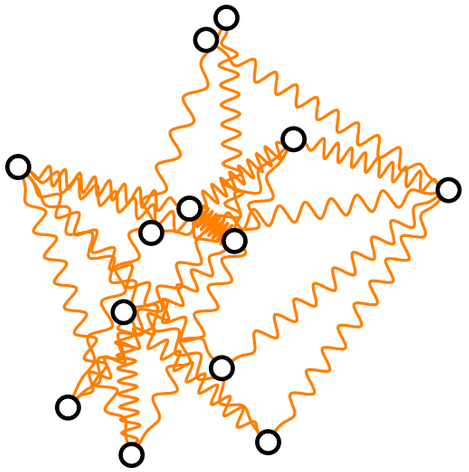
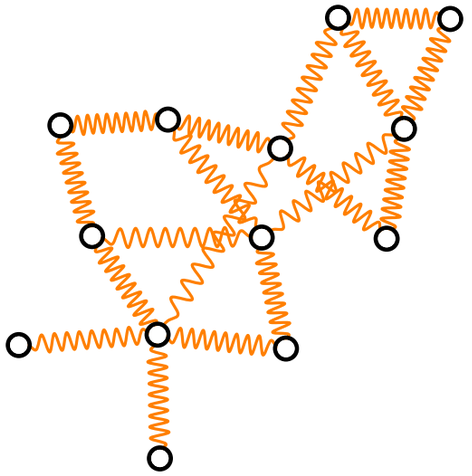
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part I:
Spring Embedders



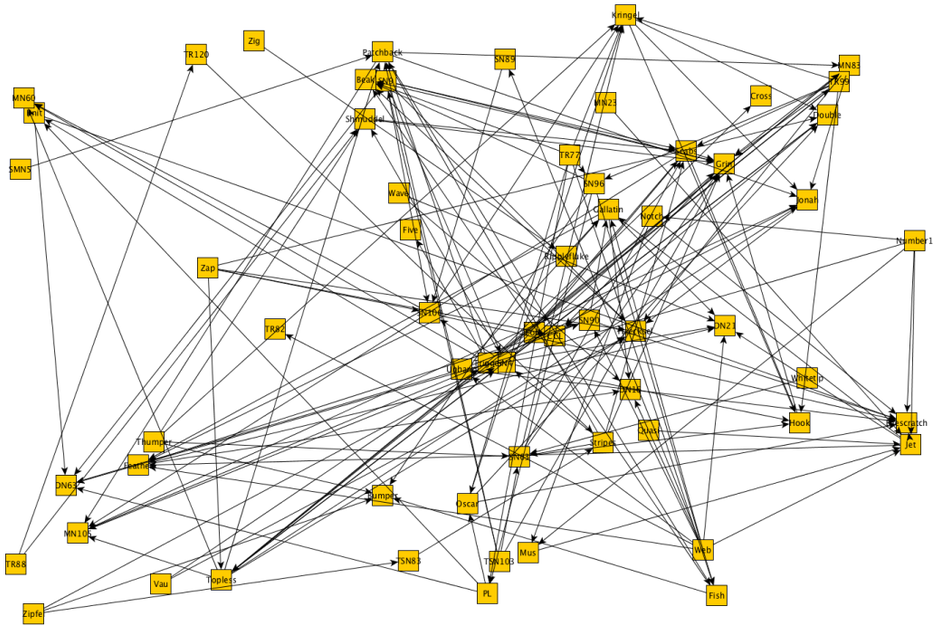
Alexander Wolff

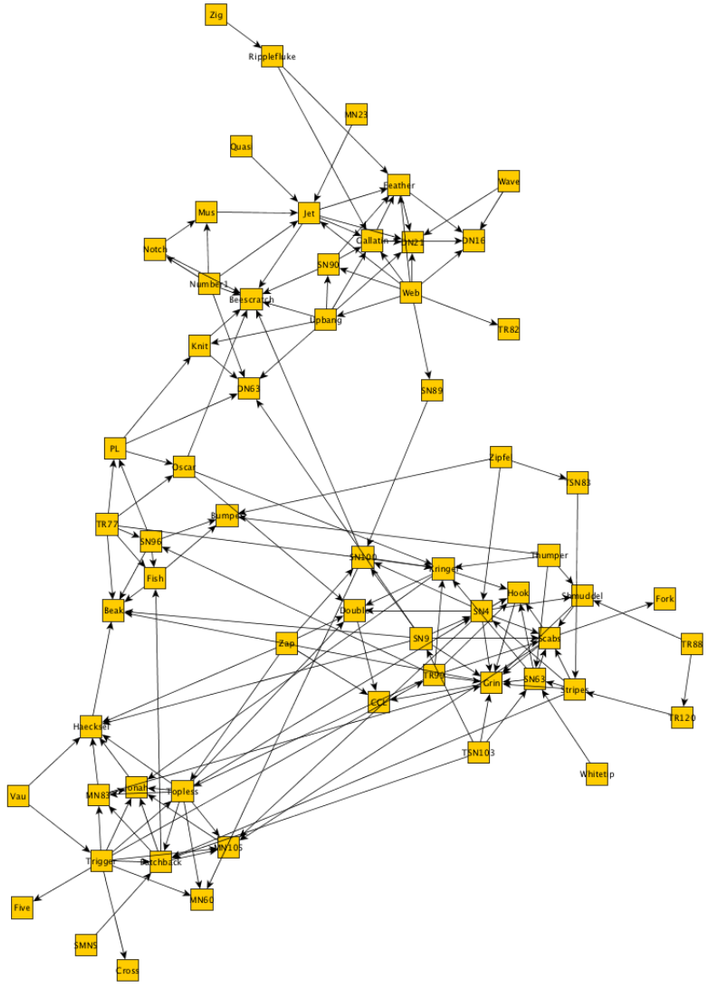
Summer term 2026

General Layout Problem

Input: Graph G

Output: Clear and readable straight-line drawing of G





General Layout Problem

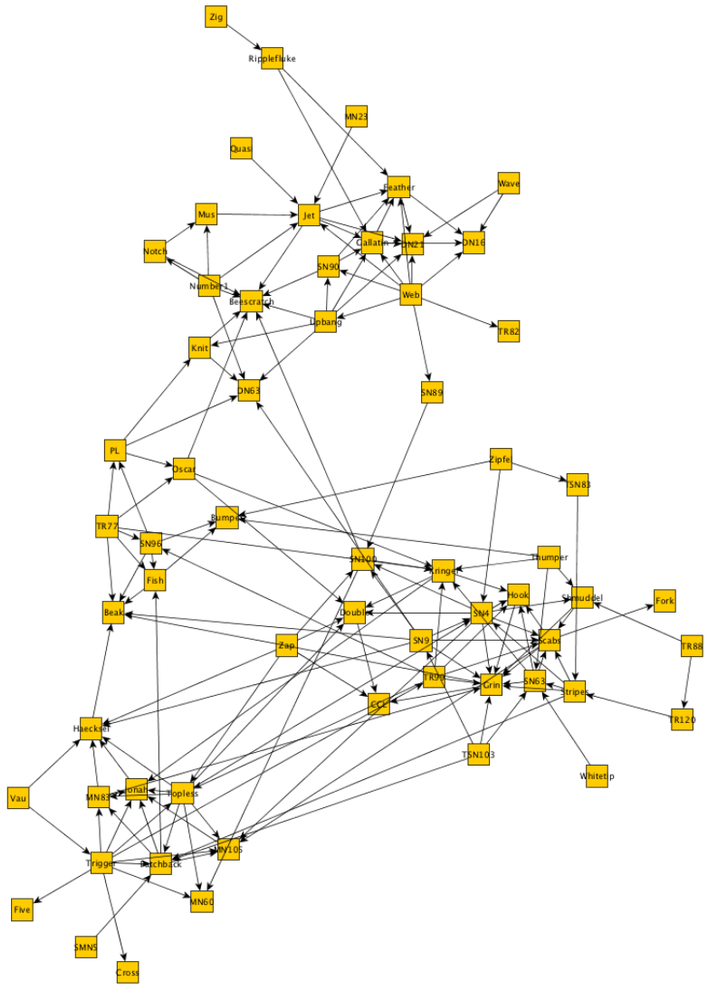
Input: Graph G

Output: Clear and readable straight-line drawing of G

Drawing aesthetics to optimize:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**
- densely connected parts (clusters) form communities
- as few crossings as possible
- nodes distributed evenly

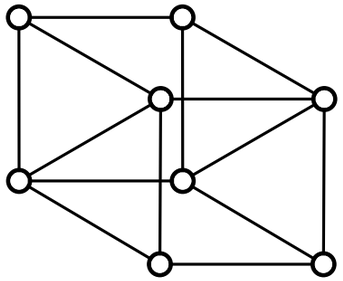
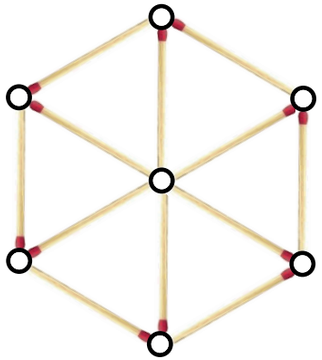
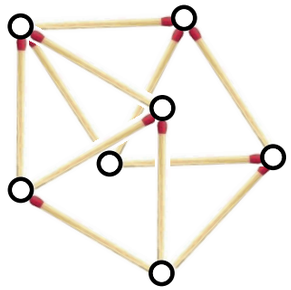
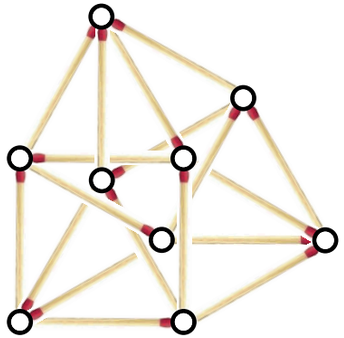
Optimization criteria partially contradict each other.



