Coloring Mixed and Directional Interval Graphs

GD 2022, Tokyo

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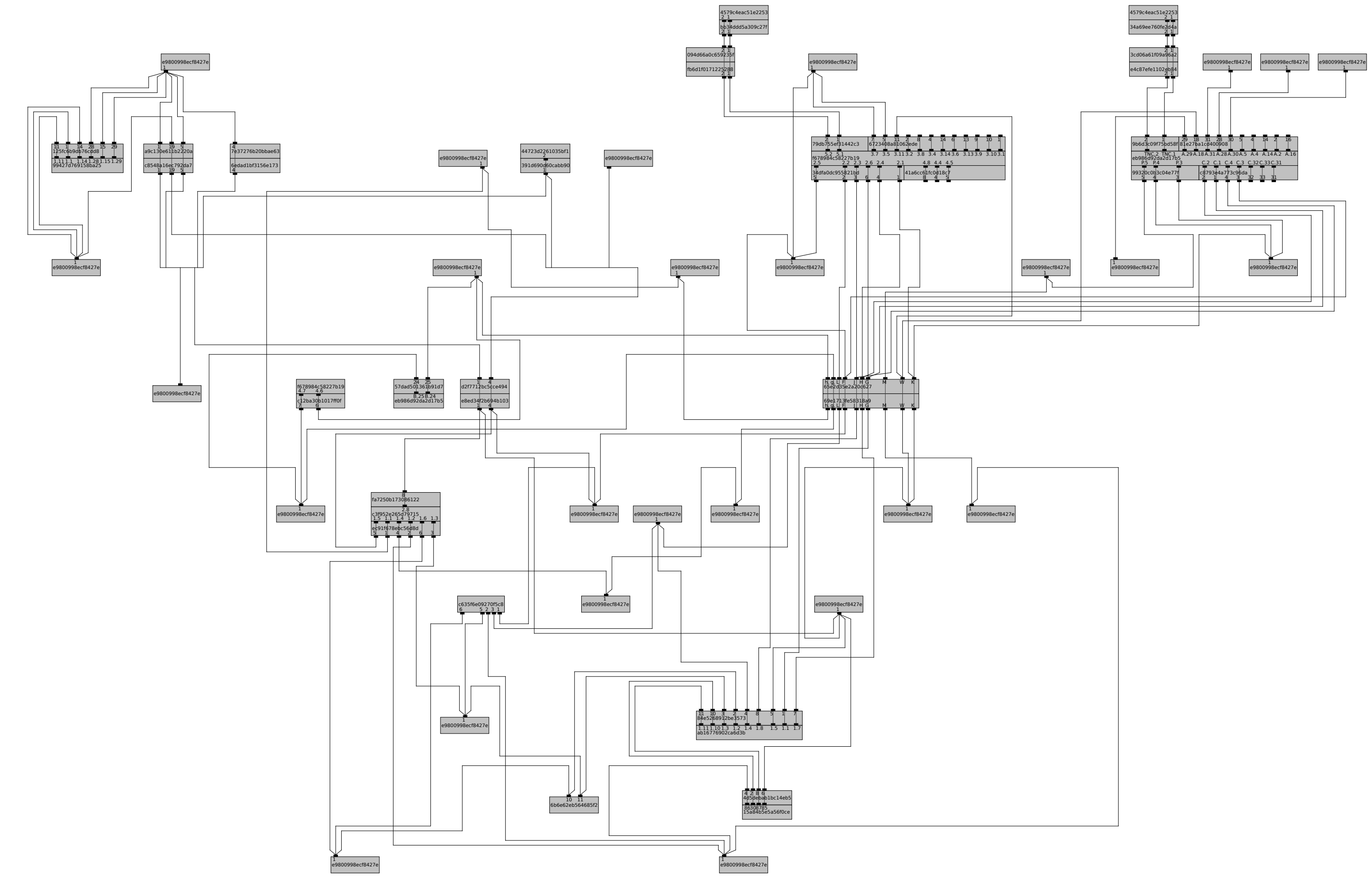
Motivation

Framework for layered g

Input: directed graph G

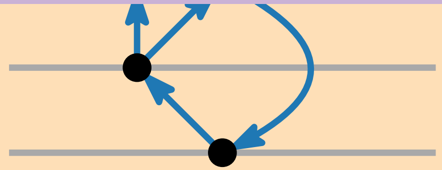
Consists of five phases:

1. cycle elimination
2. layer assignment
3. crossing minimiza
4. node placement
5. edge routing



cable plan

[Zink, Walter, Baumeister, Wolff; CGTA'22]

