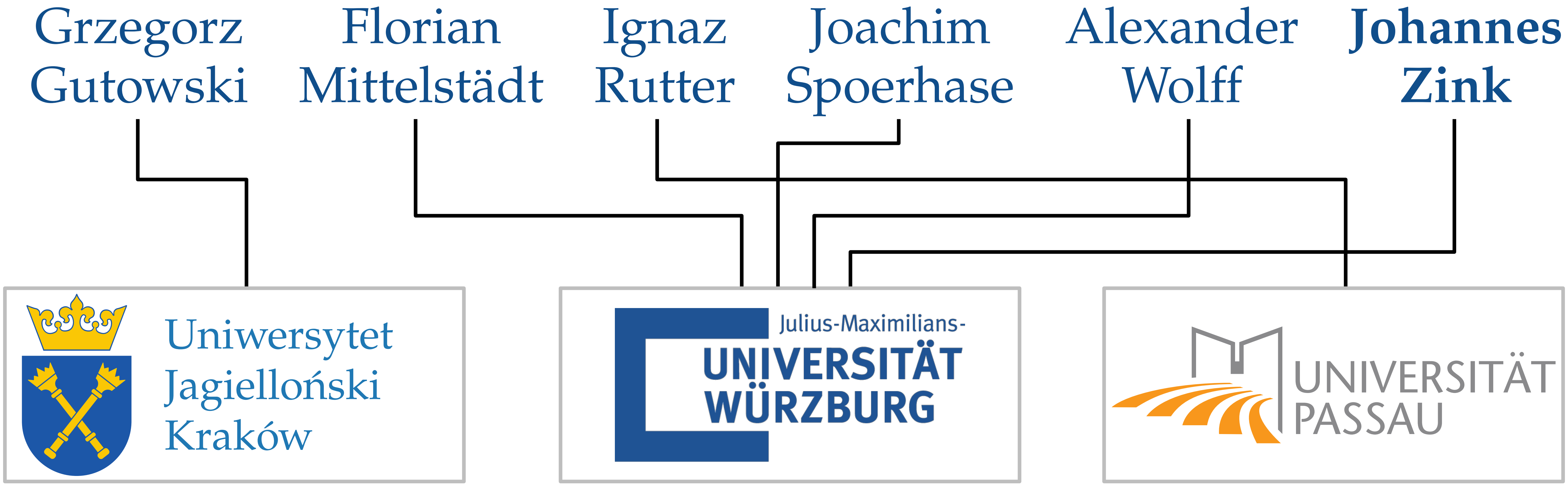
Coloring Mixed and Directional Interval Graphs

GD 2022, Tokyo



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UNIVERSITÄT PASSAU

Motivation

Framework for layered graph drawing by Sugiyama, Tagawa, and Toda (1981).

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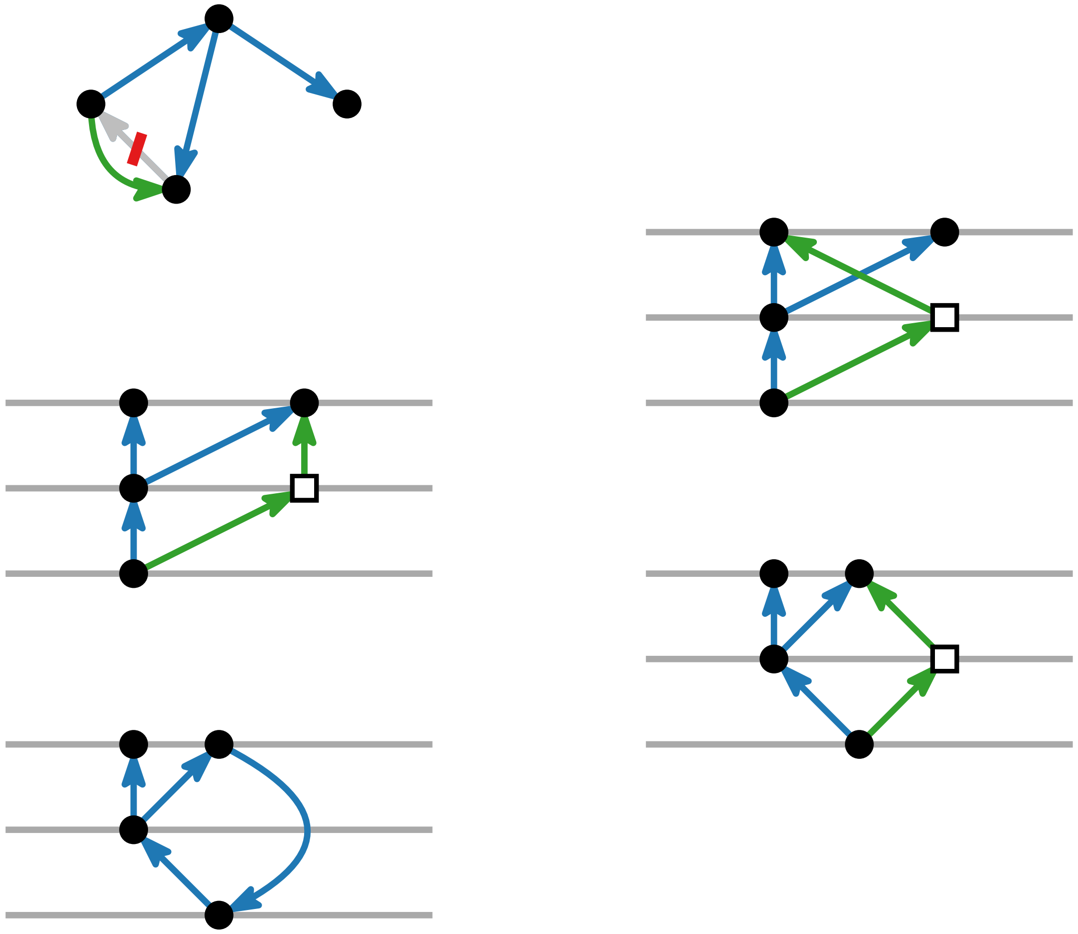
Consists of five phases:

1. cycle elimination
2. layer assignment
3. crossing minimization
4. node placement
5. edge routing



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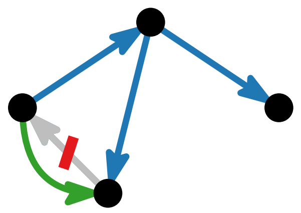
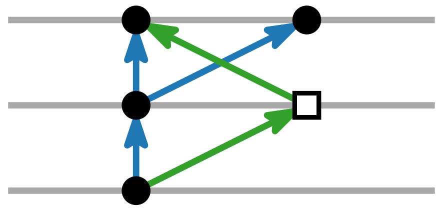
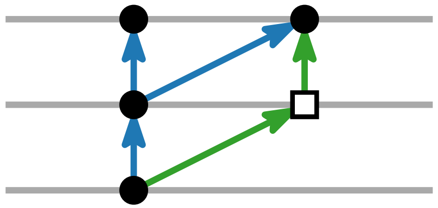


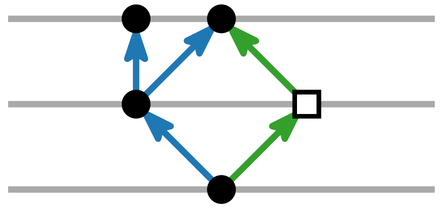
2. layer assignment



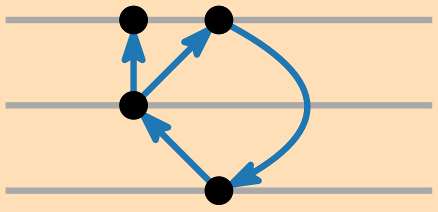
3. crossing minimization



4. node placement



5. edge routing



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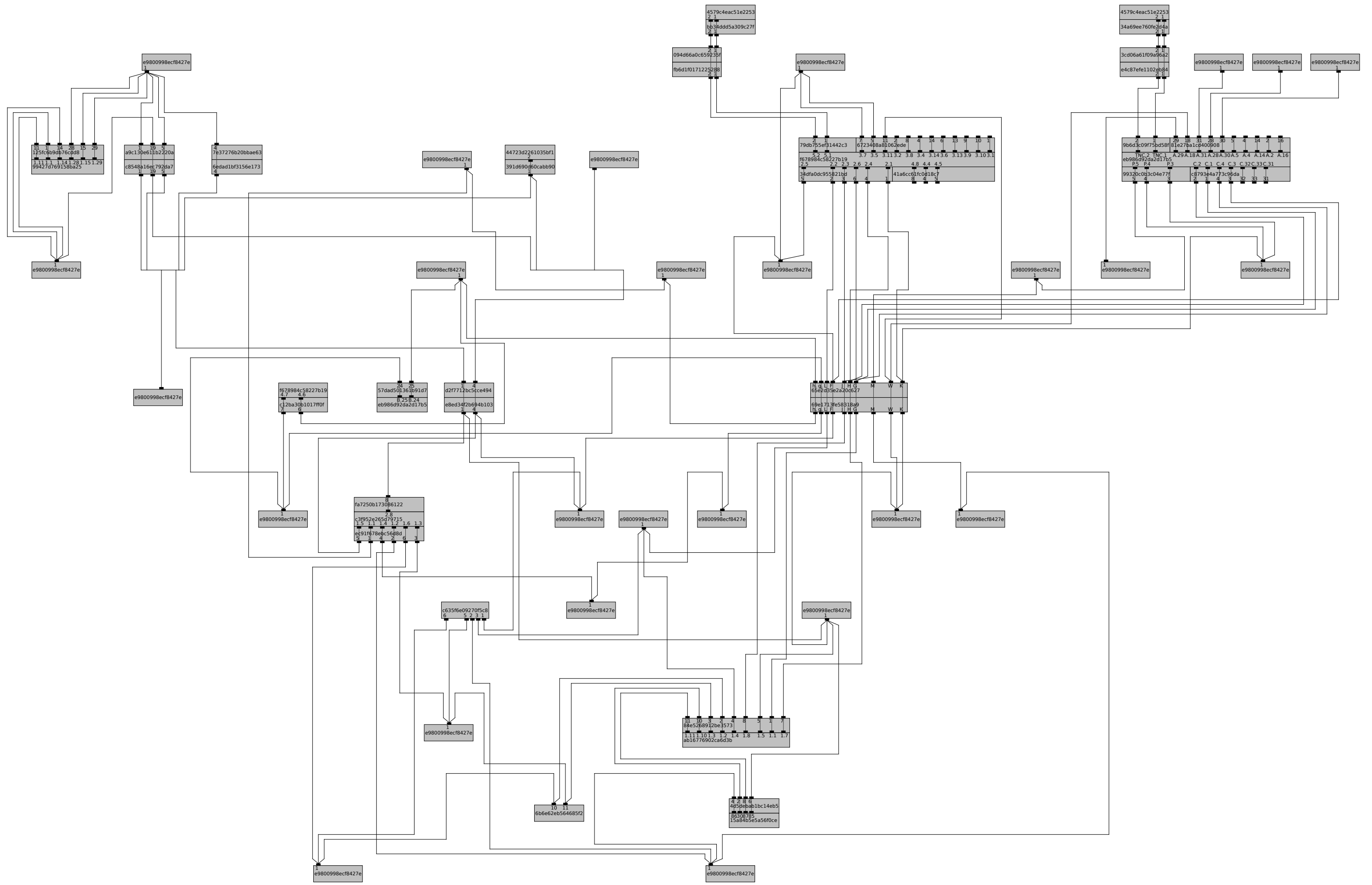
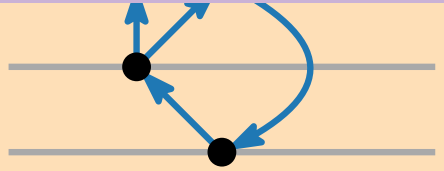


cable plan

[Zink, Walter, Baumeister, Wolff; CGTA'22]



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Motivation – Layered Orthogonal Edge Routing

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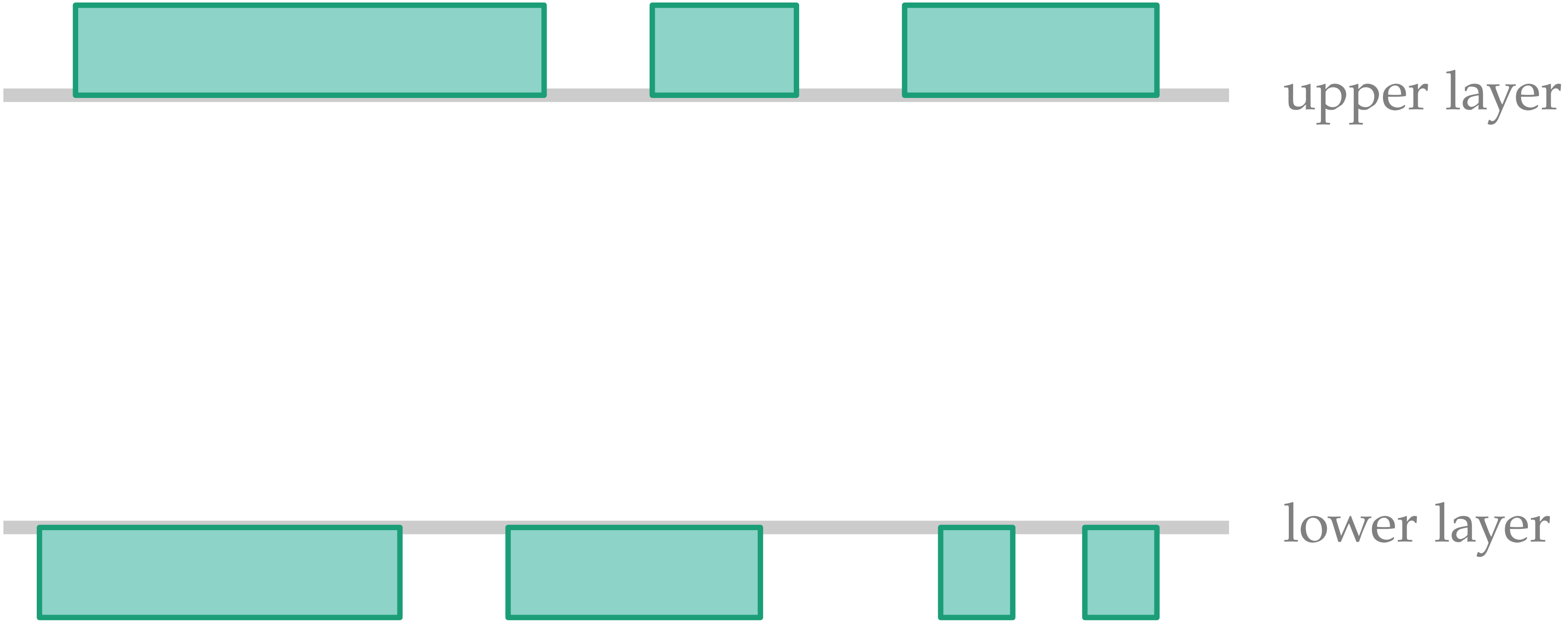
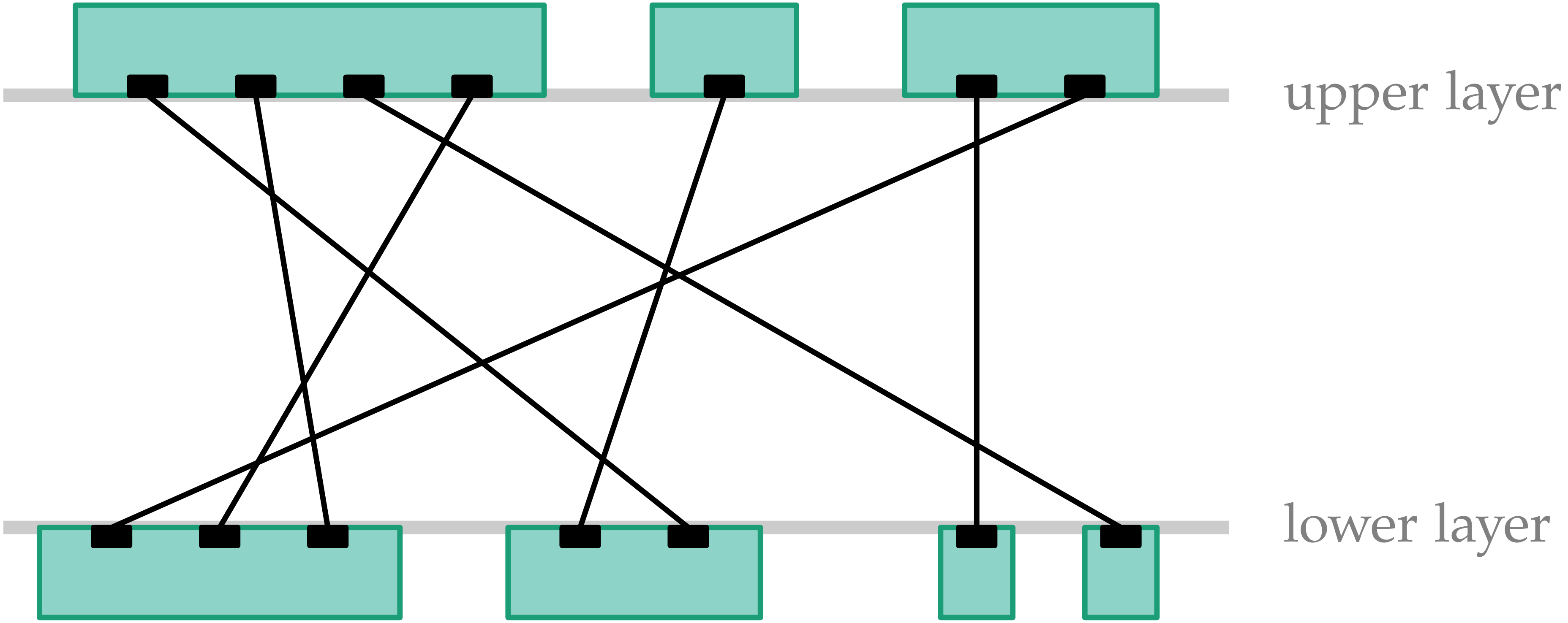


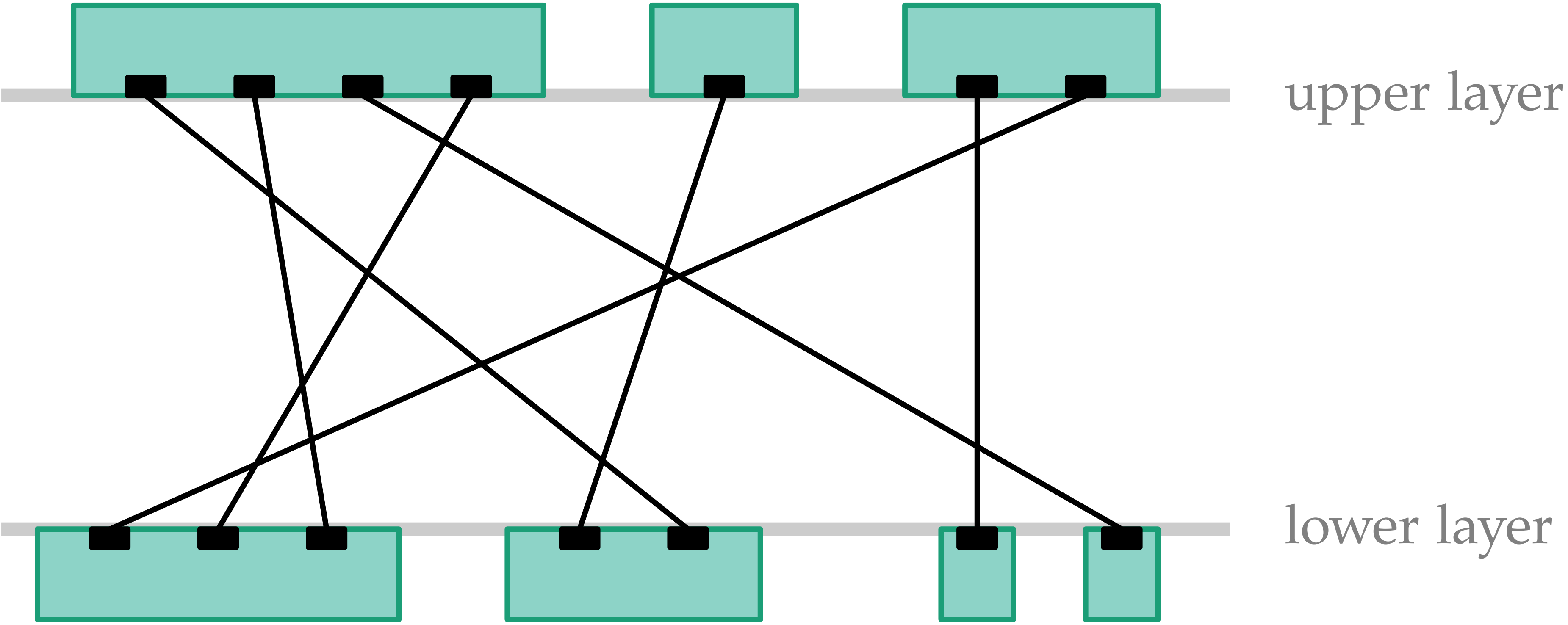
Motivation – Layered Orthogonal Edge Routing

- It suffices to consider each pair of consecutive layers individually.
- Positions of vertices are fixed.
- No two edges share a common end point (vertices have distinct ports).



Motivation – Layered Orthogonal Edge Routing

- Draw each edge with at most two vertical and one horizontal line segments.



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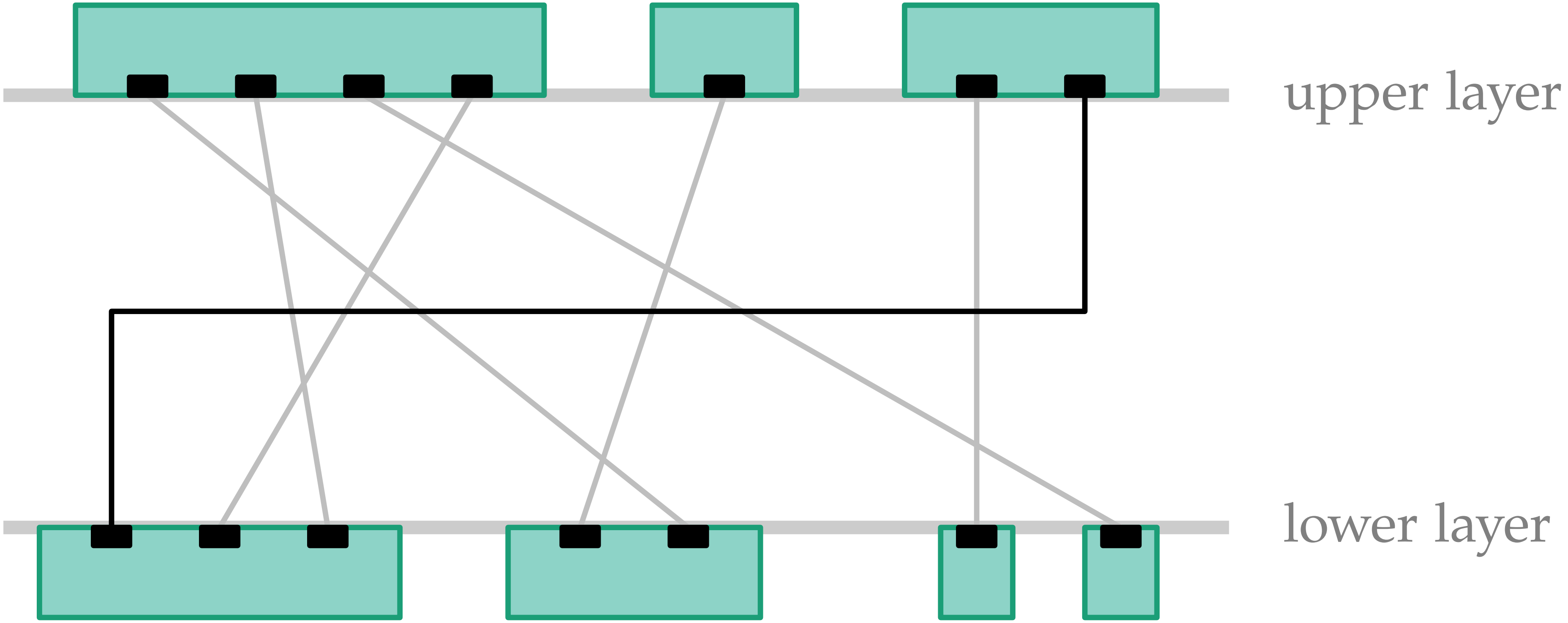

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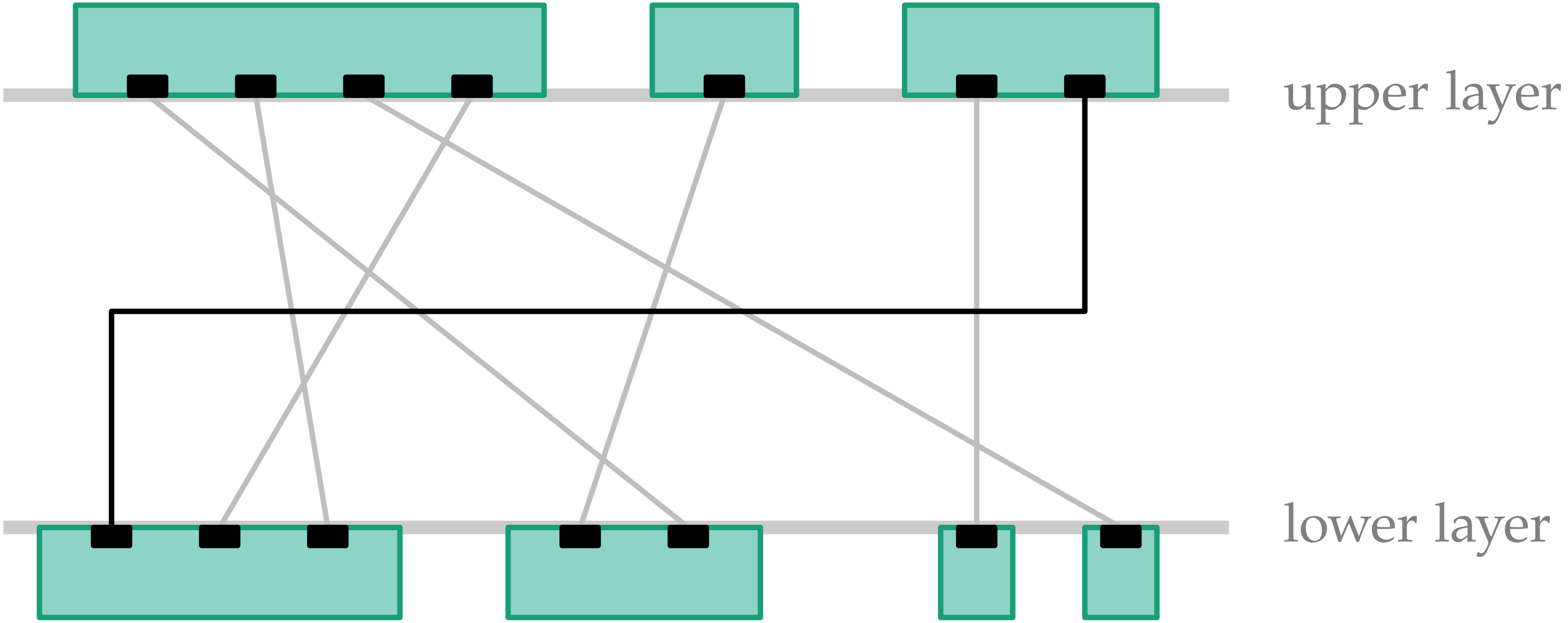
- Draw each edge with at most two vertical and one horizontal line segments.
- Avoid overlaps and double crossings between the same pair of edges.



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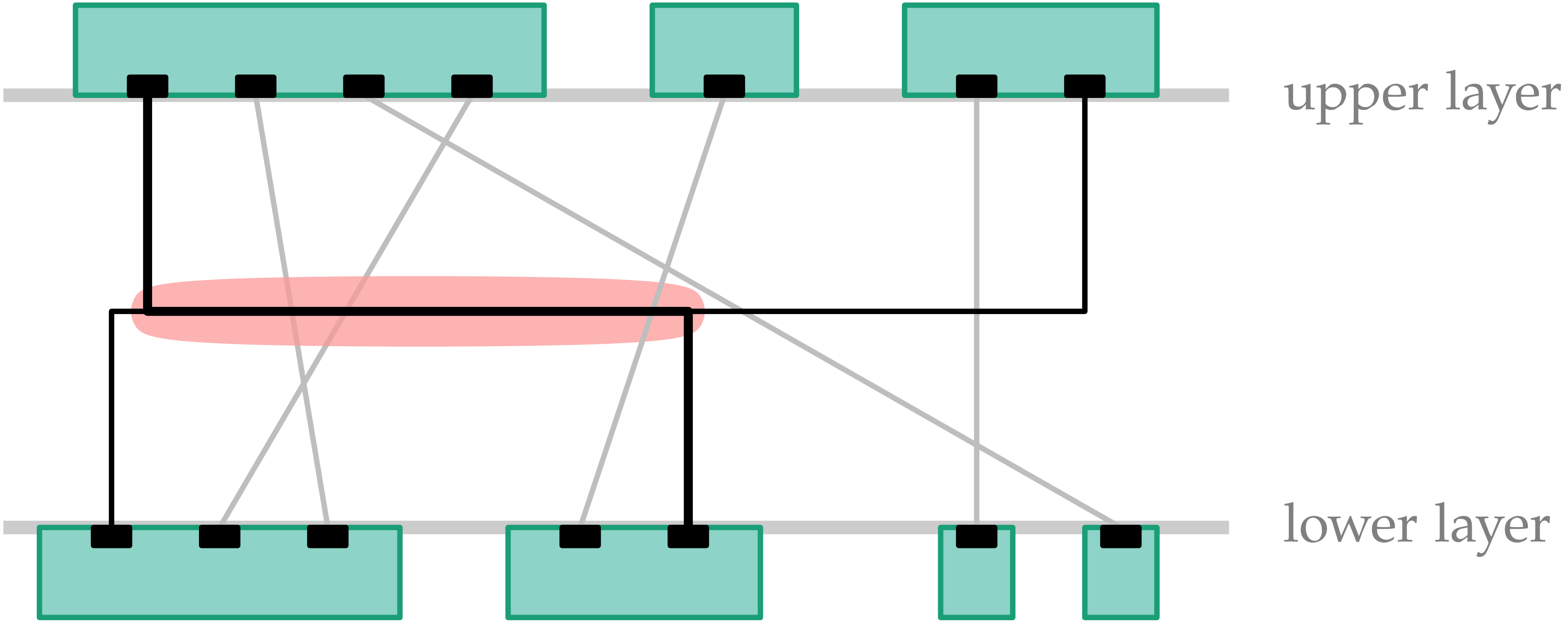

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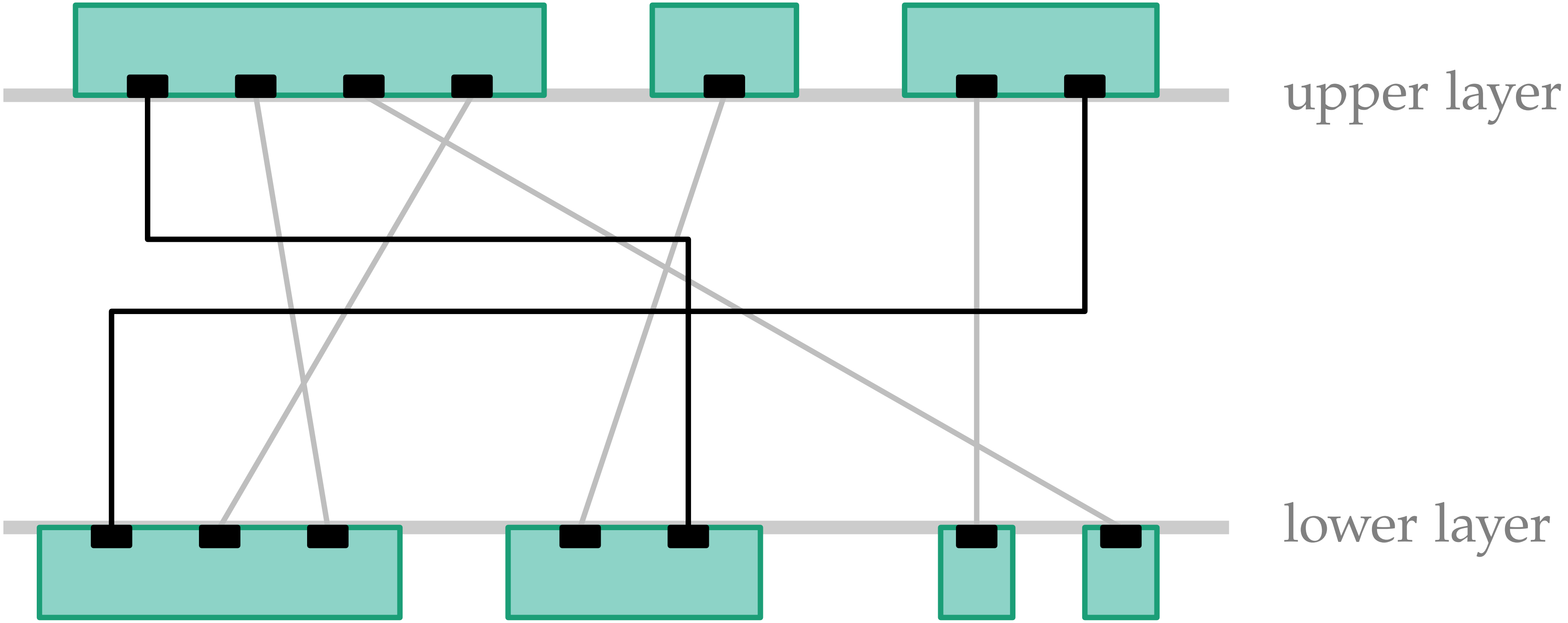

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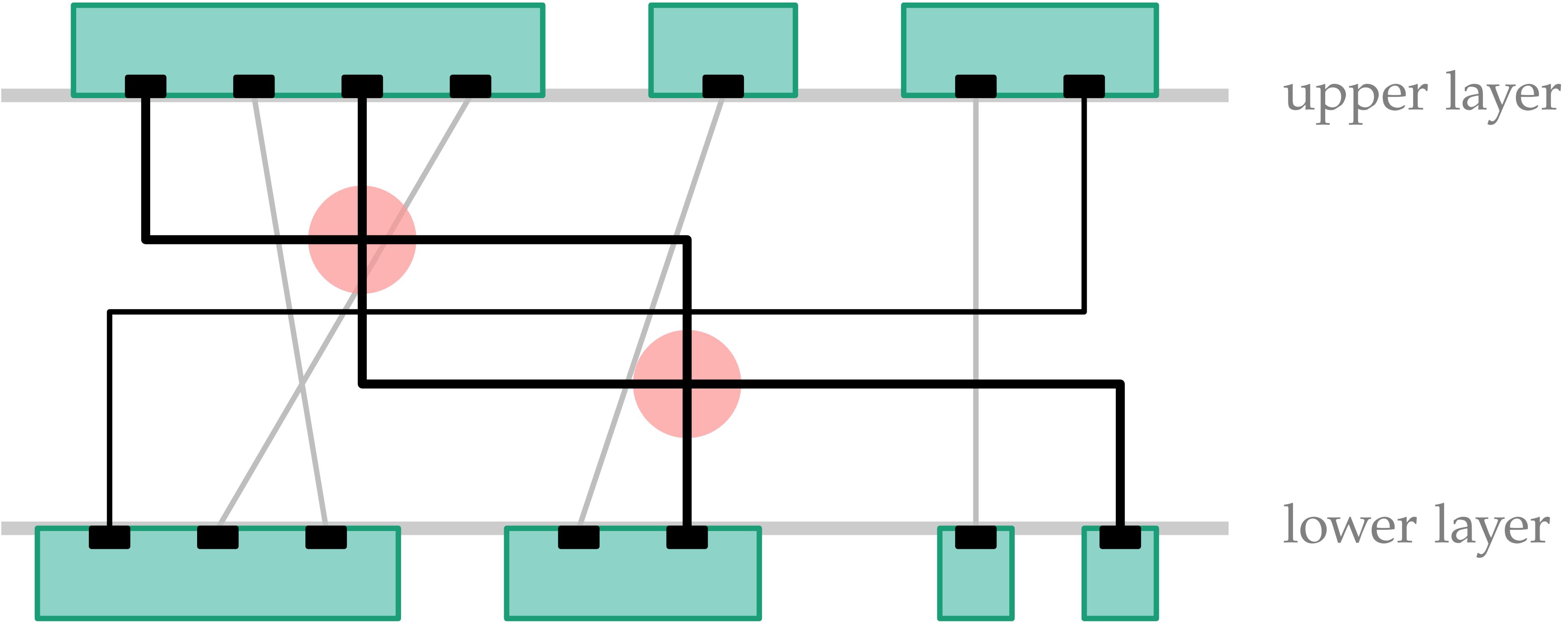

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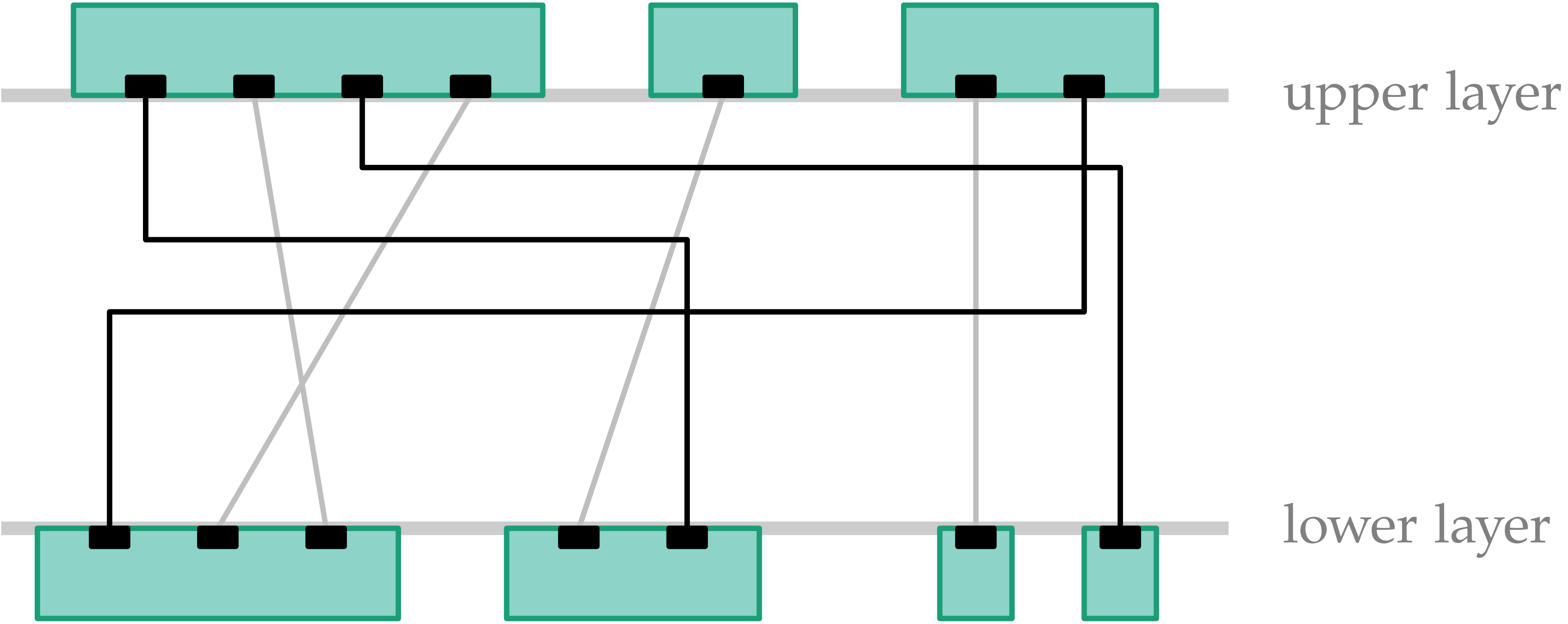

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- Draw each edge with at most two vertical and one horizontal line segments.
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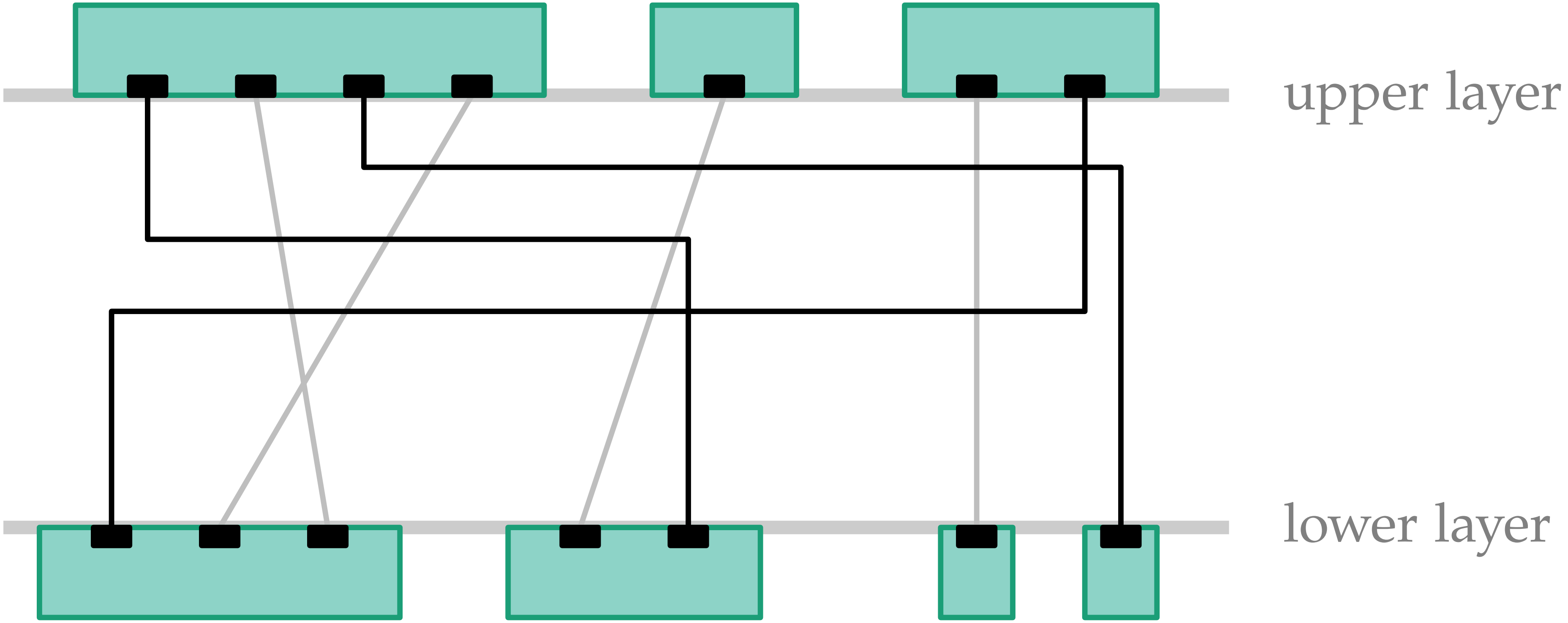
- Use as few horizontal intermediate layers (tracks) as possible.



