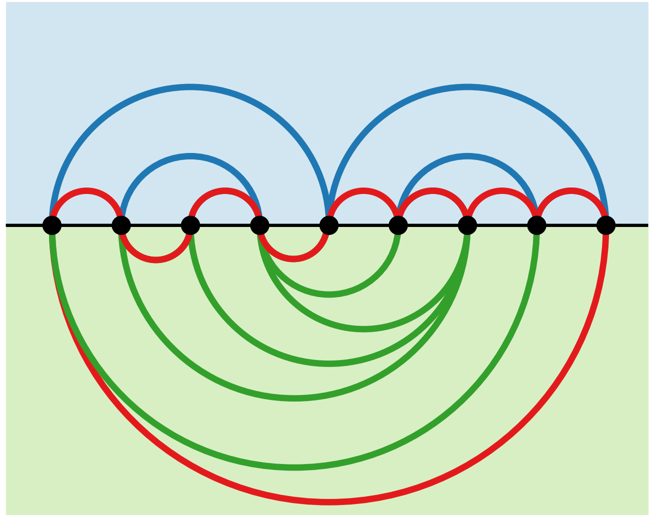
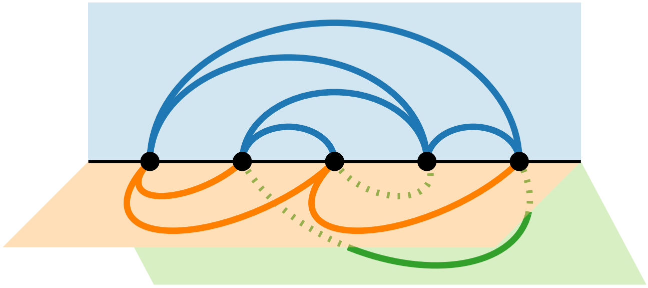
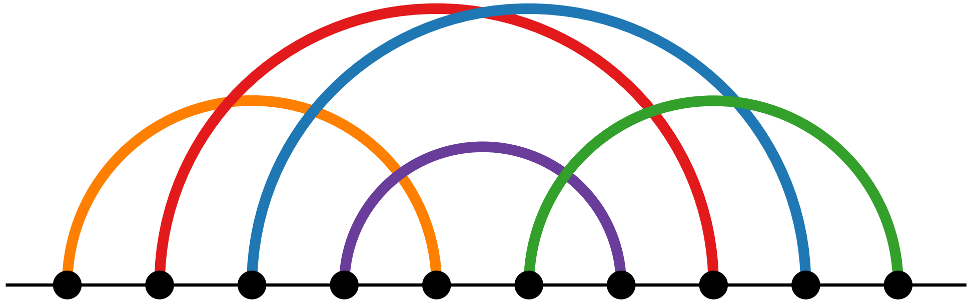
Visualization of Graphs

Lecture 12:
Linear Layouts
(Book Embeddings)



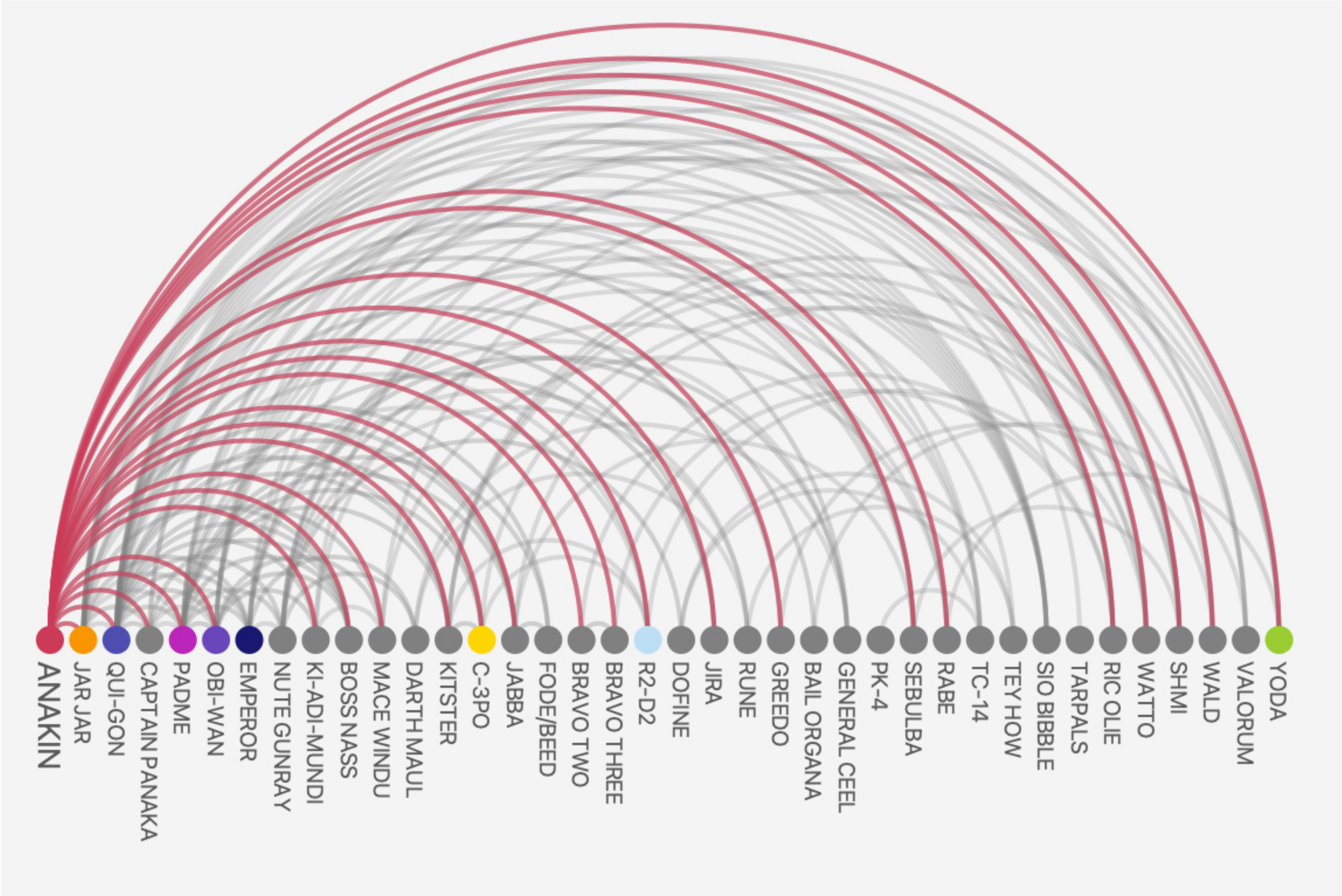




Alexander Wolff

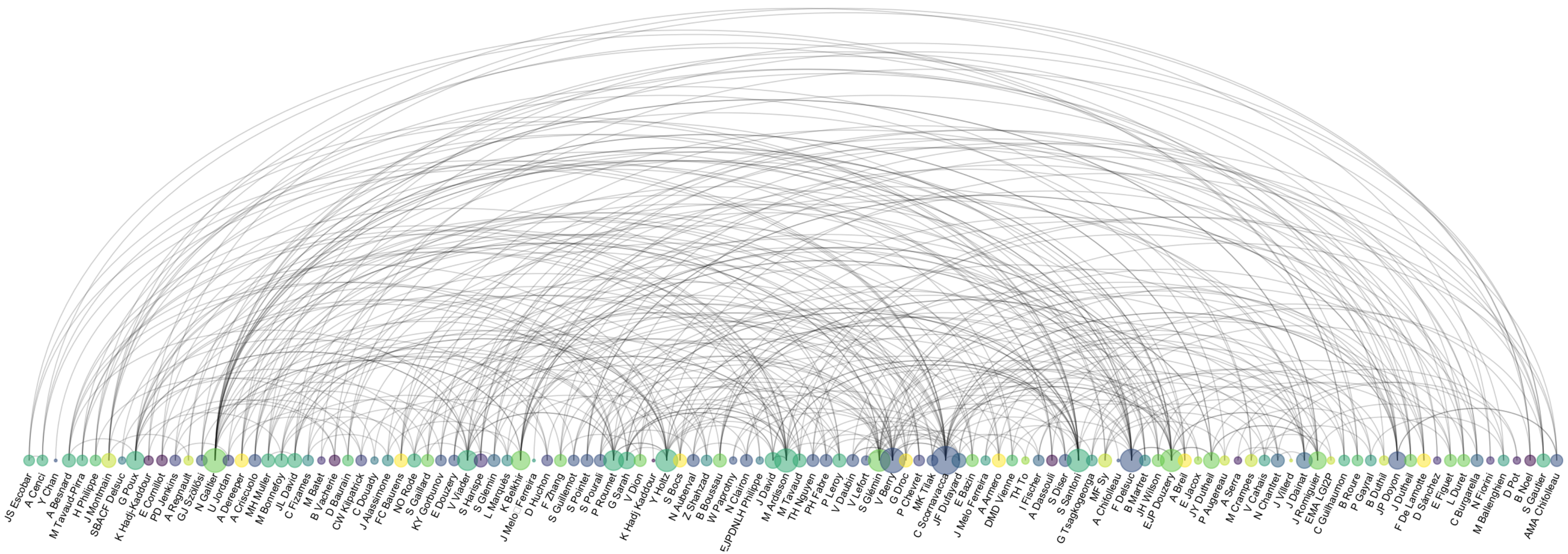
Summer term 2025

Drawing Style: Arc Diagrams



Interactions in Star Wars Episode I
[https://harmoniccode.blogspot.com/2020/11/arc-charts.html]

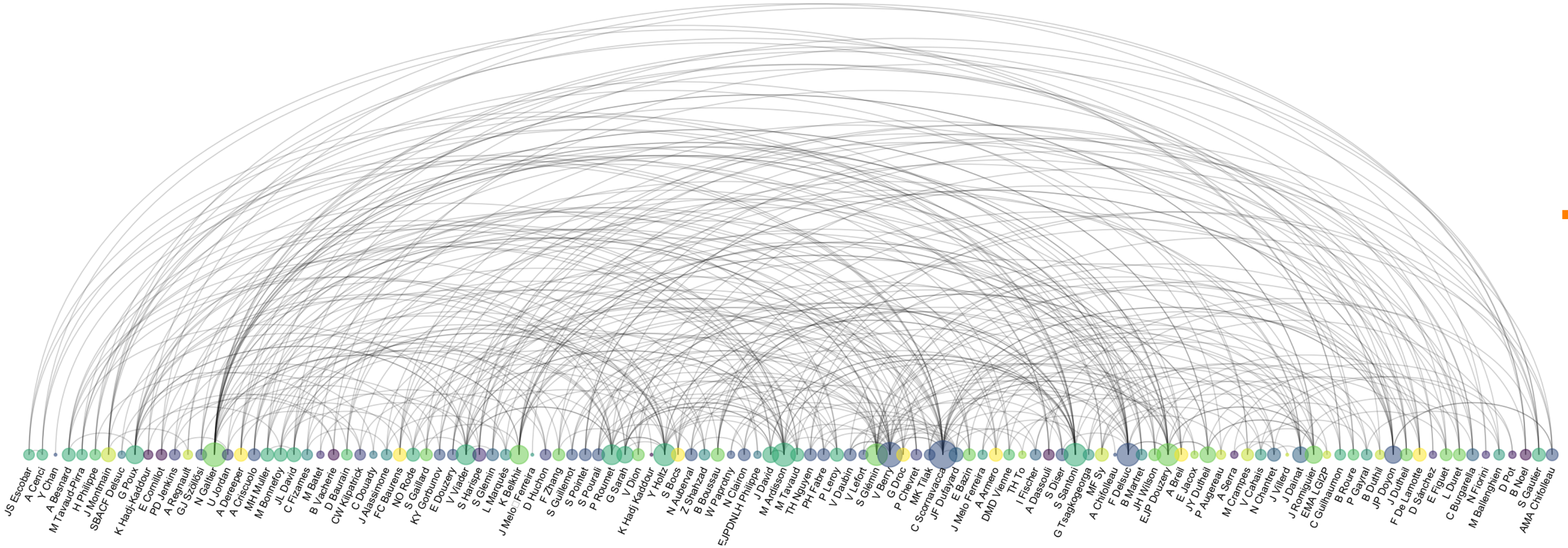
Drawing Style: Arc Diagrams



Network of co-authors of Vincent Ranwez (edge ⇔ co-authors)

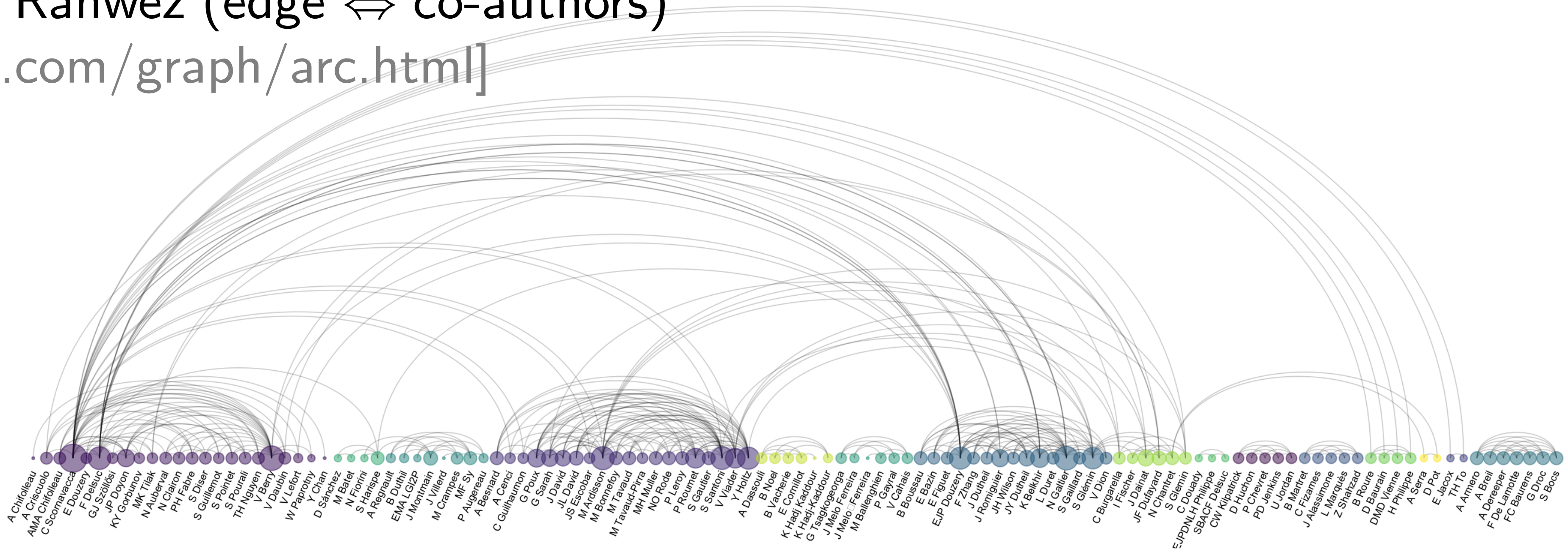
[https://www.data-to-viz.com/graph/arc.html]

Drawing Style: Arc Diagrams

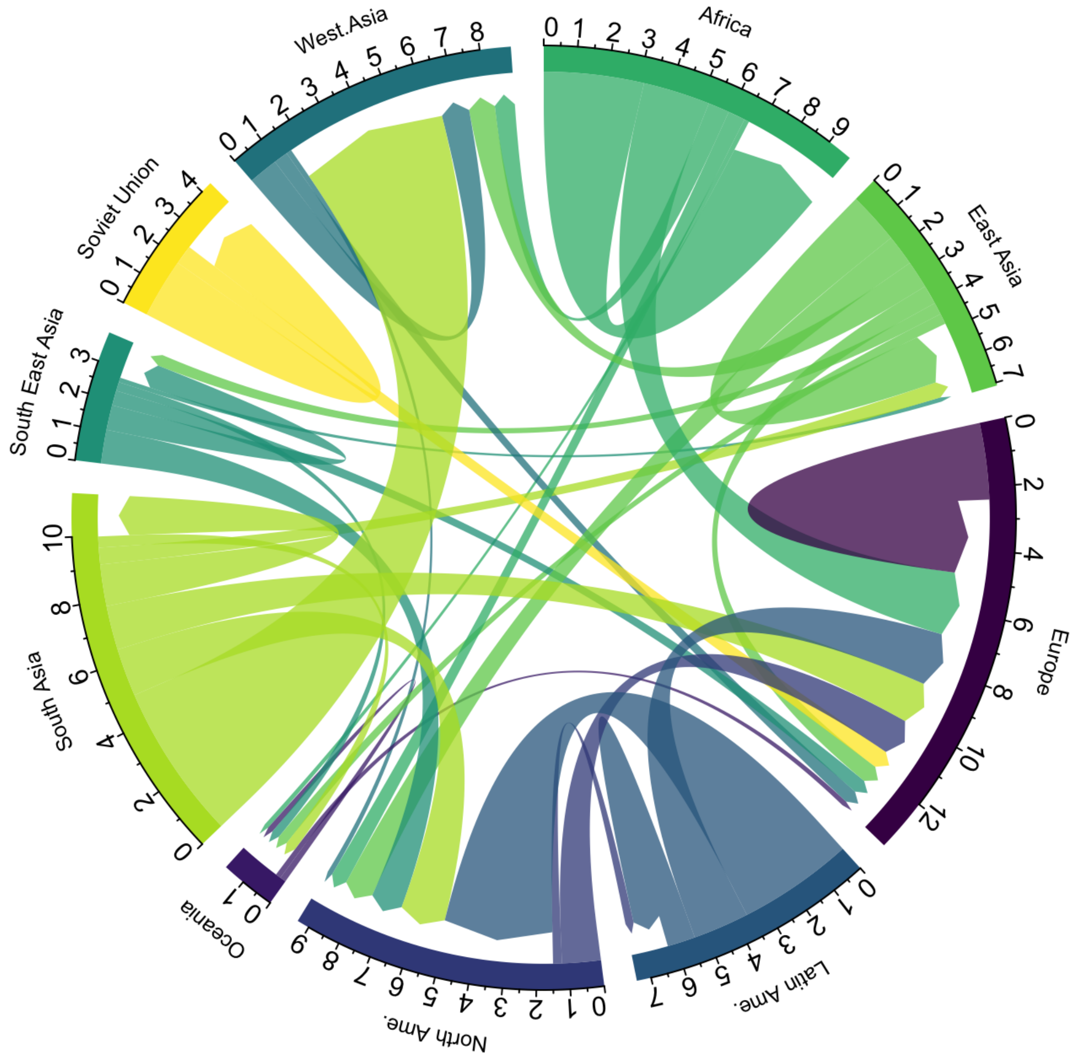


re-order vertices

Network of co-authors of Vincent Ranwez (edge $\Leftrightarrow$ co-authors)
[https://www.data-to-viz.com/graph/arc.html]



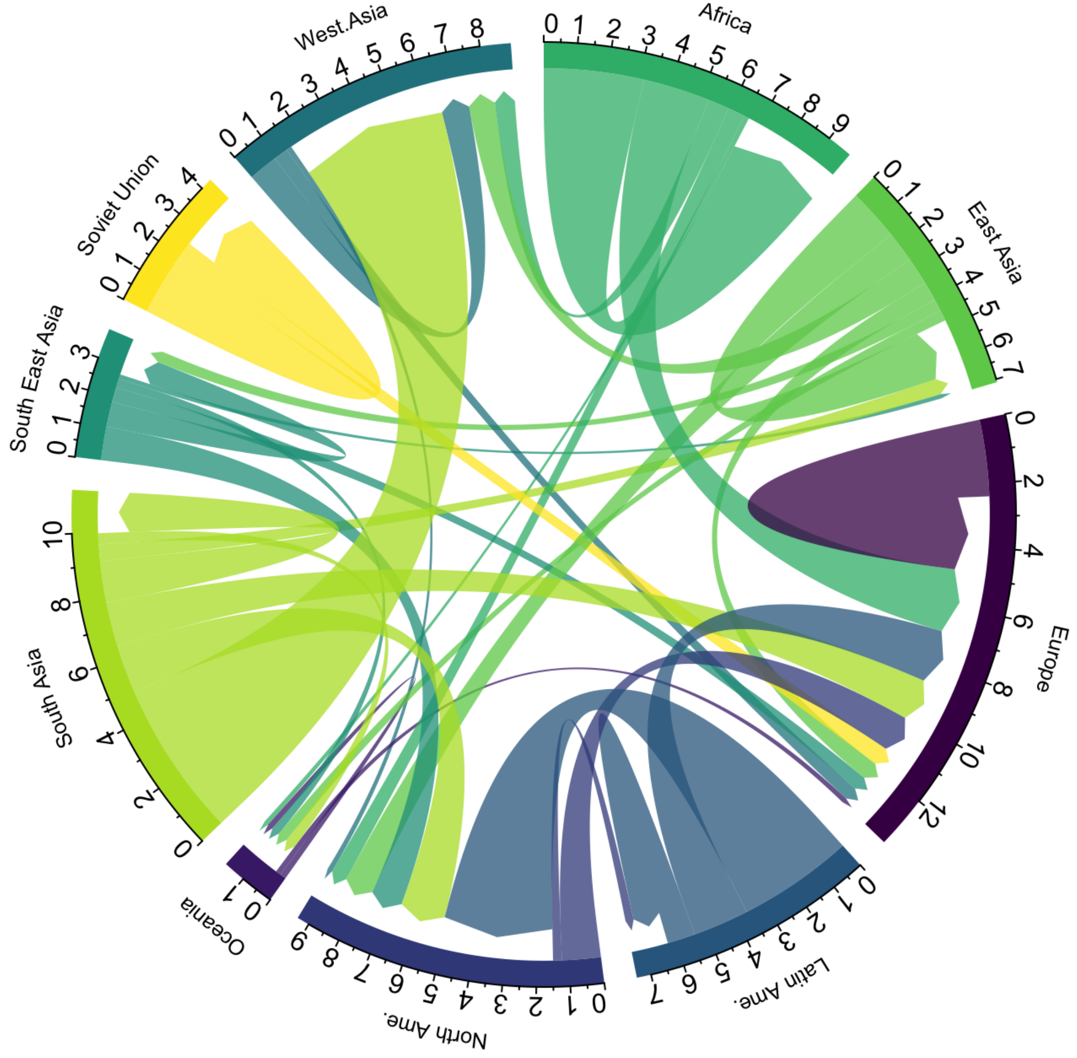
Drawing Style: Chord Diagrams



Migration between continents

[https://www.data-to-viz.com/story/AdjacencyMatrix.html]

Drawing Style: Chord Diagrams



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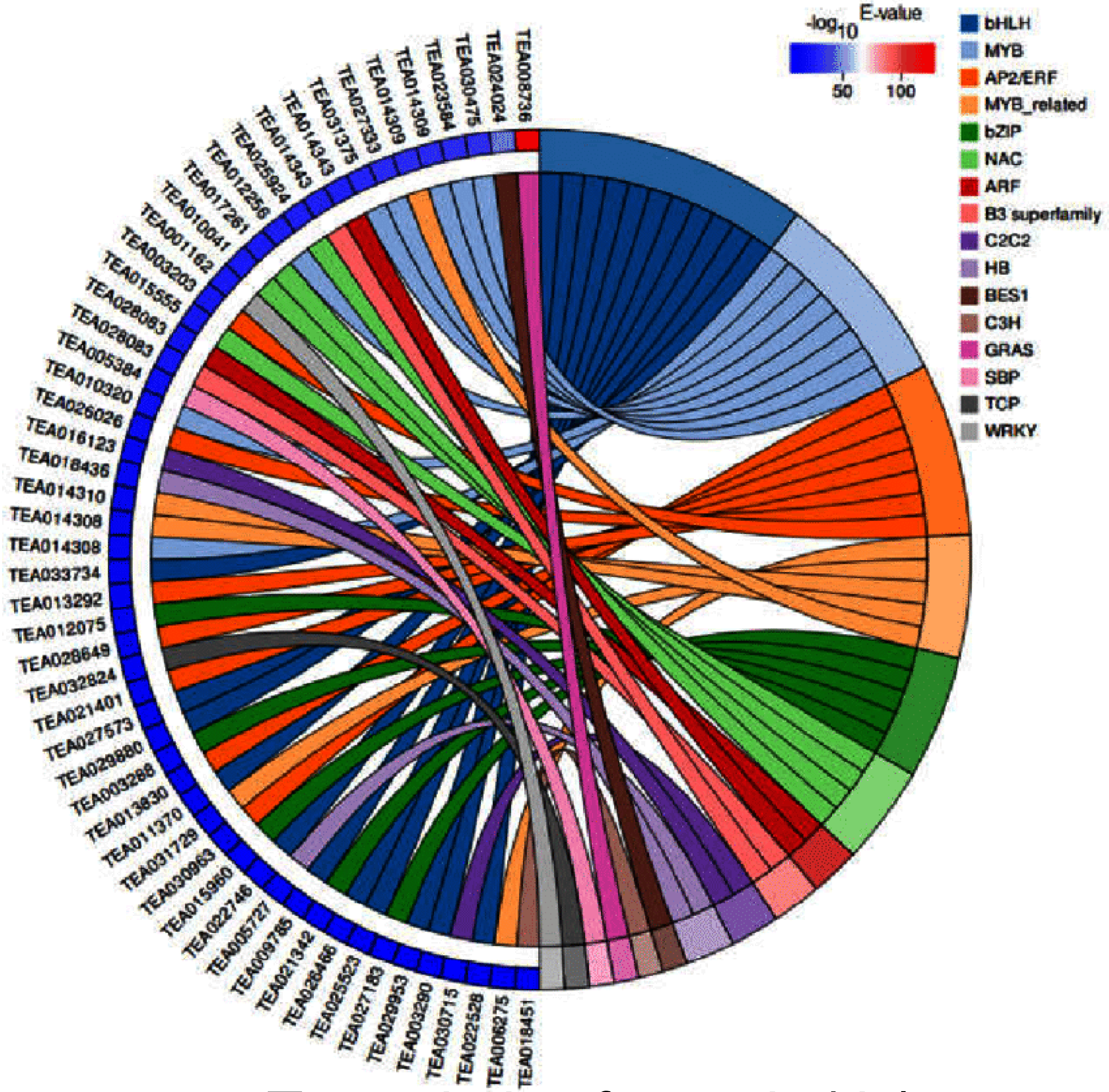
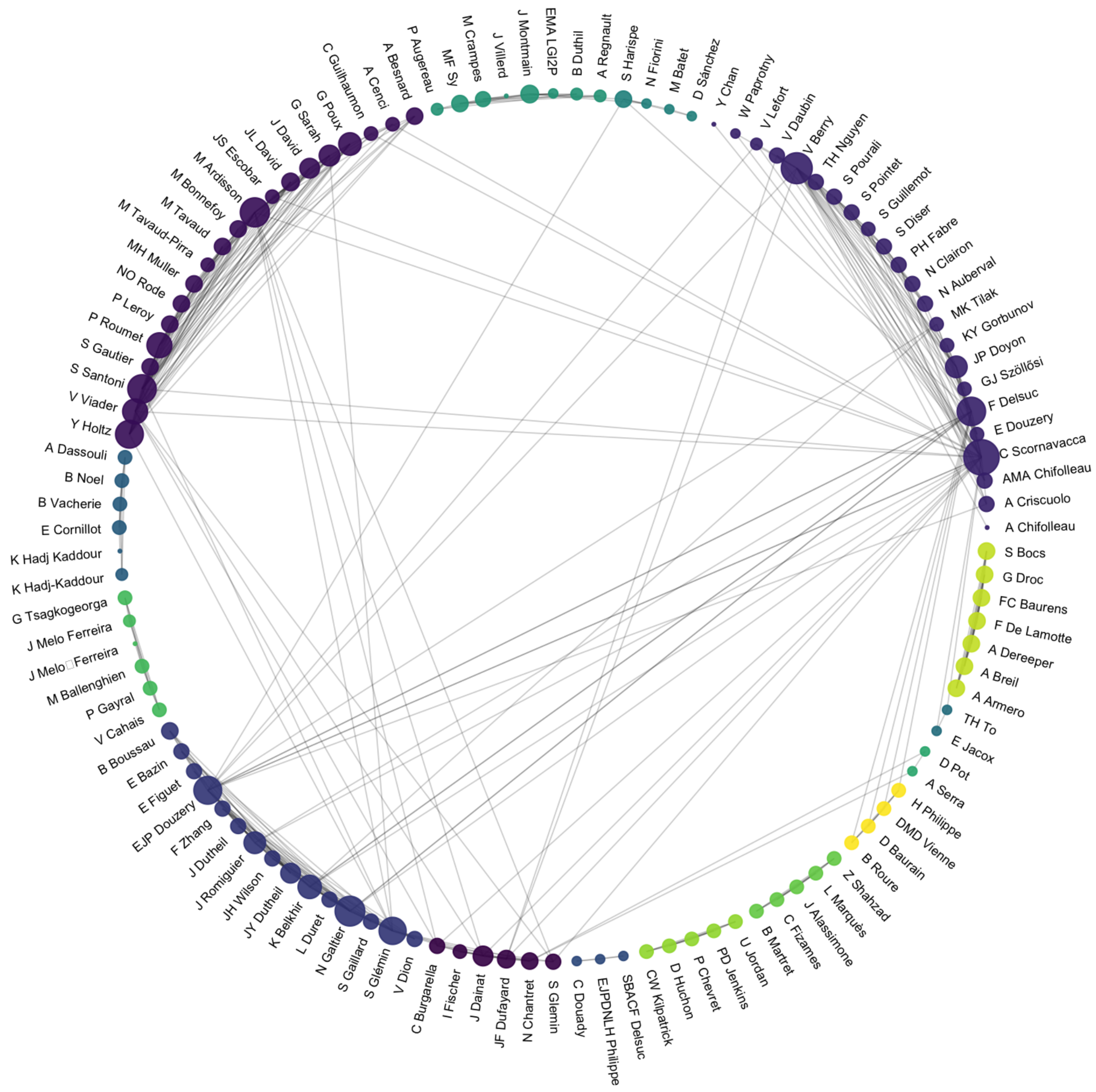


Transcription factors in biology

[Wang, Xuejin, Zhao 2020]

Exploration of the Effects of Different Blue LED Light Intensities on Flavonoid and Lipid Metabolism in Tea Plants via Transcriptomics and Metabolomics.

Drawing Style: Chord Diagrams



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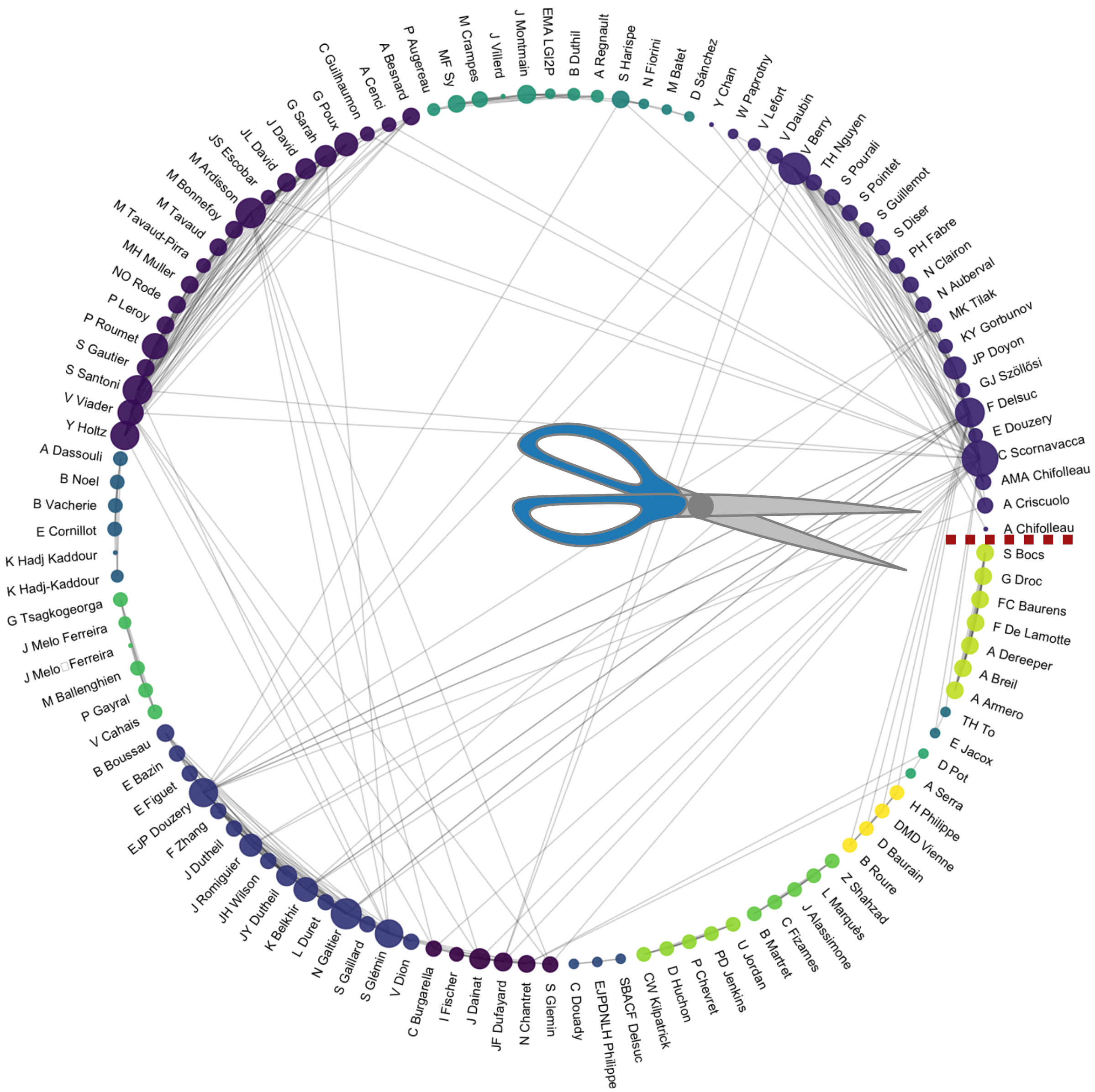
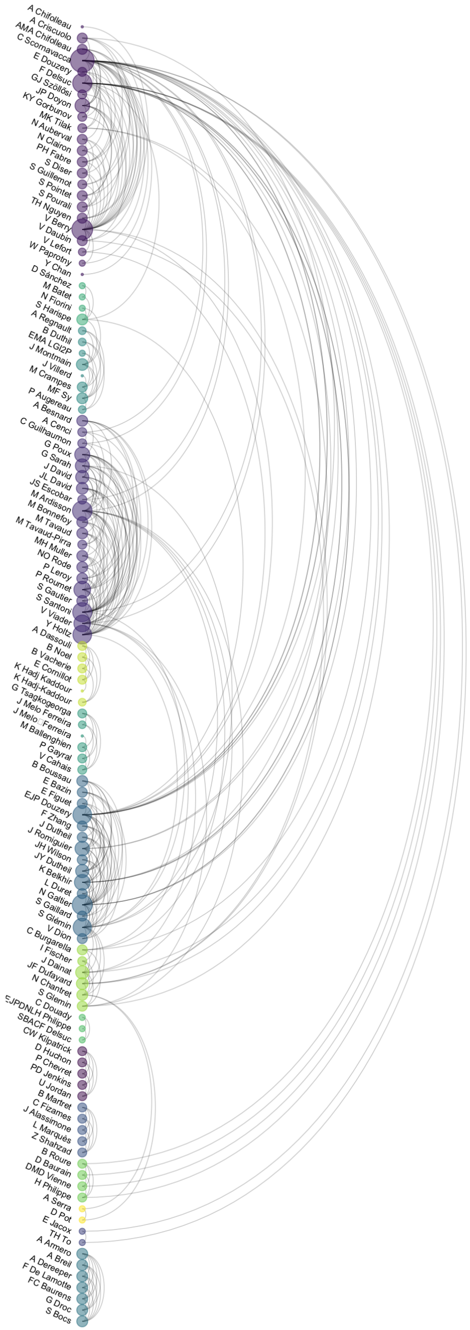
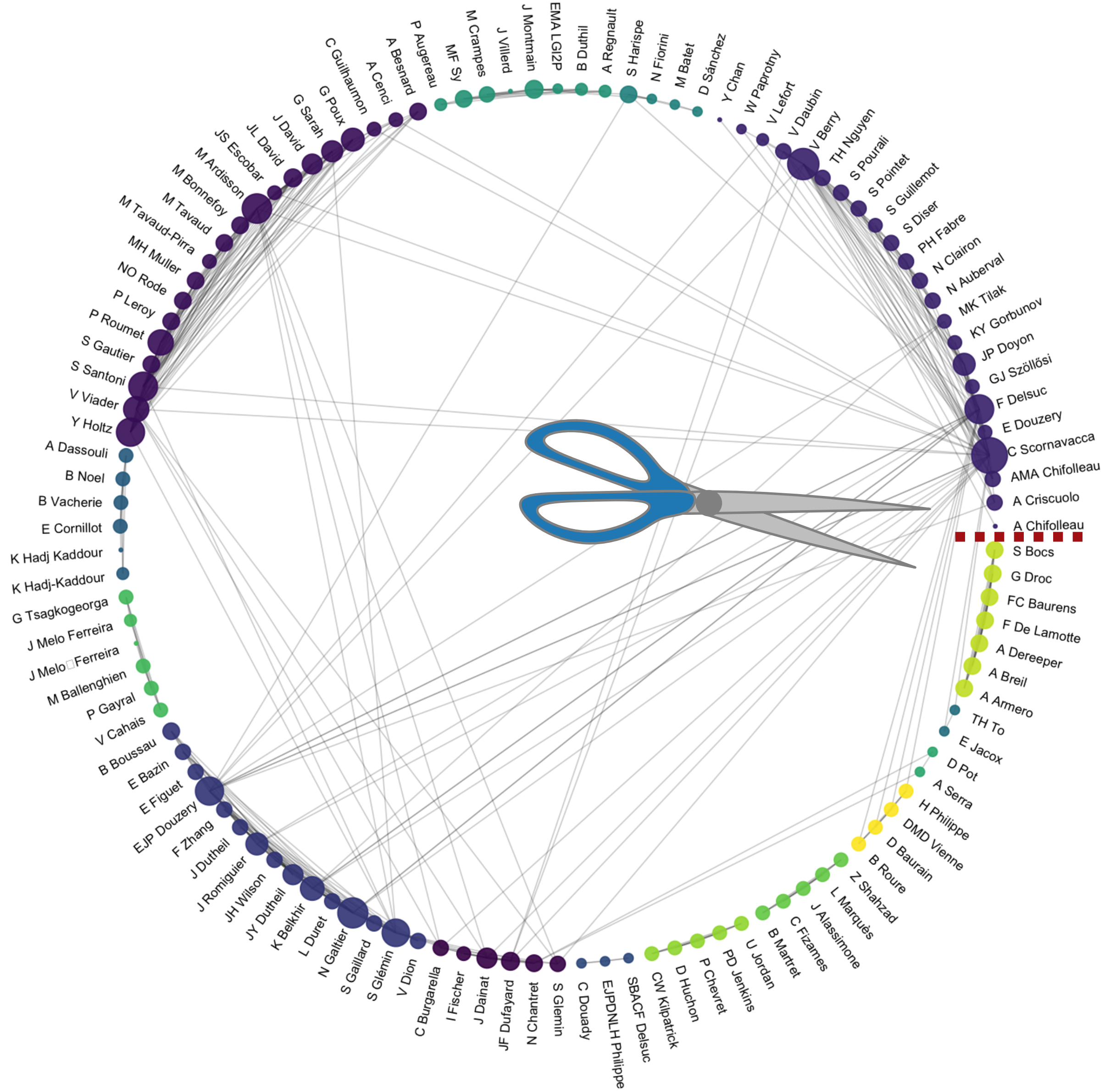


Drawing Style: Chord Diagrams



“cut” circle



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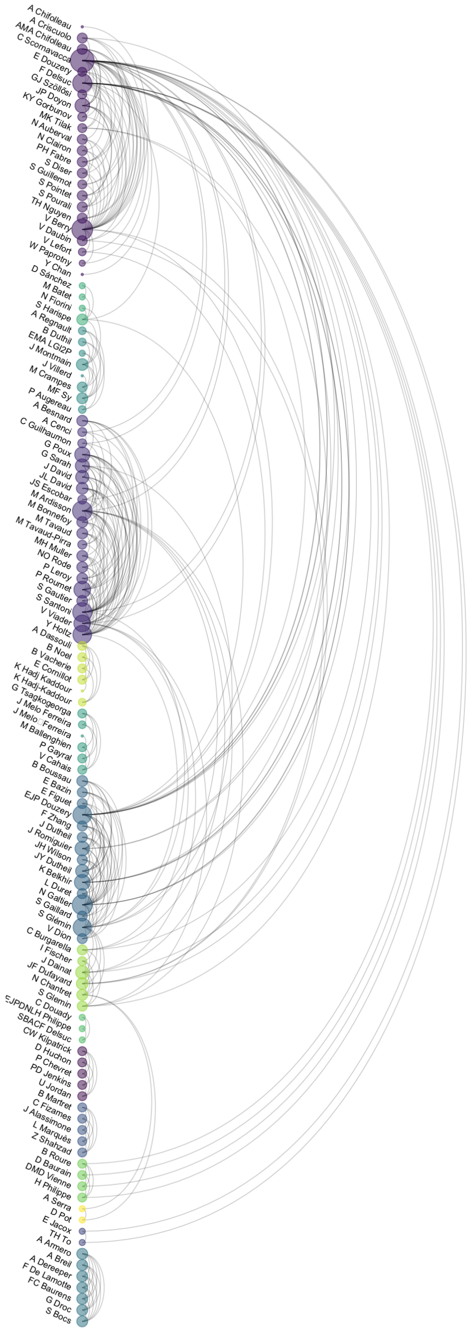
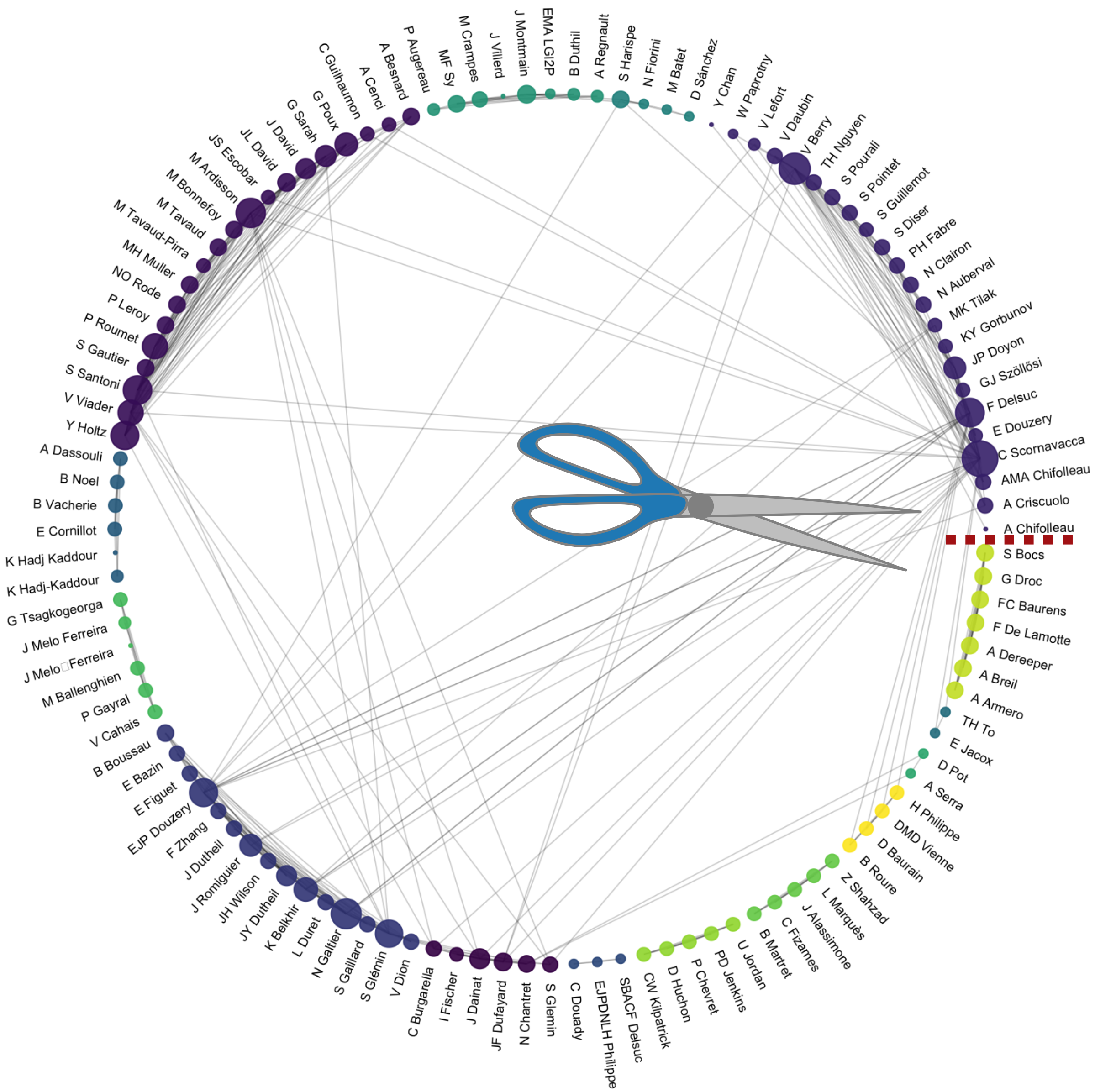
Drawing Style: Chord Diagrams



"cut" circle

"glue" sequence



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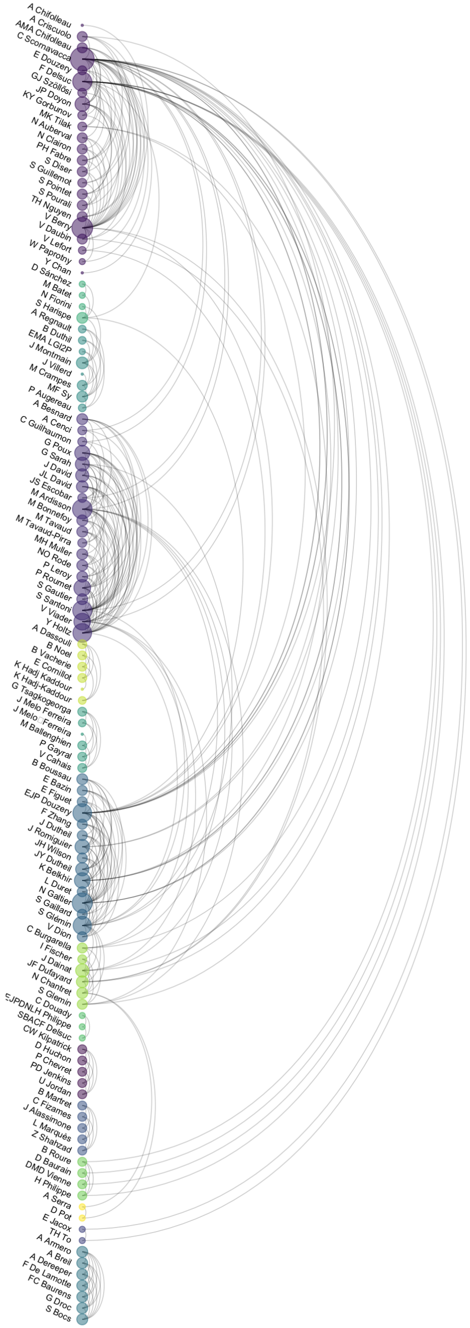
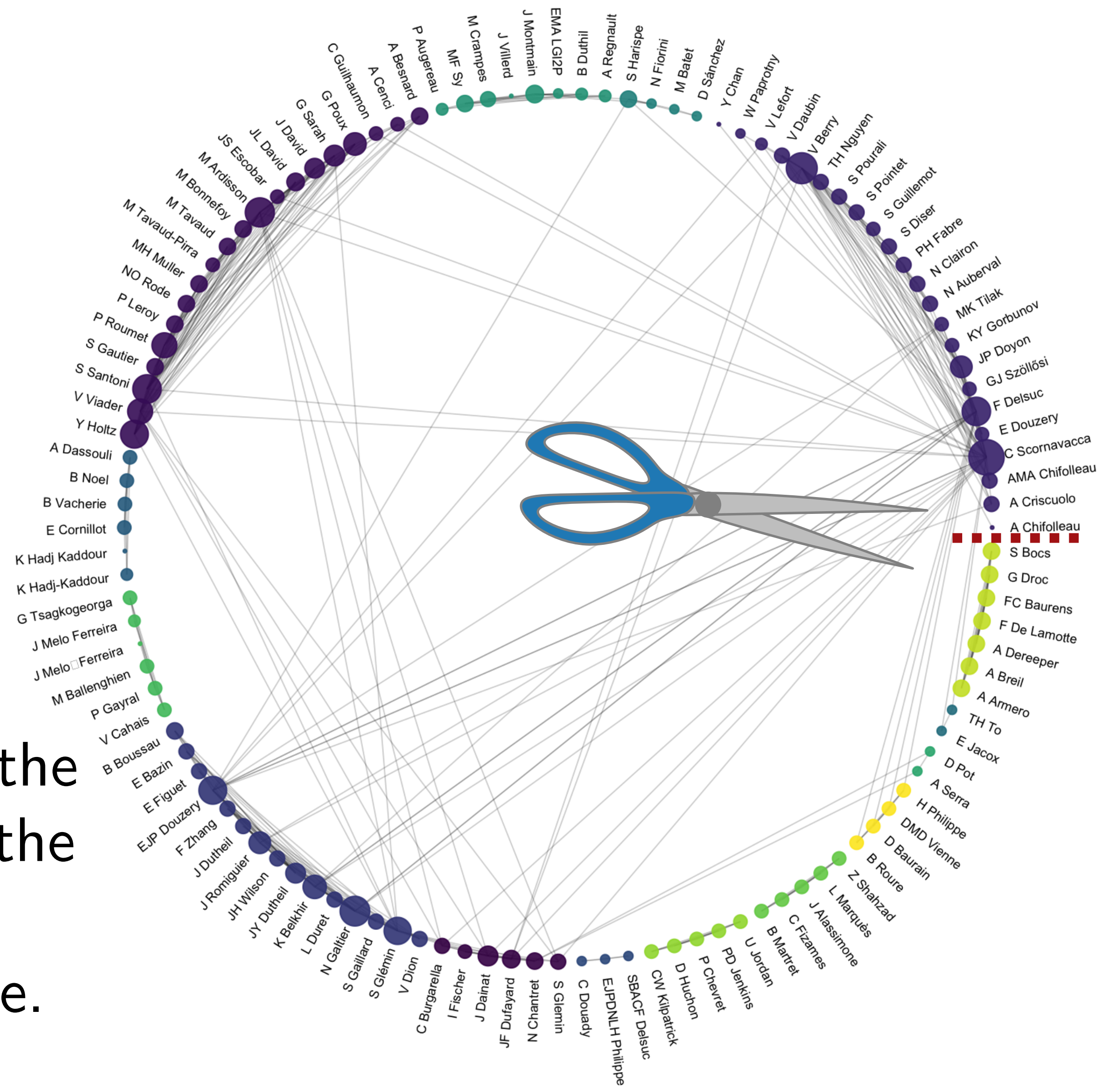


“cut” circle

“glue” sequence



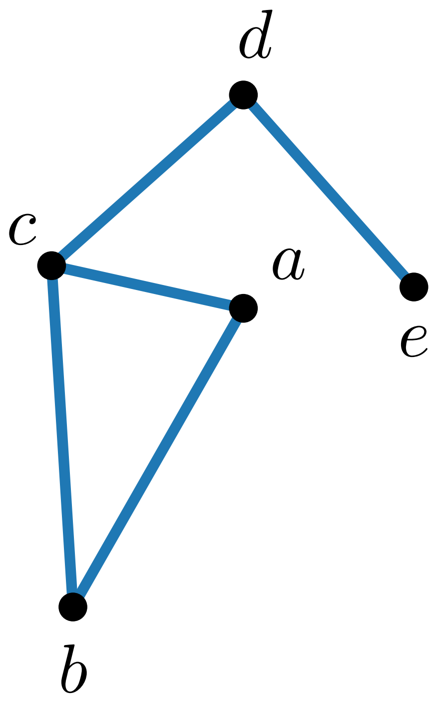
Note that, if we keep the order of the vertices, the crossings between the edges remain the same.