Fixed Edge Lengths?

Input: Graph G, required length $\ell(e)$ for each edge $e \in E(G)$.

Output: Drawing of G that realizes the given edge lengths.

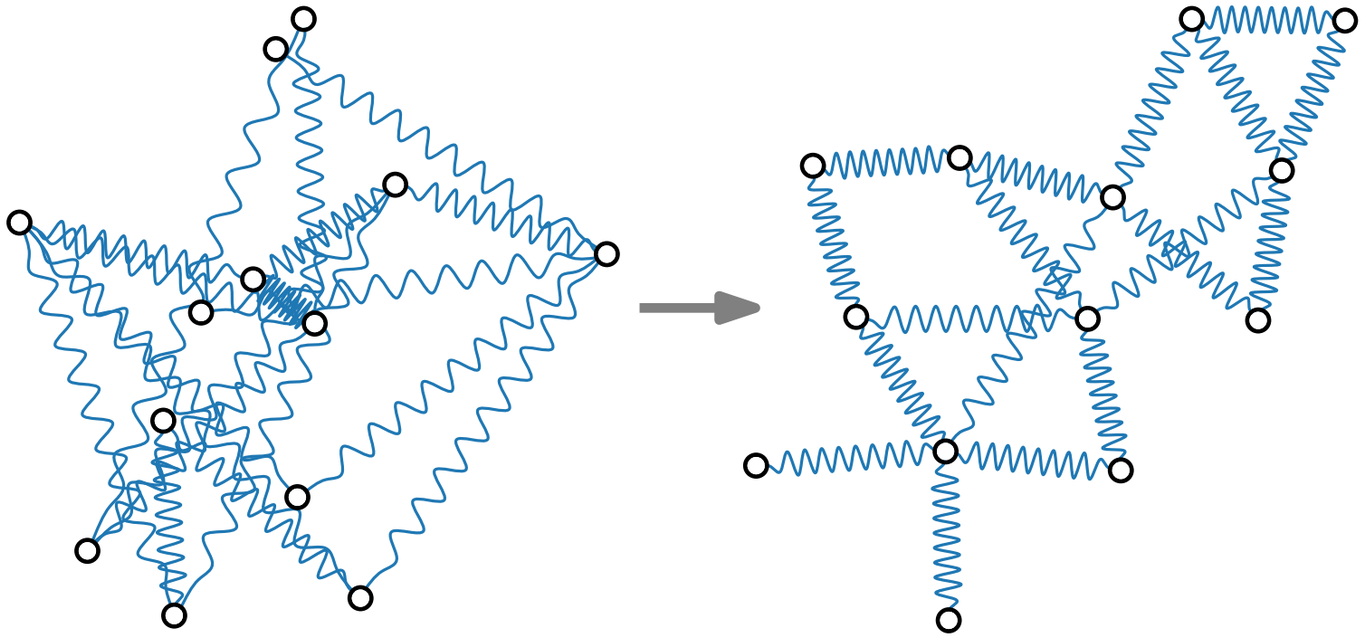
NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths in $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

"To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system... The vertices are placed in some initial layout and let go so that the spring forces on the rings move the system to a minimal energy state."



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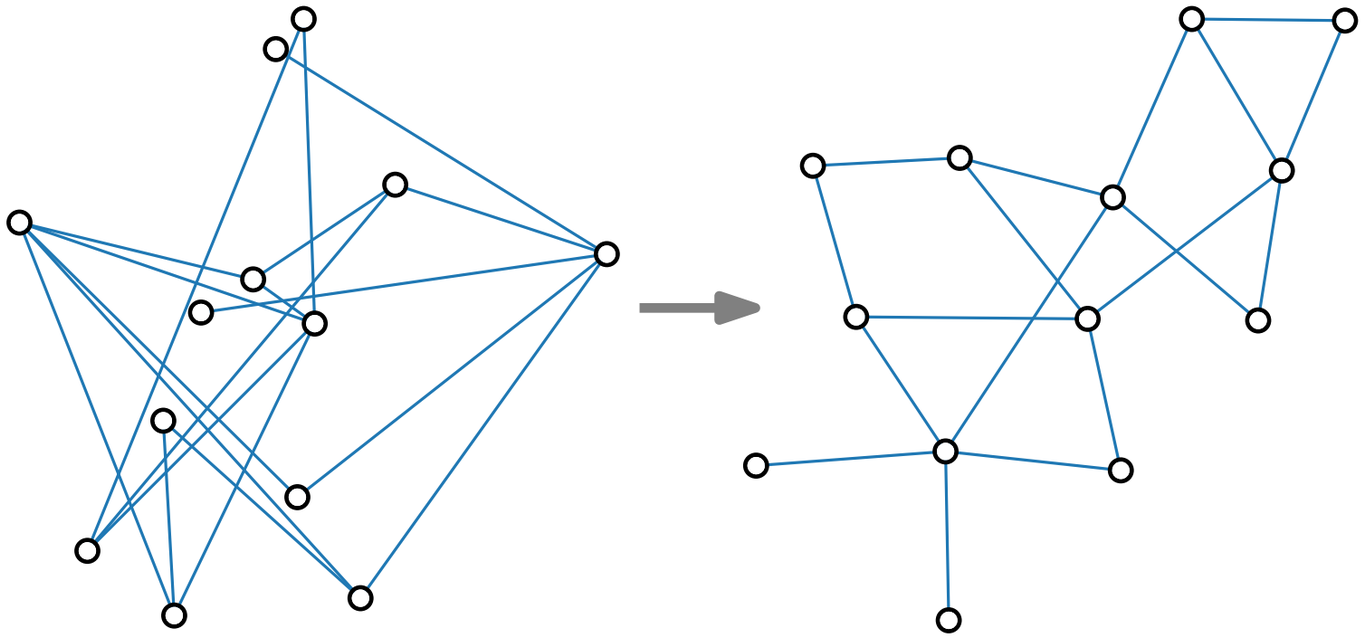

Physical Analogy

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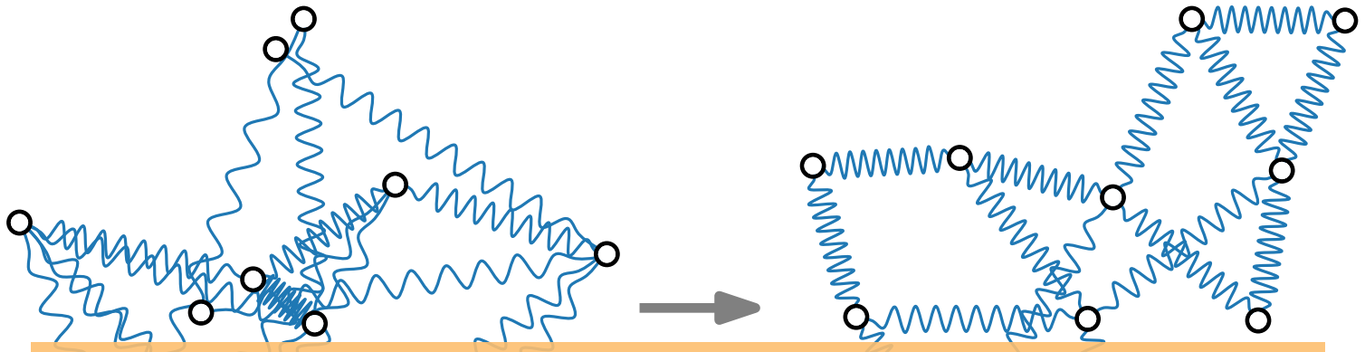


So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

Attractive forces.

pairs $\{u, v\}$ of adjacent vertices:

Repulsive forces.

any pair $\{x, y\}$ of vertices:

Force-Directed Algorithms

initial layout; may be randomly chosen positions

max # iterations

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

threshold (assume $F_v(0) = \infty$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

vertices adjacent to u

foreach $u \in V(G)$ **do**

$F_u(t) \leftarrow \sum_{v \in V(G)} f_{\mathsf{rep}}(p_u, p_v) + \sum_{v \in \mathsf{Adj}[u]} f_{\mathsf{attr}}(p_u, p_v)$

foreach $u \in V(G)$ **do**

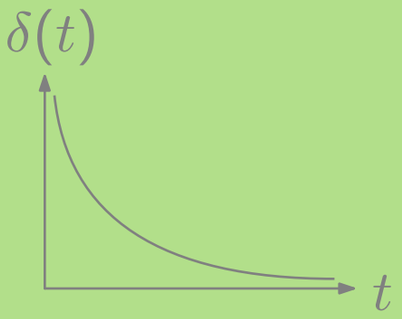
$p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