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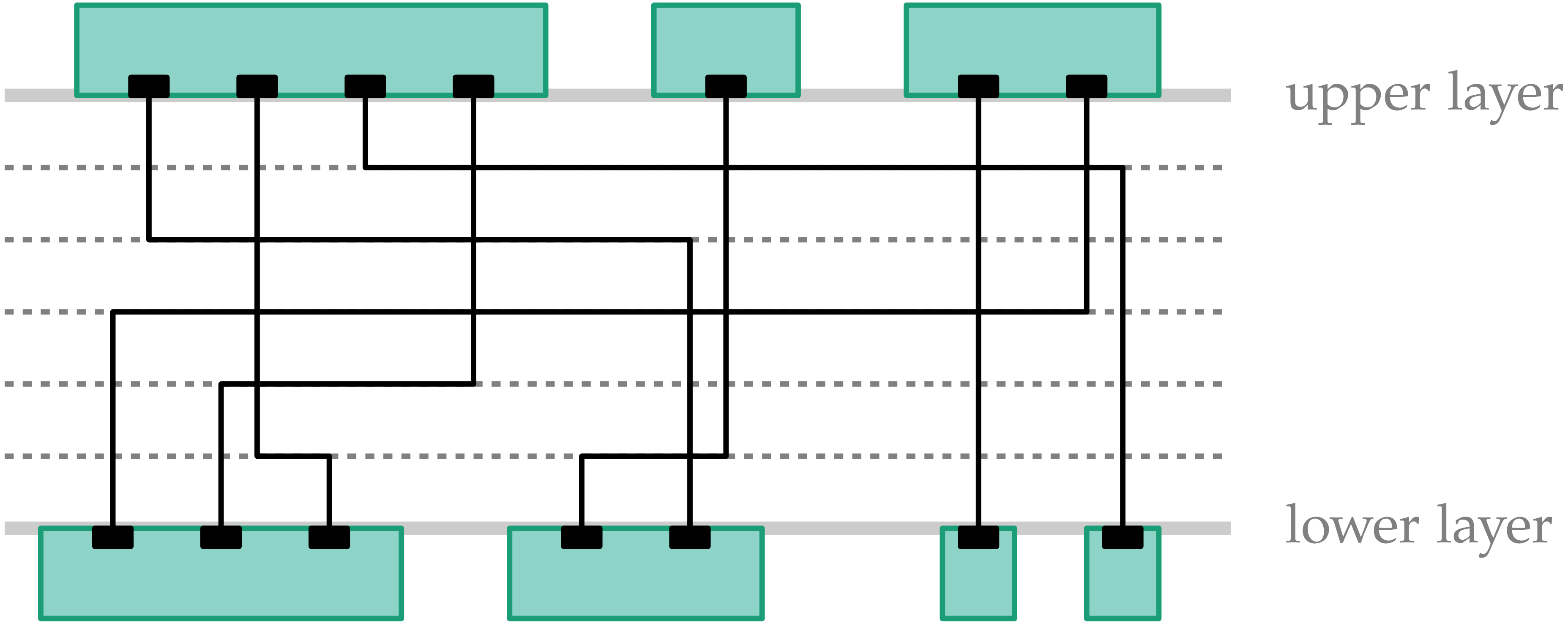

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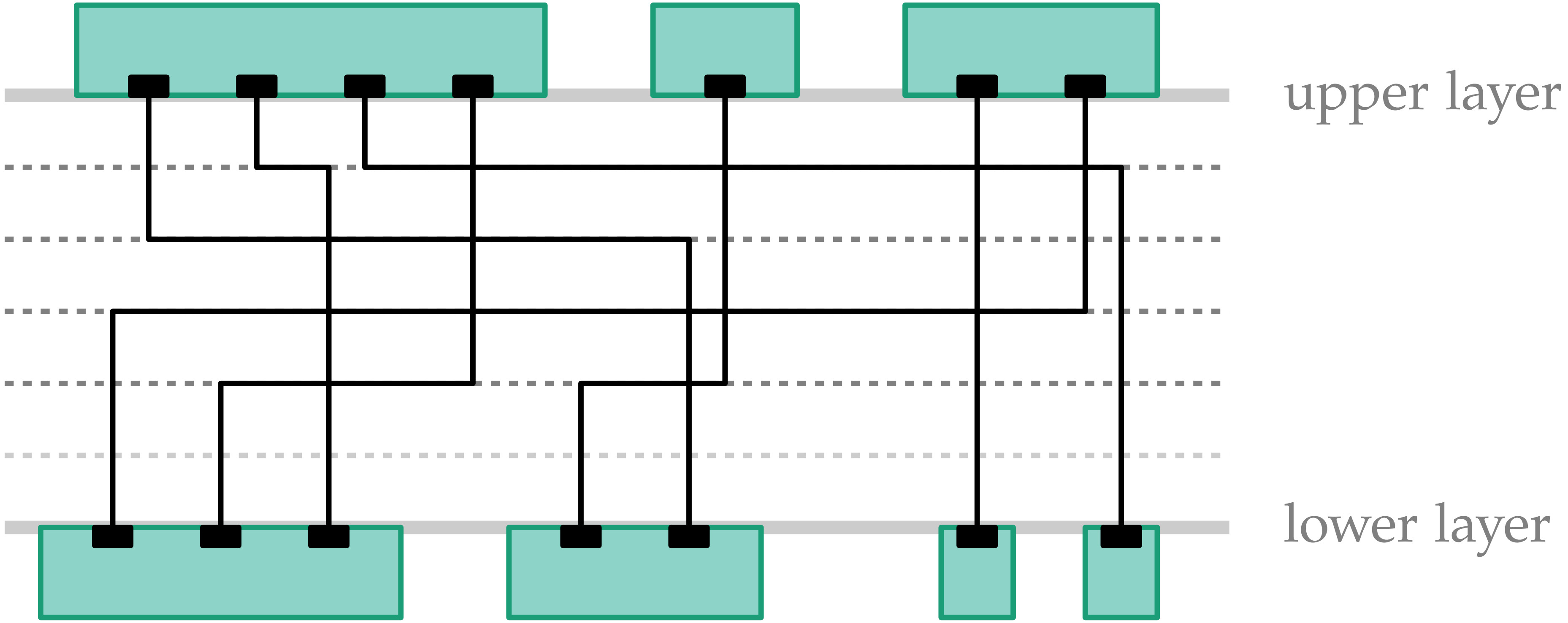

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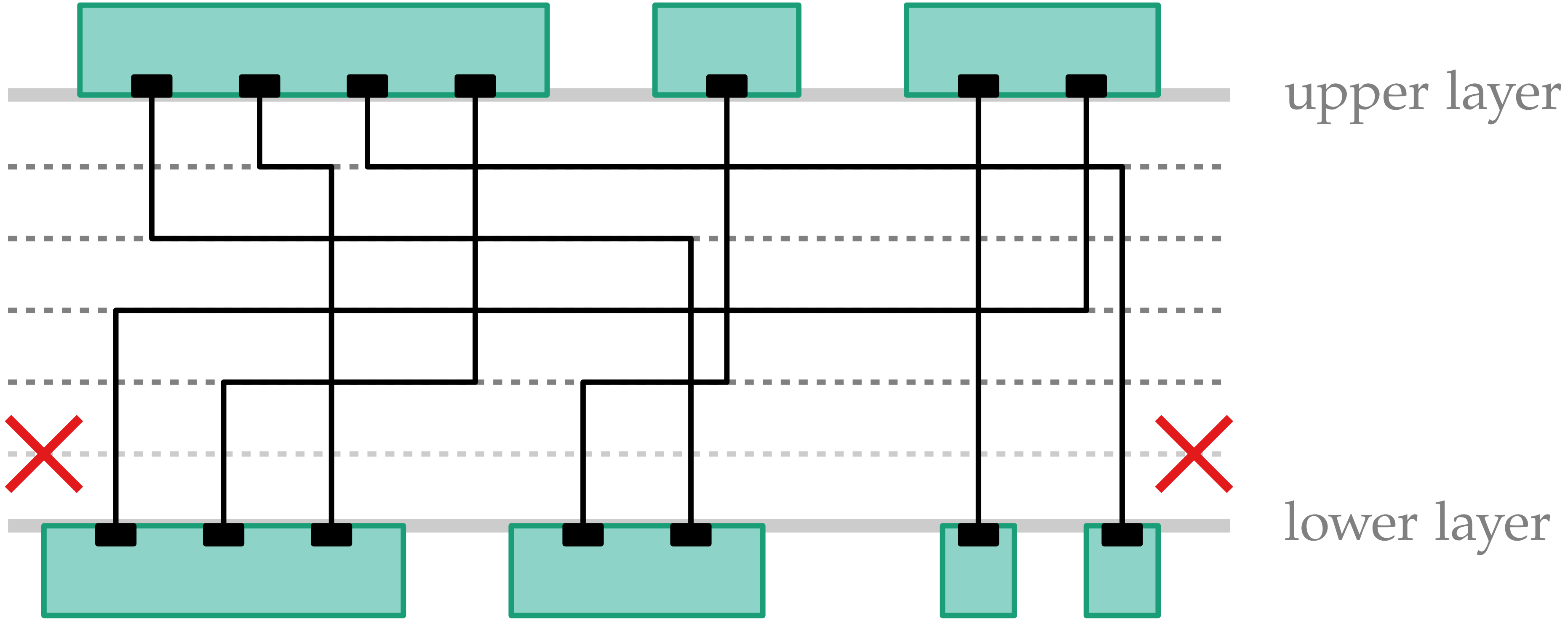
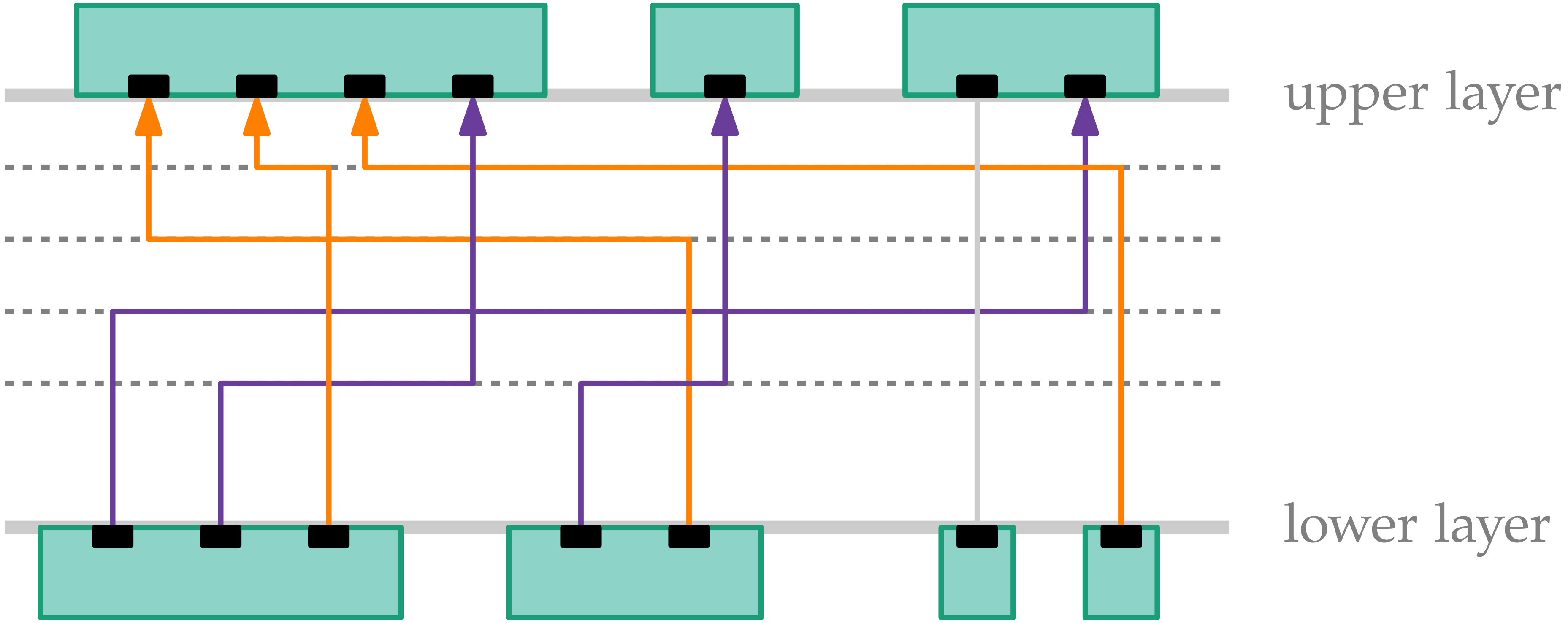


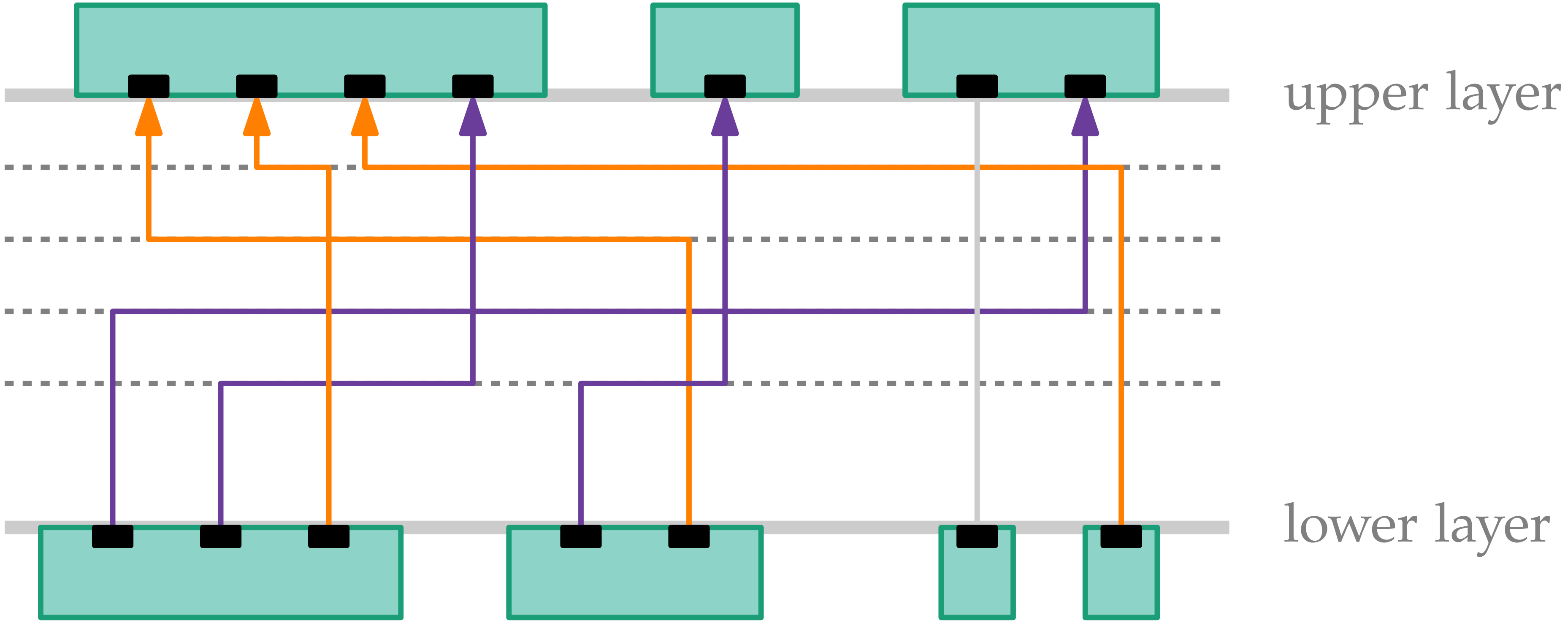
Motivation – Layered Orthogonal Edge Routing

- Distinguish between *left-going* and *right-going* edges.



Motivation – Layered Orthogonal Edge Routing

- Distinguish between *left-going* and *right-going* edges.
- Only edges going in the same direction and overlapping partially in x-dimension can cross twice.



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- Distinguish between *left-going* and *right-going* edges.
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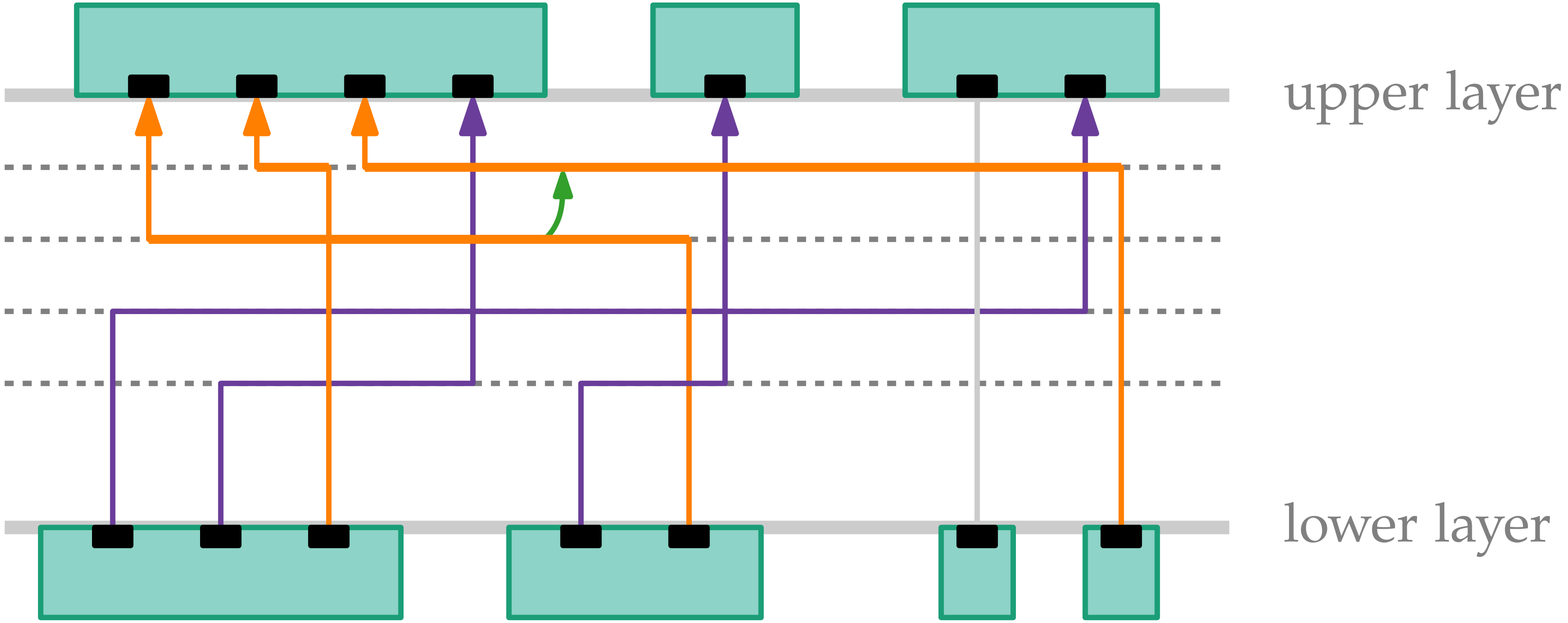
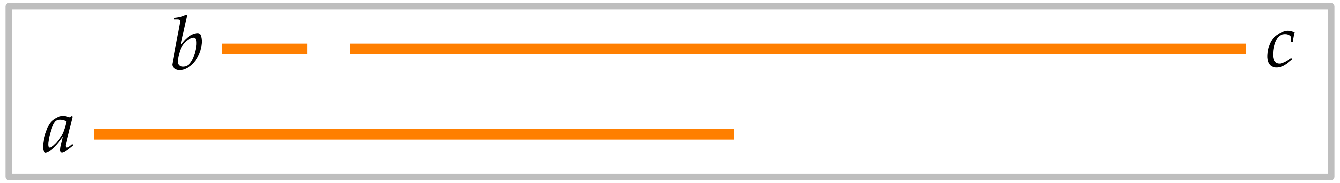
  ⇒ They induce a vertical order for the horizontal middle segments.



Definition – Directional Interval Graphs

Interval representation: set of intervals



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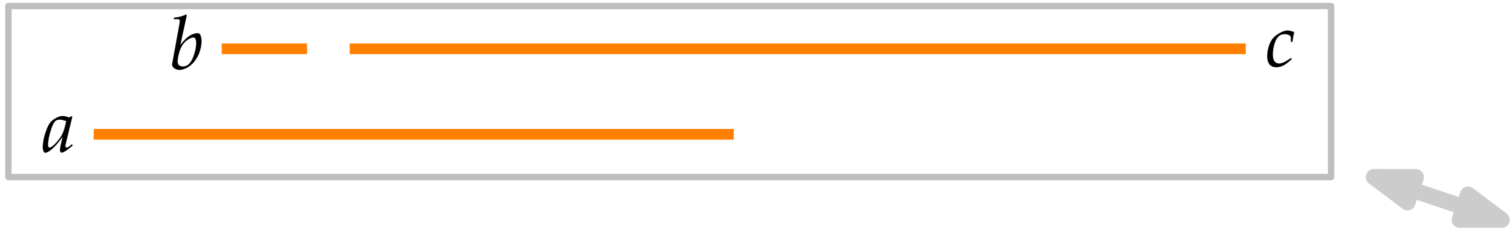
Directional interval graph:



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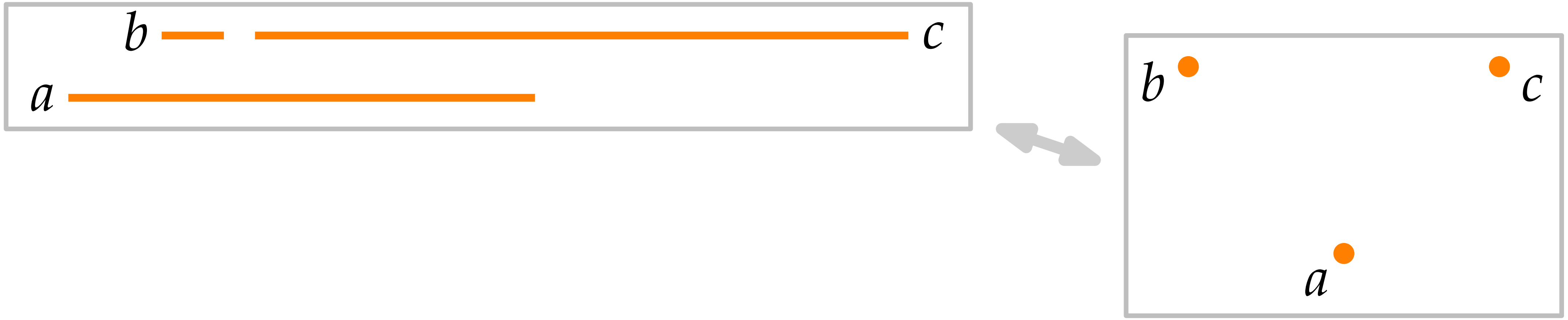

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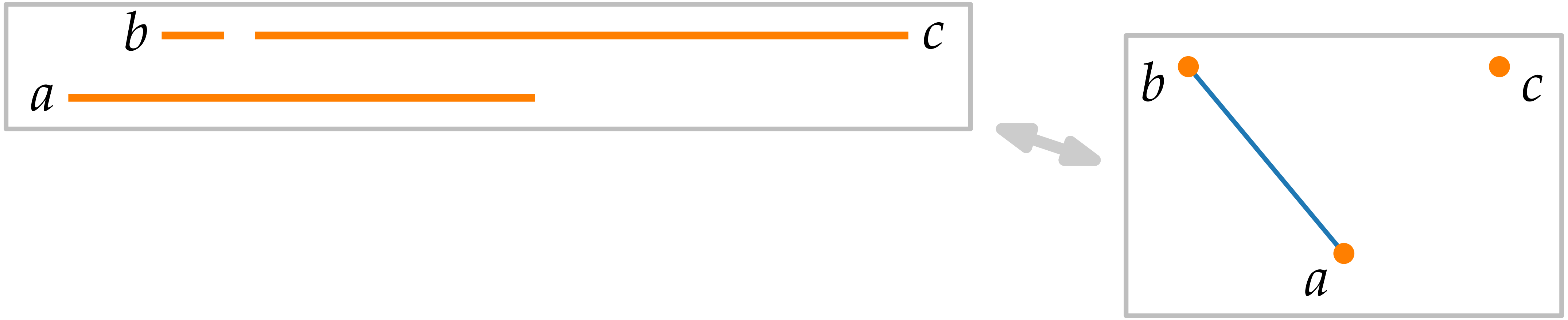
- undirected edge if one interval contains another



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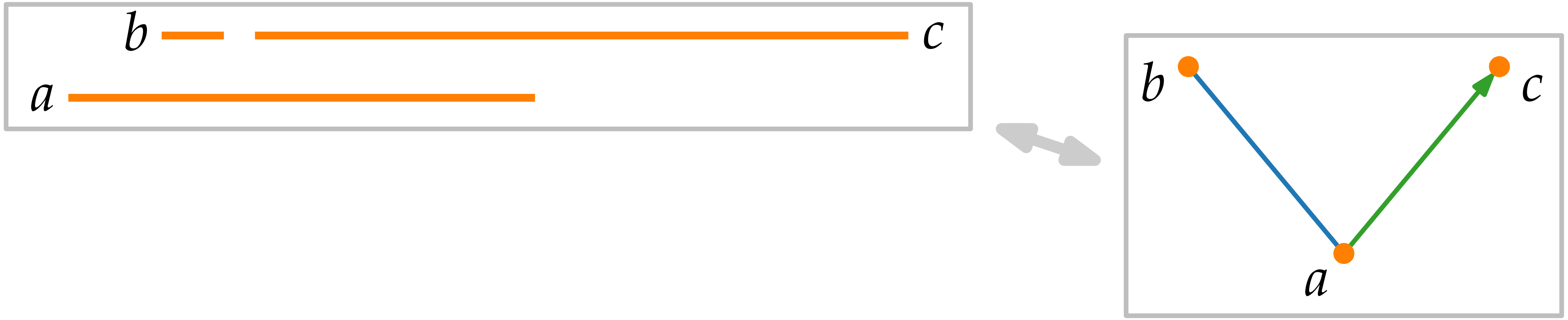

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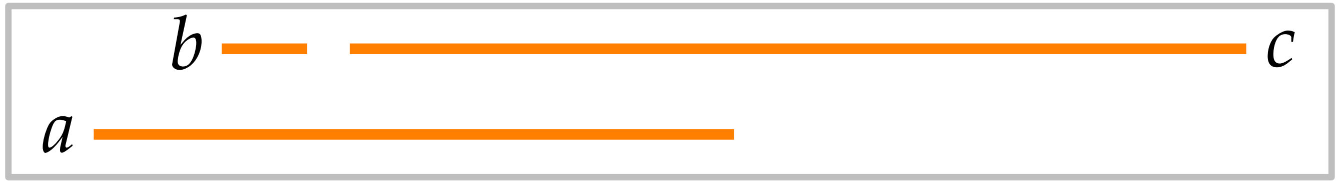

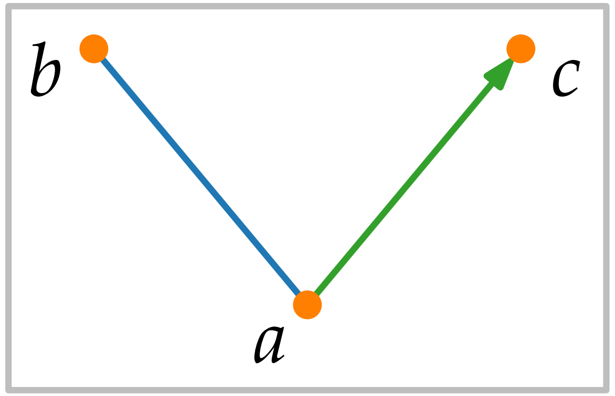

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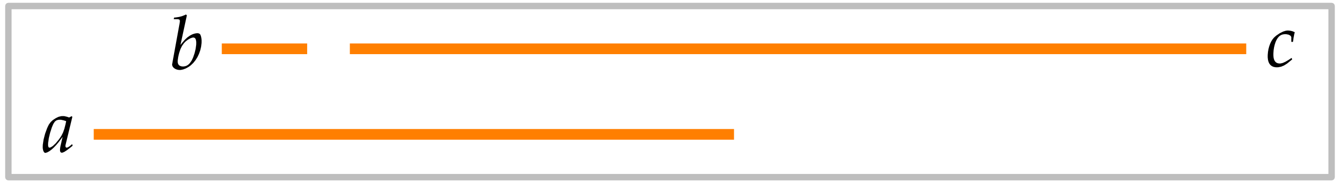

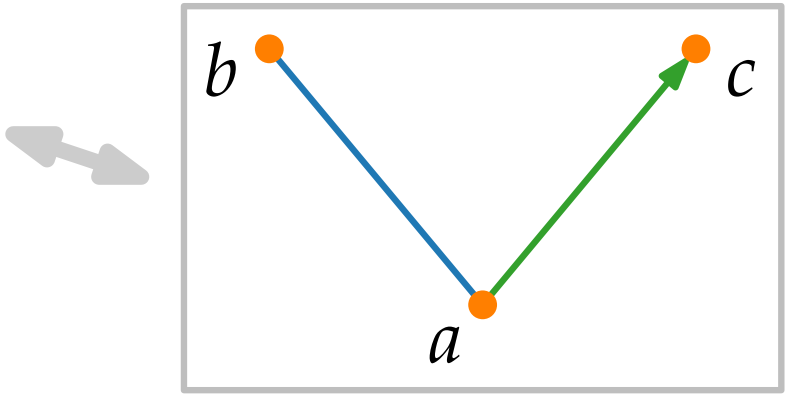

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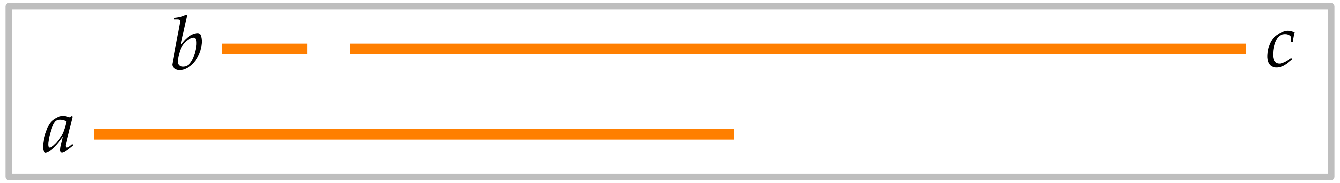

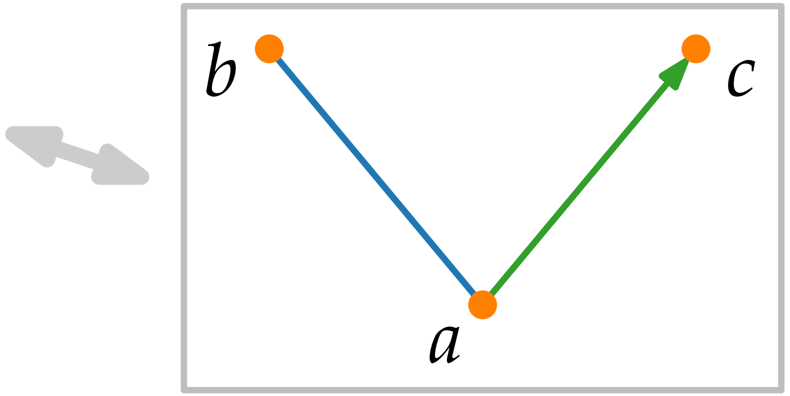

Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

Given a graph G, find a coloring $c\colon V(G) \to \mathbb{N}$ s.t.

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
- ⋆ directed edge uv: $c(u) < c(v)$,
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[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

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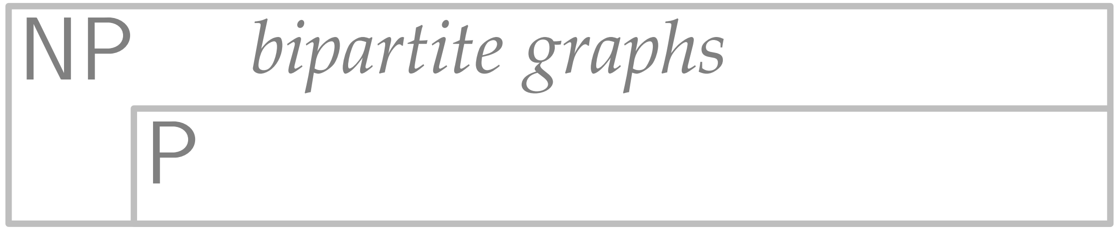

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bipartite graphs

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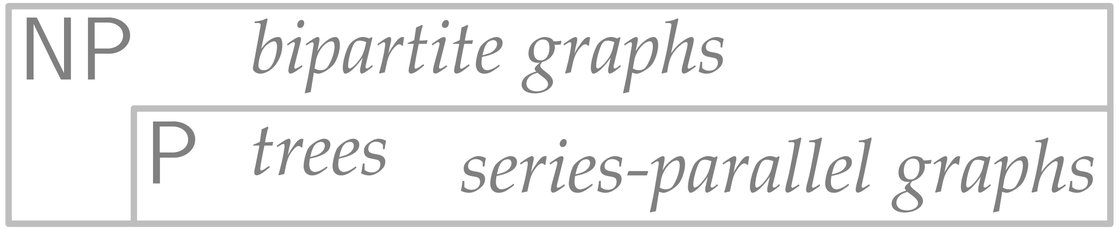

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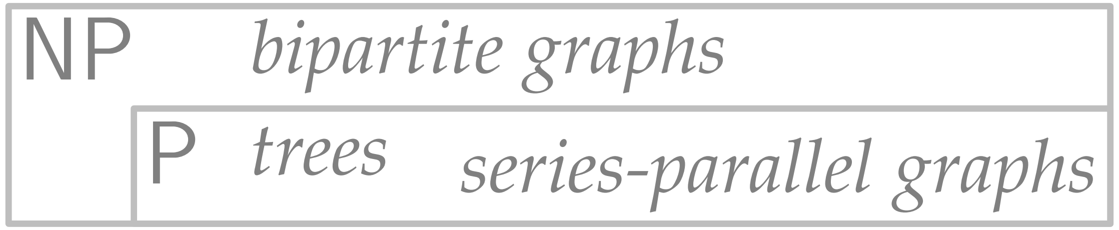

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Interval graphs (no directed edges):

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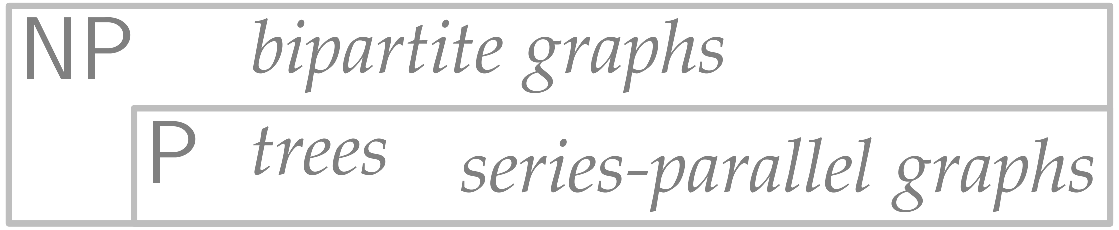

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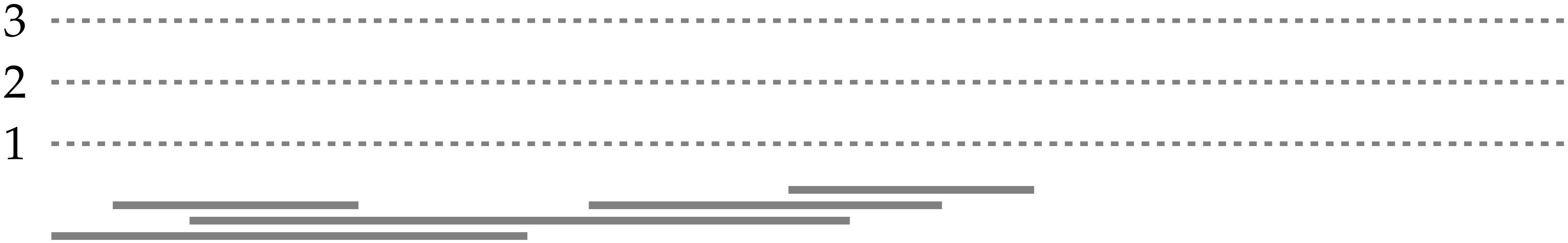

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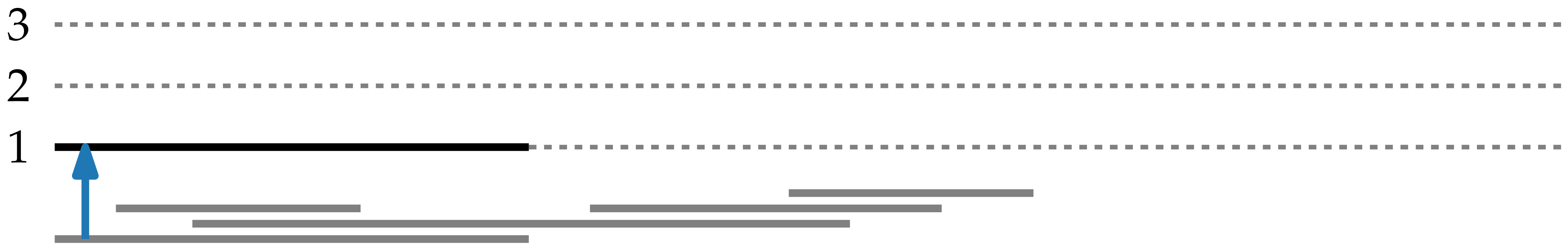

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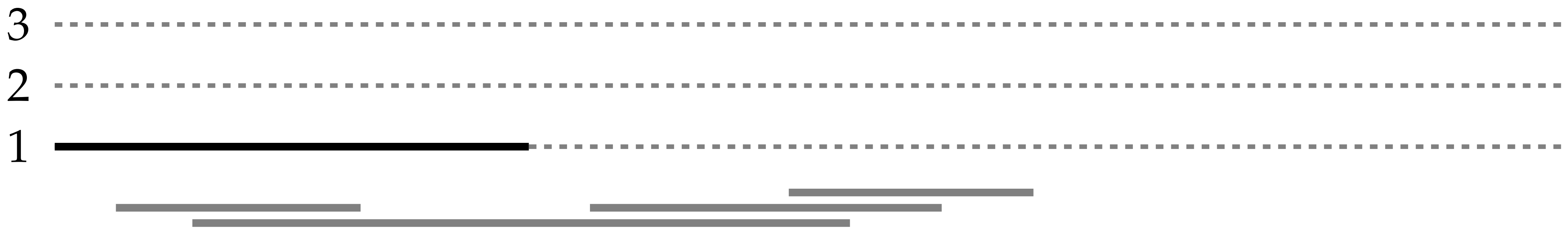

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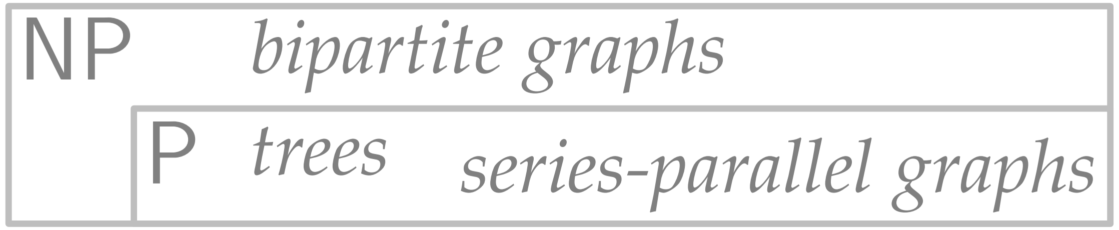

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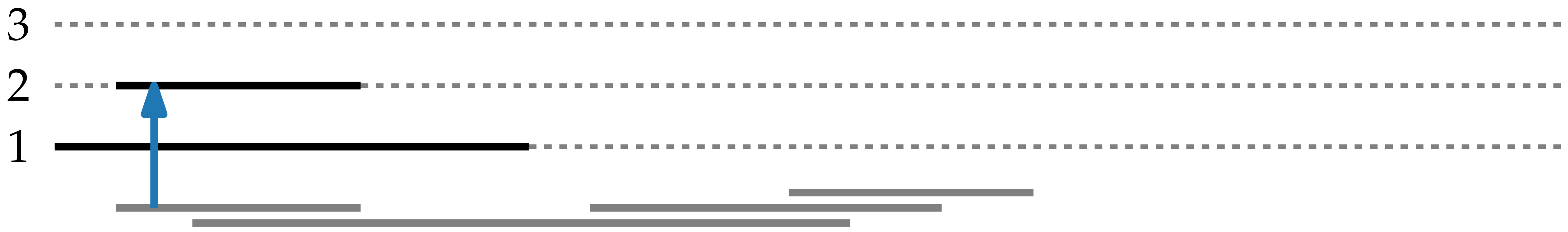

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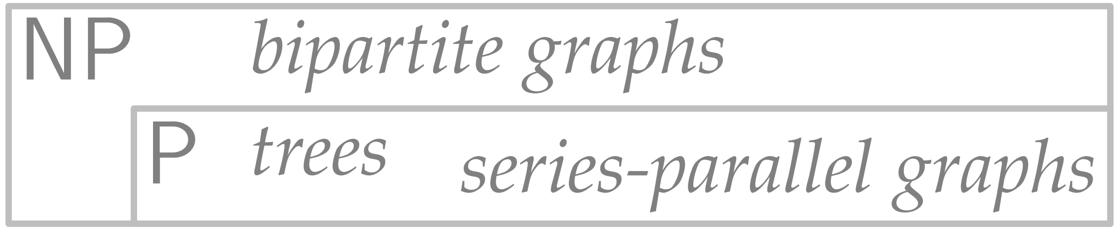
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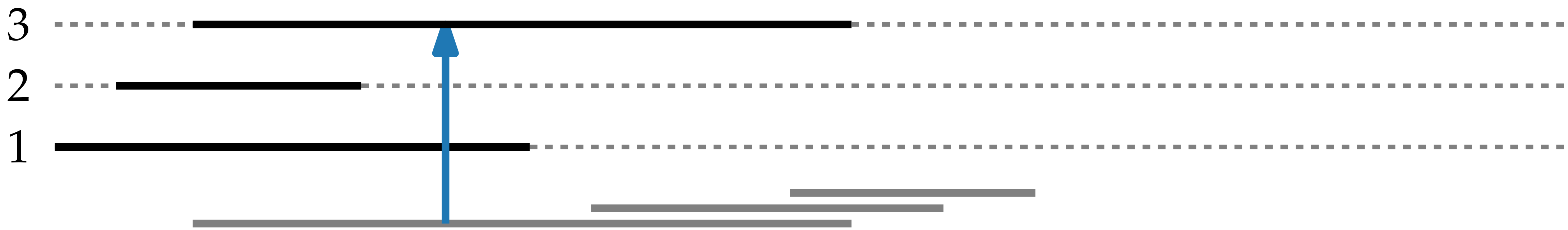