Network of co-authors of Vincent Ranwez (edge ⇔ co-authors)
[https://www.data-to-viz.com/story/AdjacencyMatrix.html]

Planarity + Arc/Chord Diagrams?

Planarity + Arc/Chord Diagrams?

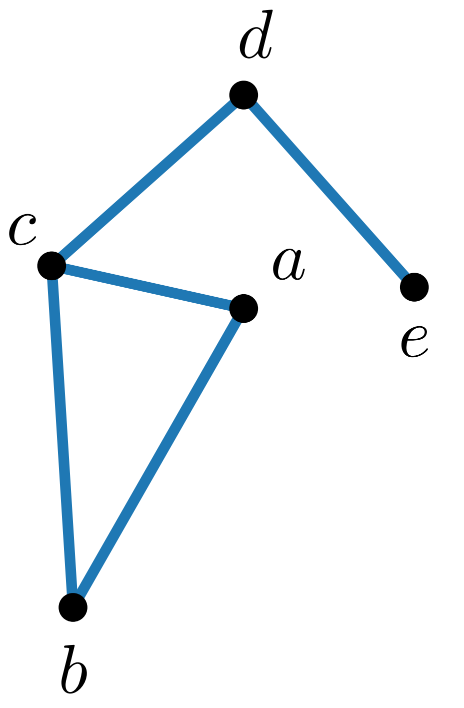
Given: ■ graph G



Planarity + Arc/Chord Diagrams?

Given: ■ graph G

Task: Find a linear order $\prec$ of $V(G)$

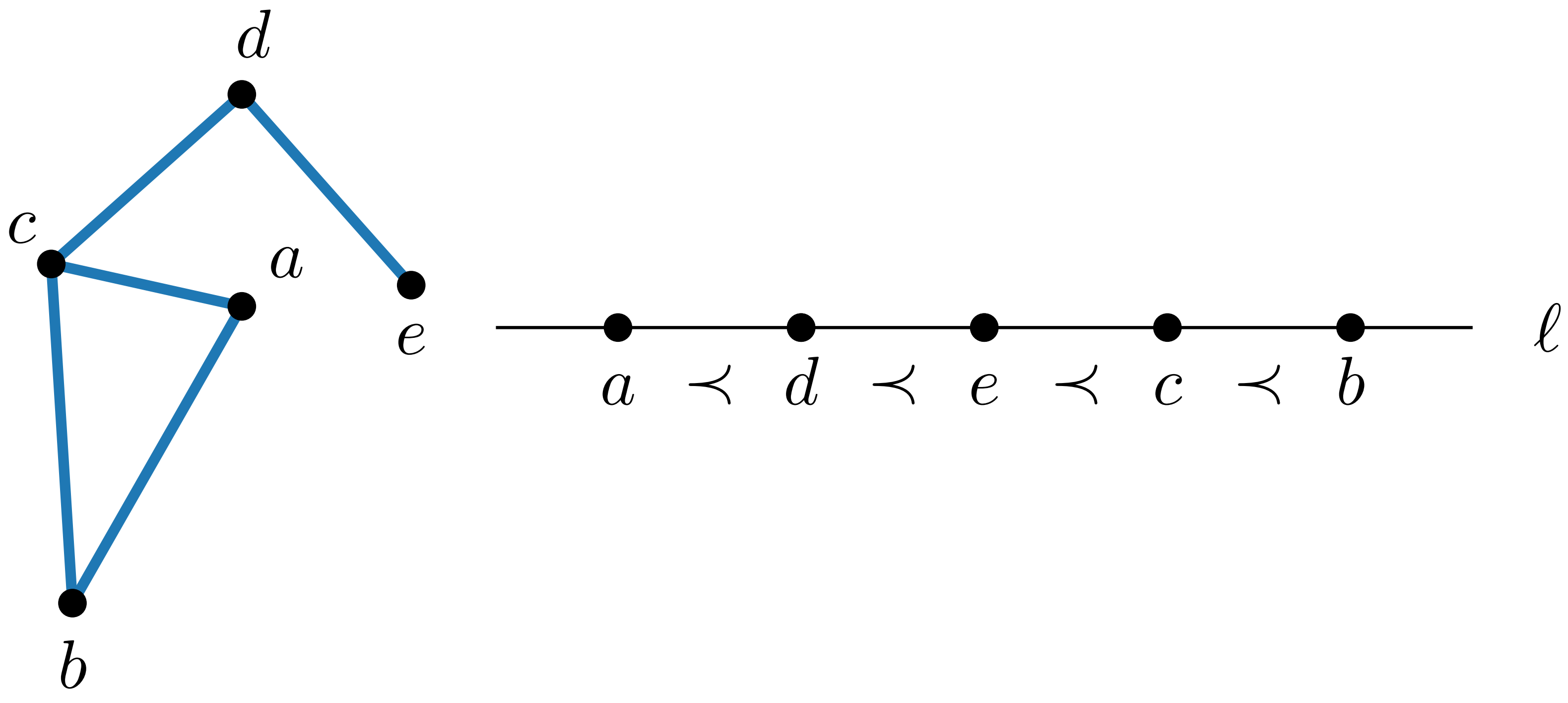


$$a \prec d \prec e \prec c \prec b$$

Planarity + Arc/Chord Diagrams?

Given:
- graph G

Task: Find a linear order $\prec$ of $V(G)$ such that there is a planar drawing where
- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ

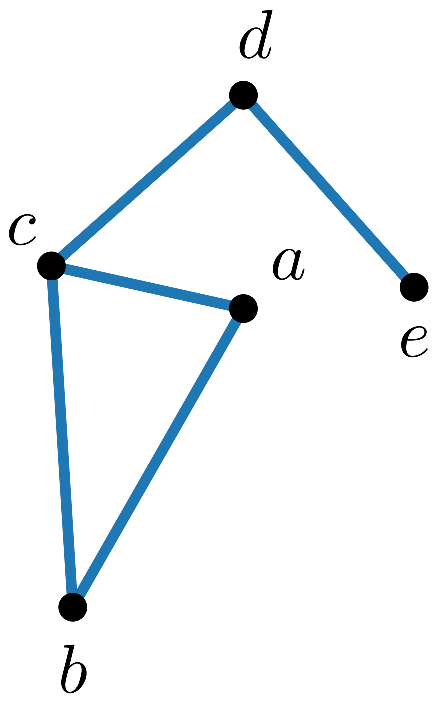


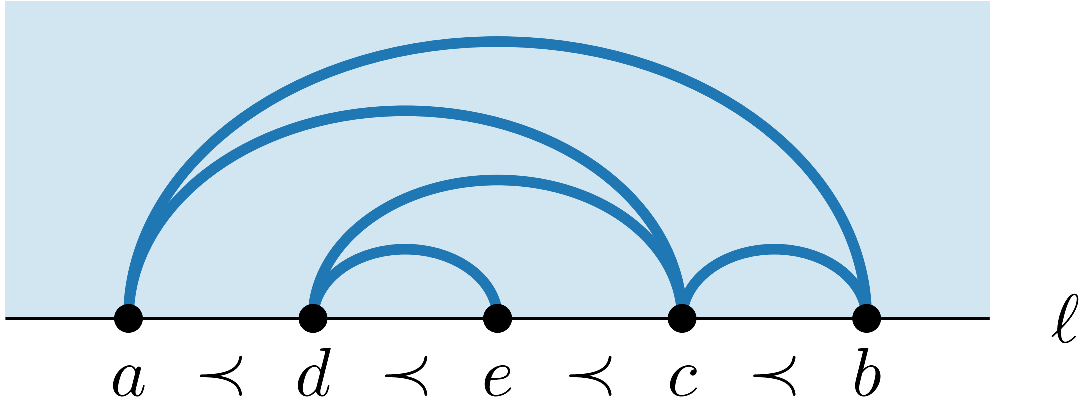
Planarity + Arc/Chord Diagrams?

Given: graph G

Task: Find a linear order $\prec$ of $V(G)$ such that there is a planar drawing where

- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ and
- the edges $E(G)$ are drawn as x-monotone arcs in the half plane above ℓ.





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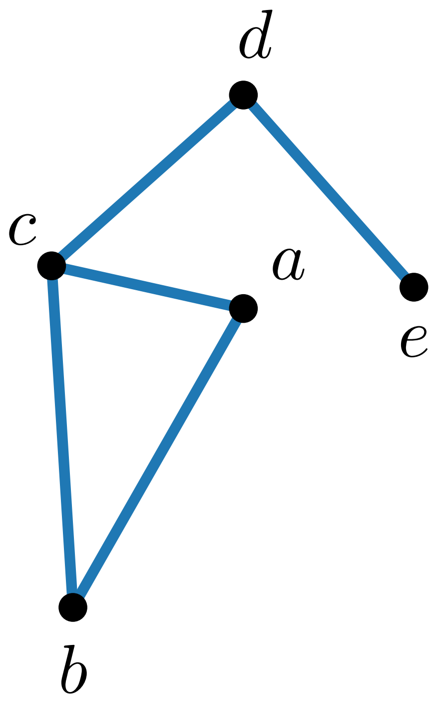

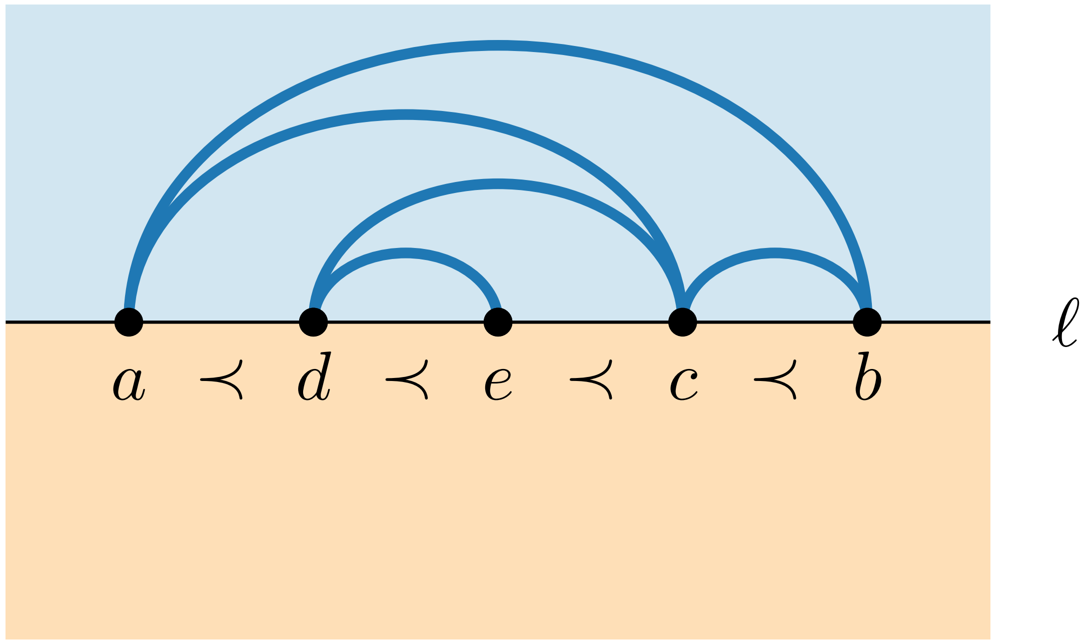

Why not using the half plane below ℓ?

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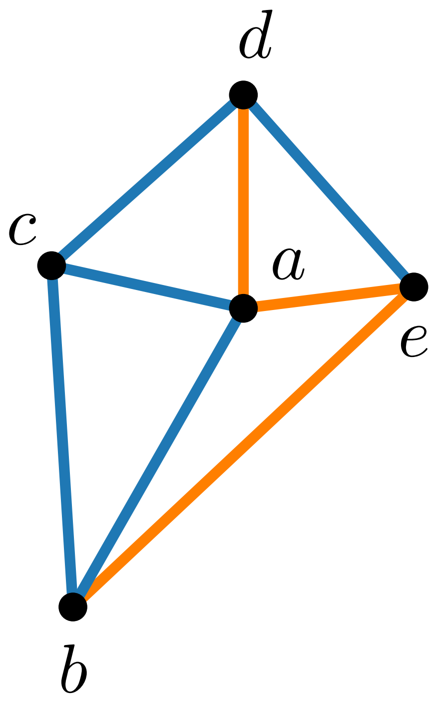

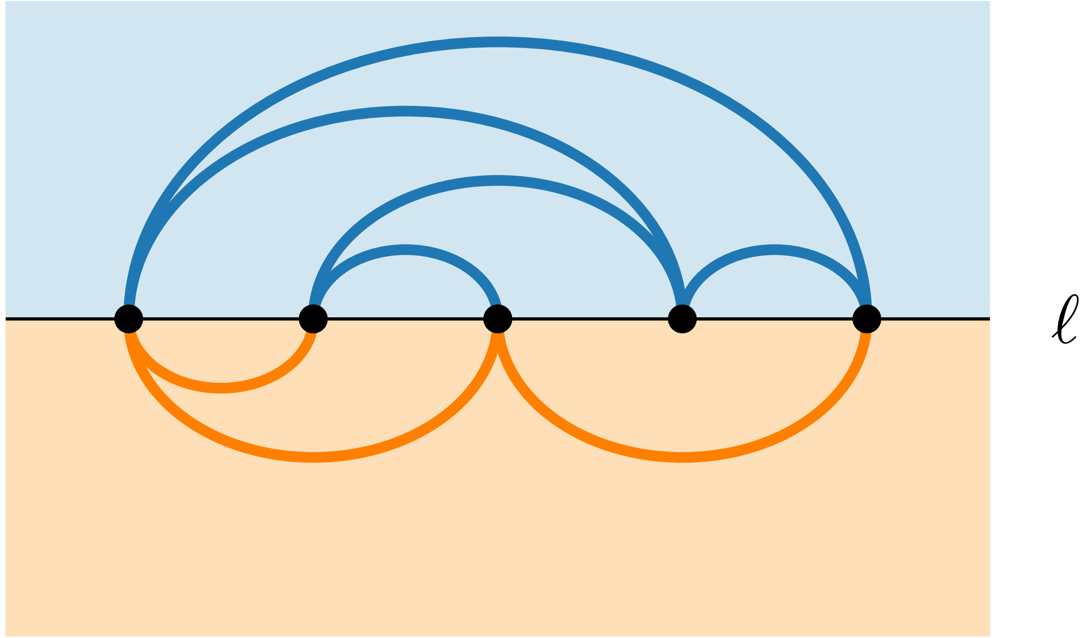

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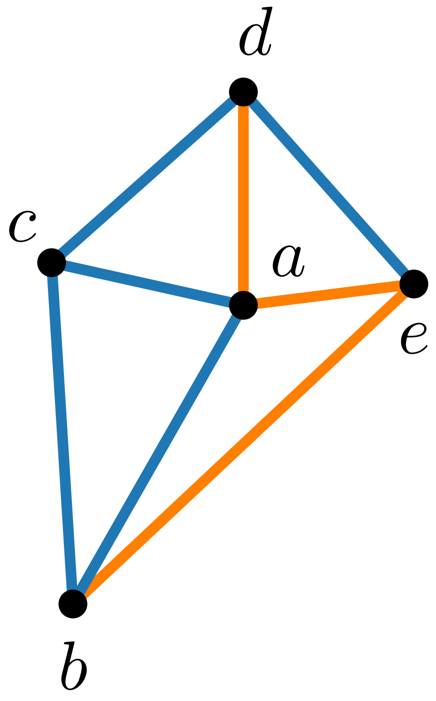

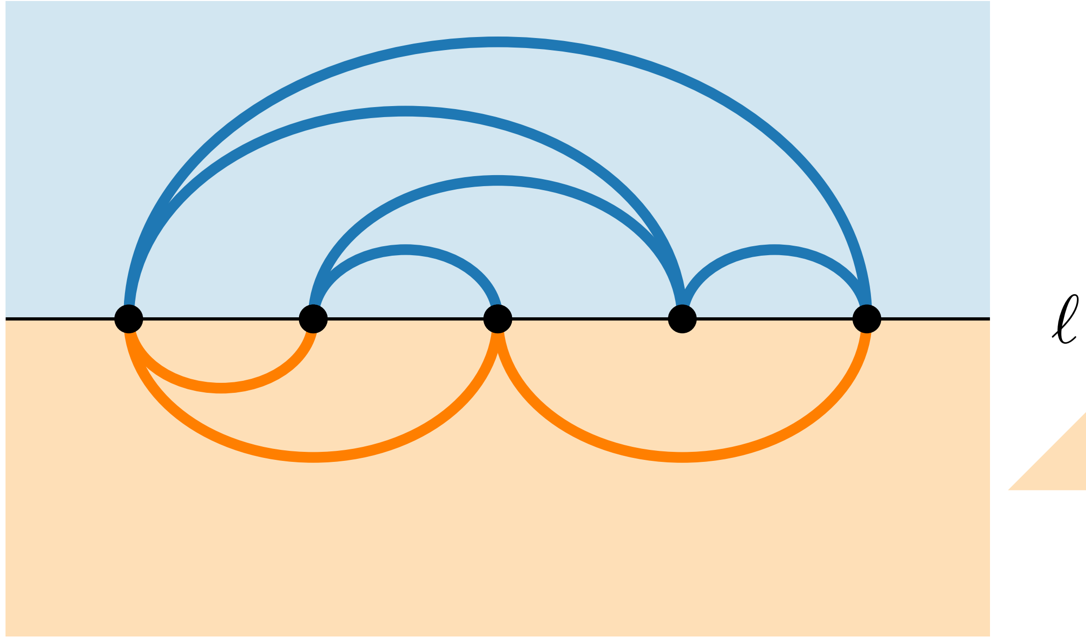

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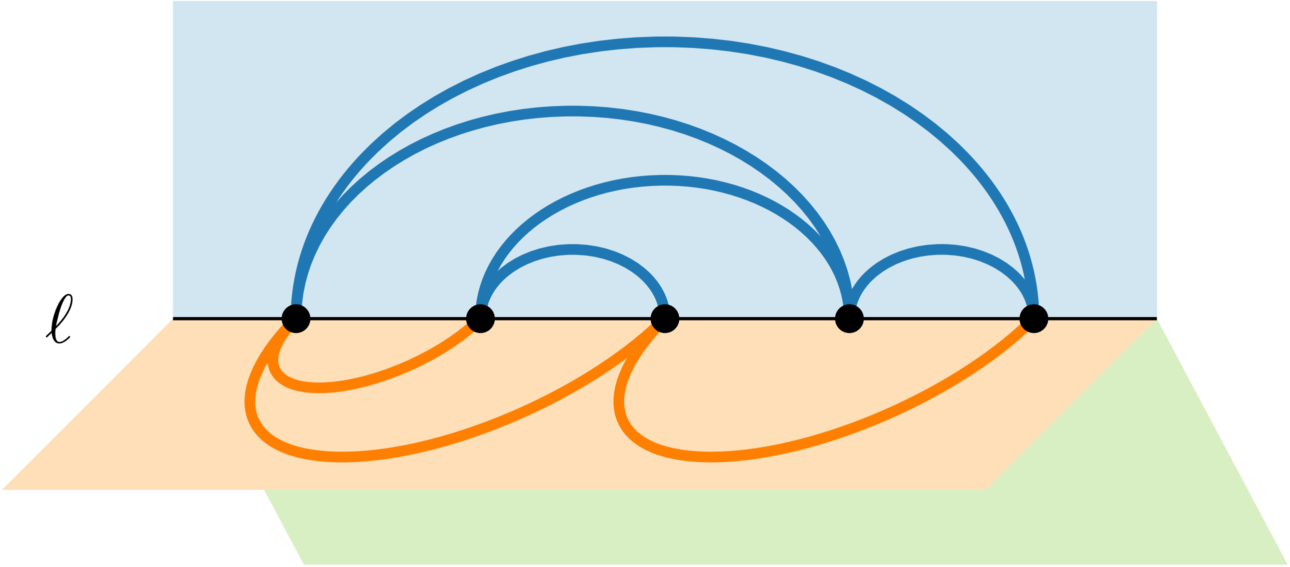

Or even more half planes?

Planarity + Arc/Chord Diagrams?

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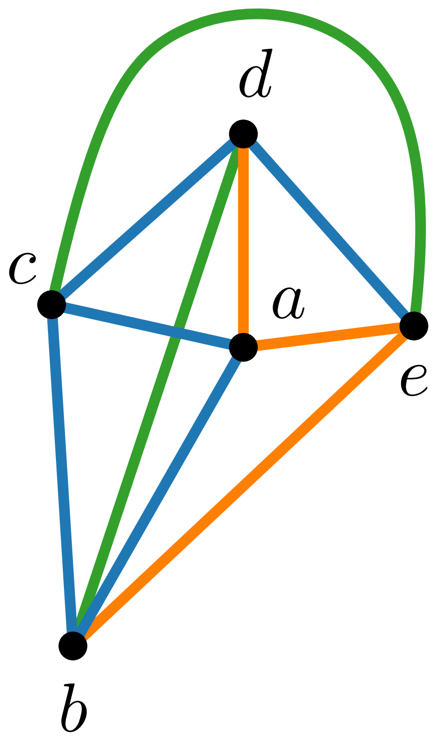

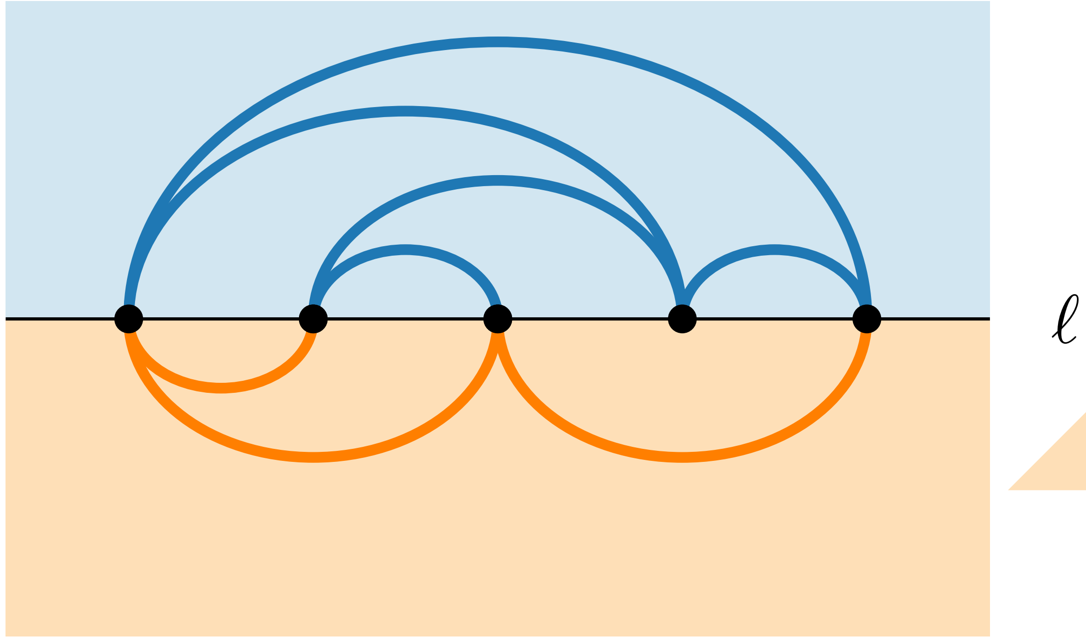

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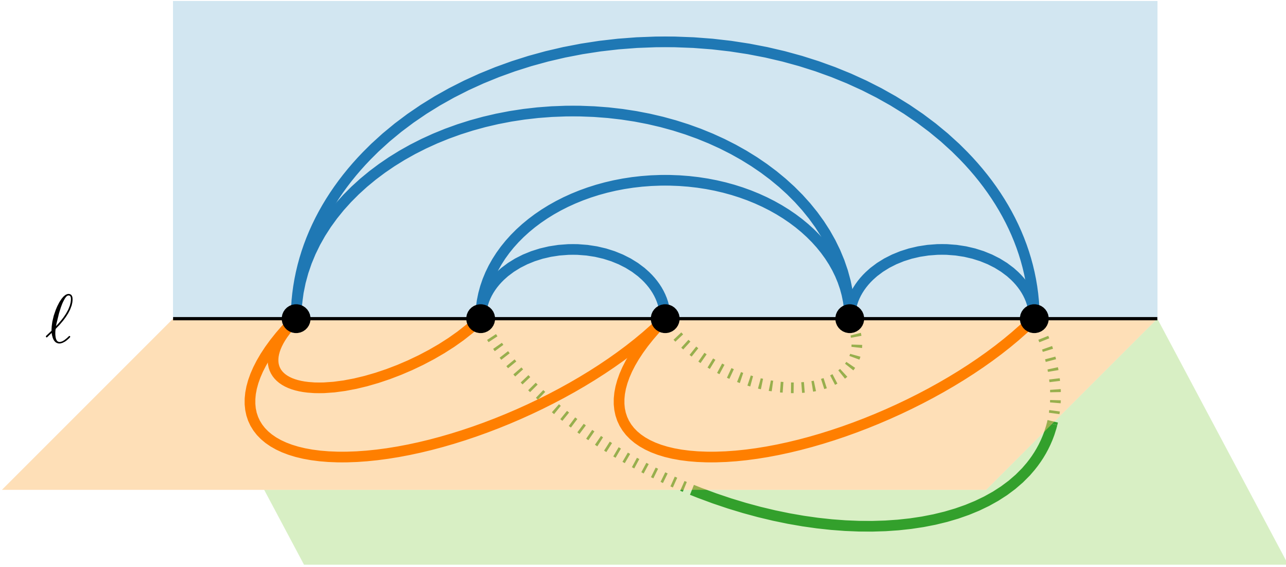

Or even more half planes?

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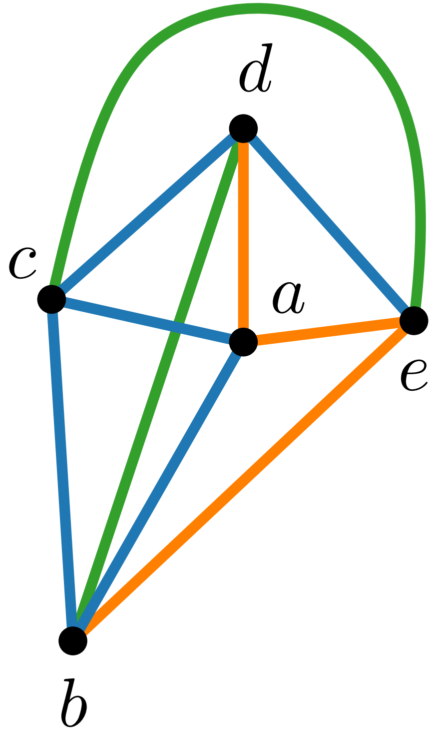

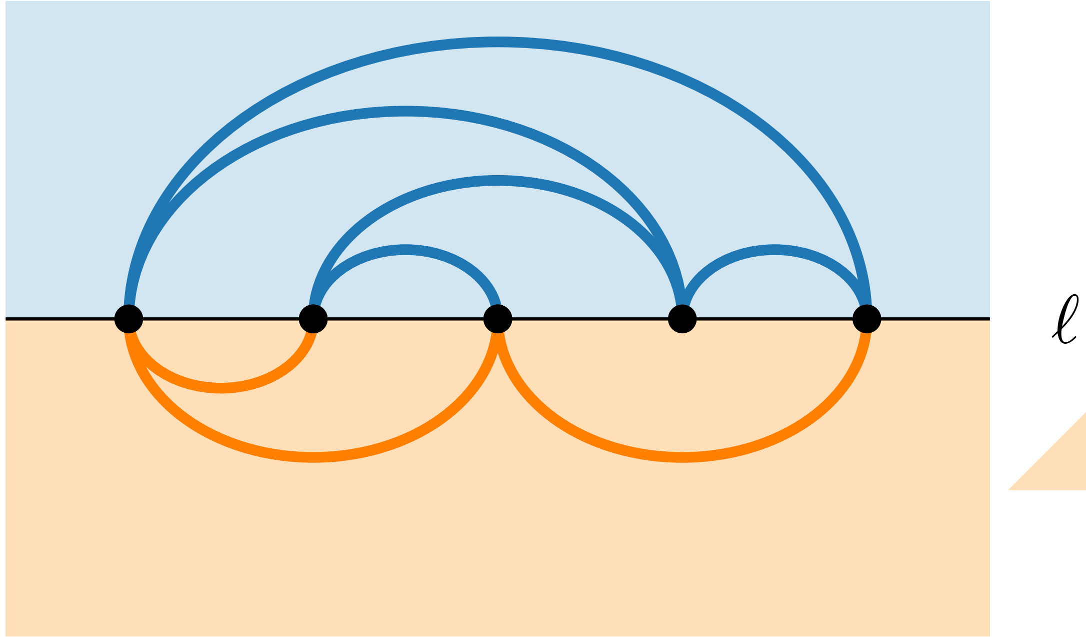

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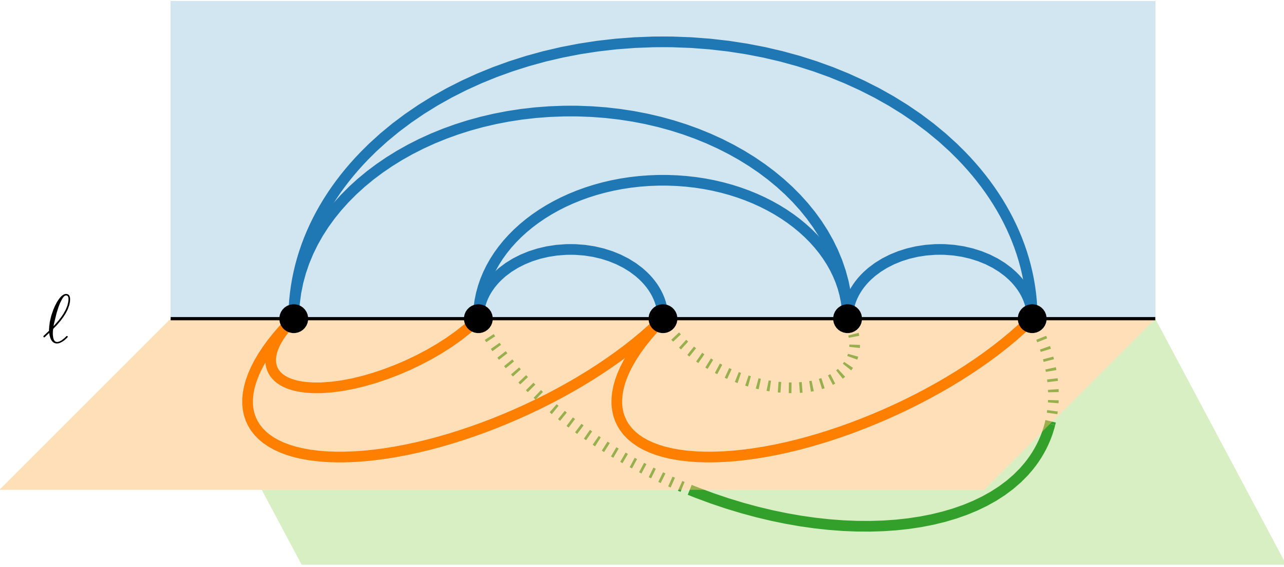
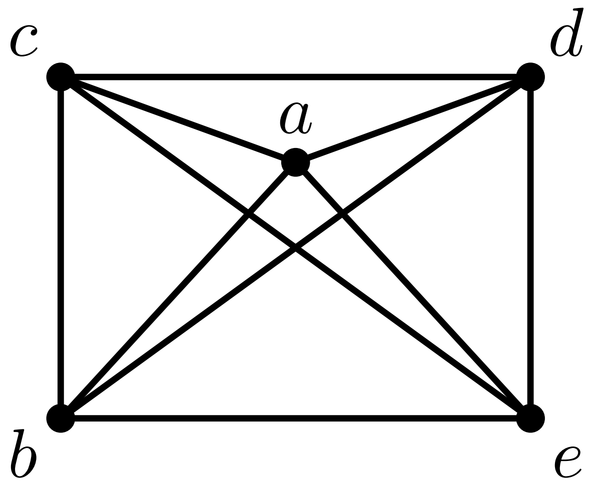


Or even more half planes?
→ **book embeddings**

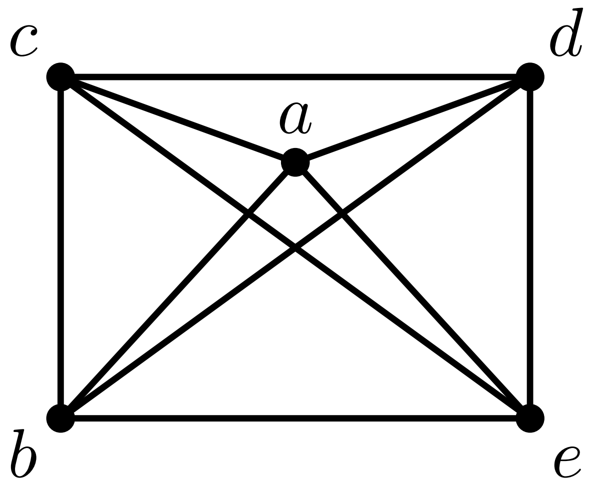
Book Embeddings

Given: ■ graph G



Book Embeddings

Given:
- graph G
- integer k



$k = 3$

Book Embeddings

Given:
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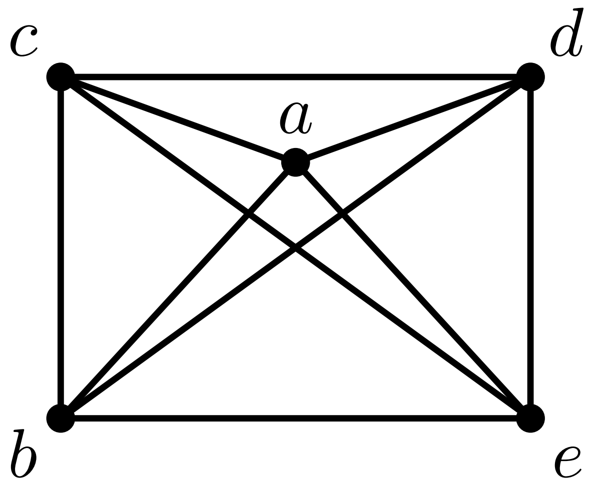
Task: Find a linear order $\prec$ of $V(G)$ such that . . .

- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ and



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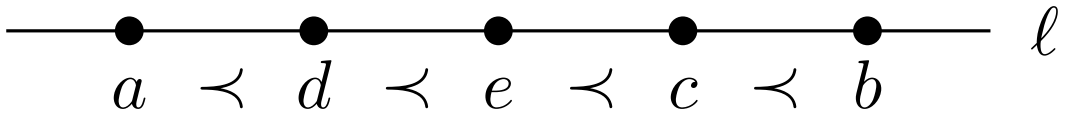

Book Embeddings

Given:
- graph G
- integer k

Task: Find (i) a linear order $\prec$ of $V(G)$ and (ii) an assigment $p\colon E(G) \to \{1, \ldots, k\}$ such that ...

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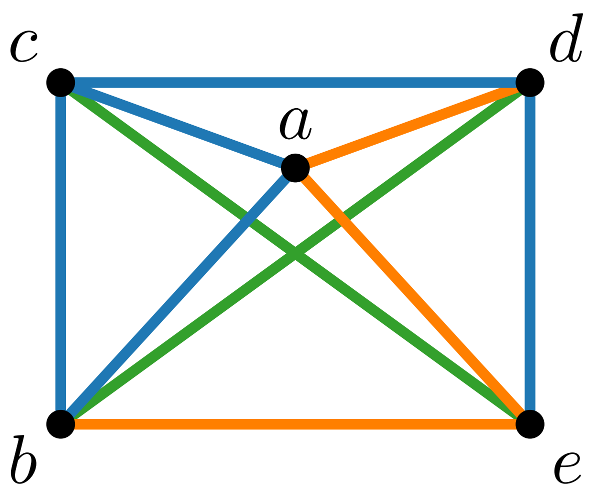

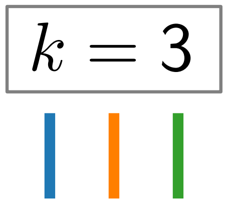

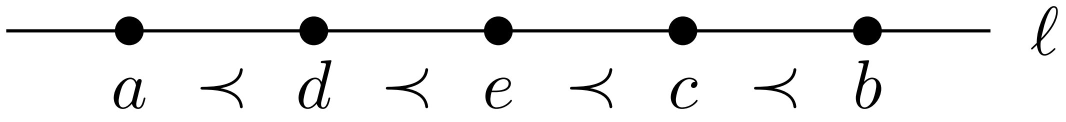

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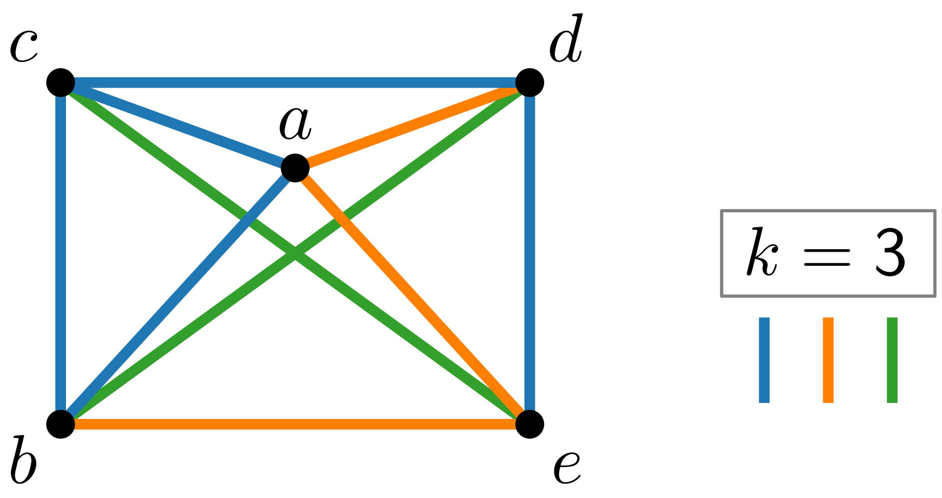

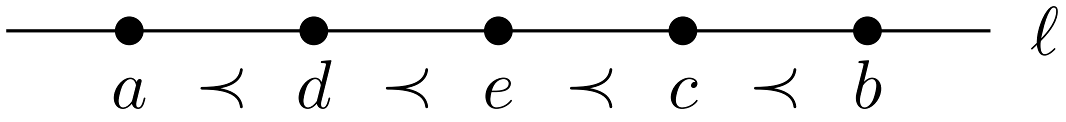

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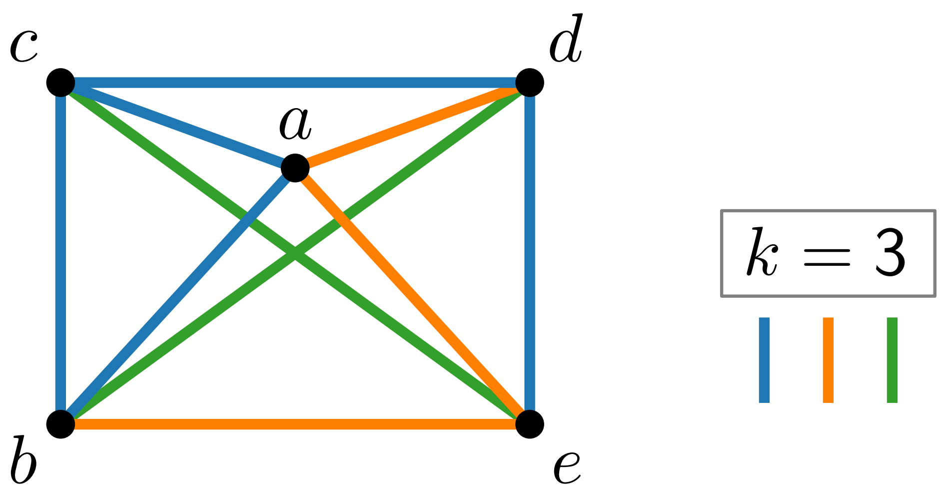

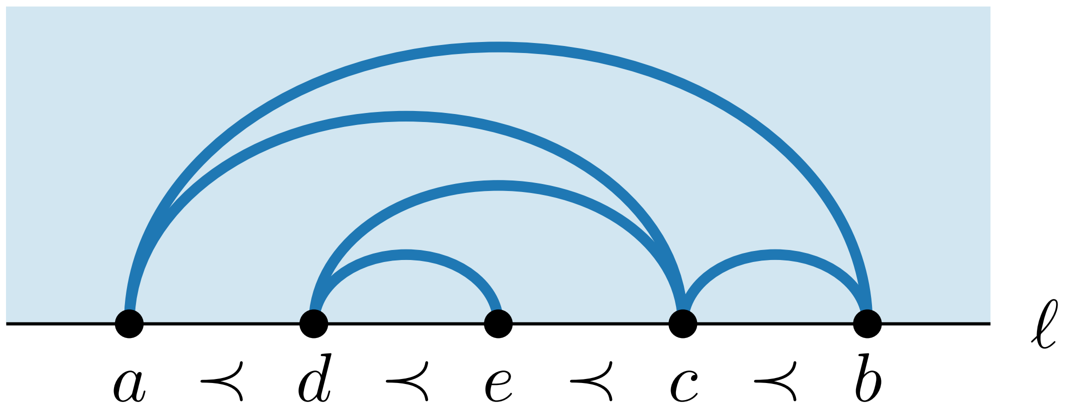

Book Embeddings

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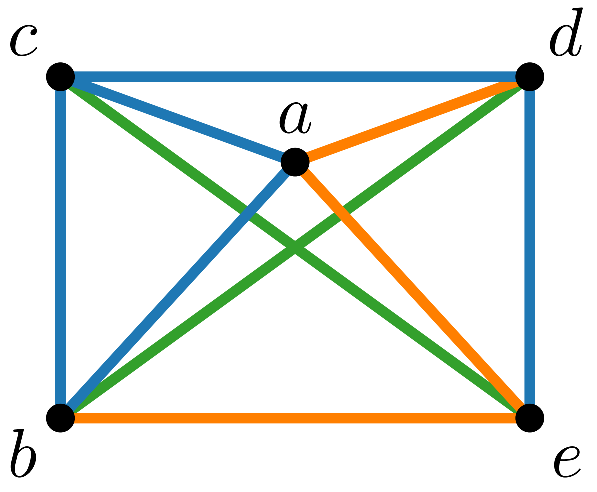
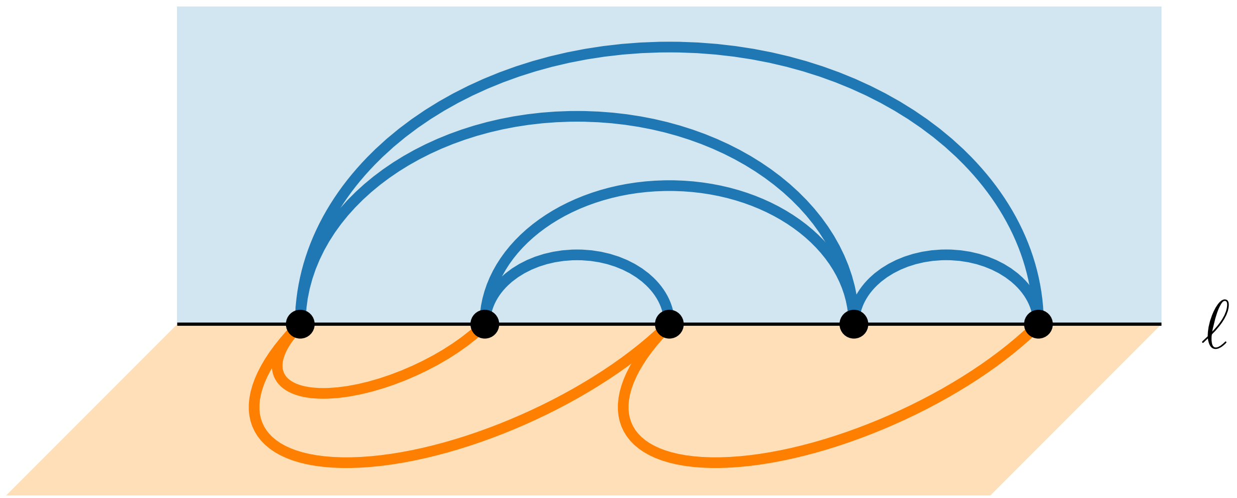


