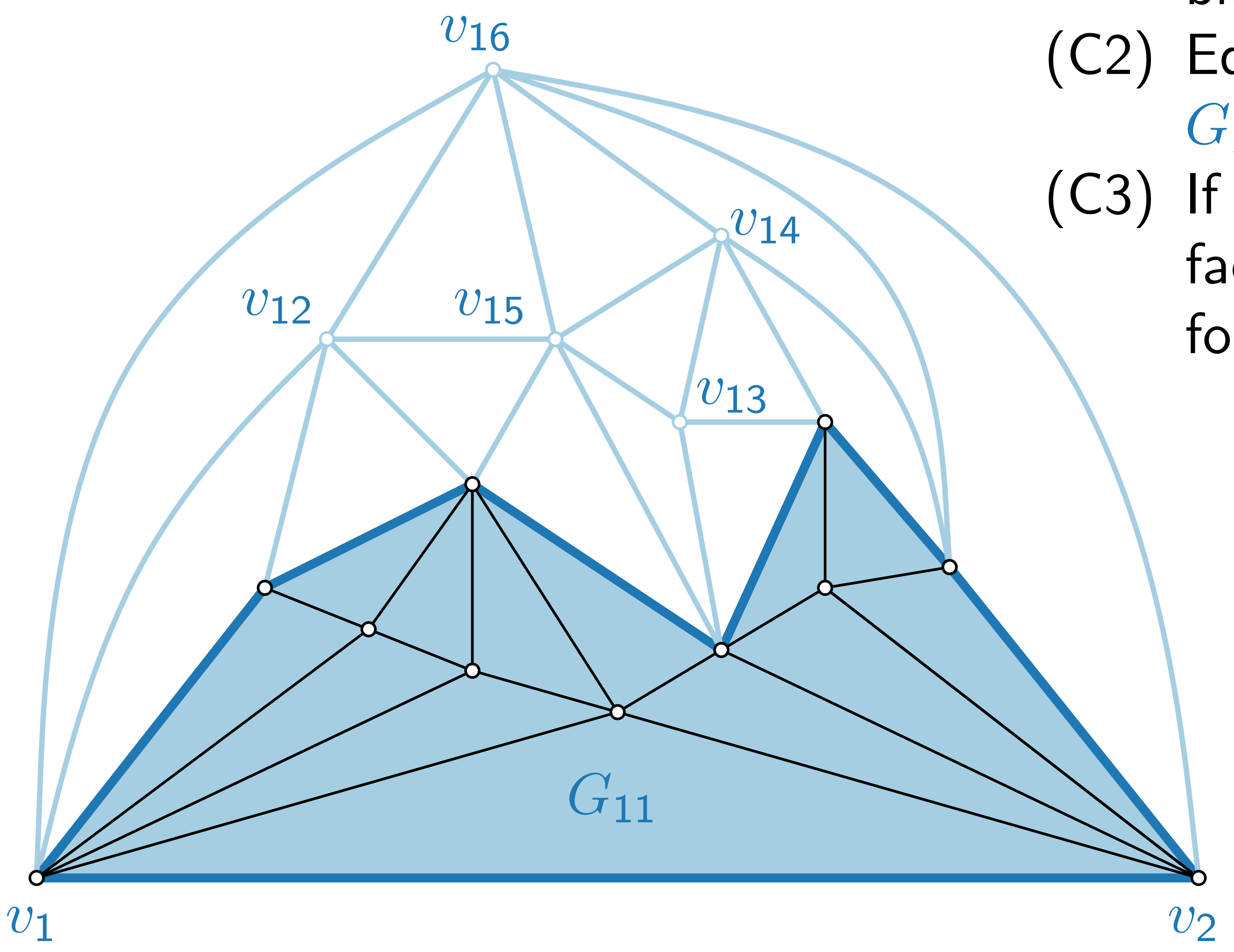

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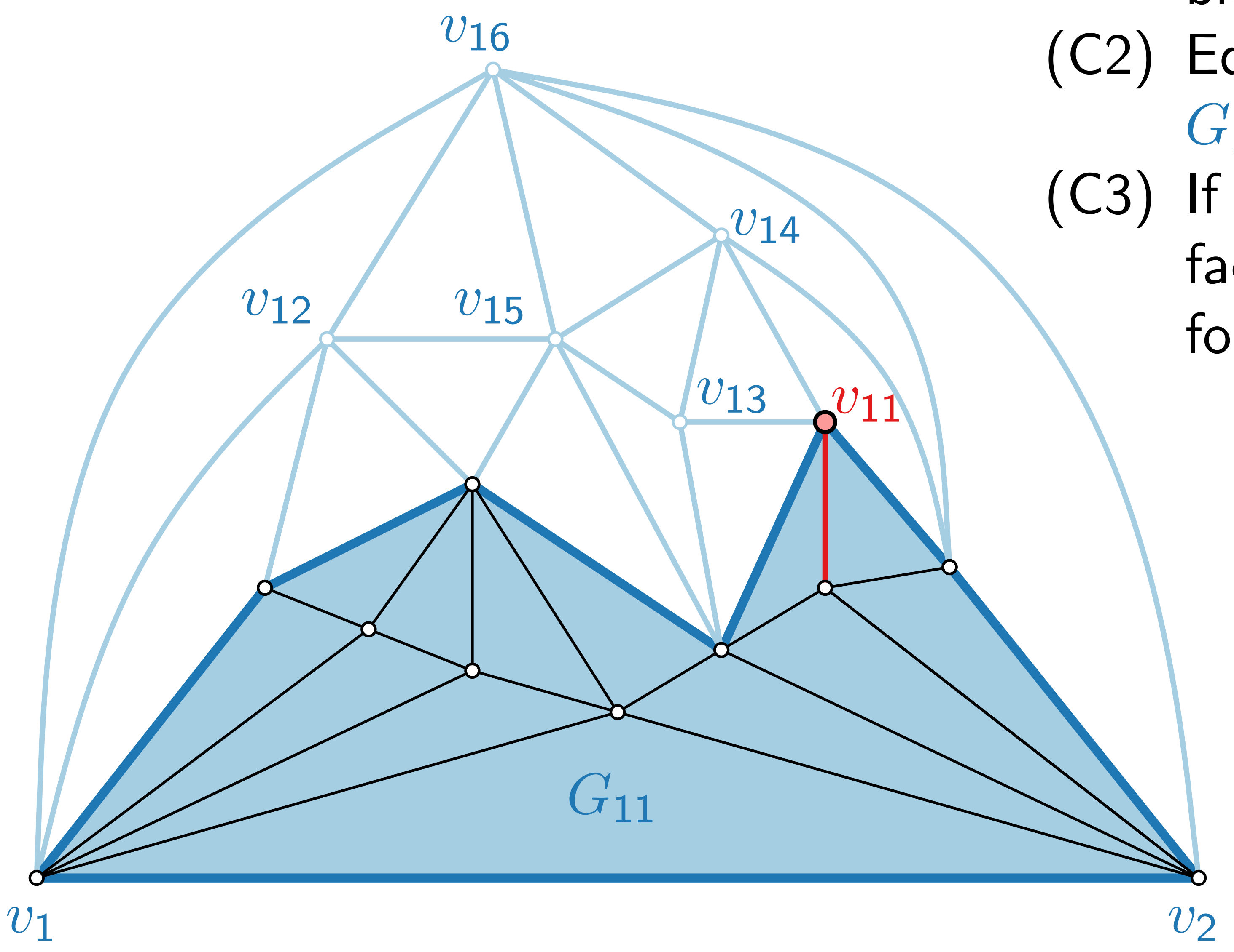

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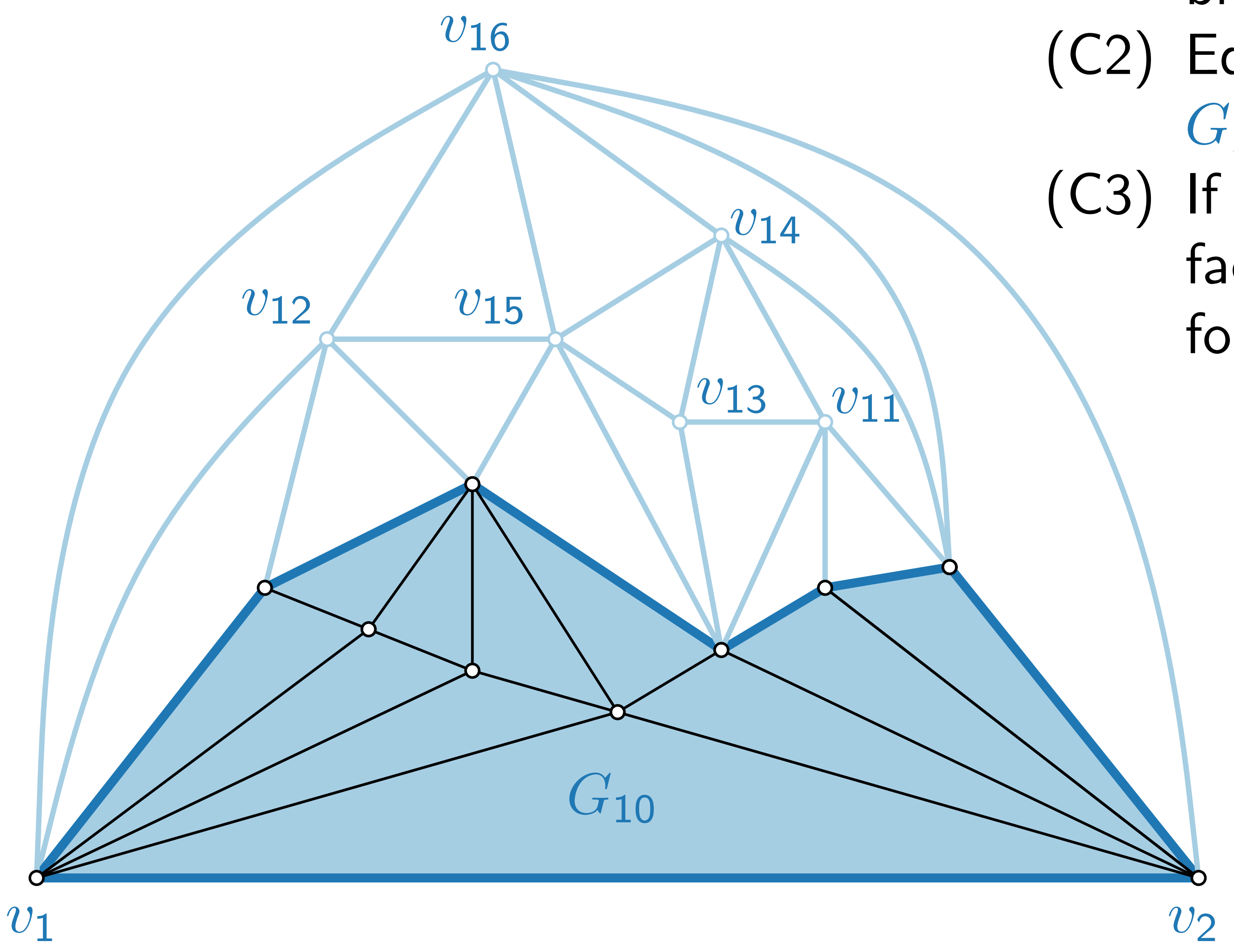

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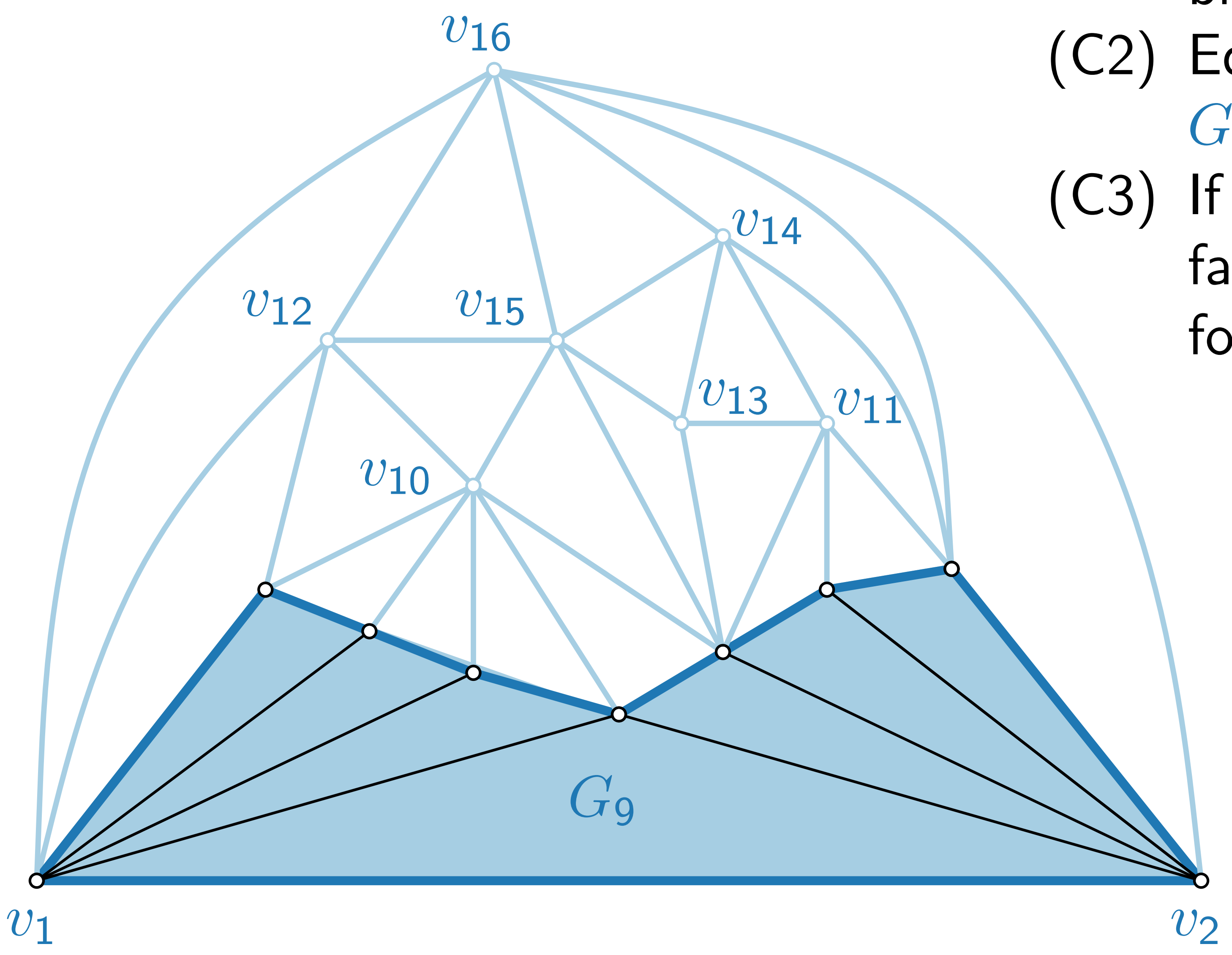

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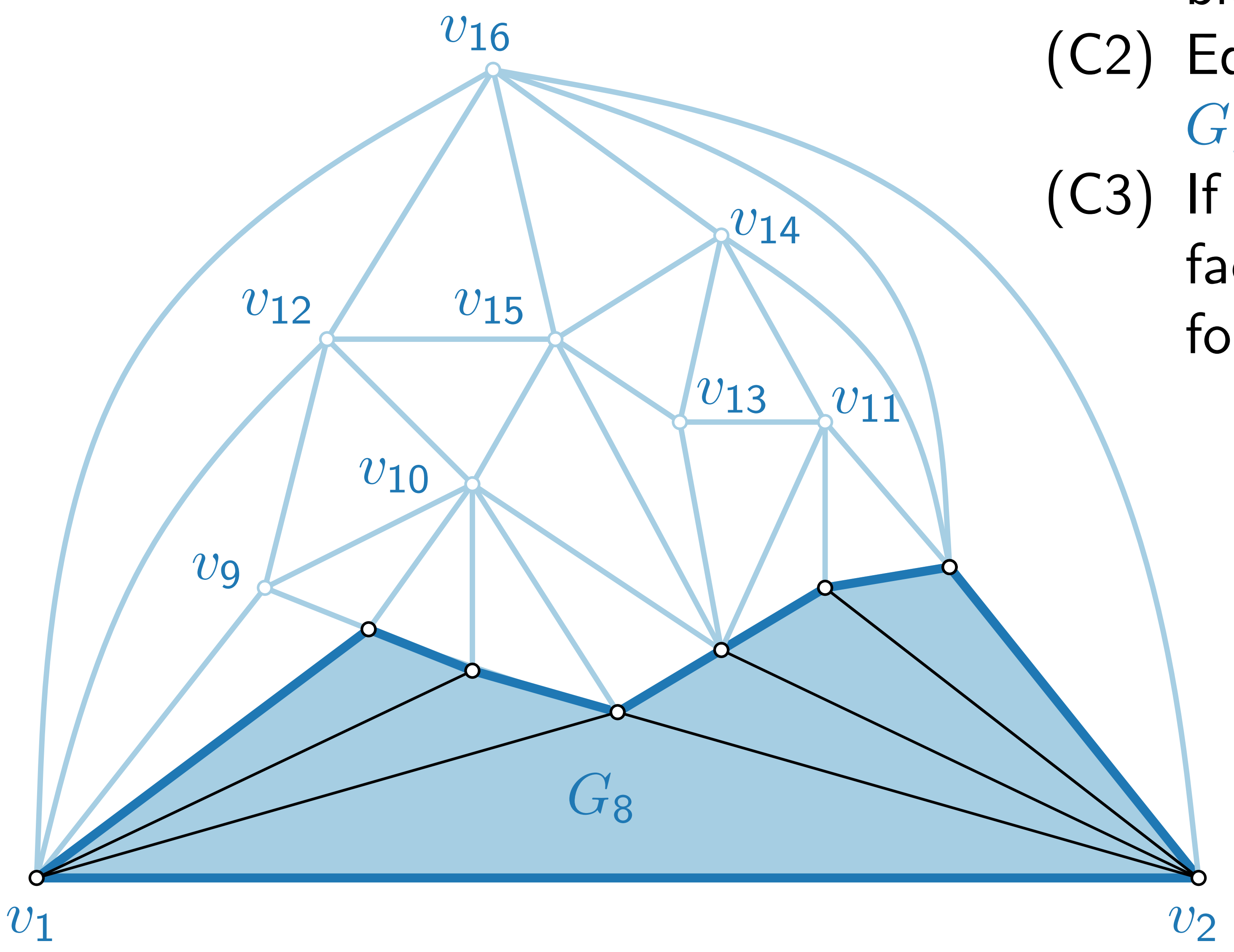

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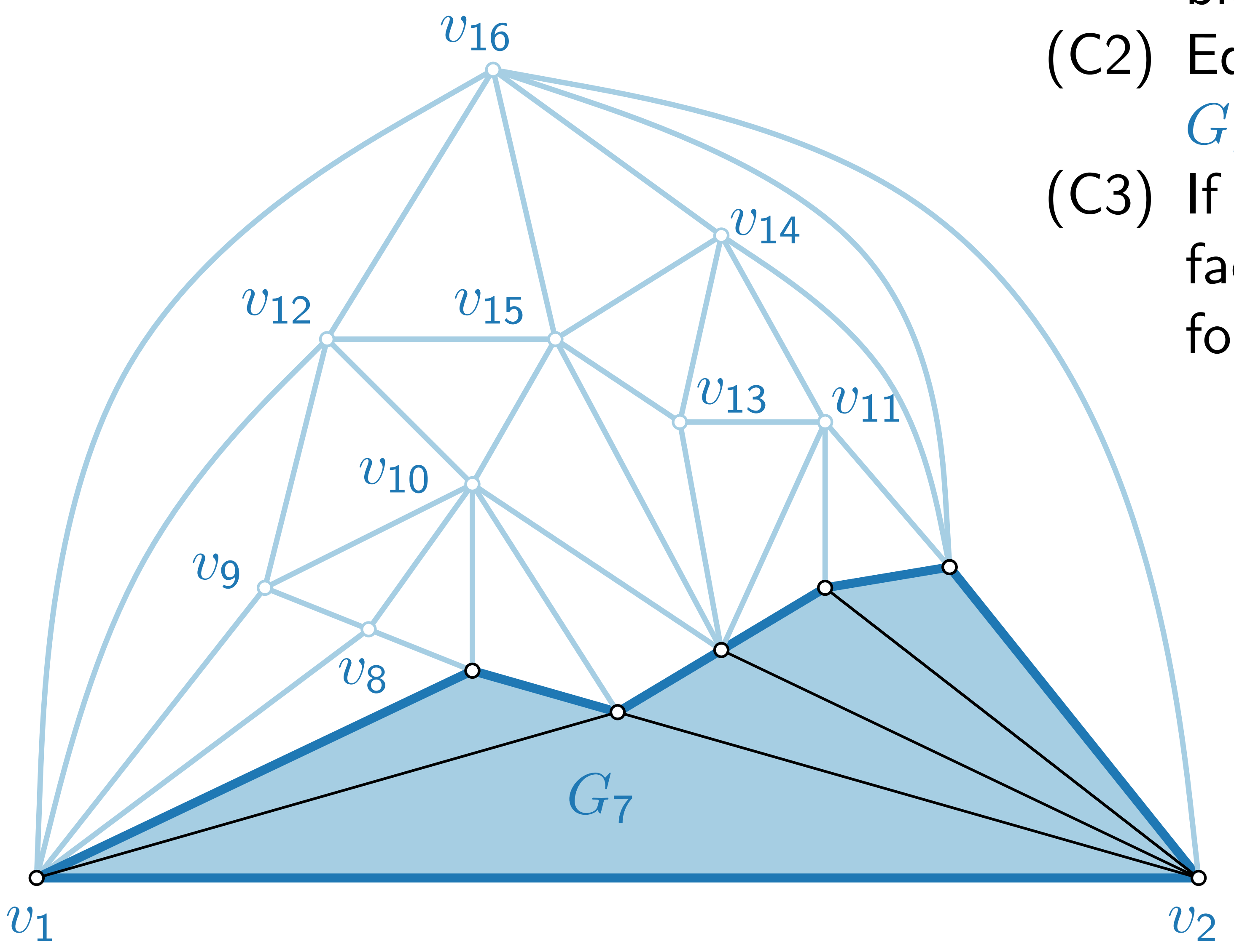

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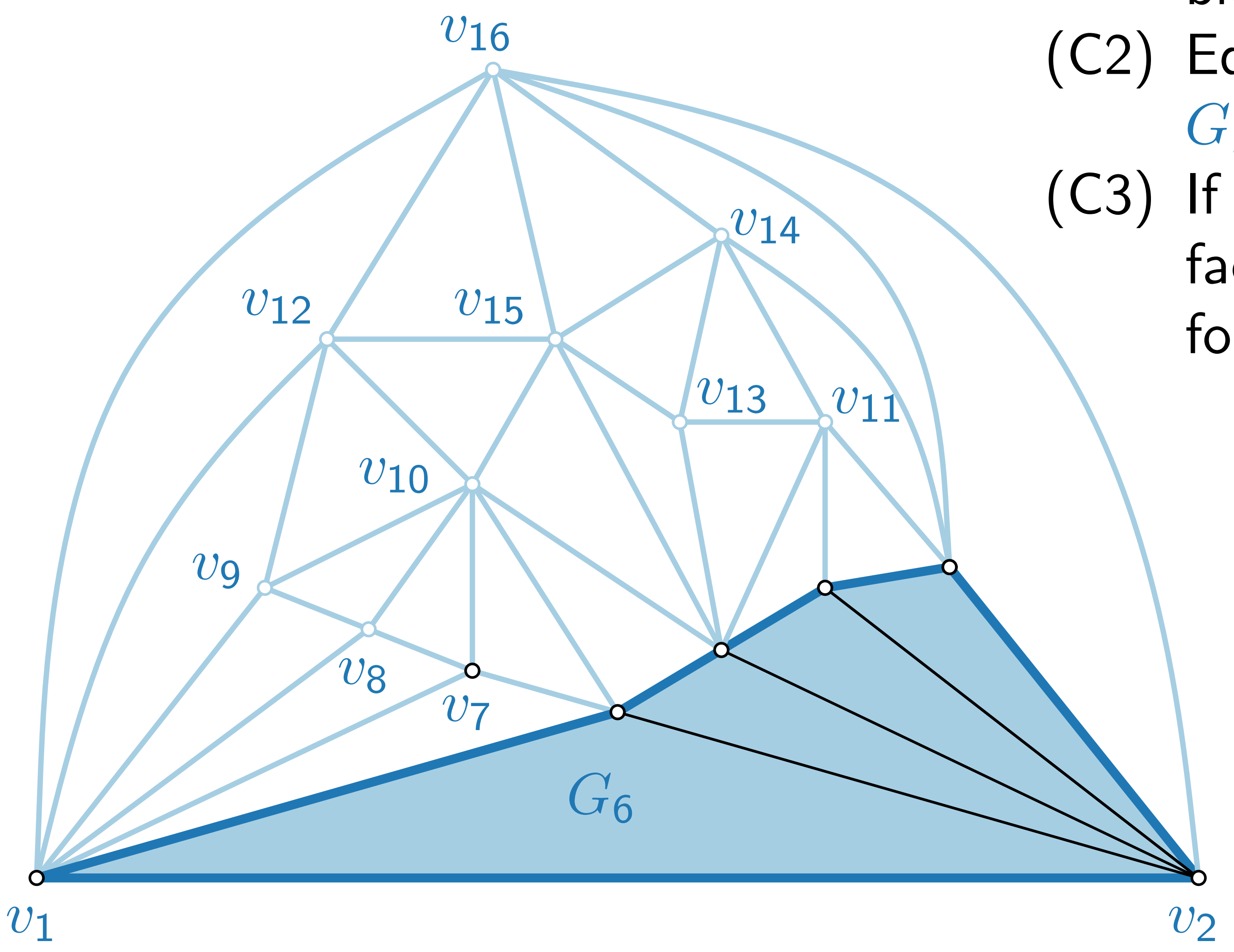

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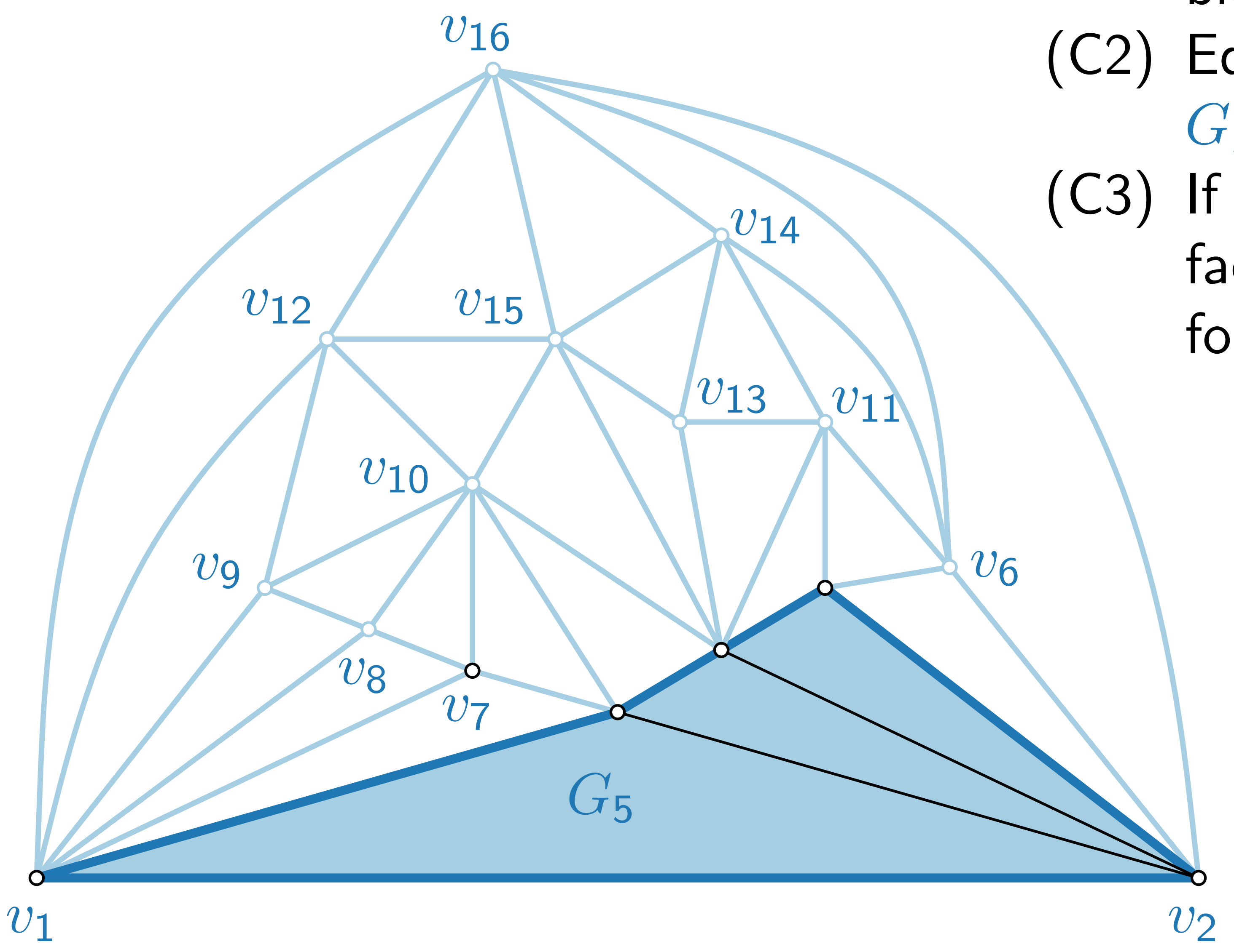

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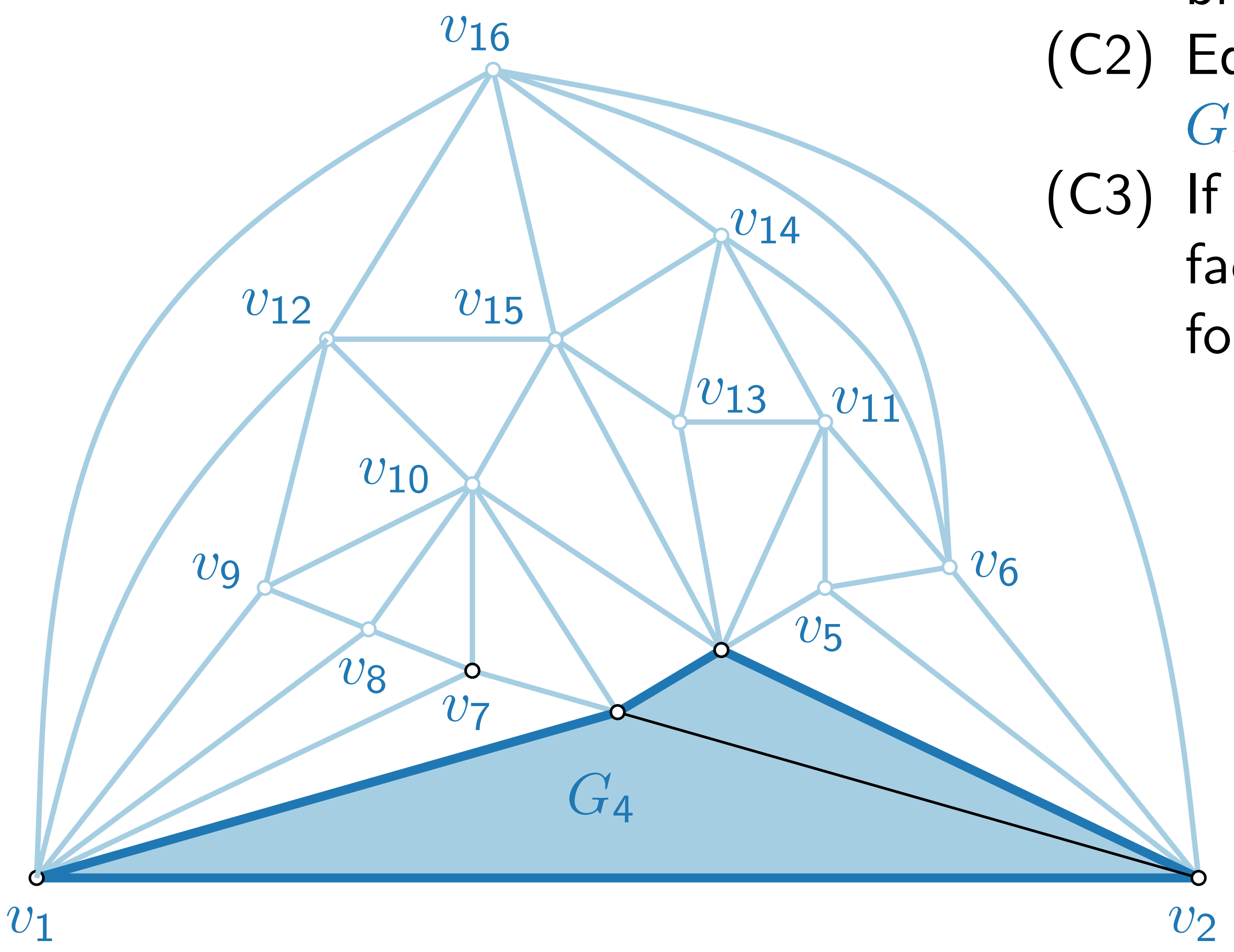

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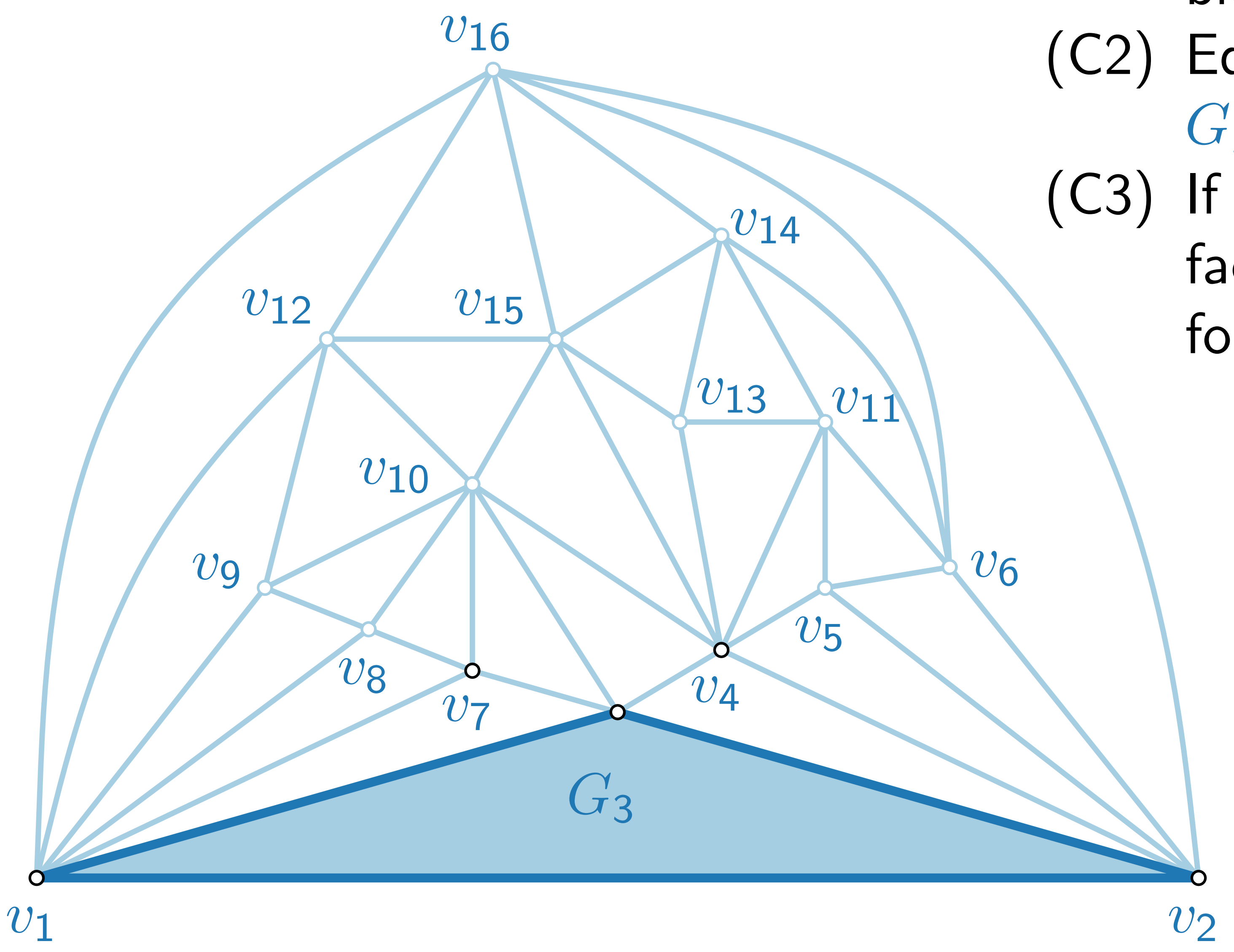

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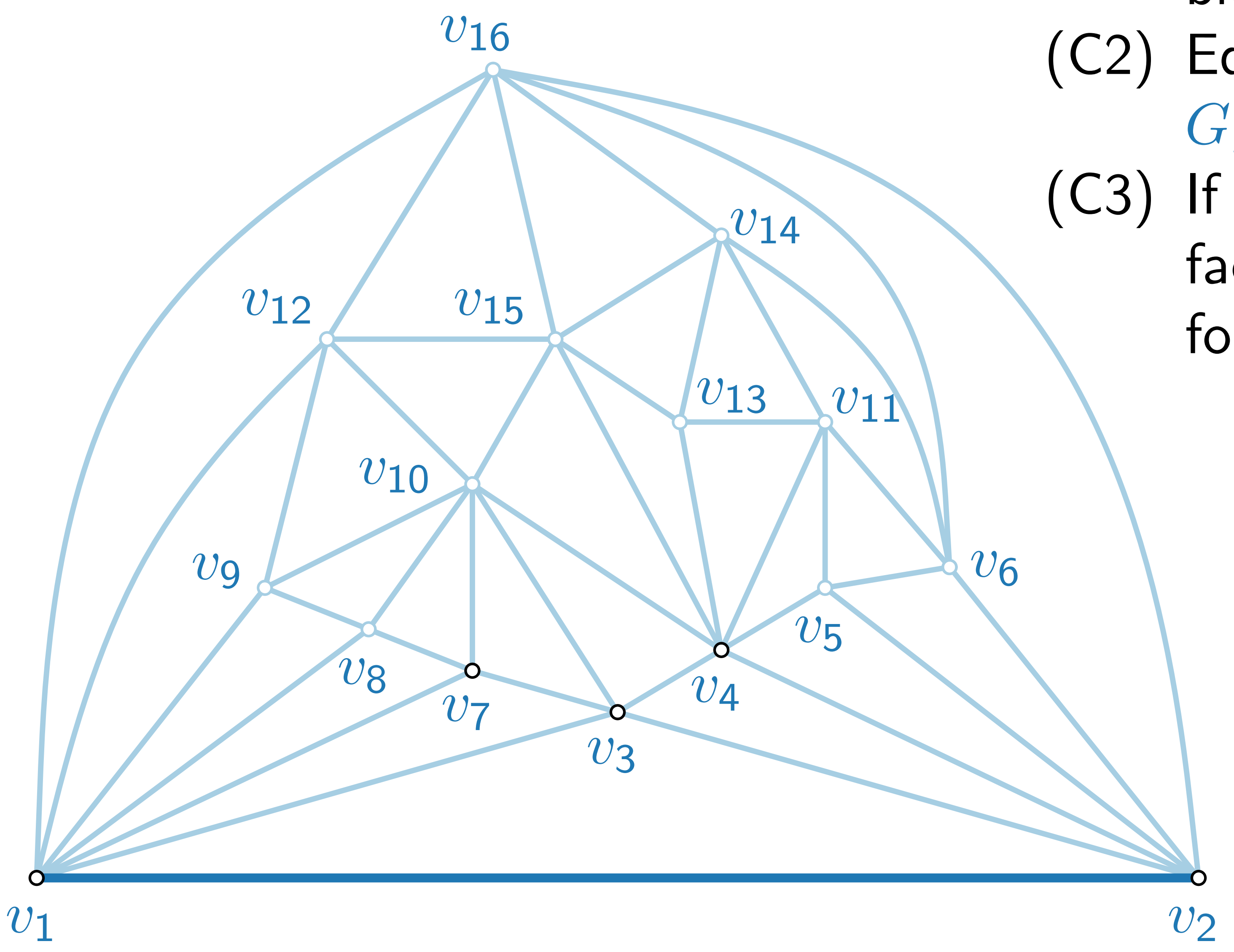

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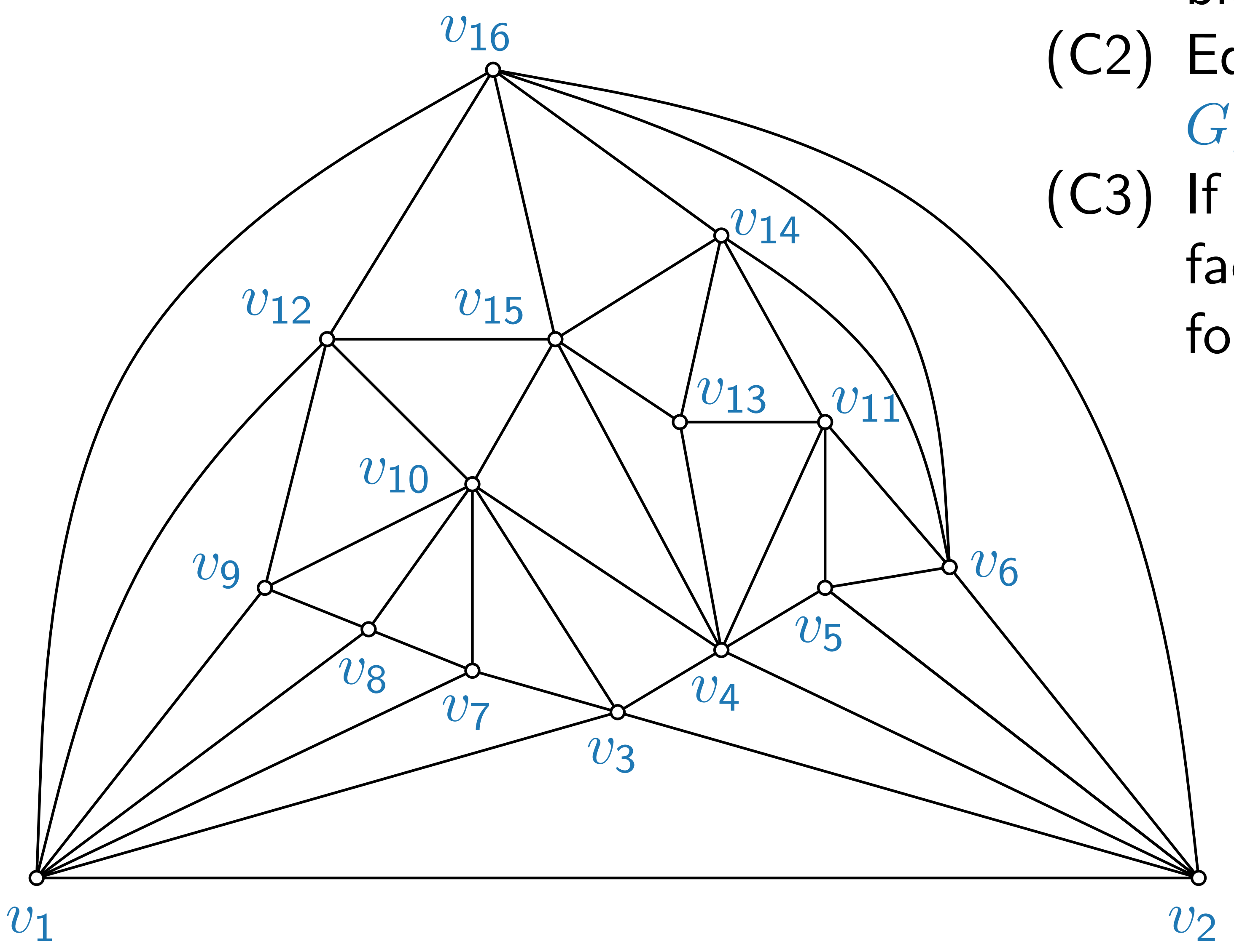

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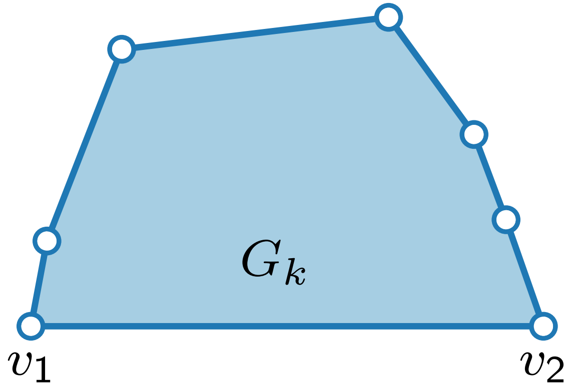

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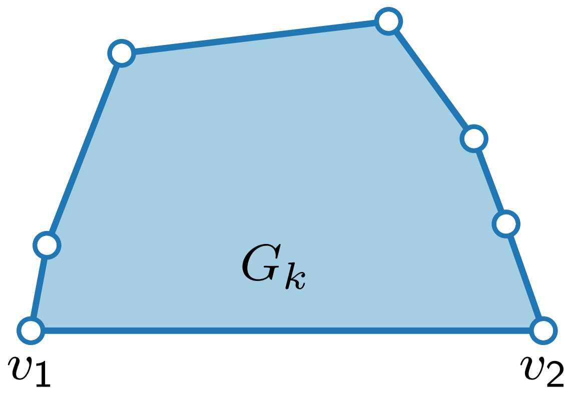

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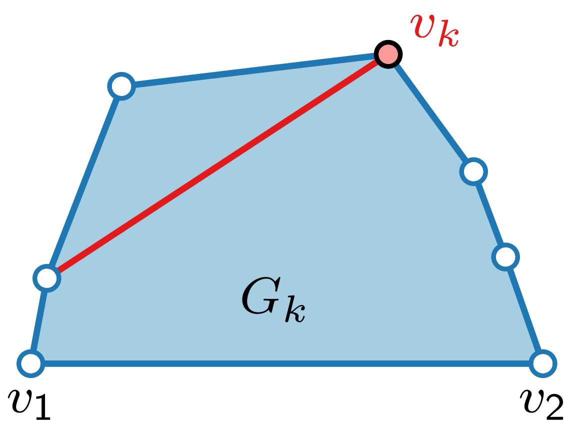

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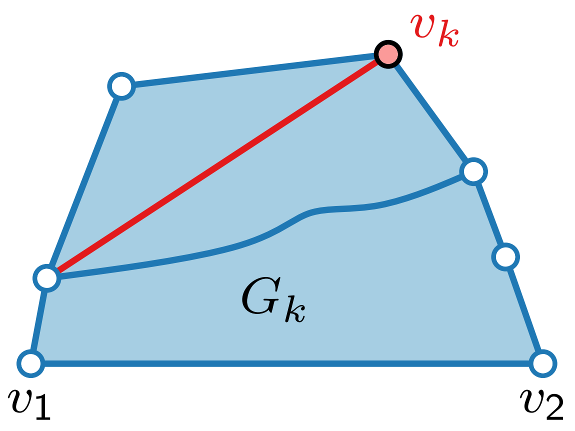

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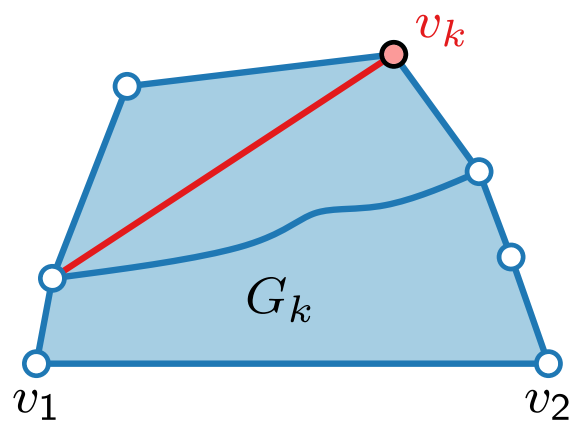

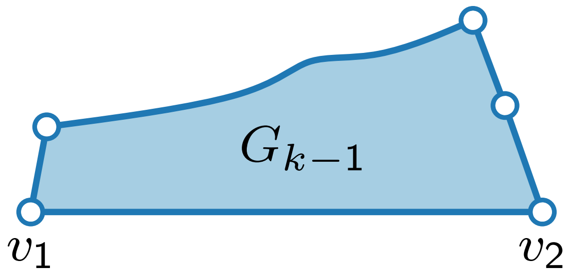

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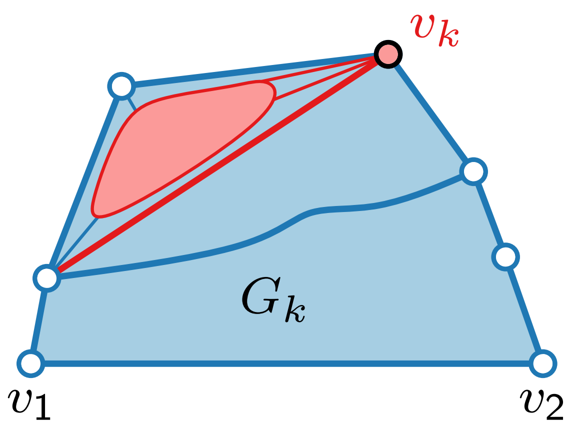

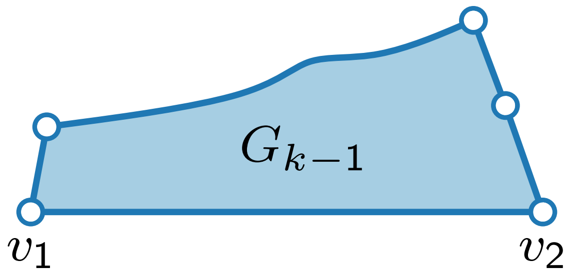

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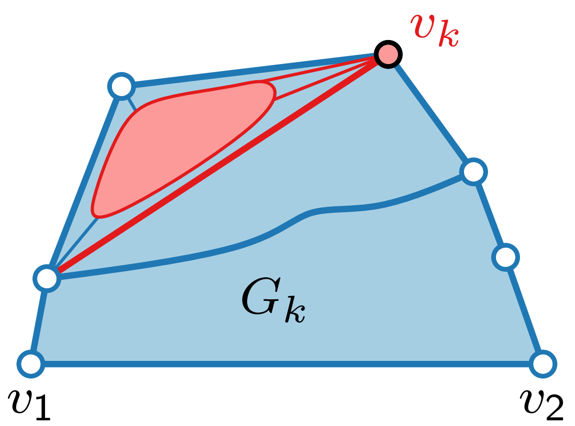

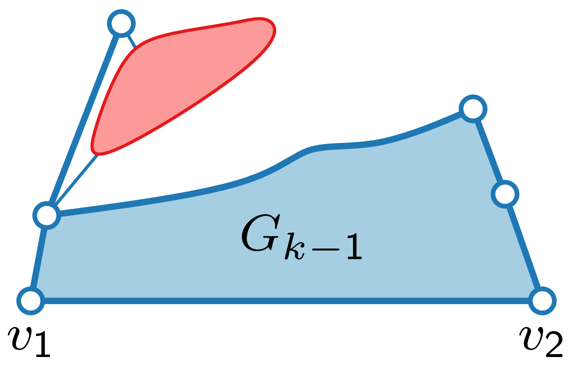

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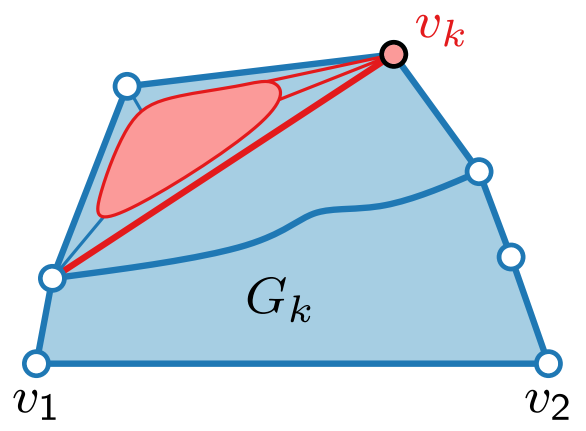

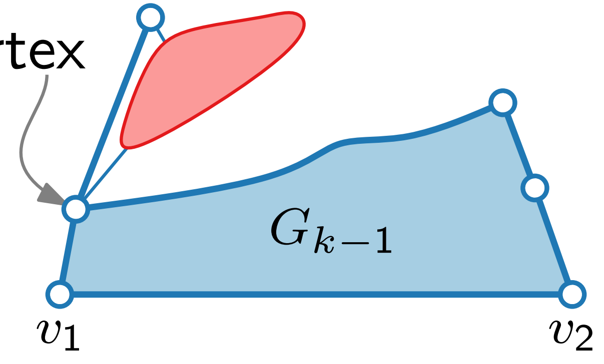

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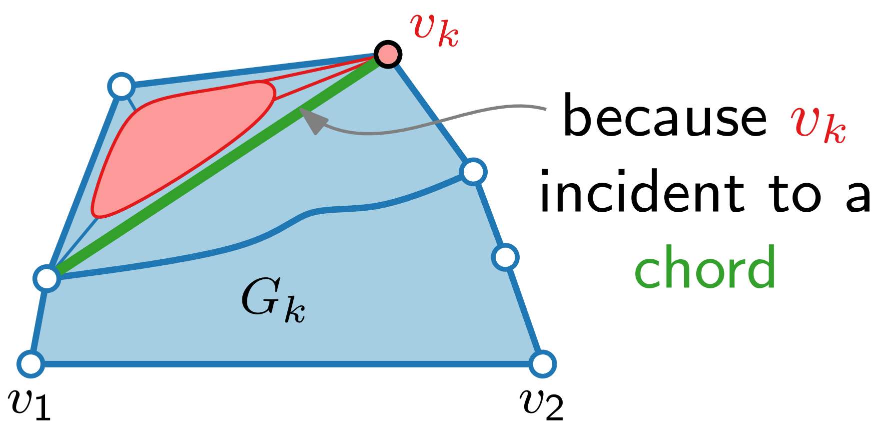

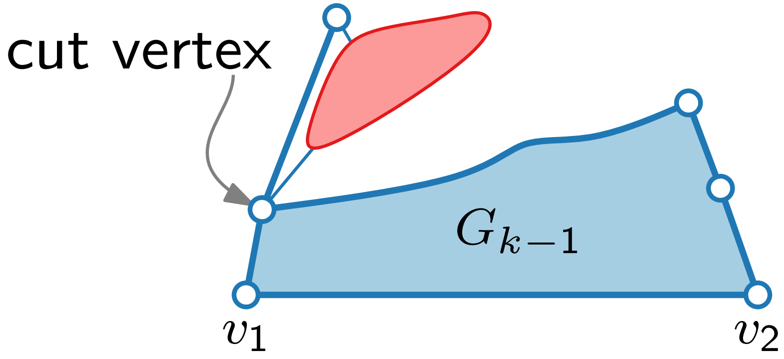

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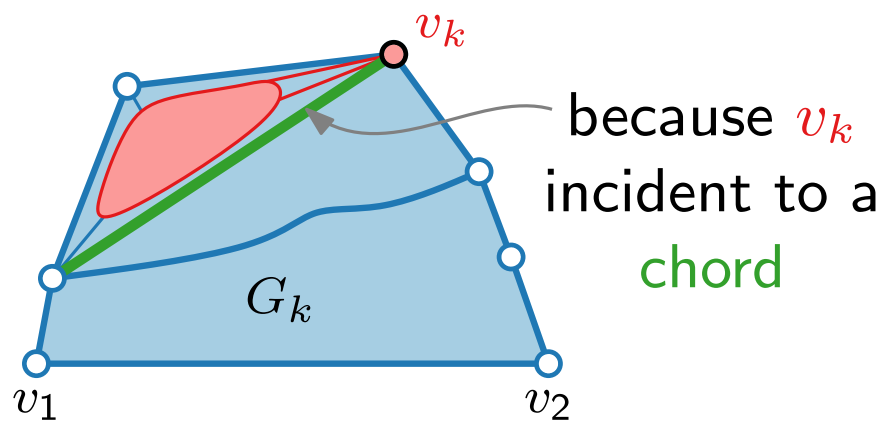
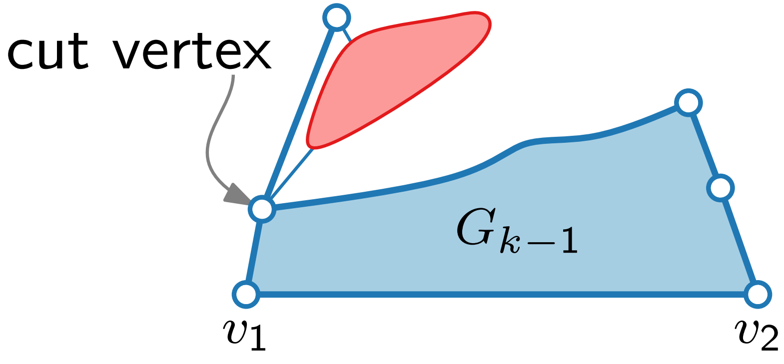




Canonical Order – Existence

(C1) G_k biconnected inner triangulation

(C2) (v_1, v_2) on outer face of G_k

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of G_k, neighbors of v_{k+1} form path on boundary of G_k

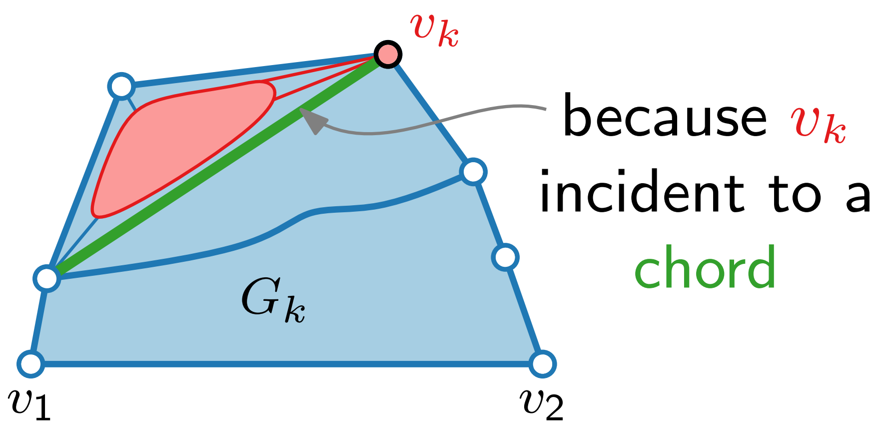
Lemma.
Every plane triangulation has a canonical order.

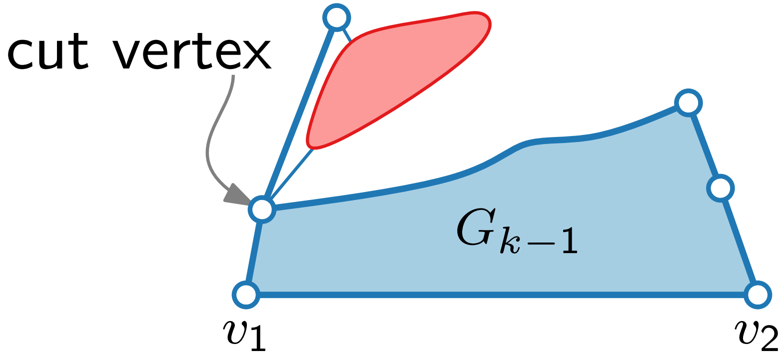
Consider any n-vertex plane triangulation. We show this statement by induction on k from n down to 3.

Induction base $(k = n)$: Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions (C1)–(C3) hold.

Induction hypothesis: Vertices $v_{n-1}, \dots, v_{k+1}$ have been chosen such that conditions (C1)–(C3) hold for every $i \in \{k+1, \dots, n\}$.

Induction step: Consider G_k. We search for v_k.





We need to show:

1. v_k not incident to chord is sufficient.

Canonical Order – Existence

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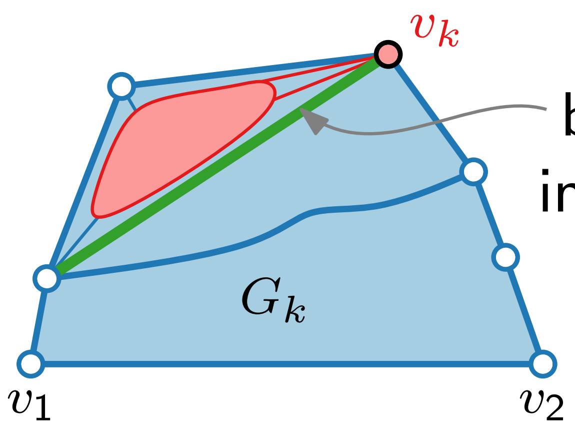

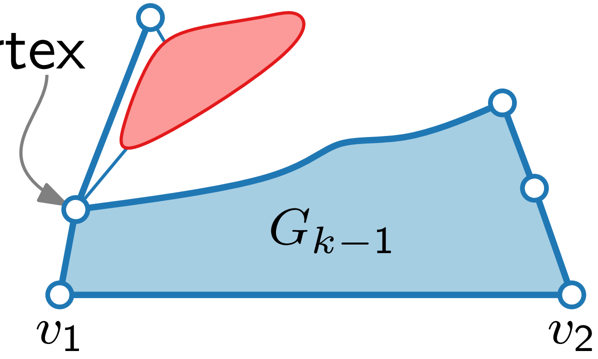

We need to show:

1. v_k not incident to chord is sufficient.
2. Such v_k exists.

