Julius-Maximilians-
UNIVERSITÄT
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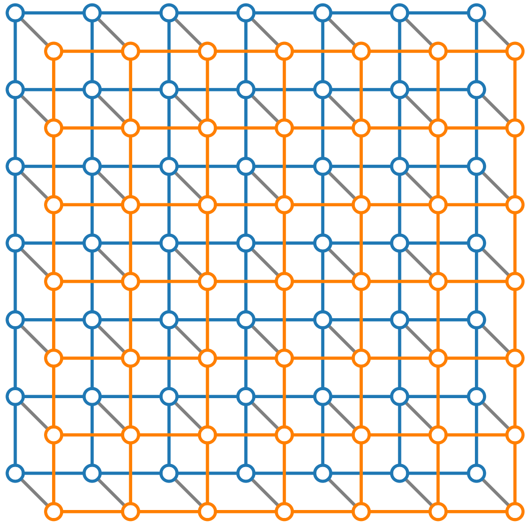
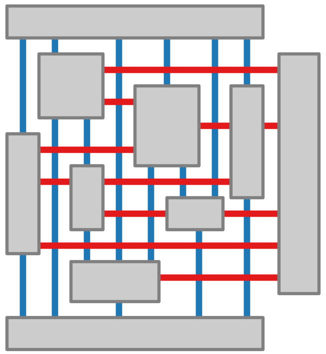
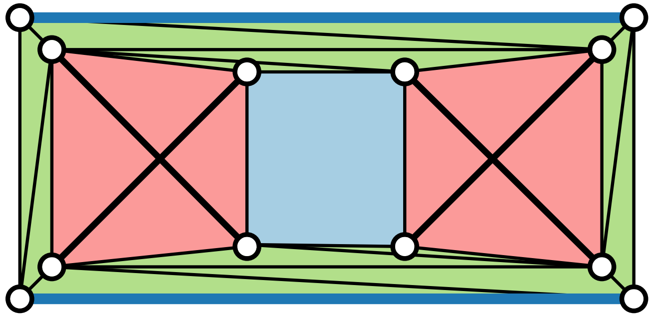


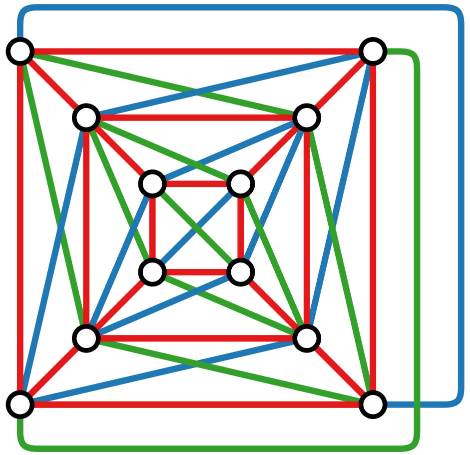
Visualization of Graphs



Lecture 11:
Beyond Planarity

Drawing Graphs with Crossings



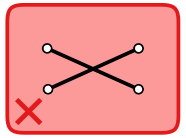


Alexander Wolff

Summer semester 2025

Planar Graphs

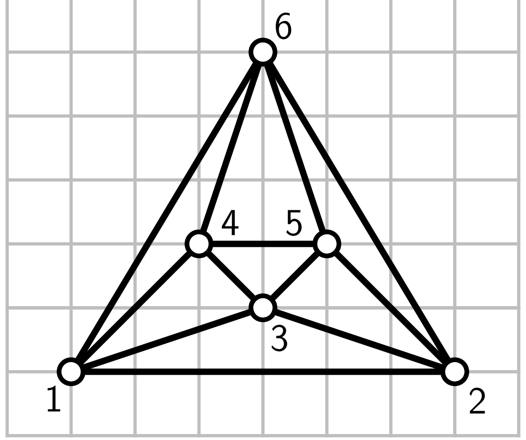
Planar graphs admit drawings in the plane without crossings.



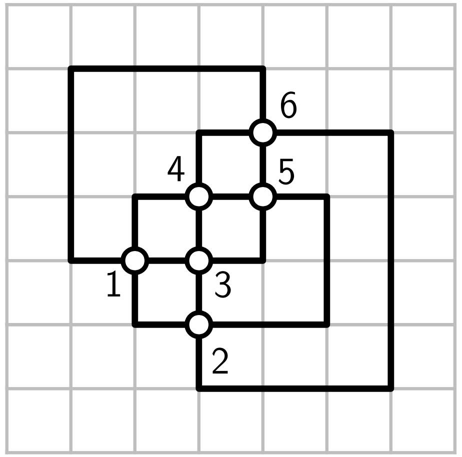
Plane graph is a planar graph with an embedding (fixed rotation system and fixed outer face).

Planarity is recognizable in linear time.

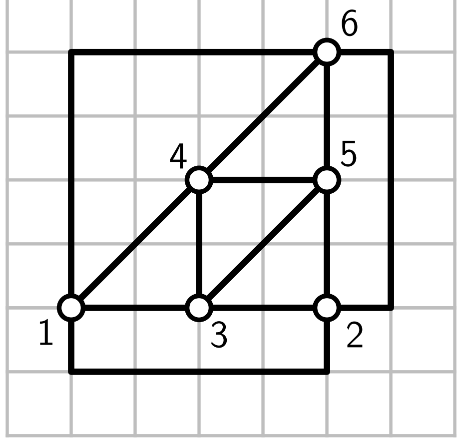
Different drawing styles . . .



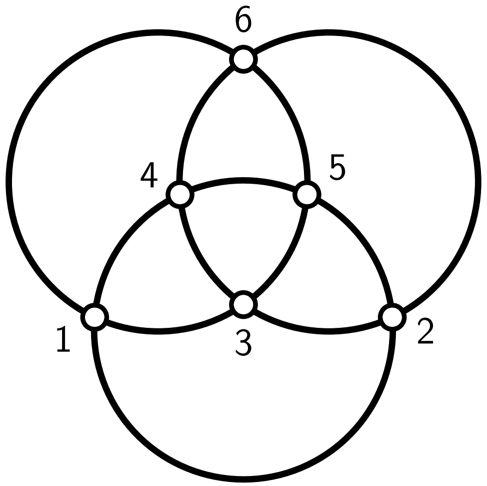
straight-line drawing



orthogonal drawing



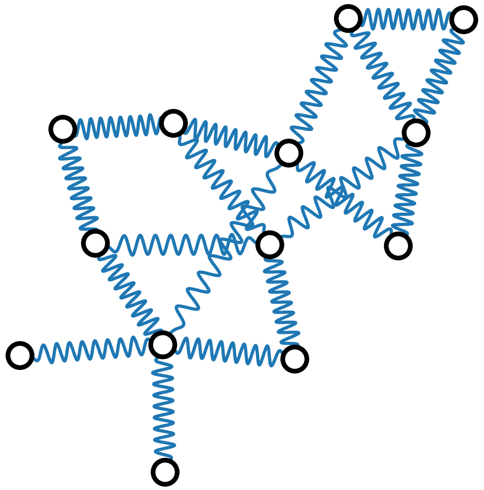
grid drawing with bends & 3 slopes



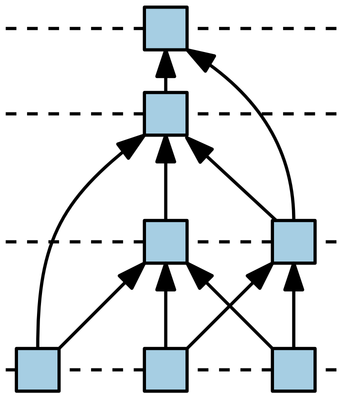
circular-arc drawing

And Non-Planar Graphs?

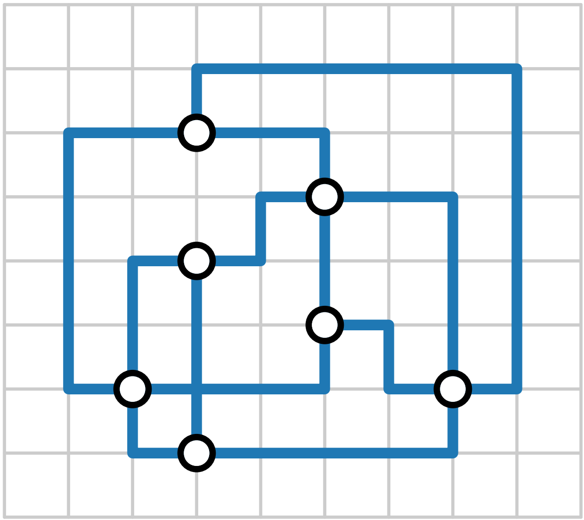
We have seen a few drawing styles:



force-directed drawing

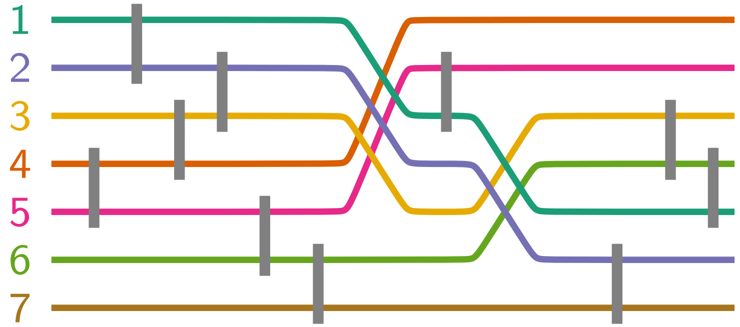


hierarchical drawing

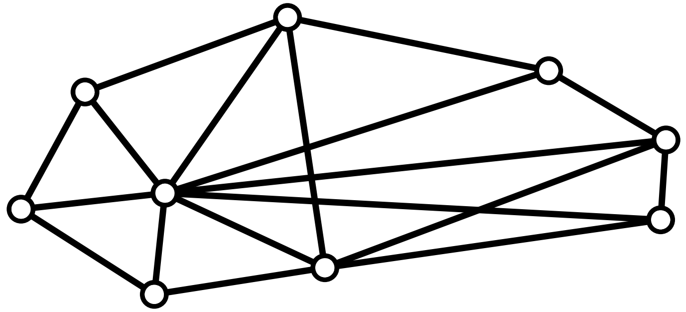


orthogonal layouts
(via planarization)

Maybe not all crossings are equally bad?



block crossings



Which crossings feel worse?

Eye-Tracking Experiment

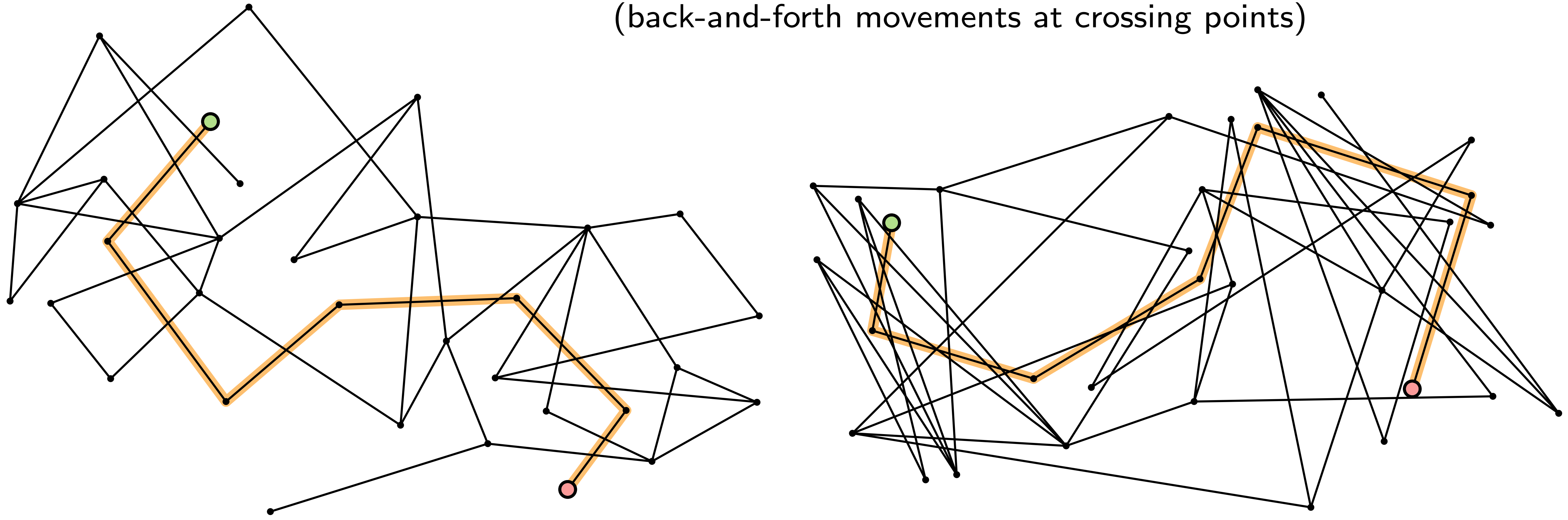
[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

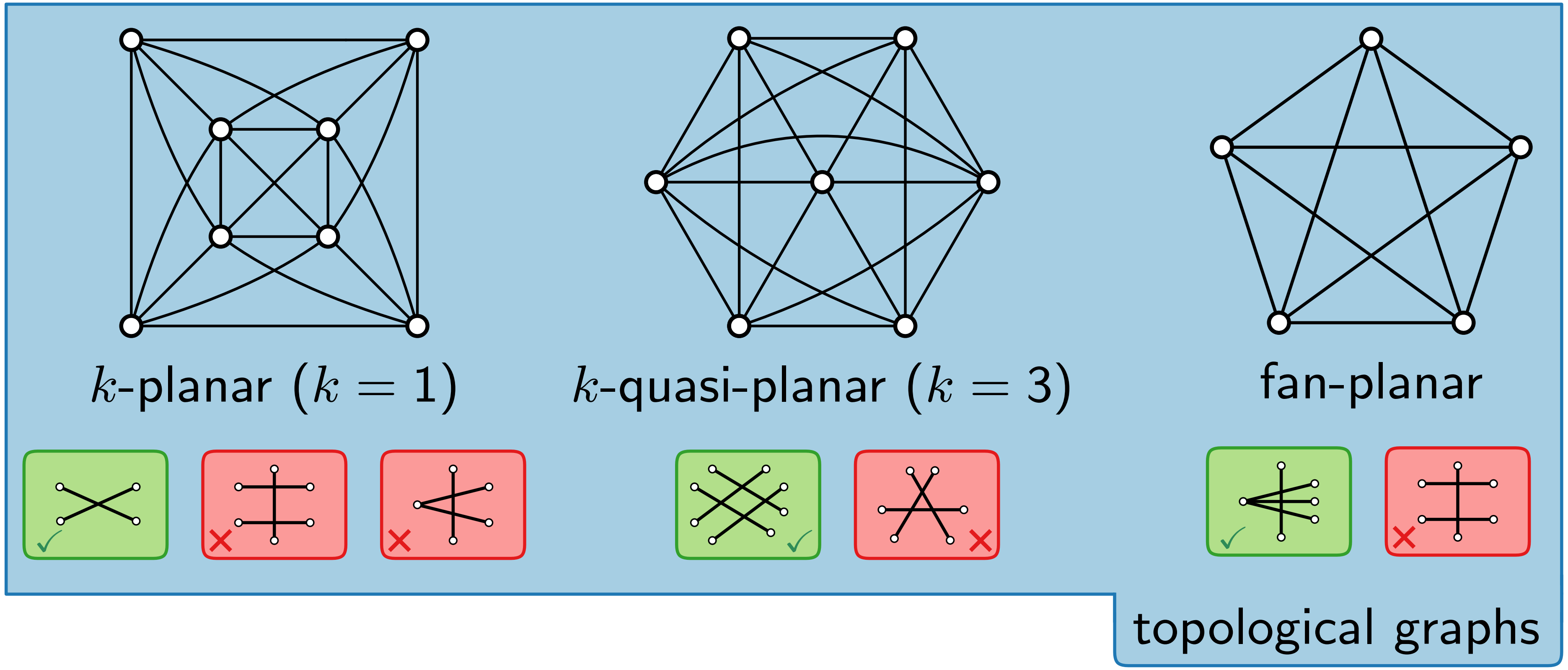
Results:

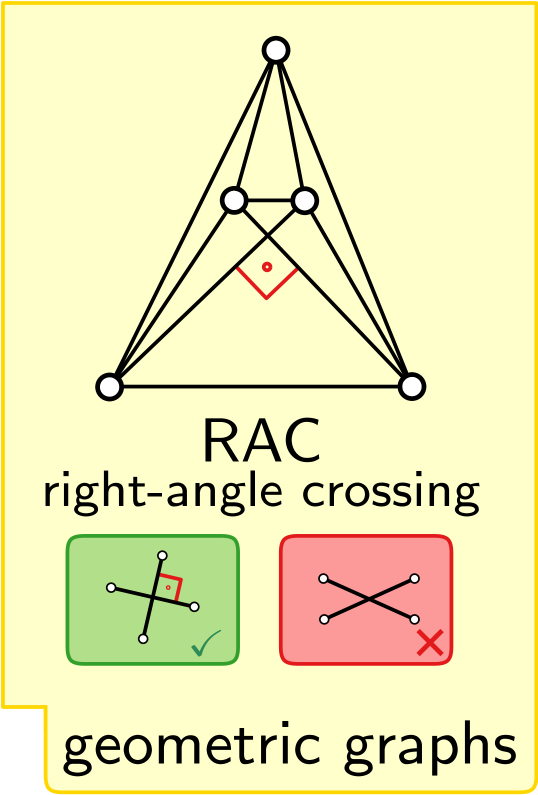
no crossings	eye movements smooth and fast
large crossing angles	eye movements smooth but slightly slower
small crossing angles	eye movements no longer smooth and very slow (back-and-forth movements at crossing points)



Some Beyond-Planar Graph Classes

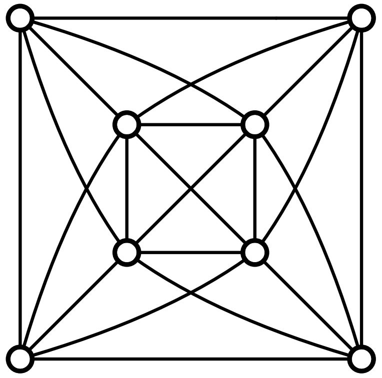
We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.



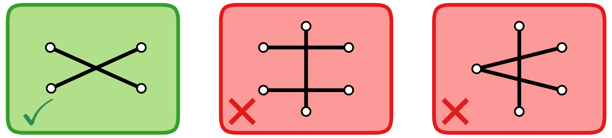


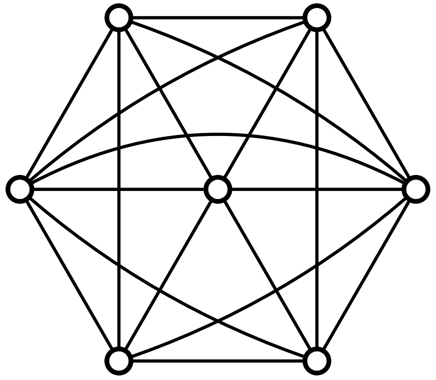
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

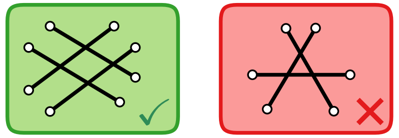


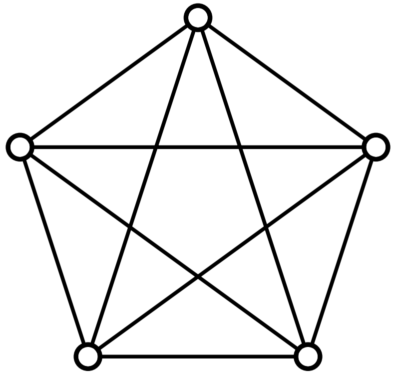
k-planar ($k = 1$)



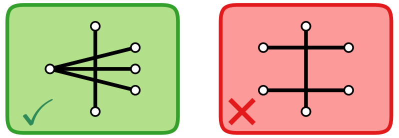


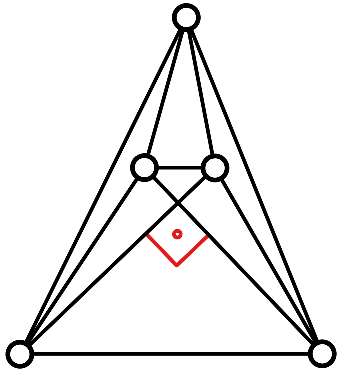
k-quasi-planar ($k = 3$)



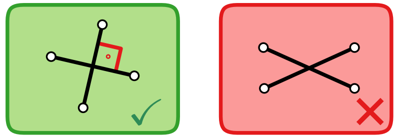


fan-planar

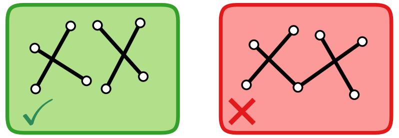




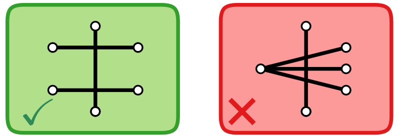
RAC
right-angle crossing



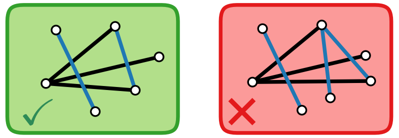
There are many more beyond-planar graph classes...



IC (independent crossing)



fan-crossing-free

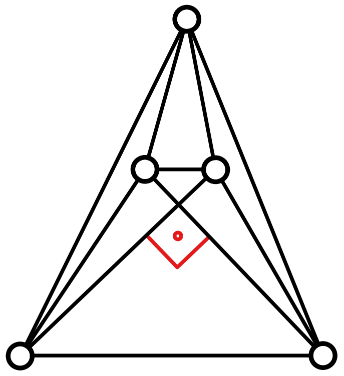


skewness-k ($k = 2$)

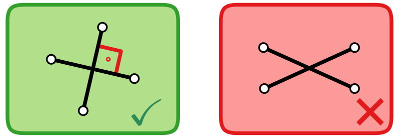
remove $\leq k$ edges to make it planar

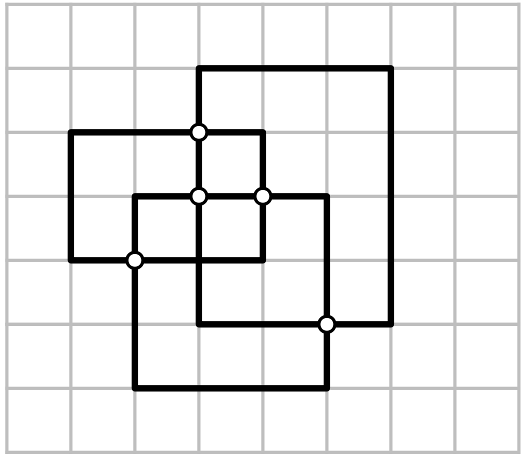
combinations, ...