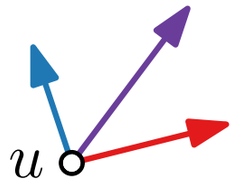
cooling factor

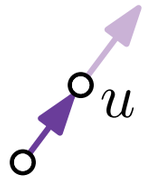
$t \leftarrow t + 1$

return p

end layout







Spring Embedder by Eades – Model

- **Repulsive forces**

repulsion constant (e.g., 2.0)

$$f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

spring constant (e.g., 1.0)

$$f_{\text{spring}}(p_u, p_v) = c_{\text{spring}} \cdot \log \frac{\|p_v - p_u\|}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{attr}}(p_u, p_v) = f_{\text{spring}}(p_u, p_v) - f_{\text{rep}}(p_u, p_v)$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$$

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

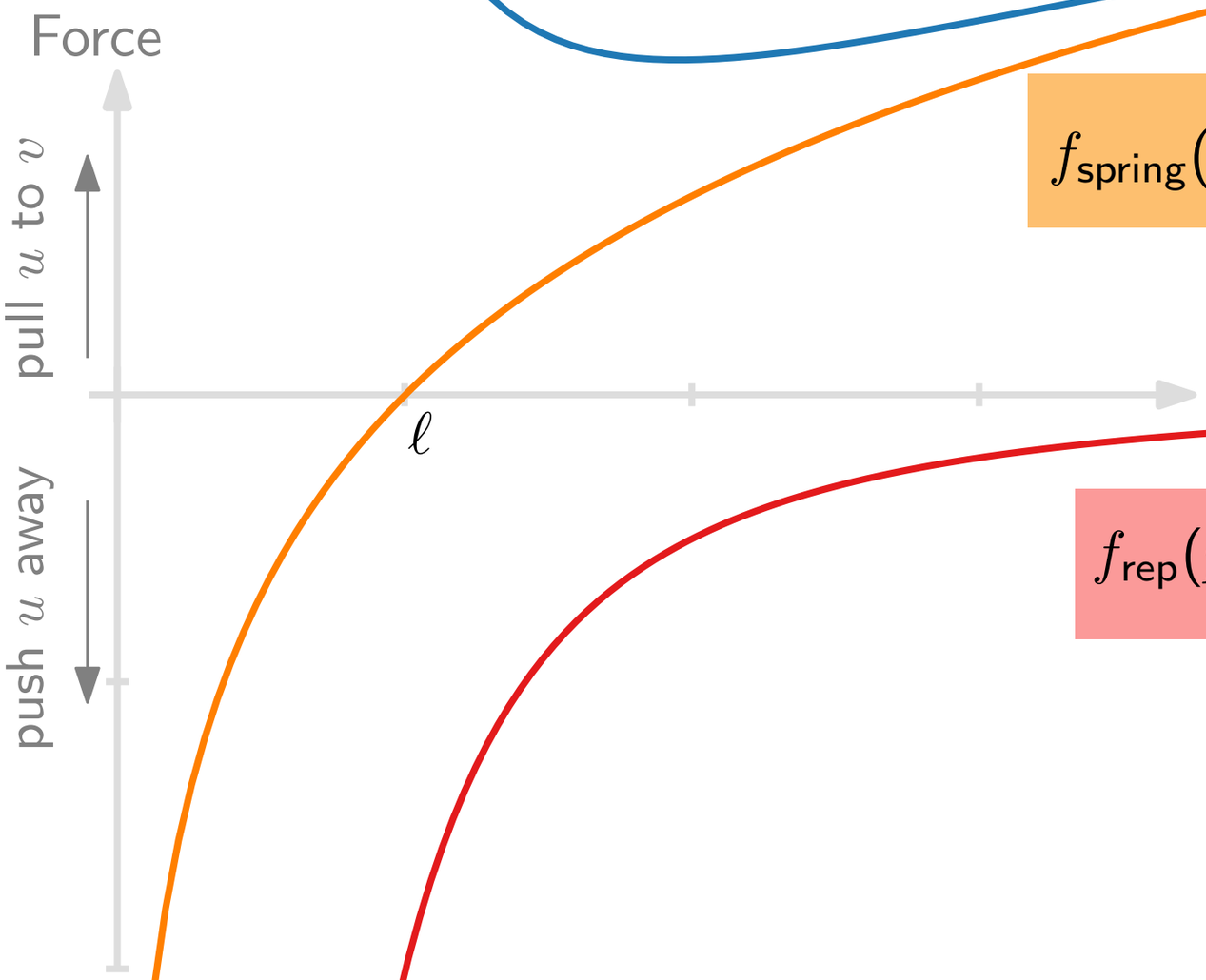
```
t ← 1
while t ≤ K and max_{v∈V(G)} ‖F_v(t−1)‖ > ε do
    foreach u ∈ V(G) do
        F_u(t) ← Σ_{v∈V(G)} f_rep(p_u, p_v) + Σ_{v∈Adj[u]} f_attr(p_u, p_v)
    foreach u ∈ V(G) do
        p_u ← p_u + δ(t) · F_u(t)
    t ← t + 1
return p
```

Notation.

- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- $\|p_v - p_u\|$ = Euclidean distance between u and v
- ℓ = ideal spring length for edges

Spring Embedder by Eades – Force Diagram

$$f_{\text{attr}}(p_u, p_v) = f_{\text{spring}}(p_u, p_v) - f_{\text{rep}}(p_u, p_v)$$



$$f_{\text{spring}}(p_u, p_v) = c_{\text{spring}} \cdot \log \frac{\|p_v - p_u\|}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{\|p_v - p_u\|^2} \cdot \overrightarrow{p_v p_u}$$

Spring Embedder by Eades – Discussion

Advantages.

- very simple algorithm
- good results for small and medium-sized graphs
- empirically good representation of symmetry and structure

Disadvantages.

- System may not be stable at the end.
- May converge to a local minimum that is not a global minimum.
- Computing f_{spring} takes $\mathcal{O}(|E(G)|)$ time; computing f_{rep} takes $\mathcal{O}(|V(G)|^2)$ time.

Influence.

- The original paper by Peter Eades [Eades '84] has been cited more than 2000 times.
- It has become the basis for many further ideas.

Variant by Fruchterman & Reingold

- **Repulsive forces**

$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

- **Resulting displacement vector**

$$F_u = \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
- $t \leftarrow 1$
- **while** $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
  - **foreach** $u \in V(G)$ **do**
    - $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
  - **foreach** $u \in V(G)$ **do**
    - $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
  - $t \leftarrow t + 1$
- **return** p

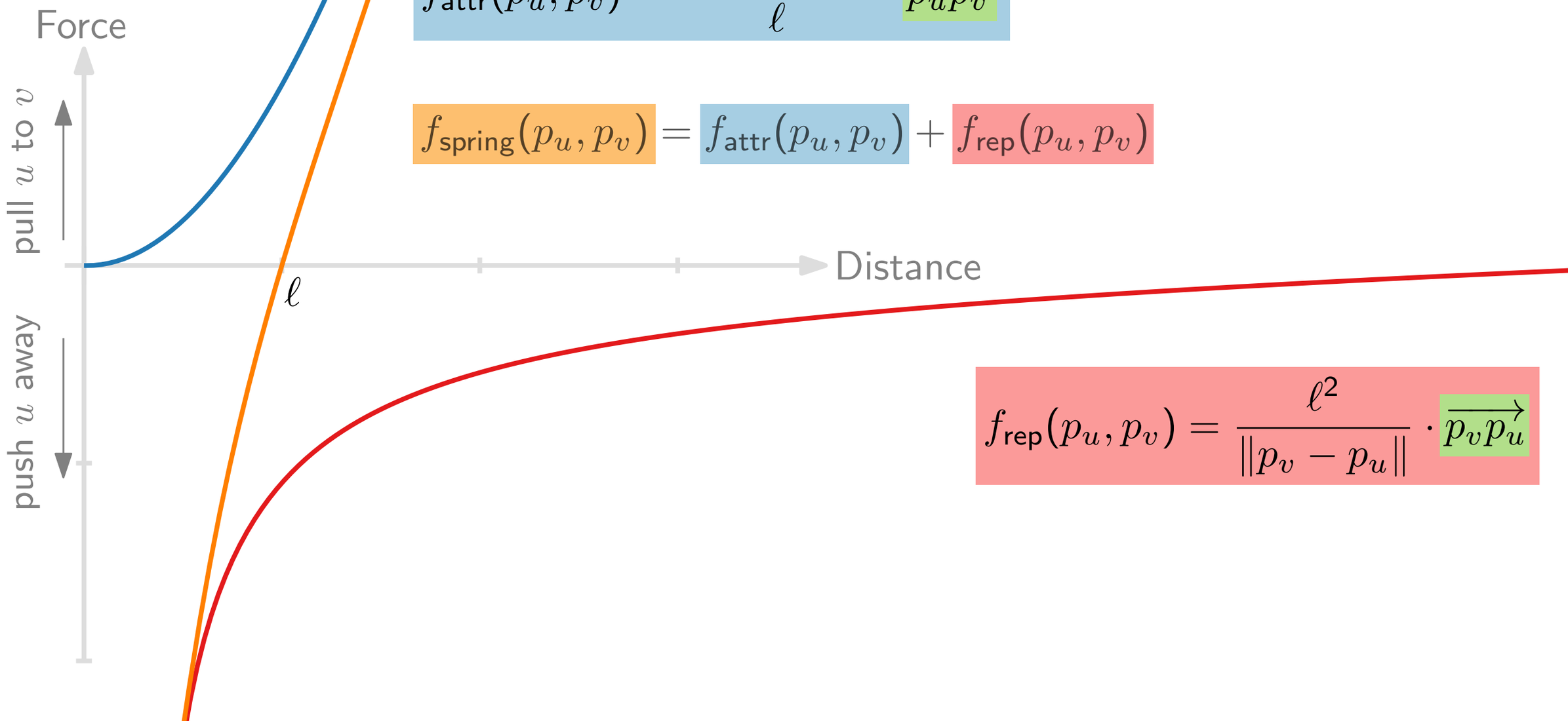
Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(p_u, p_v) = f_{\text{attr}}(p_u, p_v) + f_{\text{rep}}(p_u, p_v)$$