Canonical Order – Existence

Claim 1.
If v_k is not incident to a chord,
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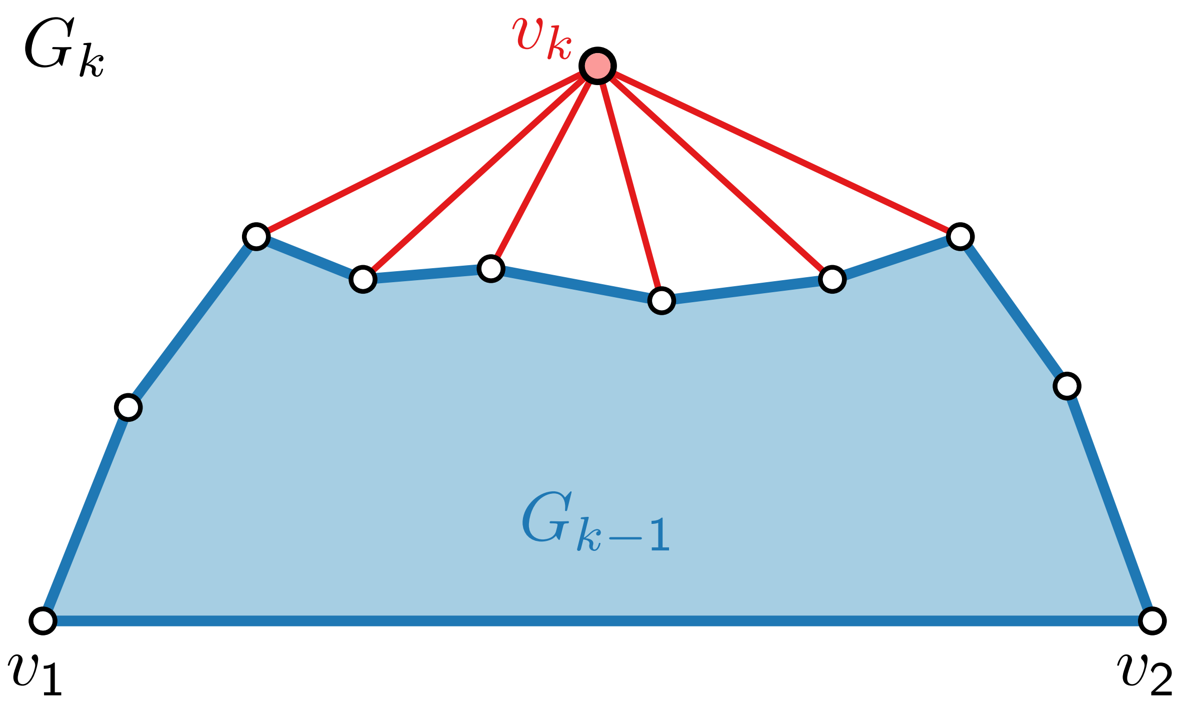

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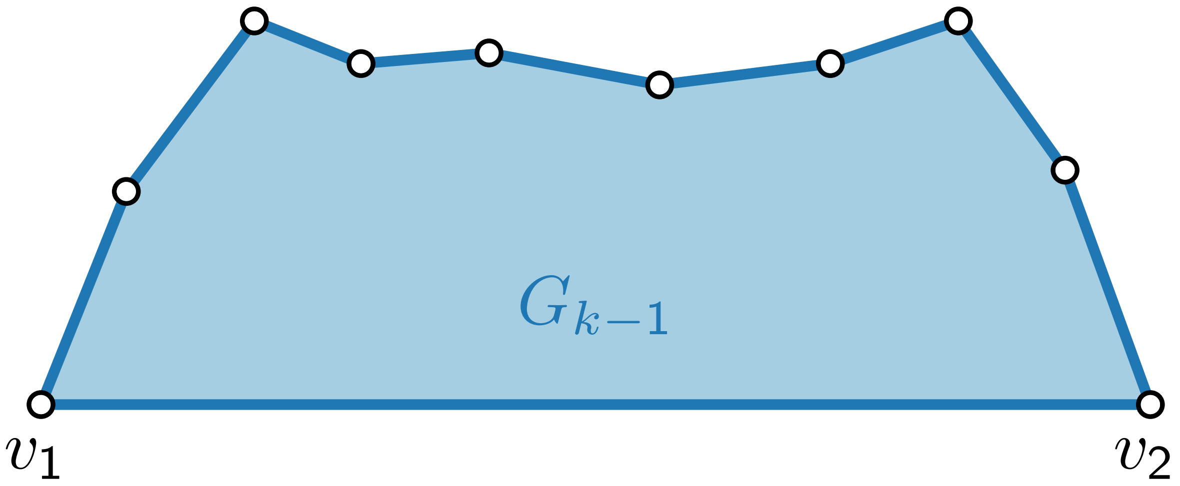

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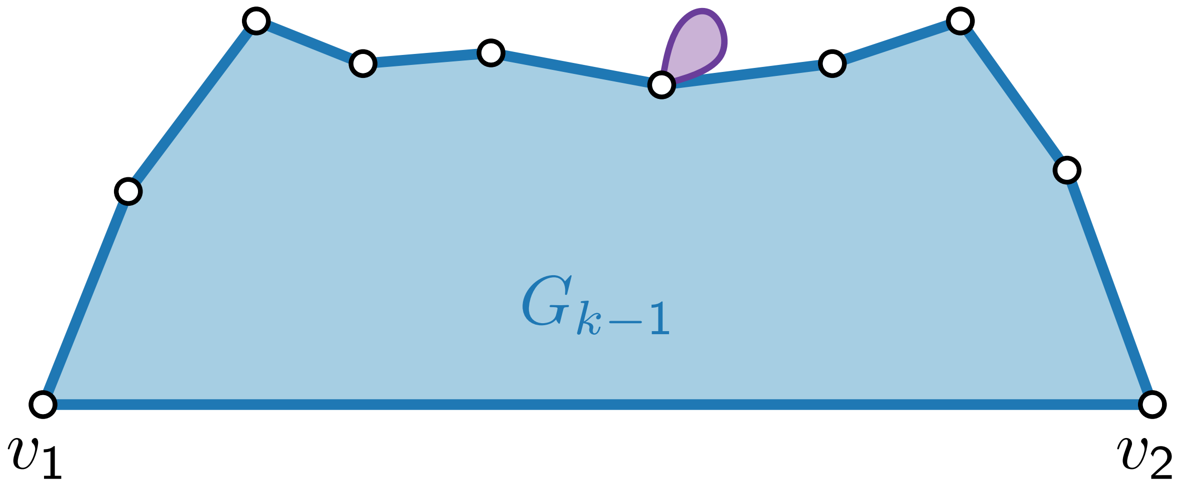

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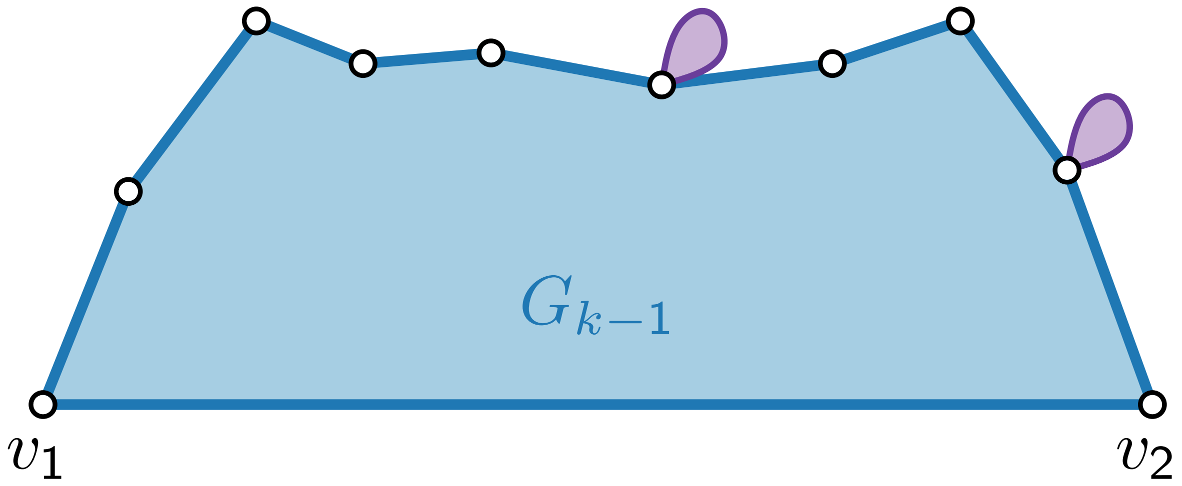

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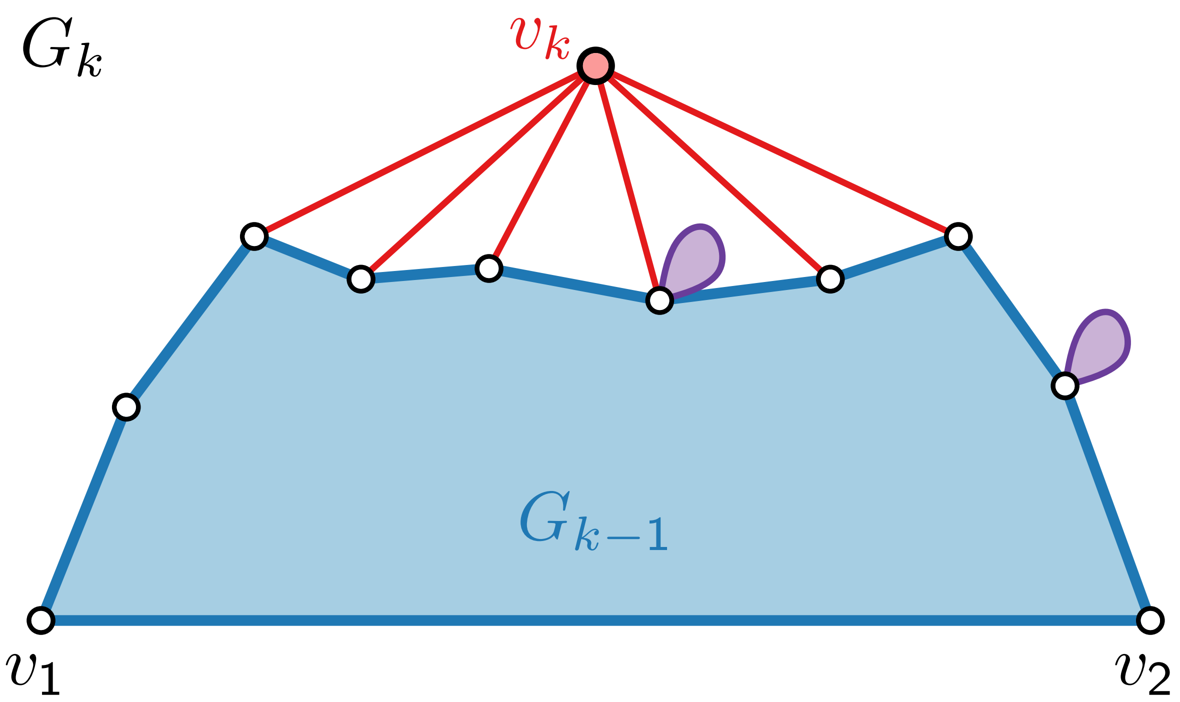

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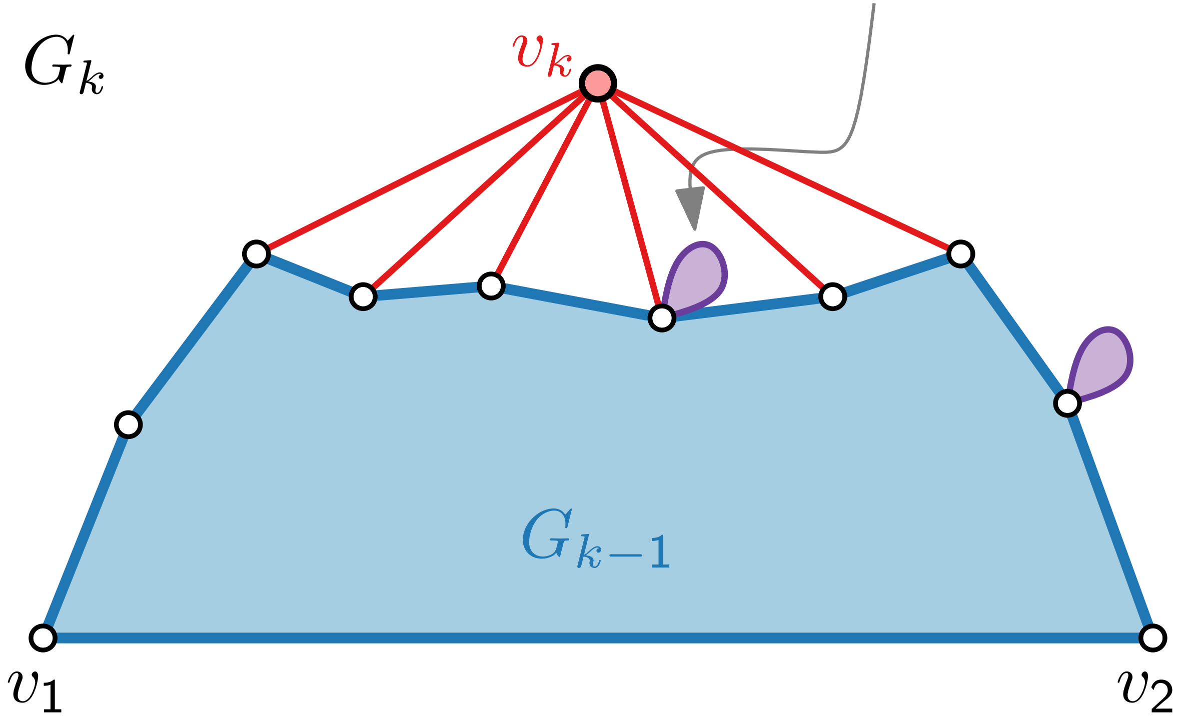
Contradiction to neighbors of v_k forming a path on ∂G_{k-1}!



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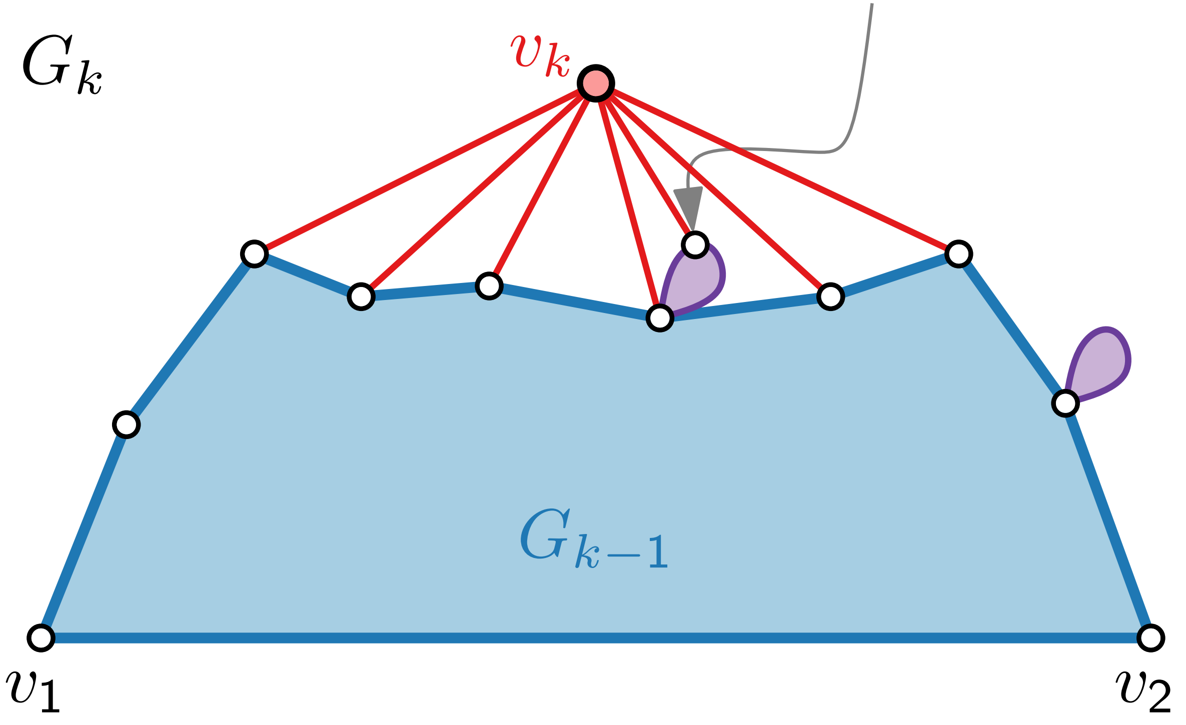

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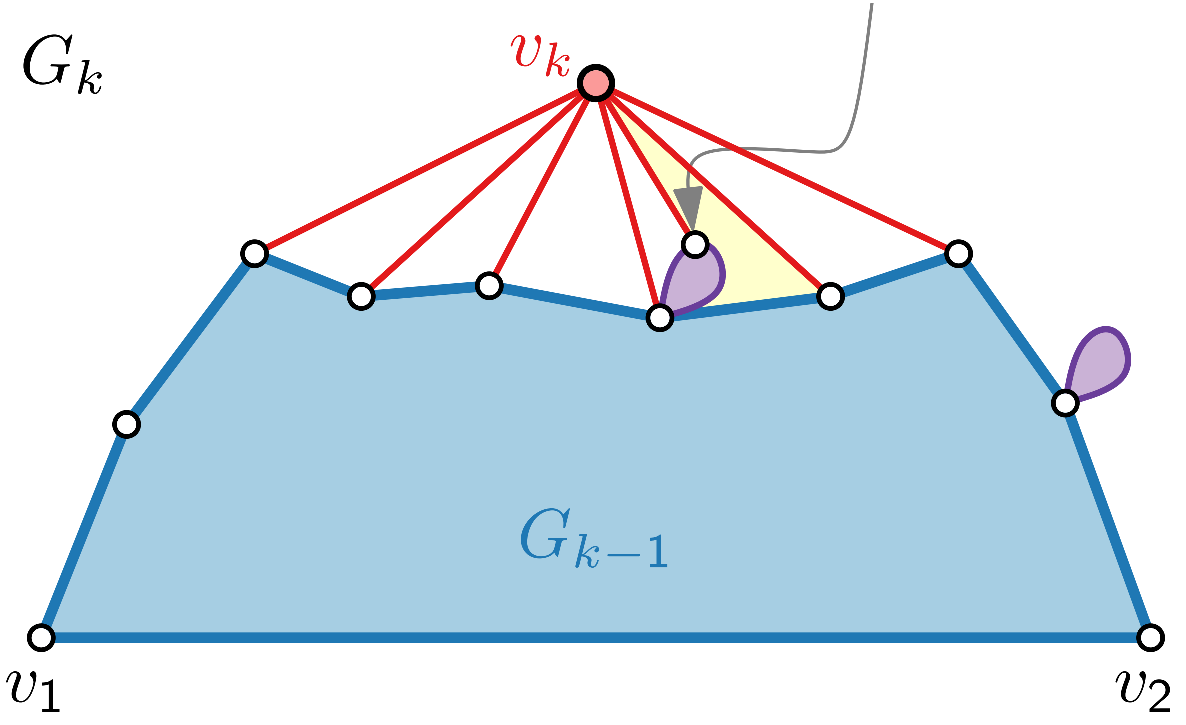

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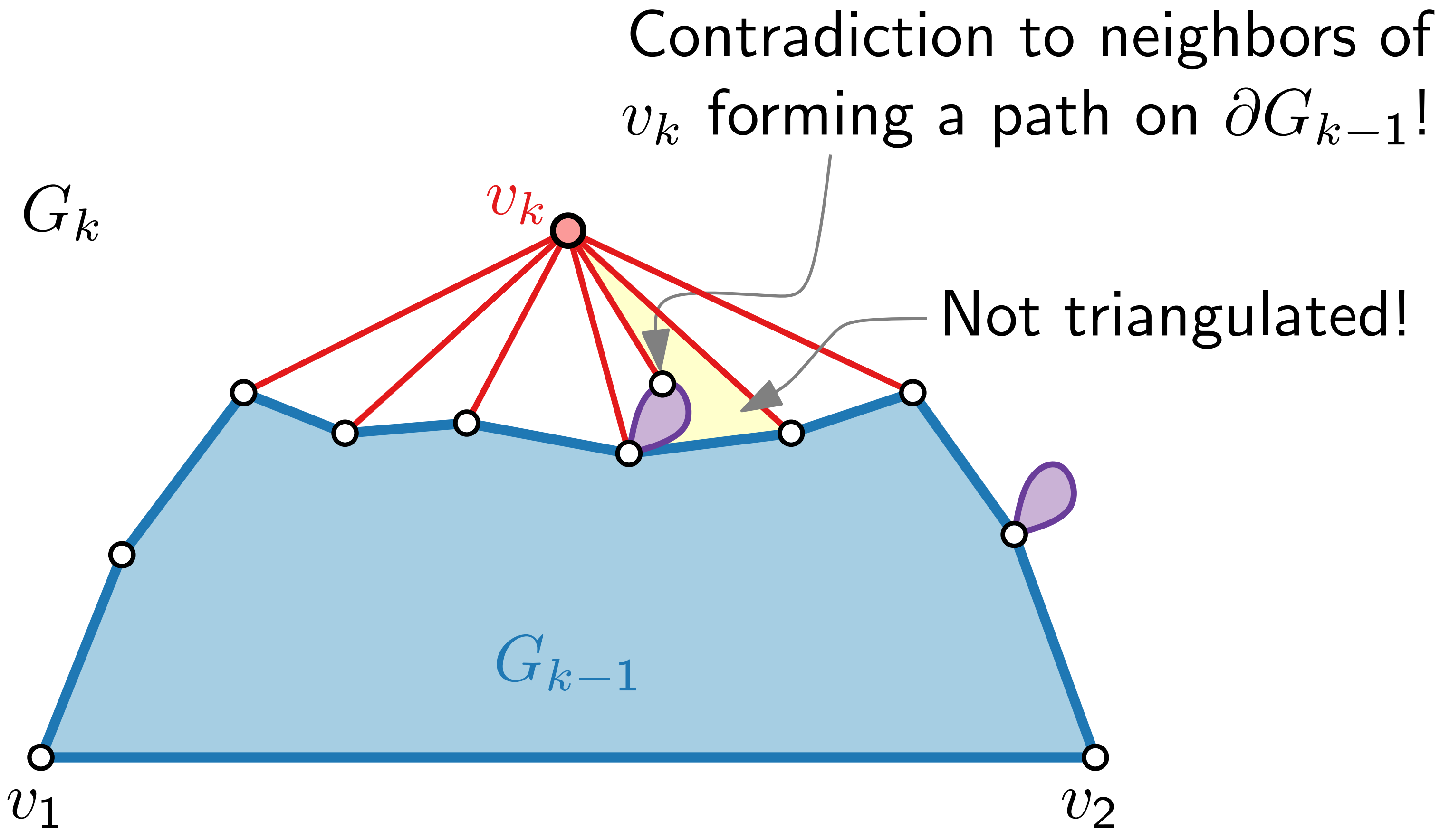

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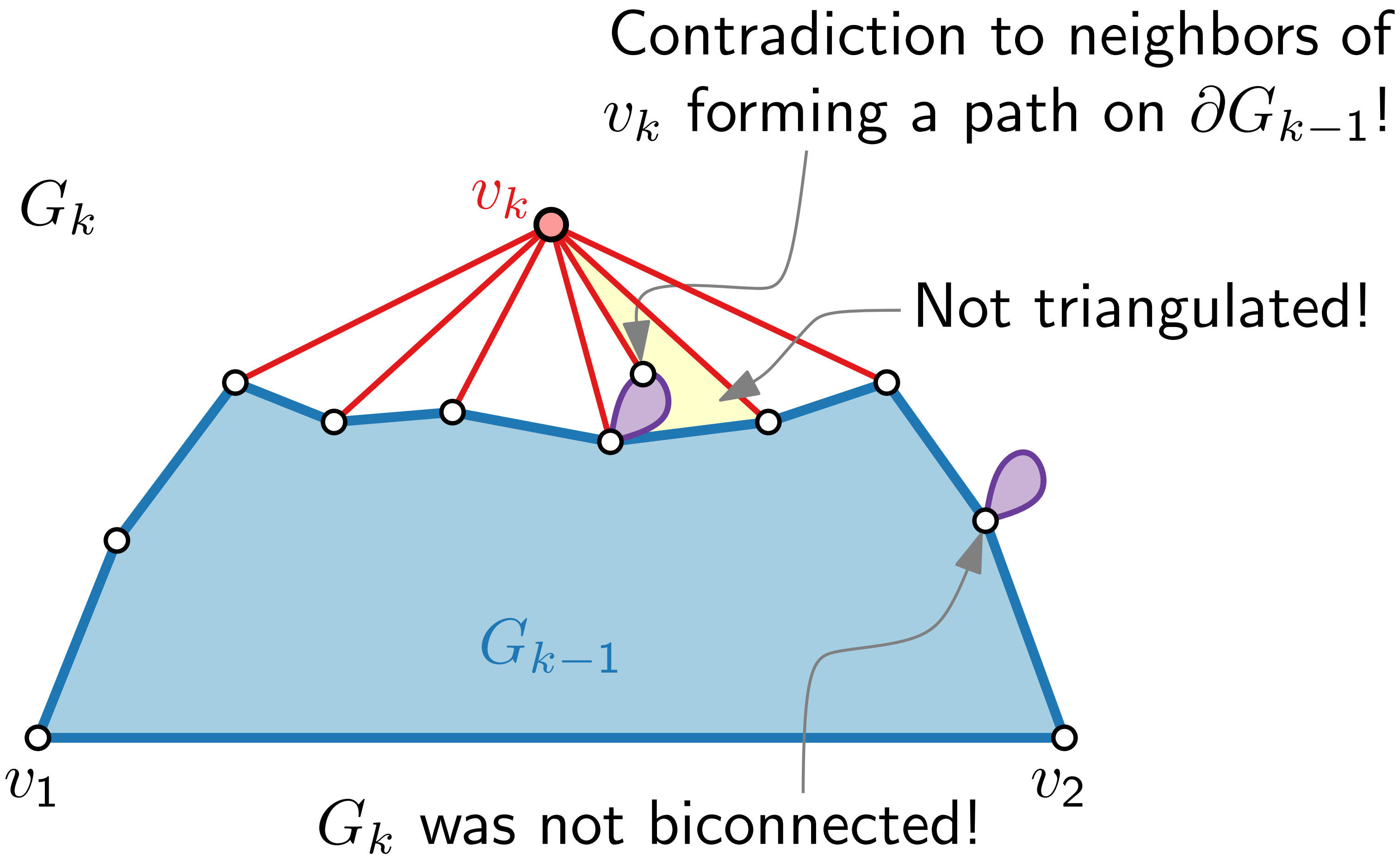

Canonical Order – Existence

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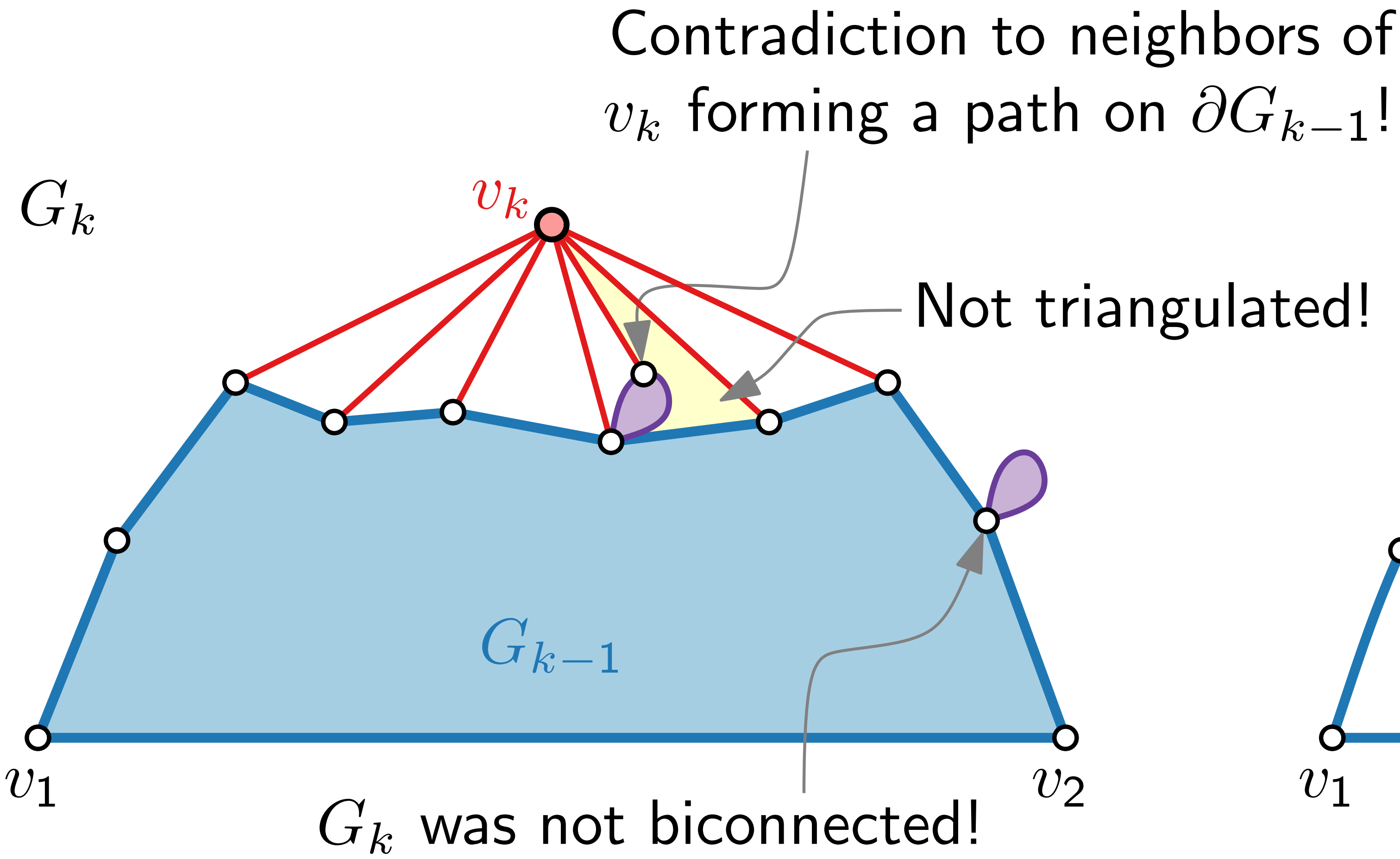
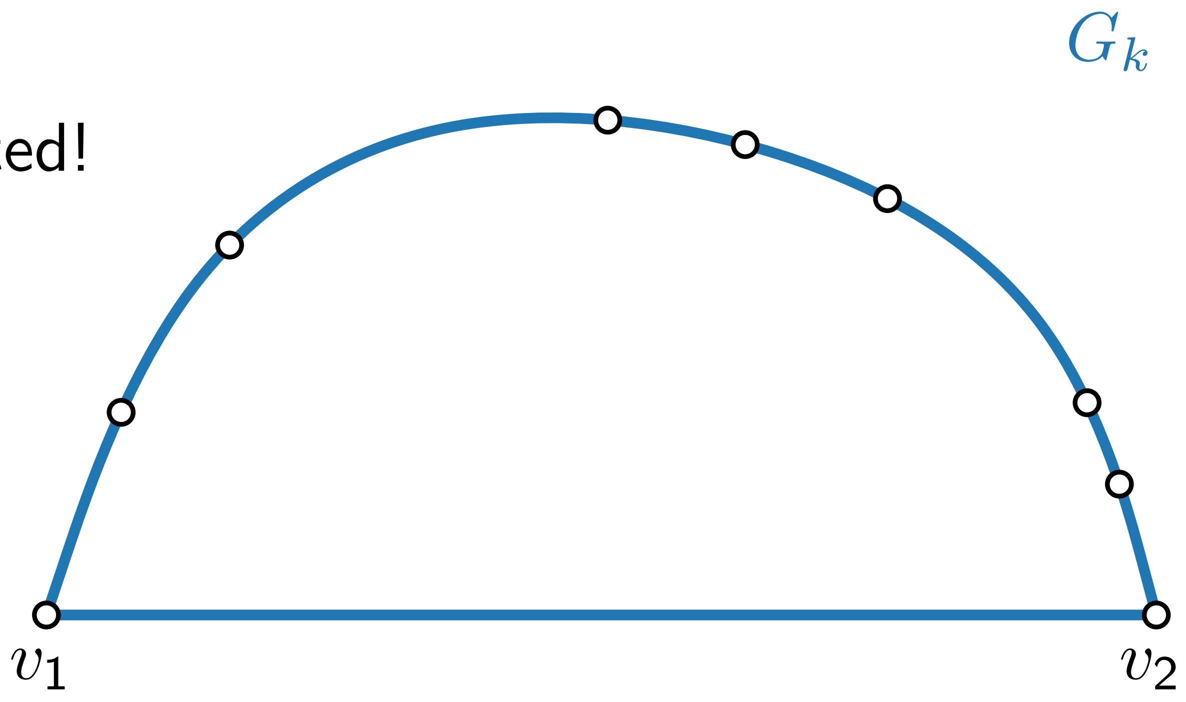
There exists a vertex in G_k that is not incident to a chord as choice for v_k.





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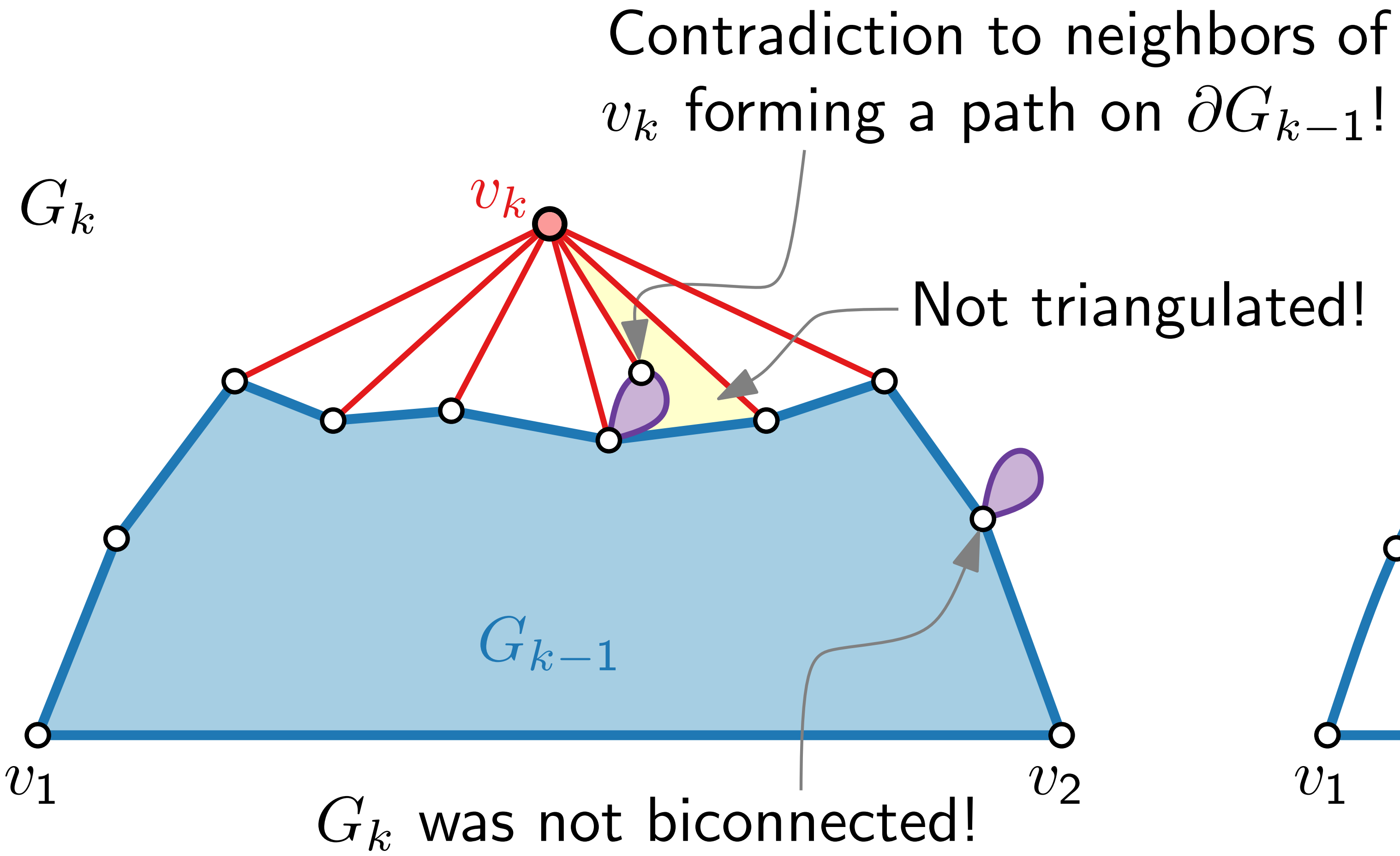

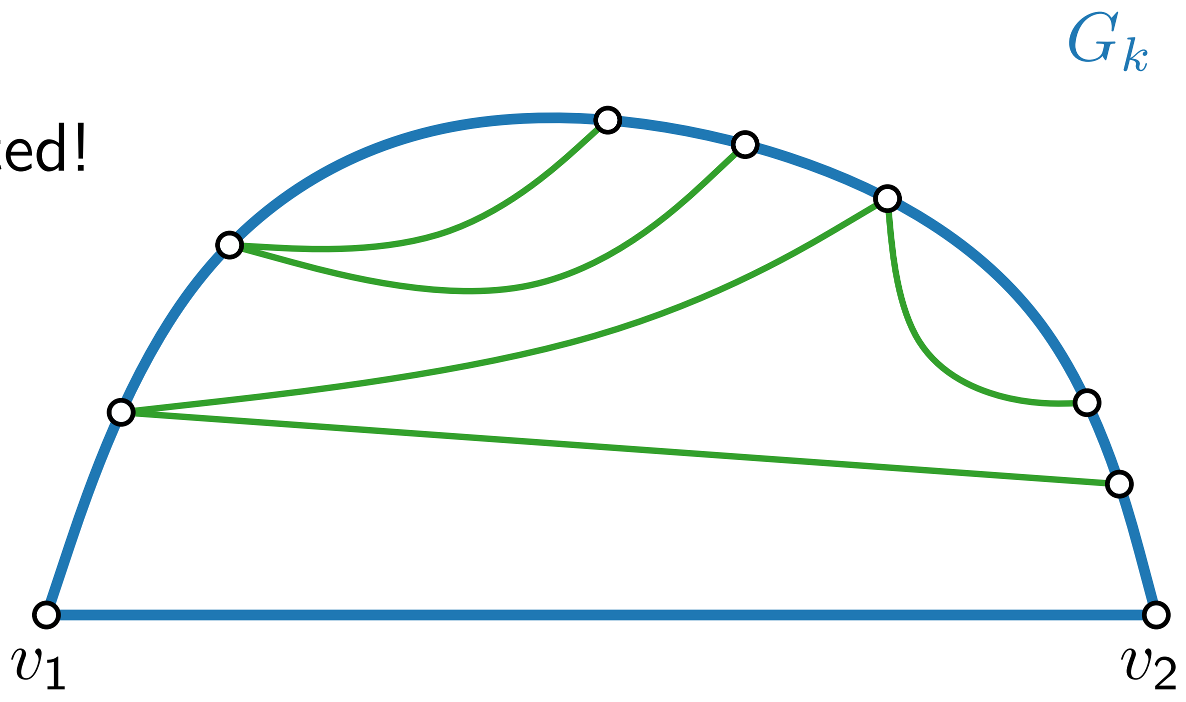

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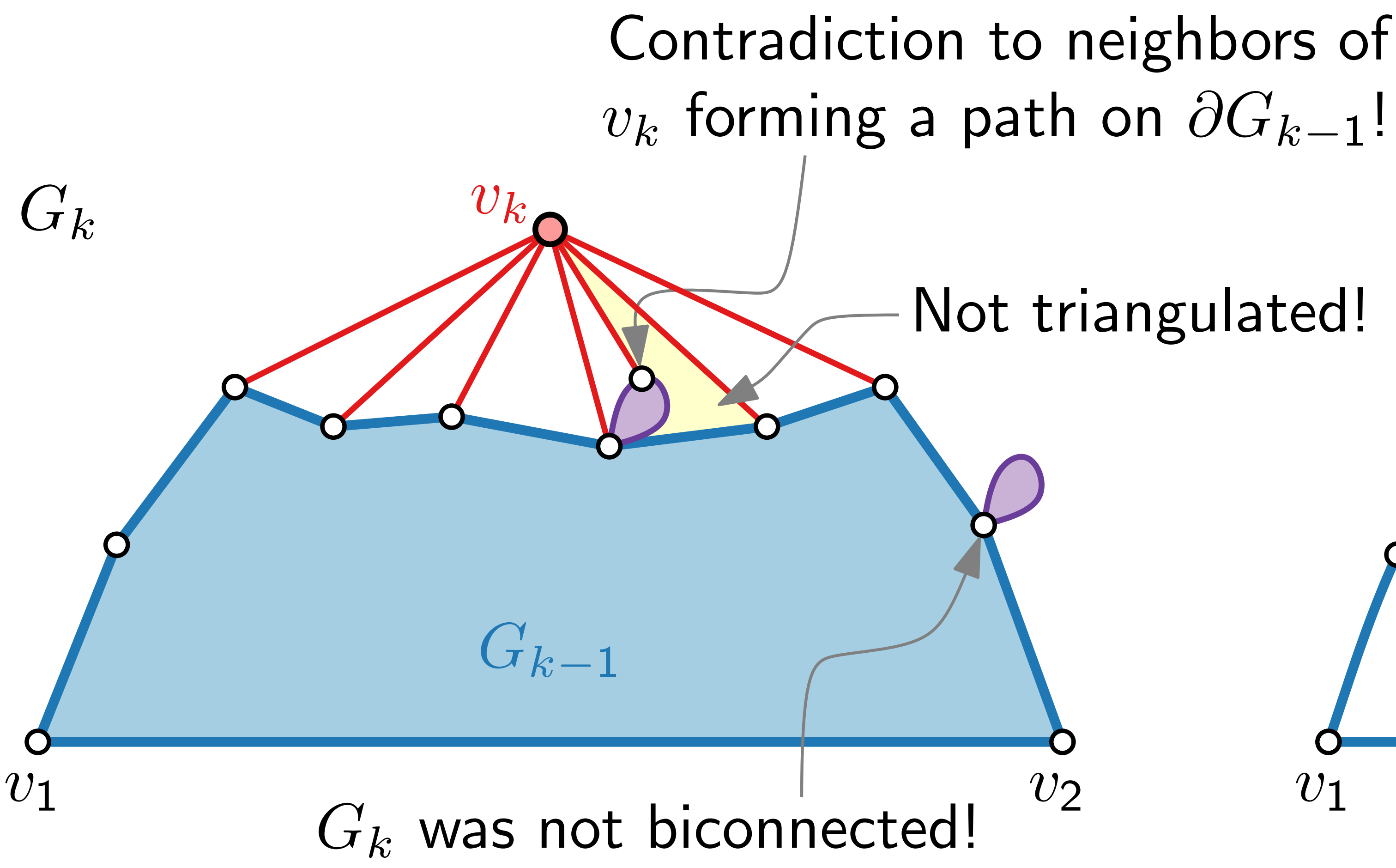

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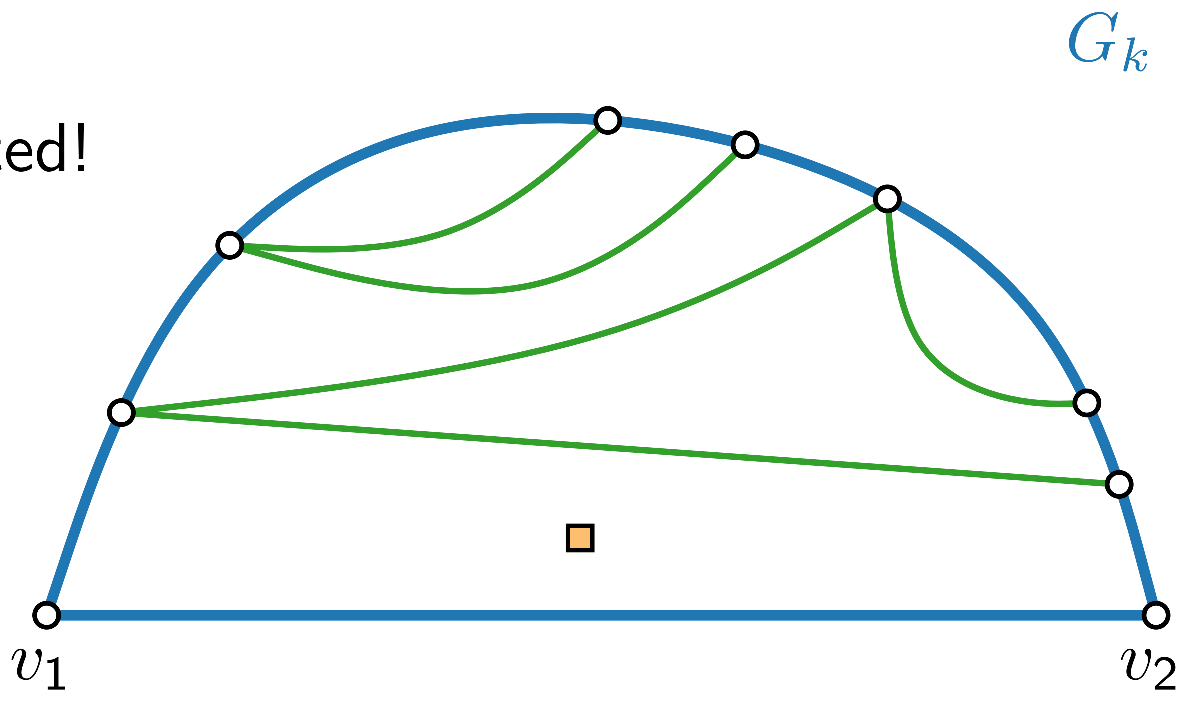

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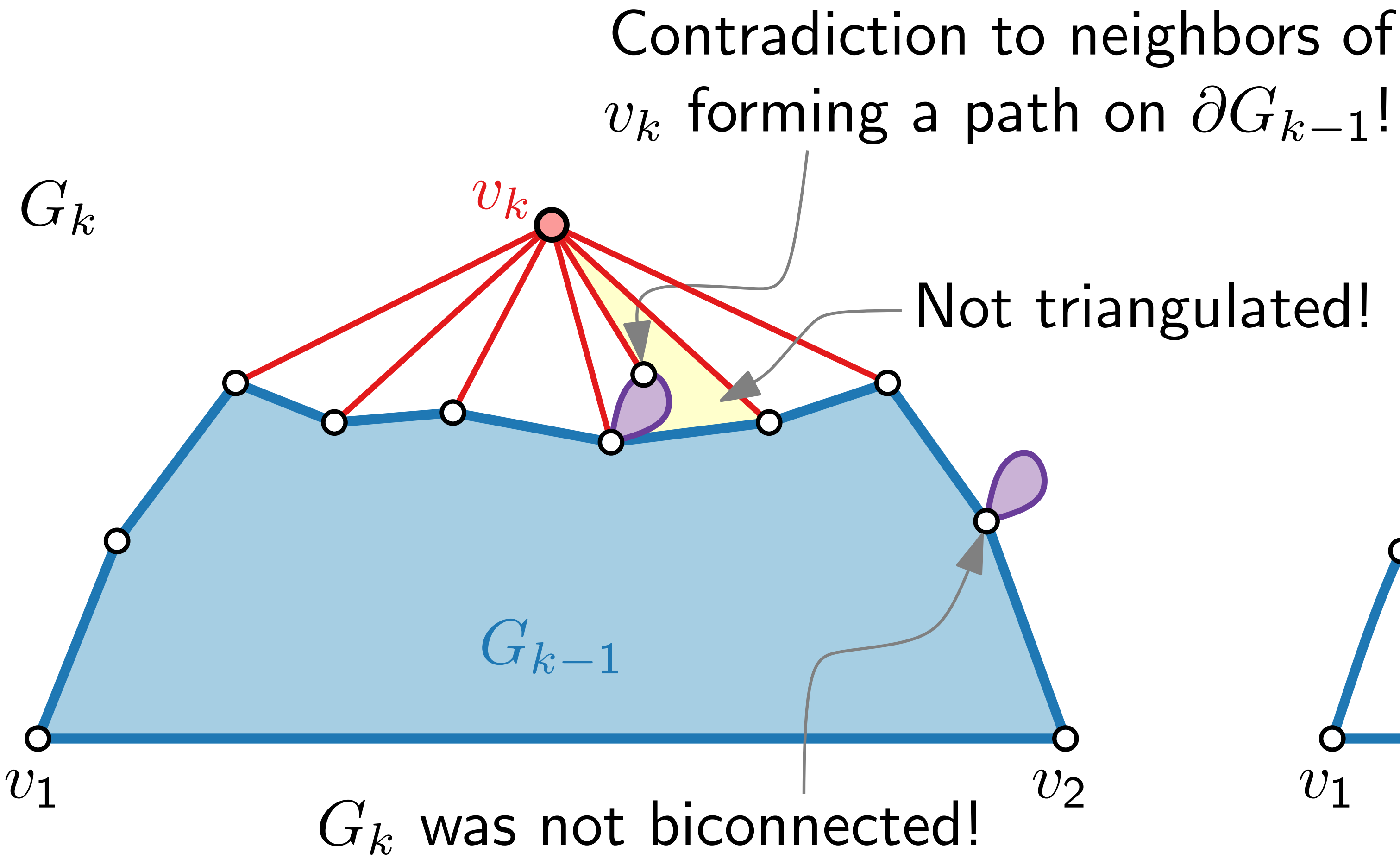

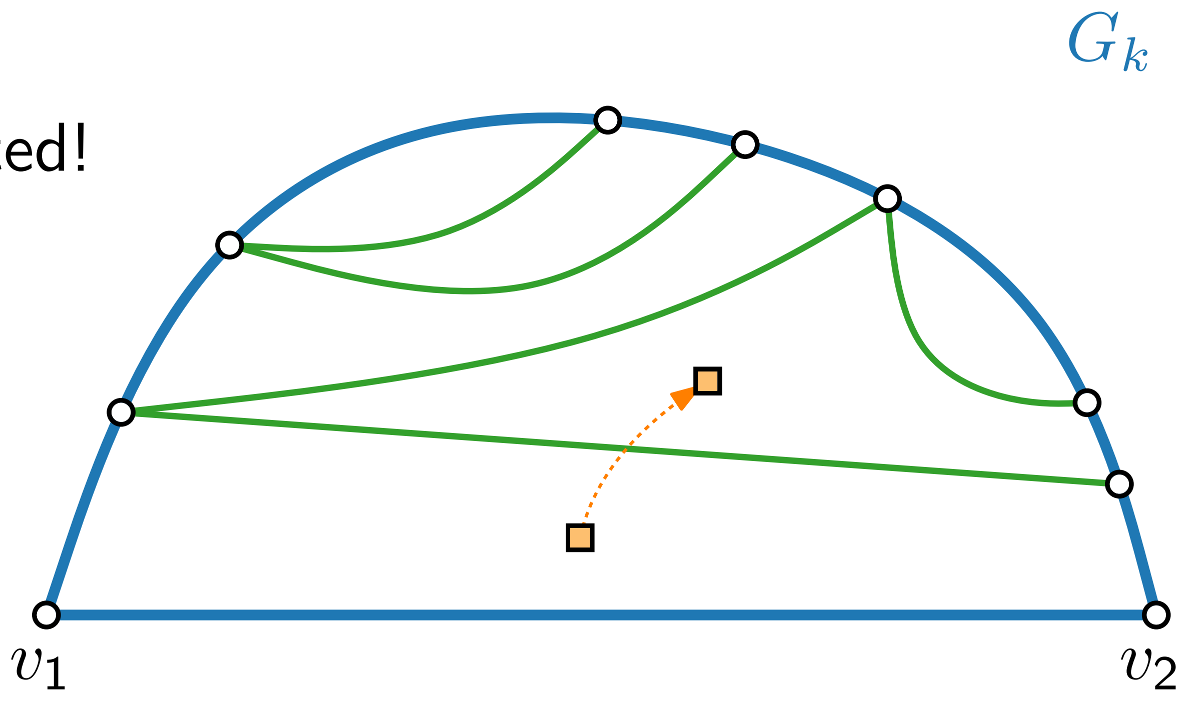

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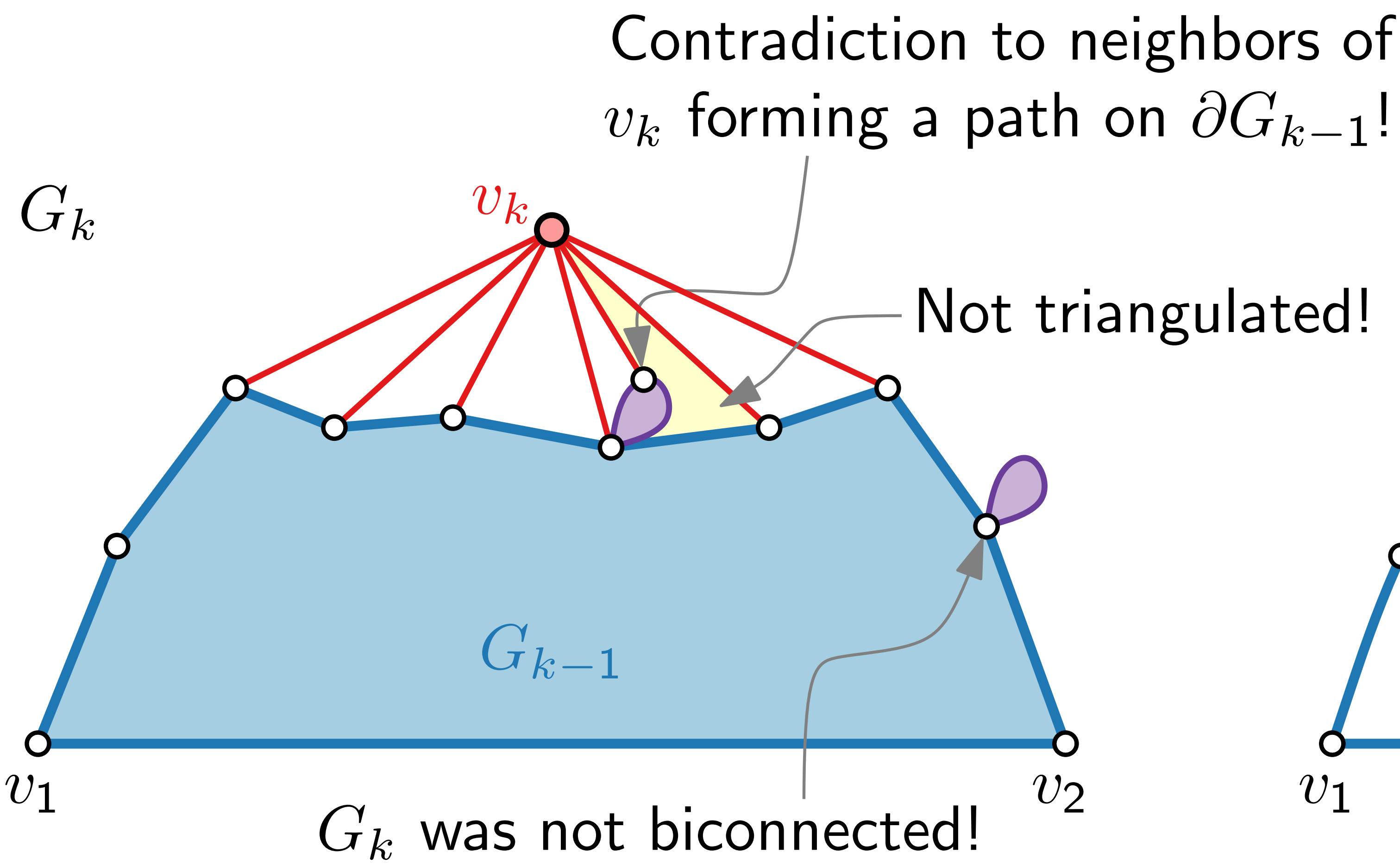

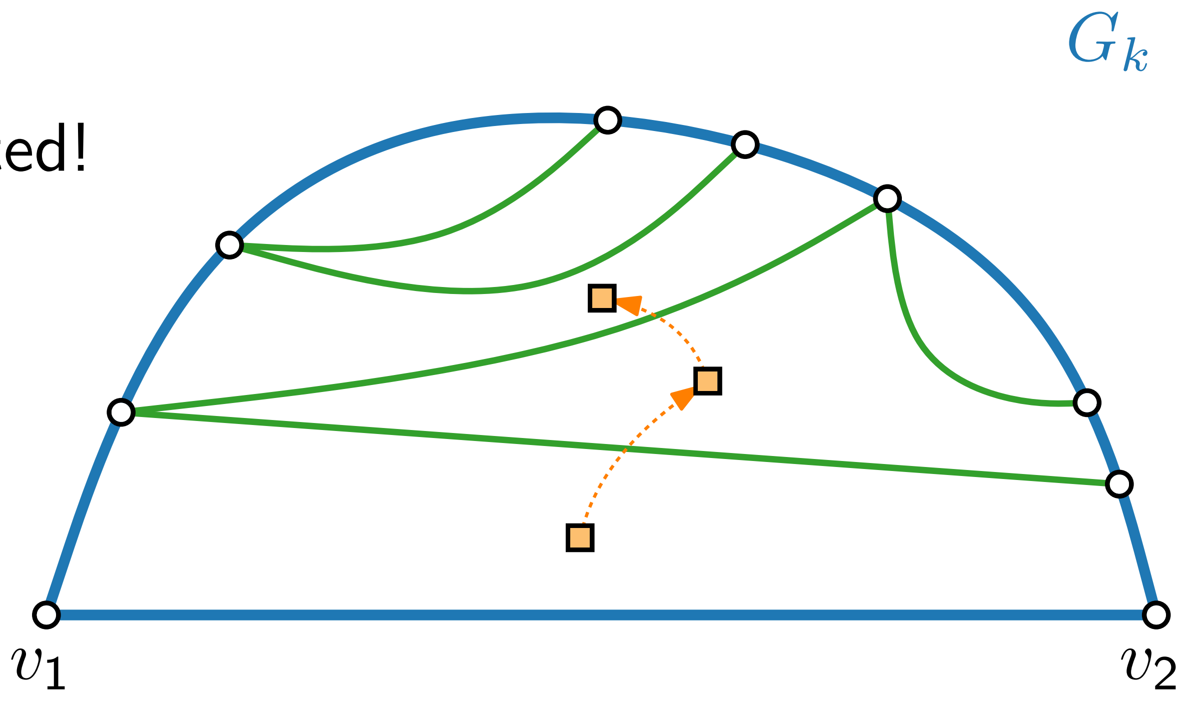

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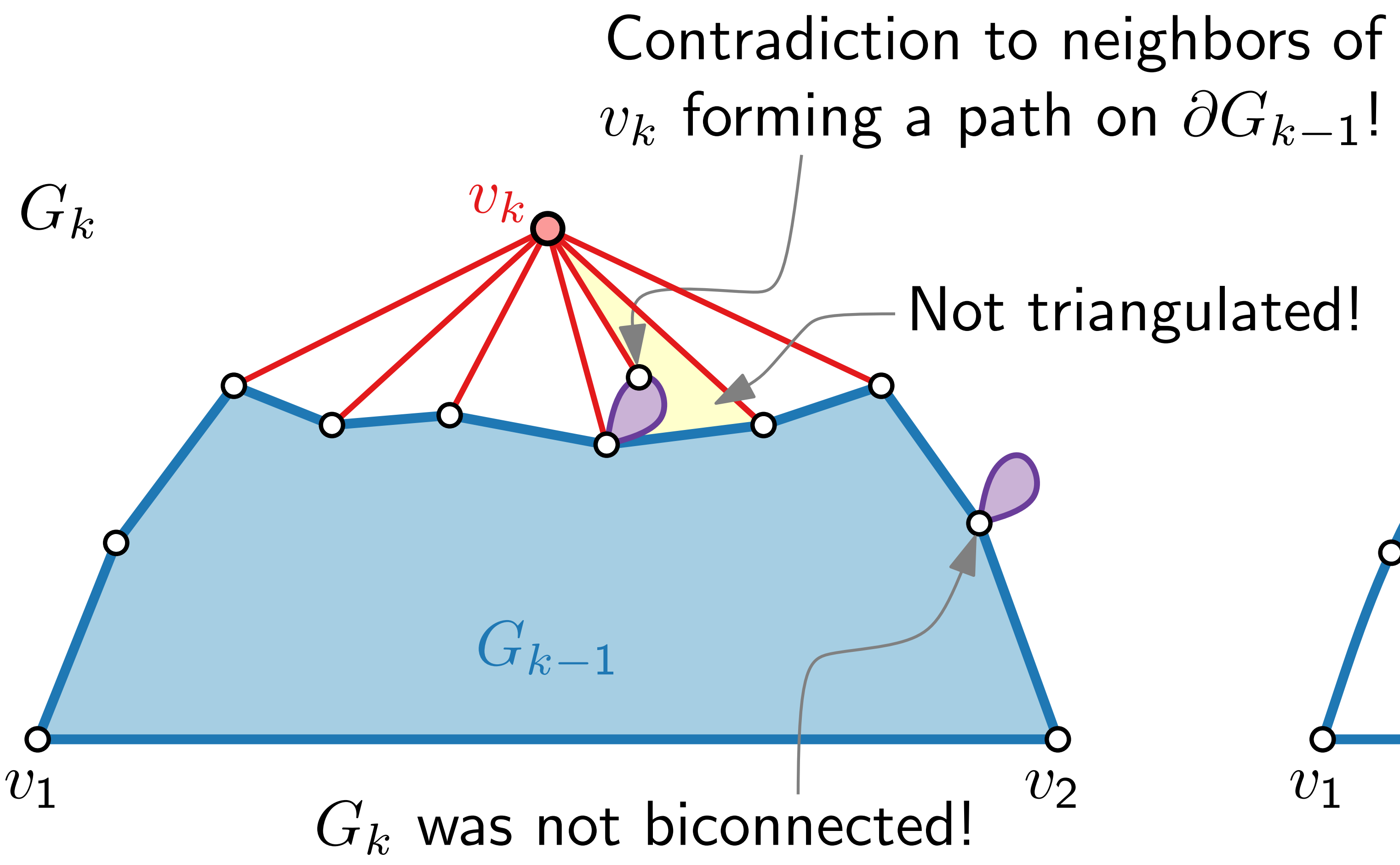

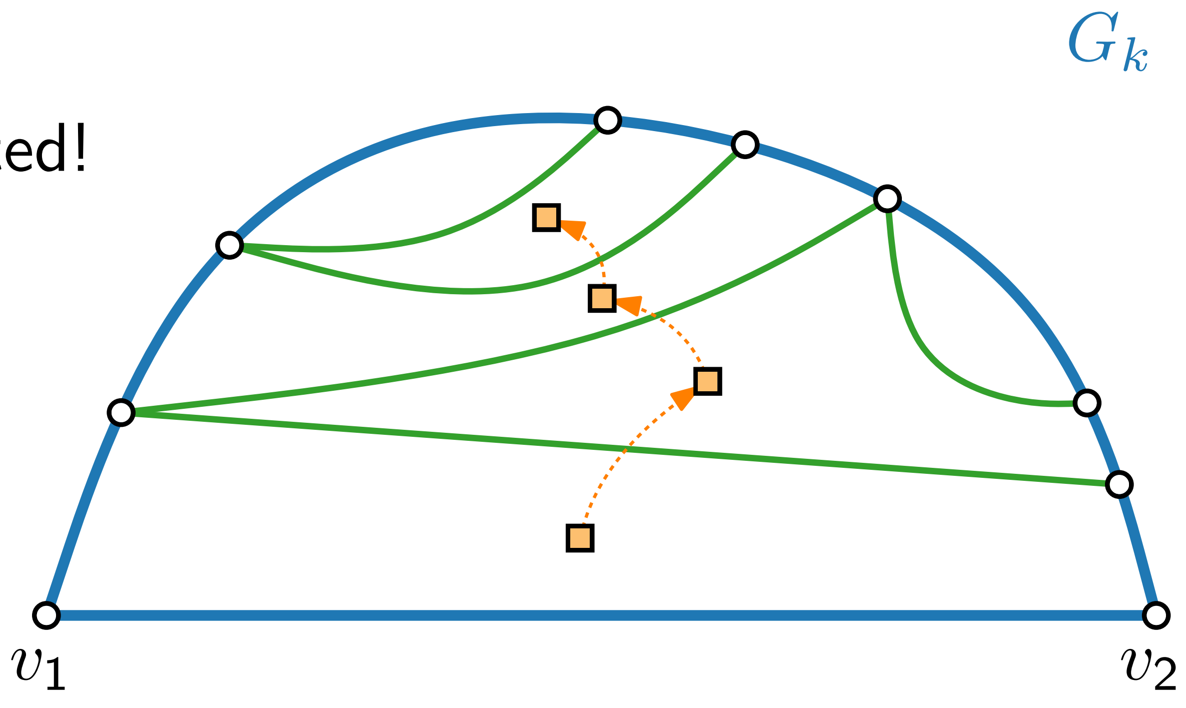

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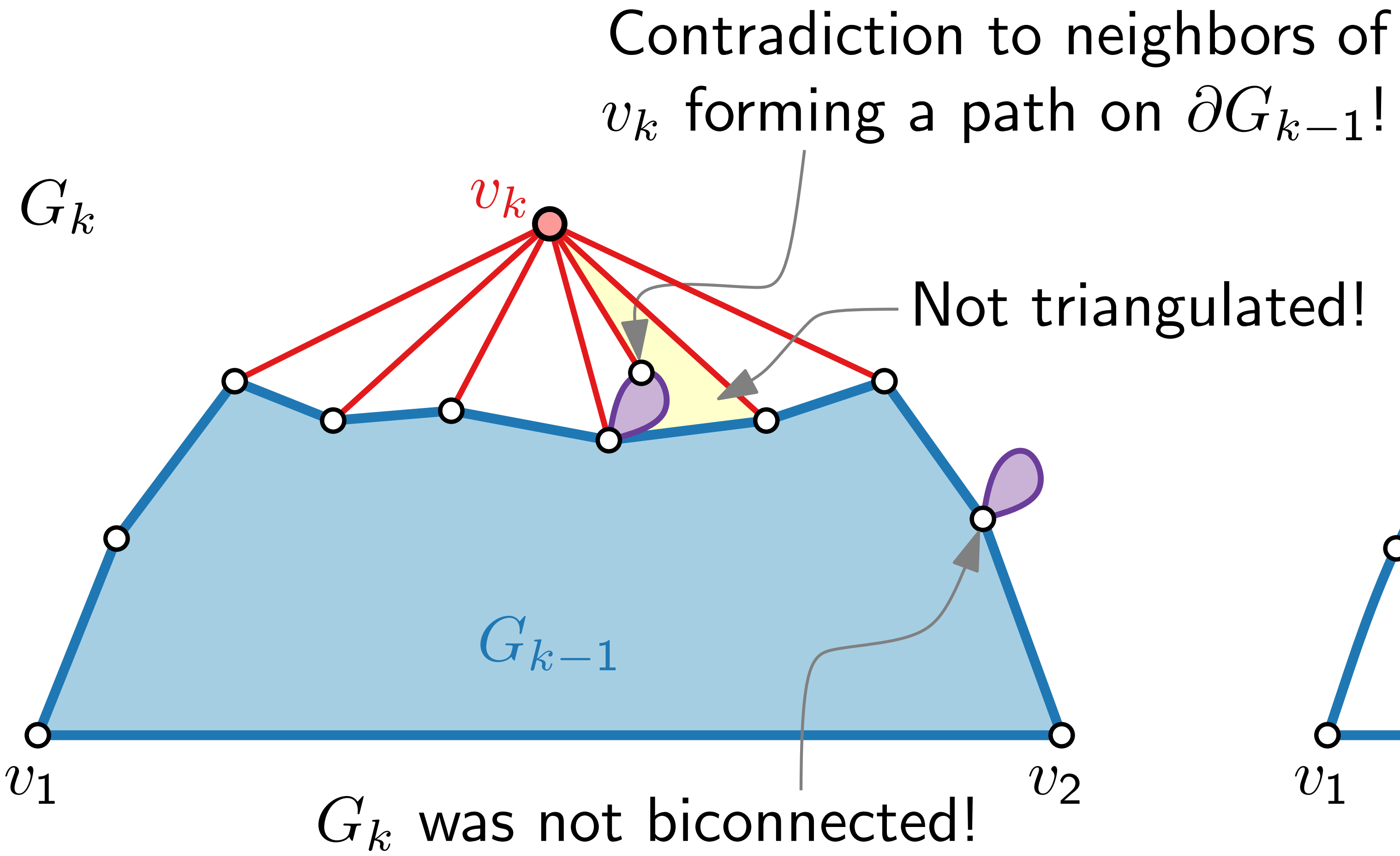