Drawing Styles for Crossings

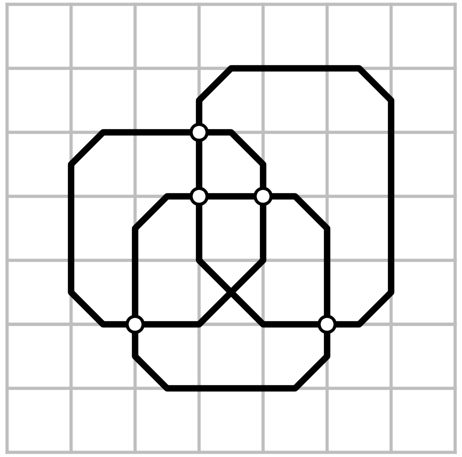


RAC
right-angle crossing

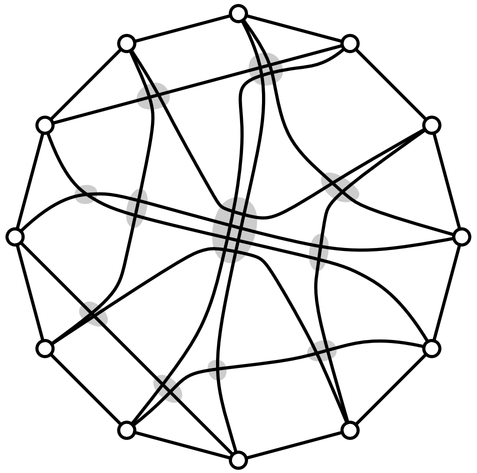




orthogonal

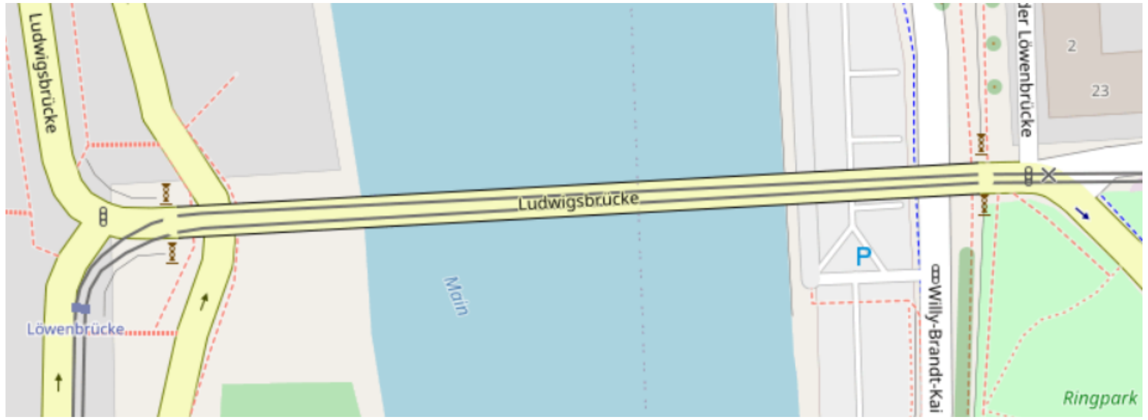


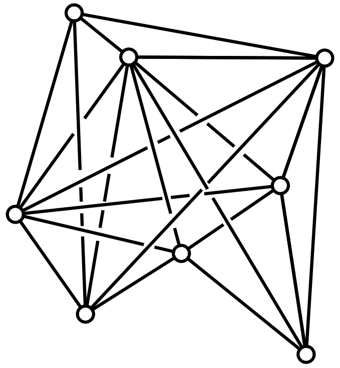
slanted orthogonal



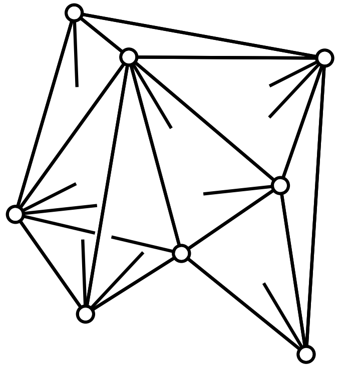
block / bundled crossings

circular layout: 28 invididual
vs. 12 bundle crossings

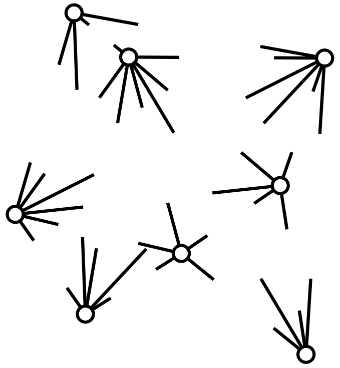




cased crossings



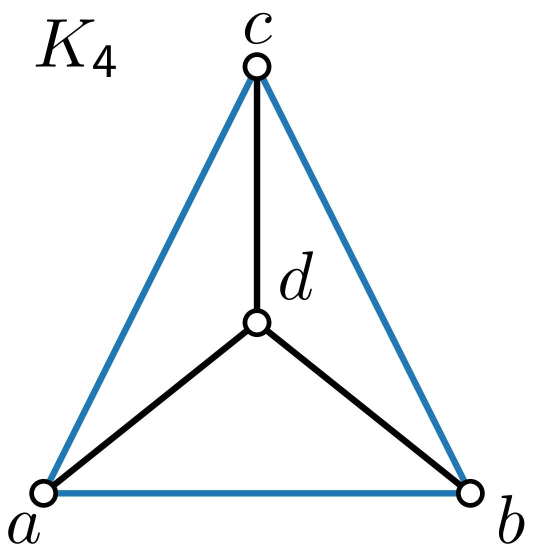
symmetric partial
edge drawing

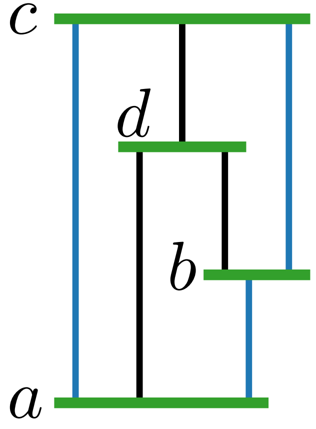


1/4-SHPED

symmetric homogenous
partial edge drawing

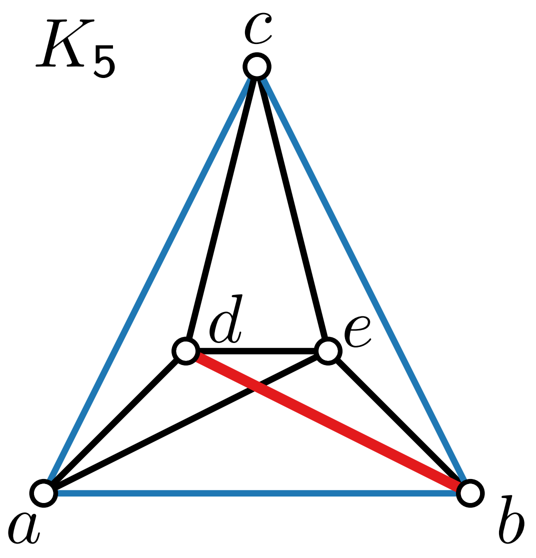
Geometric Representations

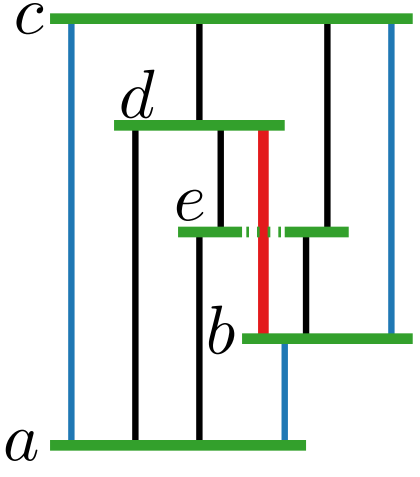




bar visibility
representation

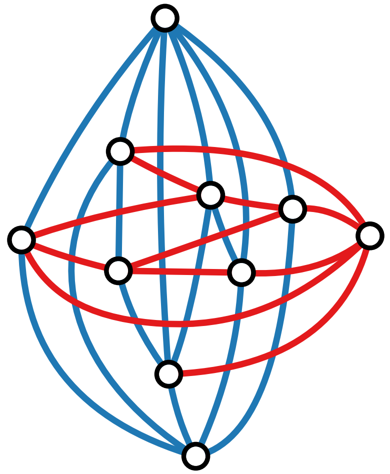
Geometric Representations





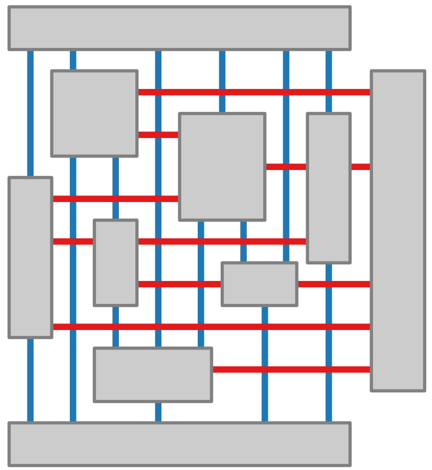
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

decompose into 2 planar graphs

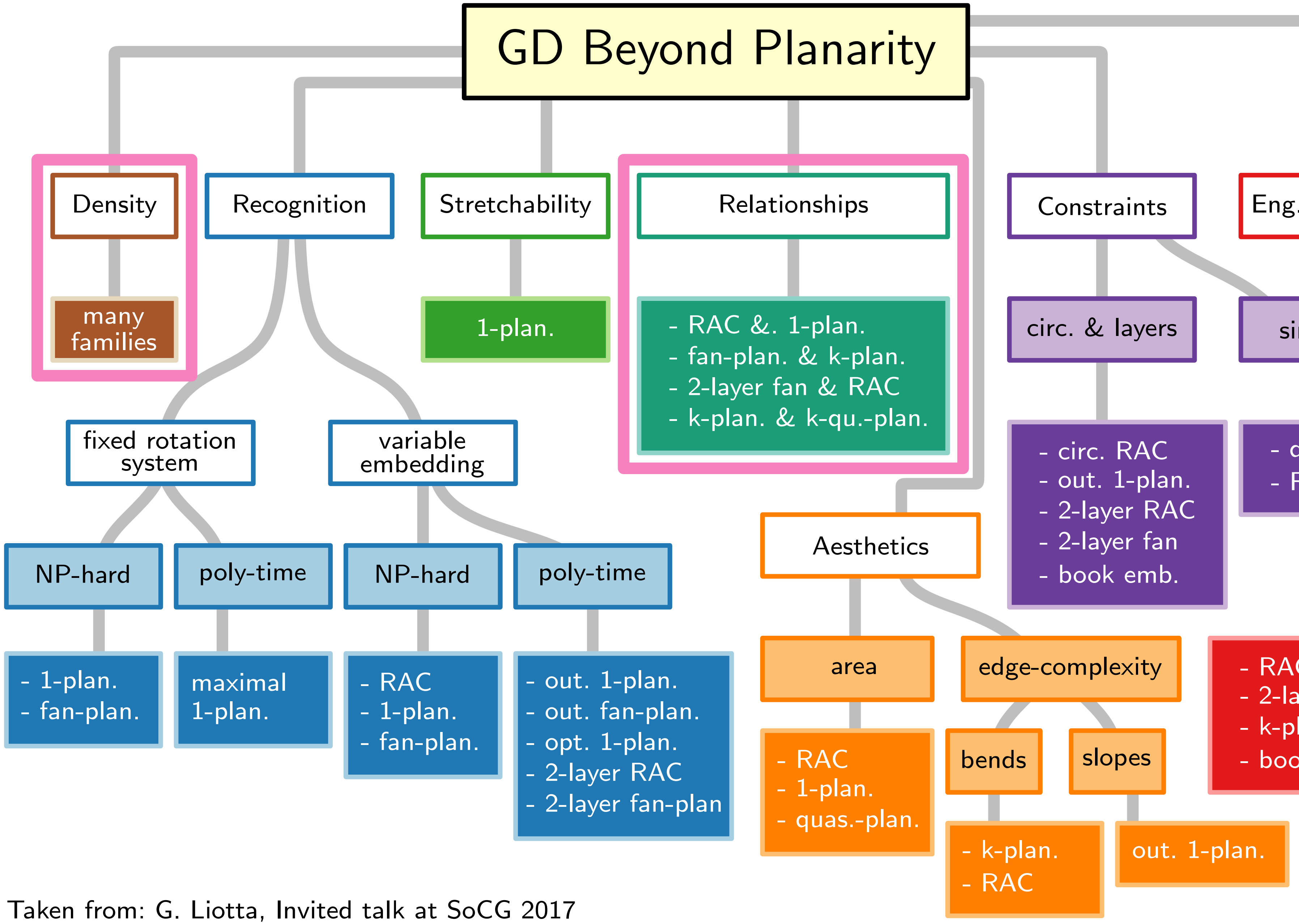


rectangle visibility representation

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

- Rectangle visibility graphs (RVGs) have $\leq 6n - 20$ edges. [Hutchinson, Shermer, Vince 1996]
- Recognizing thickness-2 graphs and RVGs is NP-hard. [Mansfields 1983] [Shermer 1996]
- RVGs can be recognized efficiently if embedding is fixed. [Biedl, Liotta, Montecchiani 2018]

GD Beyond Planarity: a Taxonomy



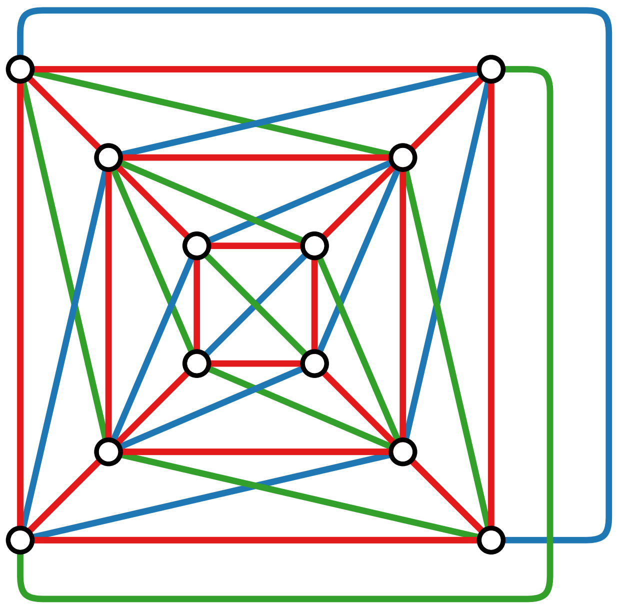
Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.

Proof sketch.

- Let the red edges be those that do not cross.
- Each blue edge crosses a green edge.
- This yields a red-blue plane graph G_{rb} with
  $m_{rb} \leq 3n - 6$
- and a green plane graph G_g with
  $m_g \leq 3n - 6 \quad \Rightarrow \quad m \leq m_{rb} + m_g \leq 6n - 12$
- Observe that each green edge joins two faces in G_{rb}.
  $m_g \leq f_{rb}/2 \leq (2n - 4)/2 = n - 2$
  $\Rightarrow \quad m = m_{rb} + m_g \leq 3n - 6 + n - 2 = 4n - 8$

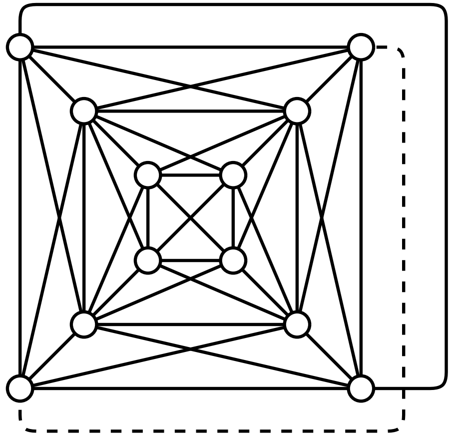


Lower-bound construction:
$2n - 4$ edges
$n - 2$ faces
Edges per face: 2 edges
Total: $4n - 8$ edges

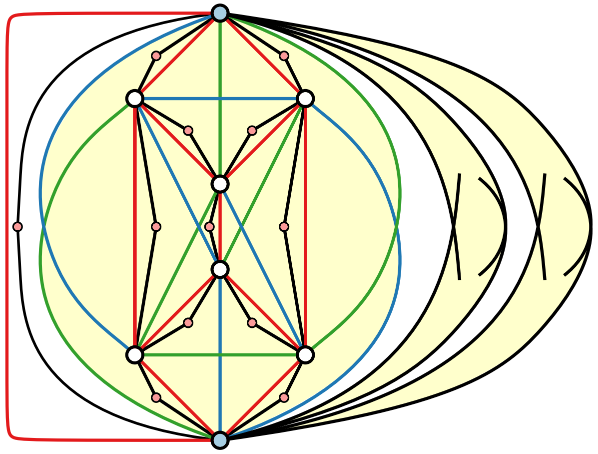
Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.

A 1-planar graph with n vertices is called **optimal** if it has exactly $4n - 8$ edges.
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.



Theorem. [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with n vertices and $45/17n - O(1) \approx 2.65n - O(1)$ edges.



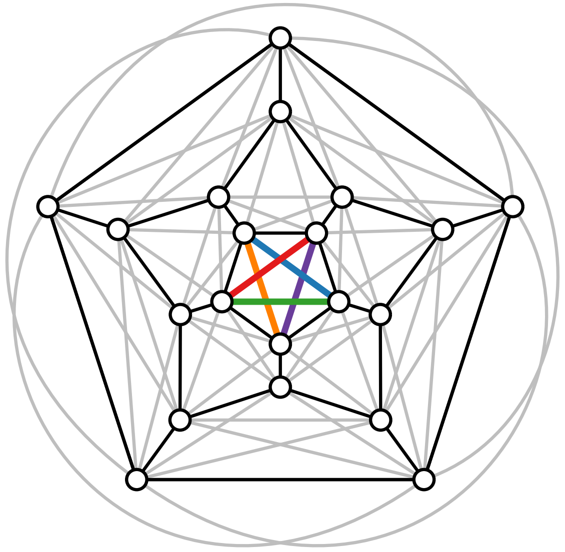
Theorem. [Didimo 2013]
A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n - 9$ edges.

Idea: in a drawing of an optimal 1-planar graph, we cannot realize the crossing on the outer face with two straight-line edges.

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]



optimal 2-planar

$n - m + f = 2$

$m = c \cdot f$?

$m = \frac{5}{2} f$

Planar structure:

$\frac{5}{3}(n-2)$ edges

$\frac{2}{3}(n-2)$ faces

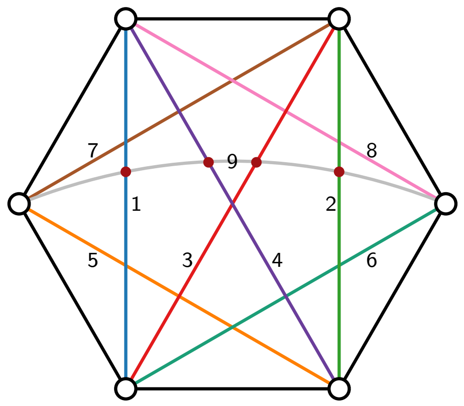
Edges per face: 5 edges

Total: $5(n-2)$ edges

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]

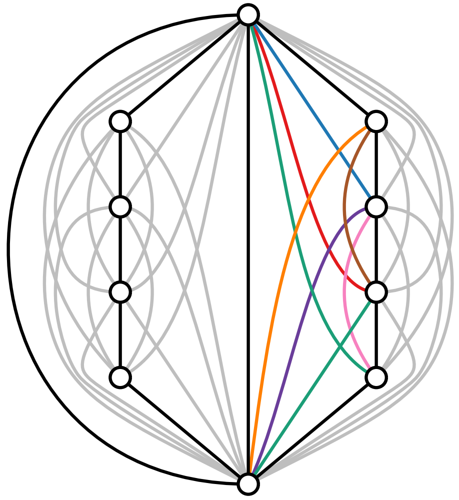


optimal 3-planar

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]



optimal 3-planar

Planar structure:
$\frac{3}{2}(n-2)$ edges
$\frac{1}{2}(n-2)$ faces

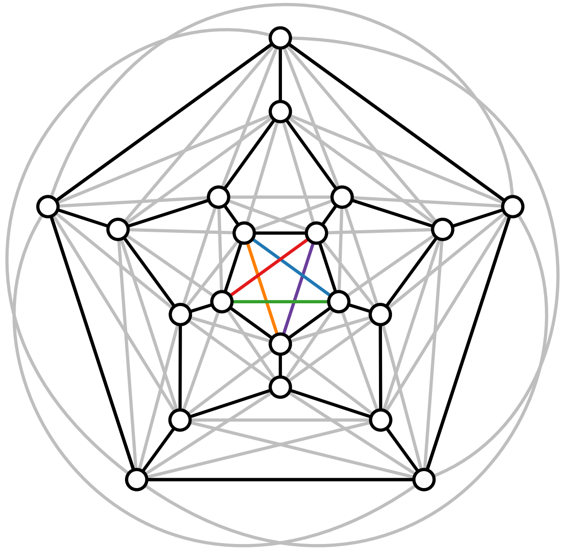
Edges per face: 8 edges

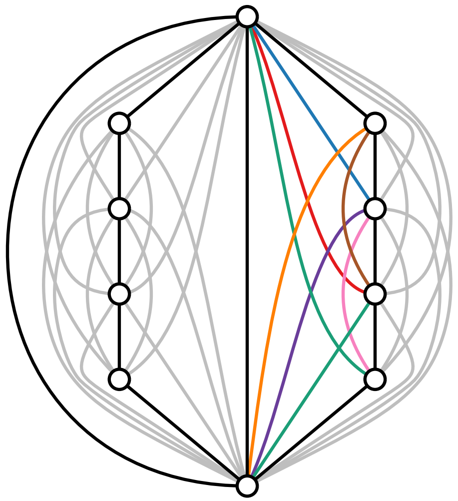
Total: $5.5(n-2)$ edges

Density of k-Planar Graphs

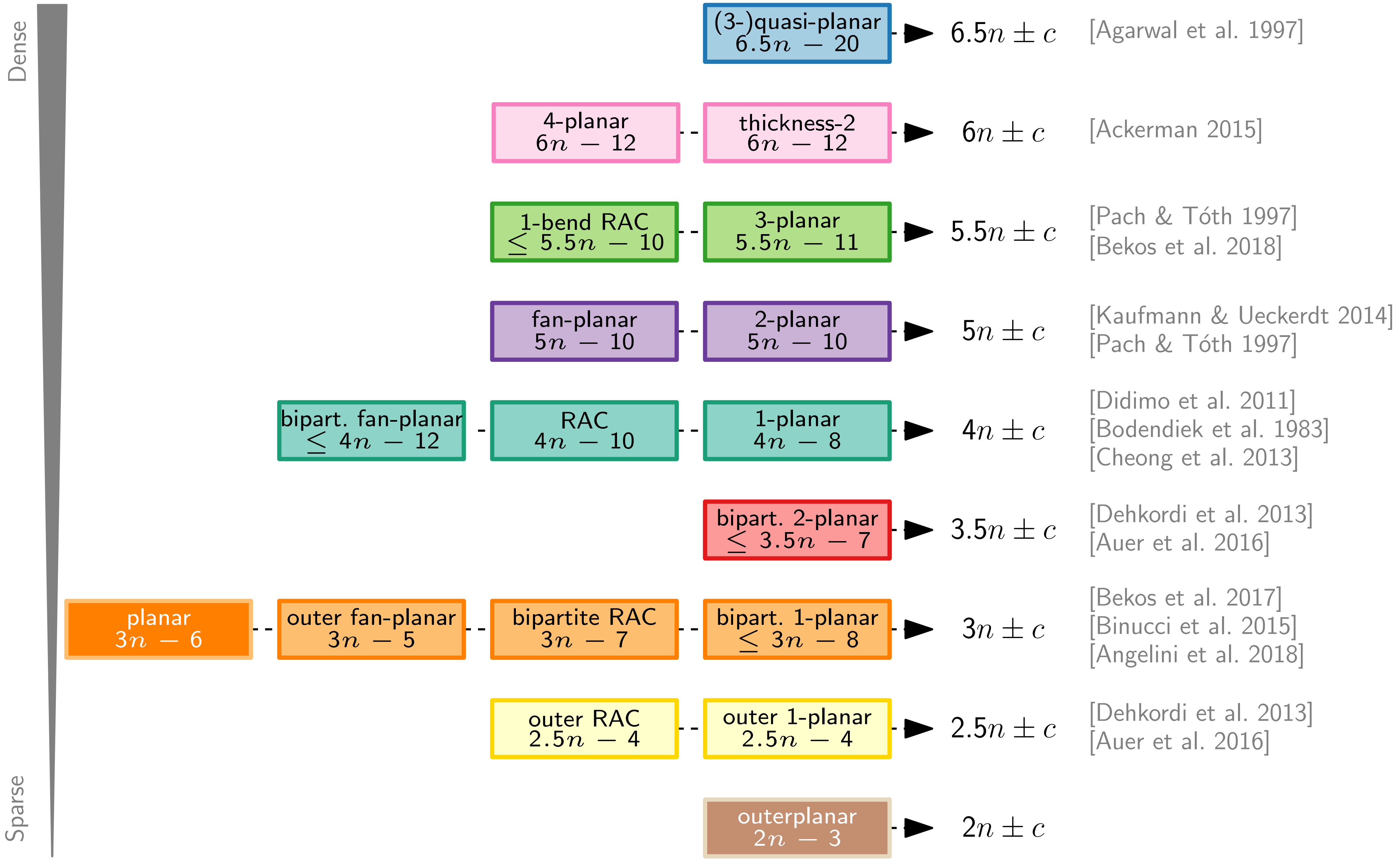
Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]
2	$5(n-2)$	[Pach and Tóth 1997]
3	$5.5(n-2)$	[Pach et al. 2006]
4	$6(n-2)$	[Ackerman 2015]
> 4	$4.108\sqrt{kn}$	[Pach and Tóth 1997]