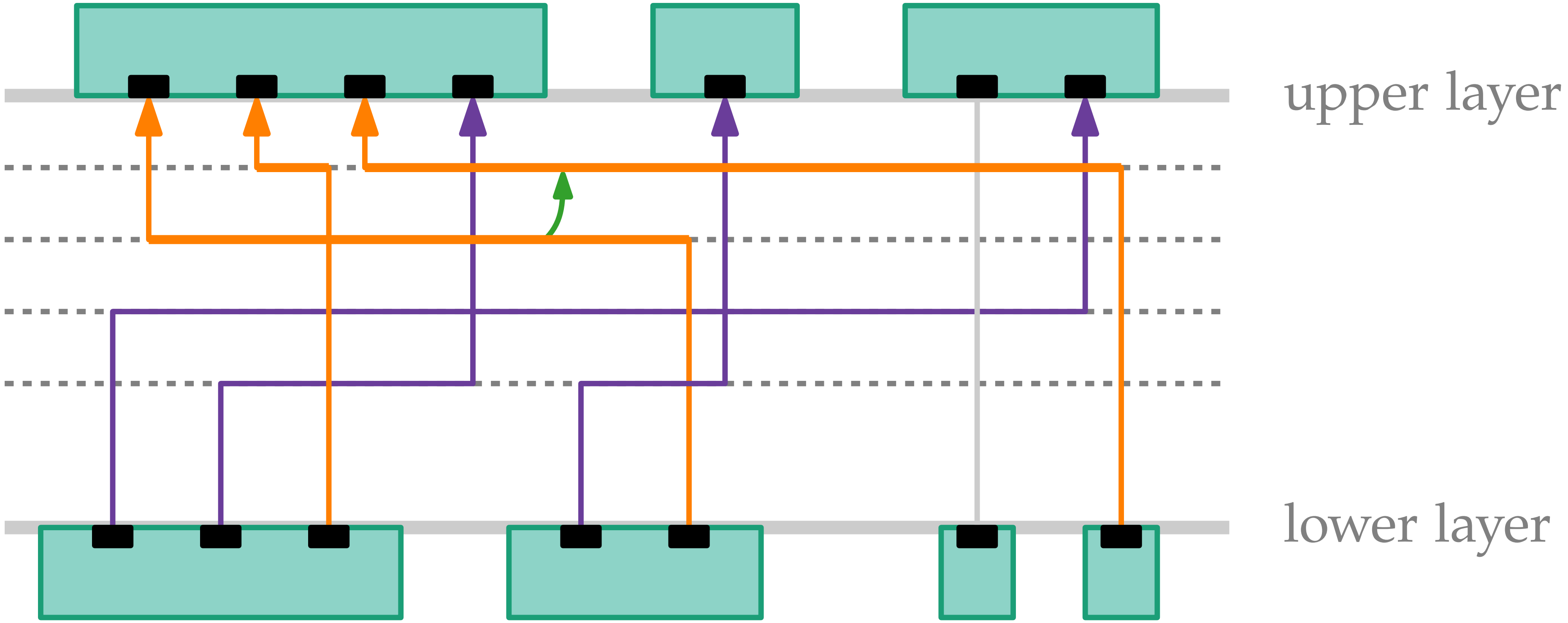


we want orthogonal edges!

Motivation – Layered Orthogonal Edge Routing

- Distinguish between *left-going* and *right-going* edges.
- Only edges going in the same direction and overlapping partially in x-dimension can cross twice.

  ⇒ They induce a vertical order for the horizontal middle segments.

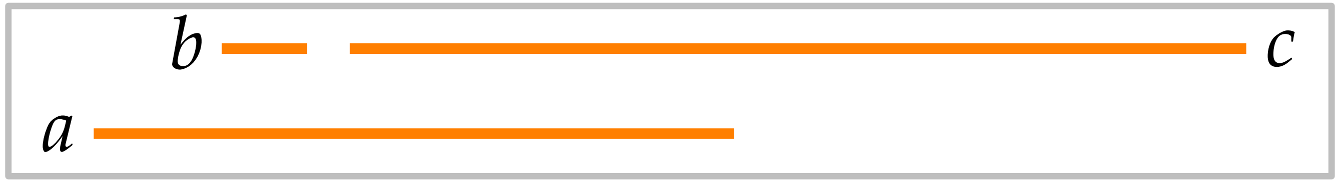


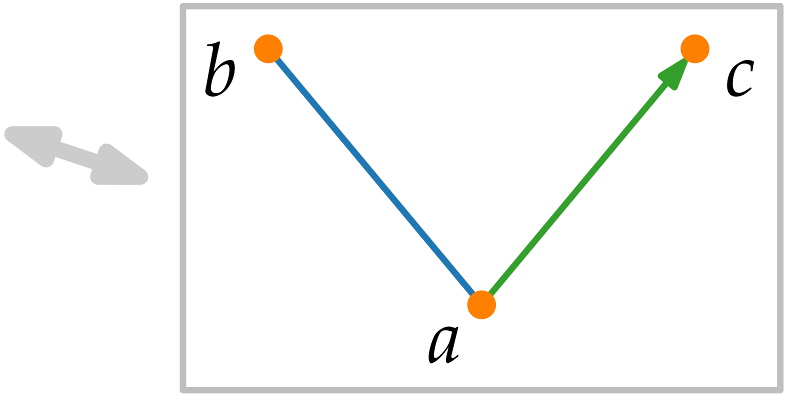
Definition – Directional Interval Graphs

Interval representation: set of intervals

Directional interval graph:

- vertex for each interval
- undirected edge if one interval contains another
- directed edge (towards the right interval) if the intervals overlap partially





Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

NP *bipartite graphs*

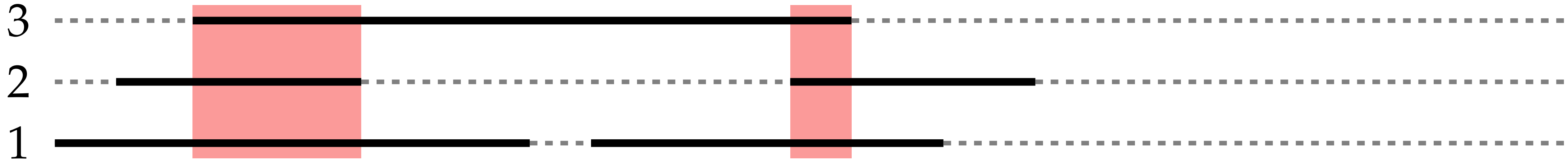
P *trees* *series-parallel graphs*

Given a graph G, find a coloring $c\colon V(G) \to \mathbb{N}$ s.t.
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
- ⋆ directed edge uv: $c(u) < c(v)$,
- ⋆ $\max_{v \in V(G)} c(v)$ is minimized.

Interval graphs (no directed edges):

- sort by left endpoints, color greedily (in linear time given sorted intervals)



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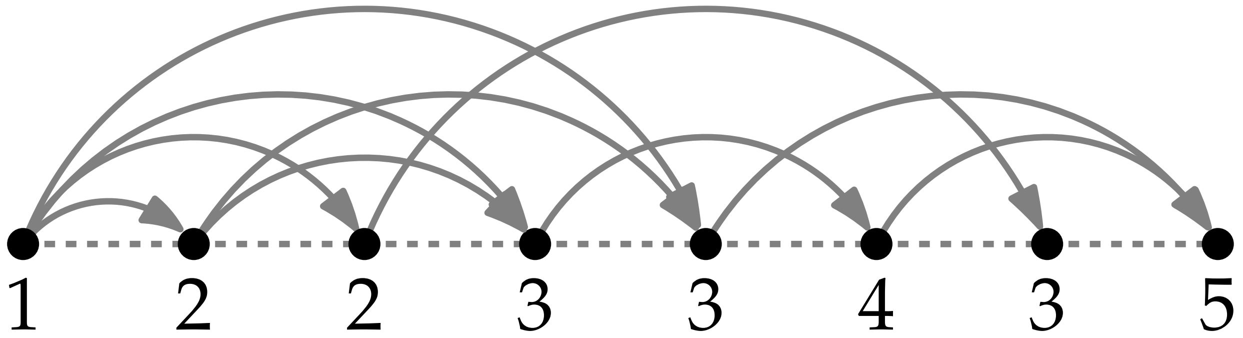