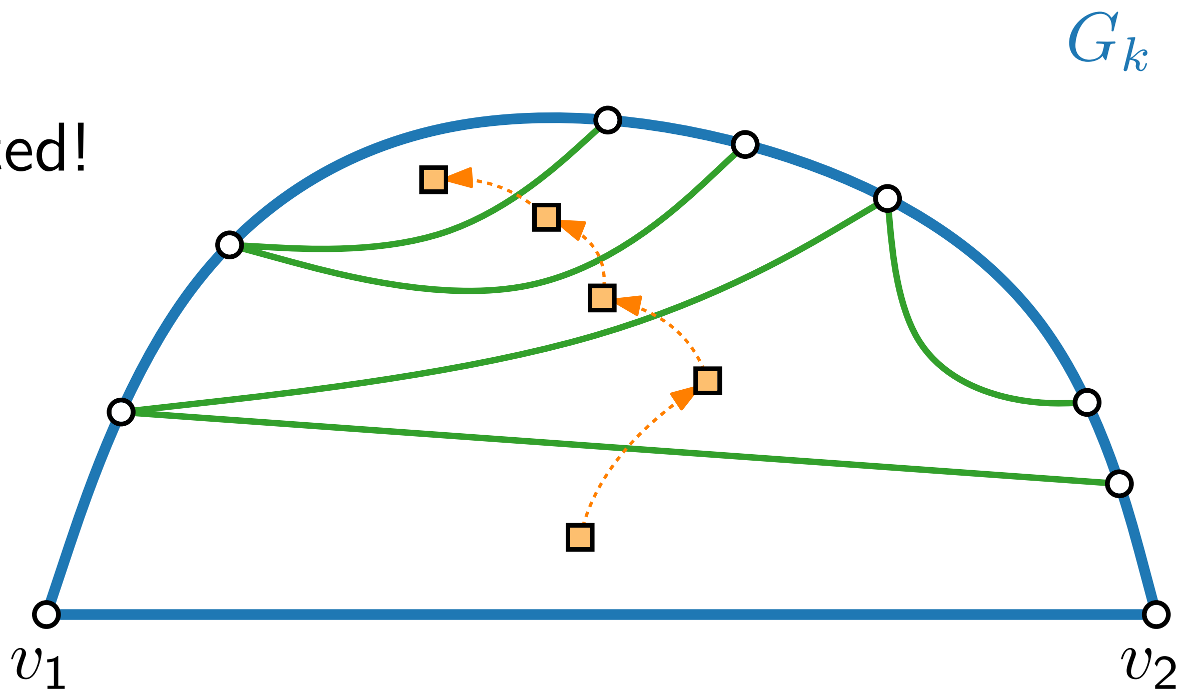

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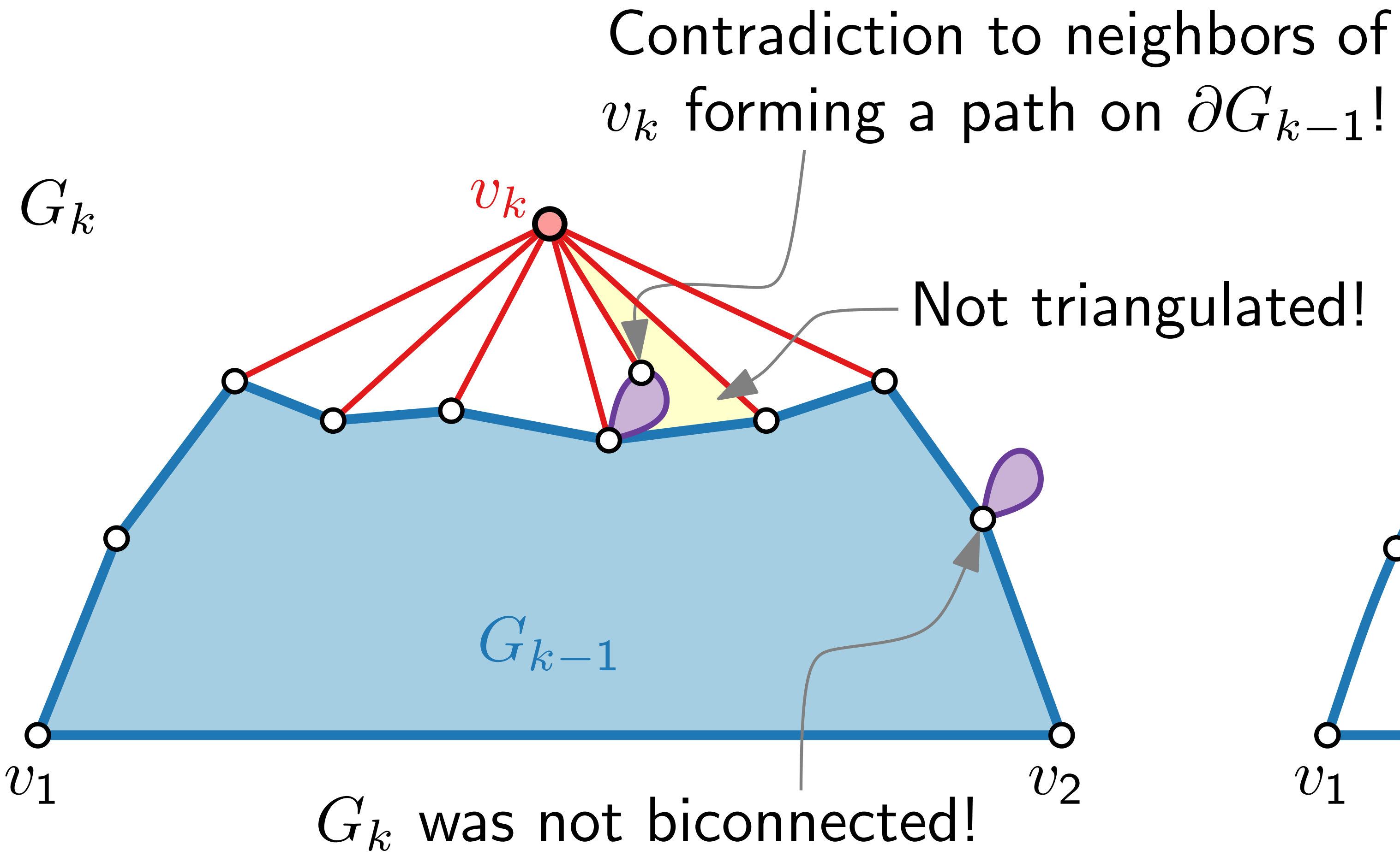

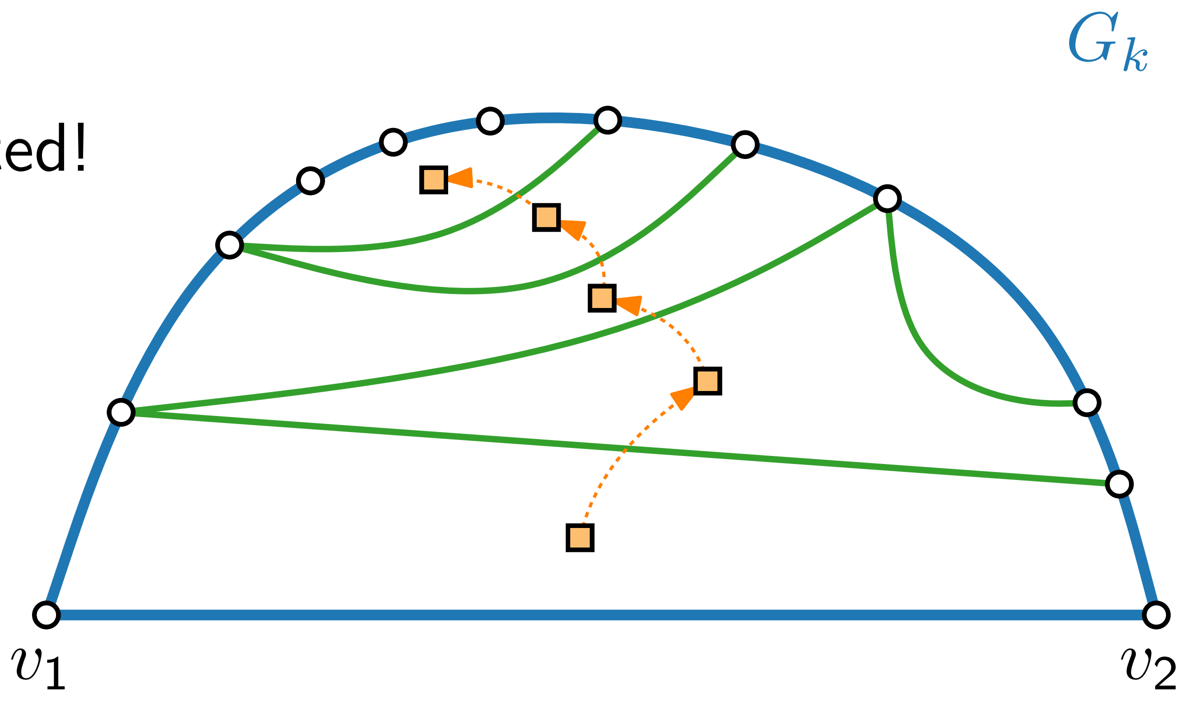

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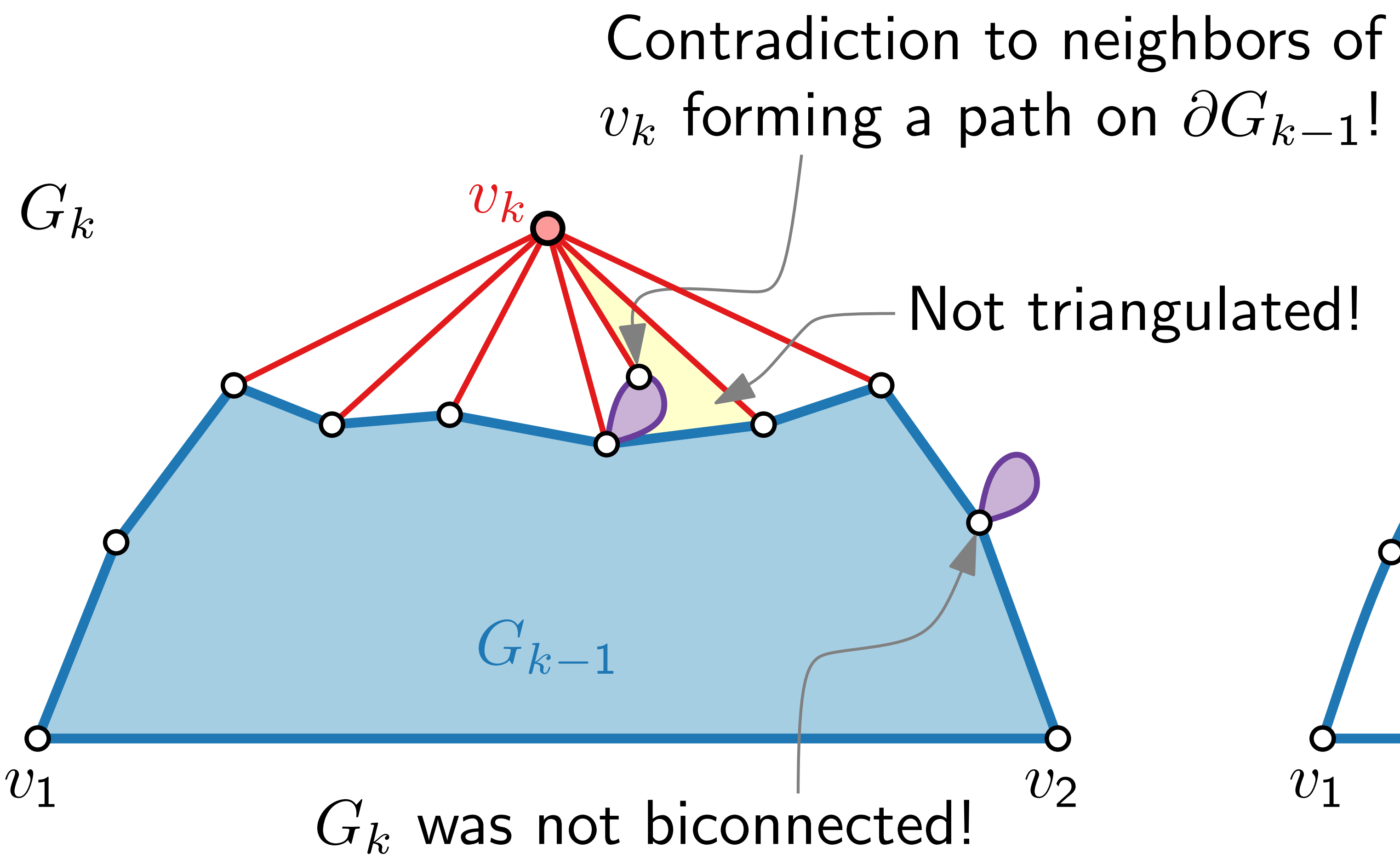

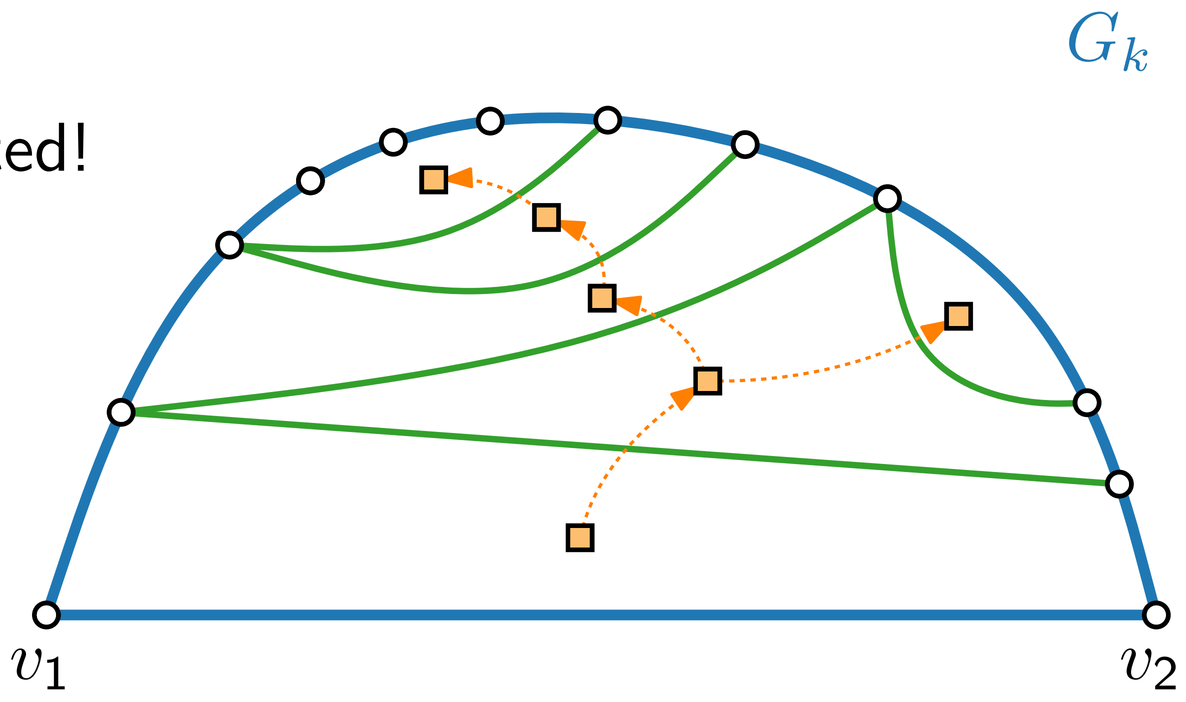

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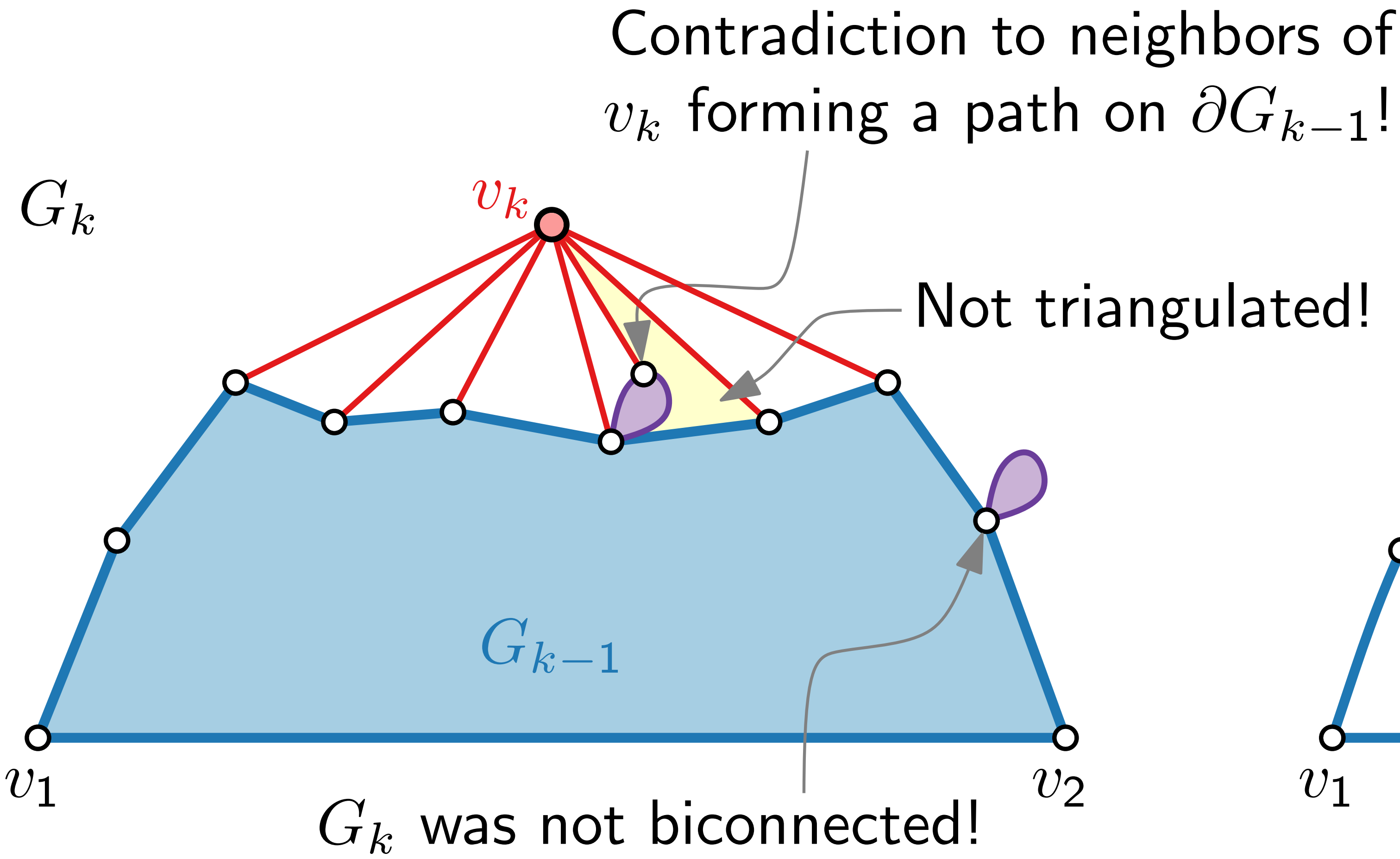

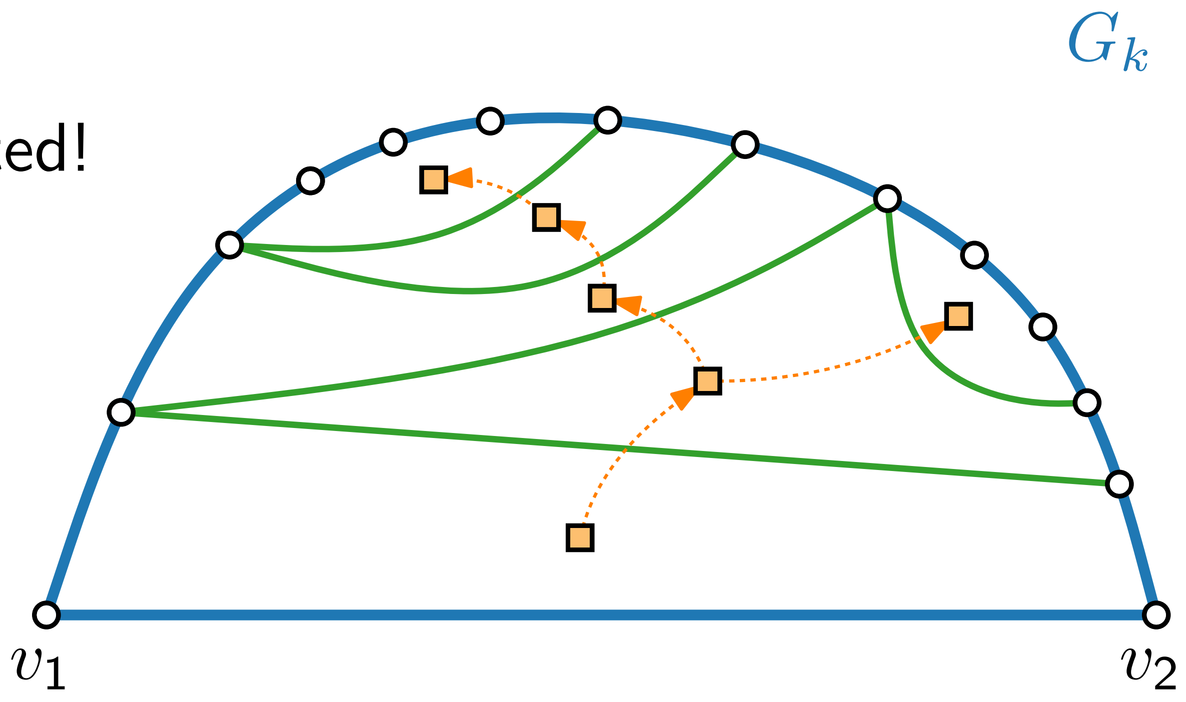

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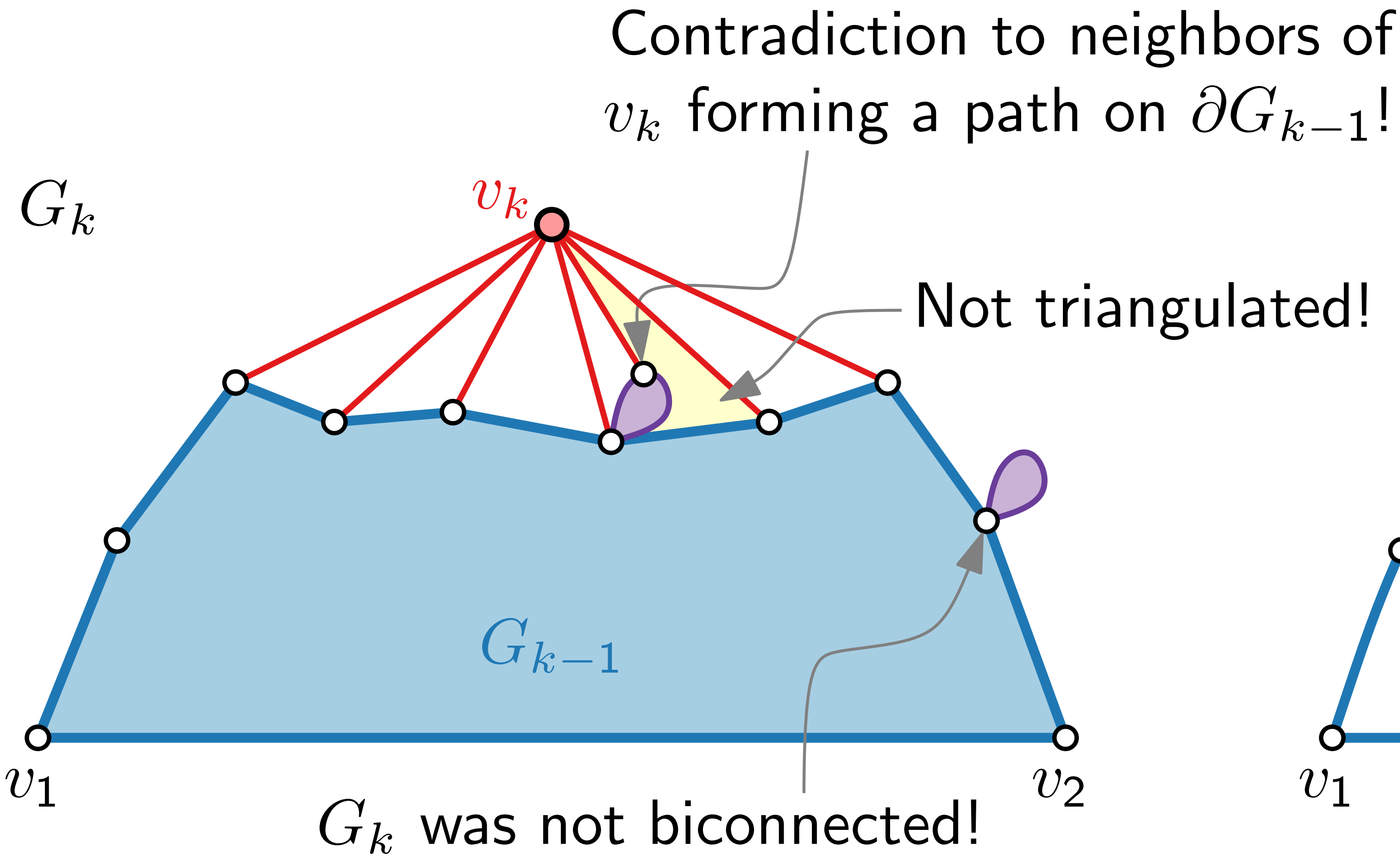

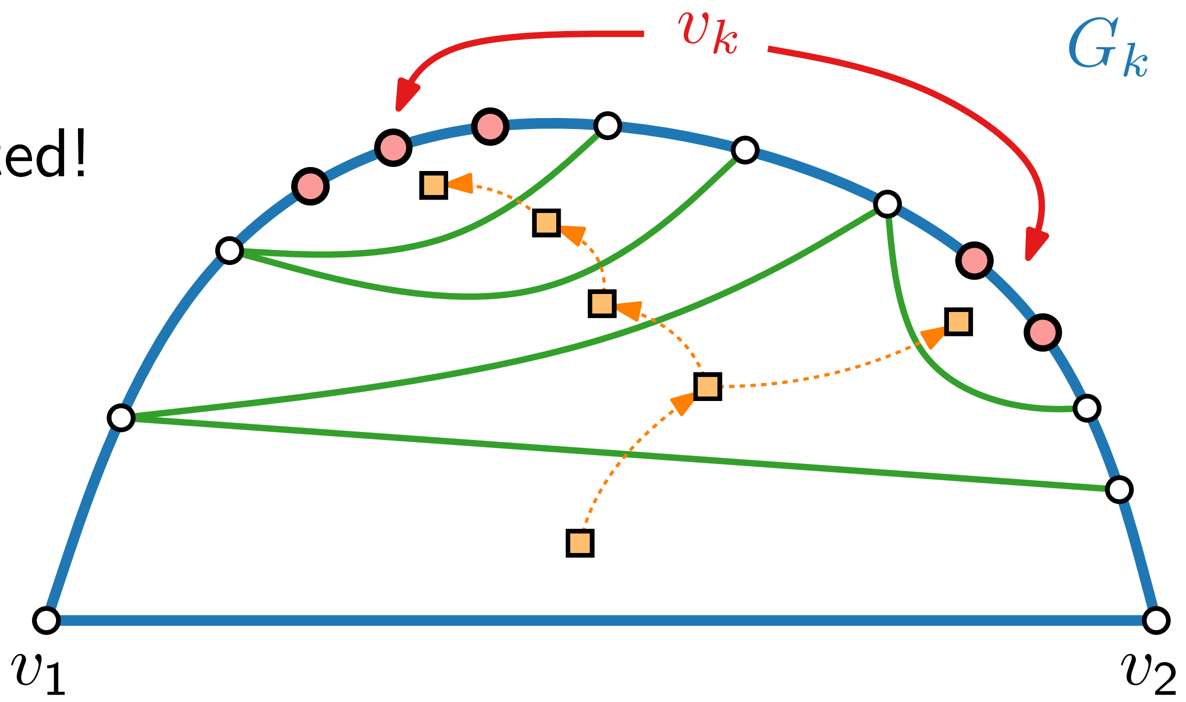

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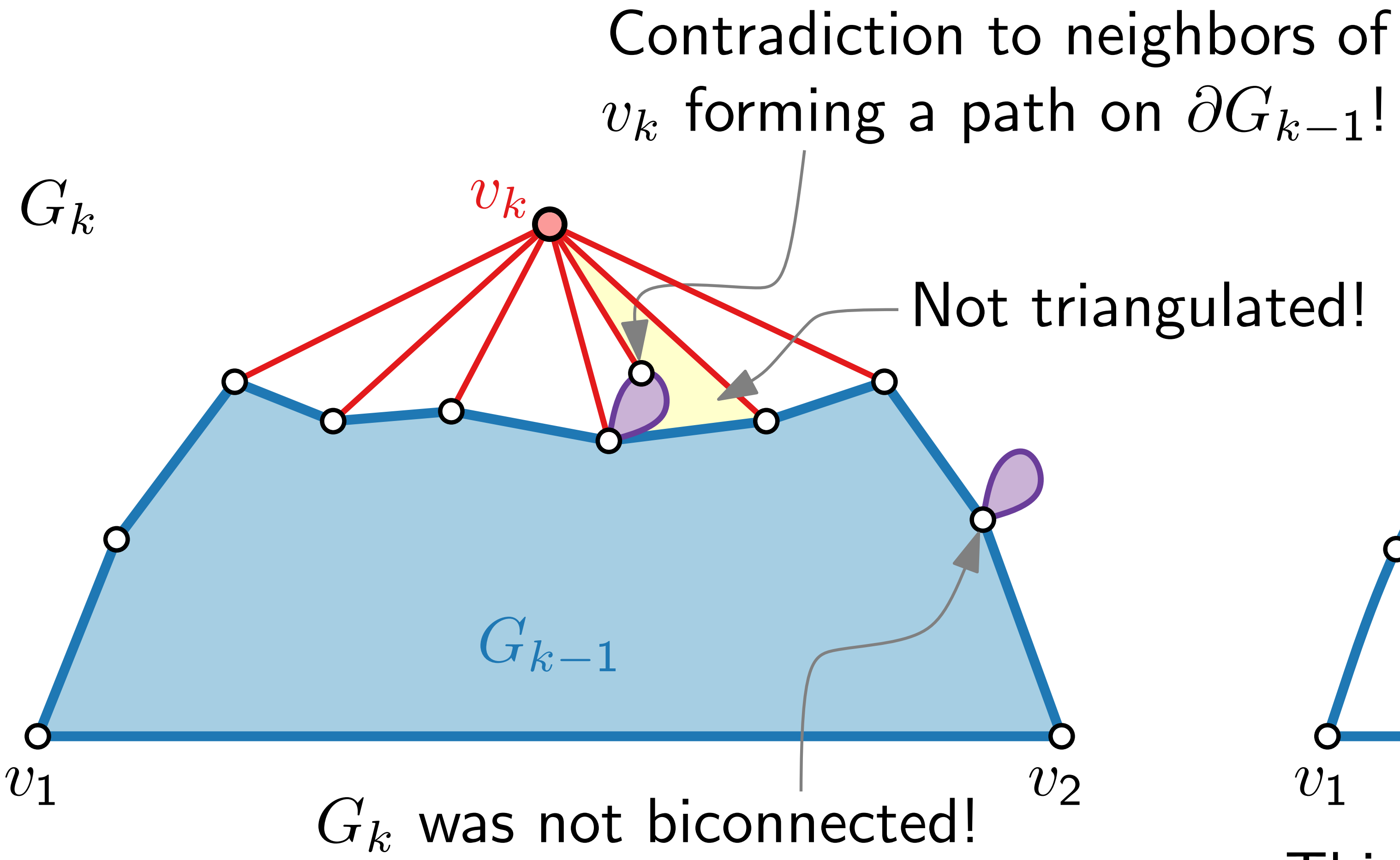

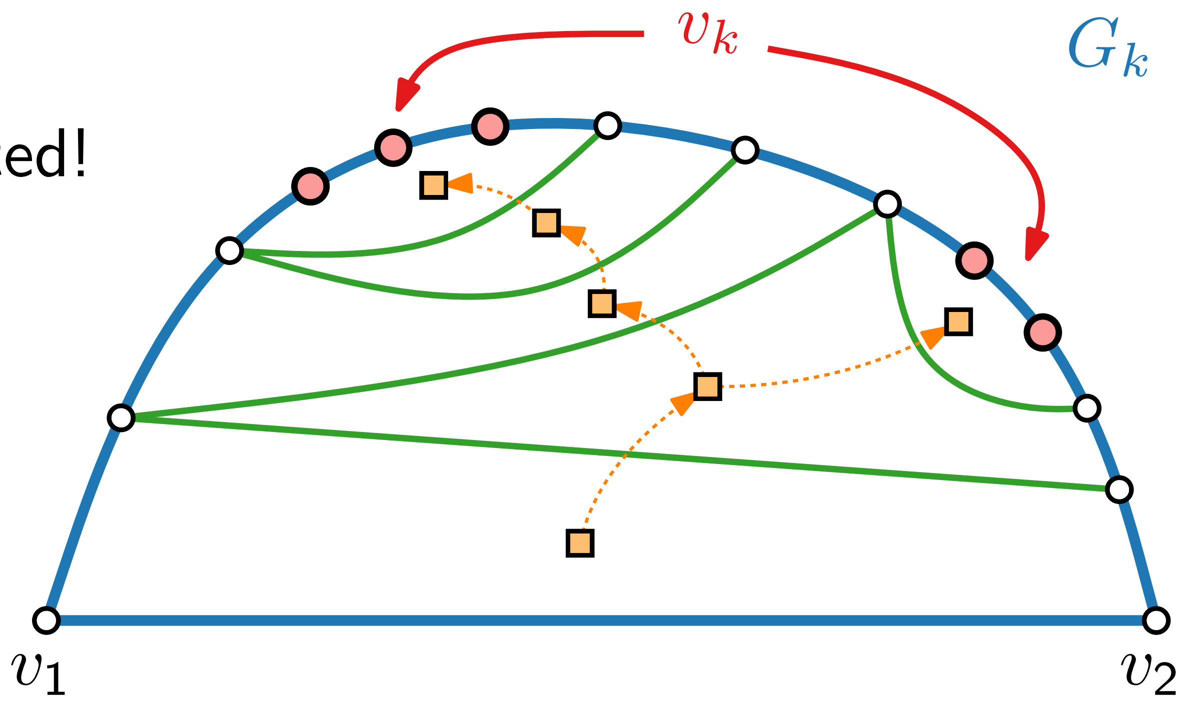

This completes the proof of the lemma. □

Canonical Order – Implementation

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

Canonical Order – Implementation

outer face

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Canonical Order – Implementation

outer face

CanonicalOrder$(G, \langle v_1, v_2, v_n \rangle)$

foreach $v \in V(G)$ **do**

Canonical Order – Implementation

outer face

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**
  chords(v) $\leftarrow$ 0;

Canonical Order – Implementation

outer face

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foreach $v \in V(G)$ **do**
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- chord(v)= # chords incident to v

Canonical Order – Implementation

outer face

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foreach $v \in V(G)$ **do**

  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false;

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Canonical Order – Implementation

outer face

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**
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- chord(v)= # chords incident to v
- out(v) = true iff v on boundary of current outer face

Canonical Order – Implementation

outer face

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Canonical Order – Implementation

outer face

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outer face

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  choose $v \in V(G) \setminus \{v_1, v_2\}$ such that:

    mark(v) $=$ false, out(v) $=$ true, chords(v) $= 0$

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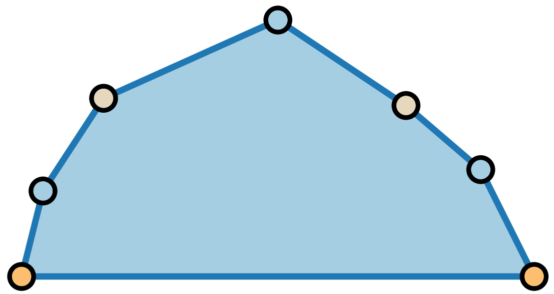

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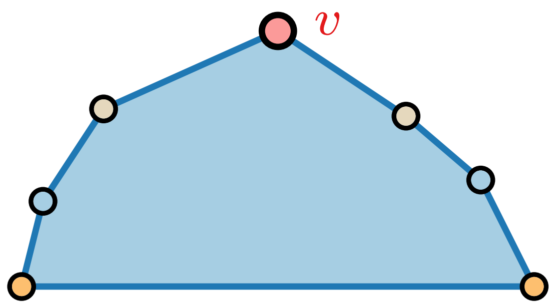

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$v_k \leftarrow v$; mark(v_k) $\leftarrow$ true; out(v_k) $\leftarrow$ false

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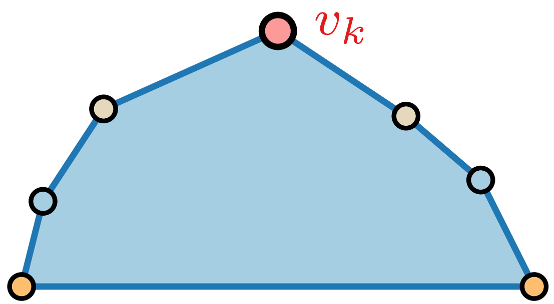

Canonical Order – Implementation

outer face

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**
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  $v_k \leftarrow v$; mark(v_k) $\leftarrow$ true; out(v_k) $\leftarrow$ false
  let $w_p, \ldots, w_q$ be the ordered unmarked neighbors of v_k

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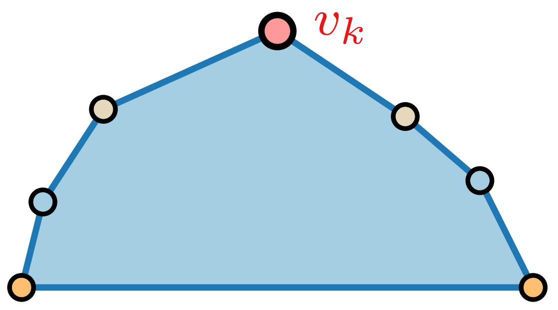

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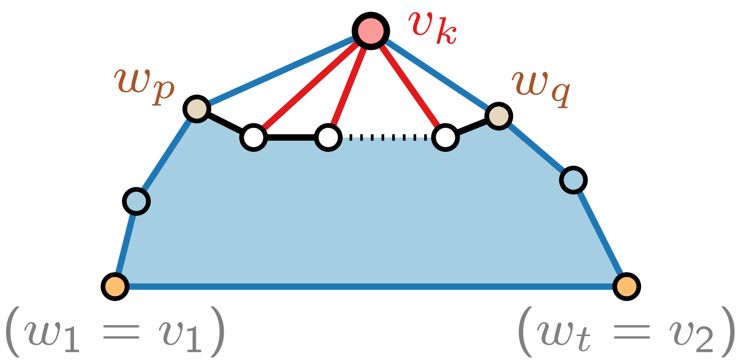

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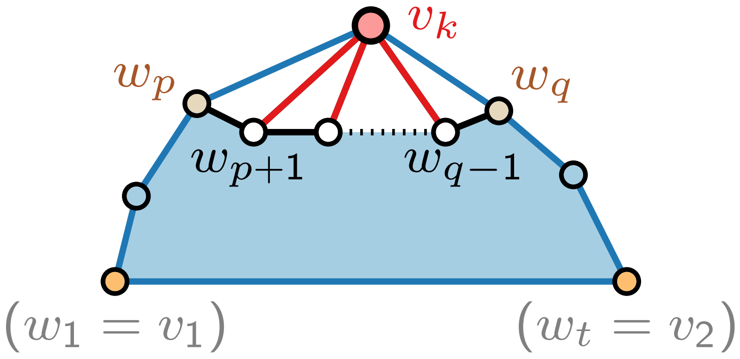

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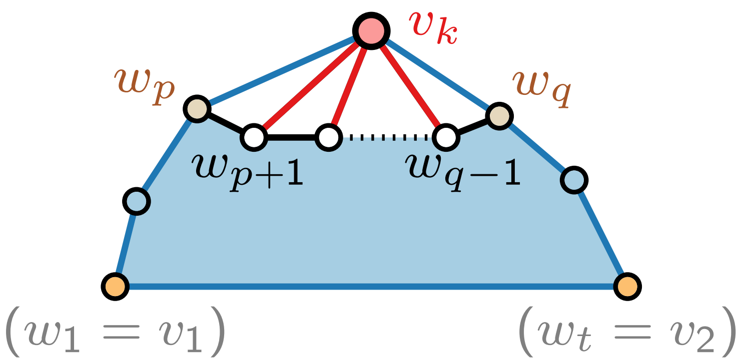

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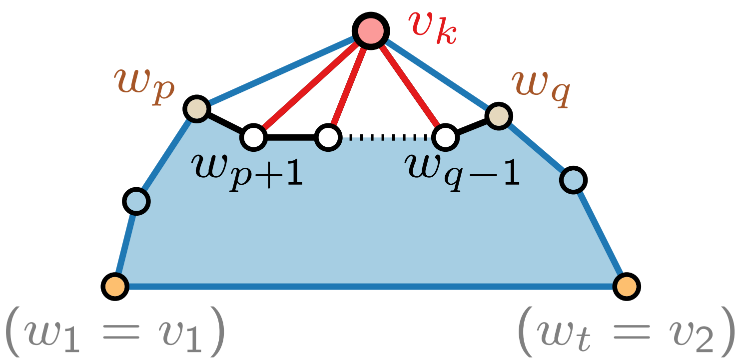

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- **for** $i = p + 1$ **to** $q - 1$ **do**
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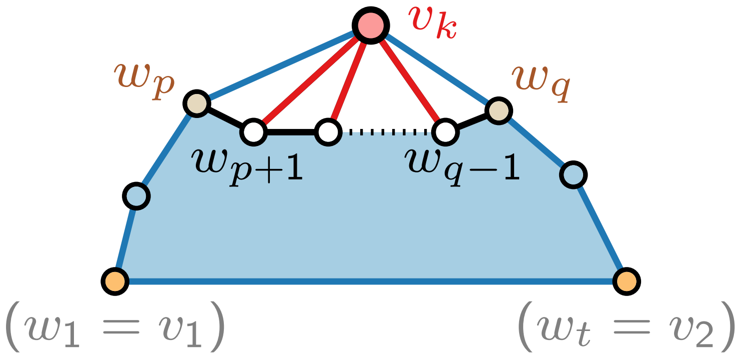

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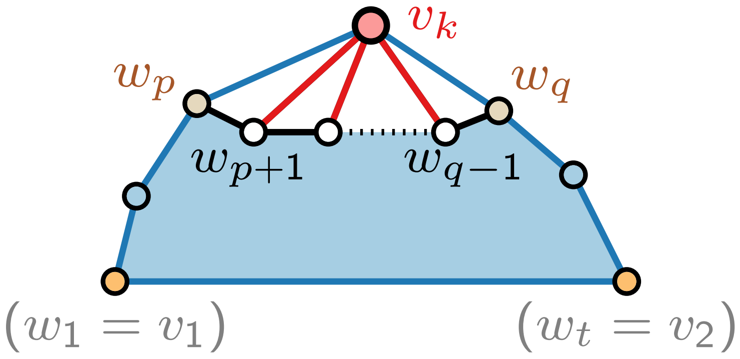

Canonical Order – Implementation

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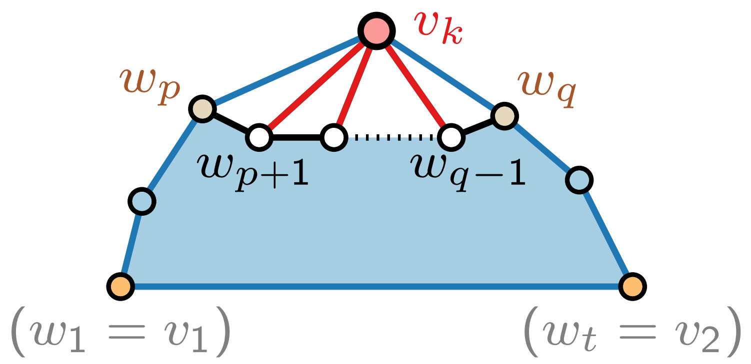

Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

Canonical Order – Implementation

outer face

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foreach $v \in V(G)$ **do**
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for $k = n$ **downto** 3 **do** *// use list of candidates*
  choose $v \in V(G) \setminus \{v_1, v_2\}$ such that:
    mark(v) = false, out(v) = true, chords(v) = 0
  $v_k \leftarrow v$; mark(v_k) $\leftarrow$ true; out(v_k) $\leftarrow$ false
  let $w_p, \dots, w_q$ be the ordered unmarked neighbors of v_k
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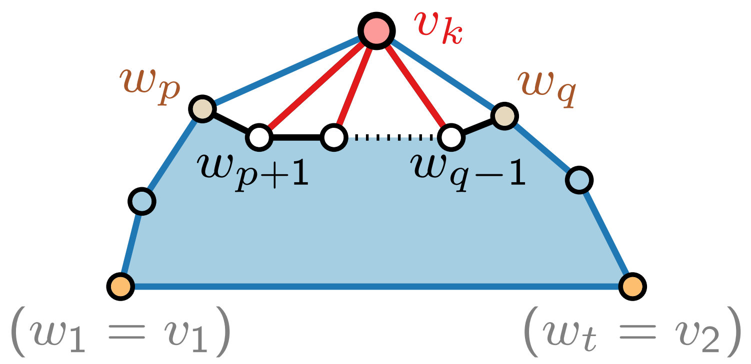

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  for i = p + 1 to q − 1 do   // O(n) time in total
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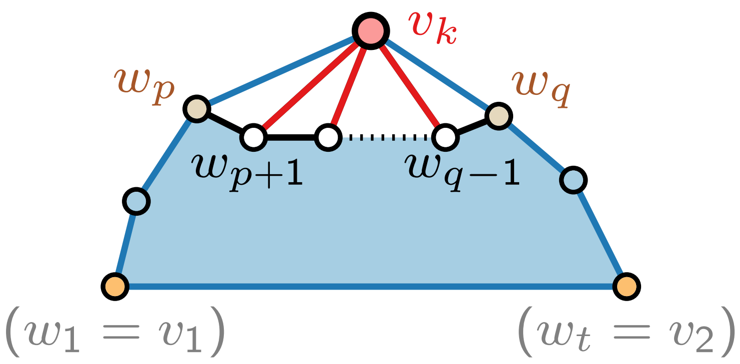

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  for i = p + 1 to q − 1 do   // O(n) time in total
    out(w_i) ← true                  // O(m) = O(n) in total
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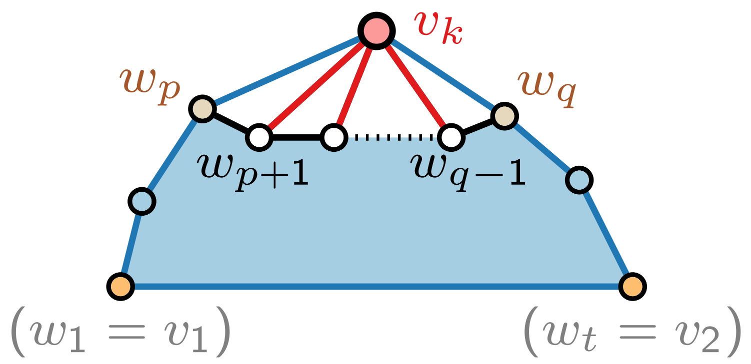

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Shift Method – Idea

Drawing invariants:

G_k is drawn such that

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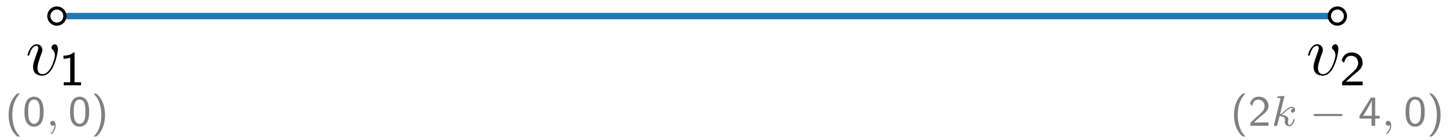

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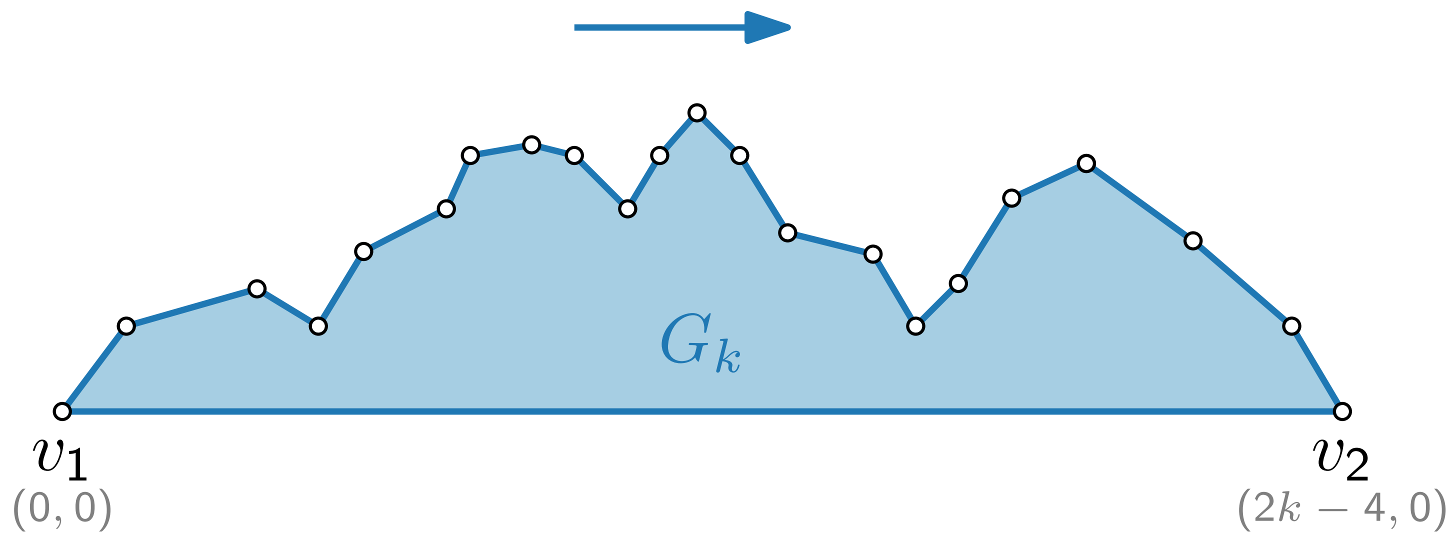

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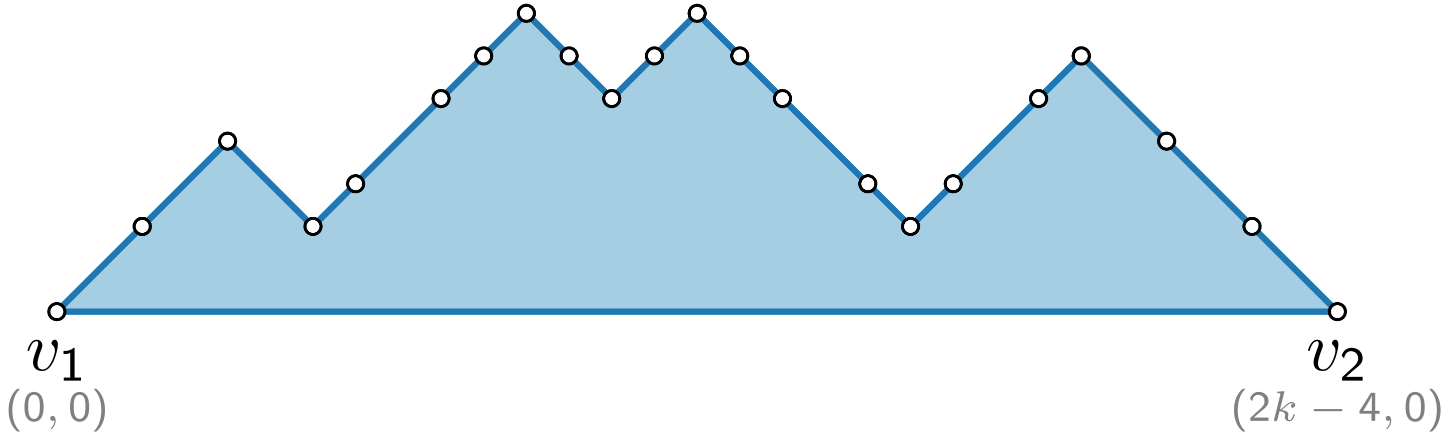

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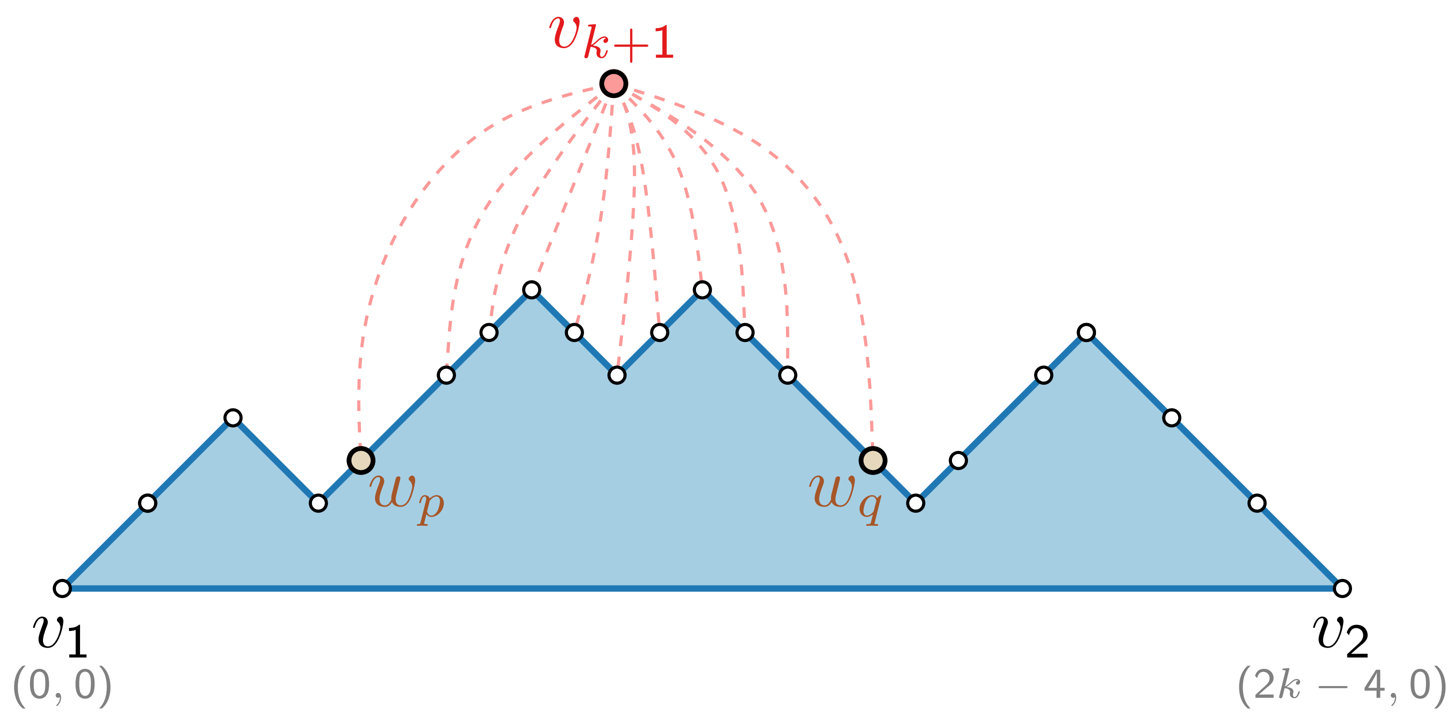

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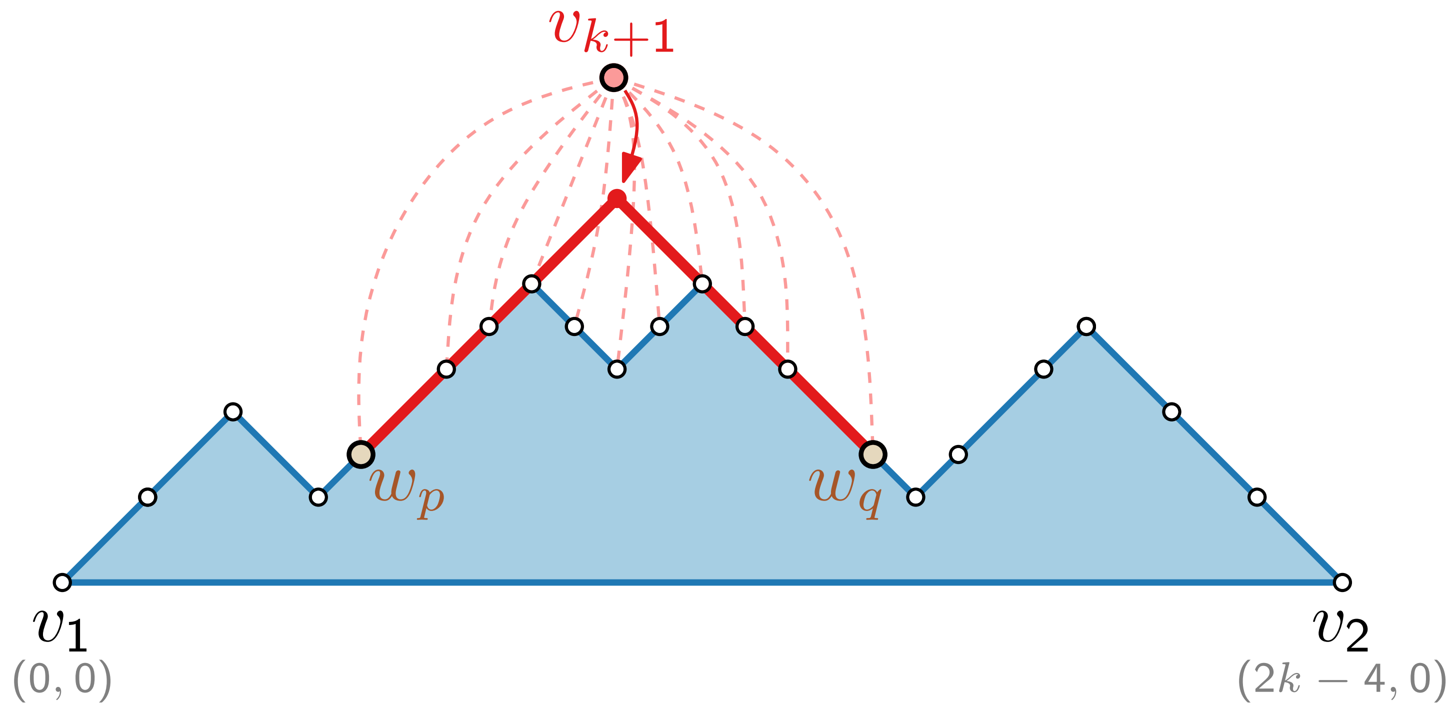

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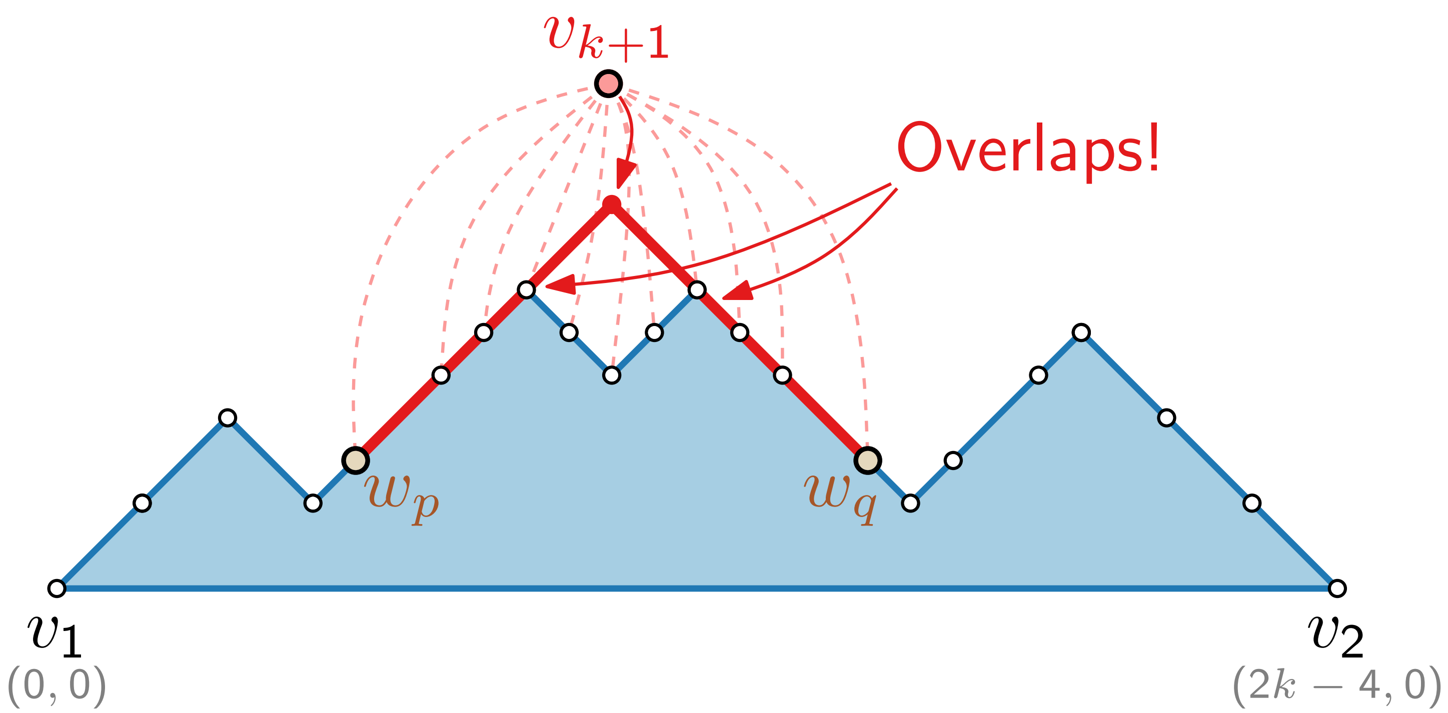