$k = 3$



Book Embeddings

Given:
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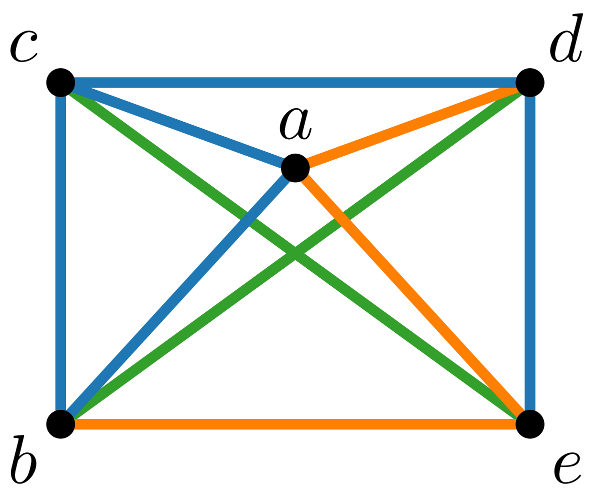

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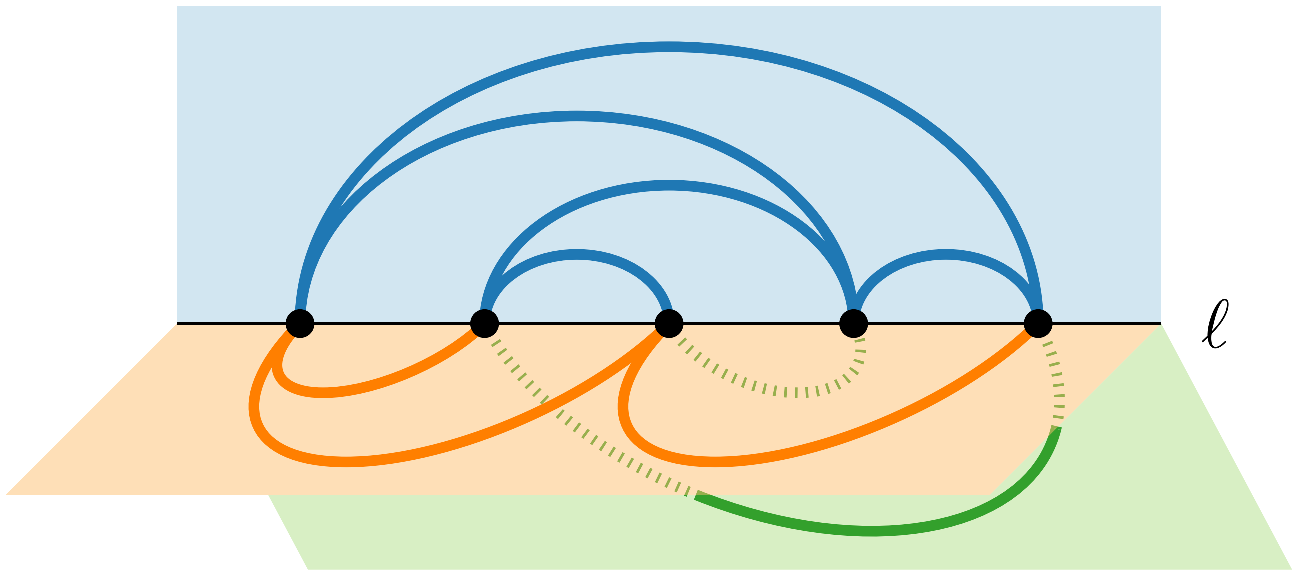

Book Embeddings

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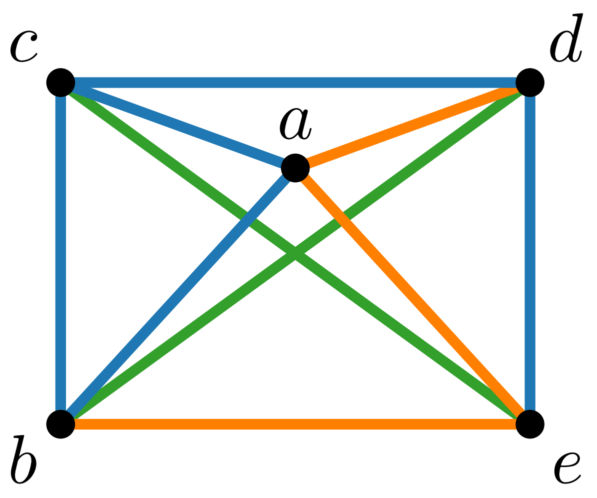

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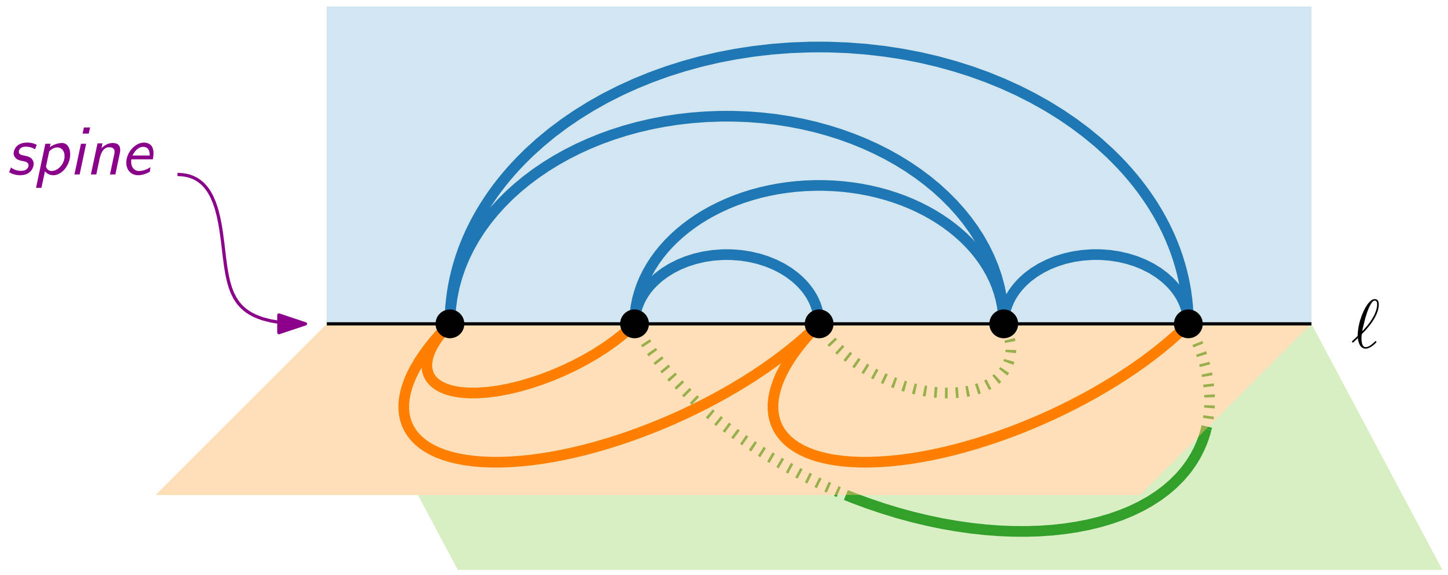

Book Embeddings

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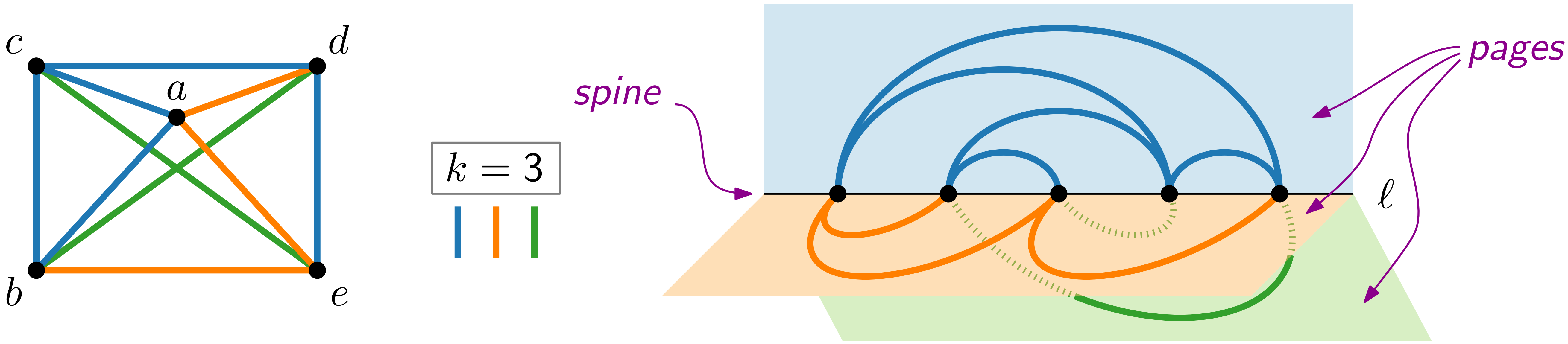
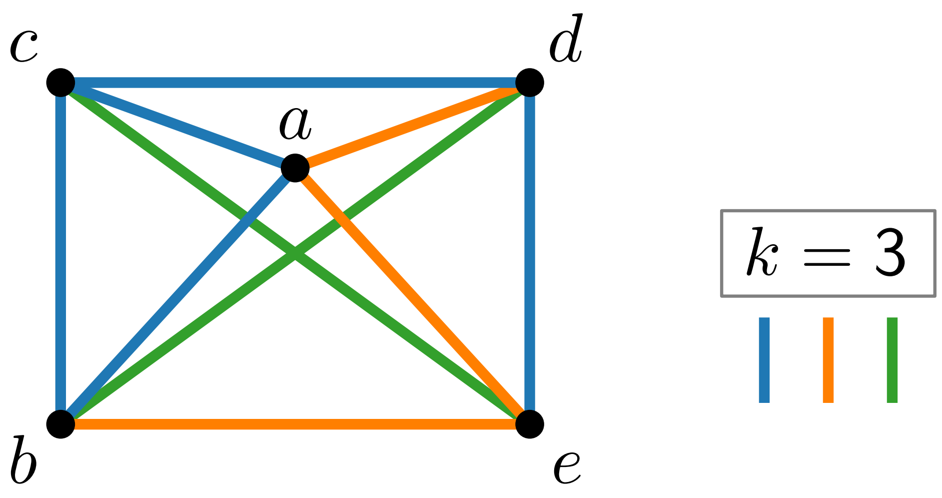


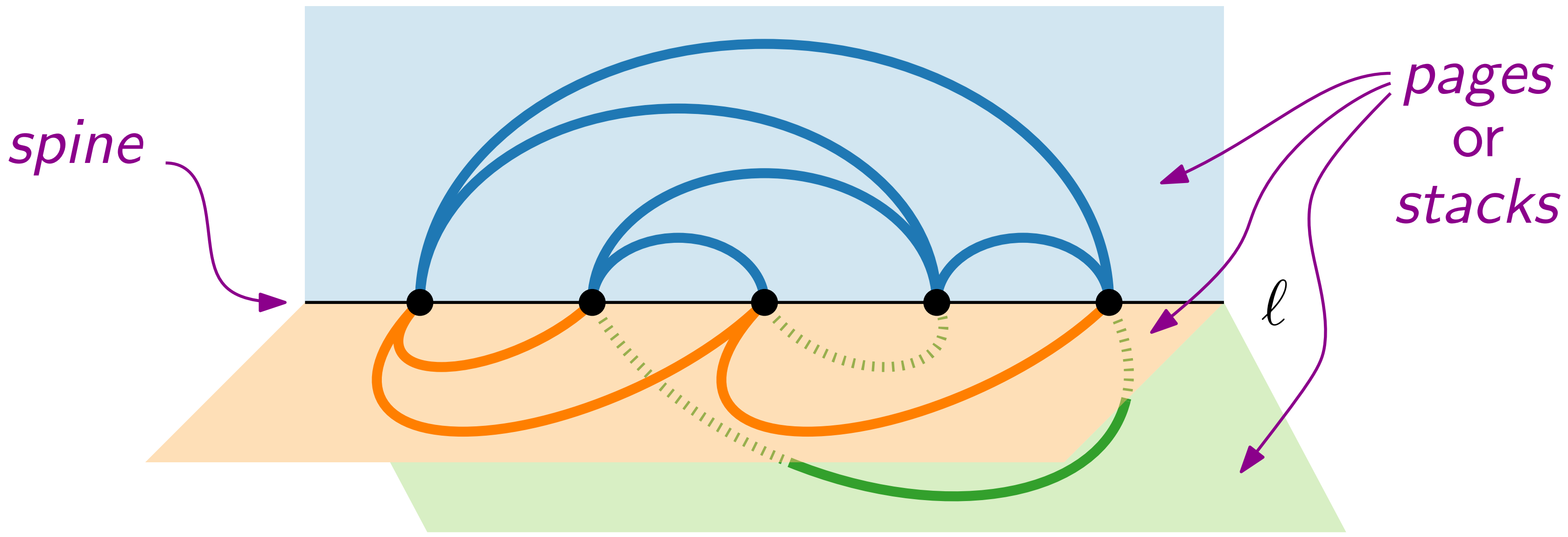
Book Embeddings (Stack Layouts)

Given:
- graph G
- integer k

Task: Find (i) a linear order $\prec$ of $V(G)$ and (ii) an assigment $p \colon E(G) \to \{1, \dots, k\}$ such that ...

- the vertices $V(G)$ in order $\prec$ are arranged along a horizontal line ℓ and
- for each $i \in \{1, \dots, k\}$, the edges in $p^{-1}(i)$ are drawn as x-monotone arcs without crossings in a (separate) half plane delimited by ℓ.





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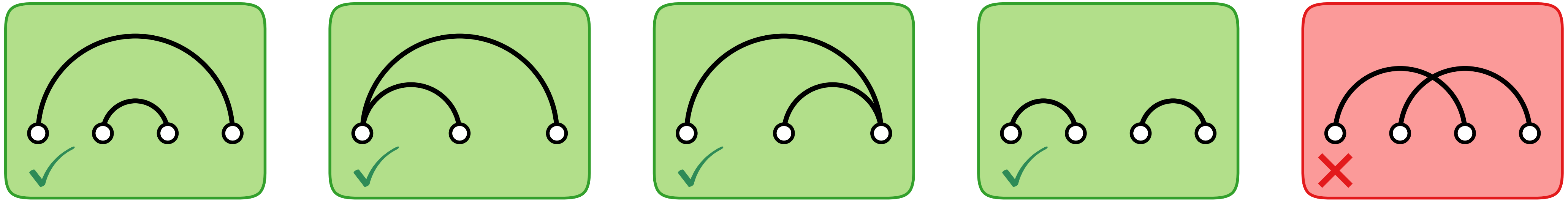

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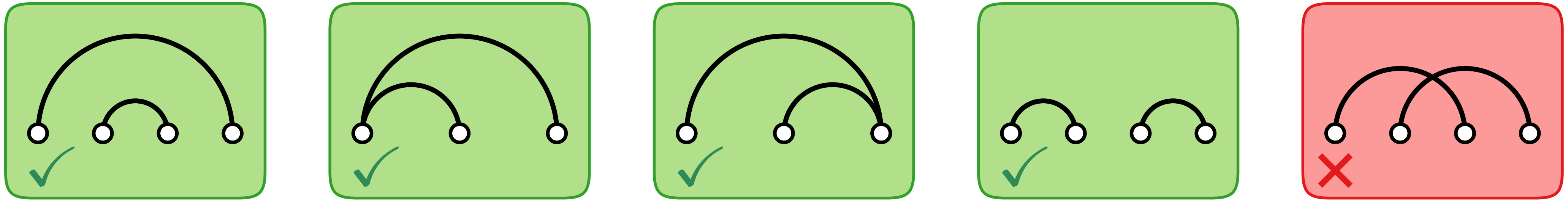

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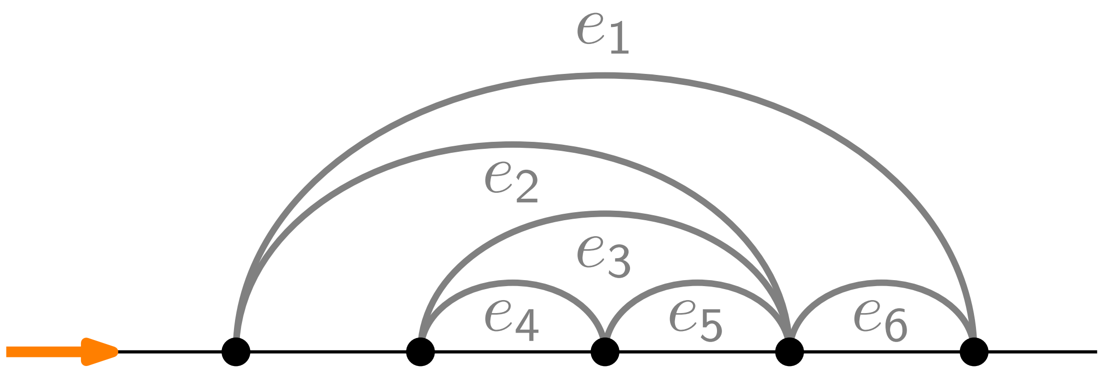

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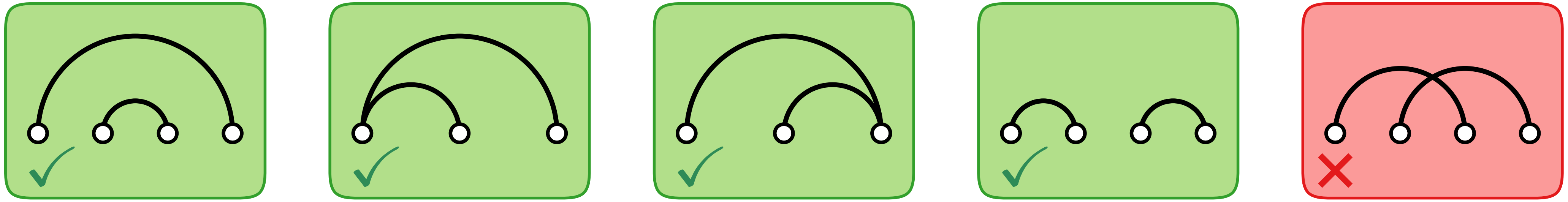

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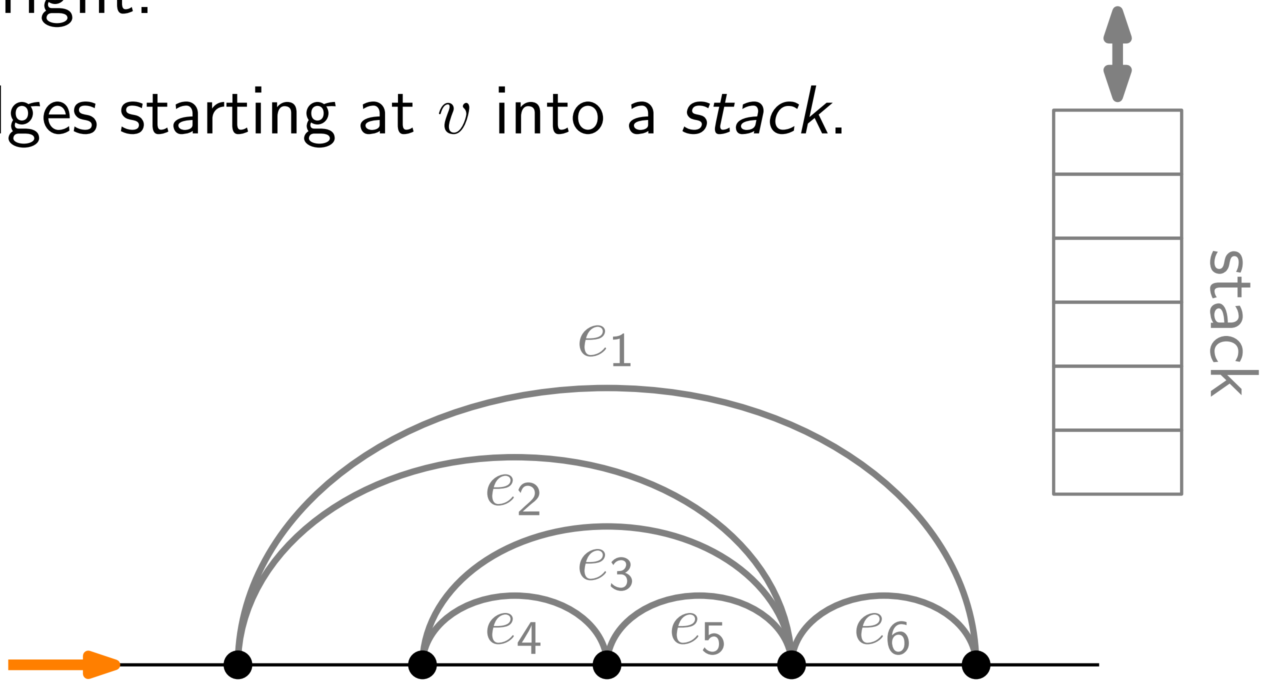

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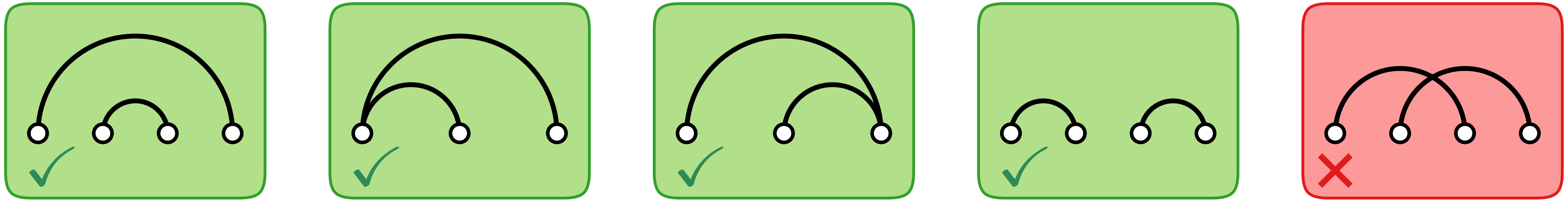

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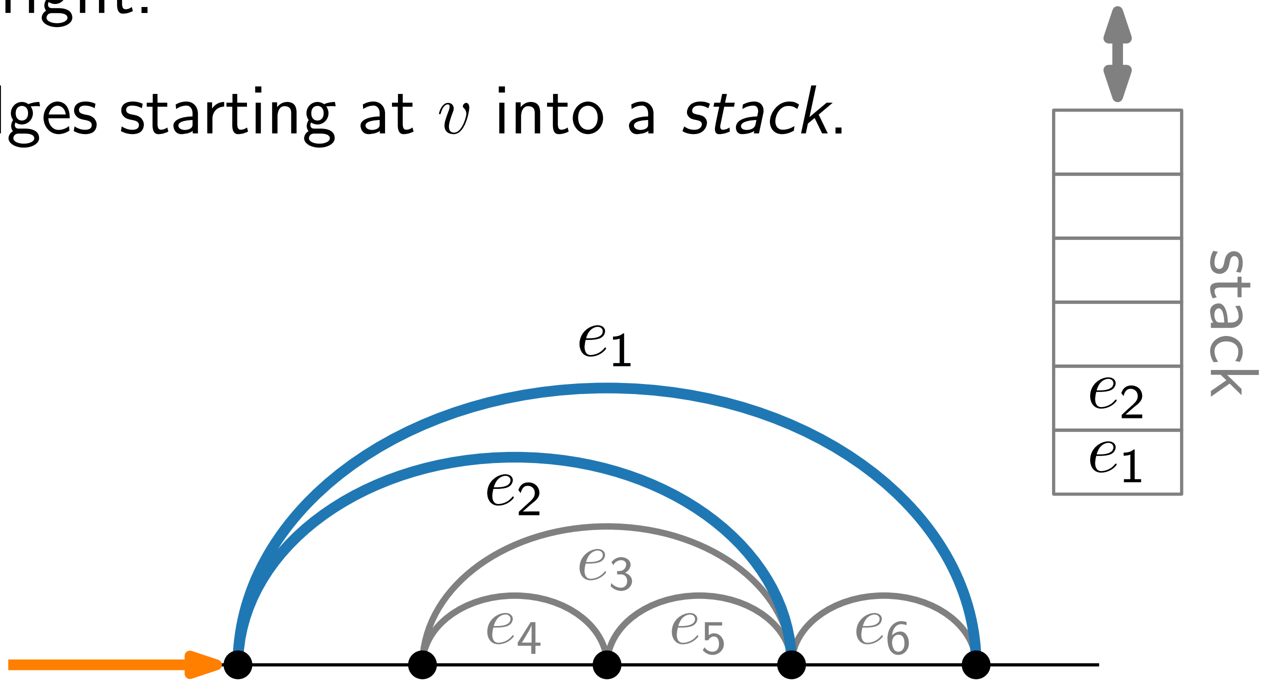

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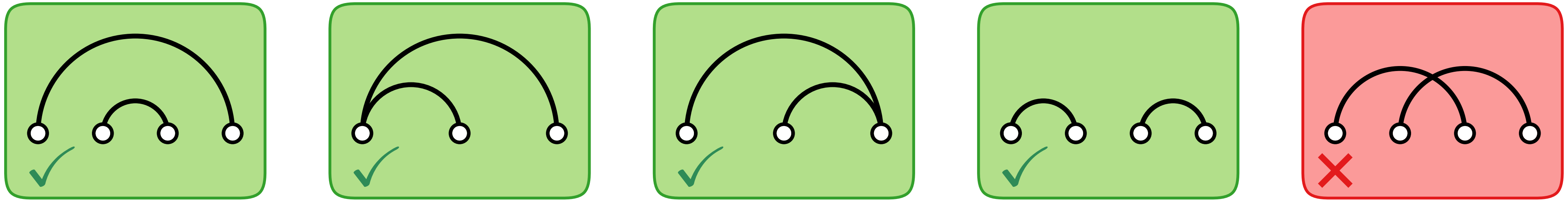

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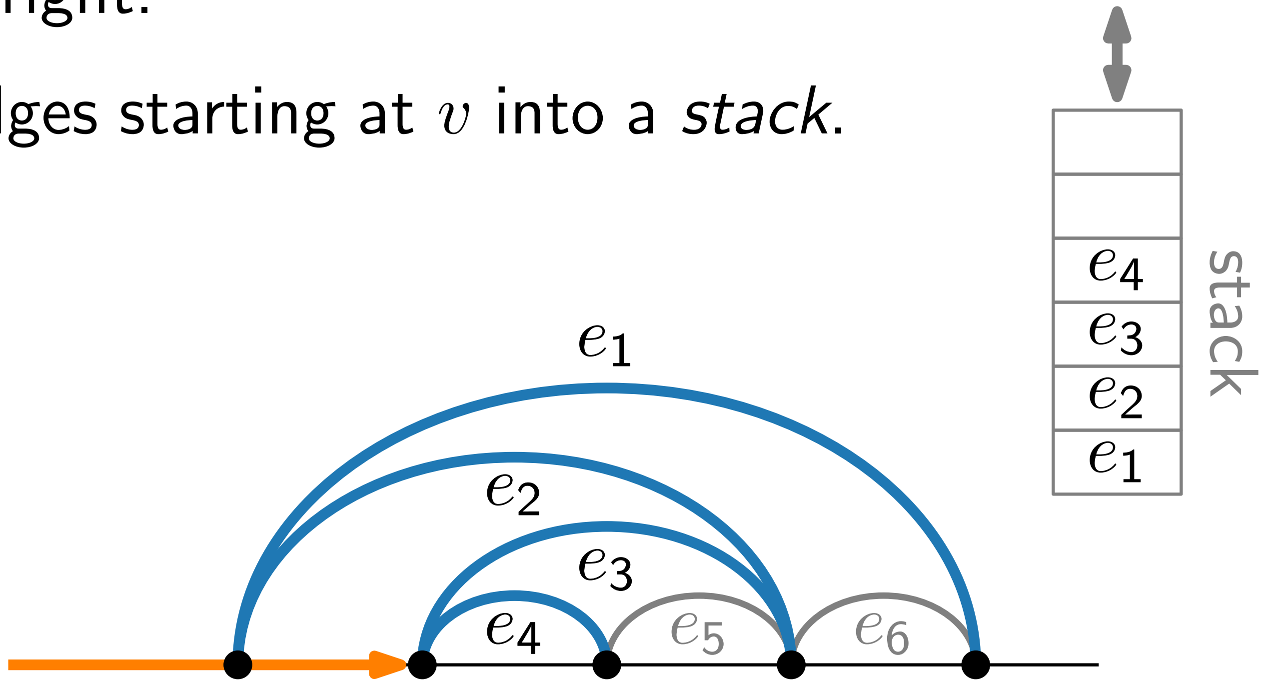

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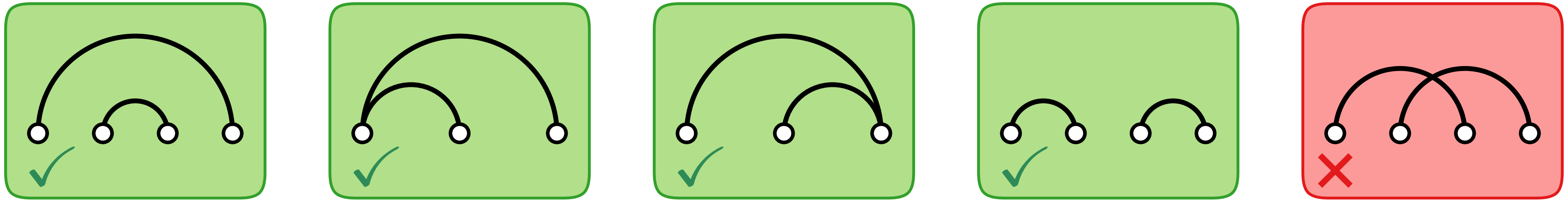

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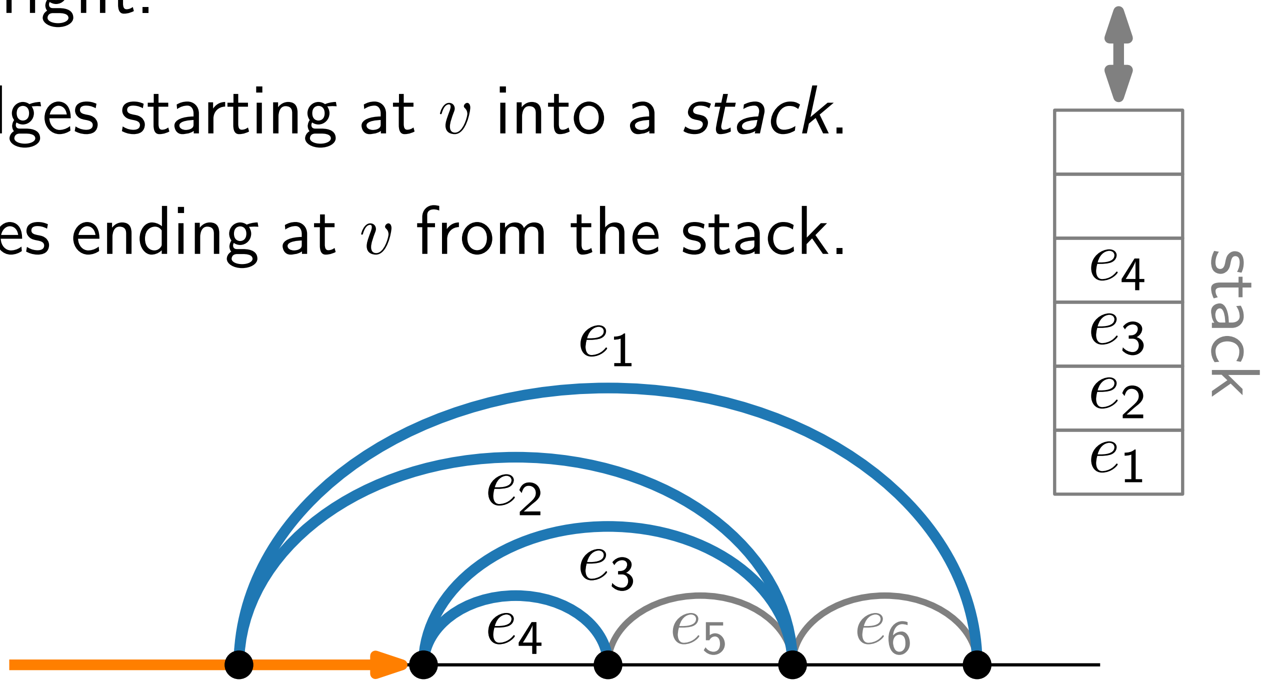

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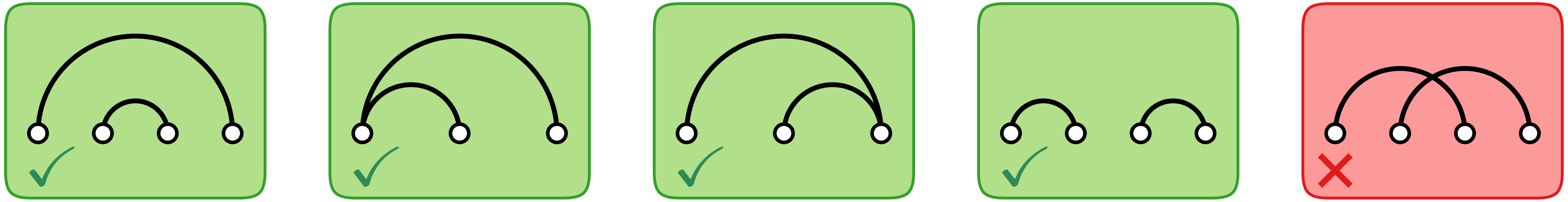

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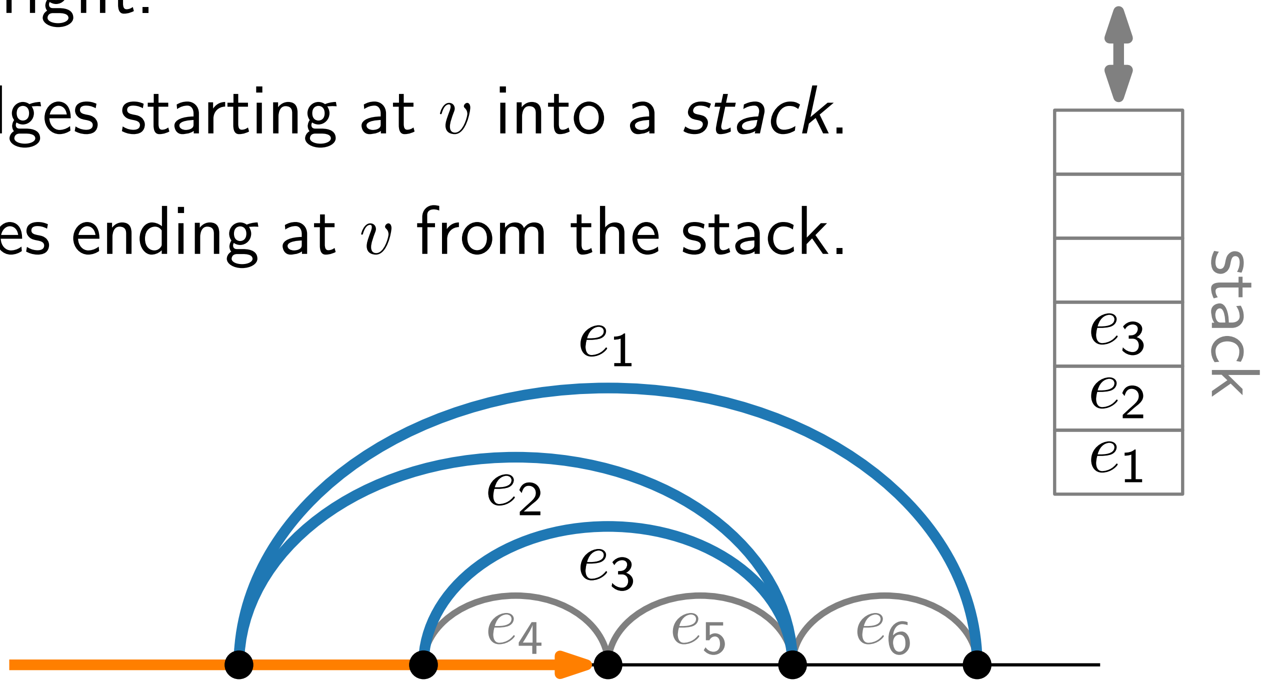

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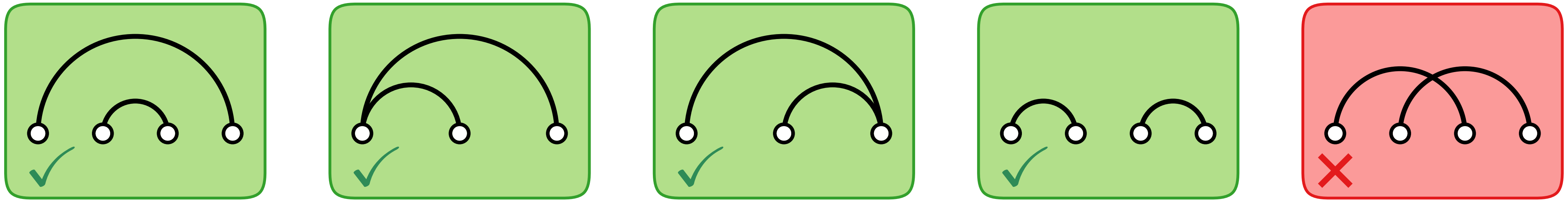

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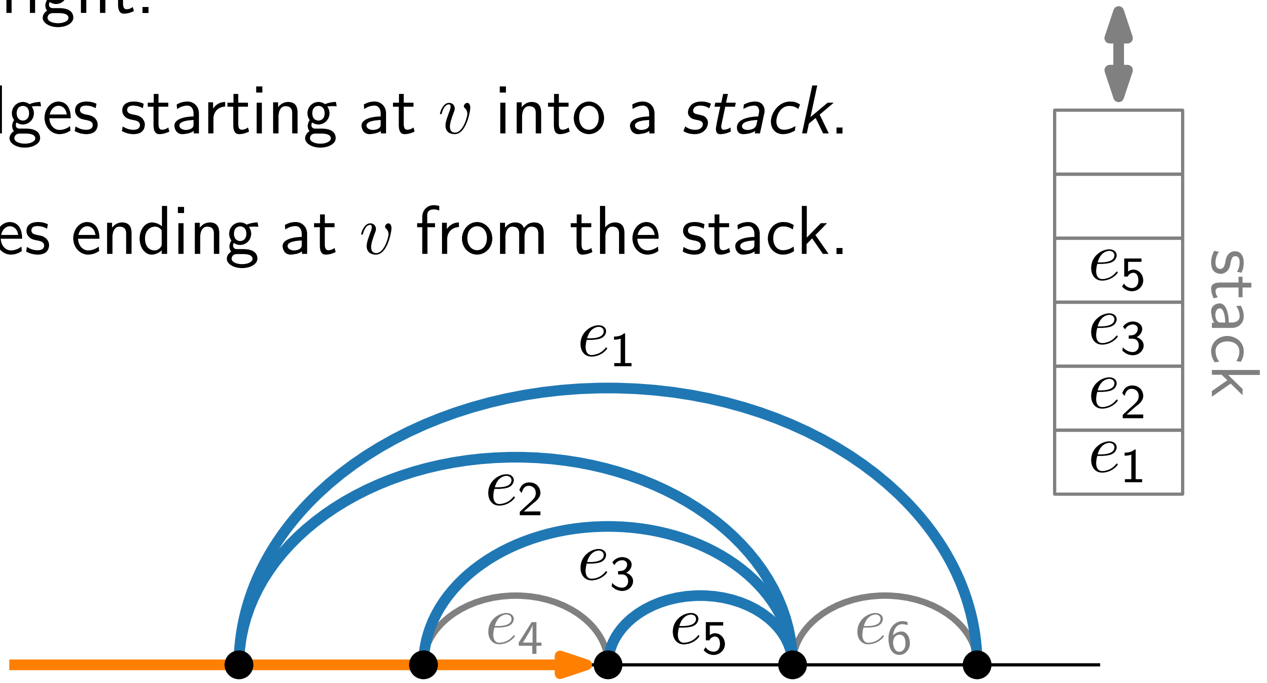

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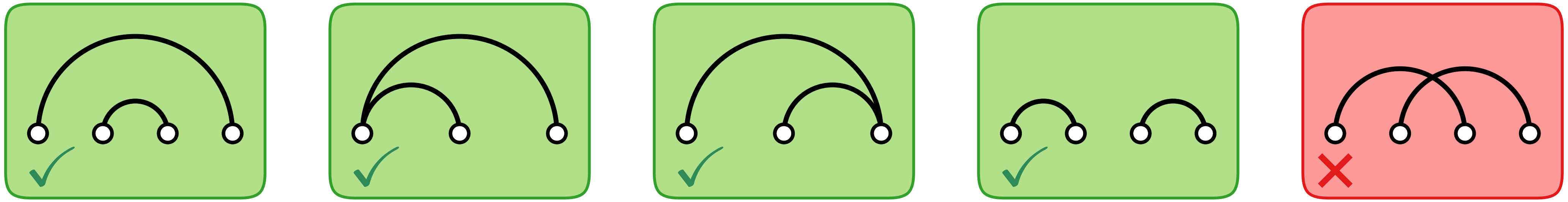

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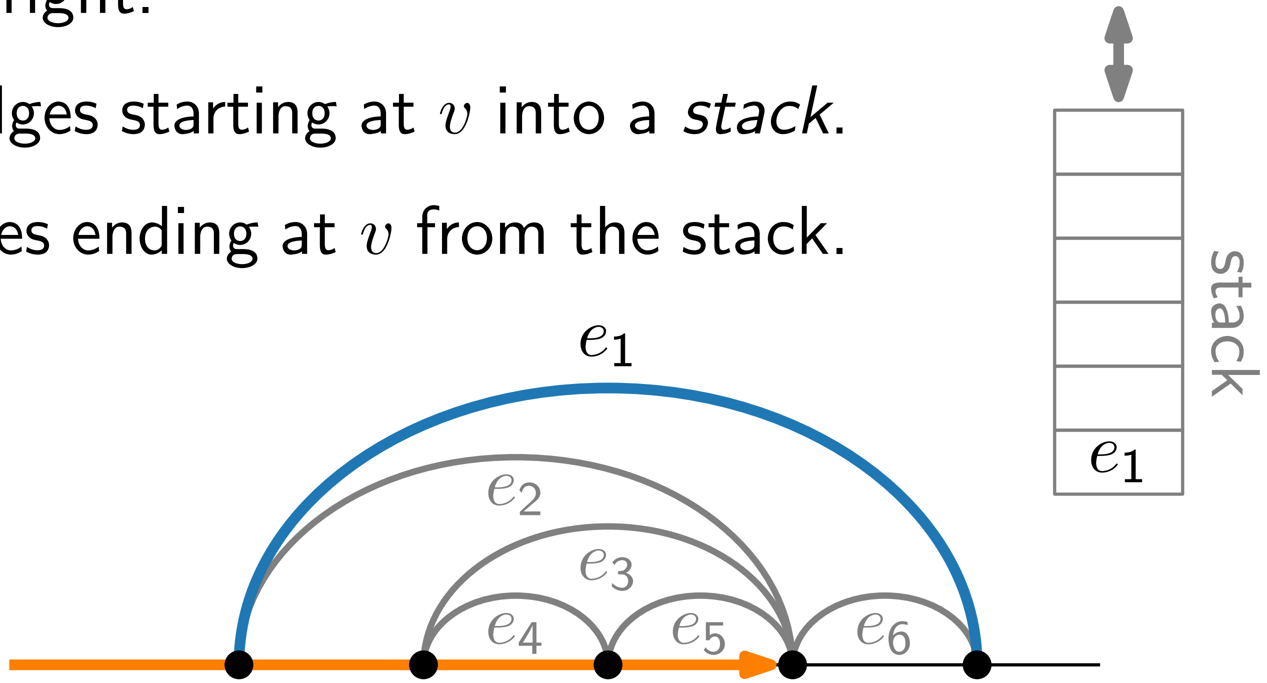

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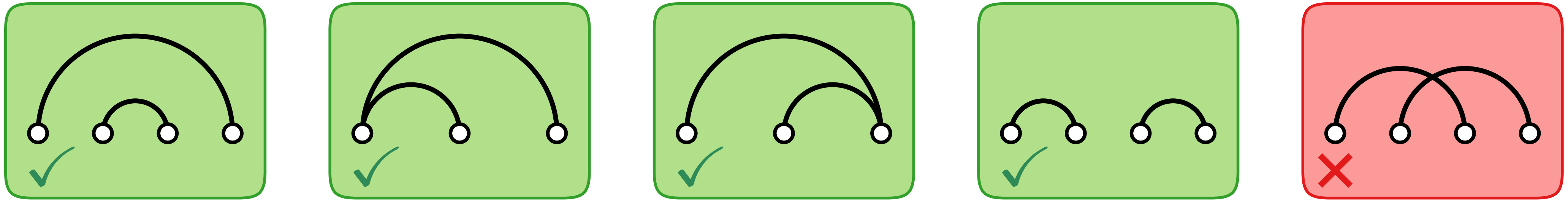

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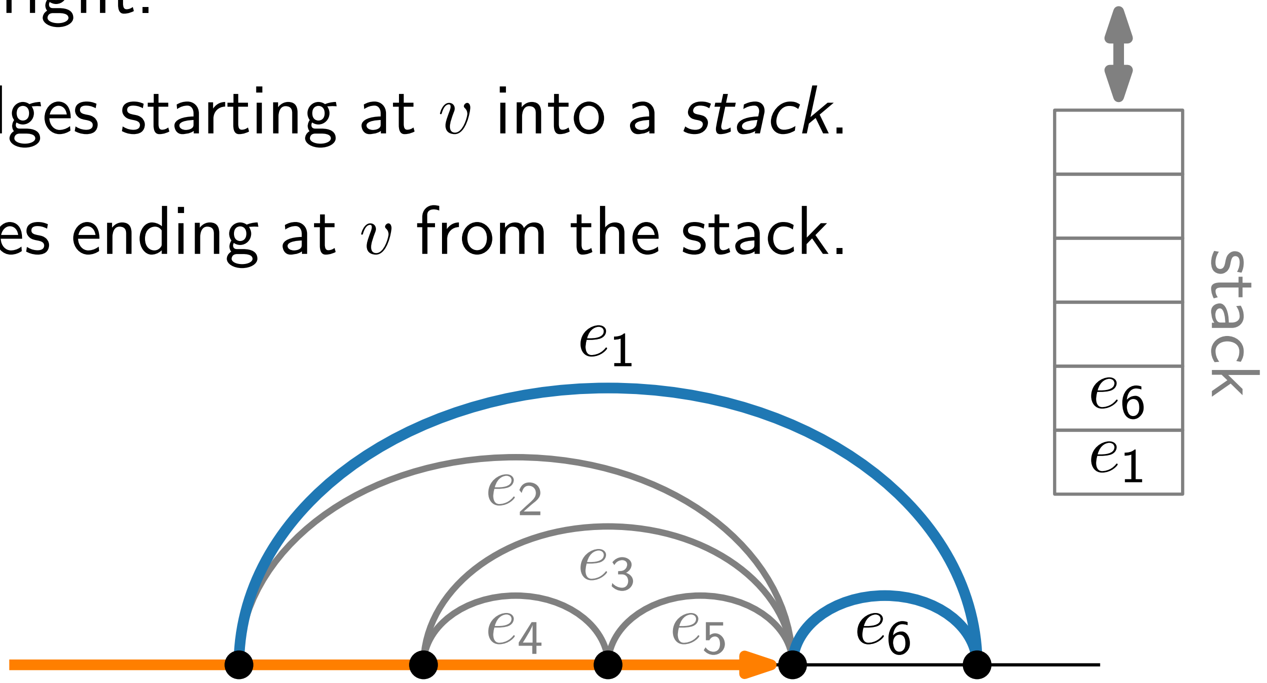