Directed acyclic graphs (only directed edges):

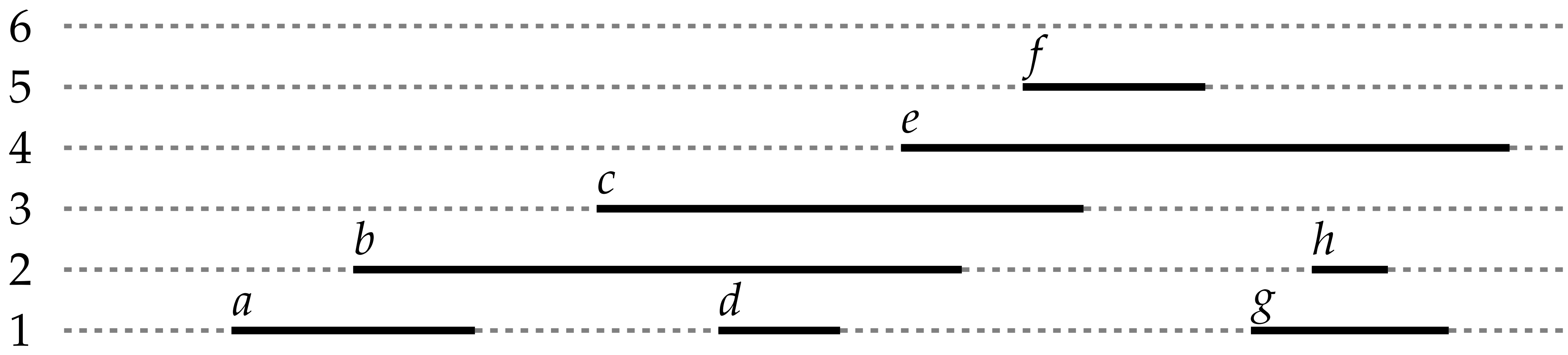
- sort topologically, color greedily (in linear time)

Coloring Directional Interval Graphs

Given: an interval representation of a directional interval graph G

`GreedyColoring:`

1. sort all intervals by left endpoint
2. for each interval, assign the smallest available color respecting incident edges



Coloring Directional Interval Graphs

Theorem 1:
A coloring c computed by `GreedyColoring` has the minimum number of colors.

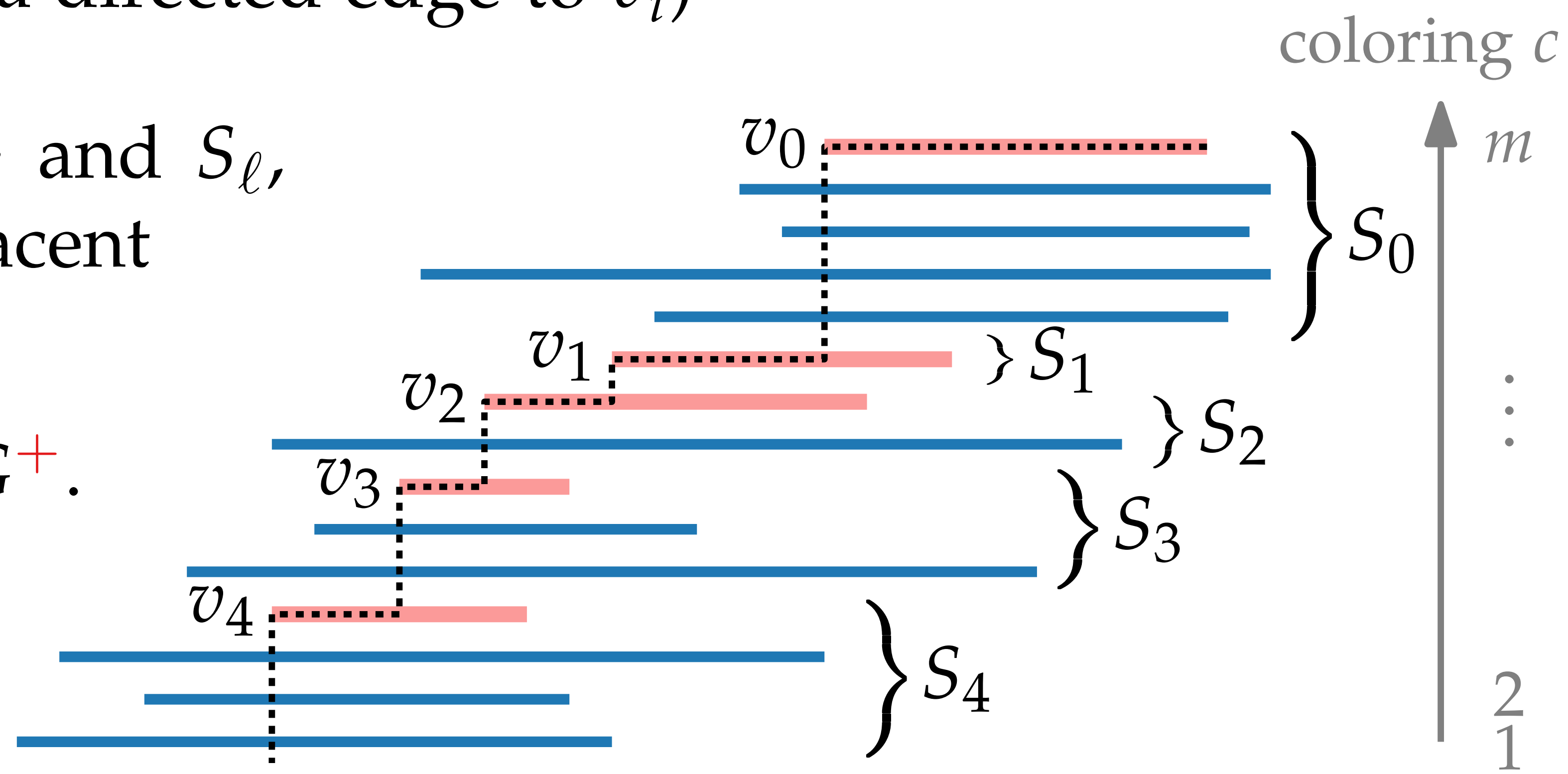
Proof sketch:

- Hence, for every *step* S_i, all intervals contain v_i. (otherwise they would have a directed edge to v_i)

- **Claim:** for any two steps S_i and S_ℓ, every pair of intervals is adjacent in the transitive closure G^+.