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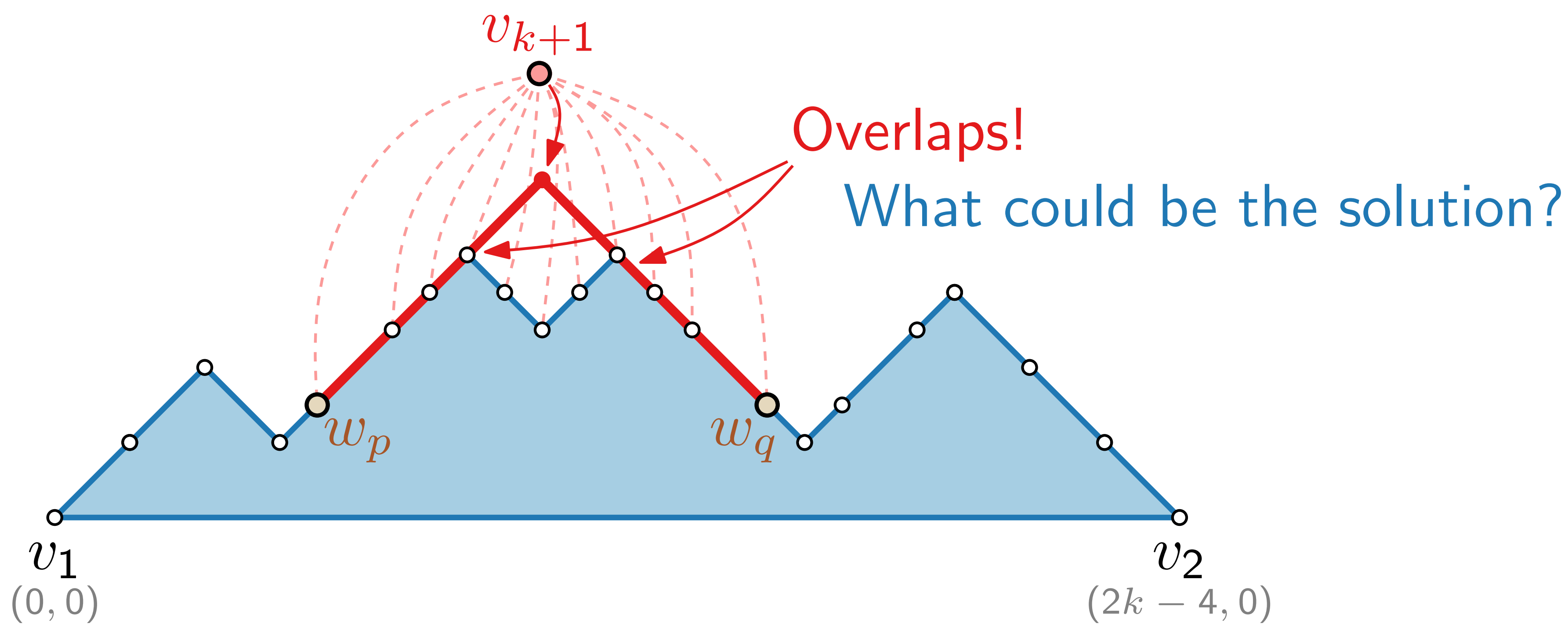

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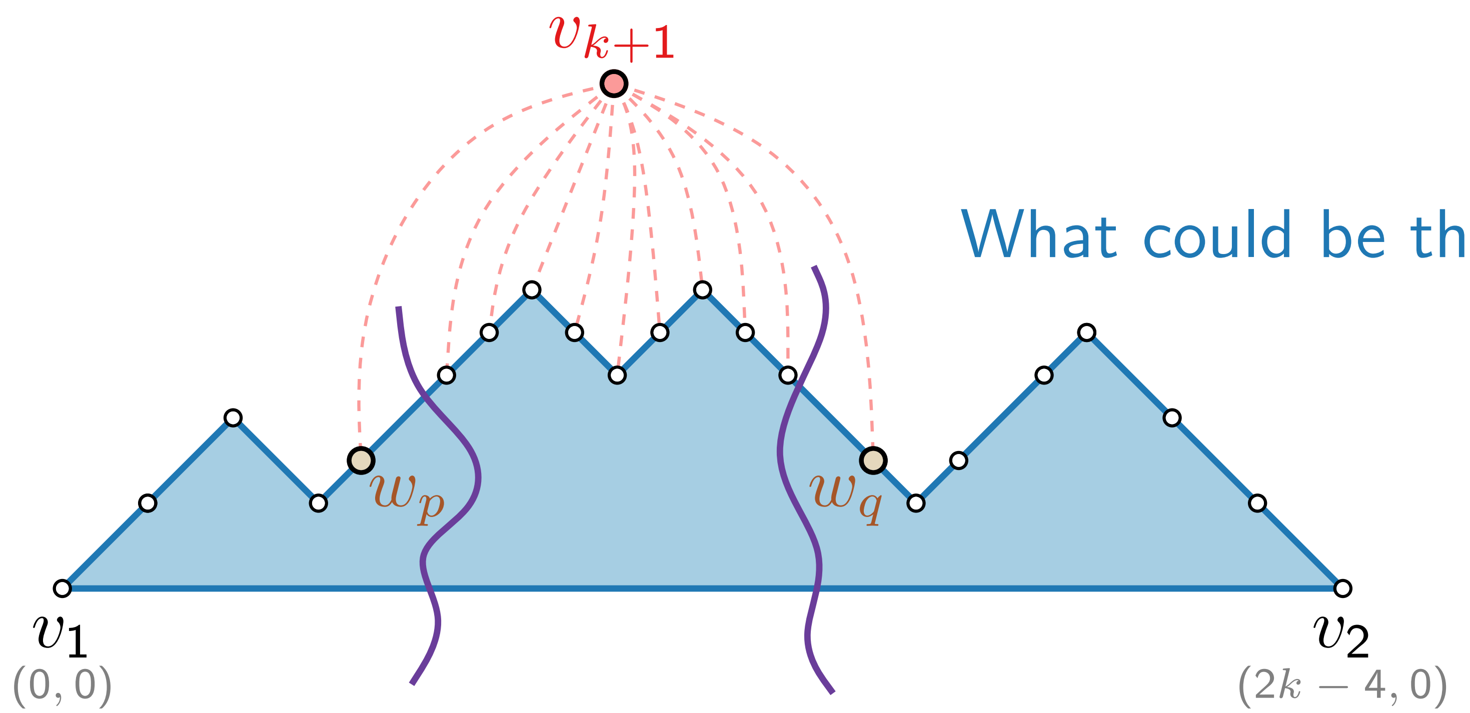

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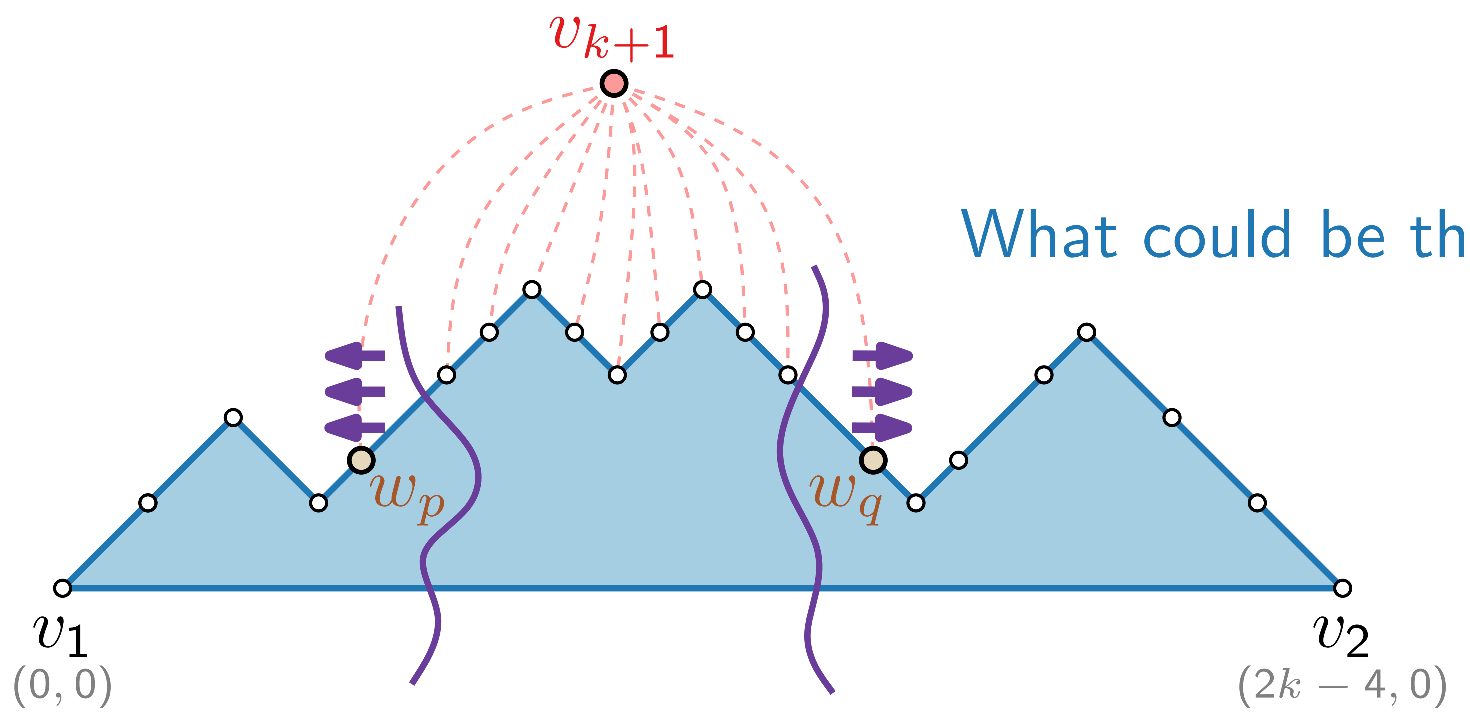

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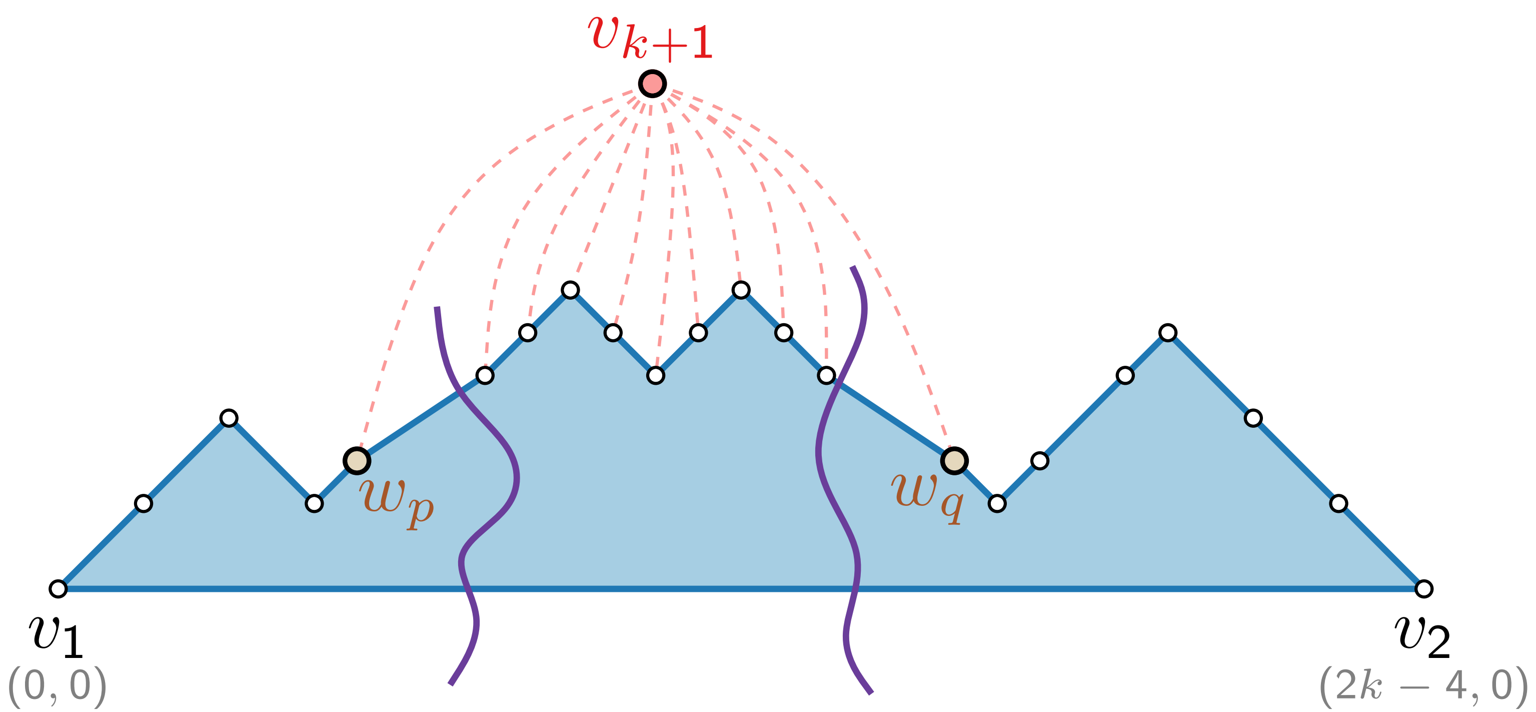

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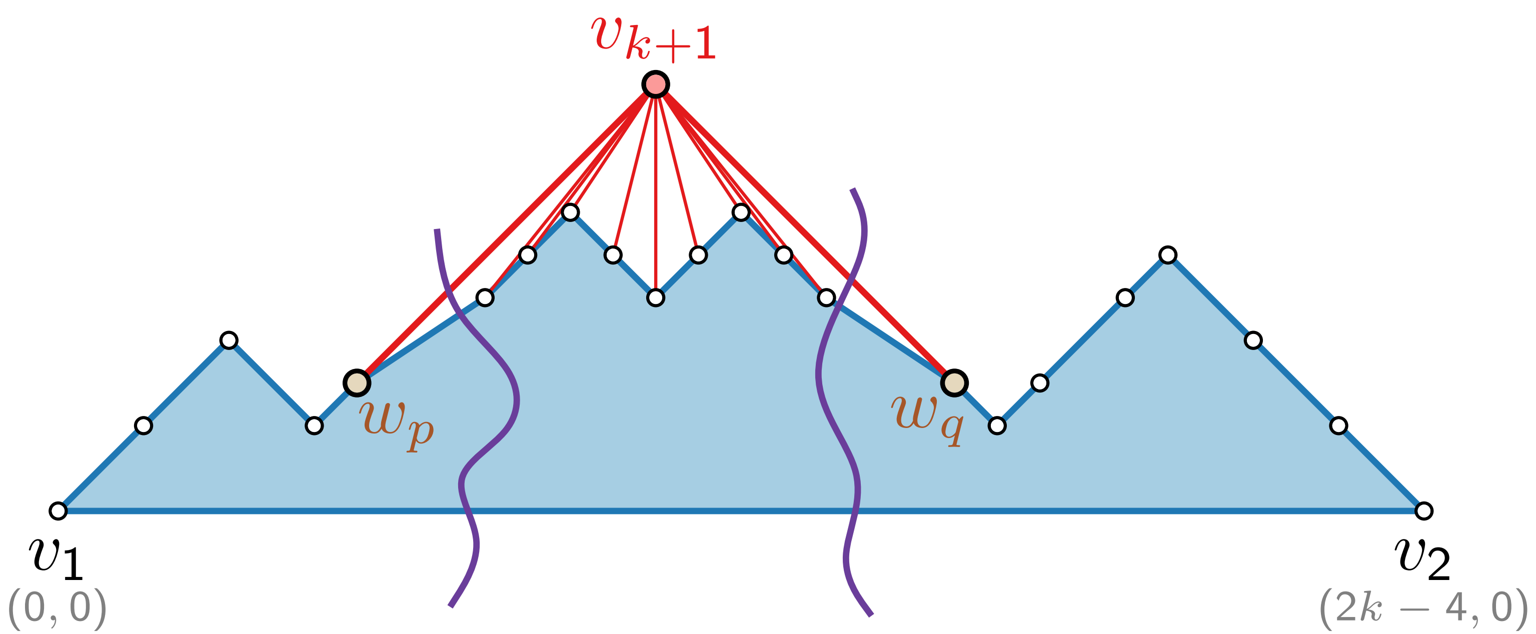

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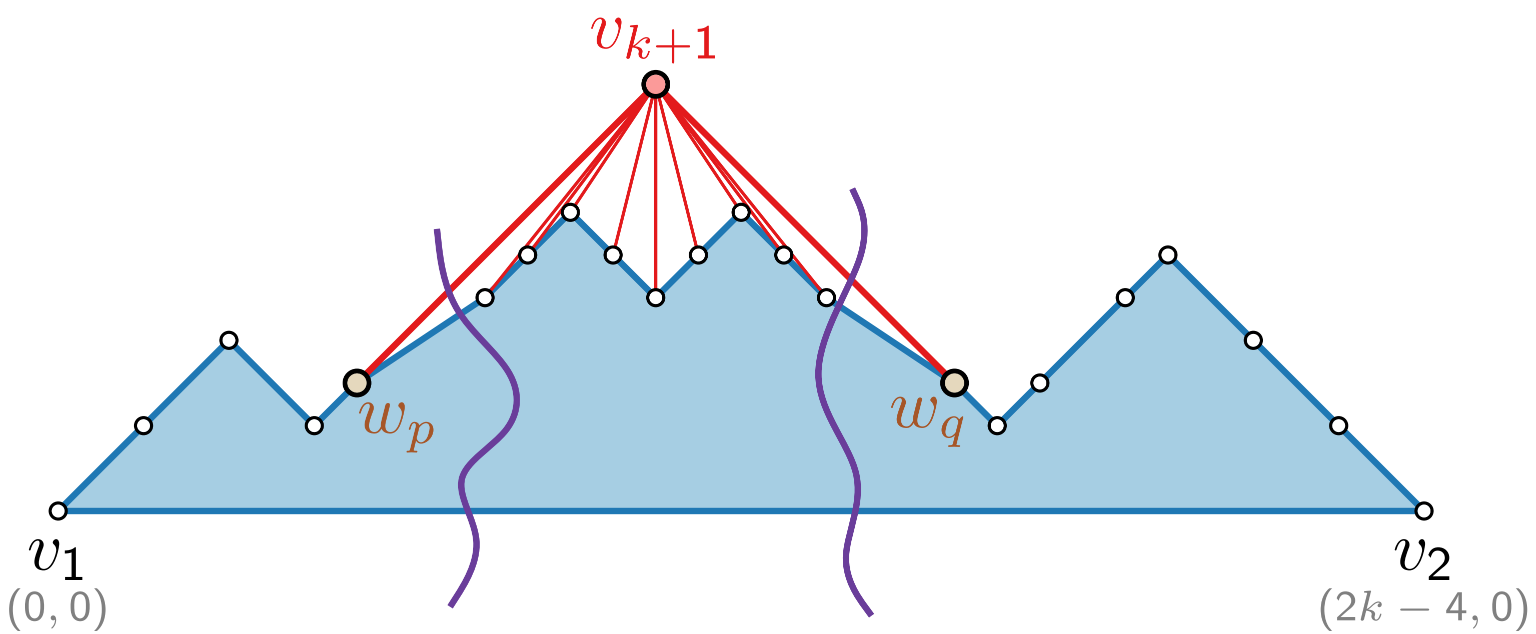
Will v_{k+1} lie on the grid?



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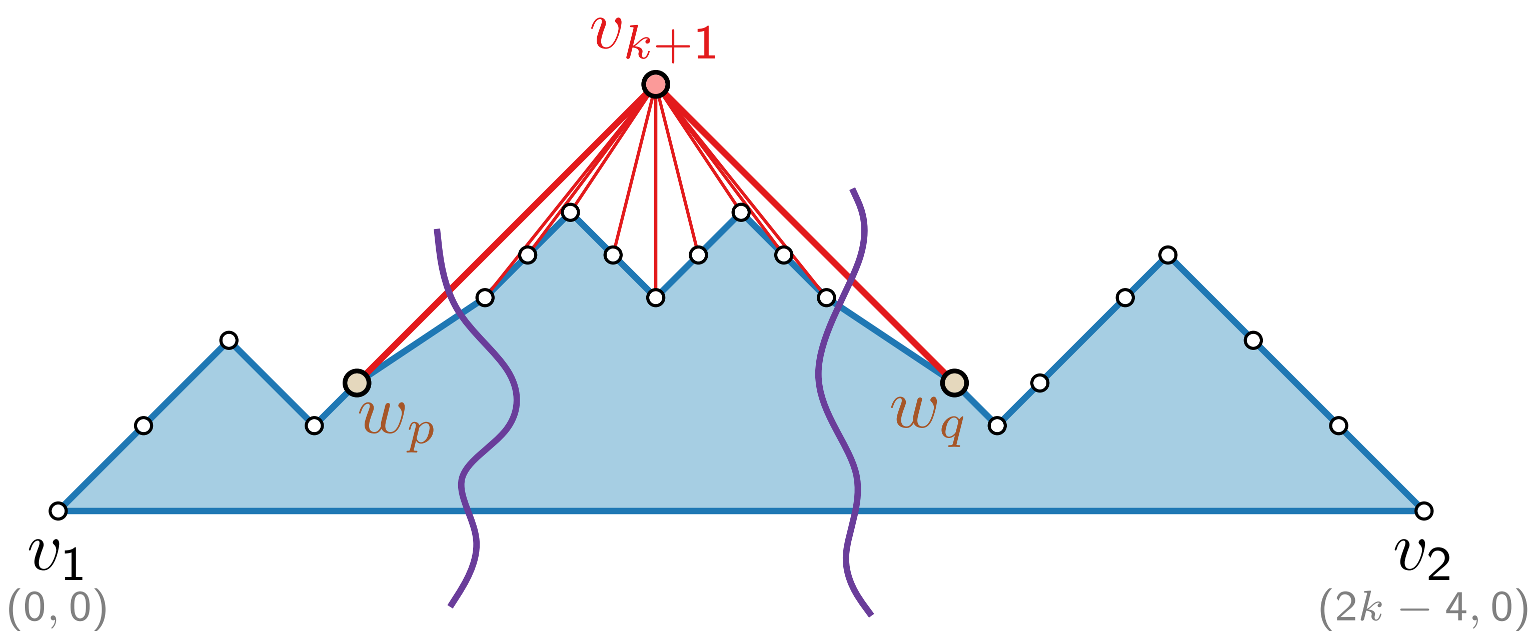

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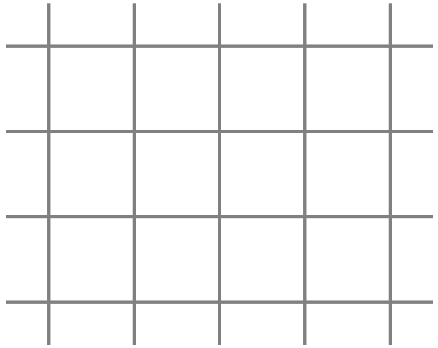

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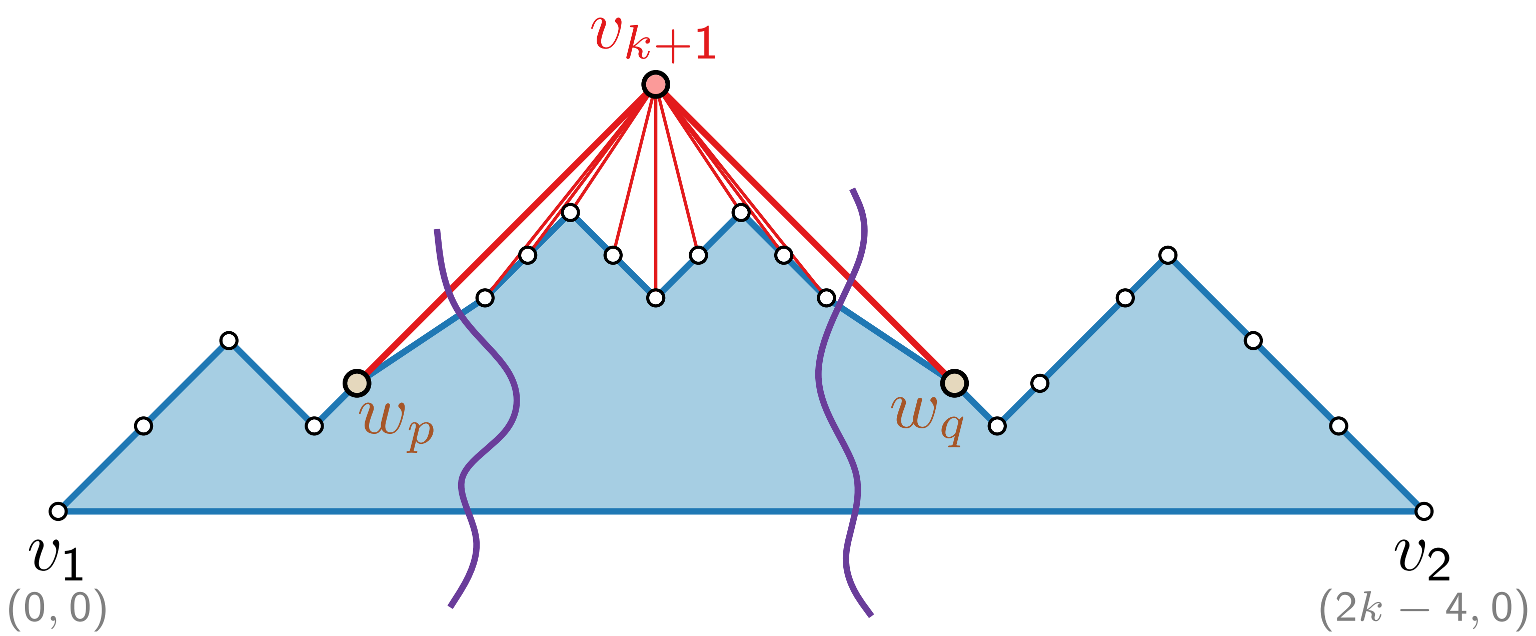
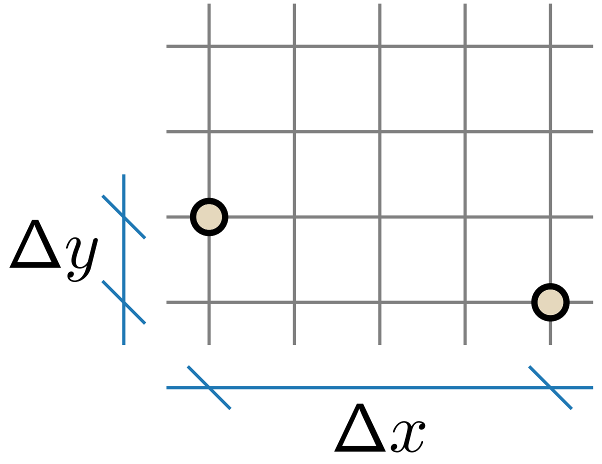


Will v_{k+1} lie on the grid?



Shift Method – Idea

Drawing invariants:

G_k is drawn such that

- v_1 is at $(0,0)$, v_2 is at $(2k-4,0)$,
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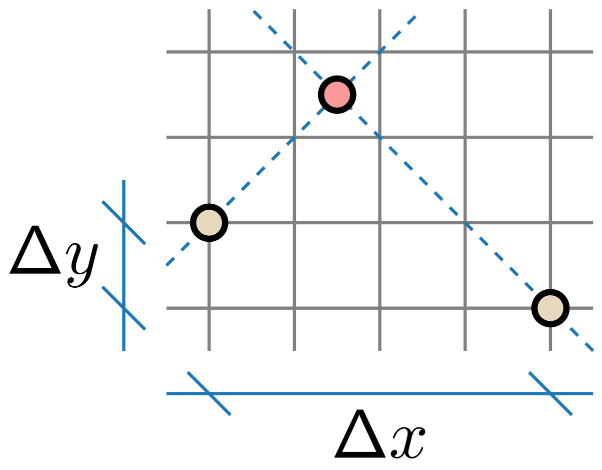

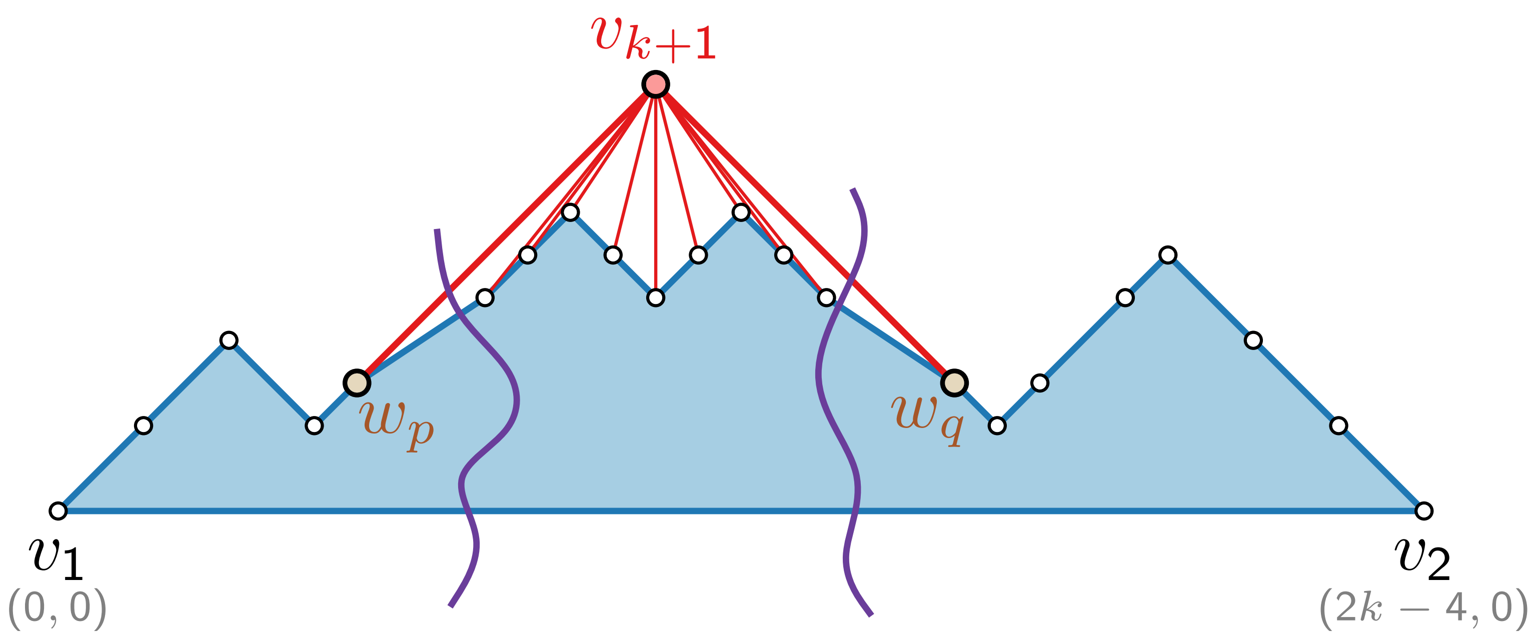

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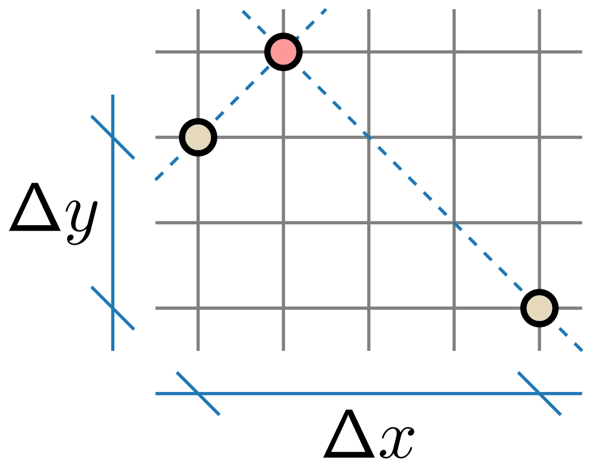

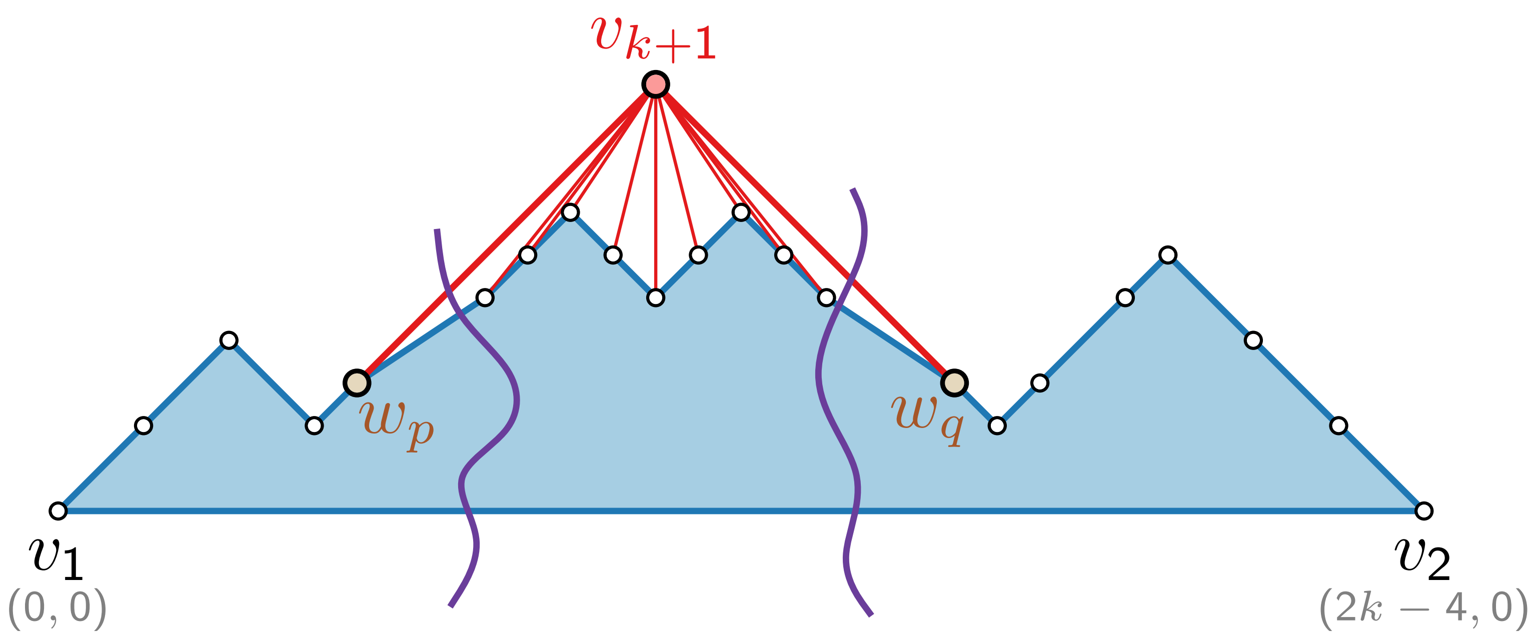

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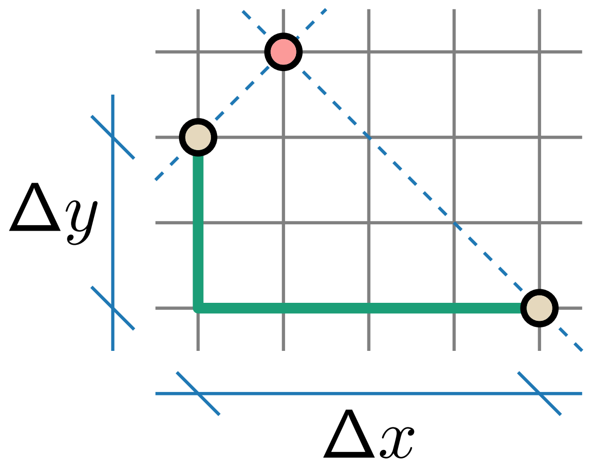

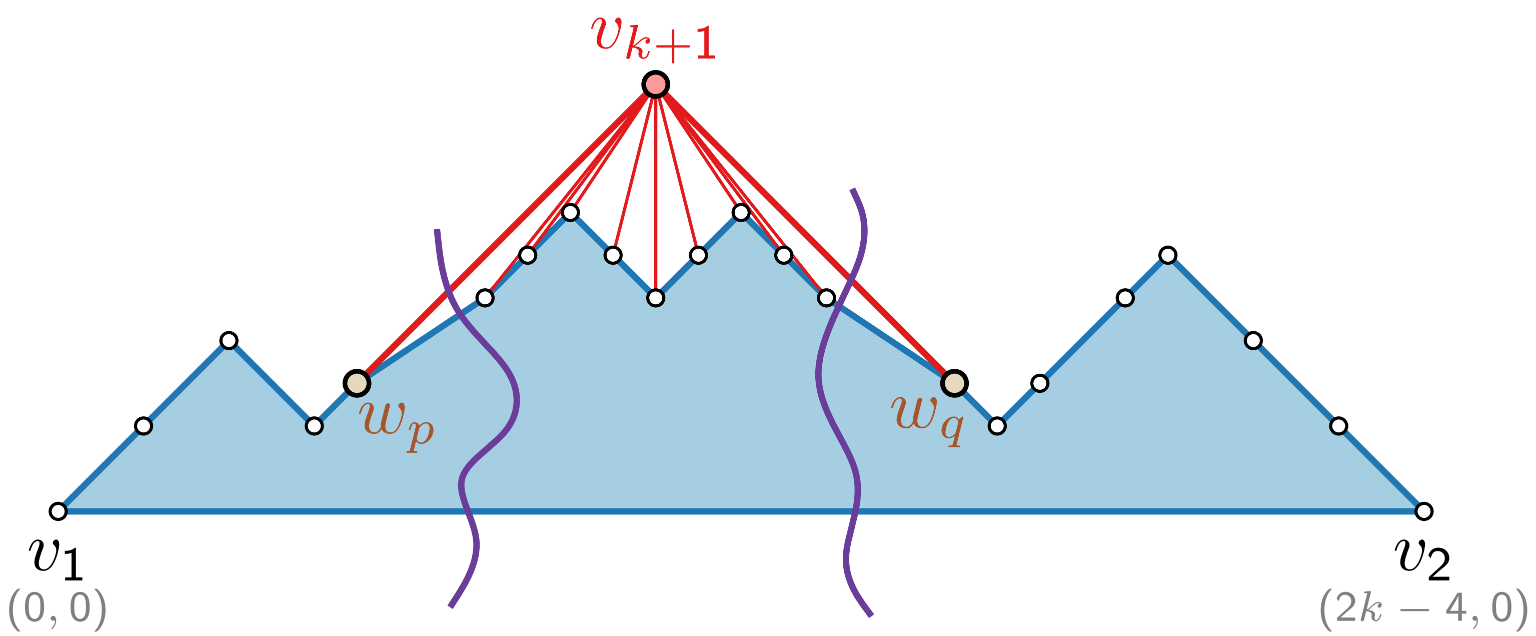

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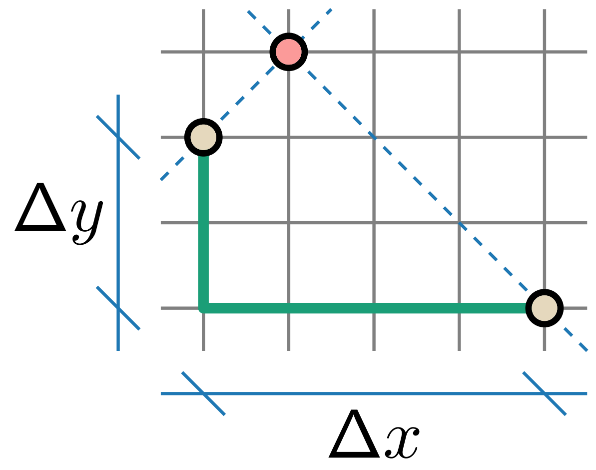
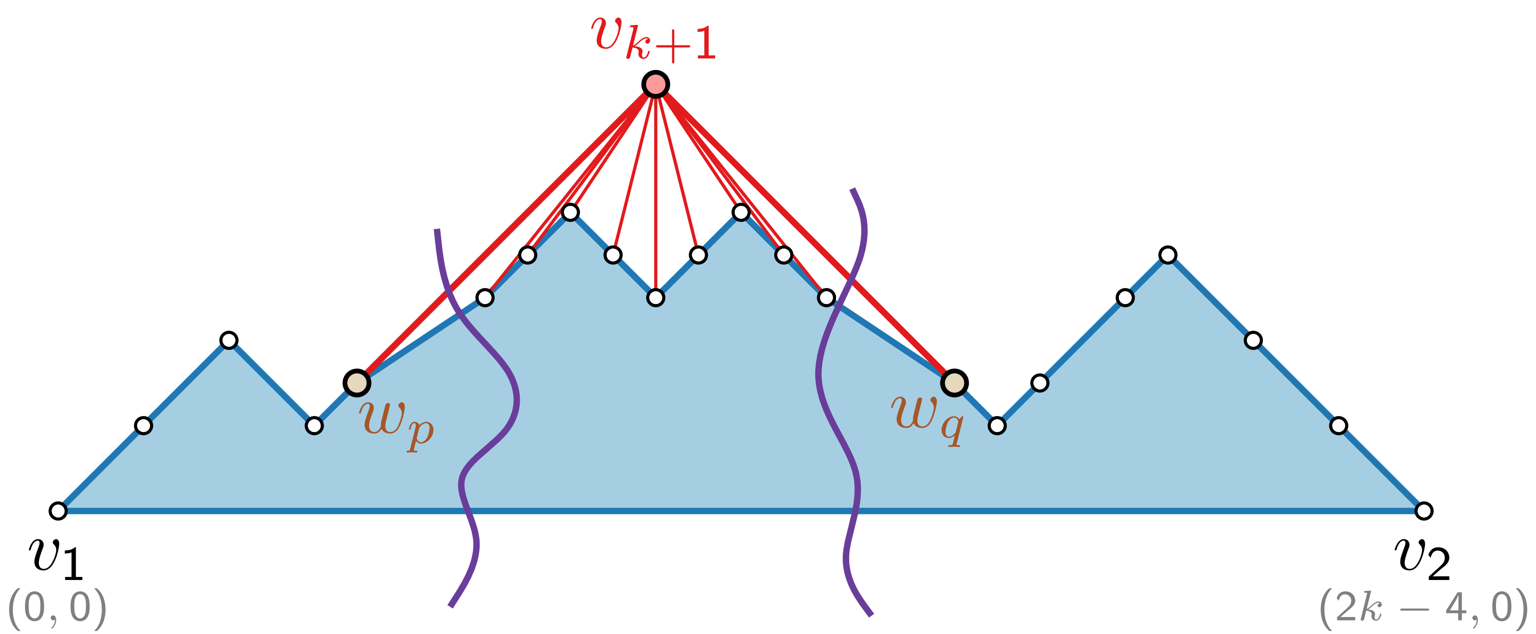


Yes, since w_p and w_q have *even* Manhattan distance $\Delta x + \Delta y$.



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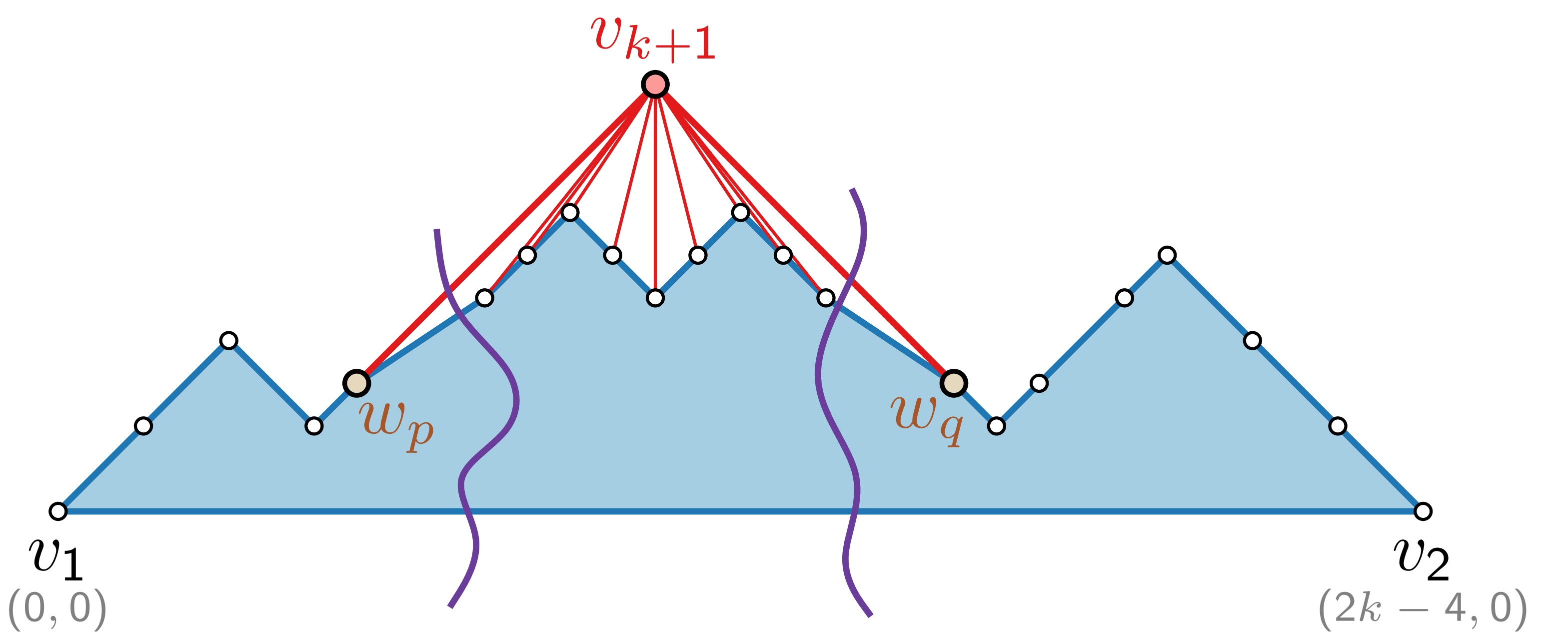

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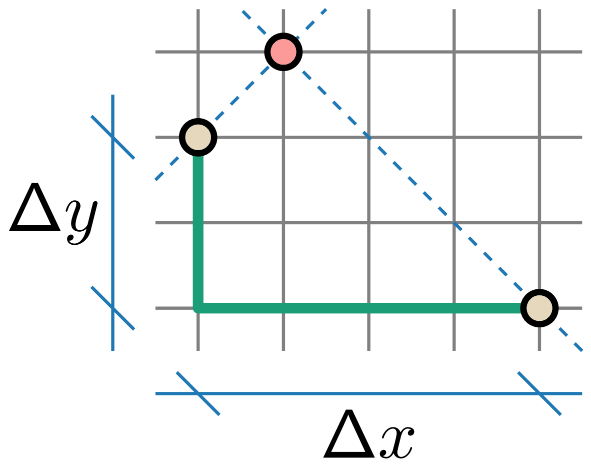

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If we follow the boundary of G_k from w_p to w_q, then from one grid point g to the next, the current Manhattan distance from w_p to g changes by 0 or 2.

Shift Method – Idea

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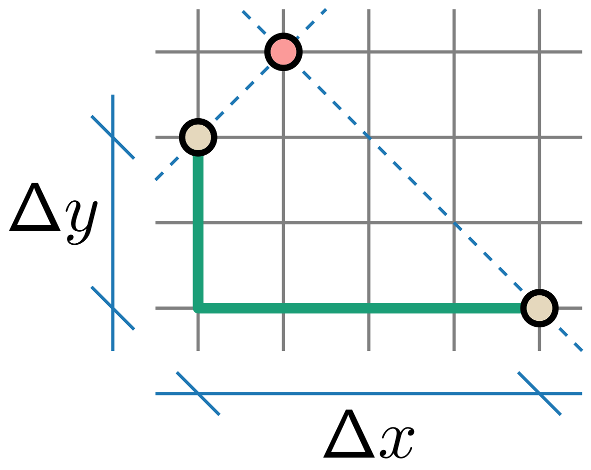

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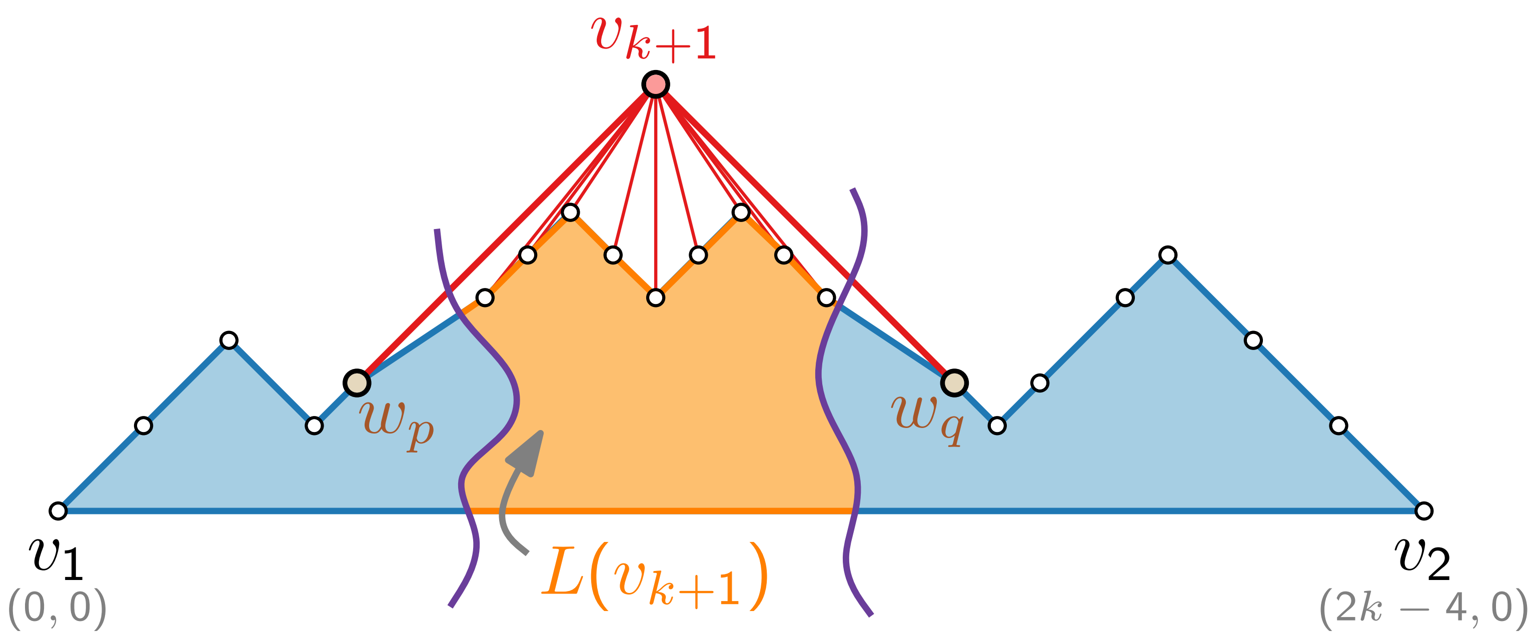

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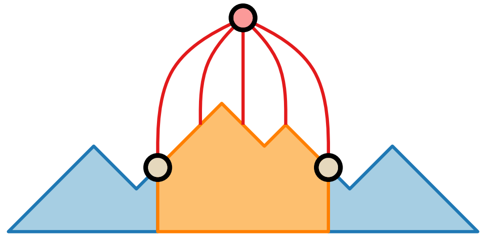

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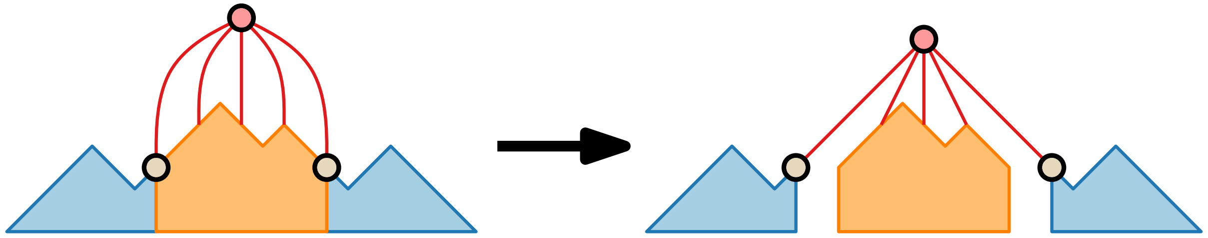

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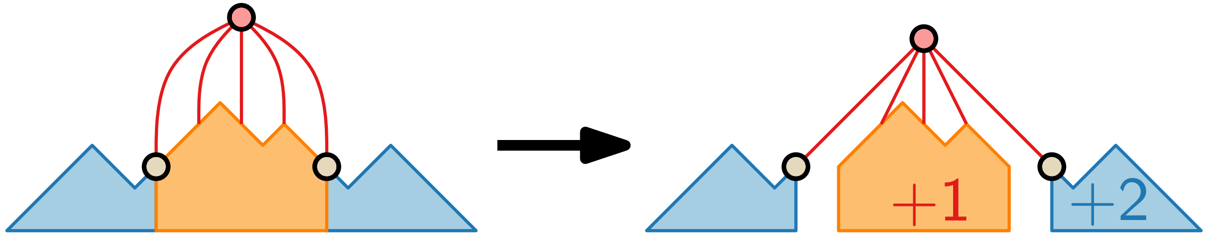

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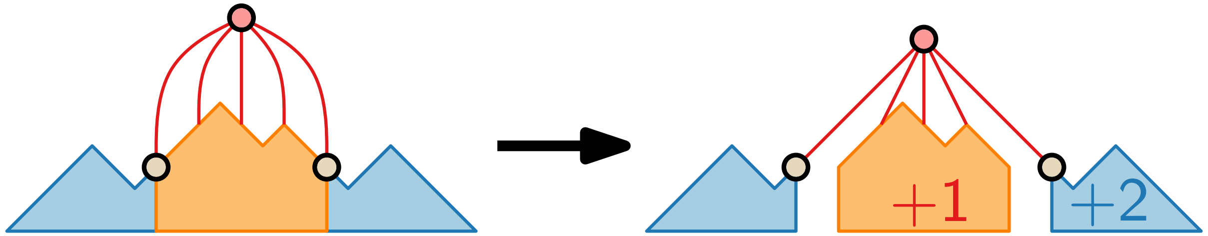

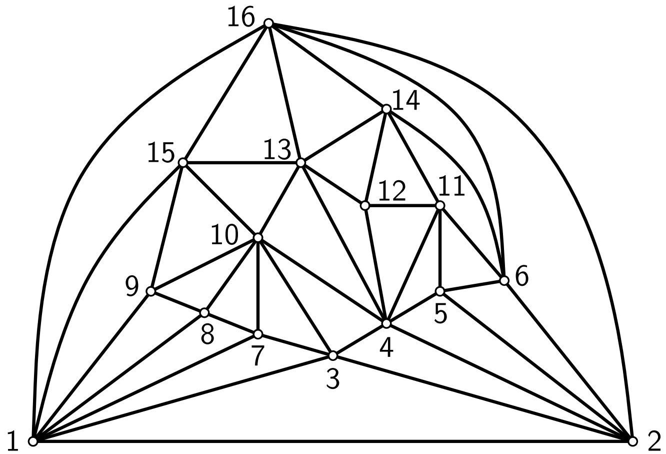

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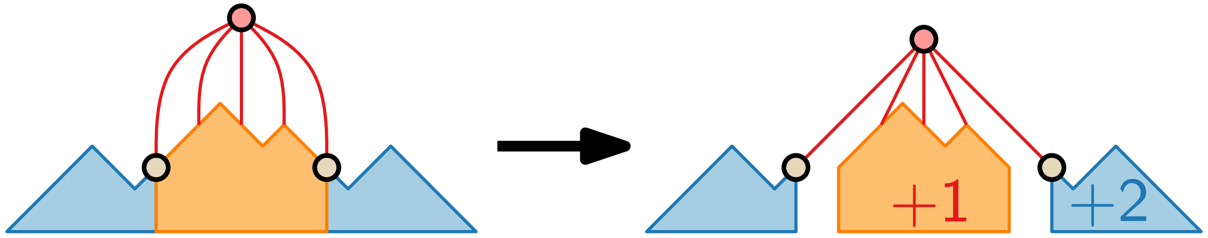

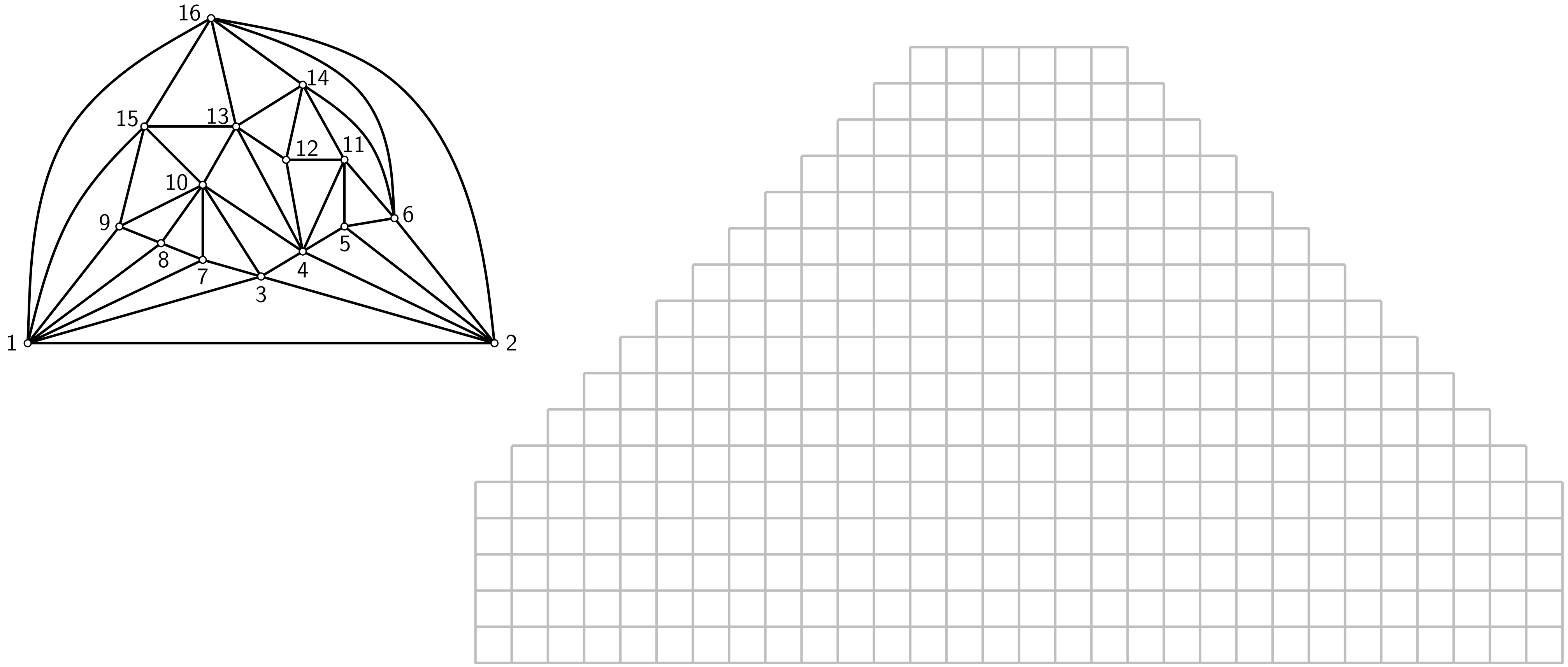

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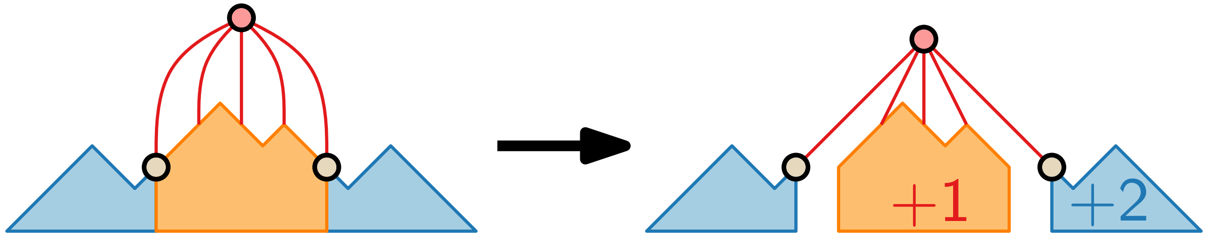

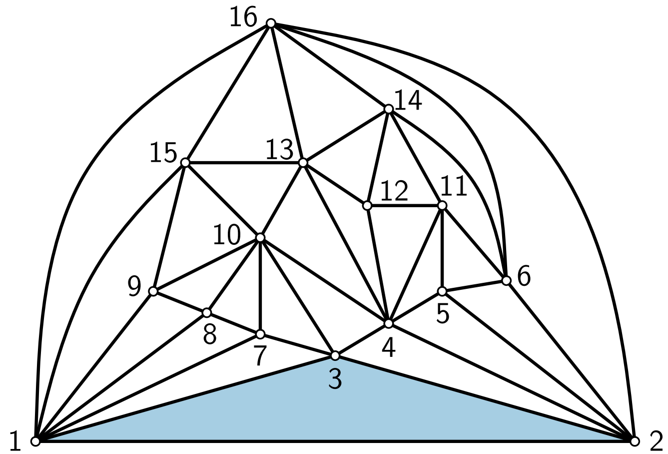

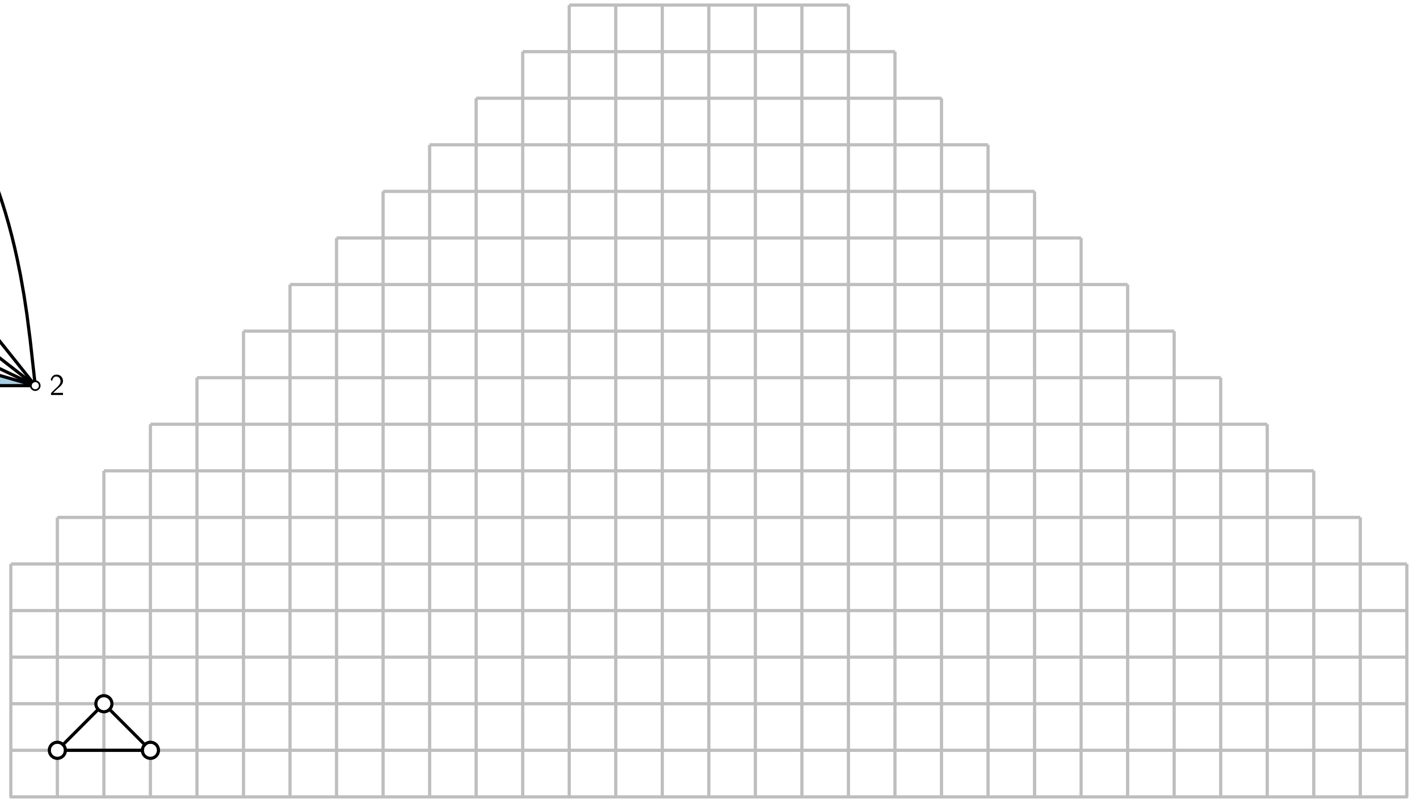

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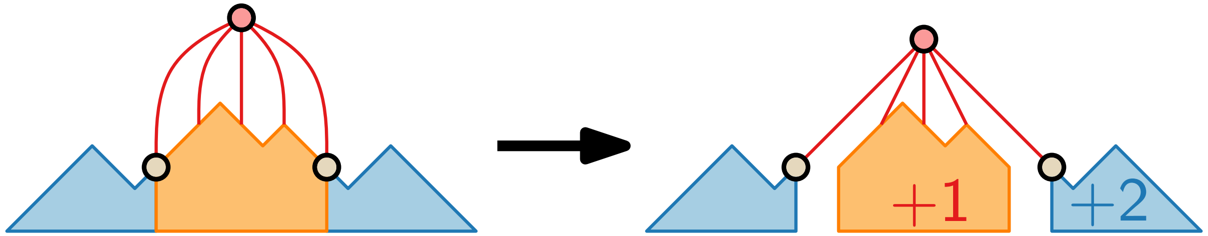