$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Adaptability

degree of vertex u, i.e., $|\text{Adj}[u]|$

Inertia. ("Trägheit")

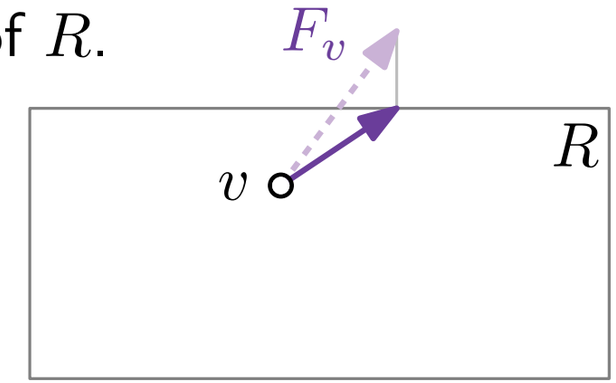
- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\text{attr}}^{\text{new}}(p_u, p_v) = f_{\text{attr}}^{\text{old}}(p_u, p_v)/\Phi(u)$

Gravitation.

- Define centroid $\sigma = 1/|V(G)| \cdot \sum_{v \in V(G)} p_v$
- Add force $f_{\text{grav}}(v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speed-ups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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  foreach $u \in V(G)$ **do**

    $p_u \leftarrow p_u + ~~\delta(t)~~ \cdot F_u(t)$

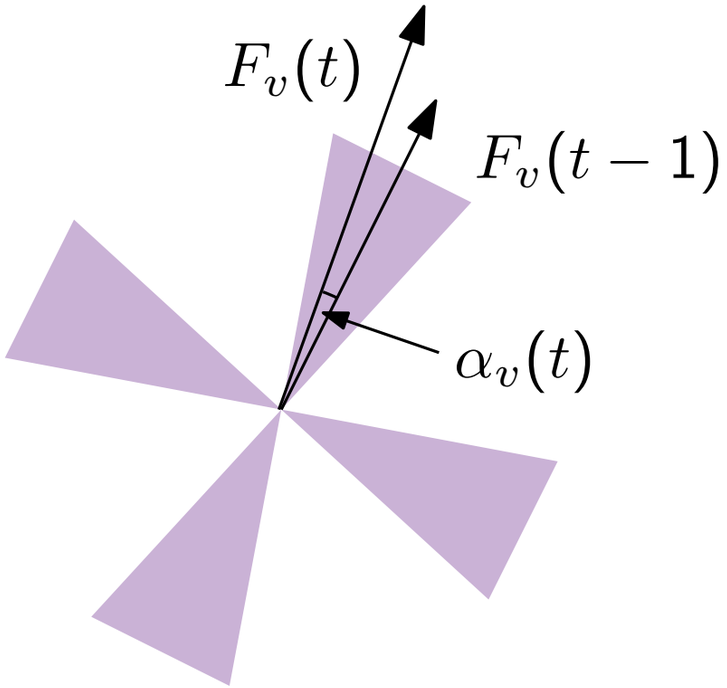
    $\delta_v(t)$

  $t \leftarrow t + 1$

return p

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

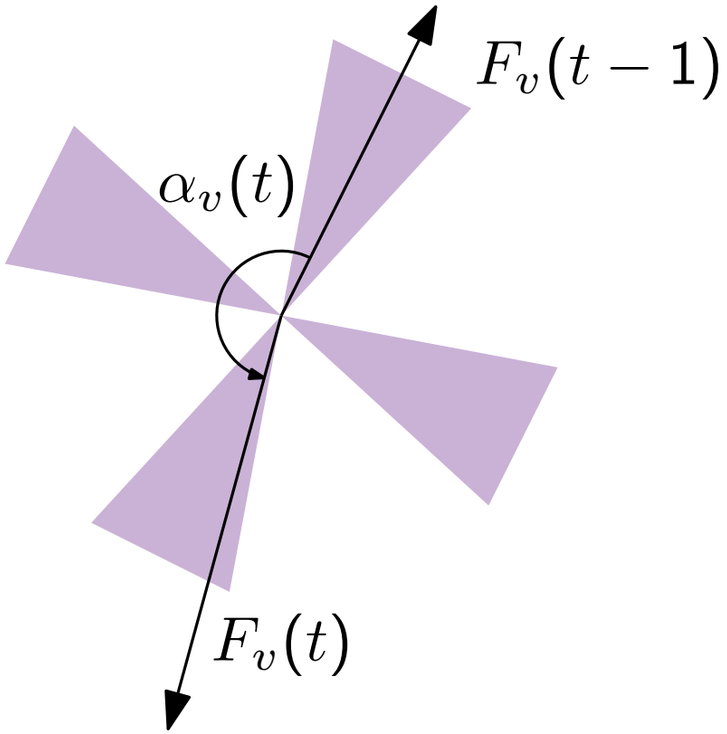


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

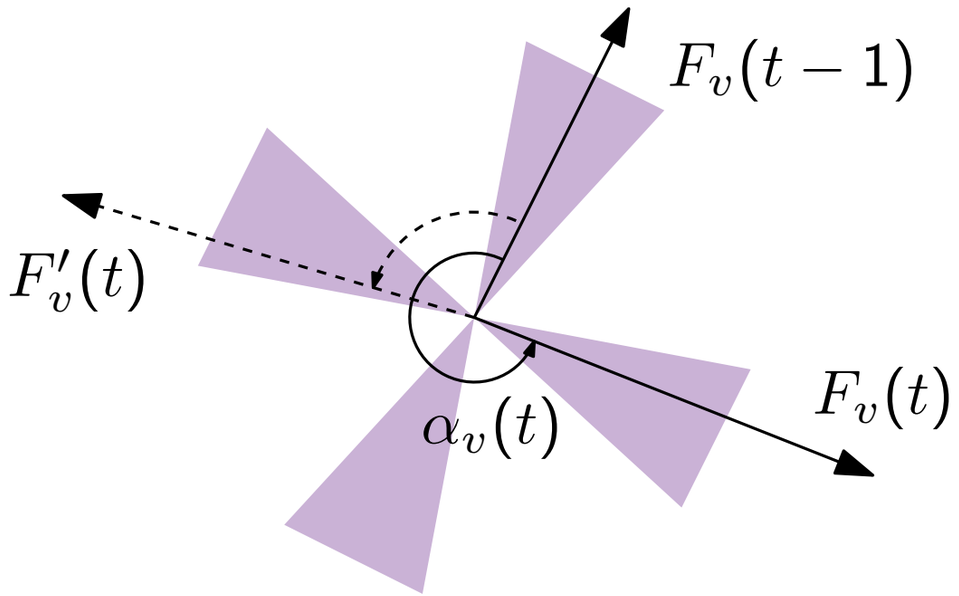
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

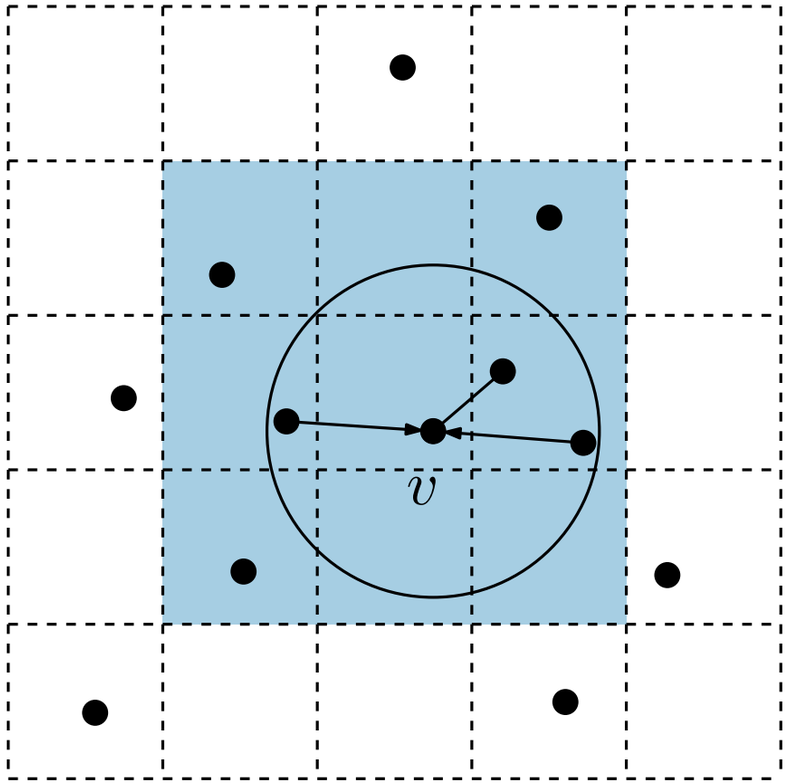
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