  $\Rightarrow$ $S = \bigcup S_i$ is a clique in G^+.

  $\Rightarrow$ S alone requires m colors in G. $\square$



Proof of the Claim

Claim: Any two intervals $u \in S_i$ and $w \in S_\ell$ are adjacent in G^+.

Proof. W.l.o.g., $u \cap w = \emptyset$ and $i < \ell$.

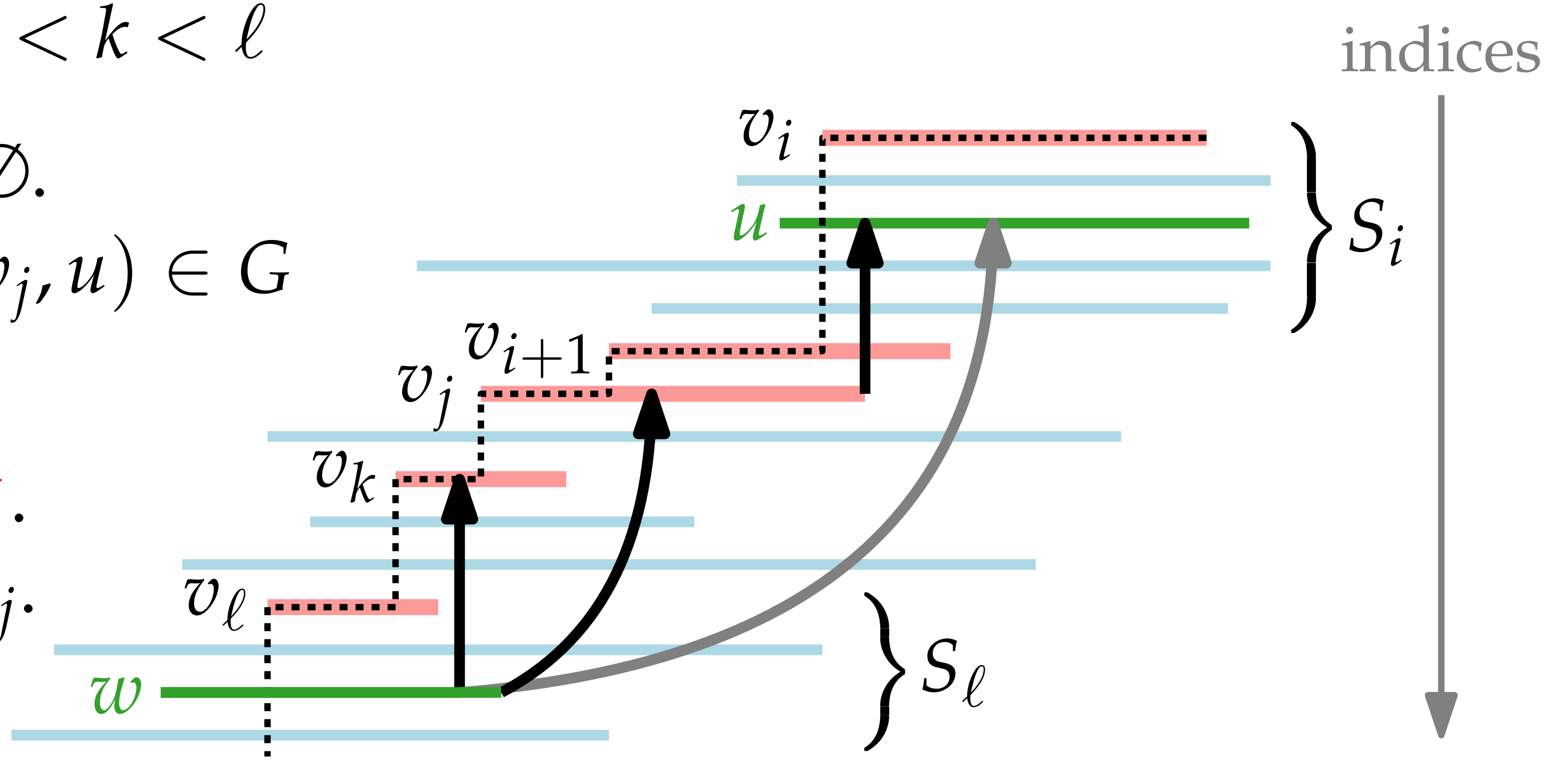
Let j be the largest index s.t. $v_j \cap u \neq \emptyset$.
Let k be the smallest index s.t. $v_k \cap w \neq \emptyset$.

$$\begin{matrix} u \cap v_{i+1} \neq \emptyset \\ w \cap v_{\ell-1} \neq \emptyset \end{matrix} \quad \underset{u \cap w = \emptyset}{\Longrightarrow} \quad \begin{matrix} i < j < \ell \\ i < k < \ell \end{matrix}$$

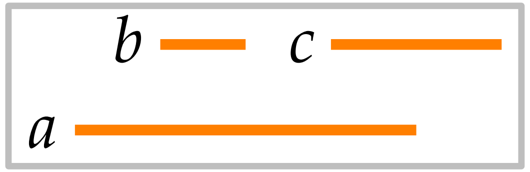
By definition, $u \cap v_{j+1} = \emptyset$.
$\Rightarrow$ u and v_j overlap $\Rightarrow (v_j, u) \in G$
Similarly, $(w, v_k) \in G$.