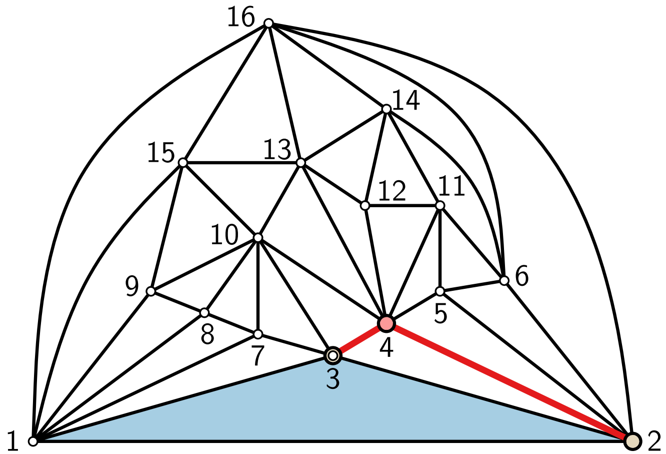

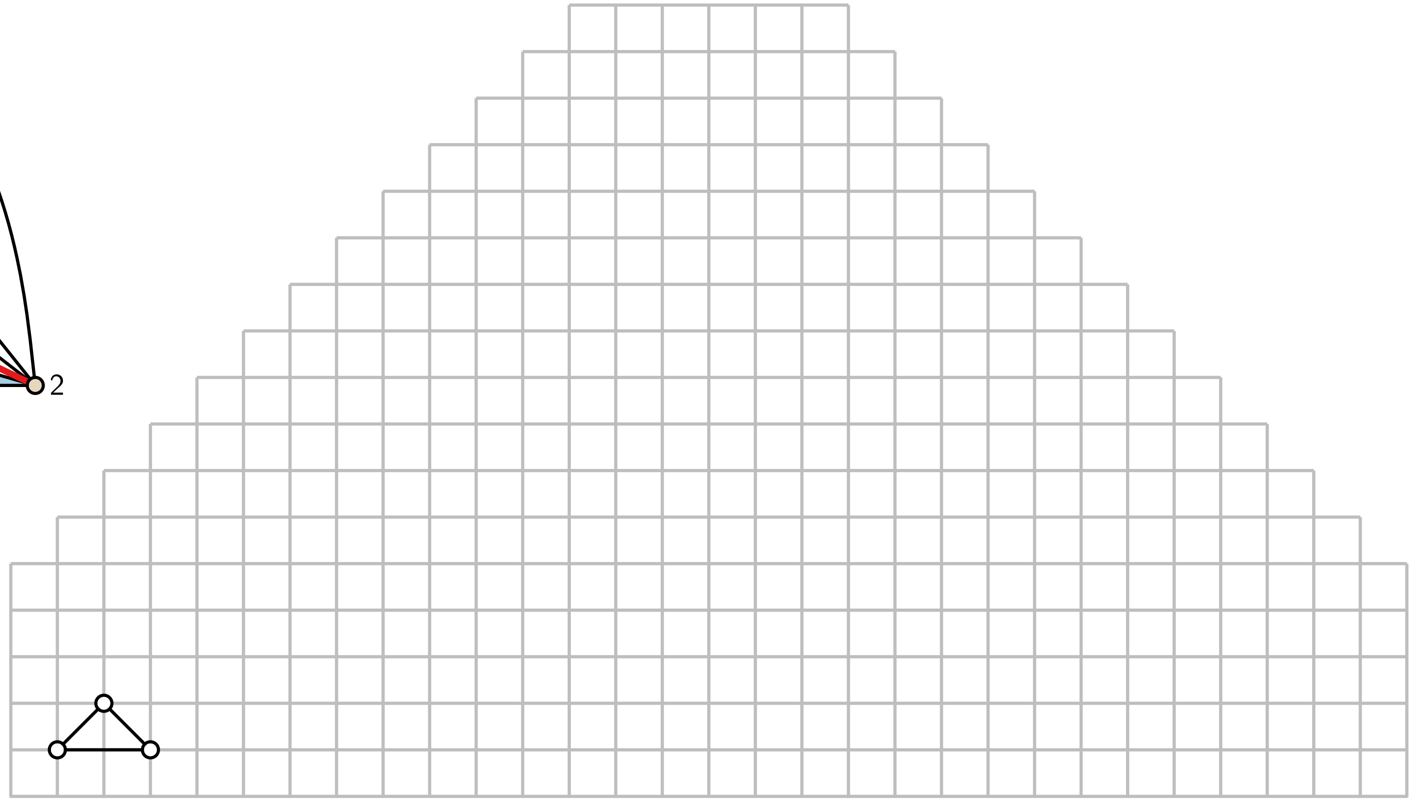

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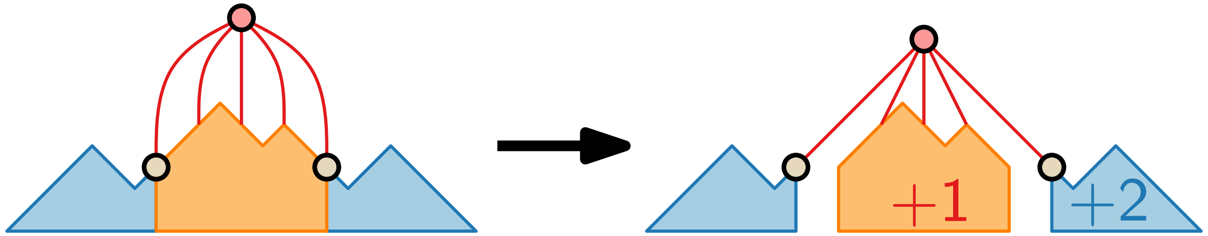

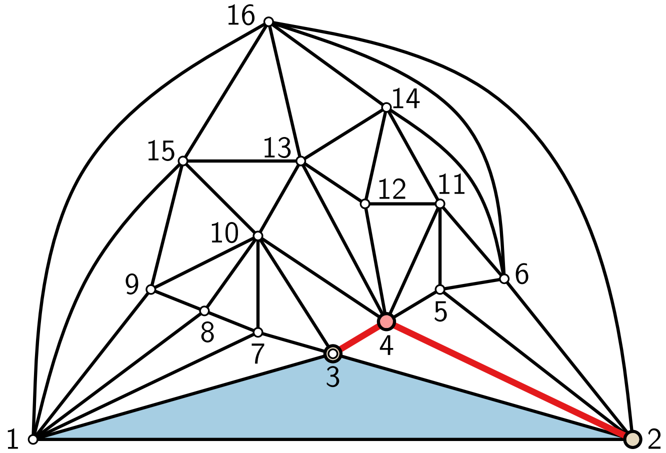

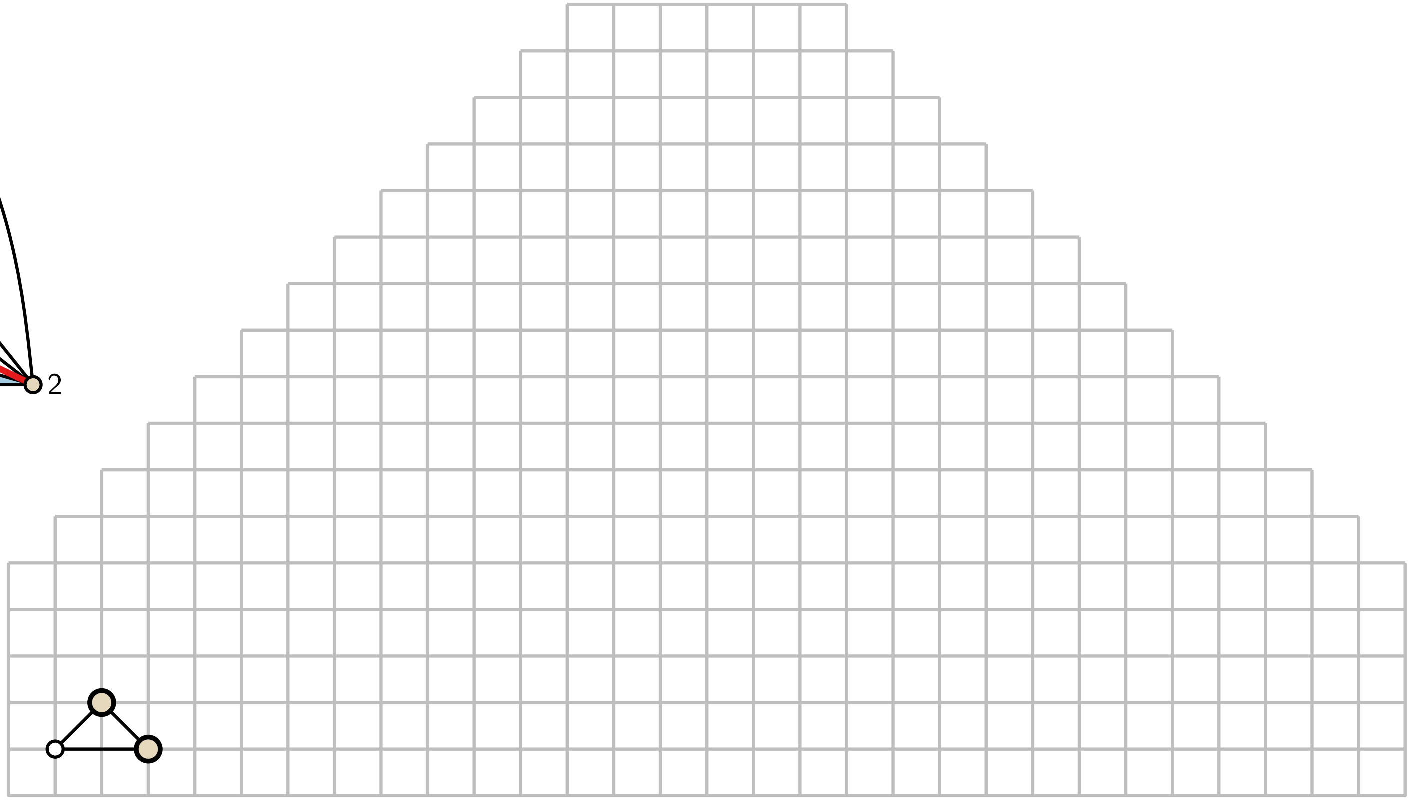

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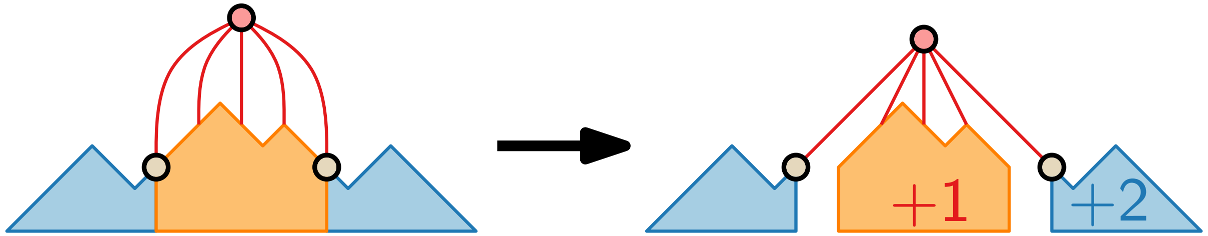

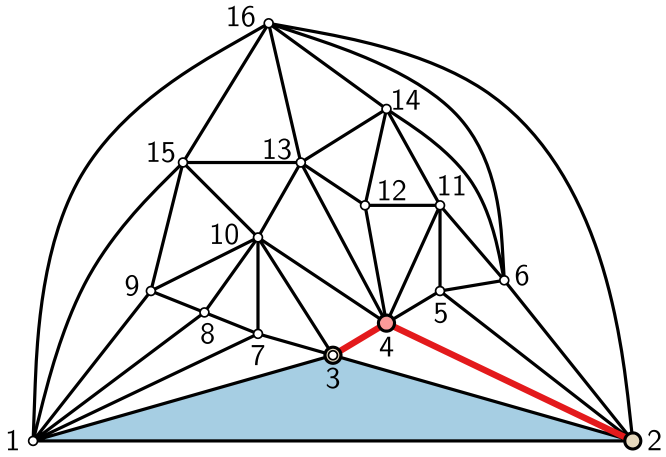

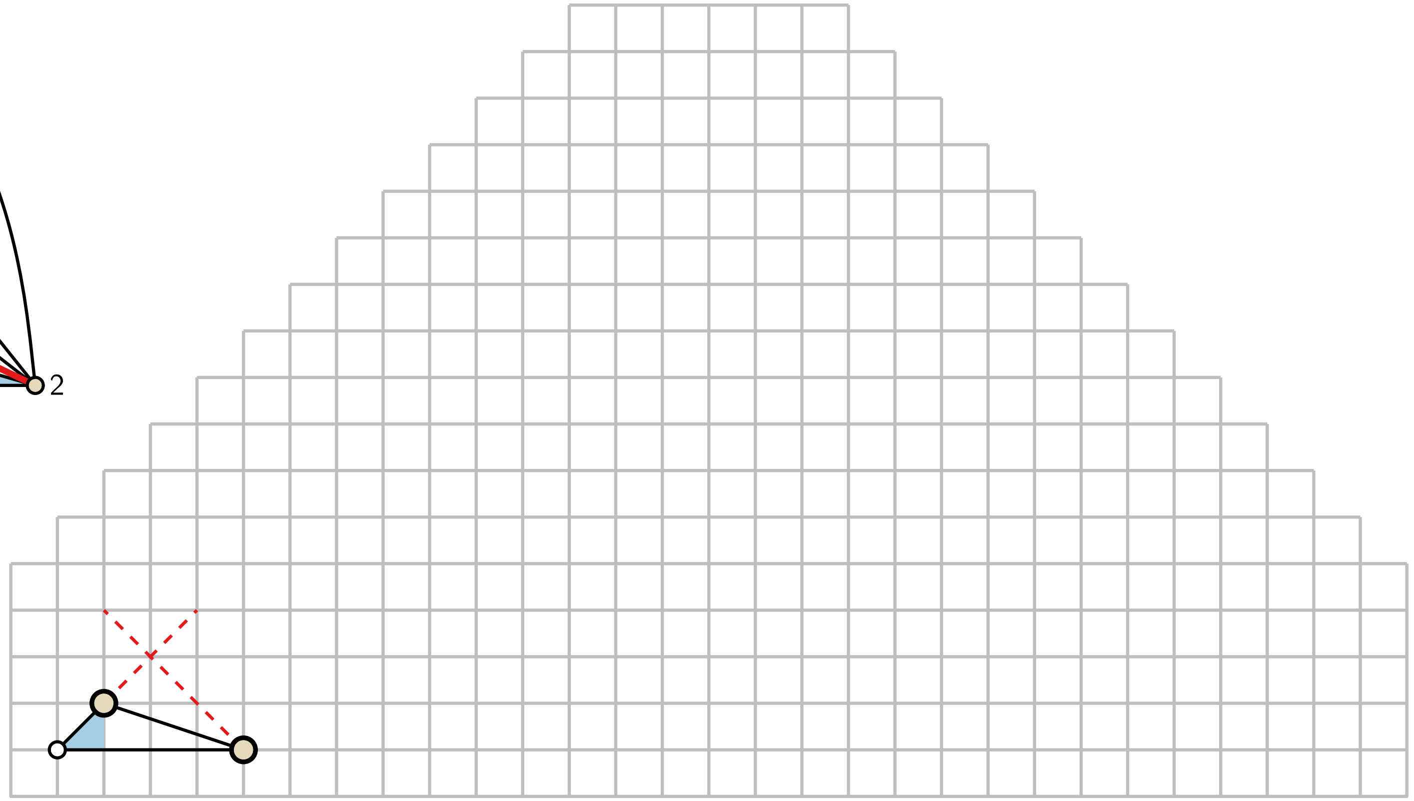

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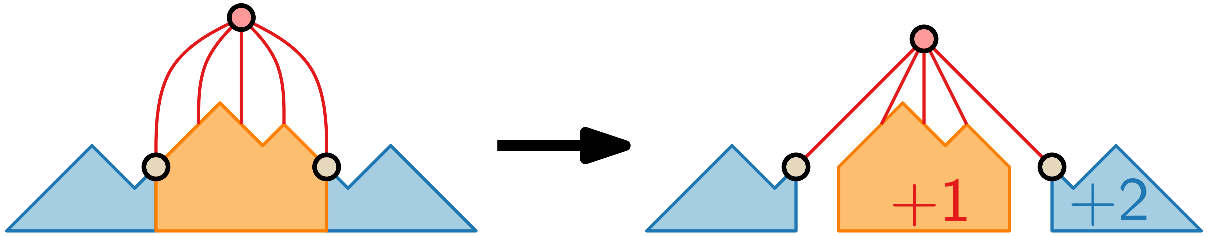

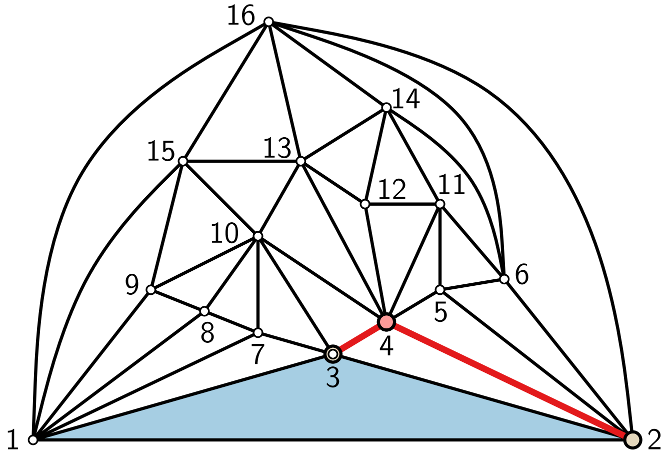

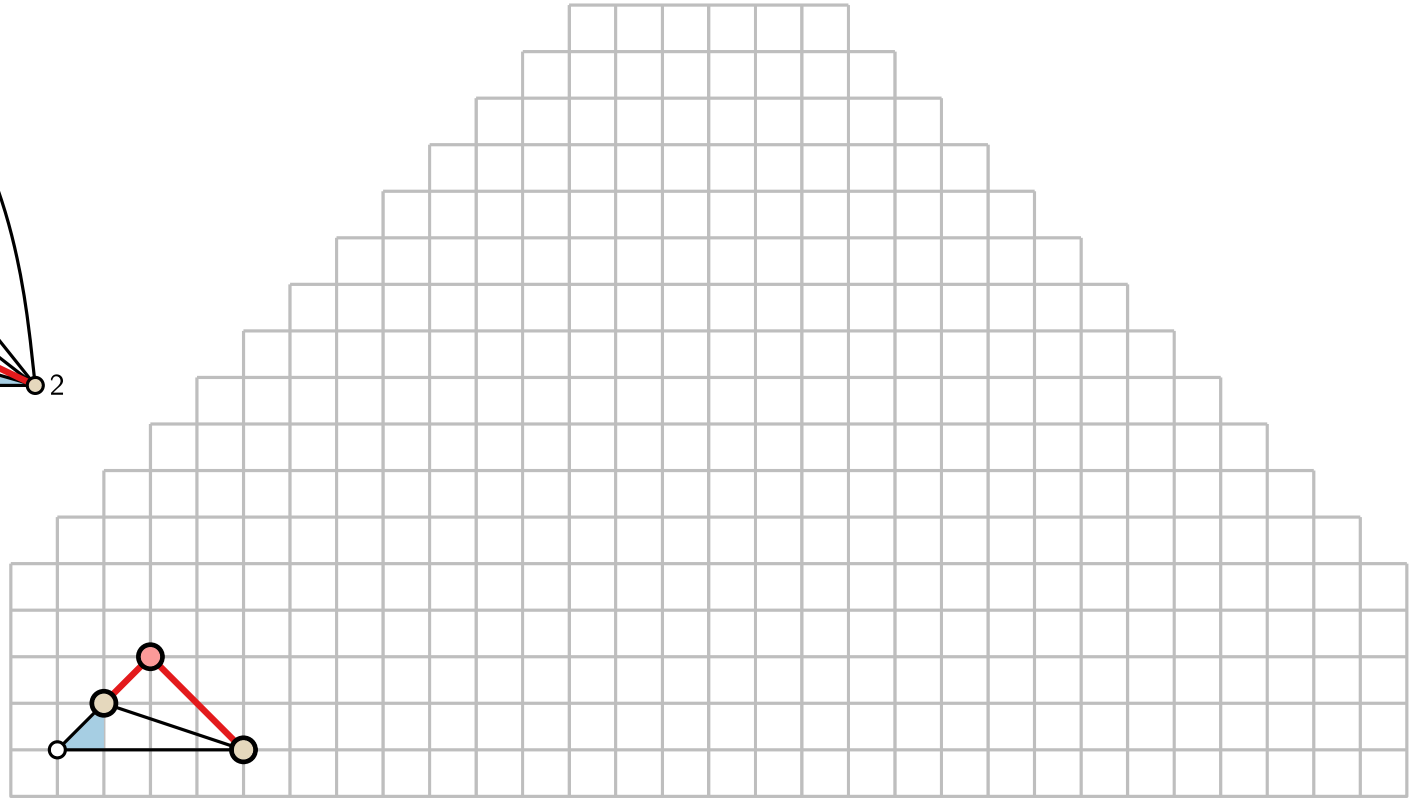

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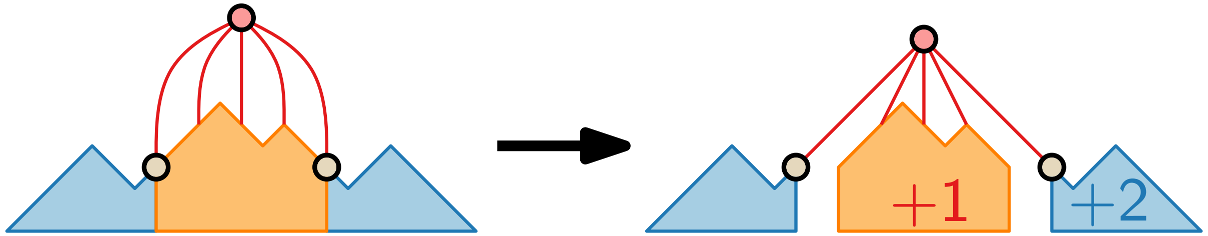

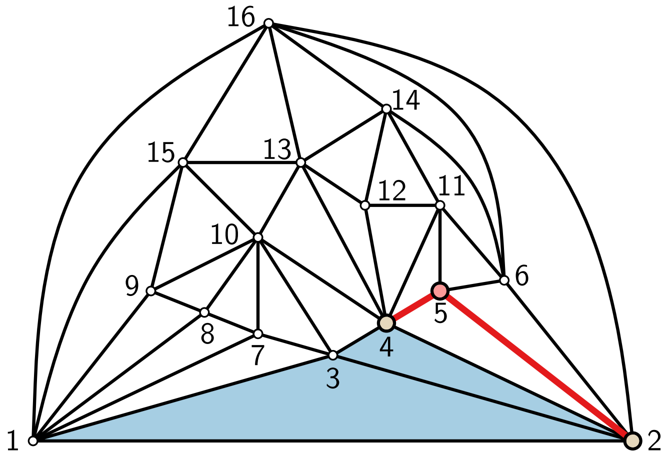

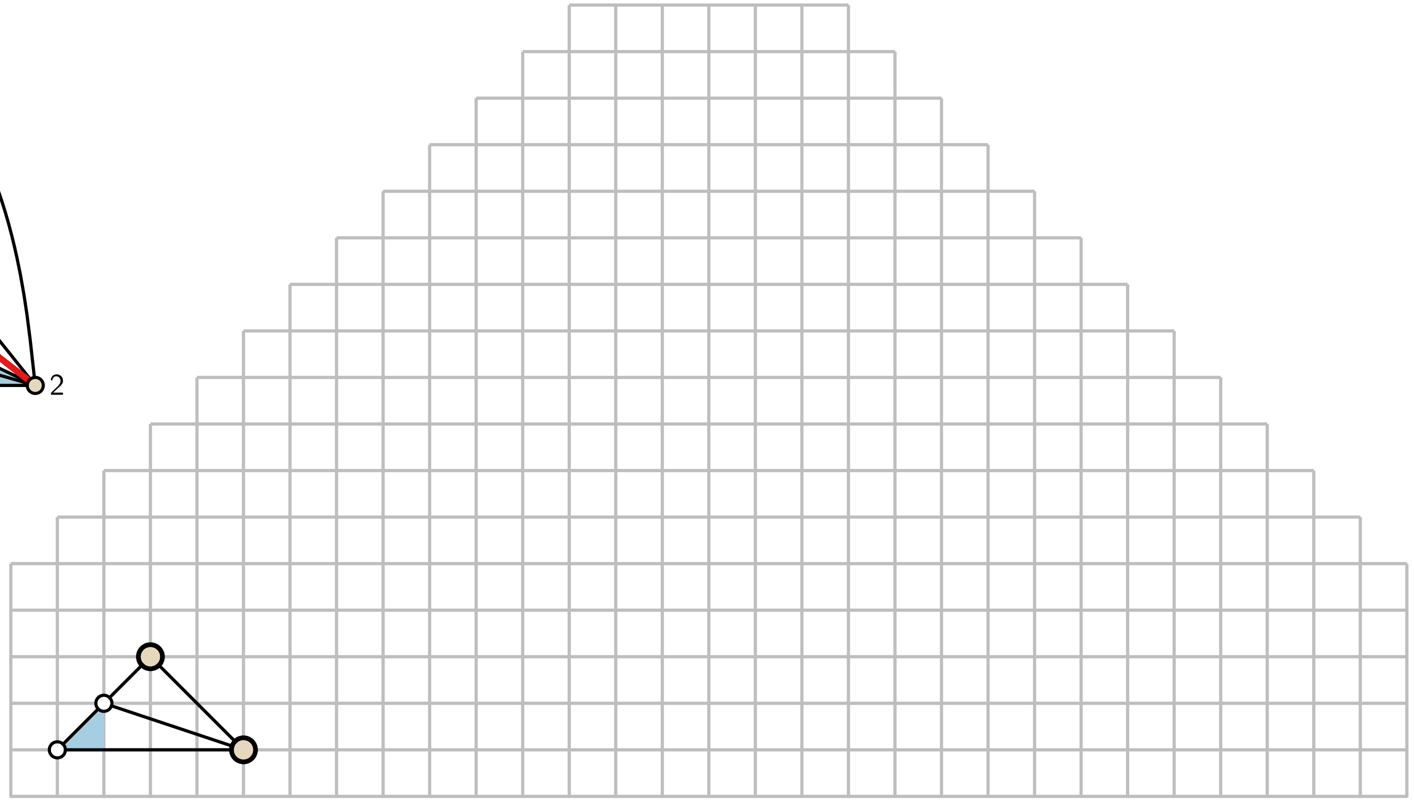

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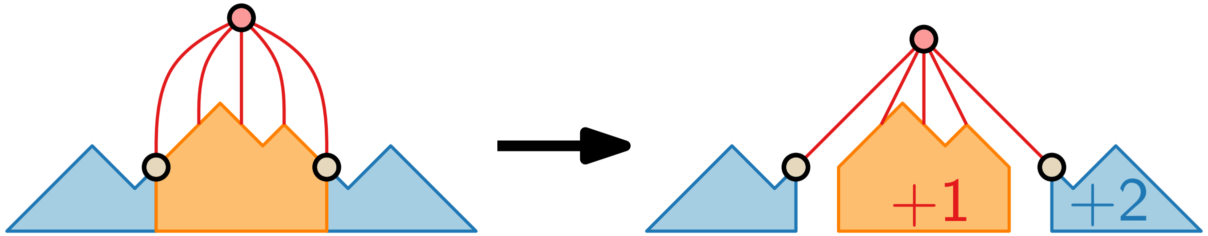

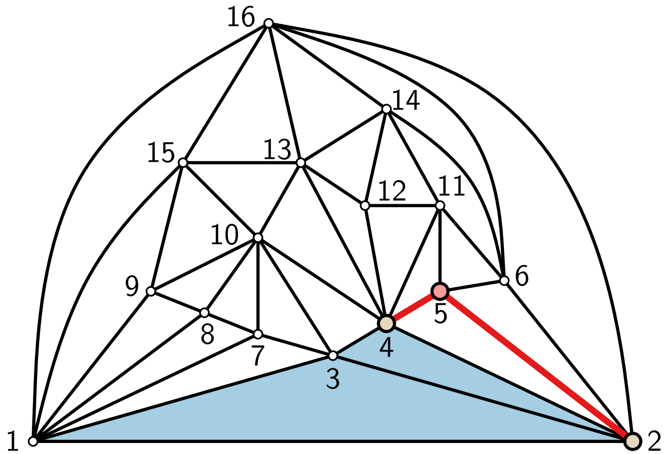

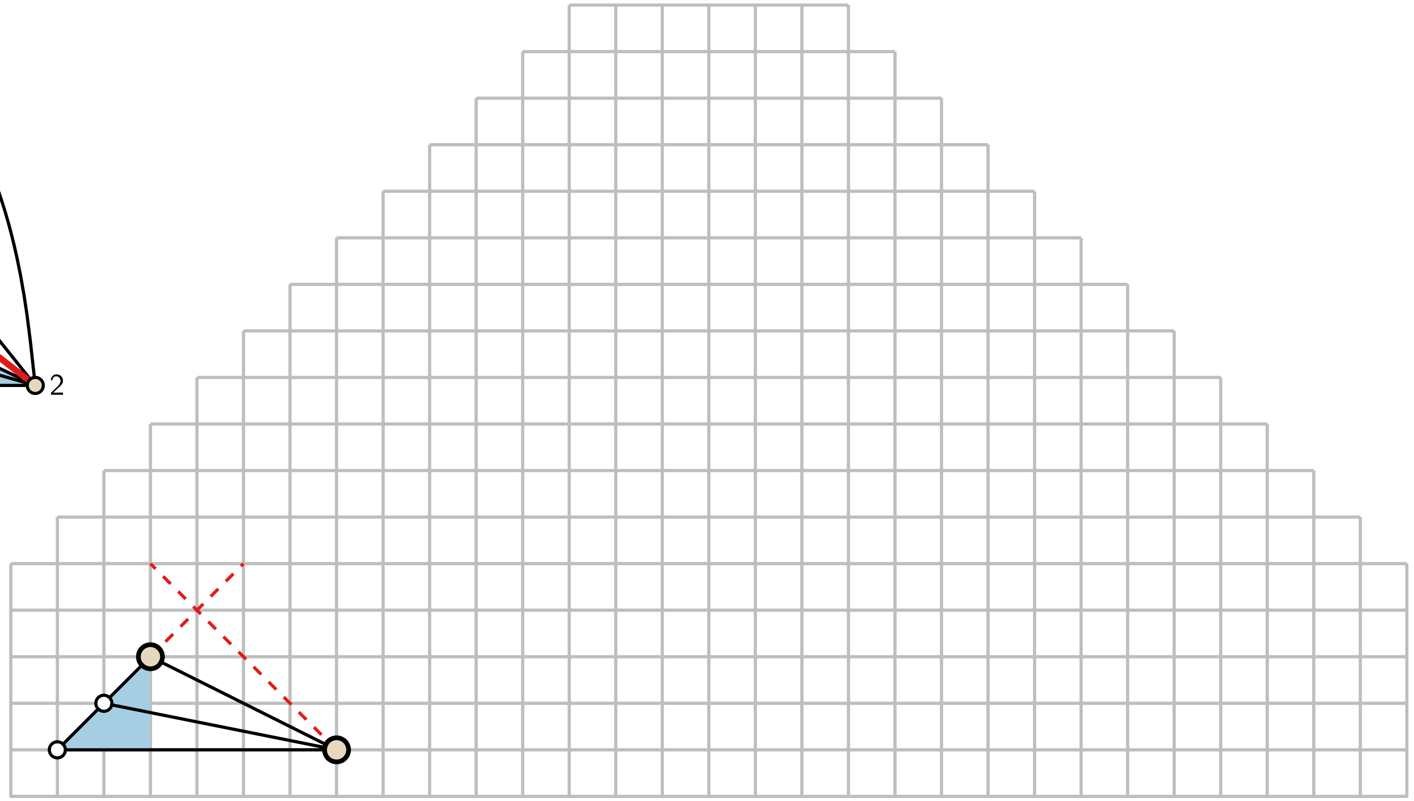

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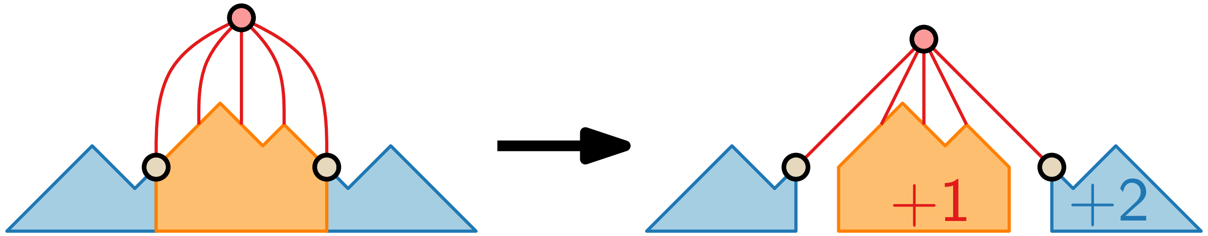

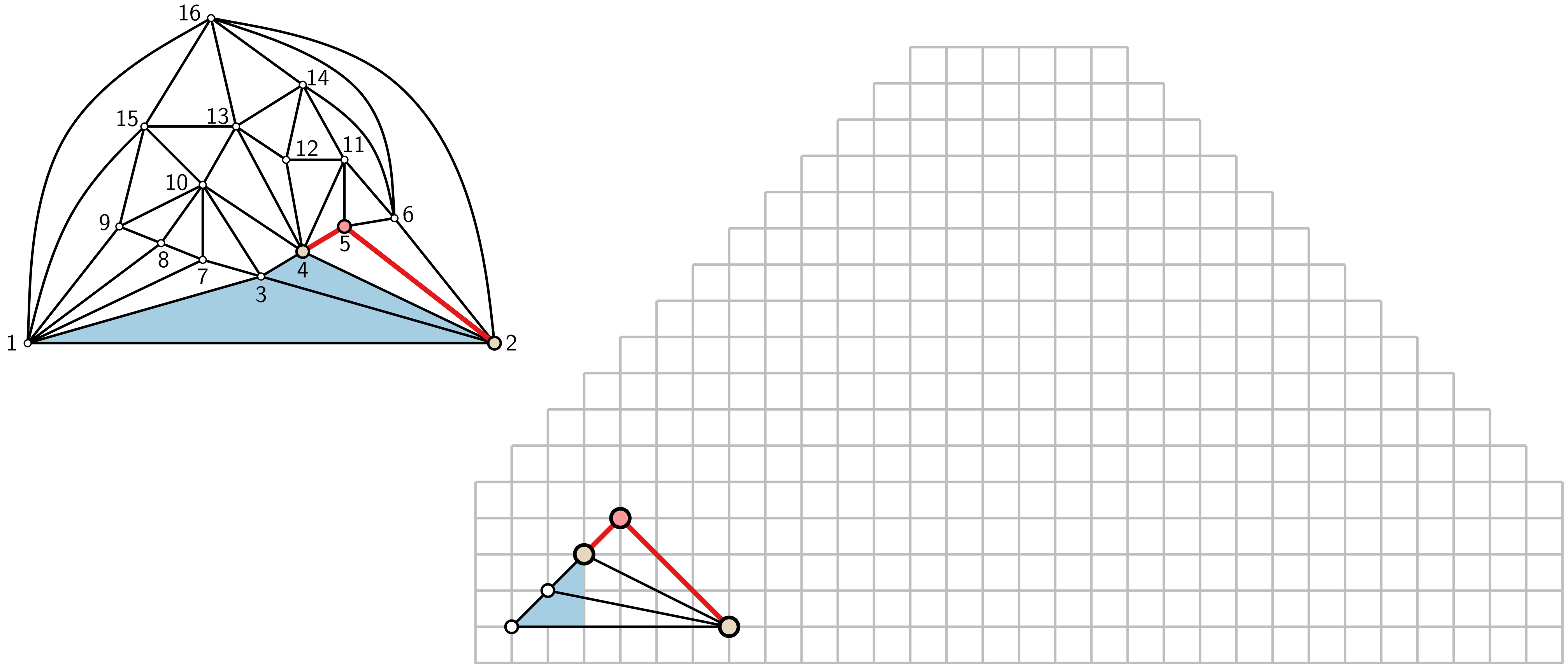

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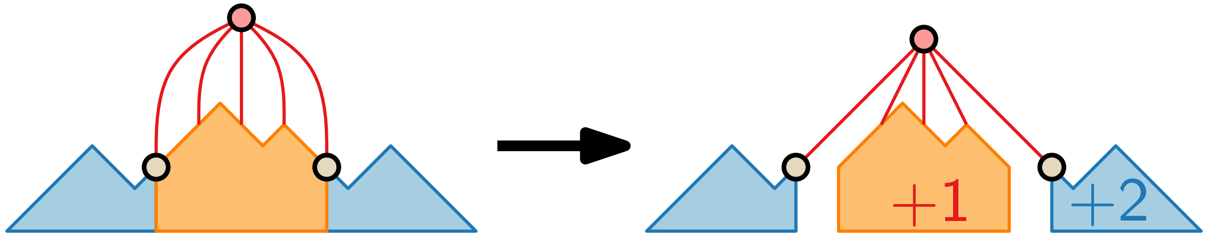

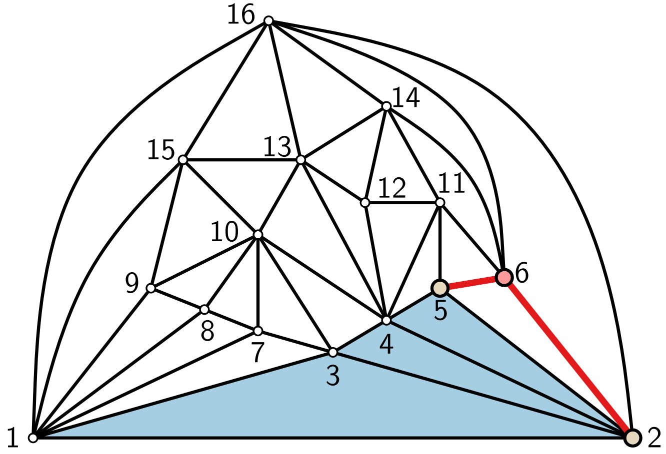

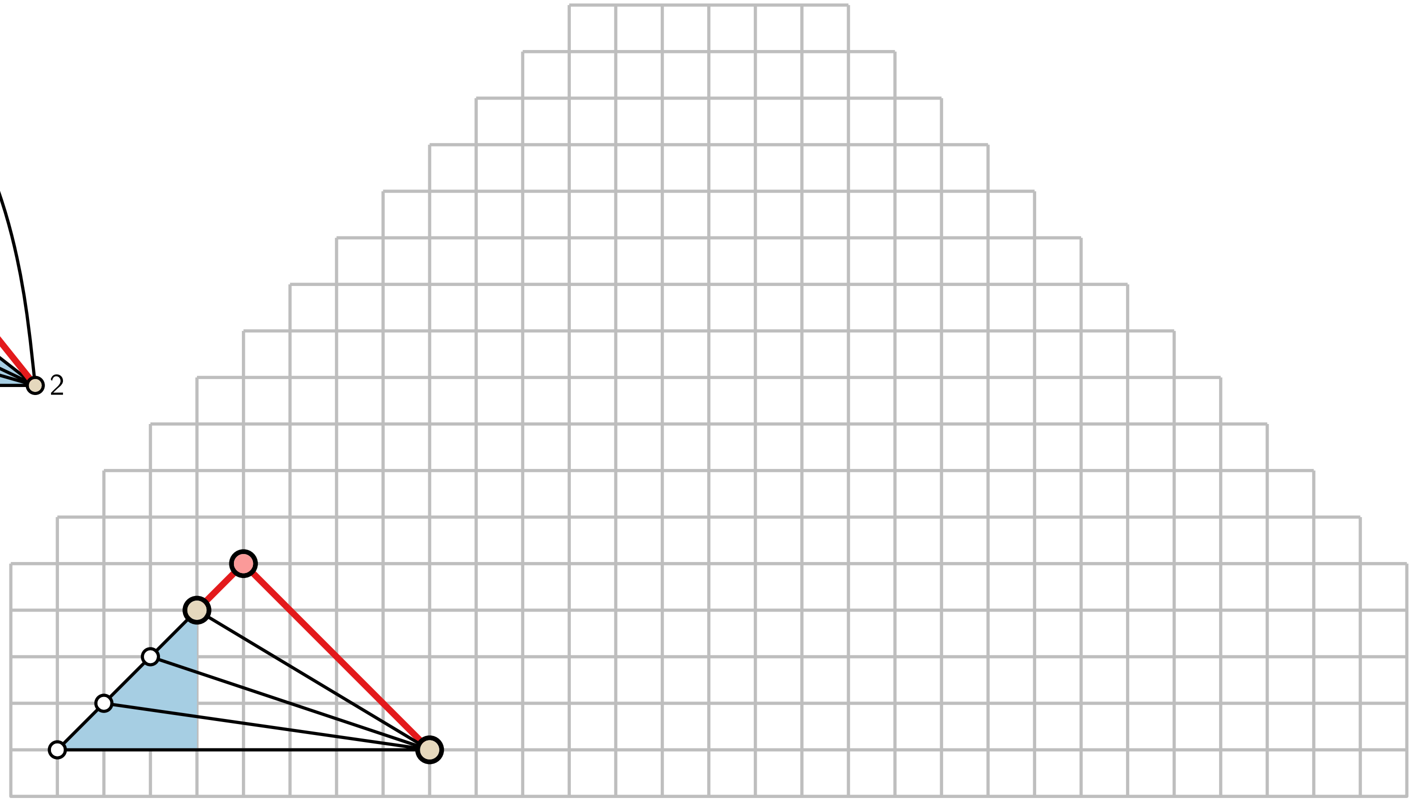

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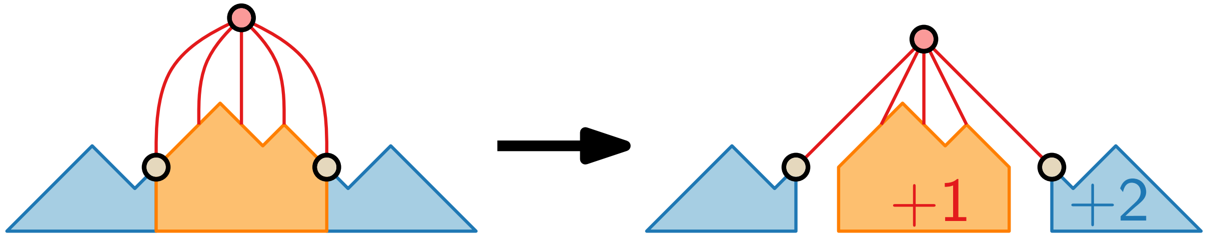

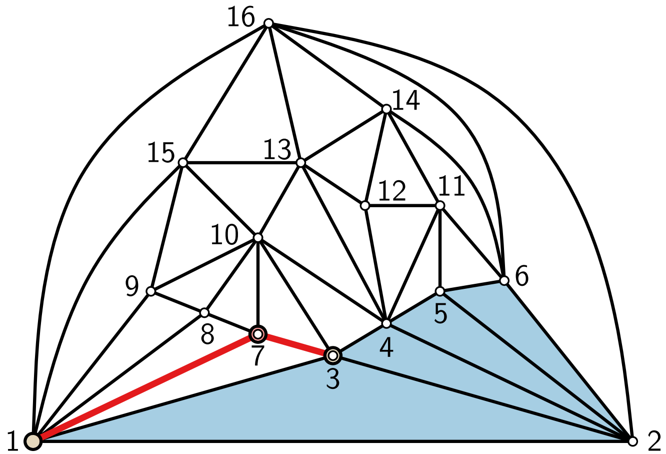

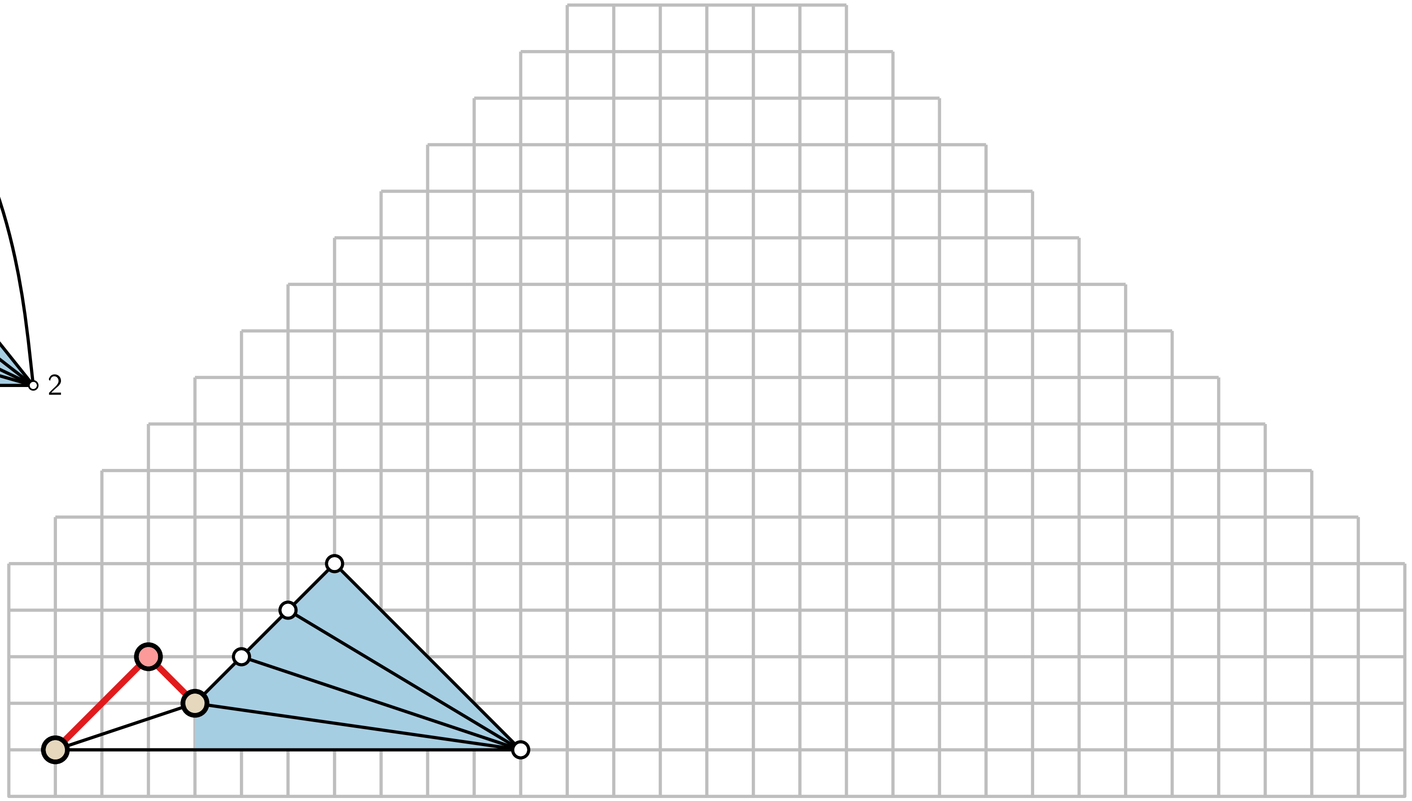

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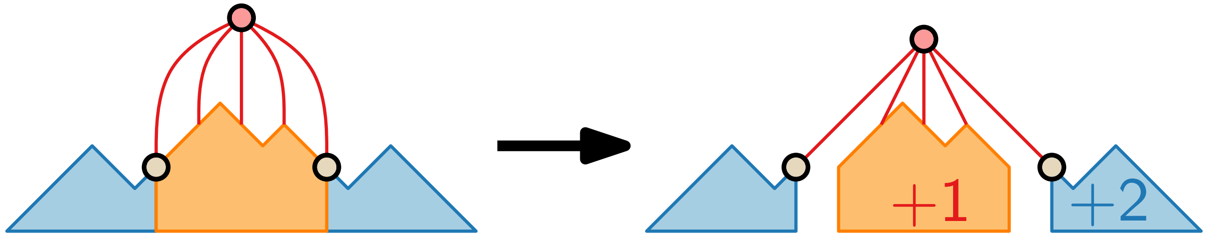

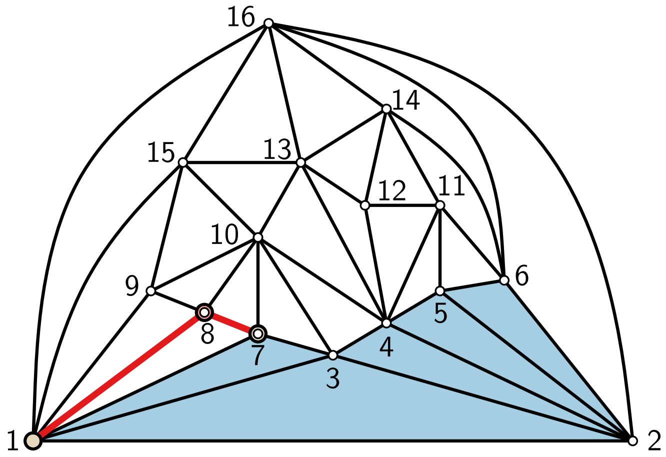

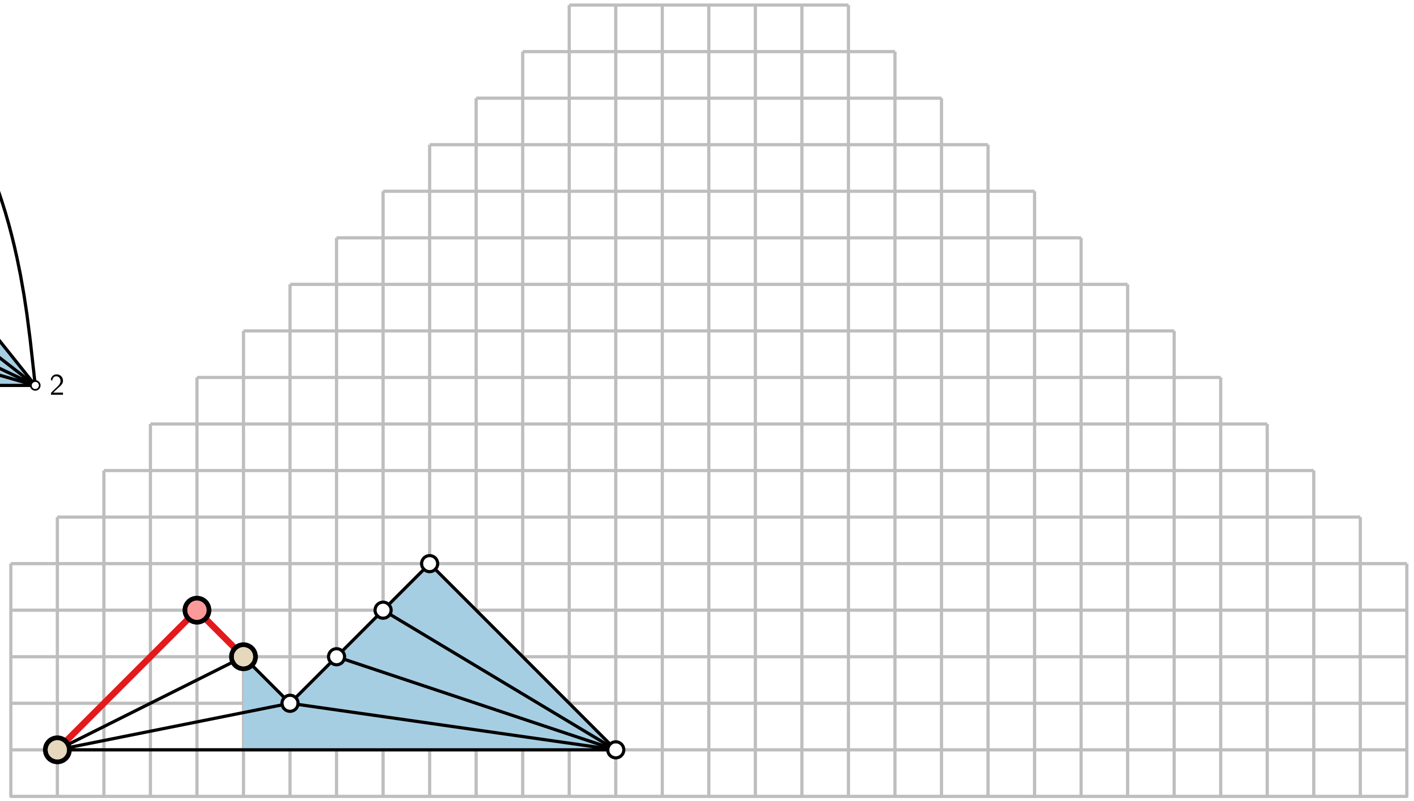

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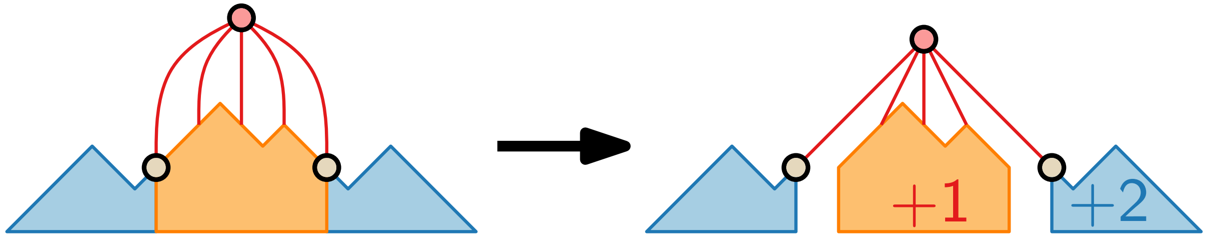

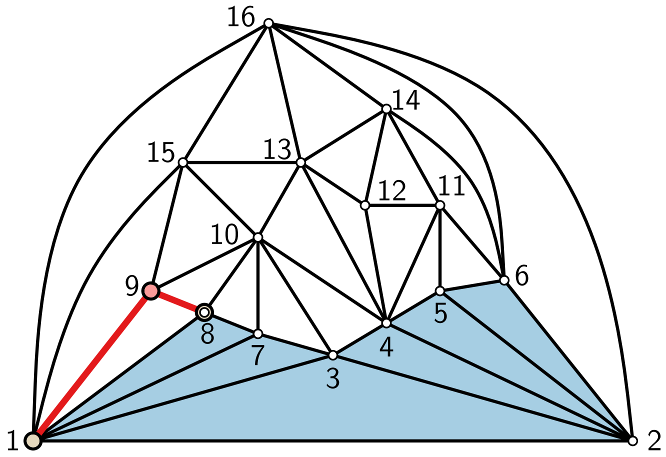

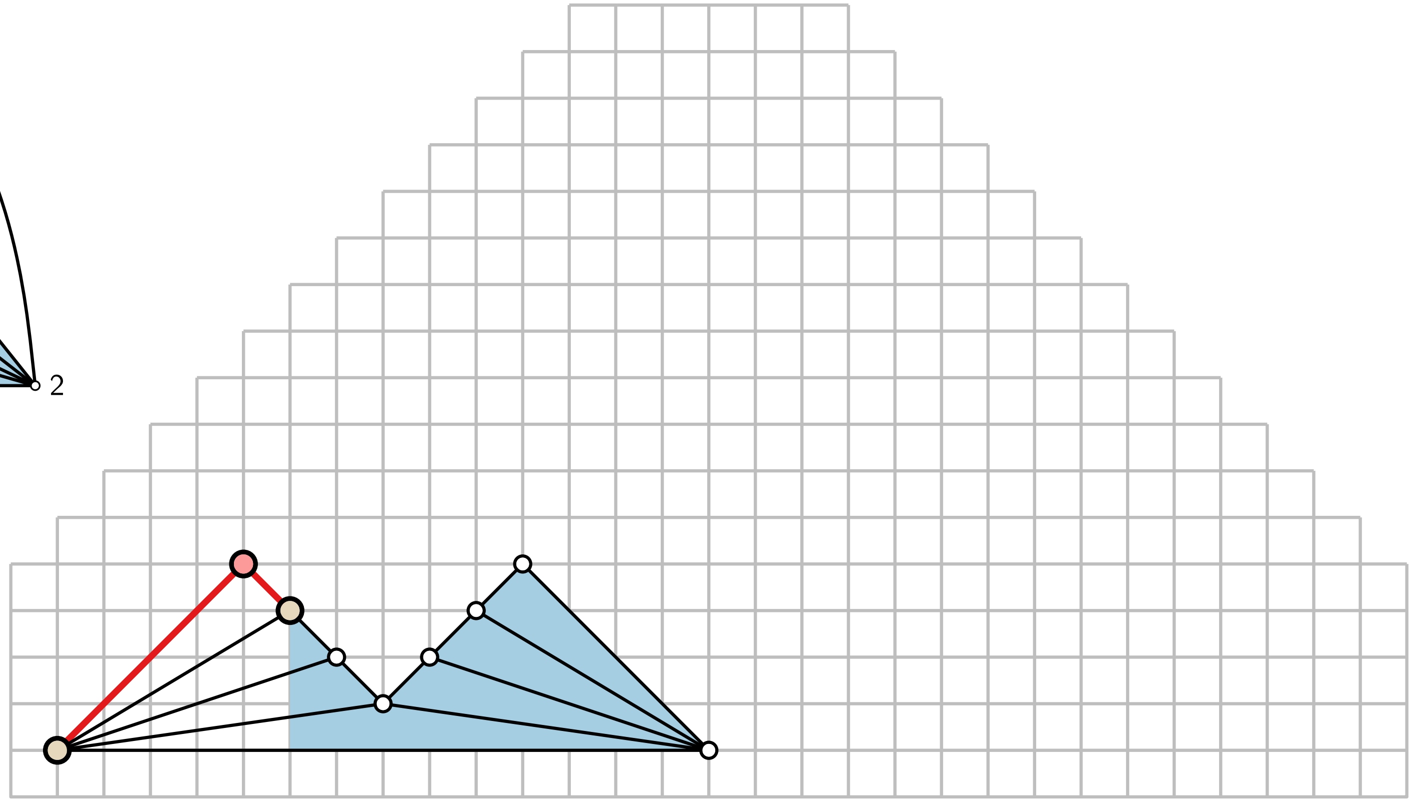

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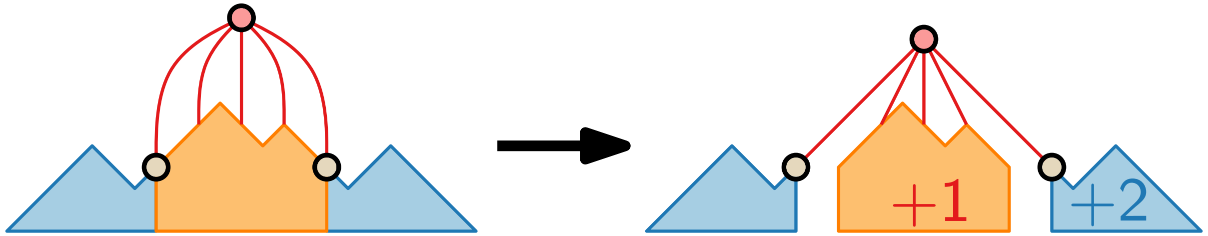

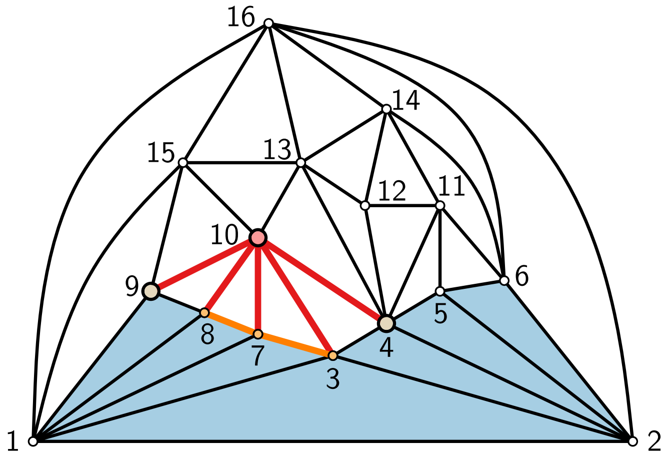

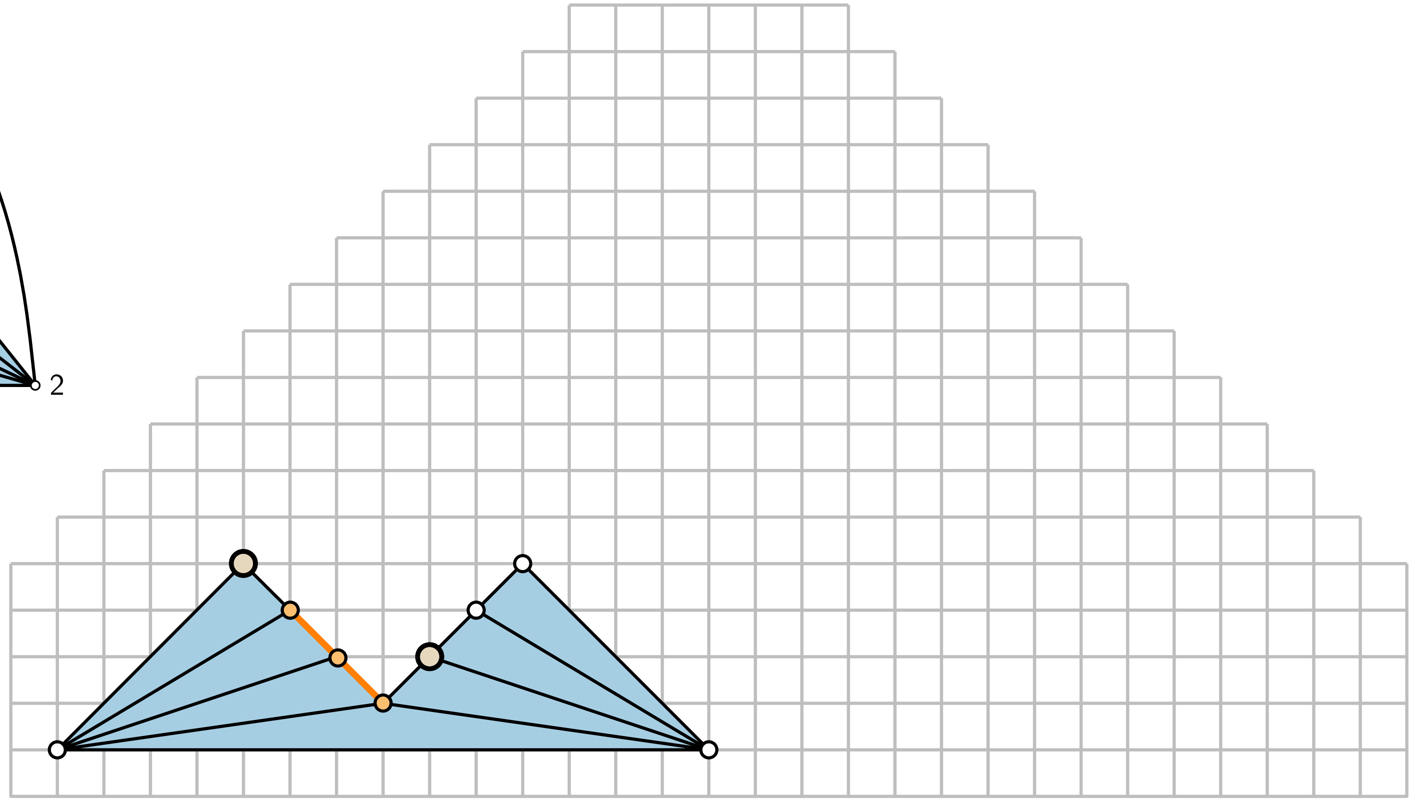

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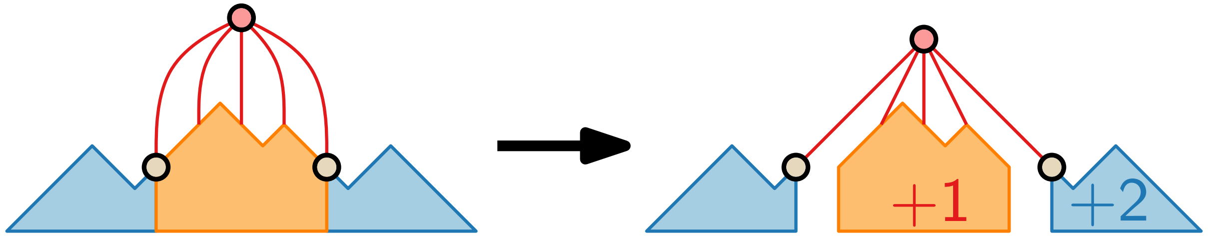