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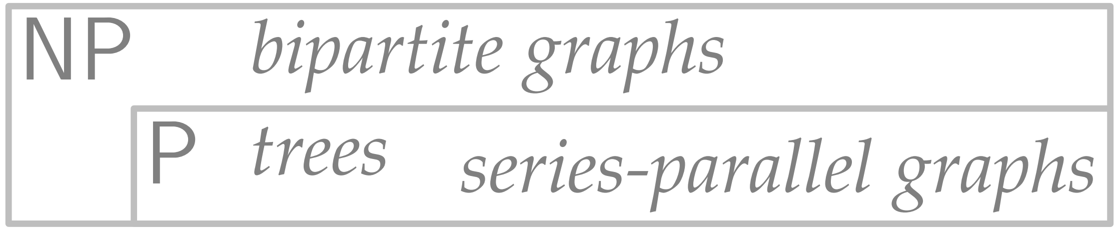

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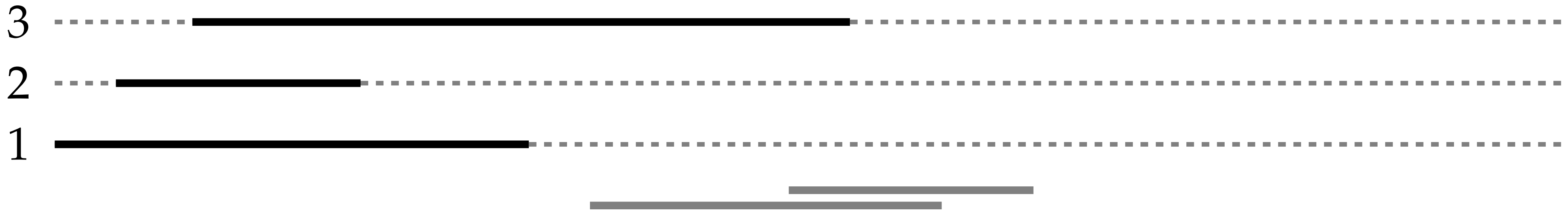

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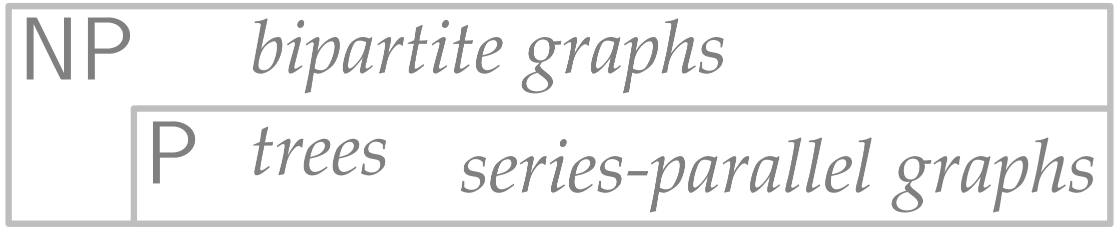

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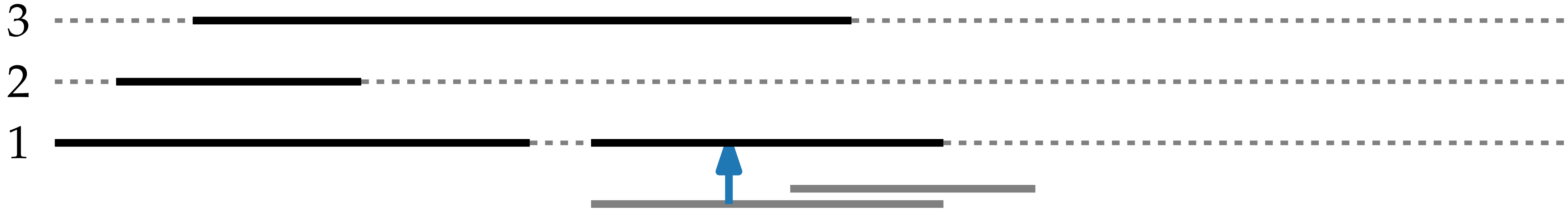

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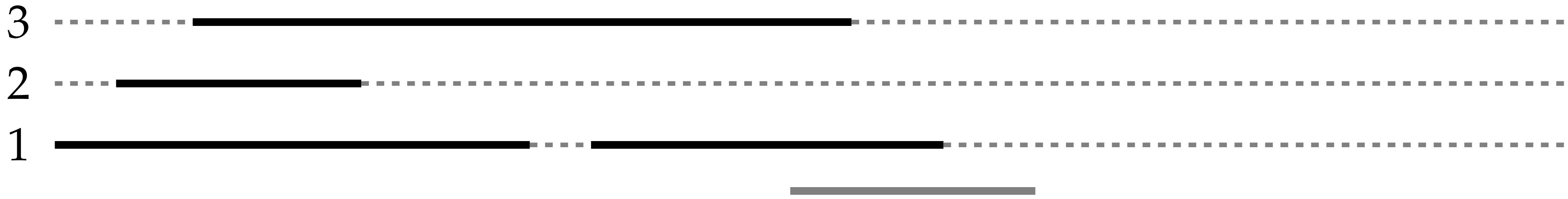

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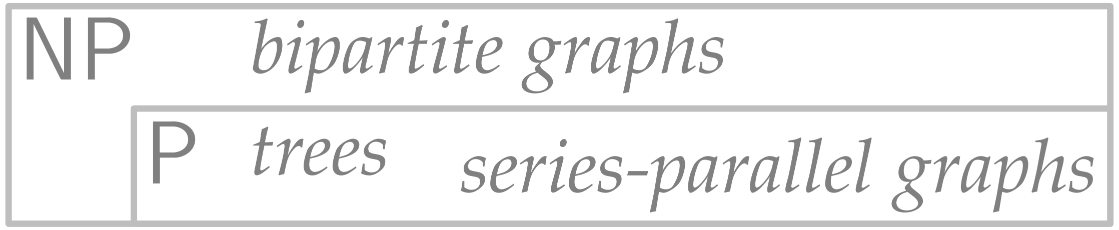

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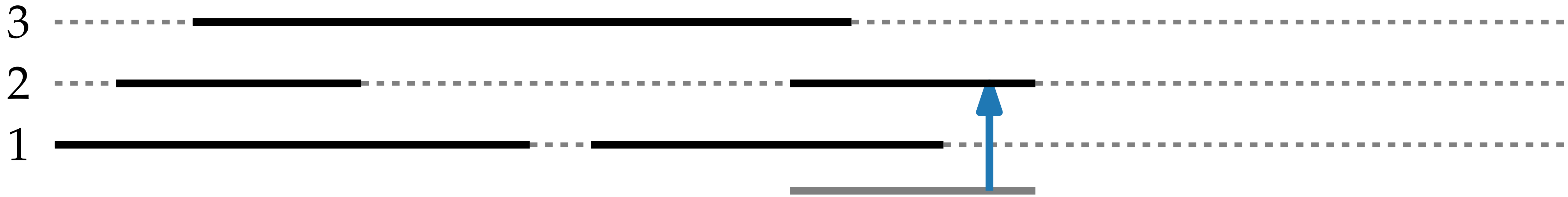

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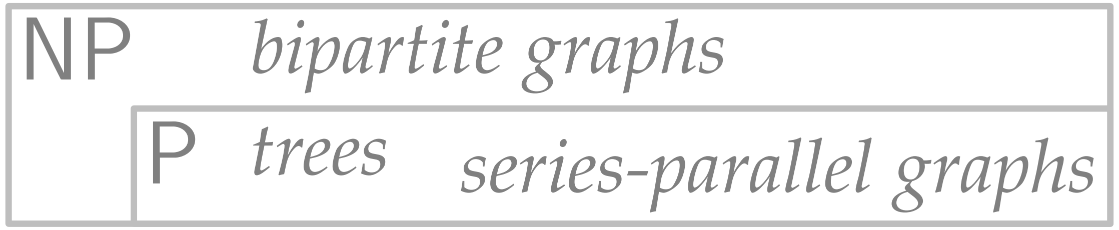

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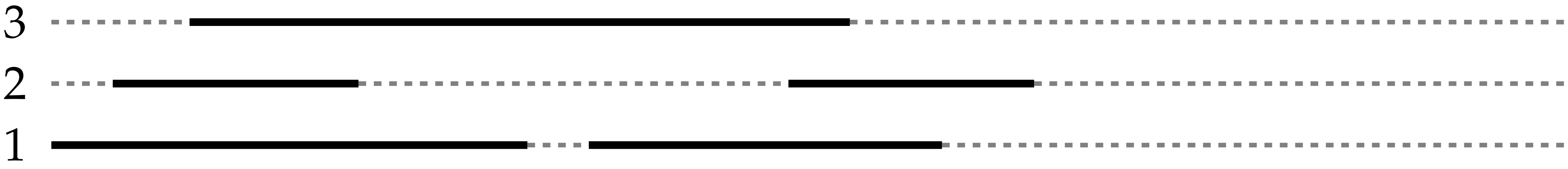

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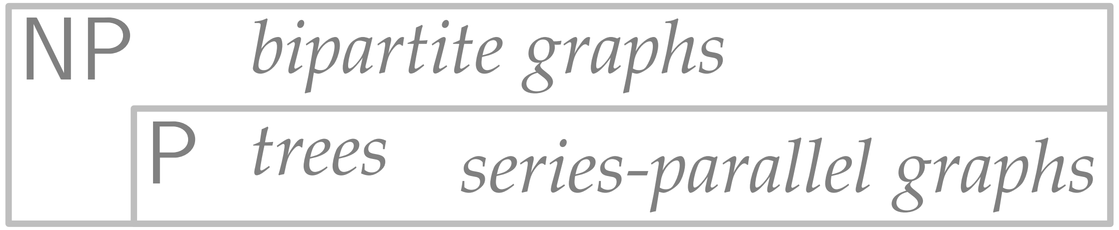
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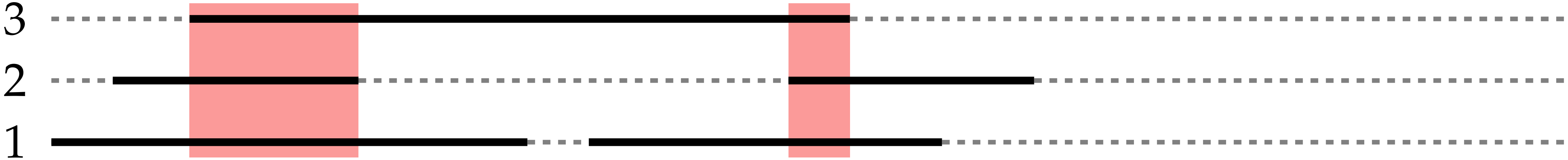

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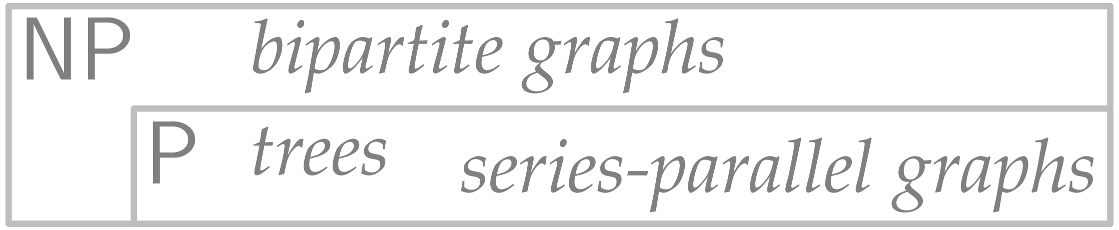
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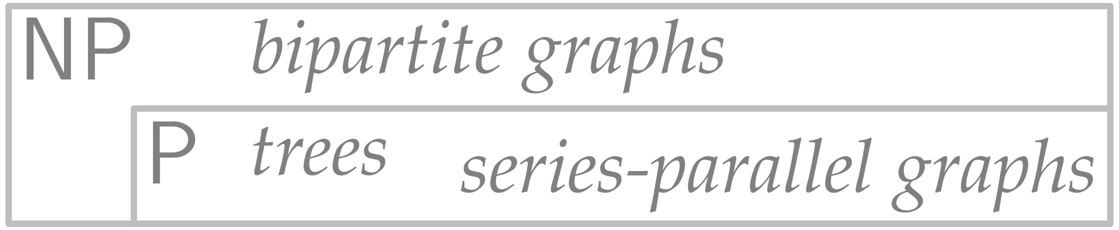

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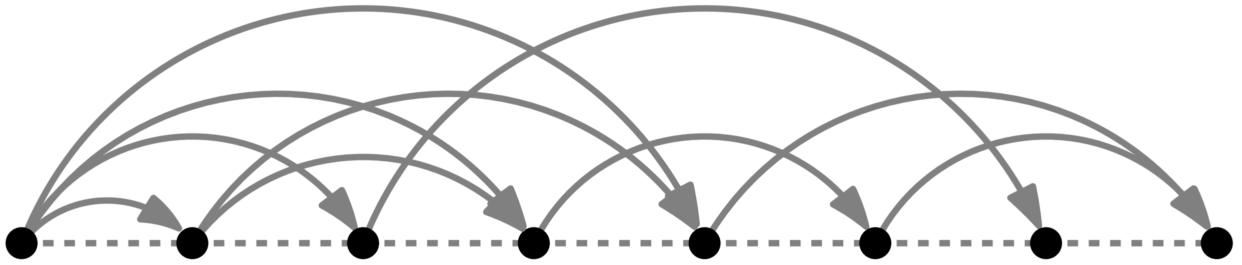

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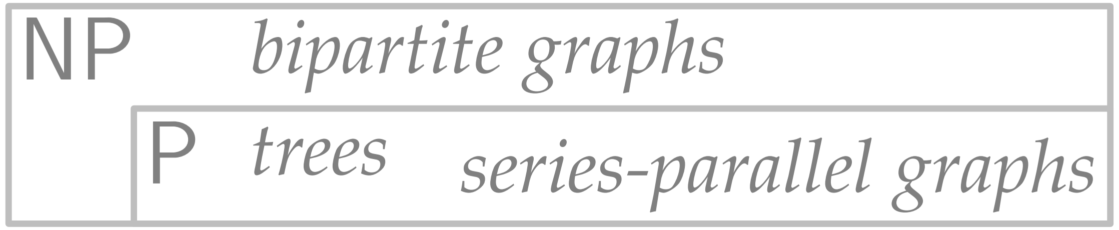

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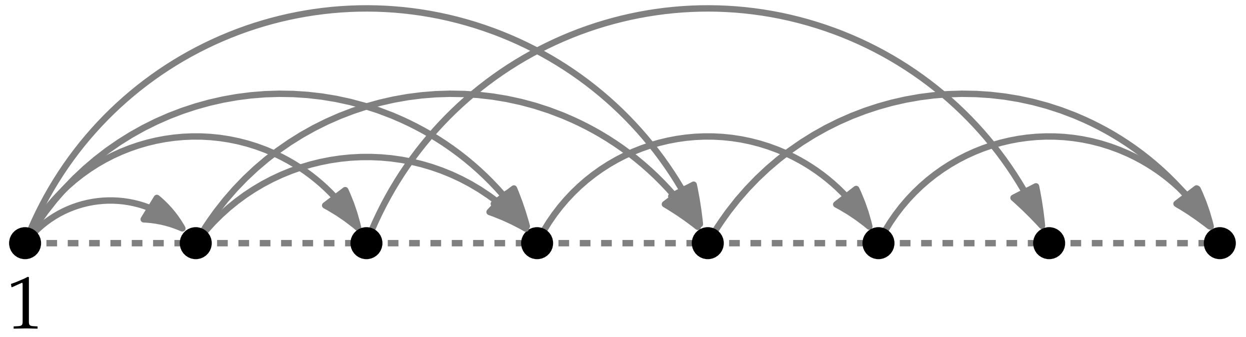

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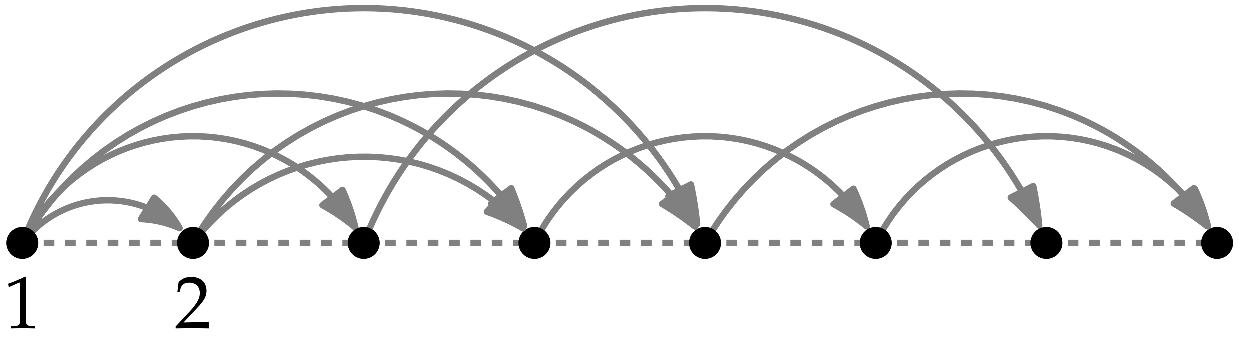

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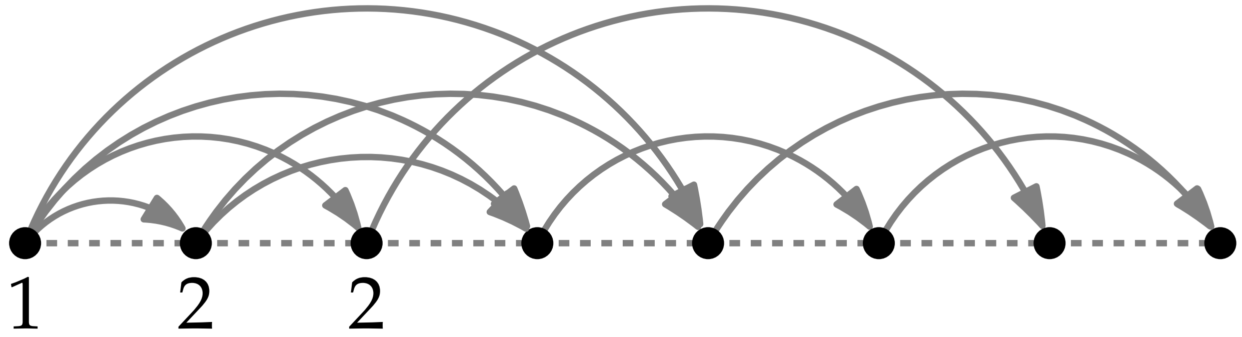

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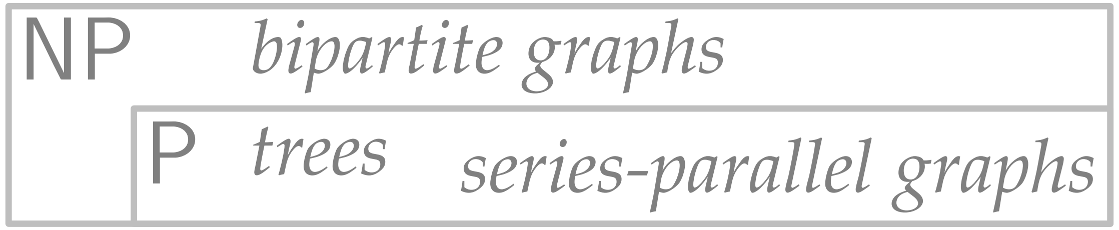

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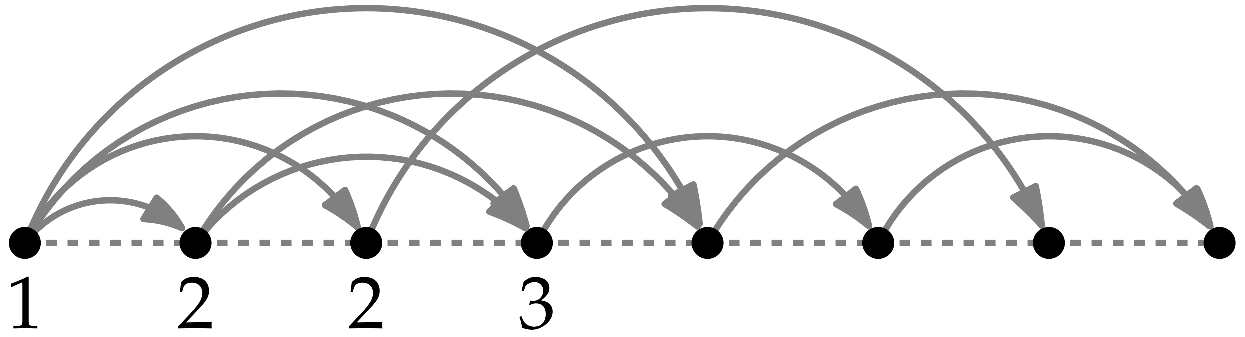

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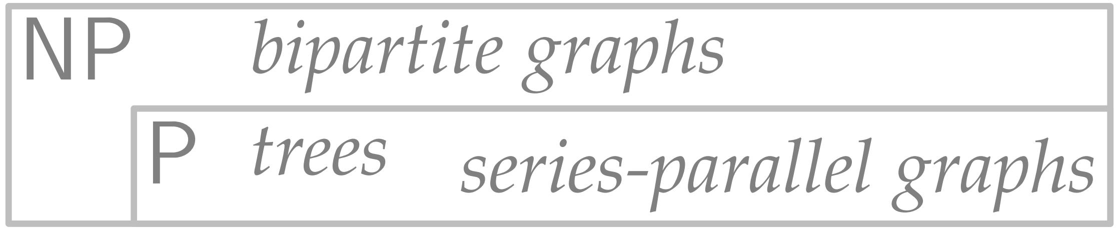
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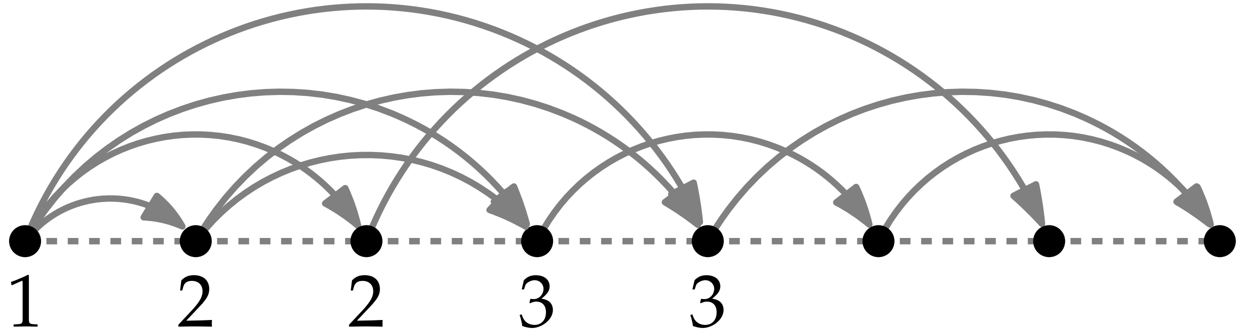

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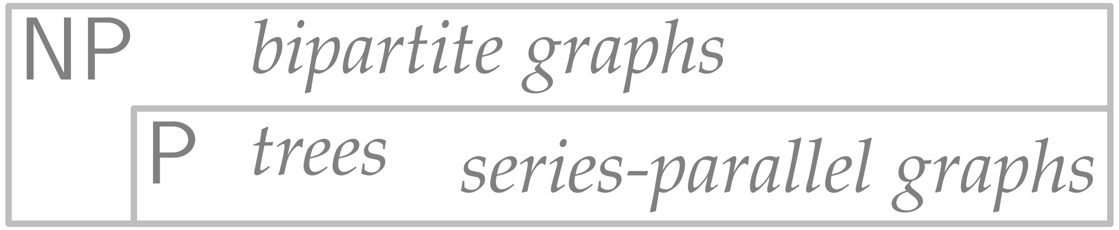
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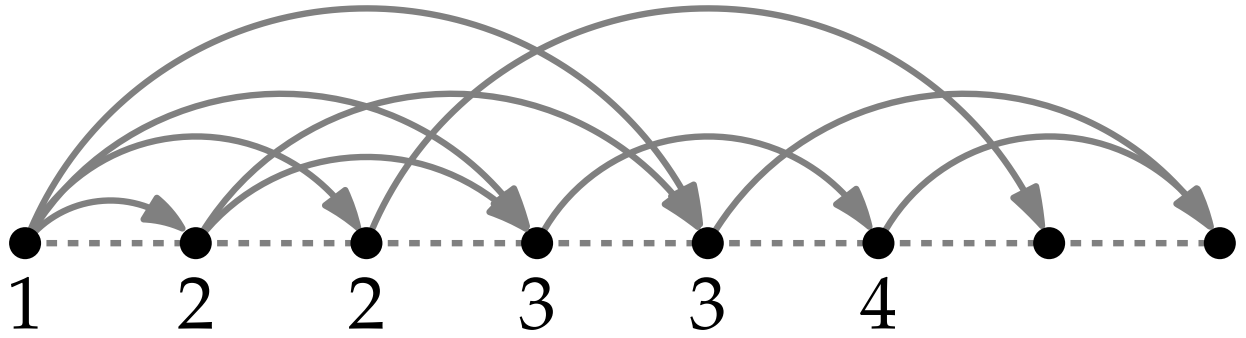

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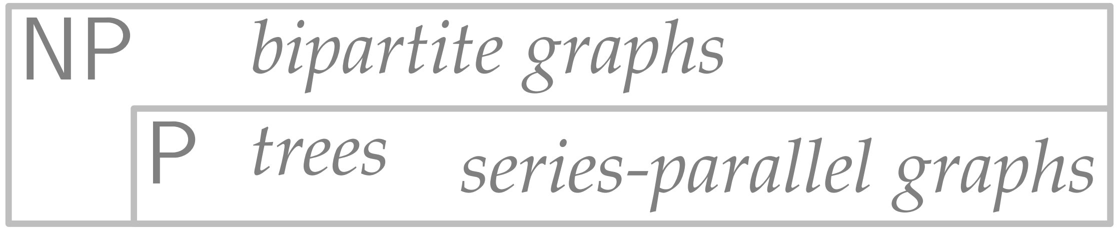

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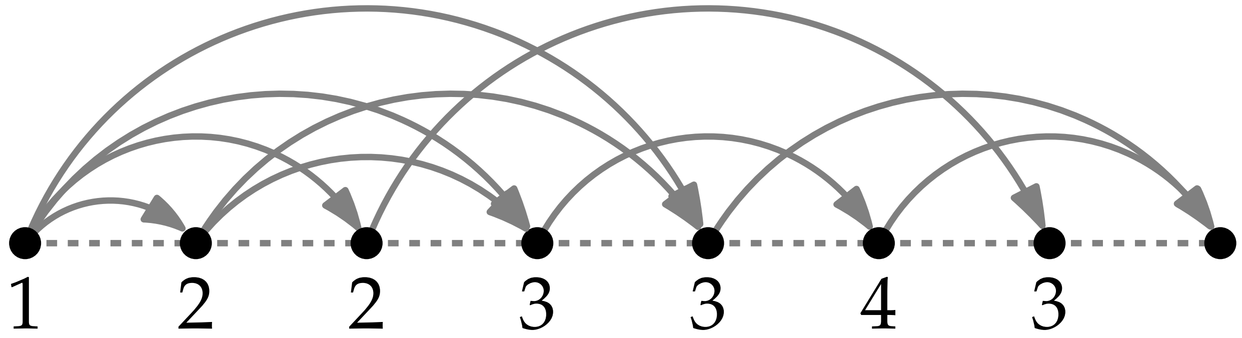

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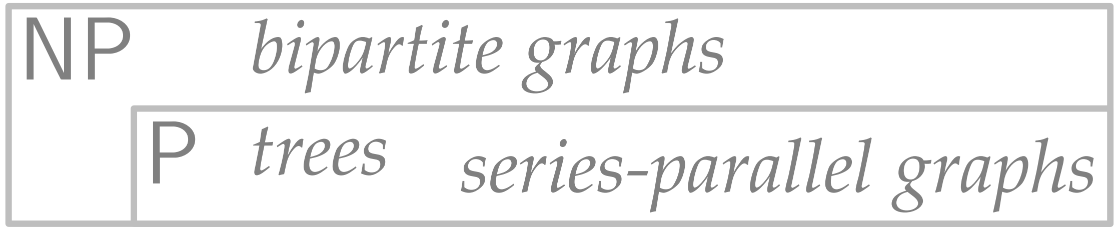
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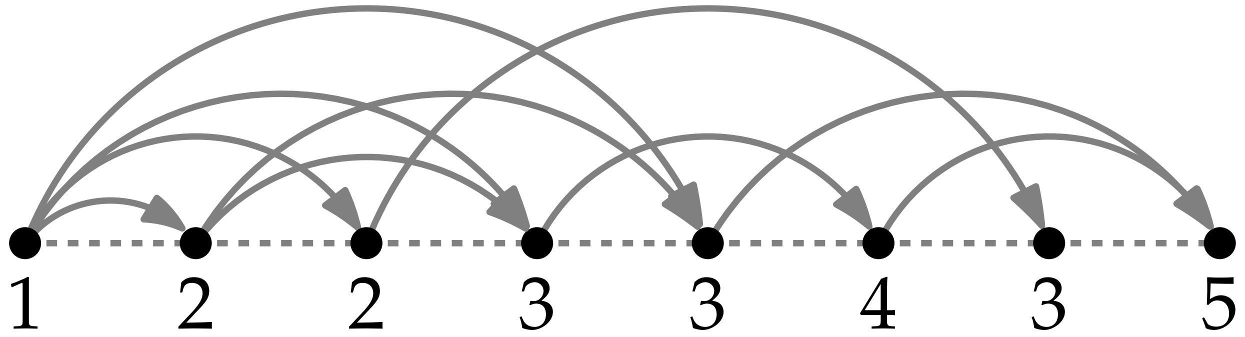

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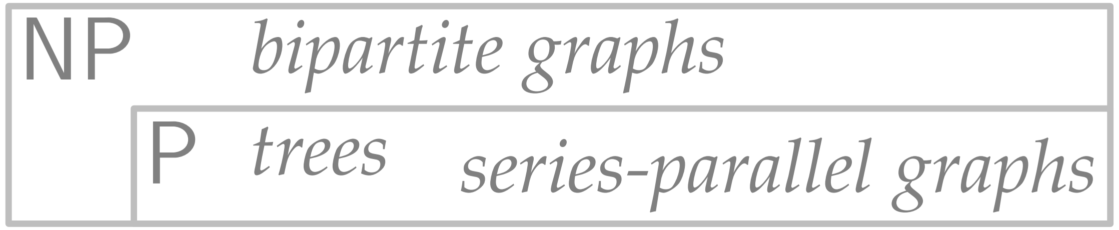
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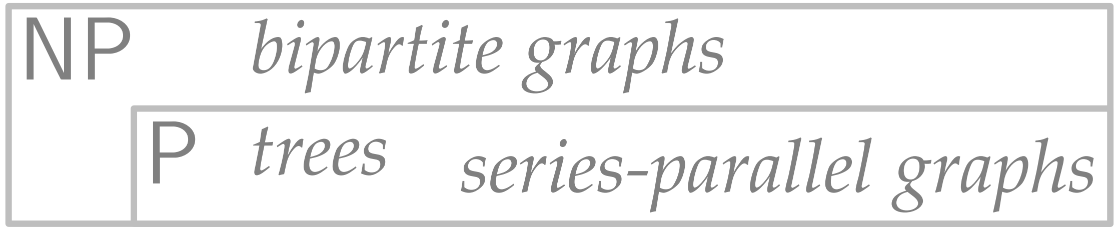
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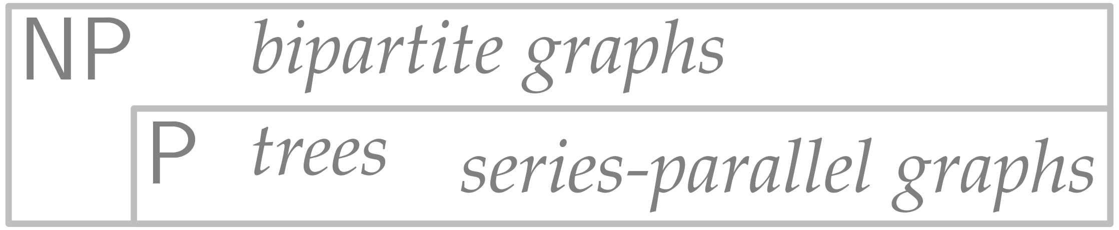
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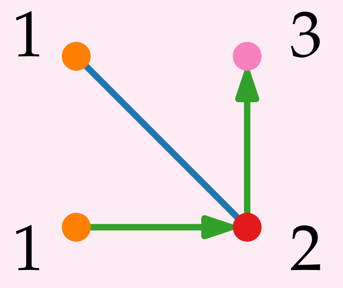

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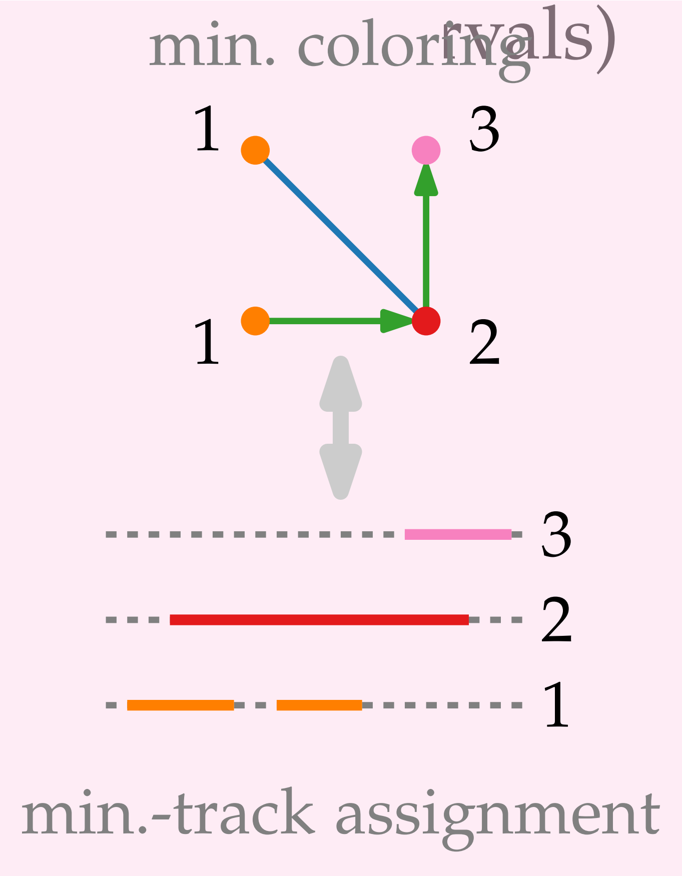

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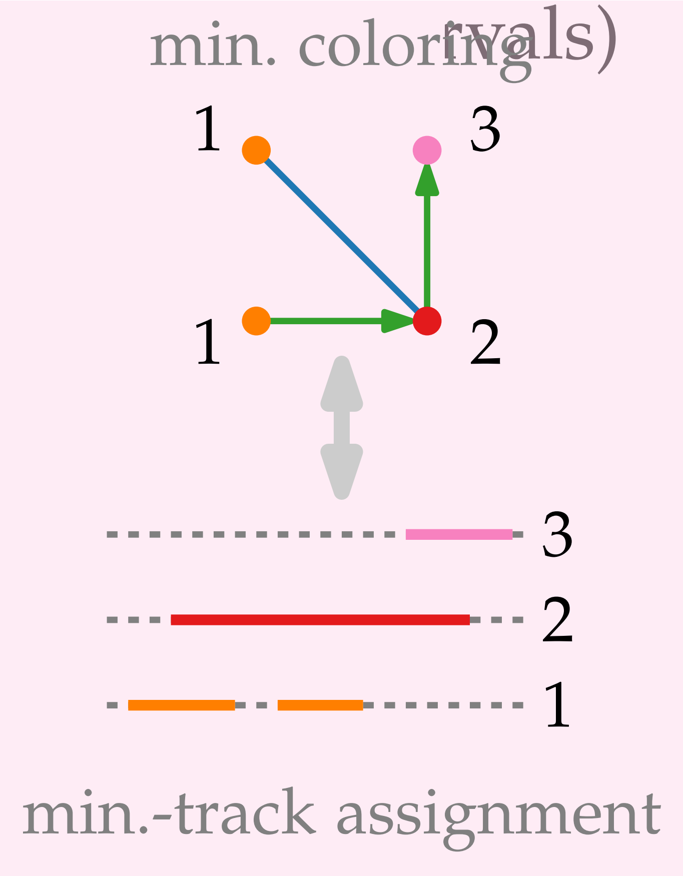

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Coloring Mixed Graphs

NP *bipartite graphs*

P *trees* *series-parallel graphs*

Given a graph G, find a coloring $c\colon V(G) \to \mathbb{N}$ s.t.

[Sotskov, Tanaev ’76; Hansen, Kuplinsky, de Werra ’97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
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Interval graphs (no directed edges):

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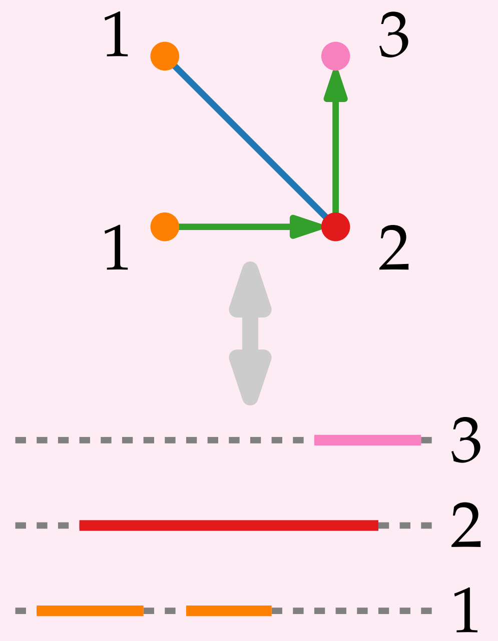

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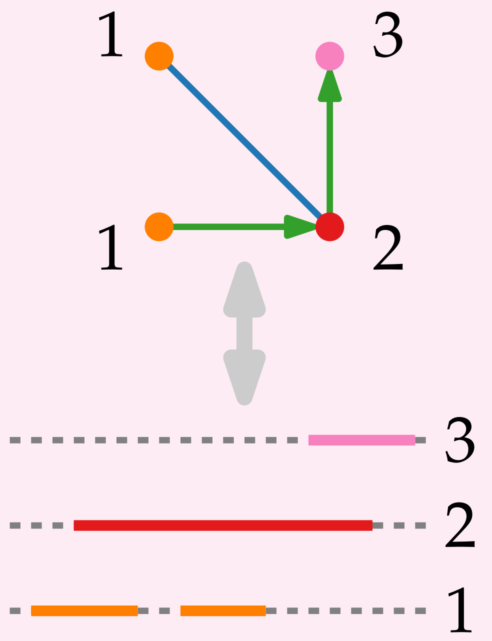

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agenda for this talk

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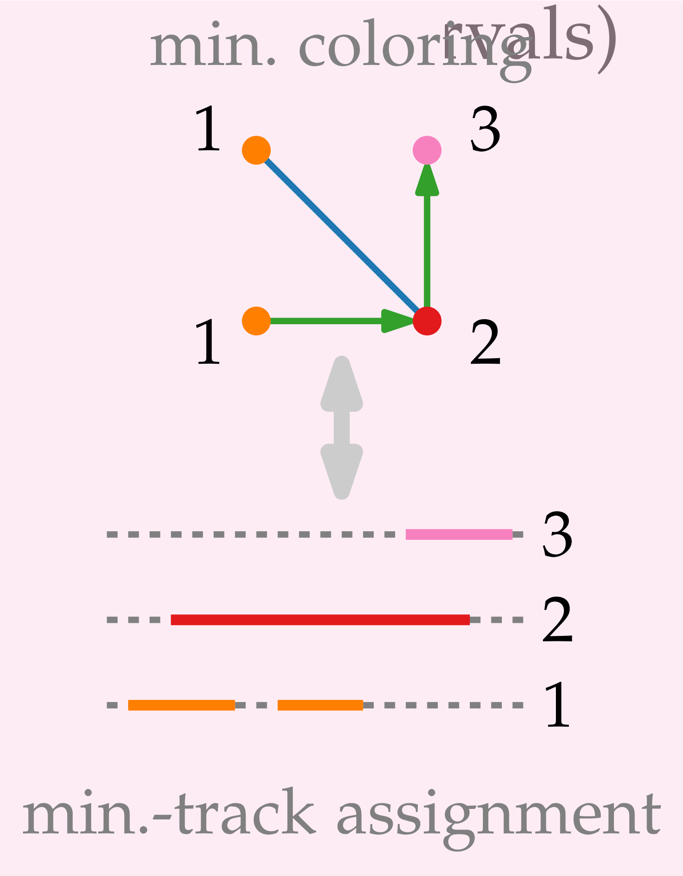

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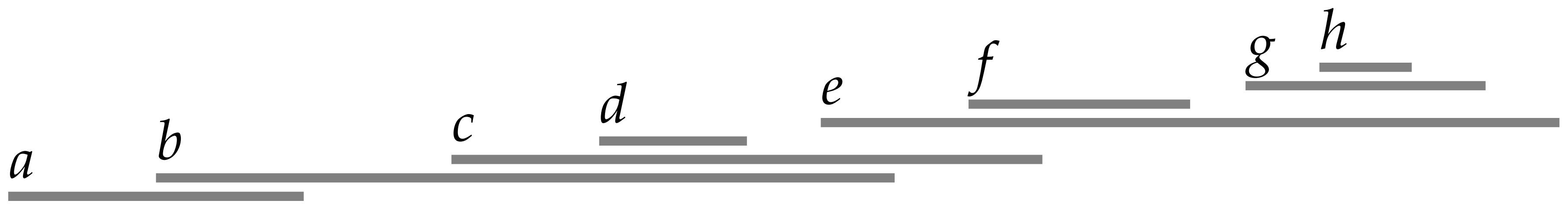

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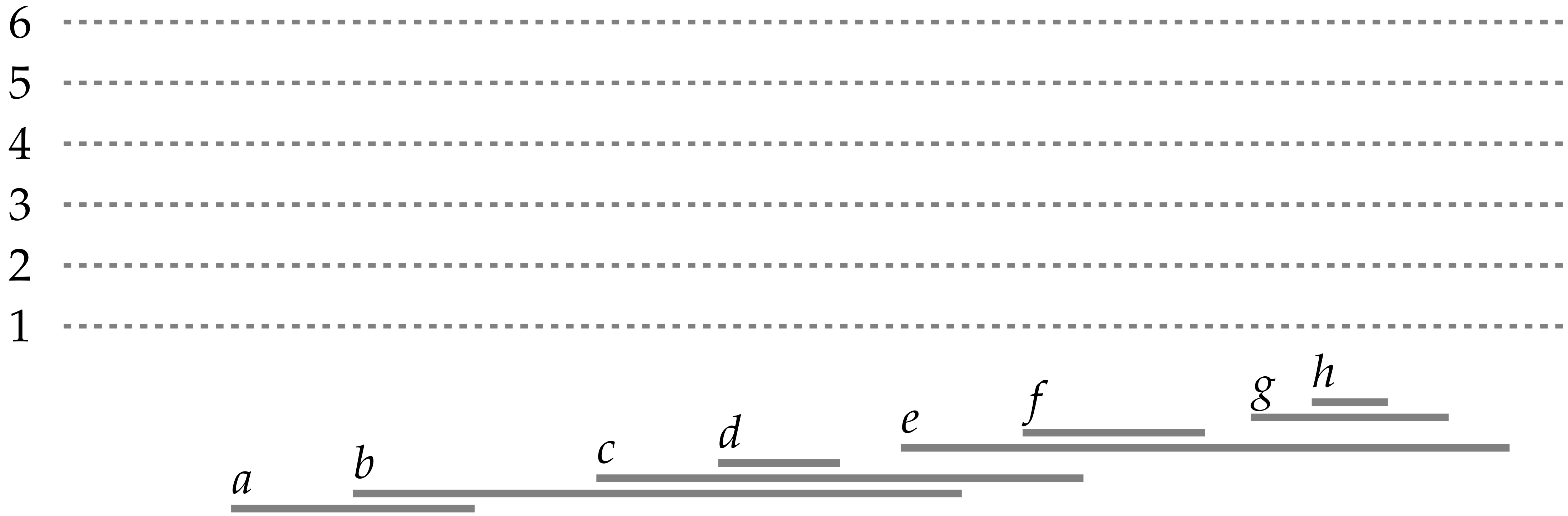