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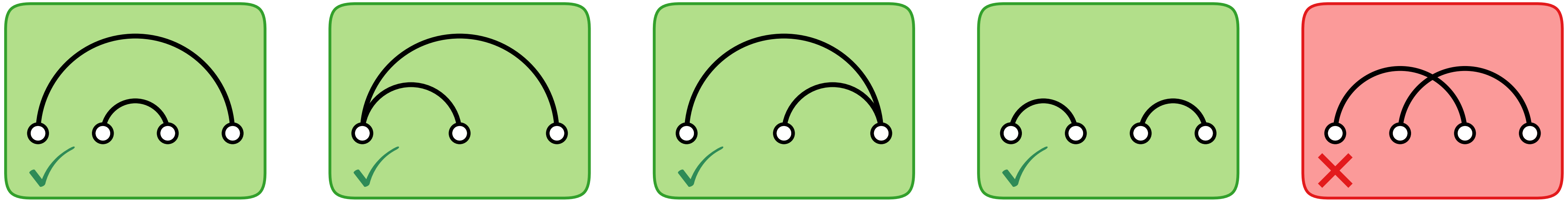

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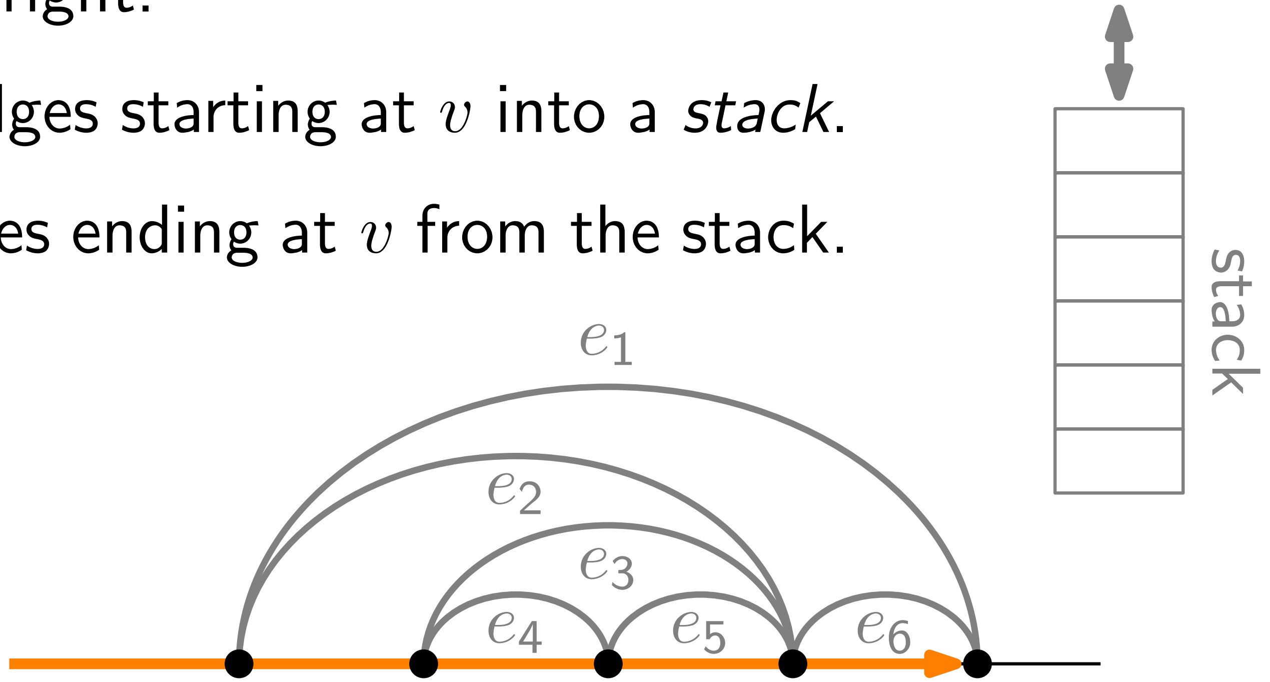

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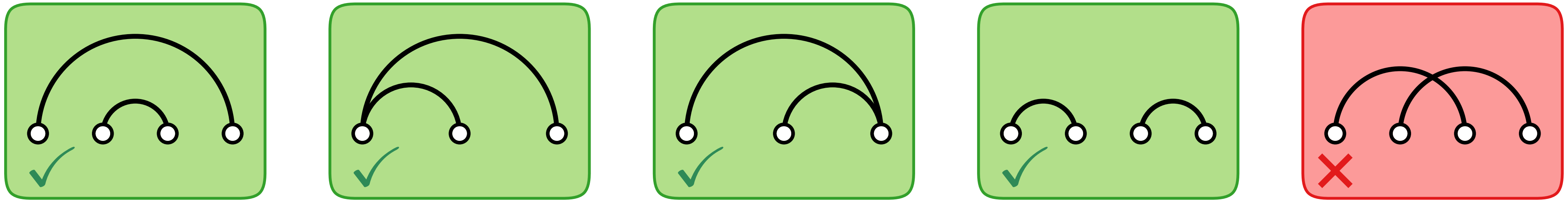

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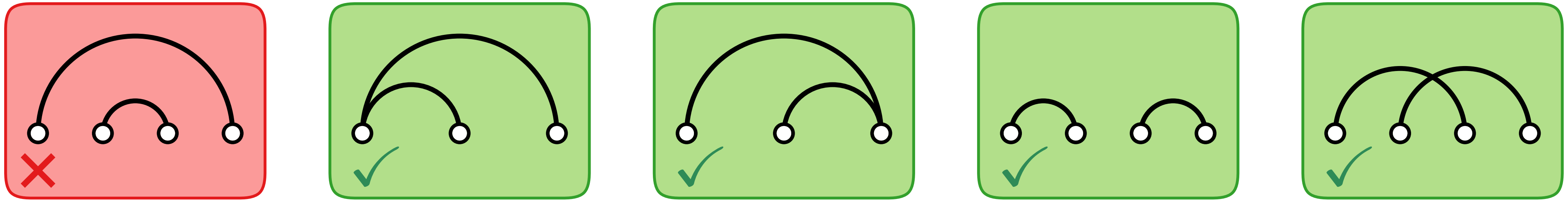

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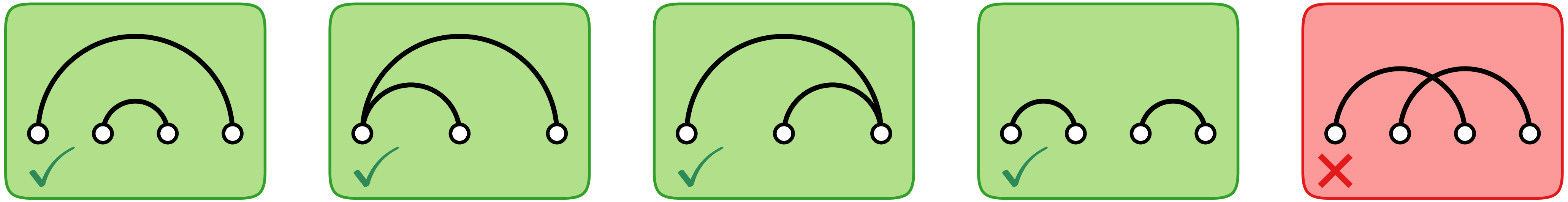

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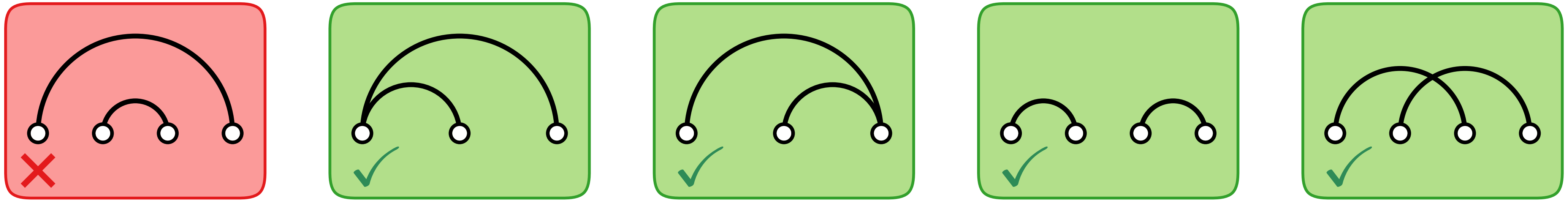

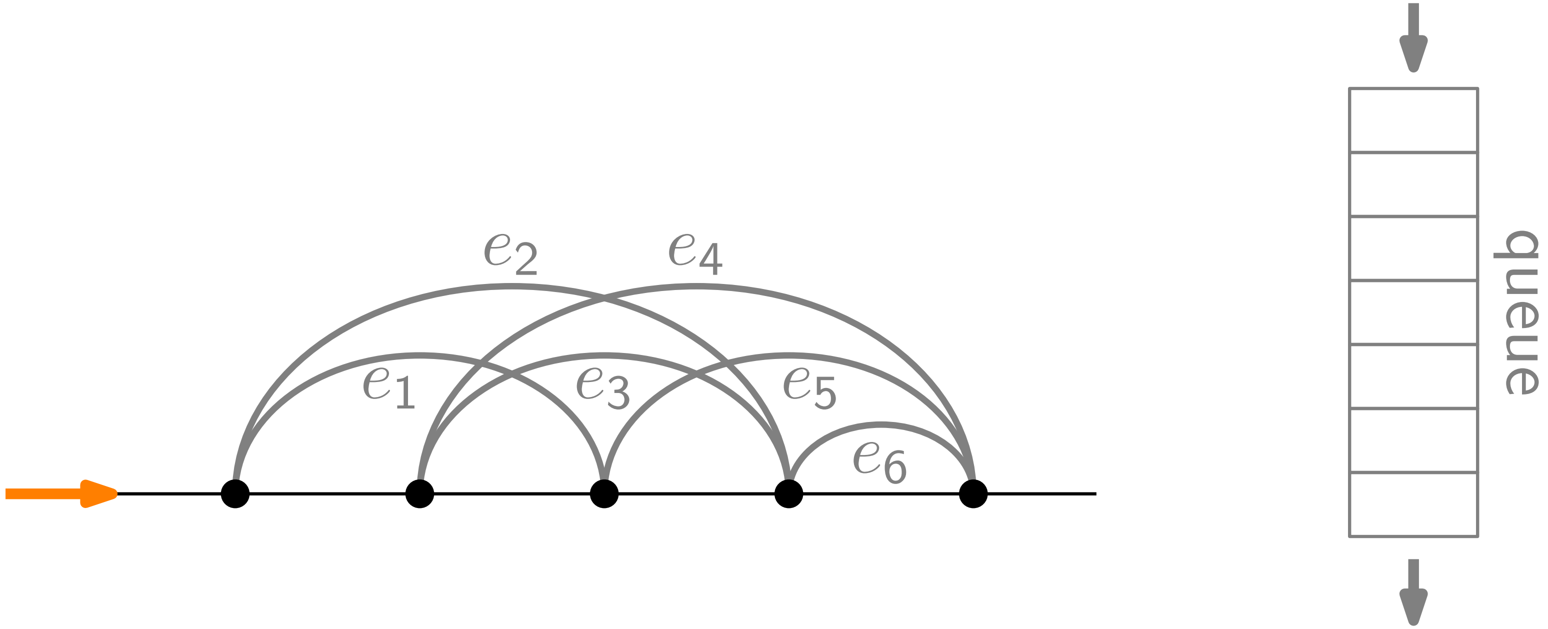

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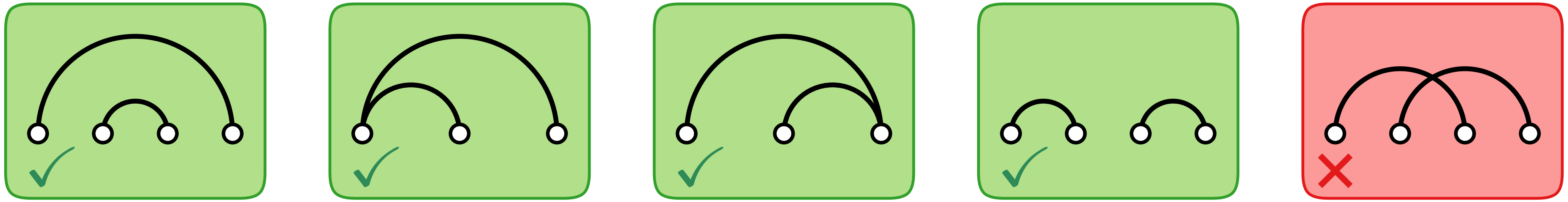

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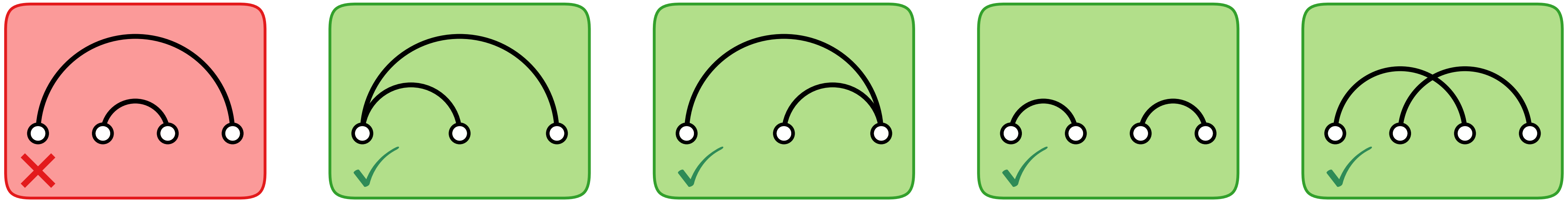

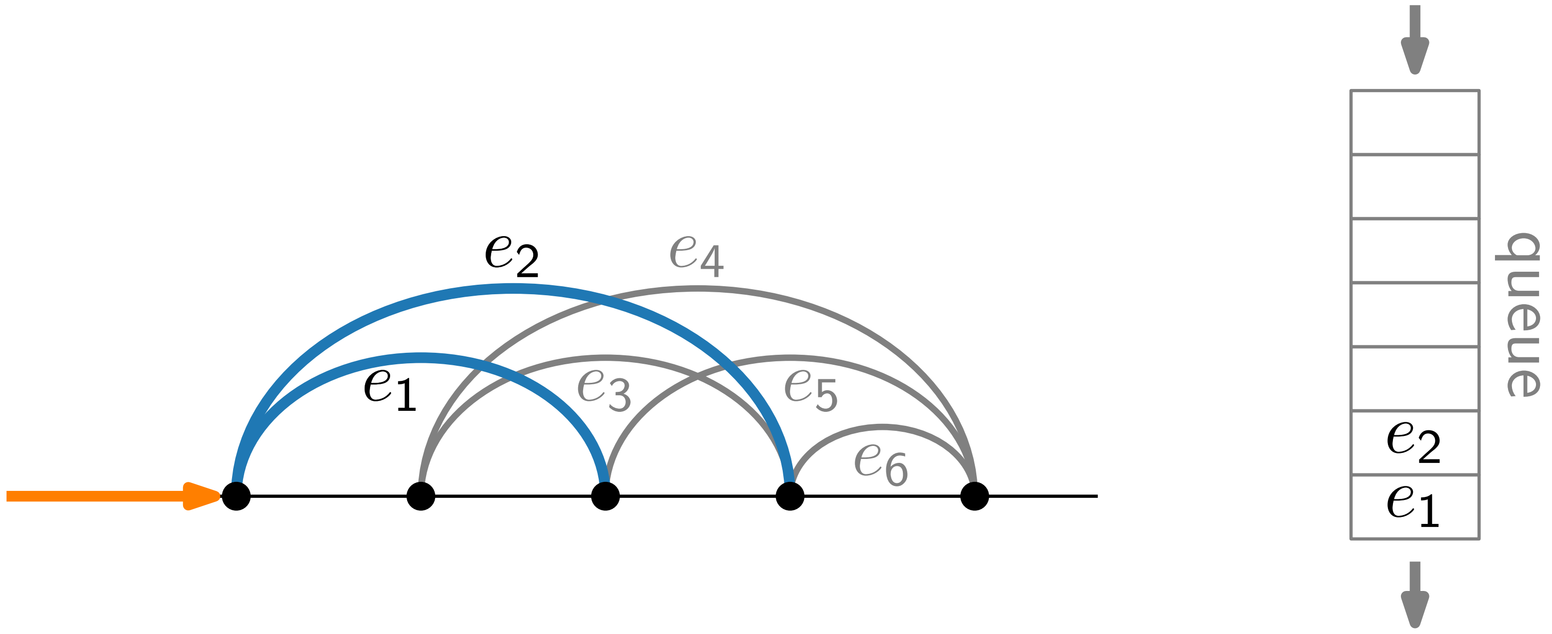

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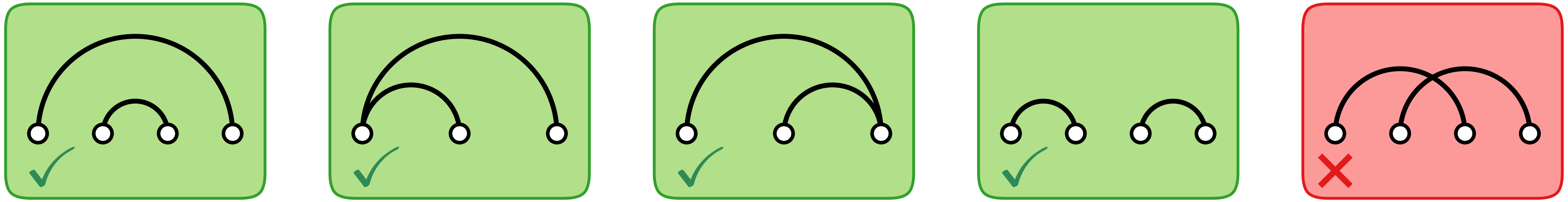

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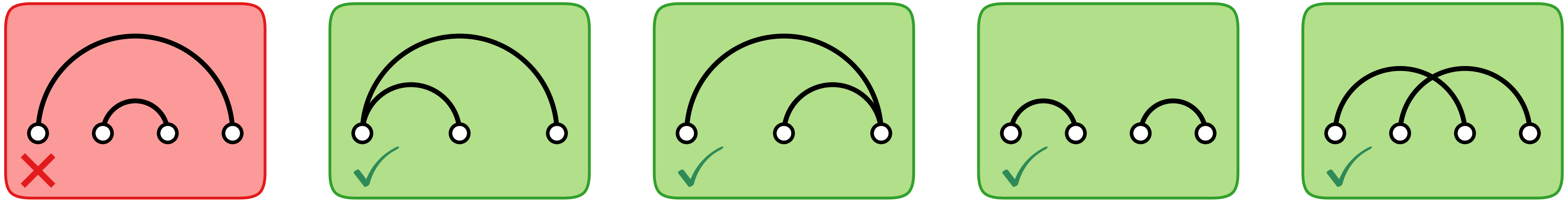

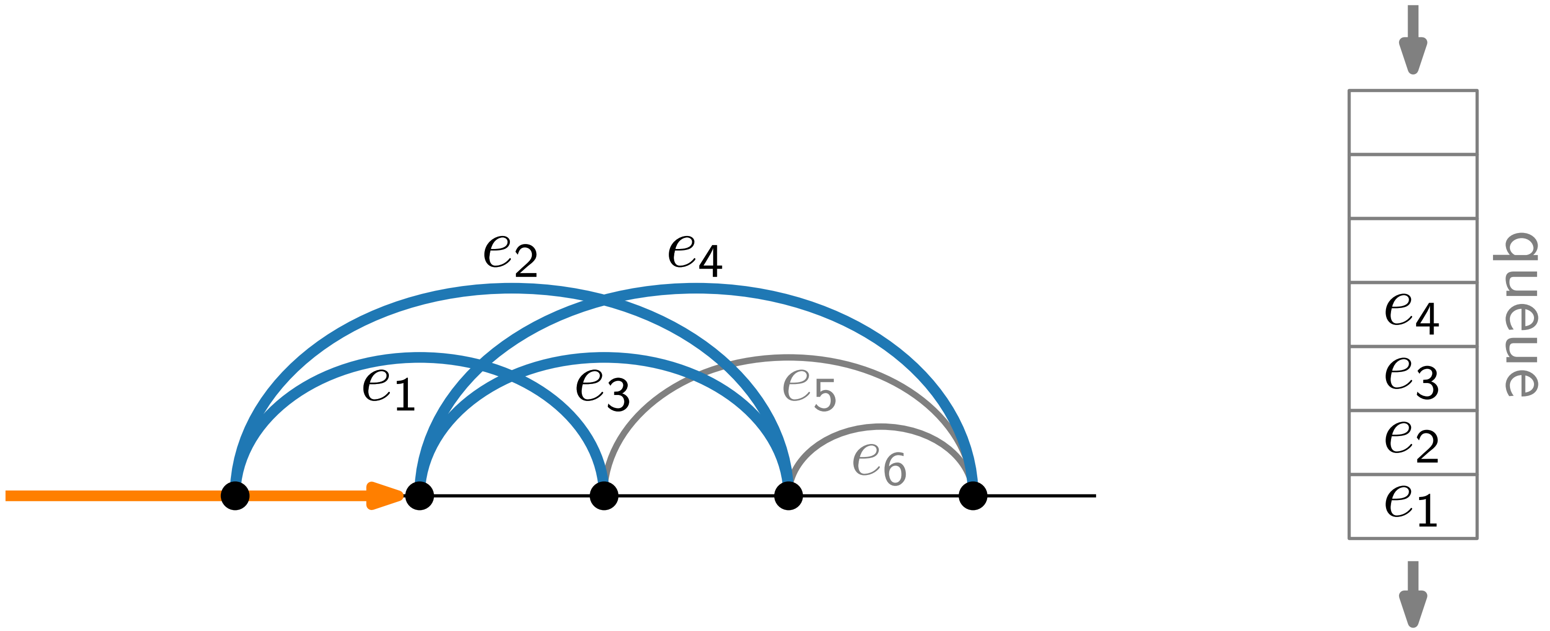

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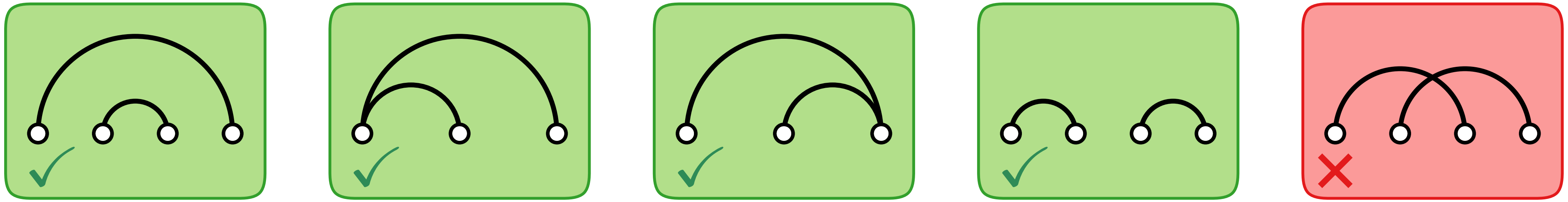

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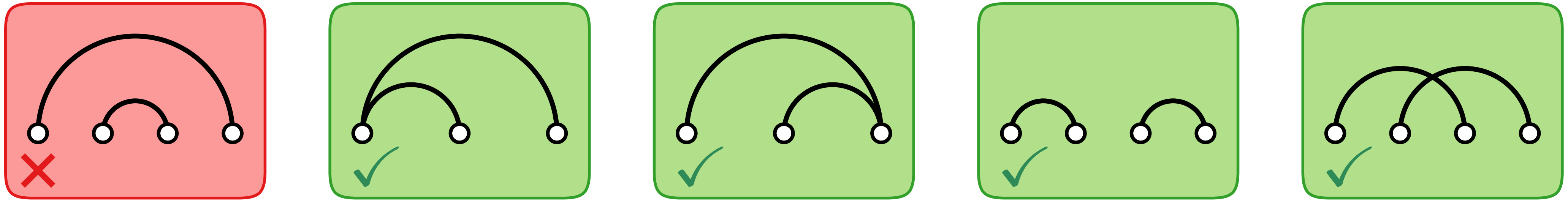

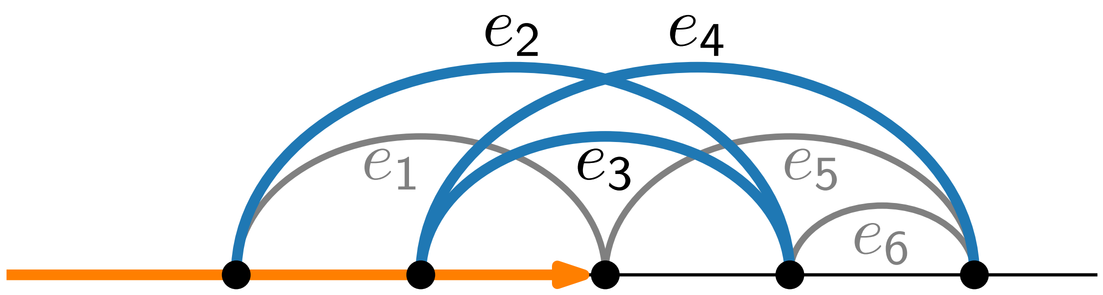

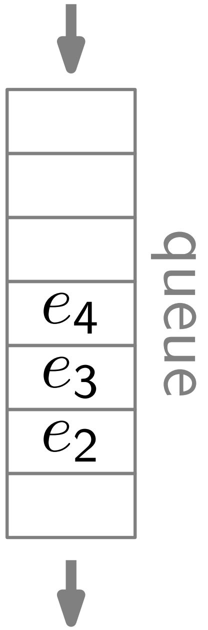

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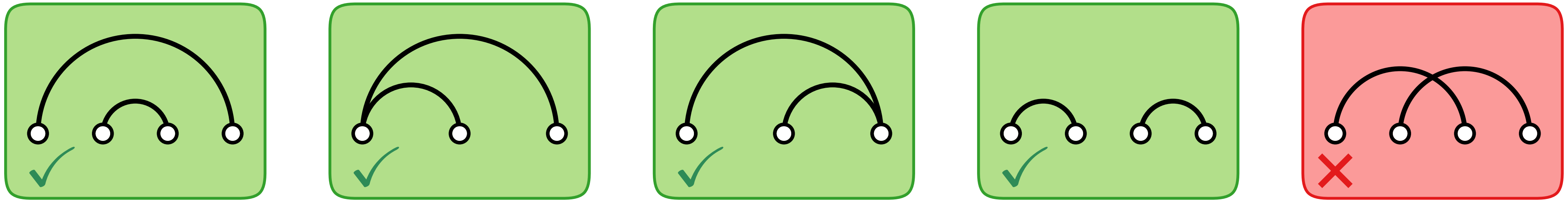

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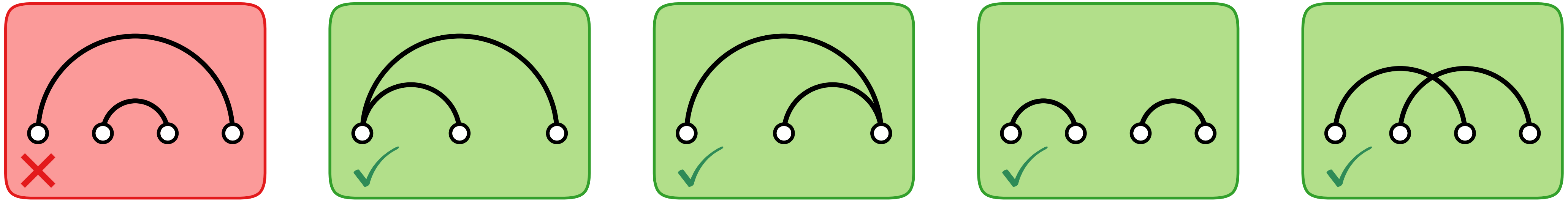

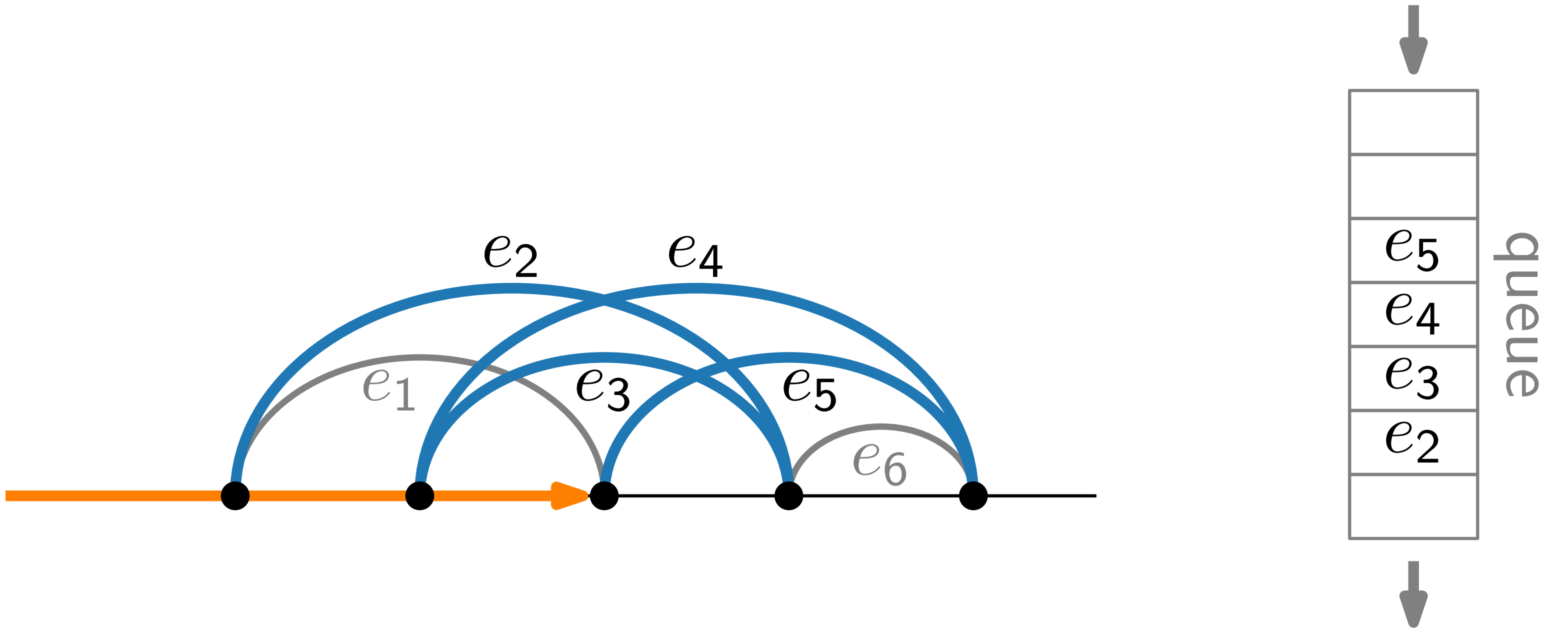

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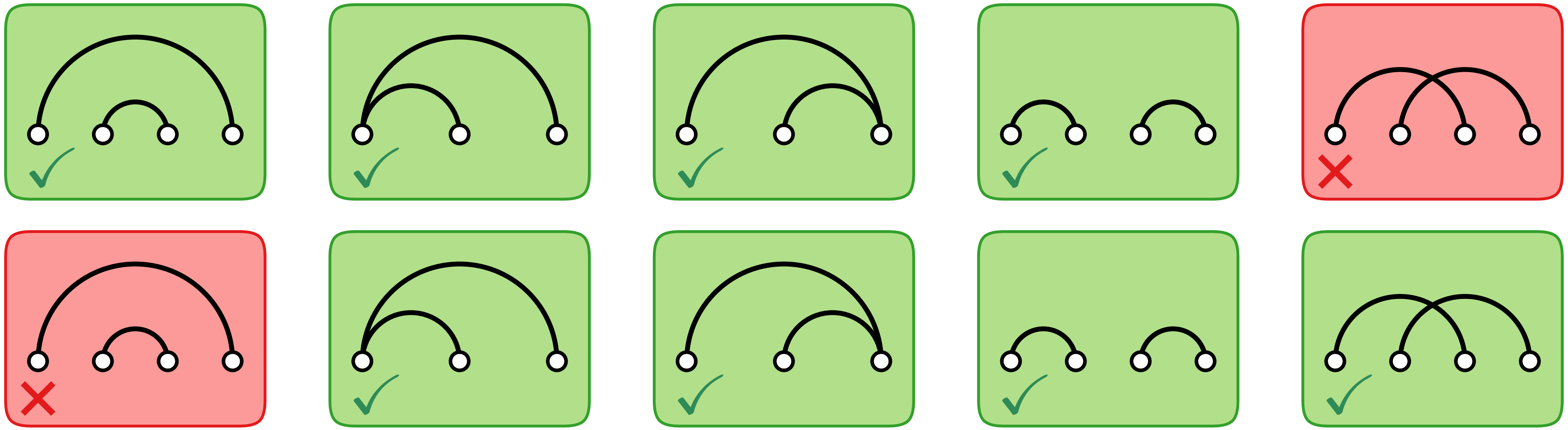

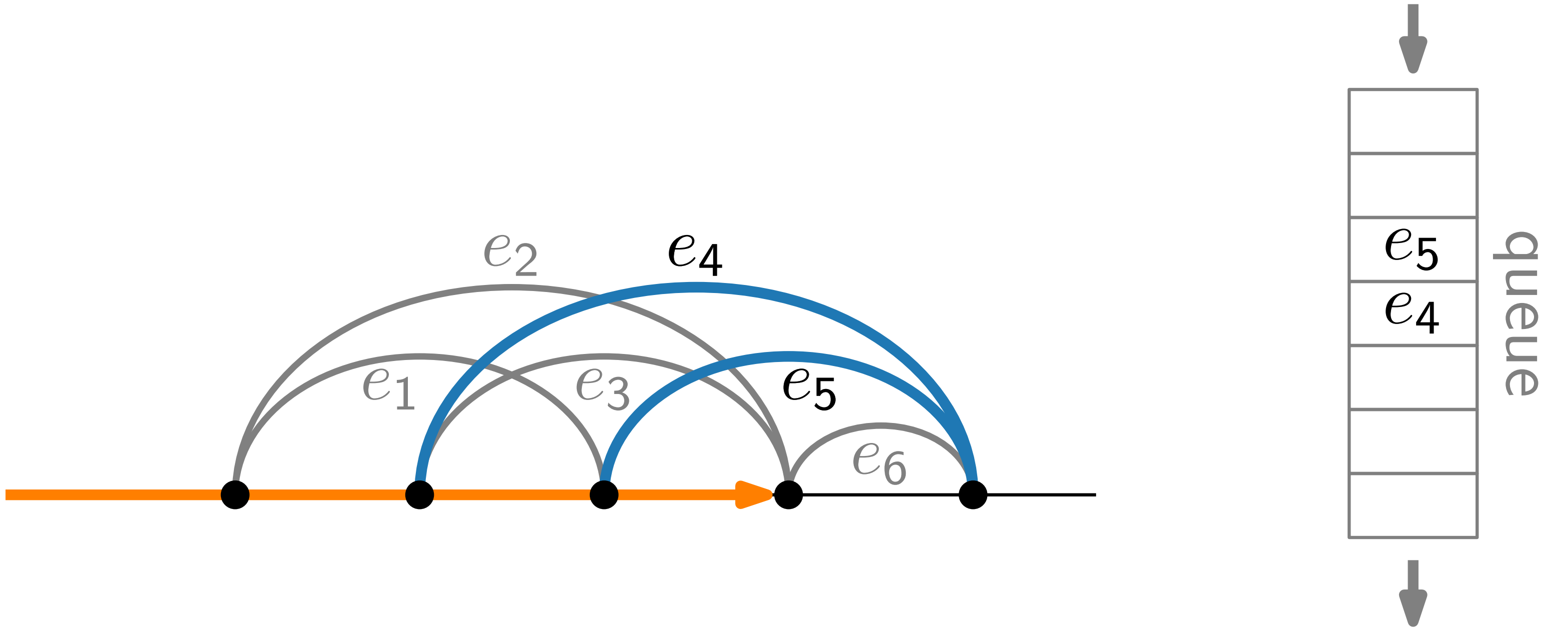

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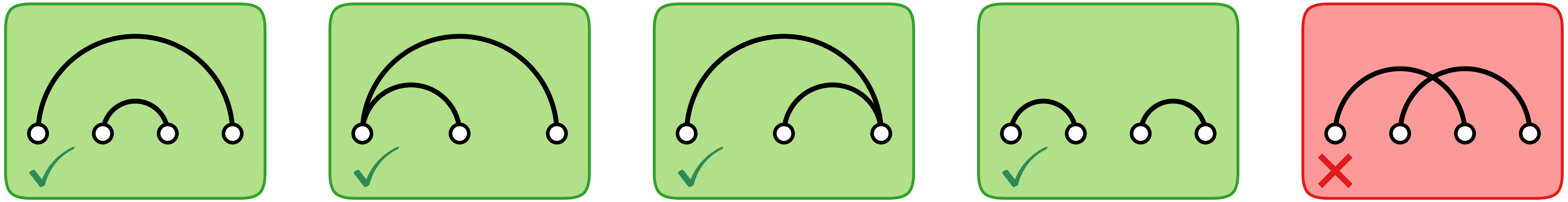

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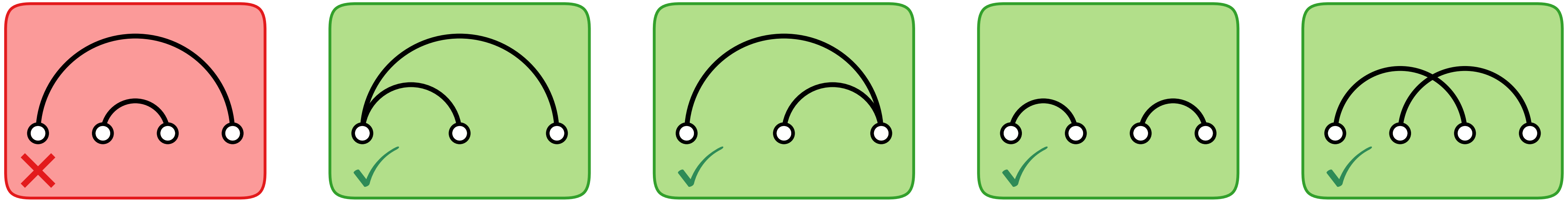

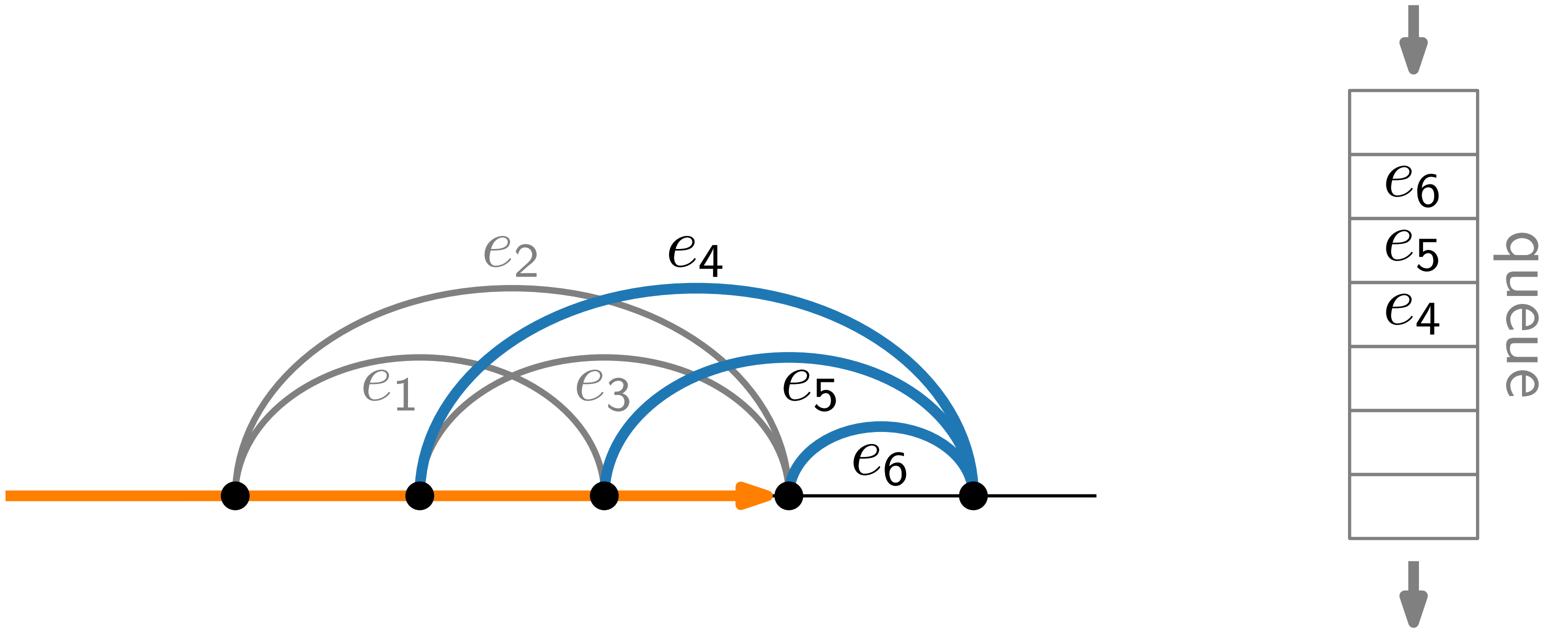

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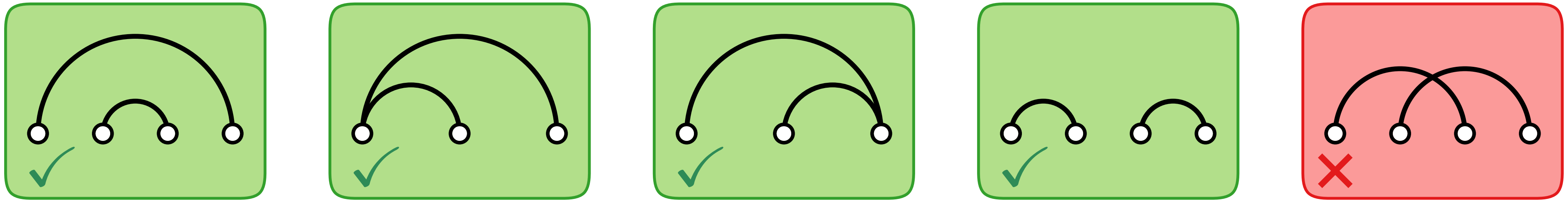

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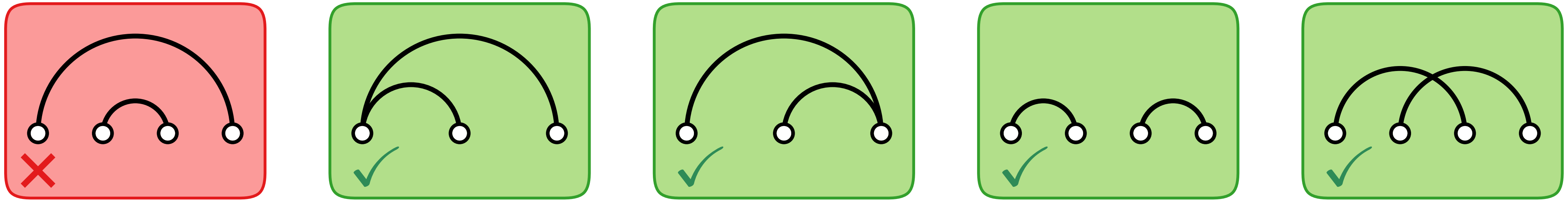

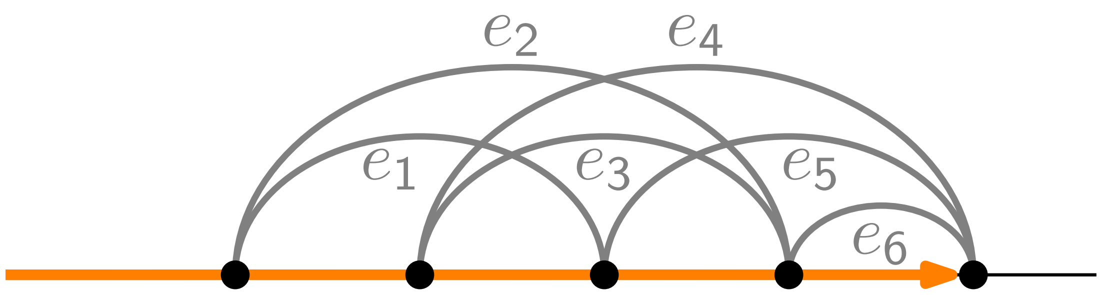

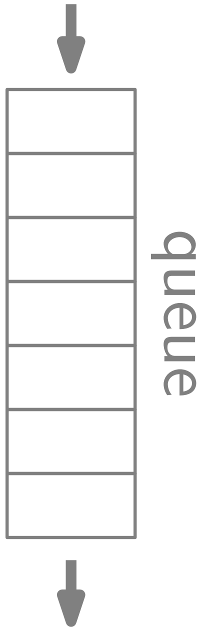

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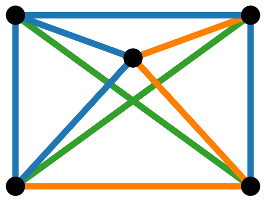

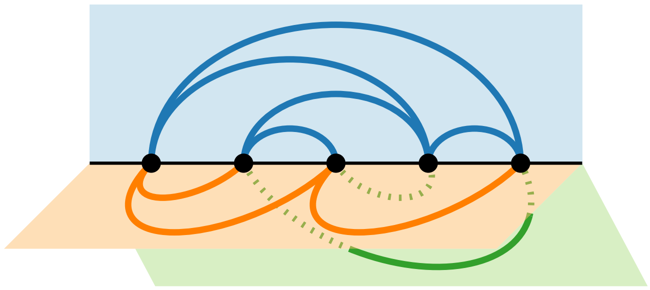

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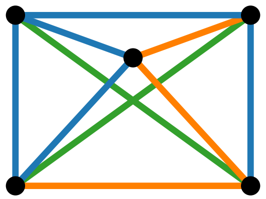

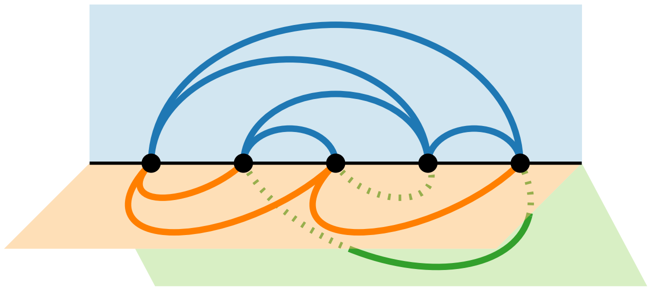

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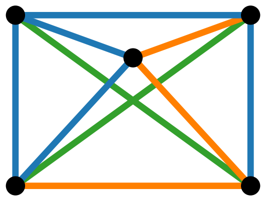

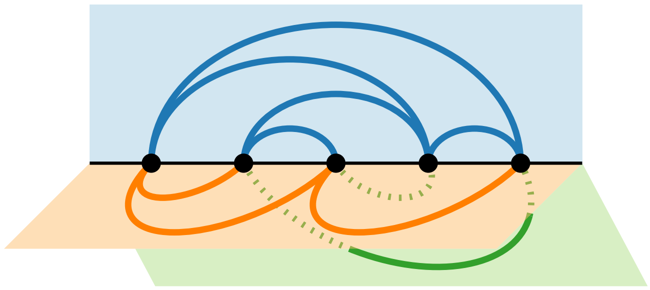

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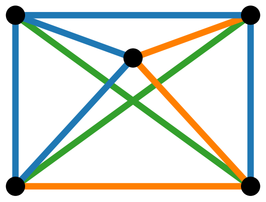