optimal 2-planar


optimal 3-planar

GD Beyond Planarity: a Hierarchy



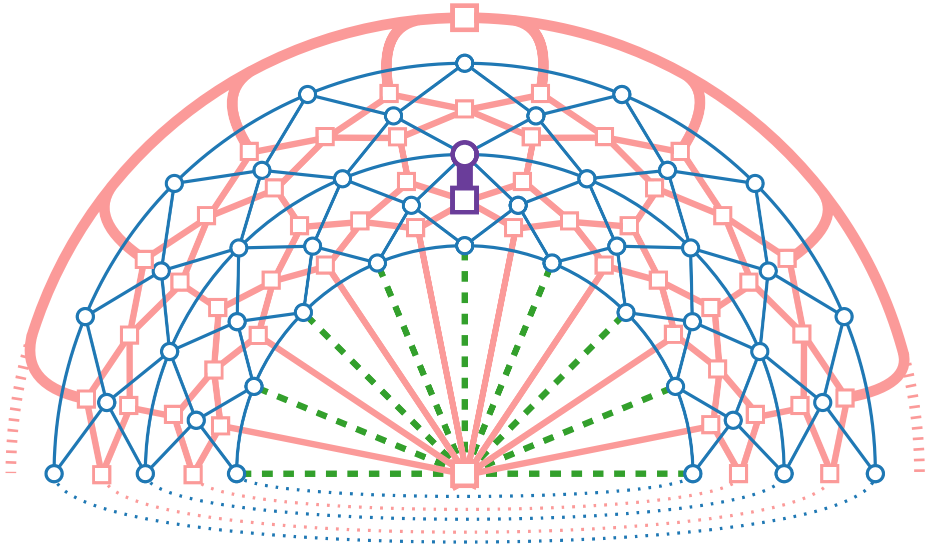
Crossing Numbers

The **k-planar crossing number** $\mathrm{cr}_{k\text{-pl}}(G)$ of a k-planar graph G is the number of crossings required in any k-planar drawing of G.

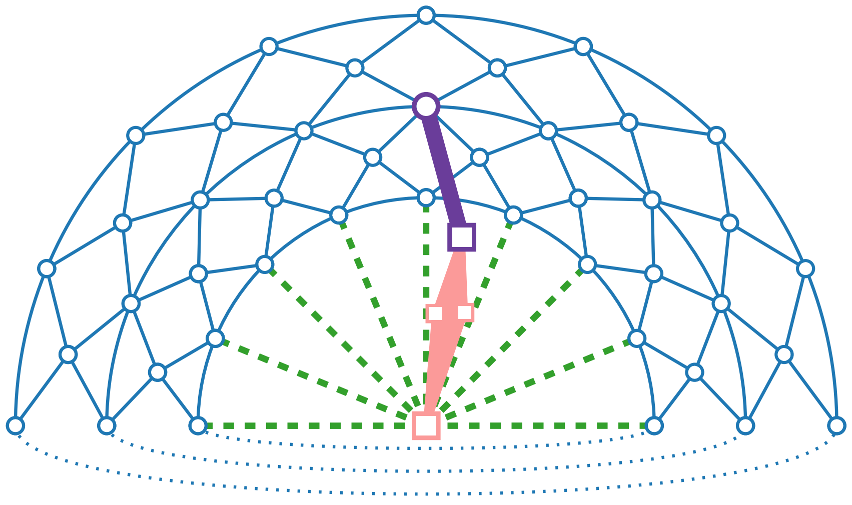
- $\mathrm{cr}_{1\text{-pl}}(G) \leq n - 2$ (there are at most $n-2$ green edges in the coloring of Theorem 1)
- $\mathrm{cr}(G) = 1 \Rightarrow \mathrm{cr}_{1\text{-pl}}(G) = 1$

Theorem. [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph G with $n = 11\ell + 2$ vertices such that $\mathrm{cr}(G) = 2$ and $\mathrm{cr}_{1\text{-pl}}(G) = n - 2$.

Crossing ratio
$\rho_{1\text{-pl}}(n) = (n-2)/2$



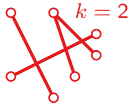
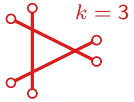
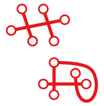
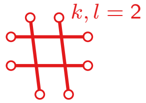
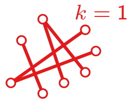
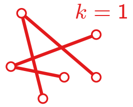
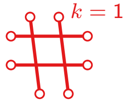
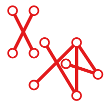
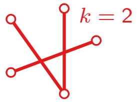
$\mathrm{cr}_{1\text{-pl}}(G) = n - 2$



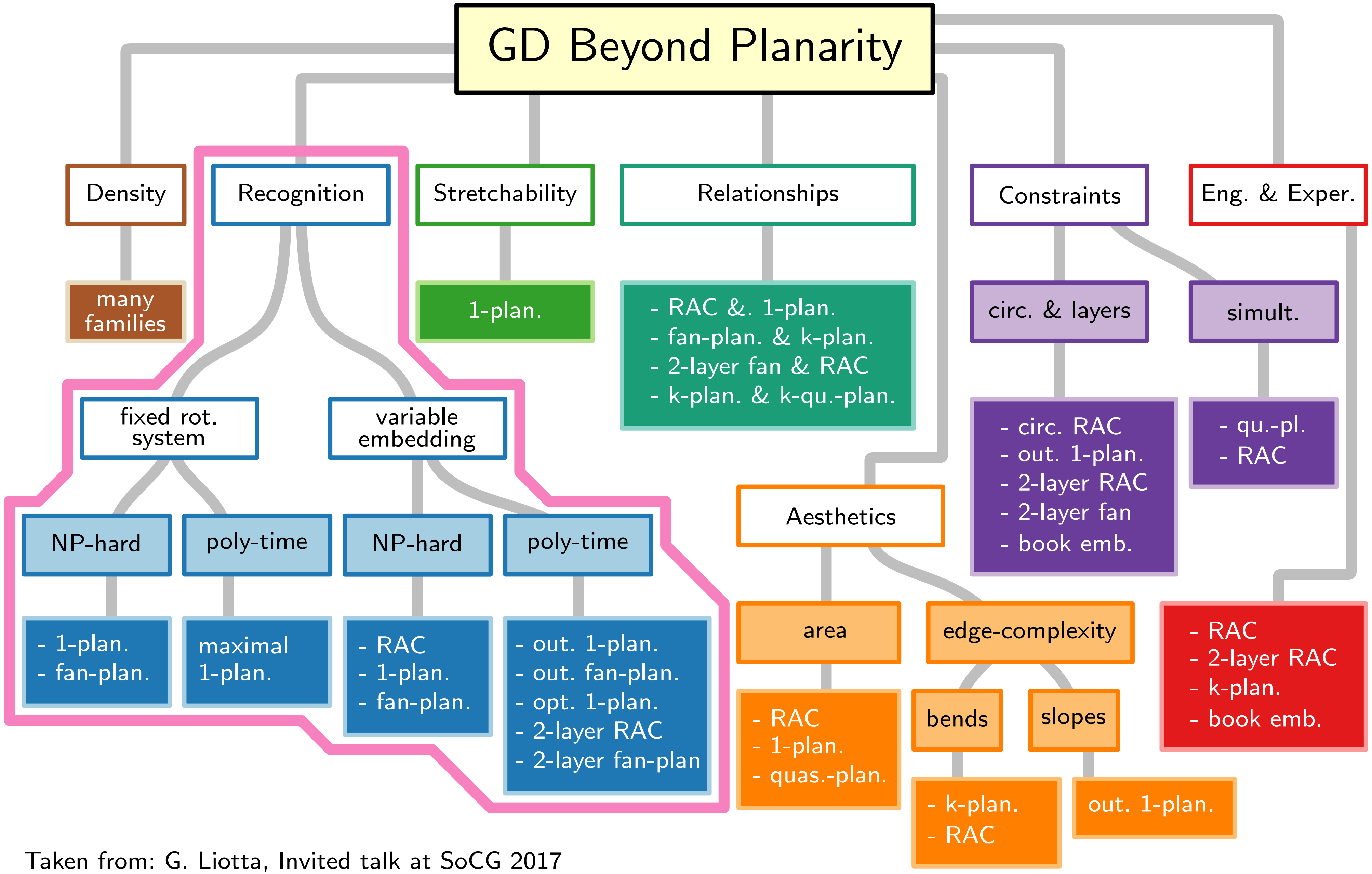
$\mathrm{cr}(G) = 2$

Crossing Ratios

Table from "Crossing Numbers of Beyond-Planar Graphs Revisited" [van Beusekom, Parada & Speckmann 2021]

Family	Forbidden Configurations	Lower	Upper
k-planar	An edge crossed more than k times 	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(k\sqrt{k}n)}$
k-quasi-planar	k pairwise crossing edges 	$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different "side" 	$\Omega(n)$	$O(n^2)$
(k, l)-grid-free	Set of k edges such that each edge crosses each edge from a set of l edges. 	$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k, l)n^2$
k-gap-planar	More than k crossings mapped to an edge in an optimal mapping 	$\boldsymbol{\Omega(n/k^3)}$	$\boldsymbol{O(k\sqrt{k}n)}$
Skewness-k	Set of crossings not covered by at most k edges 	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(kn + k^2)}$
k-apex	Set of crossings not covered by at most k vertices 	$\Omega(n/k)$	$O(k^2n^2 + k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge 	$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$
k-fan-crossing-free	An edge that crosses k adjacent edges 	$\boldsymbol{\Omega(n^2/k^3)}$	$\boldsymbol{O(k^2n^2)}$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$ 	$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017

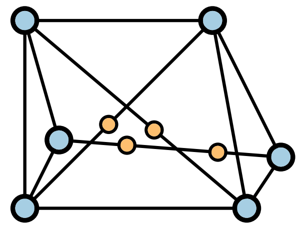
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

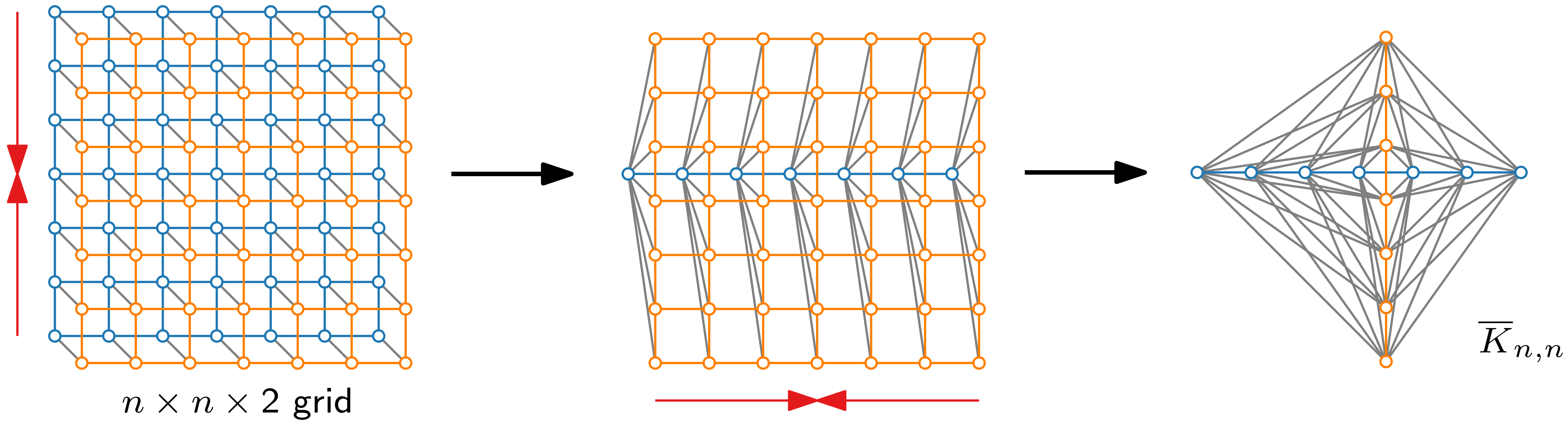
Minors of 1-Planar Graphs

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$ neither K_5 nor $K_{3,3}$ minor of G

Theorem. [Chen & Kouno 2005]
The class of 1-planar graphs is not closed under edge contraction.

For every graph there is a 1-planar subdivision.





Theorem. [Korzhik & Mohar 2013]
For any n, there exist $\Omega(2^n)$ distinct n-vertex graphs that are not 1-planar but all their proper subgraphs are 1-planar.

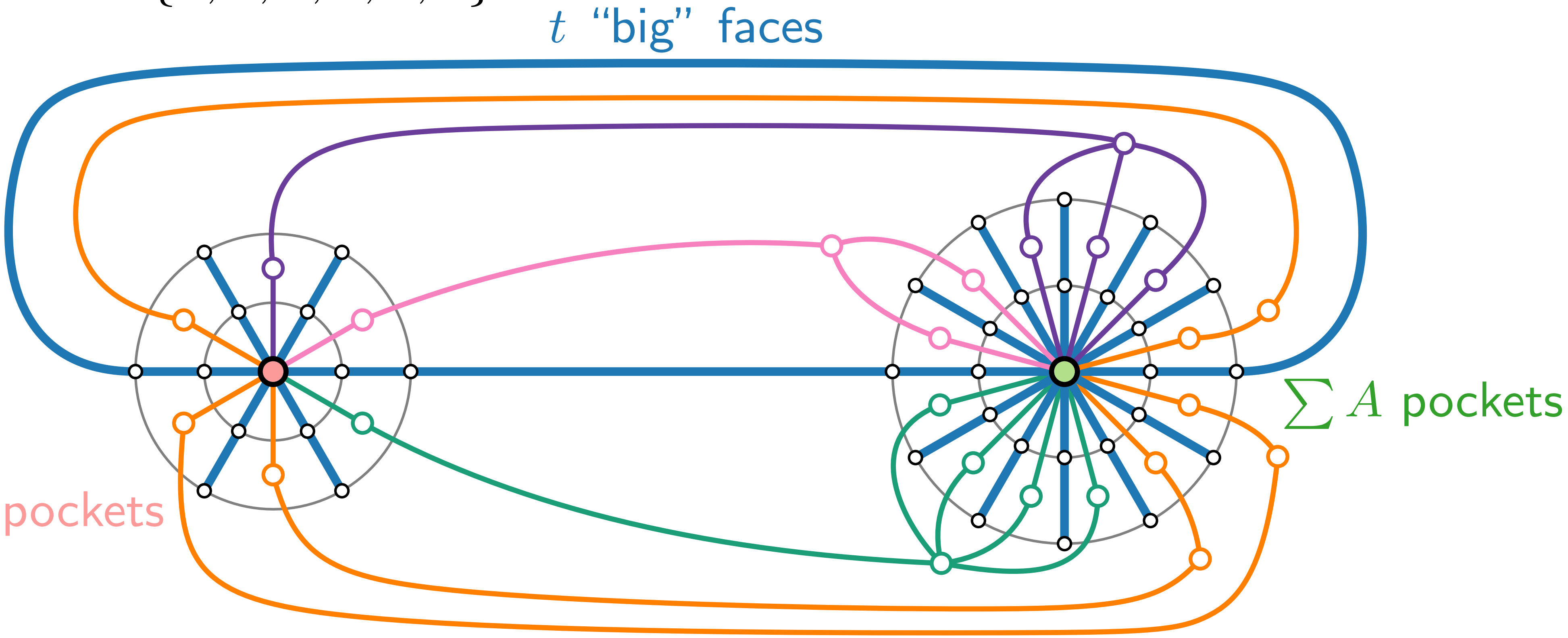
Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

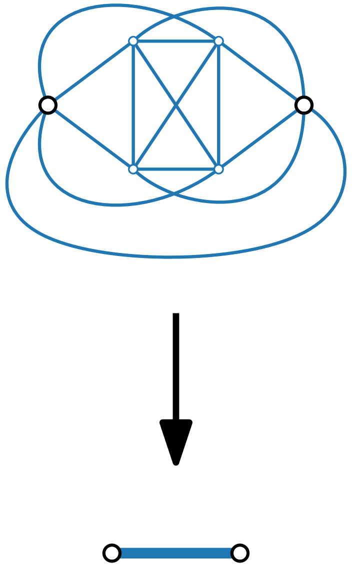
Proof Idea.
Reduction from 3-Partition.

Given a multiset $A = \{a_1, a_2, \dots, a_{3t}\}$ of $3t$ numbers, partition the numbers into t triplets such that the sum of every triplet is the same.

$$A = \{\overbrace{1, 3, 2}^{6}, \overbrace{4, 1, 1}^{6}\}$$



The only 1-planar embedding of K_6:



(cannot be crossed)

Recognition of 1-Planar Graphs

Theorem. [Grigoriev & Bodlaender 2007, Korzhik & Mohar 2013]
Testing 1-planarity is NP-complete.

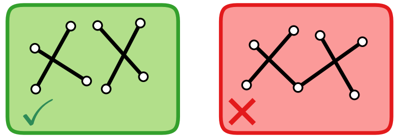
Theorem. [Cabello & Mohar 2013]
Testing 1-planarity is NP-complete –
even for almost planar graphs, i.e., planar graphs plus one edge.

Theorem. [Bannister, Cabello & Eppstein 2018]
Testing 1-planarity is NP-complete –
even for graphs of bounded bandwidth (pathwidth, treewidth).

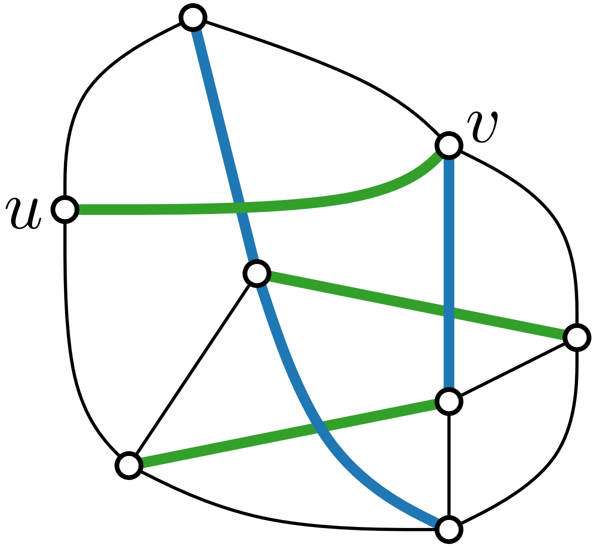
Theorem. [Auer, Brandenburg, Gleißner & Reislhuber 2015]
Testing 1-planarity is NP-complete –
even for 3-connected graphs with a fixed rotation system.

Recognition of IC-Planar Graphs

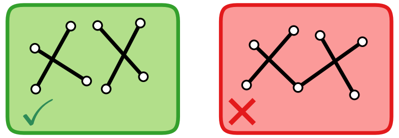
Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.



Proof.
Reduction from 1-planarity testing.



Recognition of IC-Planar Graphs



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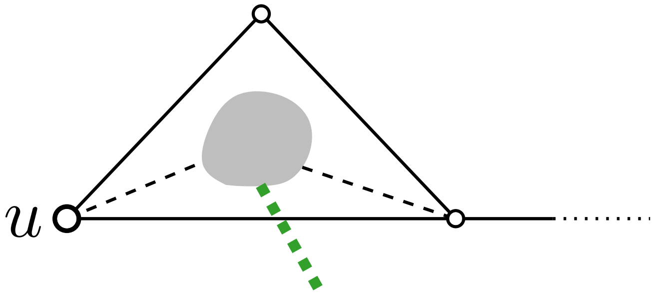
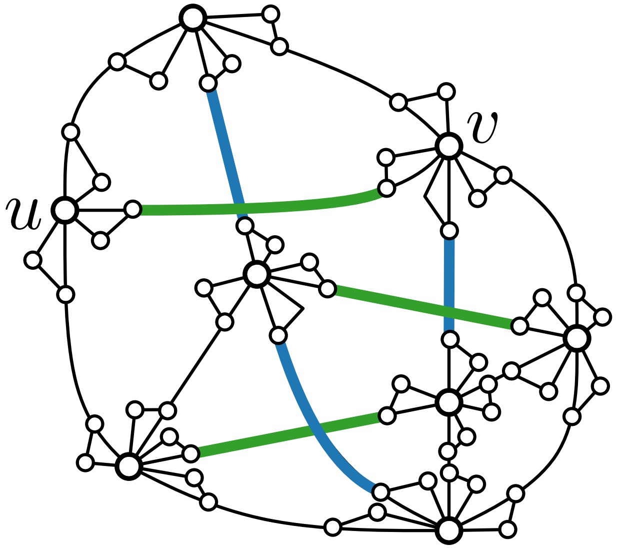
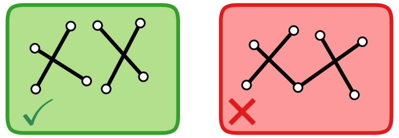




Recognition of IC-Planar Graphs



Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.