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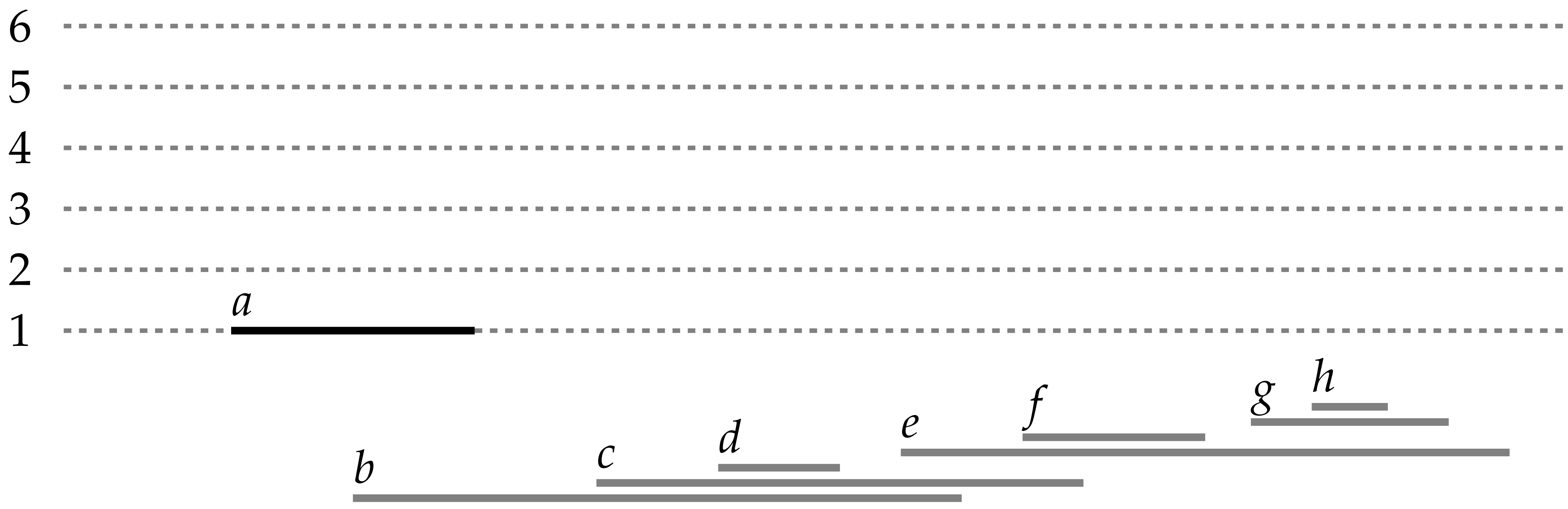

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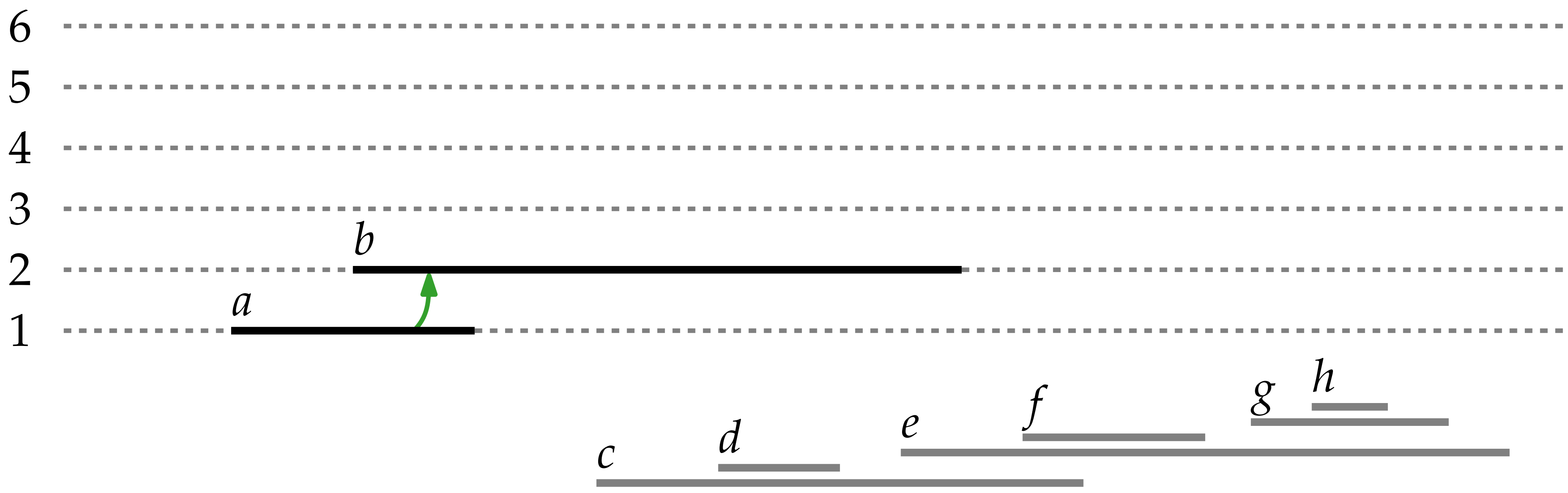

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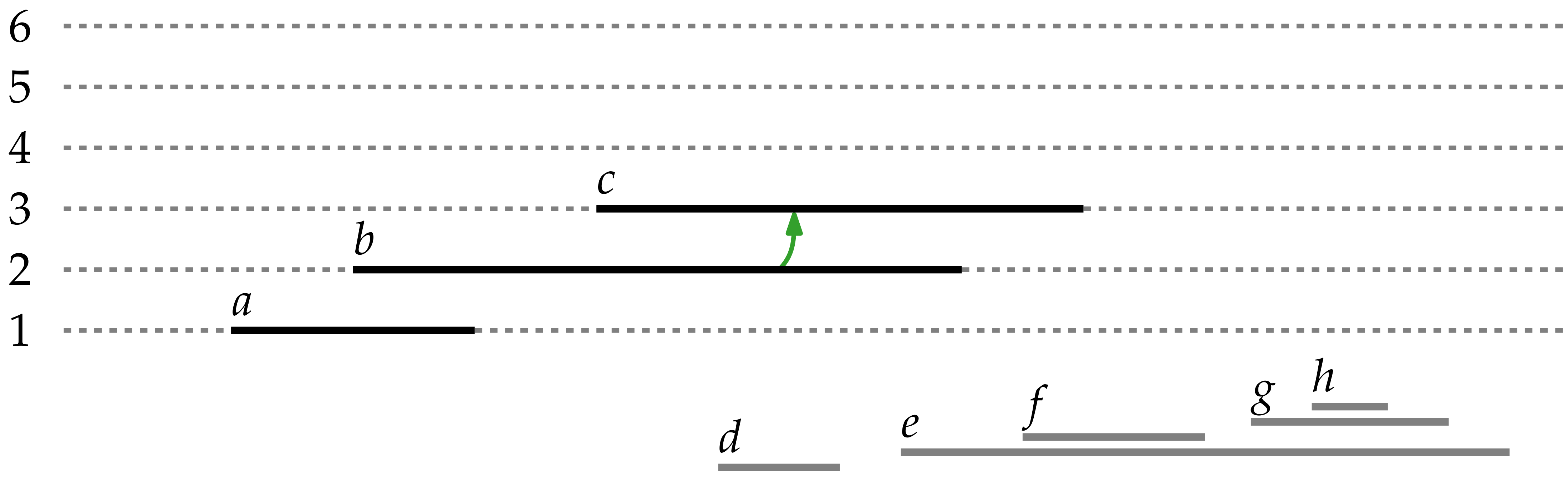

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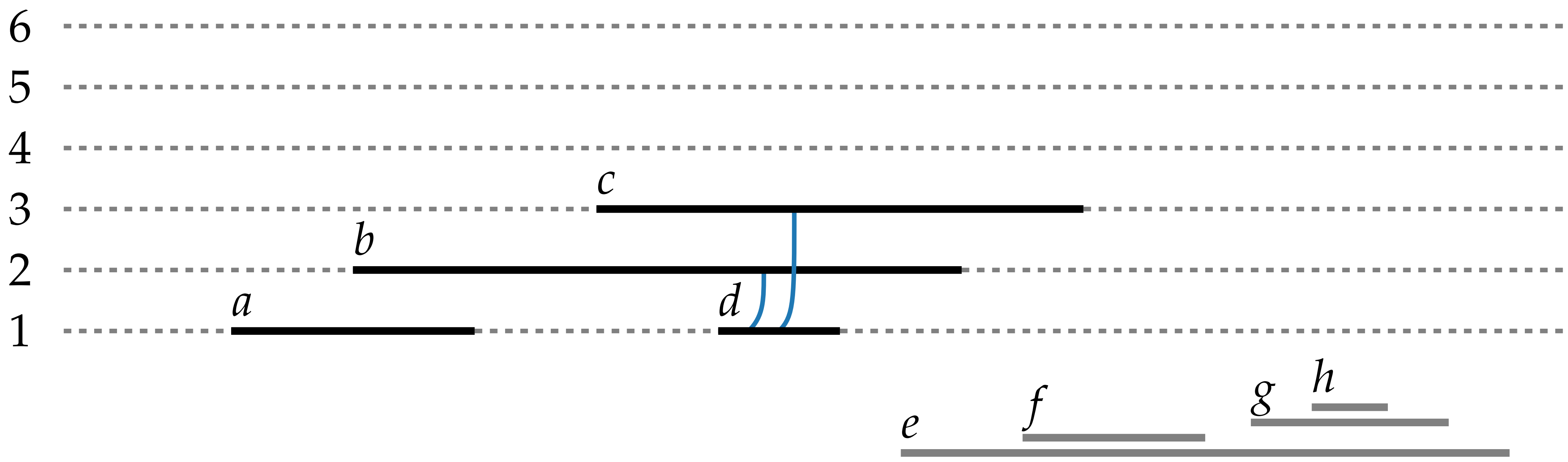

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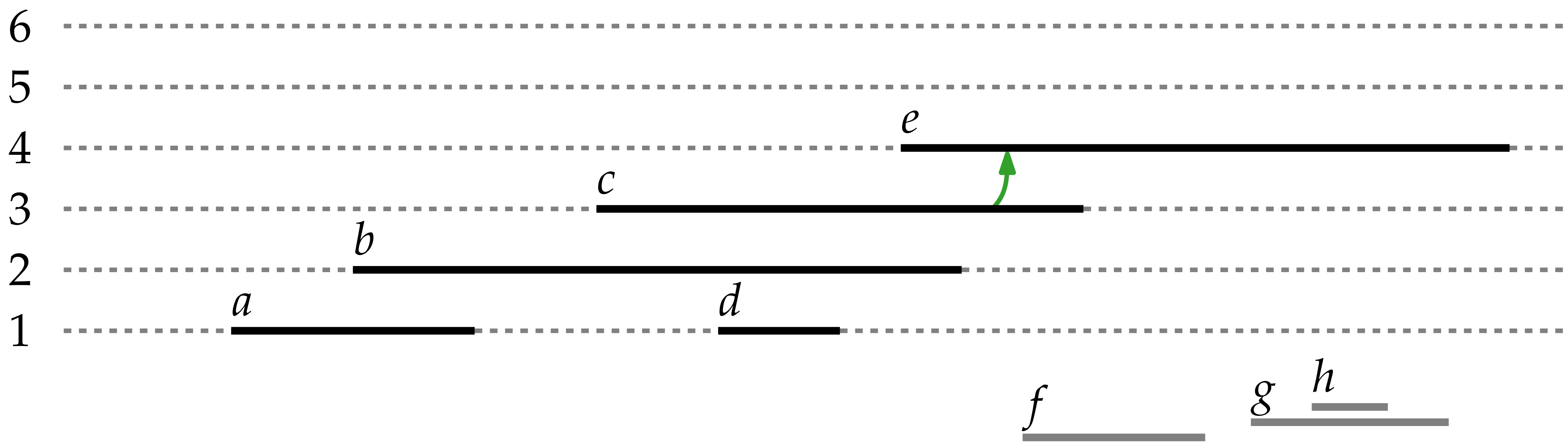

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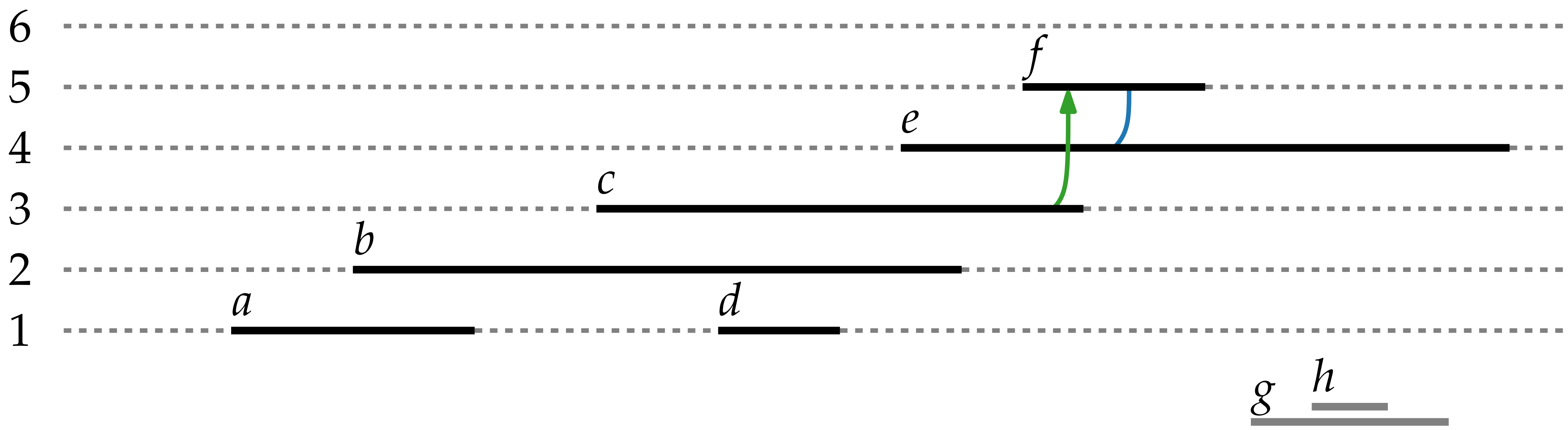

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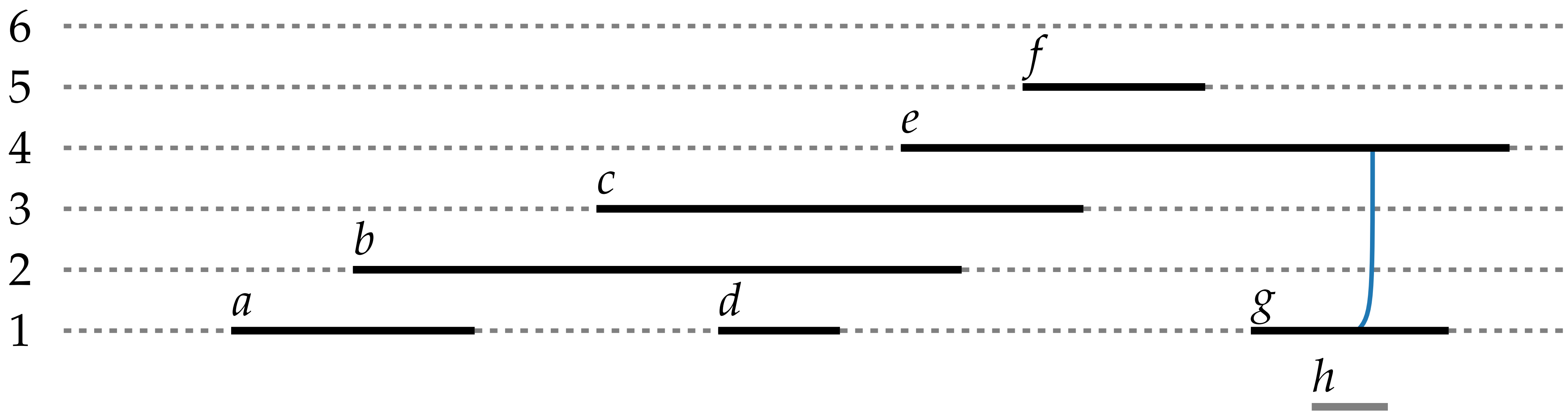

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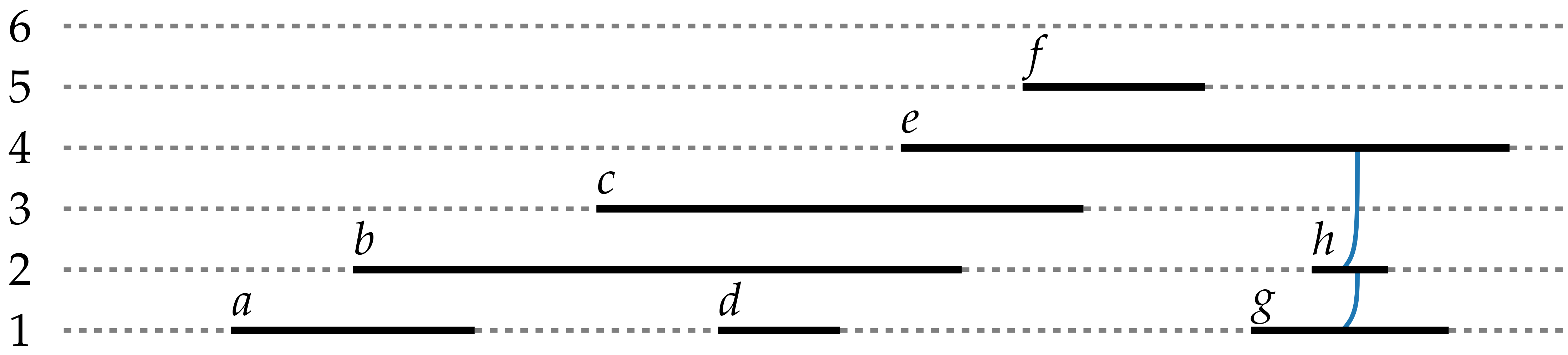

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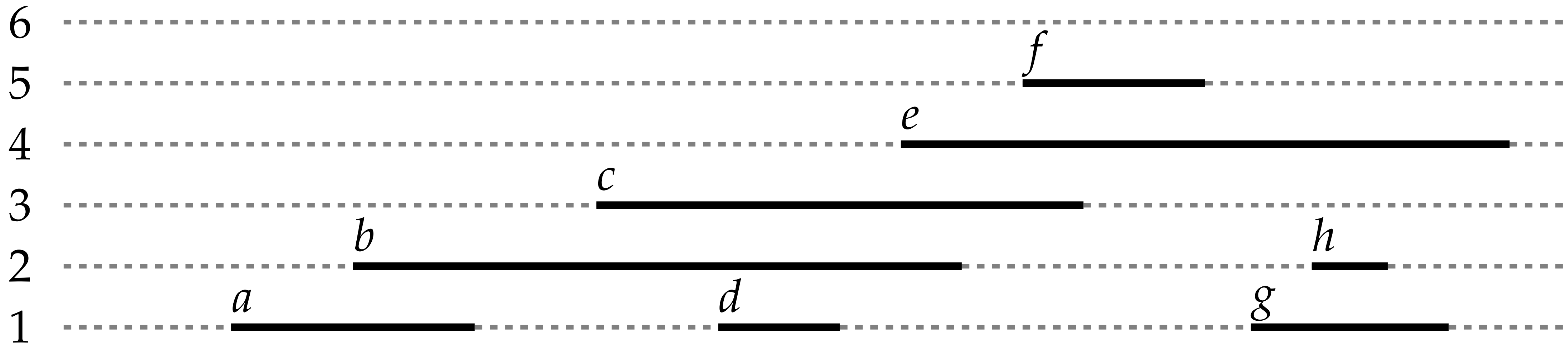

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  $\Rightarrow$ The coloring c uses the minimum number of colors.

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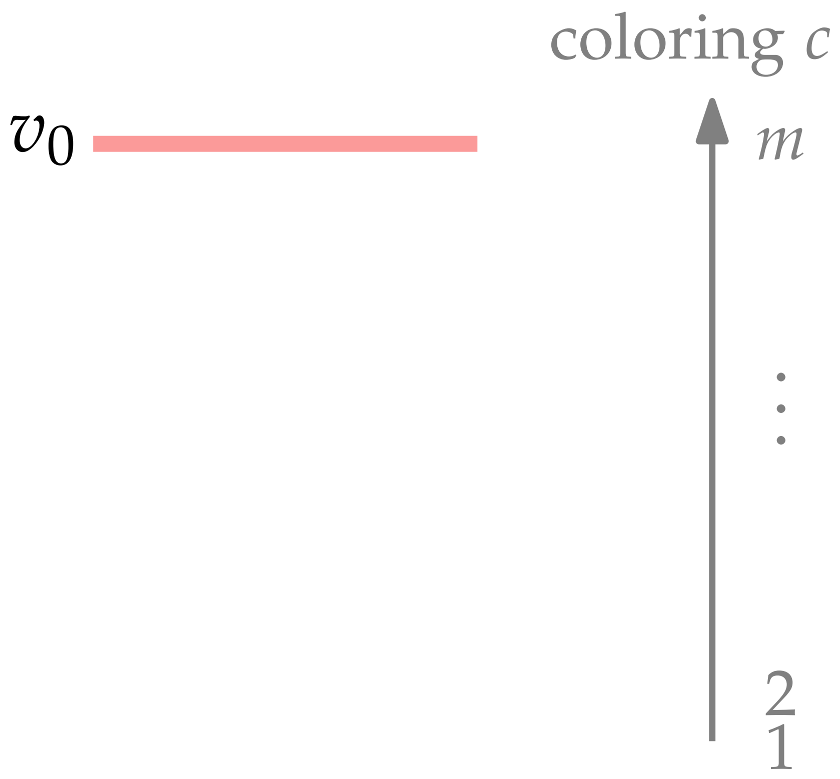

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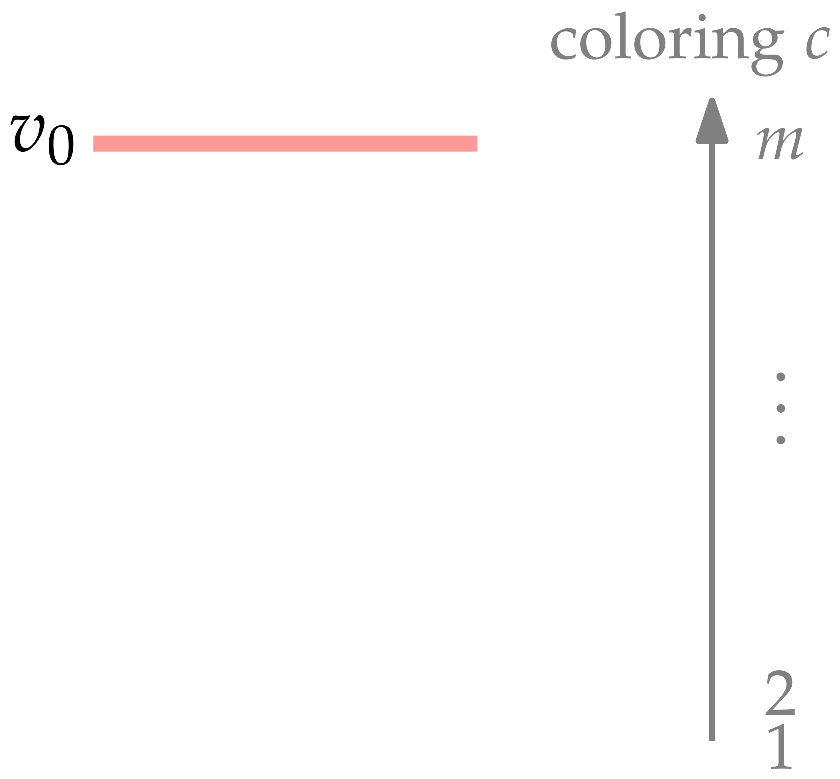

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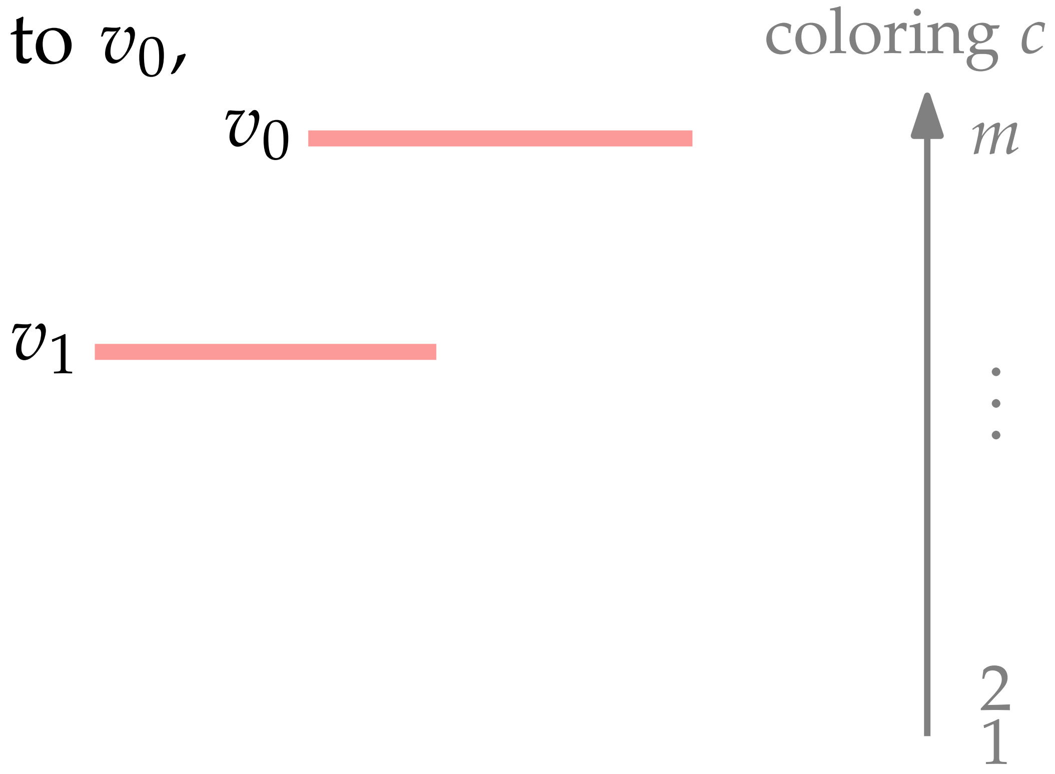

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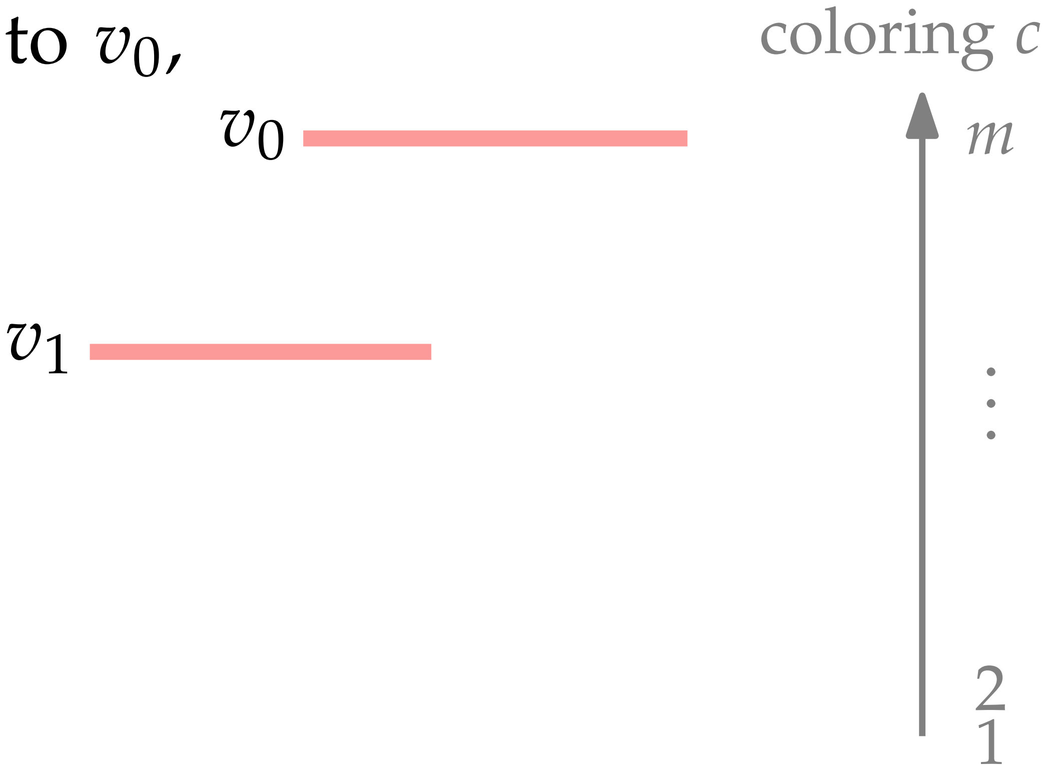

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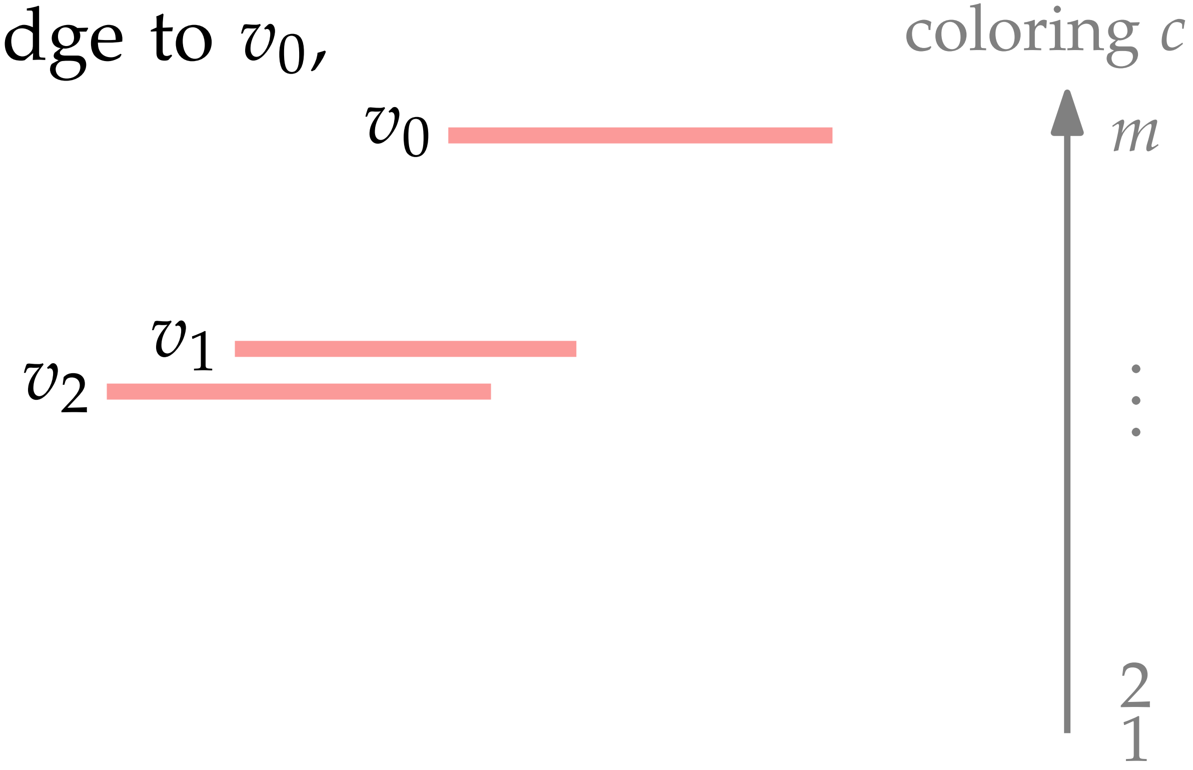

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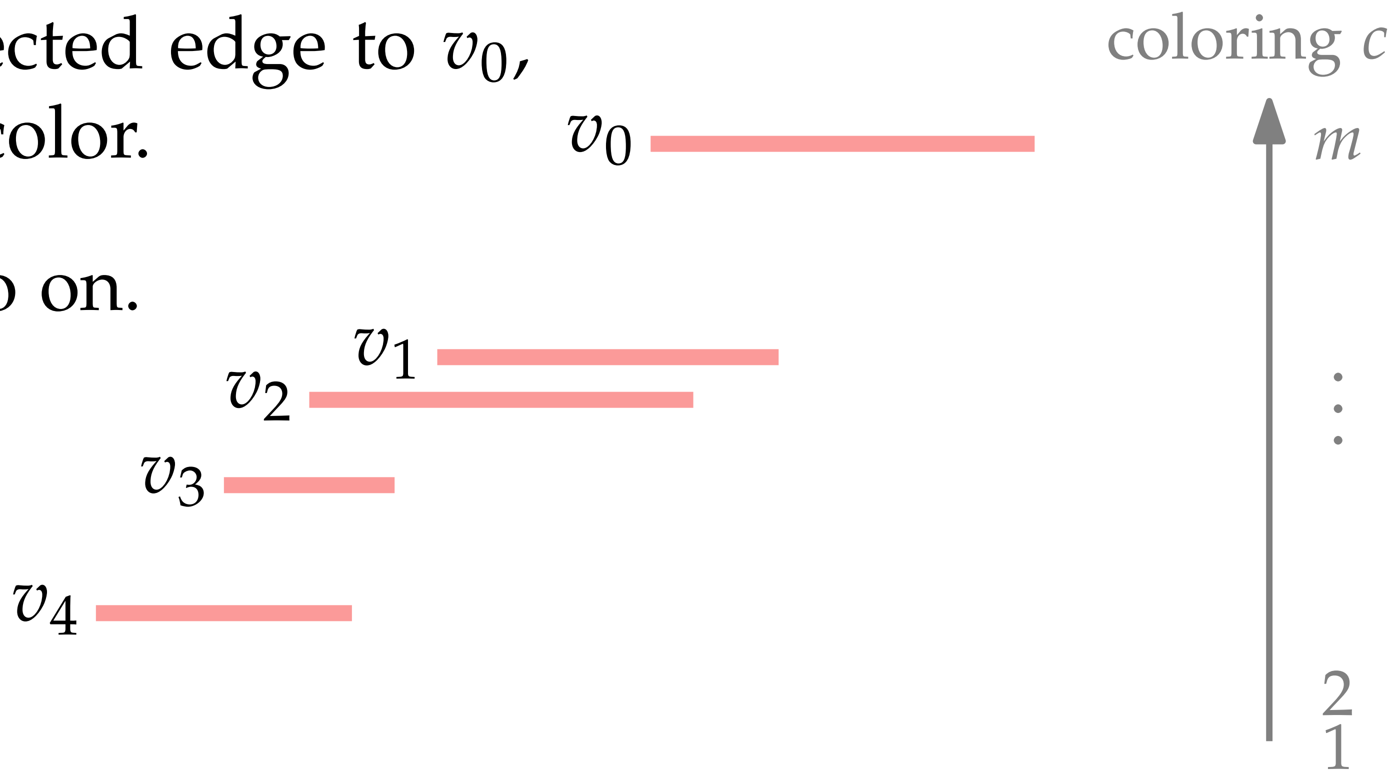

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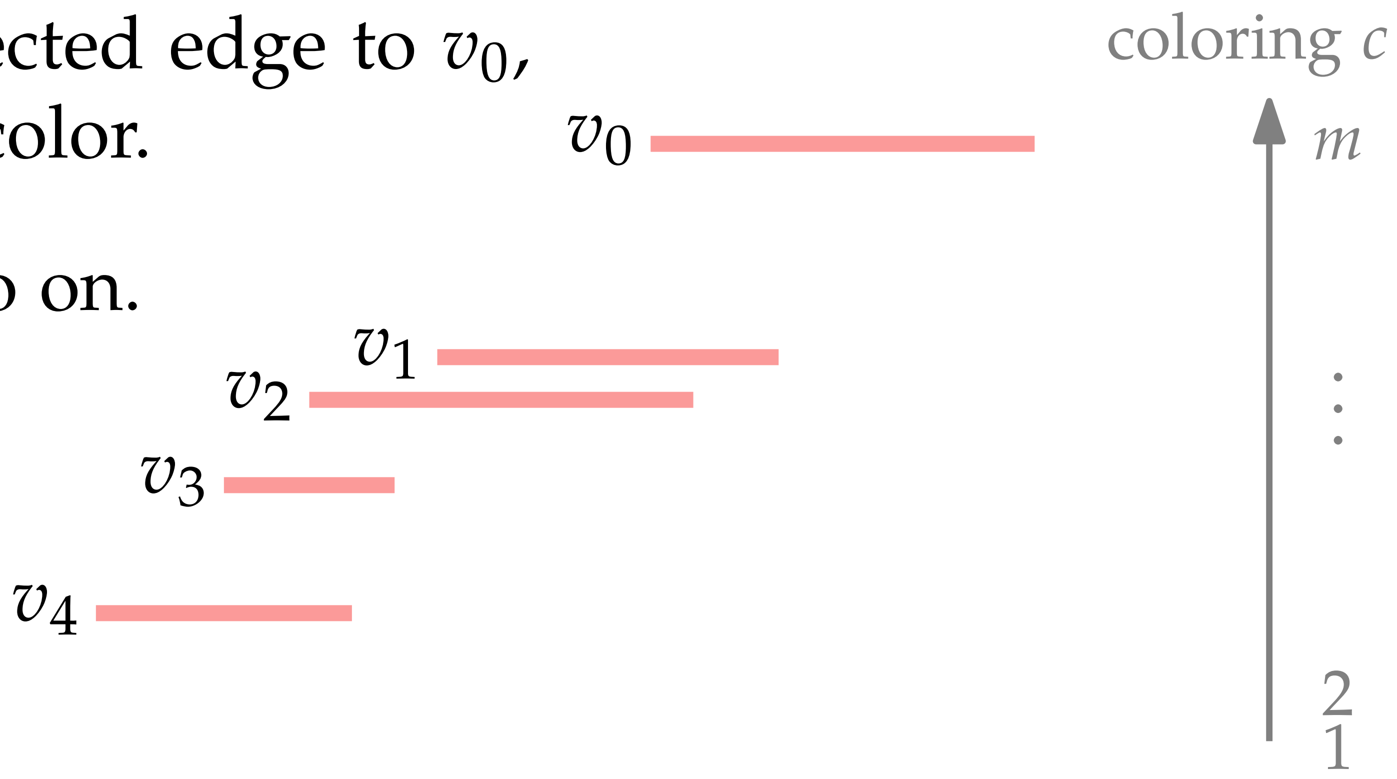

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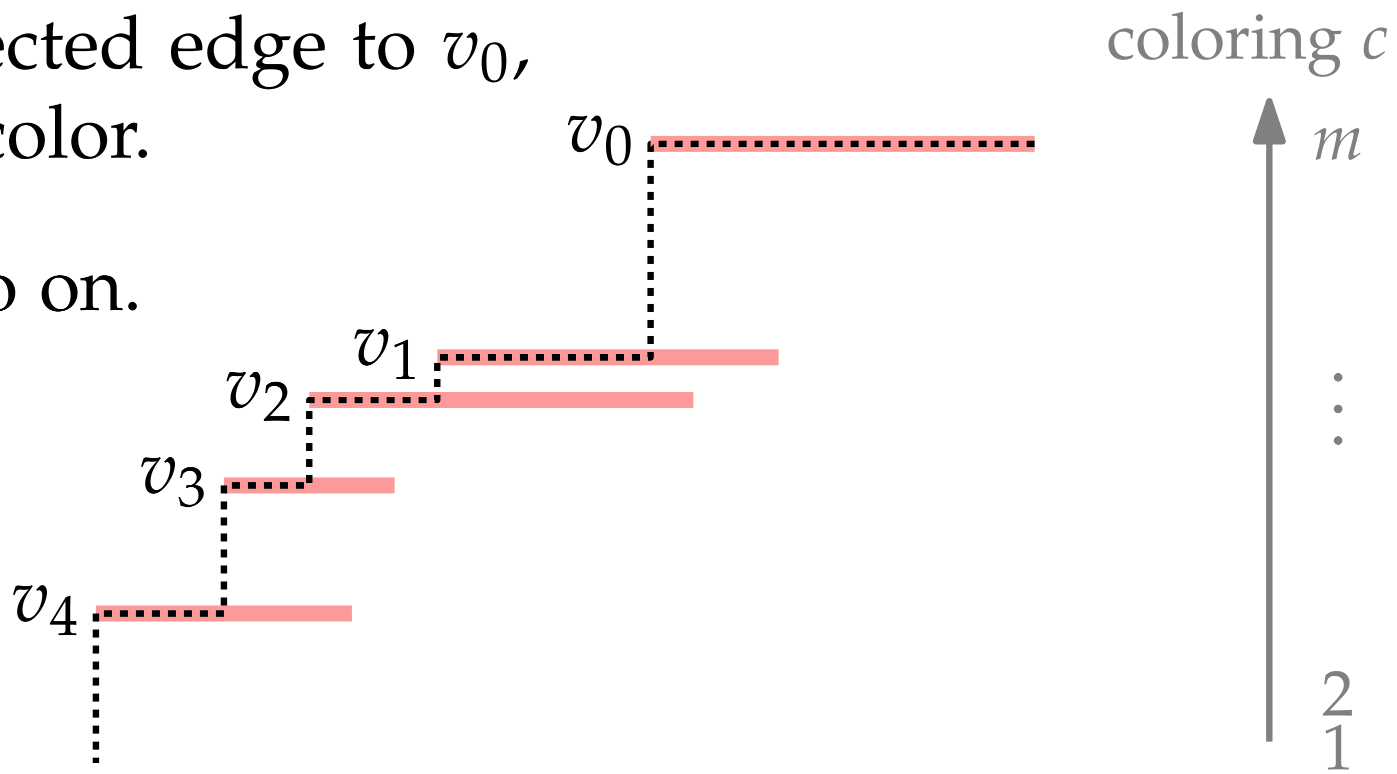