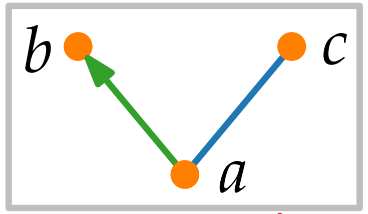
If $j < k$, then $(v_k, v_j) \in G^+$.
If $j \geq k$, then w overlaps v_j.

Transitivity $\Rightarrow$ claim.



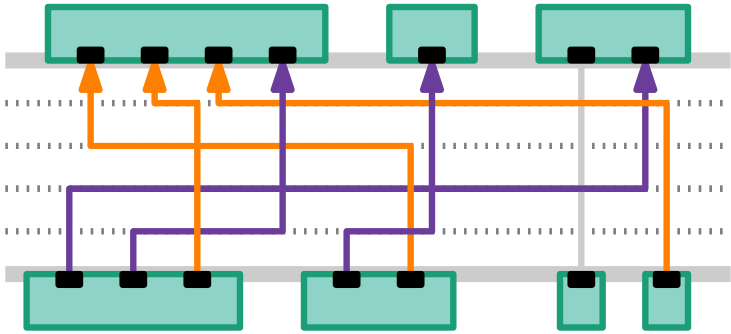
Conclusion and Open Problems





- We have introduced the natural concept of directional interval graphs.
  ??? Reviewer: Consider containment interval graphs!
- A simple greedy algorithm colors these graphs optimally in $O(n \log n)$ time.
  $n :=$ # vertices
- In layered graph drawing, this corresponds to routing "left-going" edges orthogonally to the fewest horizontal tracks. (Symmetrically "right-going".)

⇒ Combining the drawings of left-going and right-going edges yields a 2-approximation for the number of tracks. (bidirectional interval graphs)



can we do better?

- In our paper, we present a constructive $O(n^2)$-time algorithm for recognizing directional interval graphs, which is based on PQ-trees.
  bidirectional?
- For the more general case of mixed interval graphs, coloring is NP-hard.
  (Remark: NP-hardness requires both directed and undirected edges.)

Coloring and Recognizing Mixed Interval Graphs

ISAAC 2023, Kyoto

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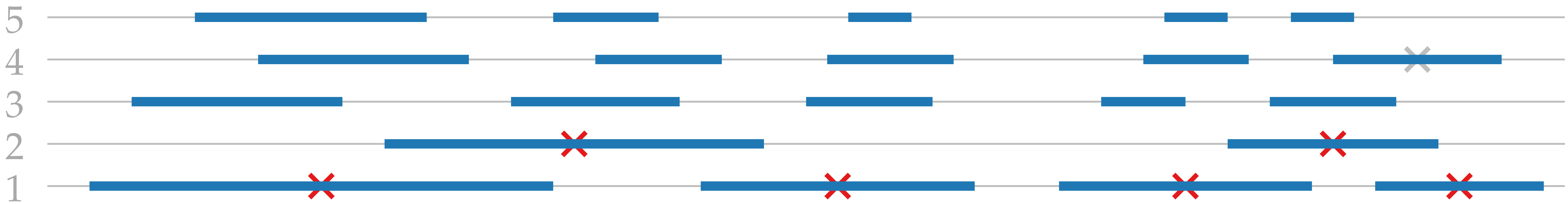
Some Observation about Interval Containment Graphs



Let $\mathcal{I}$ be a set of intervals. Let $G = \mathcal{C}[\mathcal{I}]$ be the containment graph induced by $\mathcal{I}$.

Let $M(\mathcal{I})$ be the set of inclusion-wise maximum elements in $\mathcal{I}$.

Then $\mathcal{C}[M(\mathcal{I})]$ is a *proper* interval graph – no interval contains another interval.

Also note that $\bigcup M(\mathcal{I}) = \bigcup \mathcal{I}$.

Let R be an inclusion-wise minimal subset of $M(\mathcal{I})$ such that $\bigcup R = \bigcup \mathcal{I}$.

Claim. $\mathcal{C}[R]$ is an undirected linear forest.

Proof. $\mathcal{C}[R]$ is proper $\Rightarrow$ contains no induced $K_{1,3}$ and no induced C_ℓ for $\ell \geq 4$.

It remains to show that $\mathcal{C}[R]$ contains no triangle. ✓

A 2-Approximation Algorithm for Coloring

Theorem. For any set $\mathcal{I}$ of intervals,
the graph $\mathcal{C}[\mathcal{I}]$ admits a coloring with at most $2 \cdot \overbrace{\omega(\mathcal{C}[\mathcal{I}])}^{\omega\,=\,\text{clique number}} - 1$ colors.

Since $\mathcal{C}[R]$ is a linear forest, it admits a coloring $f_1 \colon R \to \{1,2\}$.

If $R = \mathcal{I}$, we are done (using only ω many colors), so we assume $\mathcal{I} \setminus R \neq \emptyset$.

Let $G' := \mathcal{C}[\mathcal{I} \setminus R]$.

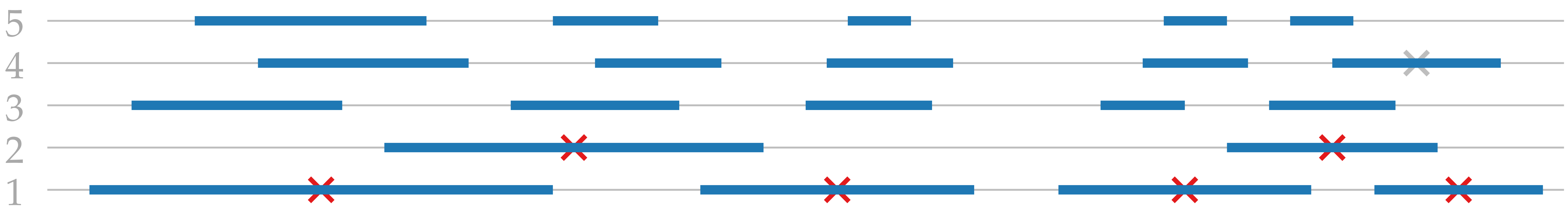
Claim. $\omega(G') \leq \omega - 1$.