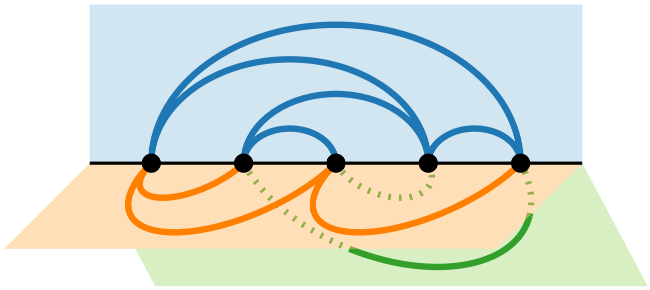

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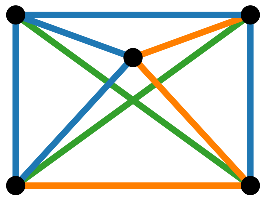

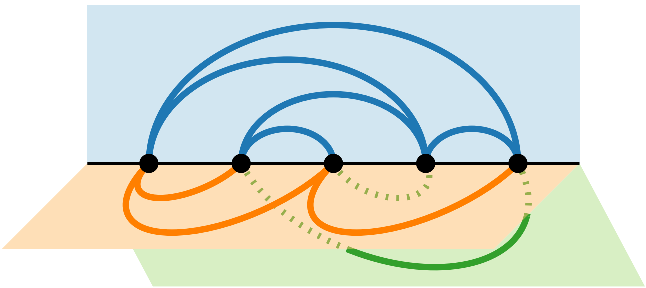

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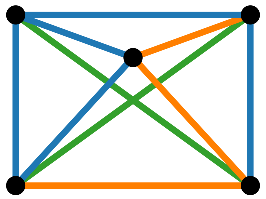

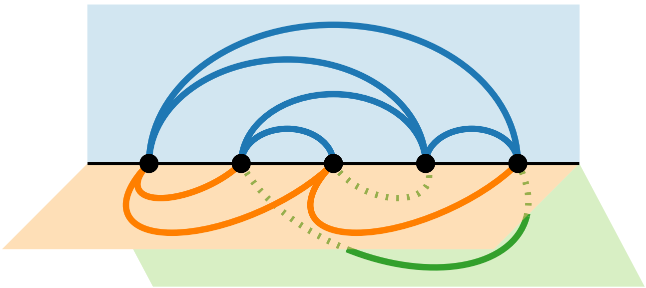

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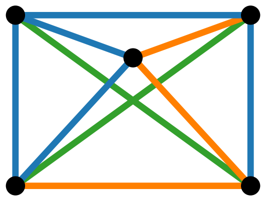

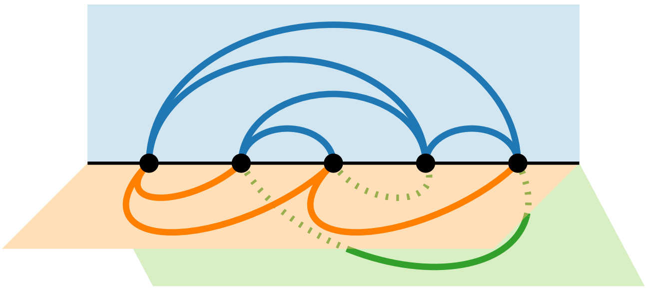

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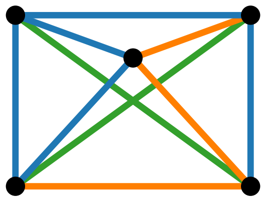

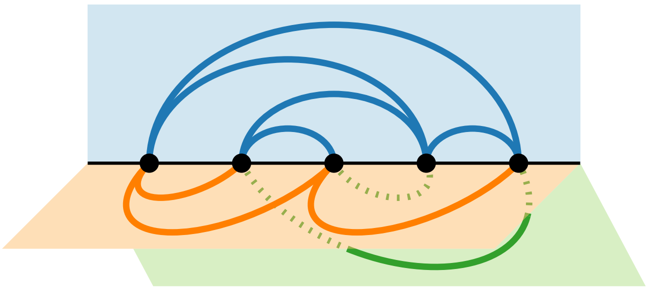

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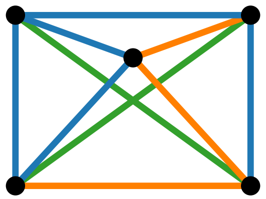

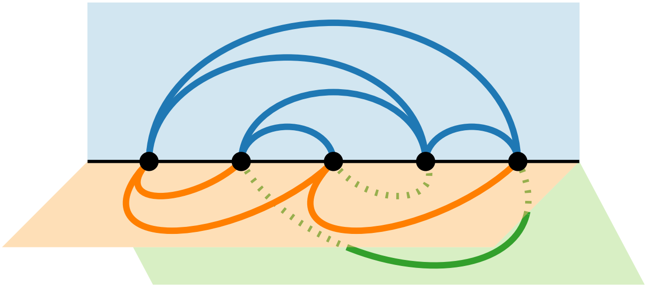

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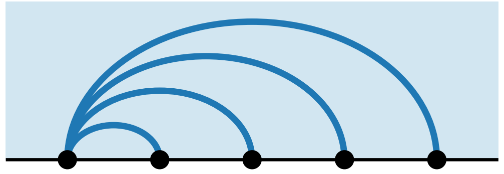

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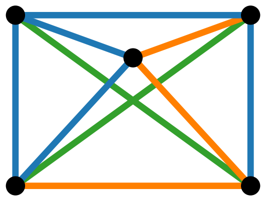

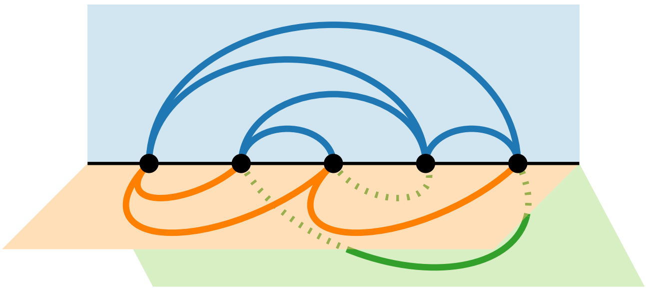

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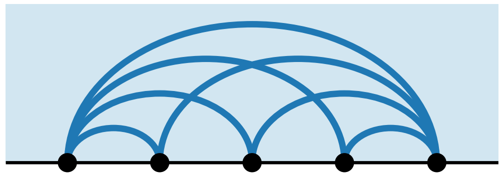

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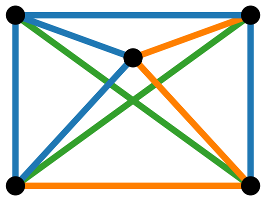

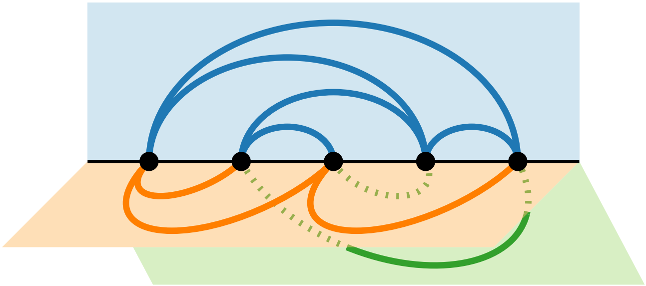

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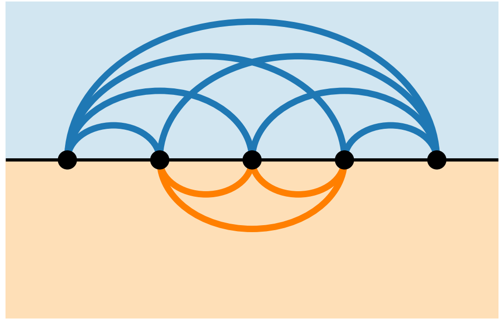

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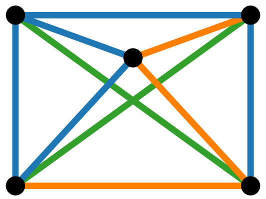

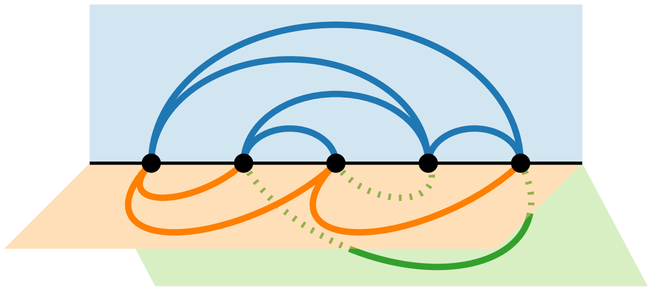

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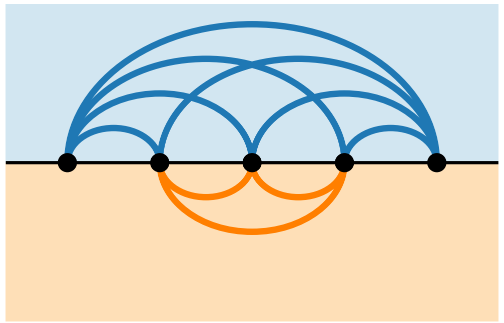

- Does K_5 have a 2-page queue layout? Yes! $\Rightarrow \mathrm{qn}(K_5) = 2$

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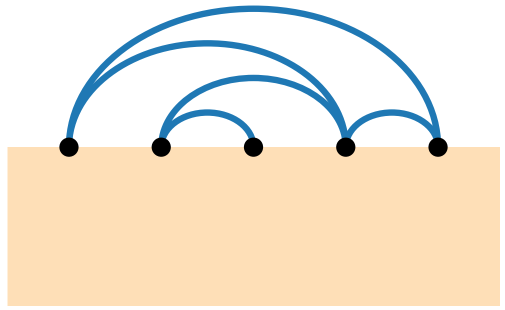

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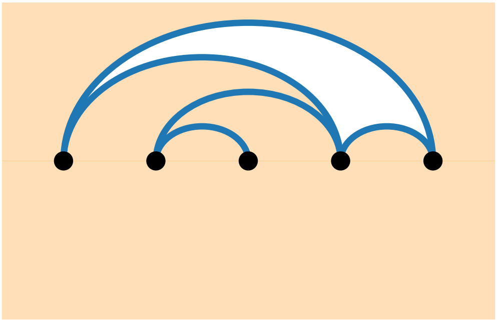

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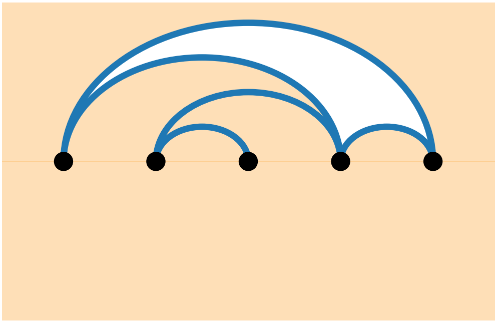

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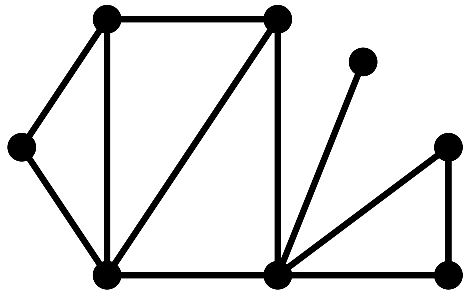

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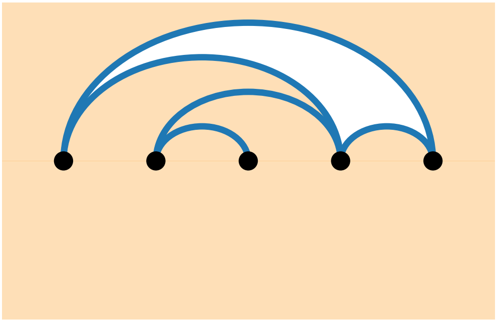

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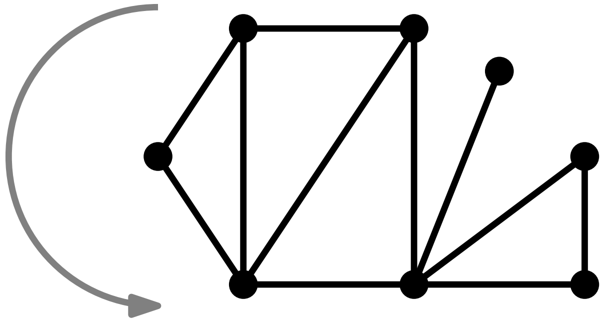

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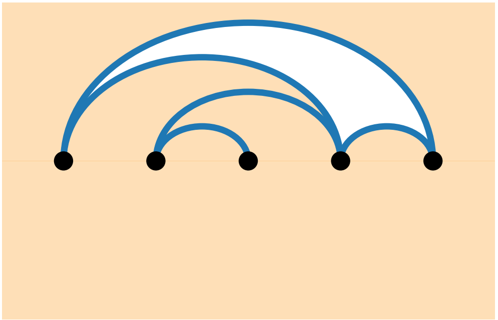

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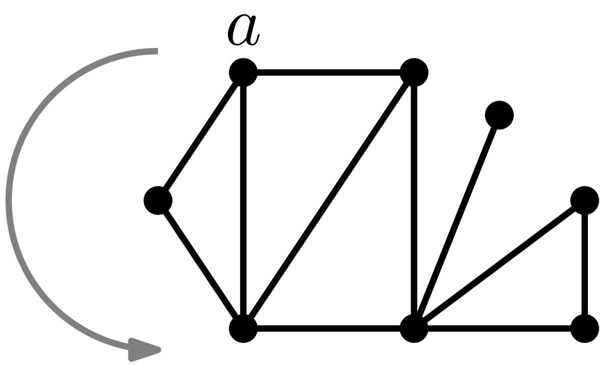

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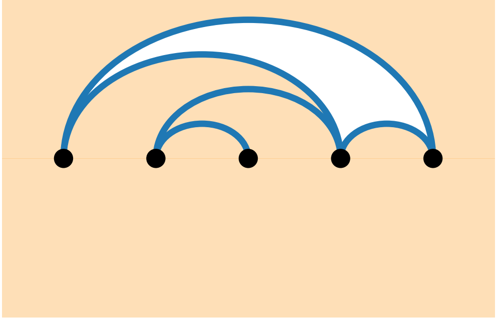

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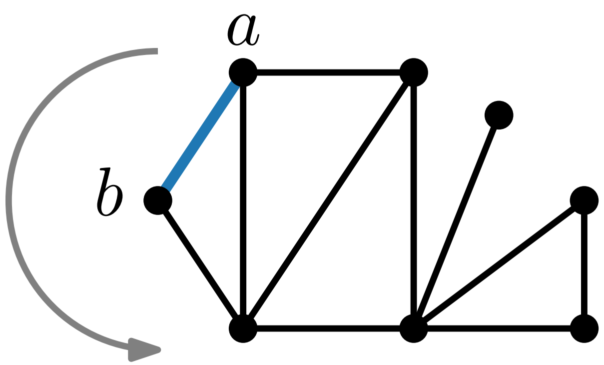

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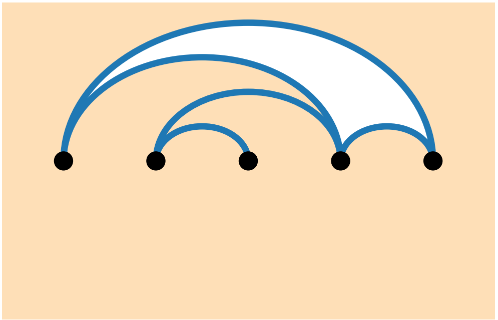

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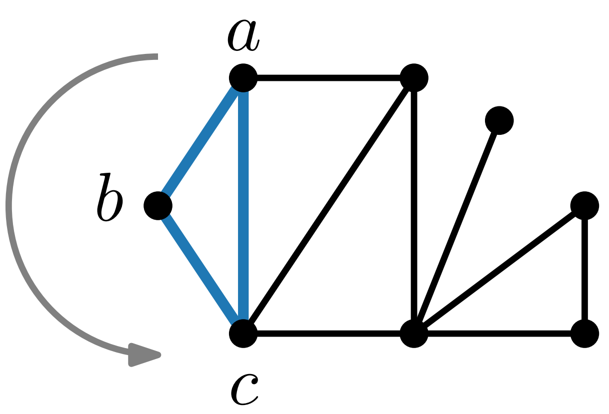

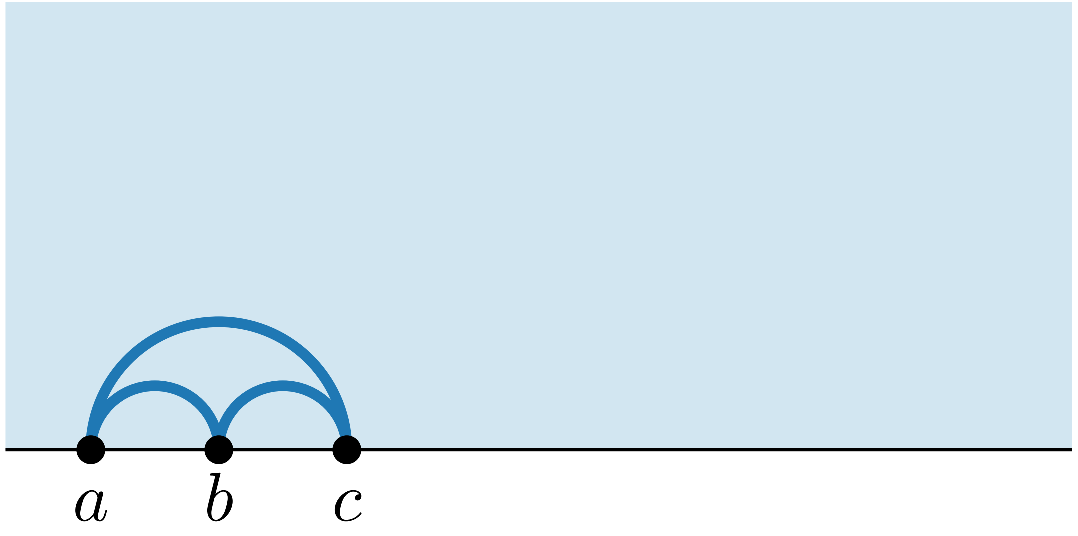

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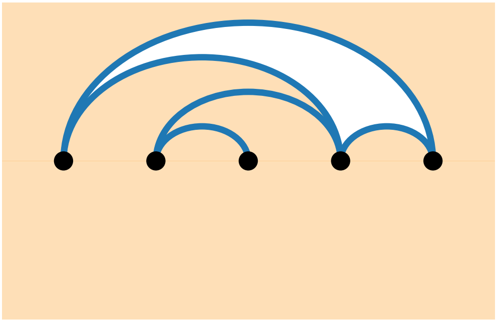

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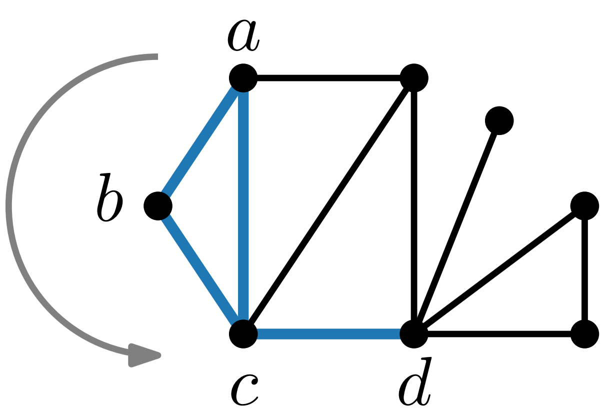

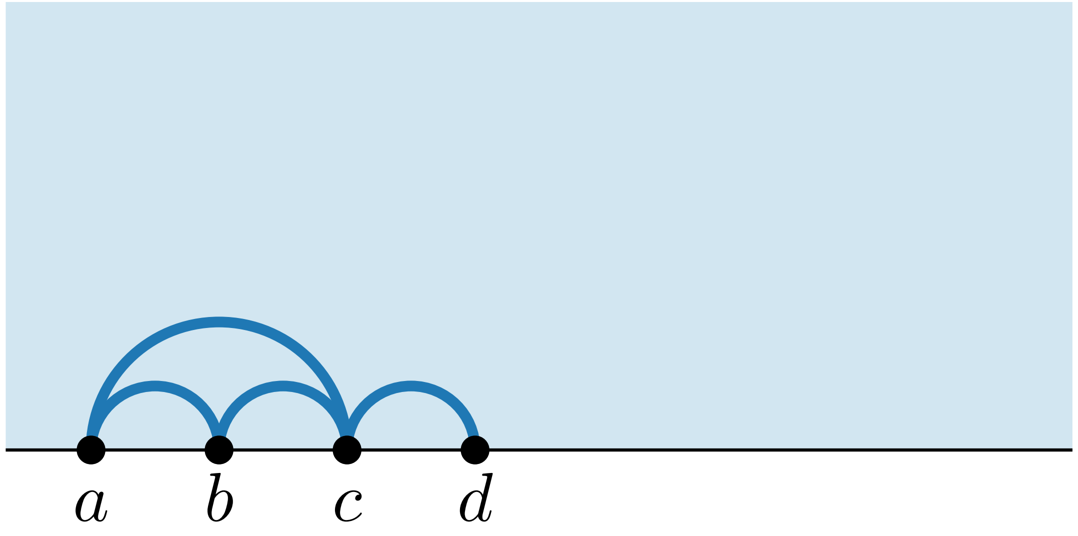

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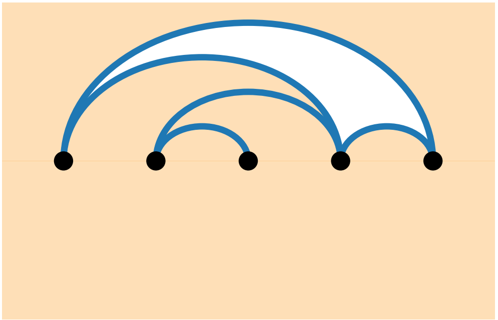

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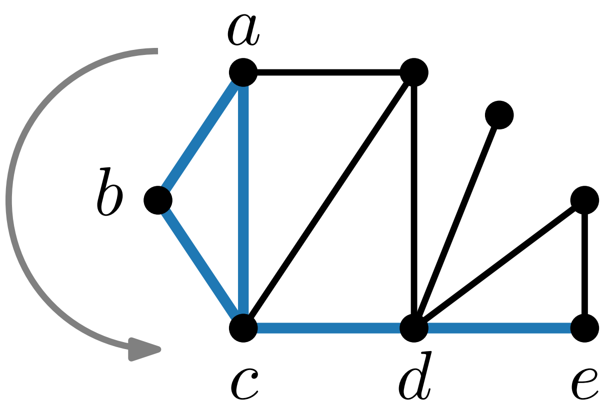

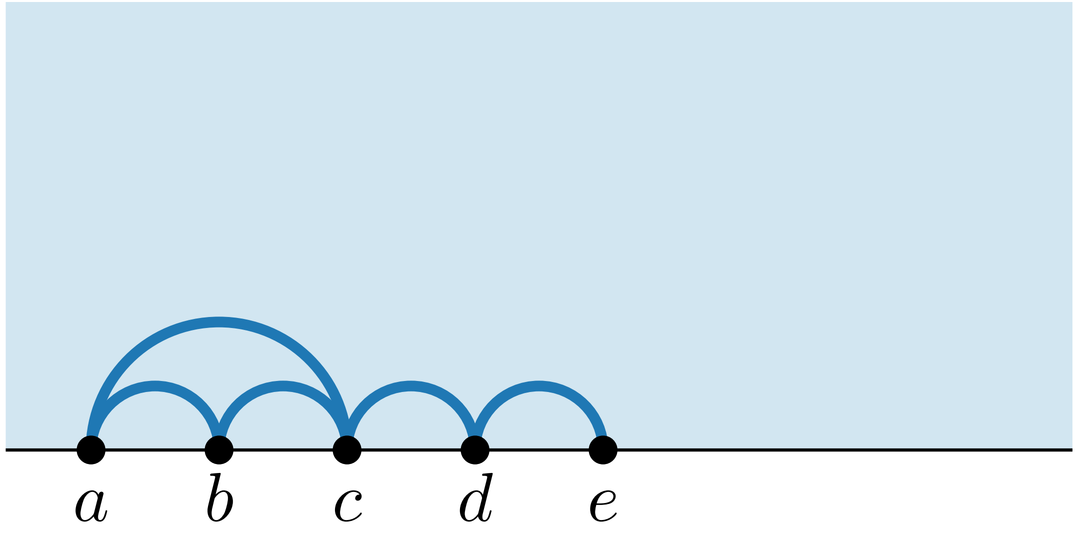

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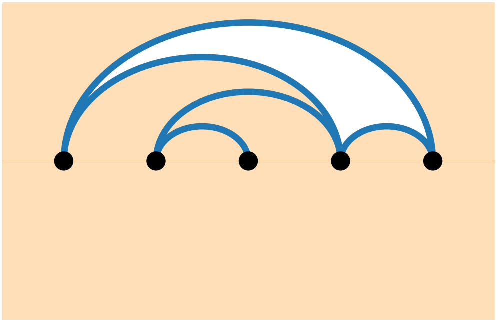

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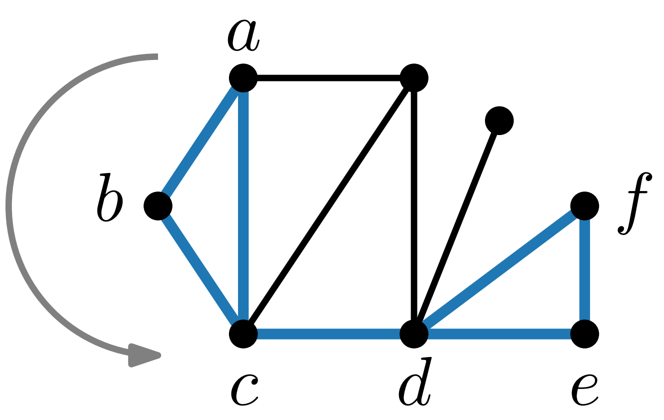

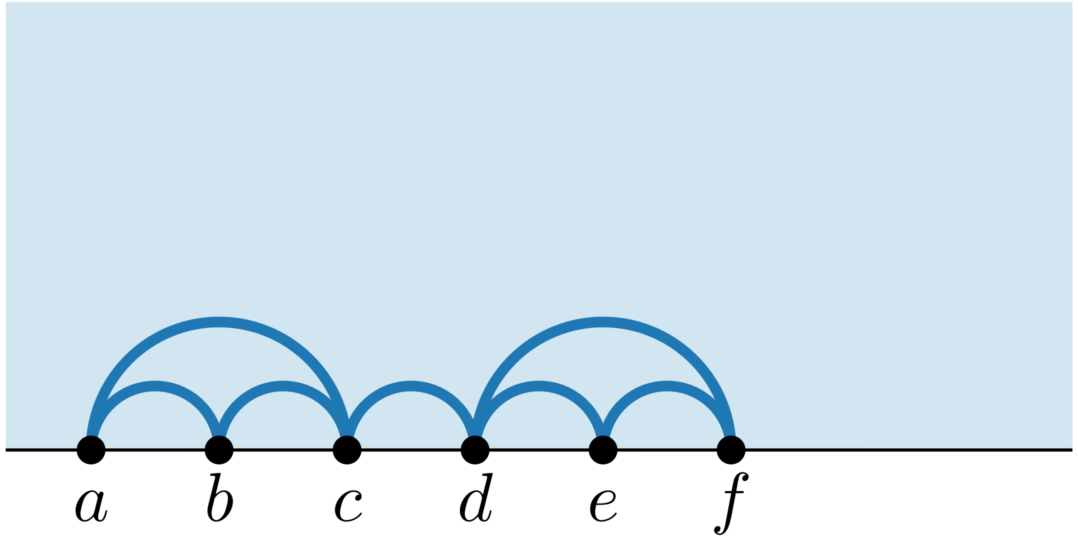

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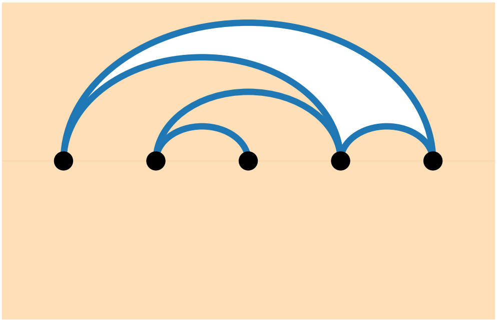

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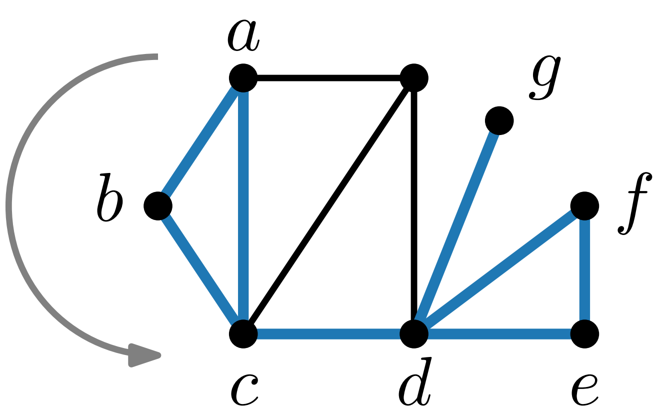
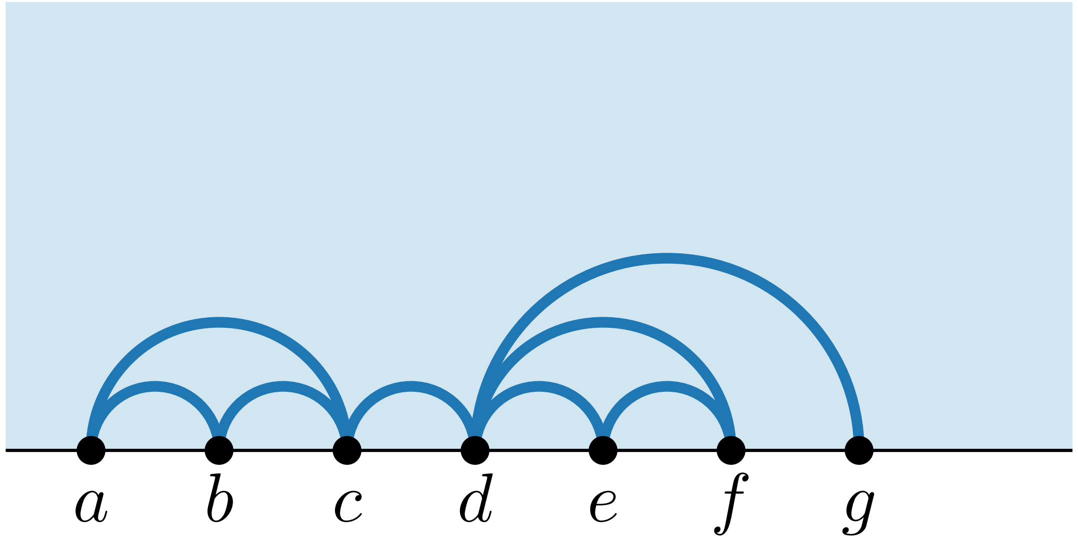
"$\Rightarrow$": Clearly, a 1-page stack layout can be perceived as a planar drawing where the vertices lie at the outer face.

"$\Leftarrow$": Given an outerplanar drawing of G, traverse the outer face in counterclockwise order and place the vertices in this order onto the spine.

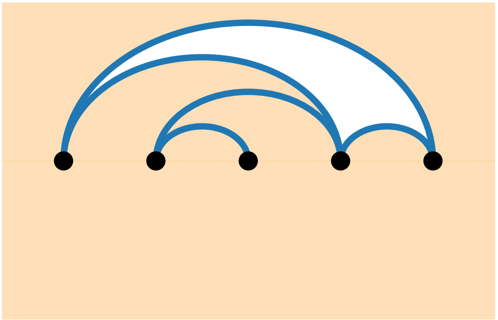




1-Page Stack Layouts

Theorem. [Bernhart & Kainen 1979]
For a graph G holds: sn$(G) = 1 \Leftrightarrow G$ is outerplanar

Recall: a graph is outerplanar if it has a planar drawing where every vertex is incident to the outer face.



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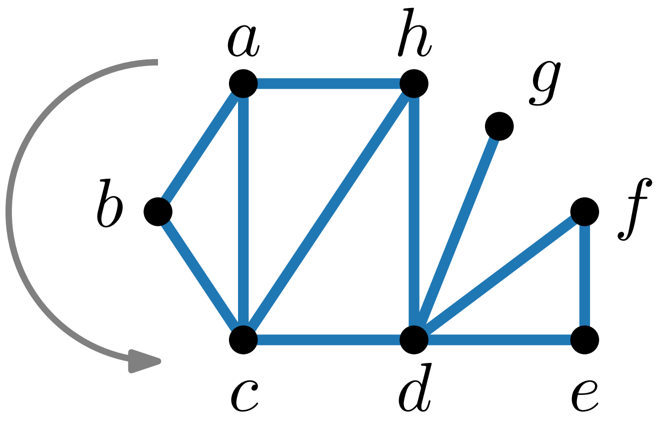

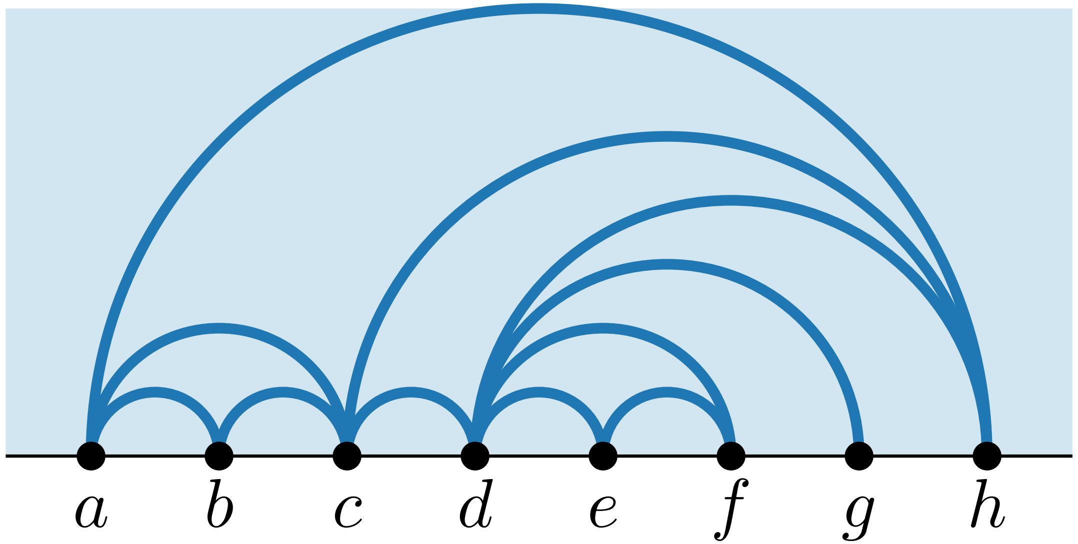

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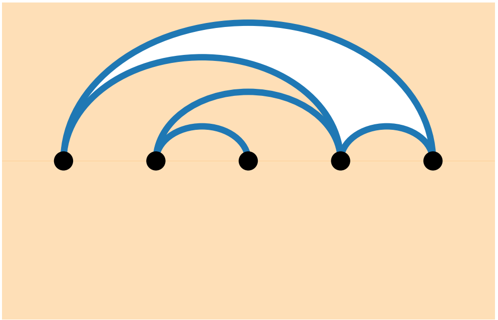

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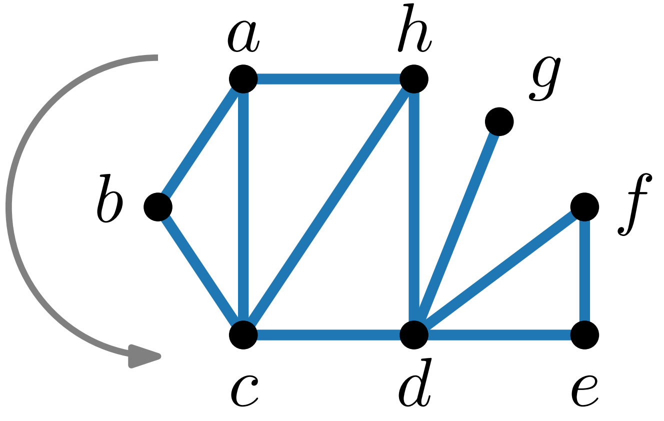

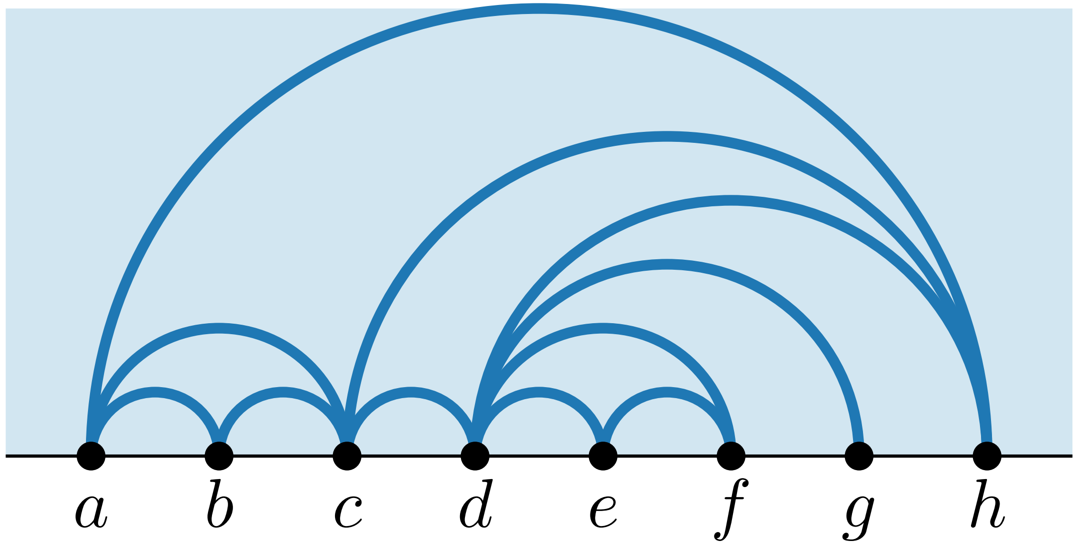

Note that the planar embedding is preserved.

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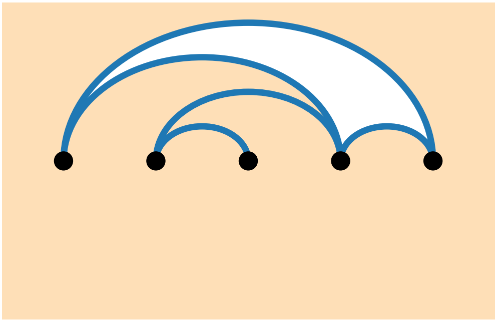

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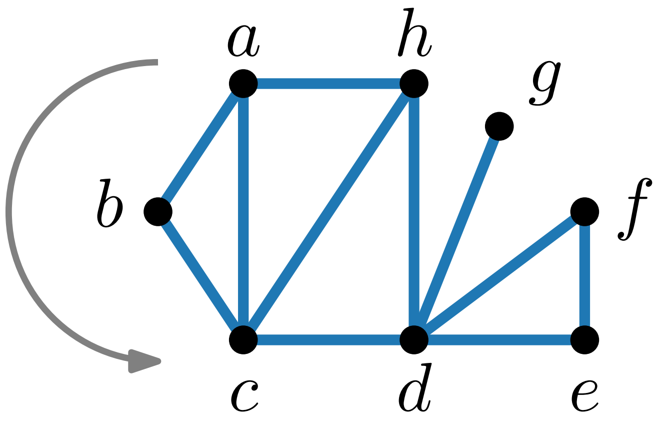

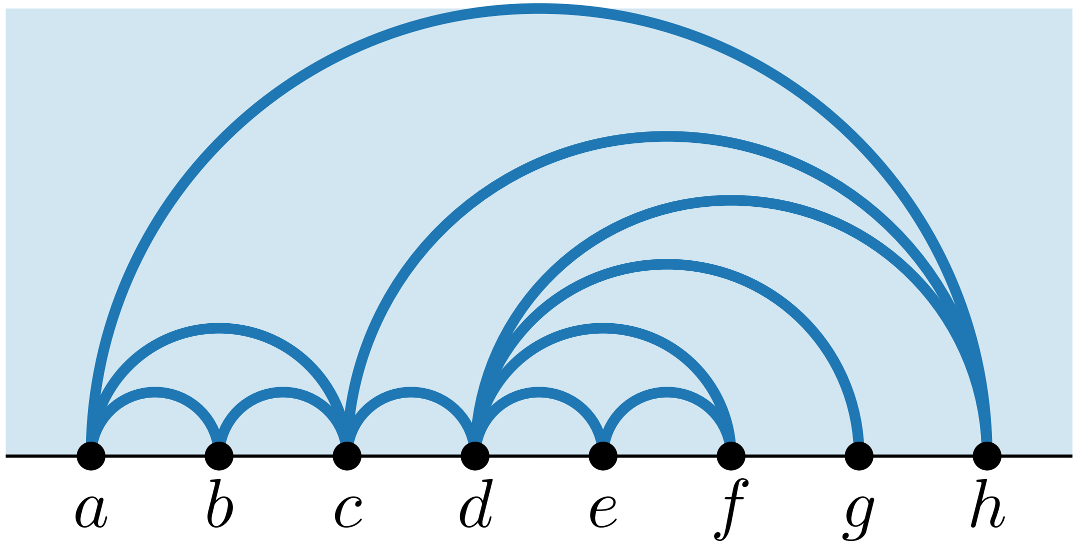

Note that the planar embedding is preserved.

We can think of "morphing" the one drawing into the other.