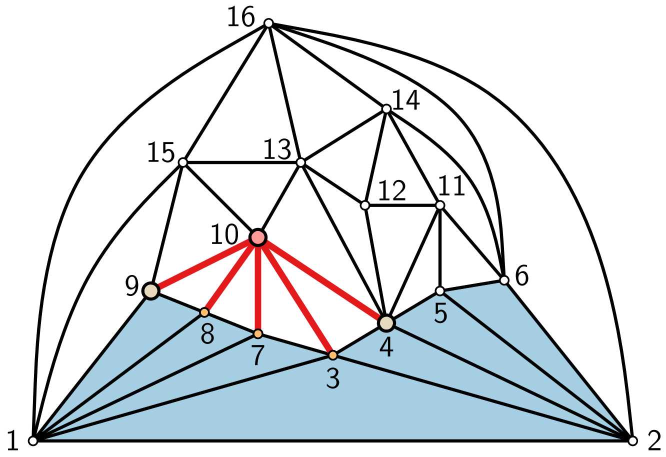

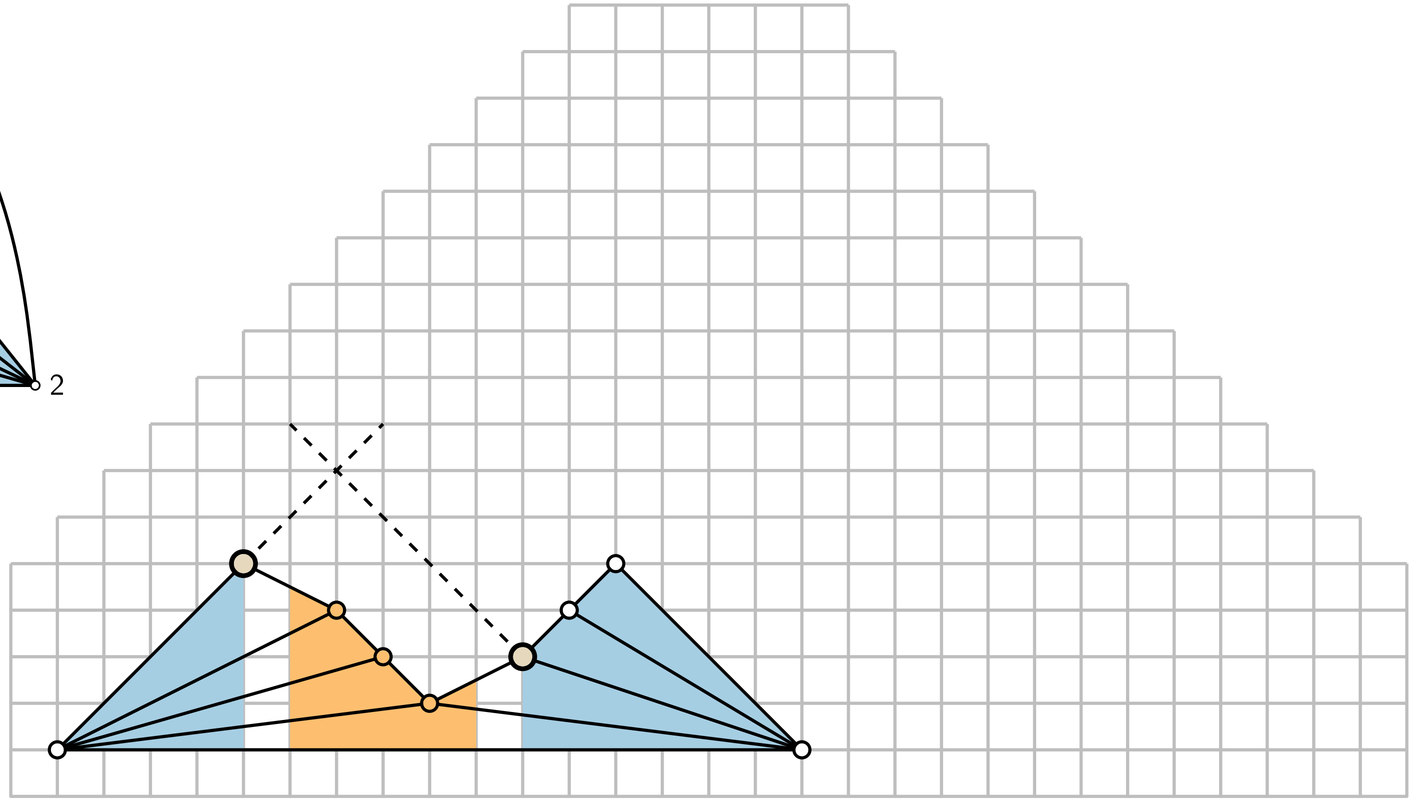

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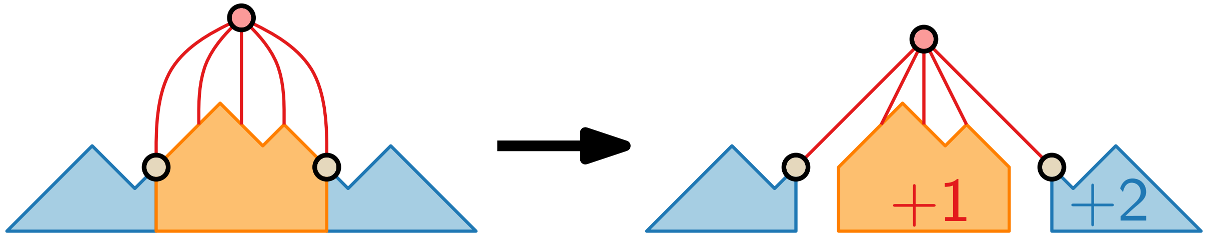

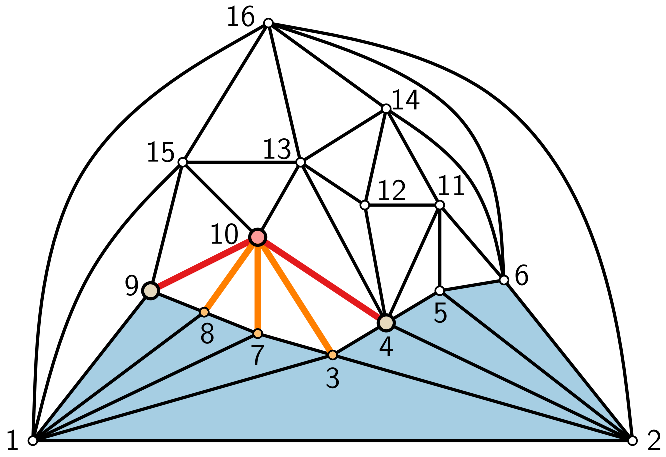

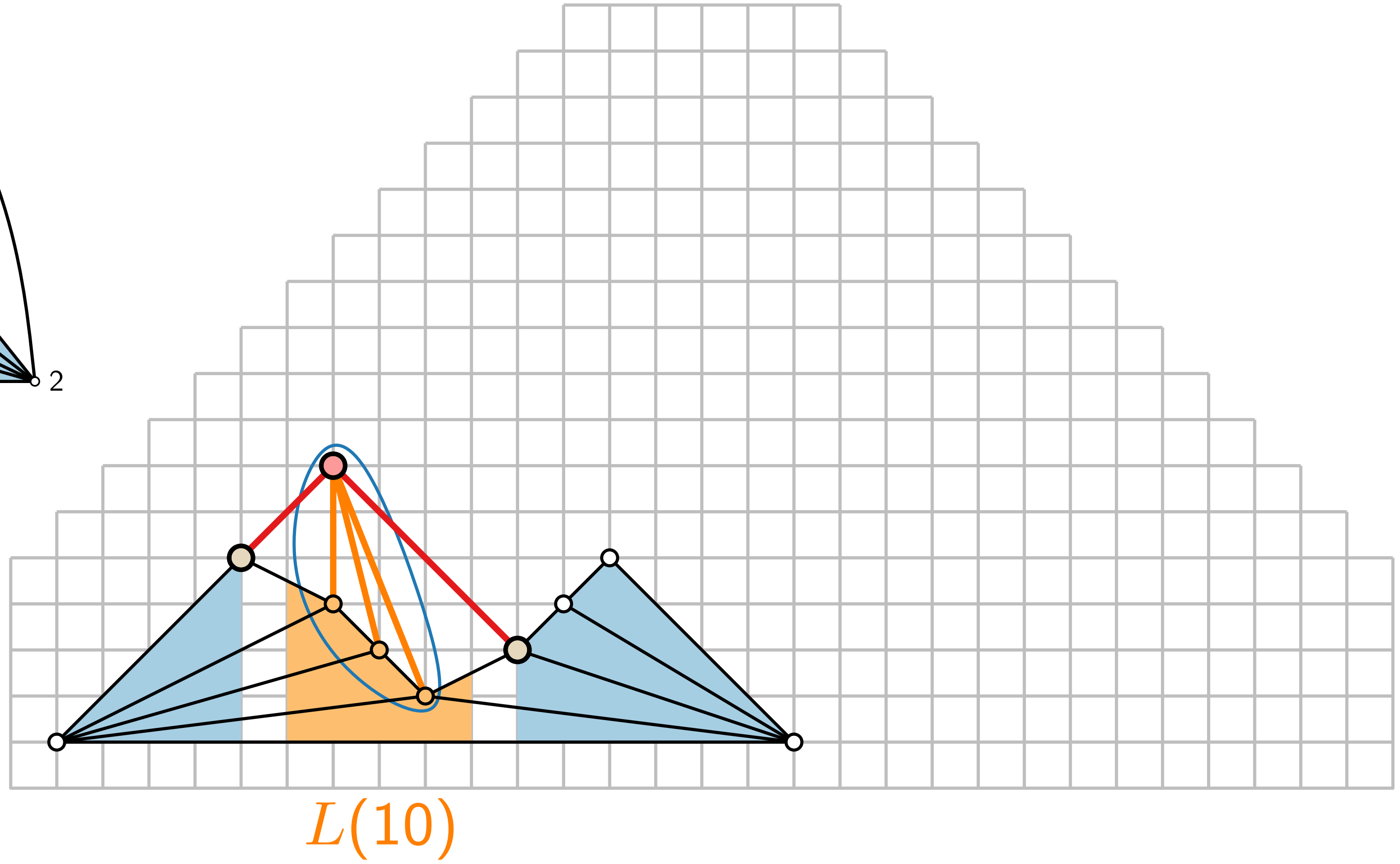

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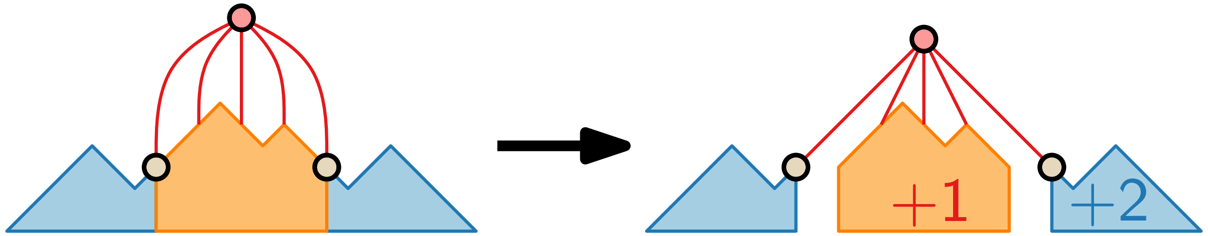
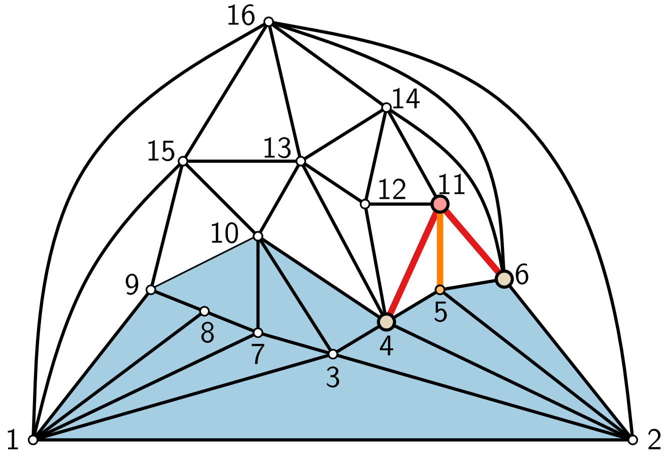
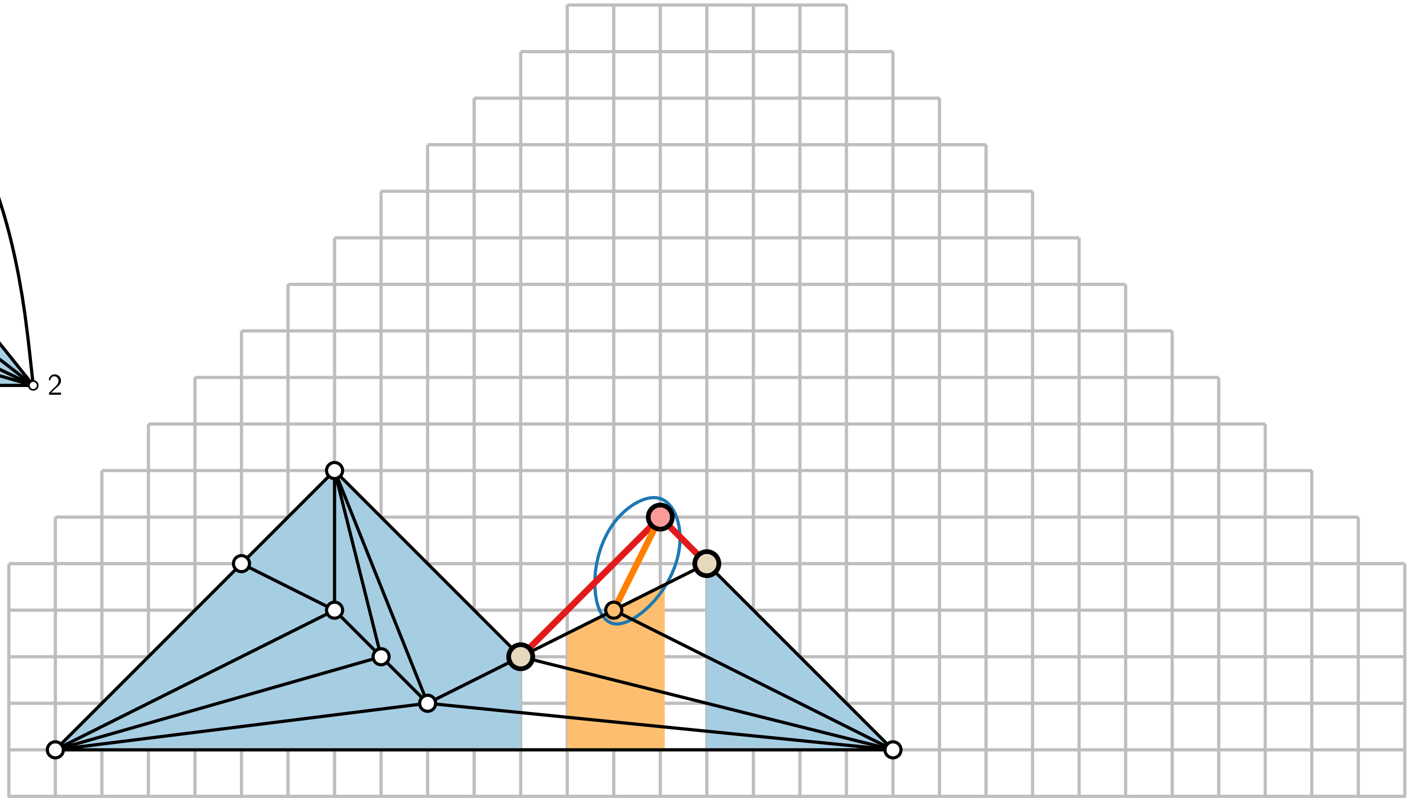
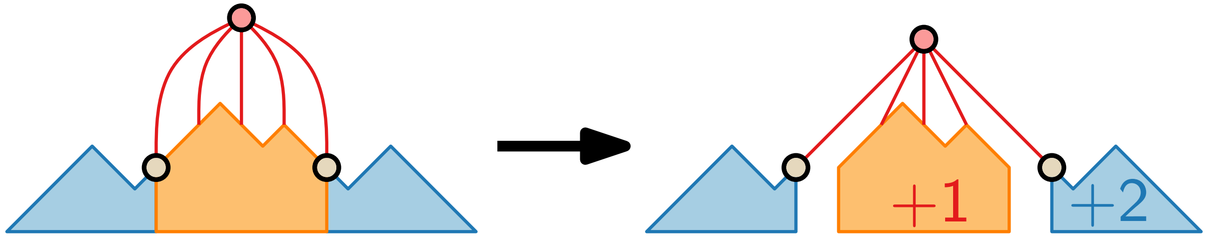
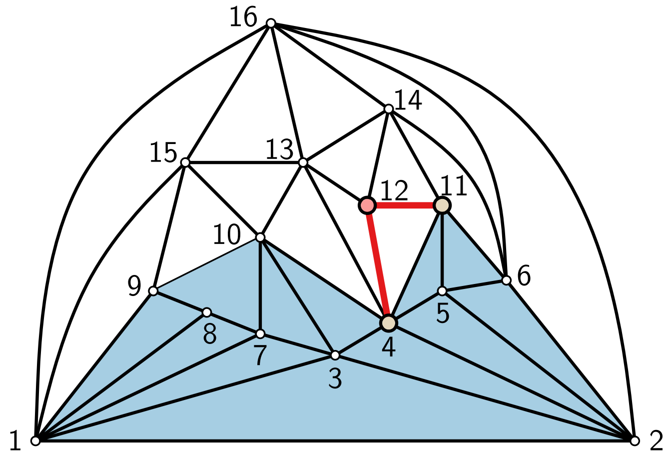


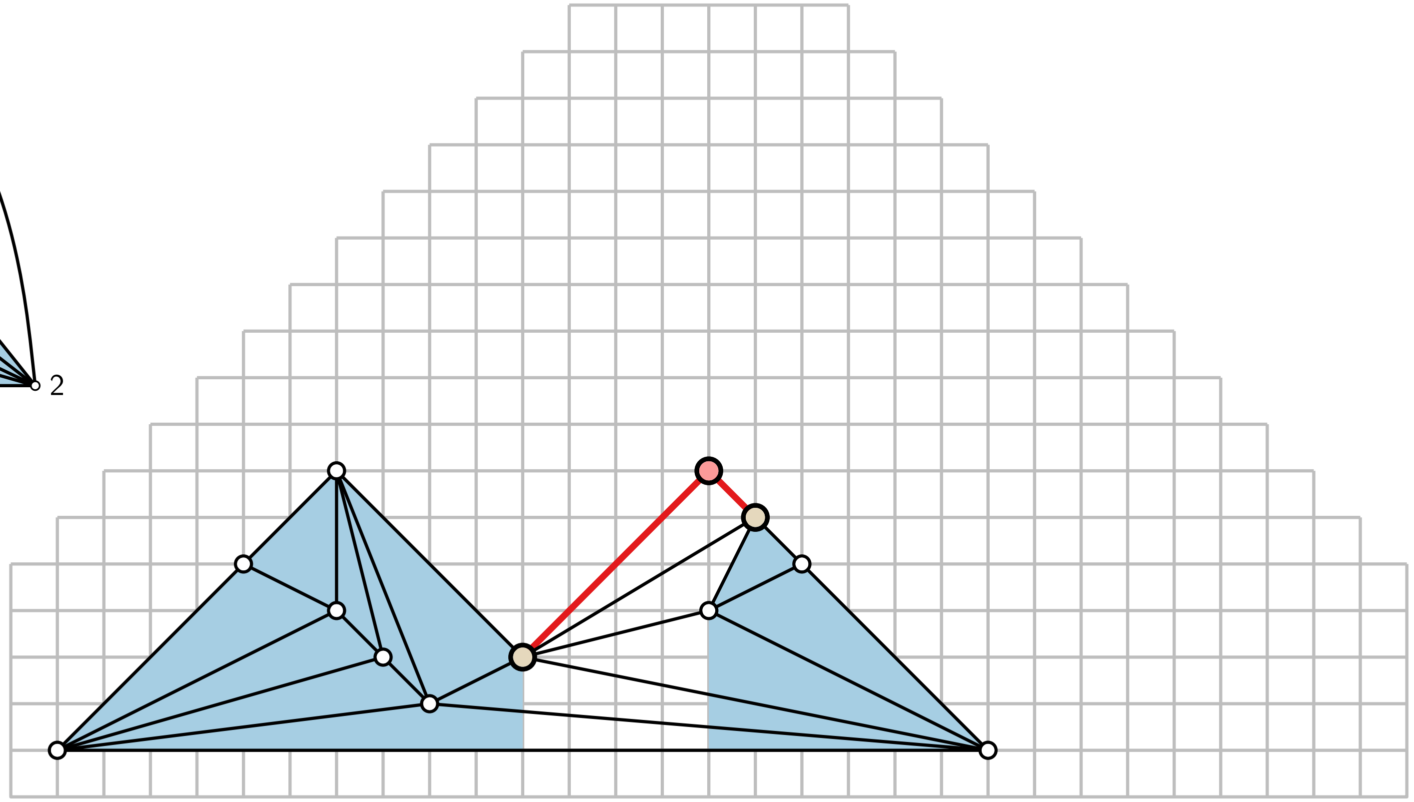




Shift Method – Example







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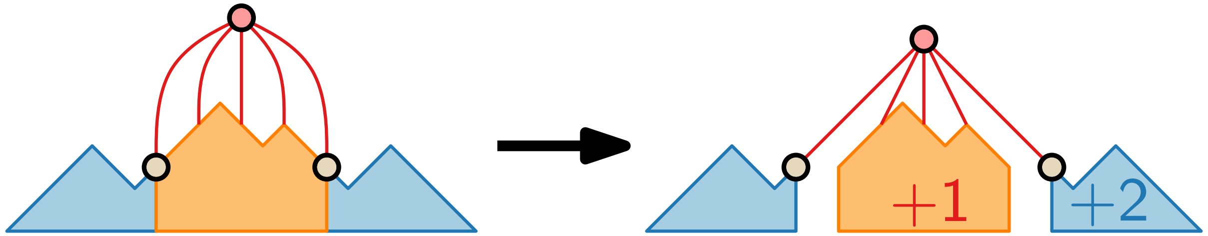

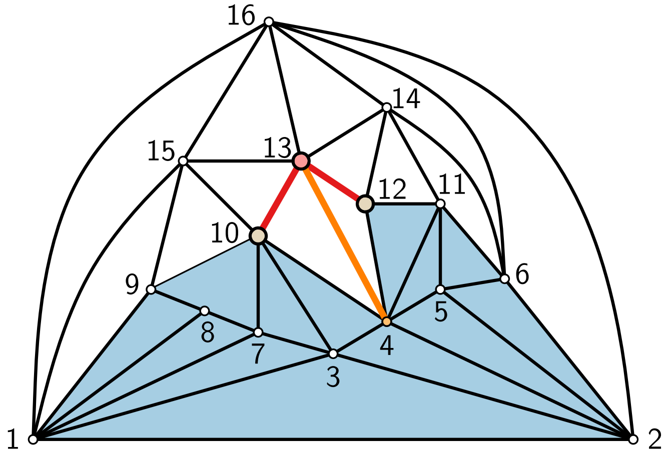

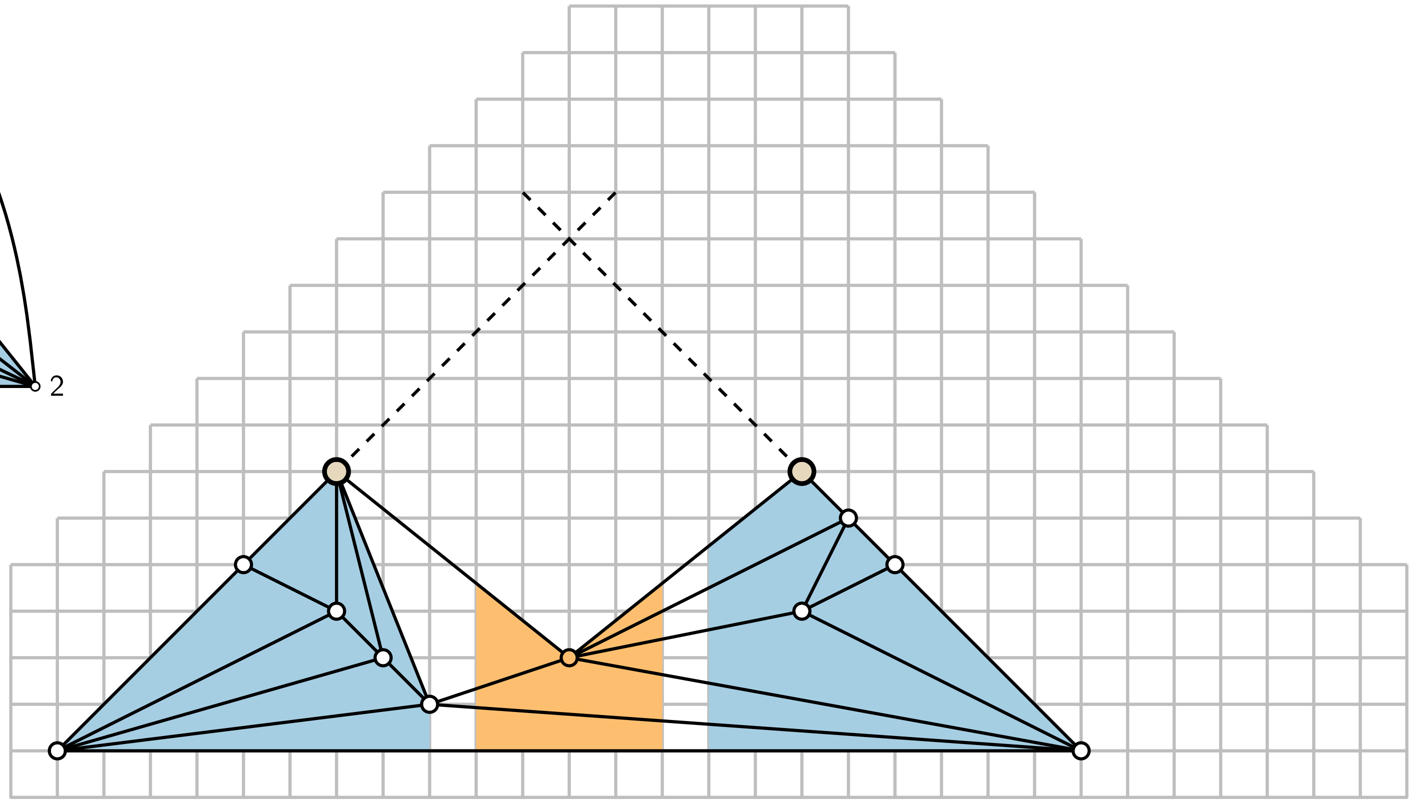

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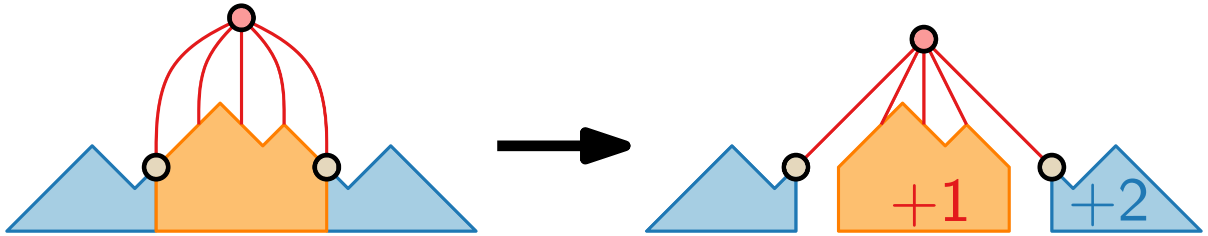

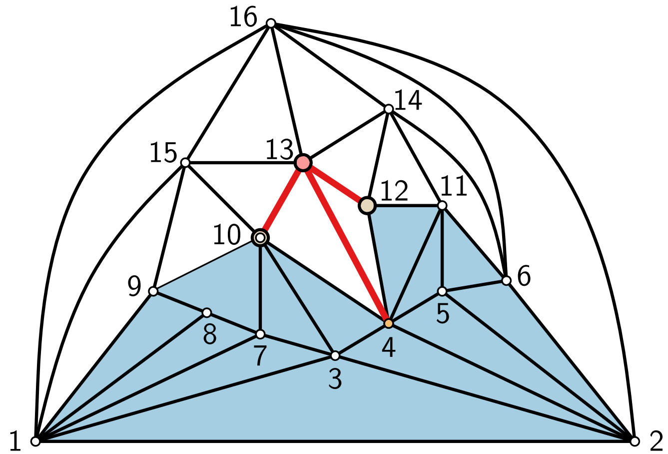

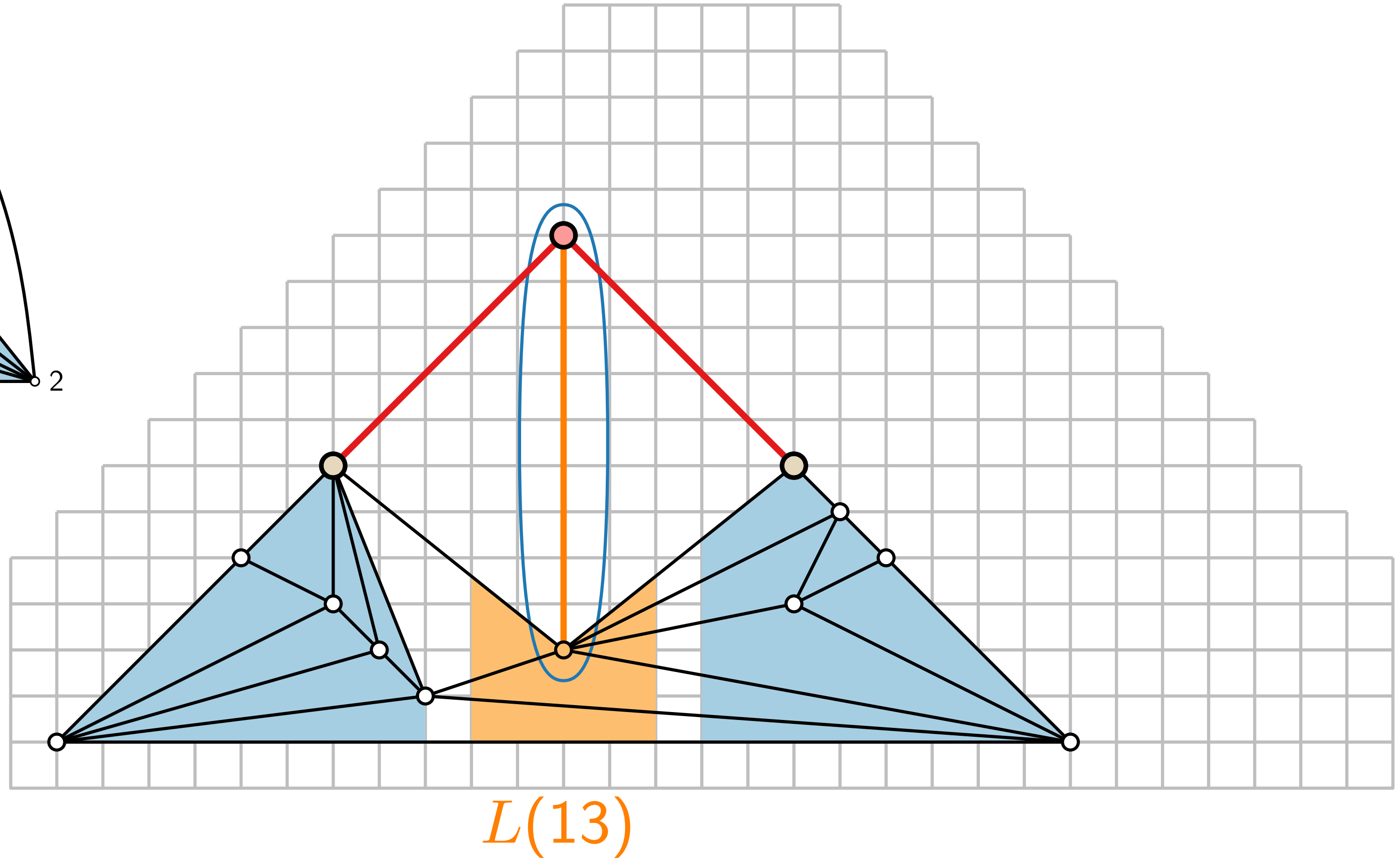

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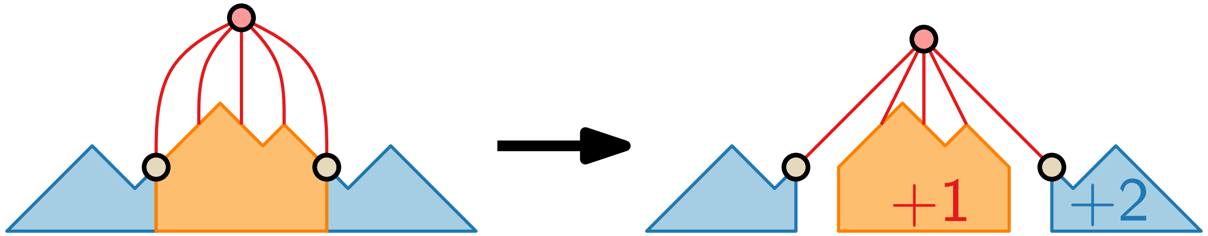

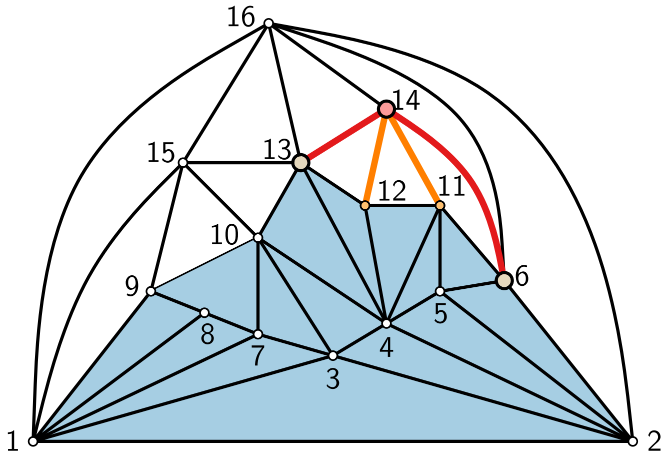

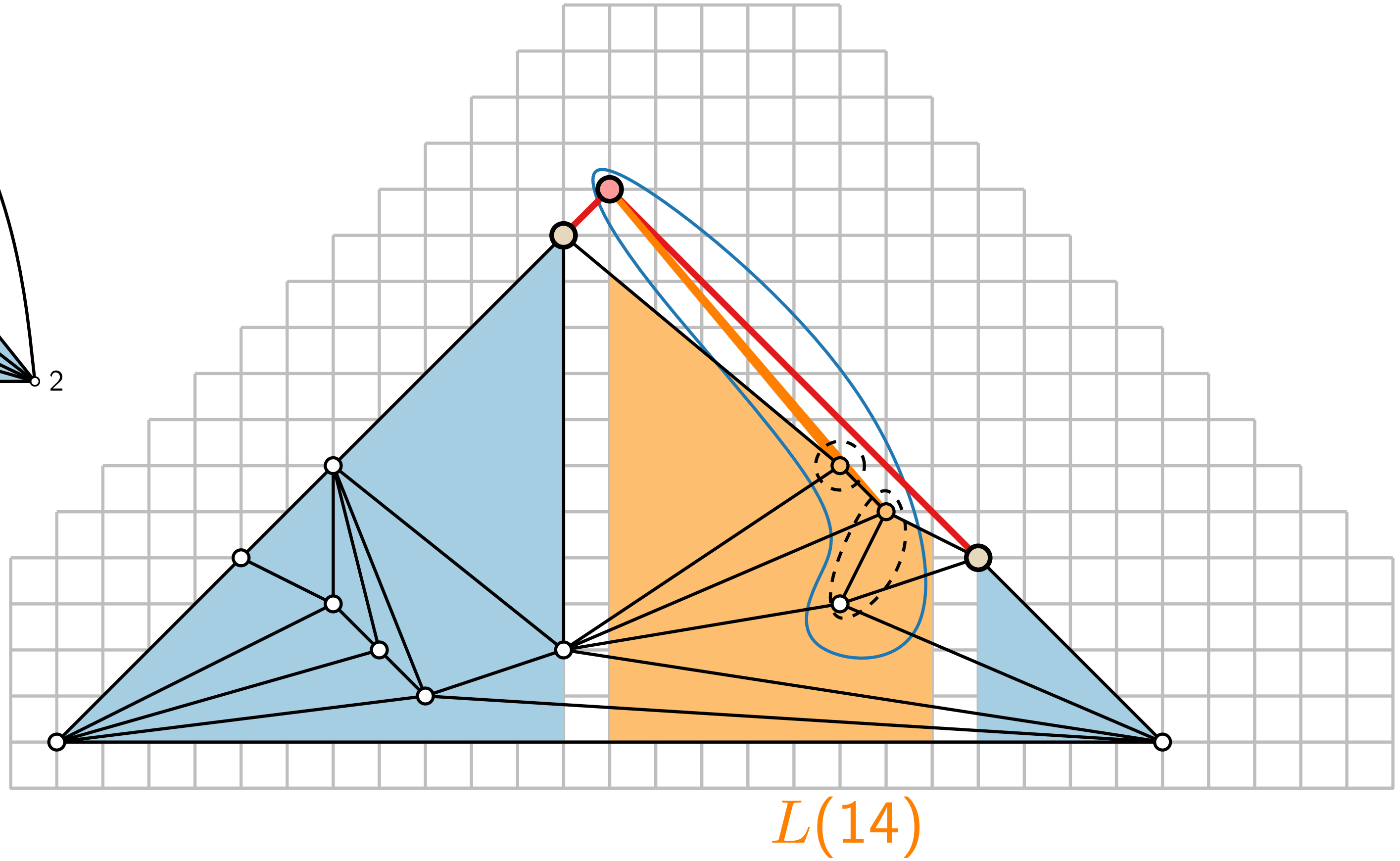

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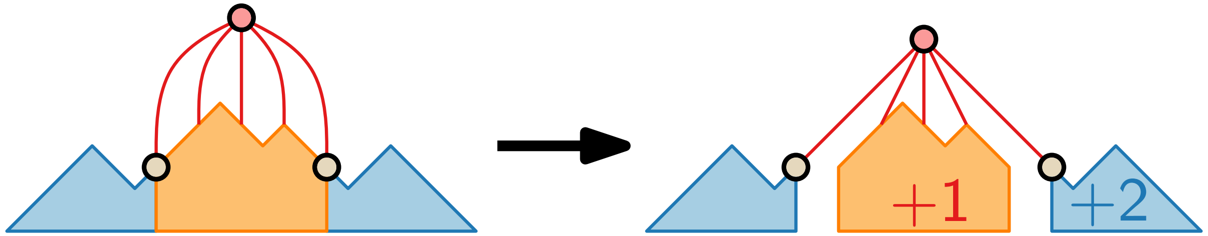

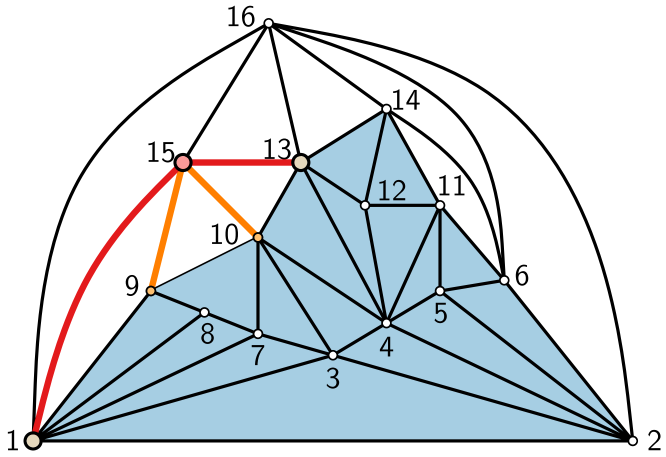

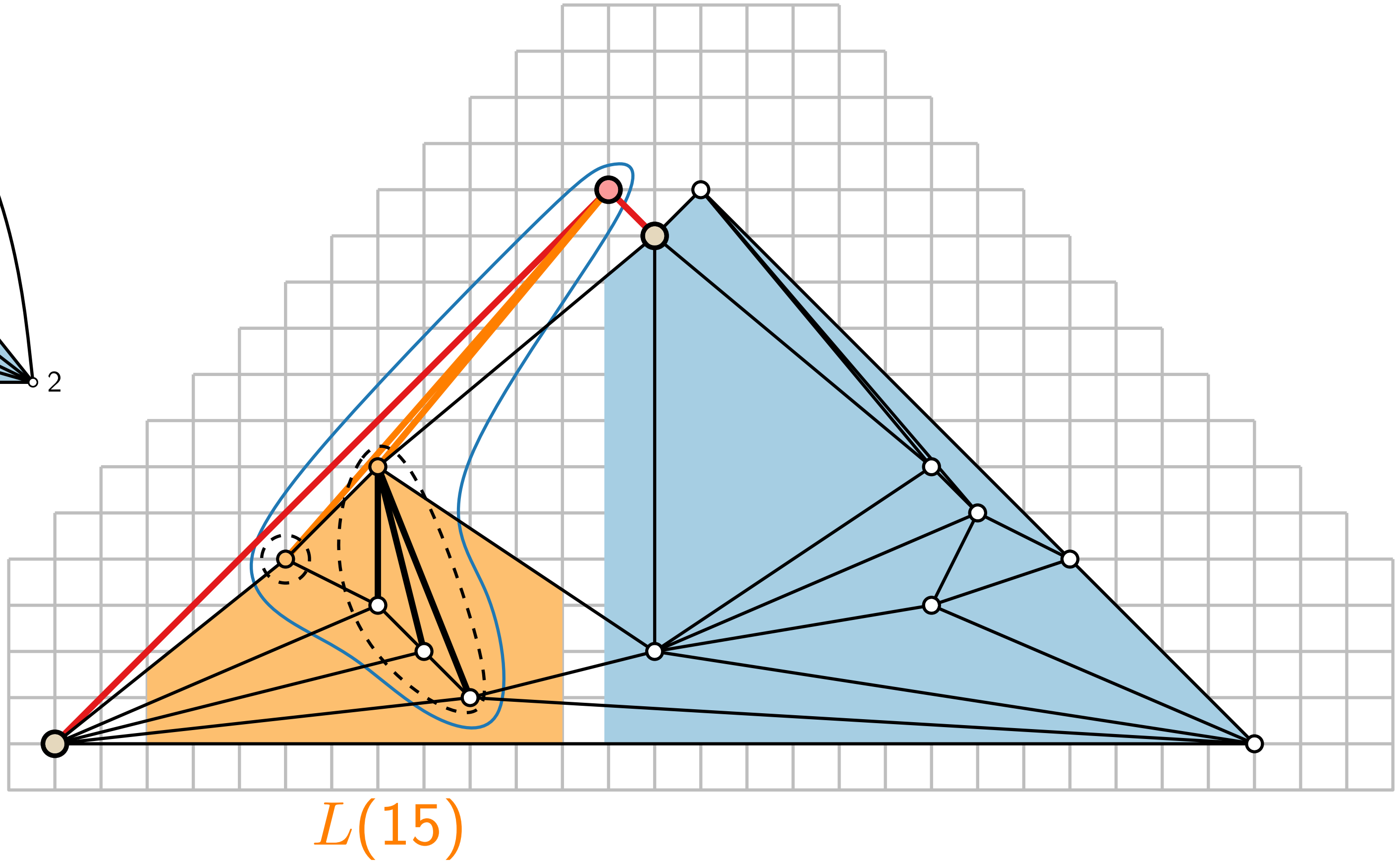

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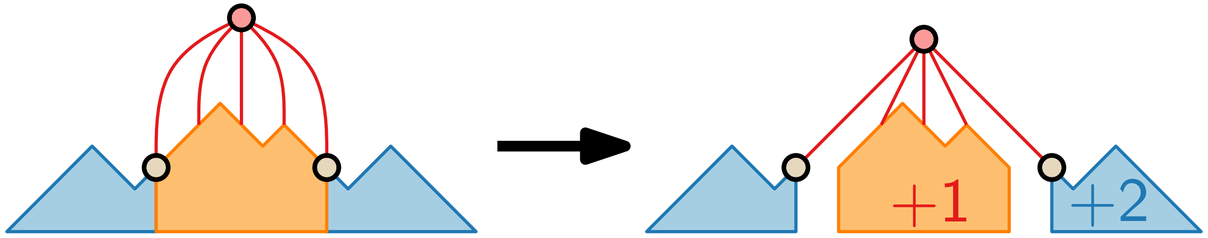
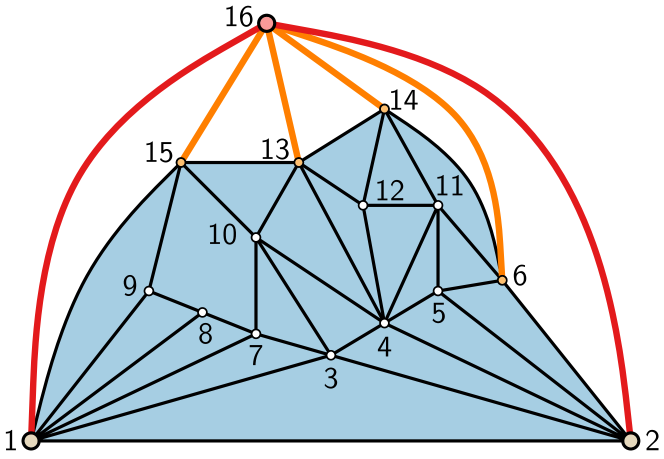
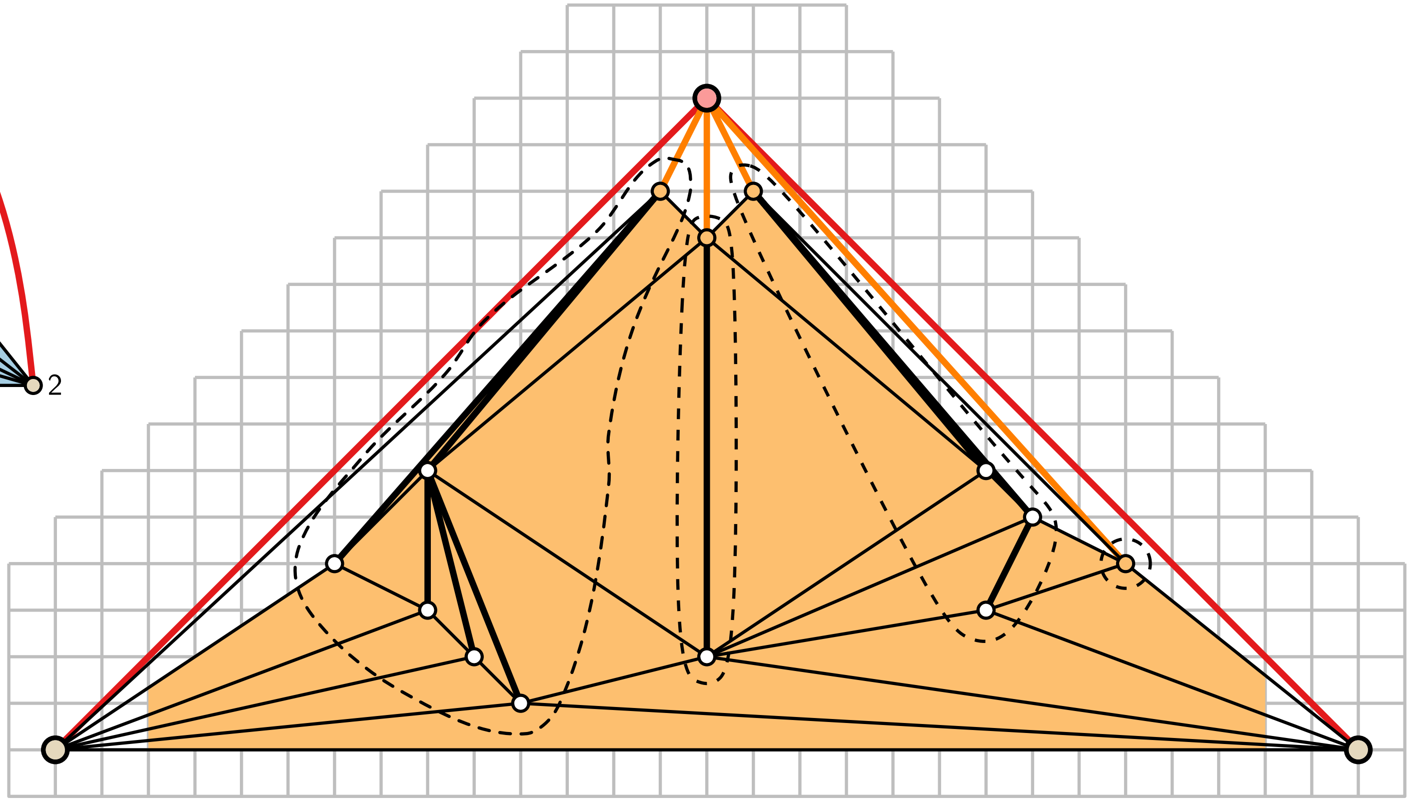
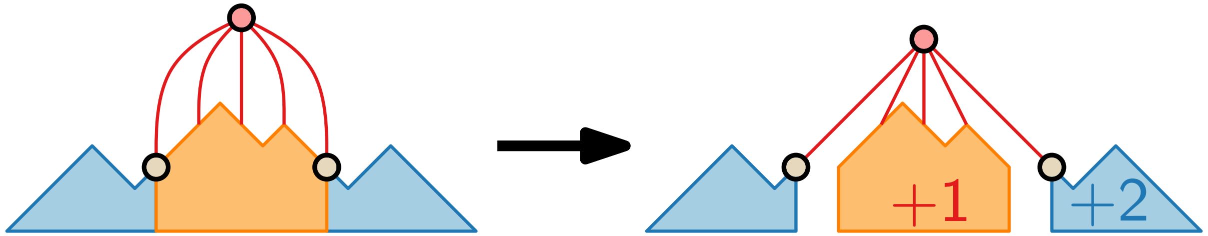


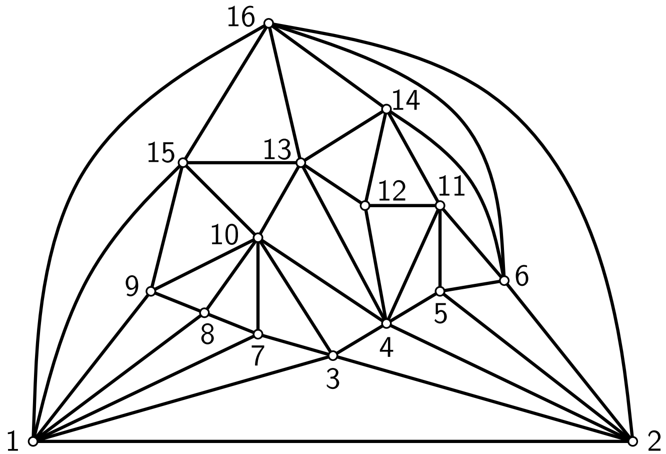


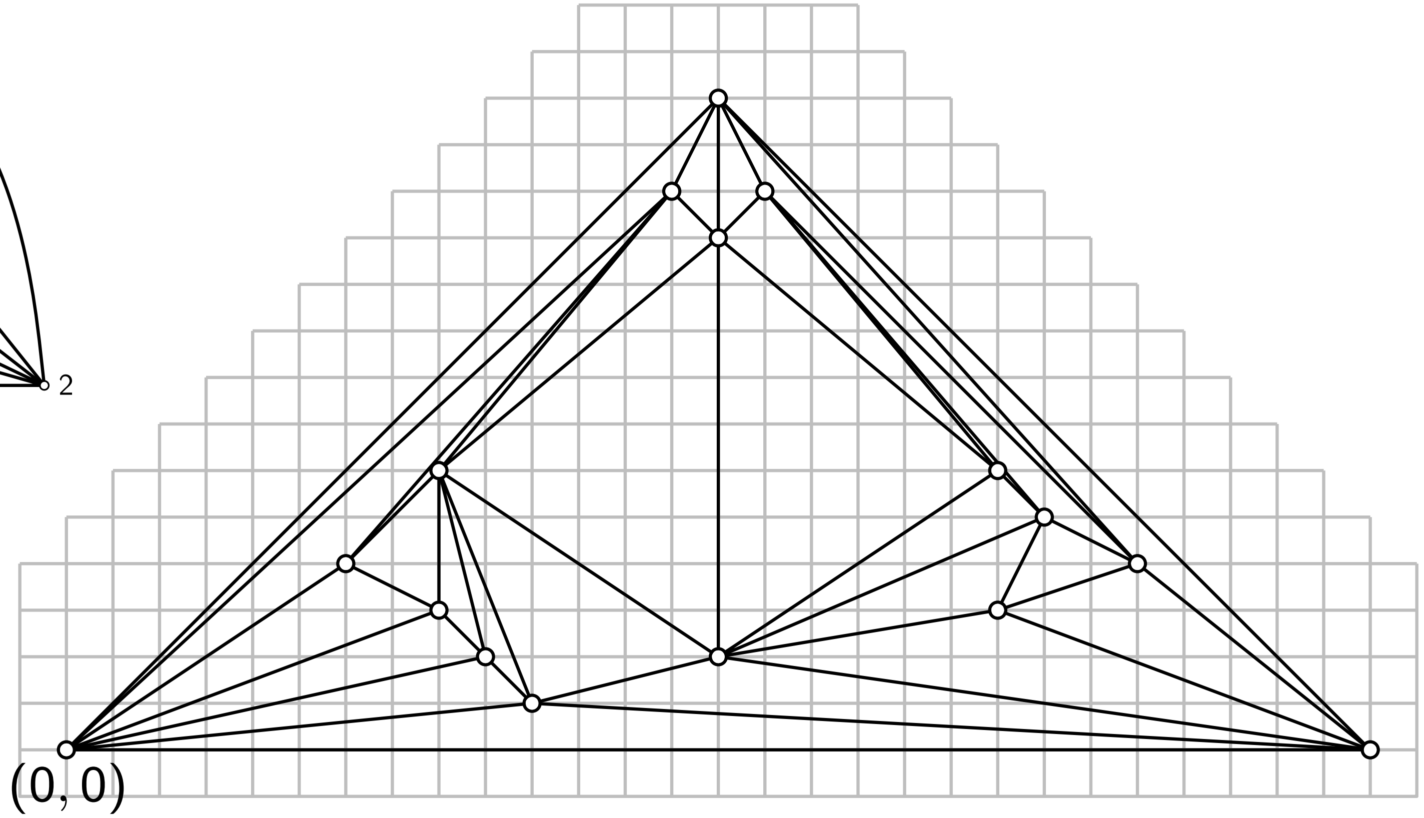


$L(16)$

Shift Method – Example







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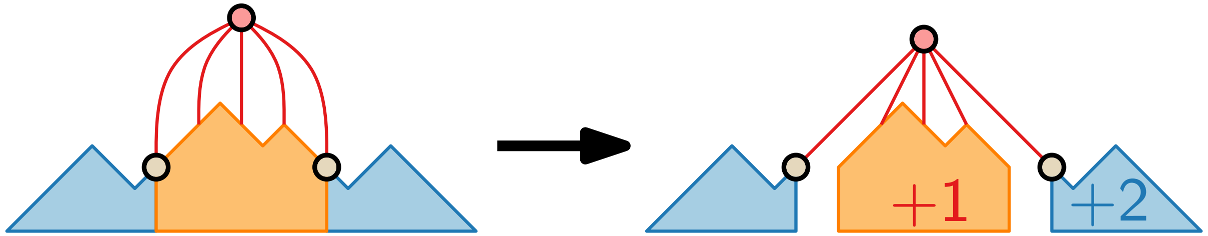

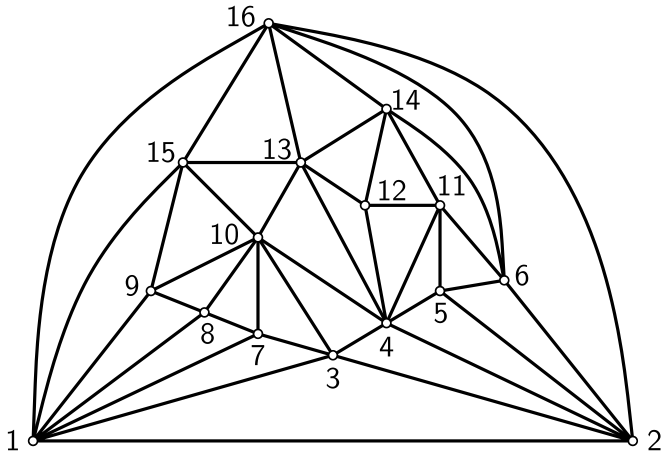

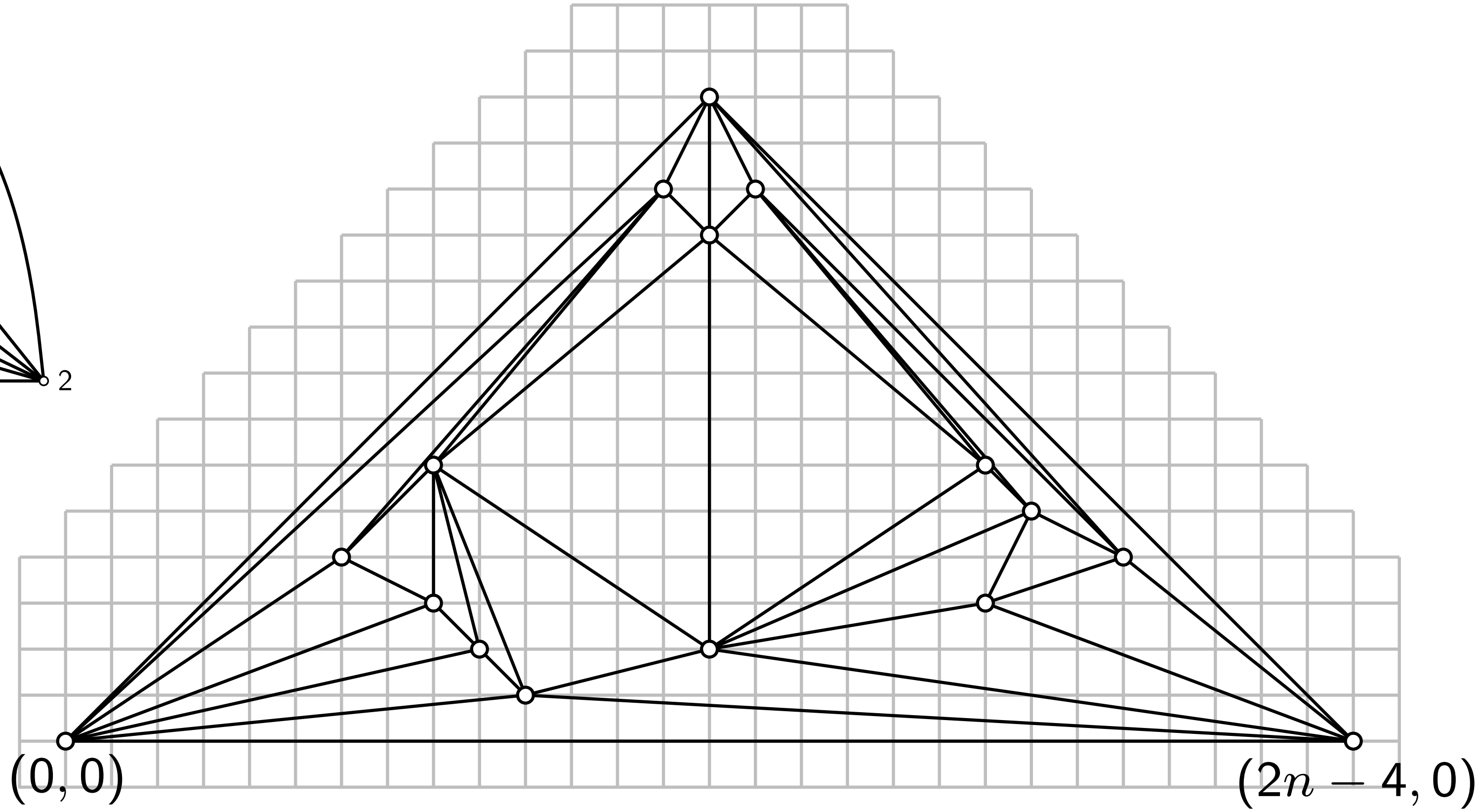

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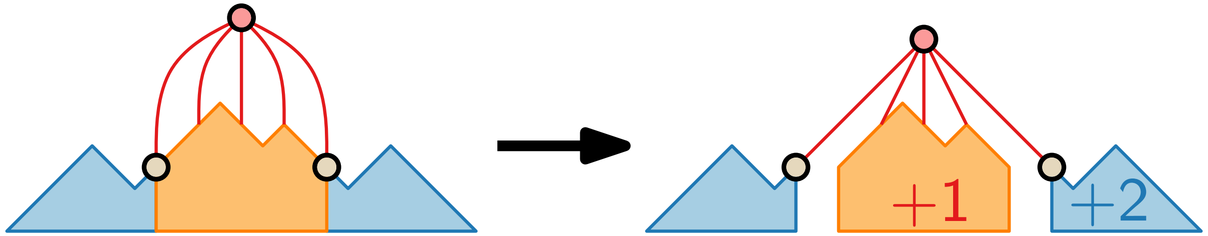
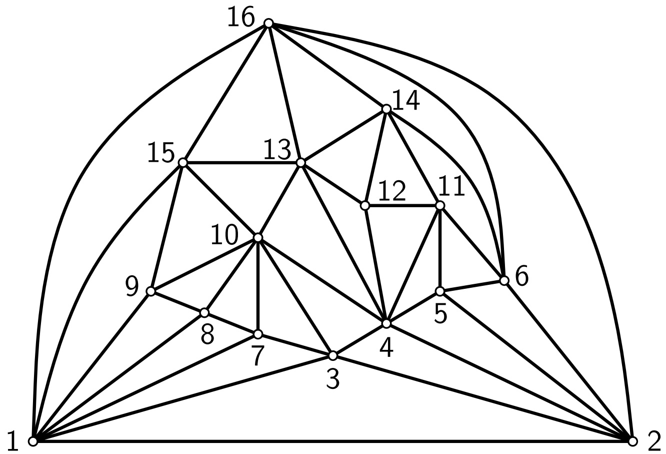
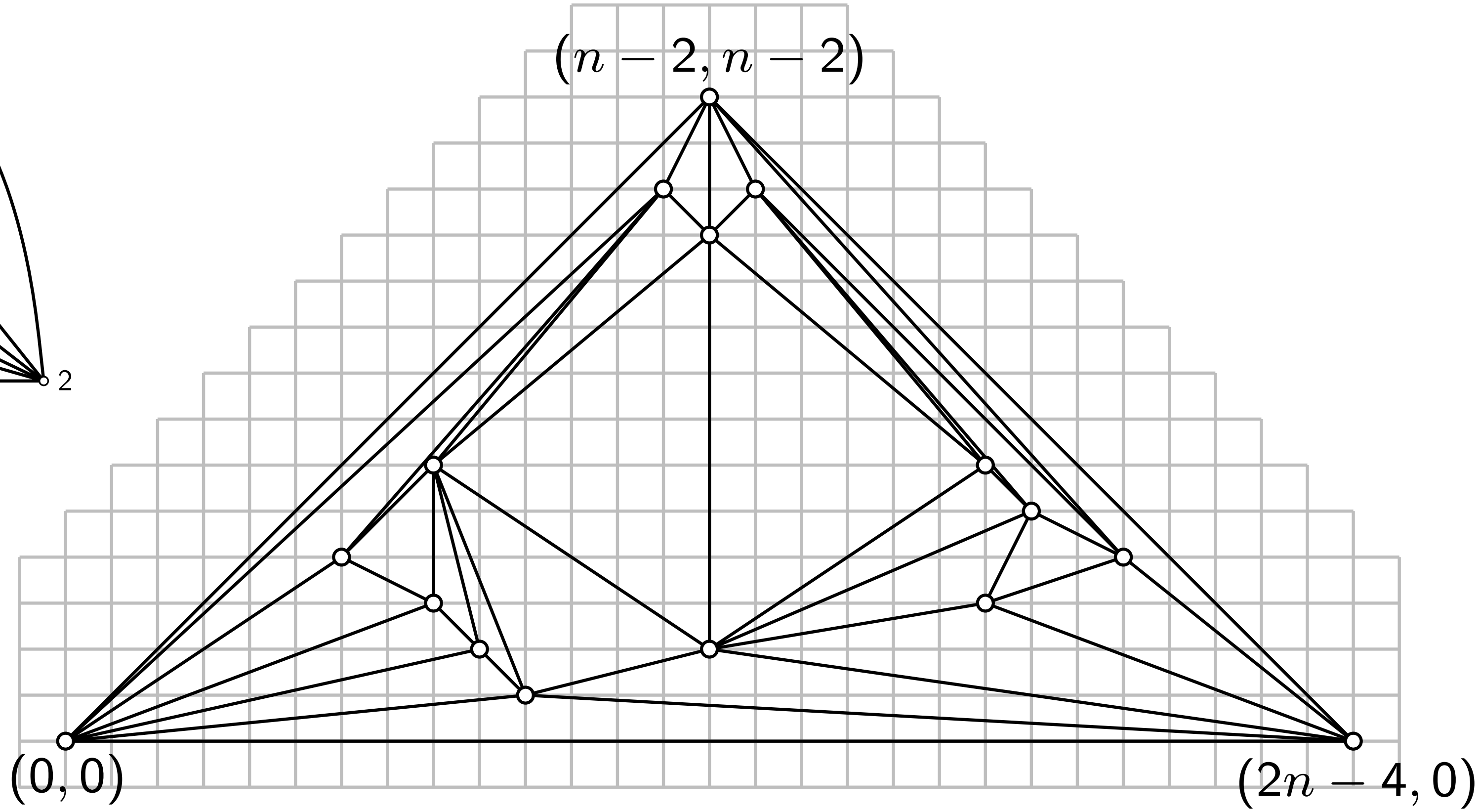
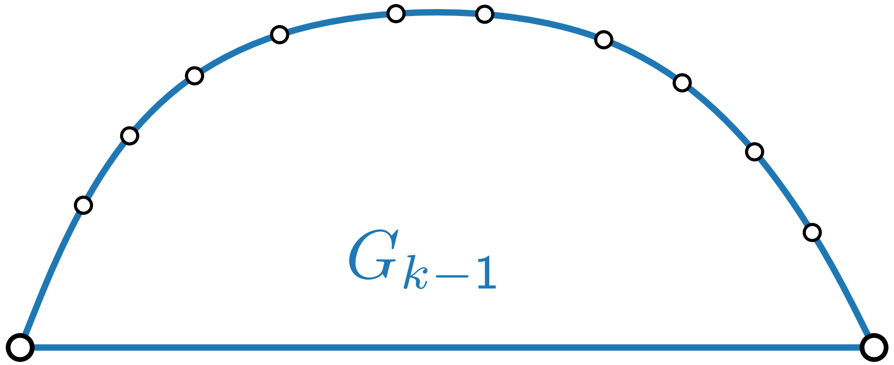