Proof. Suppose that there is a clique S in G' of size ω.
Helly property of intervals $\Rightarrow \bigcap S \neq \emptyset$. Let $p \in \bigcap S$.
Pick an $r \in R$ that contains p. $\Rightarrow S \cup \{r\}$ is a clique of size $\omega + 1$ in G. ↯

Induction $\Rightarrow G'$ admits a coloring f_2 using at most $2 \cdot \omega(G') - 1$ colors.

With f_1 and f_2, we construct a coloring f of G using colors $\{1, \ldots, 2\omega - 1\}$.

An Inductive Coloring



Let $f(x) = \begin{cases} f_1(x) & \text{if } x \in R, \\ f_2(x) + 2 & \text{else.} \end{cases}$

This defines a coloring of G:

1. If $x \cap y \neq \emptyset$, then $f(x) \neq f(y)$. Check: $x, y \in R$; $x, y \neq R$; $x \in R$ and $y \neq R$.
2. If $x \subseteq y$, then $f(x) > f(y)$. Observe that $x \neq R \Rightarrow f(x) \geq 3$

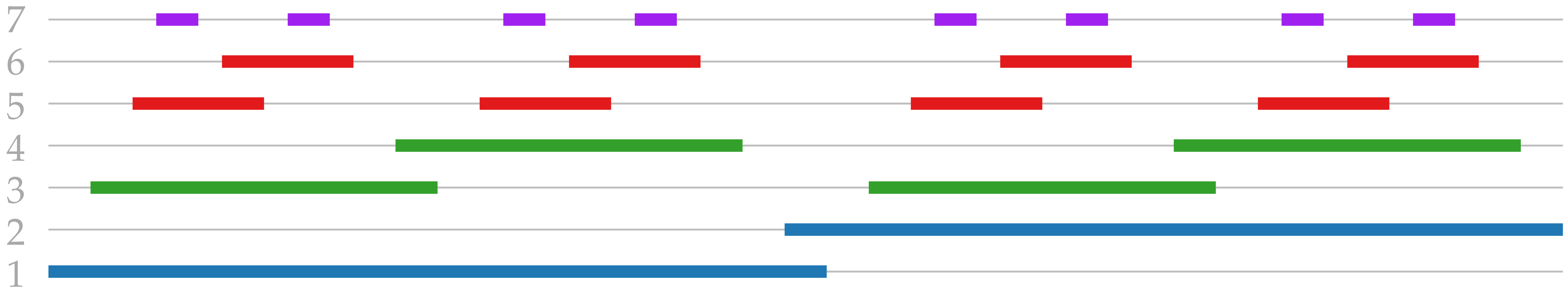
   Suppose $f(y) > f(x) \Rightarrow y \neq R$, but $f_2(x) > f_2(y)$. ↯

Corollary. There is a 2-approximation for coloring interval containment graphs. Given n intervals, the algorithm runs in $O(n \log n)$ time.

A Lower Bound Example

Proposition. There is an infinite family $(\mathcal{I}_n)_{n\geq 1}$ of sets of intervals with $|\mathcal{I}_n| = 3 \cdot 2^{n-1} - 2$, $\chi(\mathcal{C}[\mathcal{I}_n]) = 2n - 1$, and $\omega(\mathcal{C}[\mathcal{I}_n]) = n$.

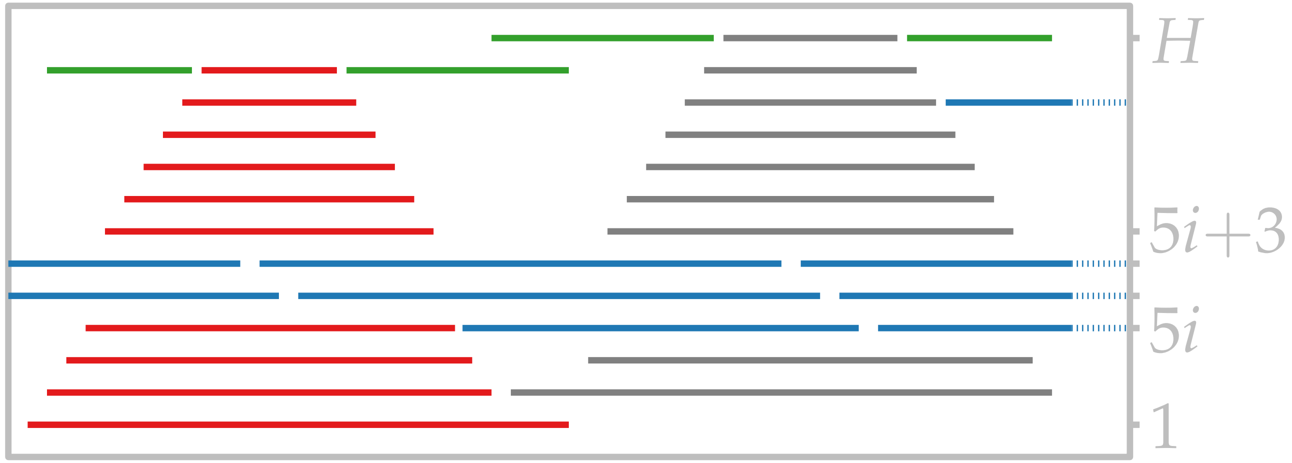
This yields $\lim_{n\to\infty} \chi(\mathcal{I}_n)/\omega(\mathcal{I}_n) = 2$.