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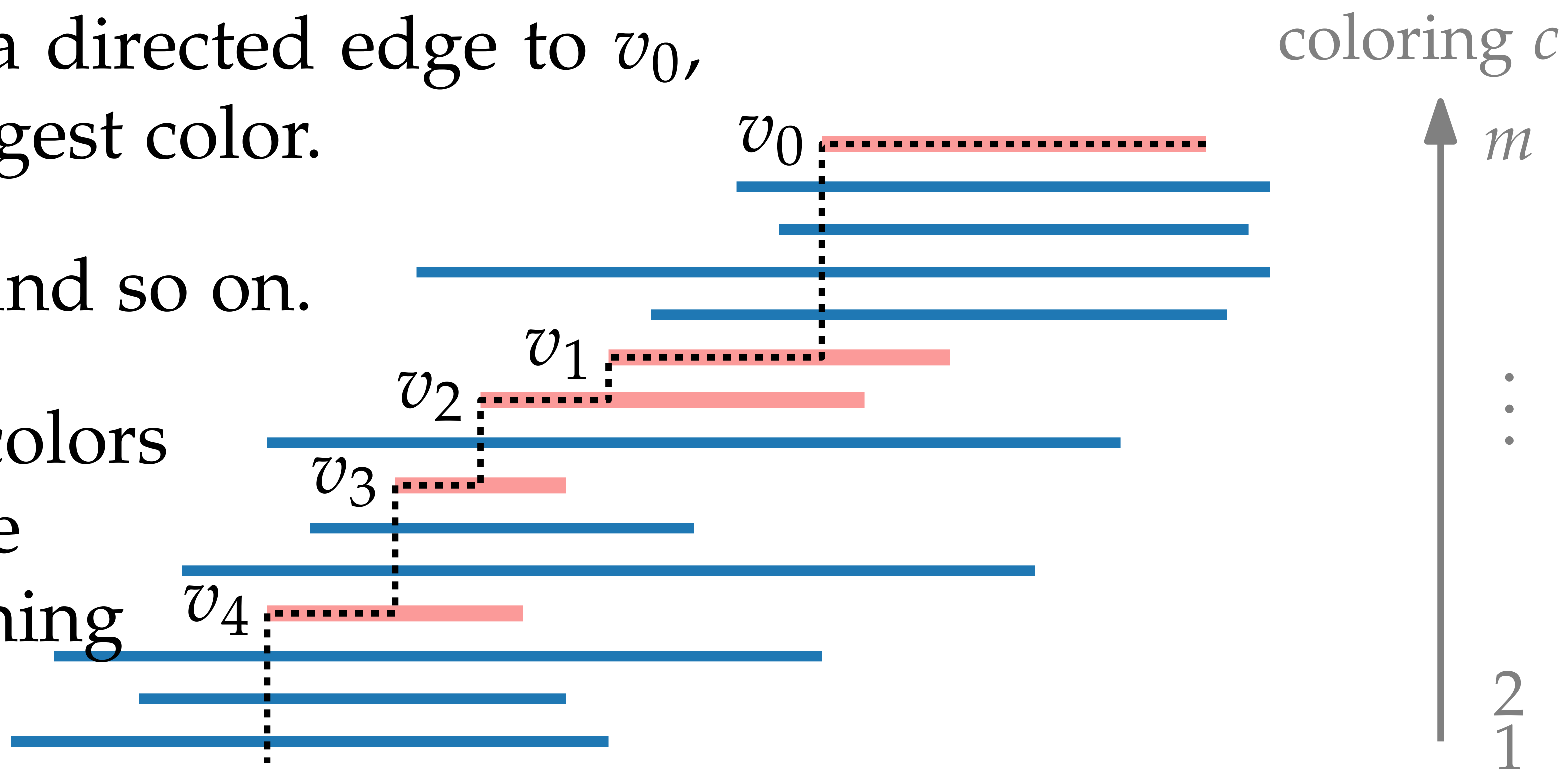

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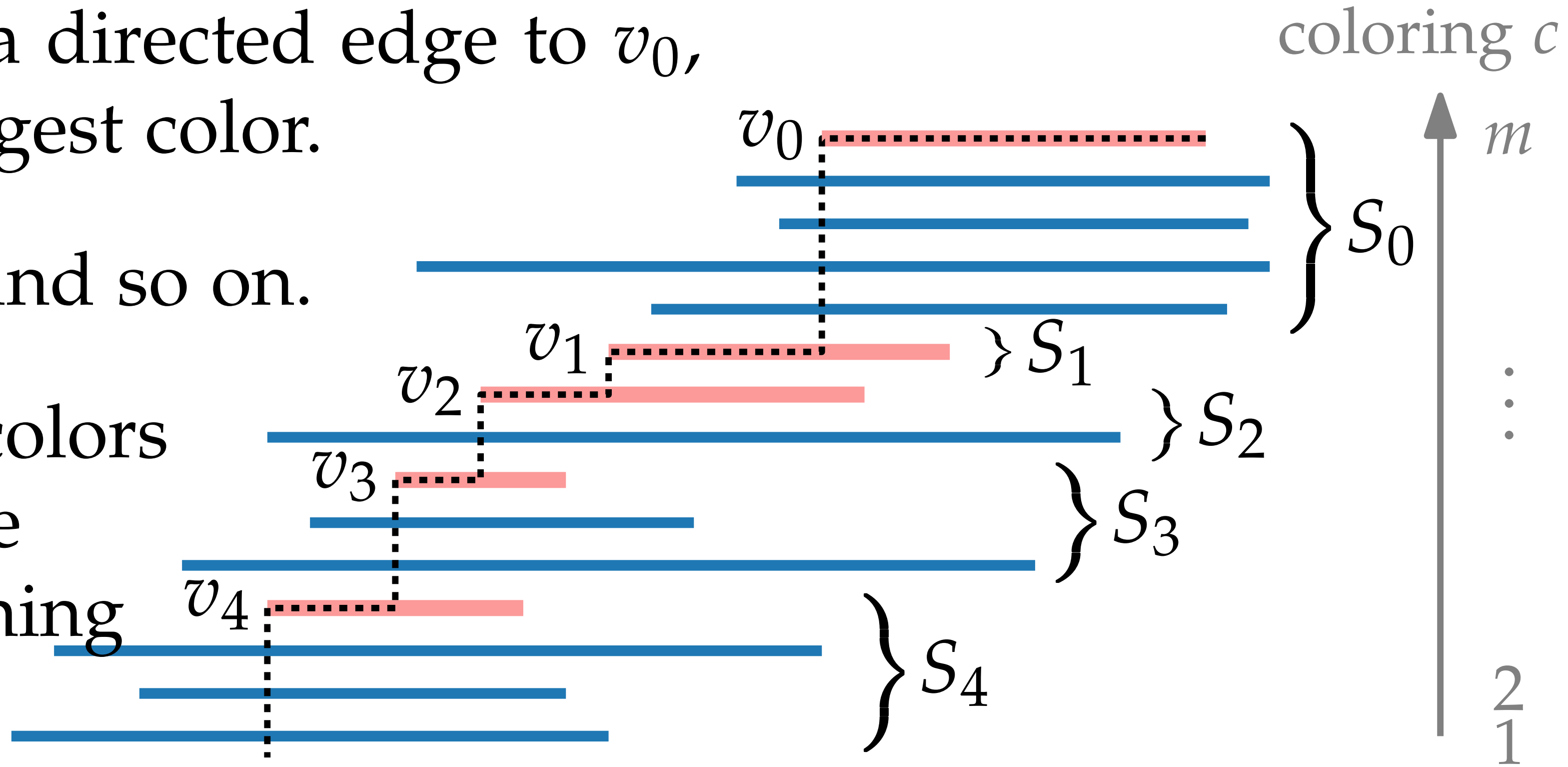

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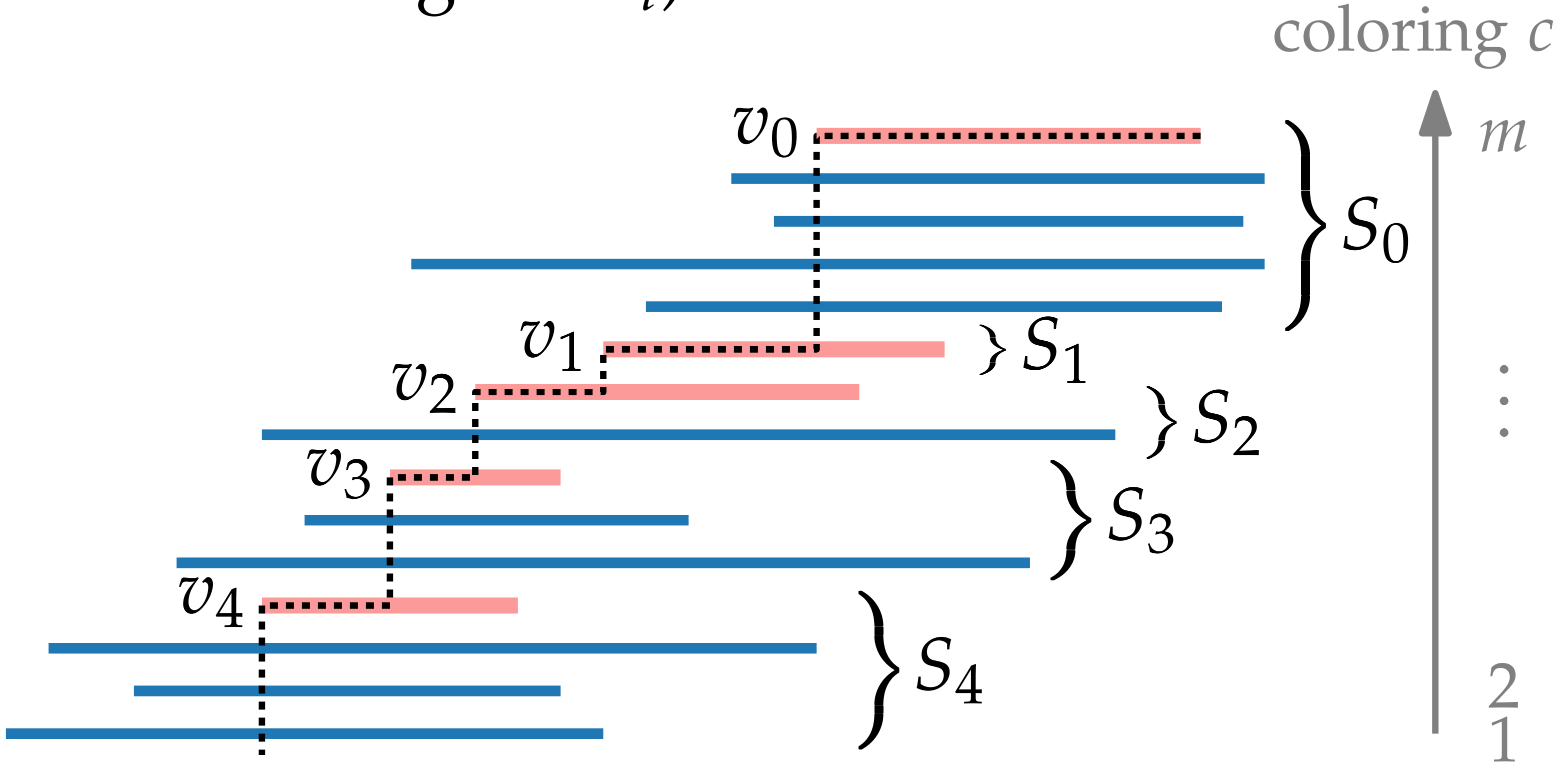

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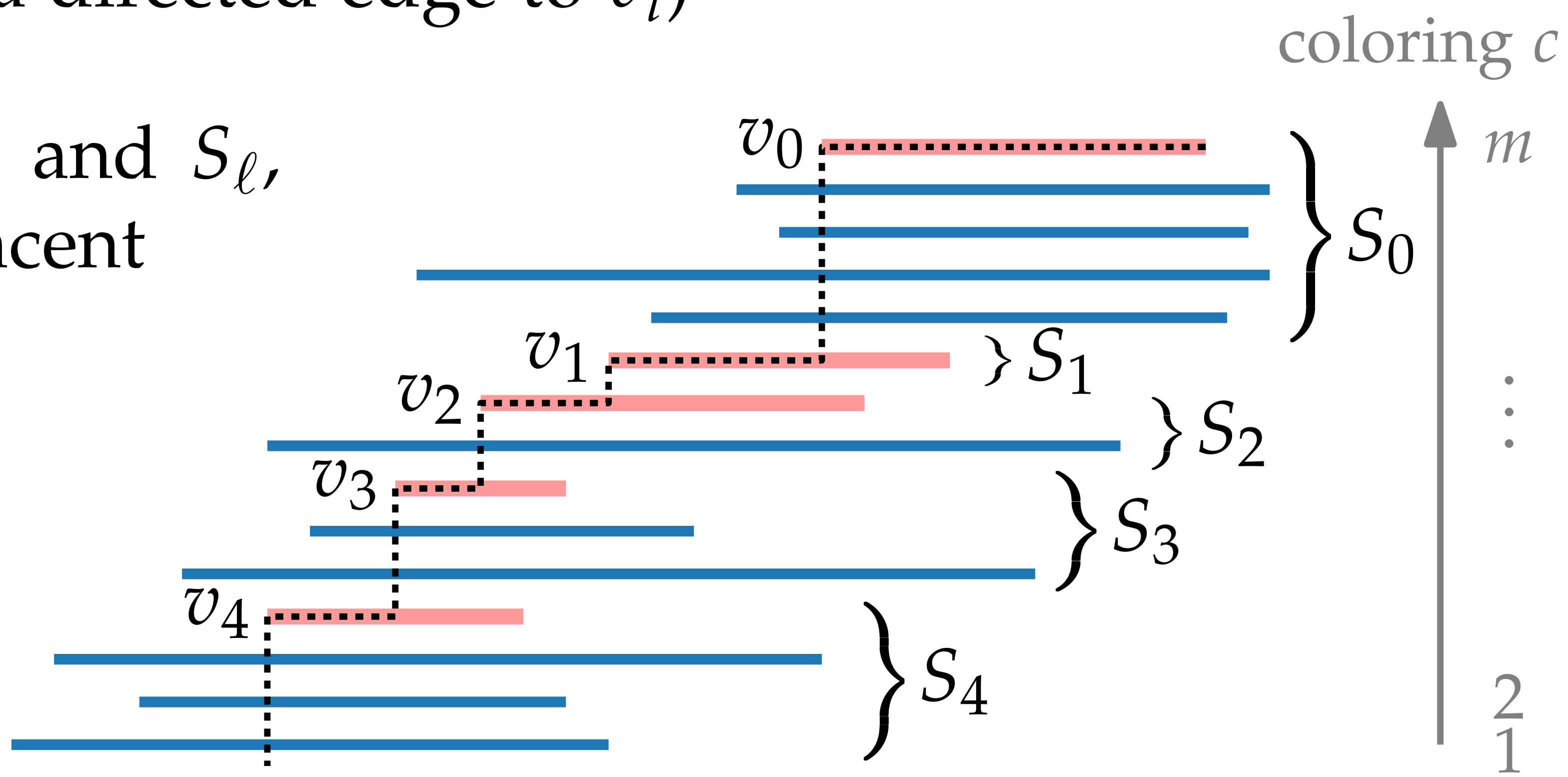

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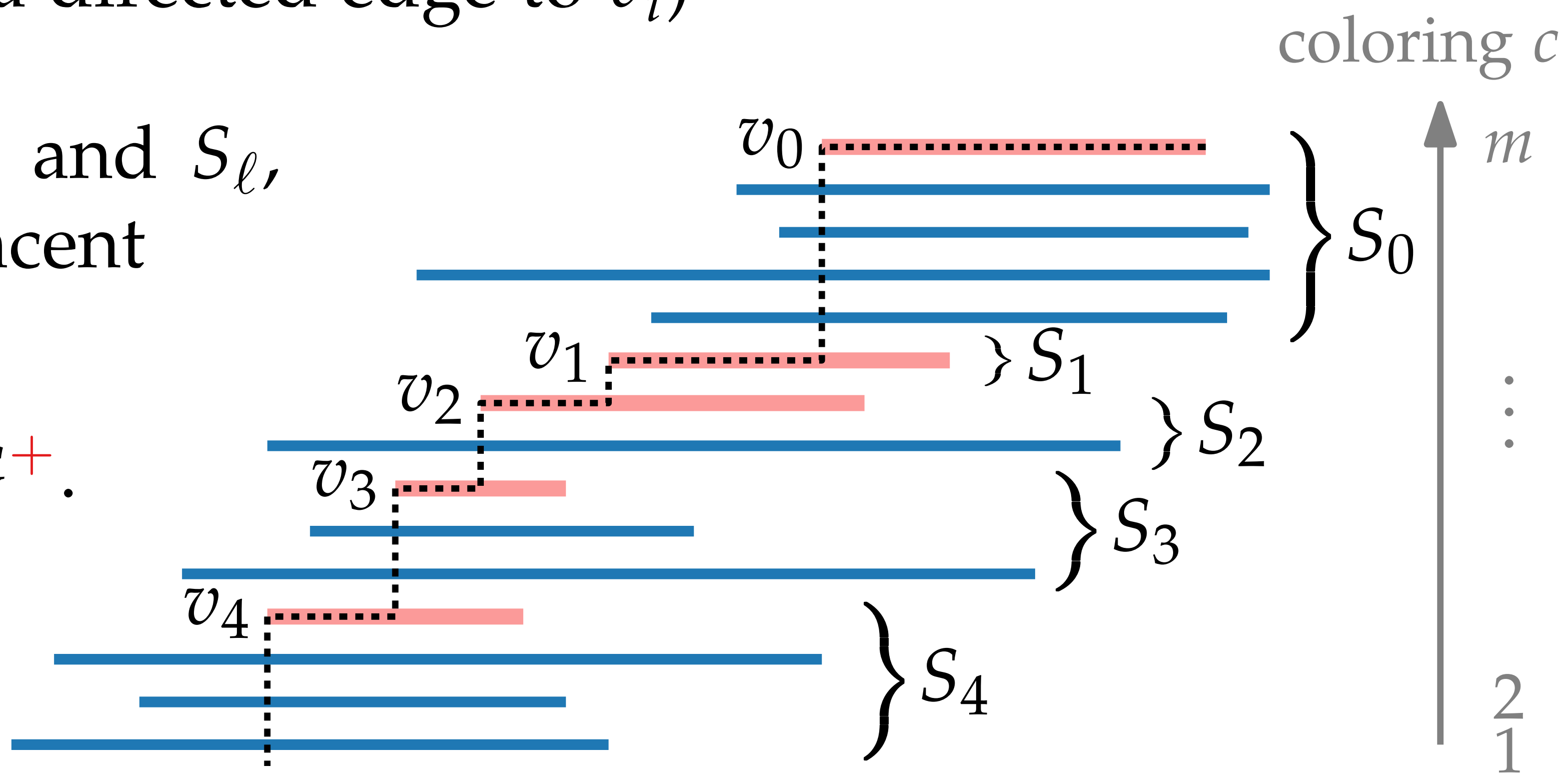

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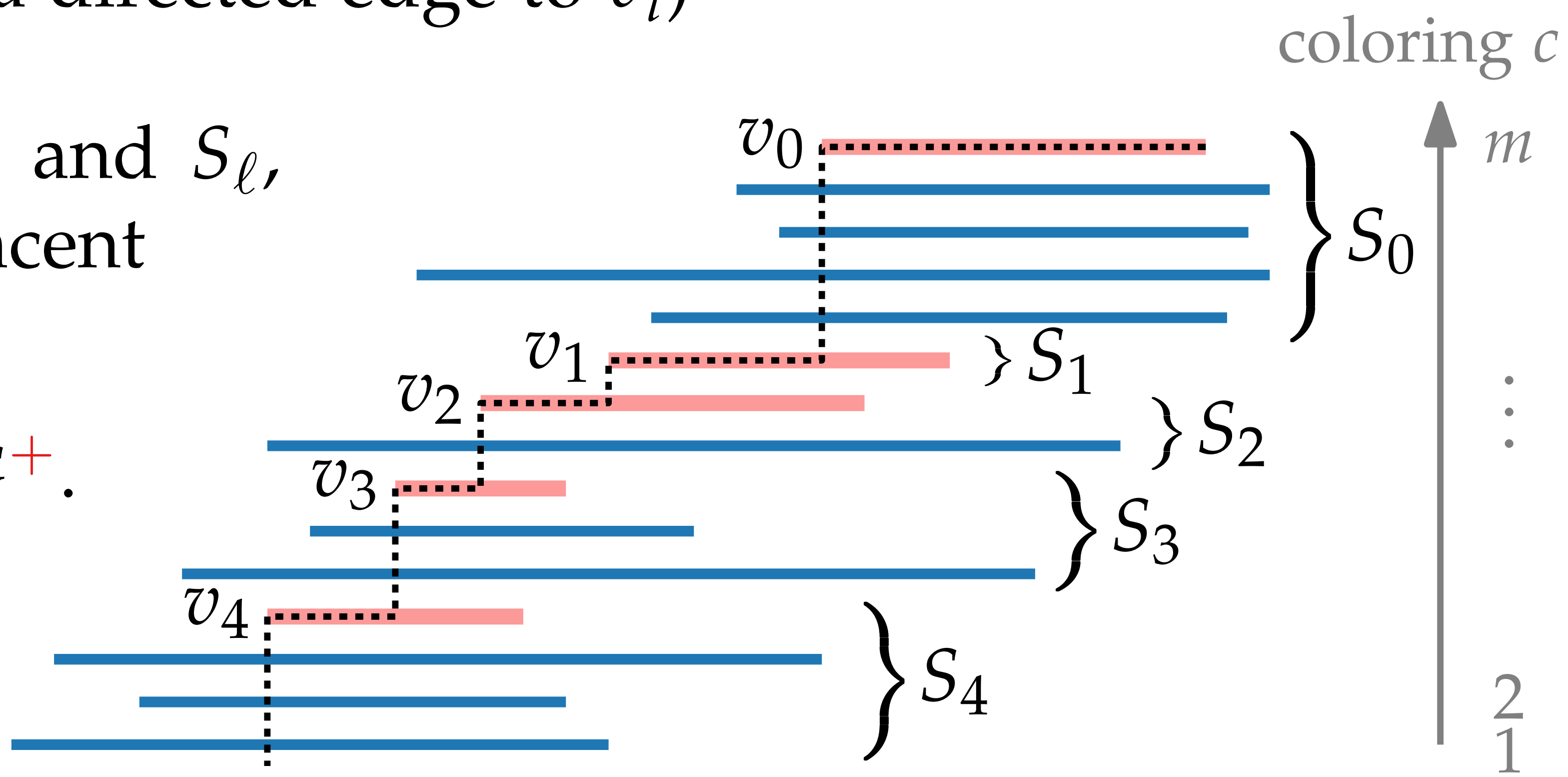

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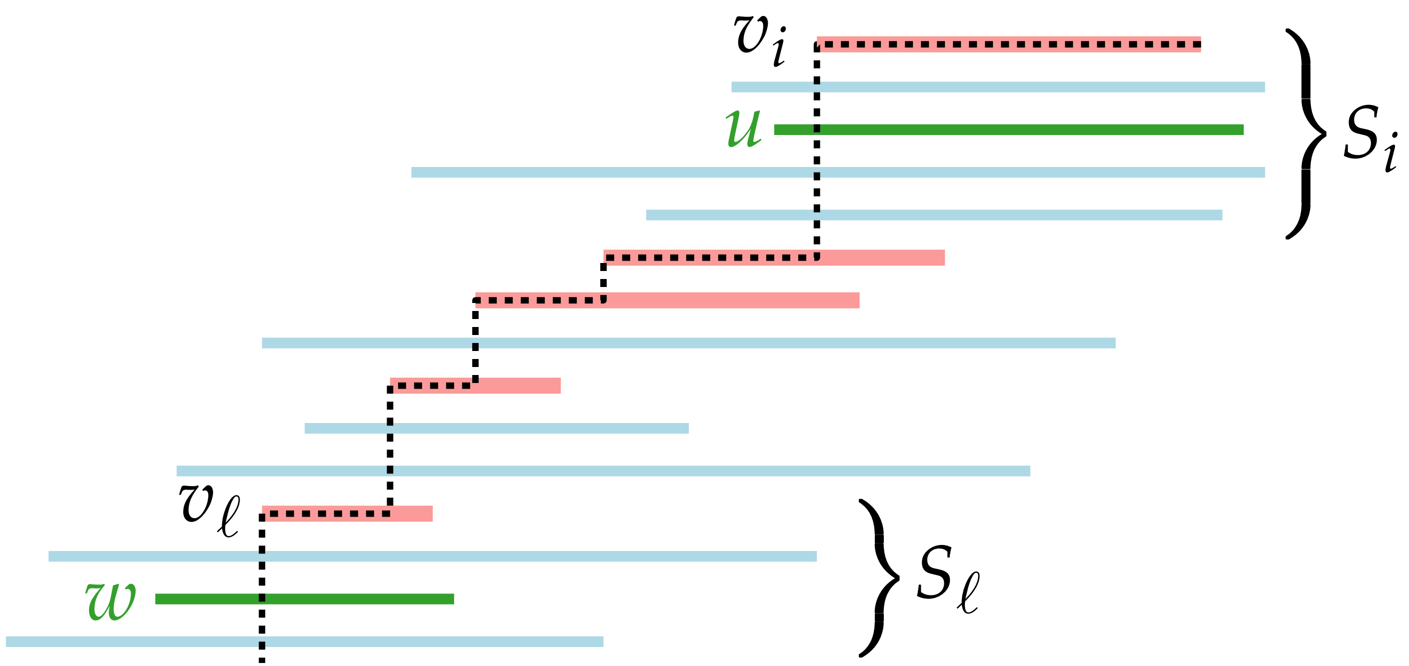

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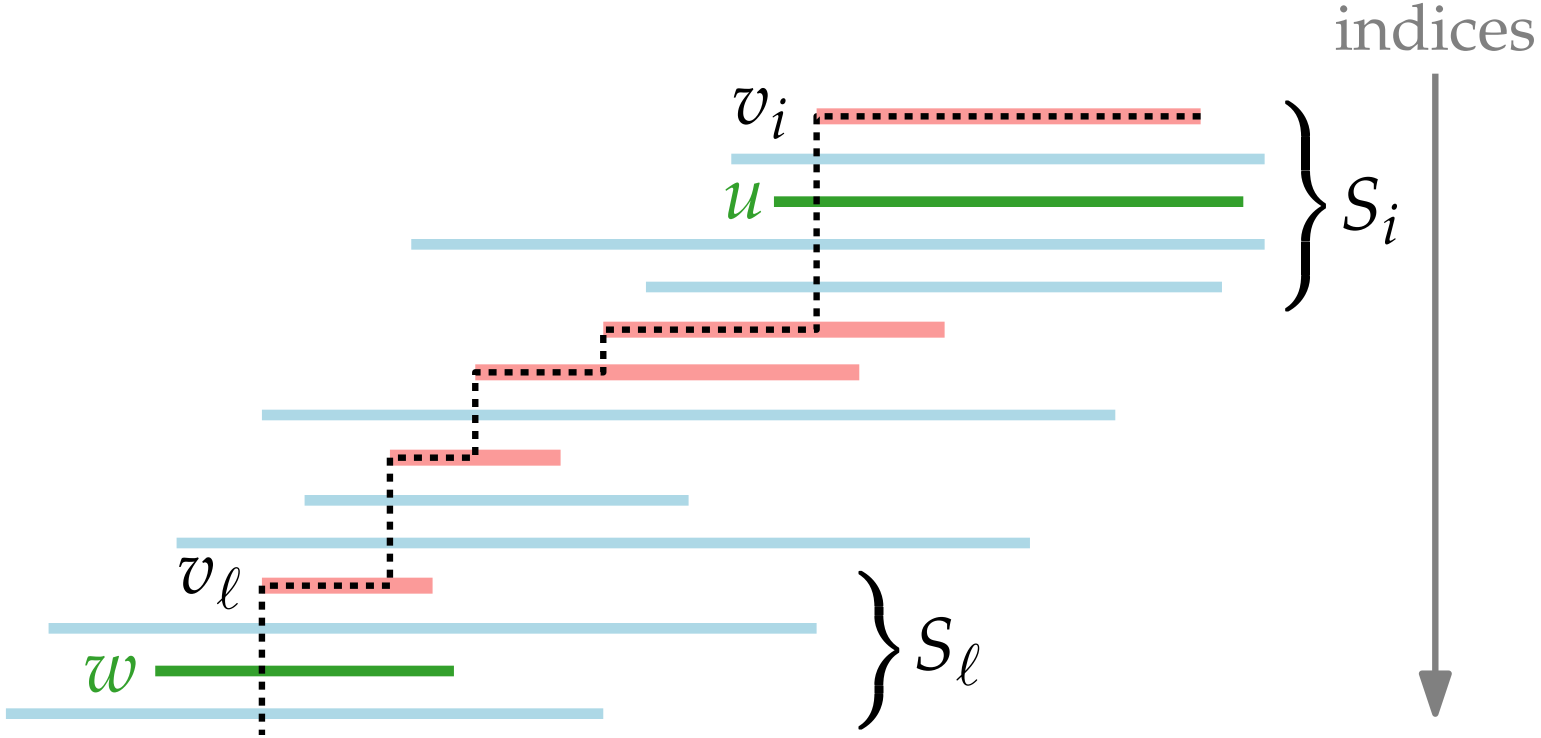

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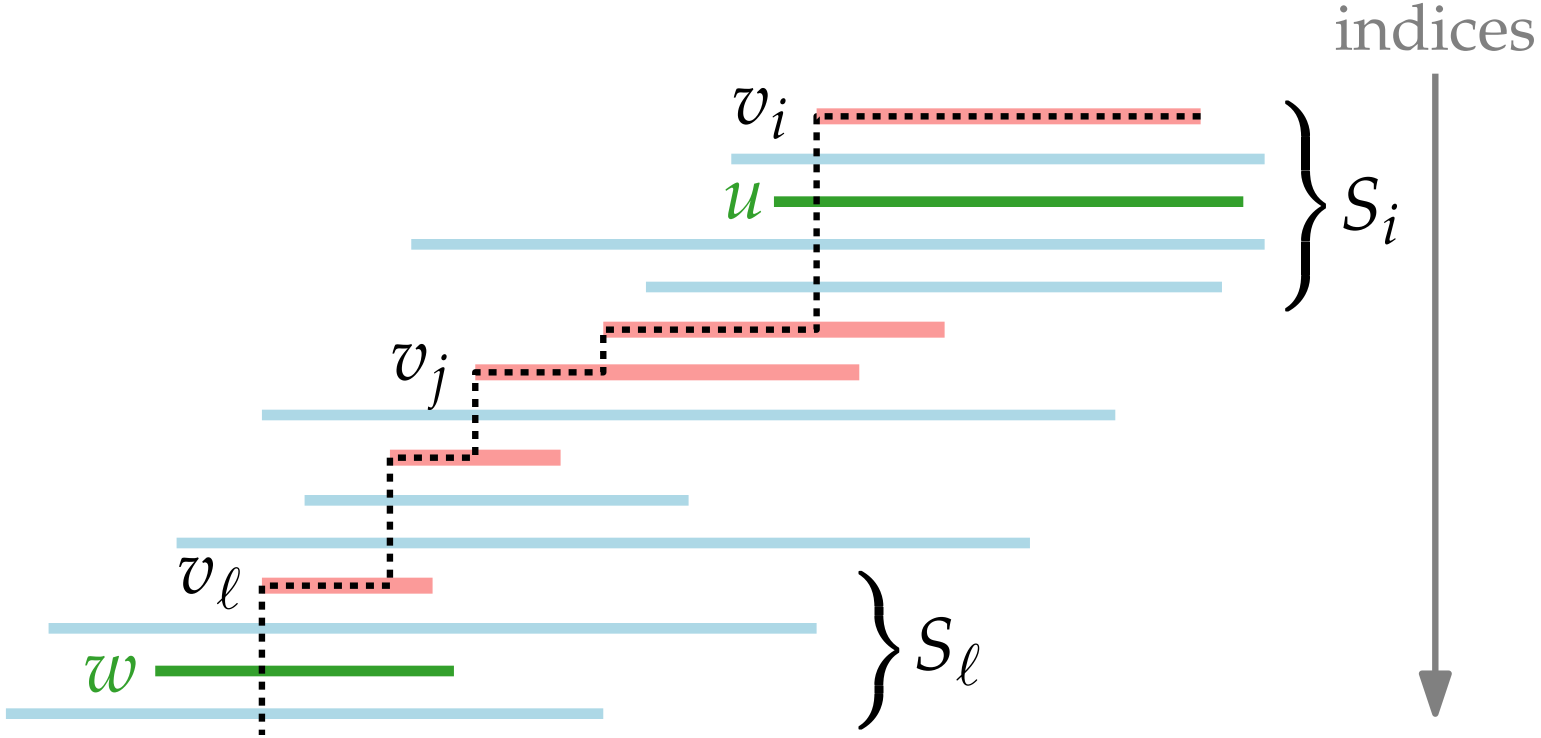

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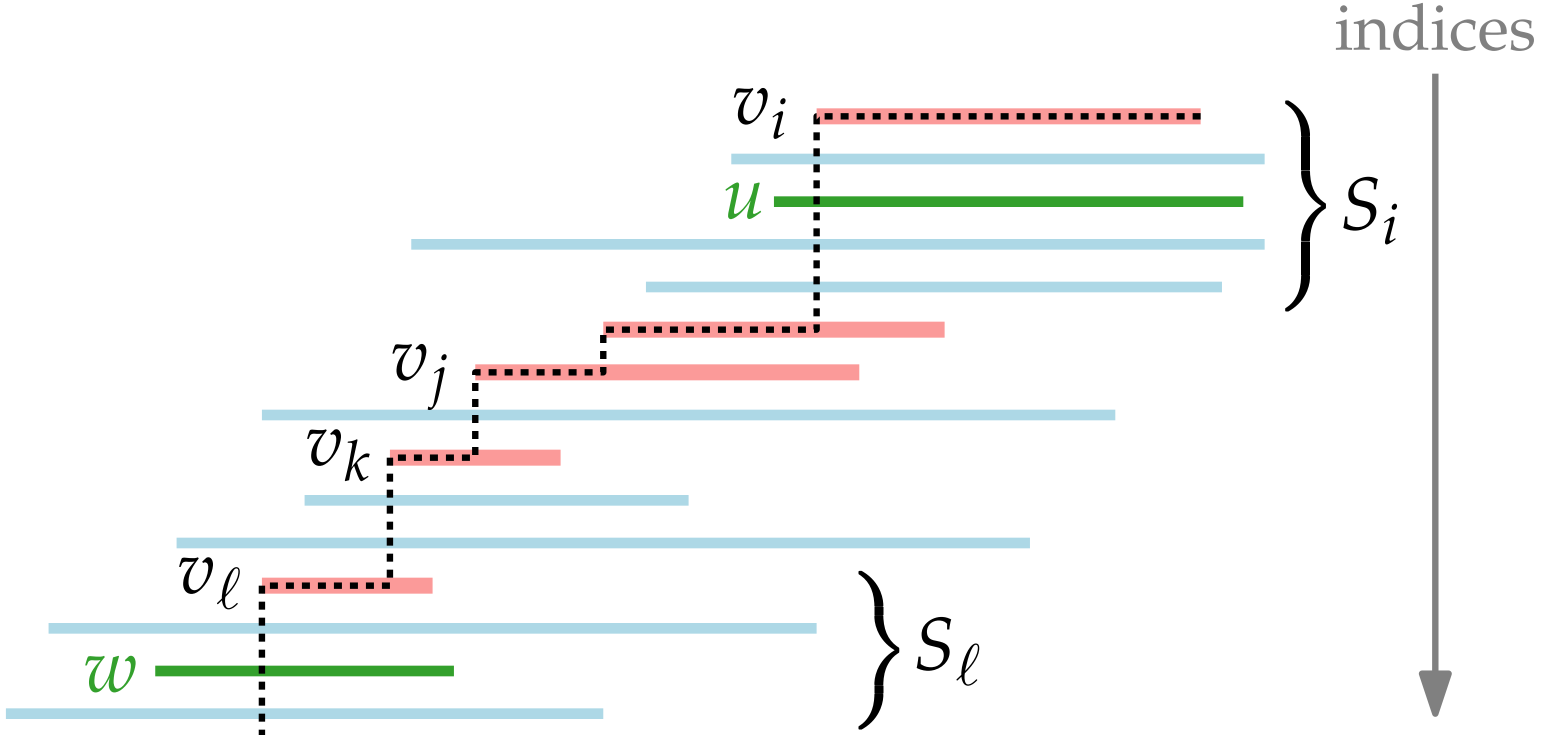

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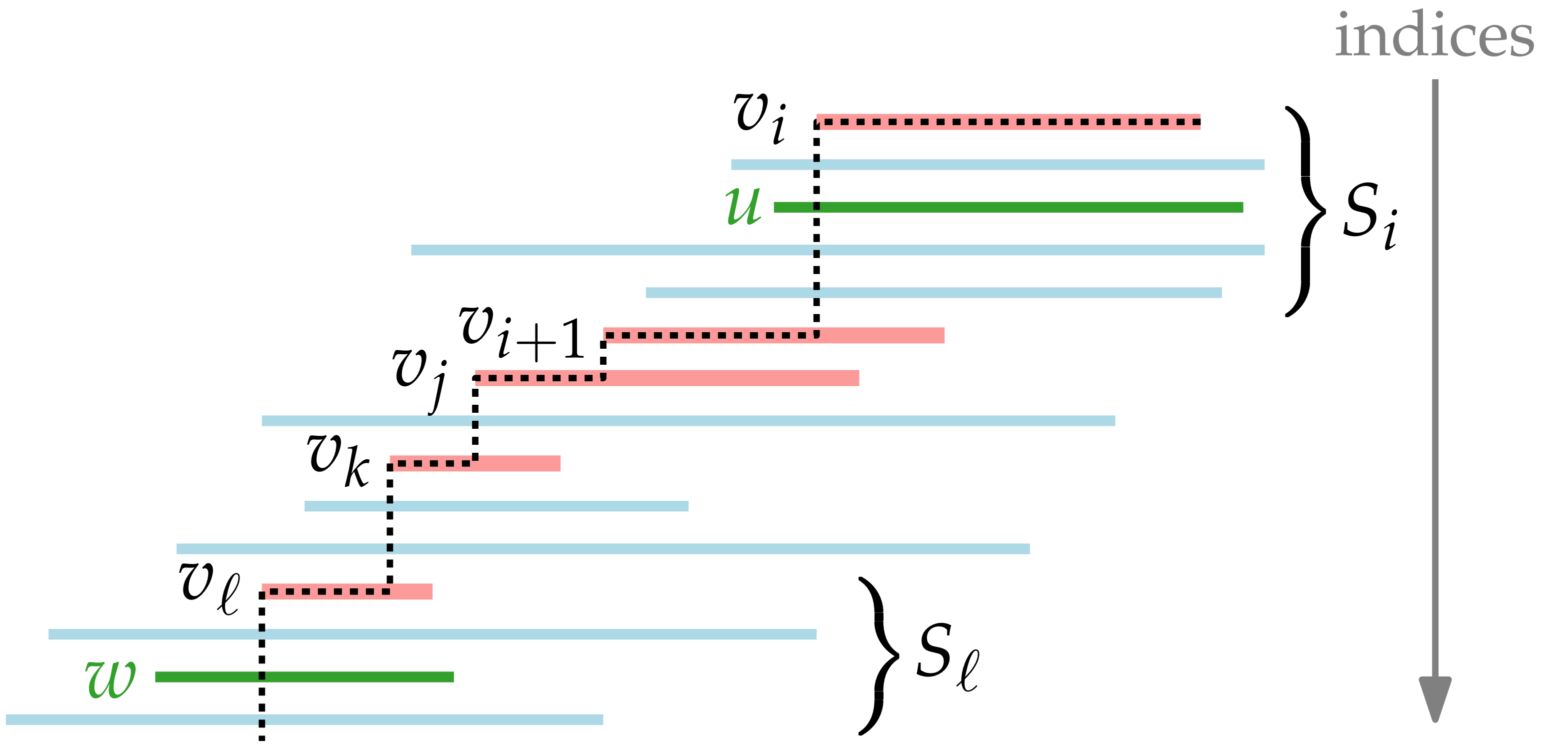

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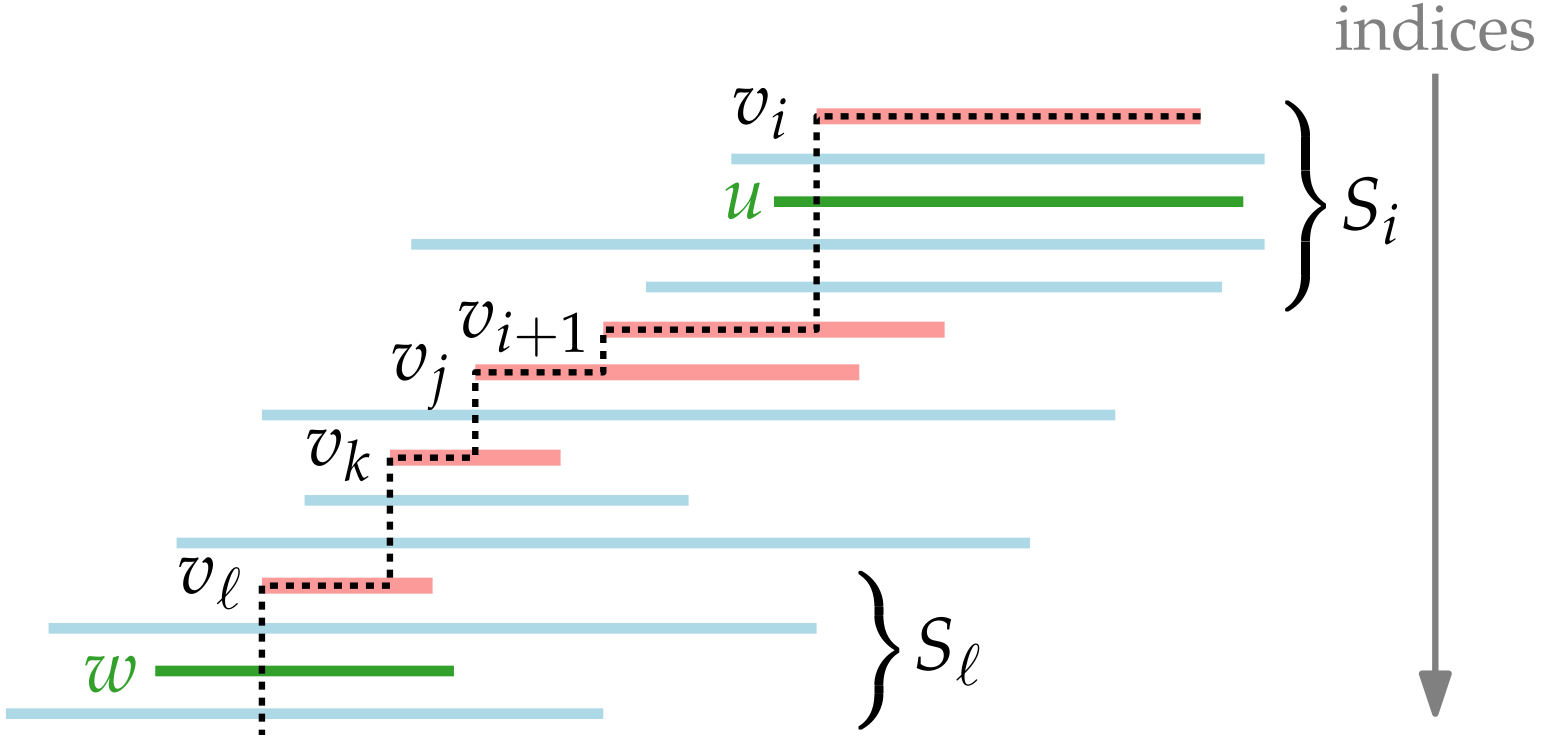