Shift Method – Planarity



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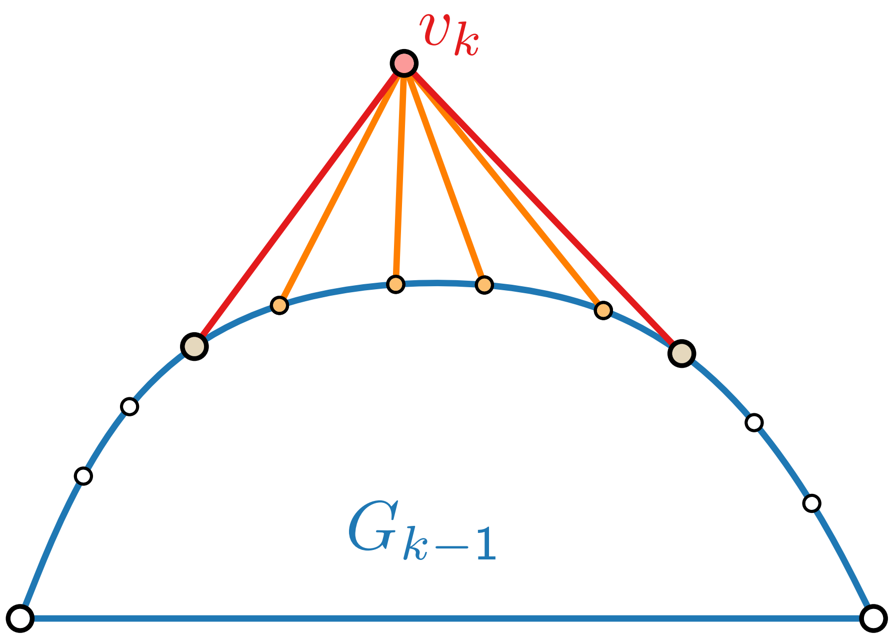

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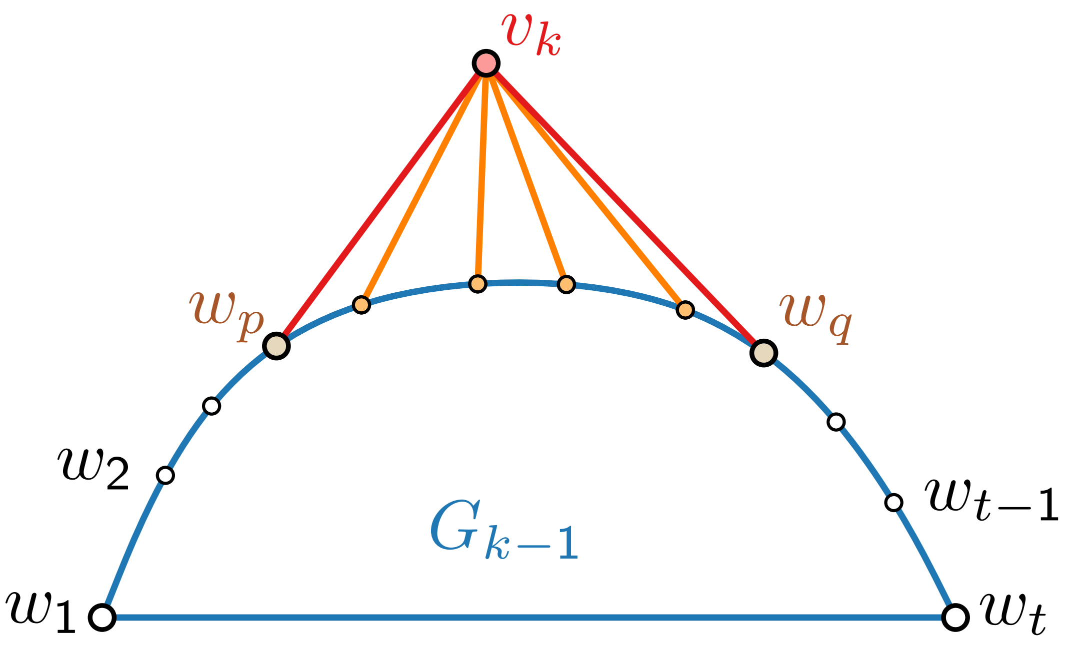

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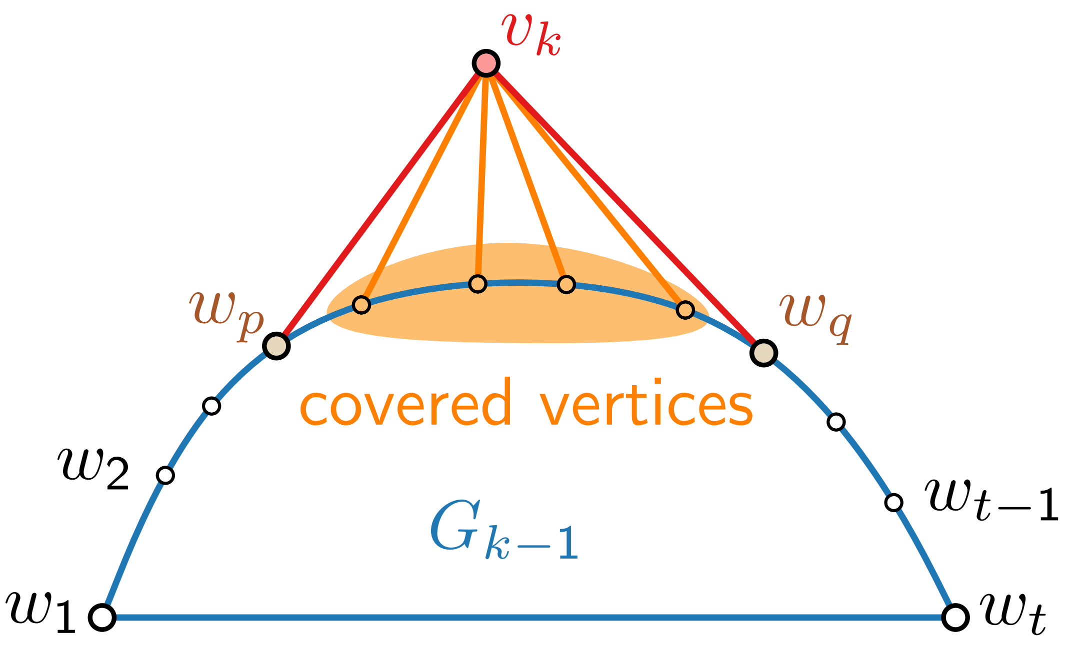
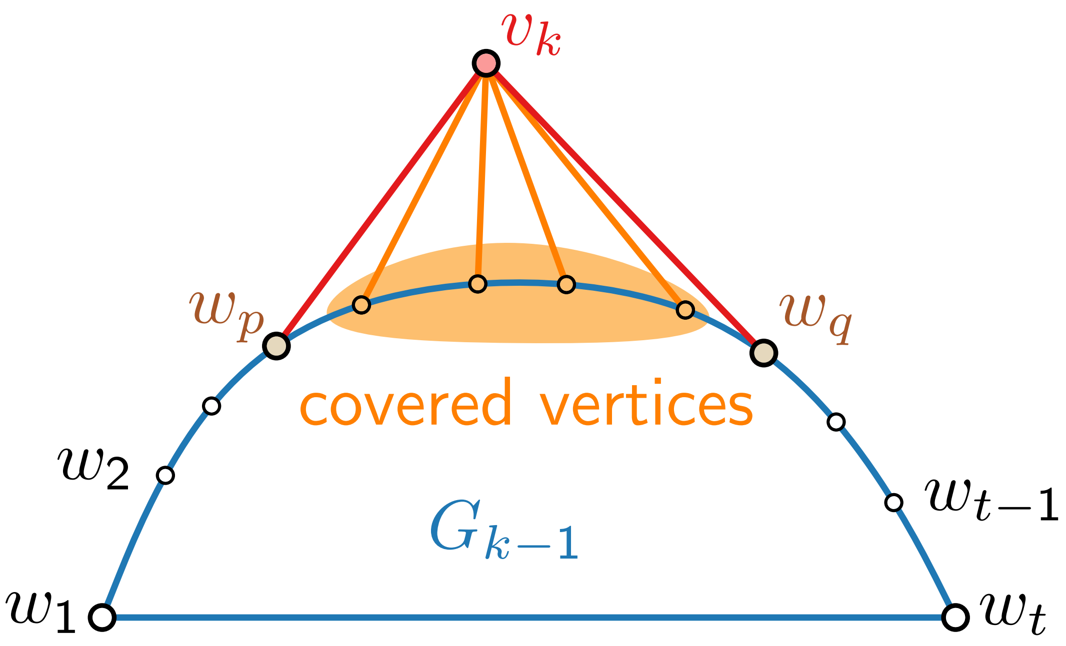


Shift Method – Planarity

Observations.

- Each internal vertex is covered exactly once.



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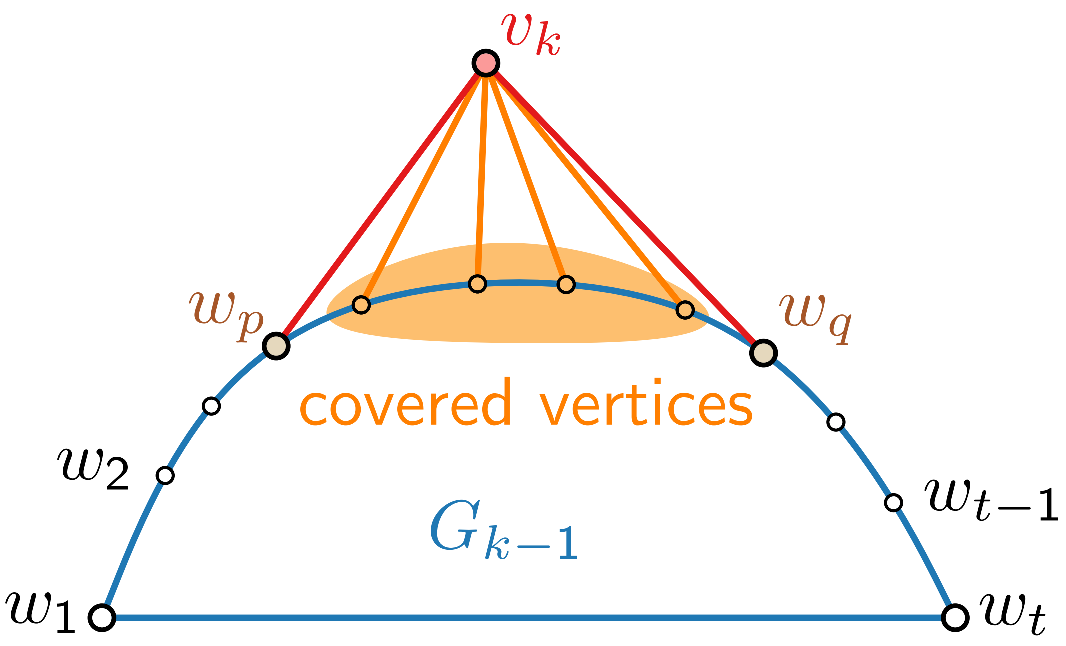
- Covering relation defines a tree in G



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Observations.

- Each internal vertex is covered exactly once.
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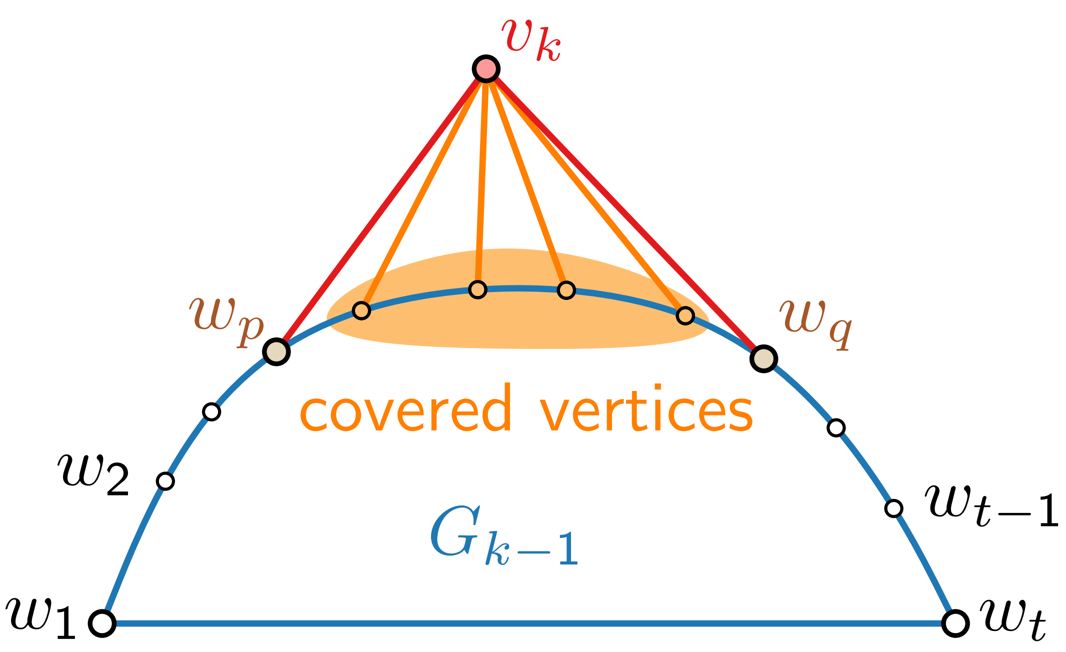

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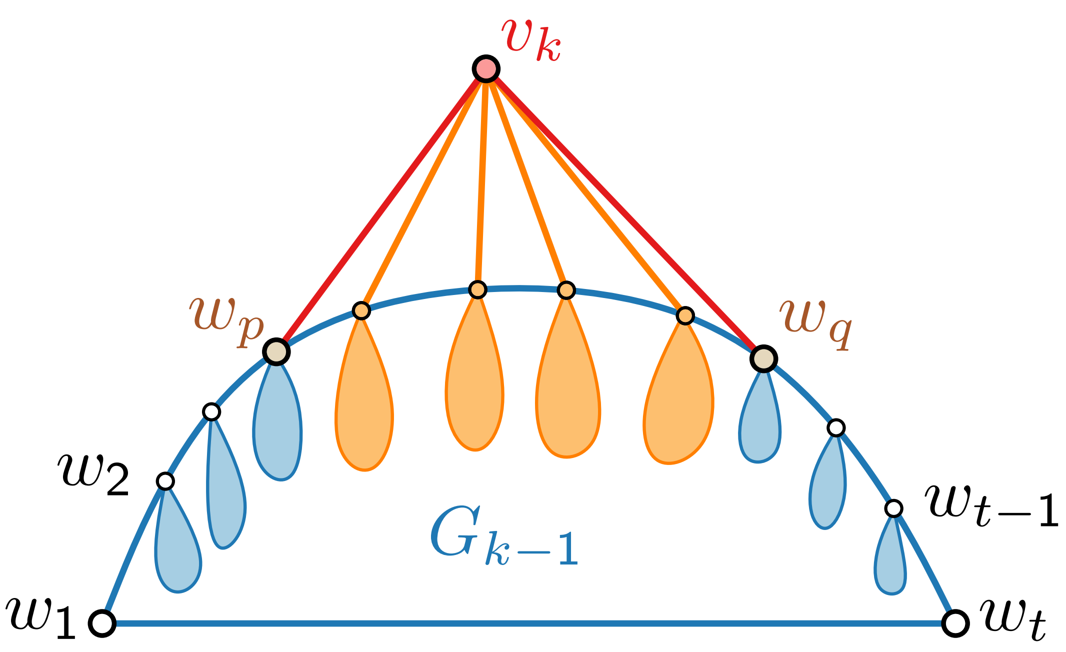

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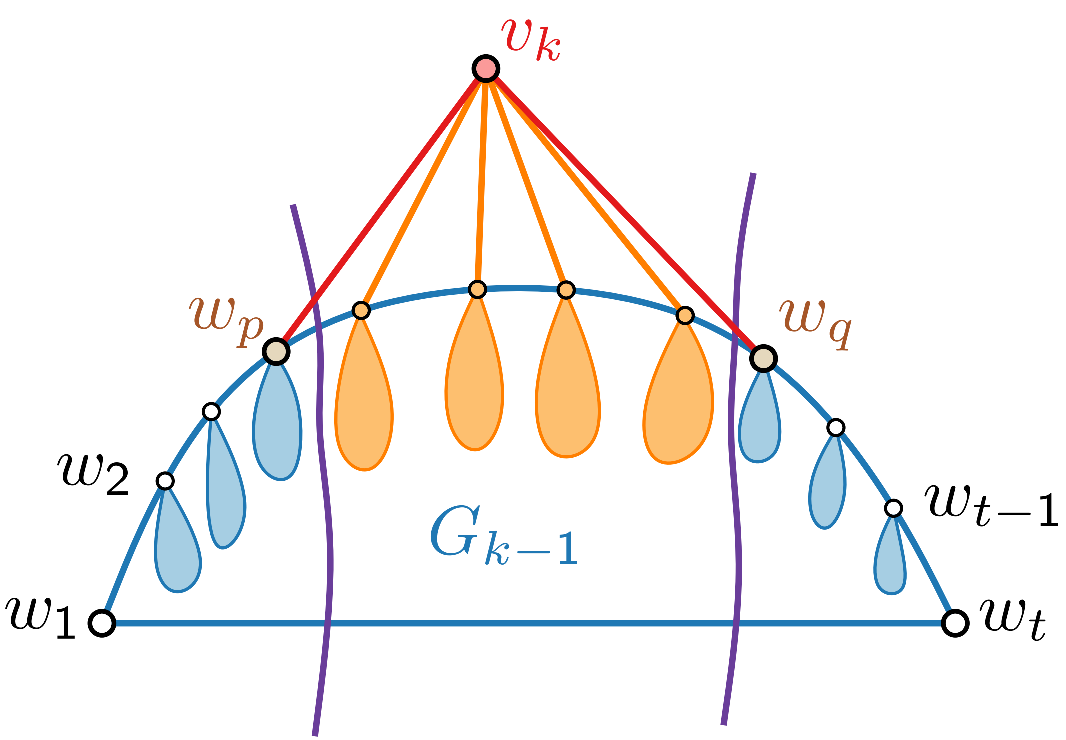

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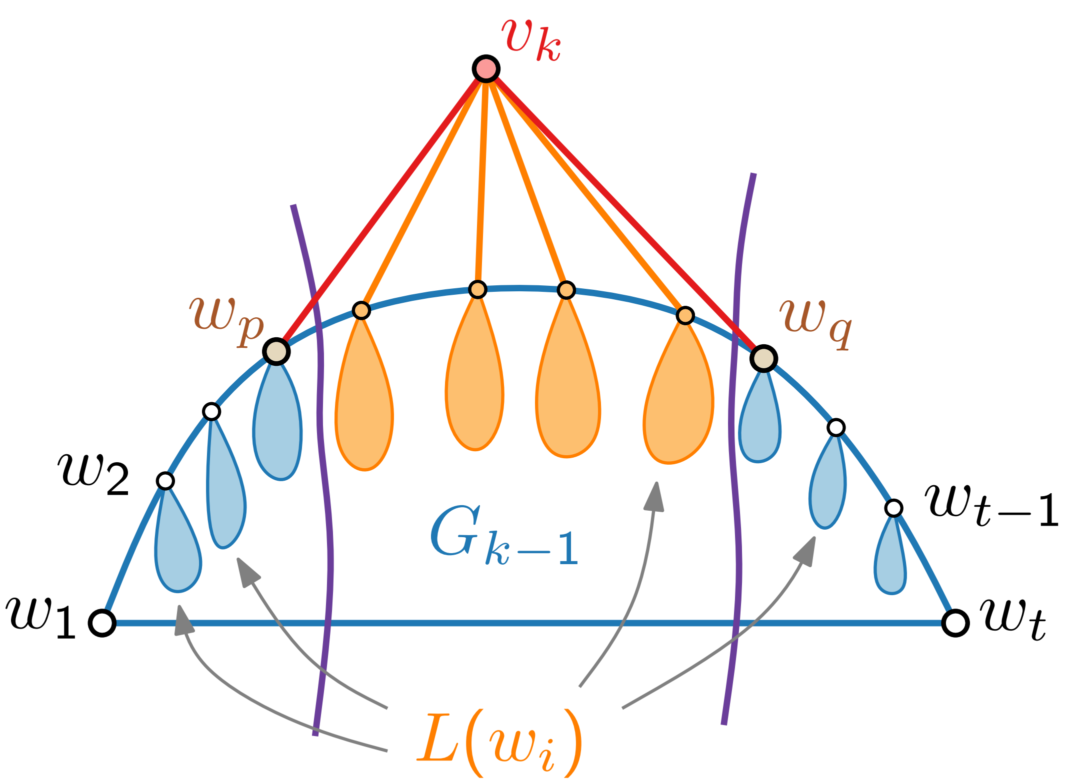

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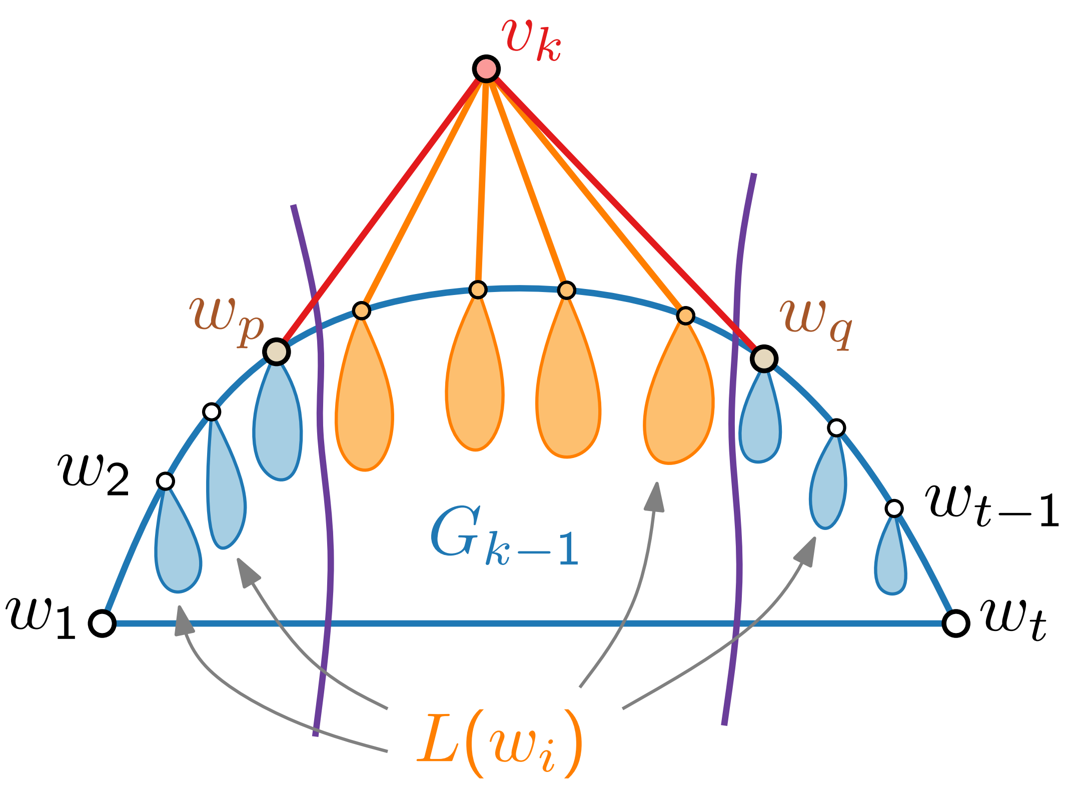
Lemma.
Let $0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, s.t. $\delta_{p+1} - \delta_p \geq 1$, $\delta_q - \delta_{q-1} \geq 1$, $\delta_q - \delta_p \geq 2$ and even.



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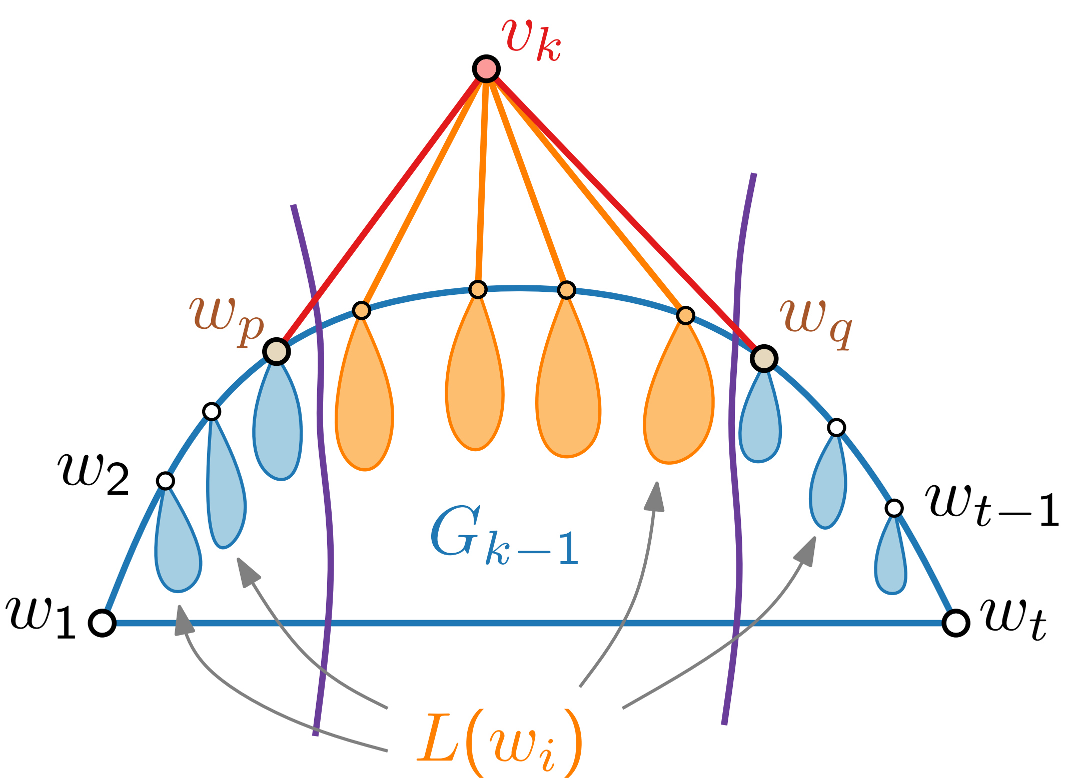

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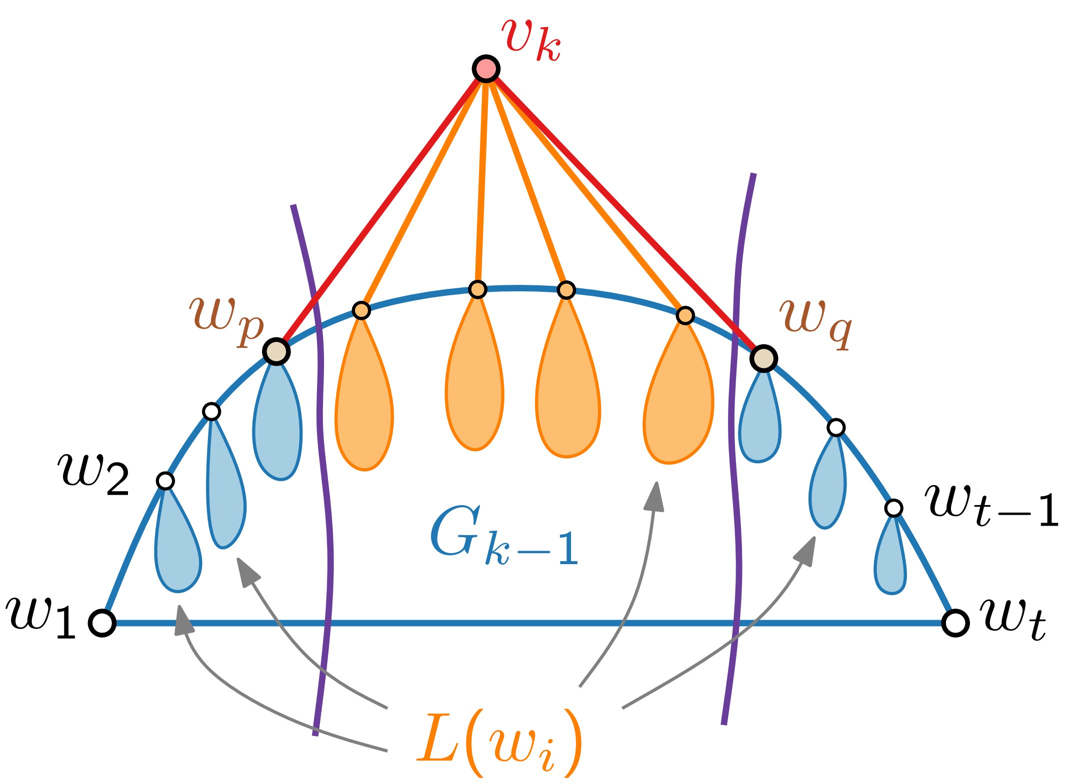

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If G_{k-1} is drawn planar and straight-line, then so is G_k.

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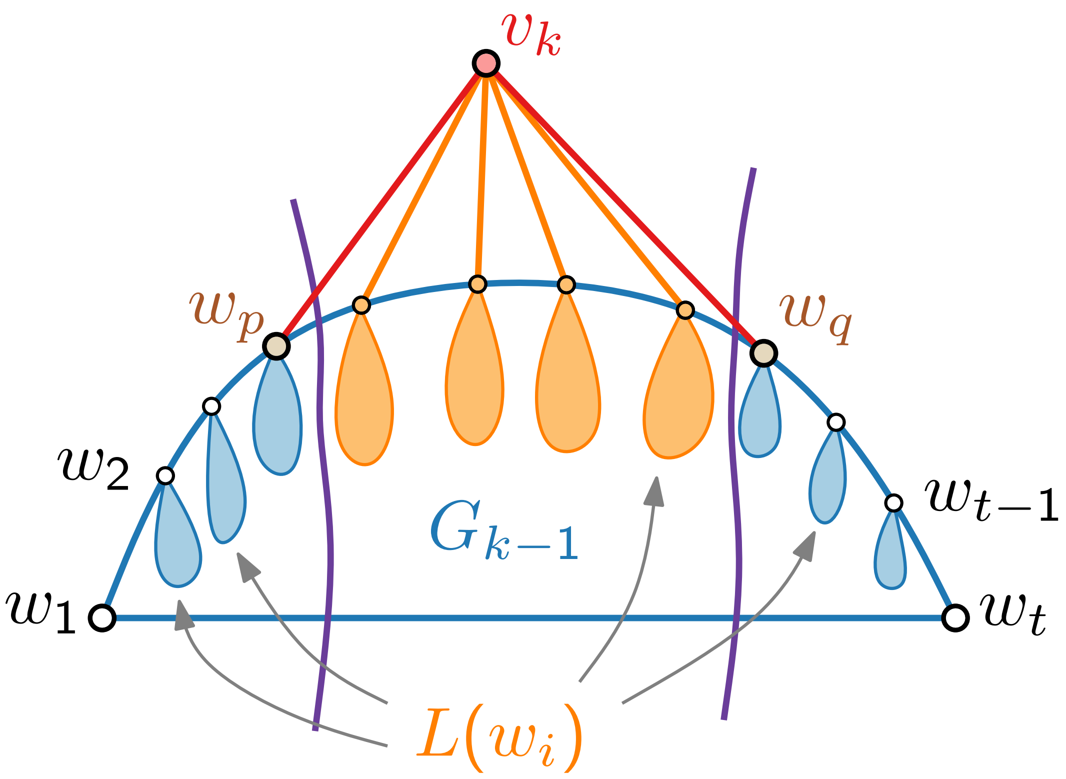

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- New edges don't intersect other edges ($\rightarrow$ invariants).
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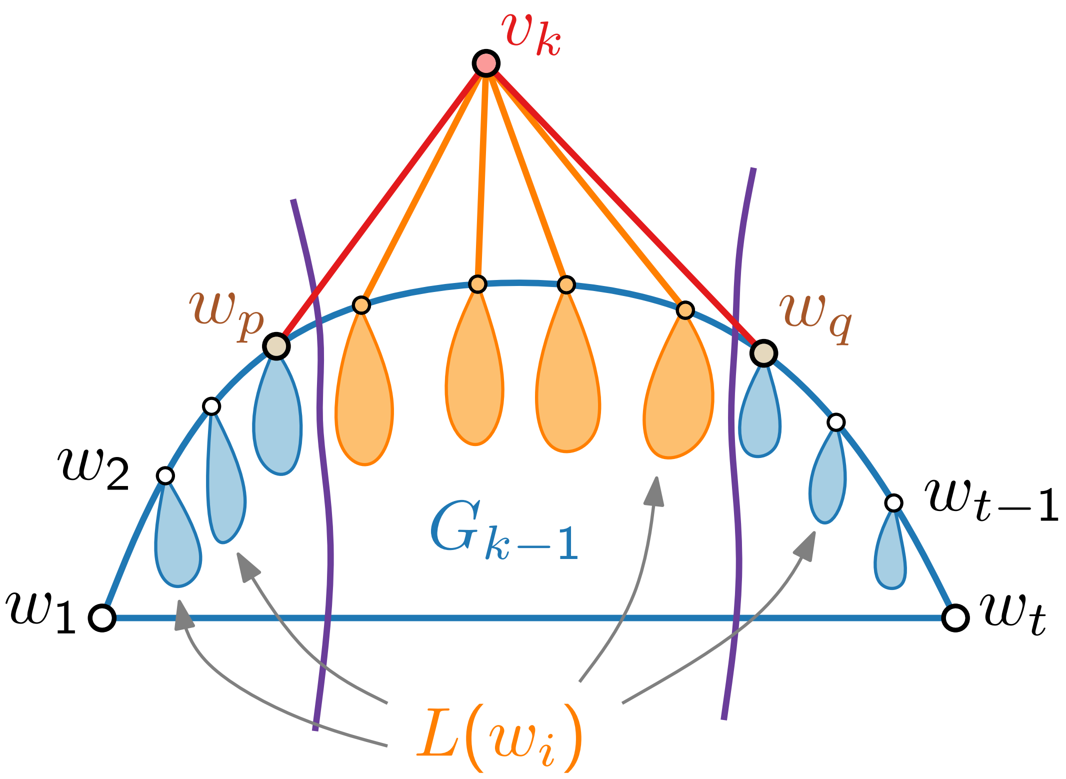

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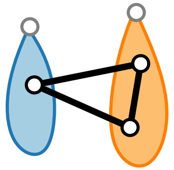

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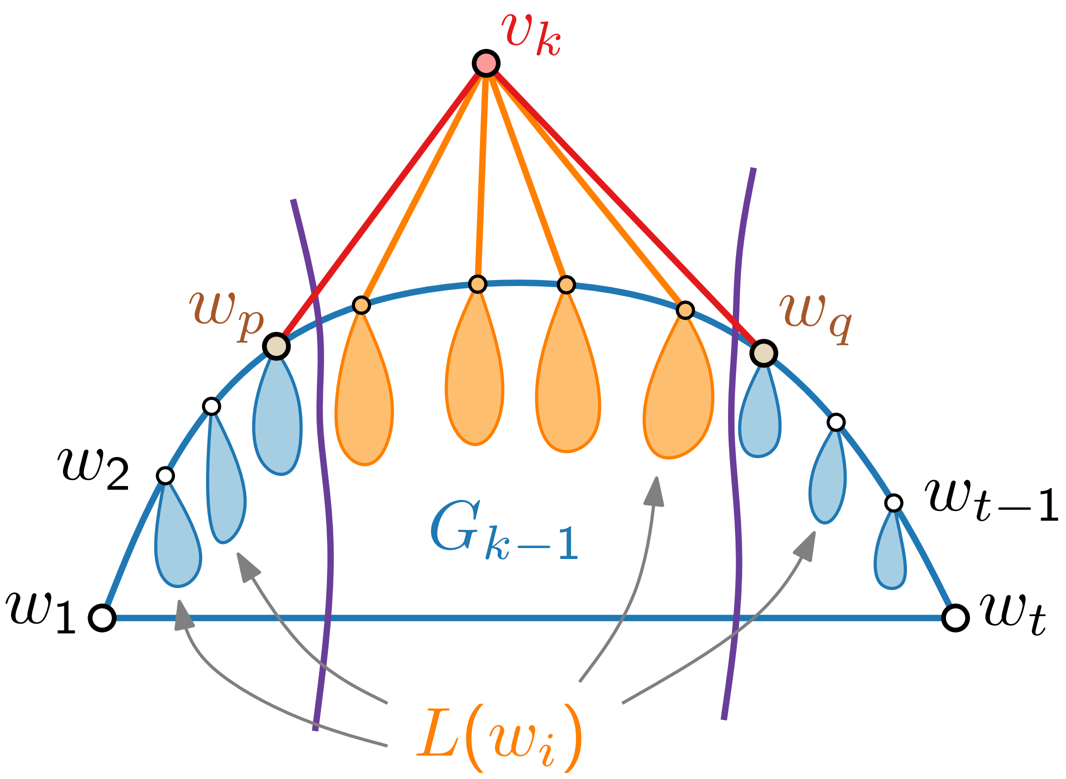

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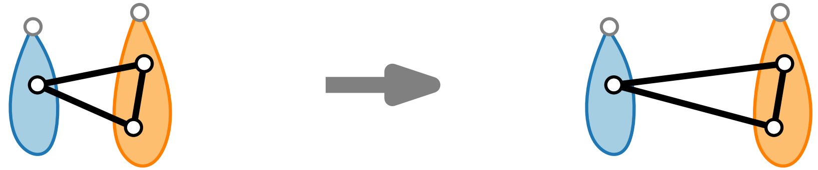

Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod($G, (v_1, v_2, \ldots, v_n)$)

for $k = 1$ to 3 **do**

for $k = 4$ to n **do**

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for $k = 1$ to 3 **do**

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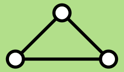
$P(v_1) \leftarrow (0, 0)$; $P(v_2) \leftarrow (2, 0)$, $P(v_3) \leftarrow (1, 1)$ 

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return $P(v_1), \ldots, P(v_n)$

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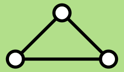

for $k = 4$ to n **do**

  Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.

  Let $w_p, \ldots, w_q$ be the neighbors of v_k.

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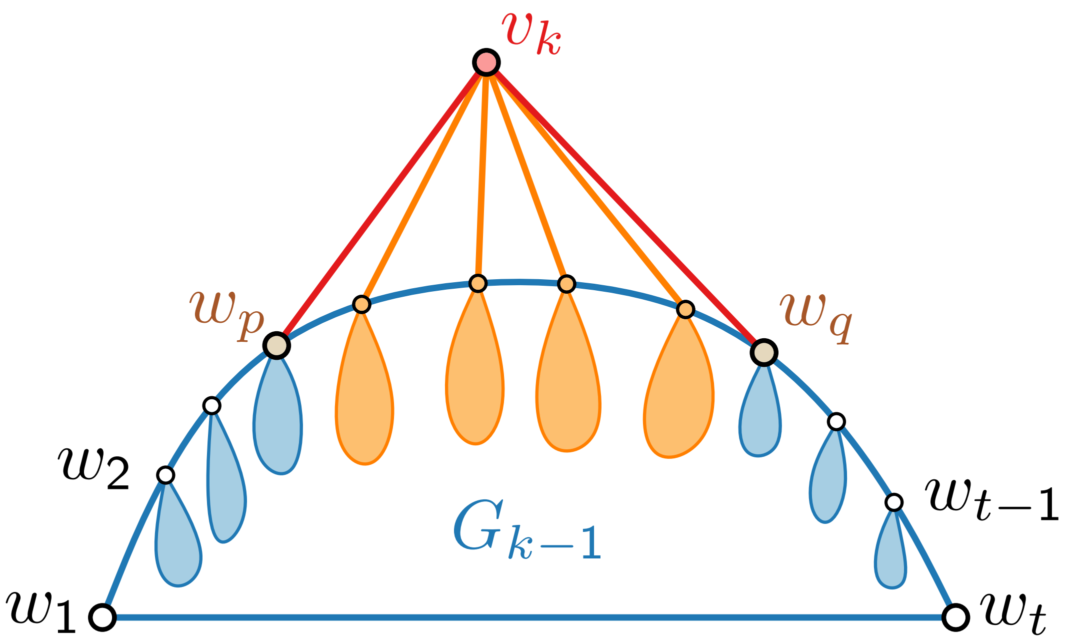

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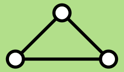

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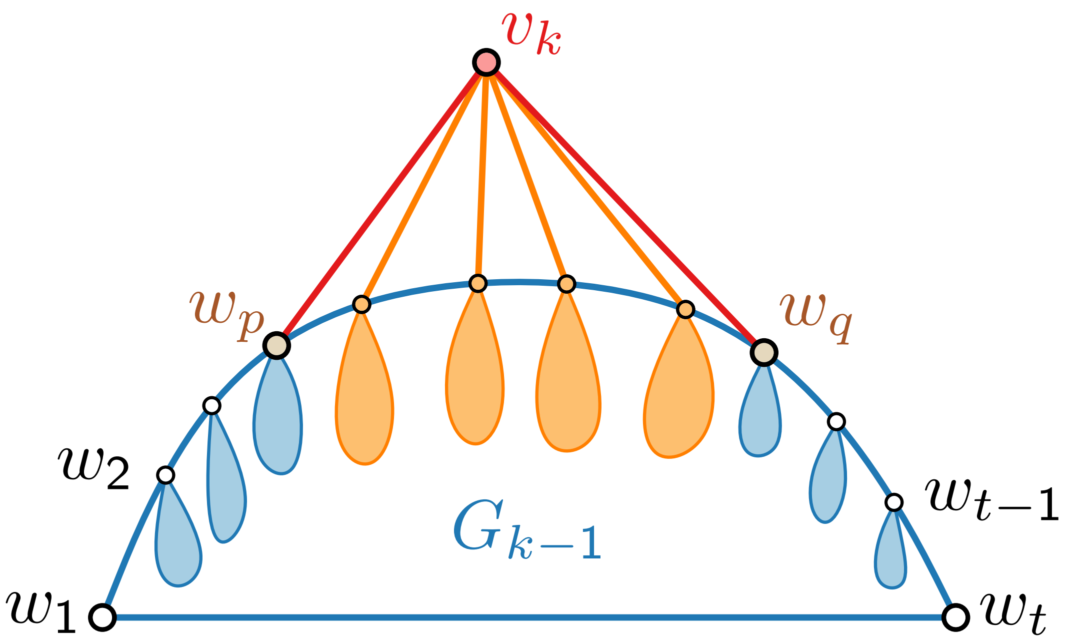

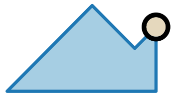

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  $L(v_k) \leftarrow \{v_k\}$

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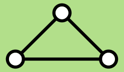
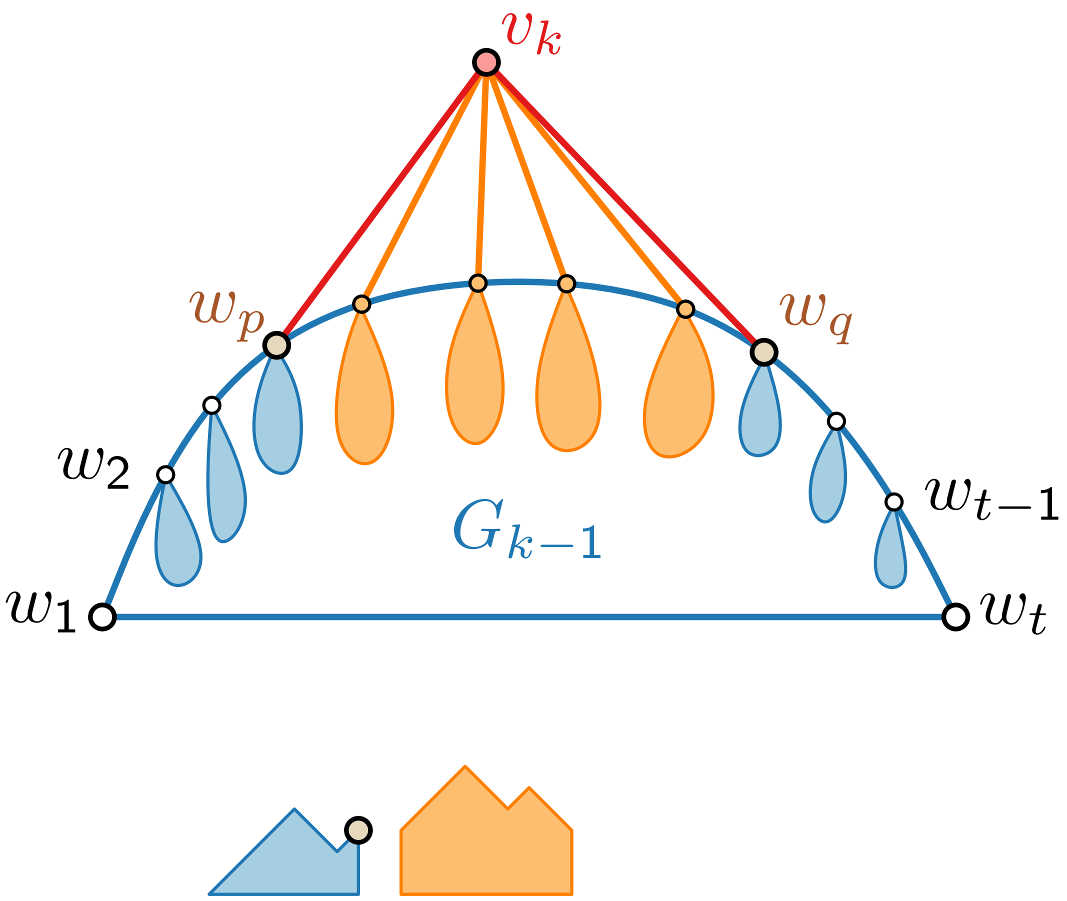


for $k = 4$ to n **do**
  Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
  Let $w_p, \ldots, w_q$ be the neighbors of v_k.
  foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**

return $P(v_1), \ldots, P(v_n)$



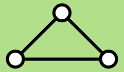
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \ldots, v_n)$)

for $k = 1$ to 3 **do**

  $L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

for $k = 4$ to n **do**

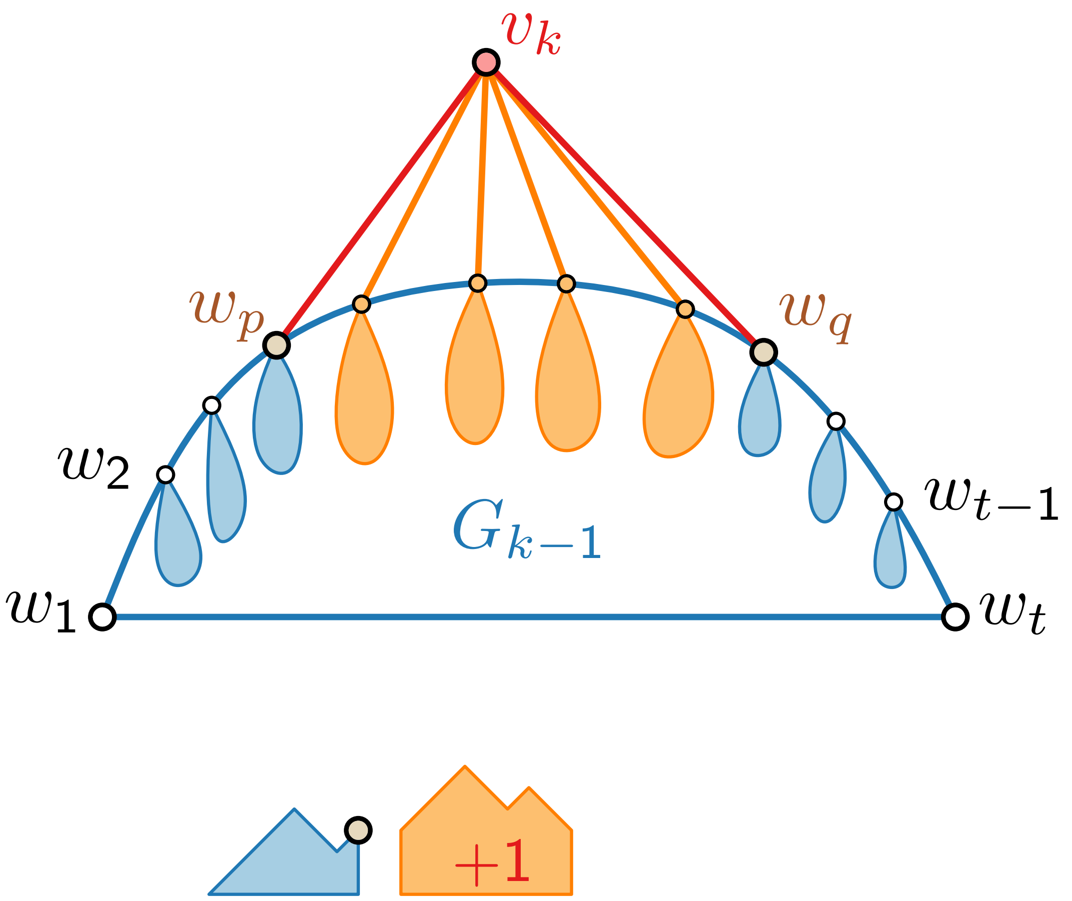
  Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.

  Let $w_p, \ldots, w_q$ be the neighbors of v_k.

  foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**

    $x(v) \leftarrow x(v) + 1$

return $P(v_1), \ldots, P(v_n)$

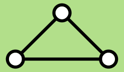


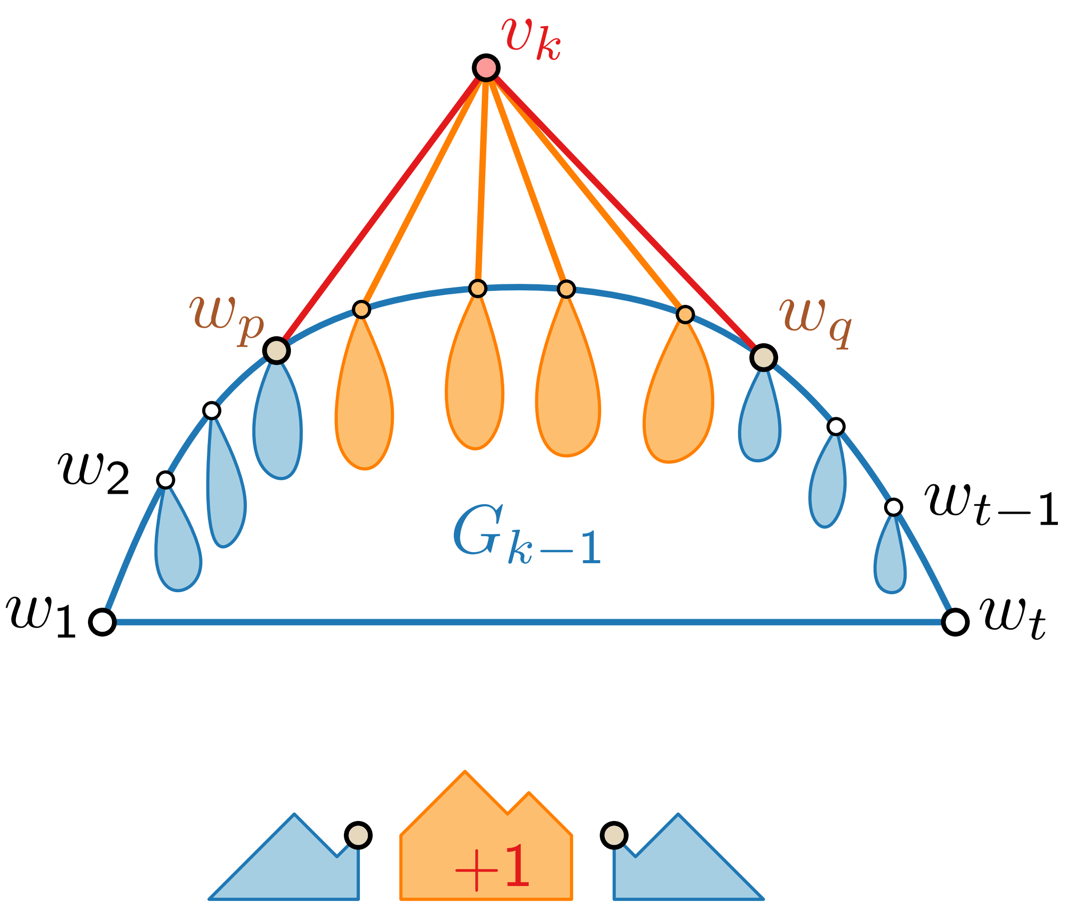
Shift Method – Pseudocode

canonical order of $V(G)$

```
ShiftMethod(G, (v_1, v_2, ..., v_n))
  for k = 1 to 3 do
    L(v_k) ← {v_k}
  P(v_1) ← (0,0); P(v_2) ← (2,0), P(v_3) ← (1,1)
  for k = 4 to n do
    Let ∂G_{k-1} be v_1 = w_1, w_2, ..., w_{t-1}, w_t = v_2.
    Let w_p, ..., w_q be the neighbors of v_k.
    foreach v ∈ ⋃_{i=p+1}^{q-1} L(w_i) do
      x(v) ← x(v) + 1
    foreach v ∈ ⋃_{i=q}^{t} L(w_i) do

  return P(v_1), ..., P(v_n)
```



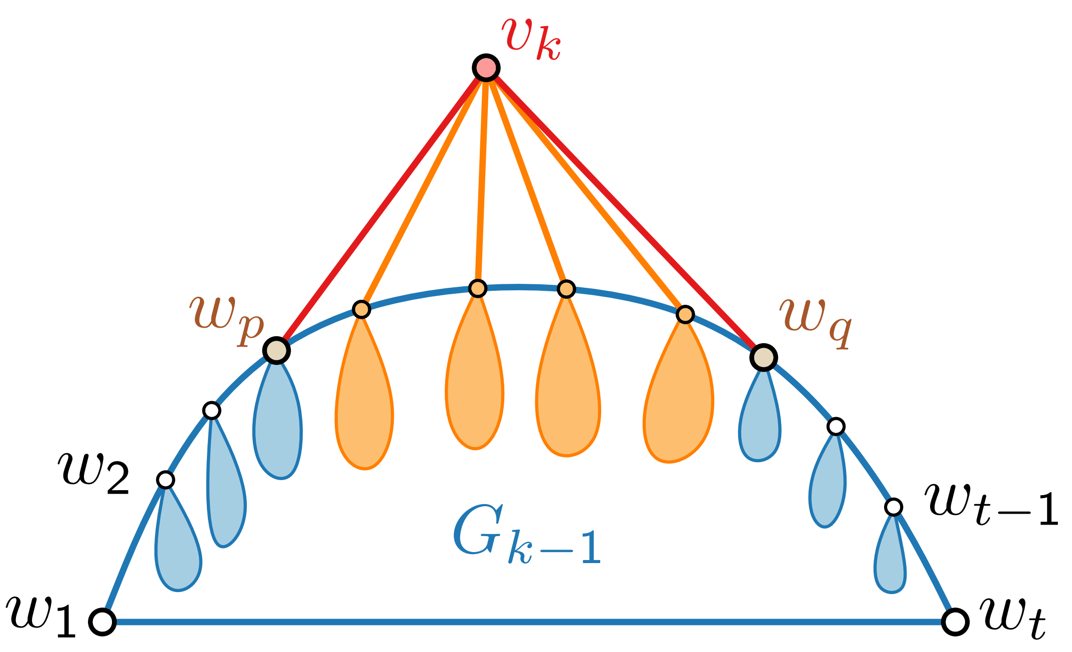


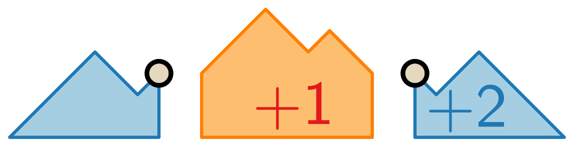
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \ldots, v_n)$)

- **for** $k = 1$ to 3 **do**
  - $L(v_k) \leftarrow \{v_k\}$
- $P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$
- **for** $k = 4$ to n **do**
  - Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
  - Let $w_p, \ldots, w_q$ be the neighbors of v_k.
  - **foreach** $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**
    - $x(v) \leftarrow x(v) + 1$
  - **foreach** $v \in \bigcup_{i=q}^{t} L(w_i)$ **do**
    - $x(v) \leftarrow x(v) + 2$
- **return** $P(v_1), \ldots, P(v_n)$





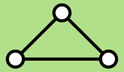
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod$(G, (v_1, v_2, \ldots, v_n))$

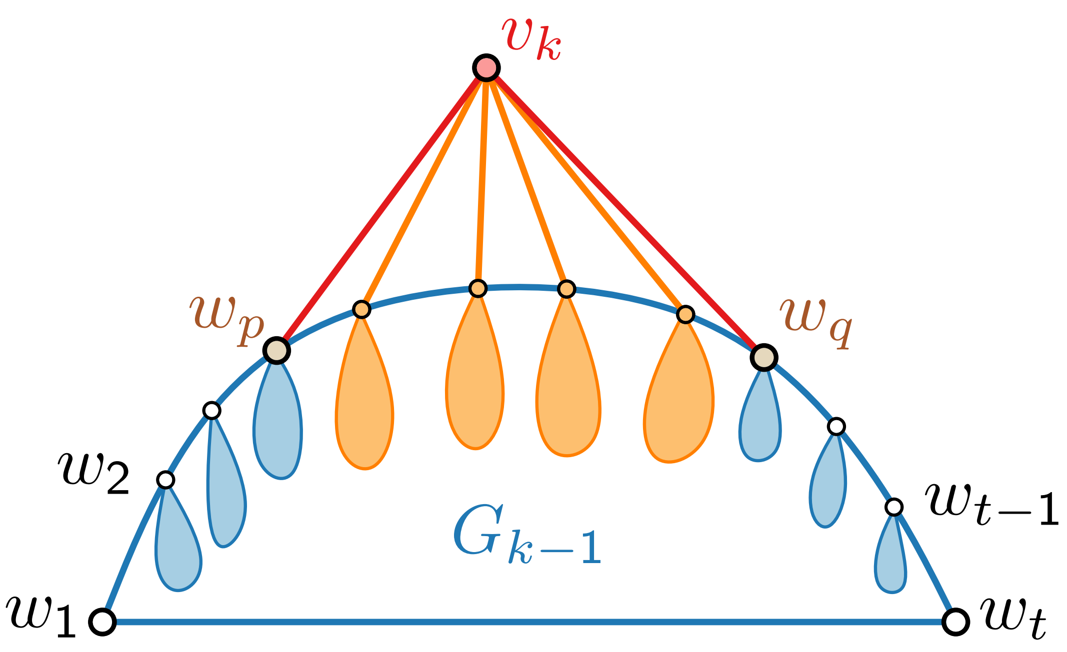
for $k = 1$ to 3 **do**
  $L(v_k) \leftarrow \{v_k\}$

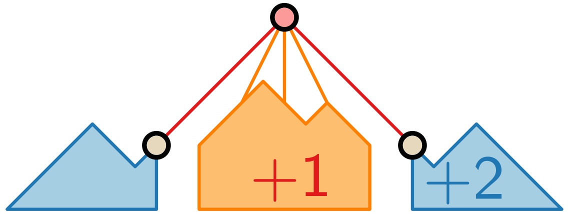
$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$



for $k = 4$ to n **do**
  Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
  Let $w_p, \ldots, w_q$ be the neighbors of v_k.
  foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**
    $x(v) \leftarrow x(v) + 1$
  foreach $v \in \bigcup_{i=q}^{t} L(w_i)$ **do**
    $x(v) \leftarrow x(v) + 2$
  $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$

return $P(v_1), \ldots, P(v_n)$

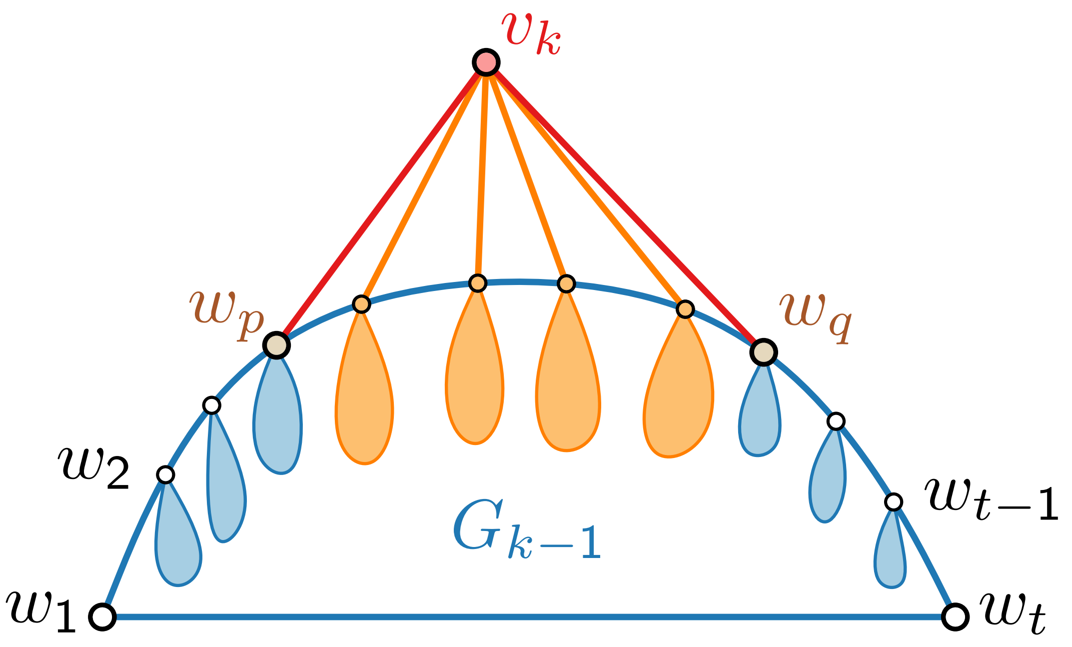


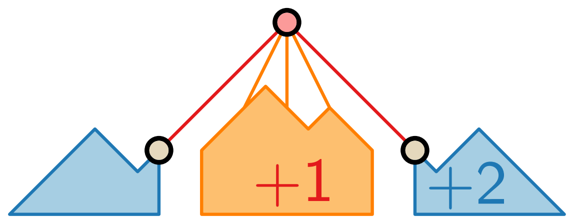


Shift Method – Pseudocode

canonical order of $V(G)$

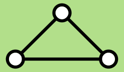
```
ShiftMethod(G, (v_1, v_2, ..., v_n))
  for k = 1 to 3 do
    L(v_k) ← {v_k}
  P(v_1) ← (0,0); P(v_2) ← (2,0), P(v_3) ← (1,1)
  for k = 4 to n do
    Let ∂G_{k-1} be v_1 = w_1, w_2, ..., w_{t-1}, w_t = v_2.
    Let w_p, ..., w_q be the neighbors of v_k.
    foreach v ∈ ⋃_{i=p+1}^{q-1} L(w_i) do
      x(v) ← x(v) + 1
    foreach v ∈ ⋃_{i=q}^{t} L(w_i) do
      x(v) ← x(v) + 2
    P(v_k) ← intersection of slope-±1 diagonals
              through P(w_p) and P(w_q)
    L(v_k) ← ⋃_{i=p+1}^{q-1} L(w_i) ∪ {v_k}
  return P(v_1), ..., P(v_n)
```

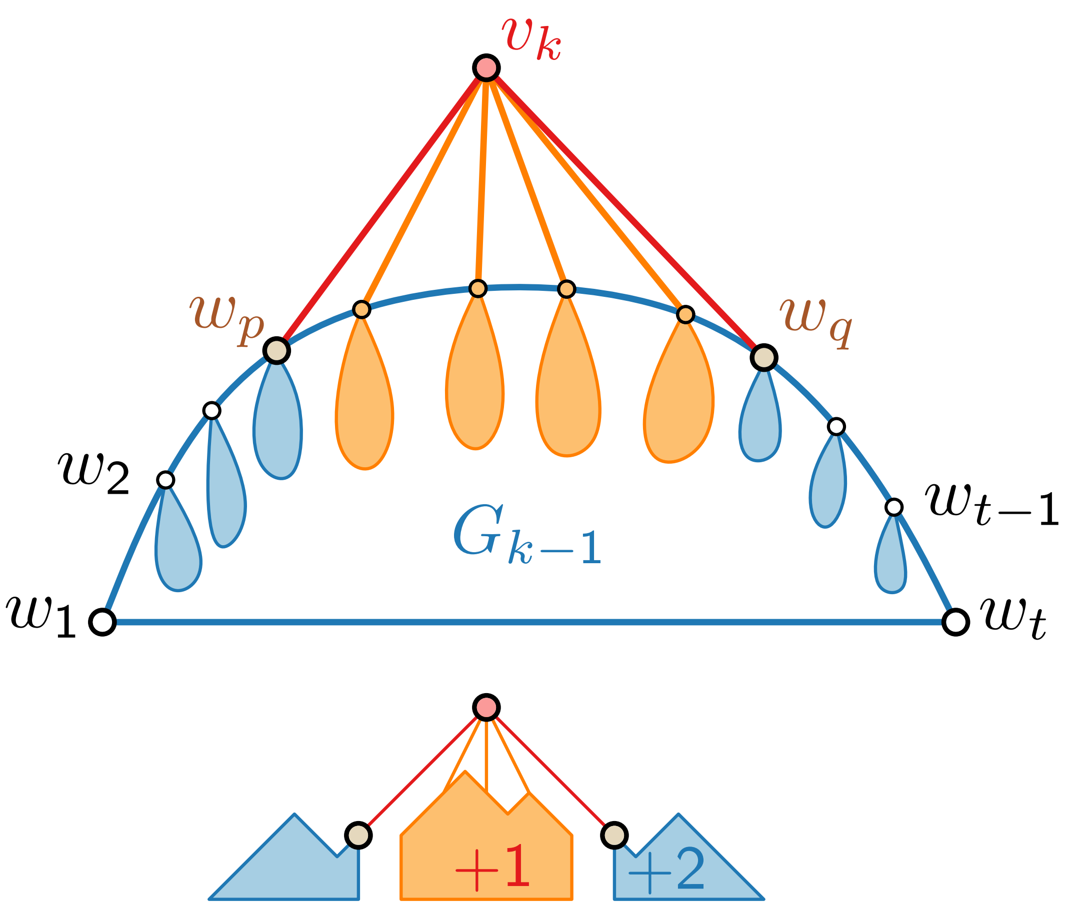




Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \ldots, v_n)$)
  for $k = 1$ to 3 **do**
    $L(v_k) \leftarrow \{v_k\}$
  $P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 
  for $k = 4$ to n **do**
    Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
    Let $w_p, \ldots, w_q$ be the neighbors of v_k.
    foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**
      $x(v) \leftarrow x(v) + 1$
    foreach $v \in \bigcup_{i=q}^{t} L(w_i)$ **do**
      $x(v) \leftarrow x(v) + 2$
    $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$
    $L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$
  return $P(v_1), \ldots, P(v_n)$



Running Time?