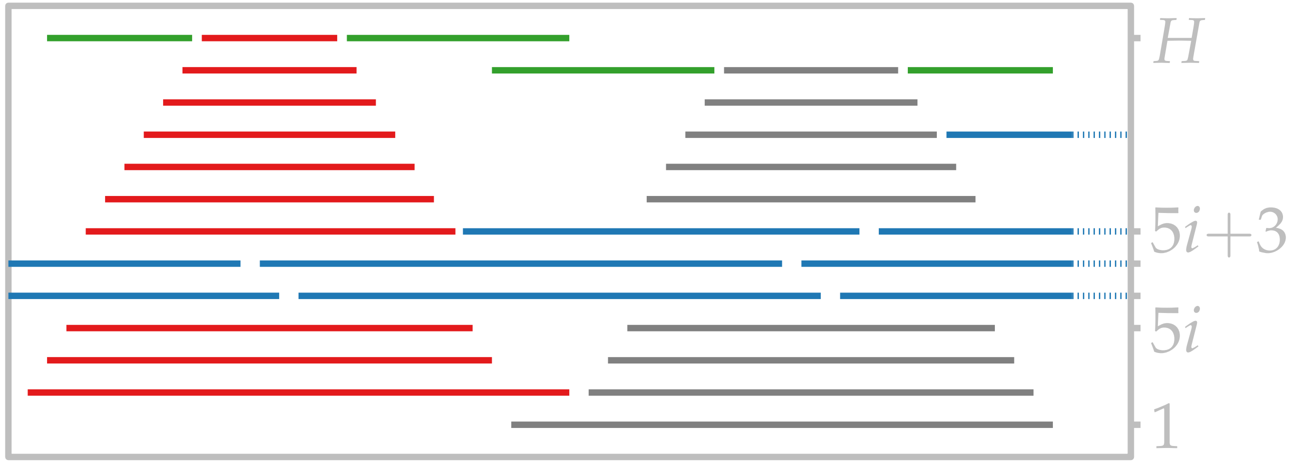
Computational Complexity

Theorem. Given a set $\mathcal{I}$ of intervals and a positive integer k, it is NP-hard to decide whether $\chi(\mathcal{C}[\mathcal{I}]) \leq k$.

Proof. By reduction from (exact) 3-SAT, where each clause has exactly 3 literals.



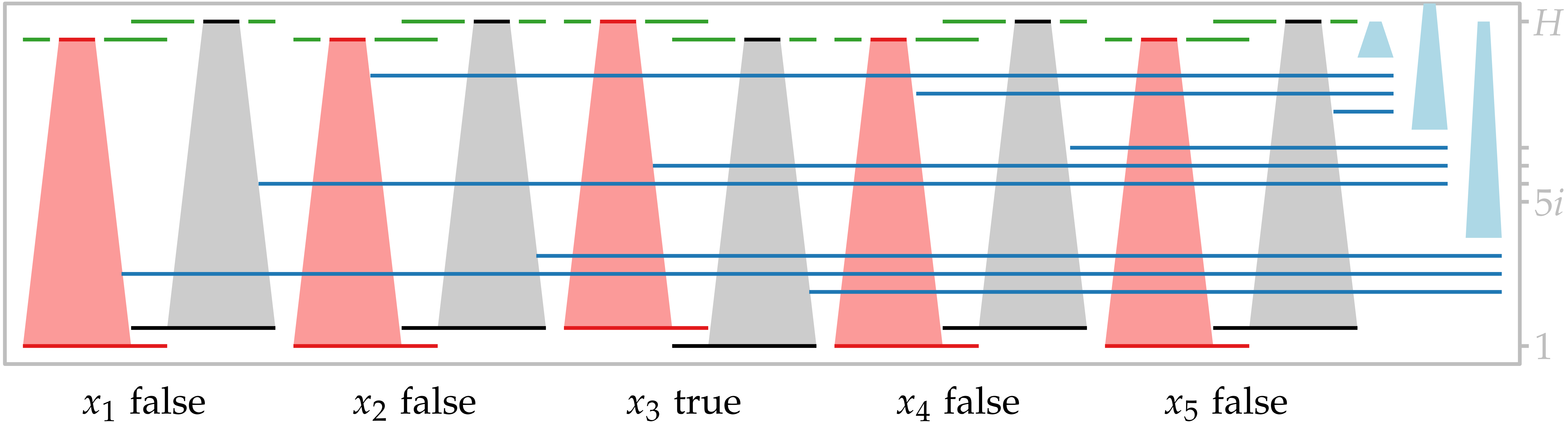
x false



x true

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an instance of 3-SAT with variables $\{x_1, x_2, \ldots, x_n\}$, and let $H = 5m + 1$. We construct a set $\mathcal{I}_\varphi$ of intervals.

Clause Gadget



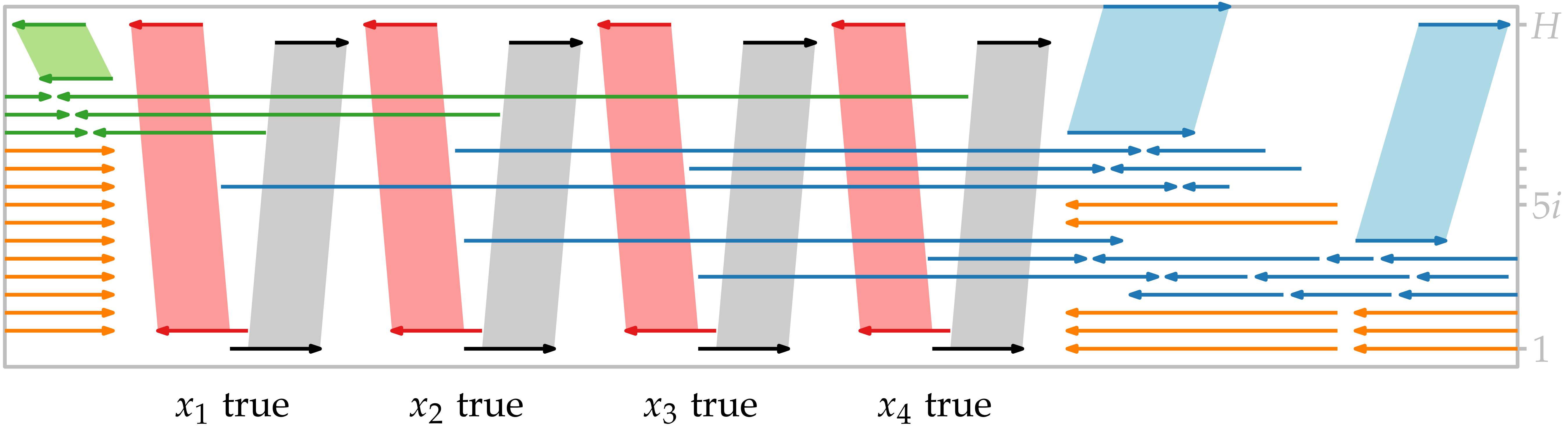
Example for $(\neg x_2 \vee \neg x_4 \vee x_5) \wedge (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3)$.