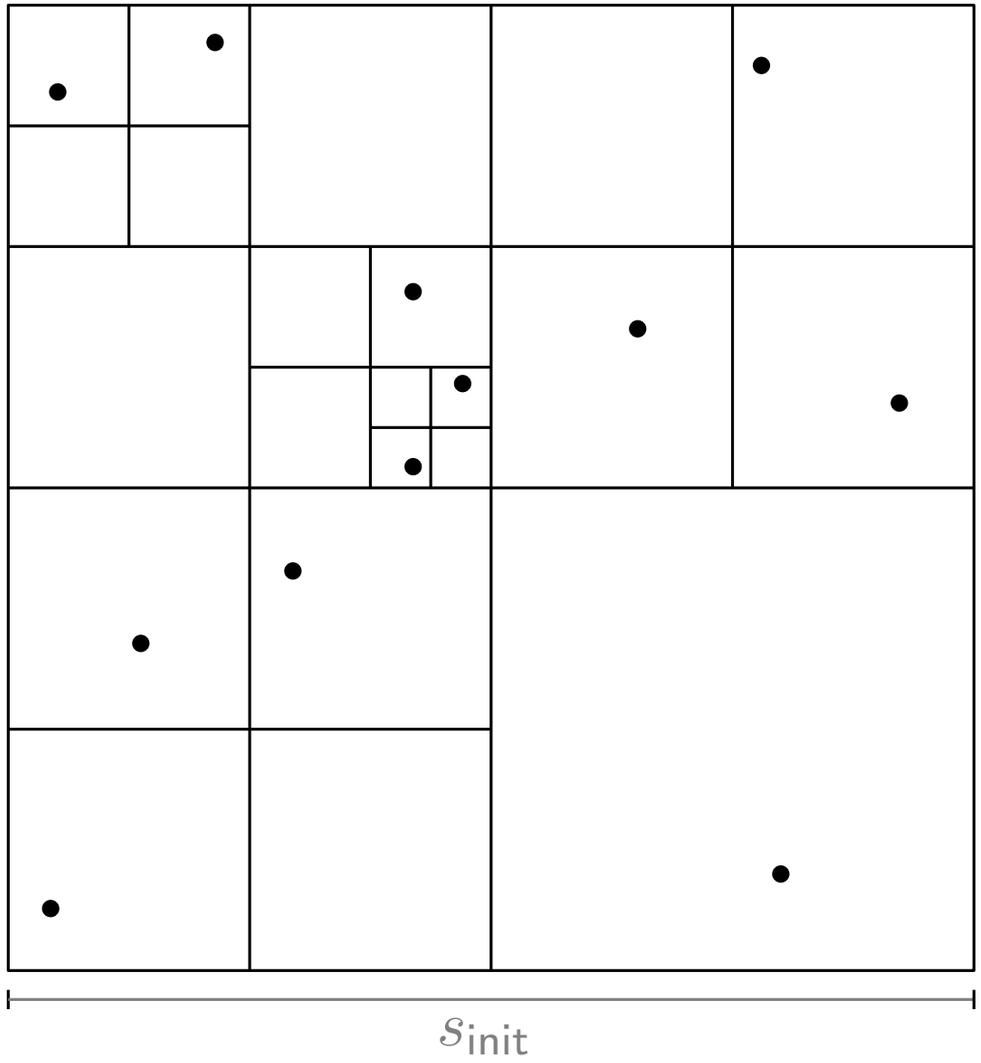
- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

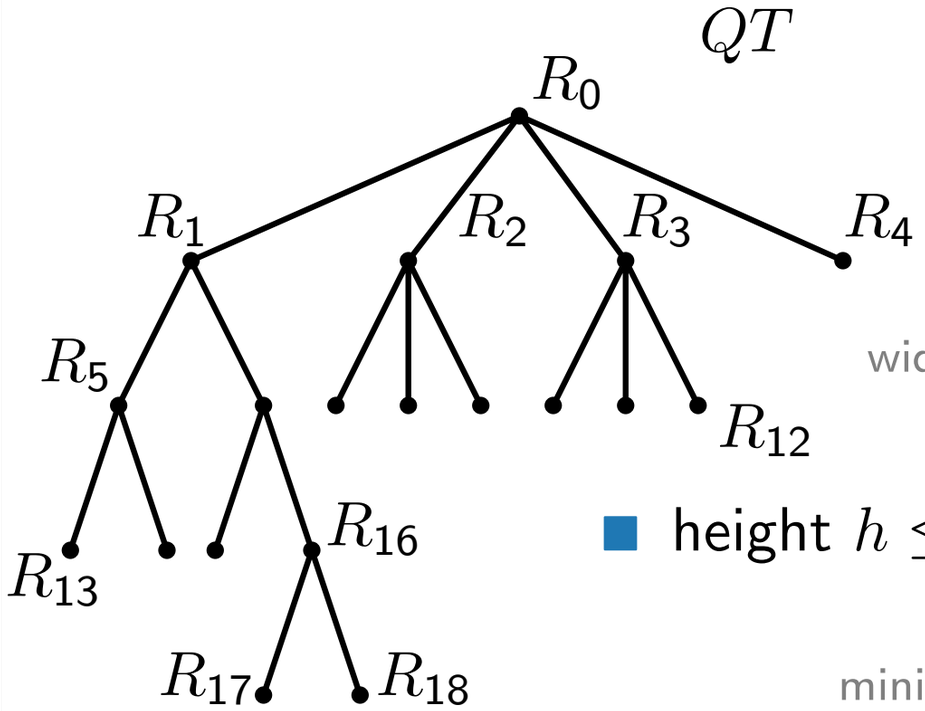
Discussion.

- good idea to improve actual runtime
- asymptotic runtime does not improve
- might introduce oscillation and thus a quality loss

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]