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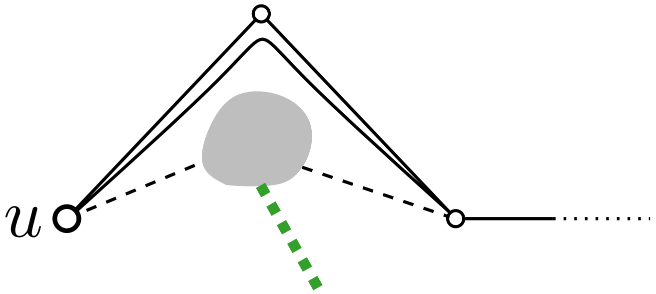
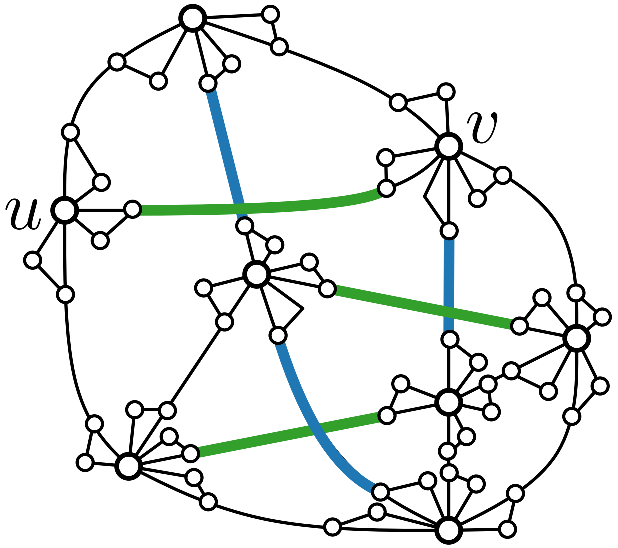
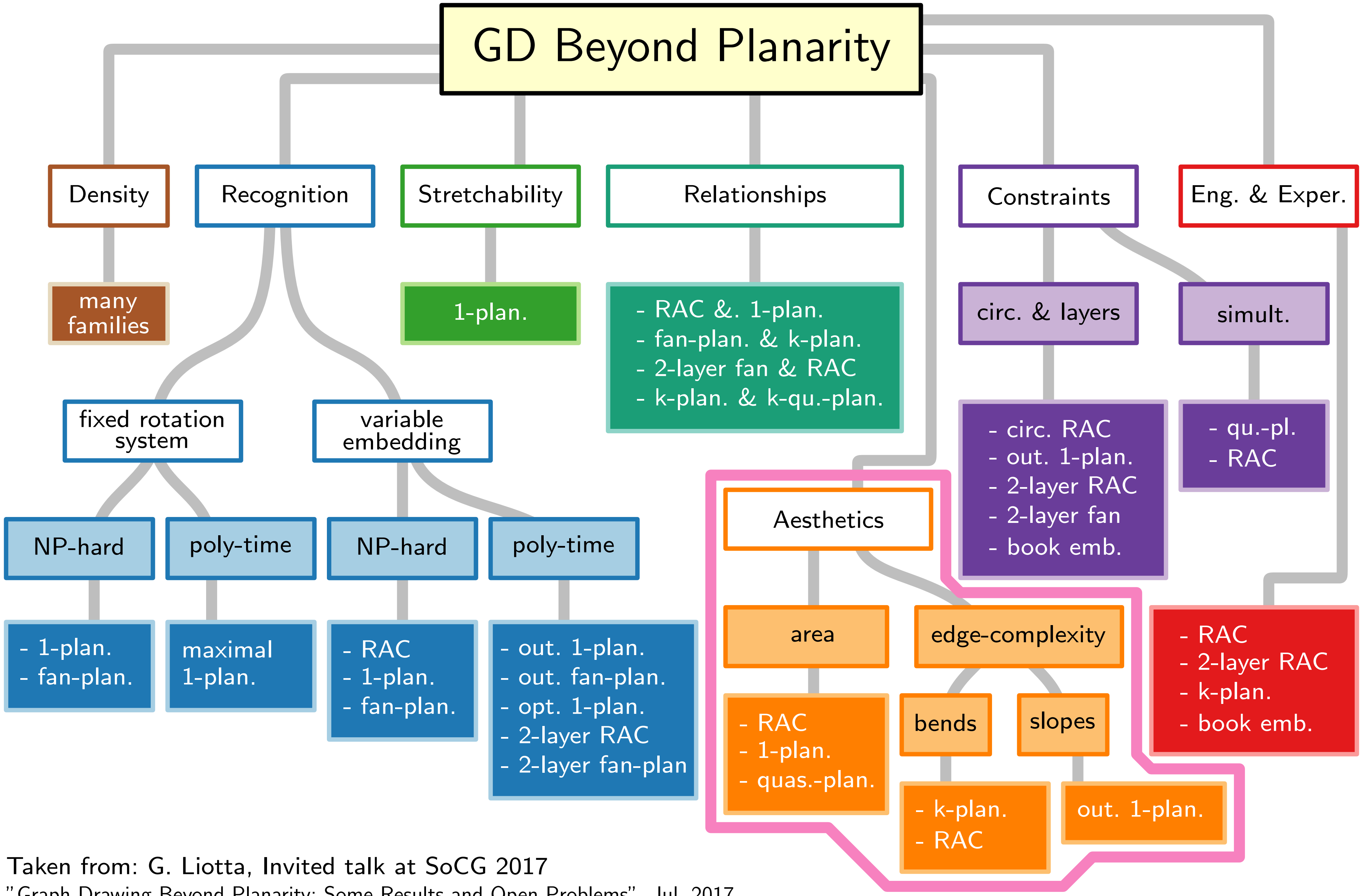




Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete,
even if the rotation system is given.

GD Beyond Planarity: a Taxonomy

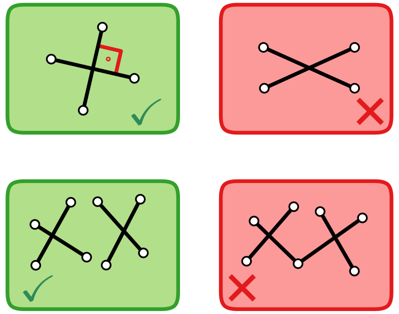


Taken from: G. Liotta, Invited talk at SoCG 2017

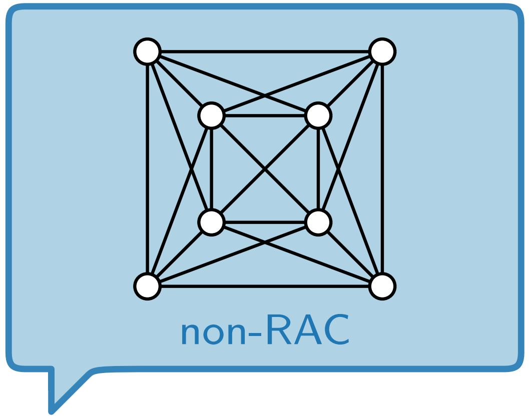
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

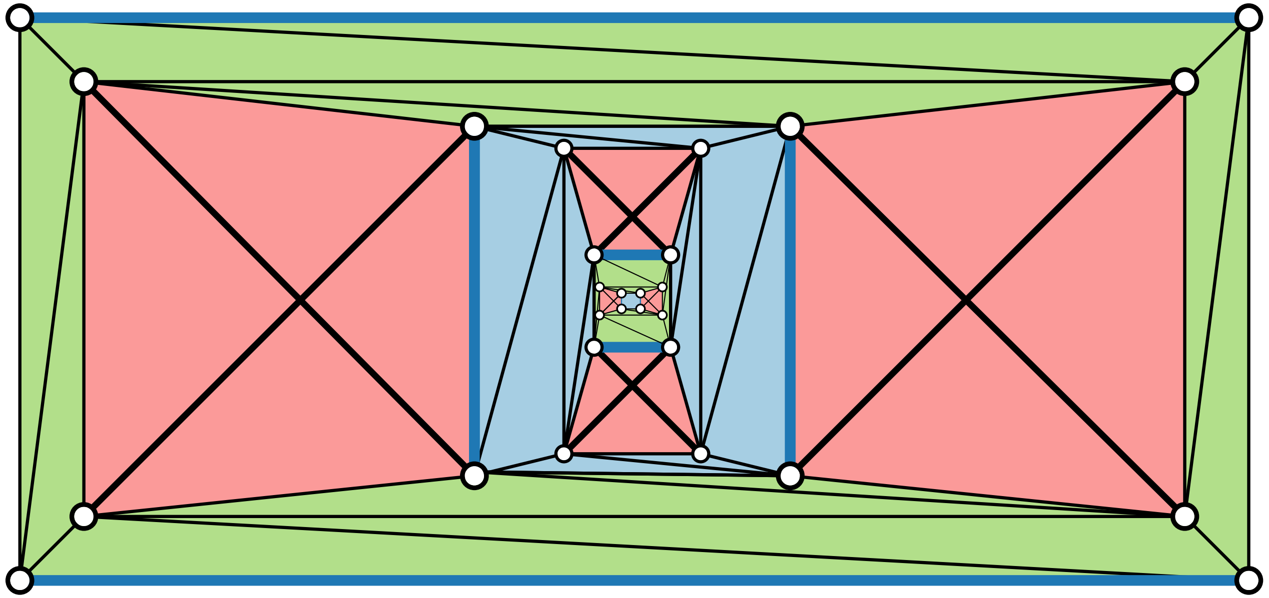
Area of Straight-Line RAC Drawings

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Some IC-planar straight-line RAC drawings require exponential area.



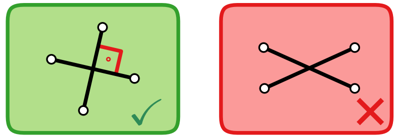
Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Every IC-planar graph has an IC-planar straight-line RAC drawing, and such a drawing can be found in polynomial time.

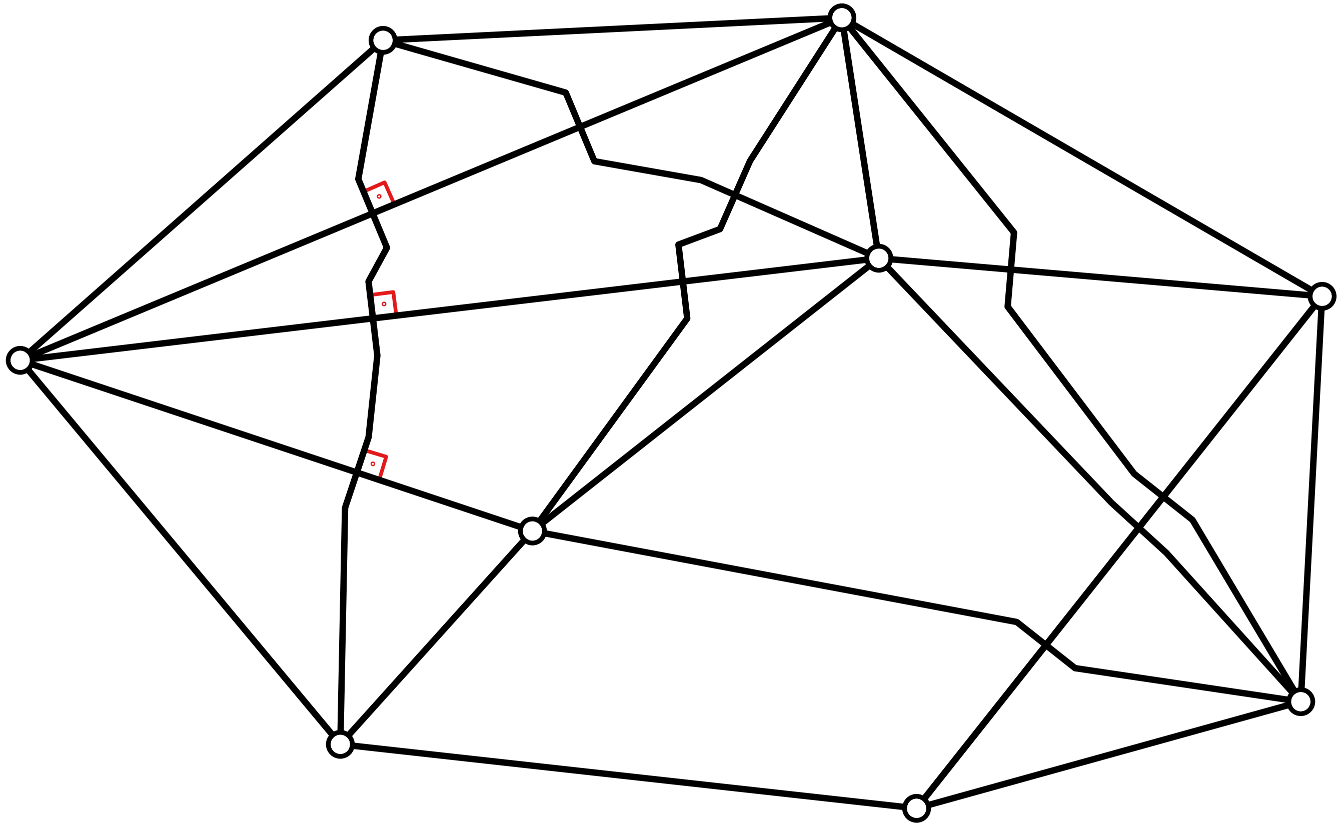




In constrast:
not every 1-planar graph admits a straight-line RAC drawing

RAC Drawings With Enough Bends





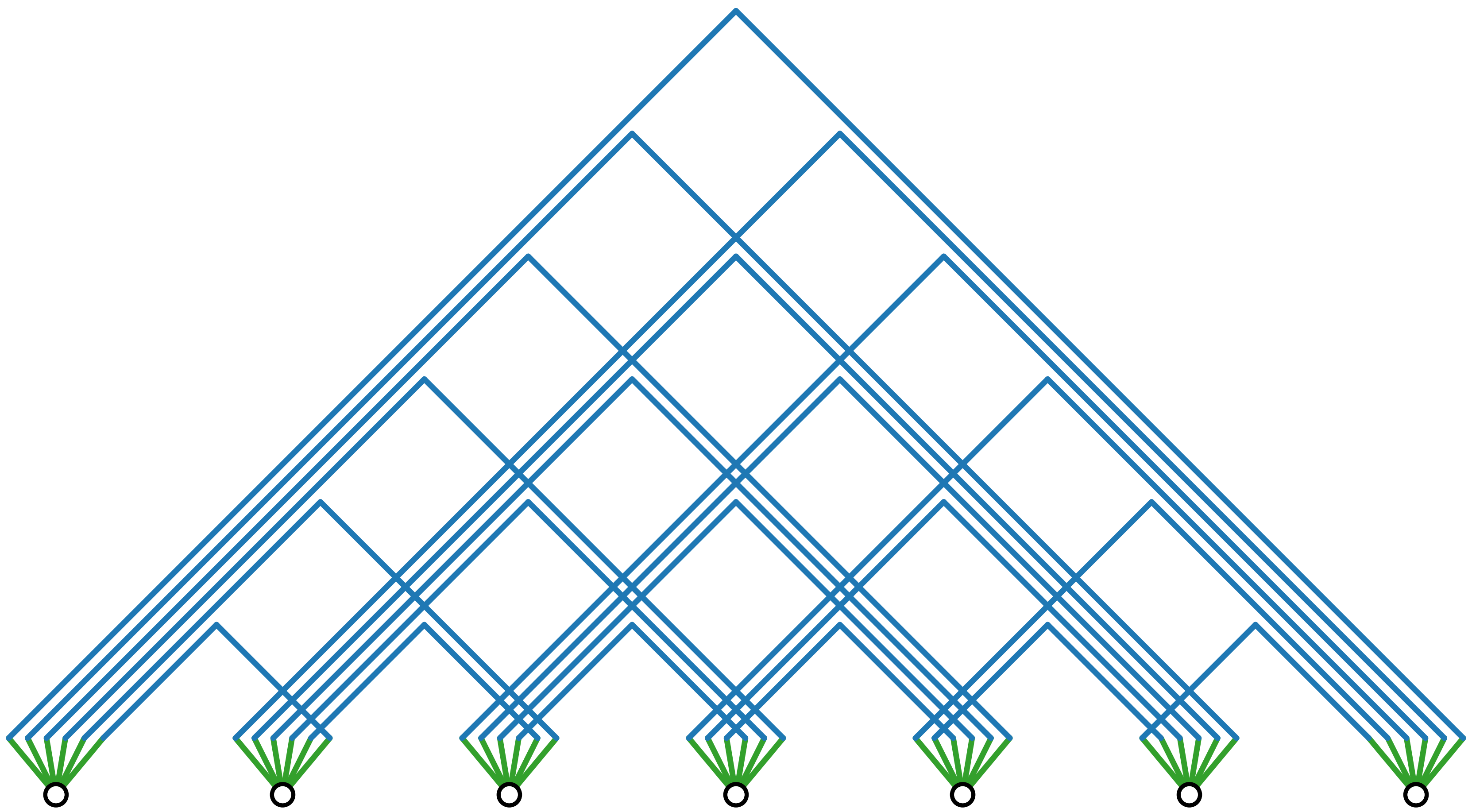
Every graph admits a RAC drawing . . .

. . . if we use enough bends.

How many do we need – in total or per edge?

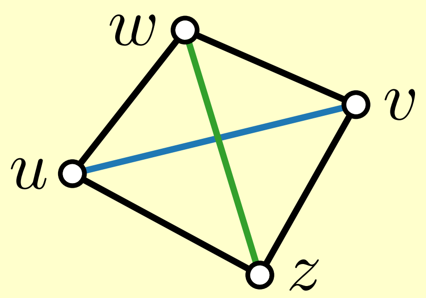
3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most three bends.

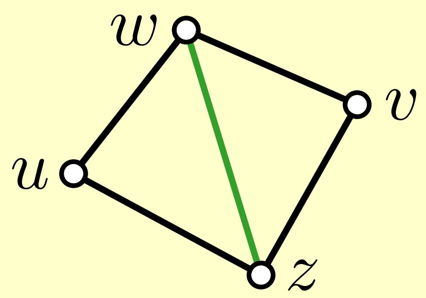


Kite Triangulations

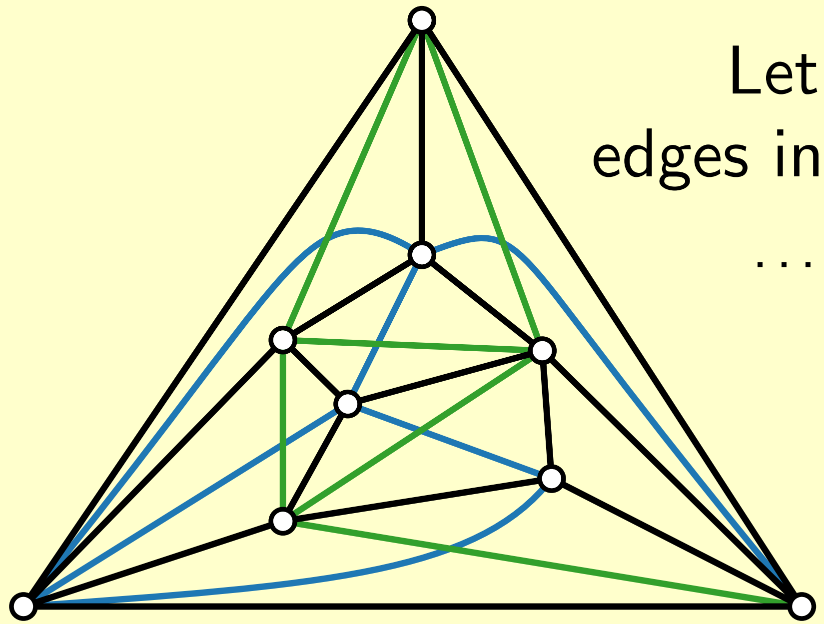
This is a **kite**:



u and v are **opposite** w.r.t. $\{z, w\}$



Let G' be a plane triangulation.



Let $S \subseteq E(G')$ s.t. no two edges in S lie on the same face

. . . and their opposite vertices do not have an edge in $E(G')$.

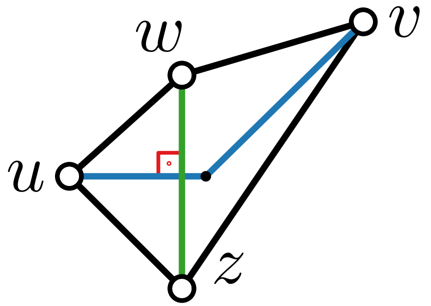
Add set T of edges connecting opposite vertices.

The resulting graph G is a **kite-triangulation**.

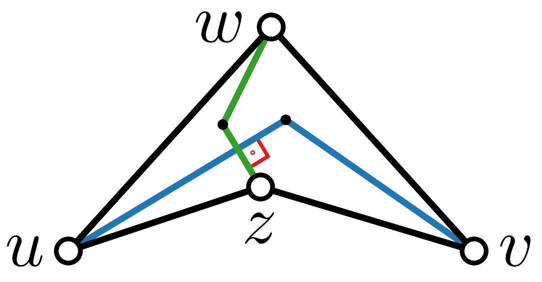
Note: optimal 1-planar graphs $\subsetneq$ kite-triangulations.

Theorem. [Angelini et al. 2011]
Every kite-triangulation G admits a 1-planar 1-bend RAC drawing, which can be constructed in linear time.

Proof.
Let G' be the underlying plane triangulation of G. Let $G'' = G' - S$.
Construct straight-line drawing of G''.
Fill faces as follows:



strictly convex face

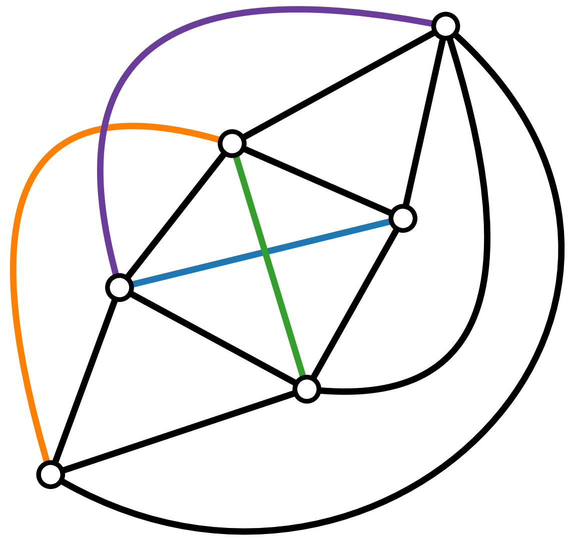


otherwise

1-Planar 1-Bend RAC Drawings

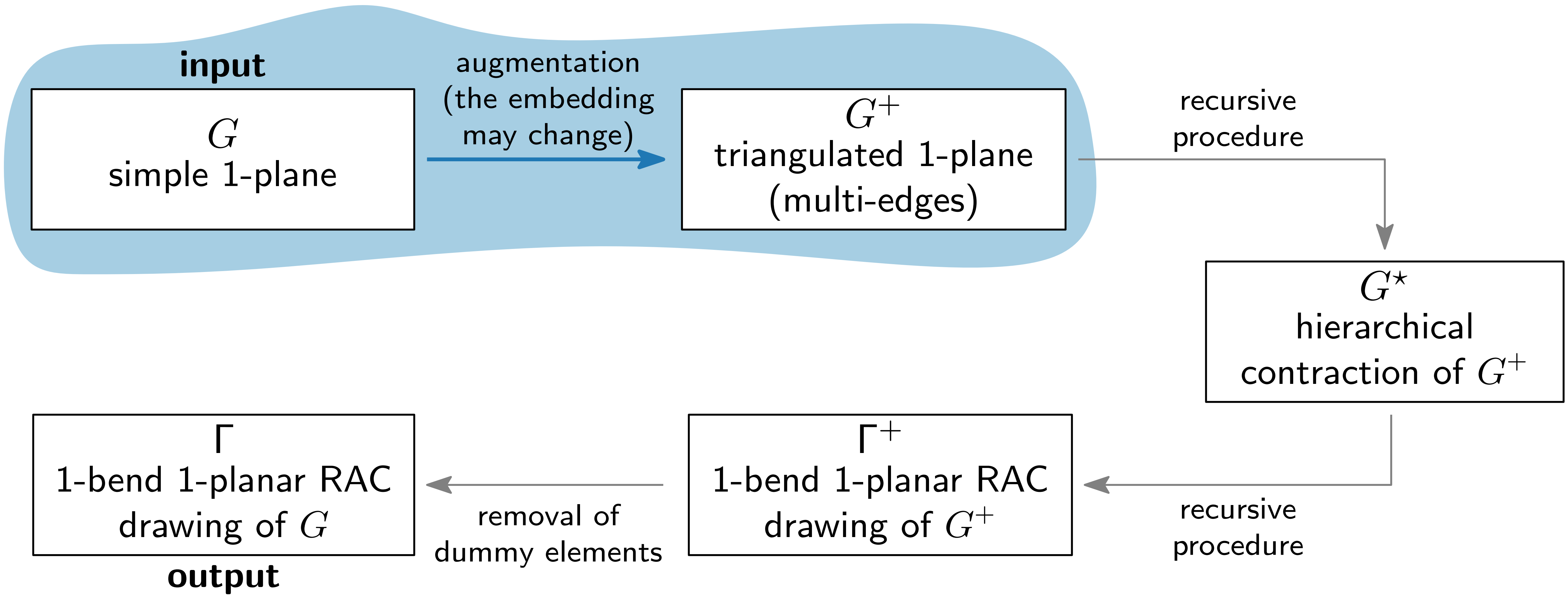
Theorem. [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph G admits a 1-planar 1-bend RAC drawing.
If a 1-planar embedding of G is given as part of the input, such a drawing can be computed in linear time.