The graph $\mathcal{C}[\mathcal{I}_\varphi]$ admits a coloring with H colors $\Leftrightarrow$ φ is satisfiable. $\square$

Bidirectional Intervals

Theorem. Given a set $\mathcal{I}$ of intervals, $\varphi\colon \mathcal{I} \to \{\mathsf{left}, \mathsf{right}\}$, and $k \in \mathbb{N}$, it is NP-hard to decide whether $\chi(\underbrace{\mathcal{B}[\mathcal{I}, \varphi]}_{\text{mixed intersection graph of bidirectional intervals}}) \leq k$.

Proof sketch.



Mixed Interval Graphs

Recall that a *mixed interval graph* is an interval graph where two intersecting intervals are connected by an edge or an arc in either direction.

If G is a mixed interval graph, then clearly $\chi(G) \geq \omega(G)$.

Let $\lambda(G)$ denote the length of a longest directed path in G.

Then clearly $\chi(G) \geq \lambda(G) + 1$. Hence, $\chi(G) \geq \max\{\omega(G), \lambda(G) + 1\}$.

Theorem. Let G be a mixed interval graph without directed cycles. Then $\chi(G) \leq (\lambda(G) + 1) \cdot \omega(G)$.

Our constructive proof yields a $\min\{\omega(G), \lambda(G) + 1\}$-approximation algorithm.

A Constructive Proof

Theorem. Let G be a mixed interval graph without directed cycles. Then $\chi(G) \leq (\lambda(G) + 1) \cdot \omega(G)$.

Proof. Let $c\colon V \to \{1, 2, \ldots, \omega(U(G))\}$ be an optimal coloring of $U(G)$.

Define a mapping f. For a vertex x of G, let $f(x) = \ell(x) \cdot \omega(G) + c(x)$.

Note that $1 \leq f(x) \leq (\lambda(G) + 1) \cdot \omega(G)$. We claim that f colors G.

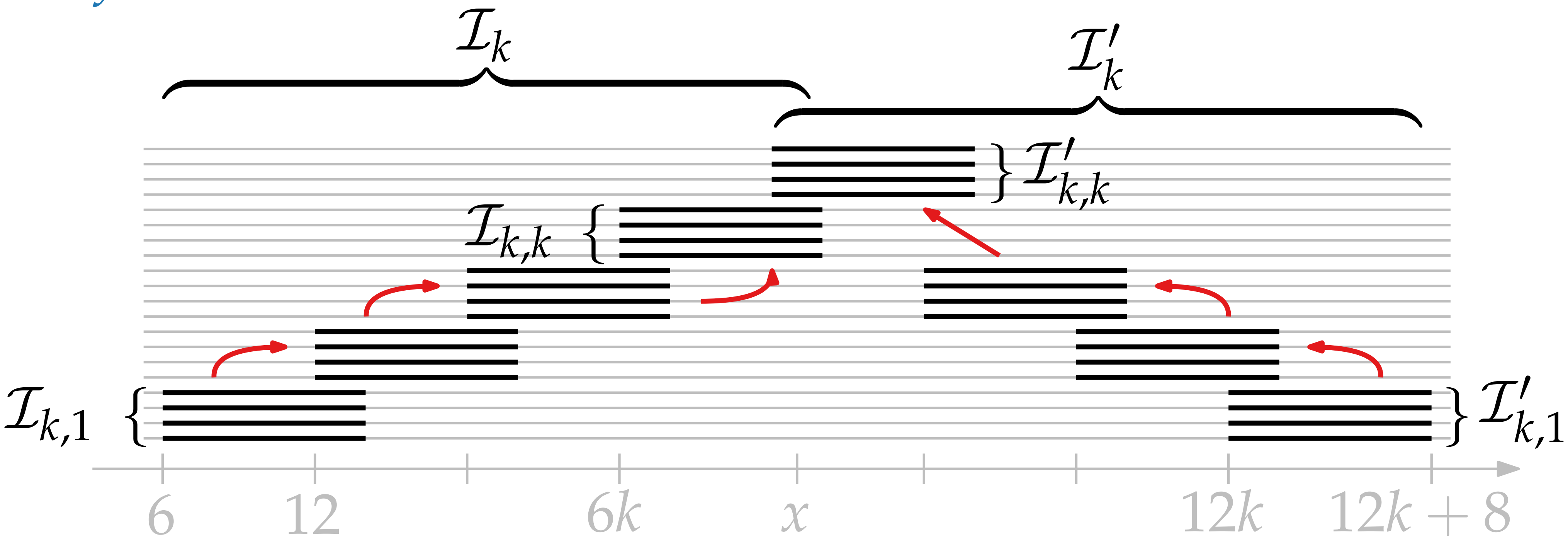
If $\{x, y\}$ is an edge of G, then $c(x) \neq c(y)$ and hence, $f(x) \neq f(y)$.
If (x, y) is an arc of G, then $\ell(x) < \ell(y)$ and hence, $f(x) < f(y)$. $\square$

A Lower Bound Example

Proposition. There is an infinite family $(G_k)_{k \geq 1}$ of mixed interval graphs with $|V(G_k)| = 2k^2$, $\lambda(G_k) = k - 1$, $\omega(G_k) = 2k$, and $\chi(G_k) = (k+1) \cdot k = (\lambda(G_k) + 2) \cdot \omega(G_k)/2$.

That is, our upper bound for $\chi(G)$, $(\lambda(G) + 1) \cdot \omega(G)$, is asymptotically tight.

Proof.



Summary

Mixed interval graph class	complexity	Coloring lower bound	upper bound	approximation	Recognition
containment	NP-hard	$2\omega-1$	$2\omega-1$	2	$O(nm)$
directional	$O(n \log n)$			1	$O(n^2)$
bidirectional	NP-hard			2	open
general	NP-hard	$(\lambda+2)\omega/2$	$(\lambda+1)\omega$	$\min\{\omega, \lambda+1\}$	$O(n+m)$ [LB79]

Follow-up Work

- Given a mixed graph G with an orientation φ, we can decide in linear time whether G admits an oriented interval representation that complies with φ.
- In particular, we can recognize directional interval graphs in linear time.