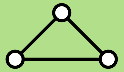
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \ldots, v_n)$)

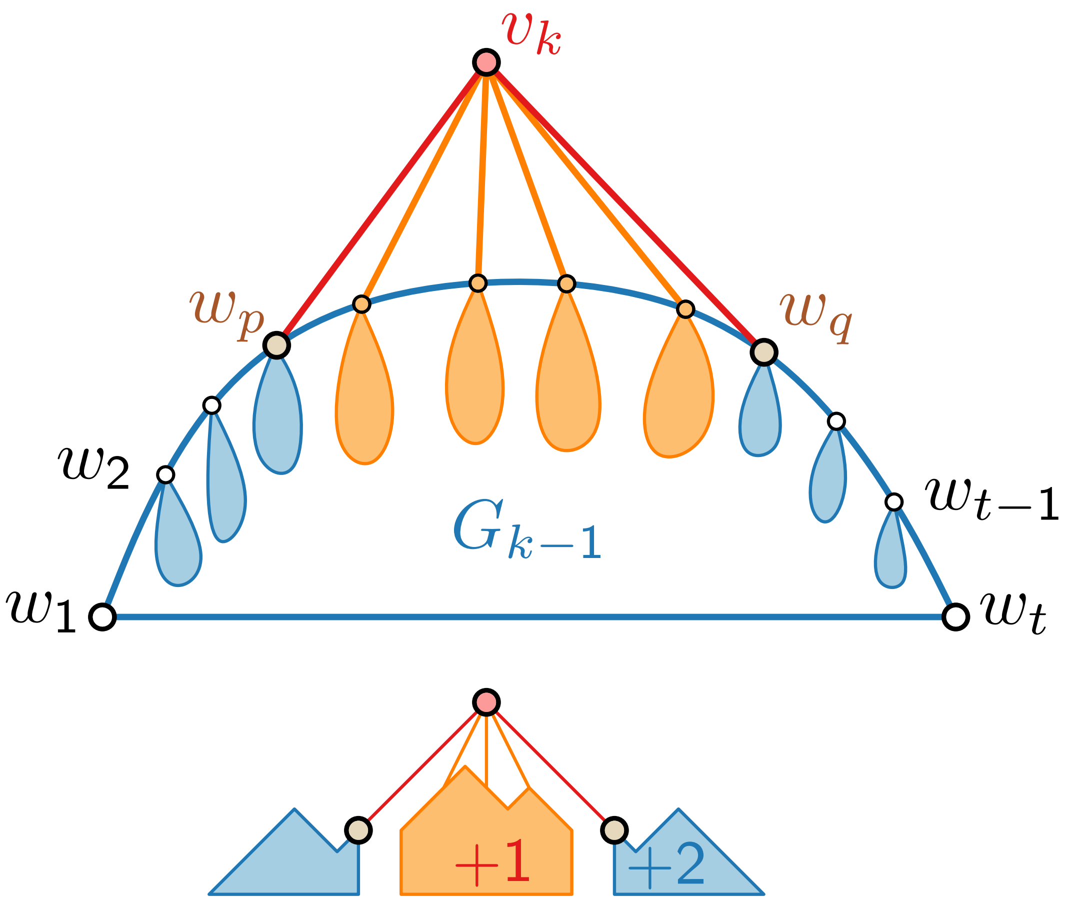
for $k = 1$ to 3 **do**
  $L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$



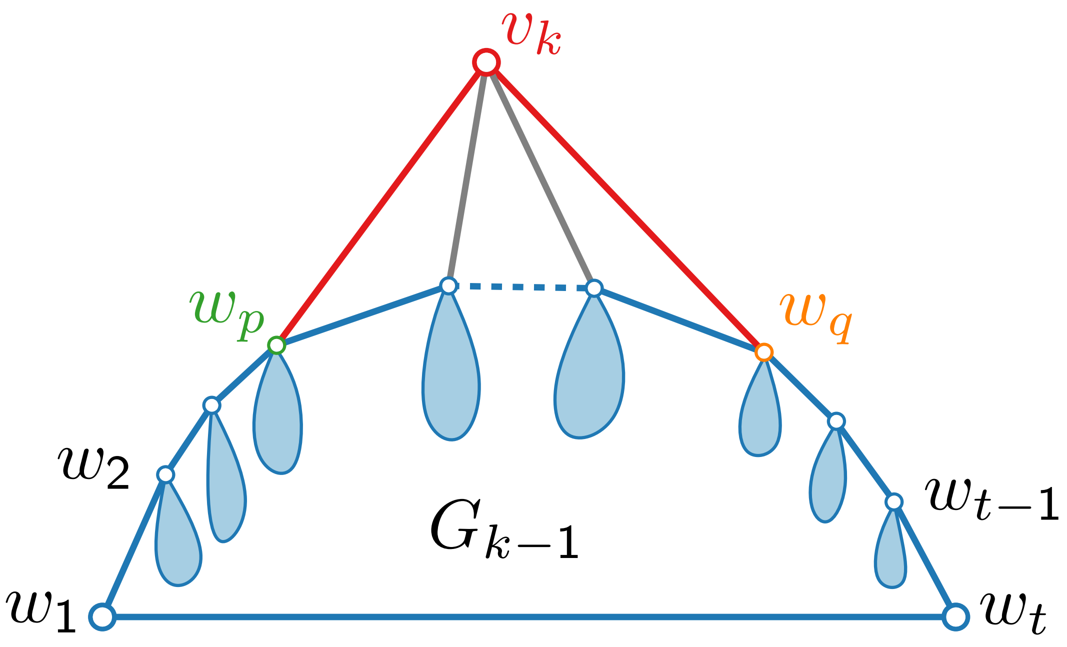
for $k = 4$ to n **do**
  Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
  Let $w_p, \ldots, w_q$ be the neighbors of v_k.
  foreach $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do** // $\mathcal{O}(n^2)$ *in total*
    $x(v) \leftarrow x(v) + 1$
  foreach $v \in \bigcup_{i=q}^{t} L(w_i)$ **do** // $\mathcal{O}(n^2)$ *in total*
    $x(v) \leftarrow x(v) + 2$
  $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$
  $L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$

return $P(v_1), \ldots, P(v_n)$



Running Time?

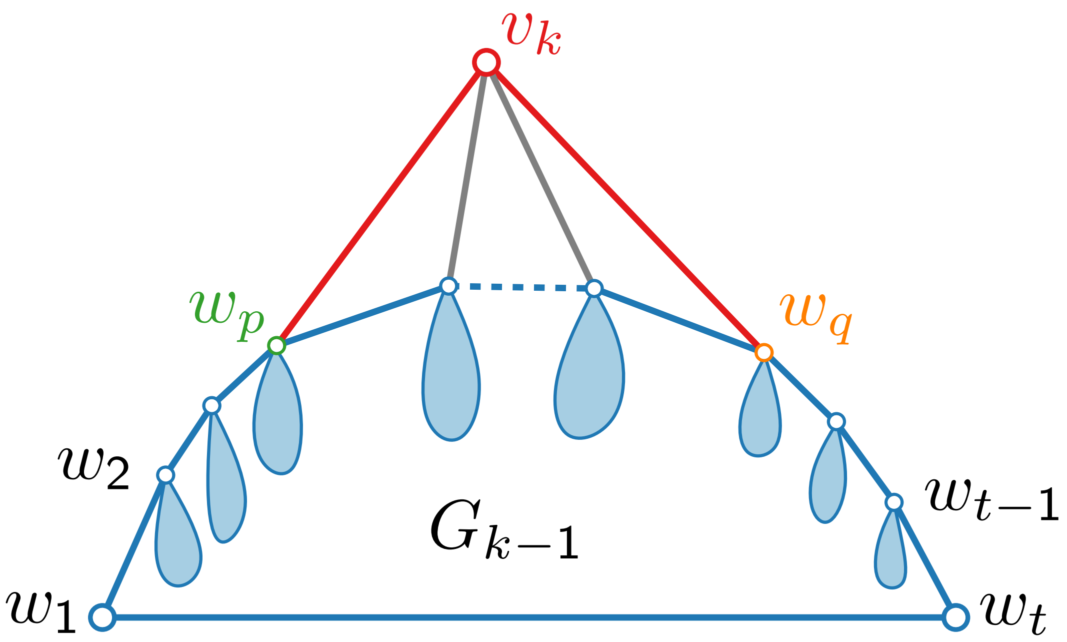
Shift Method – Linear-Time Implementation



Shift Method – Linear-Time Implementation

Idea 1.

To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$



Shift Method – Linear-Time Implementation

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To compute $x(v_k)$ and $y(v_k)$,
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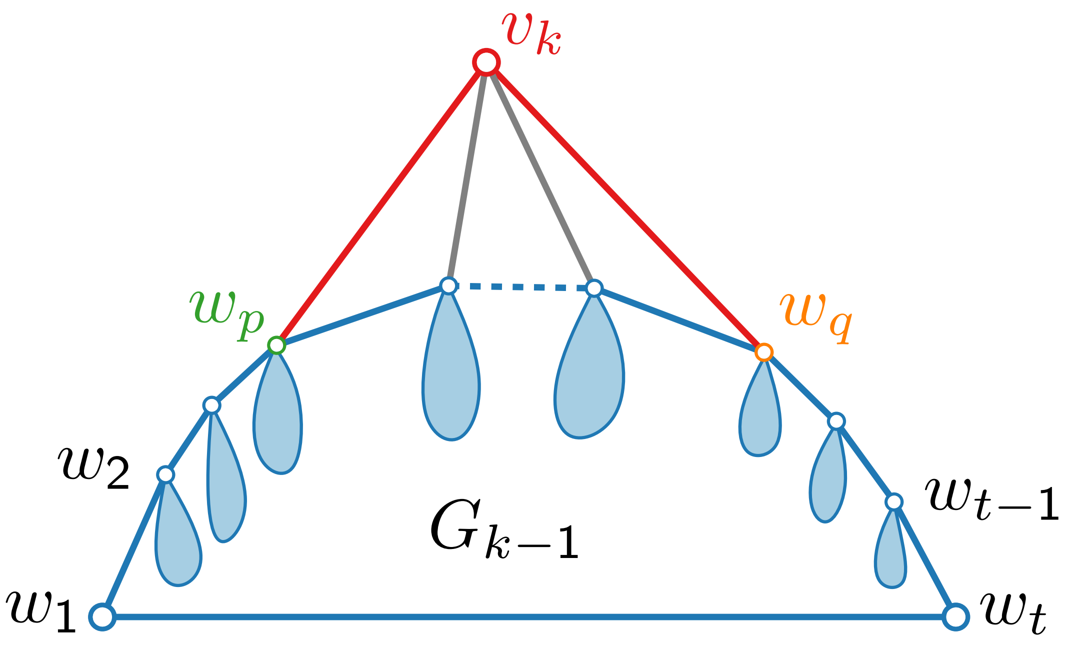
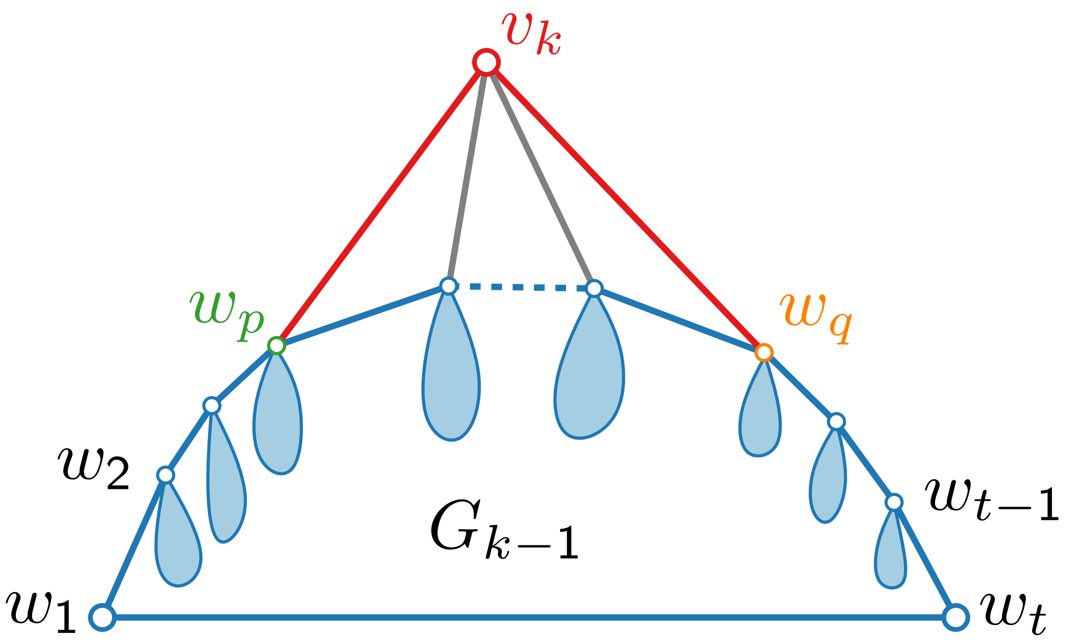


(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$



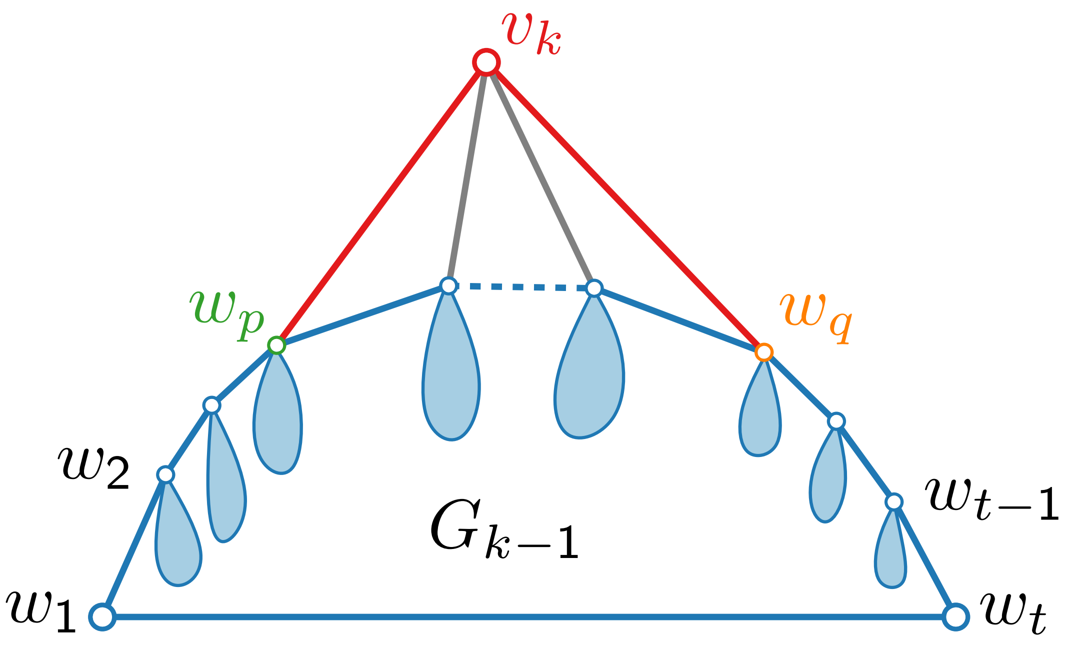
(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit (absolute) x-coordinates,
we store, for each vertex within a specific spanning tree,
the x-distance to its parent (v_1 is the root).



$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
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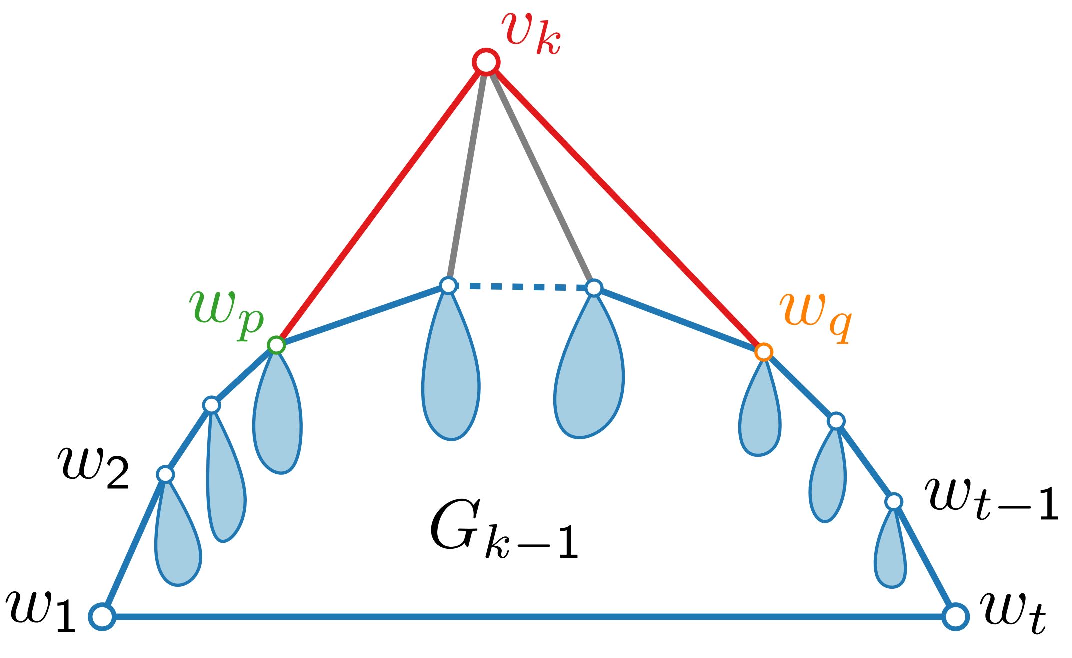


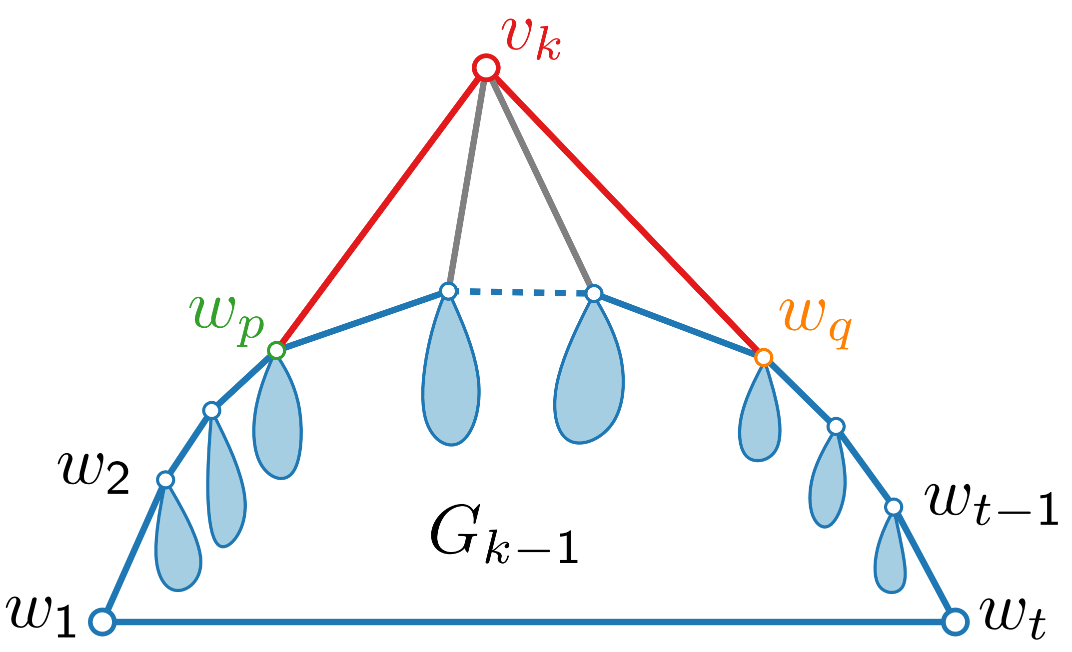
$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

$$(3)\quad x(v_k) - x(w_p) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$



Idea 2.
Instead of storing explicit (absolute) x-coordinates,
we store, for each vertex within a specific spanning tree,
the x-distance to its parent (v_1 is the root).

After all x-distances have been computed, use an additional preorder traversal to compute all x-coordinates.

$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

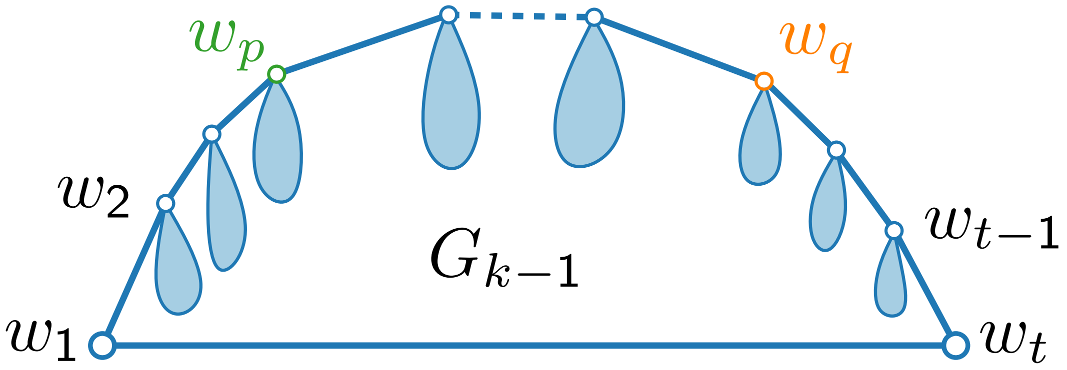
$$(3)\quad x(v_k) - x(w_p) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

$$(3)\quad x(v_k) - x(w_p) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
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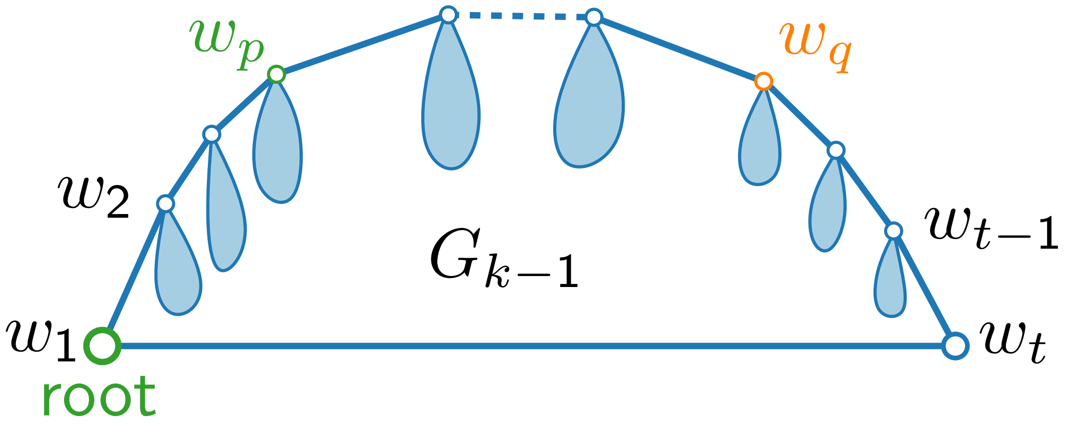

$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

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Shift Method – Linear-Time Implementation

Relative x-distance tree.

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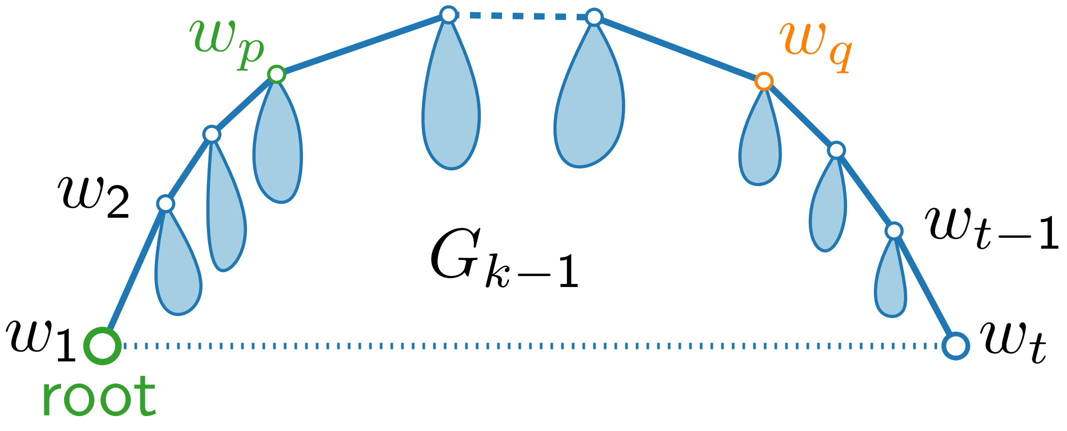

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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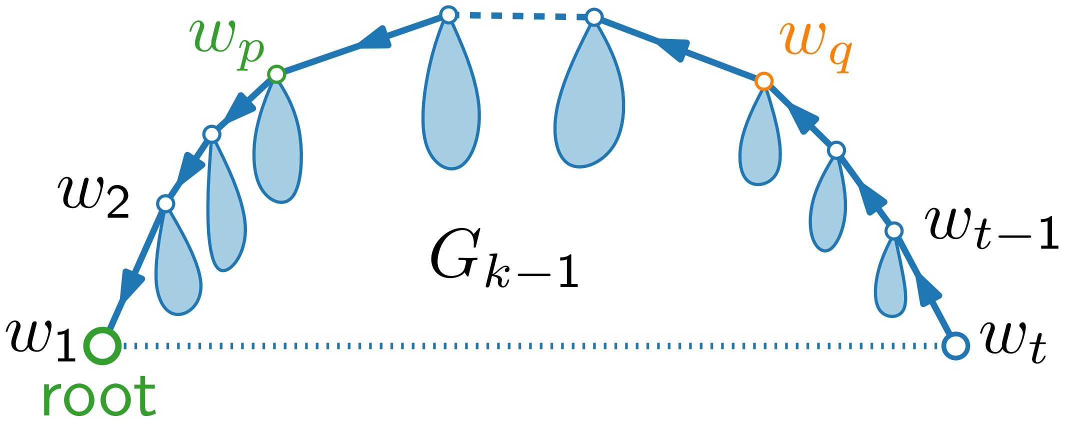
(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

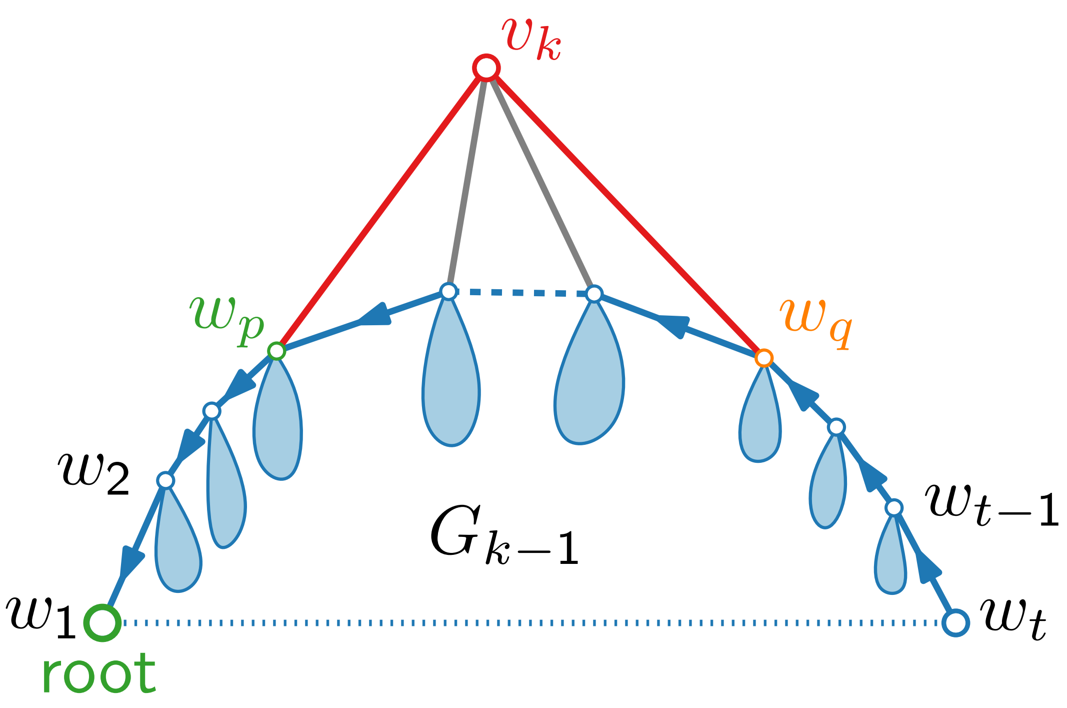
(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

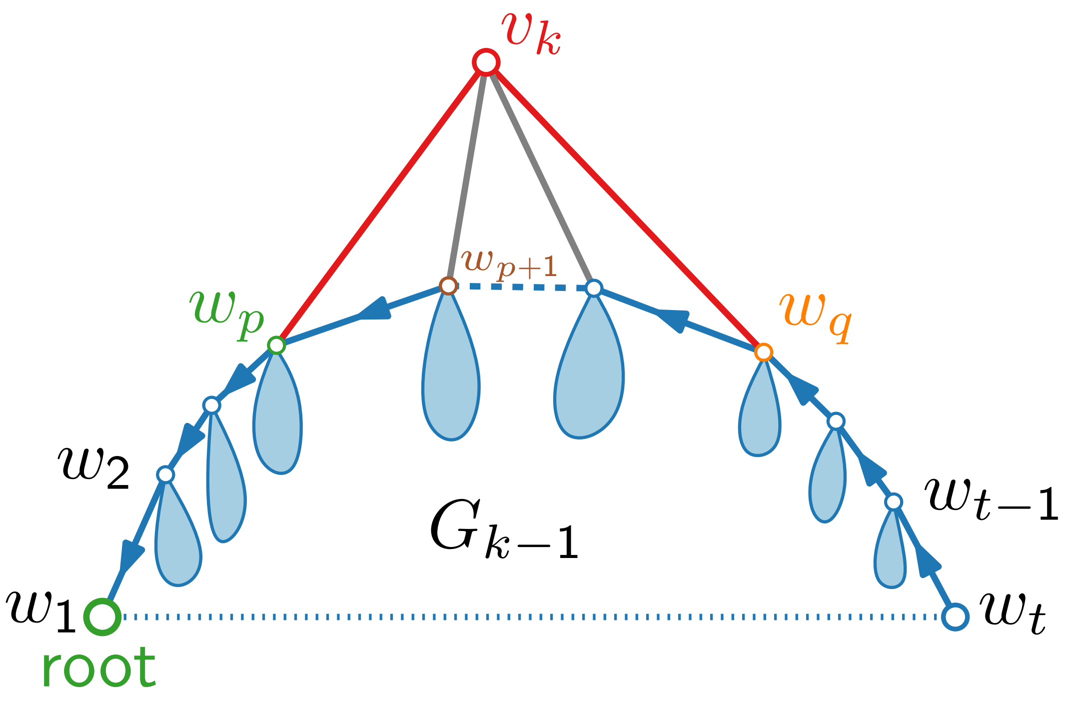
Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

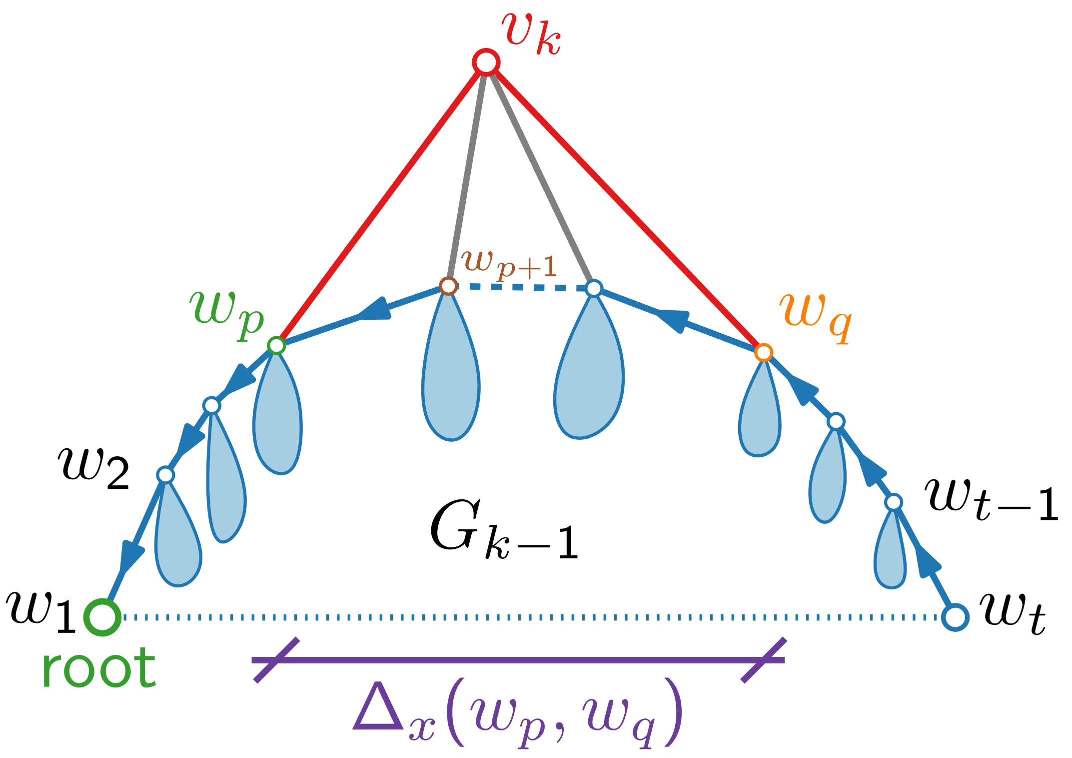
Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$



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Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
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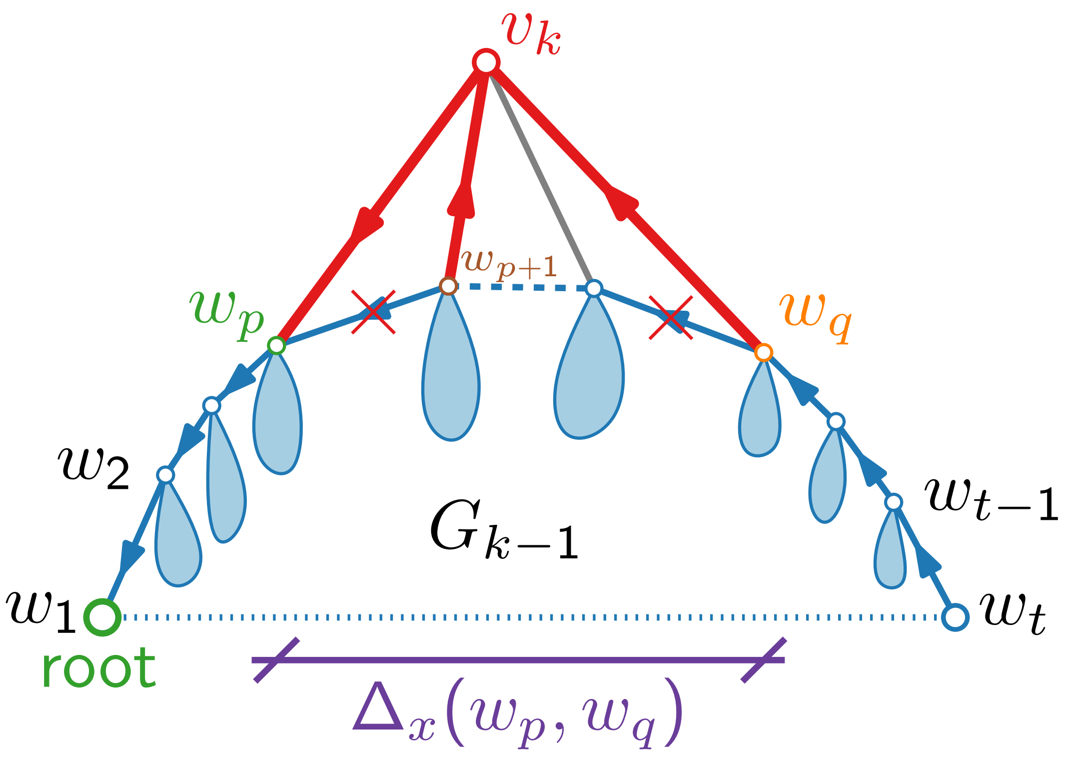

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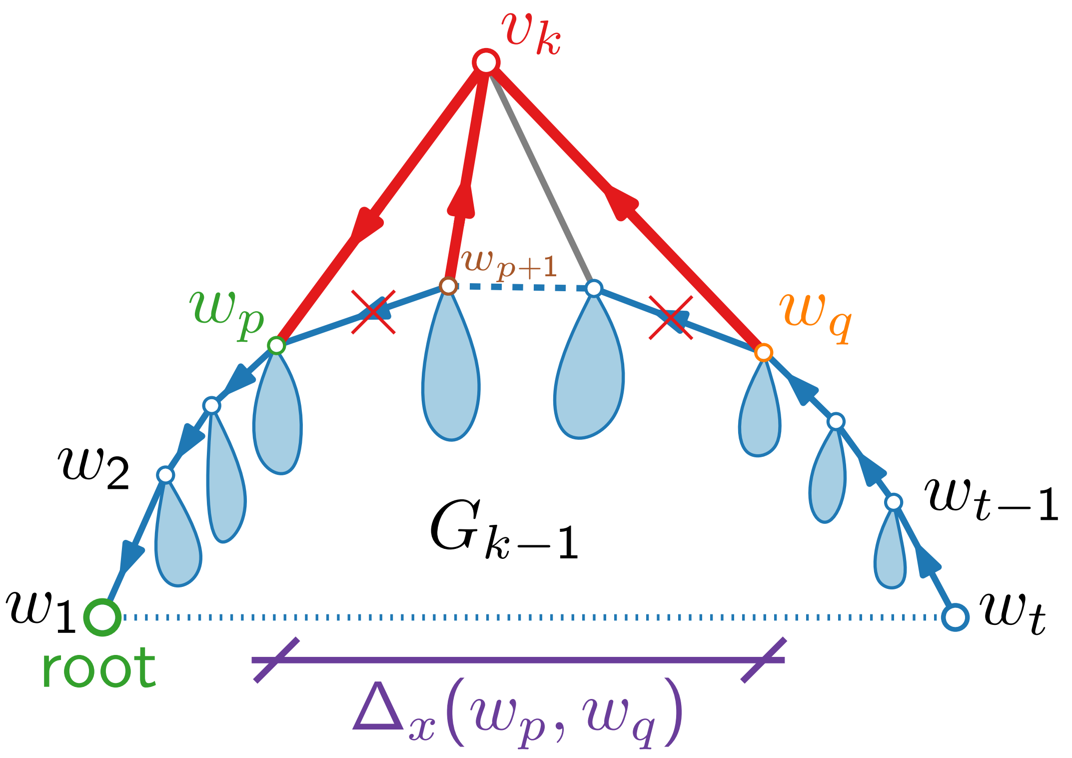

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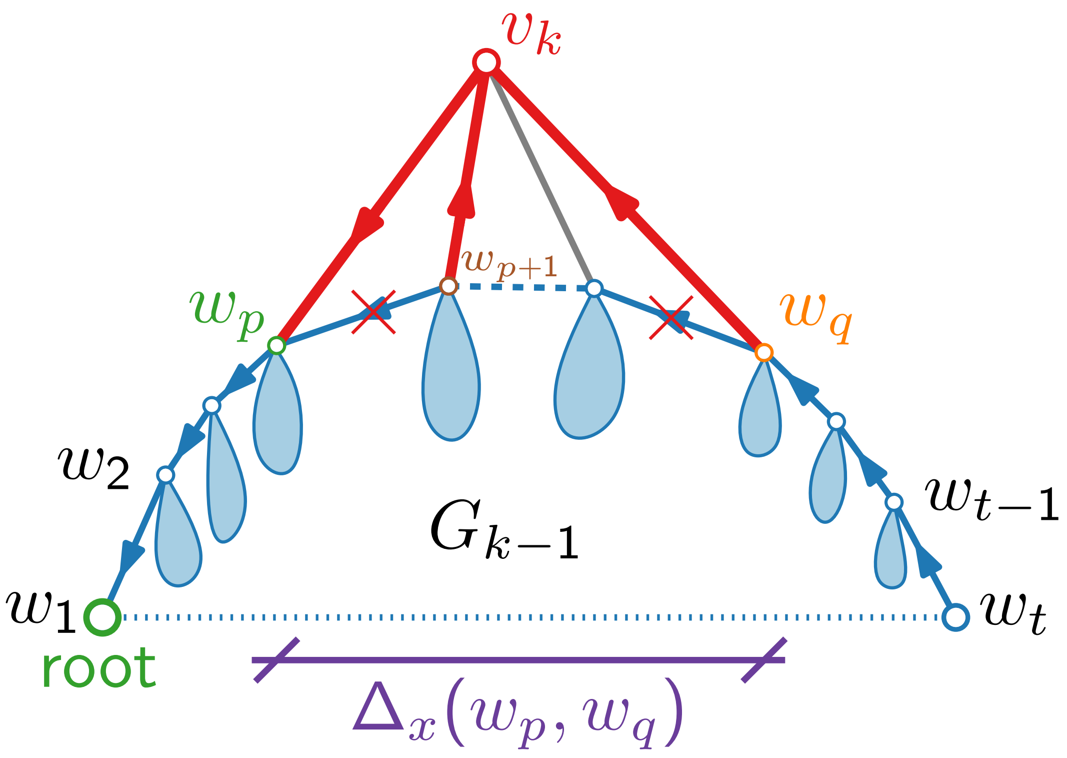

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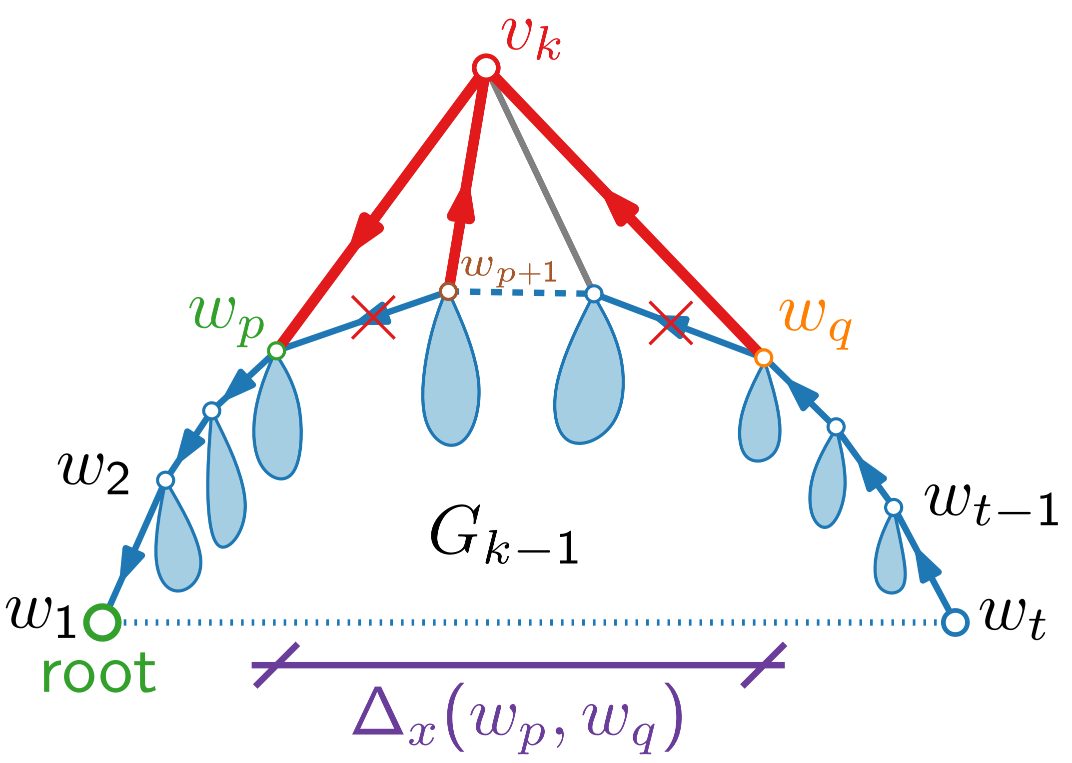

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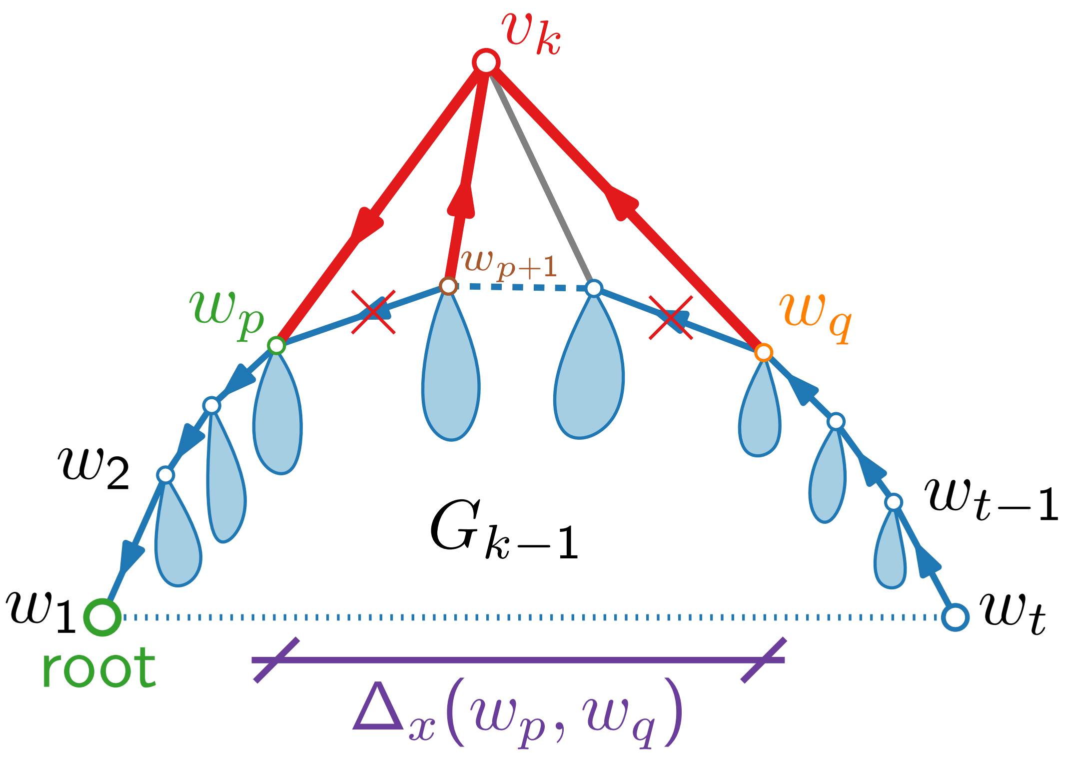

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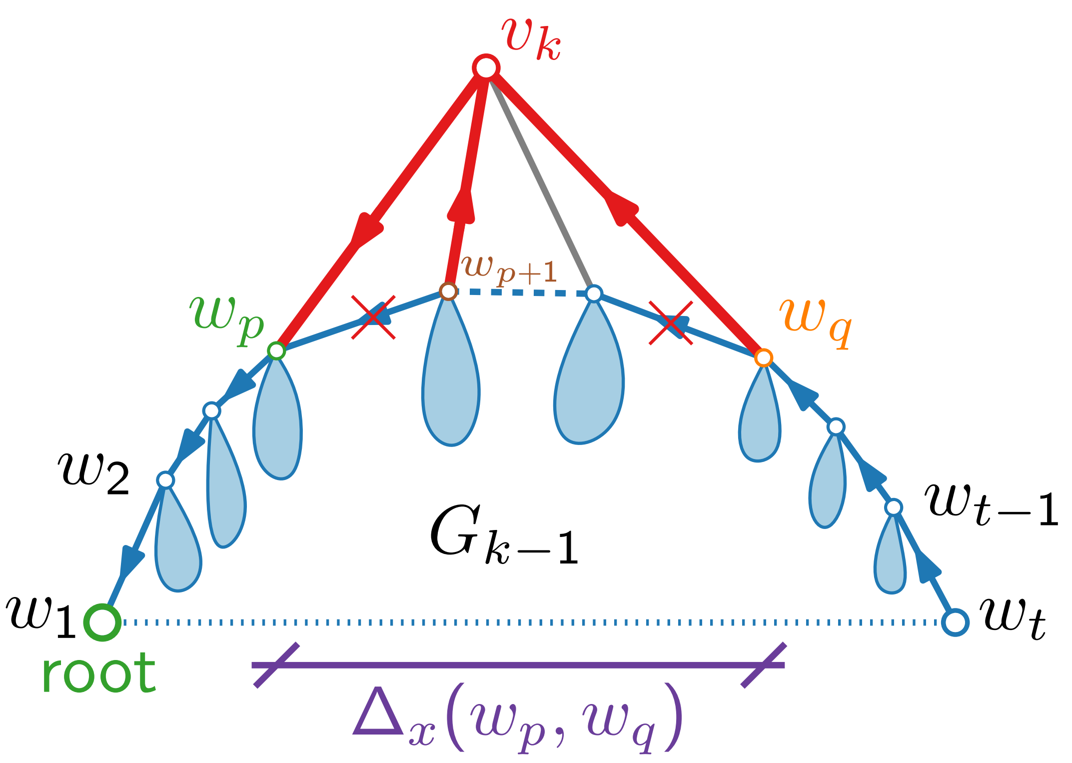

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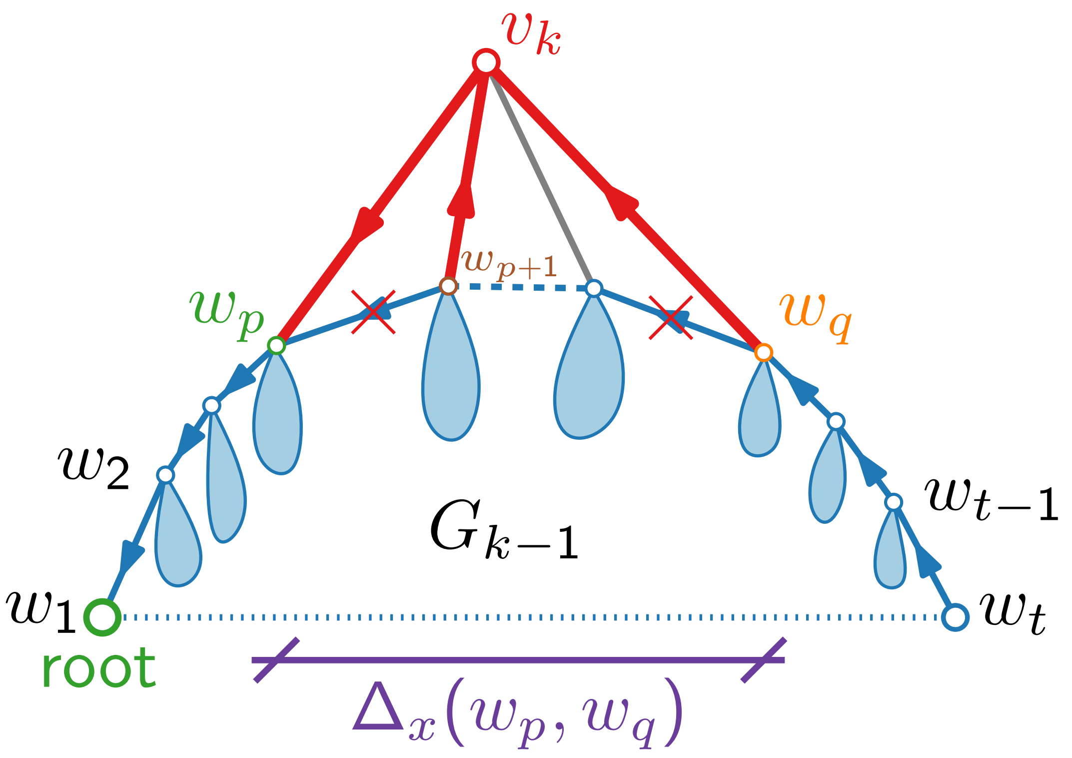
takes ? time



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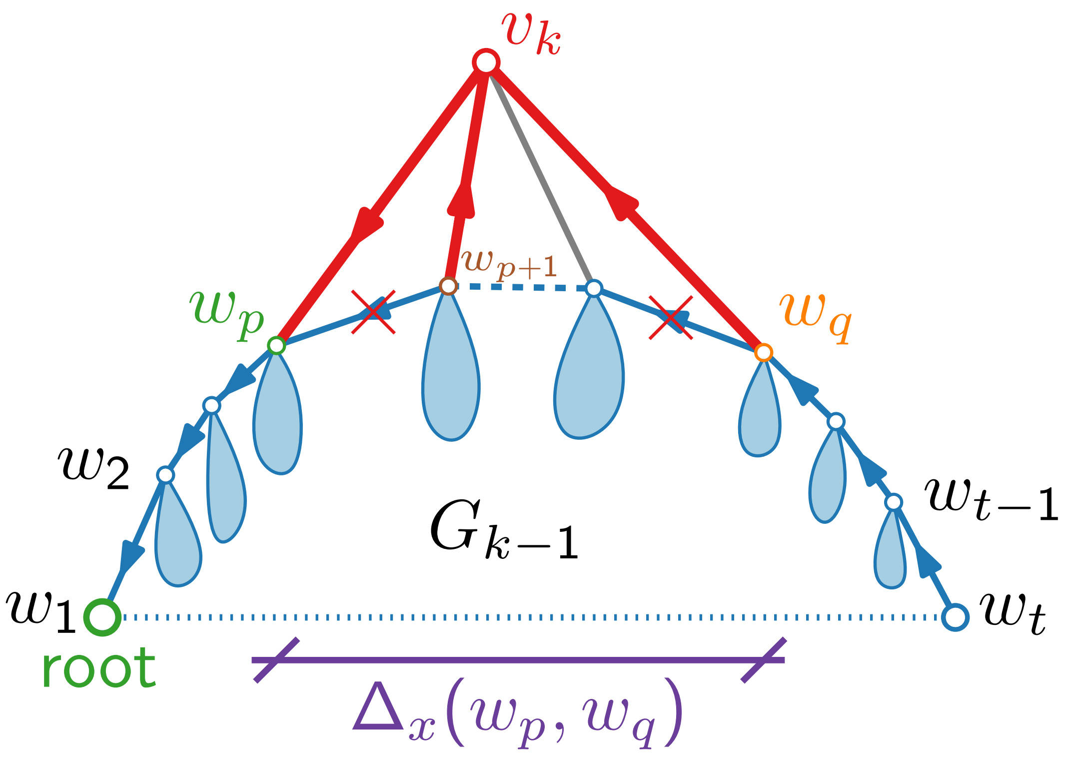

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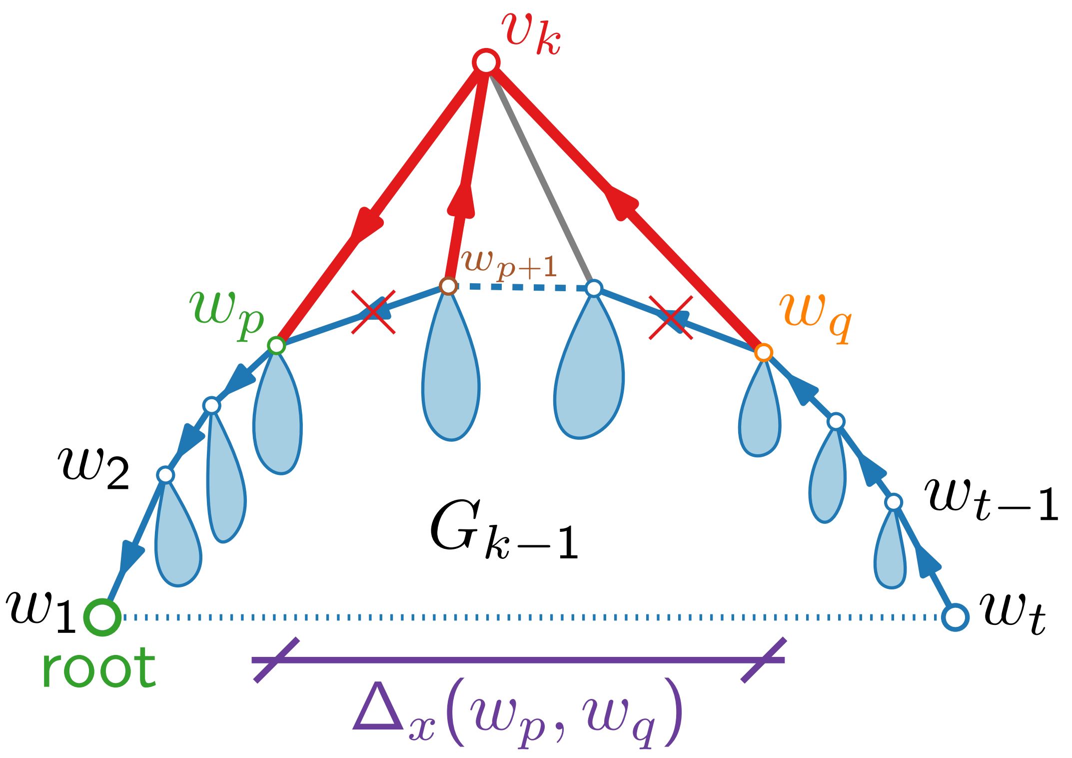
takes $\mathcal{O}(n)$ time in total ☺



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- A quite different approach yielding similar results is by Schnyder ($\rightarrow$ next lecture).

Literature

- [PGD Ch. 4.2] for detailed explanation of the shift method
- [de Fraysseix, Pach, Pollack 1990] "How to draw a planar graph on a grid"
  – original paper introducing the shift method
- [Chrobak, Payne 1995] "A linear-time algorithm for drawing a planar graph on a grid"
  – original paper on how to implement the shift method in linear time