□

2-Page Stack Layouts

Theorem. [Bernhart & Kainen 1979]

For a graph G holds: $\mathrm{sn}(G) \leq 2 \Leftrightarrow G$ is a subgraph of a planar Hamiltonian graph

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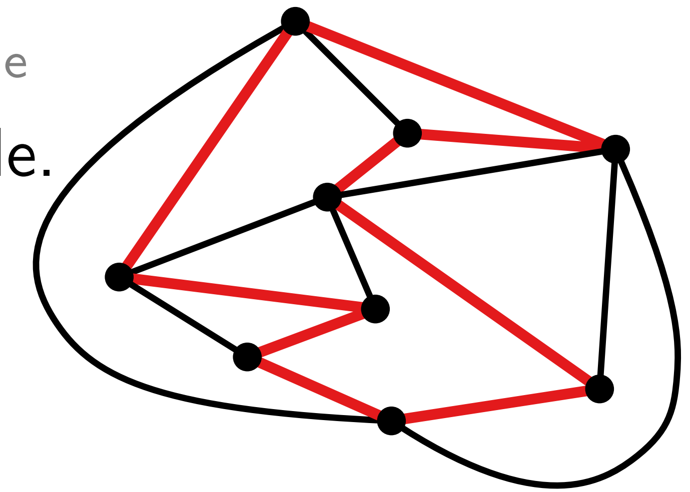

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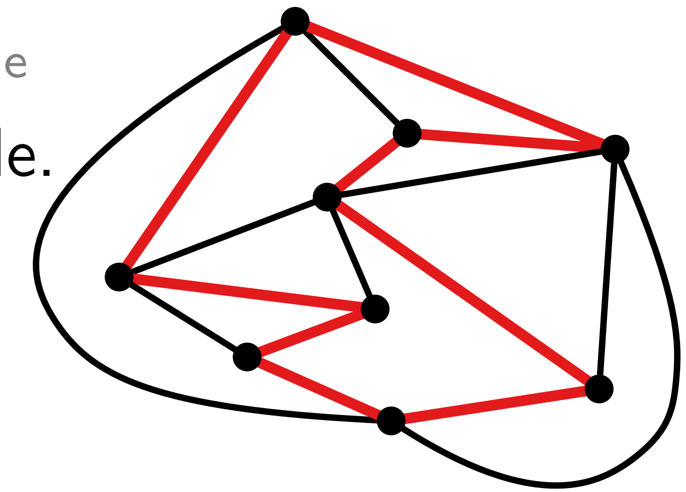

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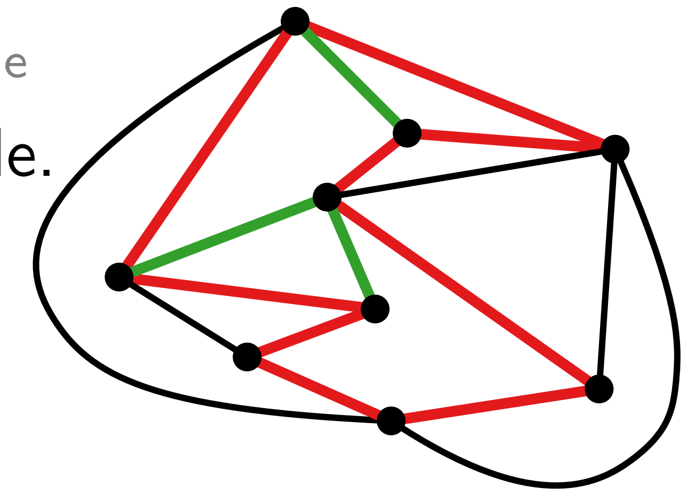

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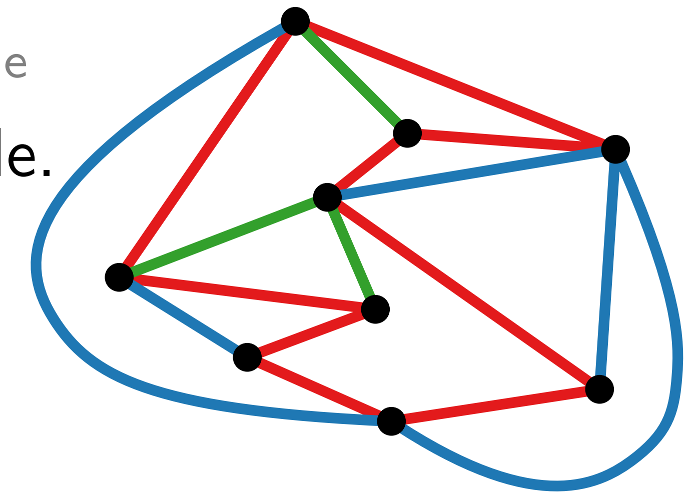

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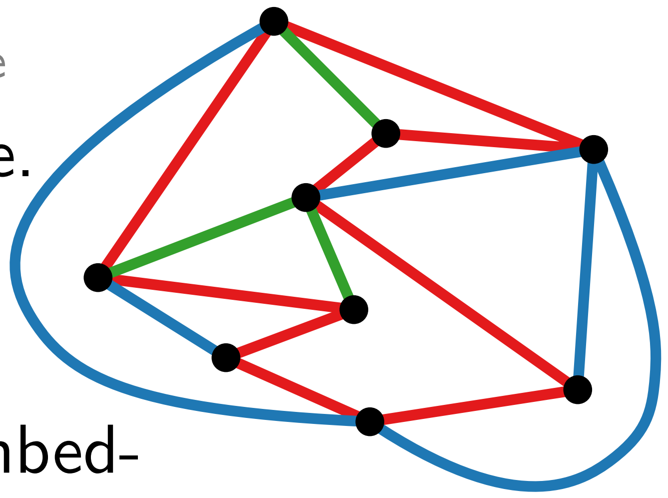

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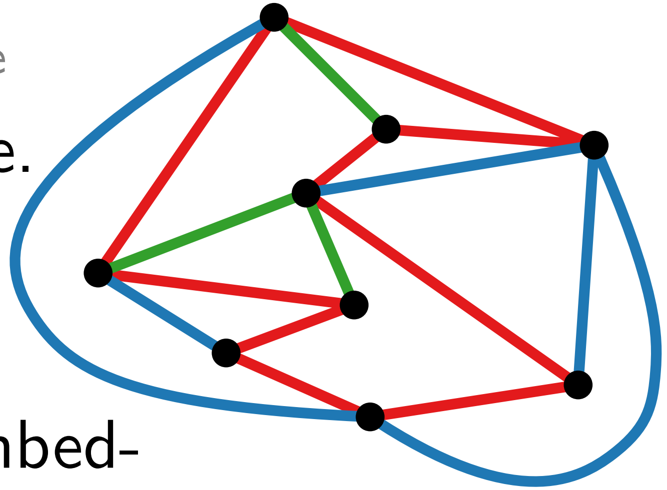

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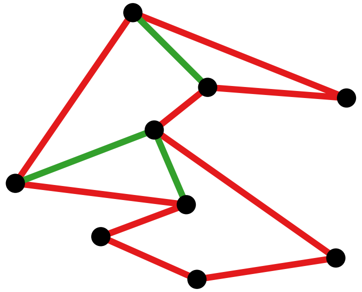

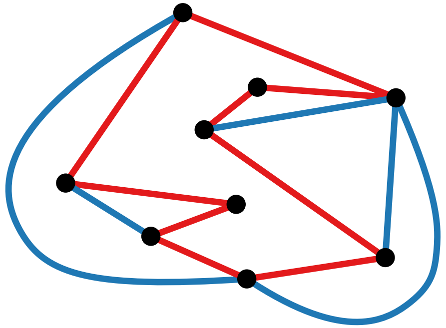

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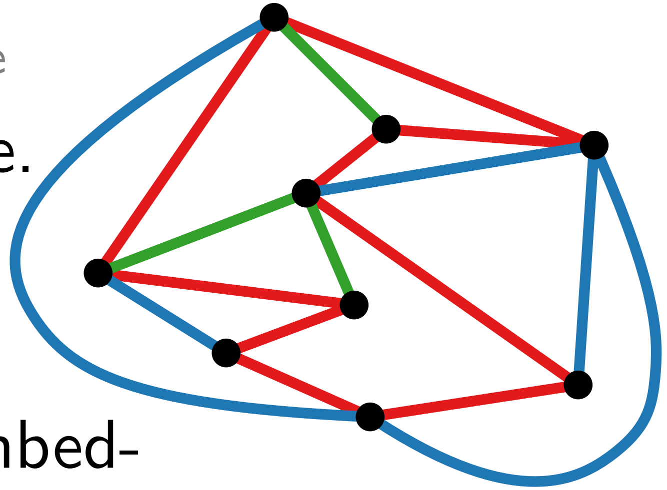

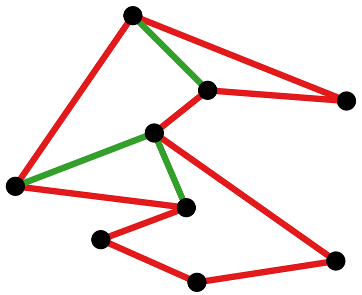

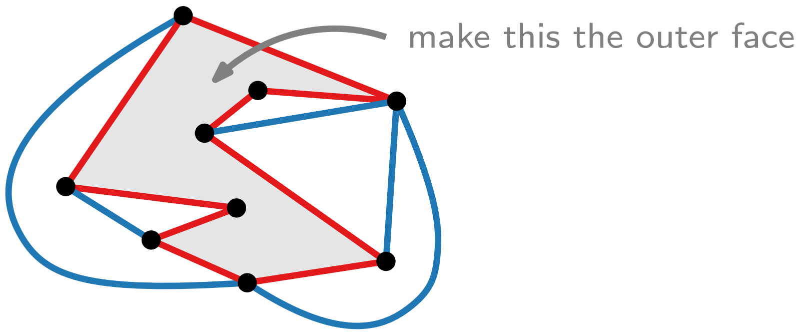

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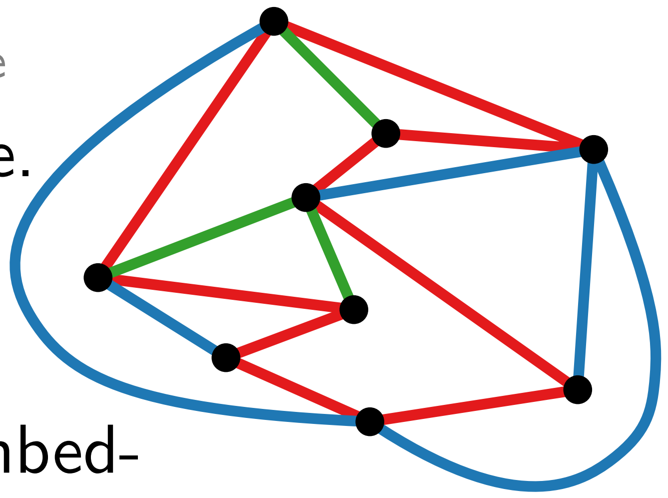

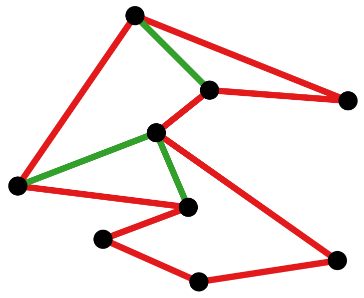

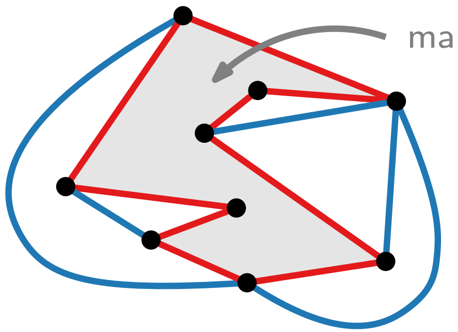

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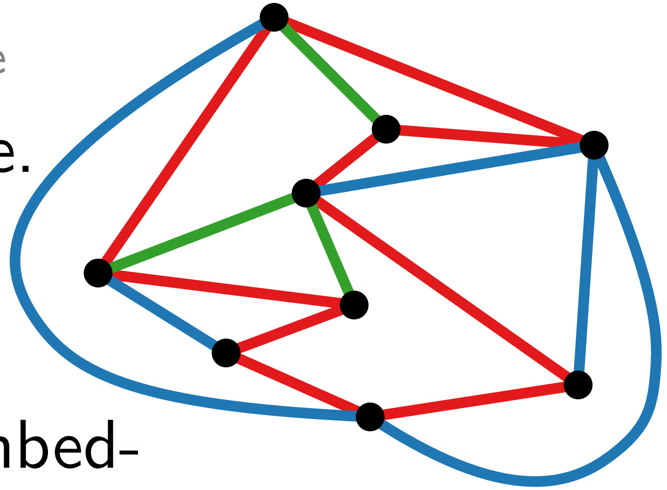

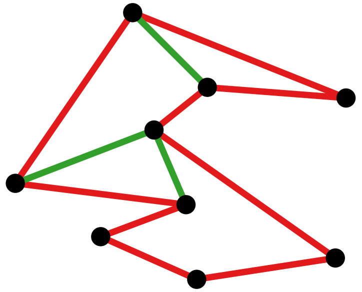

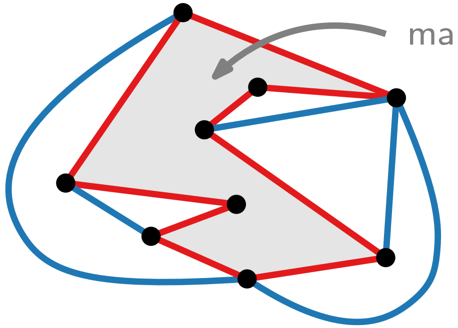

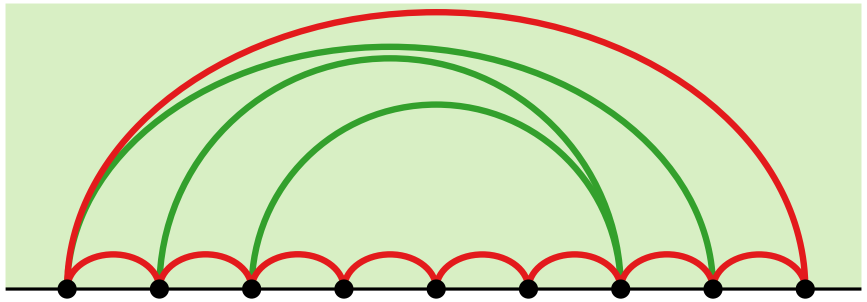

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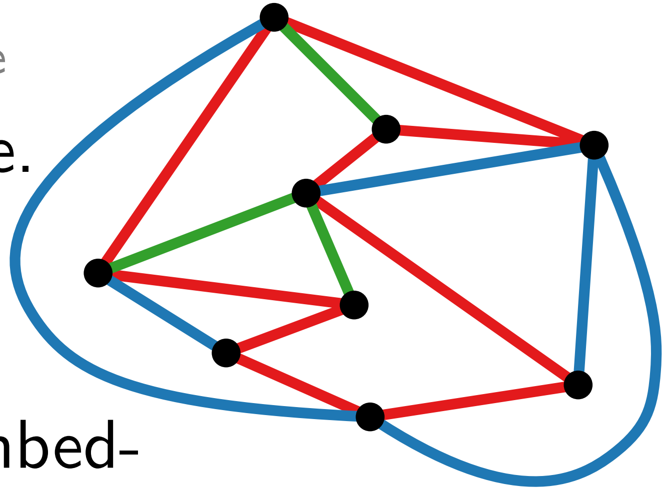

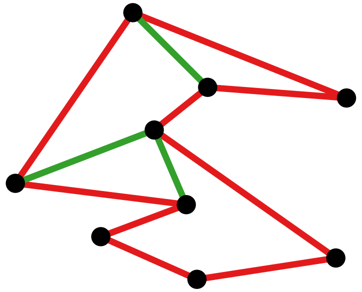

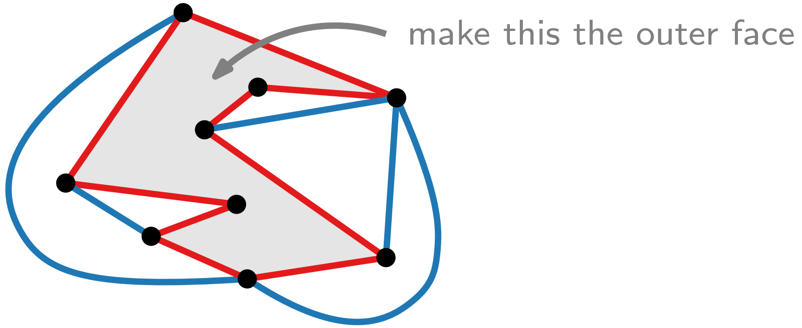

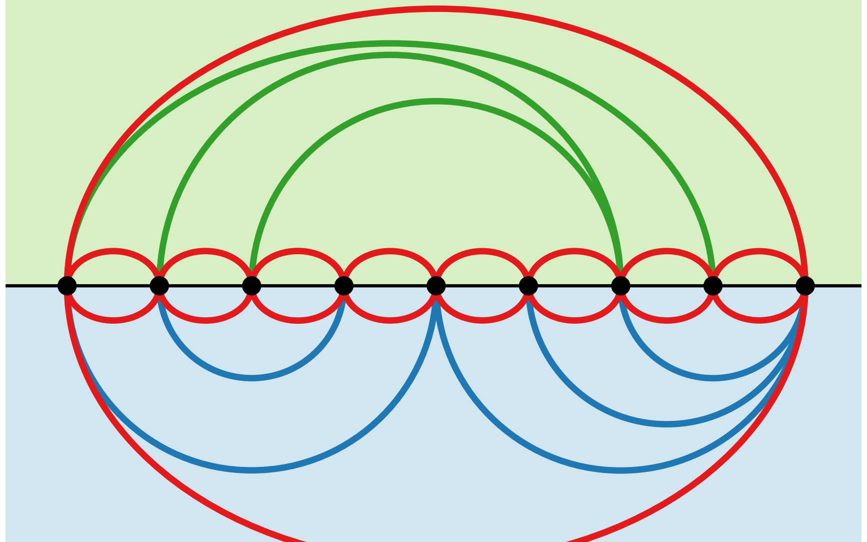

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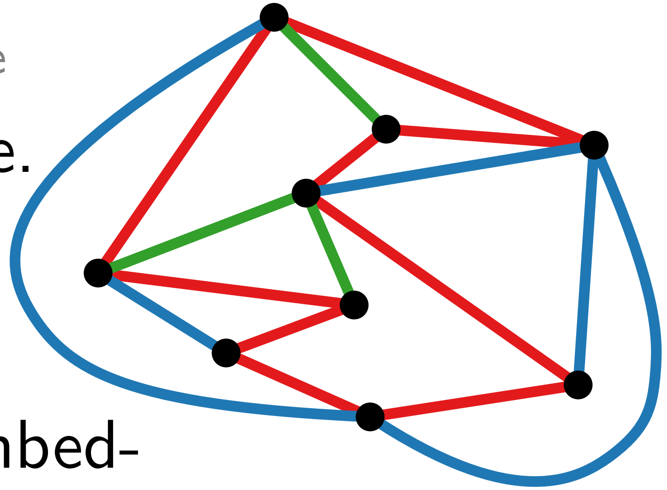

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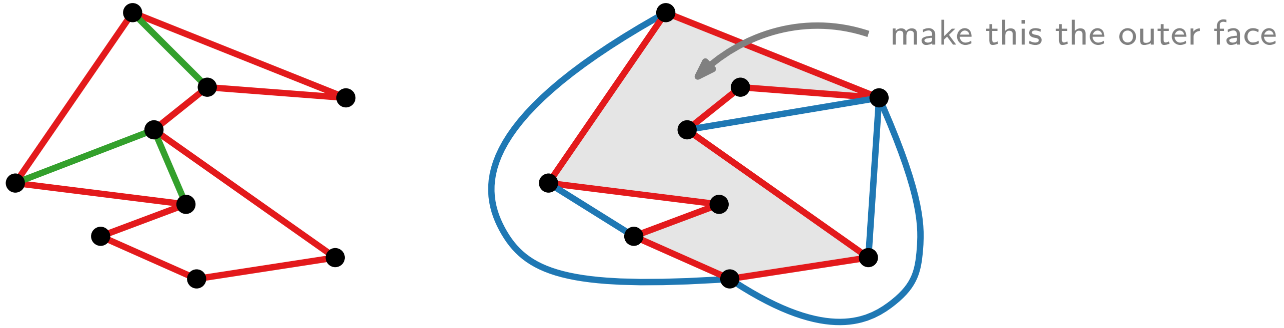

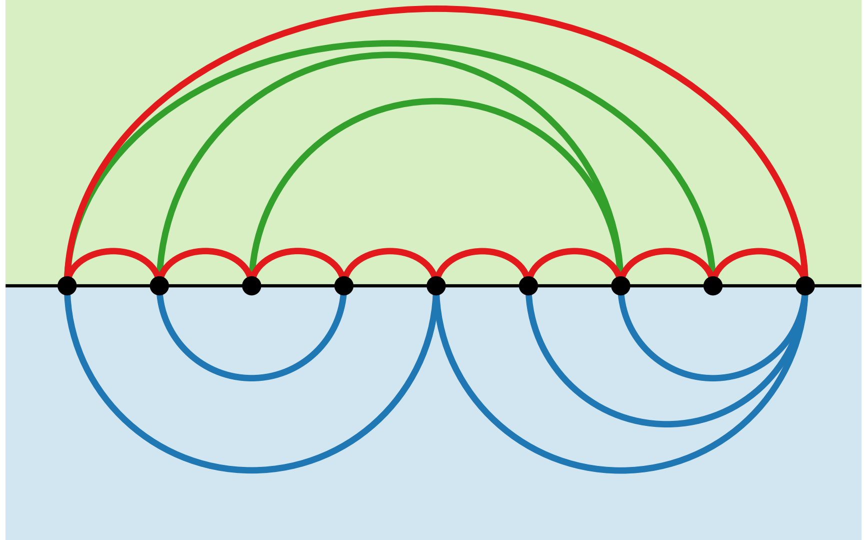

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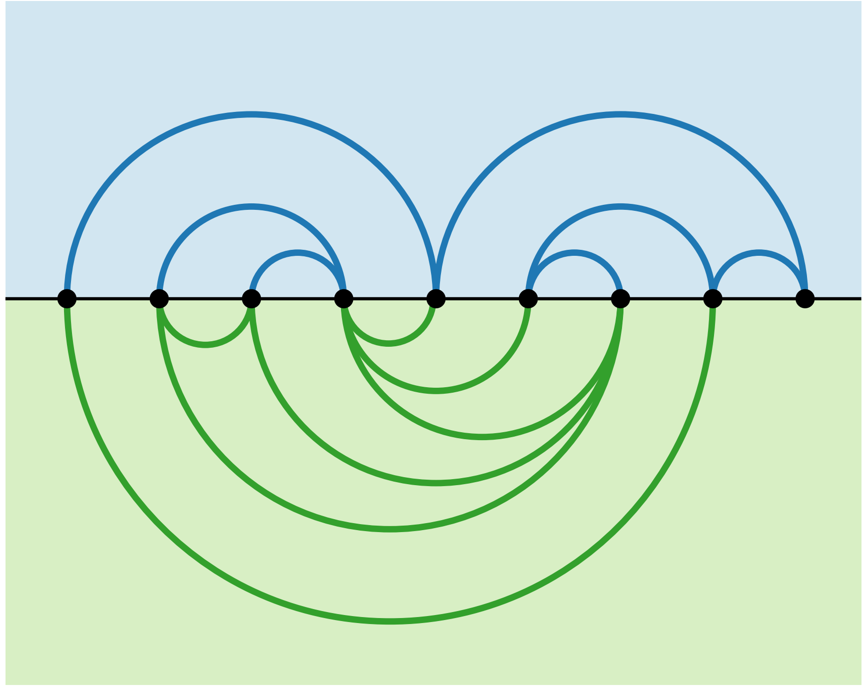

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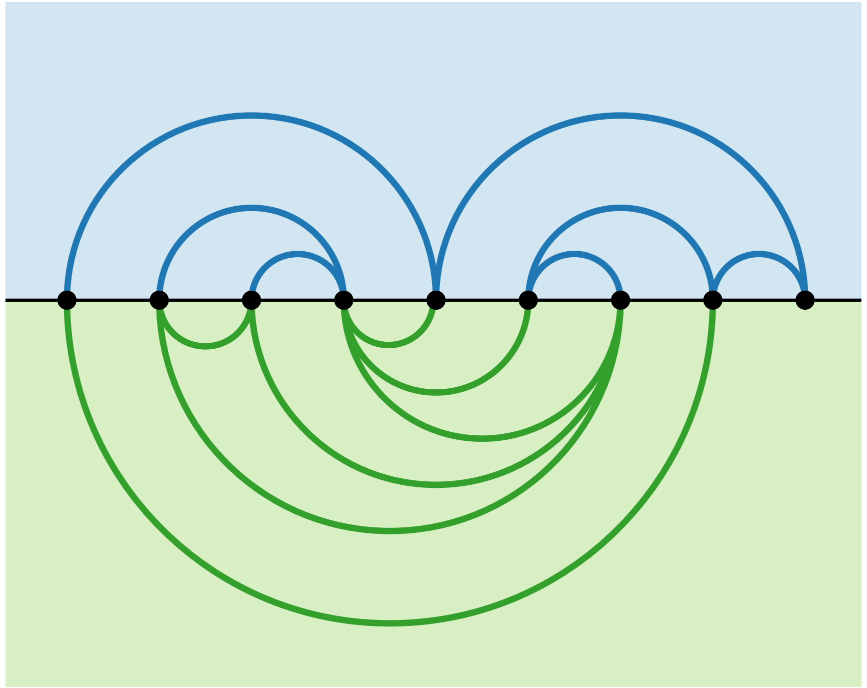

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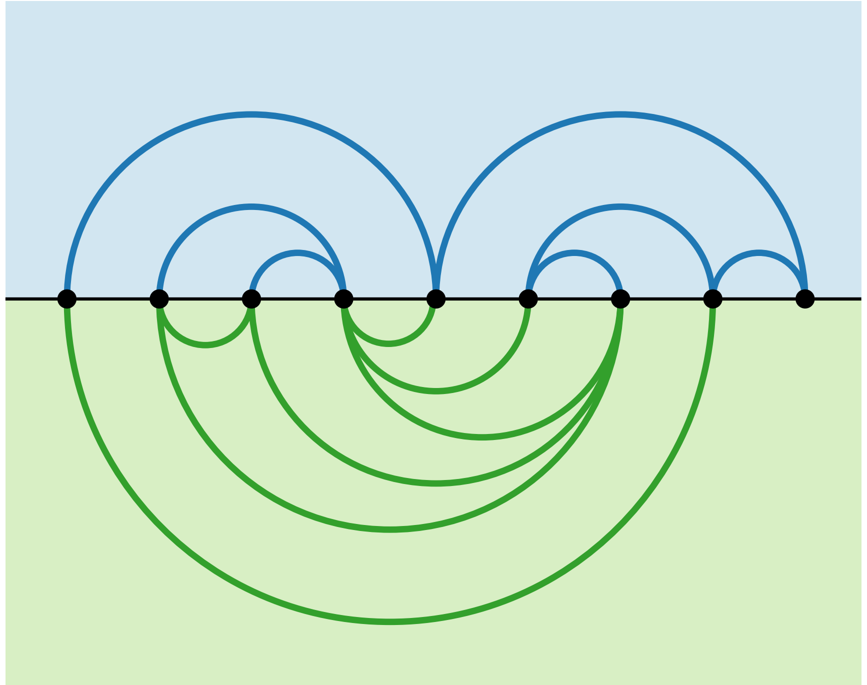

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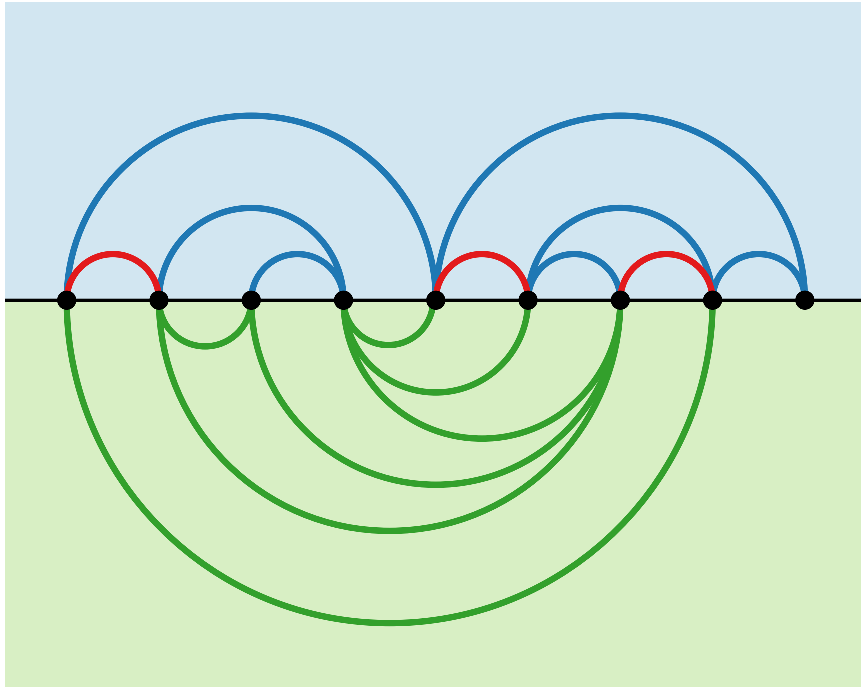

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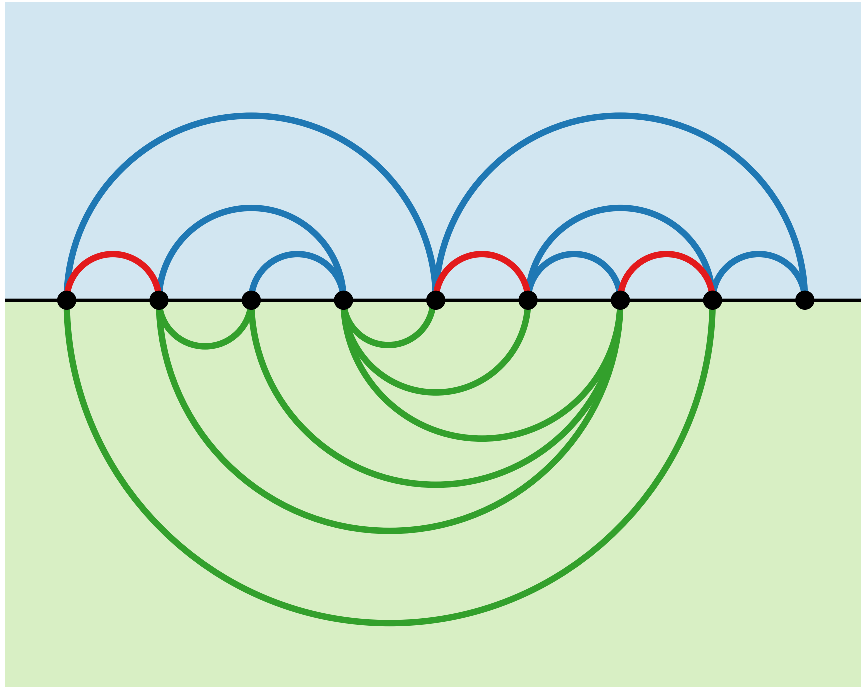

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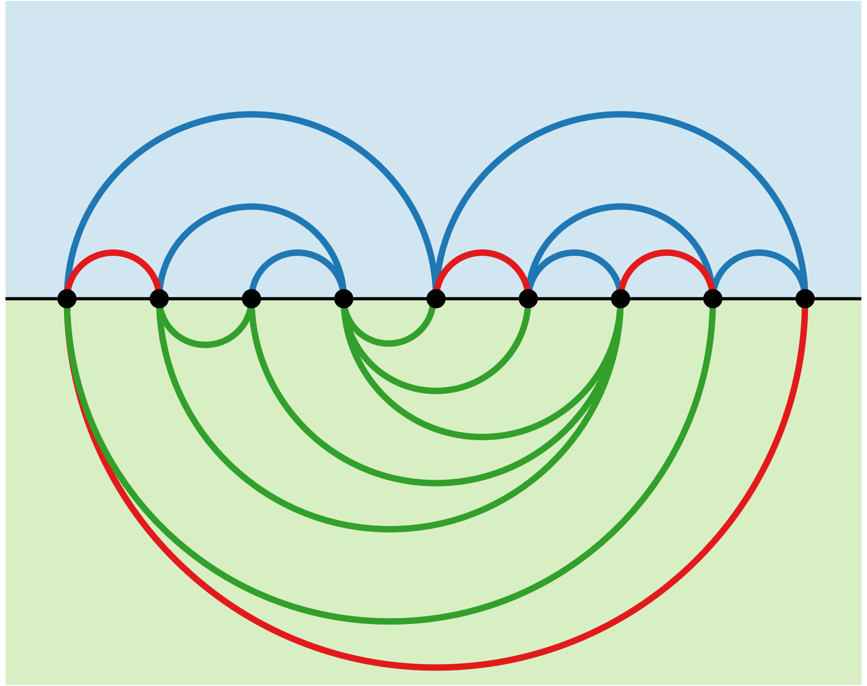

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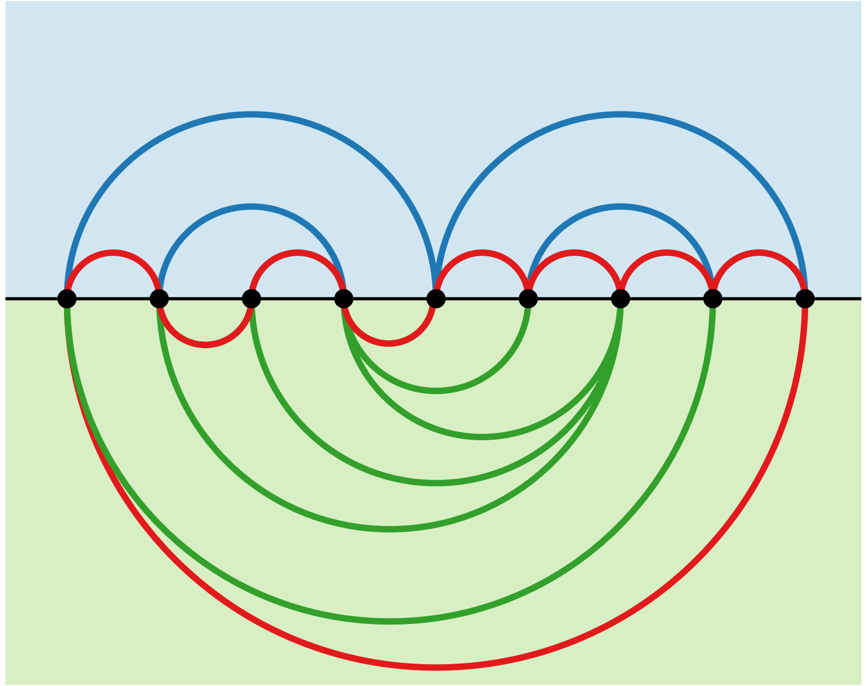

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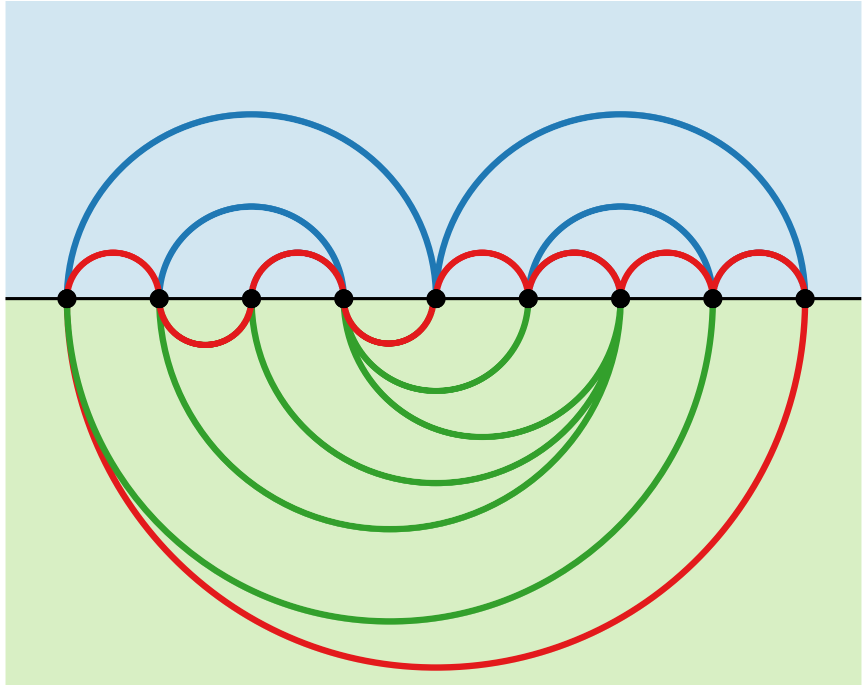

This result includes planar bipartite and series-parallel graphs.

Stack Layouts of Planar Graphs

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(not all planar graphs can be extended to have a Hamiltonian cycle)

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Theorem. [Yannakakis 2020, Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou & Ueckerdt 2020]
There is a planar graph G with $\text{sn}(G) \geq 4$.

But are there planar graphs that need 4 stacks?

Yes! (The planar graph presented by Bekos et al. has 275 vertices and 819 edges.)

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Do all graphs have constant stack number?

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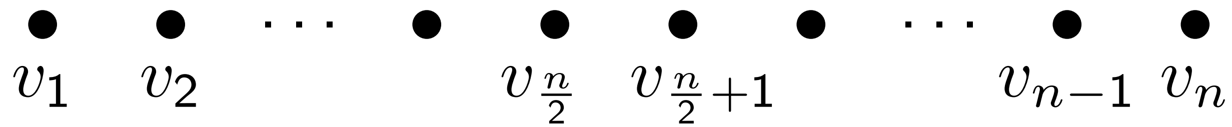

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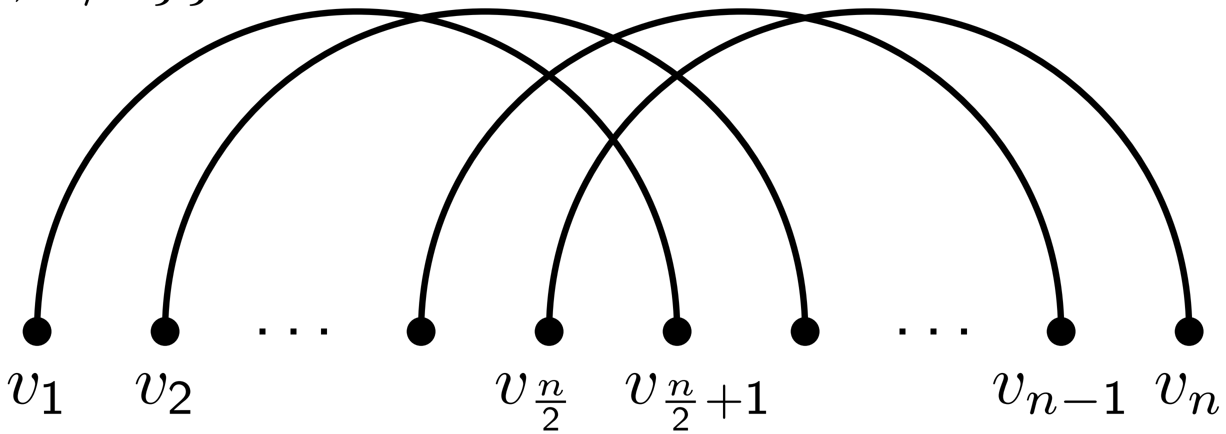

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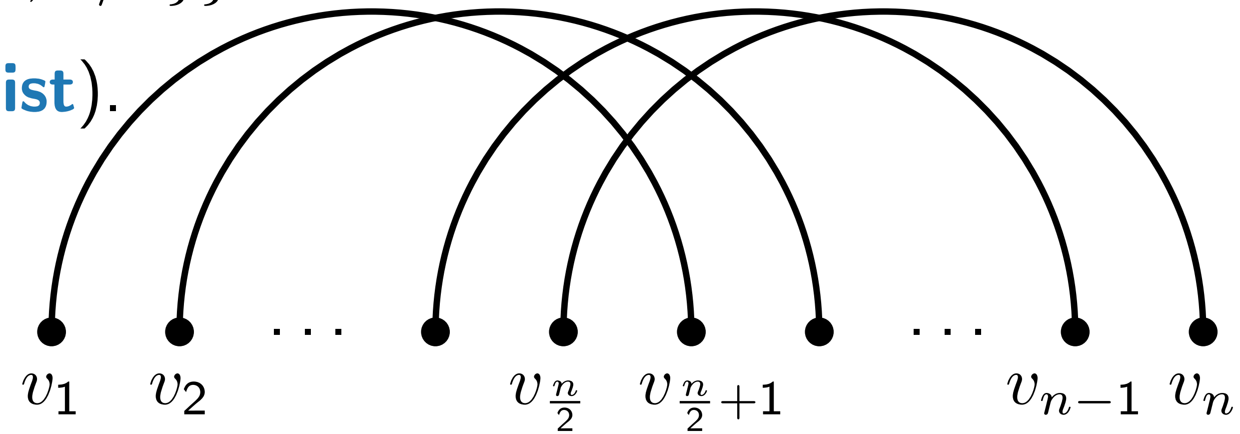

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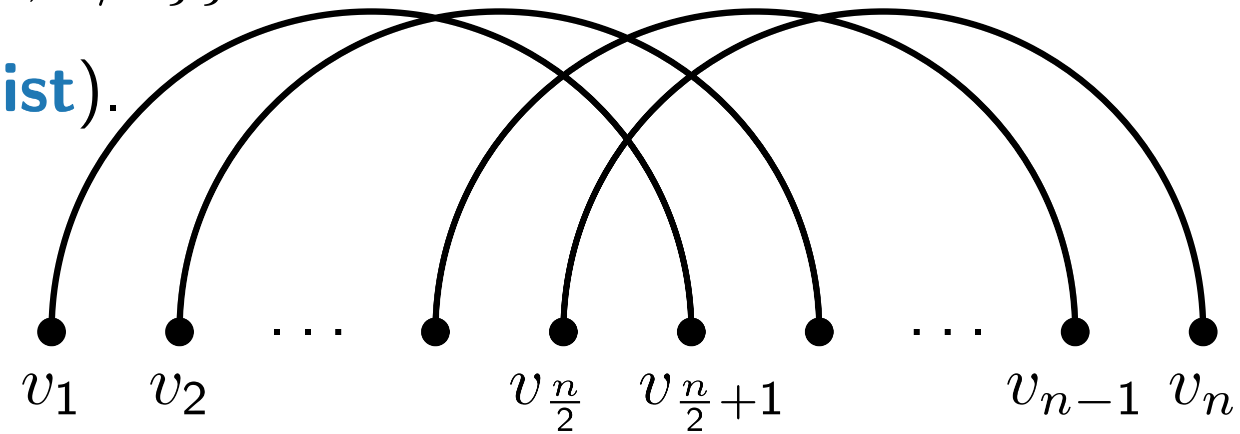

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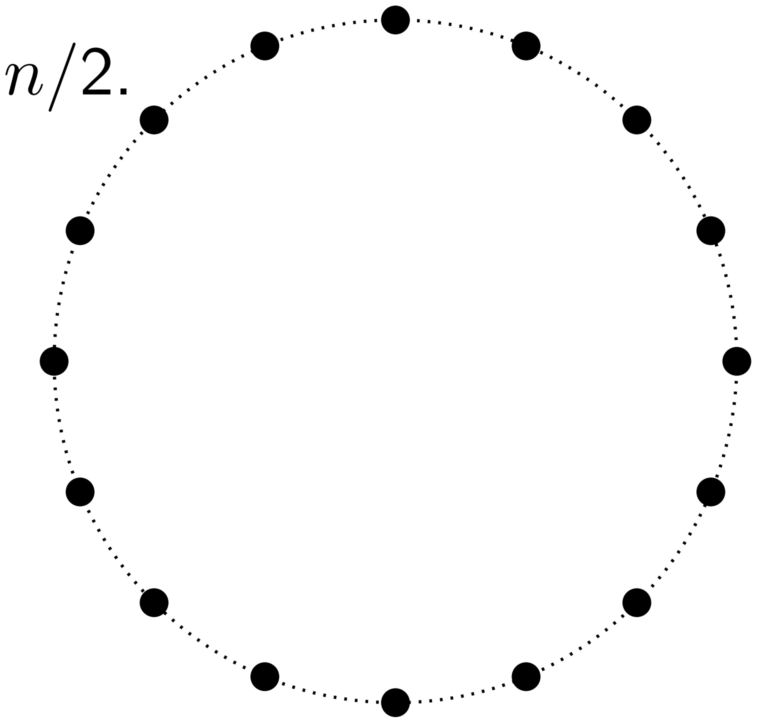

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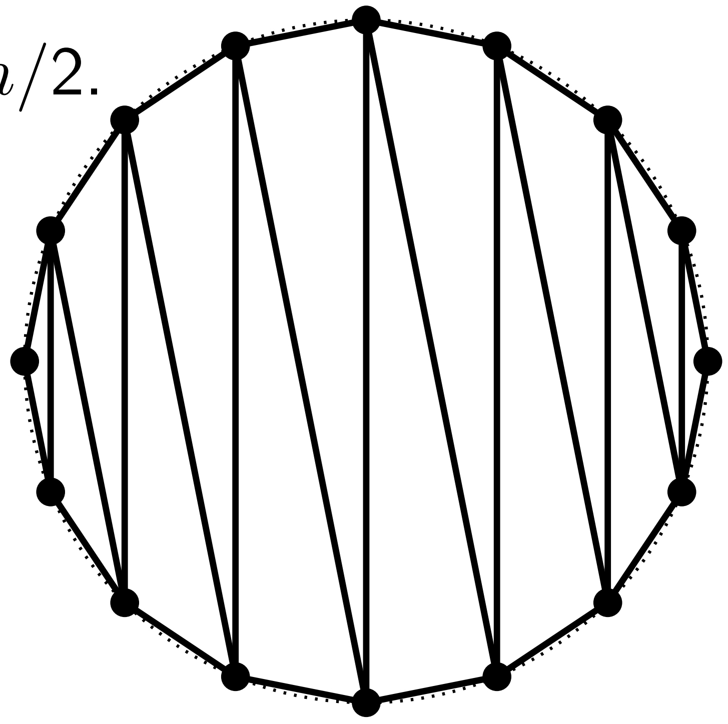

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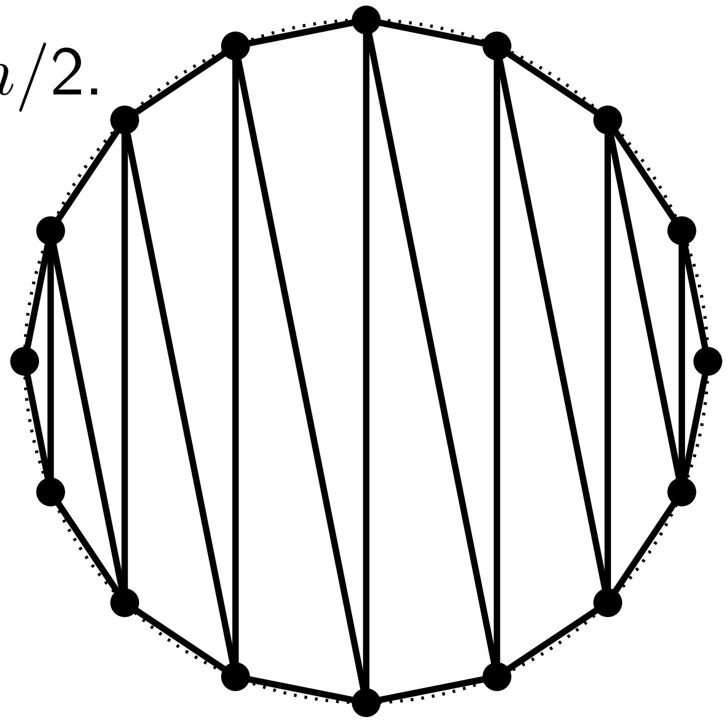

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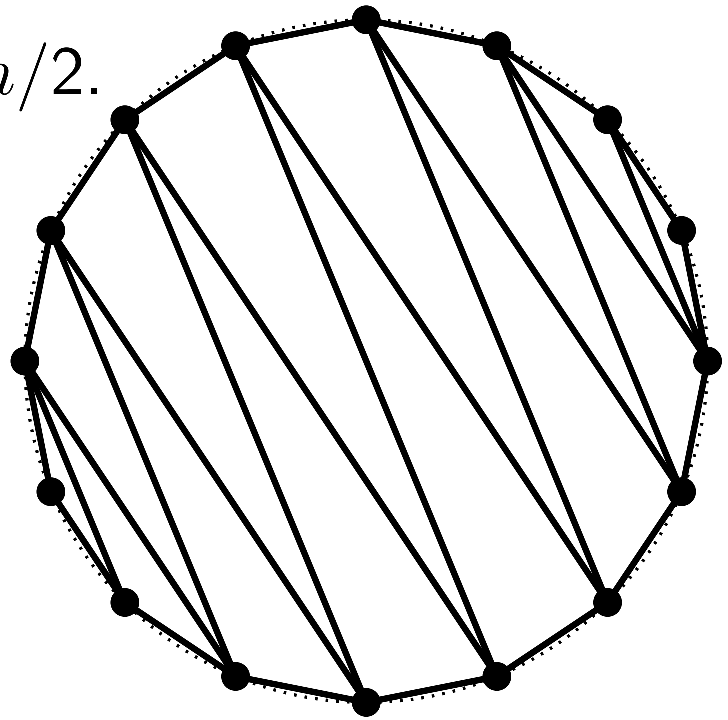

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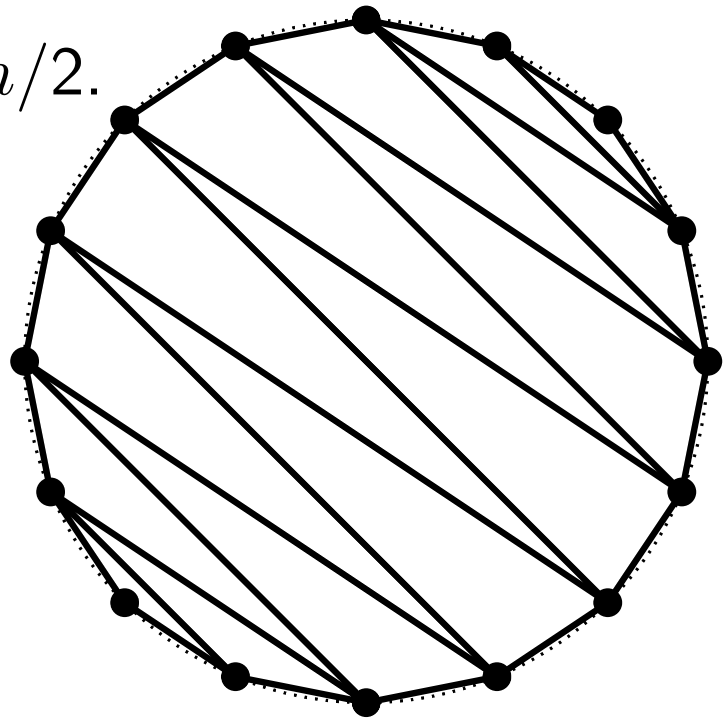

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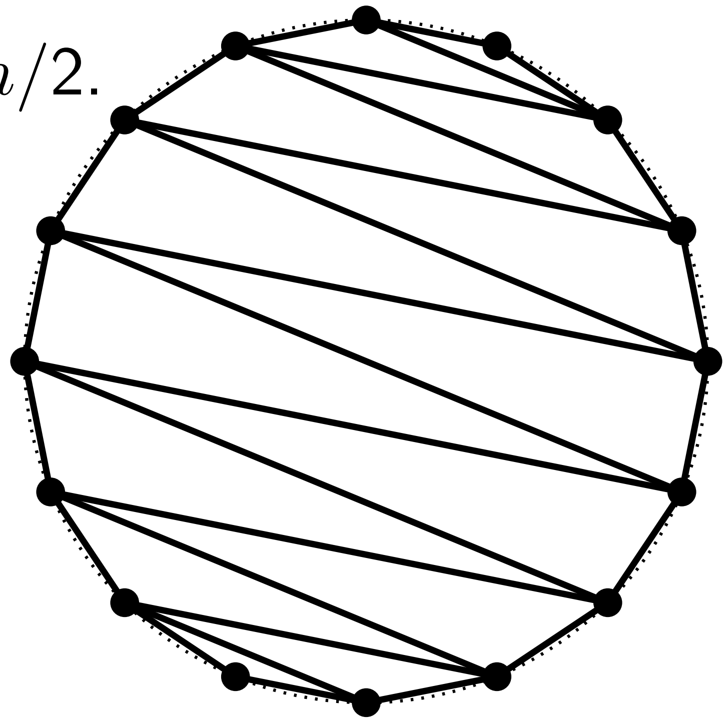