Observation.
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of G forms an empty kite, except for at most one pair if their crossing point is on the outer face of G.



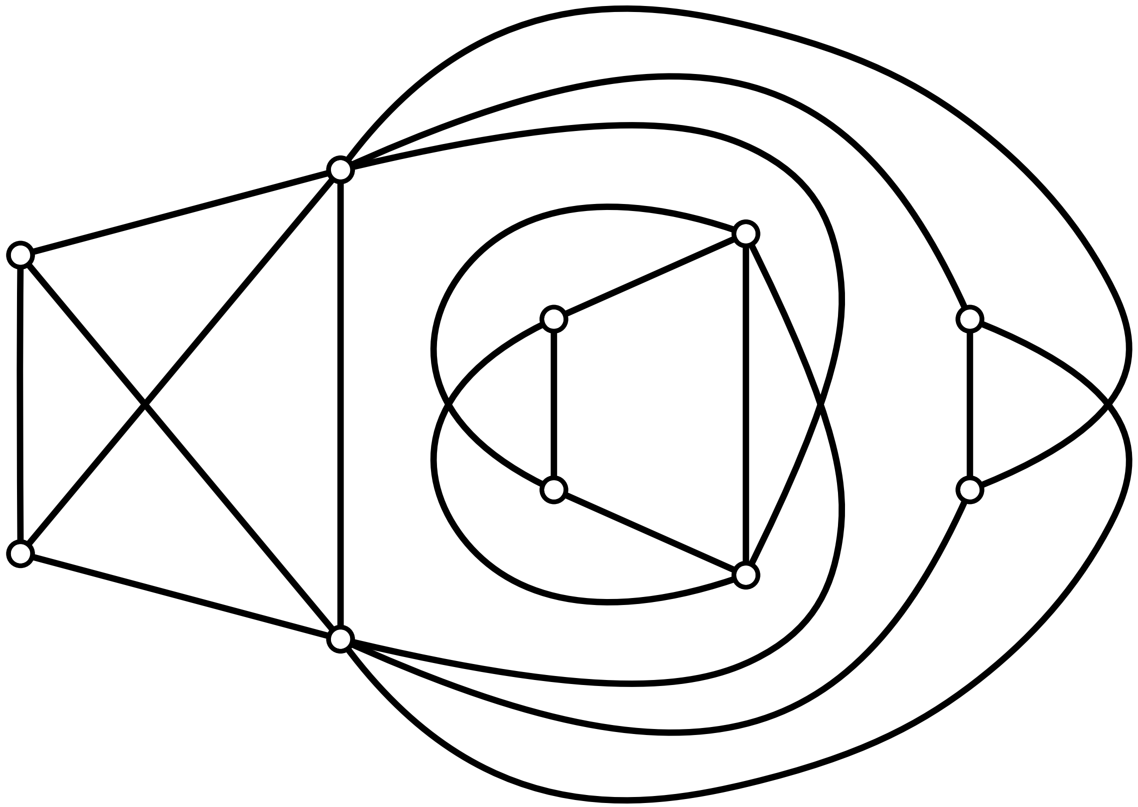
Theorem. [Chiba, Yamanouchi & Nishizeki 1984]
For every 2-connected plane graph G with outer face C_k and every convex k-gon P, there is a strictly convex planar straight-line drawing of G whose outer face coincides with P. Such a drawing can be computed in linear time.

Algorithm Outline



Algorithm Step 1: Augmentation

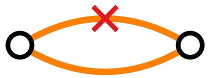
G: simple 1-plane graph



Algorithm Step 1: Augmentation

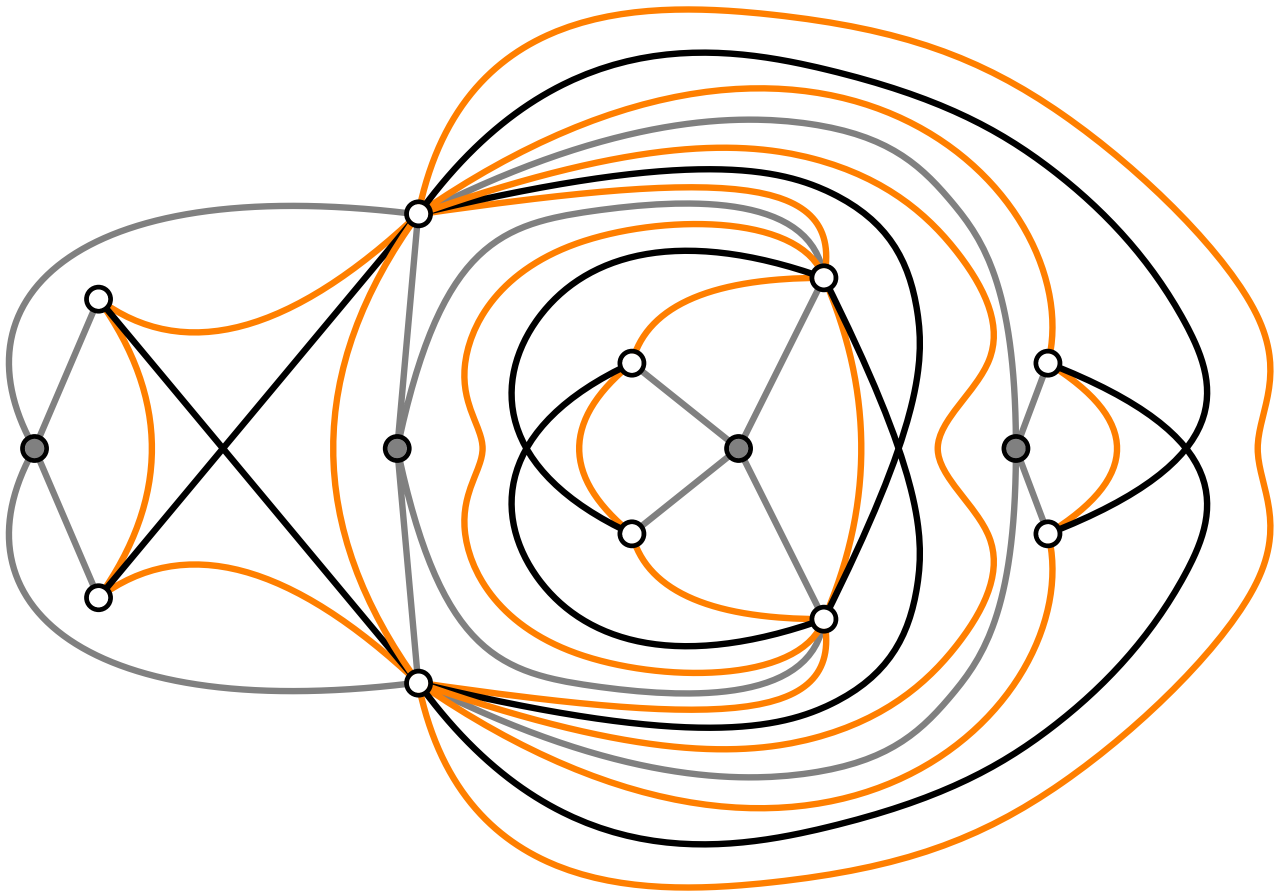
1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to G.

3. Remove one (multiple) edge from each face of degree two (if any). 

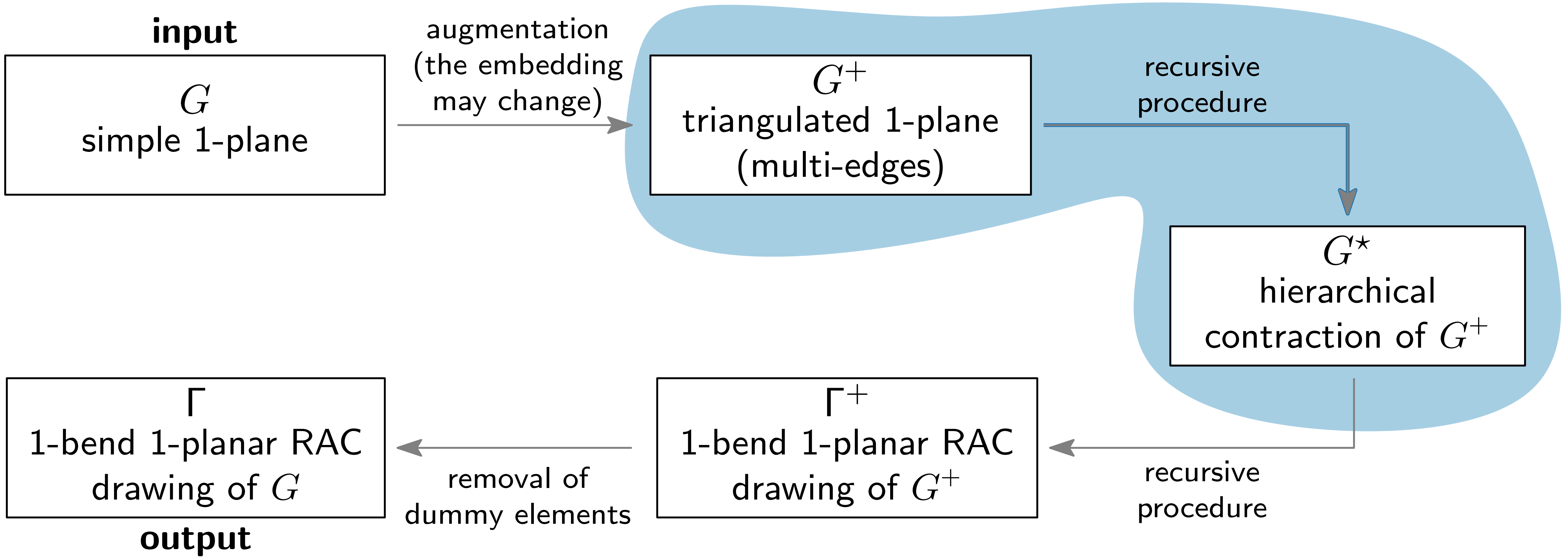
4. Triangulate faces of degree > 3 by inserting a star inside them.

G: simple 1-plane graph → G^+: triangulated 1-plane (possibly with multi-edges)



Note that we can still have parallel (orange) edges

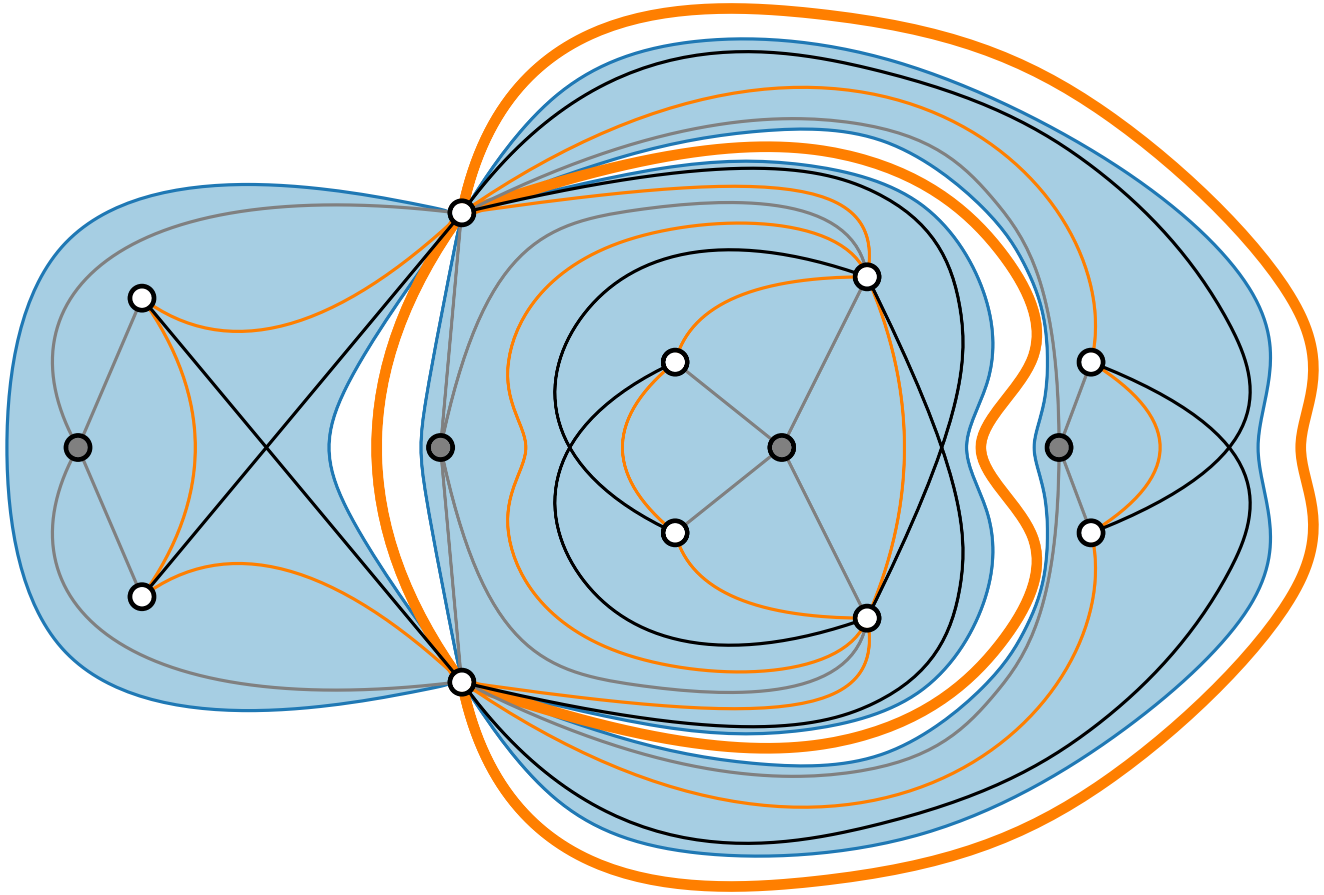
Algorithm Outline



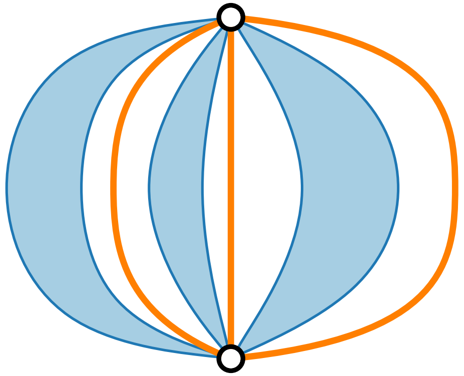
Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



$\Rightarrow$



structure of each separation pair

Algoritm Step 2: Hierarchical Contractions

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triangulated 1-plane
(multi-edges)

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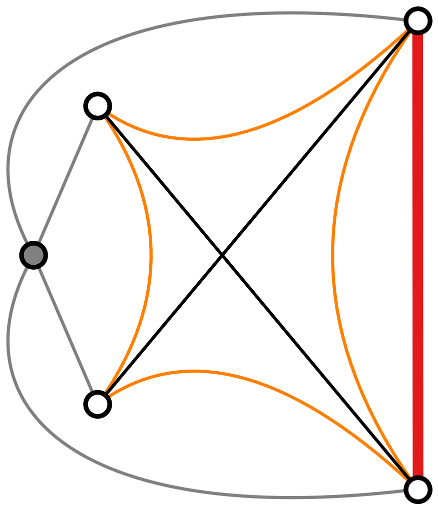
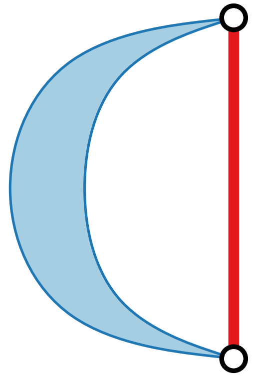


$\Rightarrow$



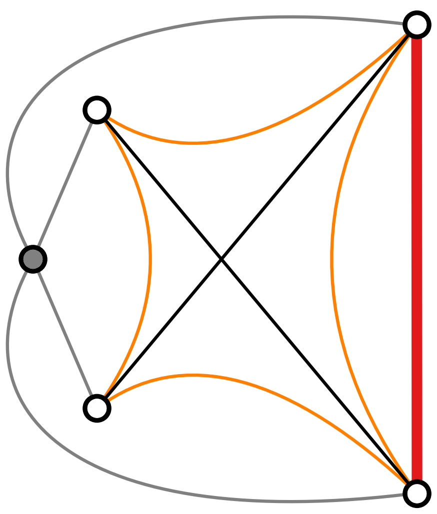
structure of each separation pair

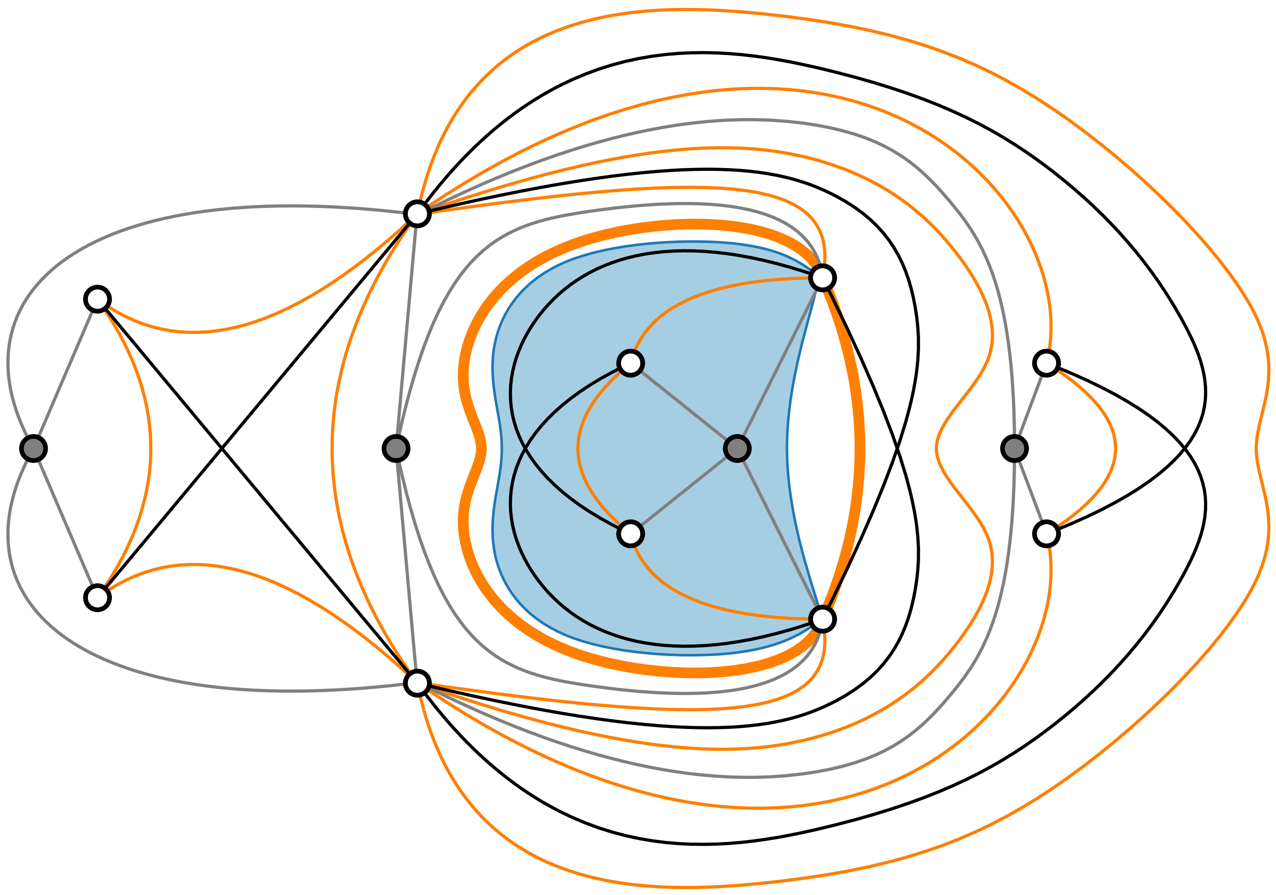
Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

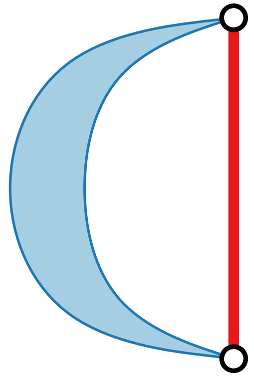
G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites





$\Rightarrow$



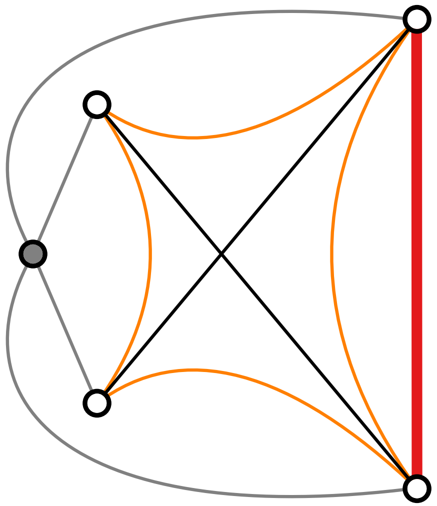
structure of each separation pair

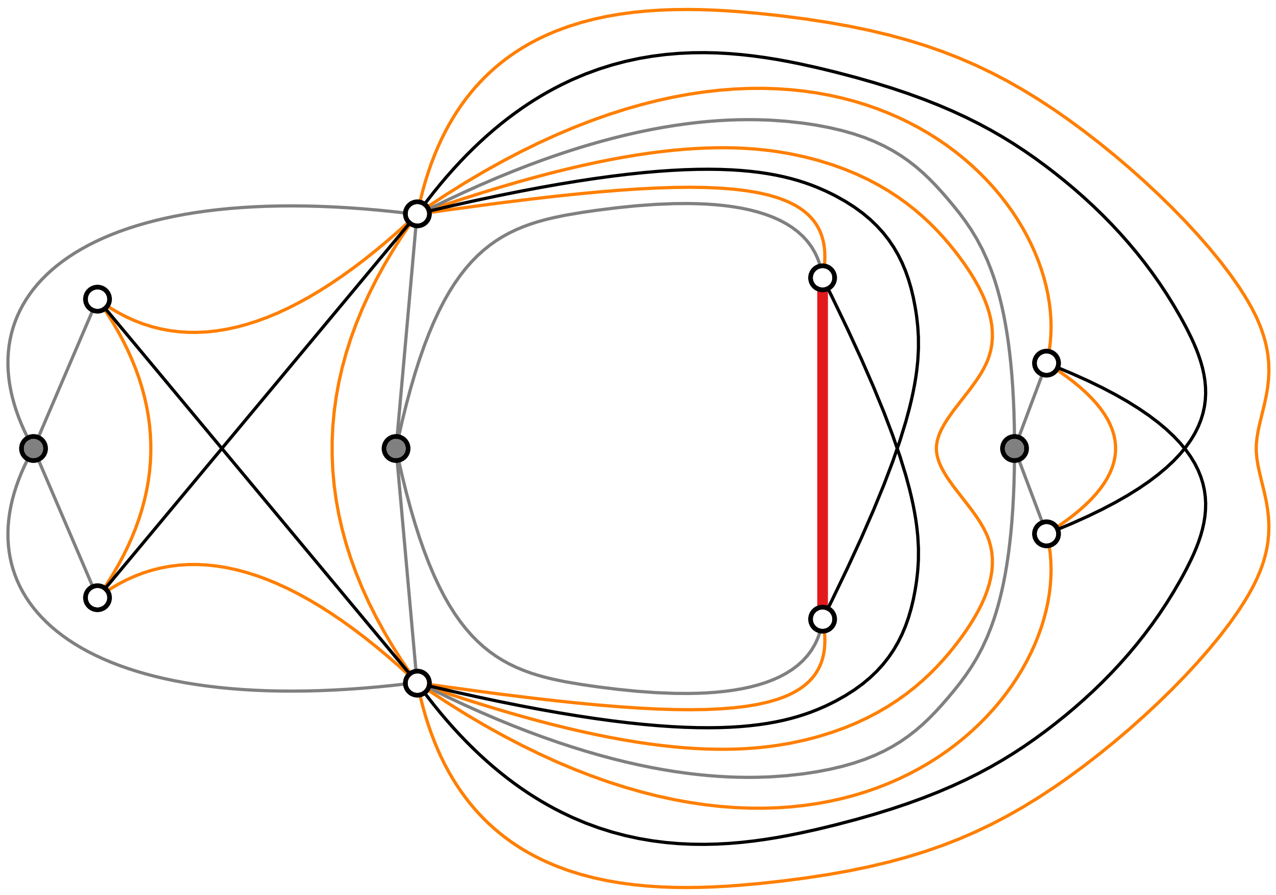
Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites





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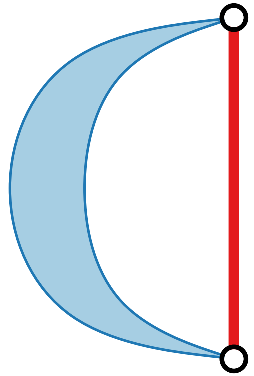
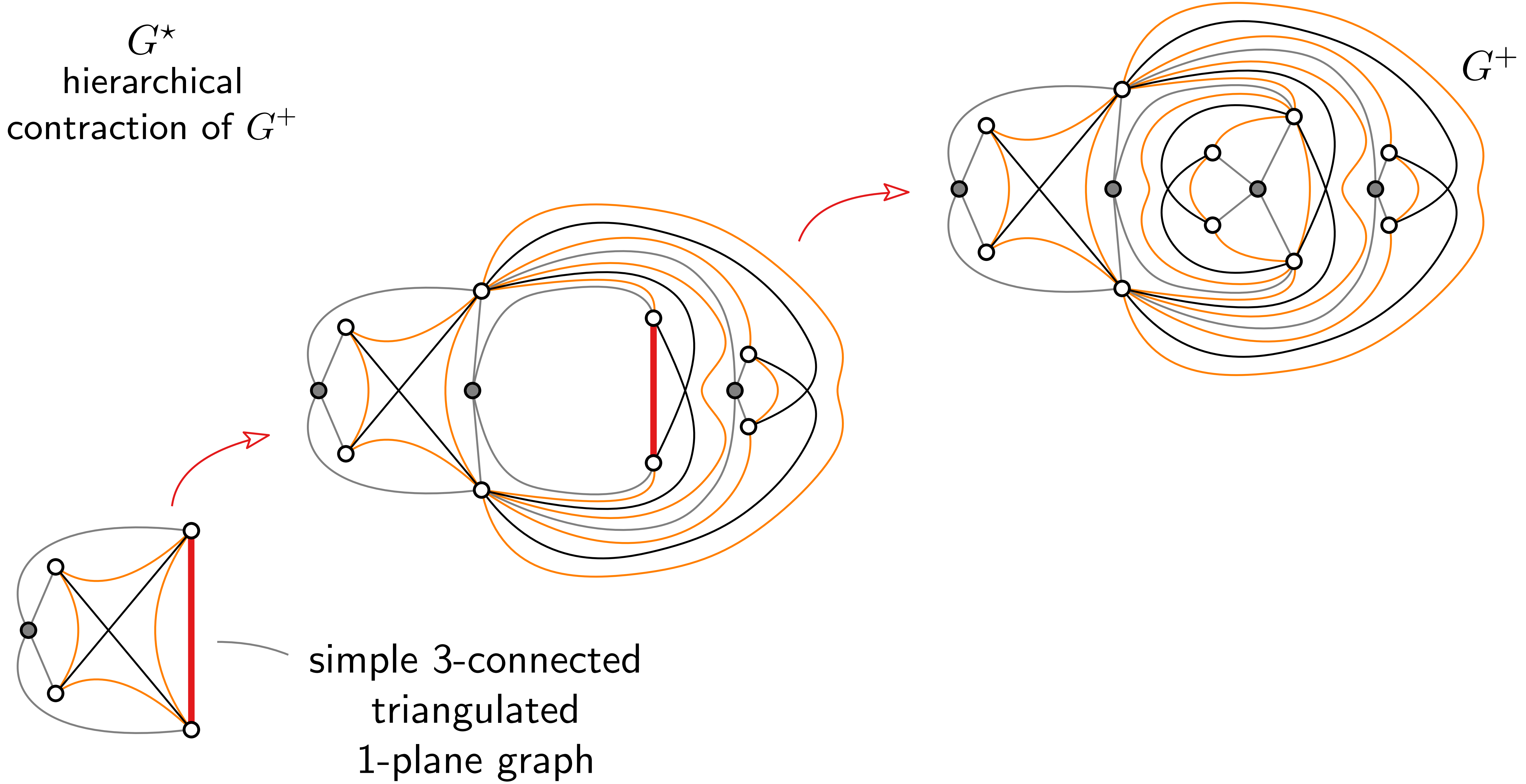


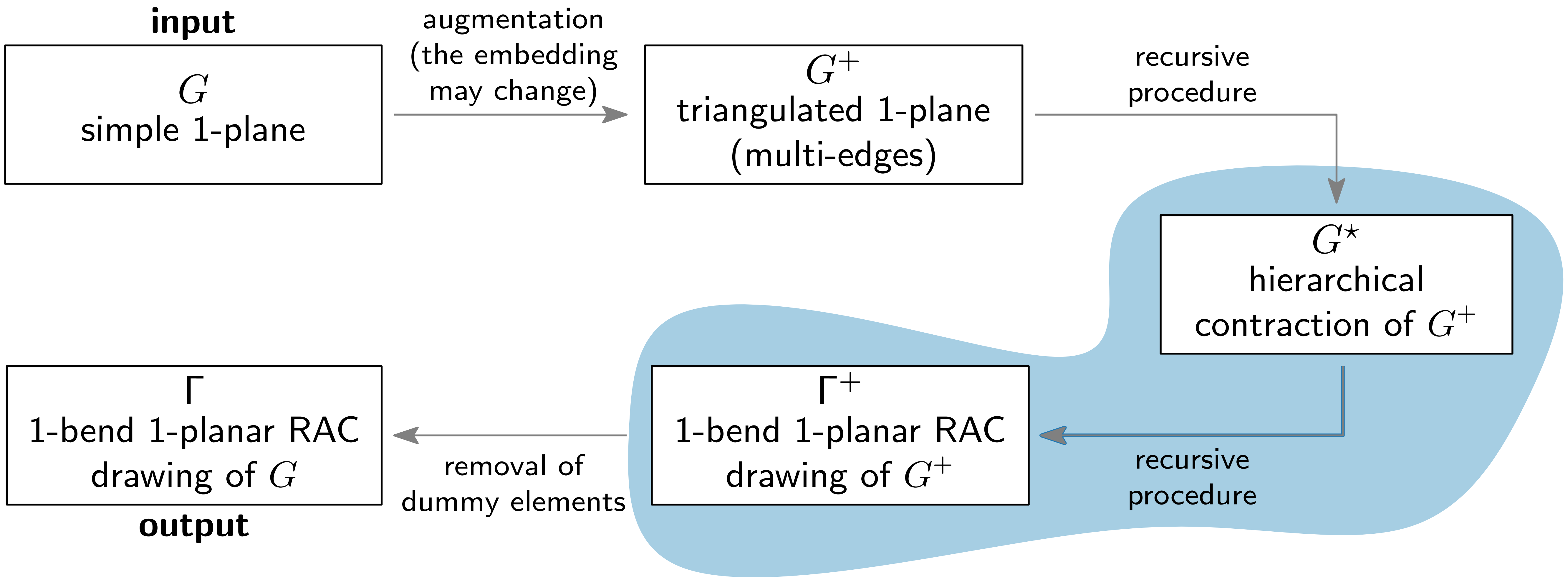
structure of each separation pair

Contract all inner components of each separation pair into a thick edge.

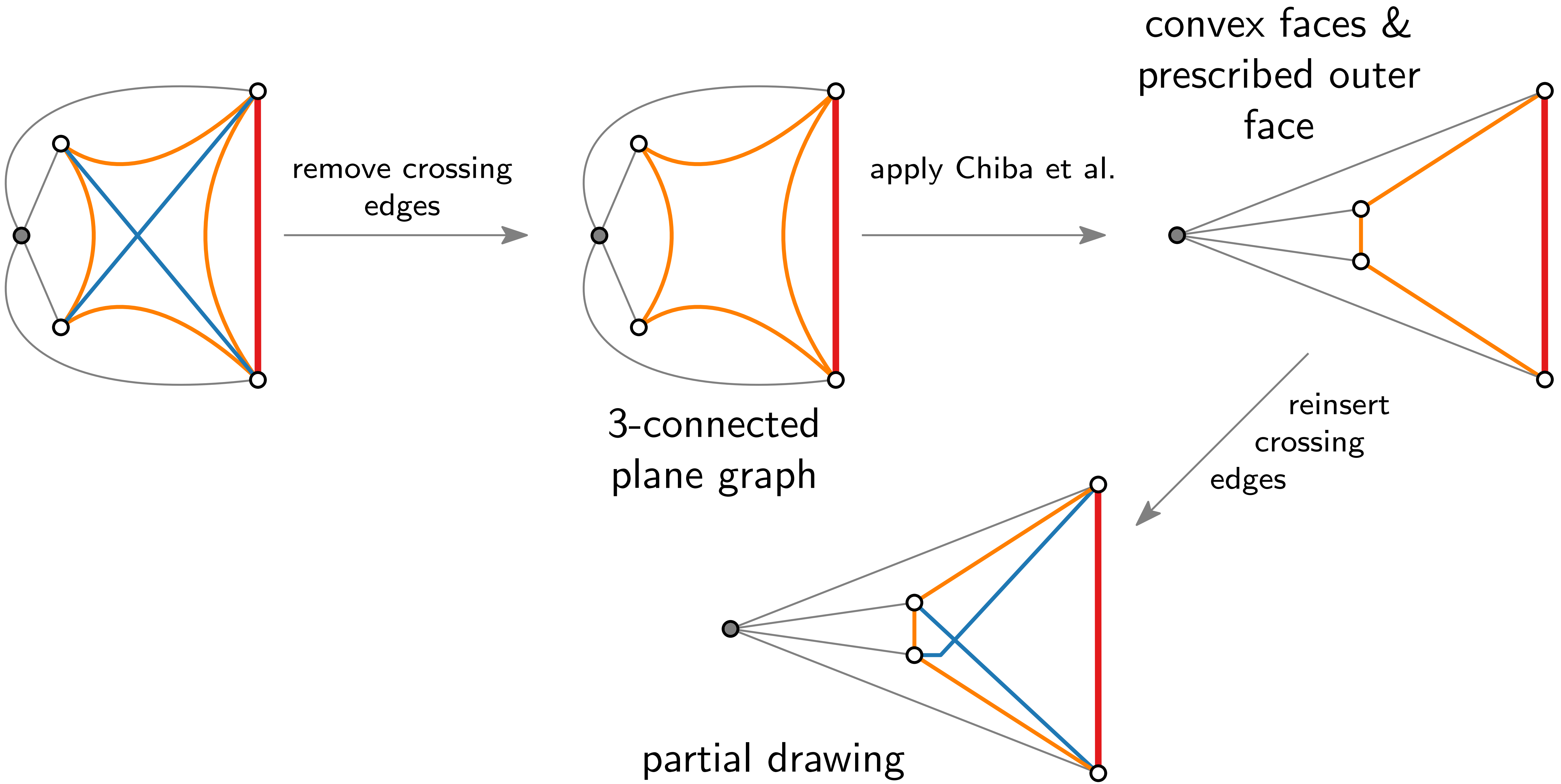
Algoritm Step 2: Hierarchical Contractions