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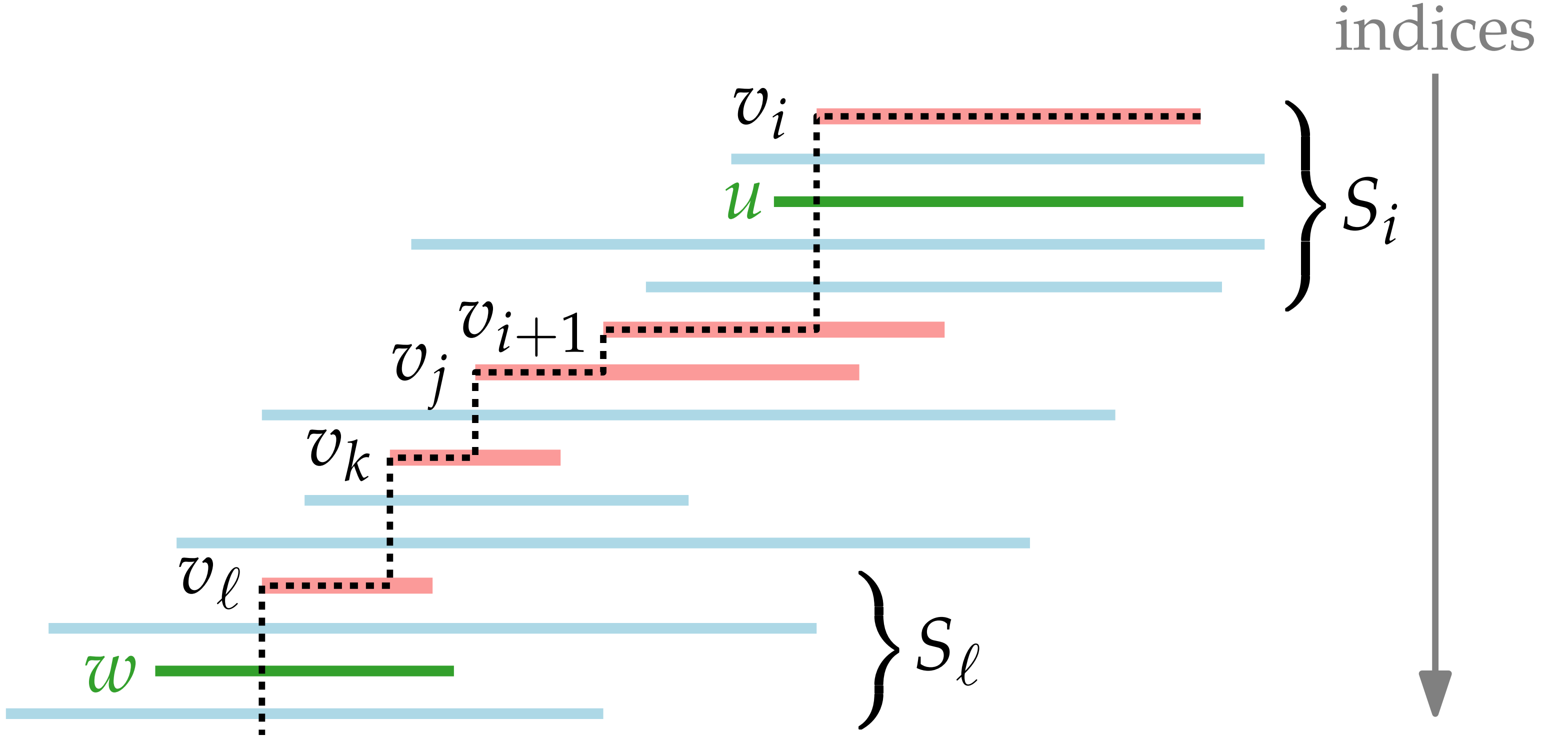

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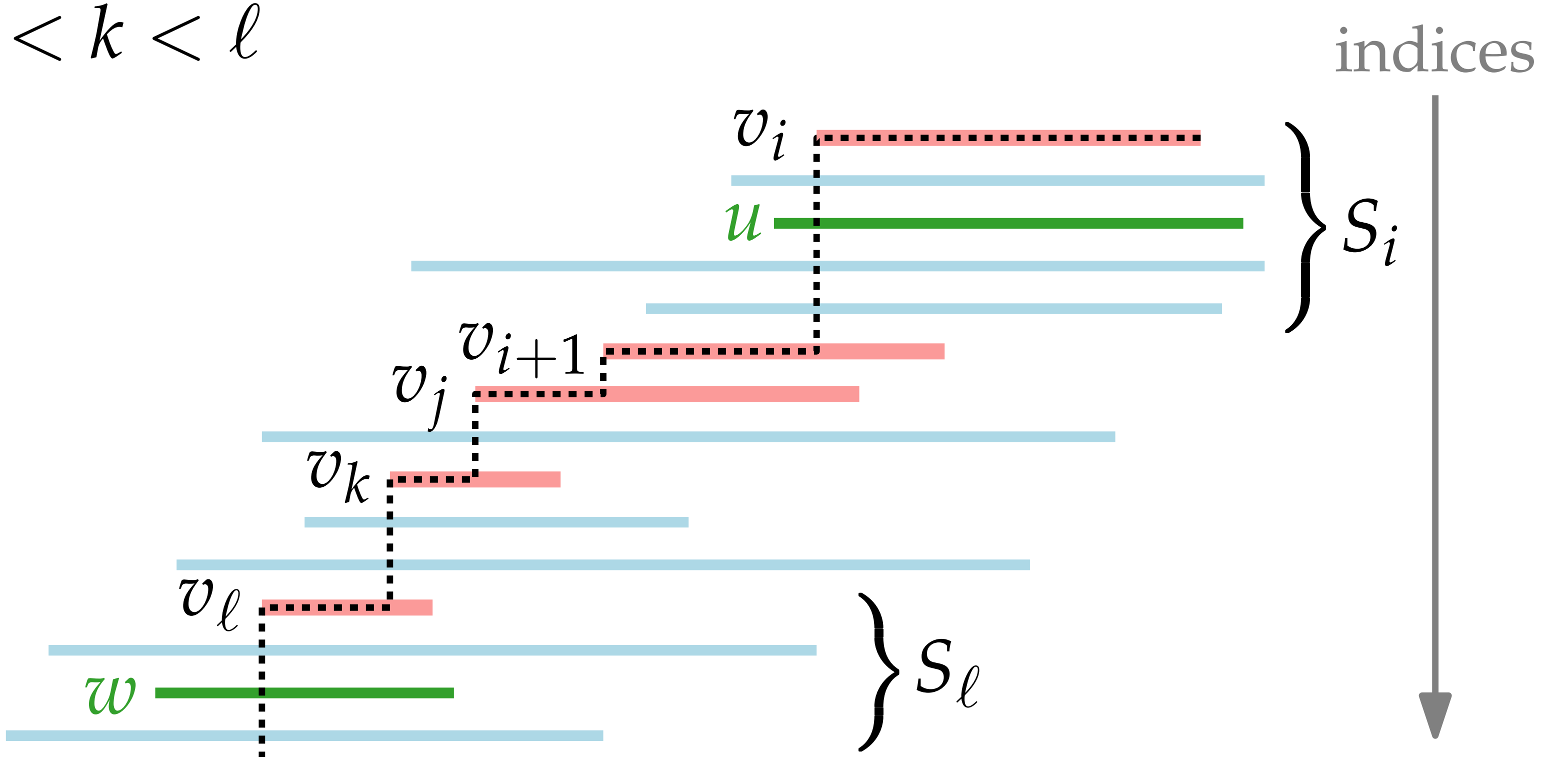

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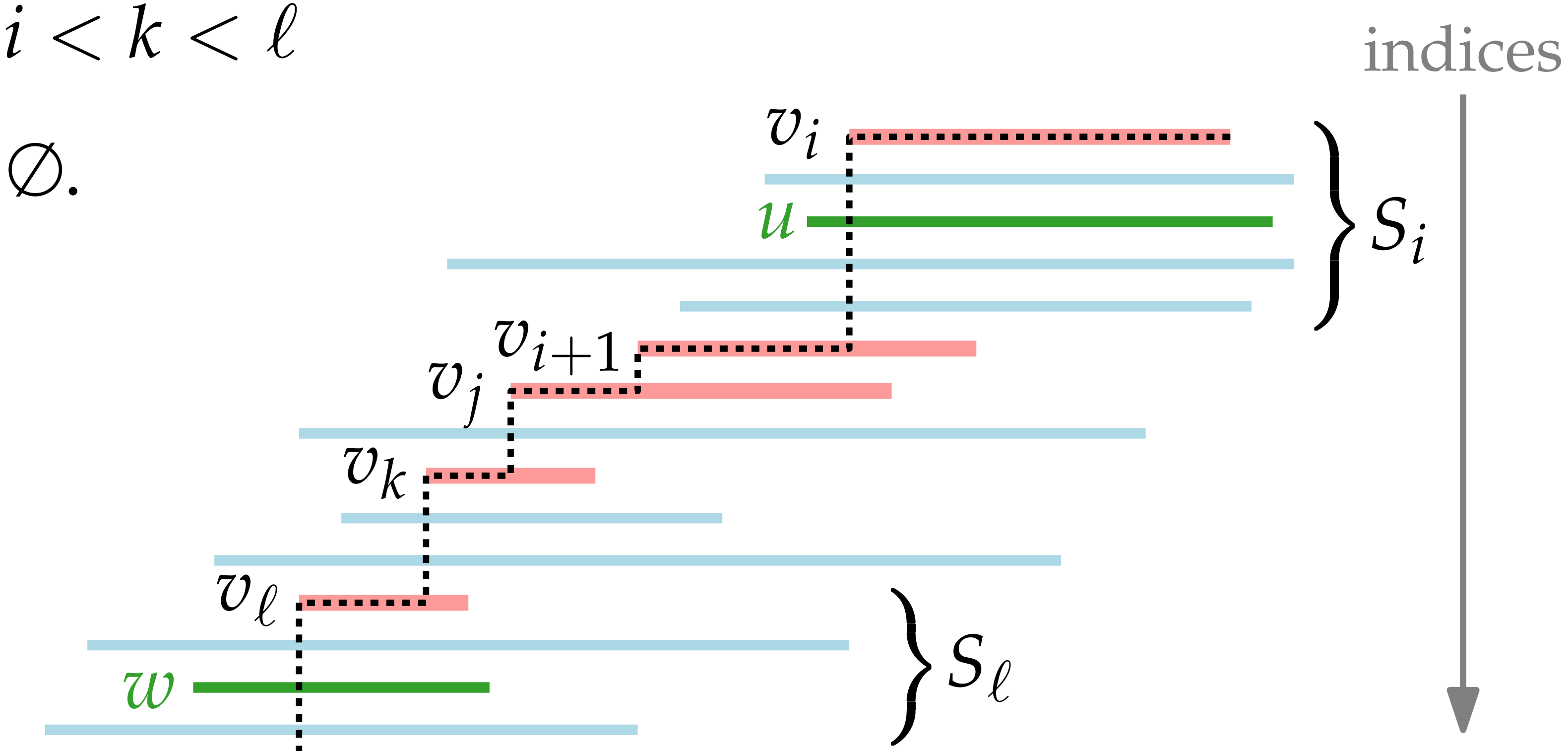

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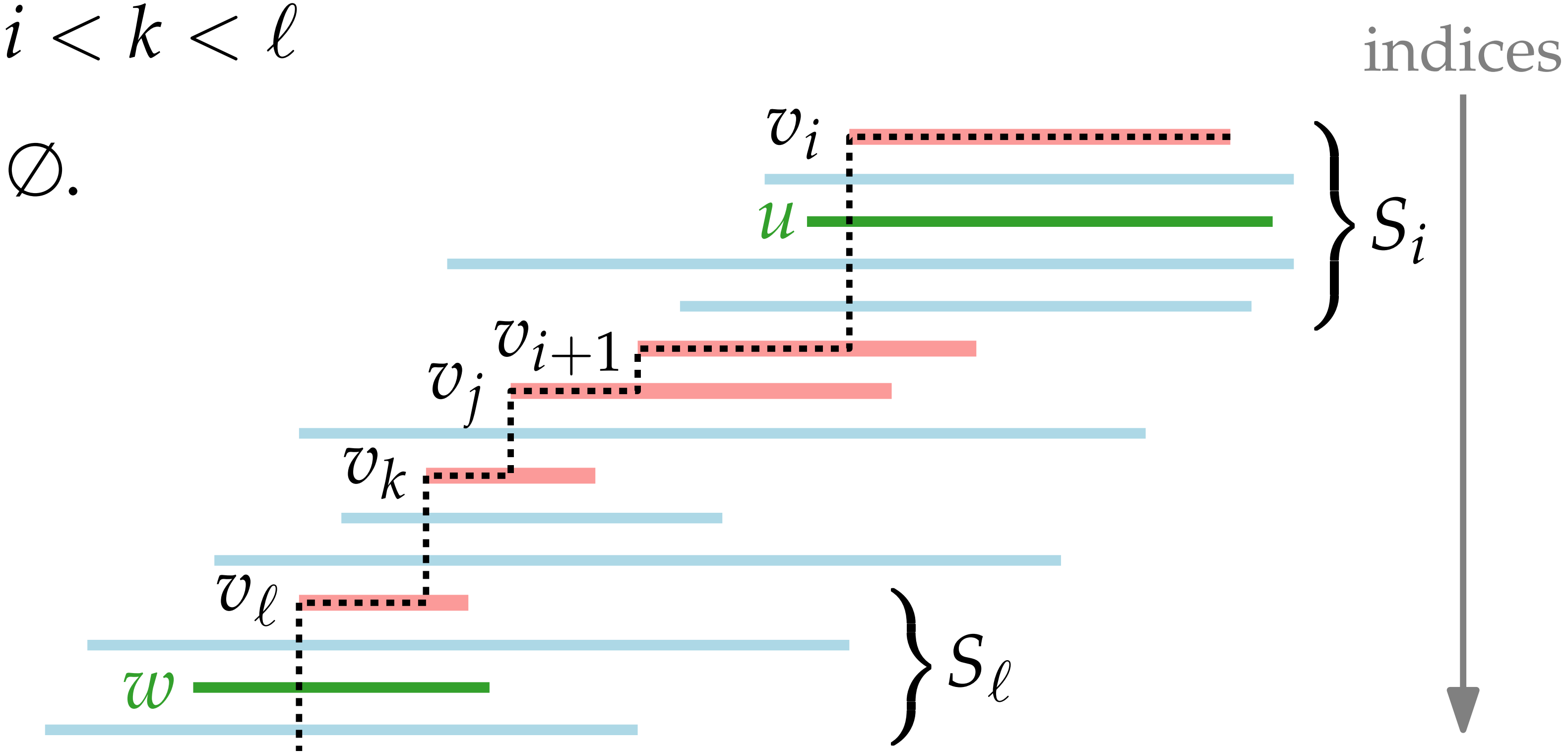

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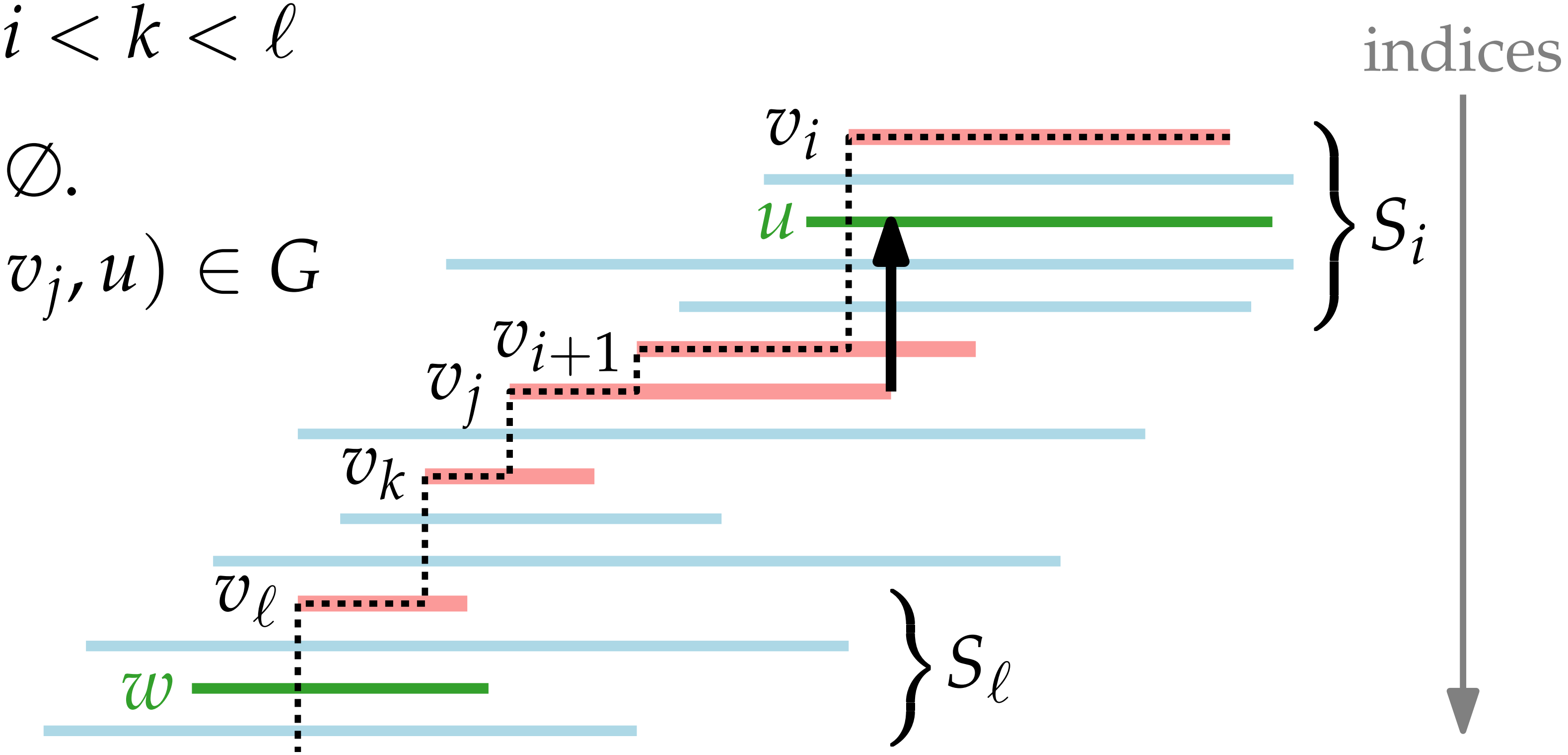

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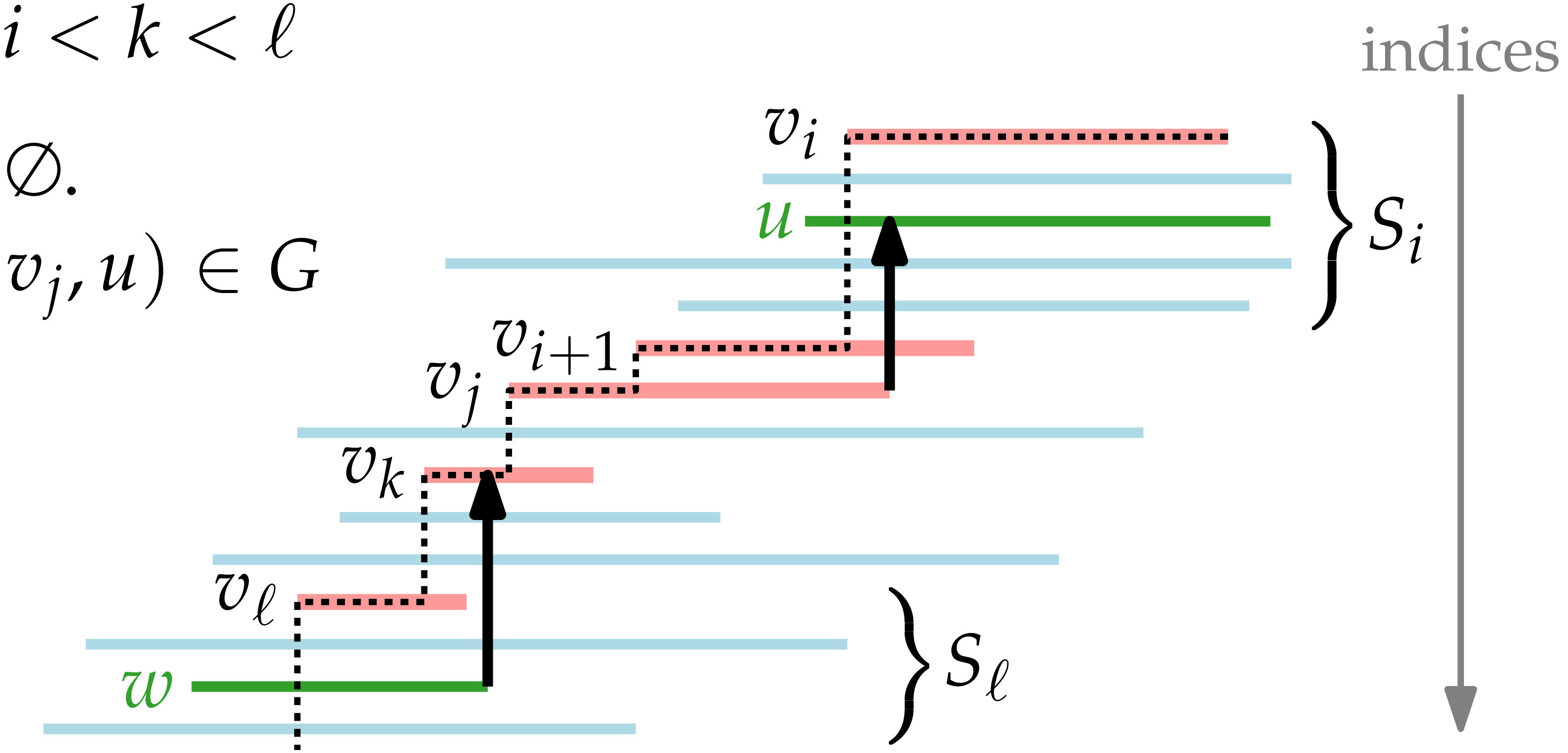

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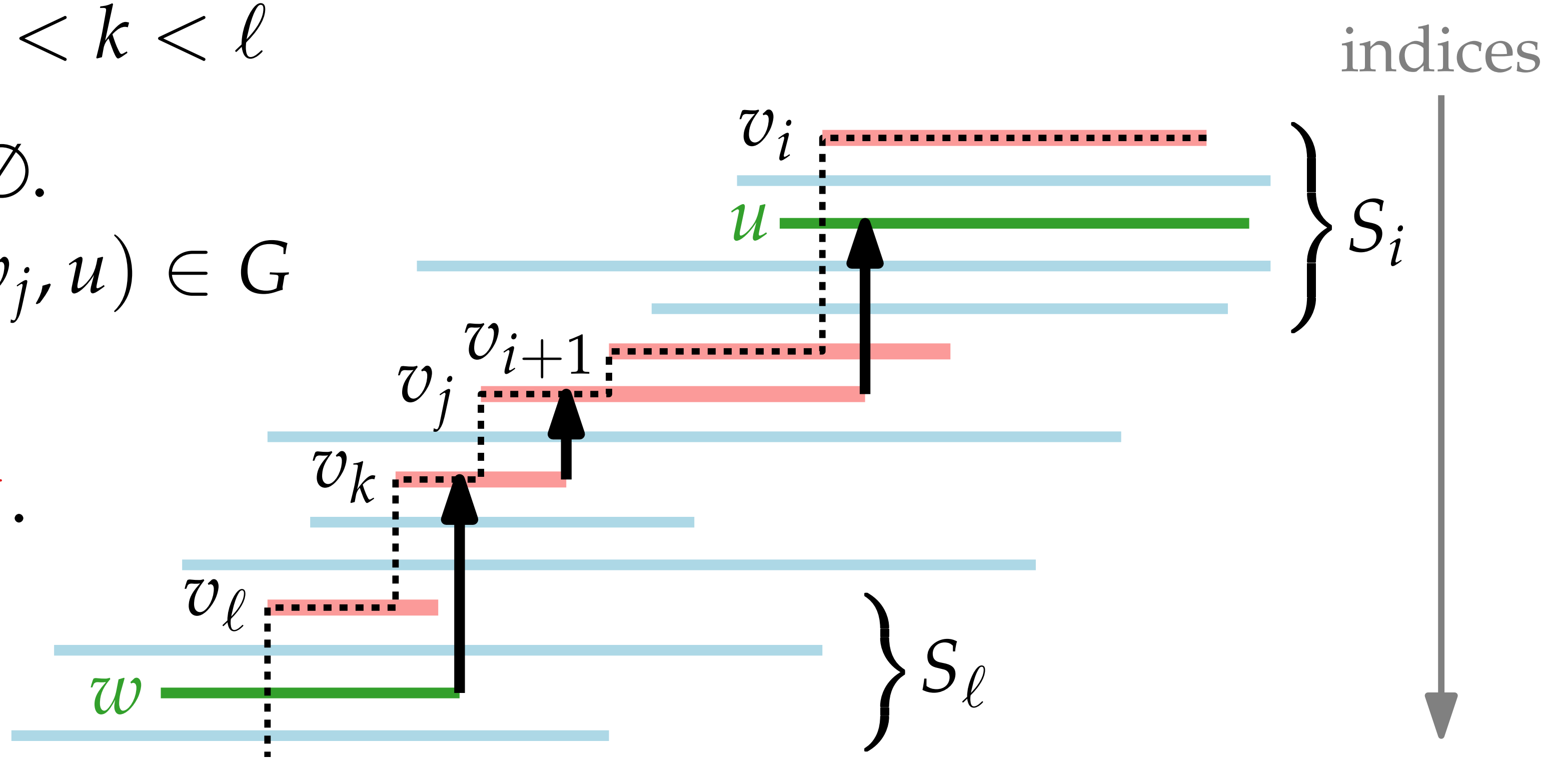

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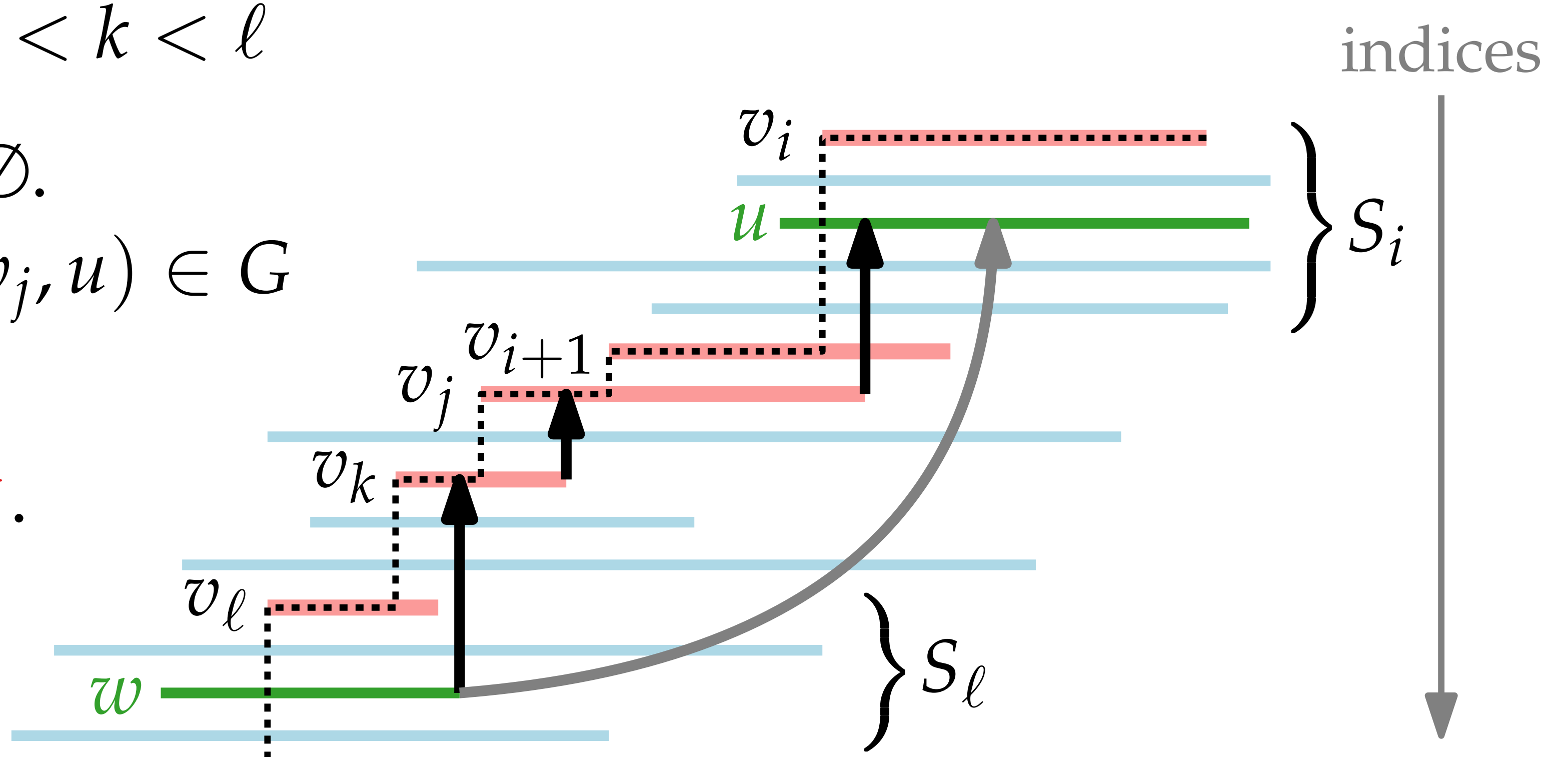

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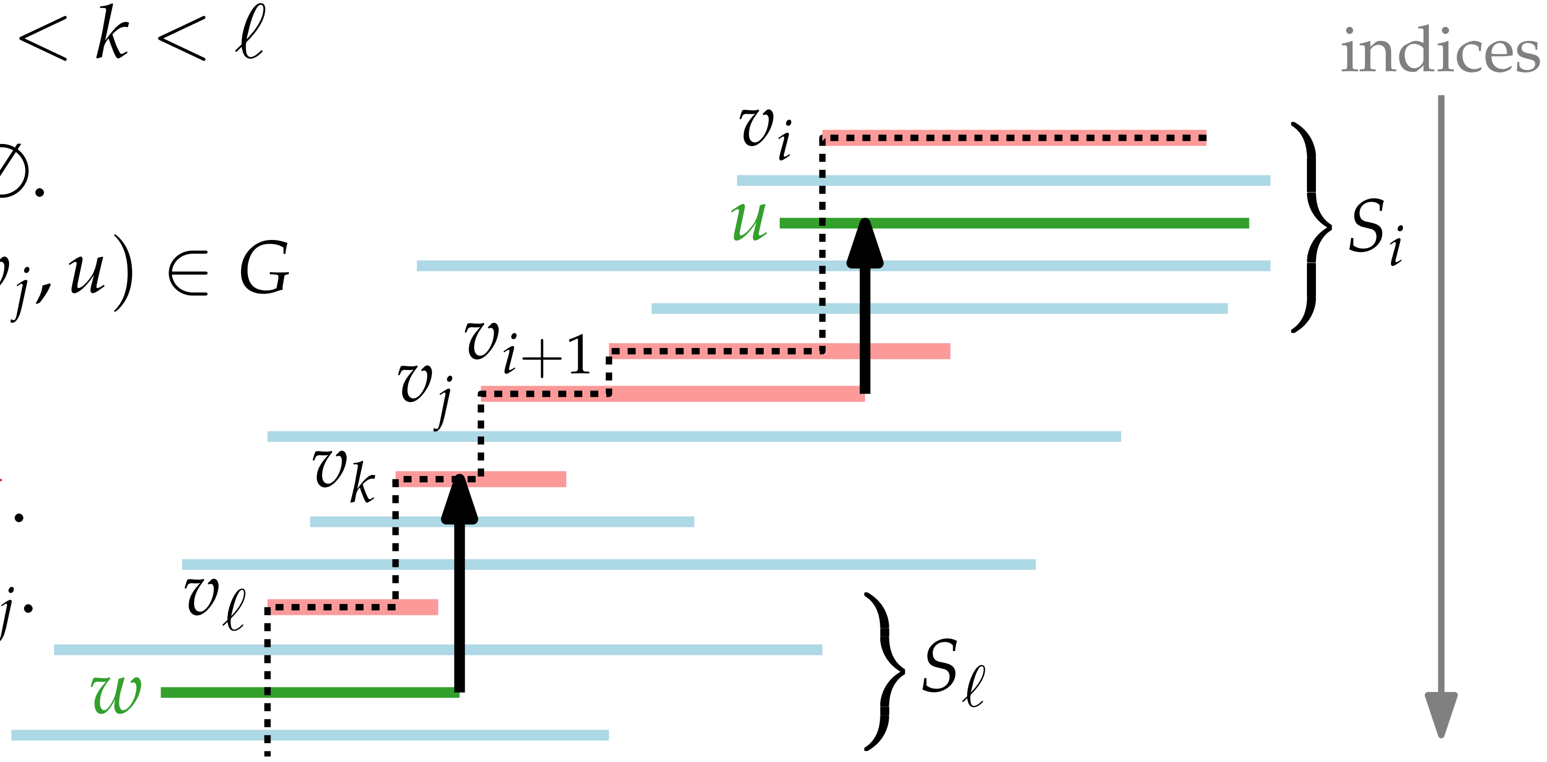

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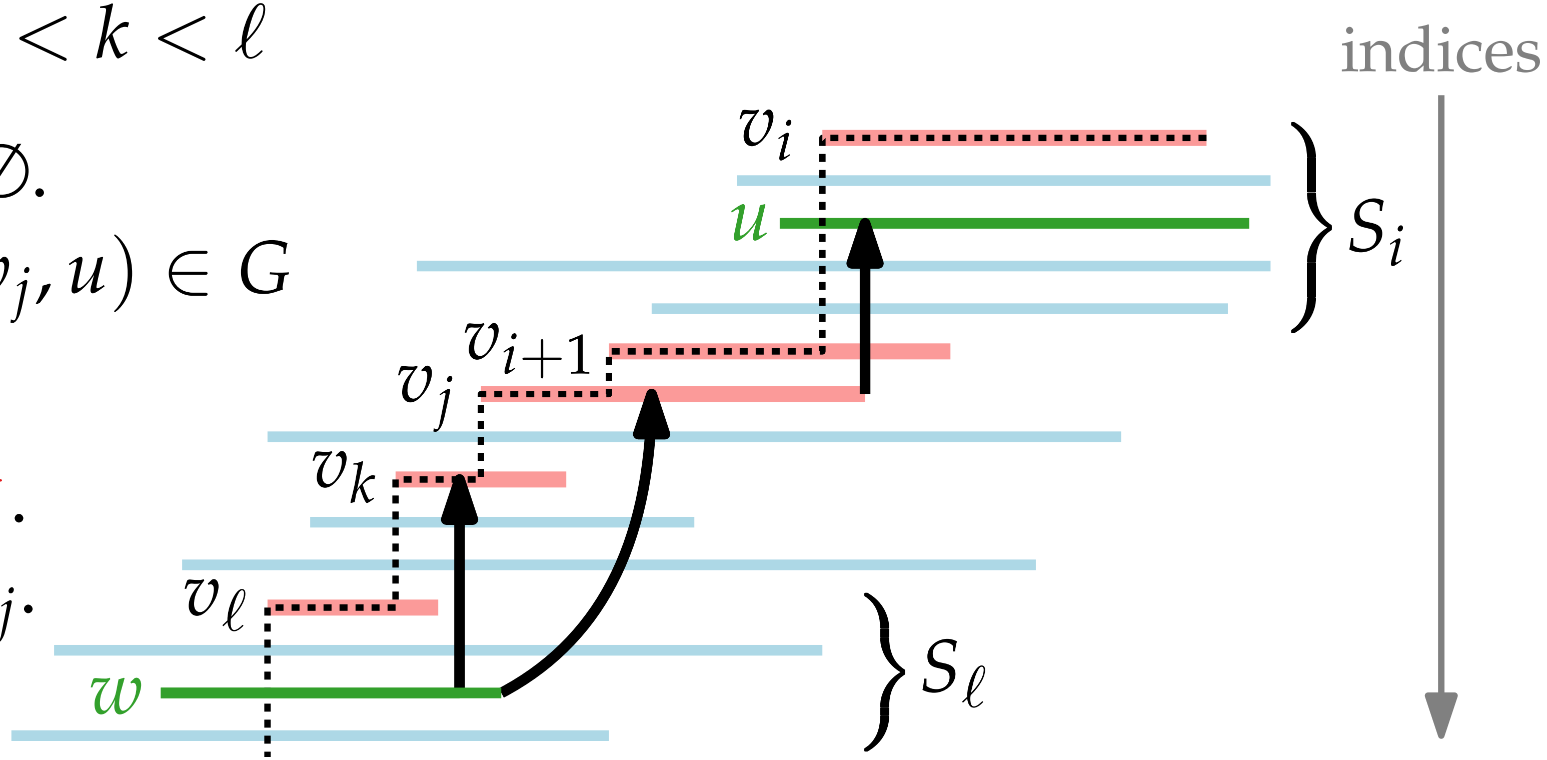

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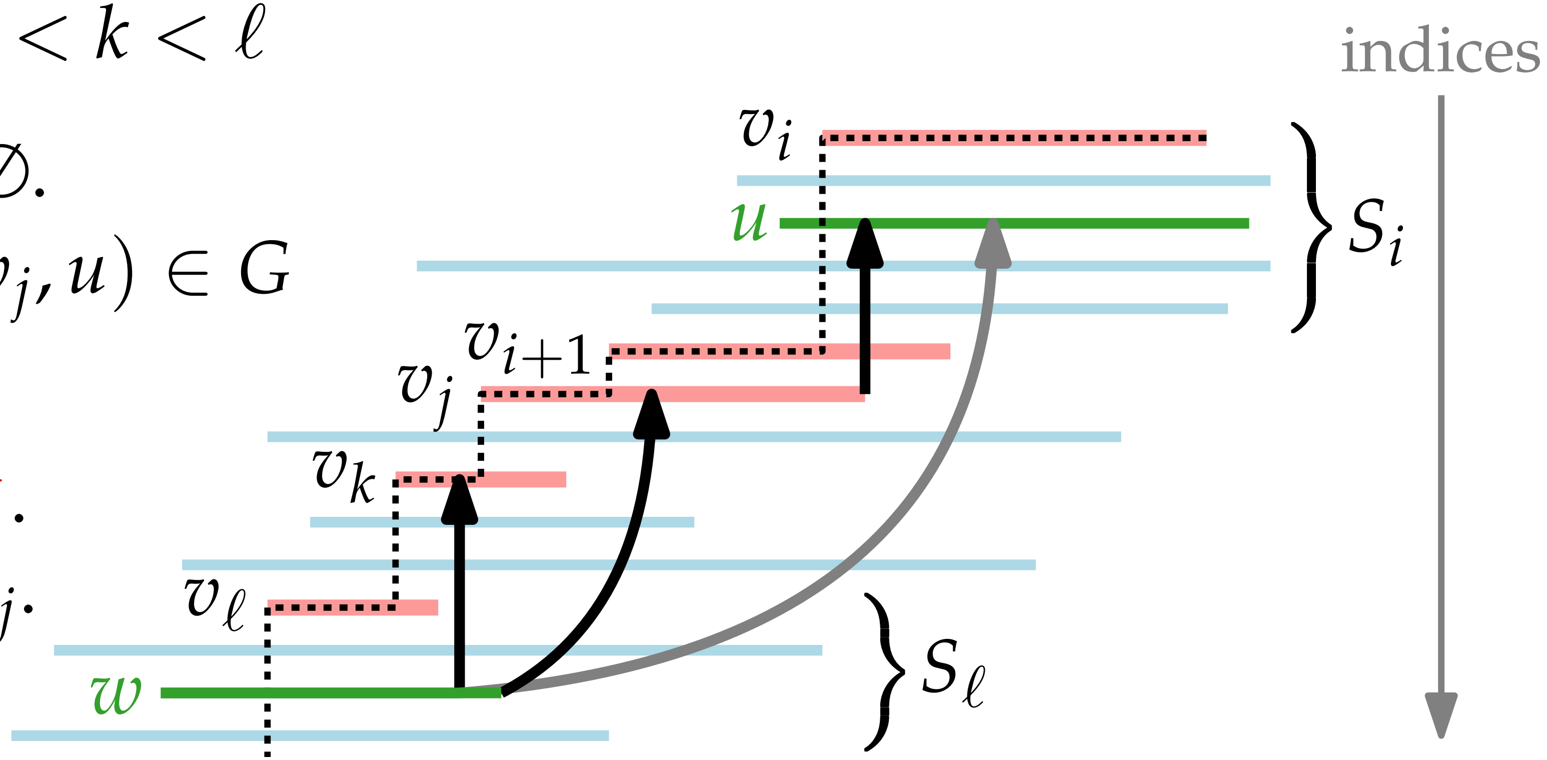
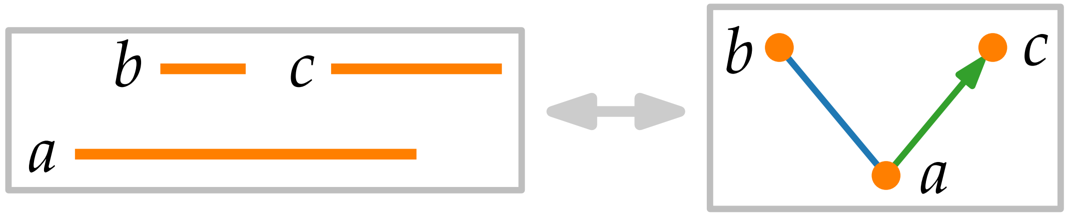


Conclusion and Open Problems



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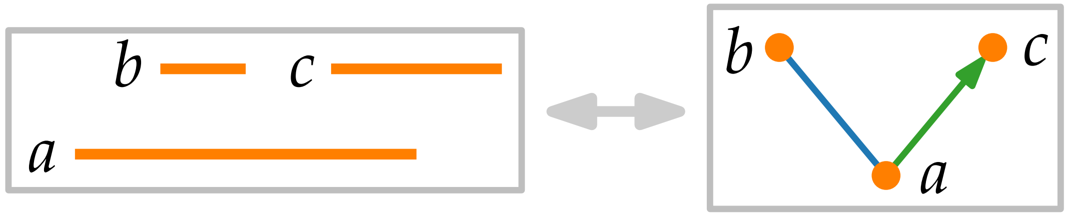

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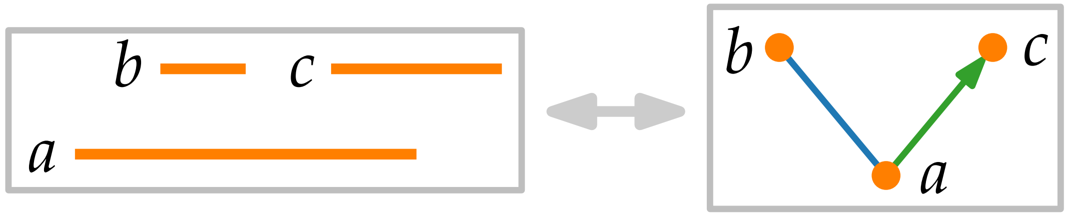