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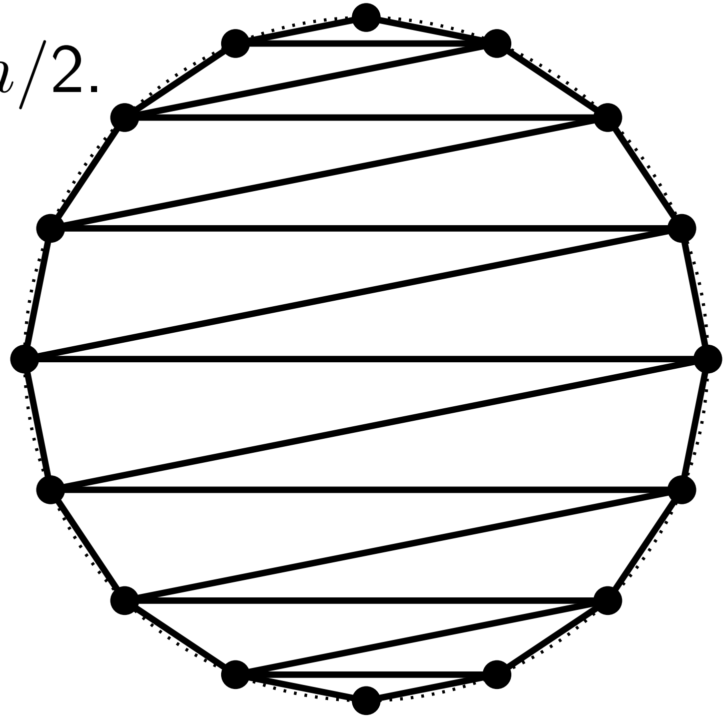

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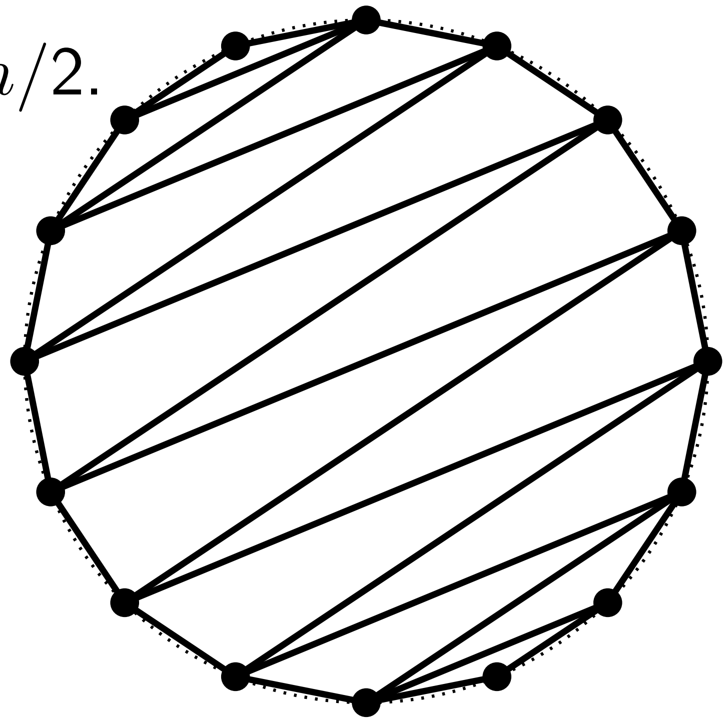

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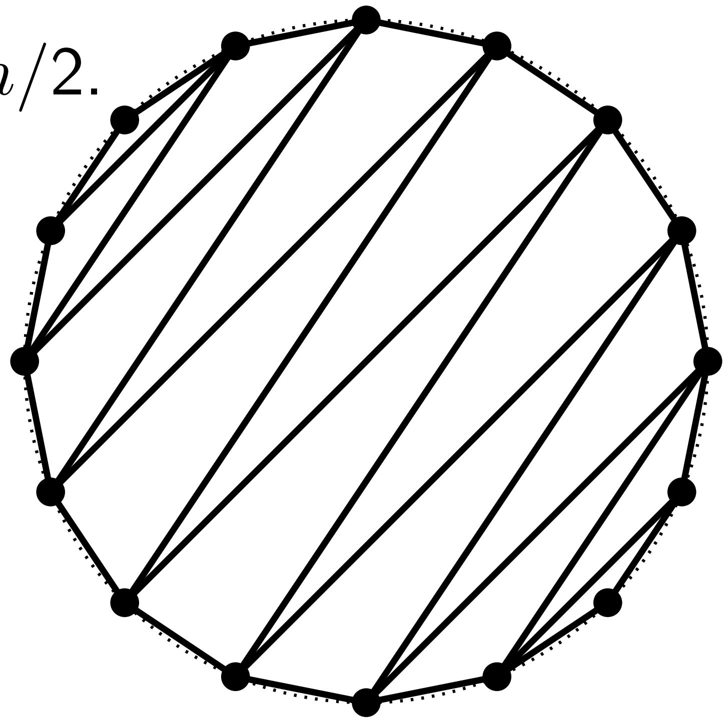

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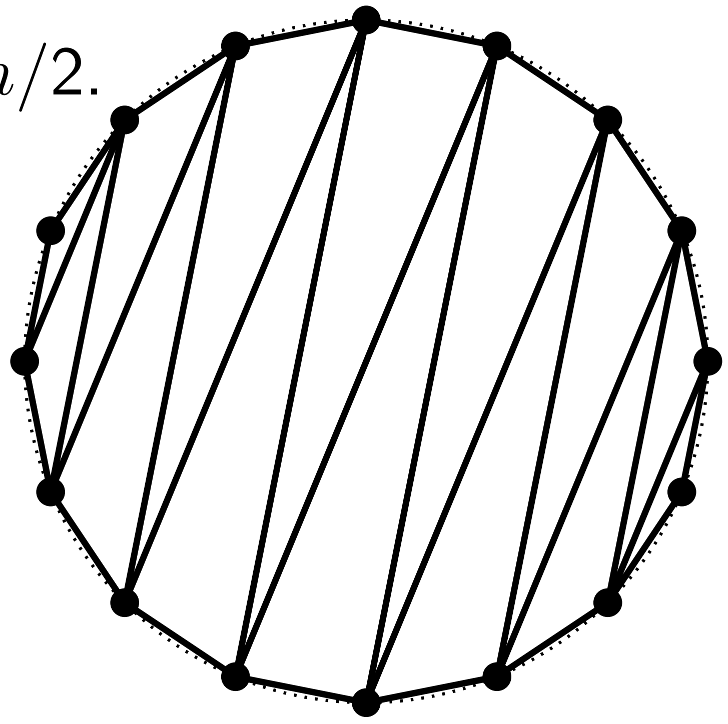

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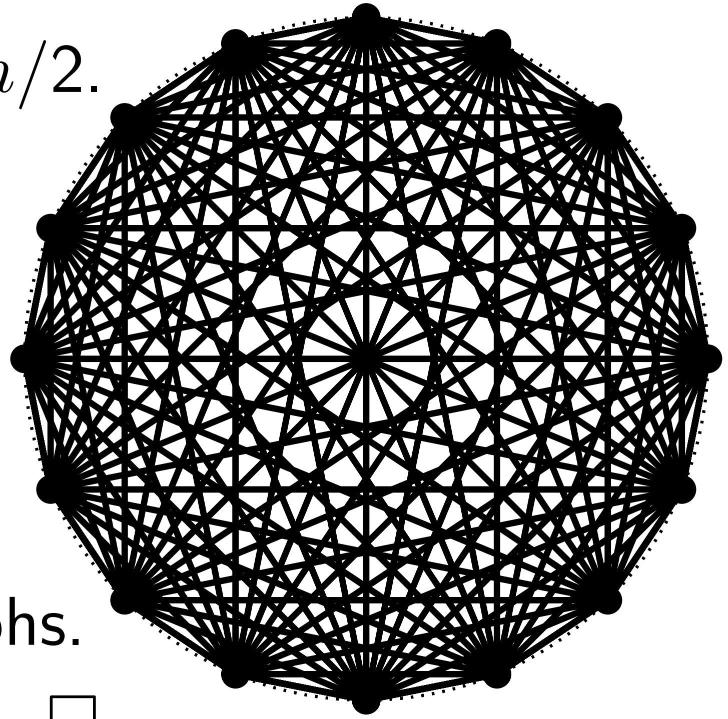

□

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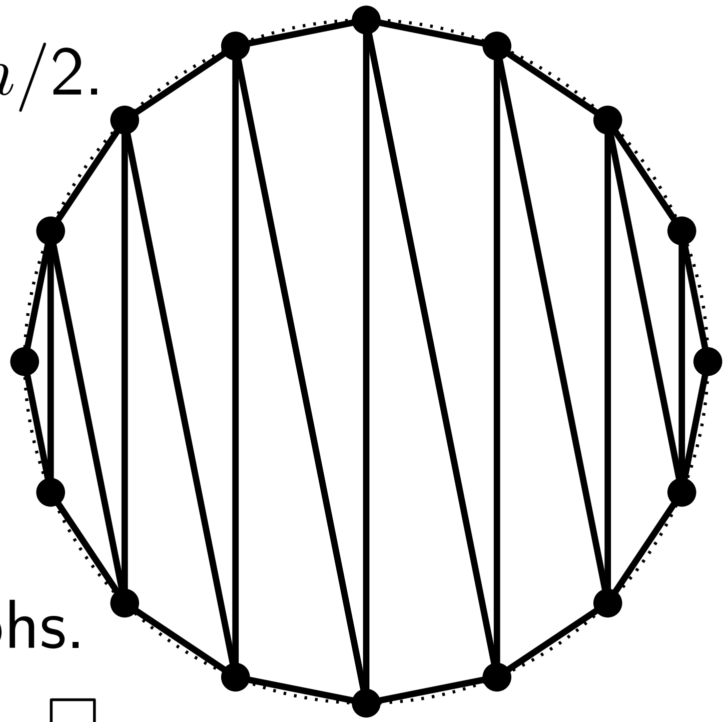

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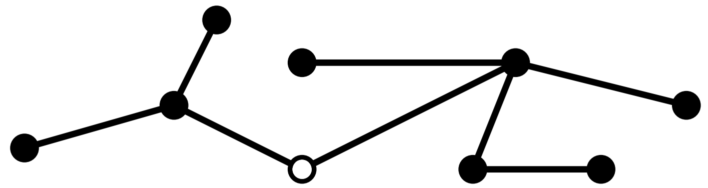

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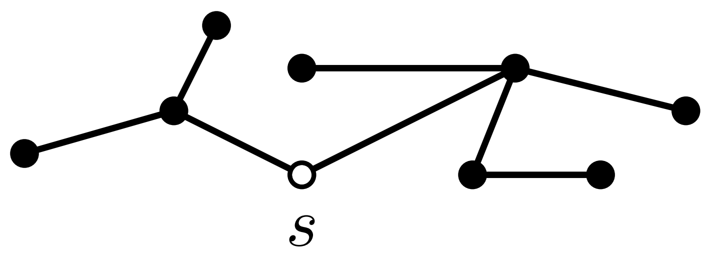

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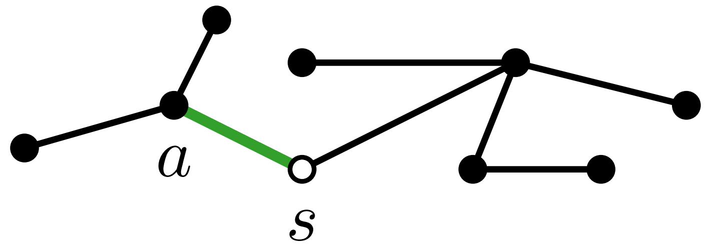

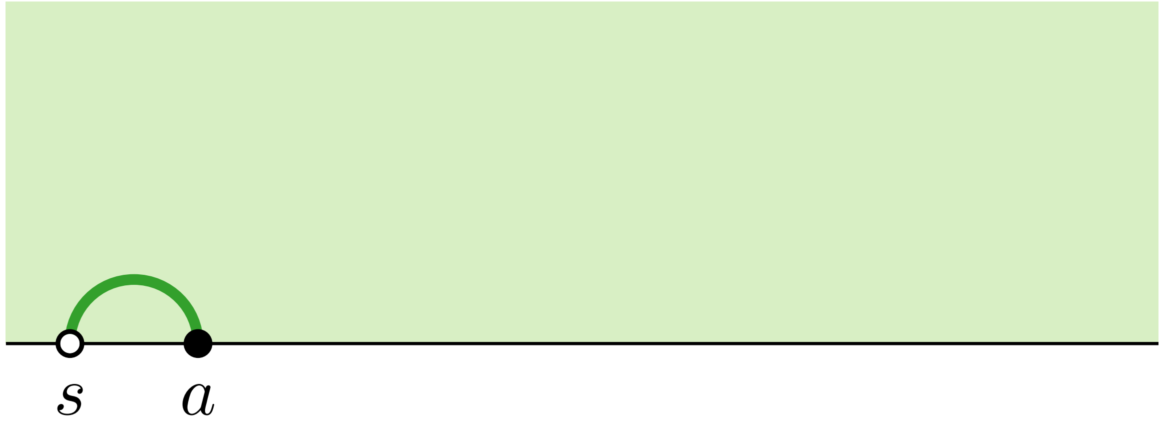

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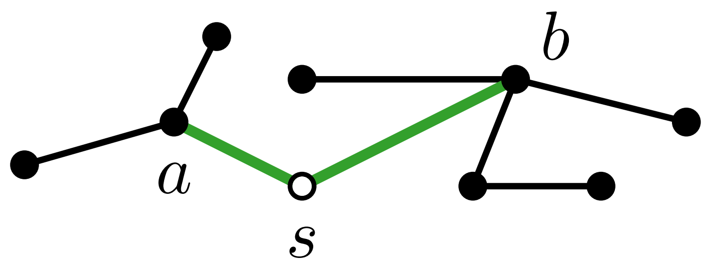

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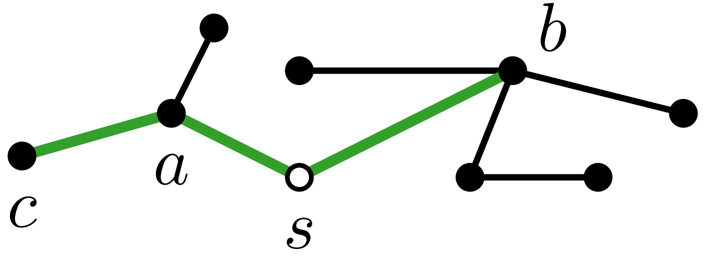

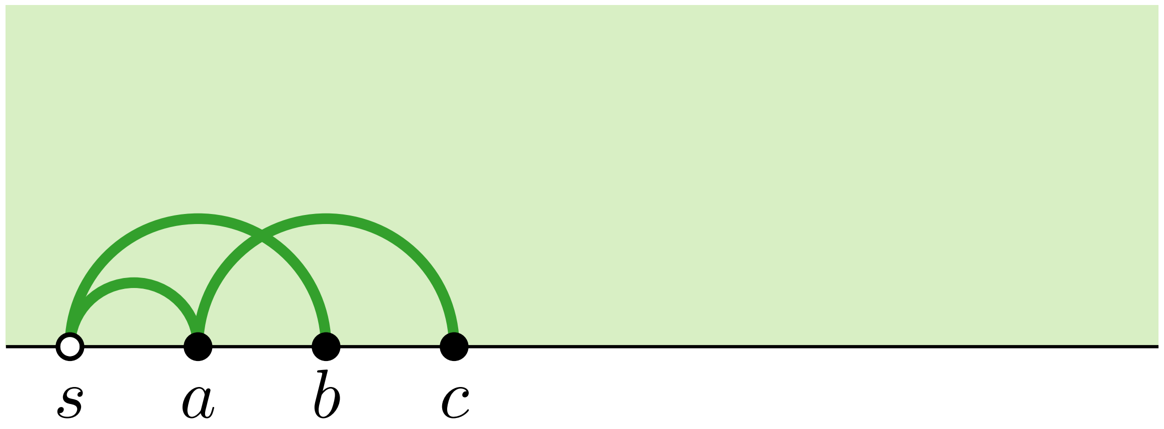

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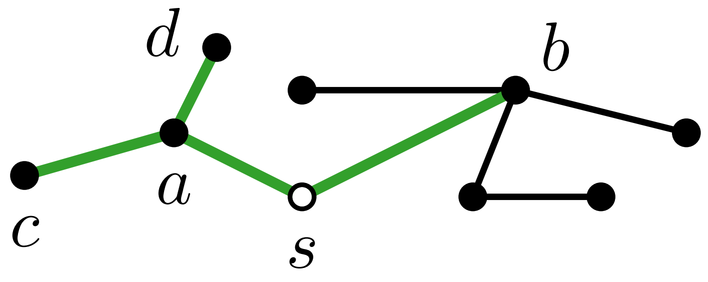

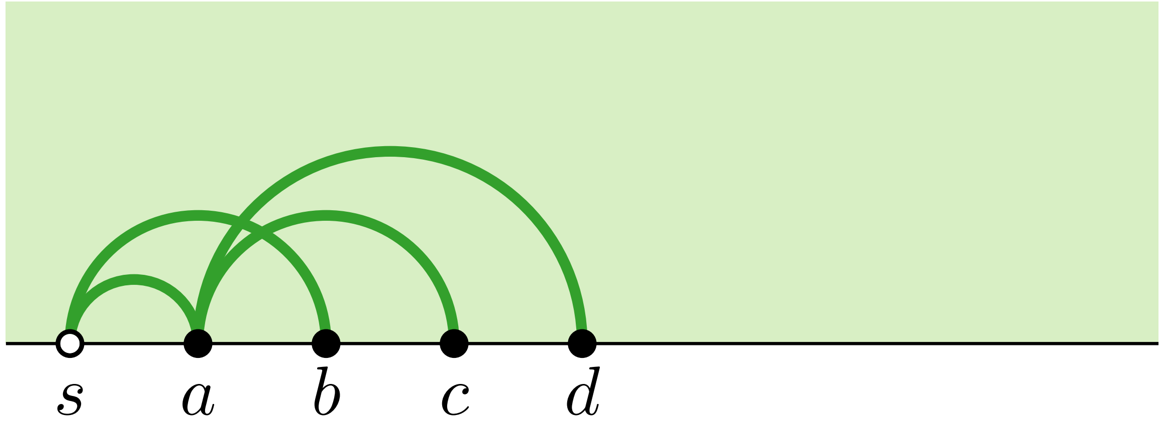

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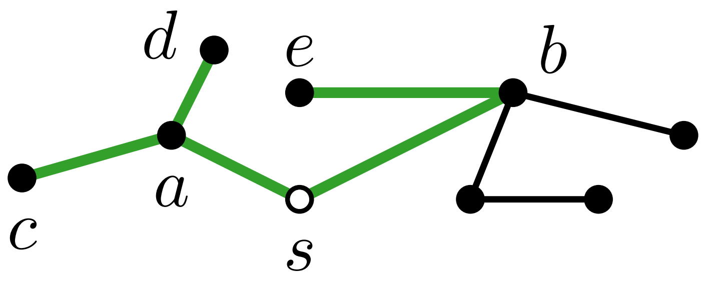

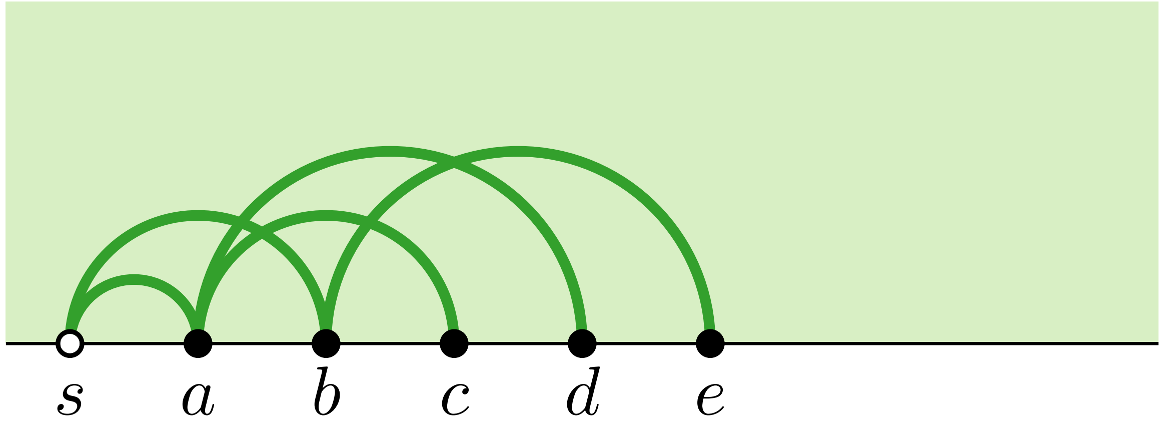

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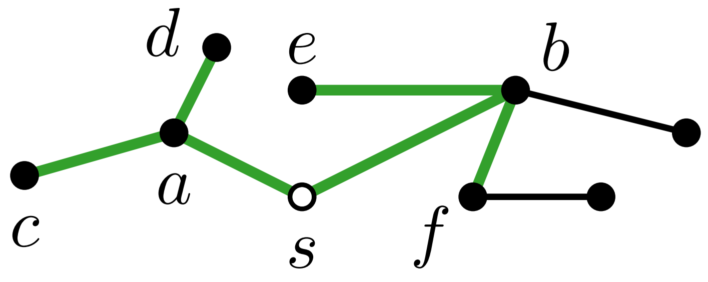

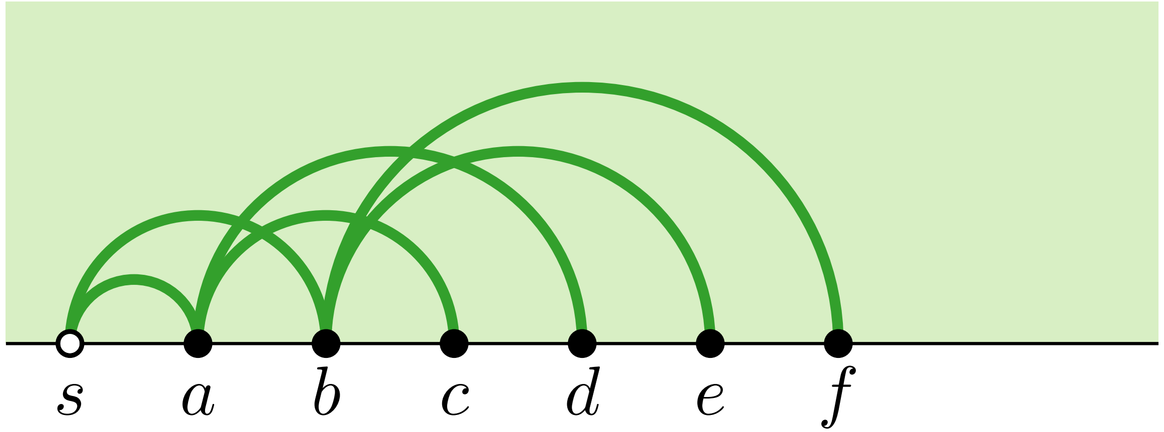

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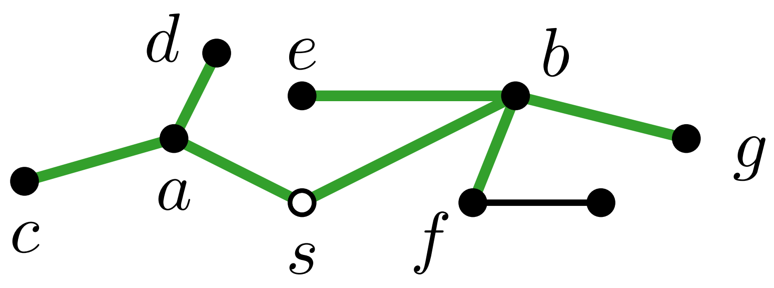

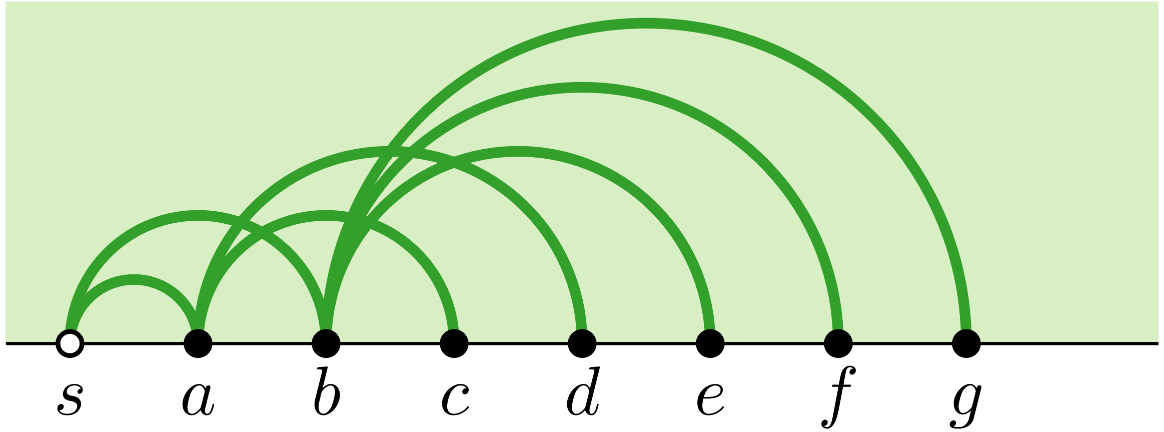

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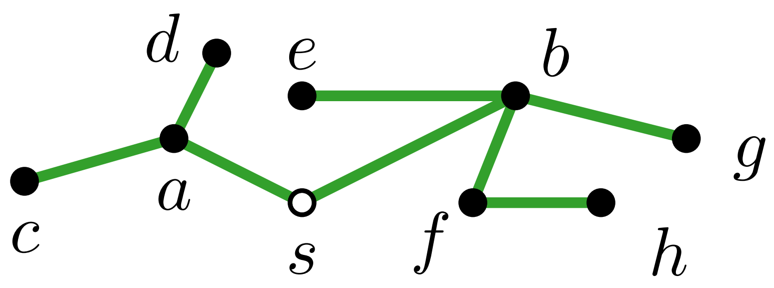

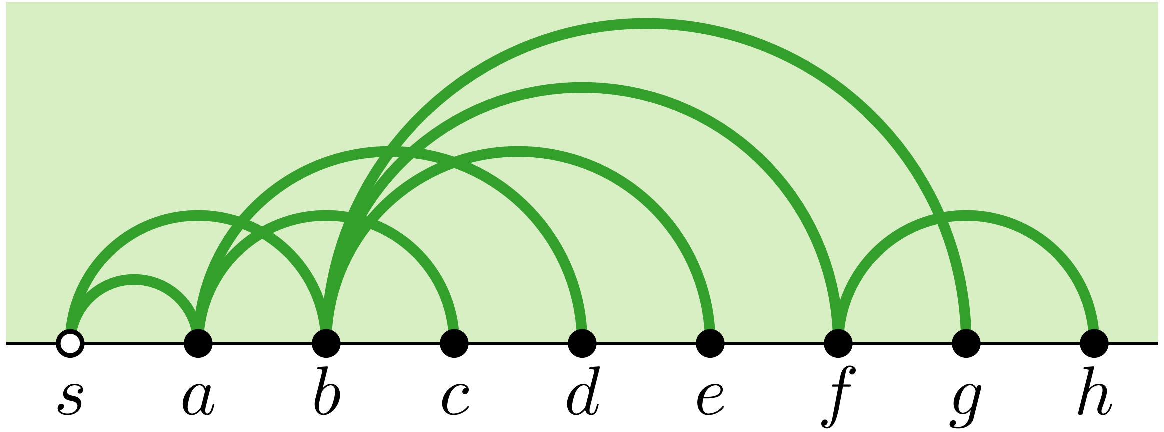

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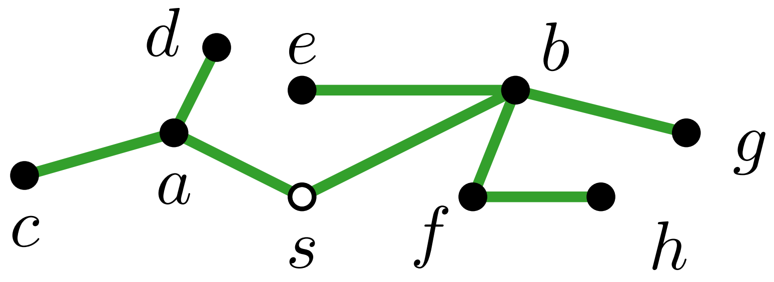
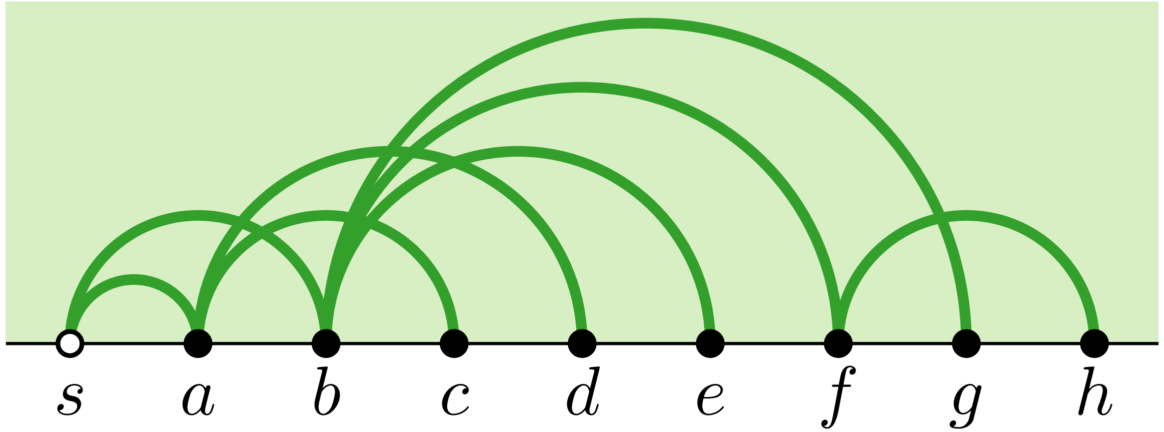
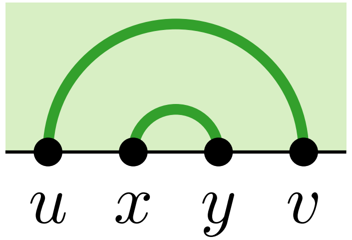




- If there was a nesting uv above xy, we would find u before x in the BFS, but discover a neighbor of x before a neighbor of u.



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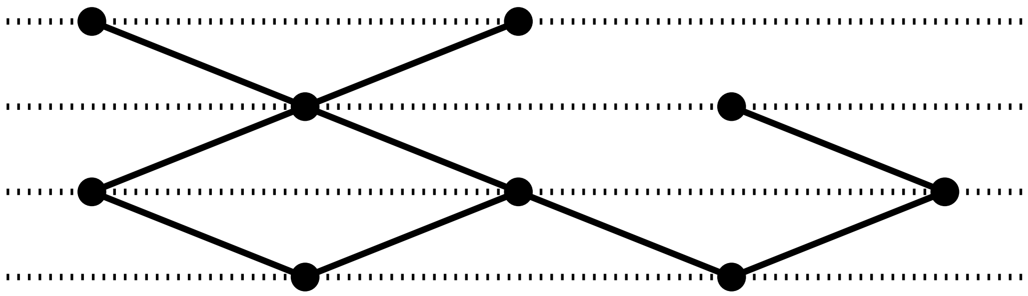

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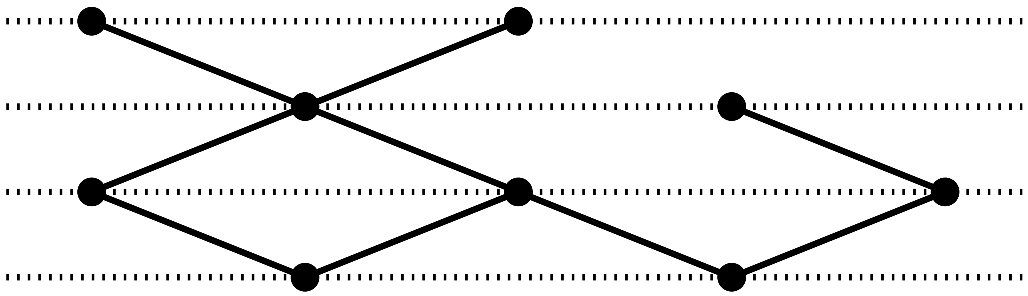

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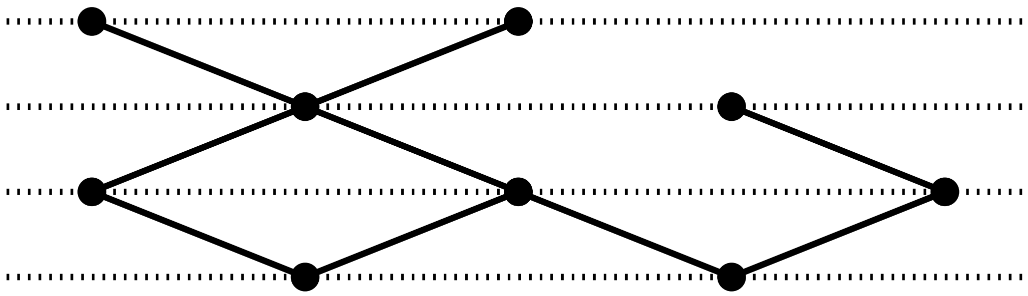

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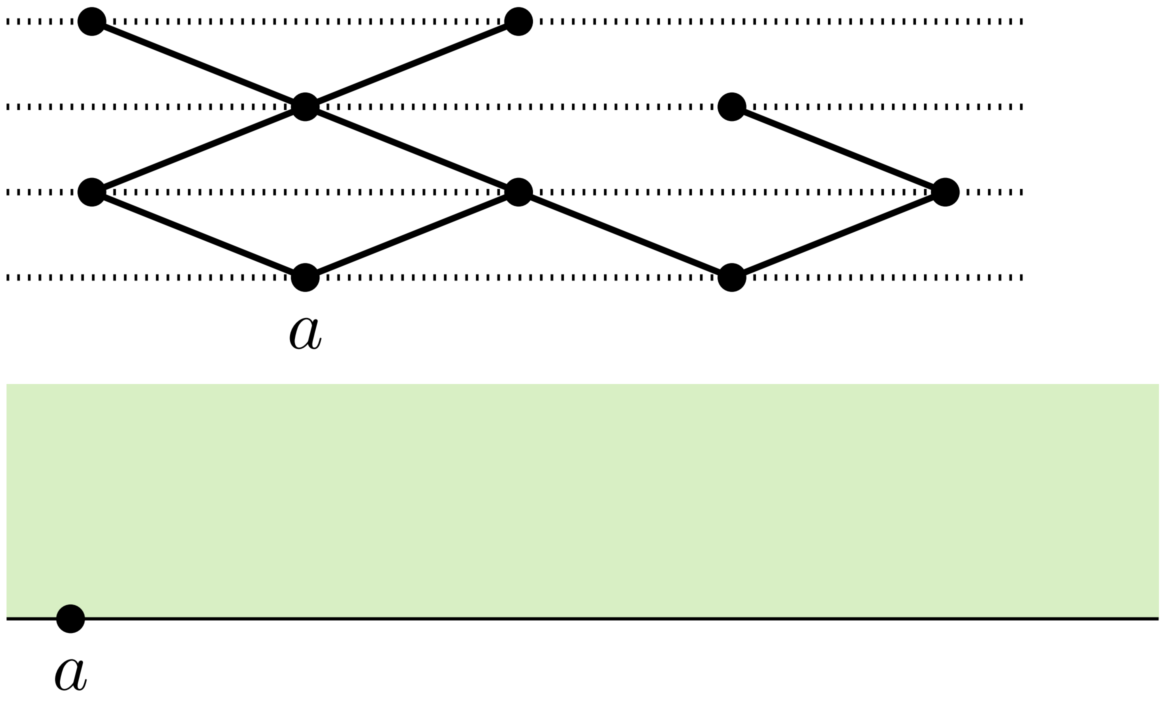

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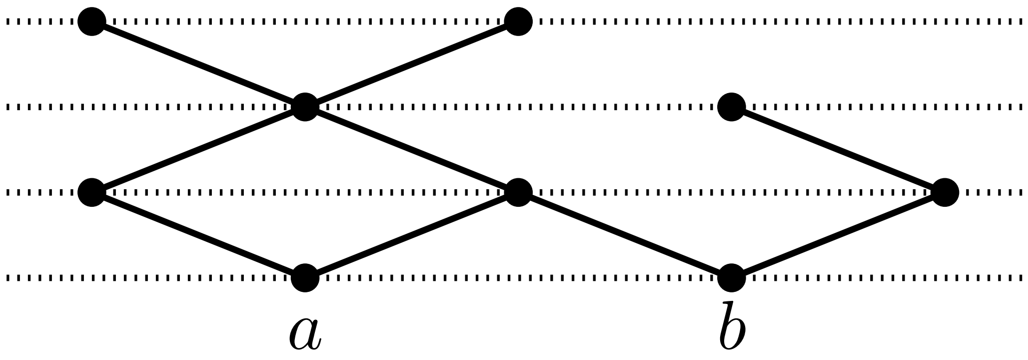

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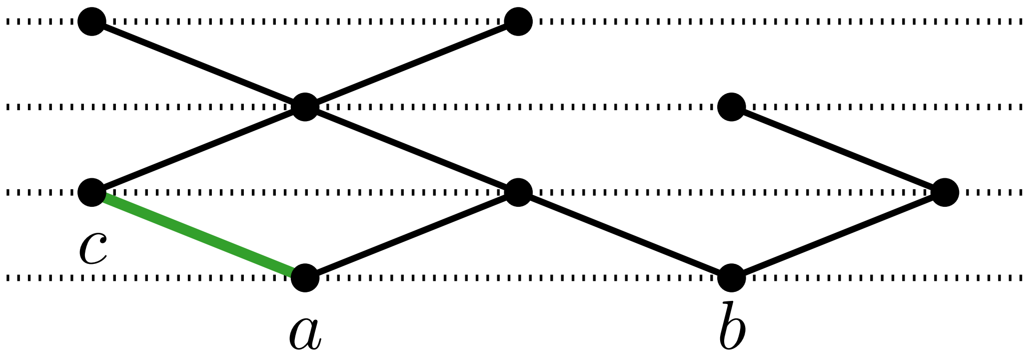

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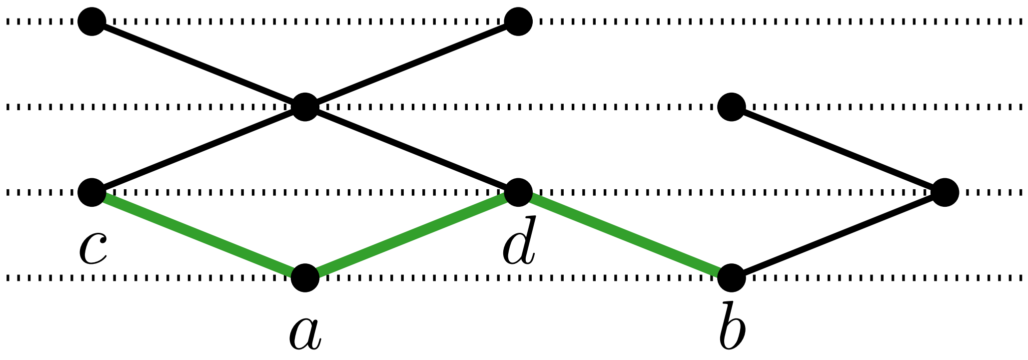

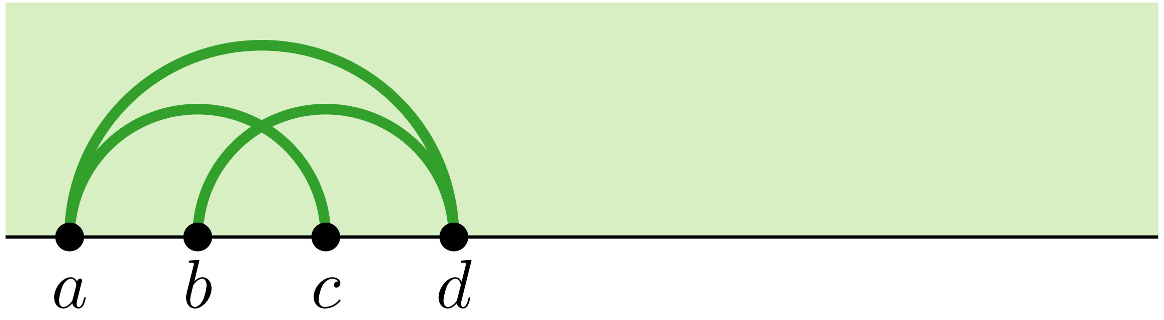

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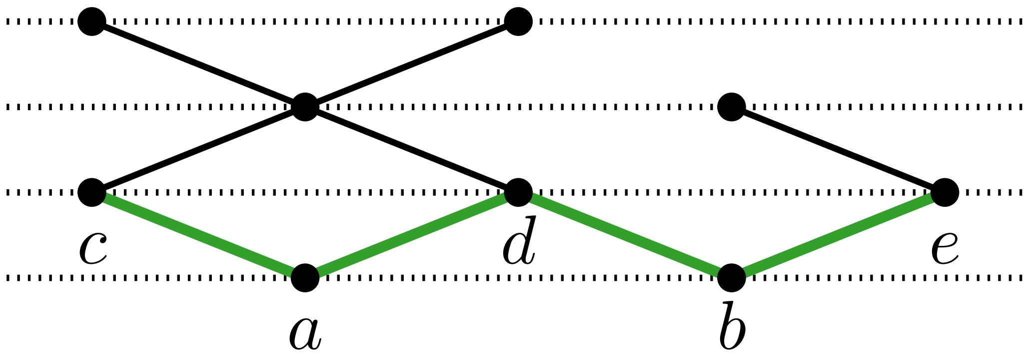

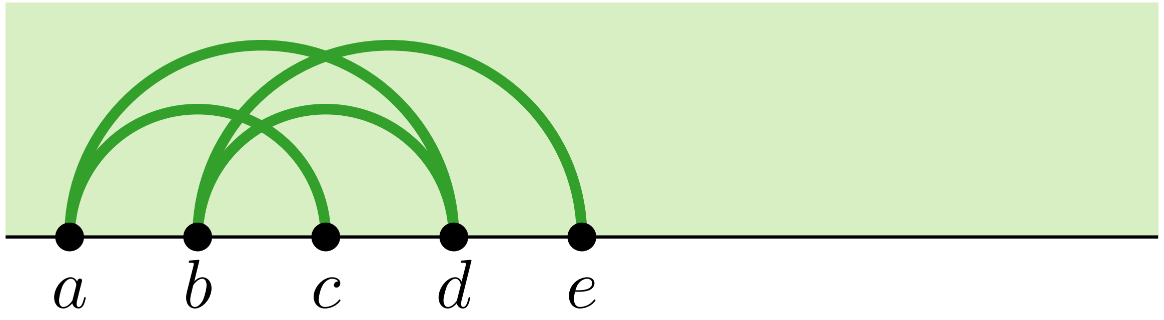

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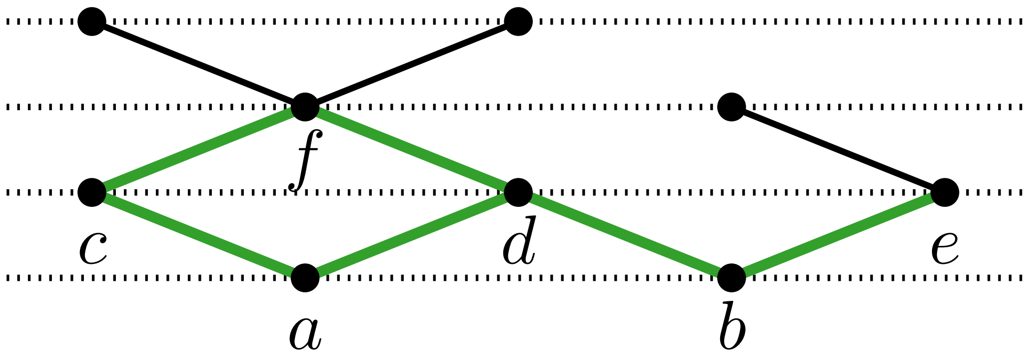

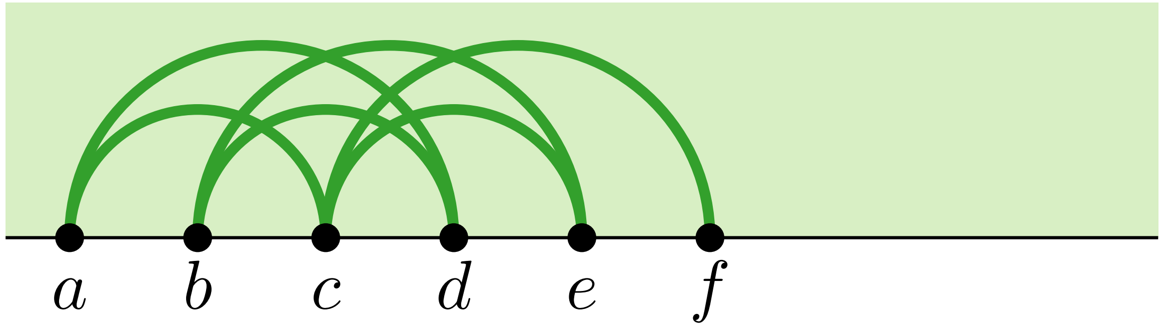

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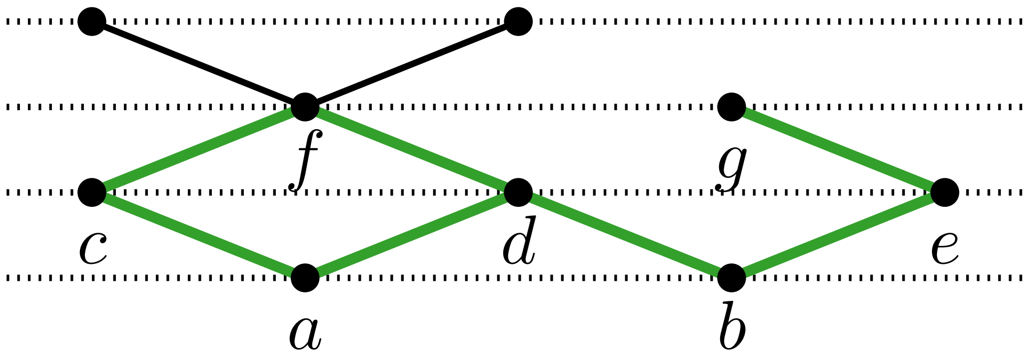

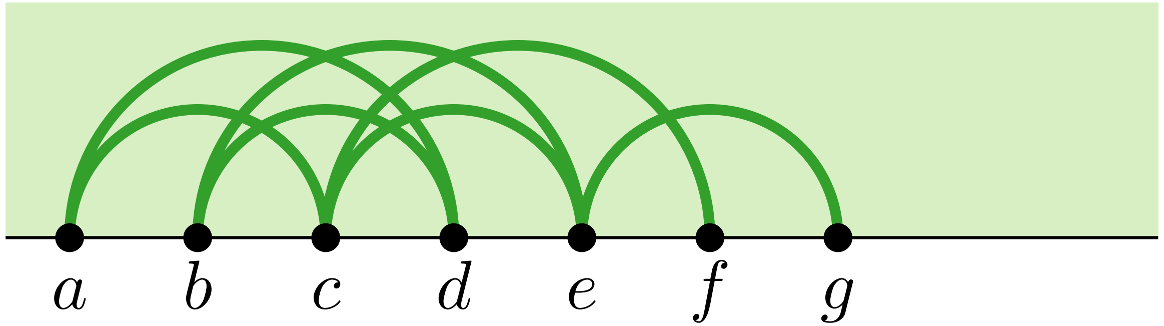

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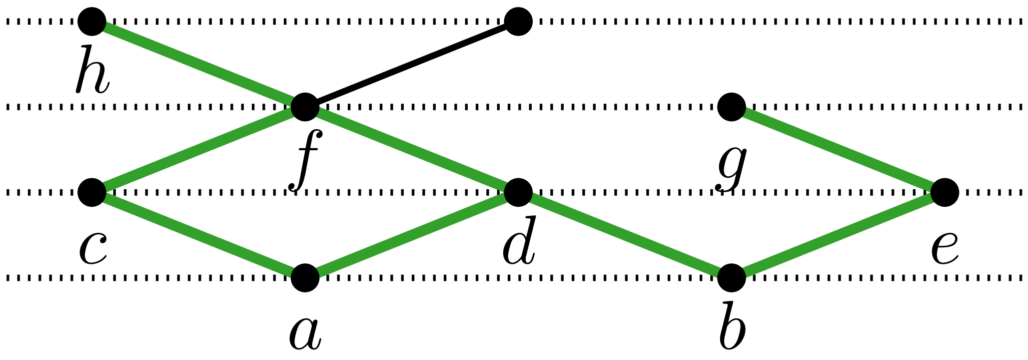