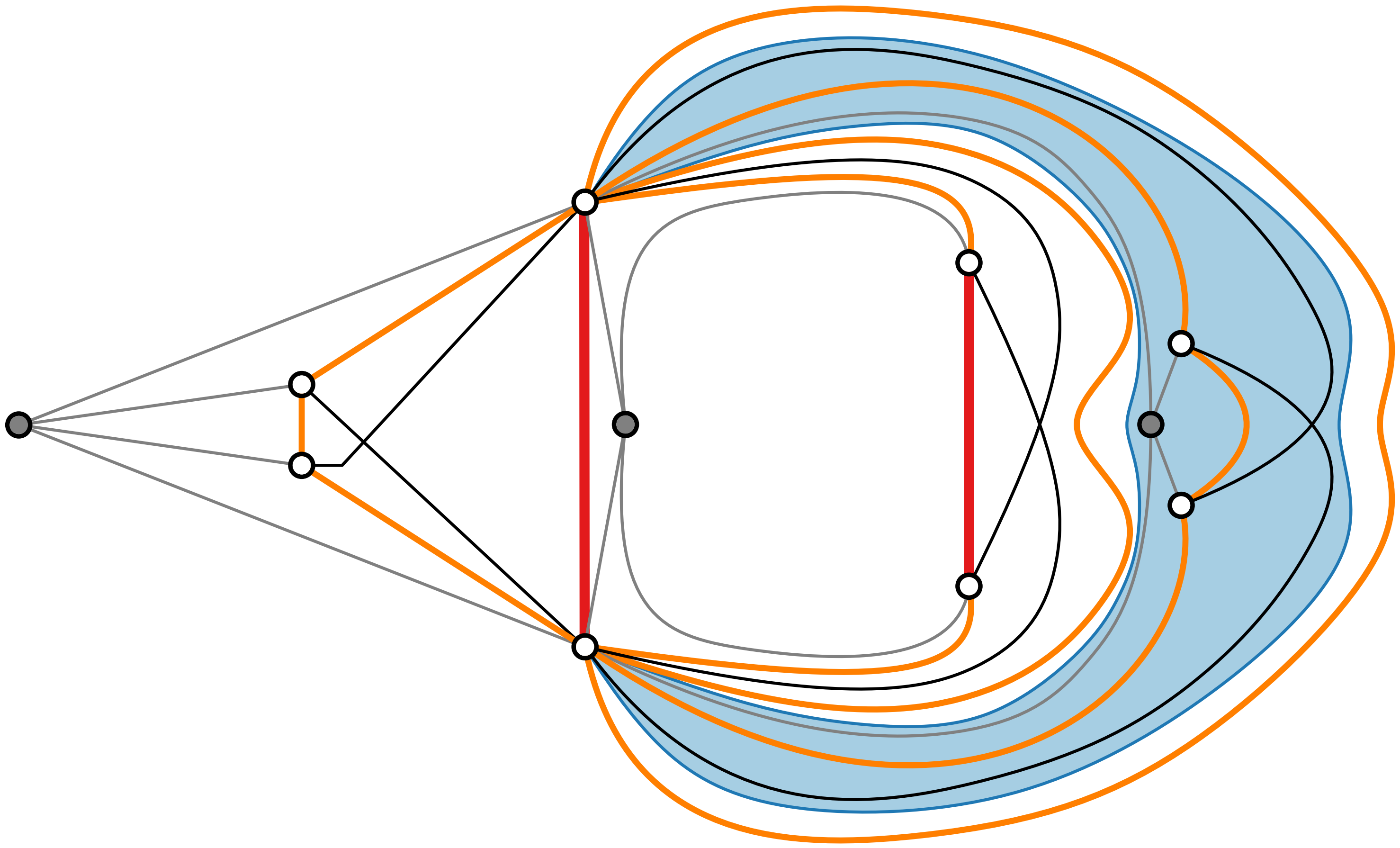
Algorithm Outline

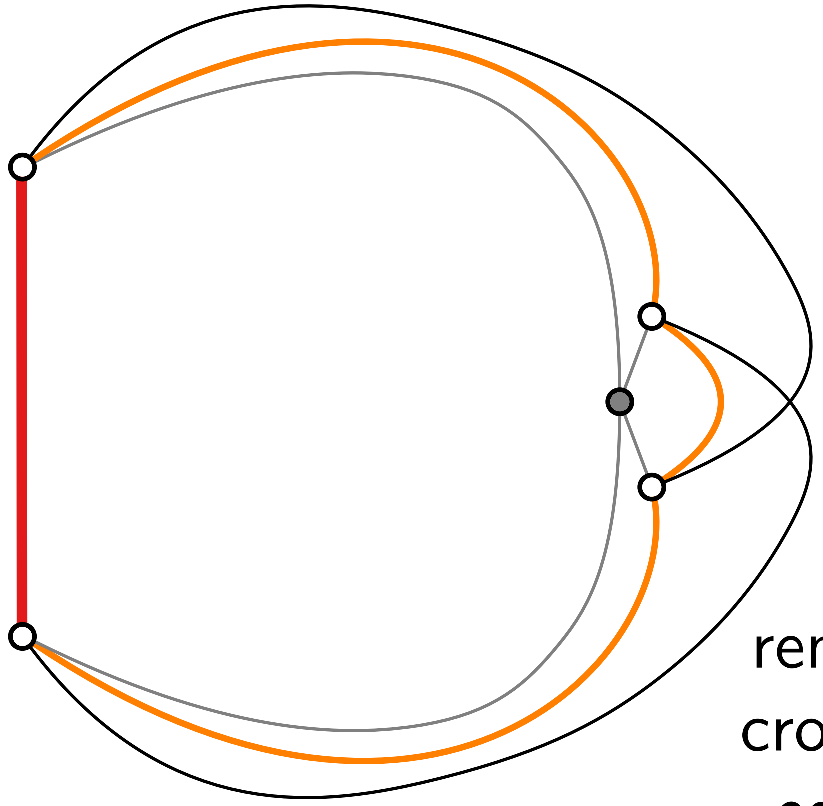


Algorithm Step 3: Drawing Procedure



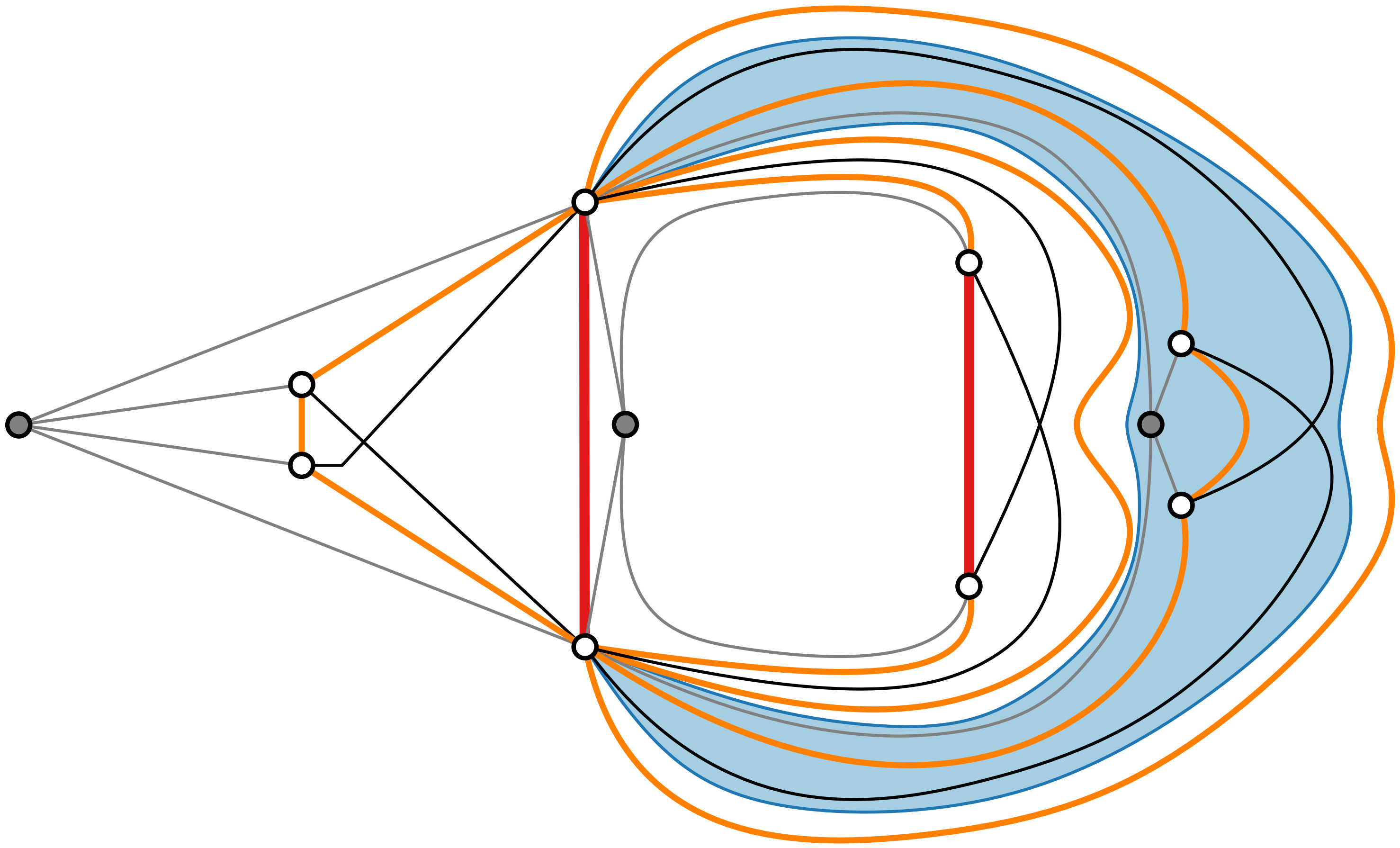
Algorithm Step 3: Drawing Procedure

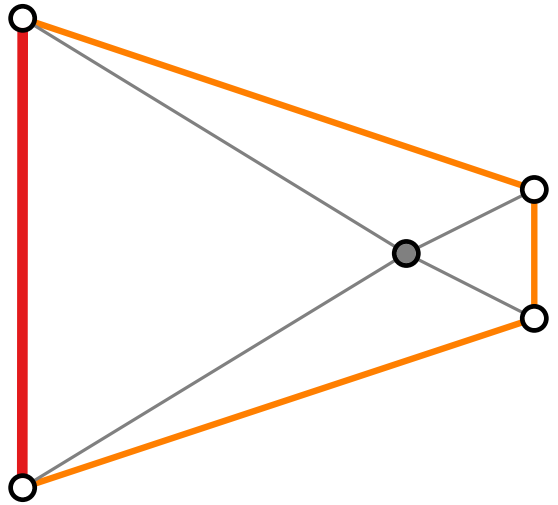




remove crossing edges

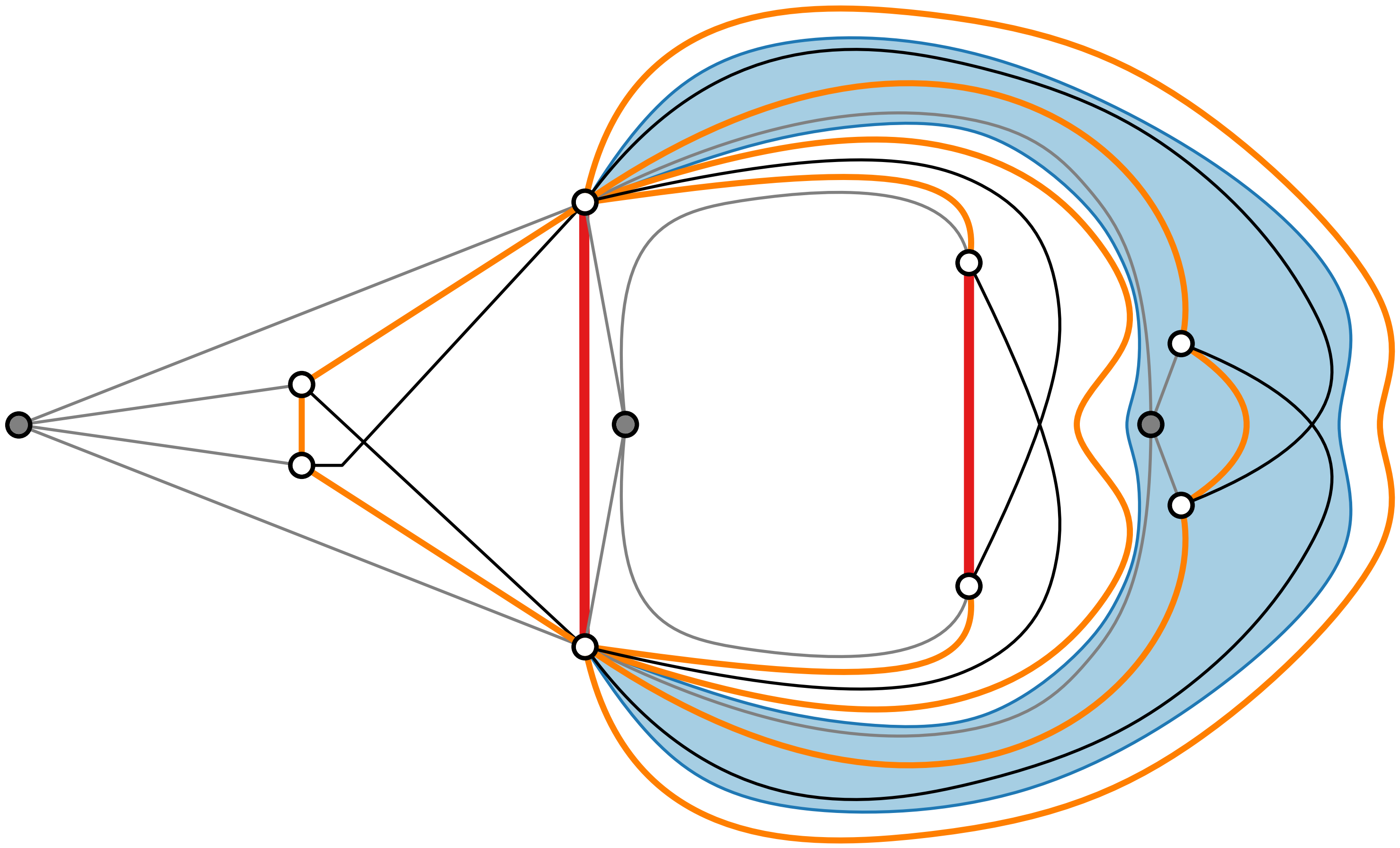
Algorithm Step 3: Drawing Procedure

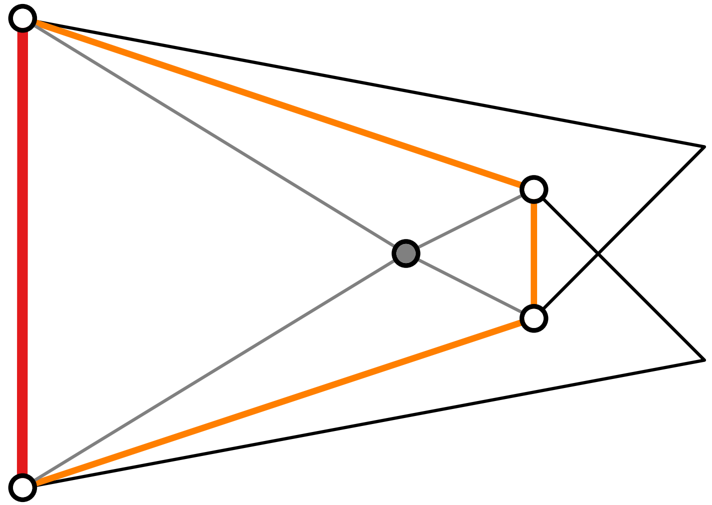




apply Chiba et al.

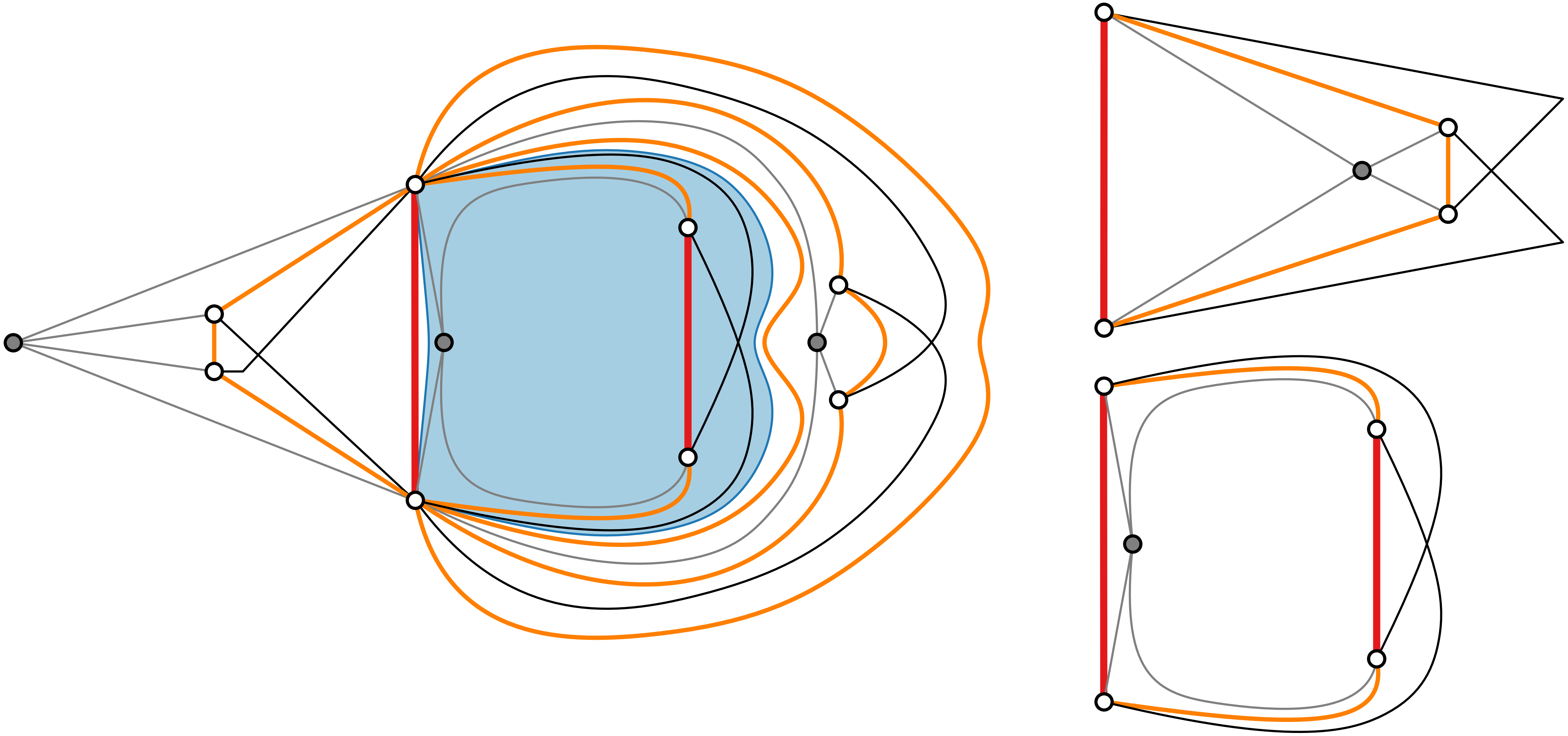
Algorithm Step 3: Drawing Procedure





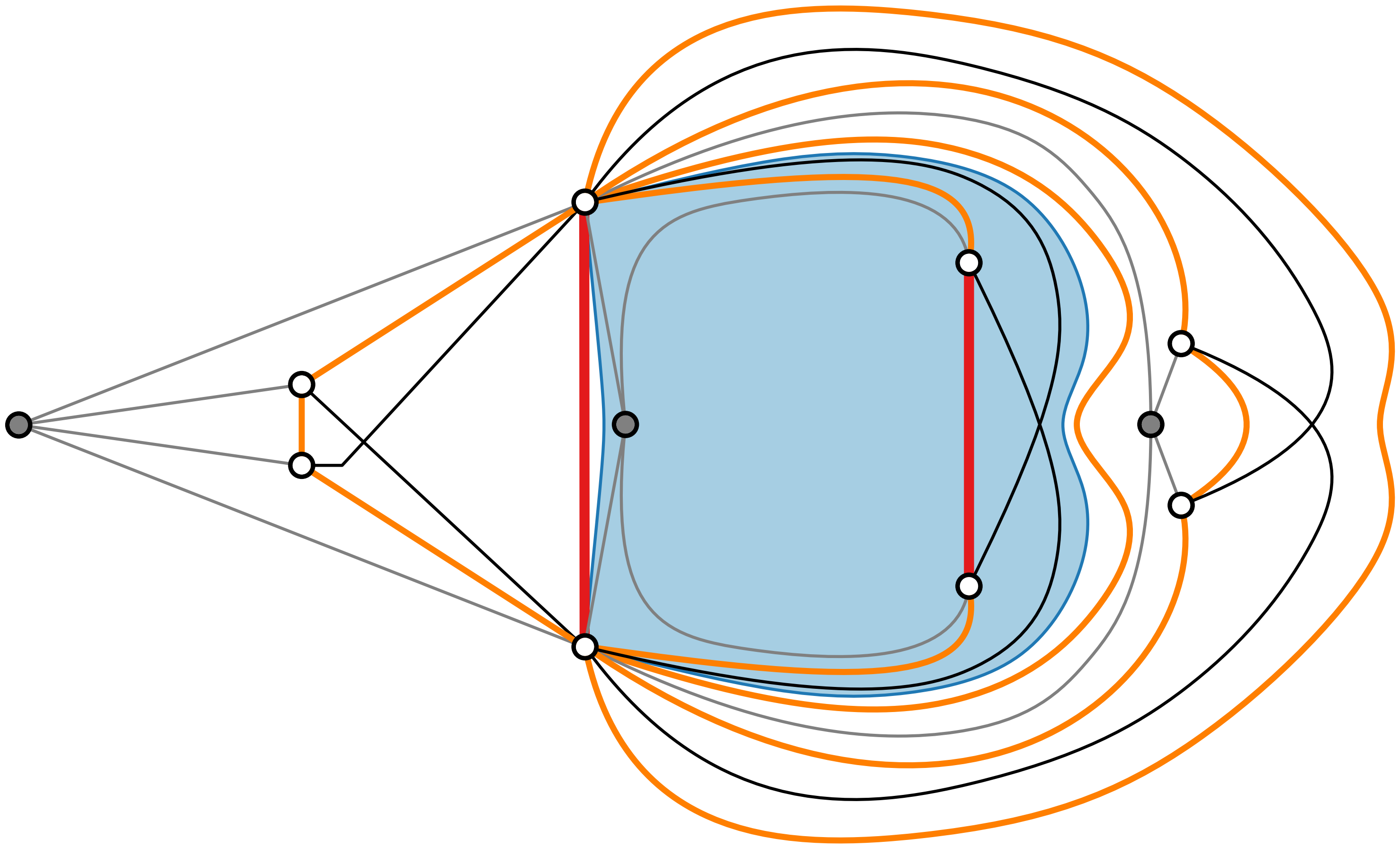
reinsert
crossing
edges

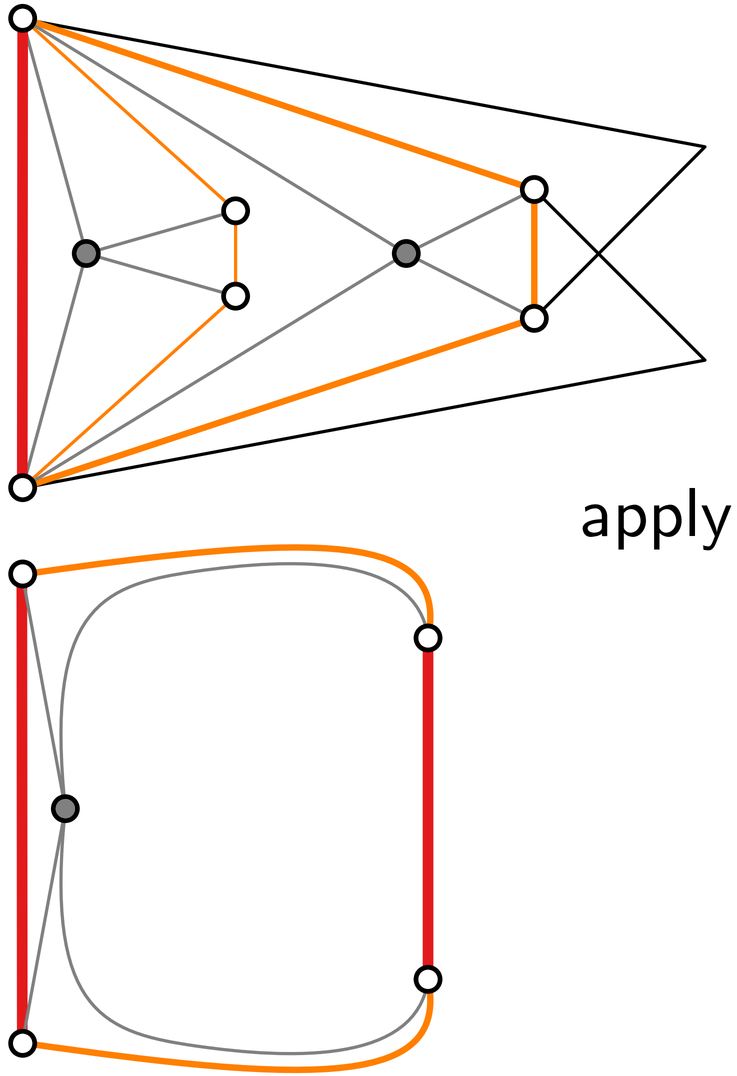
Algorithm Step 3: Drawing Procedure



remove crossing edges

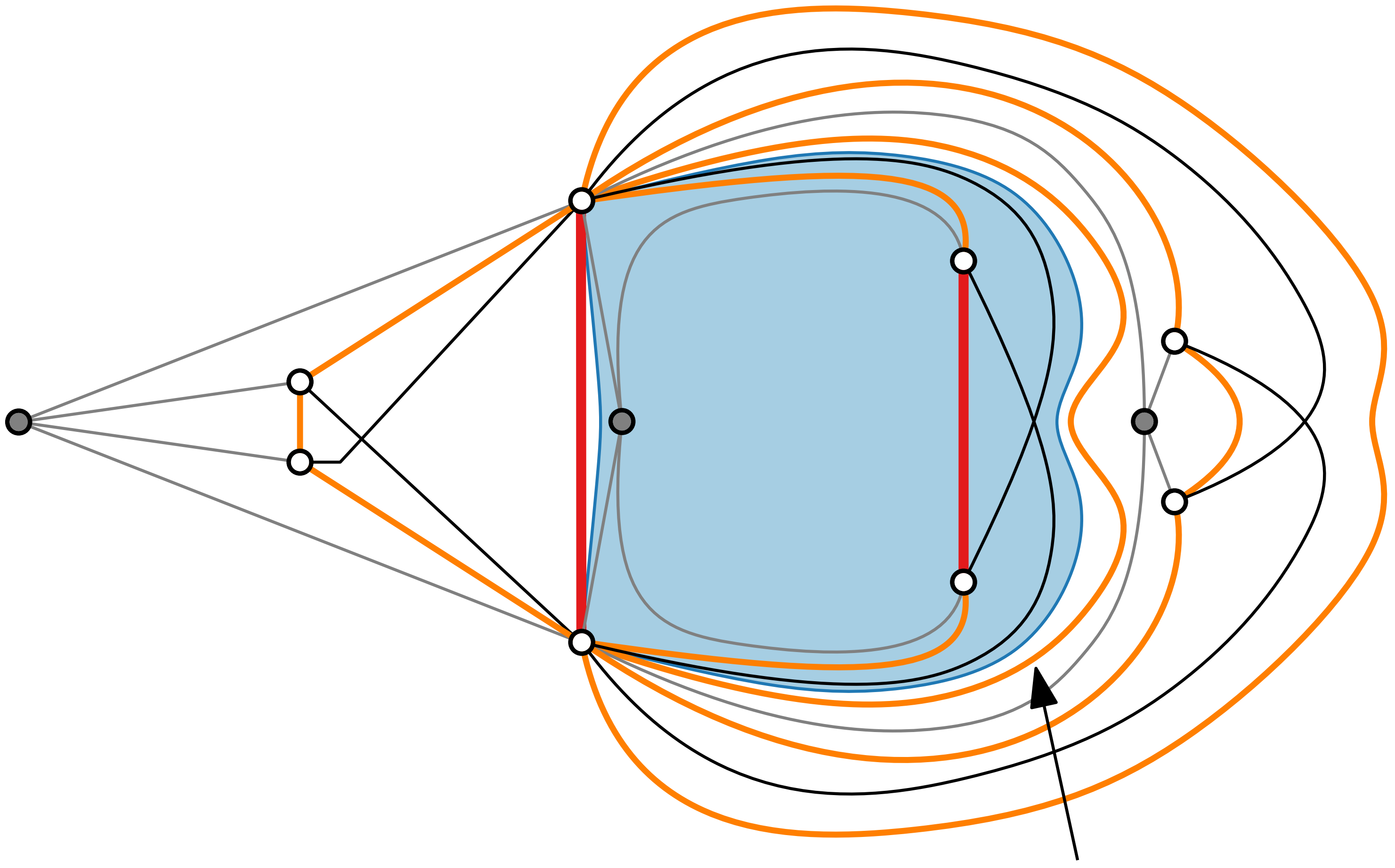
Algorithm Step 3: Drawing Procedure



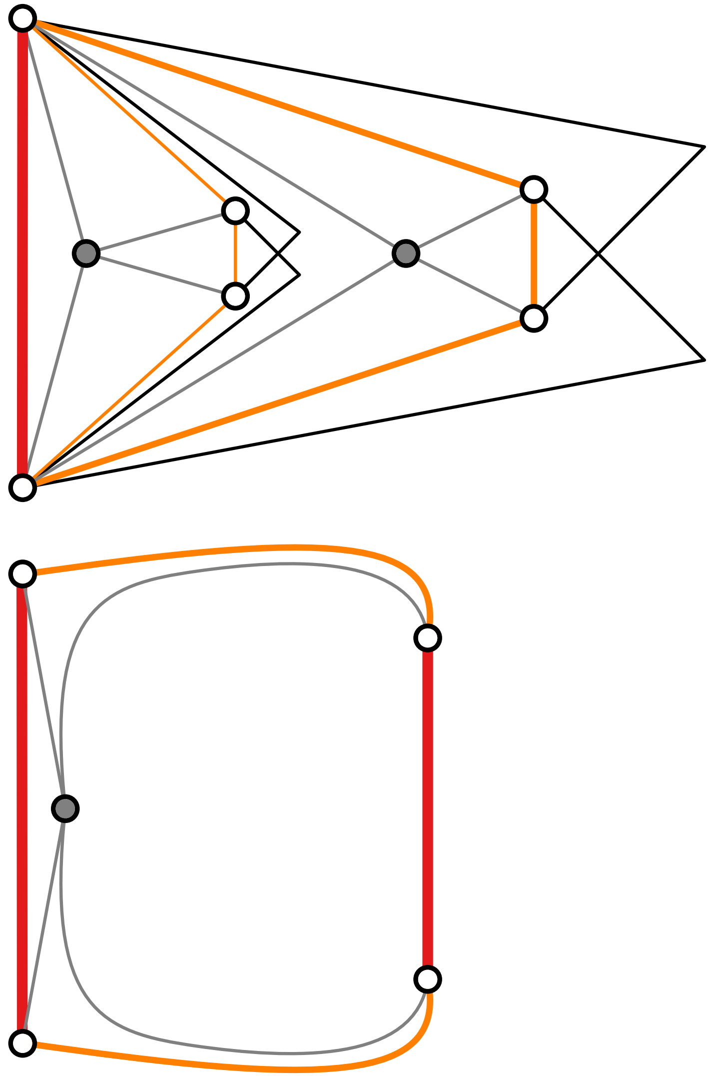


apply Chiba et al.