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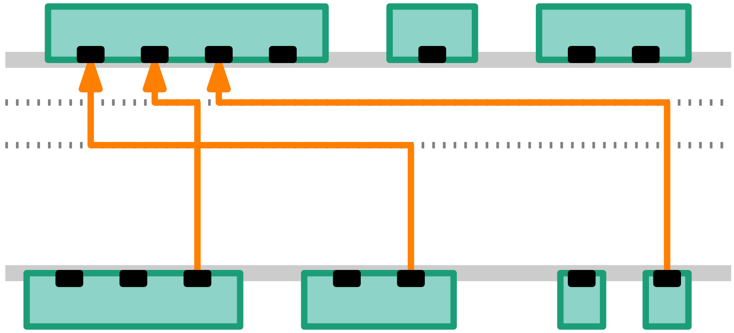

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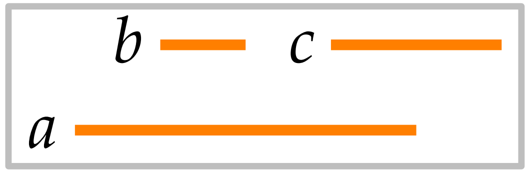

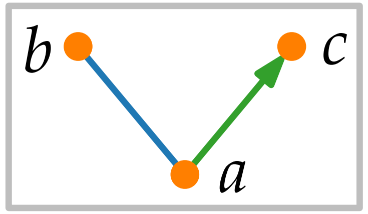

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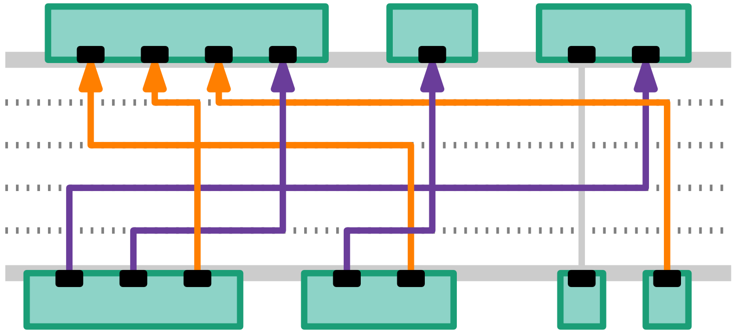

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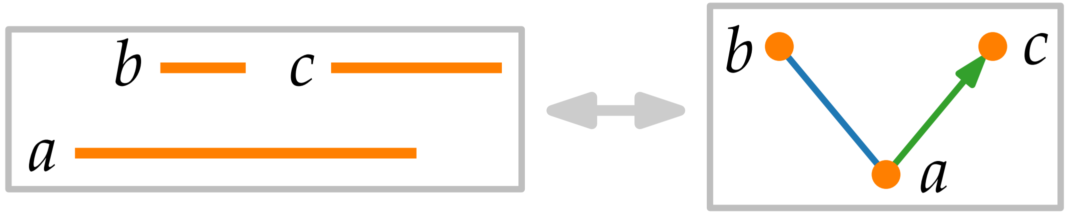

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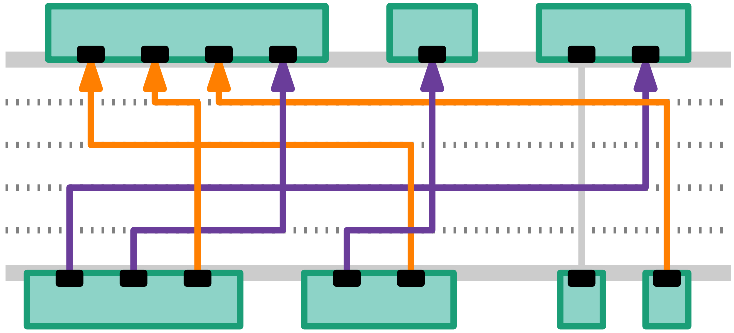

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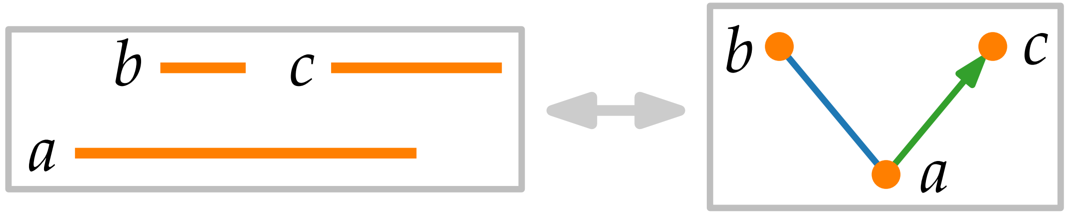

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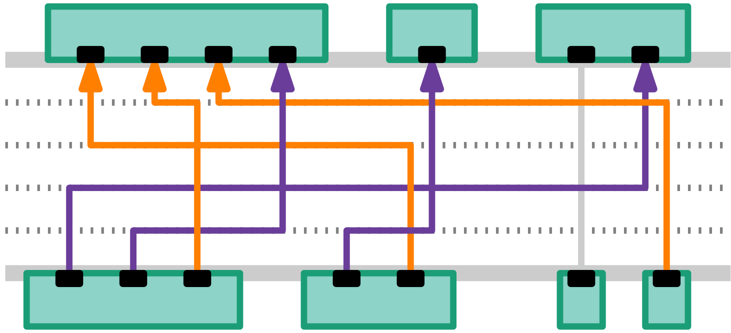

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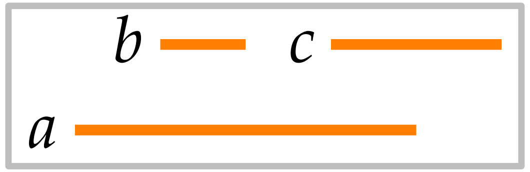

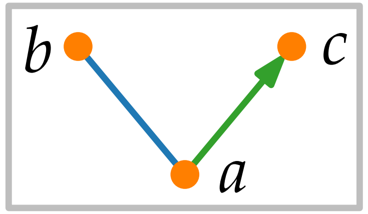

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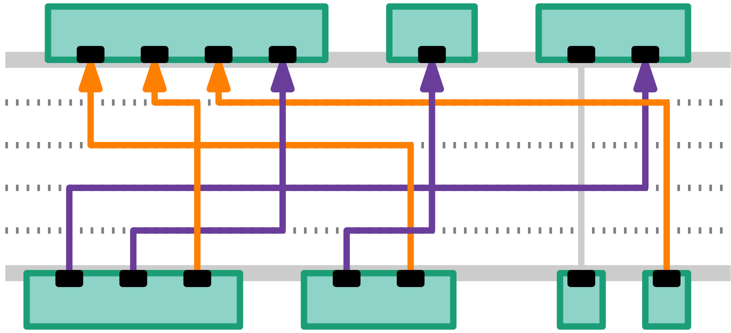

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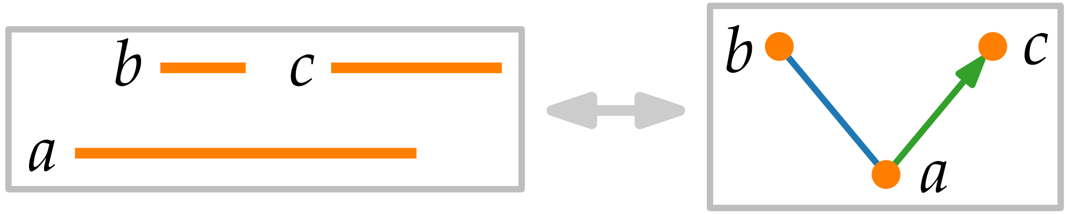

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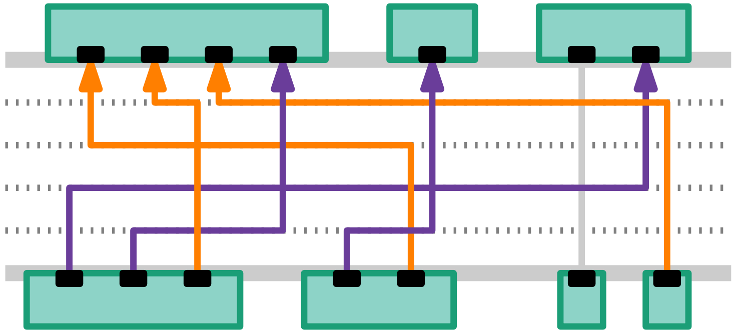

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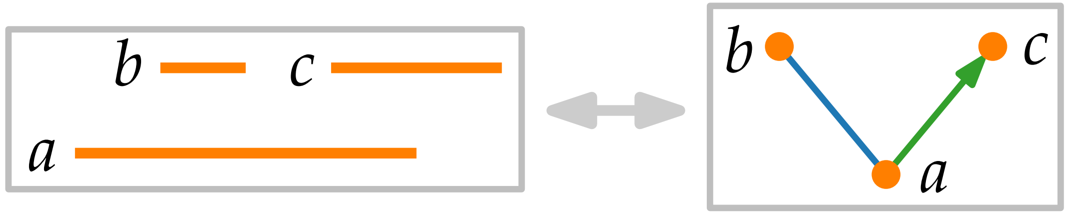

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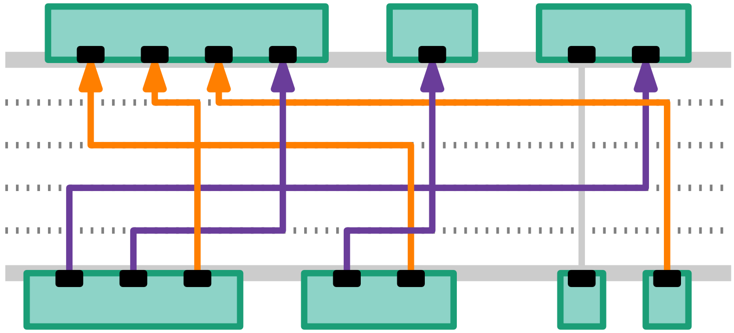

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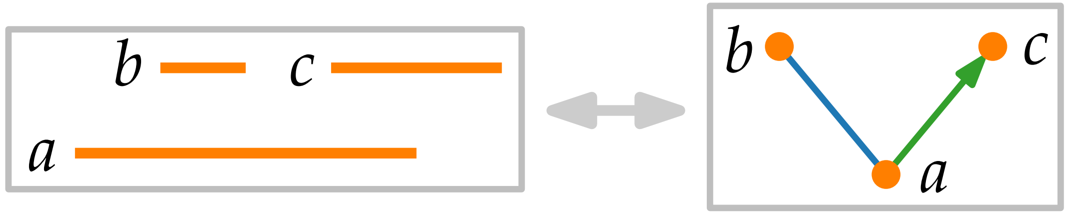

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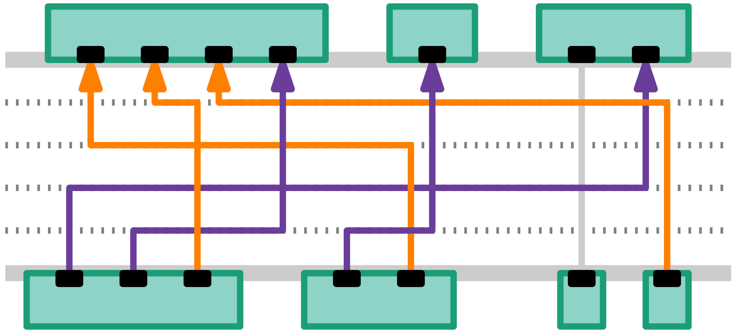

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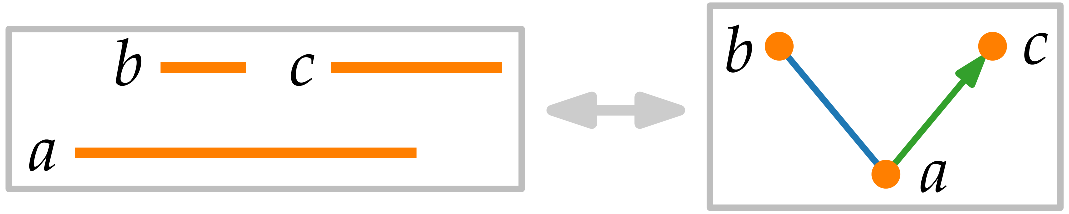

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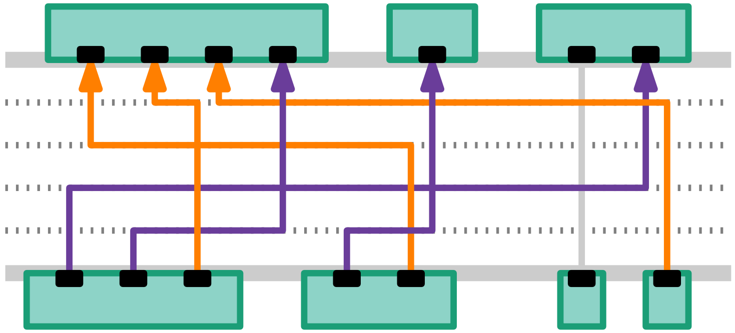

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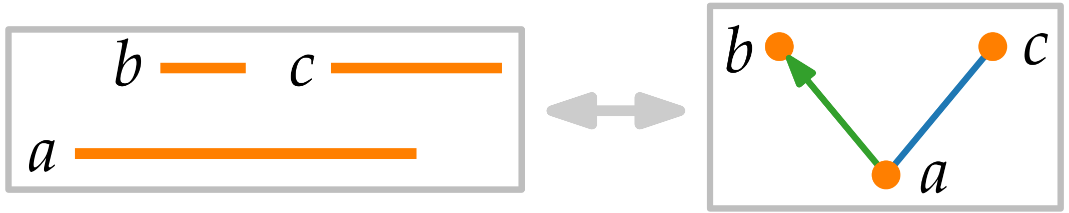

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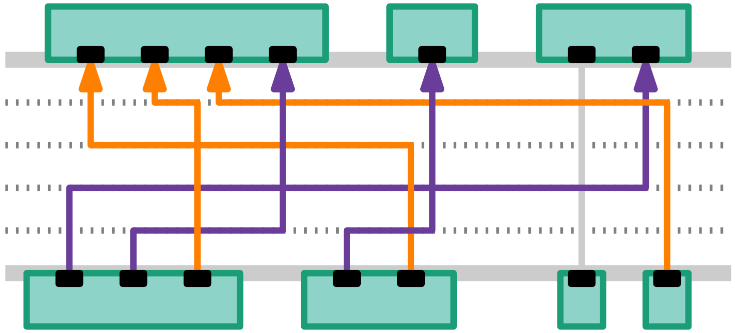

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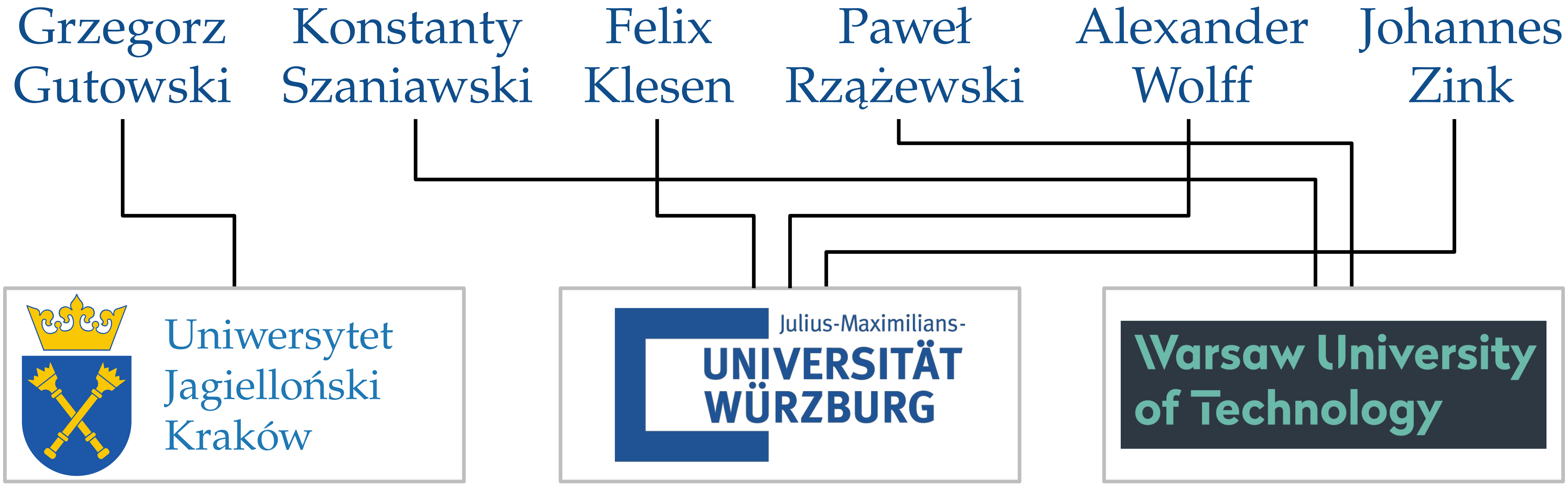
Coloring and Recognizing Mixed Interval Graphs

ISAAC 2023, Kyoto



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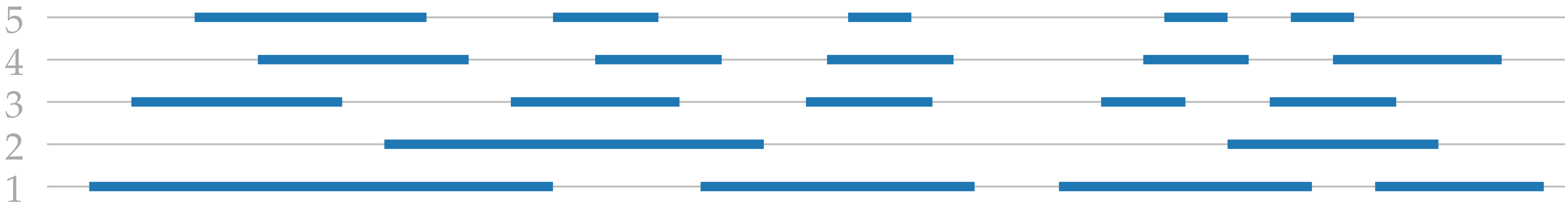
Warsaw University of Technology

Some Observation about Interval Containment Graphs



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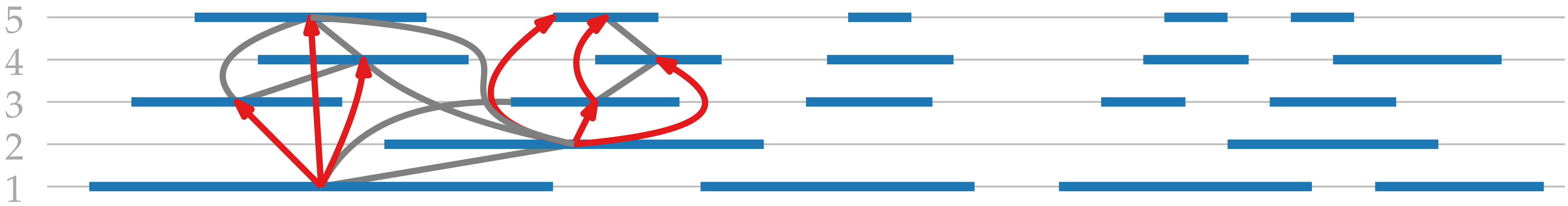

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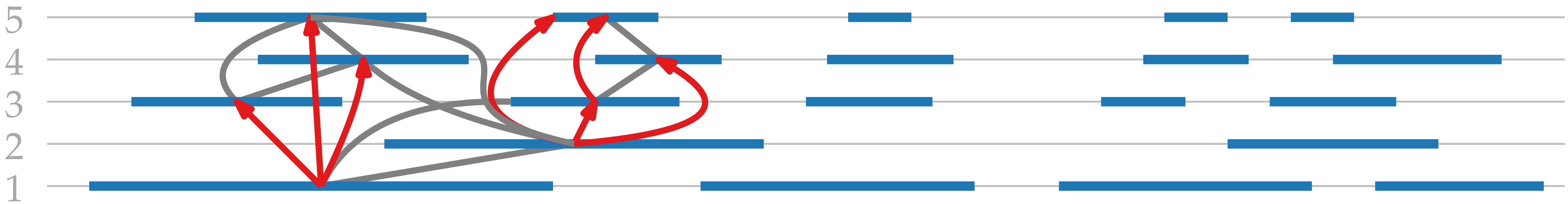

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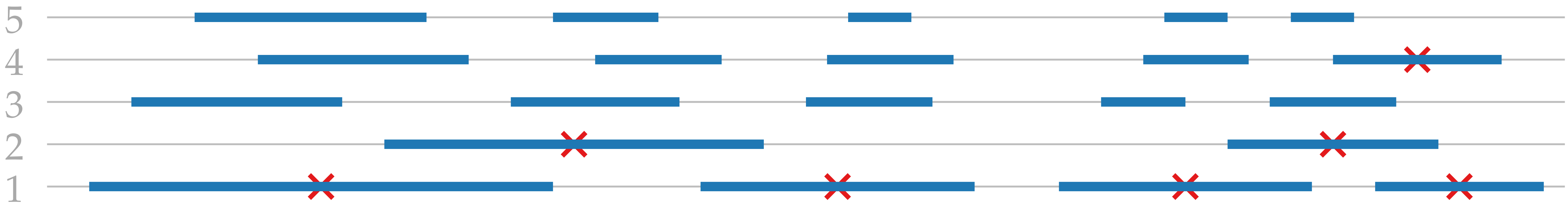

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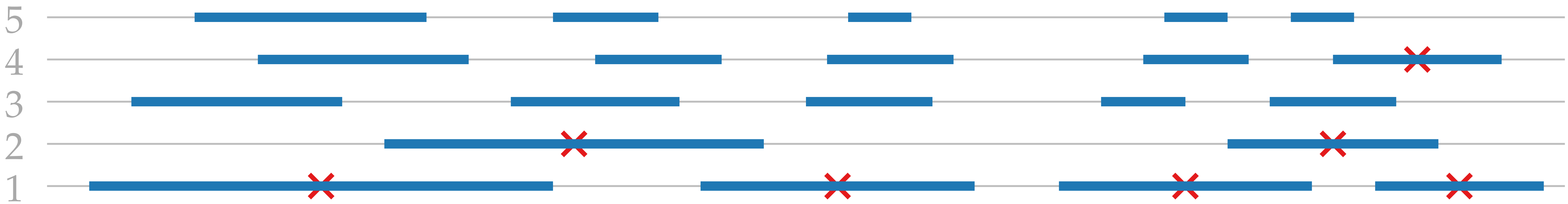

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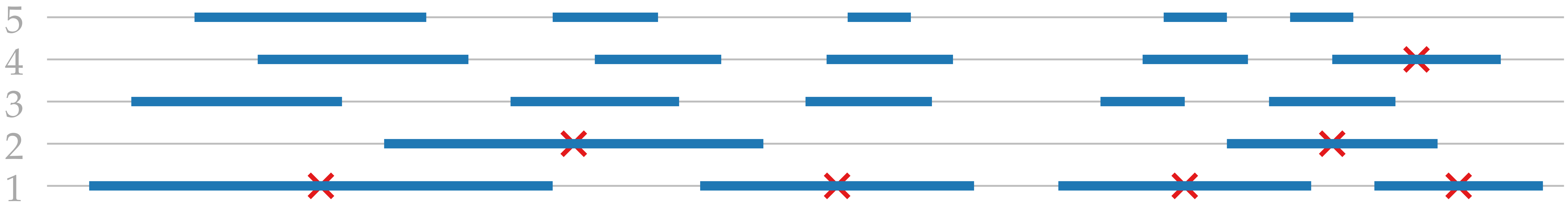

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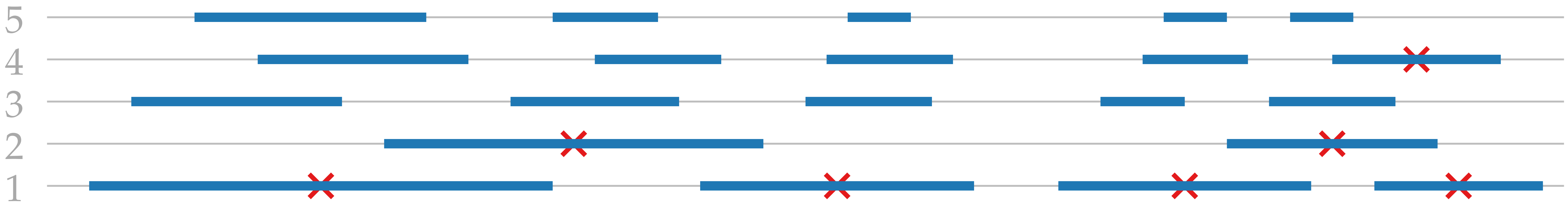

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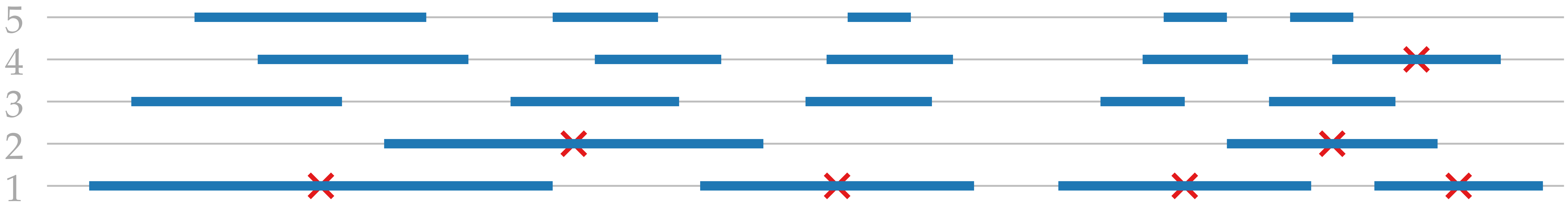

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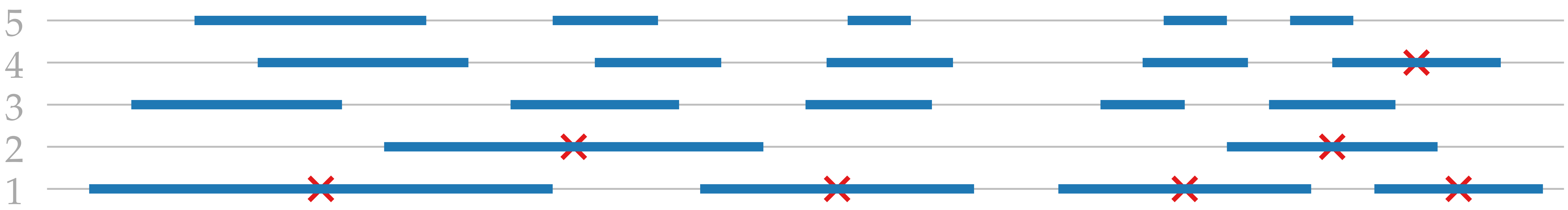

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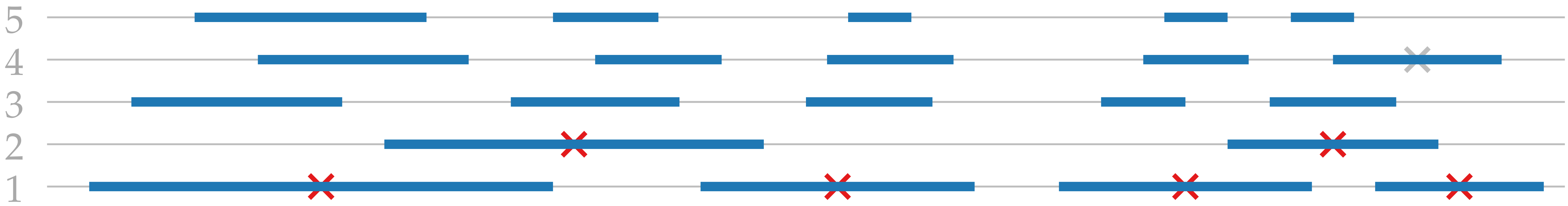

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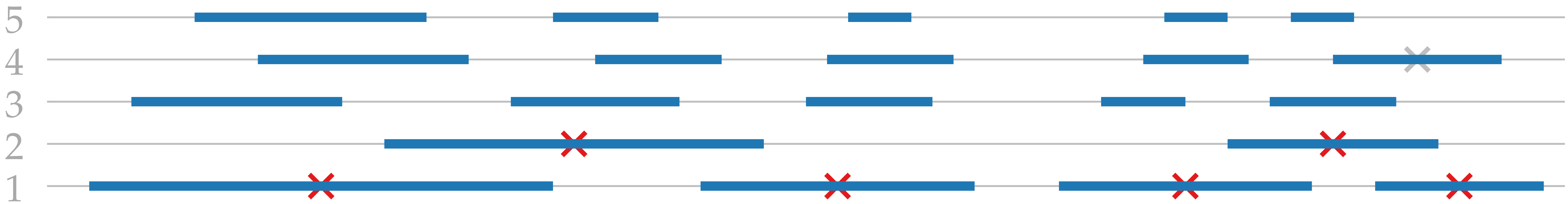

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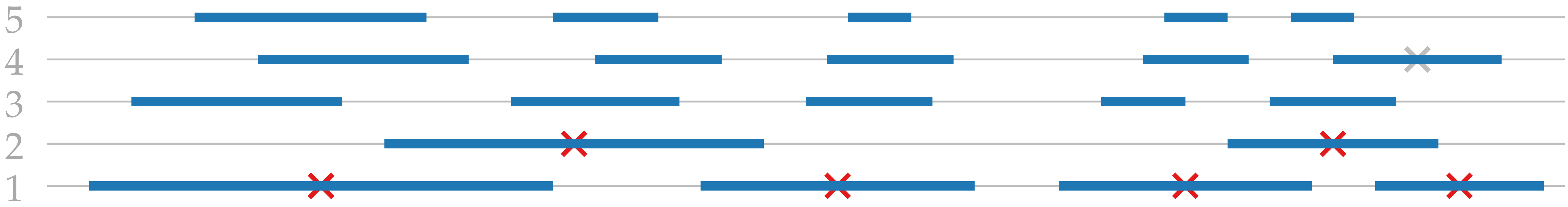

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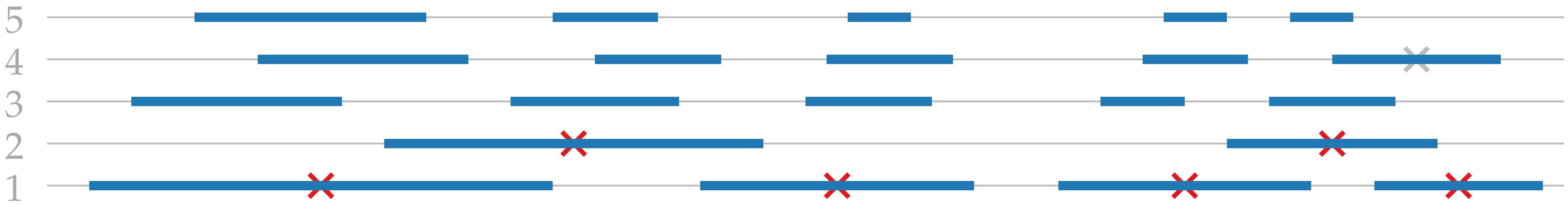

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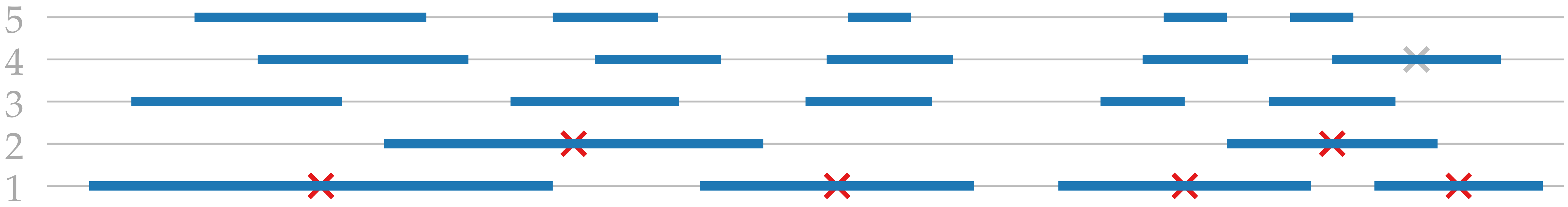

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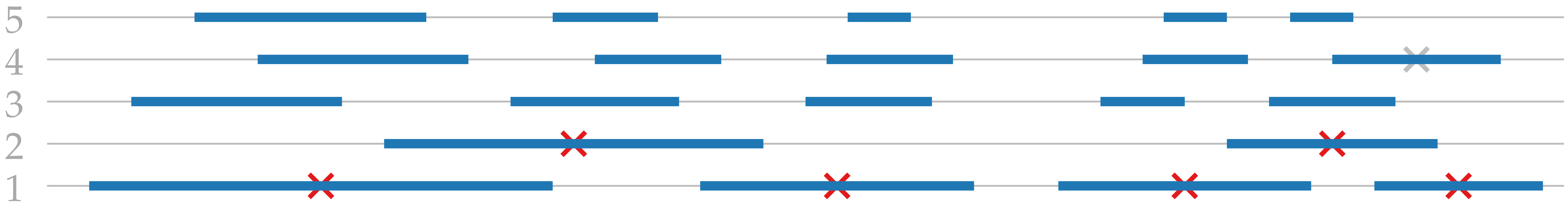

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Claim. $\mathcal{C}[R]$ is an undirected linear forest.

Proof. $\mathcal{C}[R]$ is proper $\Rightarrow$ contains no induced $K_{1,3}$ and no induced C_ℓ for $\ell \geq 4$.

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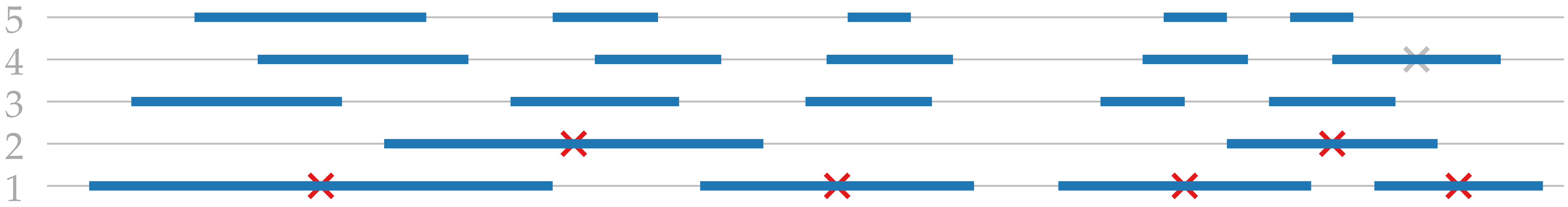

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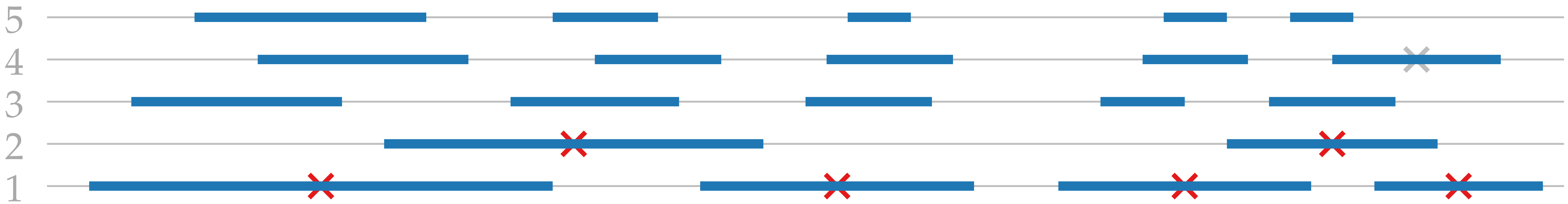

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Theorem. For any set $\mathcal{I}$ of intervals,
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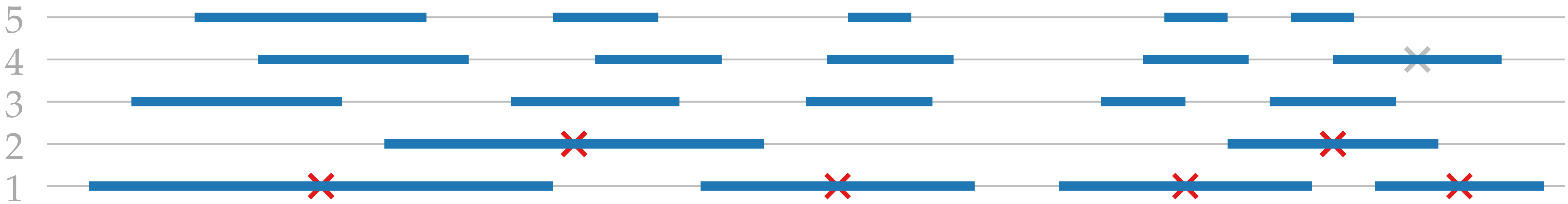
Induction $\Rightarrow$ G' admits a coloring f_2 using at most $2 \cdot \omega(G') - 1$ colors.

With f_1 and f_2, we construct a coloring f of G using colors $\{1, \ldots, 2\omega - 1\}$.

An Inductive Coloring



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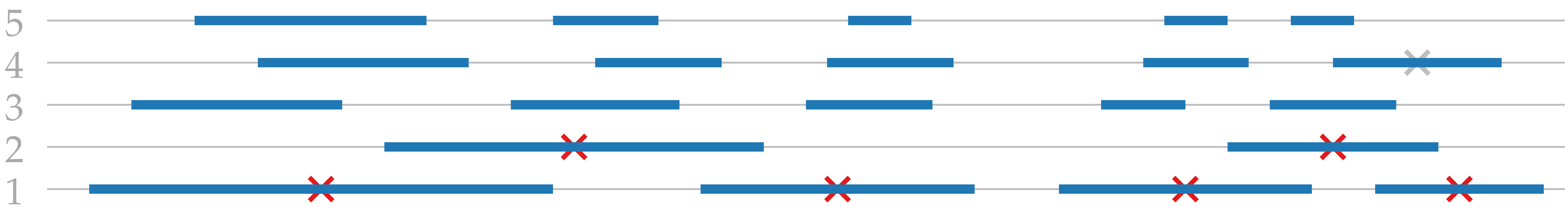

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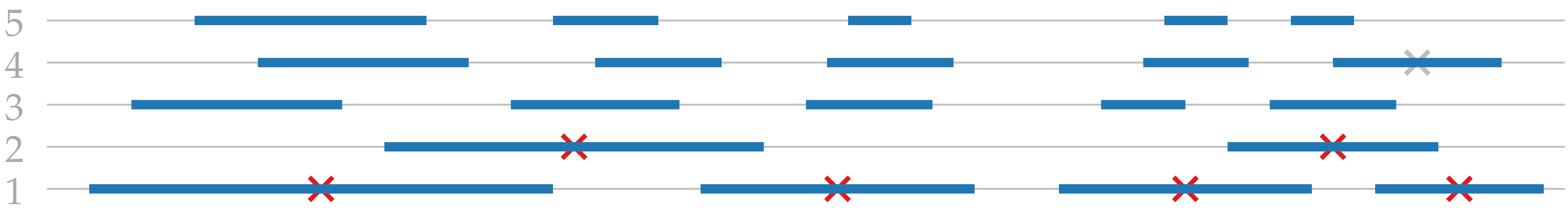

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1. If $x \cap y \neq \emptyset$, then $f(x) \neq f(y)$.