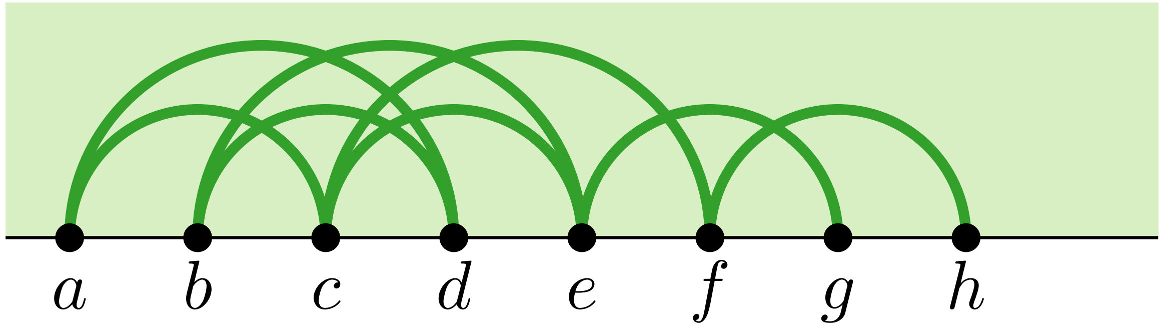

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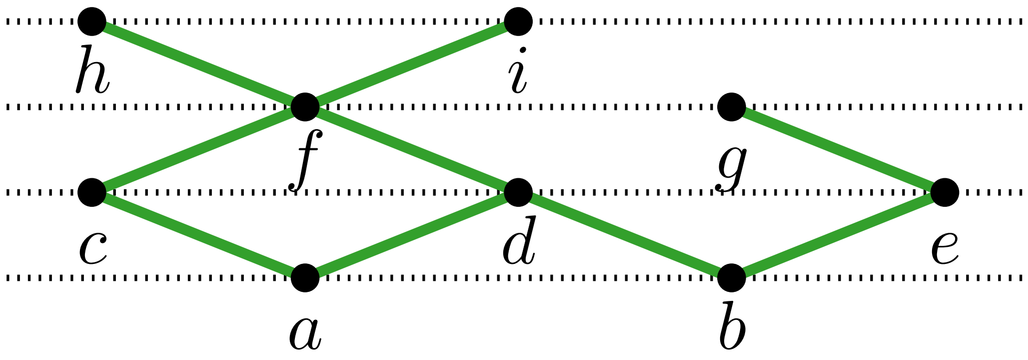

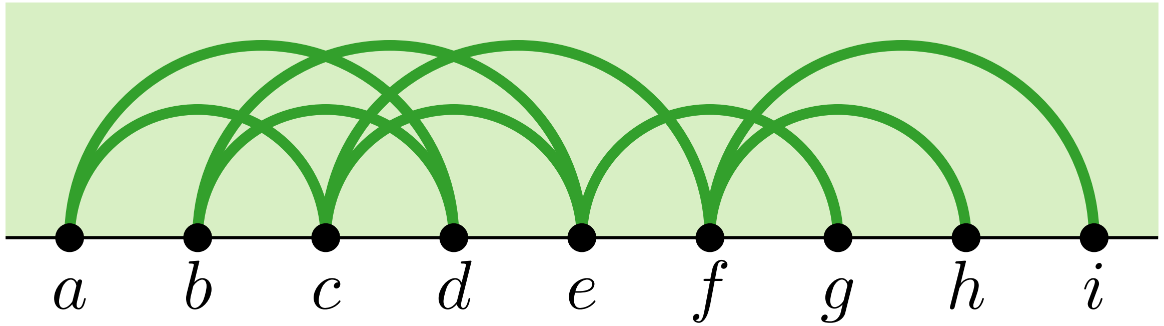

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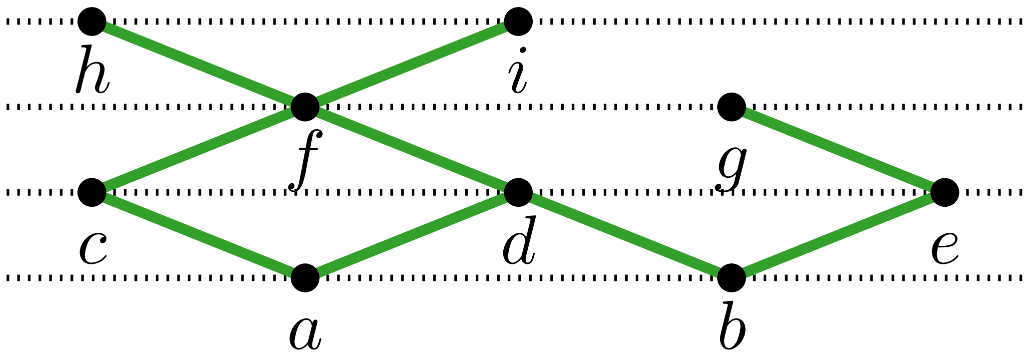

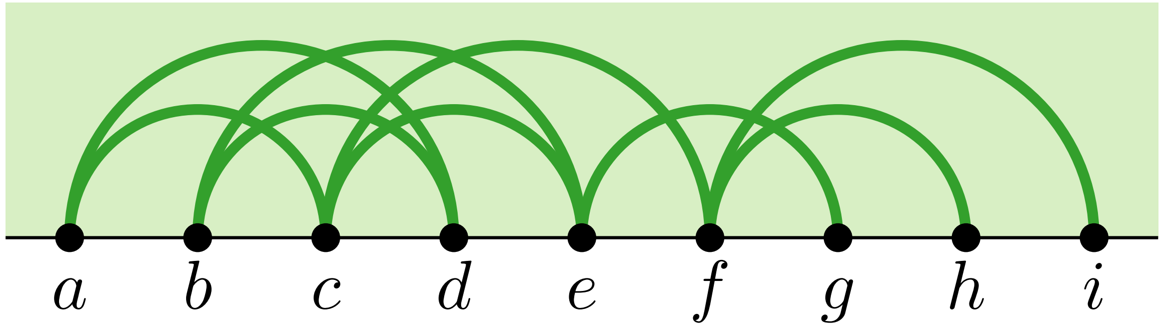

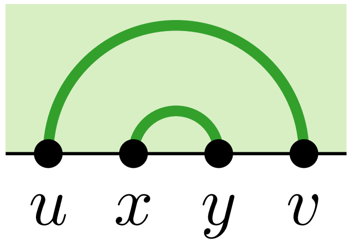

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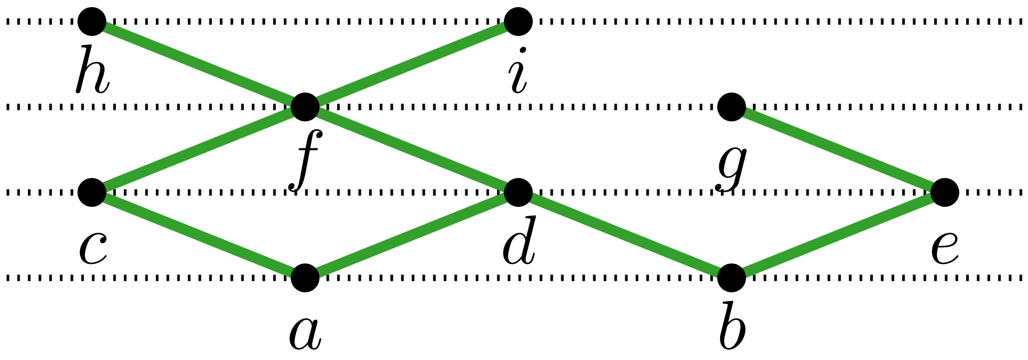

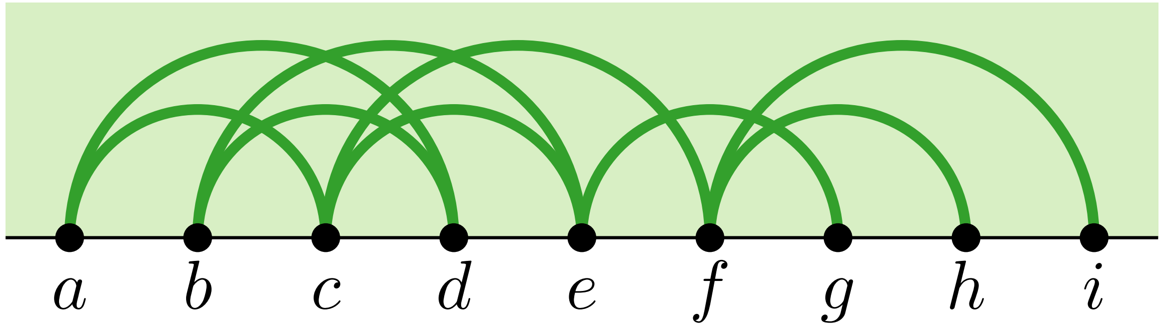

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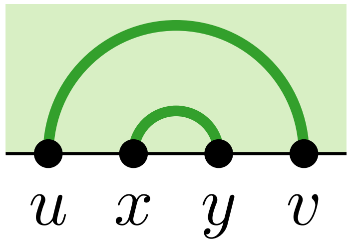
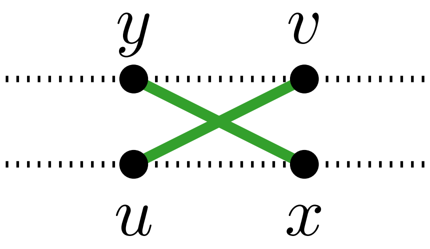
- If there was a nesting uv above xy, u would be to the left of x on one level, and y would be to the left of v on the level above; this would not be planar. □





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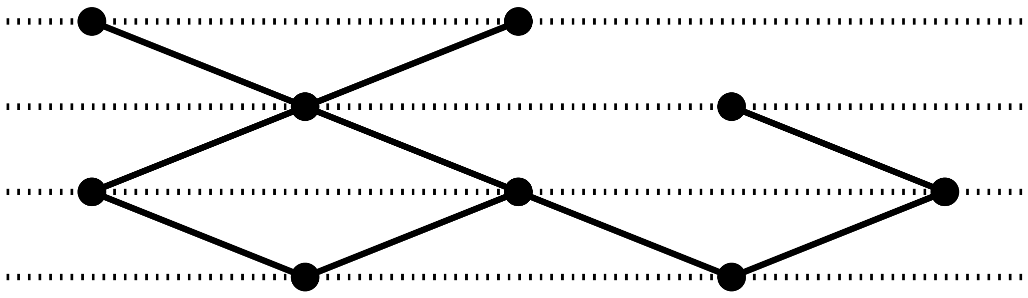

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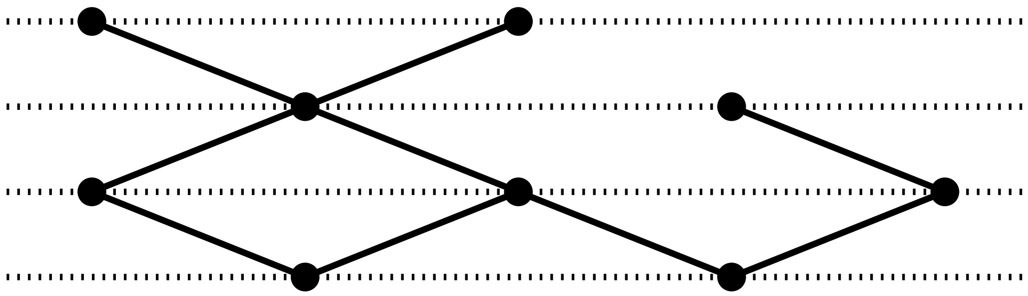

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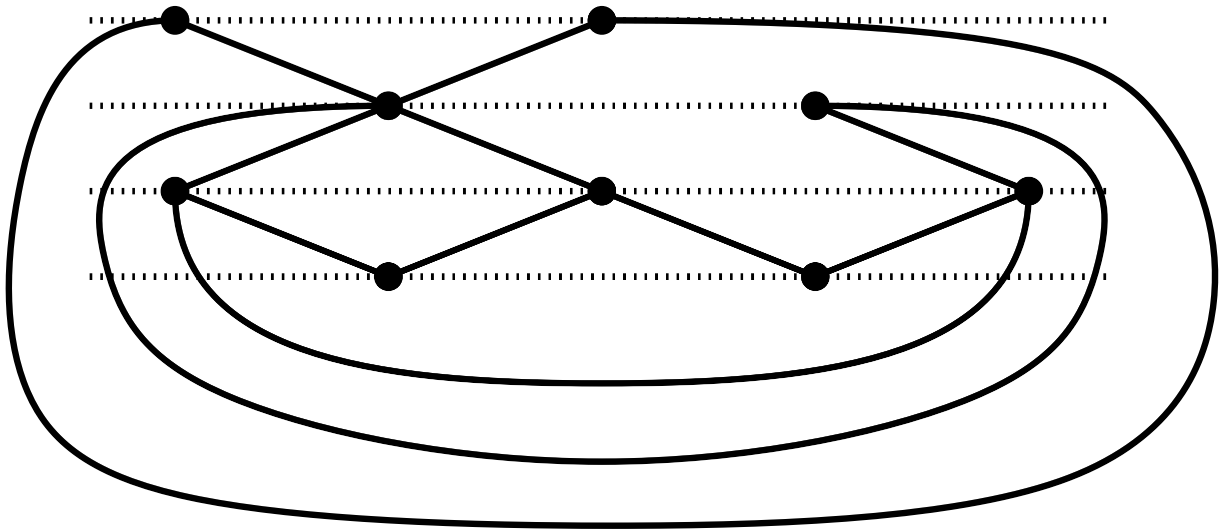
A graph is **arched leveled-planar** if it has a leveled-planar drawing where additionally vertices on the same level may be connected by edges that enclose all lower levels.



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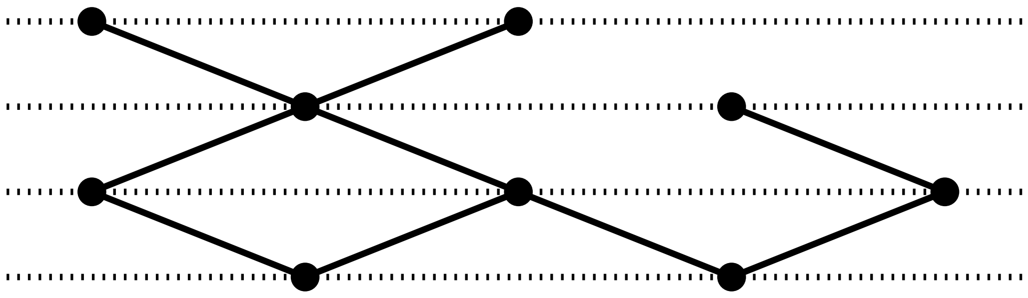

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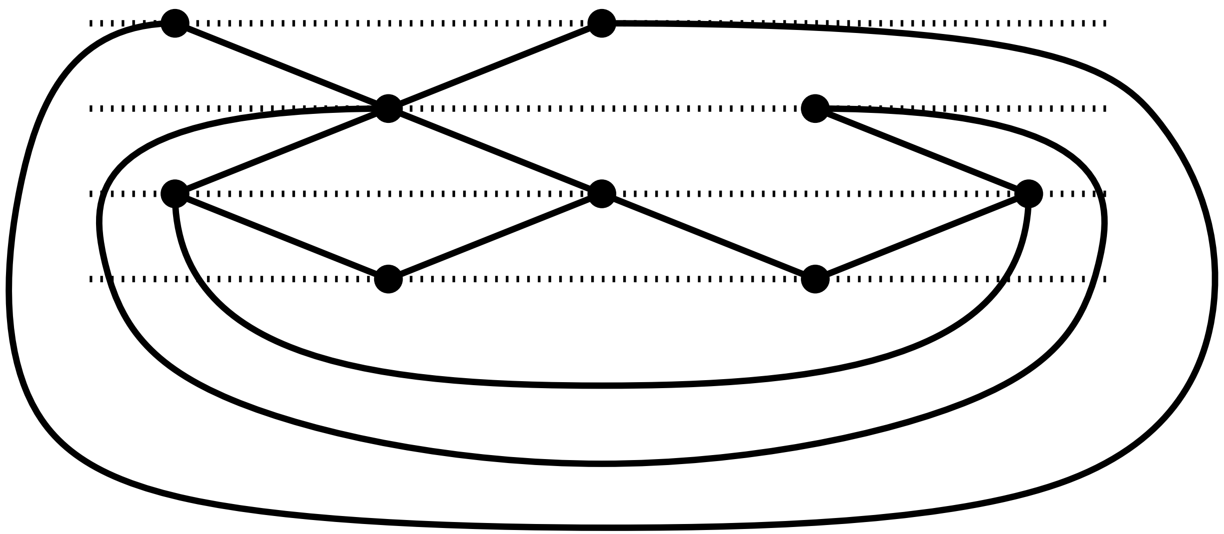
Proof. $\rightarrow$ *Exercise!*

A graph is **arched leveled-planar** if it has a leveled-planar drawing where additionally vertices on the same level may be connected by edges that enclose all lower levels.



2-Page and 3-Page Queue Layouts

Theorem. [Heath & Rosenberg 1992, Rengarajan & Veni Madhavan 1995.]

For every outerplanar graph G, $\mathrm{qn}(G) \leq 2$.

Theorem. [Rengarajan & Veni Madhavan 1996.]

For every series-parallel graph G, $\mathrm{qn}(G) \leq 3$.

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We have seen planar graphs have **stack** number at most 4. What is the max. **queue** number?

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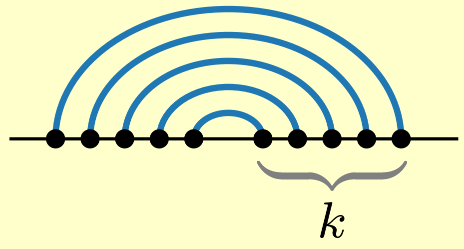

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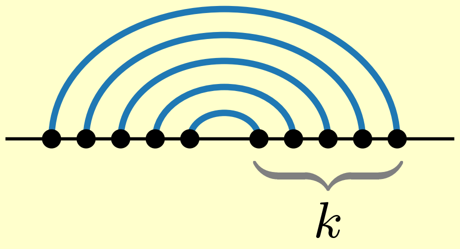

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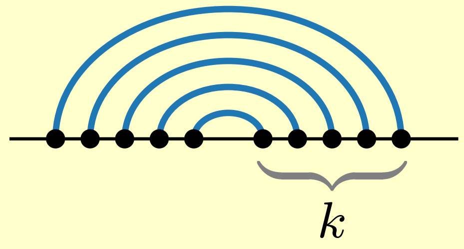

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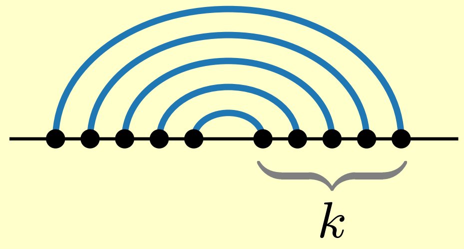

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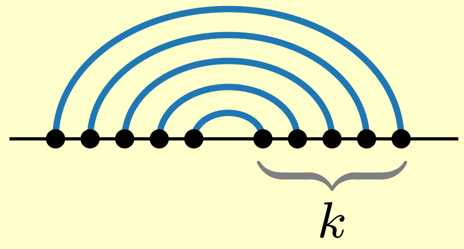

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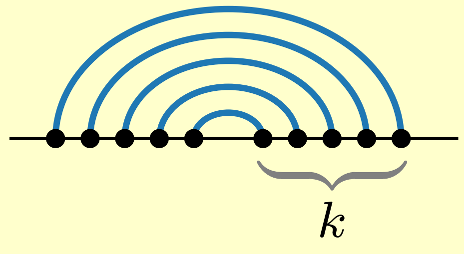

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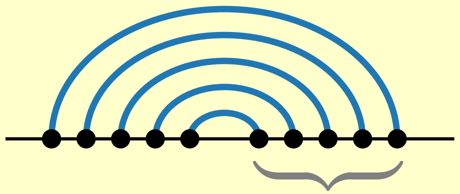

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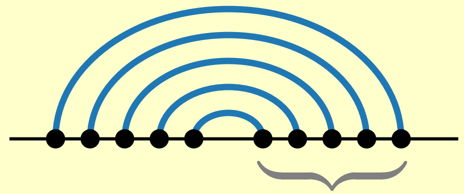

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- For the running time, see the implementation described by [Heath & Rosenberg 1992]. □

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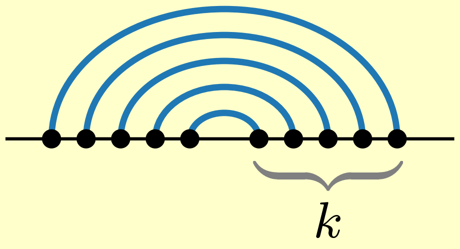

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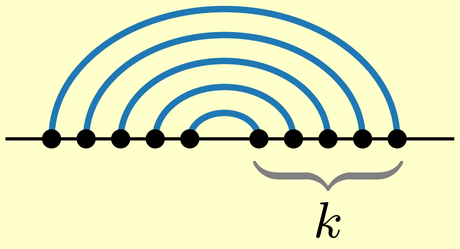

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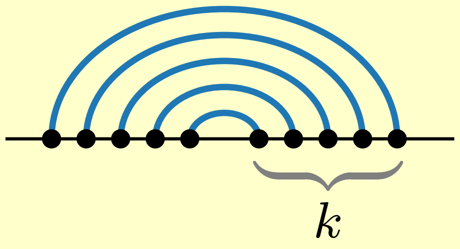

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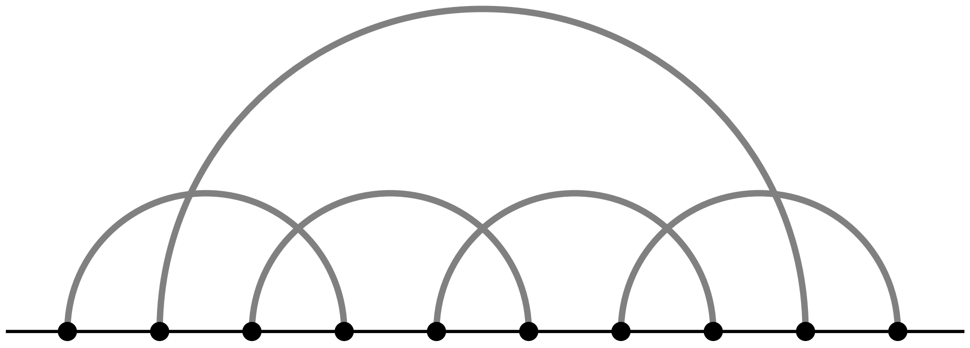

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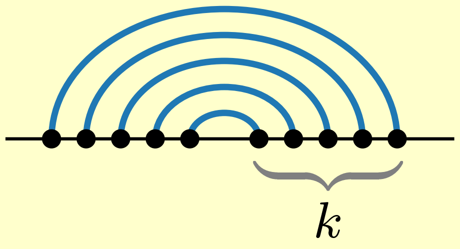

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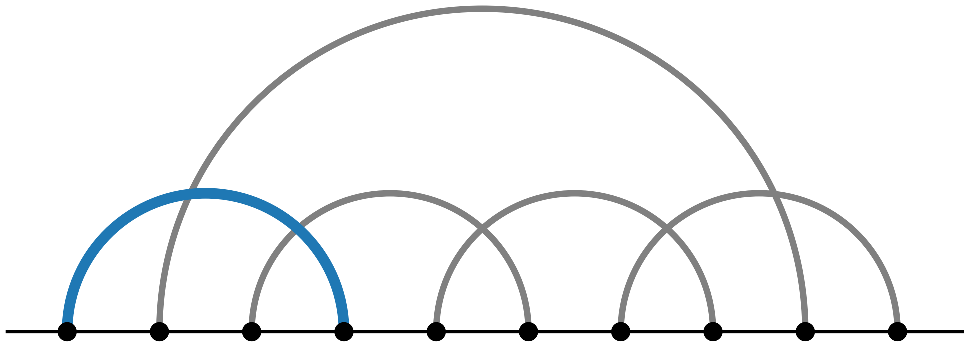

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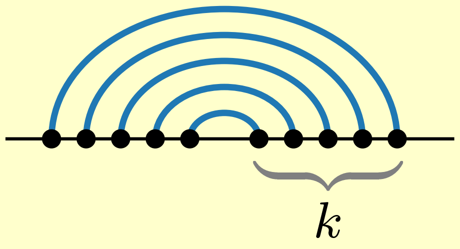

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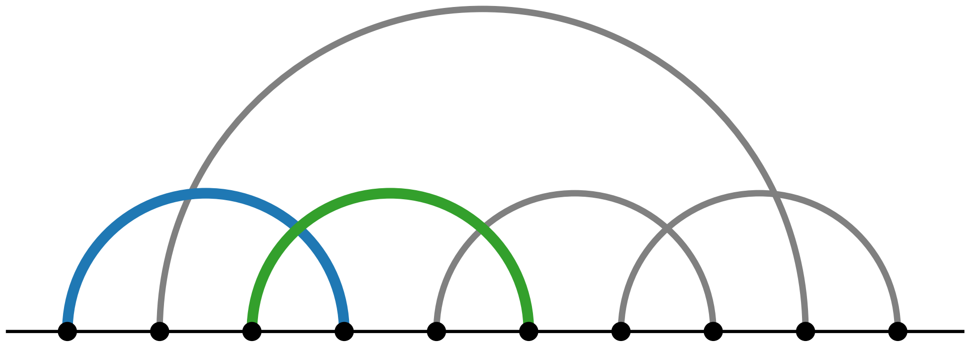

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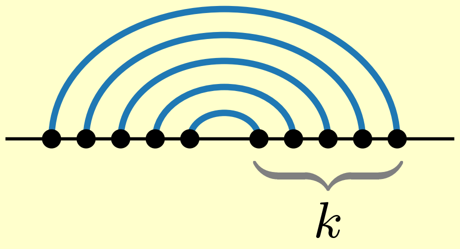

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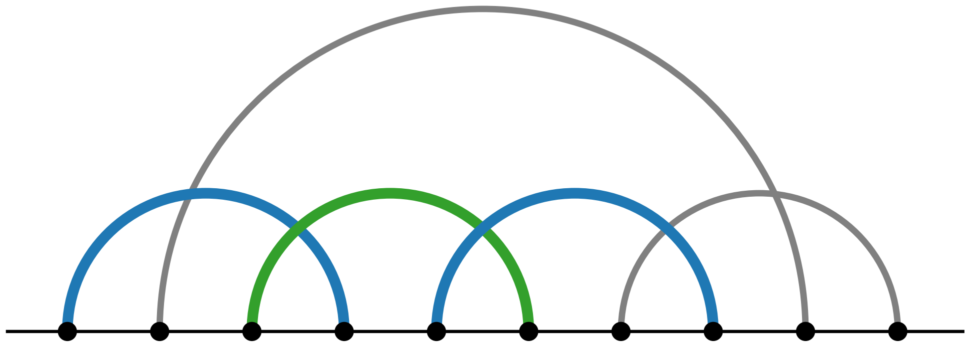

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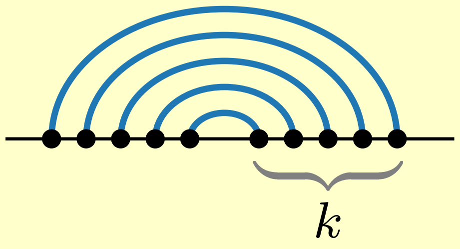

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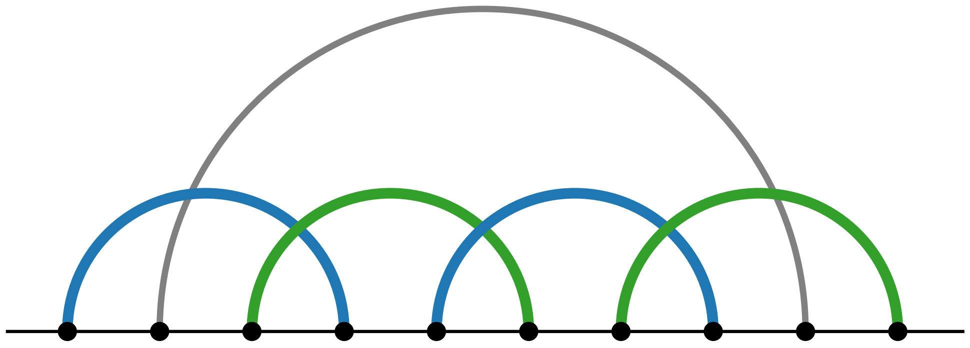

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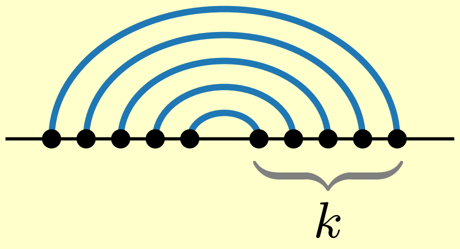

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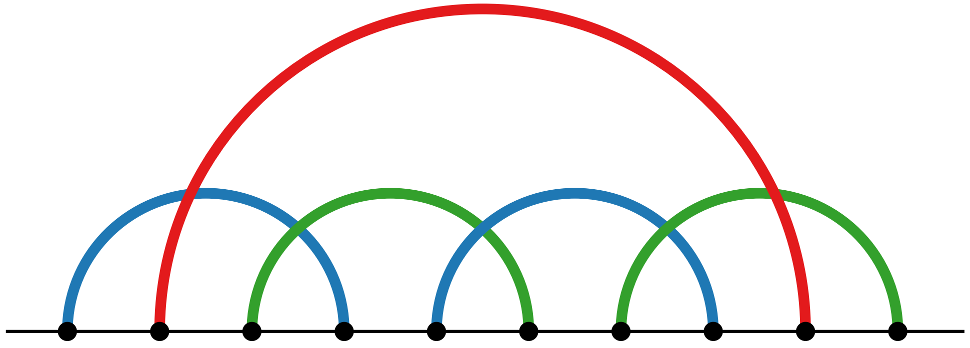

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Queue Layouts of Complete Graphs

The stack number can be linear in n. What about the the queue number of K_n?

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- Then, $\text{qn}(K_n) \leq n/2$ follows directly from Lemma 1. $\square$

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So, is determining the stack number easier if the order of the vertices on the spine is given?

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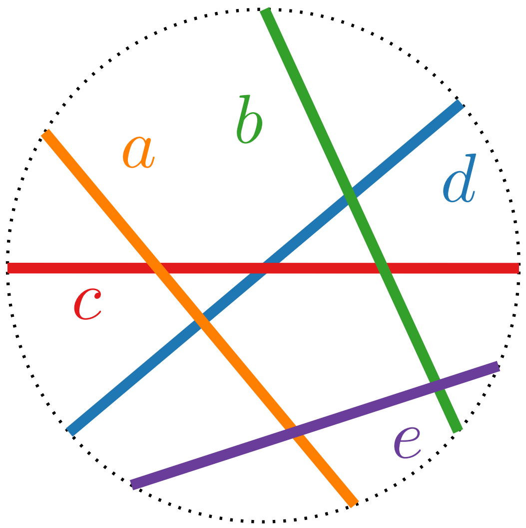

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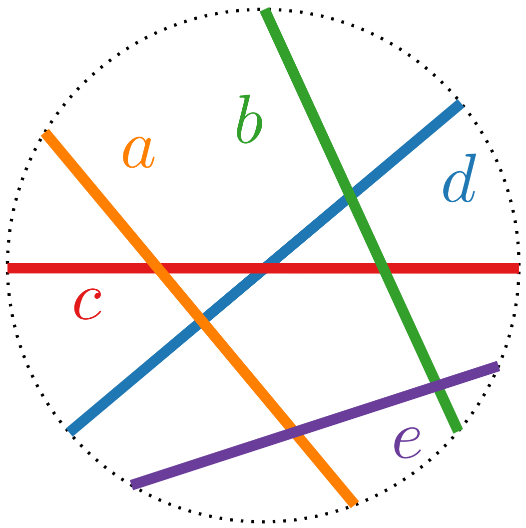

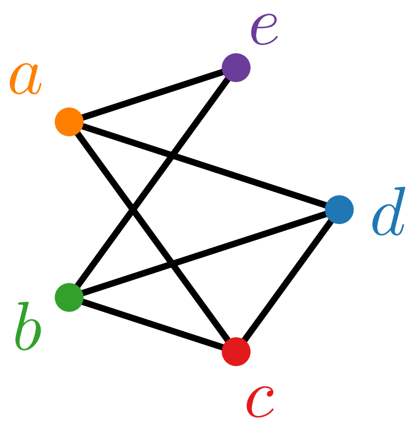

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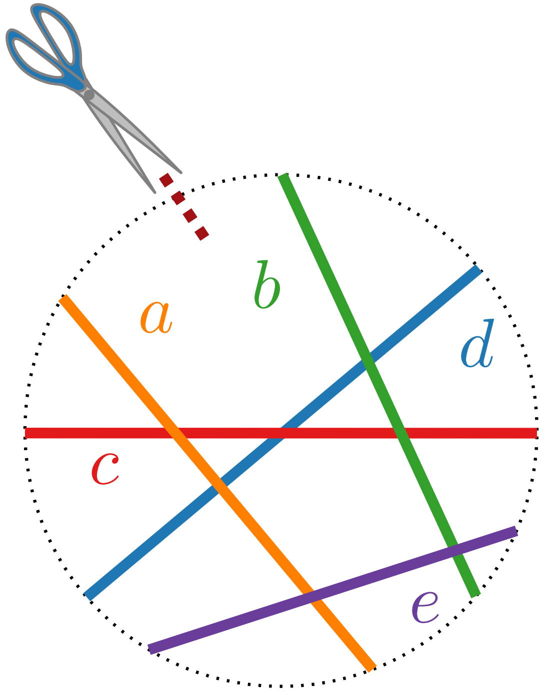

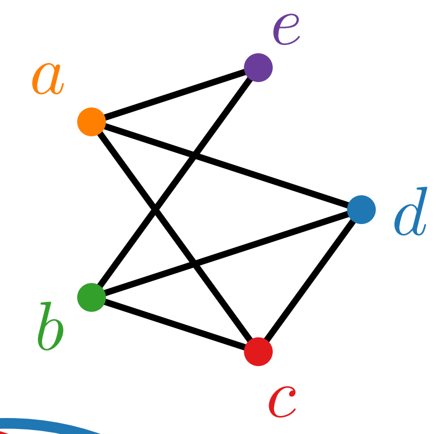

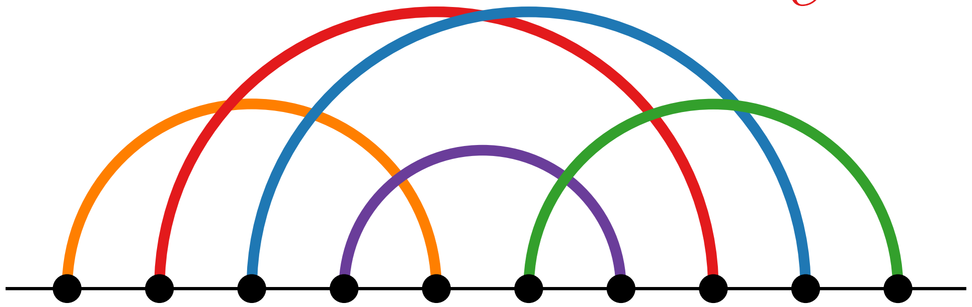

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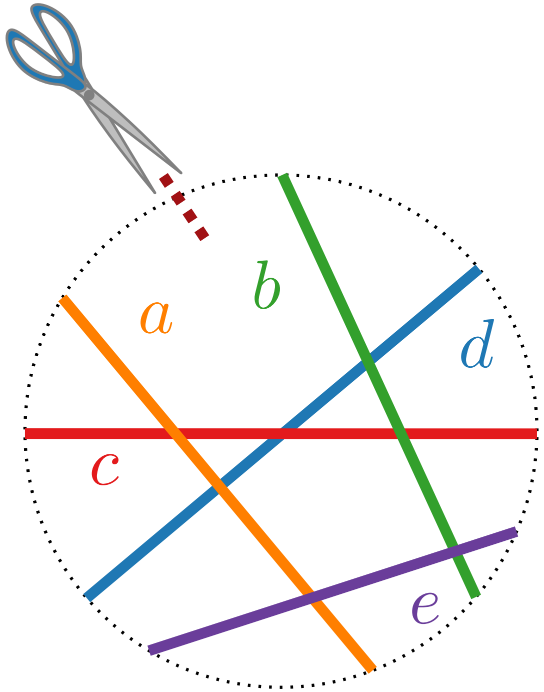

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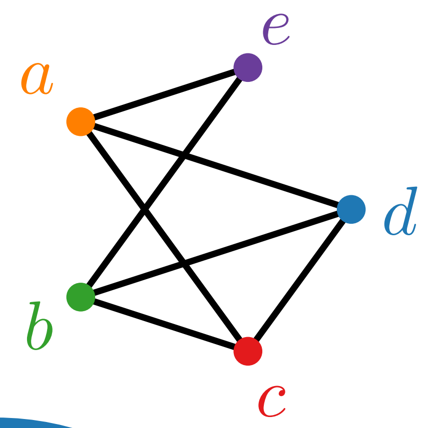

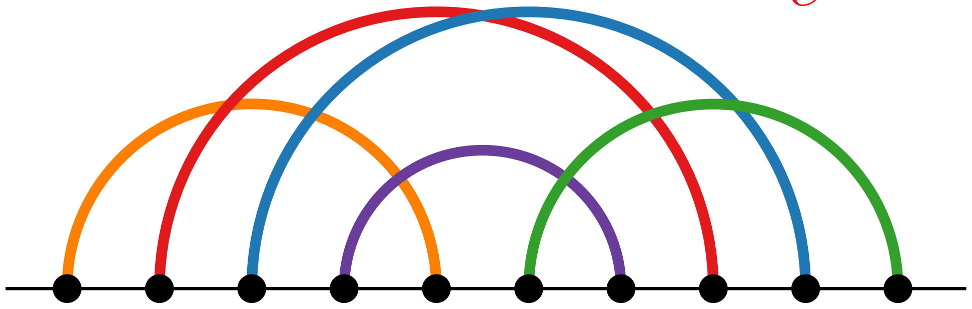

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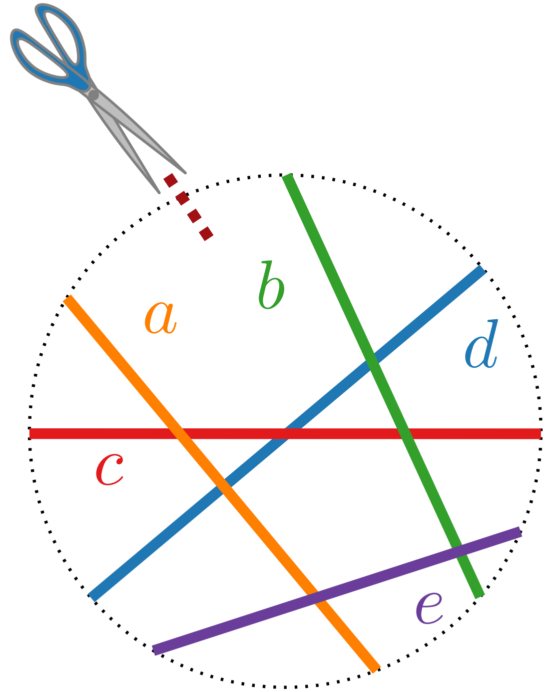

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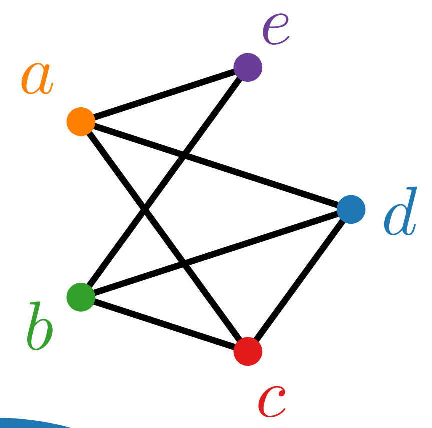

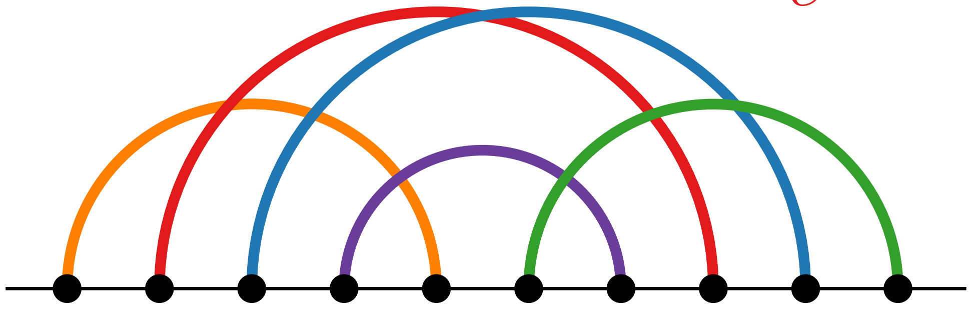

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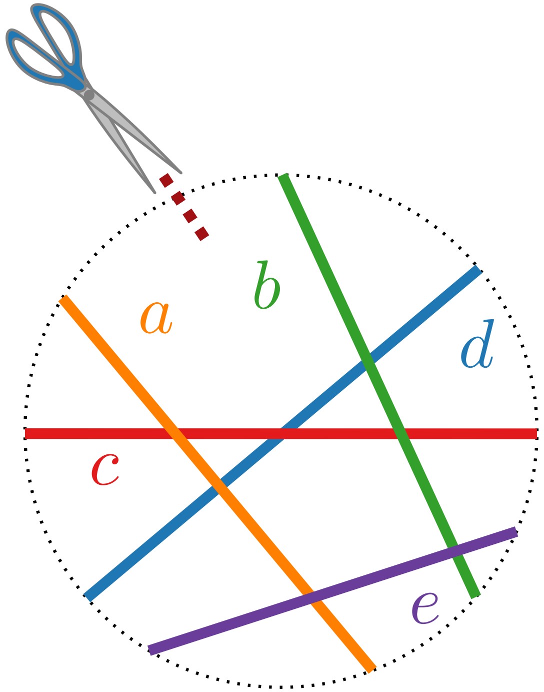

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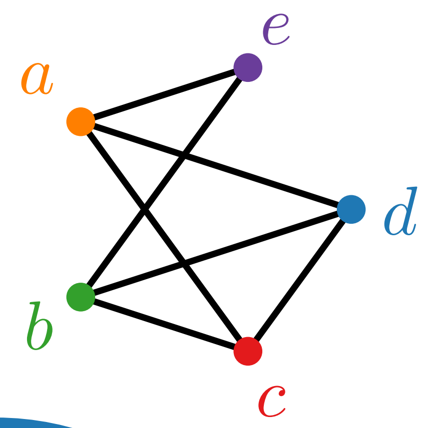

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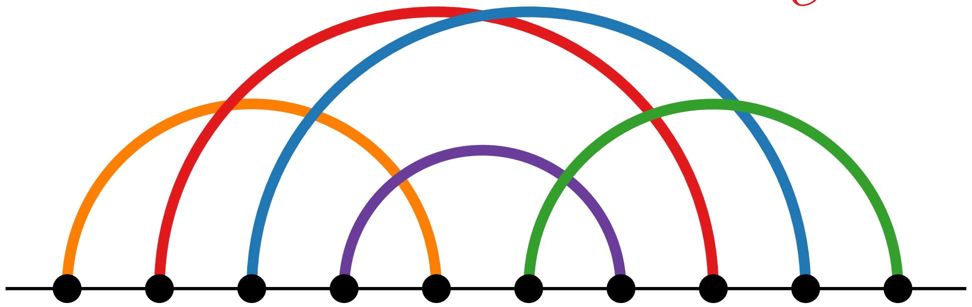

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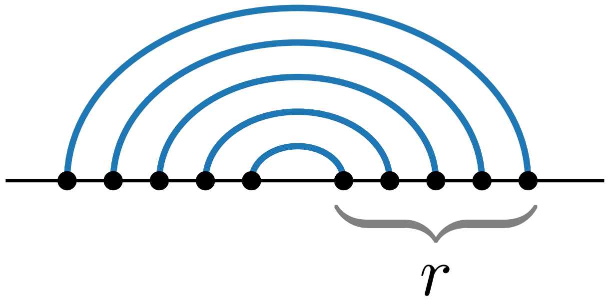

Complexity of Determining the Queue Number

Theorem. [Heath & Rosenberg 1992]
Deciding whether a graph G has queue number $\mathrm{qn}(G) \leq k$ is NP-complete for $k \geq 1$.

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- Deciding whether a given graph is arched leveled-planar is NP-complete.
  Hence, deciding whether $\mathrm{qn}(G) = 1$ is NP-hard. $\square$

Theorem. [Heath & Rosenberg 1992]
Deciding whether a graph G given with an order of the vertices on the spine has queue number $\mathrm{qn}(G) \leq k$ is polynomial-time solvable.

Proof Sketch.

Details in exercise!

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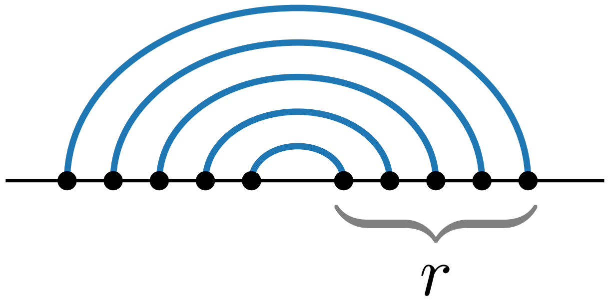

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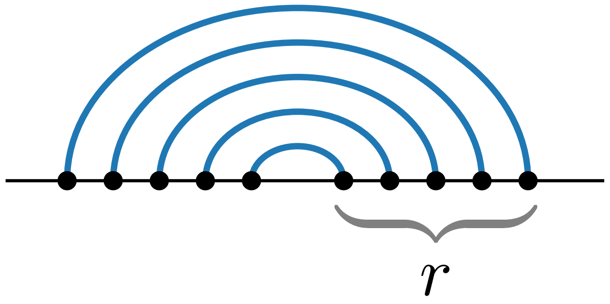

- Determine the size r of the largest rainbow in polynomial time.
- If $r \leq k$, then there is k-page queue layout due to Lemma 1. □

Discussion

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- By the book-embedding paradigm, **page number** and **book thickness** are alternative terms for *stack number*.
- There are many more variants, e.g., for fixed vertex order, directed graphs, using other data structures, . . .

Literature

Sources for the overview:

- [Ueckerdt 2022] Invited Talk on WG 2022: *Stack and queue layouts of planar graphs.*
- [Pupyrev 2024] Website on Linear Layouts:
  `https://spupyrev.github.io/linearlayouts.html`

Some of the referenced papers:

- [Bernhart & Kainen 1979] *The book thickness of a graph.*
- [Yannakakis 1986] *Embedding planar graphs in four pages.*
- [Heath & Rosenberg 1992] *Laying out graphs using queues.*
- [Bekos, Kaufmann, Klute, Pupyrev, Raftopoulou & Ueckerdt 2020]
  Four pages are indeed necessary for planar graphs.
- [Dujmović, Joret, Micek, Morin, Ueckerdt & Wood 2020]
  Planar graphs have bounded queue-number.
- [Bekos, Gronemann & Raftopoulou 2021]
  An improved upper bound on the queue number of planar graphs.