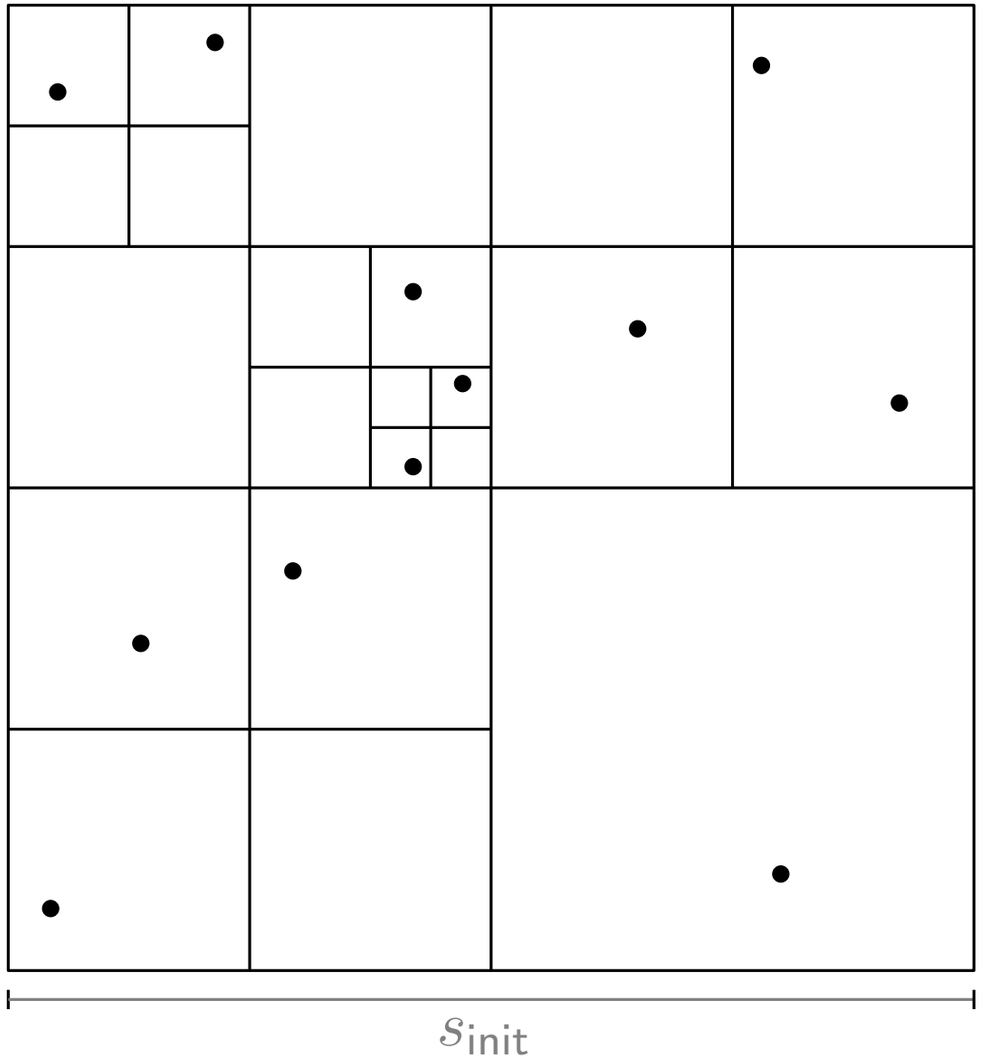
width/height of outer square

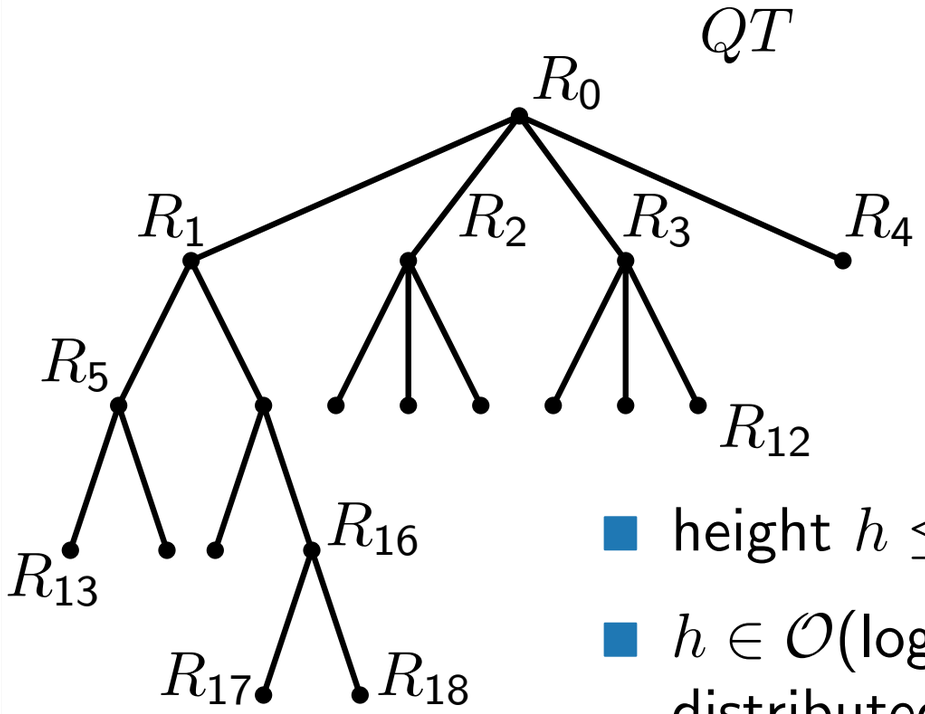
- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$

minimum distance btw. two pts.

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

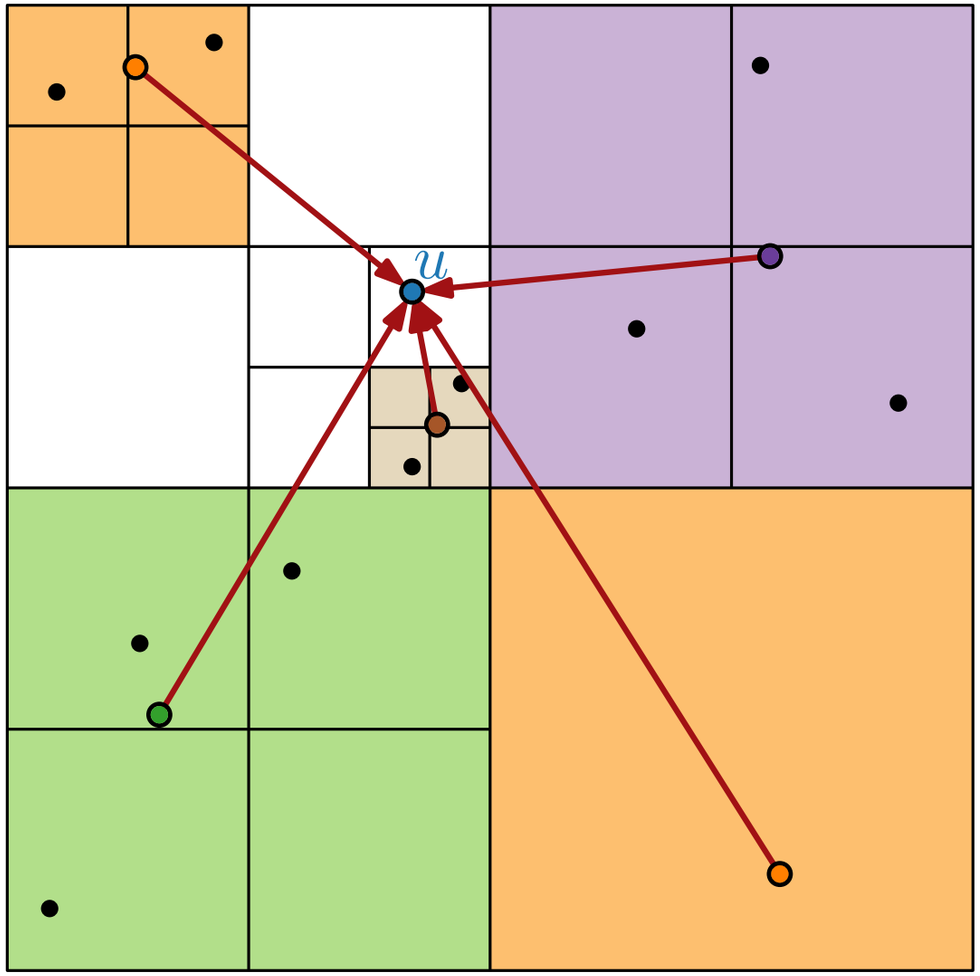


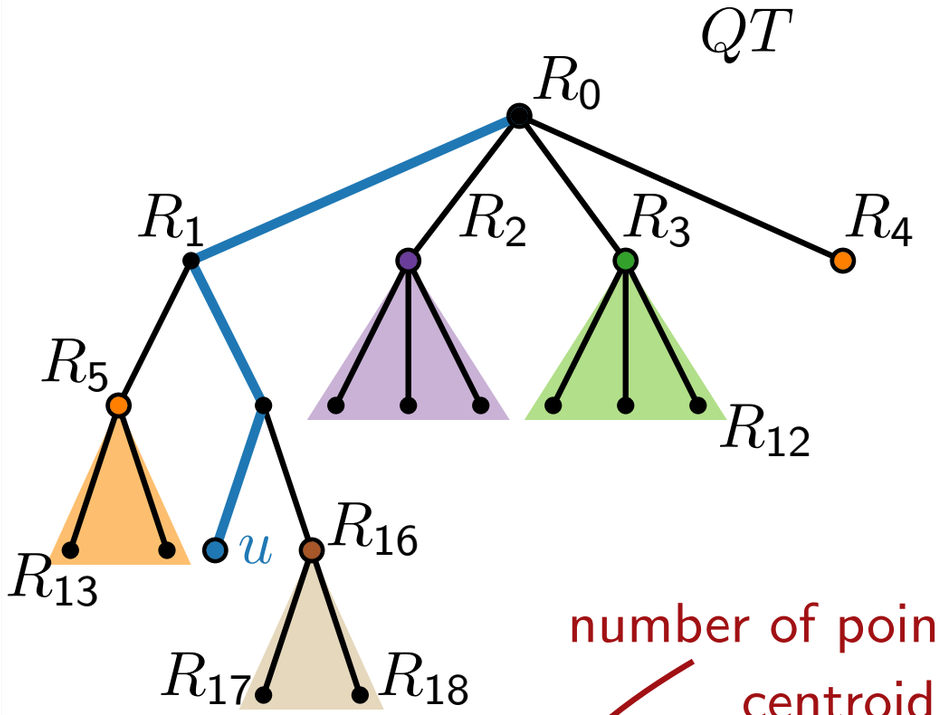


- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]





number of points in the subtree R_i

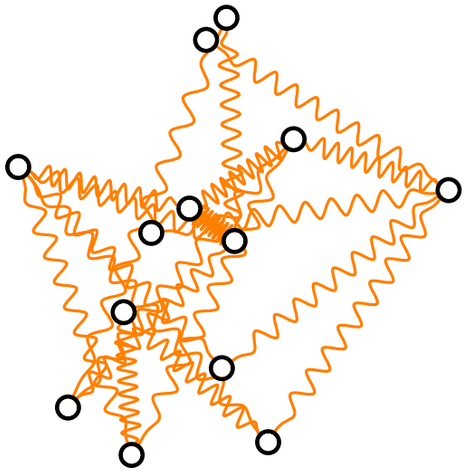
centroid of R_i (pre-computed)

$$f_{\text{rep}}(R_i, p_u) = |R_i| \cdot f_{\text{rep}}(\sigma_{R_i}, p_u)$$

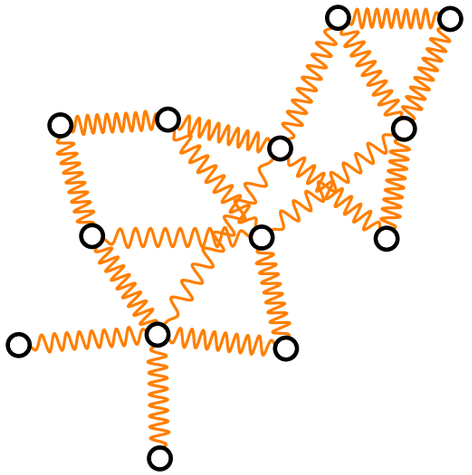
for each child R_i of a vertex on path from root to u.