2. If $x \subseteq y$, then $f(x) > f(y)$.

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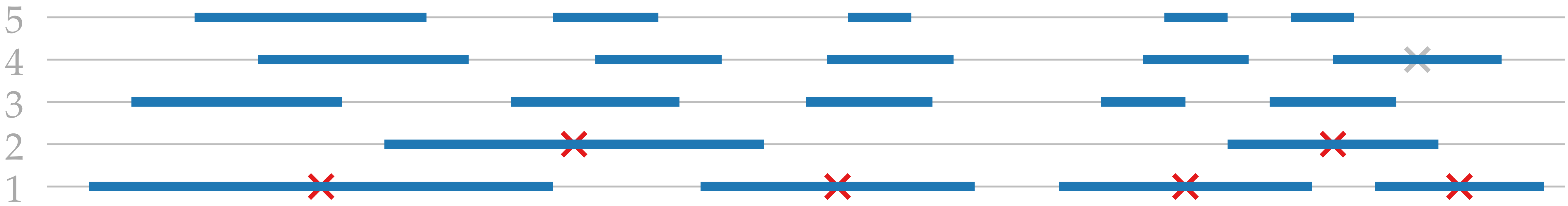

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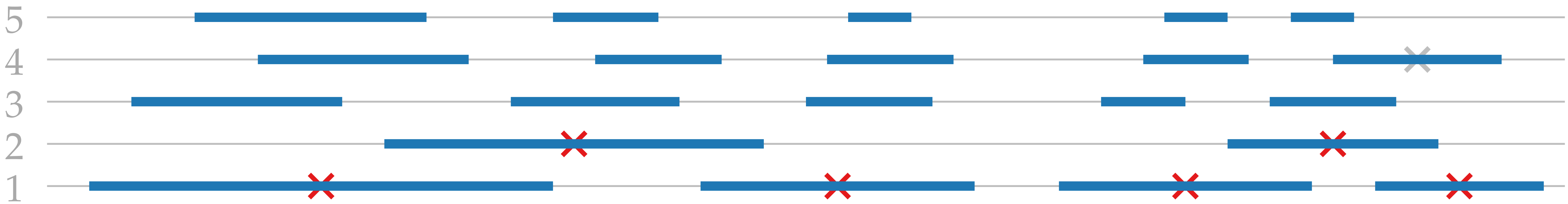

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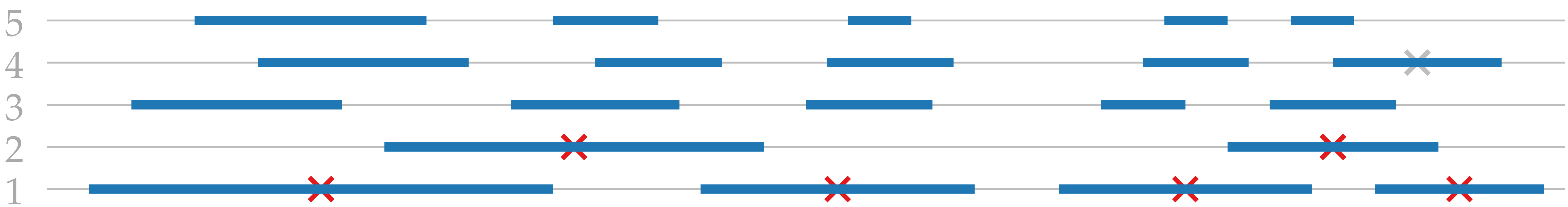

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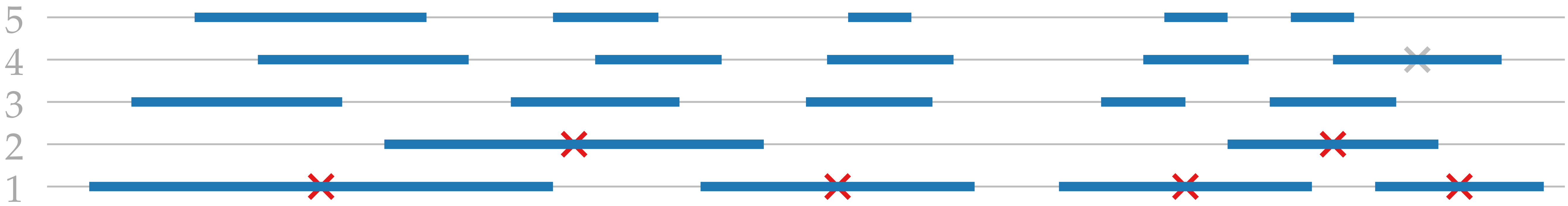

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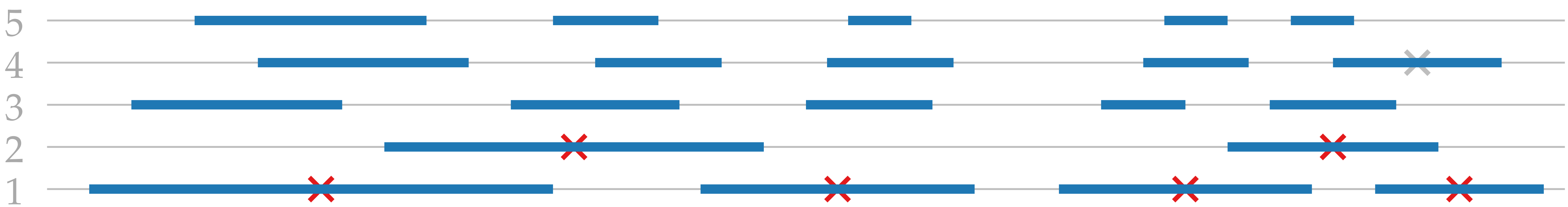

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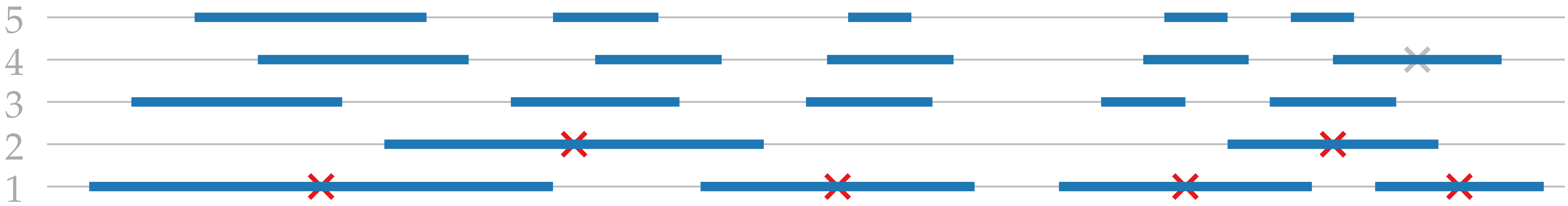

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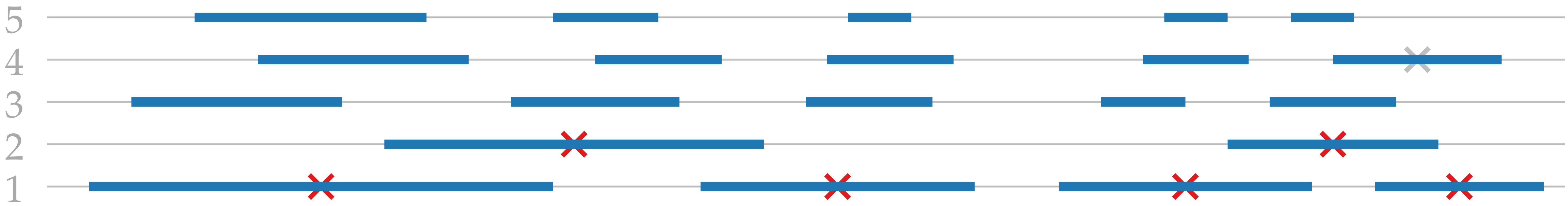

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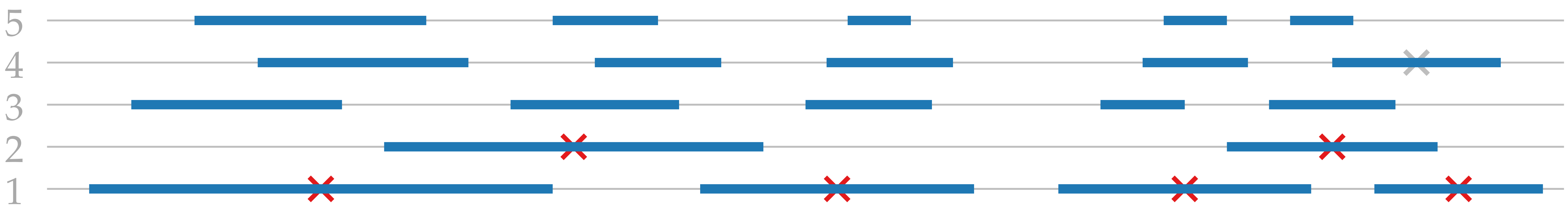

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Corollary. There is a 2-approximation for coloring interval containment graphs. Given n intervals, the algorithm runs in $O(n \log n)$ time.

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Proposition. There is an infinite family $(\mathcal{I}_n)_{n\geq 1}$ of sets of intervals with $|\mathcal{I}_n| = 3 \cdot 2^{n-1} - 2$, $\chi(\mathcal{C}[\mathcal{I}_n]) = 2n - 1$, and $\omega(\mathcal{C}[\mathcal{I}_n]) = n$.

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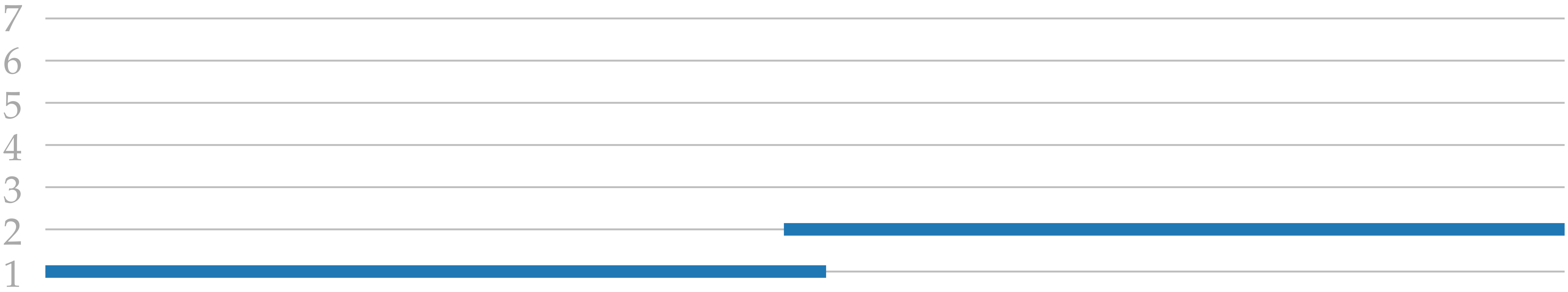

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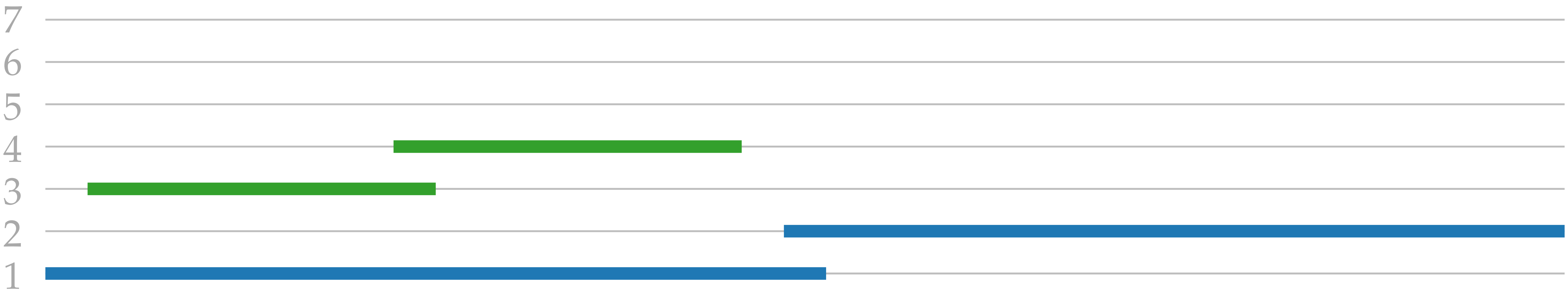

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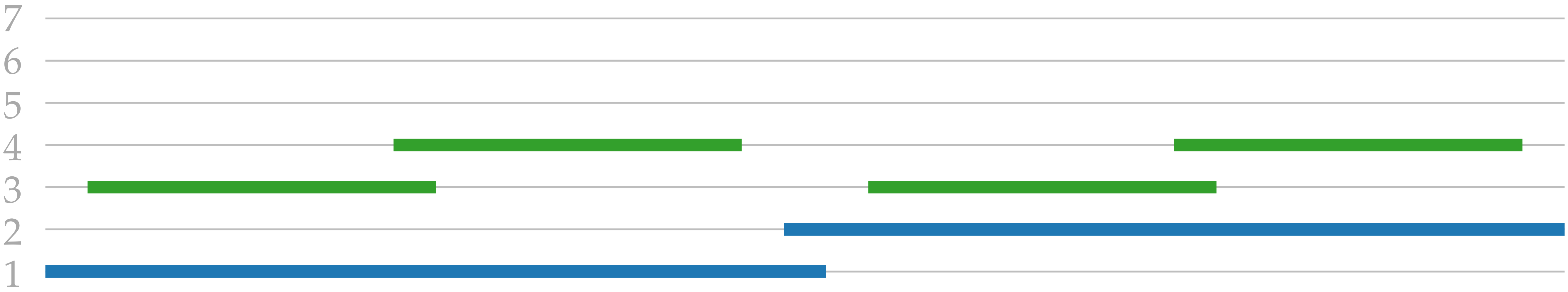

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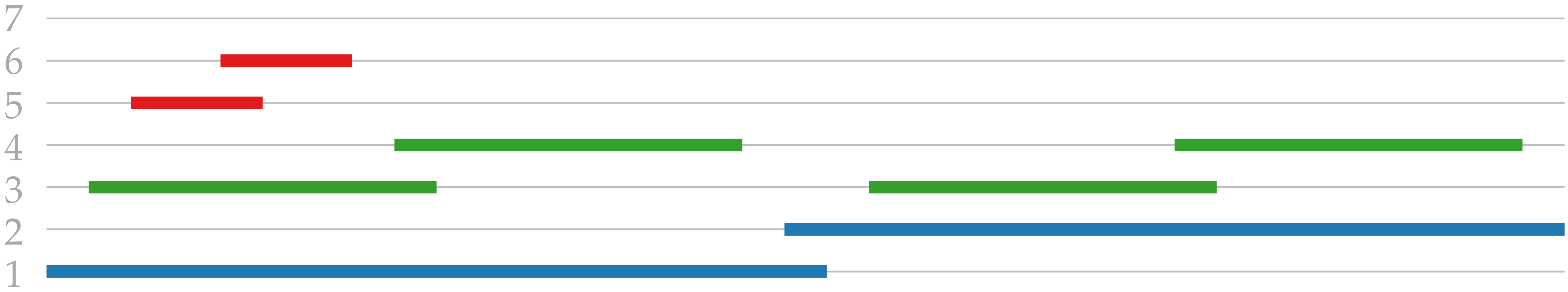

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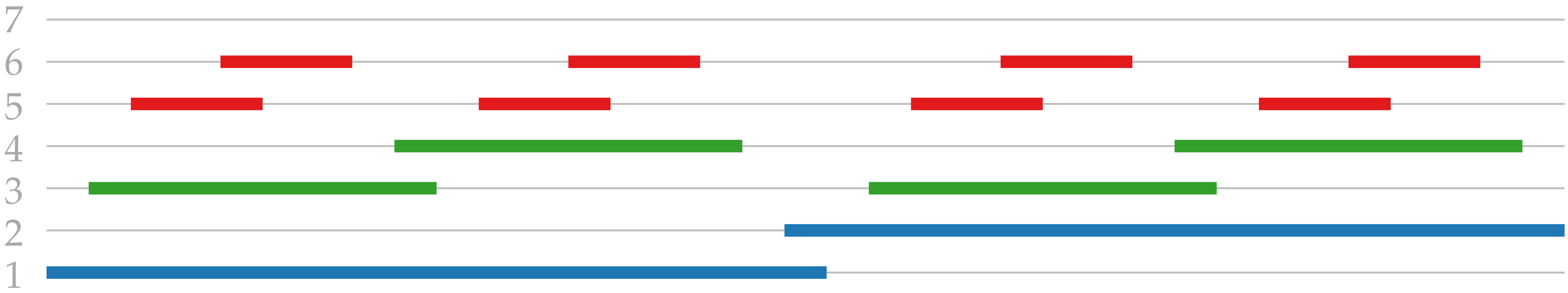

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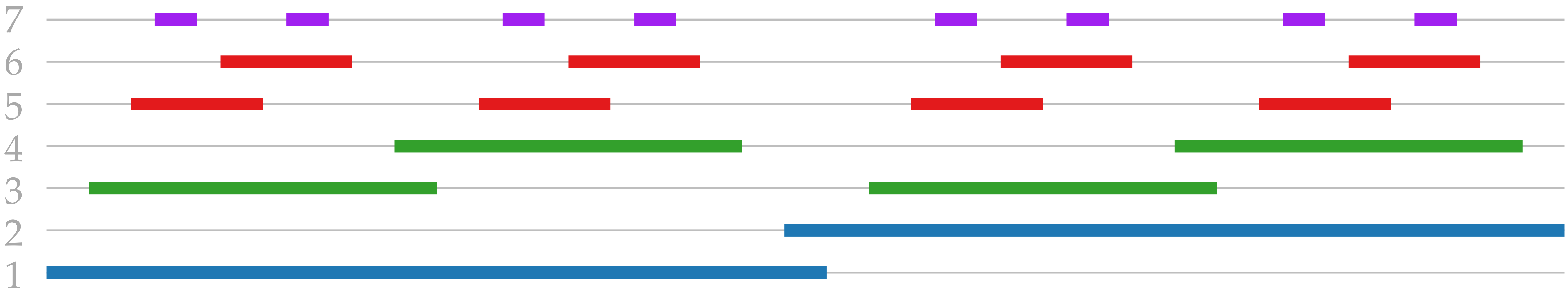

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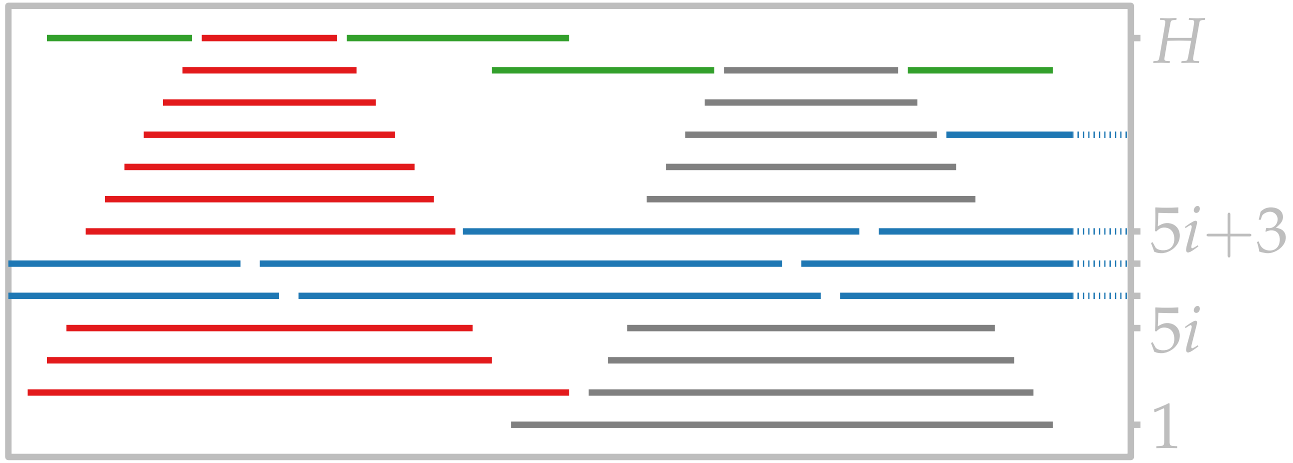

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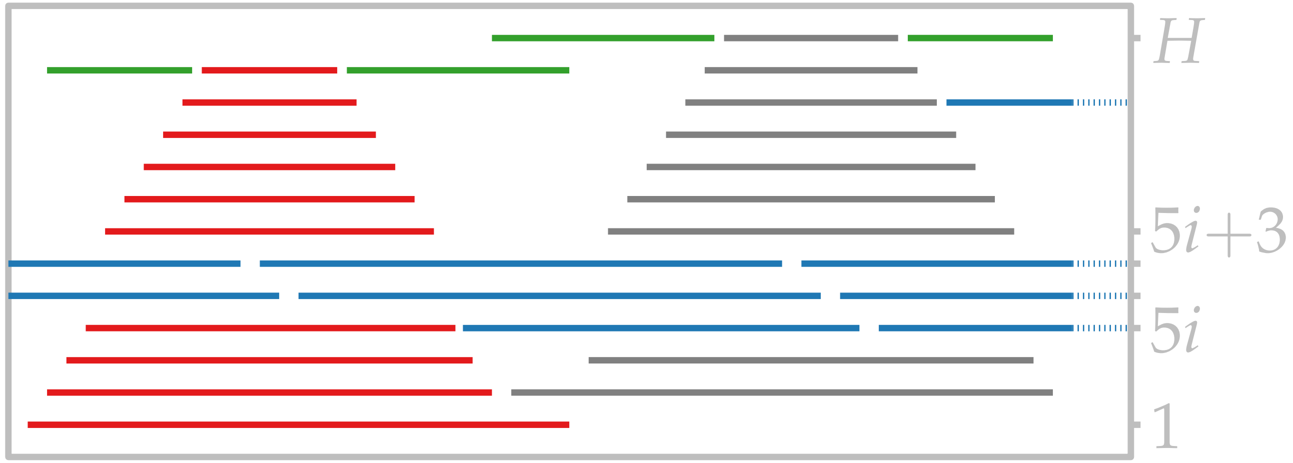

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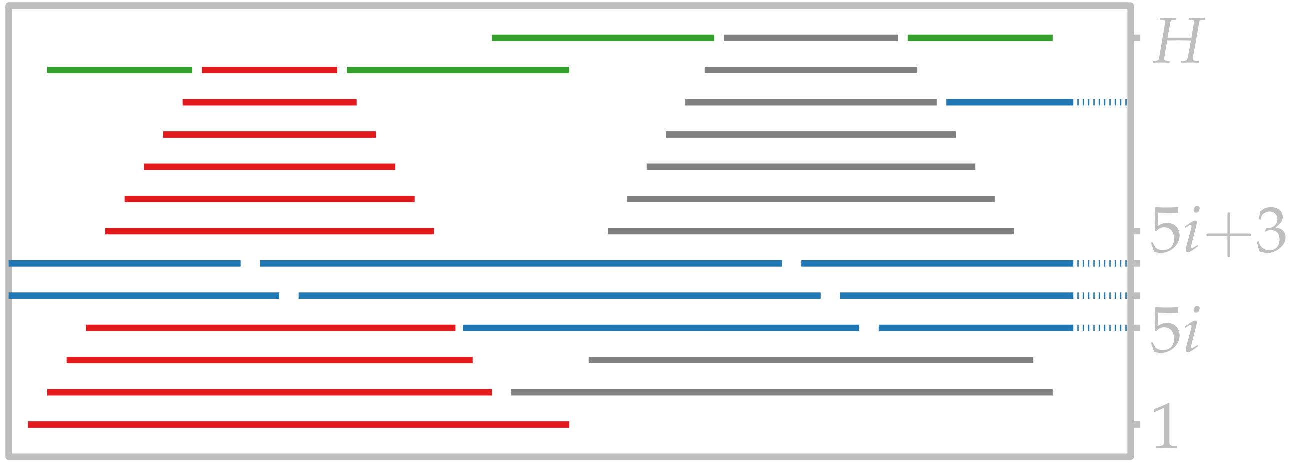

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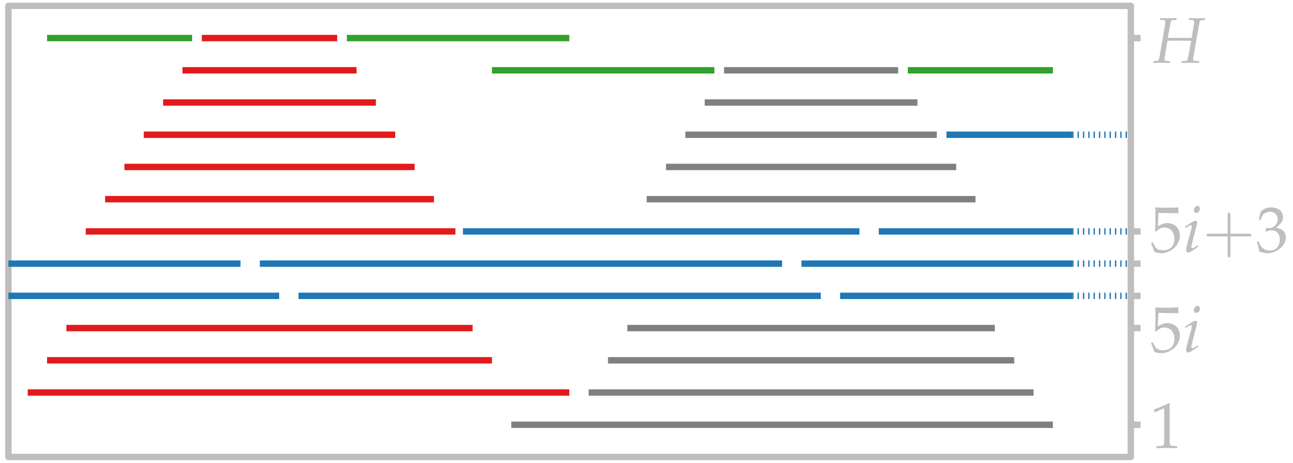

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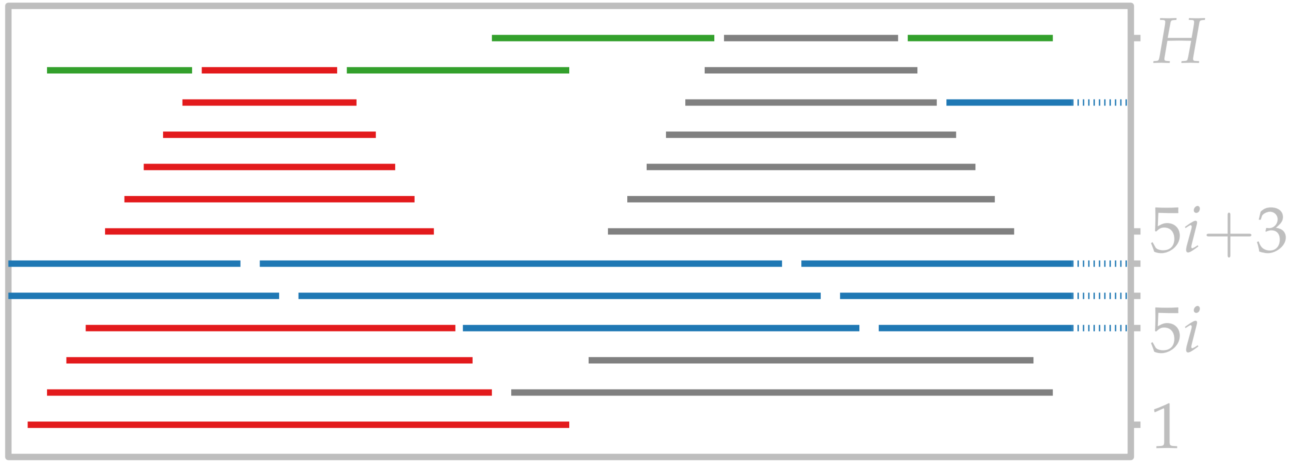

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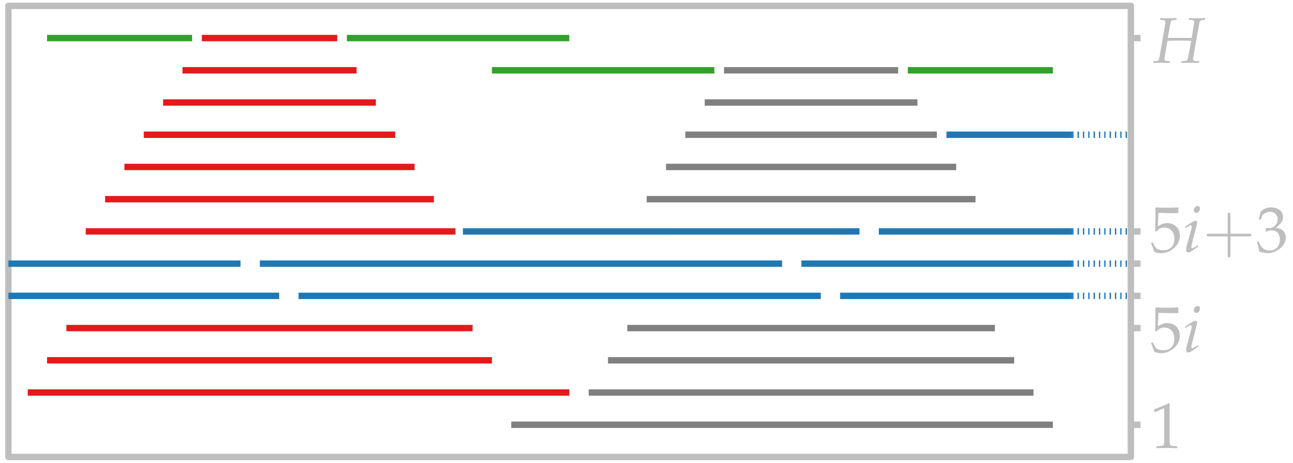

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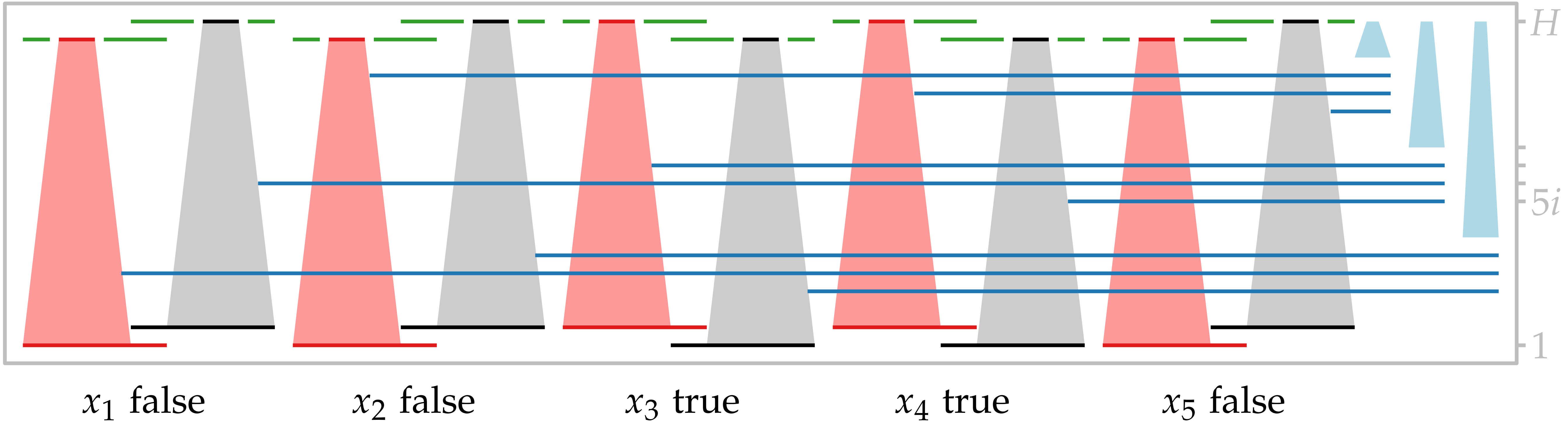
Clause Gadget



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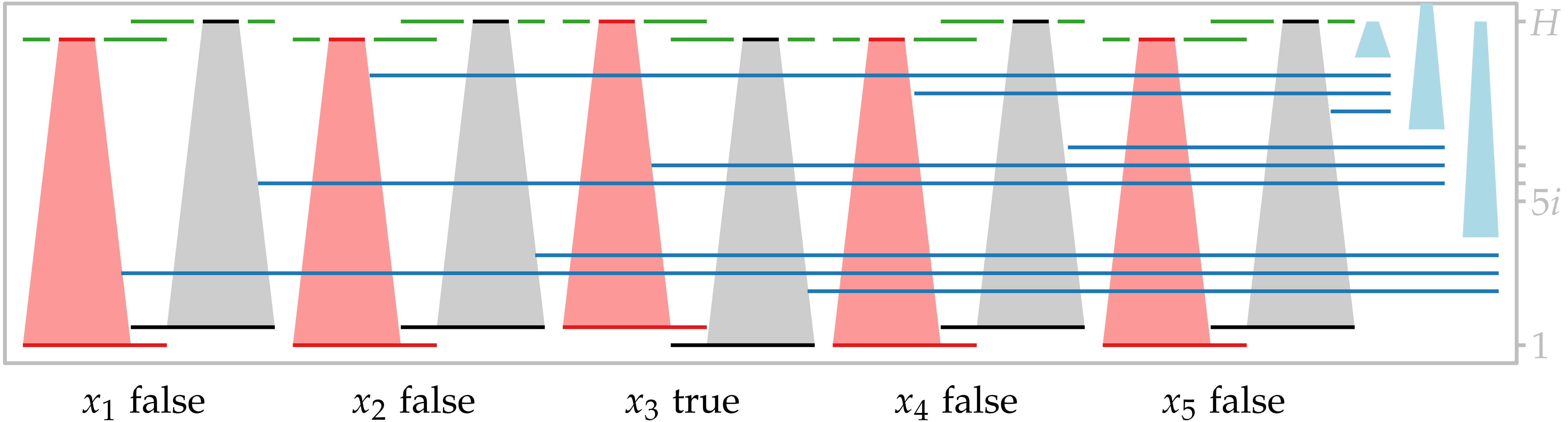

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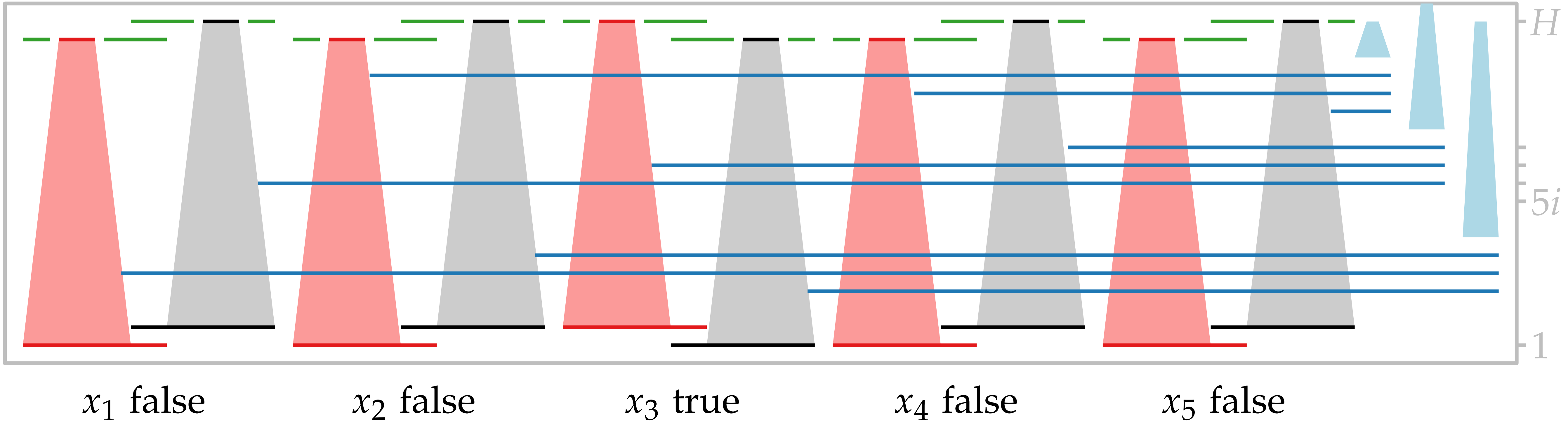

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The graph $\mathcal{C}[\mathcal{I}_\varphi]$ admits a coloring with H colors $\Leftrightarrow$ φ is satisfiable. □

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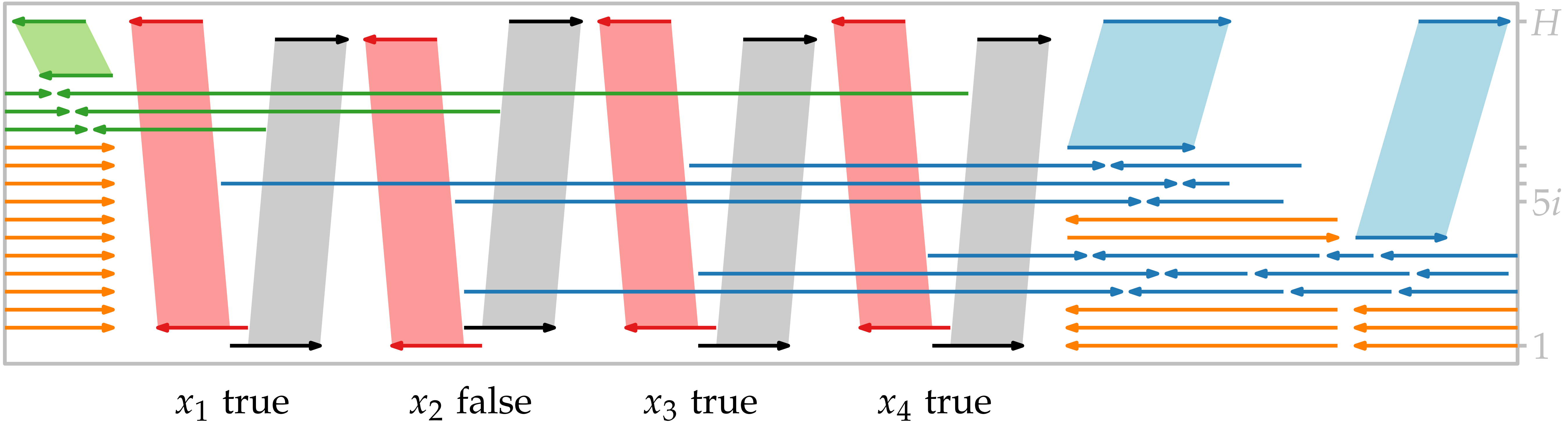

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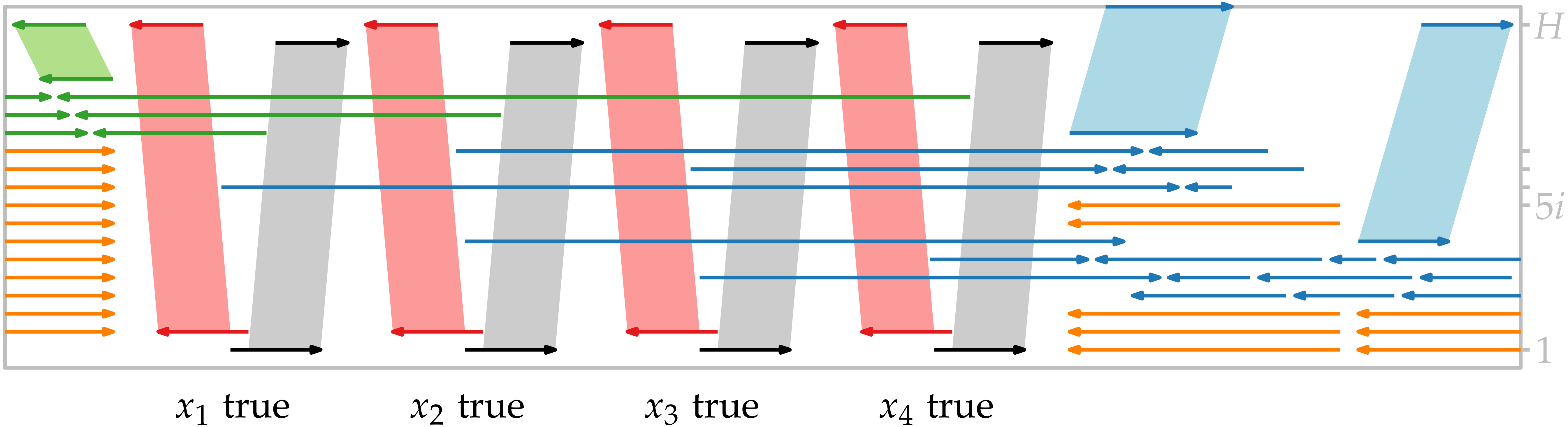

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For $i = 1, 2, \dots$, let L_i be the set of sources in $G^{\rightarrow} \setminus \bigcup_{j=0}^{i-1} L_j$.

Note that $\lambda(G) = \max\{i \colon L_i \neq \emptyset\}$.

For $x \in V(G)$, let $\ell(x) \in \{0, \dots, \lambda(G)\}$ be the *layer* of x.

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A Lower Bound Example

Proposition. There is an infinite family $(G_k)_{k \geq 1}$ of mixed interval graphs with $|V(G_k)| = 2k^2$, $\lambda(G_k) = k - 1$, $\omega(G_k) = 2k$, and $\chi(G_k) = (k+1) \cdot k = (\lambda(G_k) + 2) \cdot \omega(G_k)/2$.

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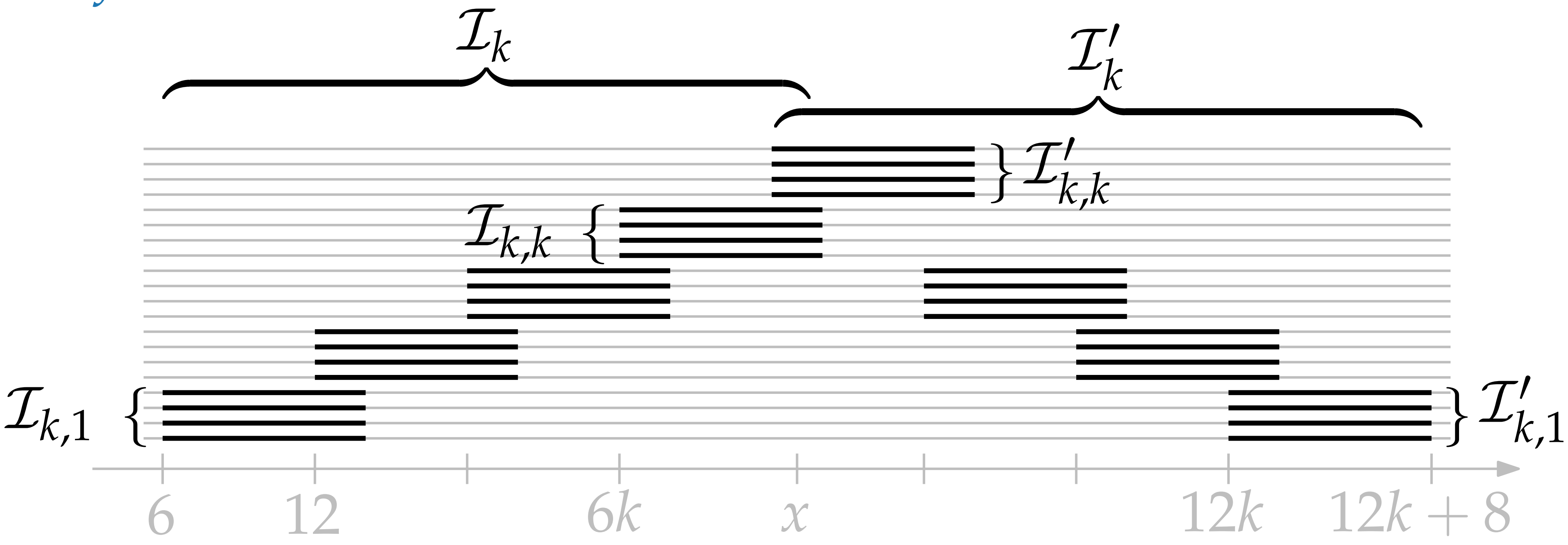

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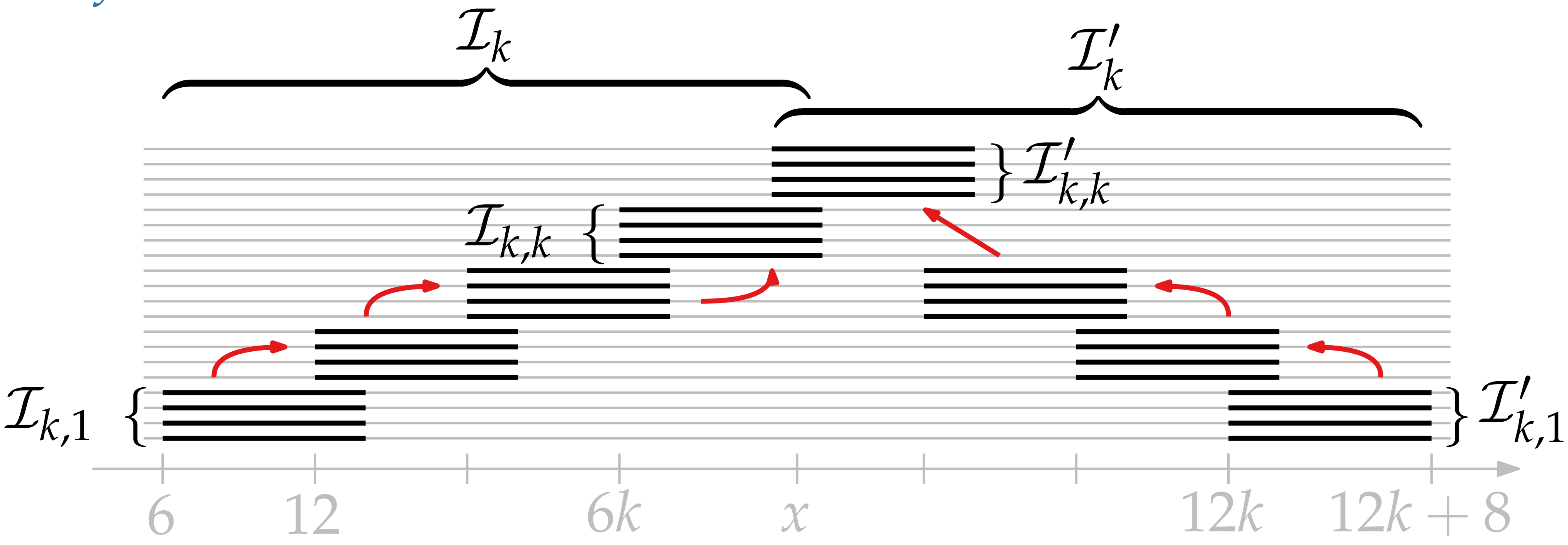

Summary

Mixed interval	Coloring				Recognition
graph class	complexity	lower bound	upper bound	approximation	
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directional	$O(n \log n)$			1	$O(n^2)$
bidirectional	NP-hard			2	open
general	NP-hard	$(\lambda+2)\omega/2$	$(\lambda+1)\omega$	$\min\{\omega, \lambda+1\}$	$O(n+m)$ [LB79]

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- Given a mixed graph G with an orientation φ, we can decide in linear time whether G admits an oriented interval representation that complies with φ.

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- In particular, we can recognize directional interval graphs in linear time.