Algorithm Step 3: Drawing Procedure

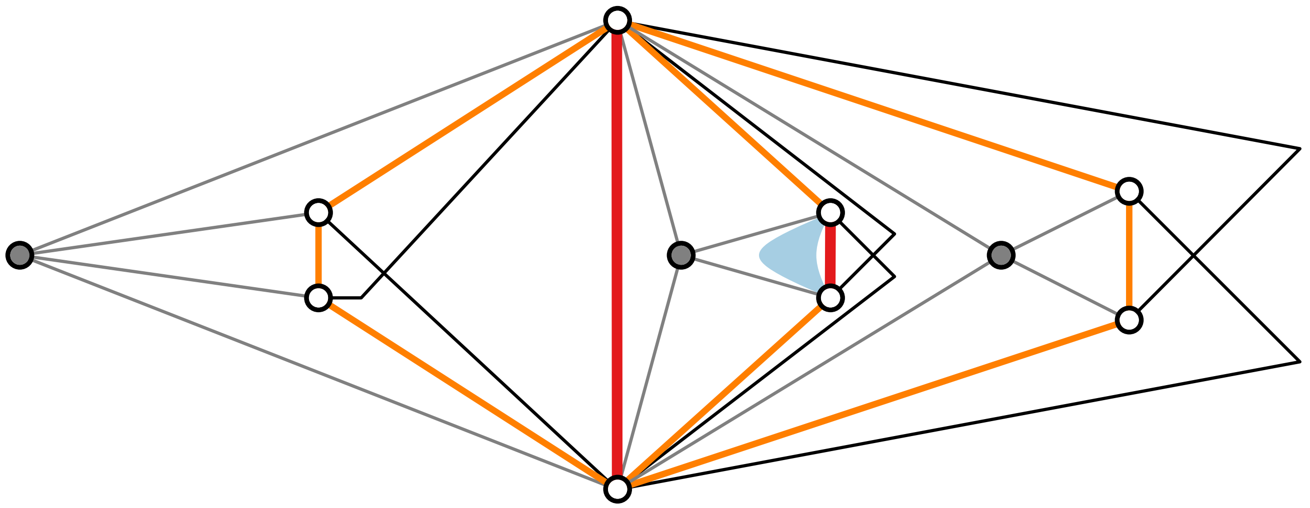


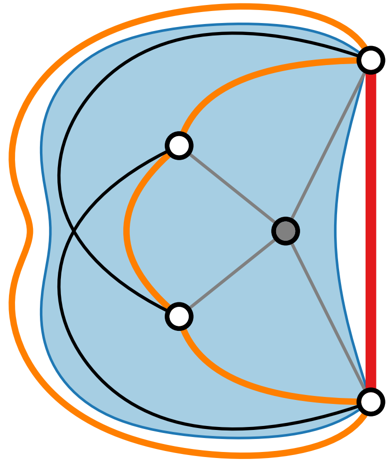
This kite edge, which is a multi-edge, is not drawn.



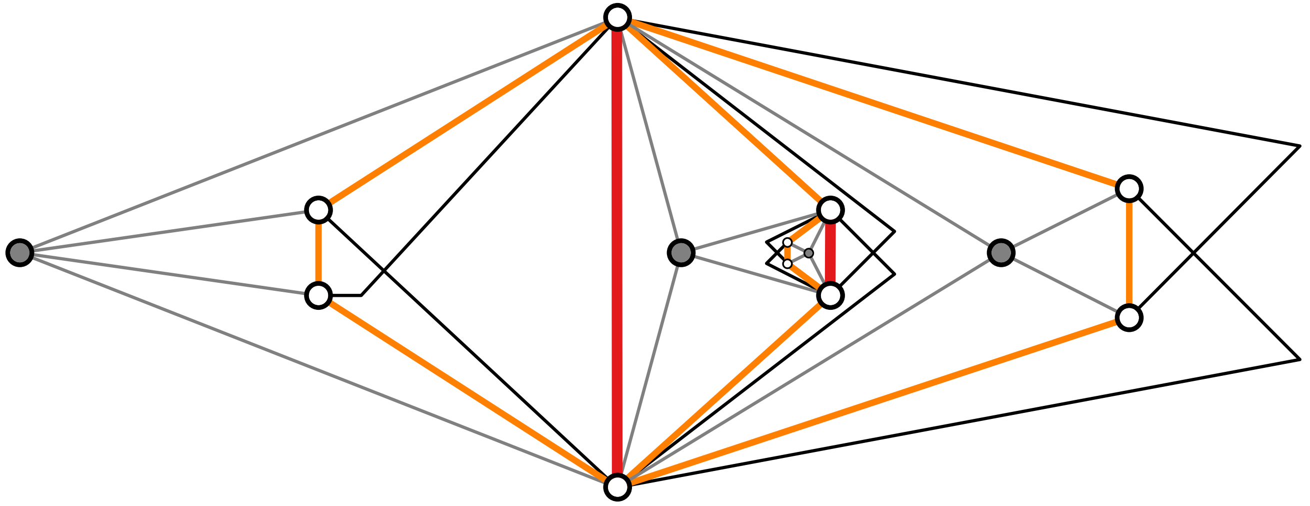
reinsert crossing edges

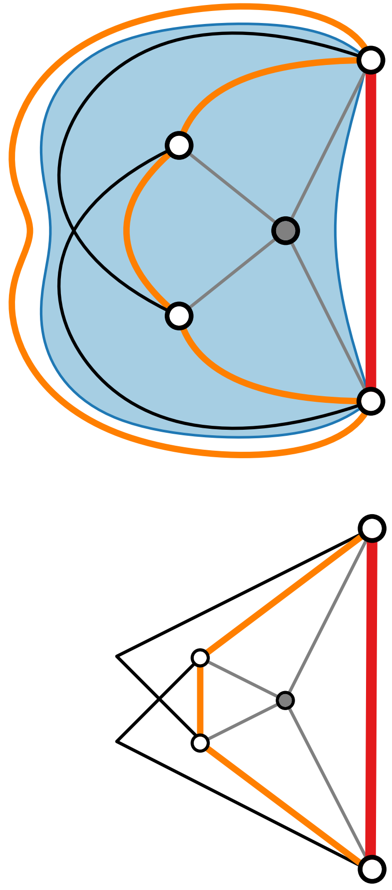
Algorithm Step 3: Drawing Procedure





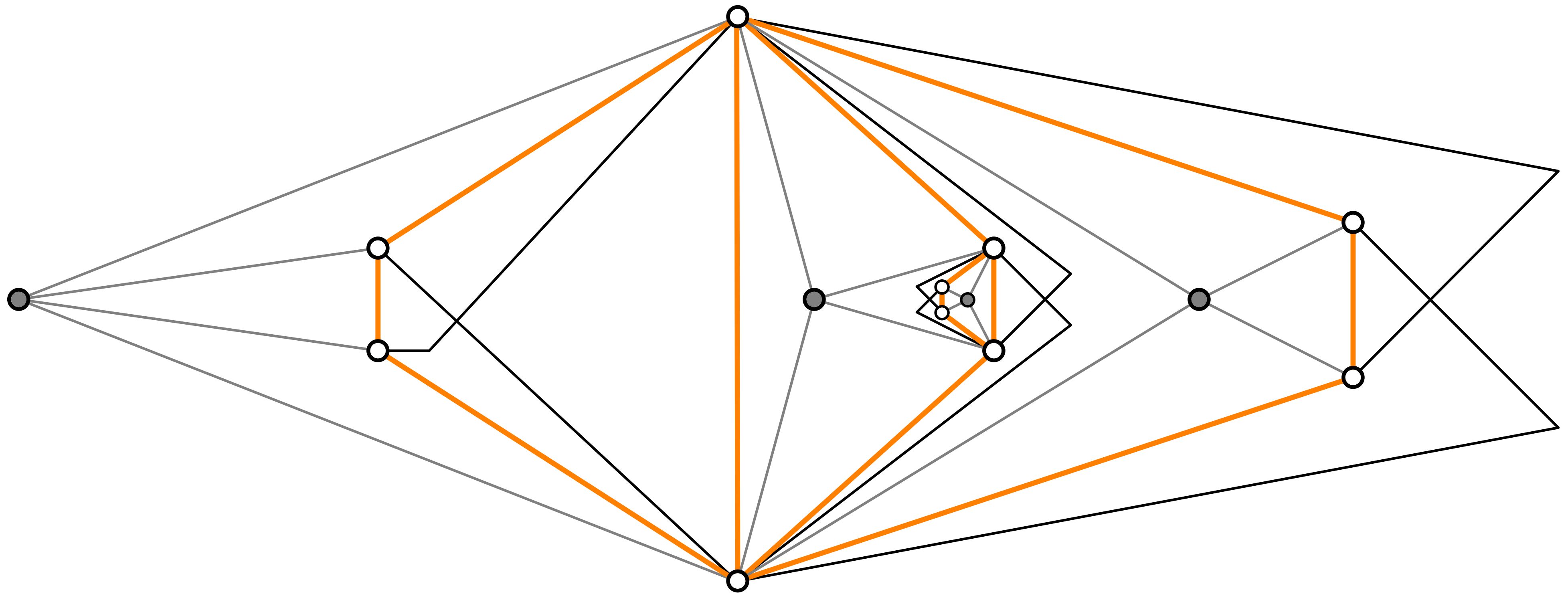
Algorithm Step 3: Drawing Procedure



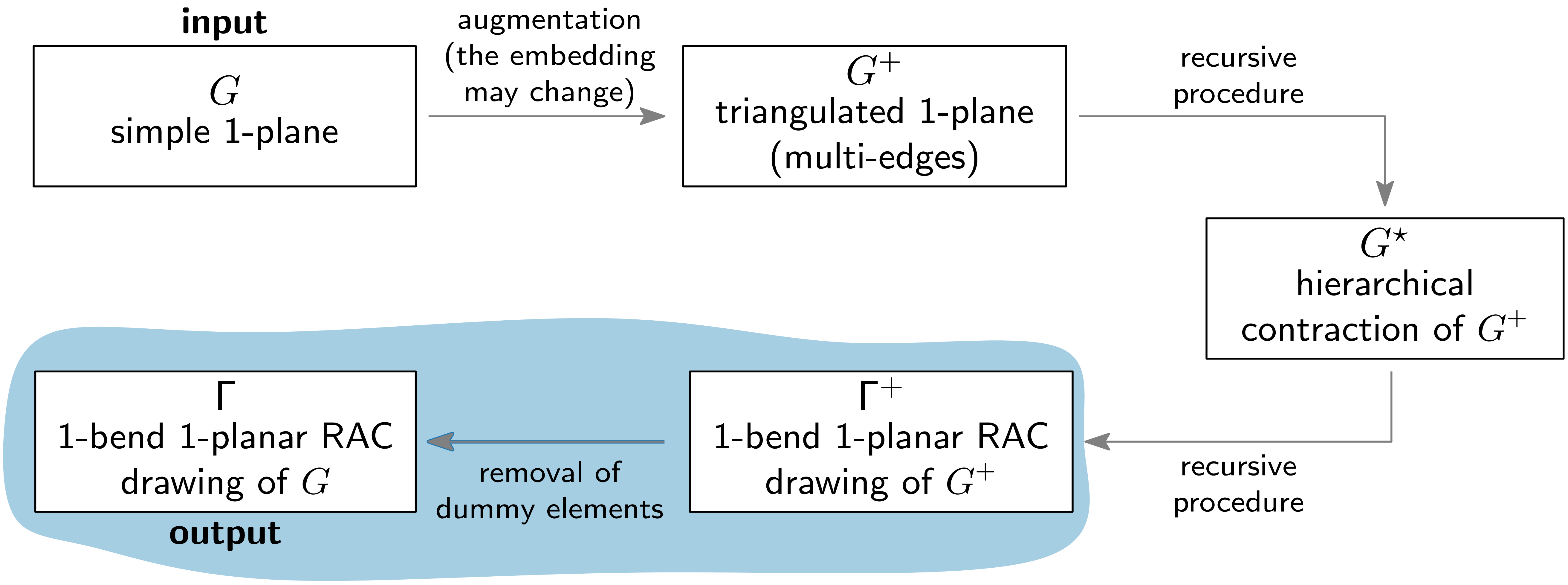


Algorithm Step 3: Drawing Procedure

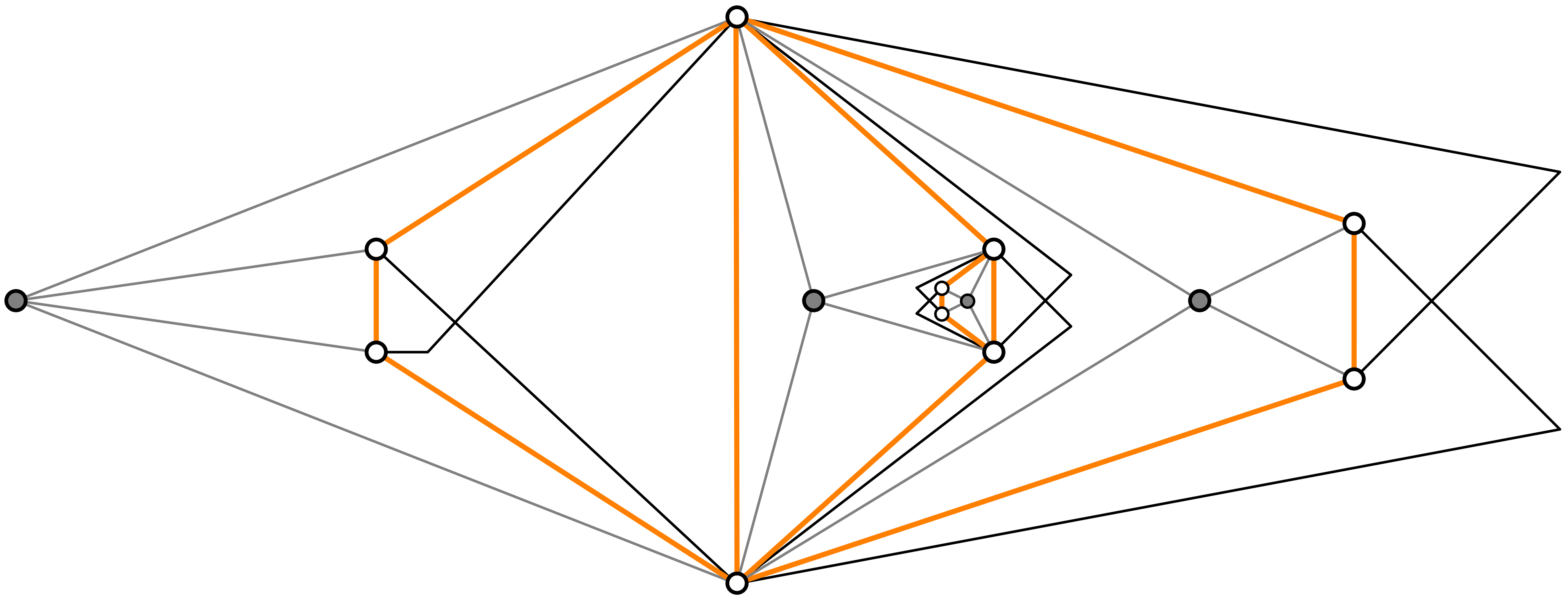
Γ^+: 1-bend 1-planar RAC drawing of G^+



Algorithm Outline

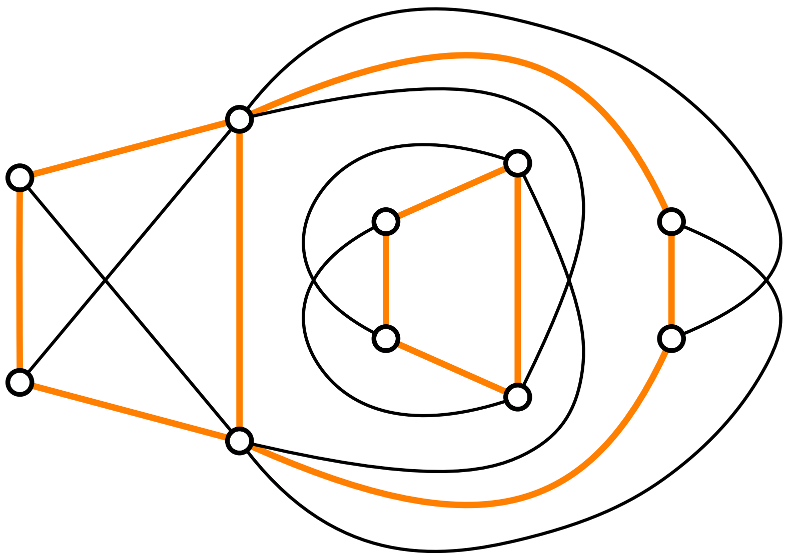


Algorithm Step 4: Removal of Dummy Vertices



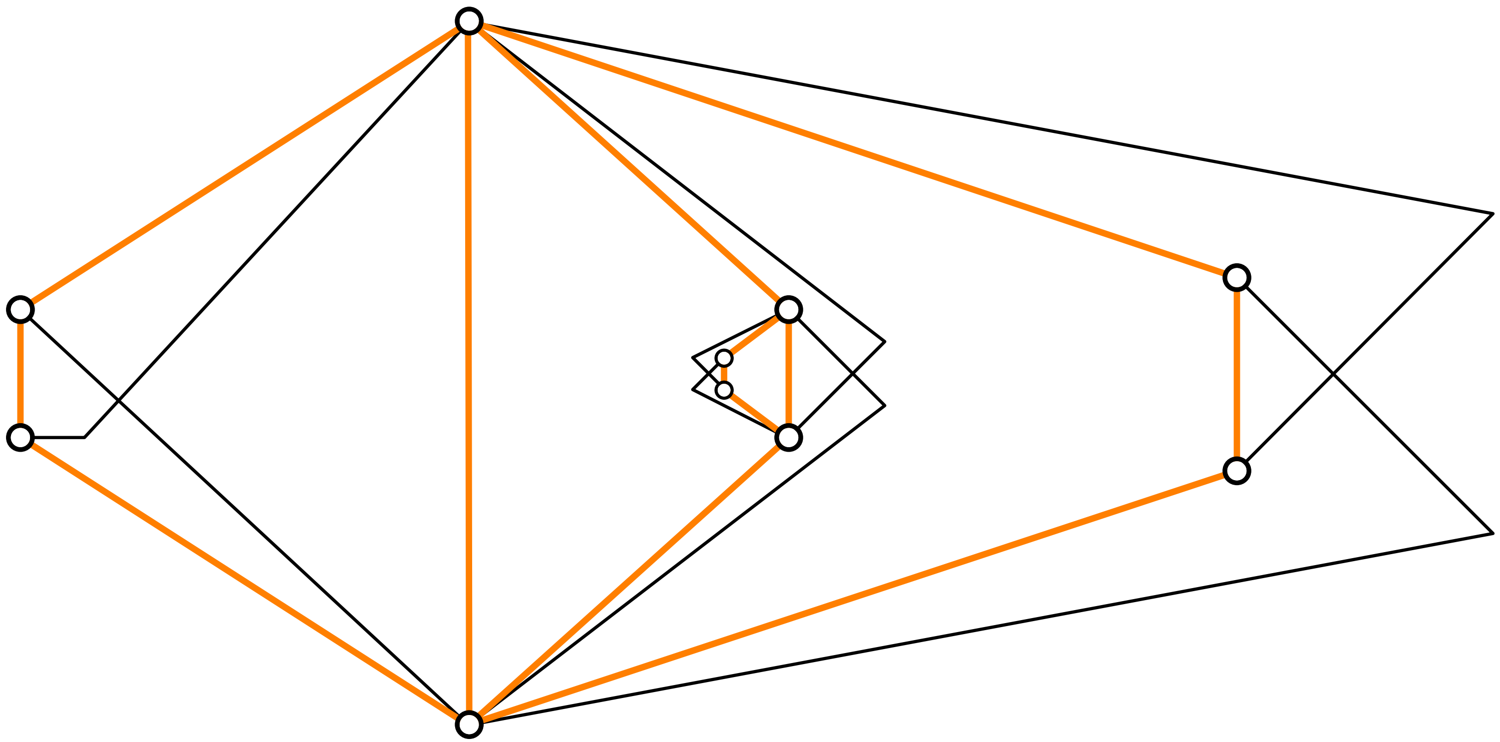
Algorithm Step 4: Removal of Dummy Vertices

G: simple 1-plane graph



Γ: 1-bend 1-planar RAC drawing of G

(embedding may differ)

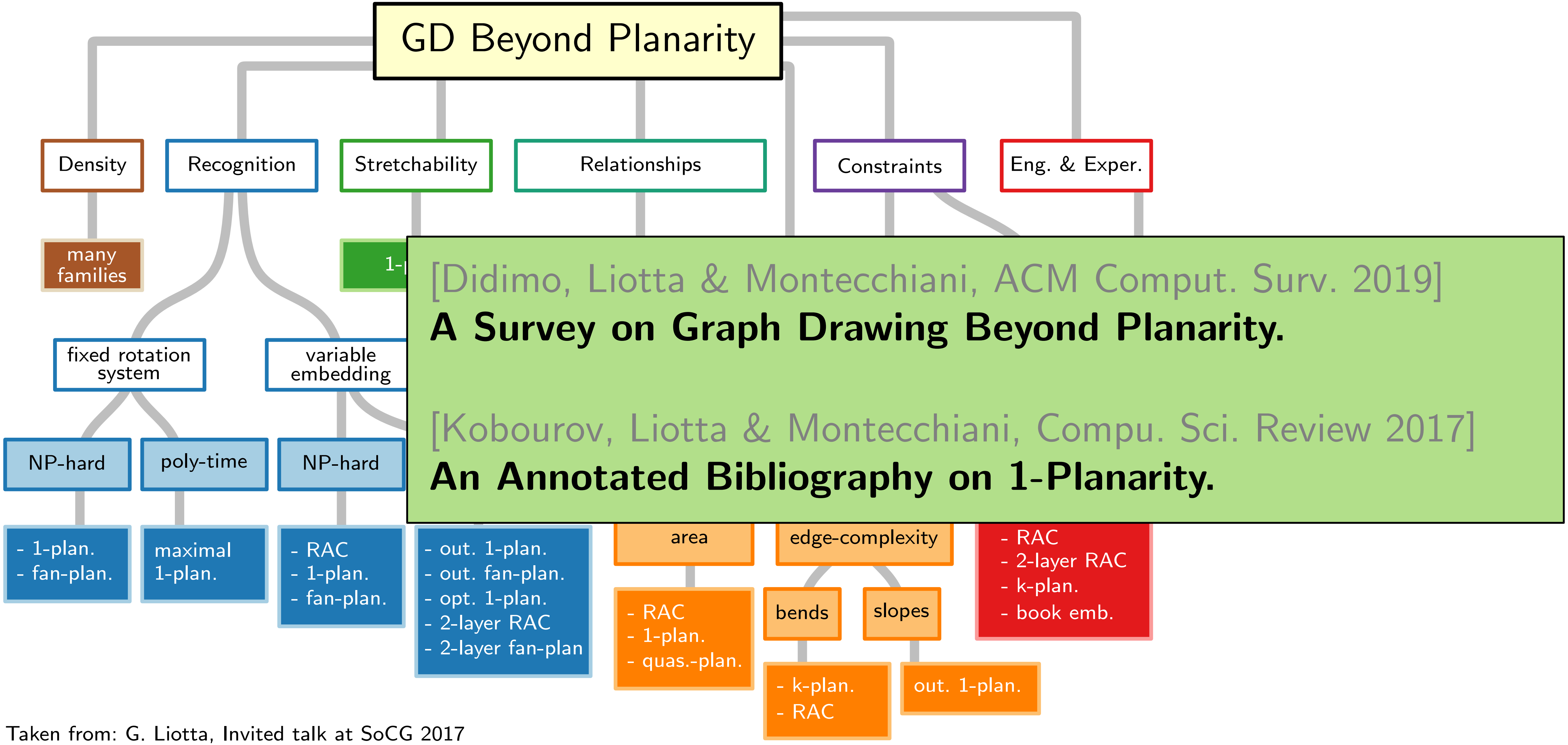


Remark.

By modifying the algorithm slightly, the given input embedding can be preserved.

[Chaplick, Lipp, Wolff, Zink 2019]

GD Beyond Planarity: a Taxonomy



[Didimo, Liotta & Montecchiani, ACM Comput. Surv. 2019]
A Survey on Graph Drawing Beyond Planarity.

[Kobourov, Liotta & Montecchiani, Compu. Sci. Review 2017]
An Annotated Bibliography on 1-Planarity.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Hong and Tokuyama, editors '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchani, Valtr '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
- [Didimo, Eades, Liotta '17] Drawing graphs with right angle crossings
- [Bekos et al. '17] On RAC drawings of 1-planar graphs