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



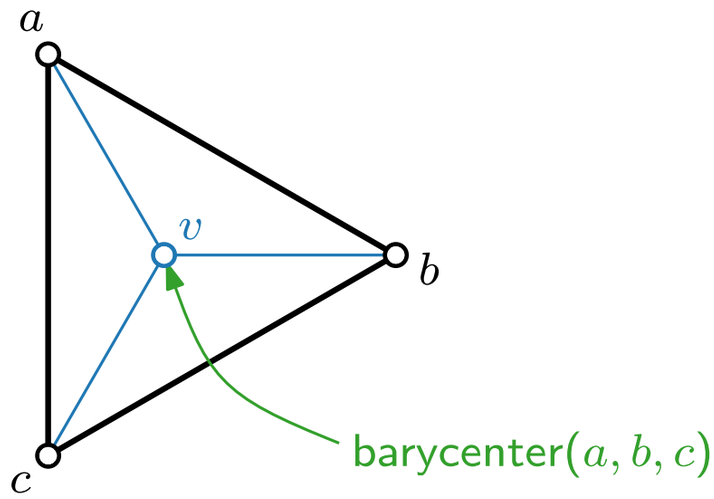
Part II:
Tutte Embeddings

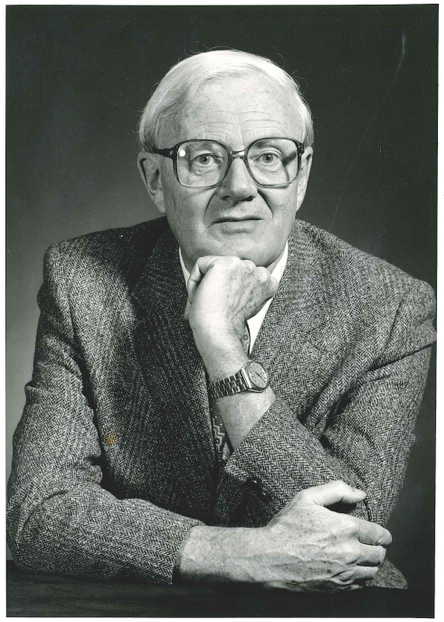


Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?





William T. Tutte
1917 – 2002

barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i/k$

Idea.
Repeatedly place every vertex at barycenter of neighbors.

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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  $t \leftarrow t + 1$

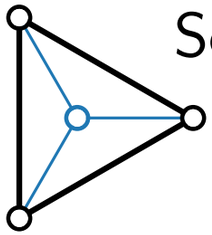
return p

$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \begin{cases} 0 & \text{if } u \text{ fixed,} \\ \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v} & \text{otherwise.} \end{cases}$$

Global minimum: $p_u = (0,0)\ \forall u \in V(G)$ ☹



Solution: fix coordinates of outer face! ☺

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

System of Linear Equations

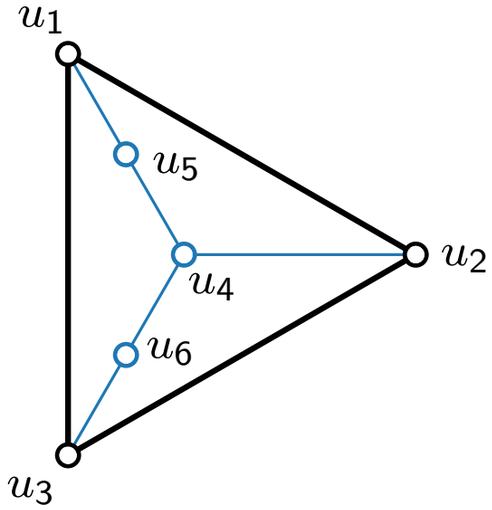
Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b \quad b = (0)_n$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$
$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$



A

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	3	−1	−1	0	−1	0
u_2	−1	3	−1	−1	0	0
u_3	−1	−1	3	0	0	−1
u_4	0	−1	0	3	−1	−1
u_5	−1	0	0	−1	2	0
u_6	0	0	−1	−1	0	2

Laplacian matrix of G

$$A_{ii} = \deg(u_i)$$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

n variables, n constraints, $\det(A) = 0$

$\Rightarrow$ no unique solution

System of Linear Equations

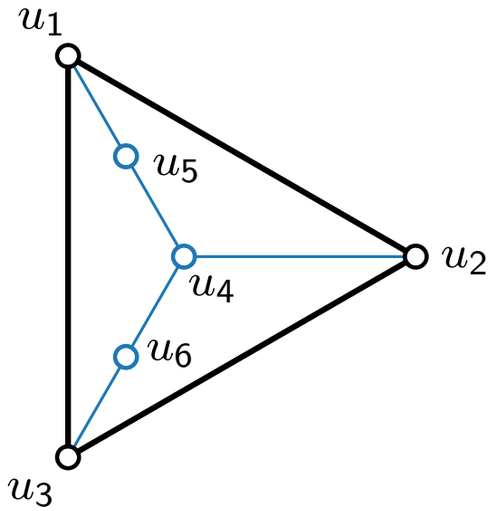
solve two systems of linear equations

Theorem. **= Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) =$

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$
$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$



$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} A'$$

Laplacian matrix of G

$$A_{ii} = \deg(u_i)$$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

Solution:
1. No need to change fixed vertices.
2. Constraints that depend on fixed vertices are constant and can be moved into b.

k variables, k constraints, $\det(A') \overset{\star}{>} 0$

$k = \#\text{free vertices}$

$\Rightarrow$ unique solution

*) because A' is *diagonally dominant*.

3-Connected Planar Graphs

(up to the choice of the outer face and mirroring)

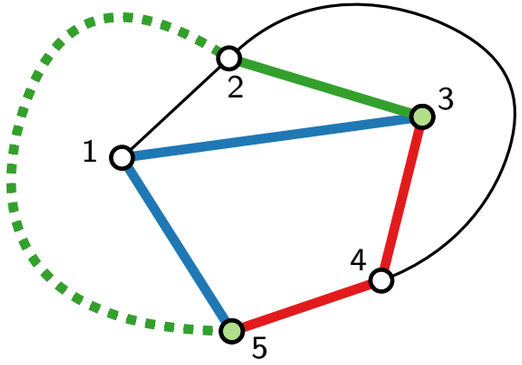
G **planar**: G can be drawn such that no two edges cross each other.

G **connected**: $\exists\ u$–v path for every vertex pair $\{u, v\}$.

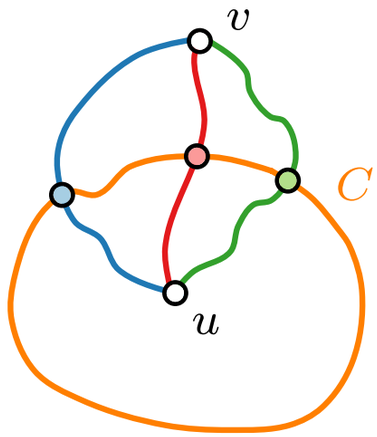
k**-connected:** $G - \{v_1, \ldots, v_{k-1}\}$ is connected for **any** $k-1$ vertices $v_1 \ldots, v_{k-1}$.
Or (equivalently if $G \neq K_k$):
There are at least k vertex-disjoint u–v paths for every vertex pair $\{u, v\}$.

Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.

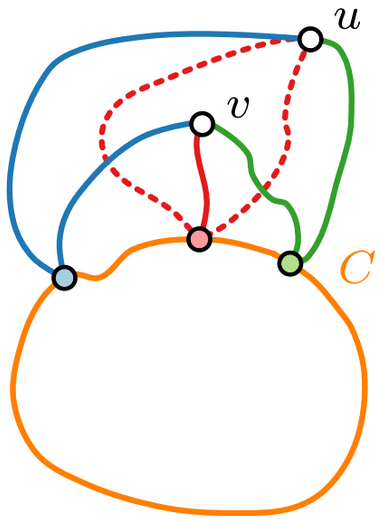
Proof sketch.
Γ_1, Γ_2 planar embeddings of G.
Let C be a face of Γ_2, but not of Γ_1.
u inside C in Γ_1, v outside C in Γ_1
both on same side in Γ_2



Γ_1



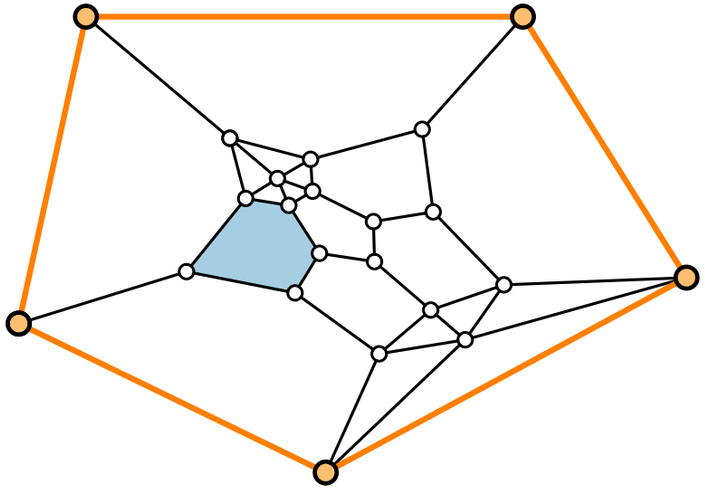
Γ_2



Tutte's Theorem

Theorem. [Tutte 1963]

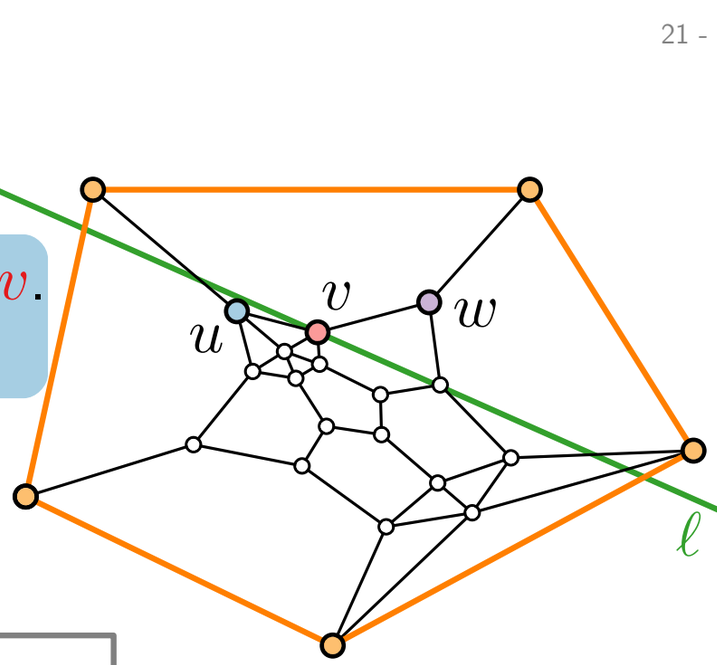
Let G be a 3-connected planar graph, and
let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar and all its faces are strictly convex.



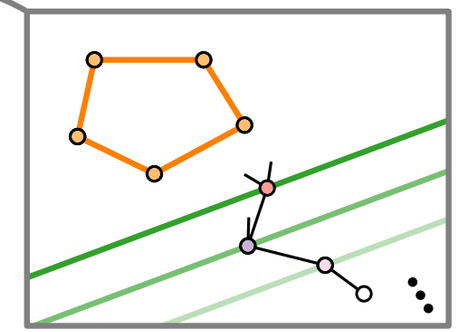
Properties of Tutte Drawings

Property 1. Let $v \in V(G)$ be free (i.e., not fixed), ℓ line through v.
$\exists u \in \mathsf{Adj}[v] \cap \ell^+ \Rightarrow \exists w \in \mathsf{Adj}[v] \cap \ell^-$.

Otherwise, all forces pull v to the same side ...

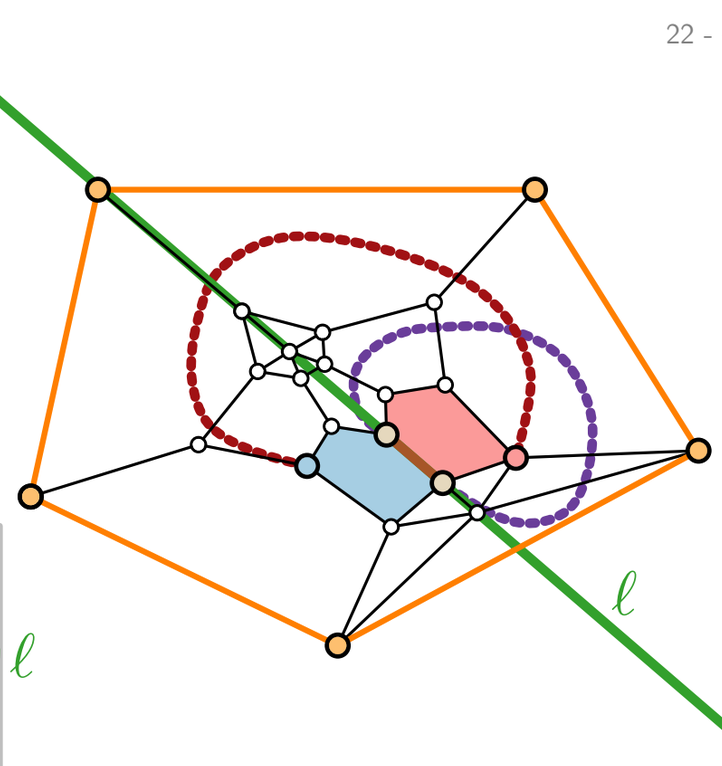


Property 2. All free vertices lie inside C.



Proof of Tutte's Theorem

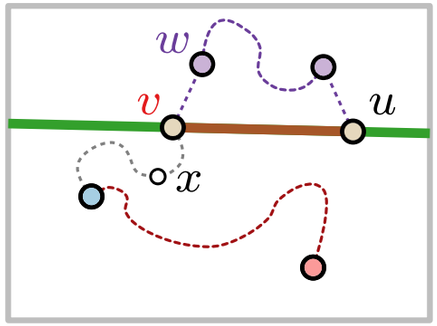
Lemma. Let uv be a non-boundary edge, ℓ line through uv. Then the two faces f_1, f_2 incident to uv lie completely on opposite sides of ℓ.



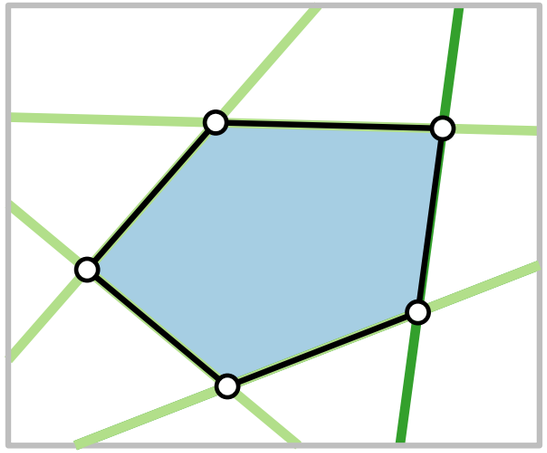
Property 1. Let v be a free vertex, ℓ line through v.
$\exists x \in \text{Adj}[v] \cap \ell^+ \Rightarrow \exists w \in \text{Adj}[v] \cap \ell^-$.

Property 3. Let ℓ be any line.
Let V_ℓ be the set of vertices on one side of ℓ.
Then $G[V_\ell]$ is connected.

Property 4. No vertex is collinear with all its neighbors.



Lemma. All internal faces are strictly convex.



Lemma. The drawing is planar.

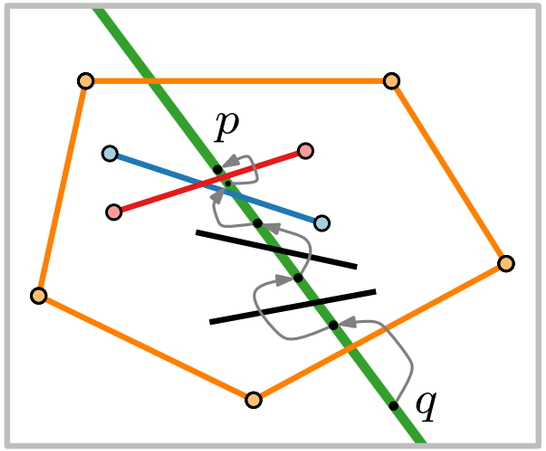
Assume that point p lies in two faces.

Property 2. All free vertices lie inside C.

$\Rightarrow q$ lies in one (i.e., the outer) face.

When jumping an edge, #faces doesn't change.

$\Rightarrow p$ lies in one face.



Discussion

- In practice, force-directed graph drawing methods are very often used.
- Numerous variants, adaptations, and extensions exist.
- They are well-suited for small and medium-size graphs (up to ≈ 100 vertices).
- A way to deal with larger graphs is to coarsen the graph by merging vertices. First draw the coarsened graph and then unpack and draw the vertices of the original graph.
- In practice, the related technique of *multidimensional scaling* (MDS) is often used, too. There, for every pair of vertices, an optimal distance (the distance in the graph) is determined and a drawing with these optimal distances is computed in high-dimensional space. Afterwards this drawing is projected into the plane.
- From a theoretical perspective, Tutte drawings posses many powerful properties.
- If a graph is not 3-connected, we can (temporarily) add sufficiently many edges.
- In practice, Tutte drawings are hardly used because the inner parts often become tiny.

Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph