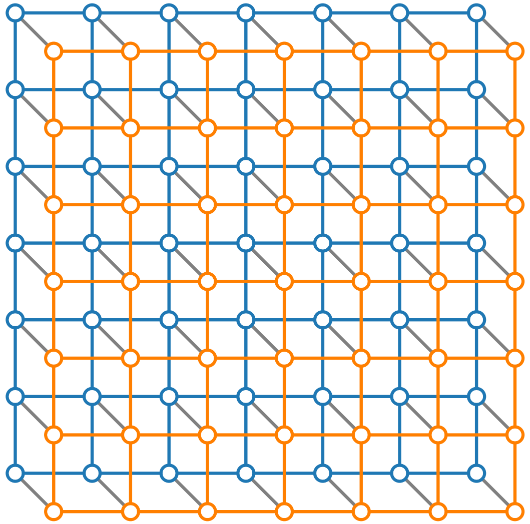
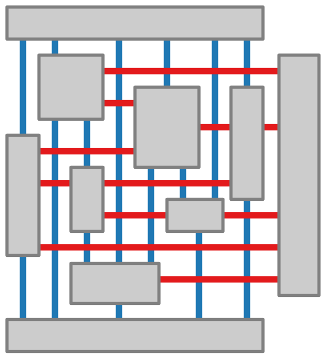
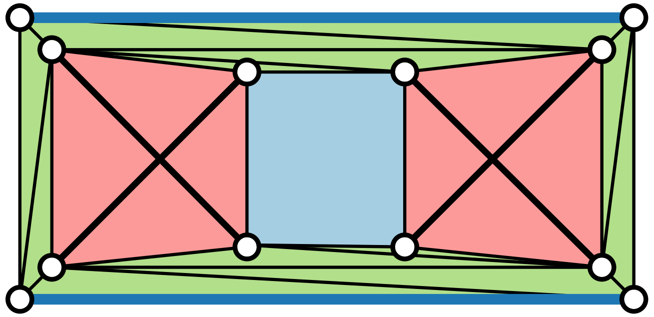
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

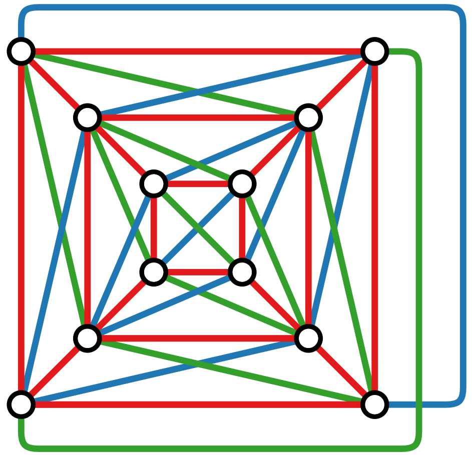
Visualization of Graphs





Lecture 11:
Beyond Planarity
Drawing Graphs with Crossings





Alexander Wolff

Summer semester 2025

Planar Graphs

Planar graphs admit drawings in the plane without crossings.

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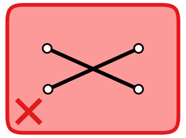

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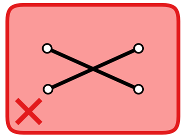

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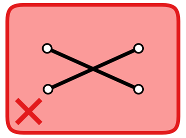

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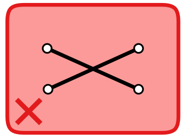

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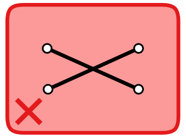

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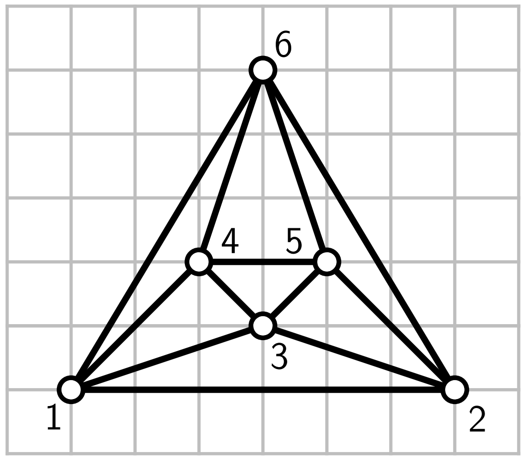

straight-line drawing

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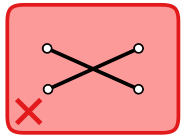

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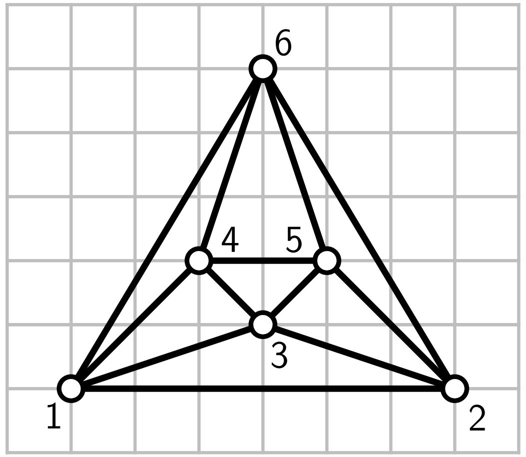

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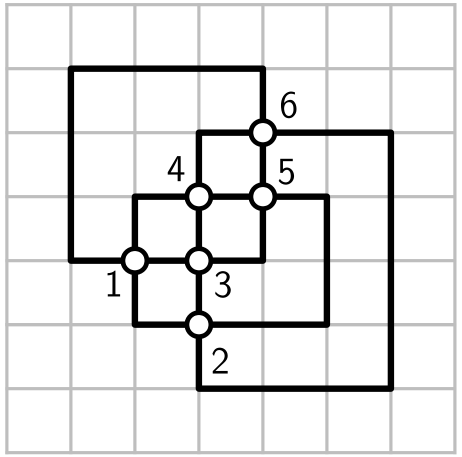

orthogonal drawing

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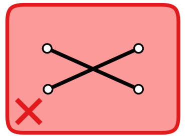

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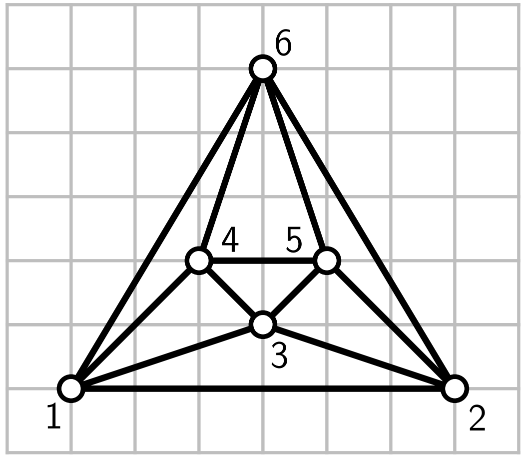

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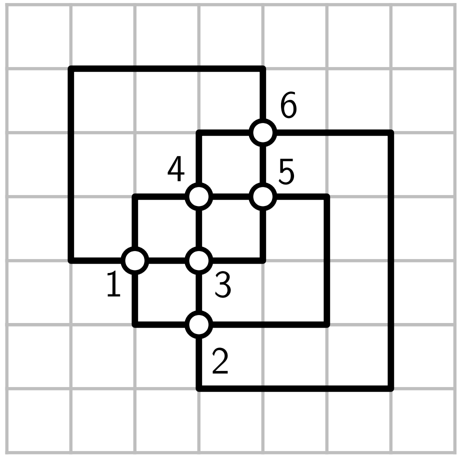

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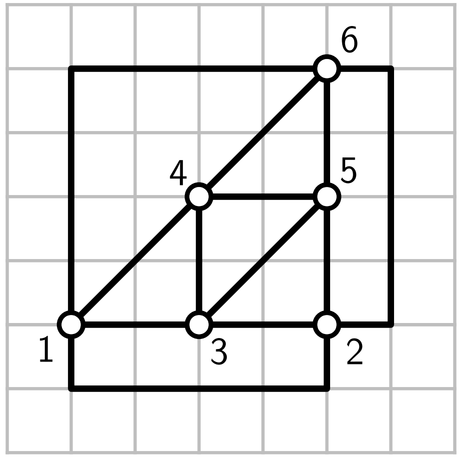

grid drawing with bends & 3 slopes

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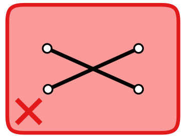

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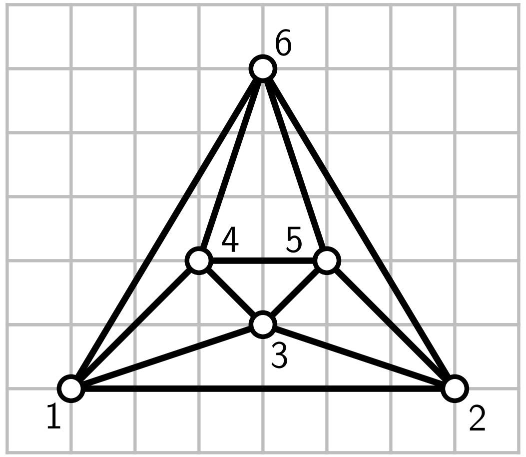

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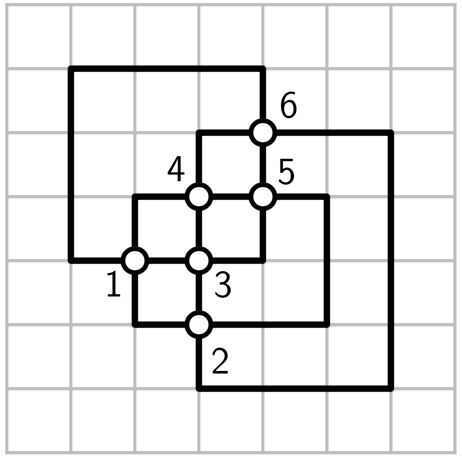

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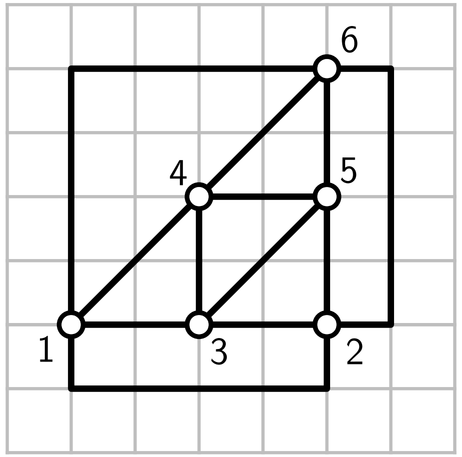

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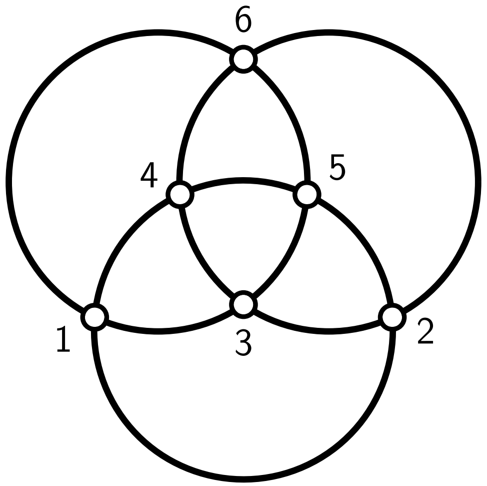

circular-arc drawing

And Non-Planar Graphs?

We have seen a few drawing styles:

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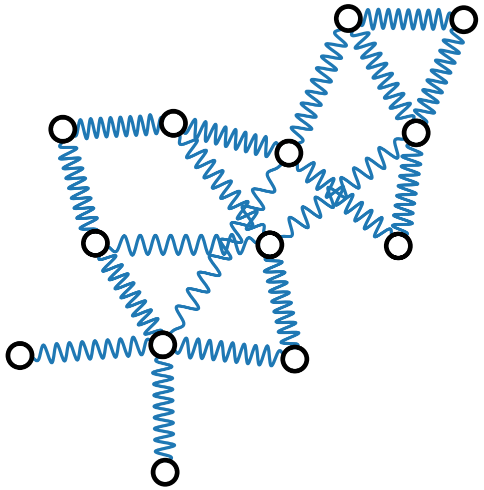

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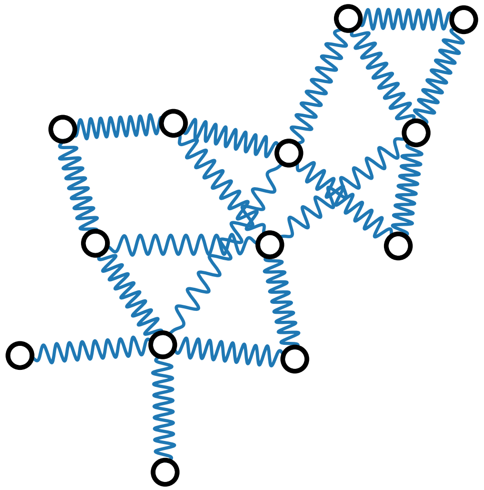

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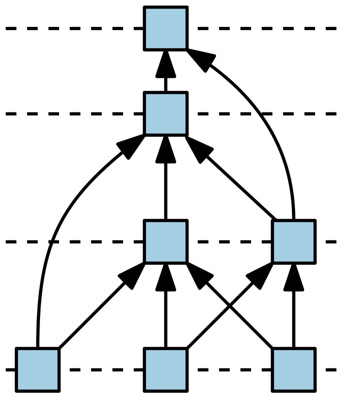

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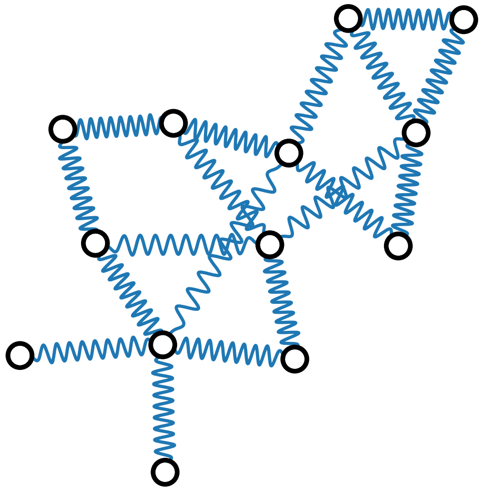

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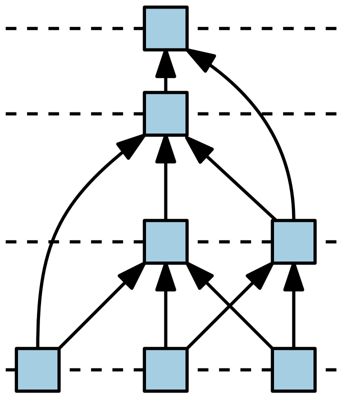

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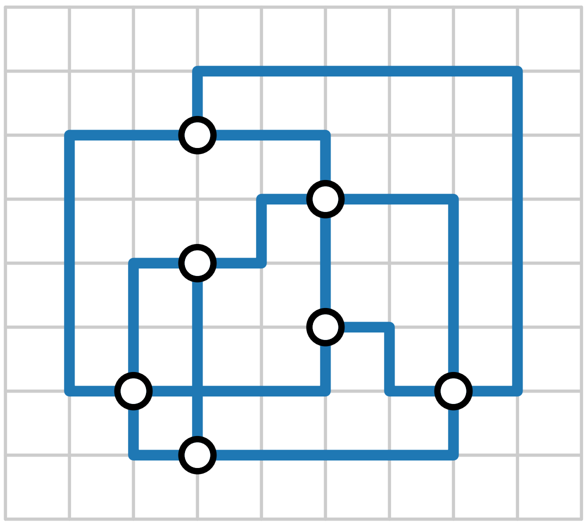

orthogonal layouts
(via planarization)

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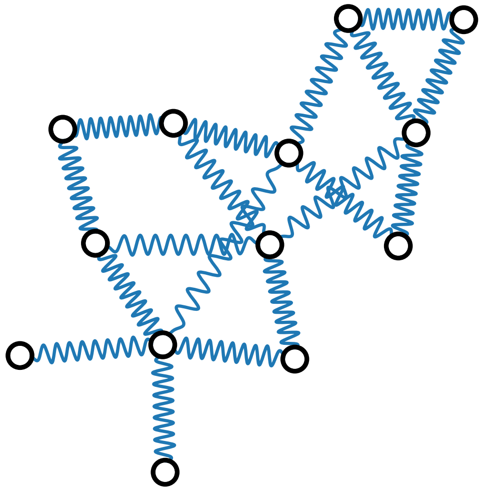

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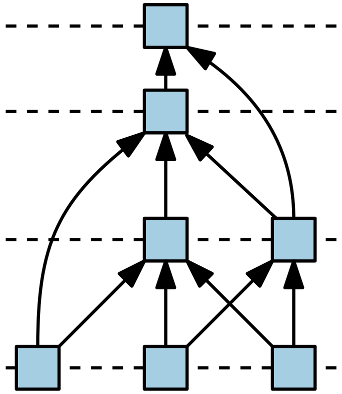

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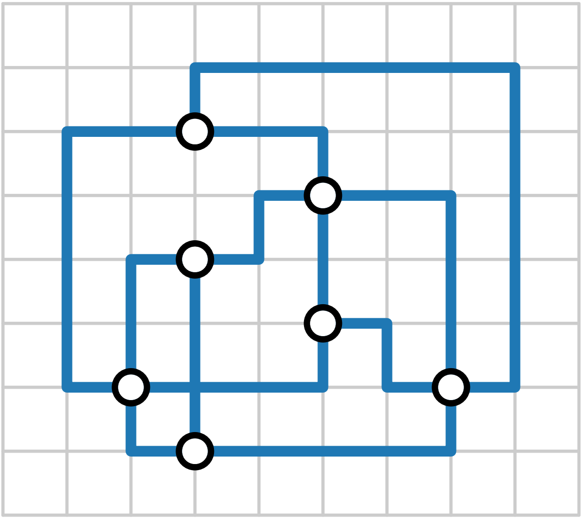

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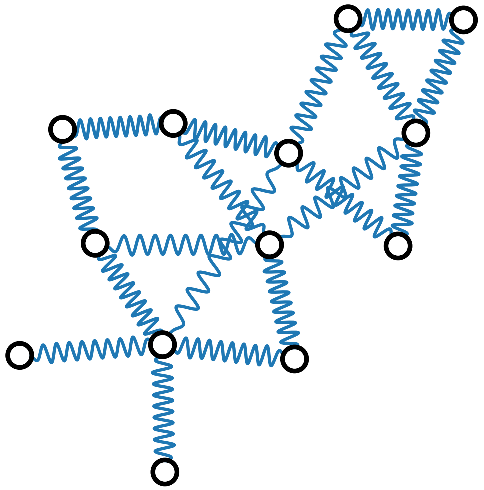

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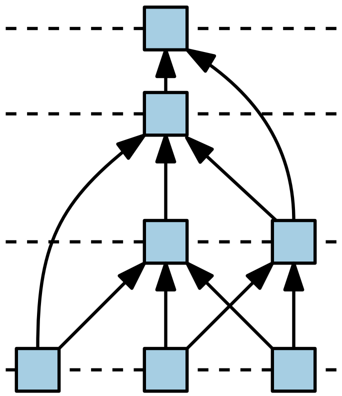

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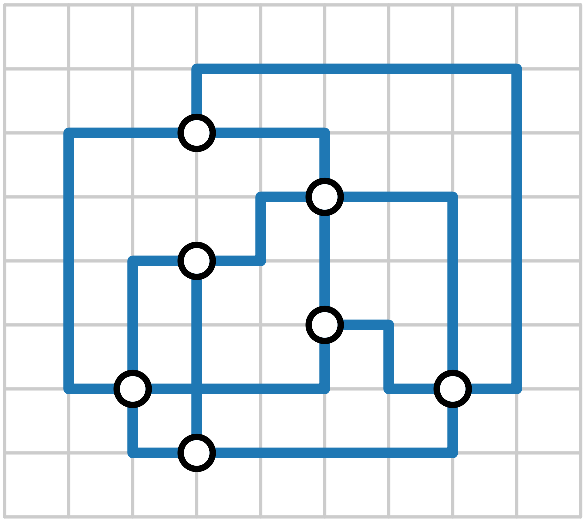

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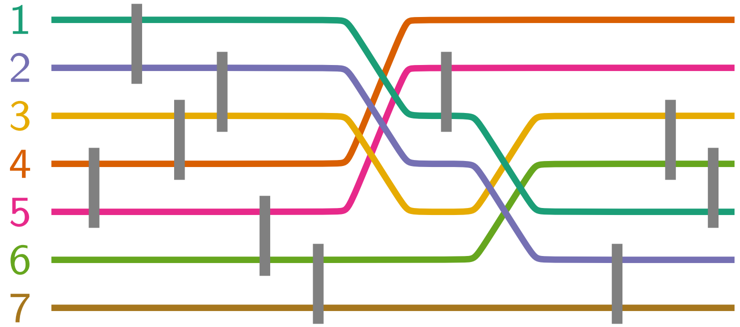

block crossings

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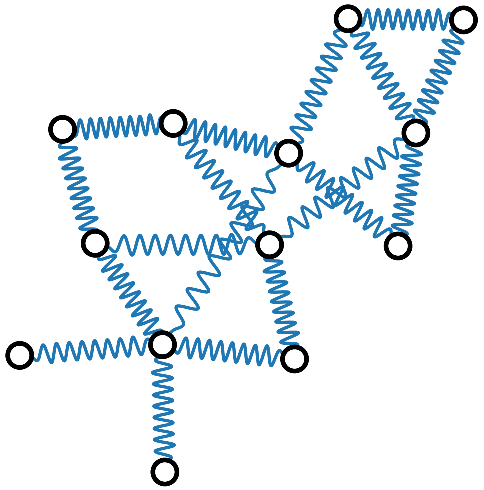

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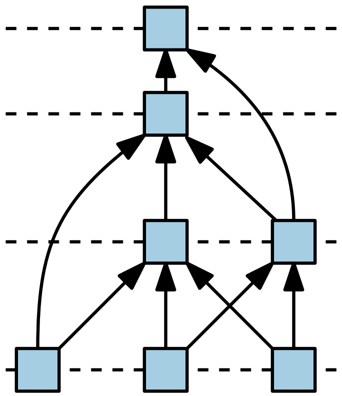

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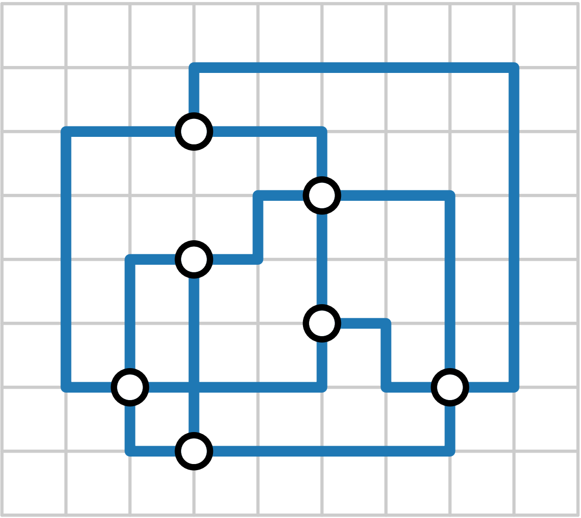

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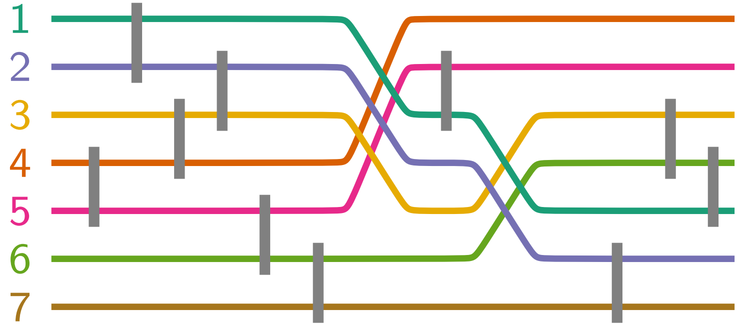

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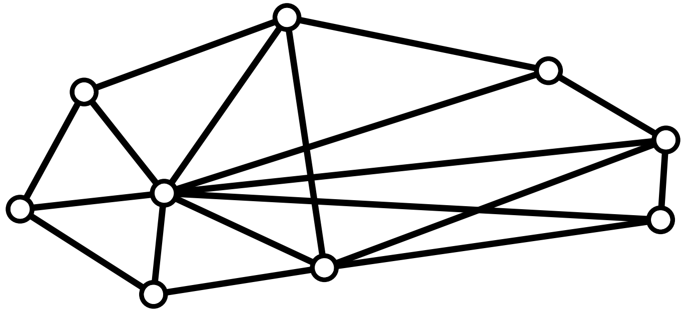
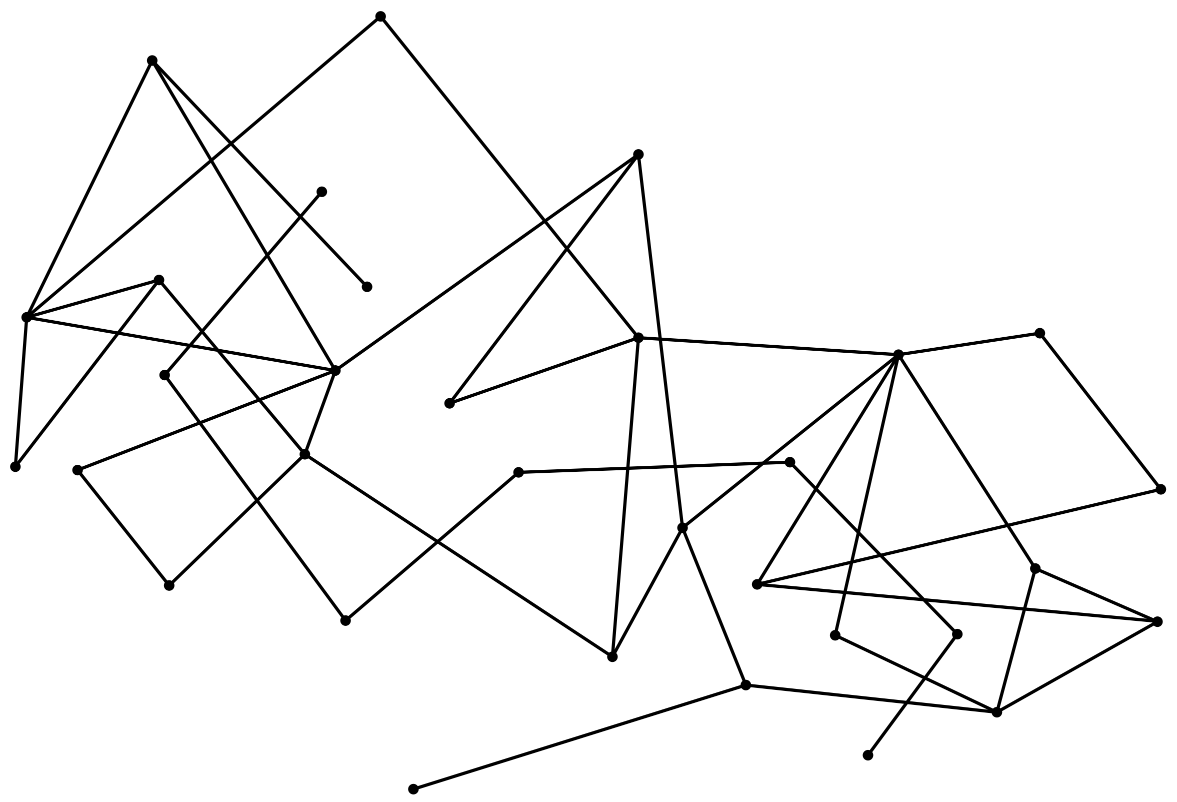


Which crossings feel worse?

Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.



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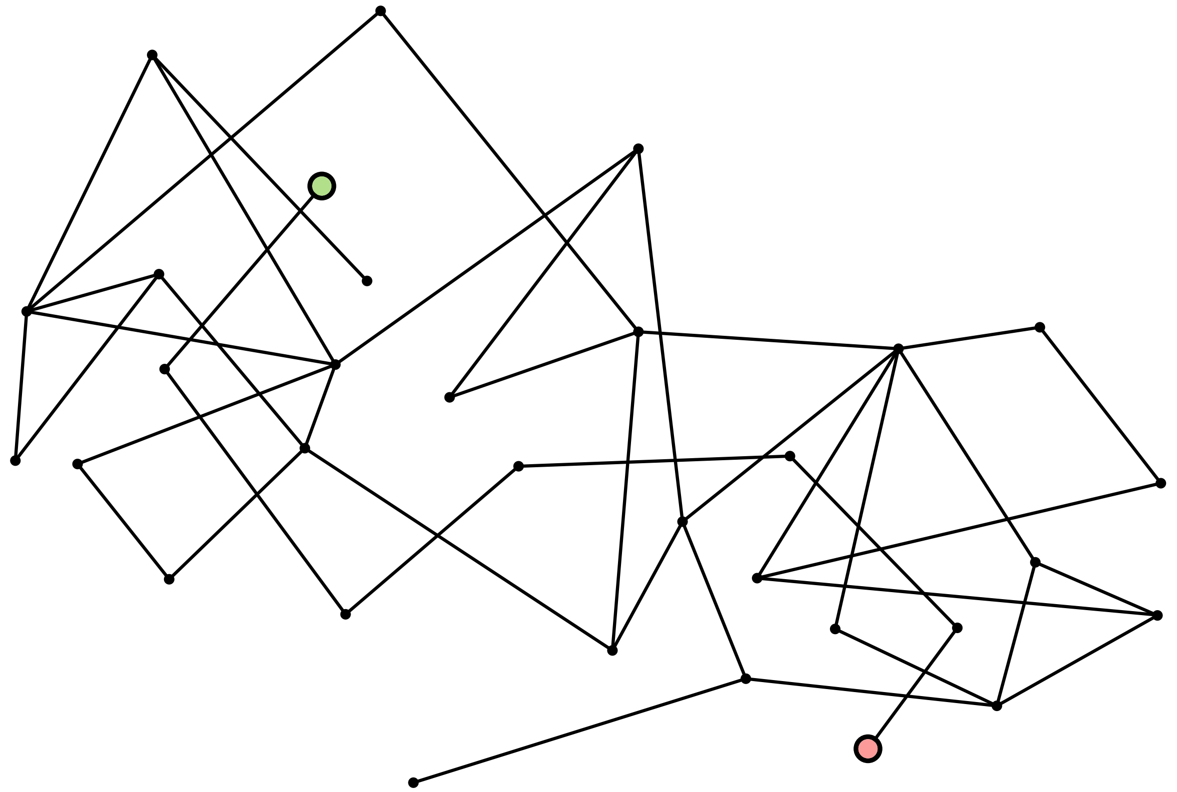

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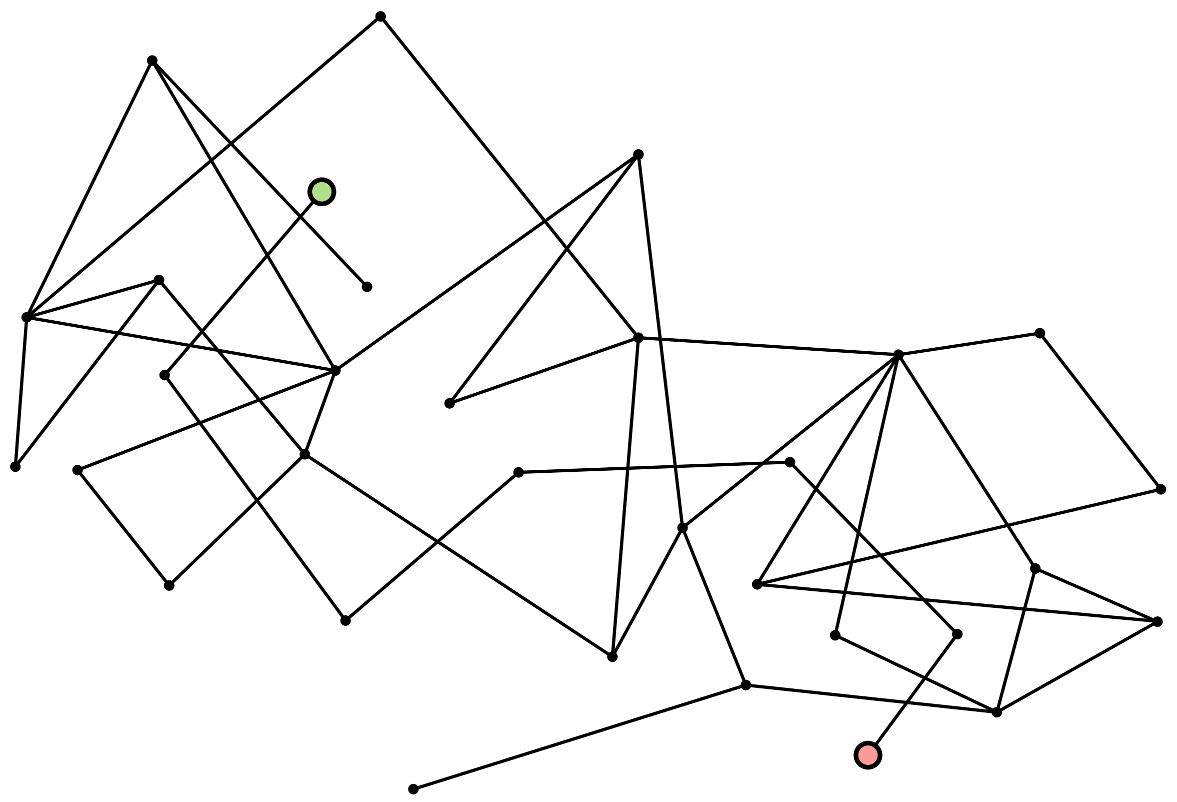

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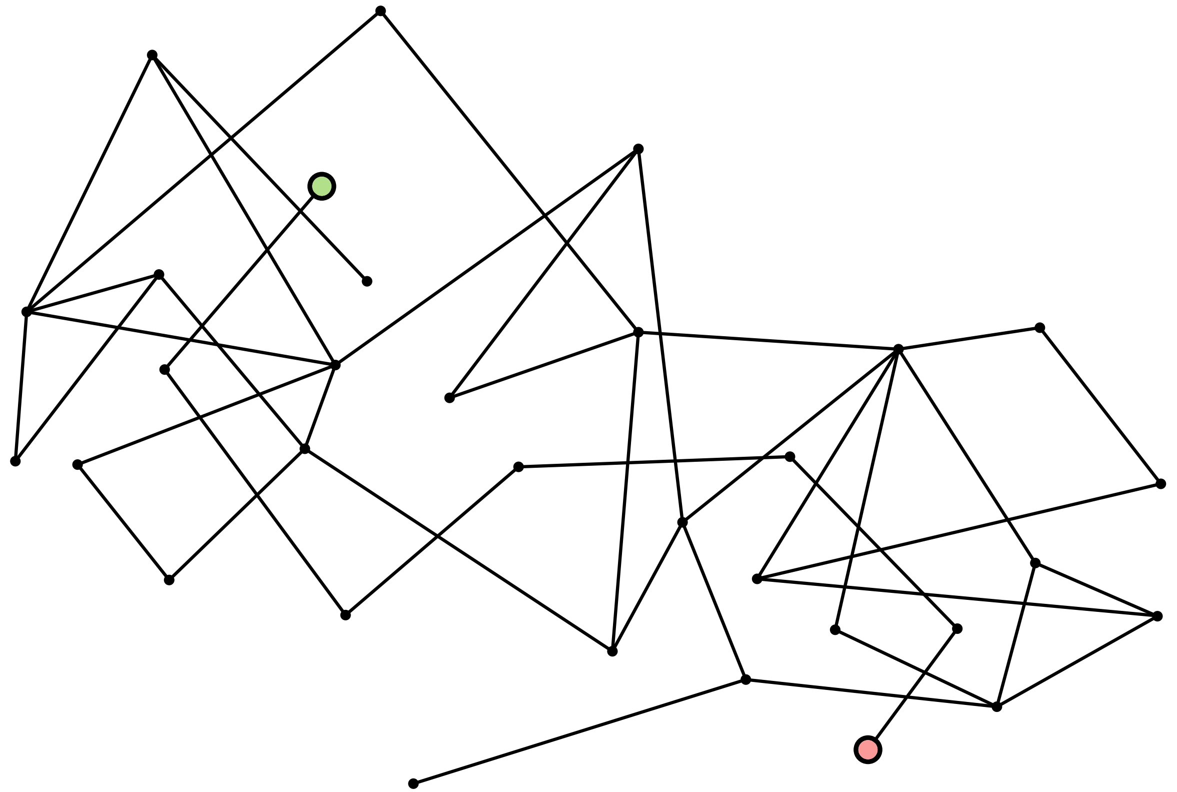

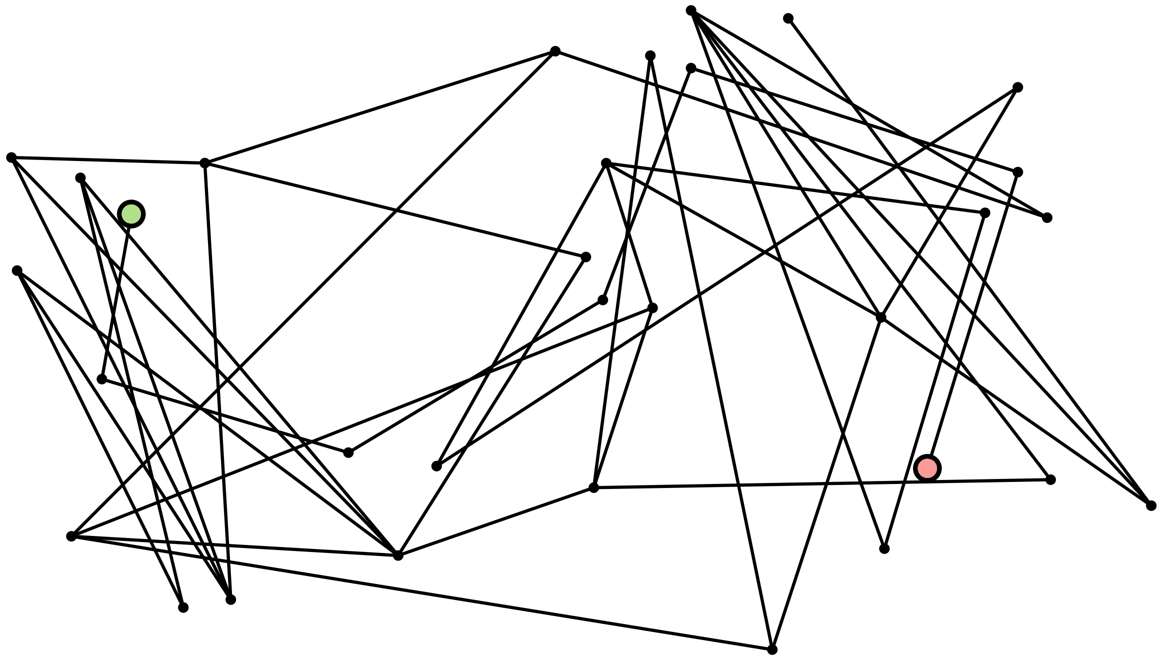

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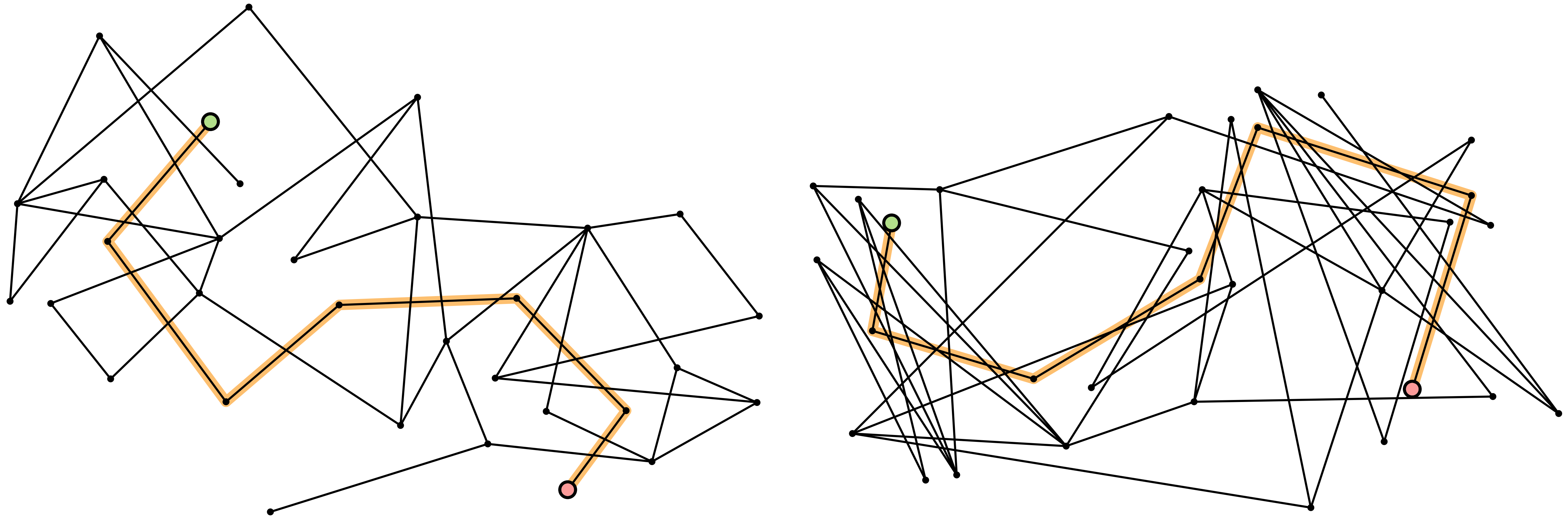

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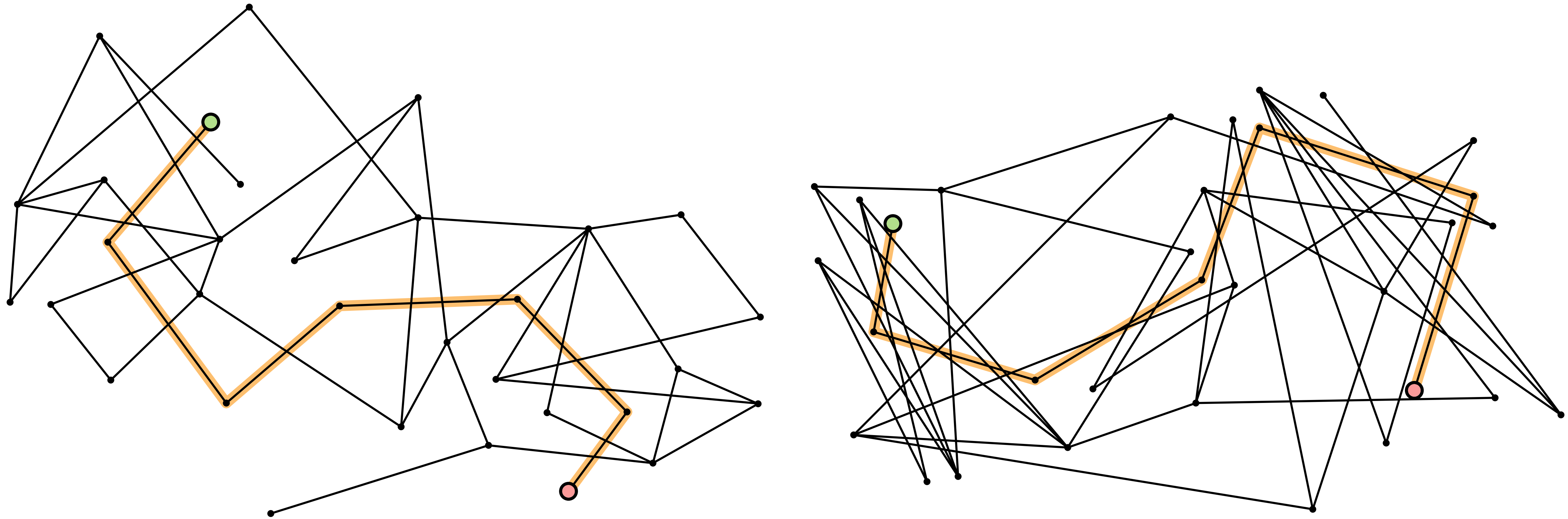
Results:



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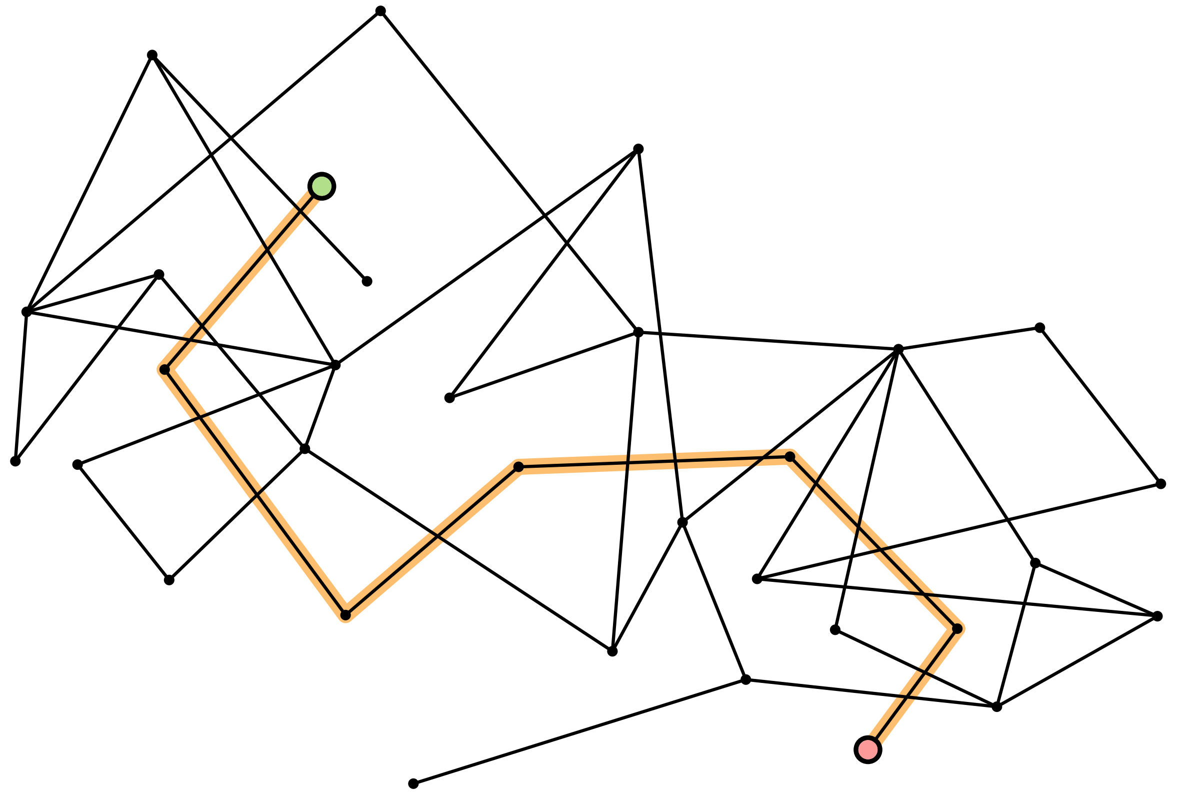
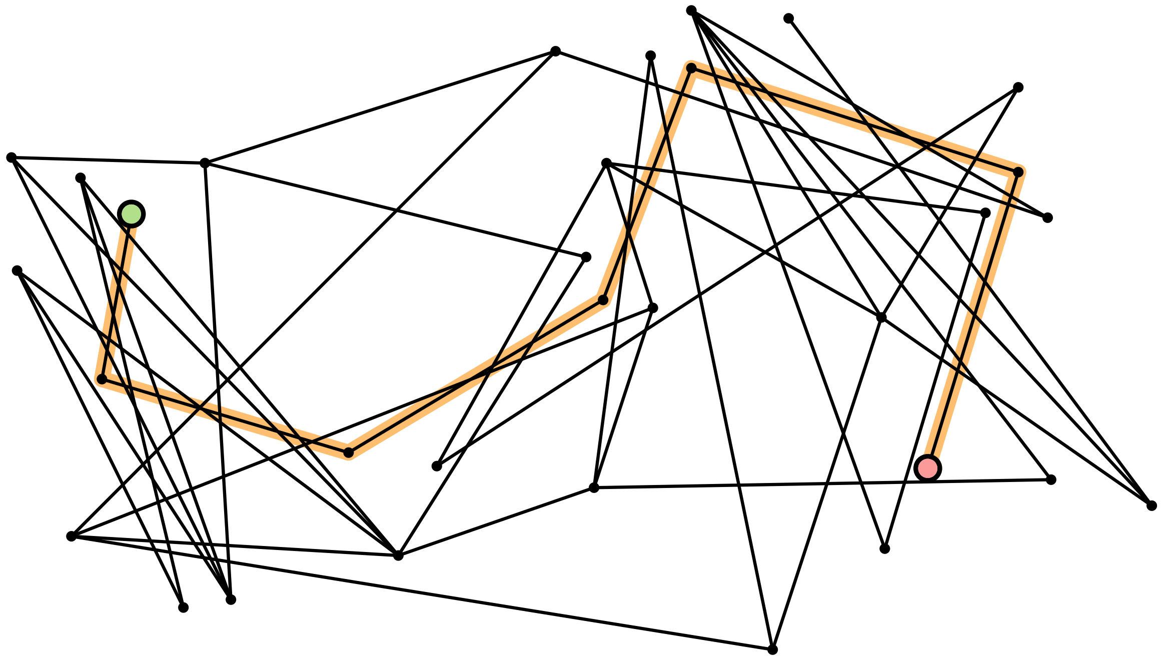
[Eades, Hong & Huang 2008]

Input: A graph drawing and designated path.

Task: Trace path and count number of edges.

Results: no crossings eye movements smooth and fast





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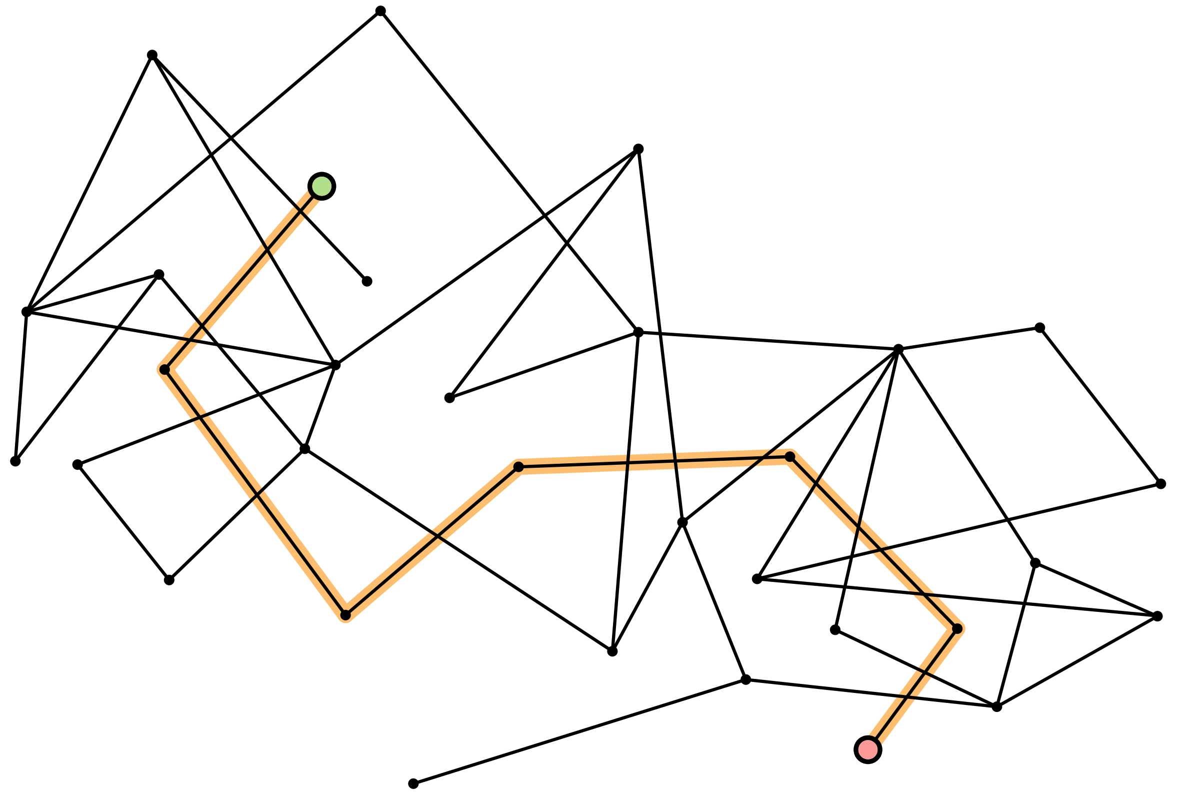
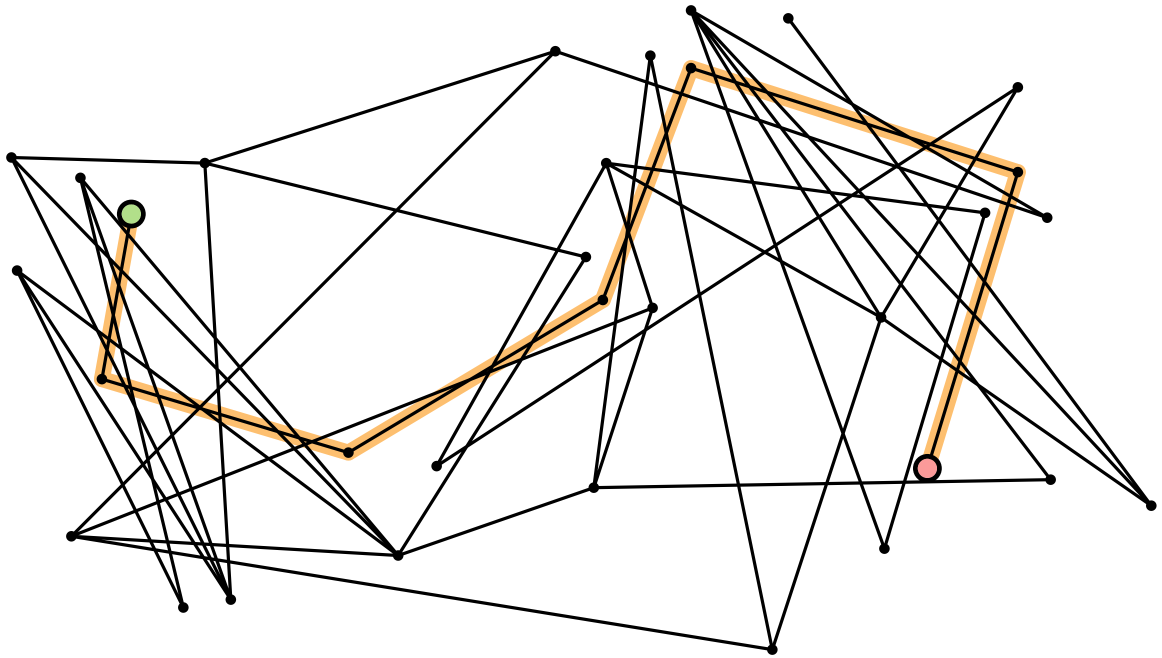
Task: Trace path and count number of edges.

Results:

no crossings	eye movements smooth and fast
large crossing angles	eye movements smooth but slightly slower





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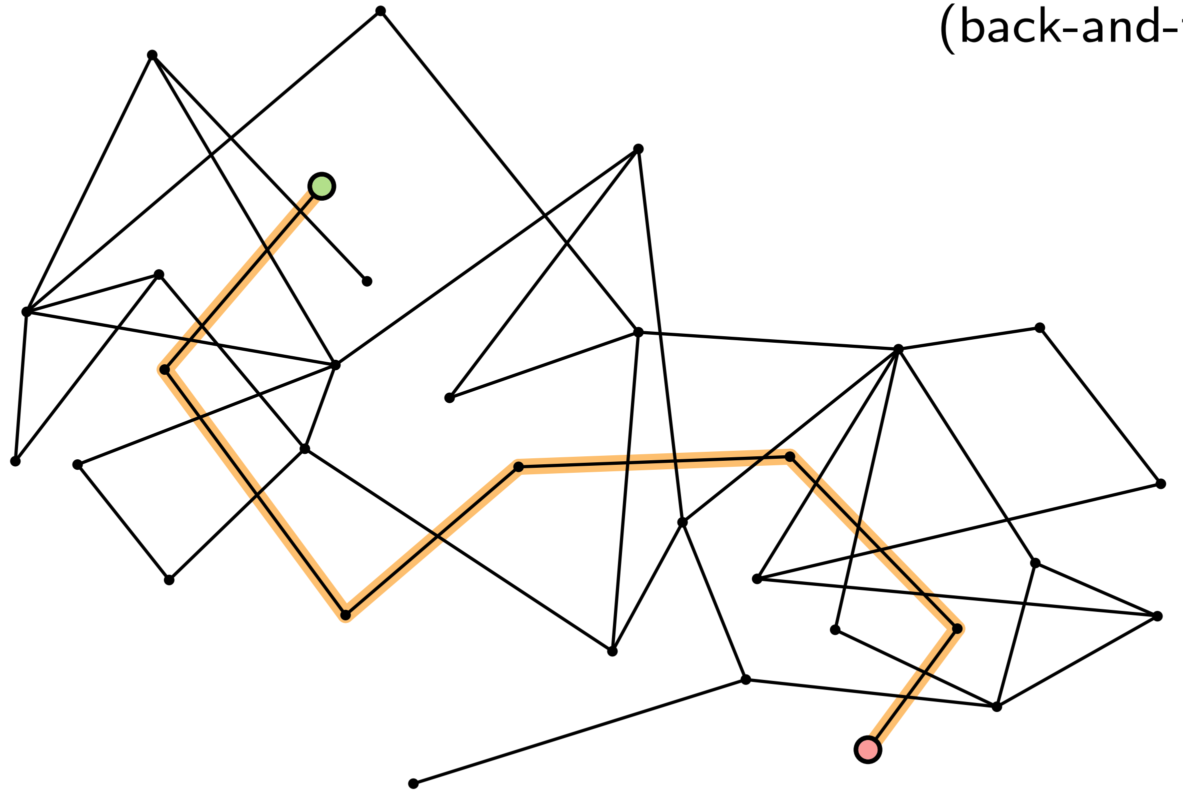
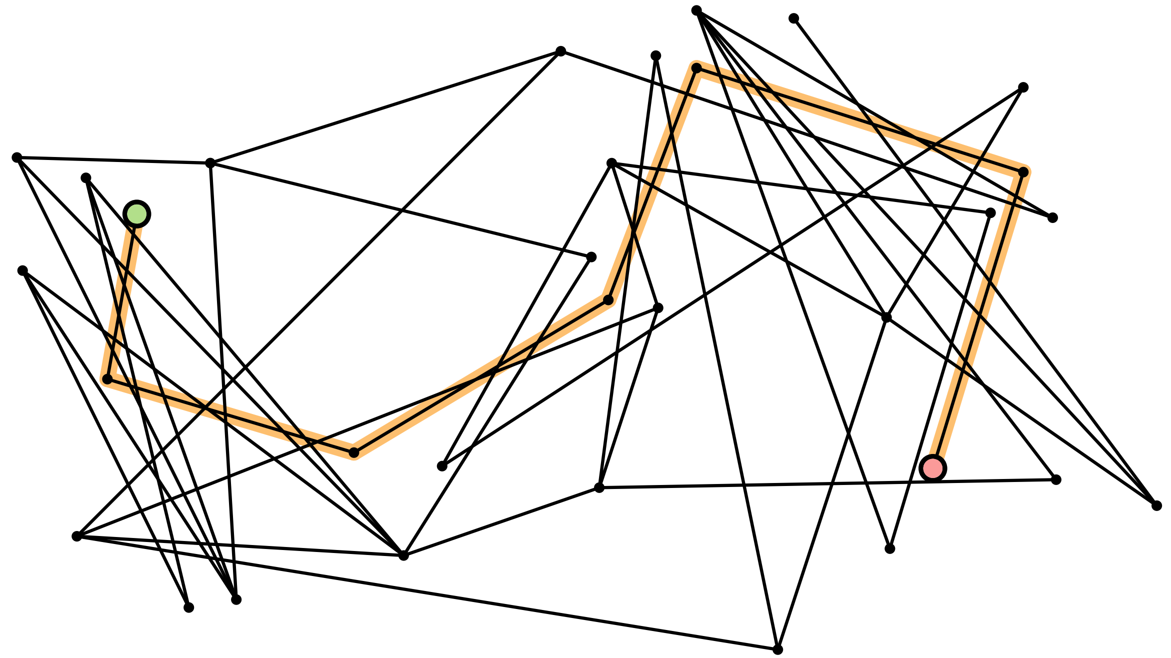
Task: Trace path and count number of edges.

Results:

no crossings	eye movements smooth and fast
large crossing angles	eye movements smooth but slightly slower
small crossing angles	eye movements no longer smooth and very slow (back-and-forth movements at crossing points)





Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

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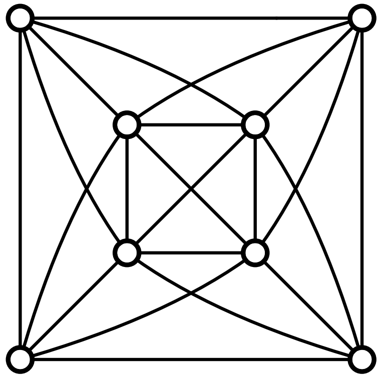

k-planar ($k = 1$)

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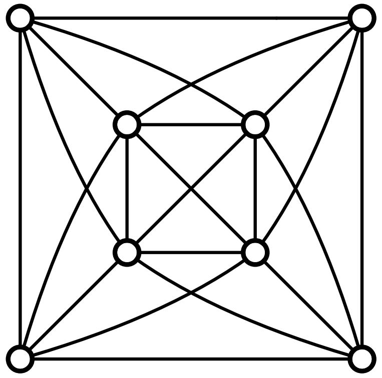

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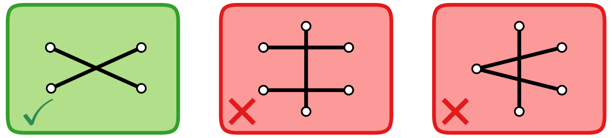

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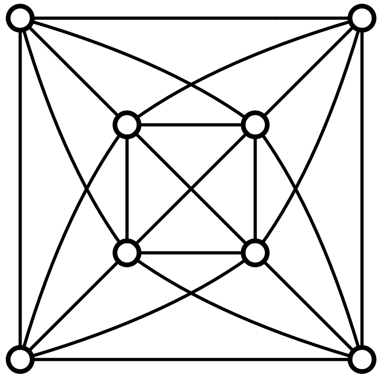

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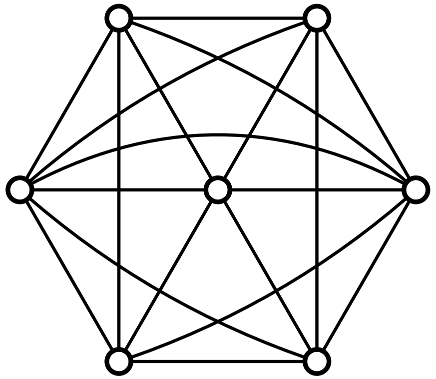
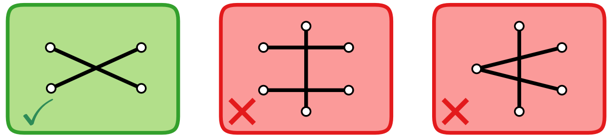


k-quasi-planar ($k = 3$)



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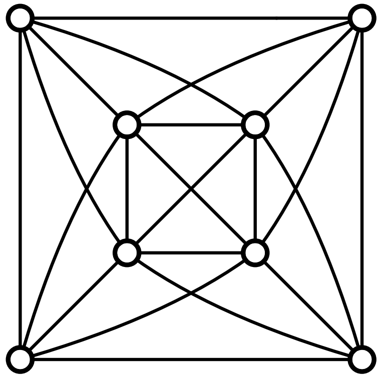

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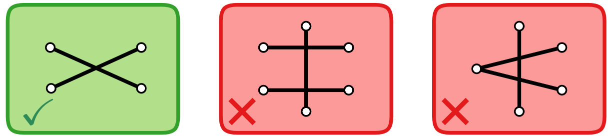

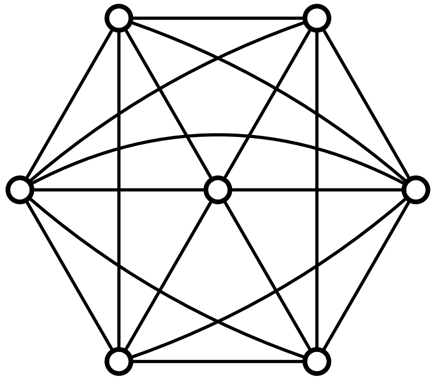

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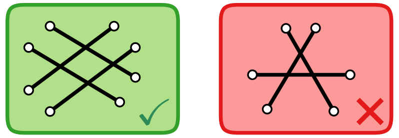

Some Beyond-Planar Graph Classes

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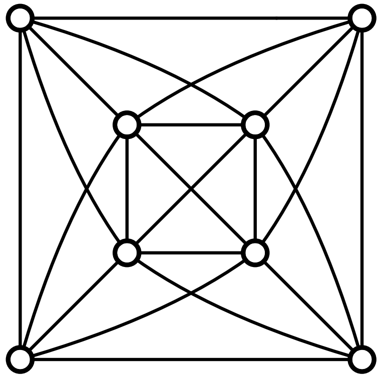
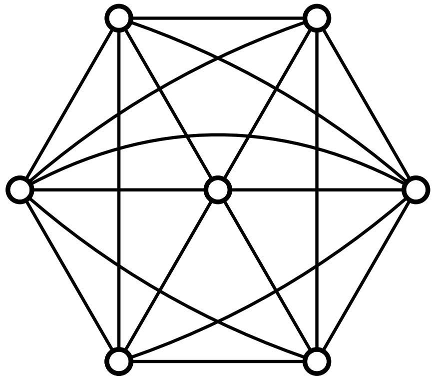
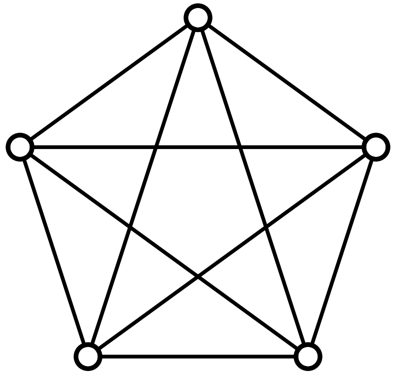


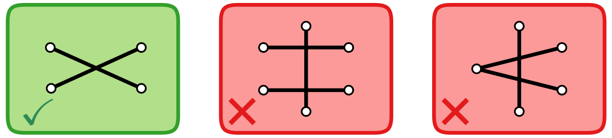
k-planar ($k = 1$)

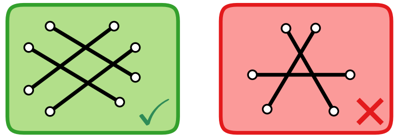


k-quasi-planar ($k = 3$)



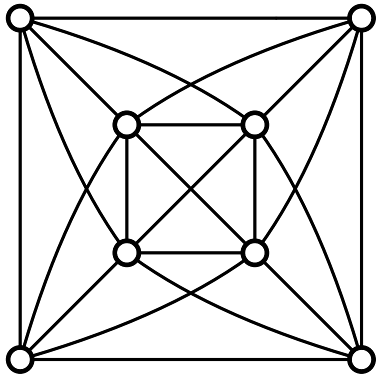
fan-planar



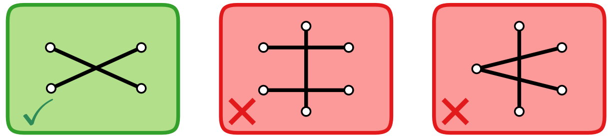


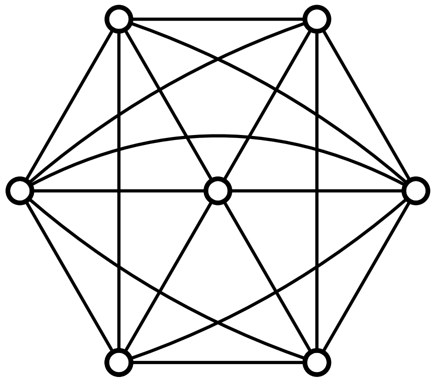
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.

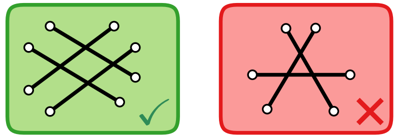


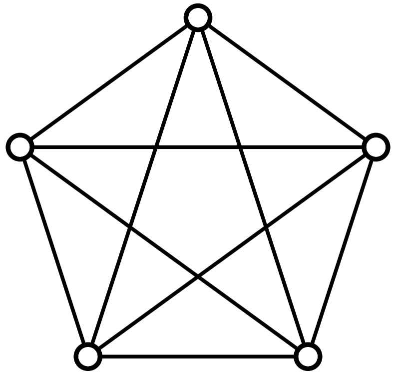
k-planar ($k = 1$)



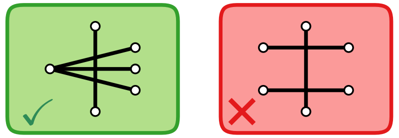


k-quasi-planar ($k = 3$)



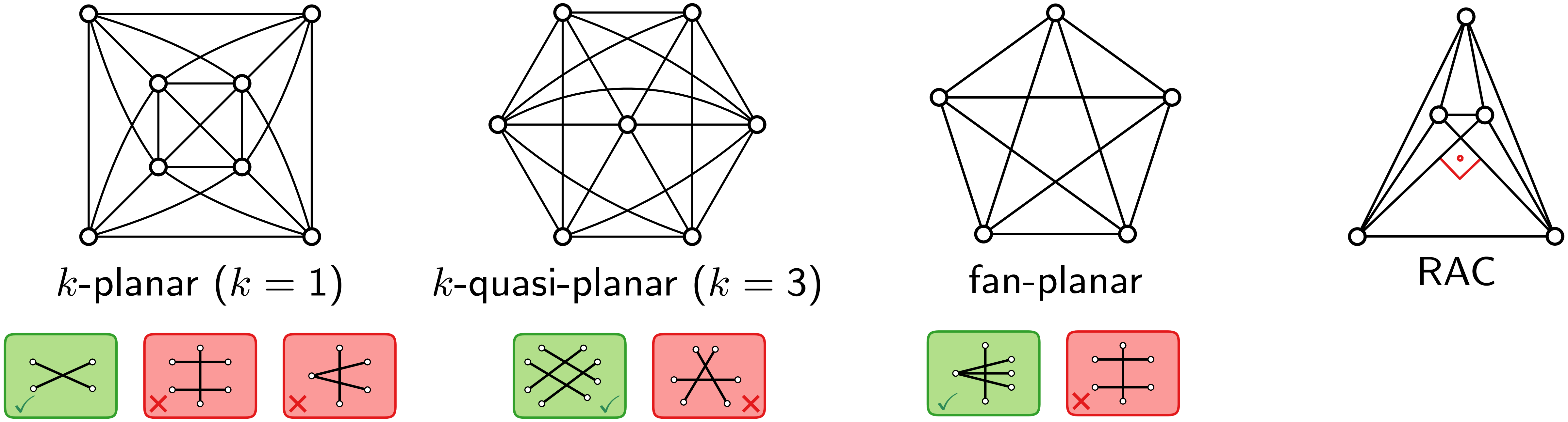


fan-planar



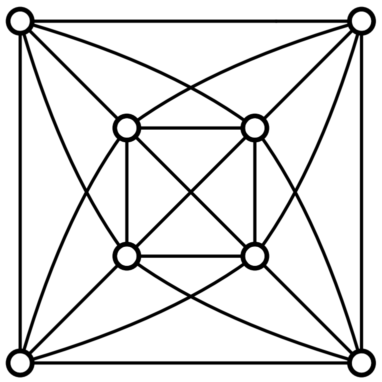
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

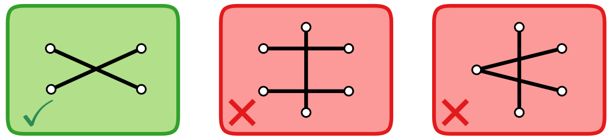


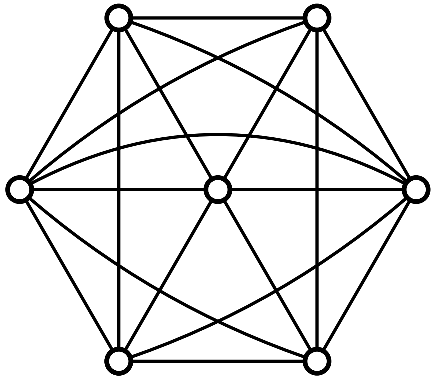
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

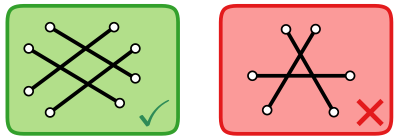


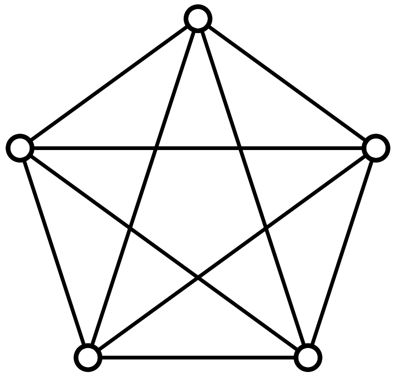
k-planar ($k = 1$)



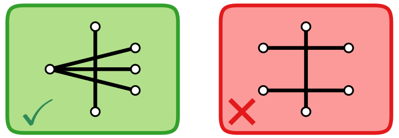


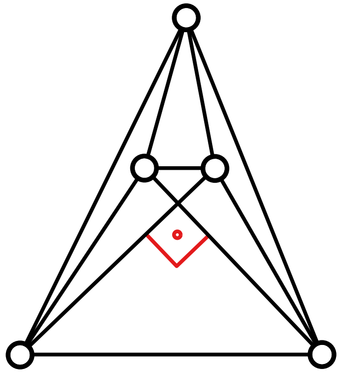
k-quasi-planar ($k = 3$)



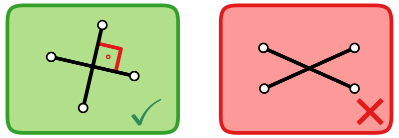


fan-planar



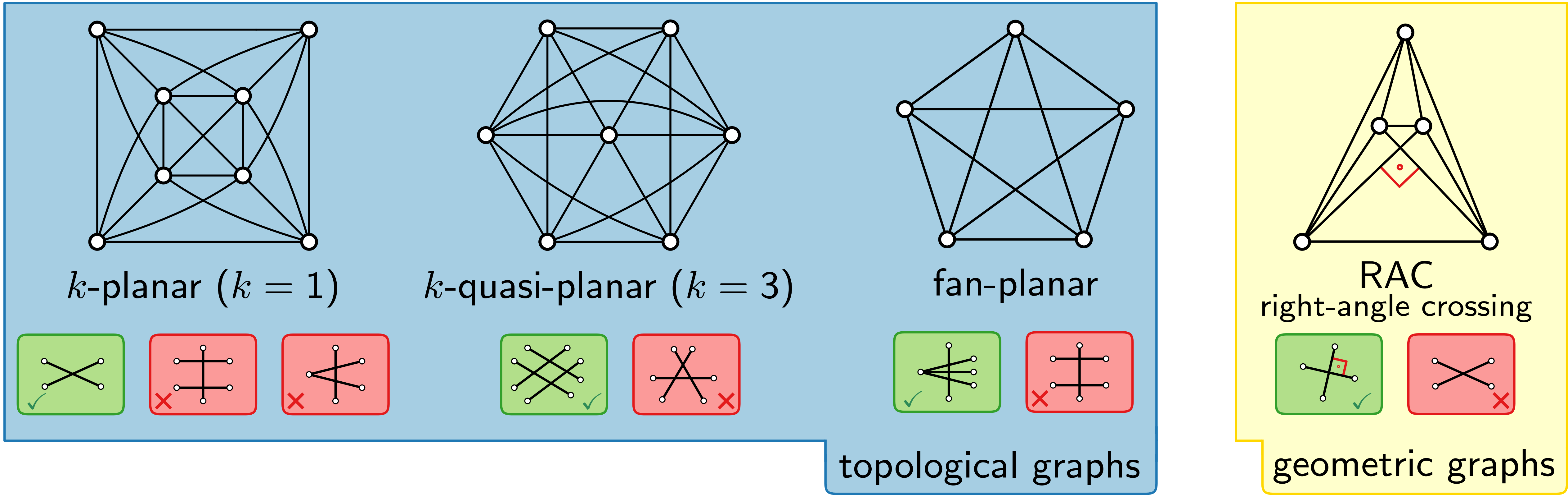


RAC
right-angle crossing



Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



Some Beyond-Planar Graph Classes

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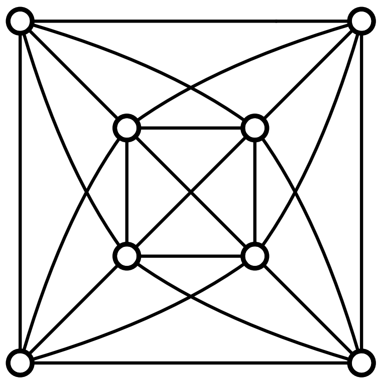
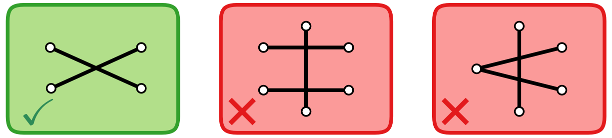
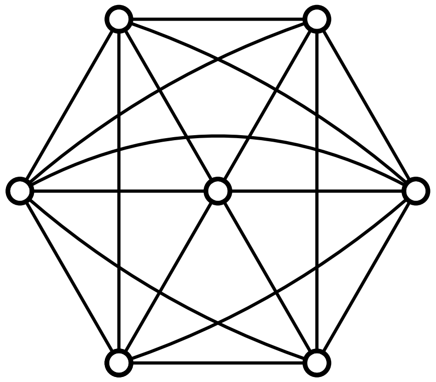
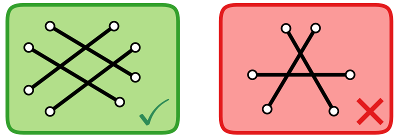


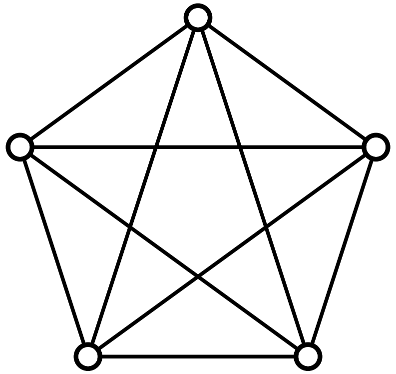
k-planar ($k = 1$)



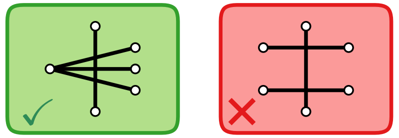


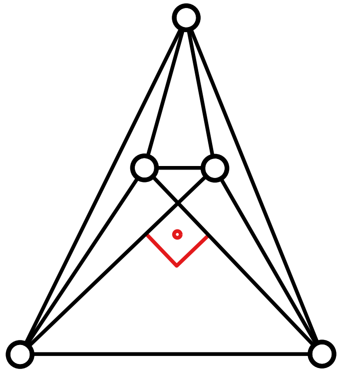
k-quasi-planar ($k = 3$)



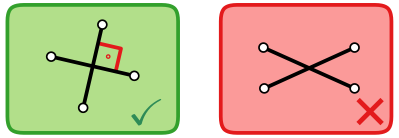


fan-planar





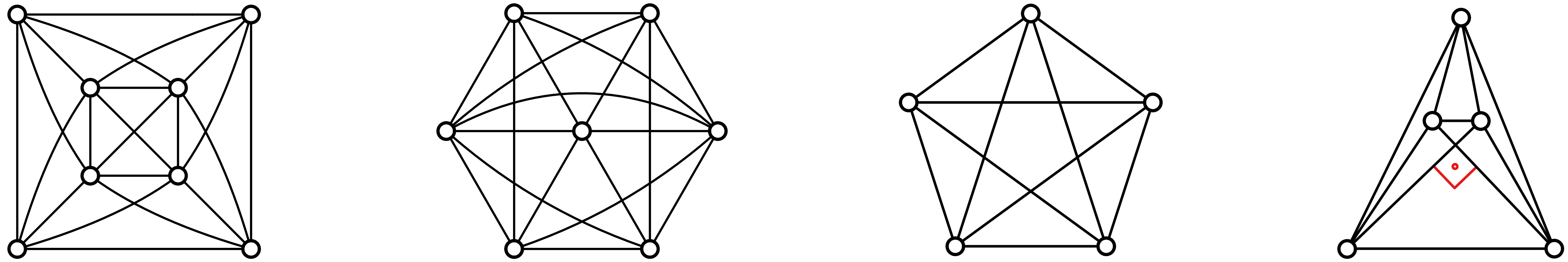
RAC
right-angle crossing



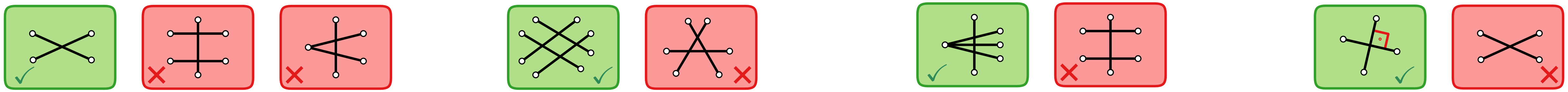
There are many more beyond-planar graph classes...

Some Beyond-Planar Graph Classes

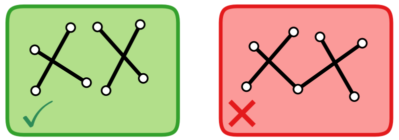
We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



k-planar ($k = 1$) k-quasi-planar ($k = 3$) fan-planar RAC right-angle crossing



There are many more beyond-planar graph classes. . .



IC (independent crossing)

Some Beyond-Planar Graph Classes

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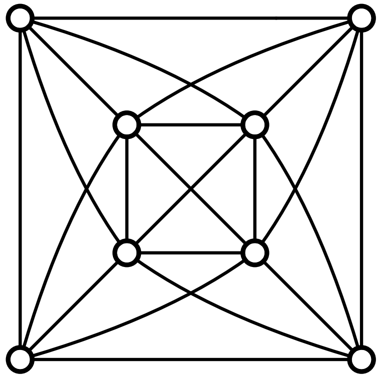
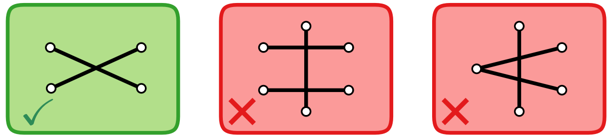
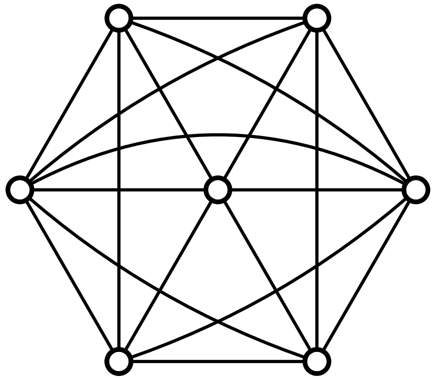
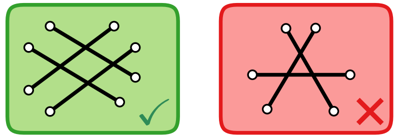


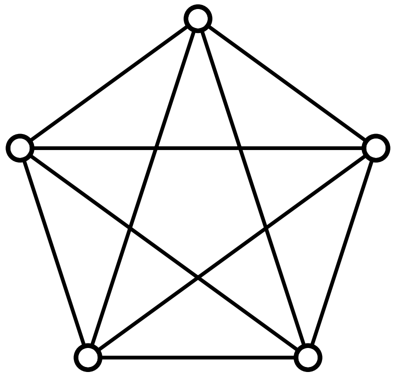
k-planar ($k = 1$)



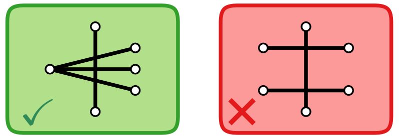


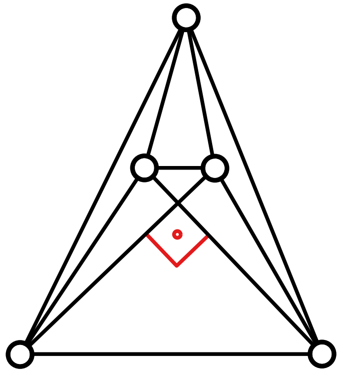
k-quasi-planar ($k = 3$)



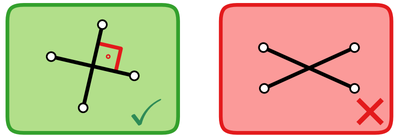


fan-planar





RAC
right-angle crossing



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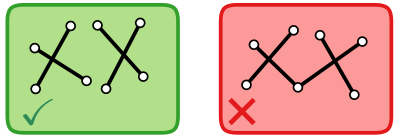
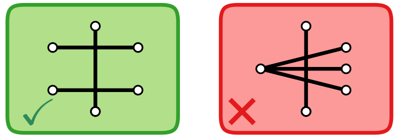


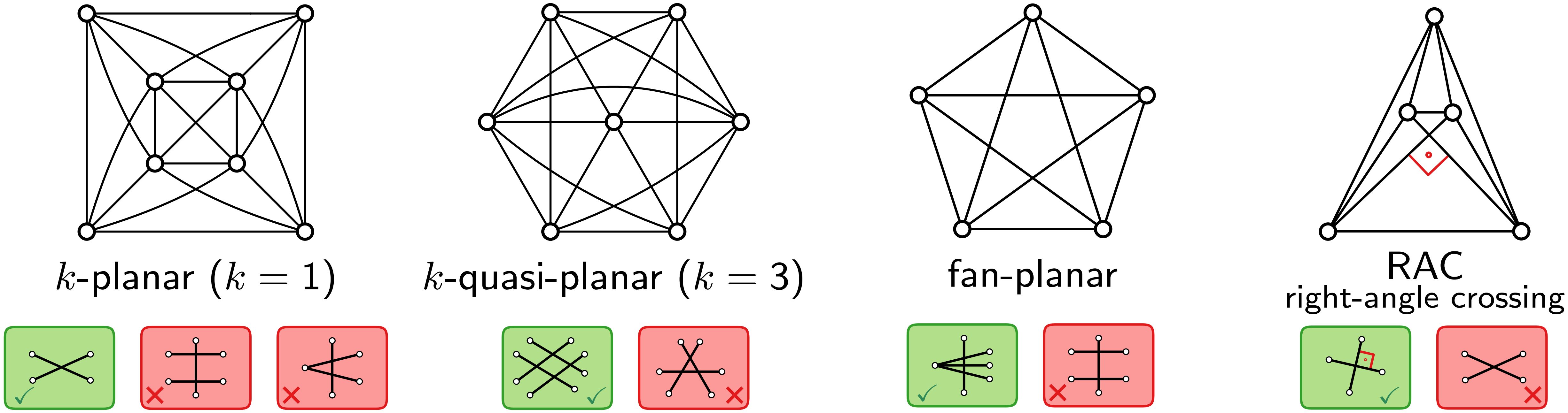
IC (independent crossing)



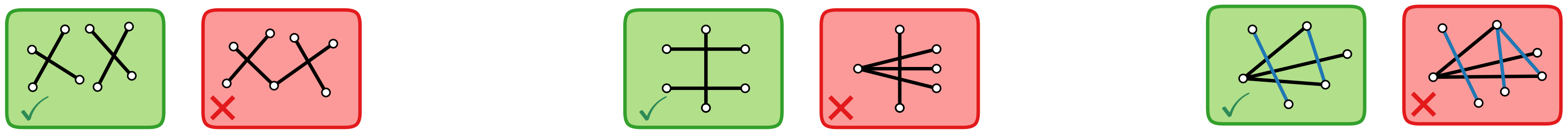
fan-crossing-free

Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



There are many more beyond-planar graph classes...



IC (independent crossing)

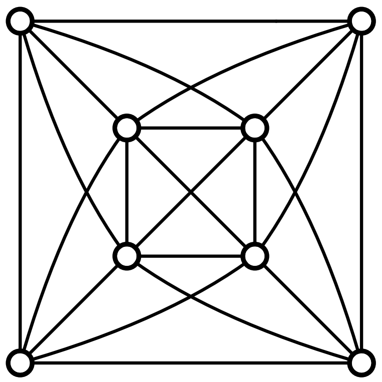
fan-crossing-free

skewness-k ($k = 2$)

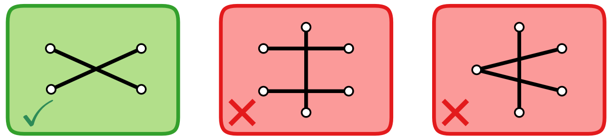
remove $\leq k$ edges to make it planar

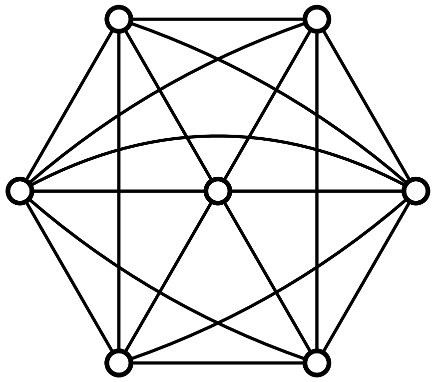
Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

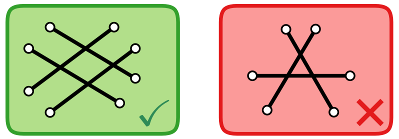


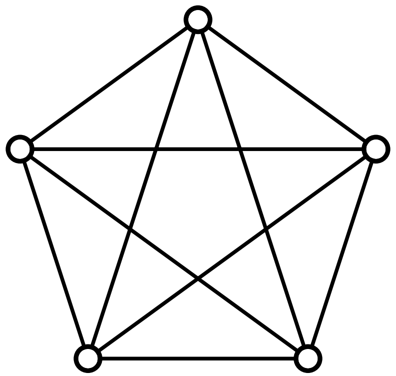
k-planar ($k = 1$)



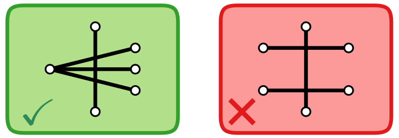


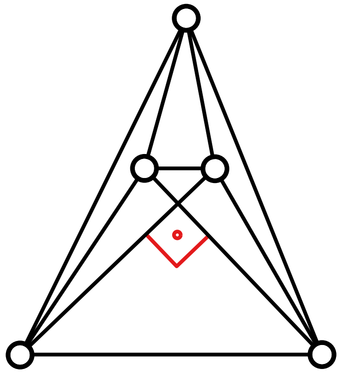
k-quasi-planar ($k = 3$)



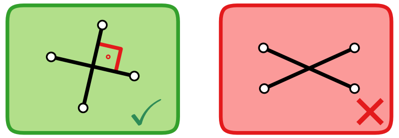


fan-planar

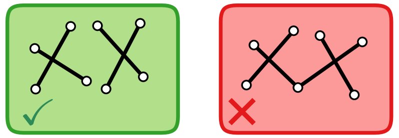




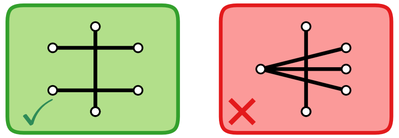
RAC
right-angle crossing



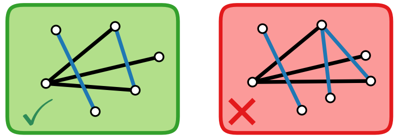
There are many more beyond-planar graph classes...



IC (independent crossing)



fan-crossing-free

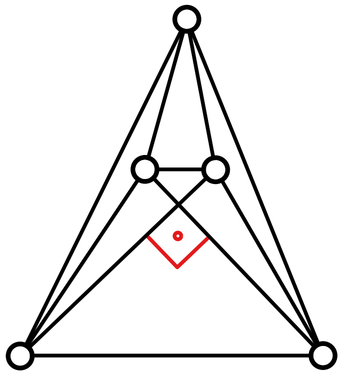


skewness-k ($k = 2$)

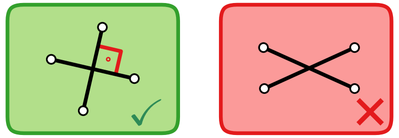
remove $\leq k$ edges to make it planar

combinations, ...

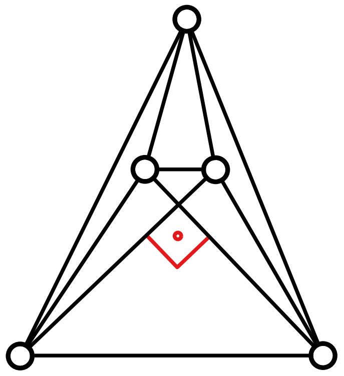
Drawing Styles for Crossings



RAC
right-angle crossing



Drawing Styles for Crossings



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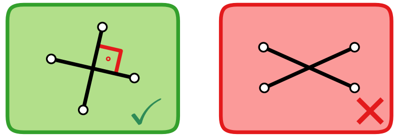
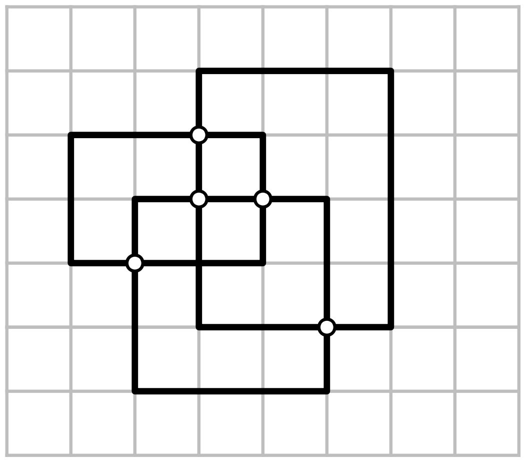
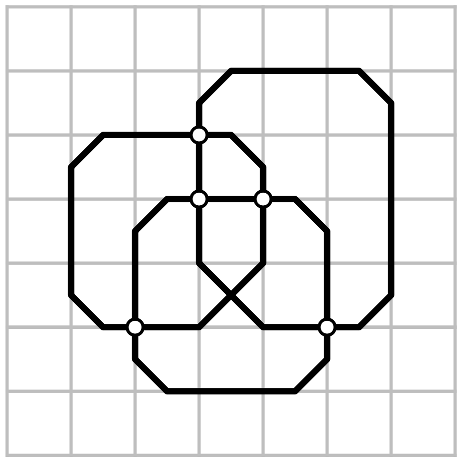
right-angle crossing



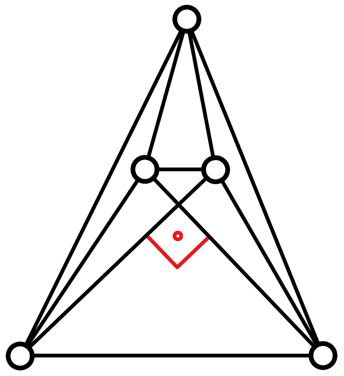


orthogonal

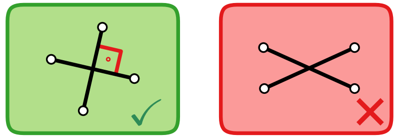


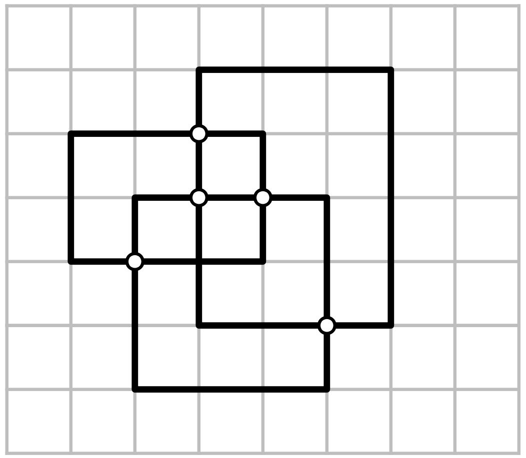
slanted orthogonal

Drawing Styles for Crossings

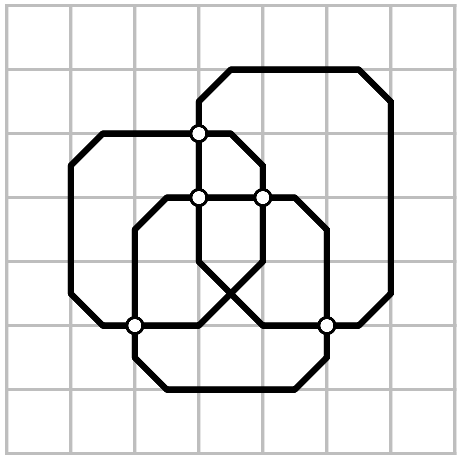


RAC
right-angle crossing

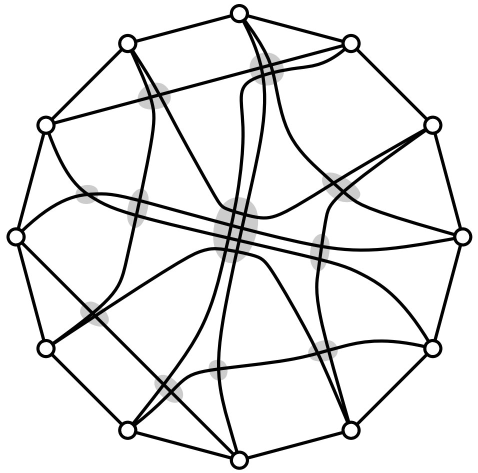




orthogonal



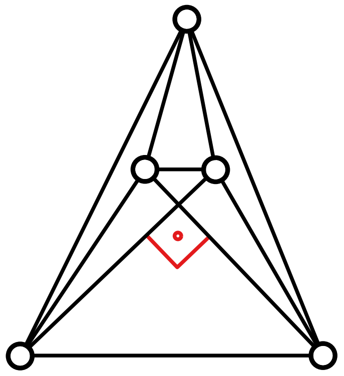
slanted orthogonal



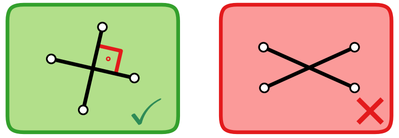
block / bundled crossings

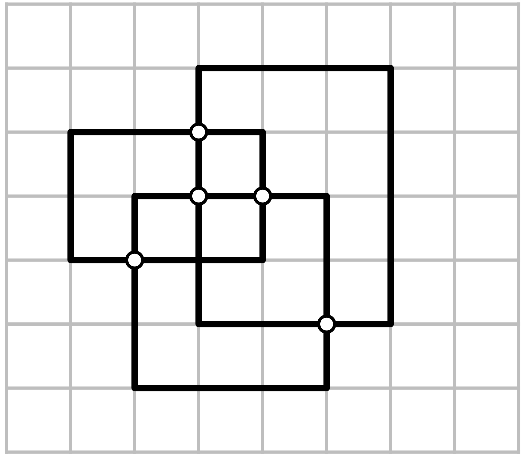
circular layout: 28 invididual
vs. 12 bundle crossings

Drawing Styles for Crossings

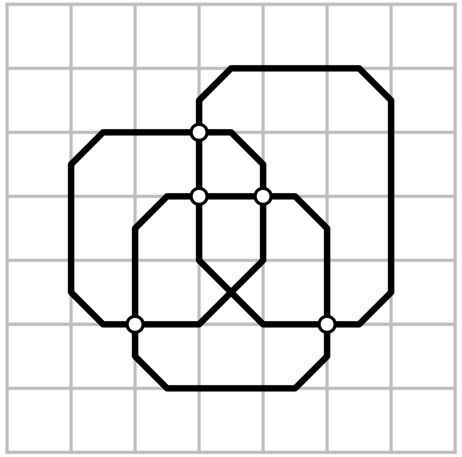


RAC
right-angle crossing

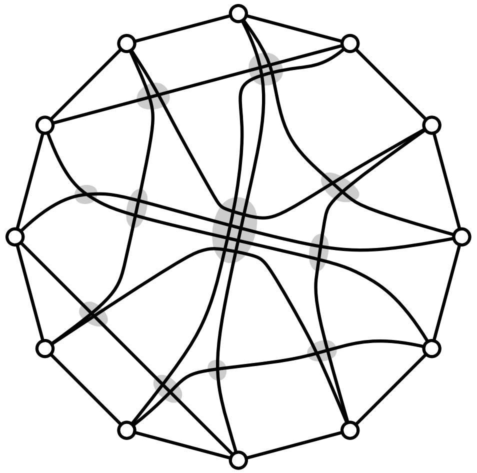




orthogonal

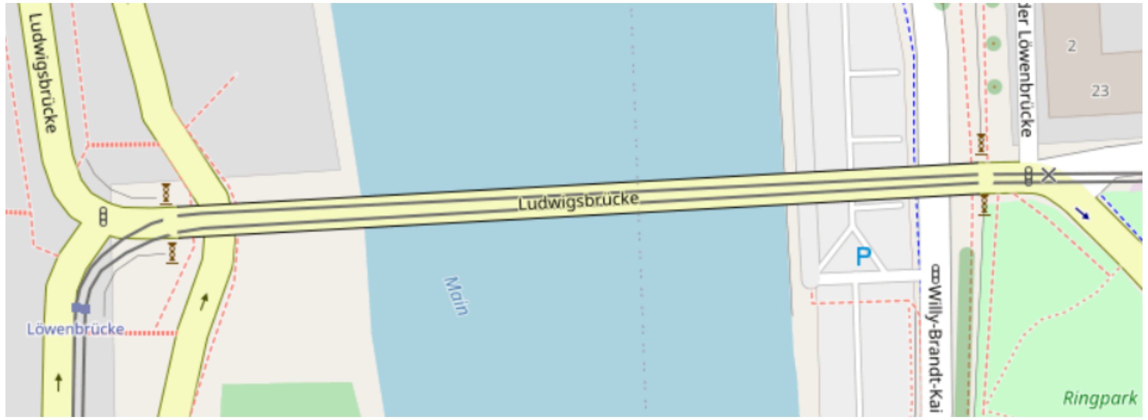


slanted orthogonal

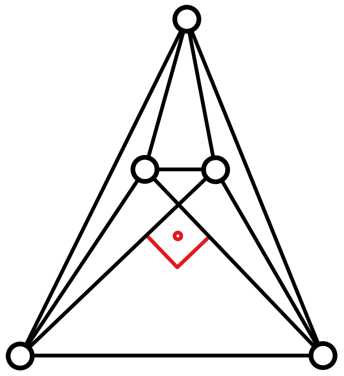


block / bundled crossings

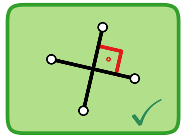
circular layout: 28 invididual
vs. 12 bundle crossings

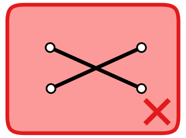


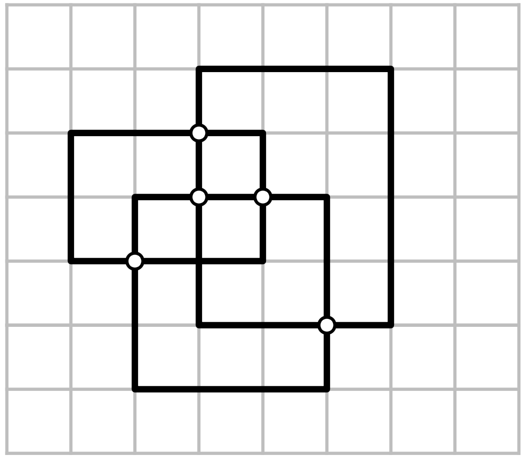
Drawing Styles for Crossings



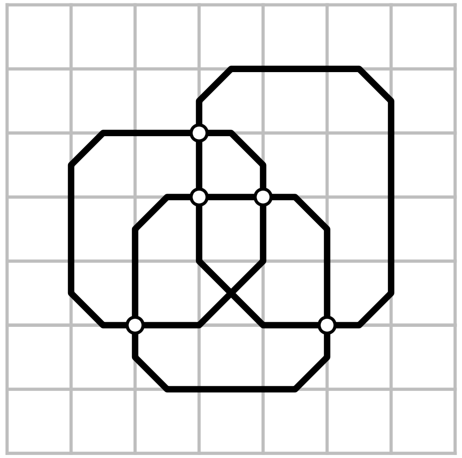
RAC
right-angle crossing



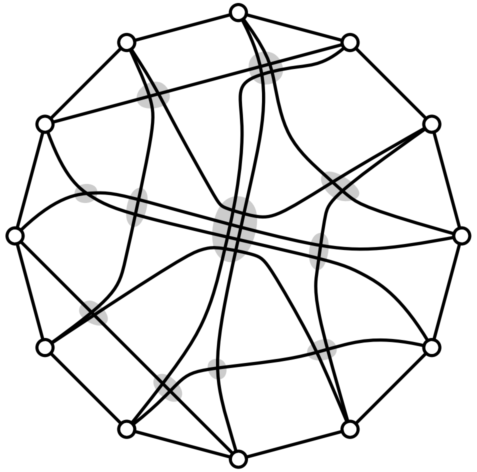




orthogonal



slanted orthogonal



block / bundled crossings

circular layout: 28 invididual
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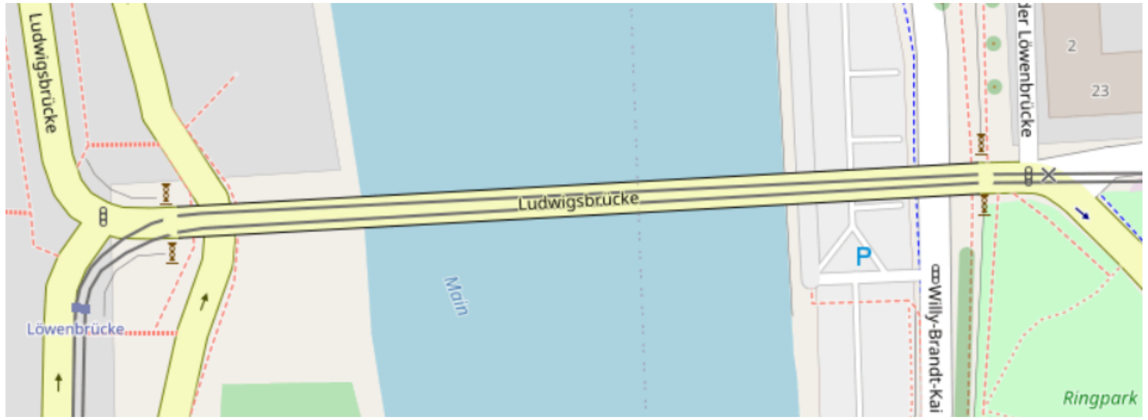
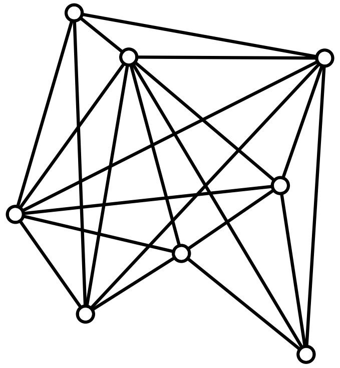
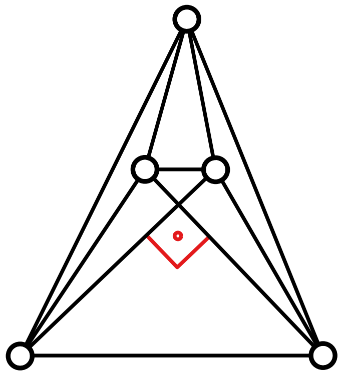
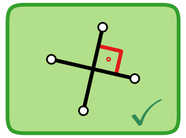


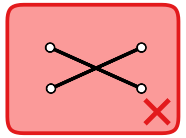


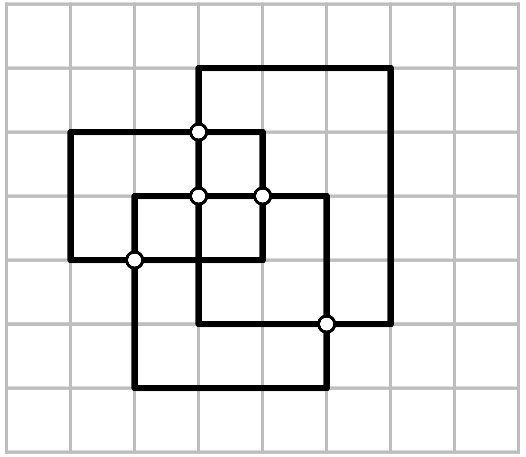
Drawing Styles for Crossings



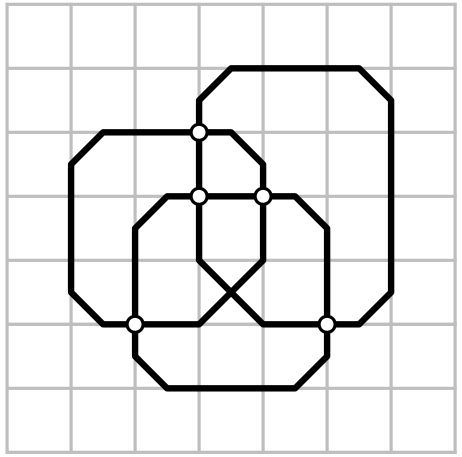
RAC
right-angle crossing



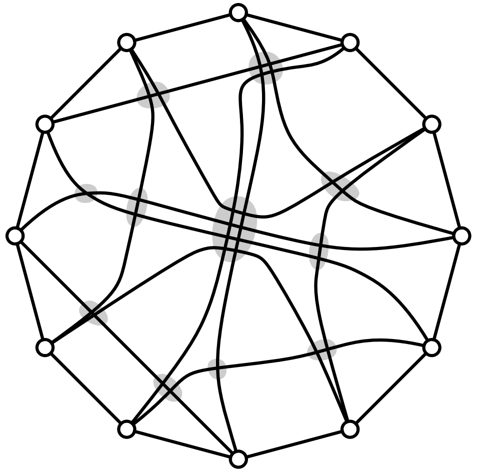




orthogonal



slanted orthogonal



block / bundled crossings

circular layout: 28 invididual
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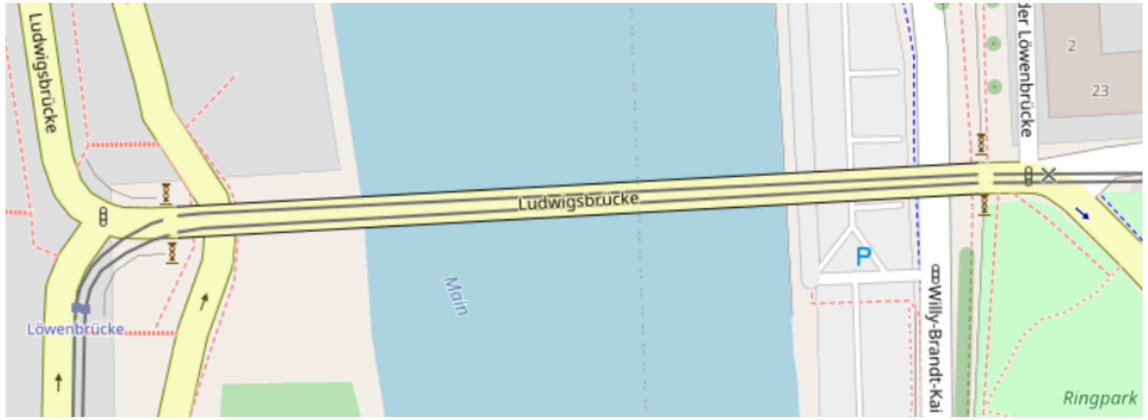
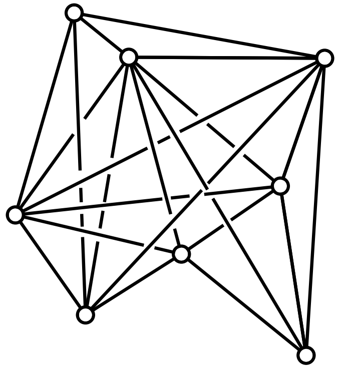
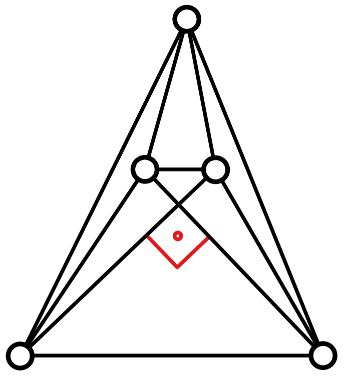


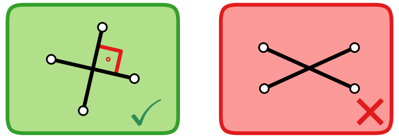


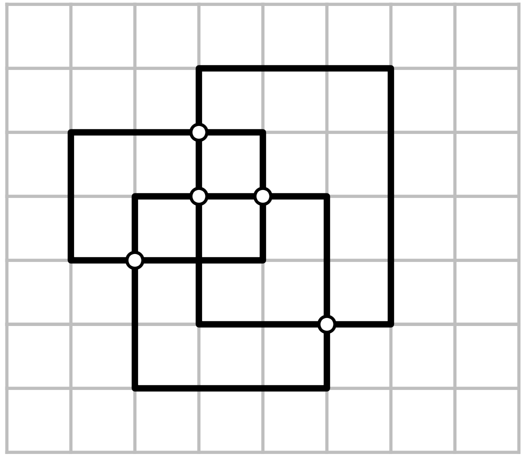
cased crossings

Drawing Styles for Crossings

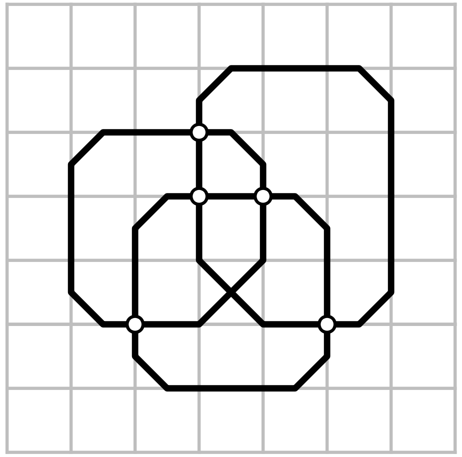


RAC
right-angle crossing

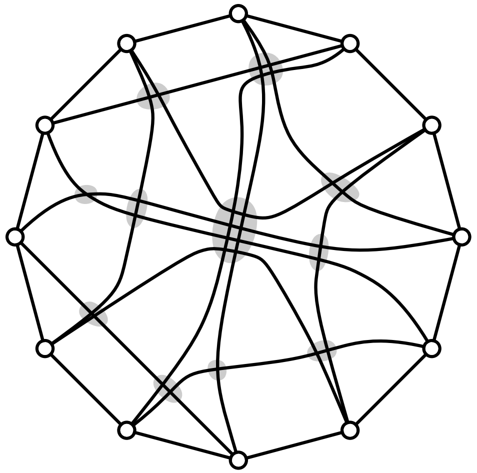




orthogonal



slanted orthogonal



block / bundled crossings

circular layout: 28 invididual
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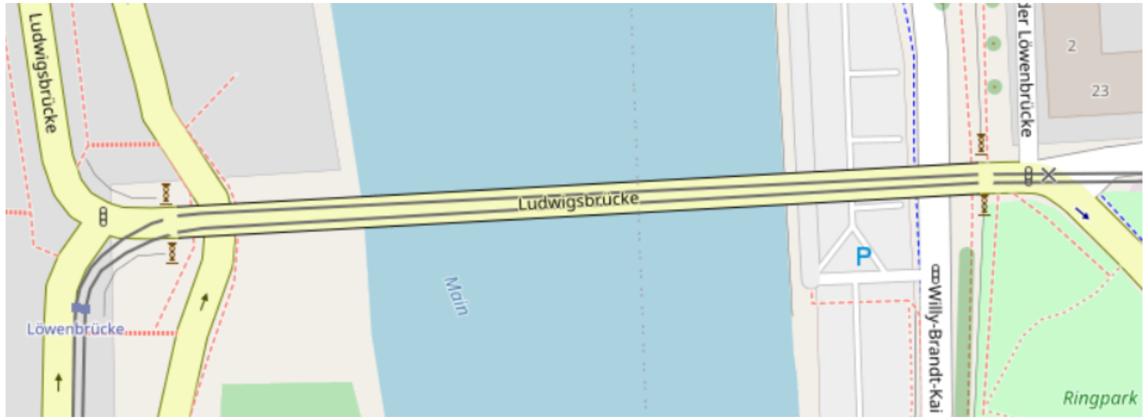
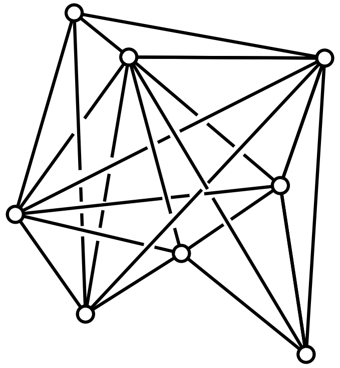
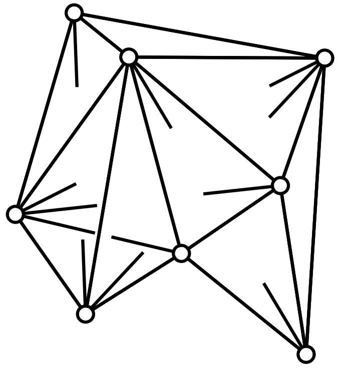


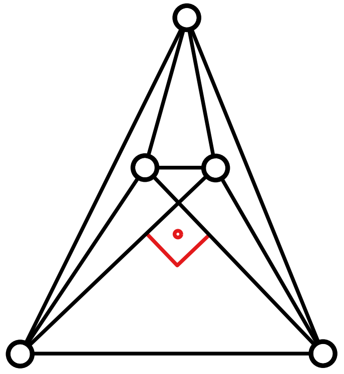


cased crossings

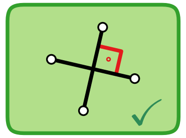


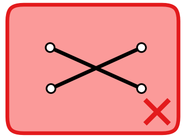
symmetric partial
edge drawing

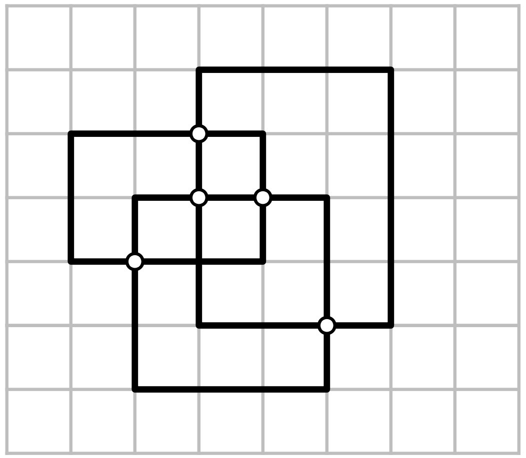
Drawing Styles for Crossings



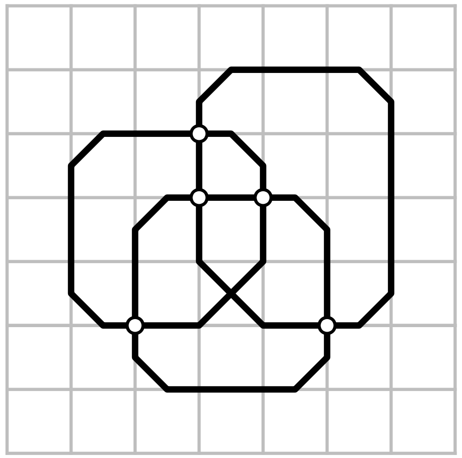
RAC
right-angle crossing



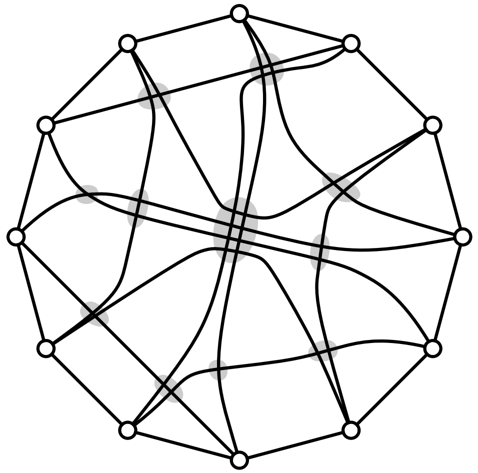




orthogonal

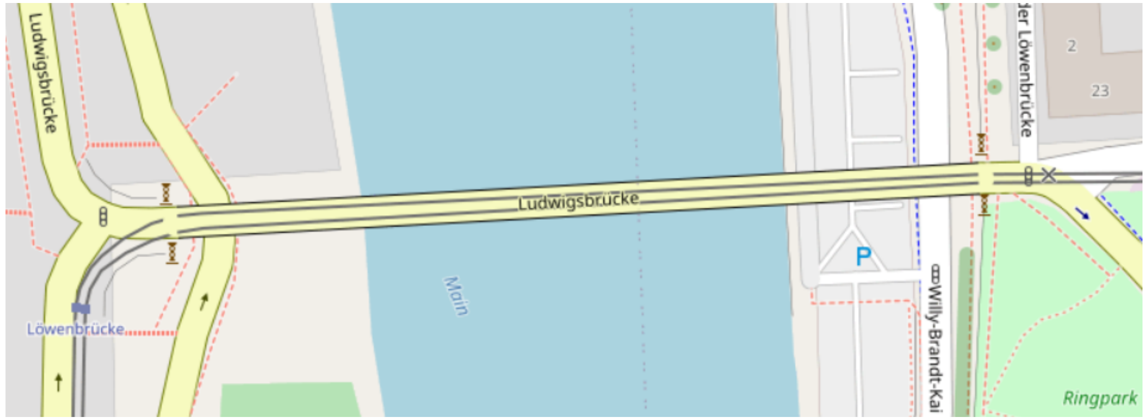


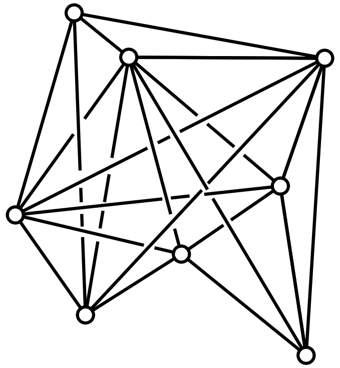
slanted orthogonal



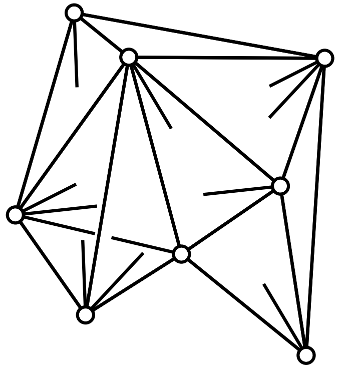
block / bundled crossings

circular layout: 28 invididual
vs. 12 bundle crossings

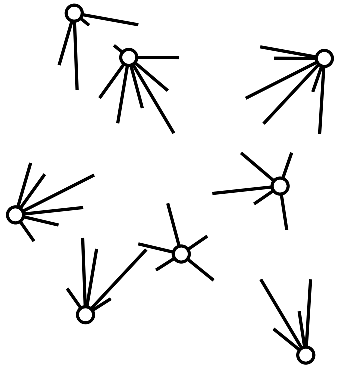




cased crossings

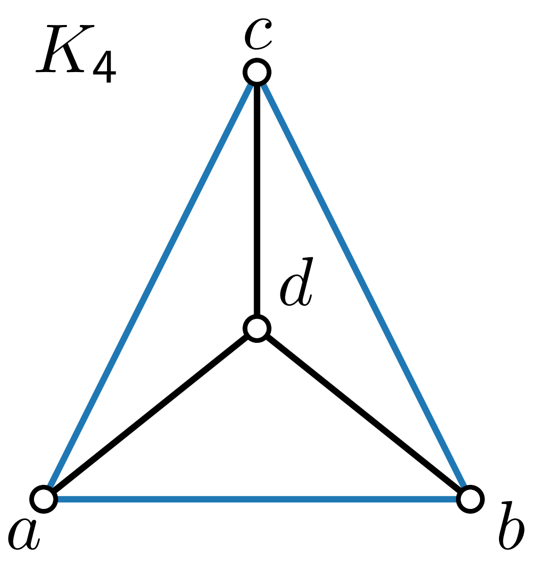


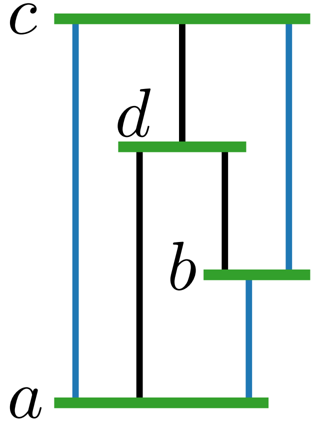
symmetric partial
edge drawing



1/4-SHPED
symmetric homogenous
partial edge drawing

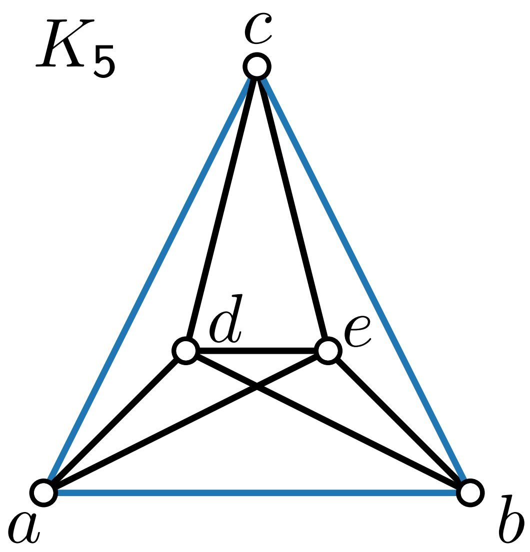
Geometric Representations

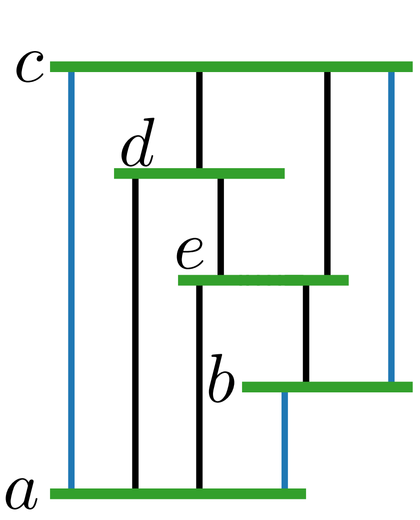




bar visibility
representation

Geometric Representations

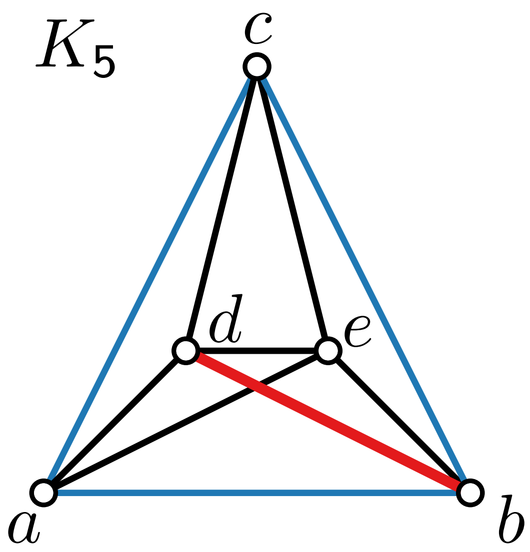


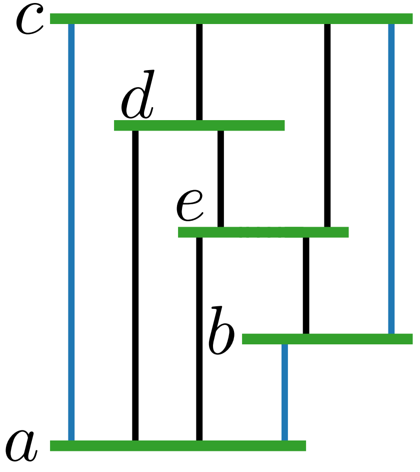


bar 1-visibility
representation (B1VR)

lines of sight through $\leq$ 1 bars

Geometric Representations





bar 1-visibility
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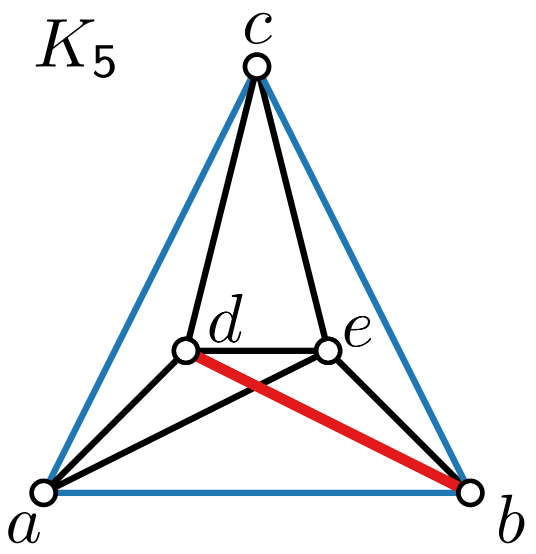
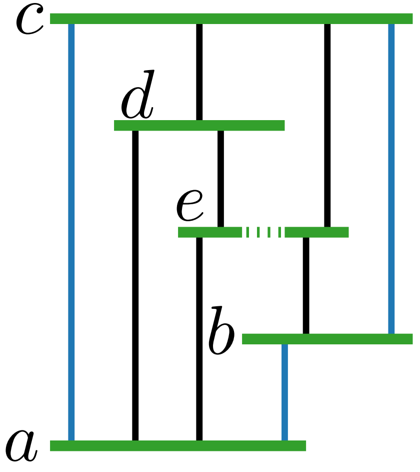
Geometric Representations





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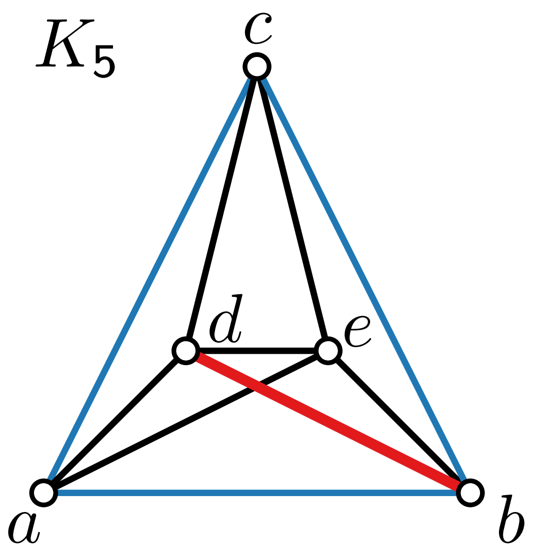
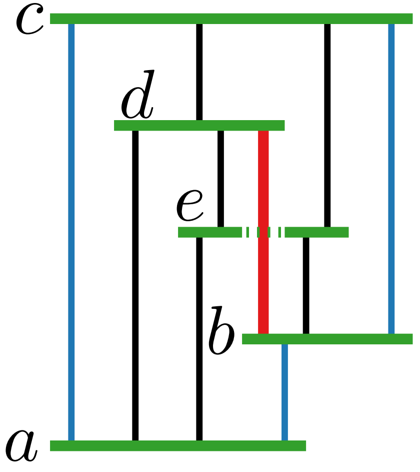
Geometric Representations

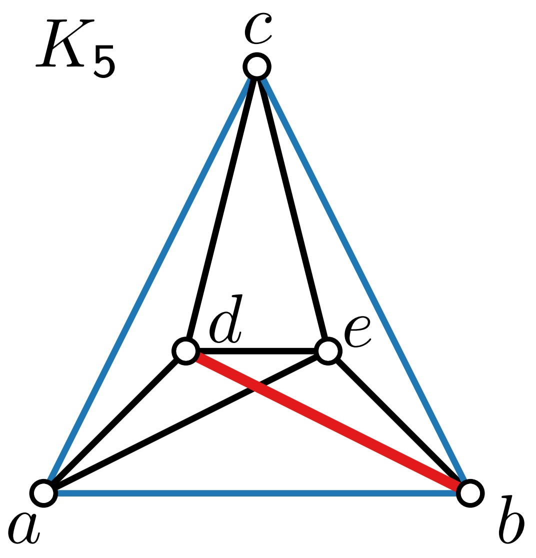


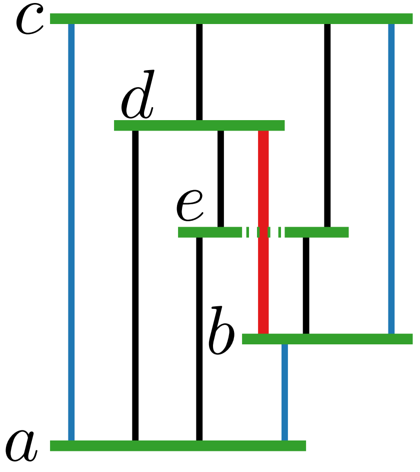


bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars

Geometric Representations



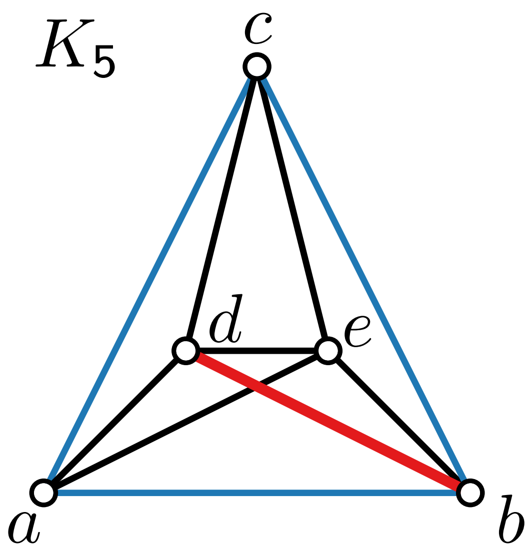


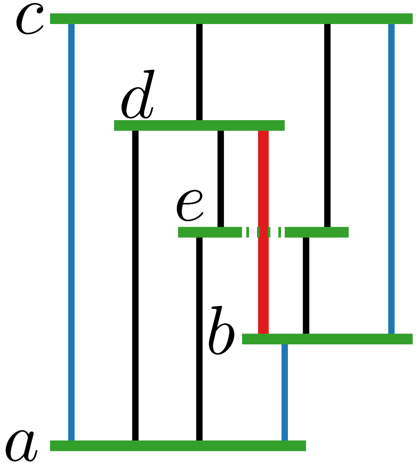
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014;
  Angelini et al. 2018]

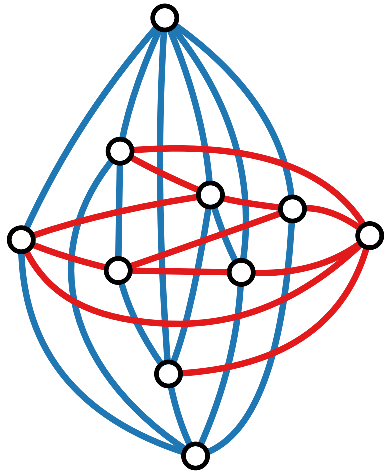
Geometric Representations





bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars

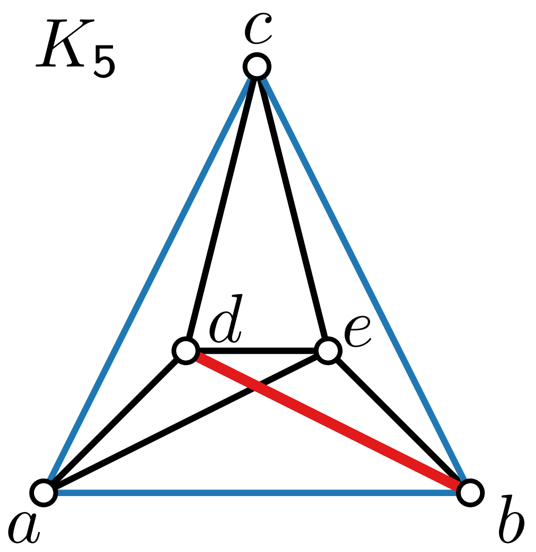


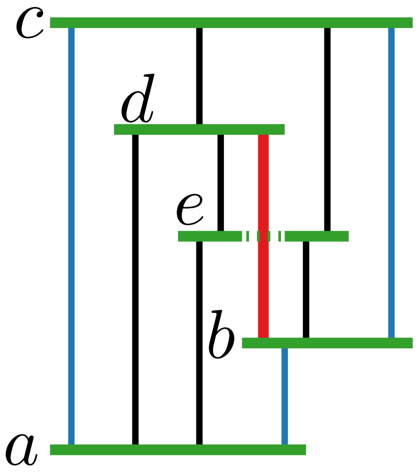
thickness-2 graph

decompose into 2 planar graphs

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

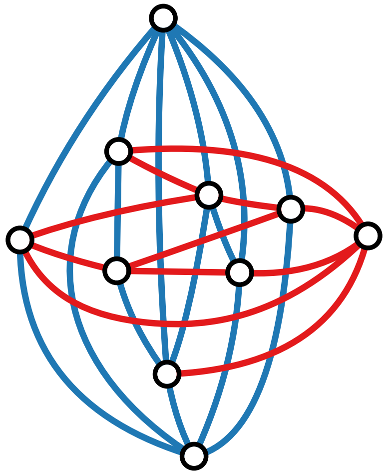
Geometric Representations





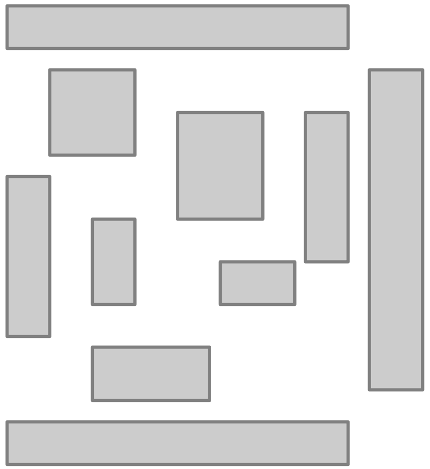
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

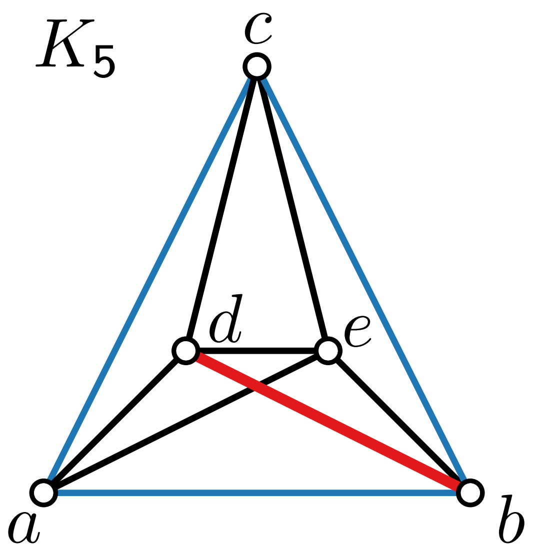
decompose into 2 planar graphs

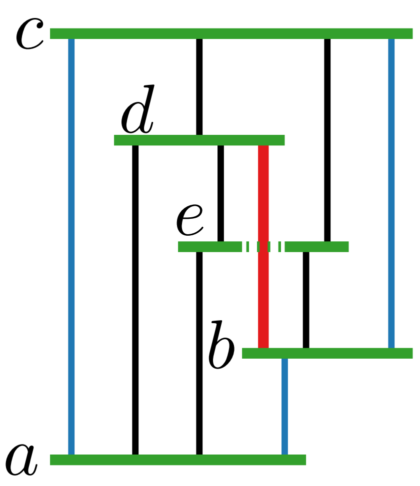


rectangle visibility representation

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

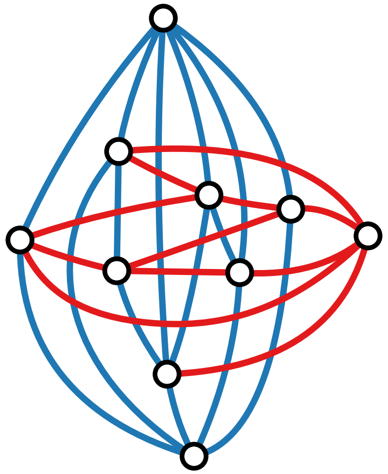
Geometric Representations





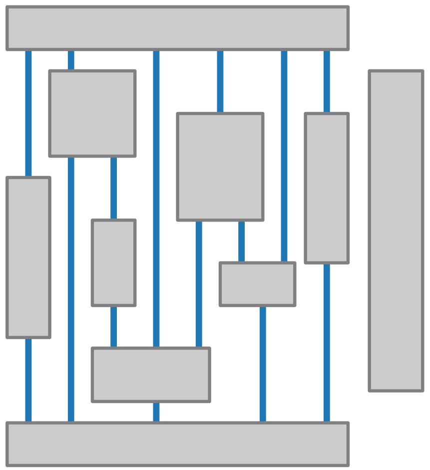
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

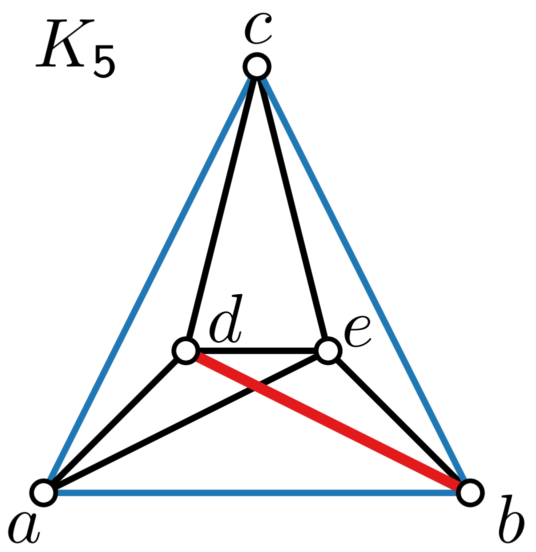
decompose into 2 planar graphs

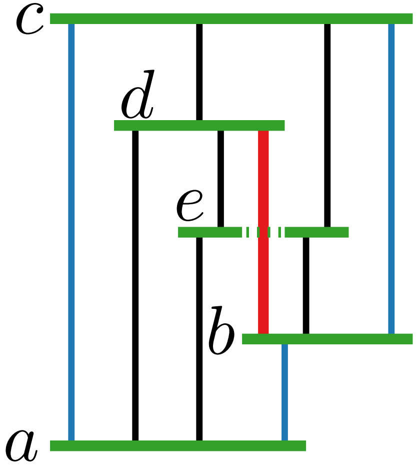


rectangle visibility representation

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

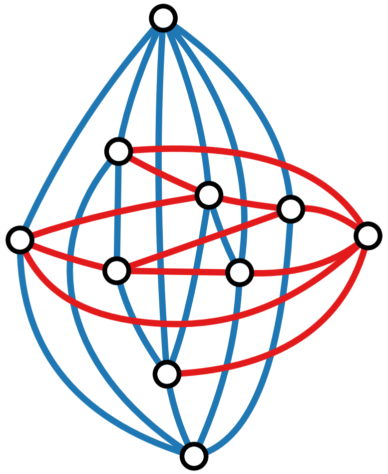
Geometric Representations





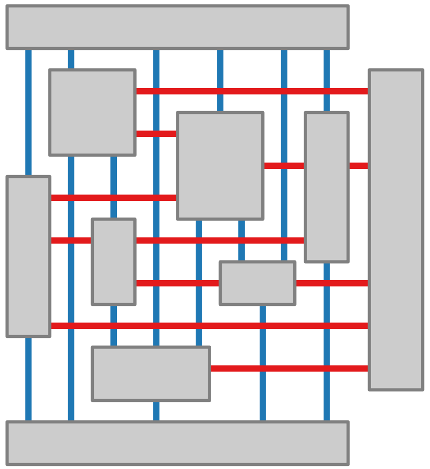
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

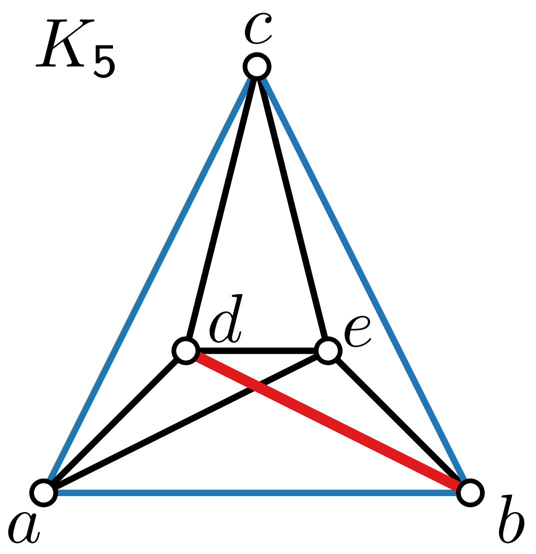
decompose into 2 planar graphs

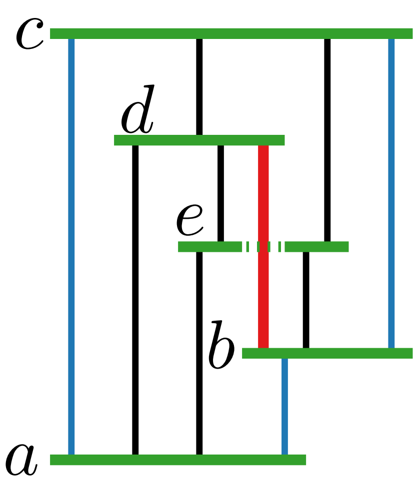


rectangle visibility representation

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]

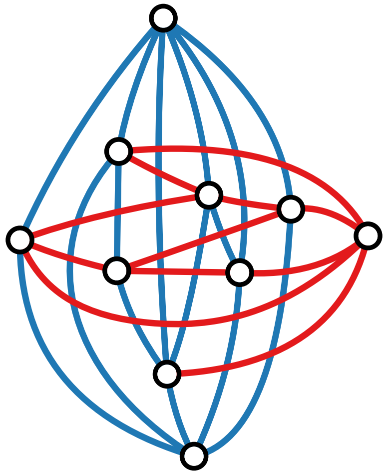
Geometric Representations





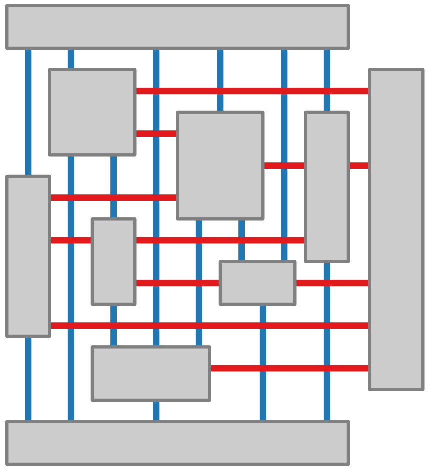
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

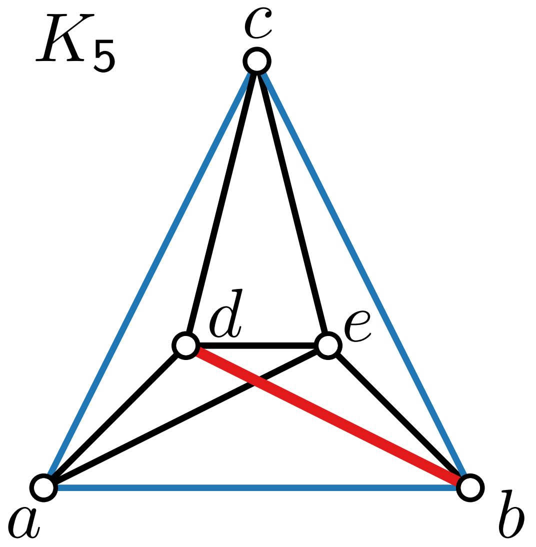
decompose into 2 planar graphs

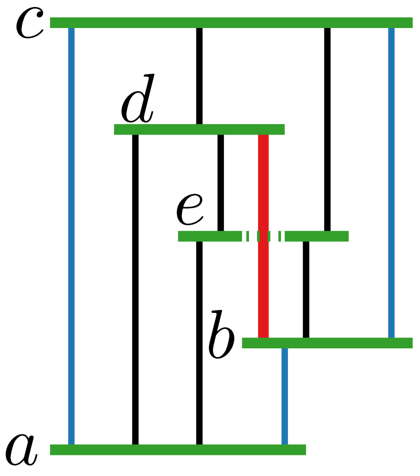


rectangle visibility representation

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- Rectangle visibility graphs (RVGs) have $\leq 6n - 20$ edges. [Hutchinson, Shermer, Vince 1996]

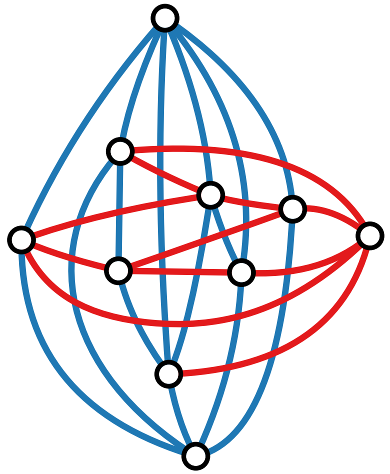
Geometric Representations





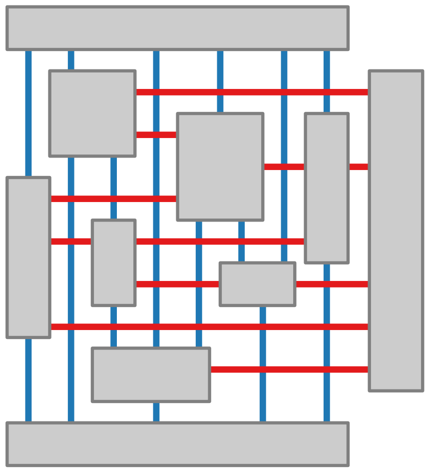
bar 1-visibility representation (B1VR)

lines of sight through $\leq$ 1 bars



thickness-2 graph

decompose into 2 planar graphs



rectangle visibility representation

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- Rectangle visibility graphs (RVGs) have $\leq 6n - 20$ edges. [Hutchinson, Shermer, Vince 1996]
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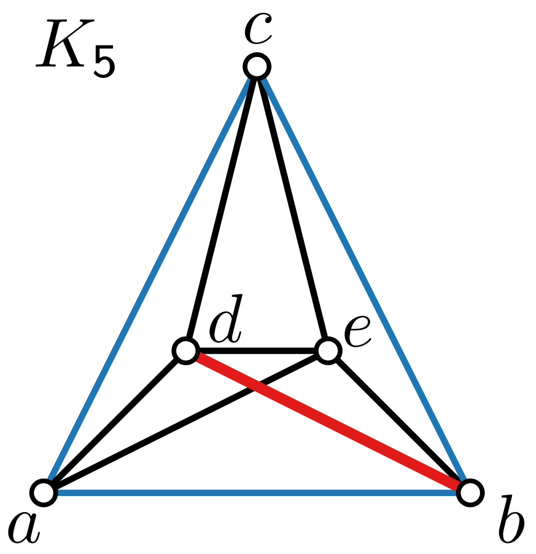
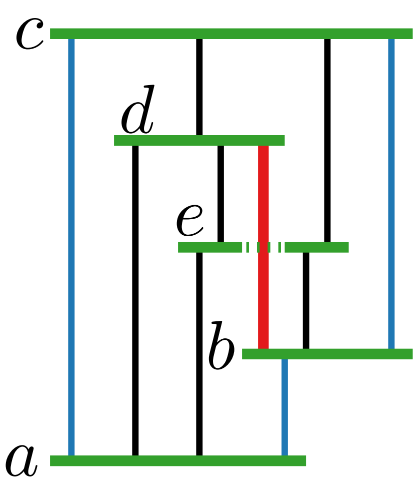
Geometric Representations





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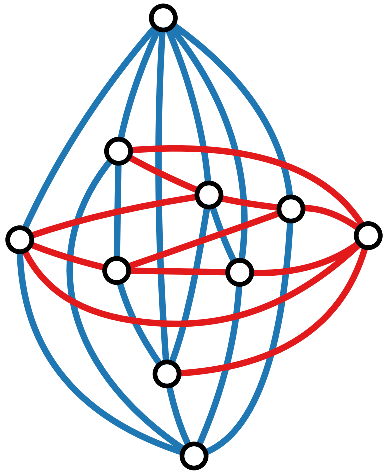

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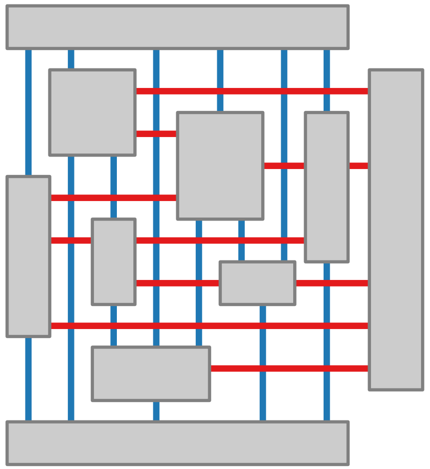

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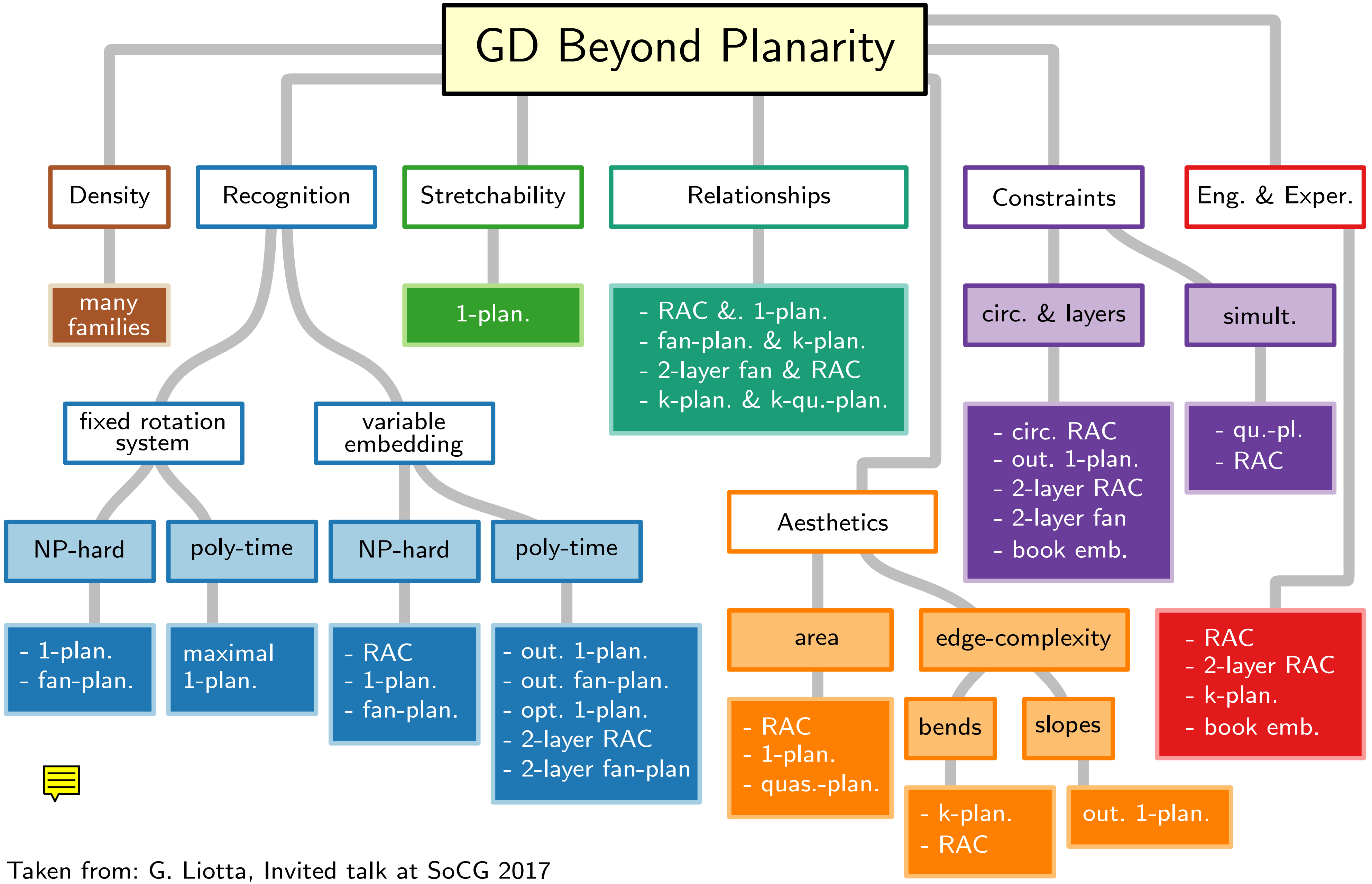
- RVGs can be recognized efficiently if embedding is fixed. [Biedl, Liotta, Montecchiani 2018]

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

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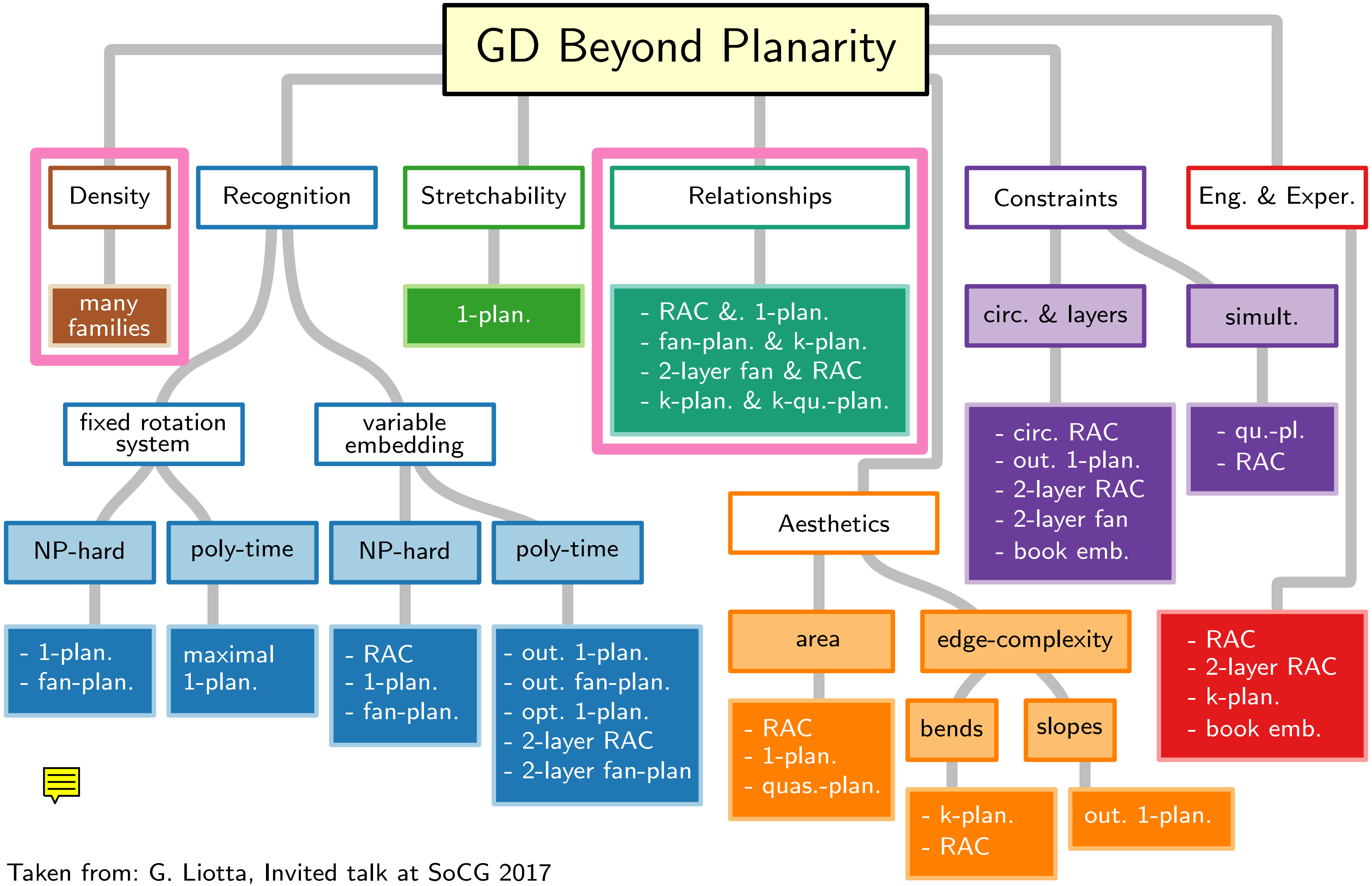

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Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n - 8$ edges.

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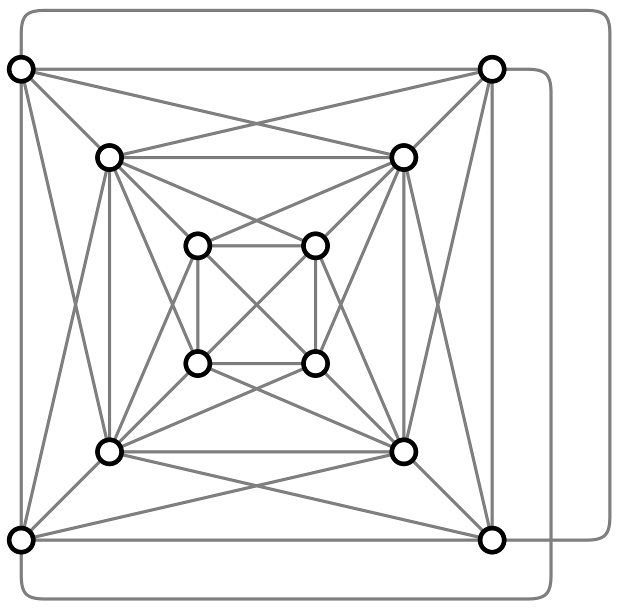
Proof sketch.



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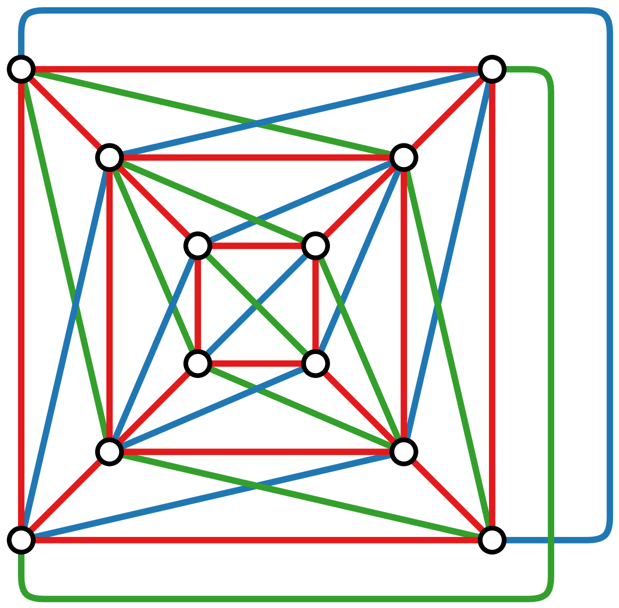

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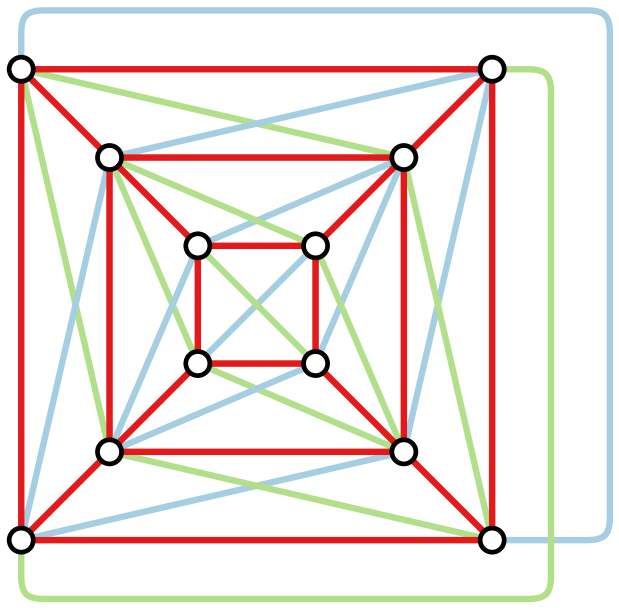

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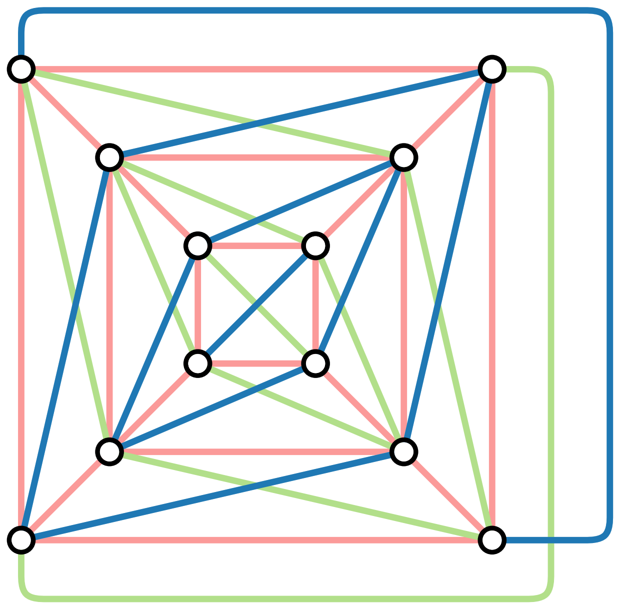

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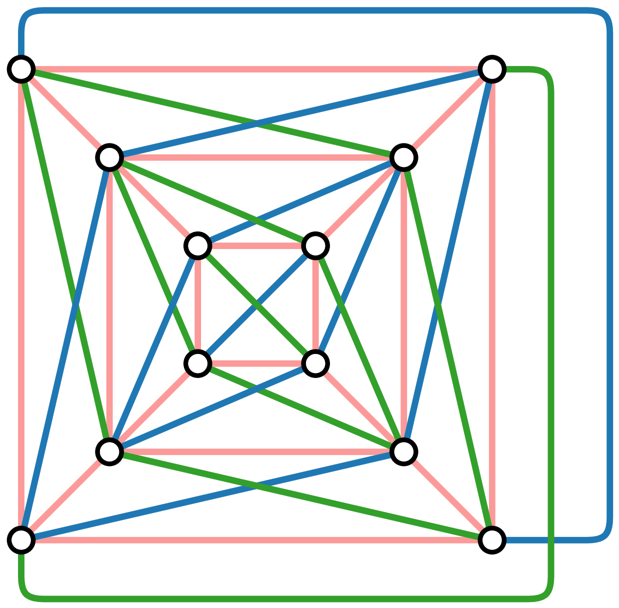

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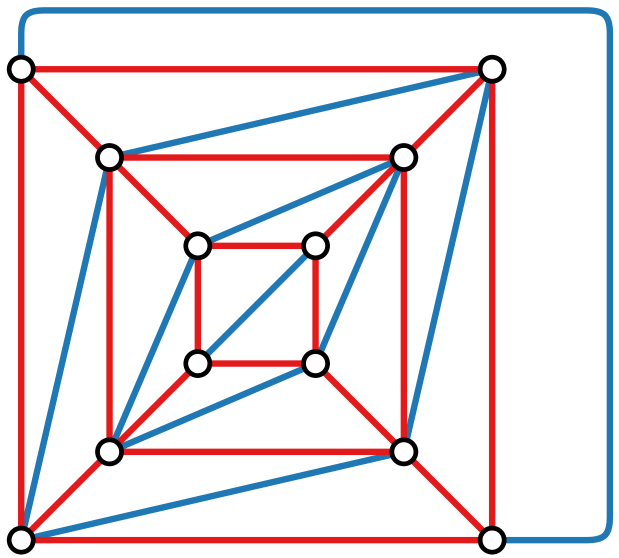

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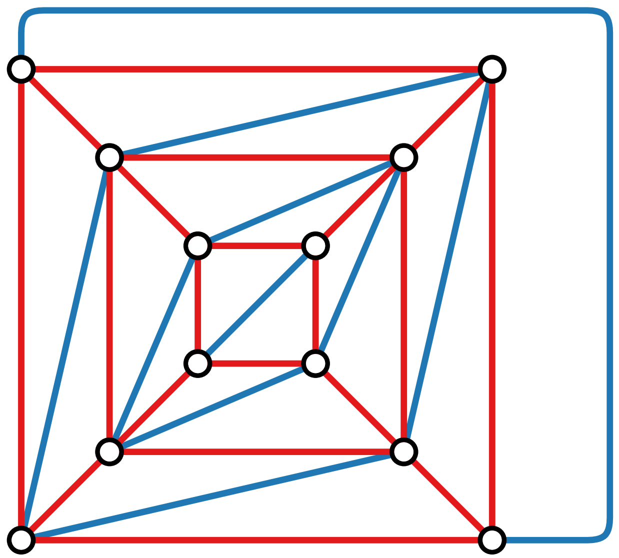

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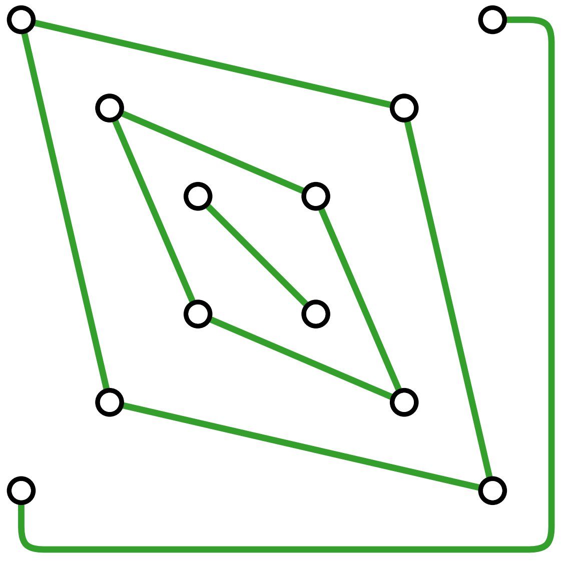

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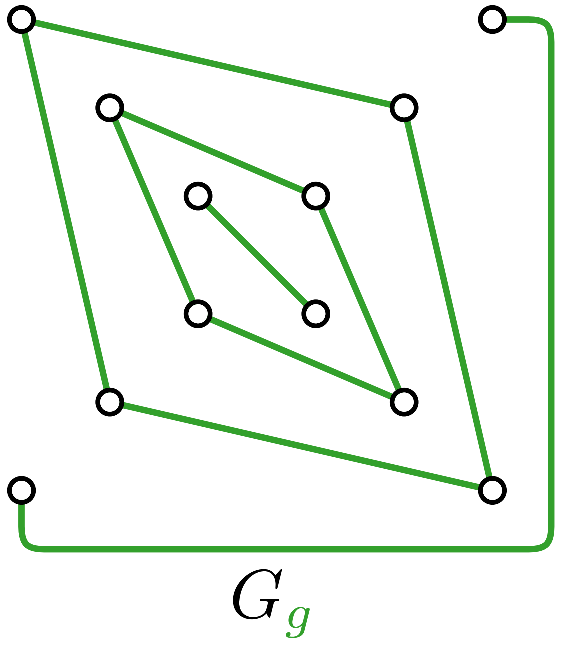

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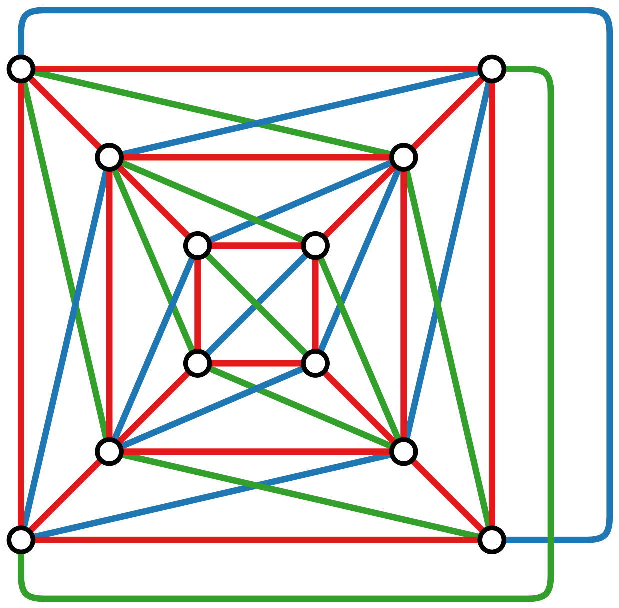

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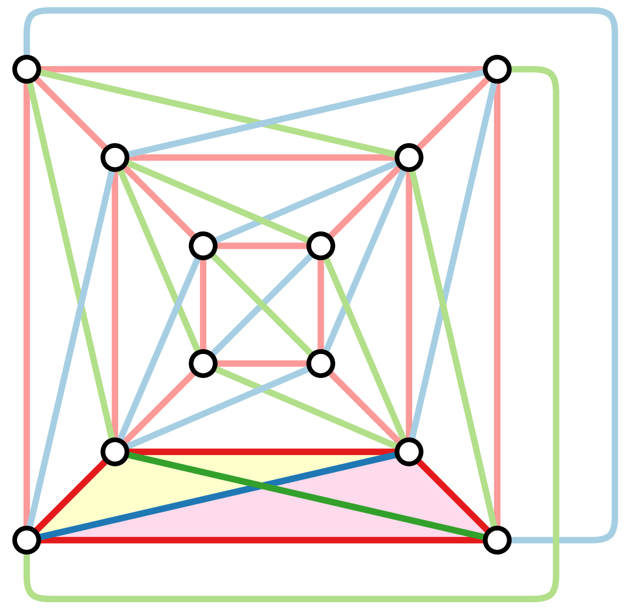

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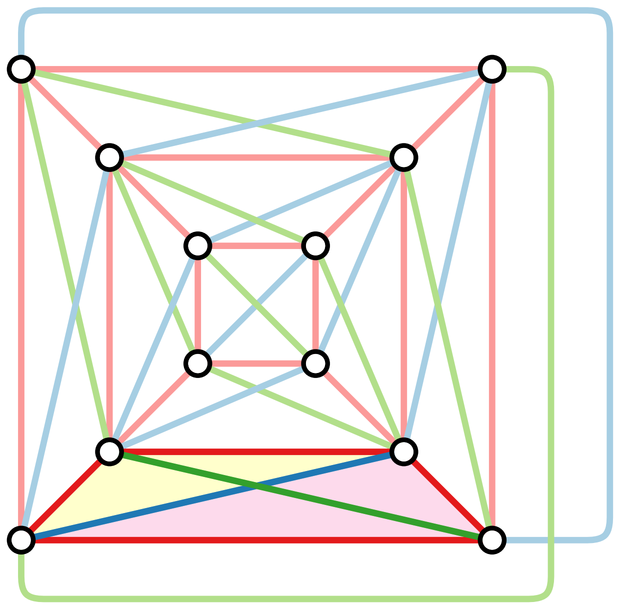

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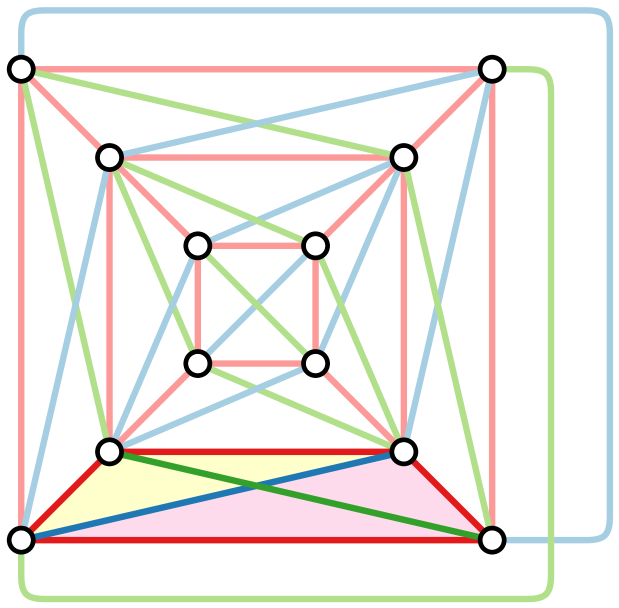

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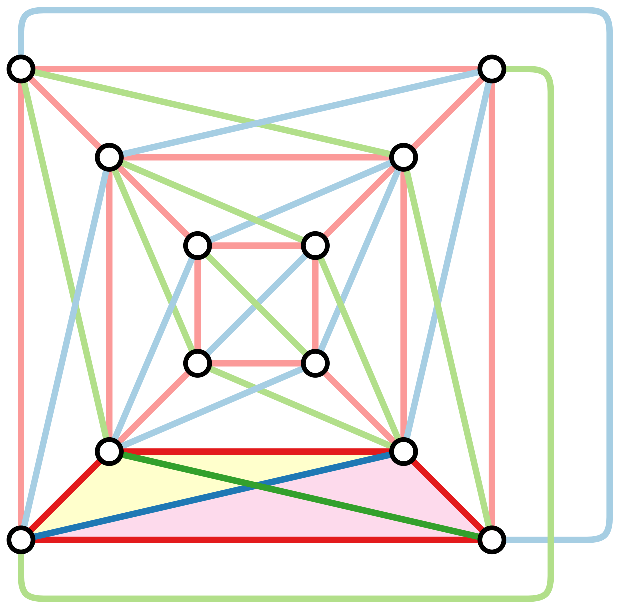

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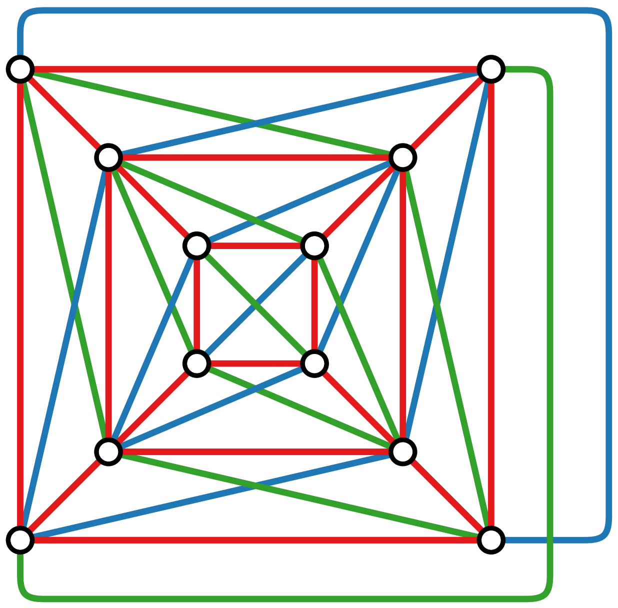

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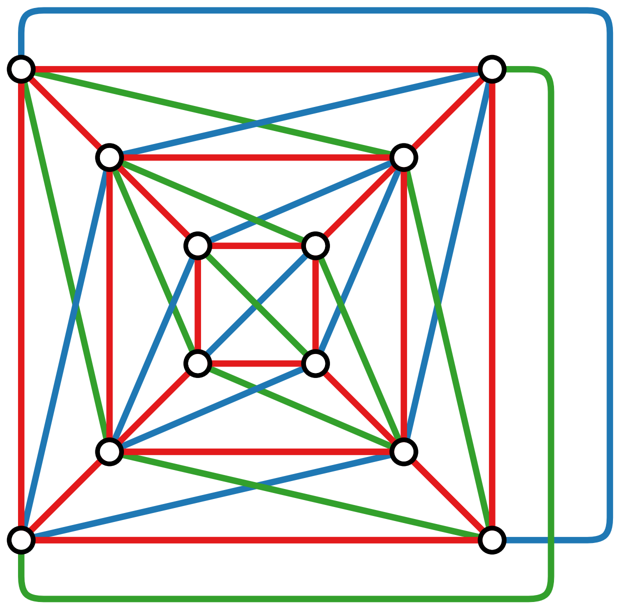

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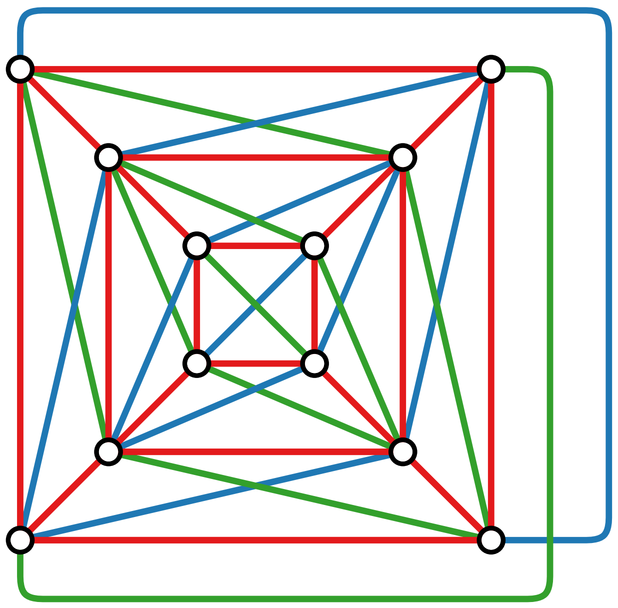
Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.



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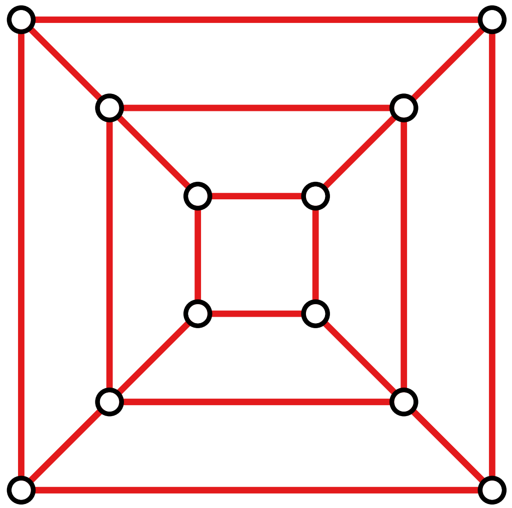

Lower-bound construction:
$2n-4$ edges
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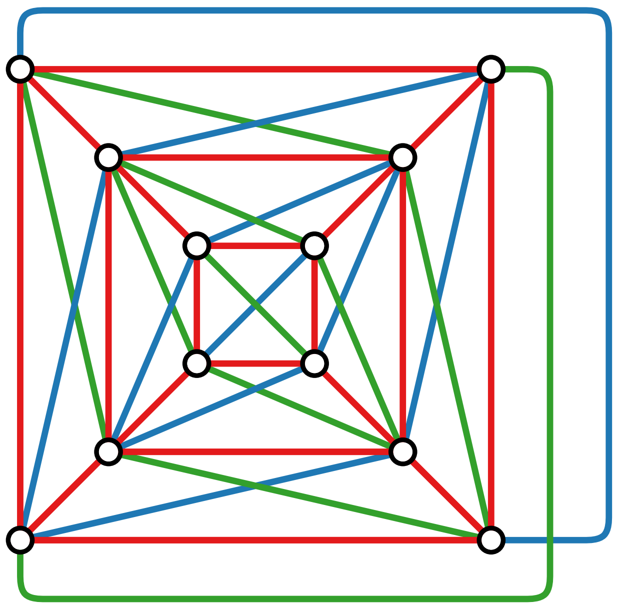

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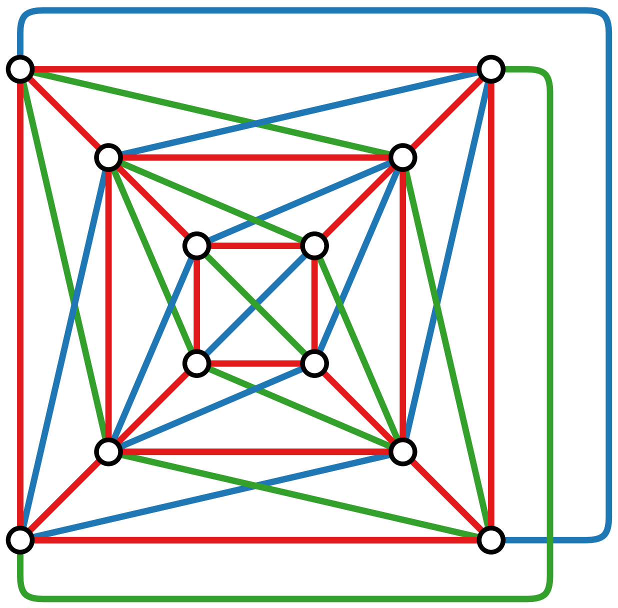

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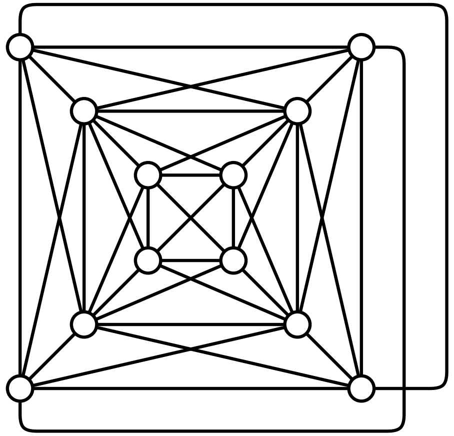
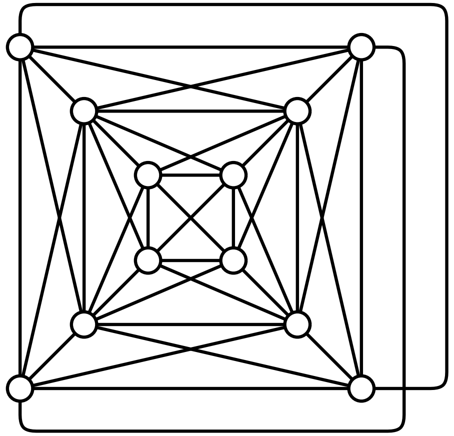


$n = 12, m = 40$

Density of 1-Planar Graphs

Theorem. [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with n vertices has at most $4n-8$ edges, which is a tight bound.

A 1-planar graph with n vertices is called **optimal** if it has exactly $4n-8$ edges.



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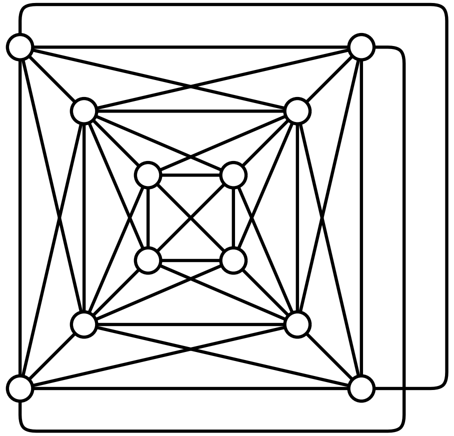

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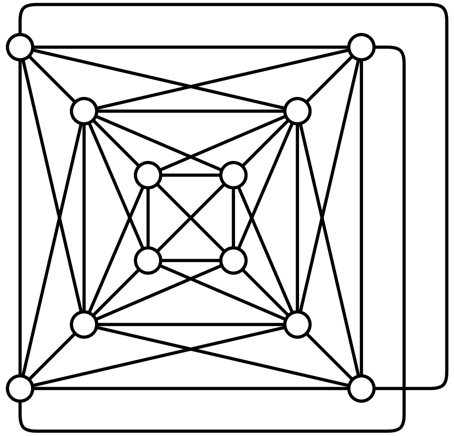
Theorem. [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with n vertices and $45/17n - O(1) \approx 2.65n - O(1)$ edges.



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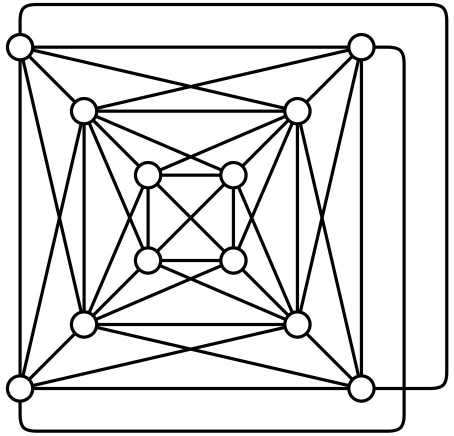

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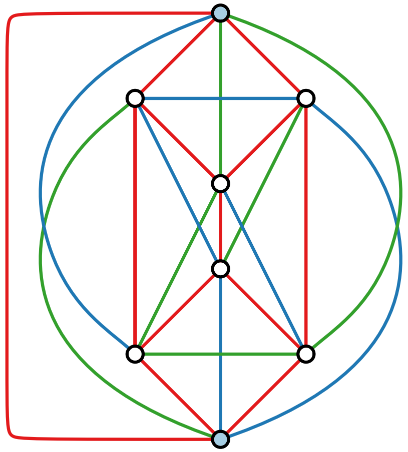

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Density of 1-Planar Graphs

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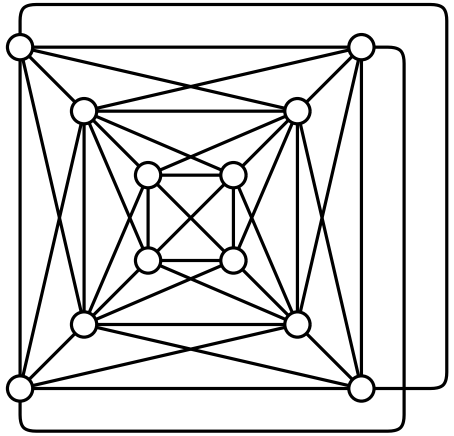
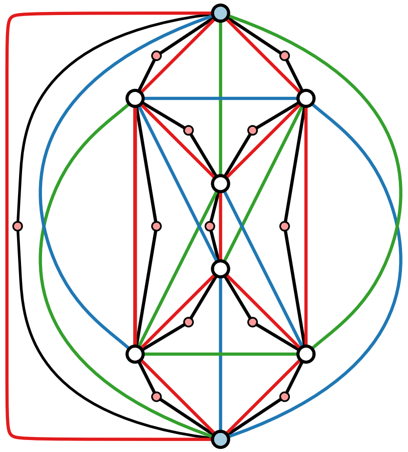
A 1-planar graph with n vertices has at most $4n - 8$ edges, which is a tight bound.



$n = 12, m = 40$

A 1-planar graph with n vertices is called **optimal** if it has exactly $4n - 8$ edges.
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.

Theorem. [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with n vertices and $45/17n - O(1) \approx 2.65n - O(1)$ edges.



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Density of 1-Planar Graphs

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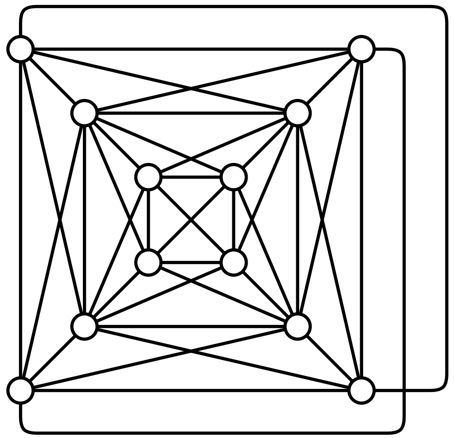

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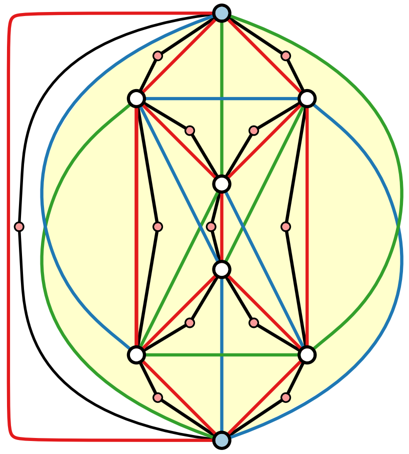

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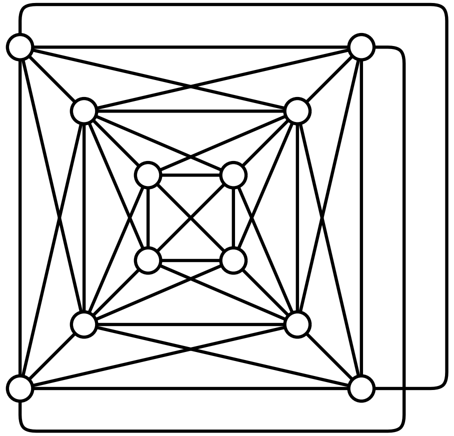

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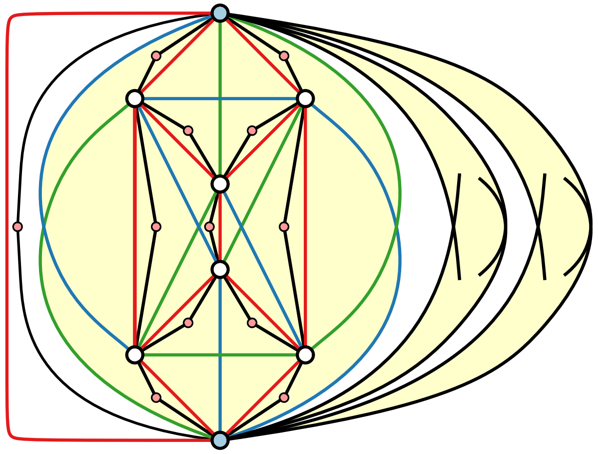

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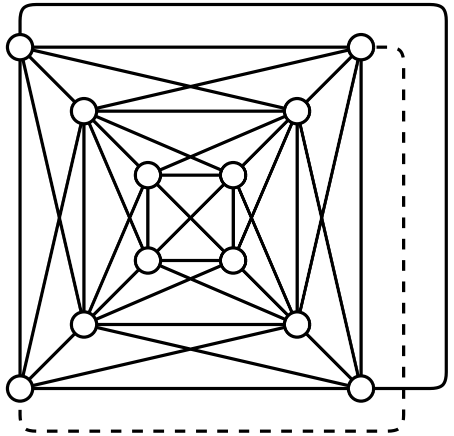

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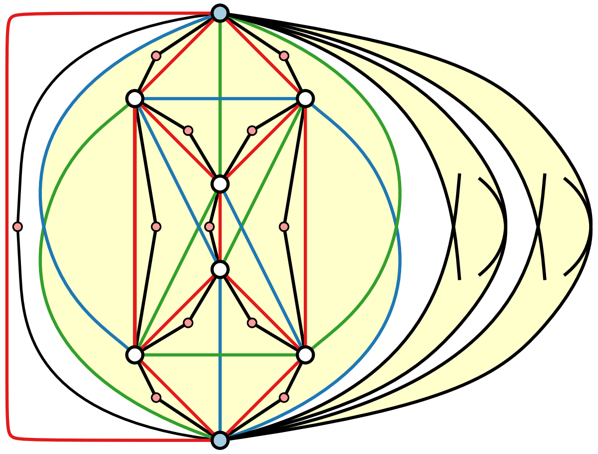

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A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n-9$ edges.

Density of 1-Planar Graphs

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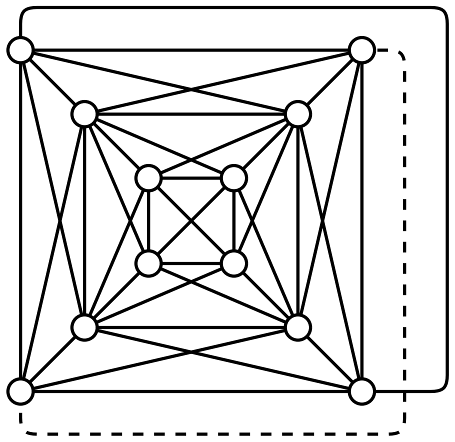

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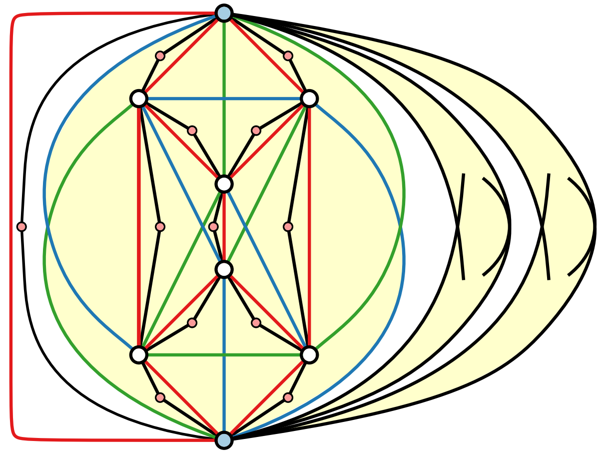

Theorem. [Didimo 2013]
A 1-planar graph with n vertices that admits a **straight-line drawing** has at most $4n-9$ edges.

Idea: in a drawing of an optimal 1-planar graph, we cannot realize the crossing on the outer face with two straight-line edges.

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0		Euler's formula

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula

Density of k-Planar Graphs

Theorem.
A k-planar graph with n vertices has at most:

k	number of edges	
0	$3(n-2)$	Euler's formula
1	$4(n-2)$	[Ringel 1965]

Density of k-Planar Graphs

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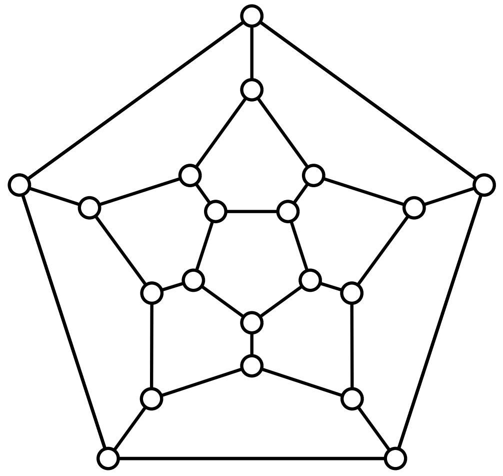

Planar structure:

Density of k-Planar Graphs

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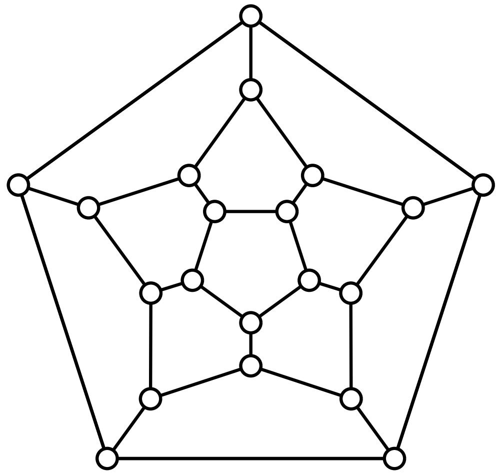

Planar structure:

Edges per face:
Total:

Density of k-Planar Graphs

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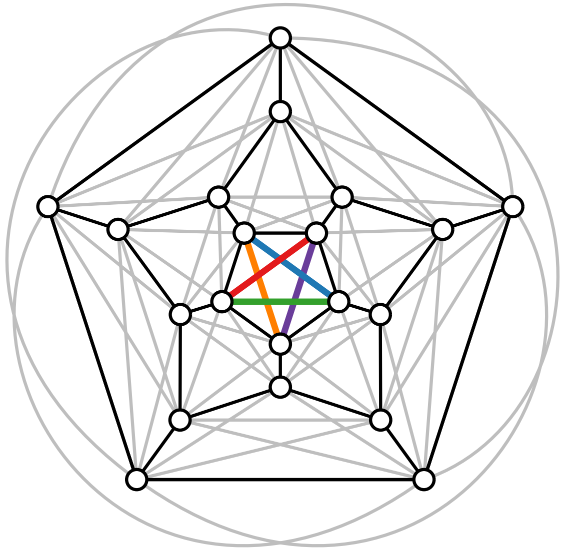

optimal 2-planar

Planar structure:

Edges per face:

Total:

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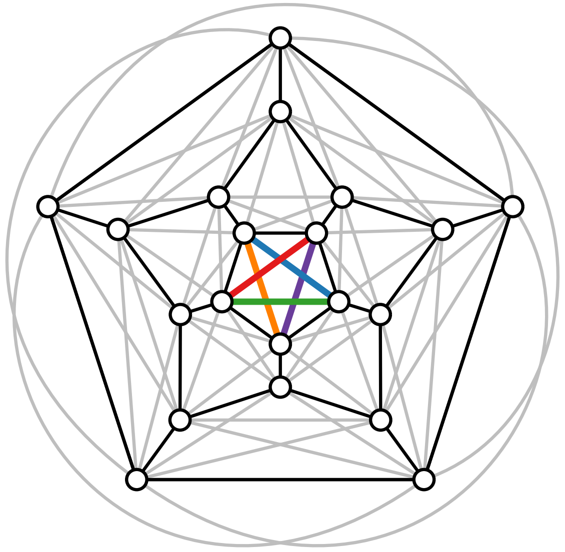

optimal 2-planar

Planar structure:

Edges per face:

Total:

$n - m + f = 2$

$m = c \cdot f$?

Density of k-Planar Graphs

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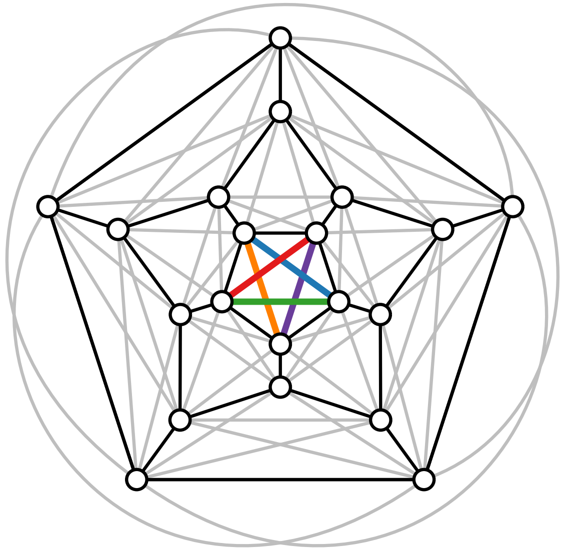

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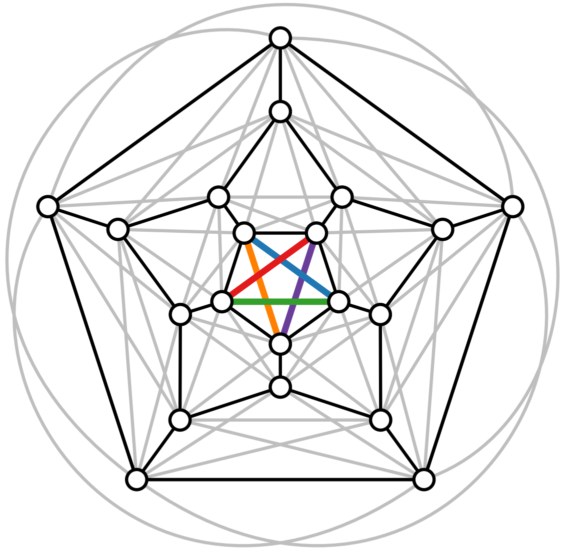

optimal 2-planar

$$n - m + f = 2$$
$$m = c \cdot f \ ?$$
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Planar structure:
$\frac{5}{3}(n-2)$ edges
$\frac{2}{3}(n-2)$ faces

Edges per face:

Total:

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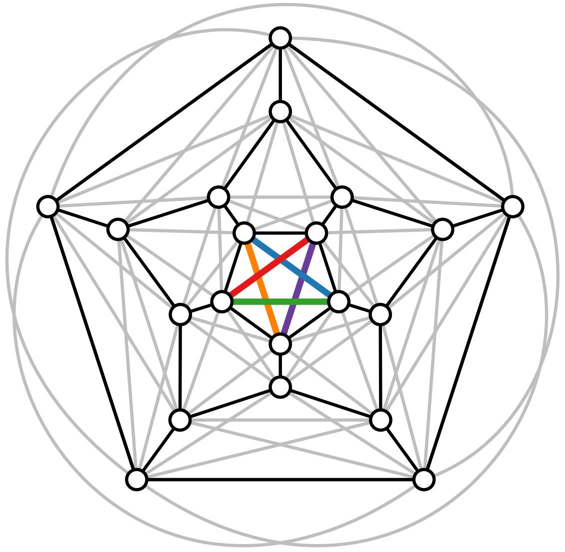

optimal 2-planar

$n - m + f = 2$
$m = c \cdot f$?
$m = \frac{5}{2} f$

Planar structure:
$\frac{5}{3}(n-2)$ edges
$\frac{2}{3}(n-2)$ faces
Edges per face: 5 edges
Total:

Density of k-Planar Graphs

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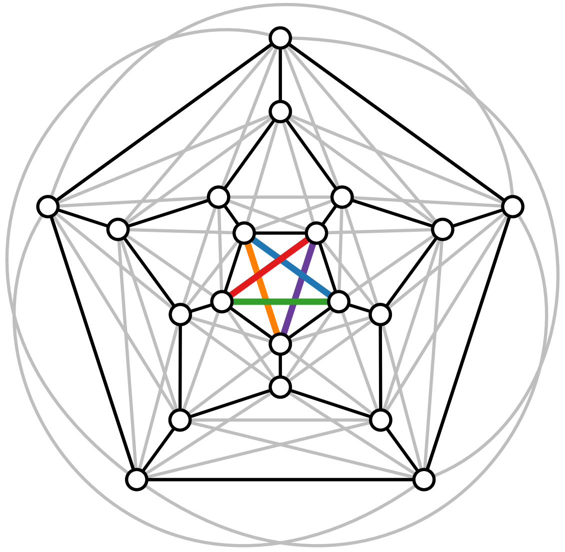

optimal 2-planar

Planar structure:

$\frac{5}{3}(n-2)$ edges

$\frac{2}{3}(n-2)$ faces

Edges per face: 5 edges

Total: $5(n-2)$ edges

Density of k-Planar Graphs

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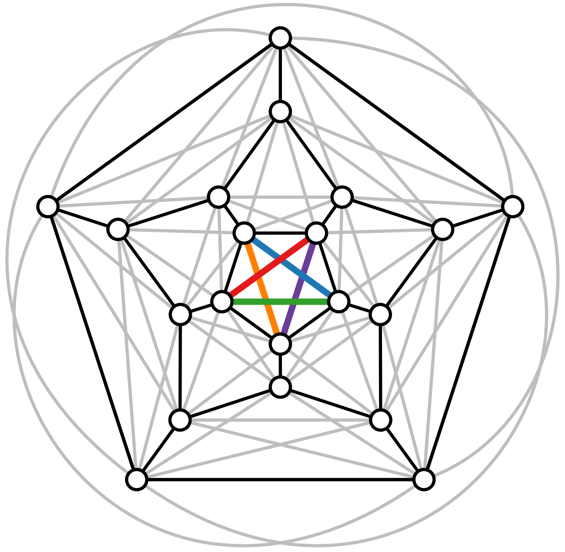

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$$n - m + f = 2$$
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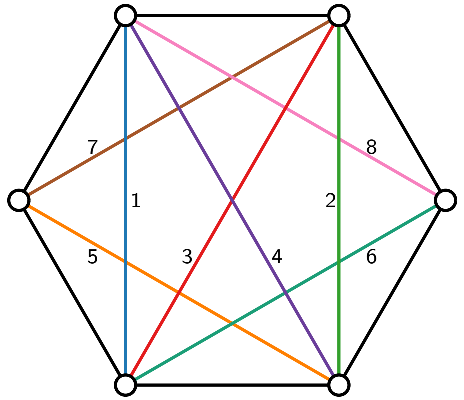

optimal 3-planar

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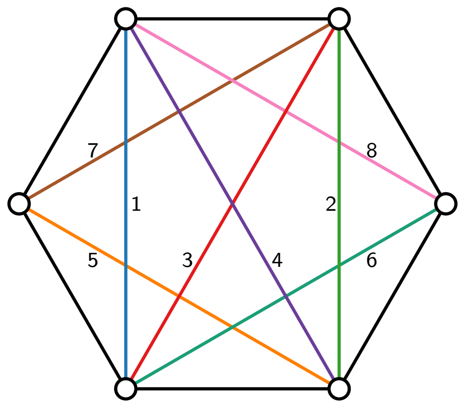

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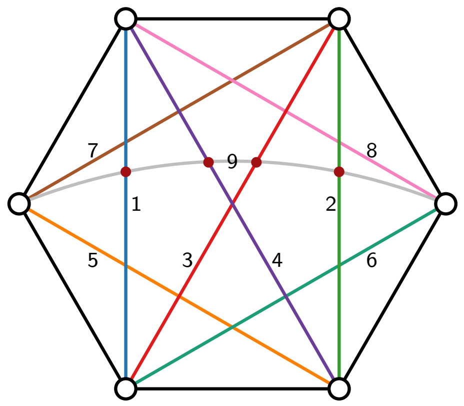

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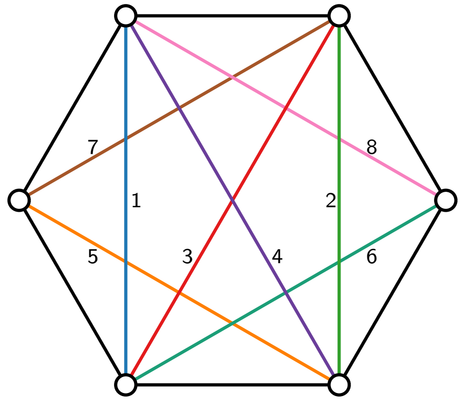

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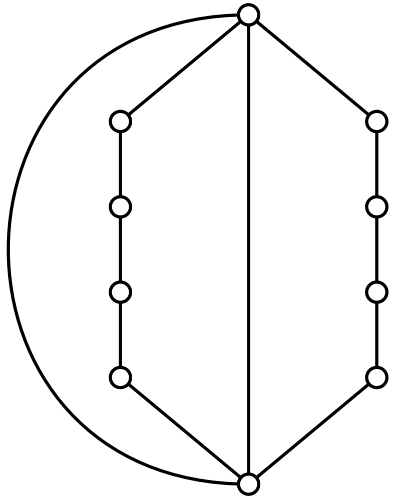

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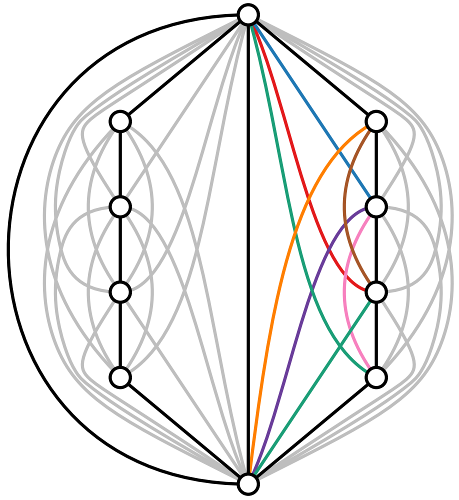

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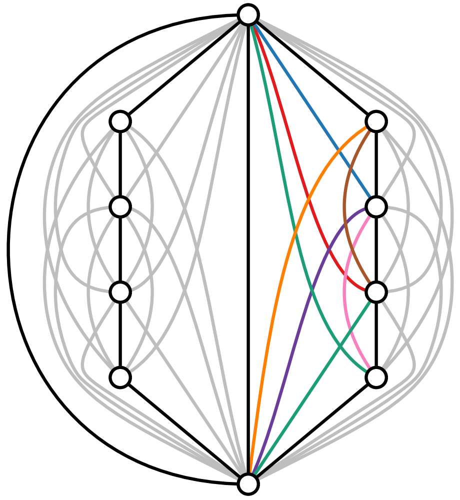

optimal 3-planar

Planar structure:
$\frac{3}{2}(n-2)$ edges
$\frac{1}{2}(n-2)$ faces

Edges per face: 8 edges

Total: $5.5(n-2)$ edges

Density of k-Planar Graphs

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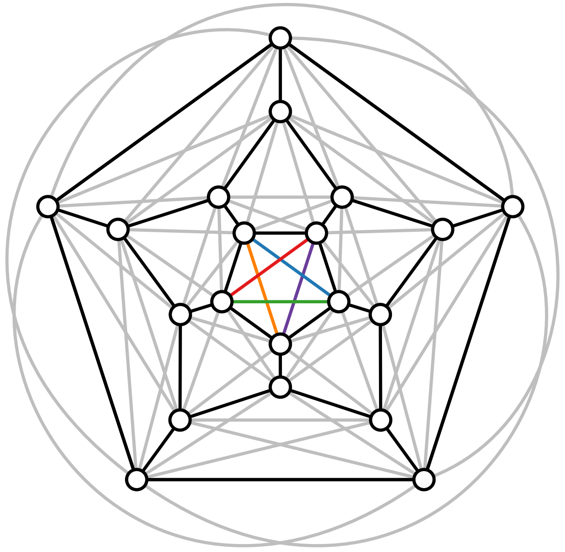
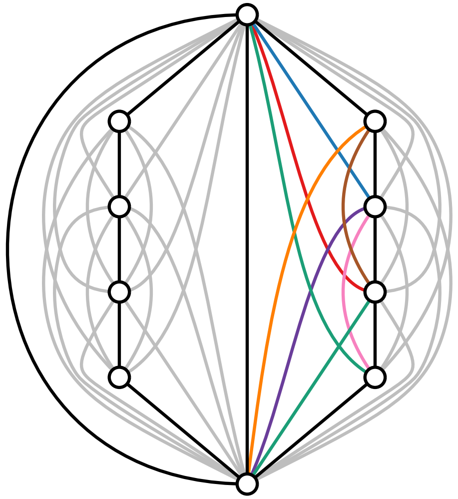
4	$6(n-2)$	[Ackerman 2015]


optimal 2-planar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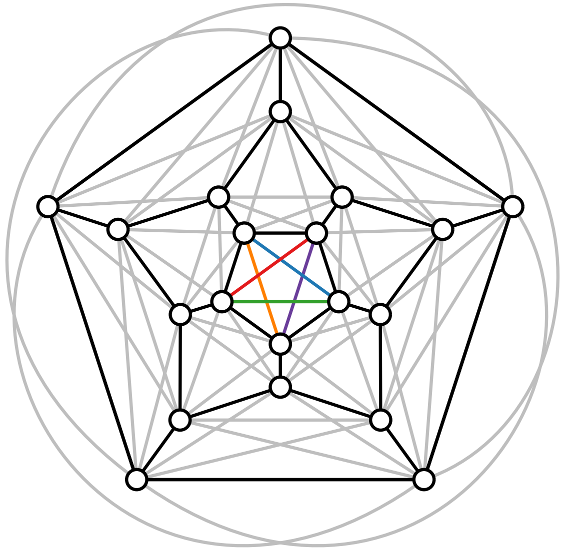

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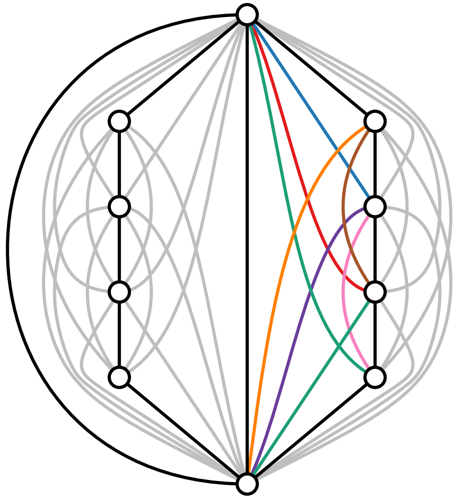

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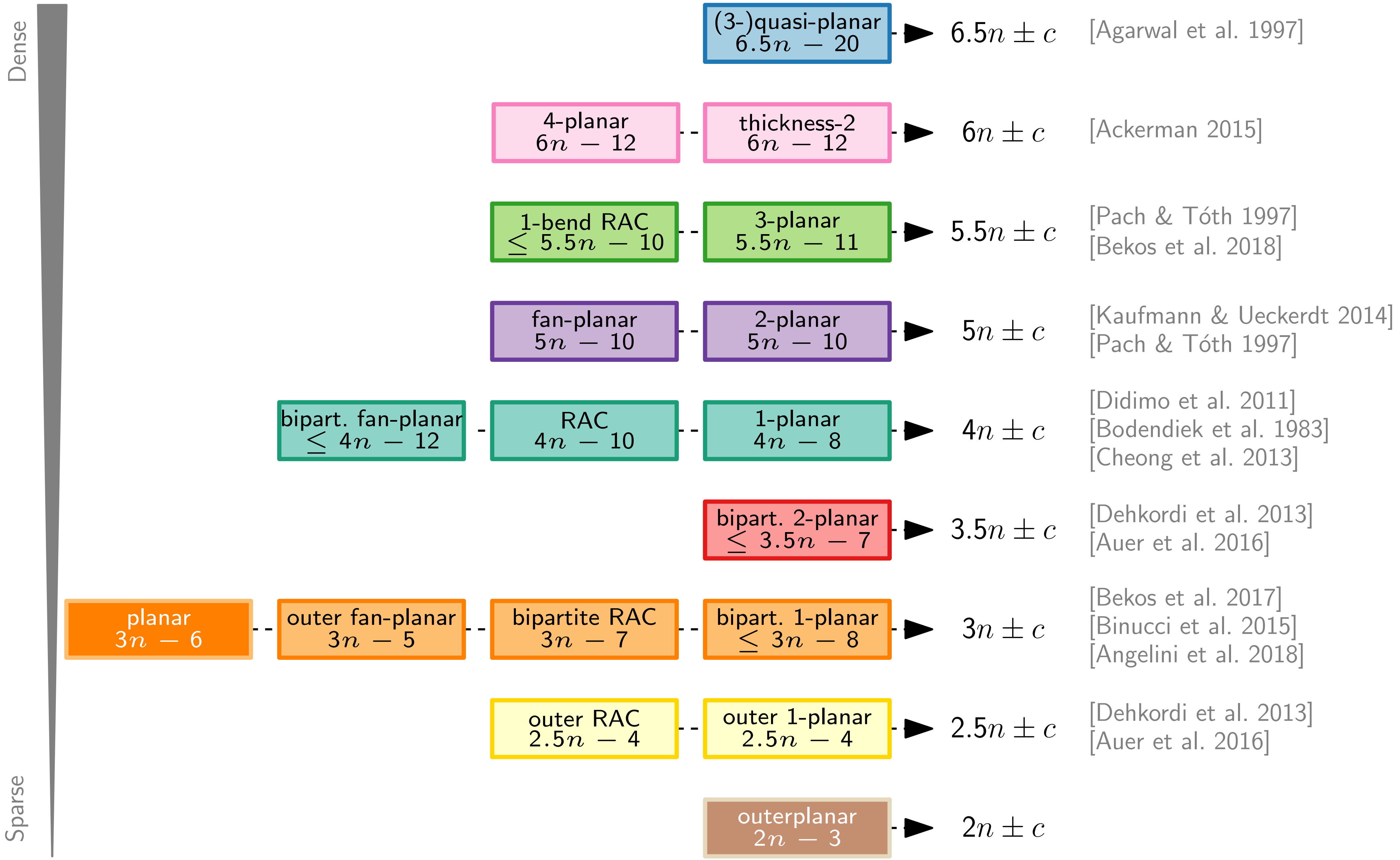
GD Beyond Planarity: a Hierarchy



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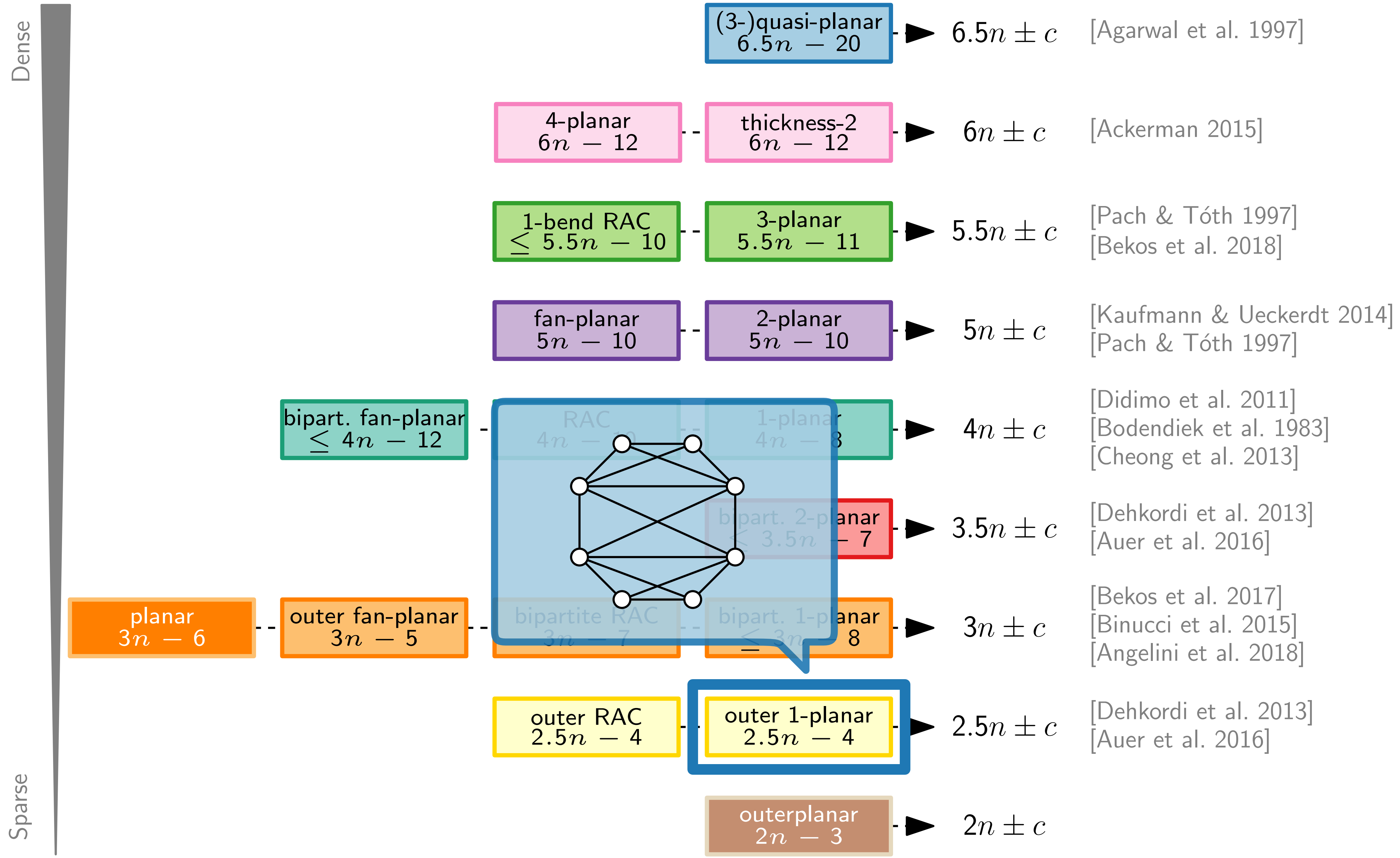

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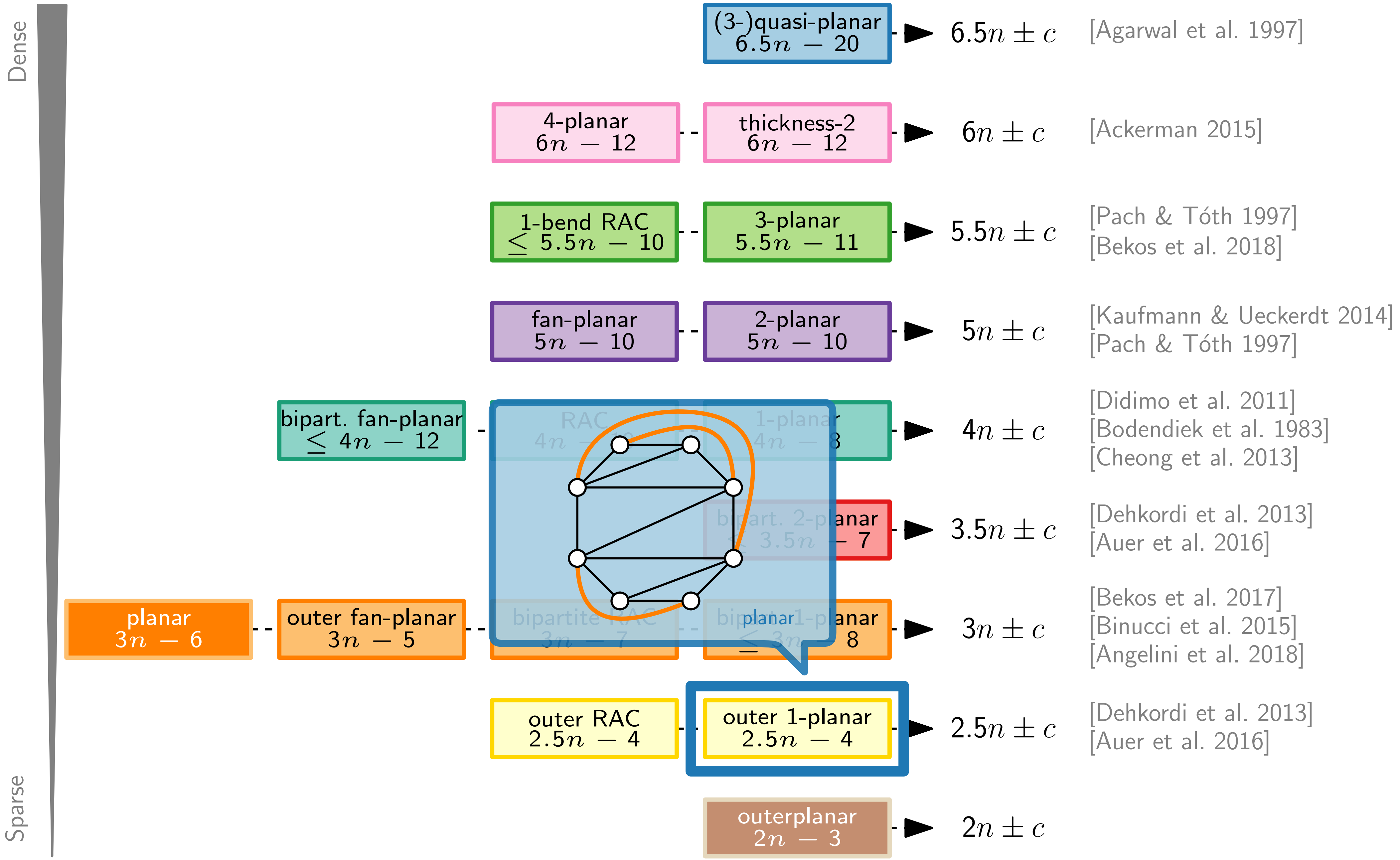

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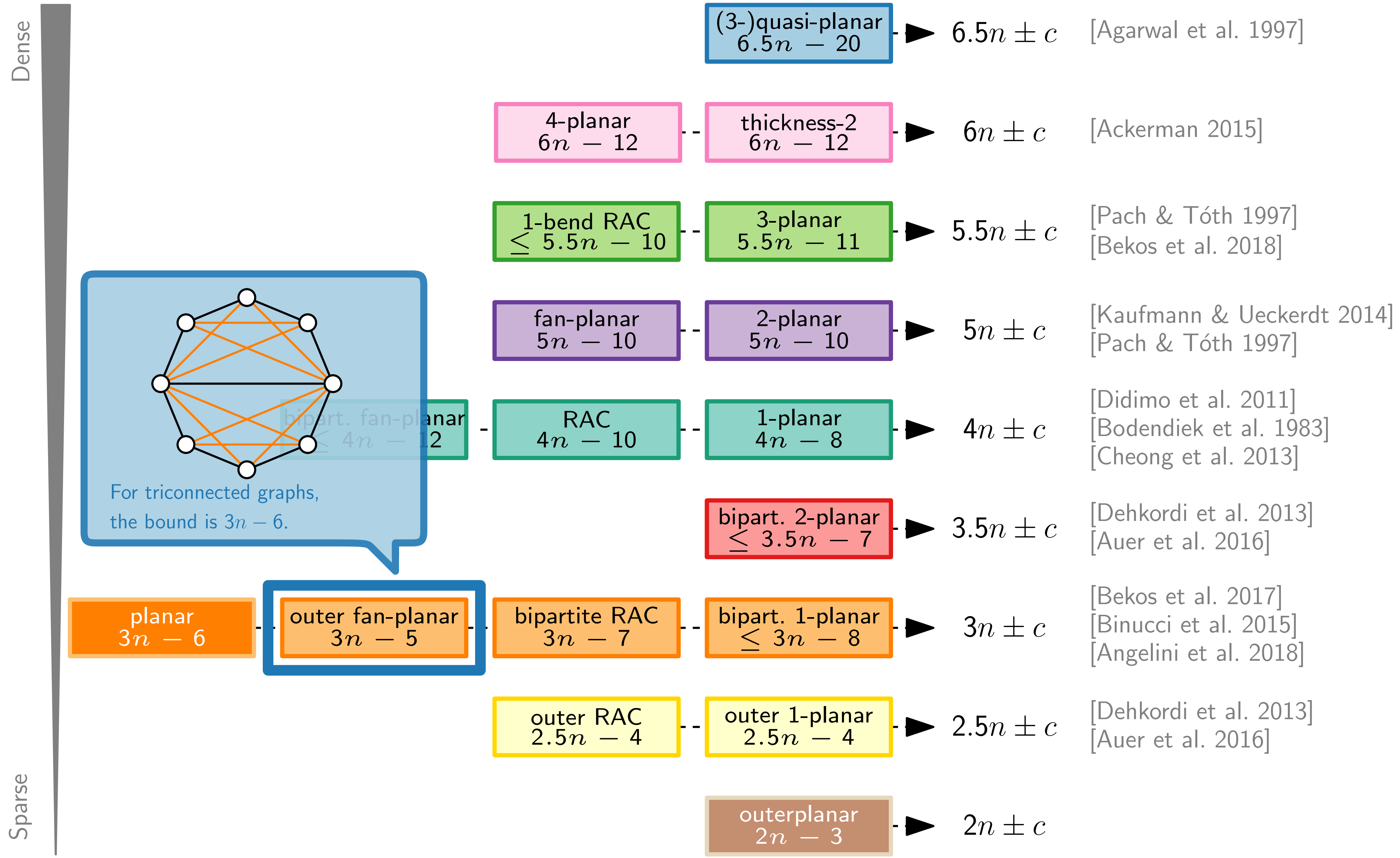

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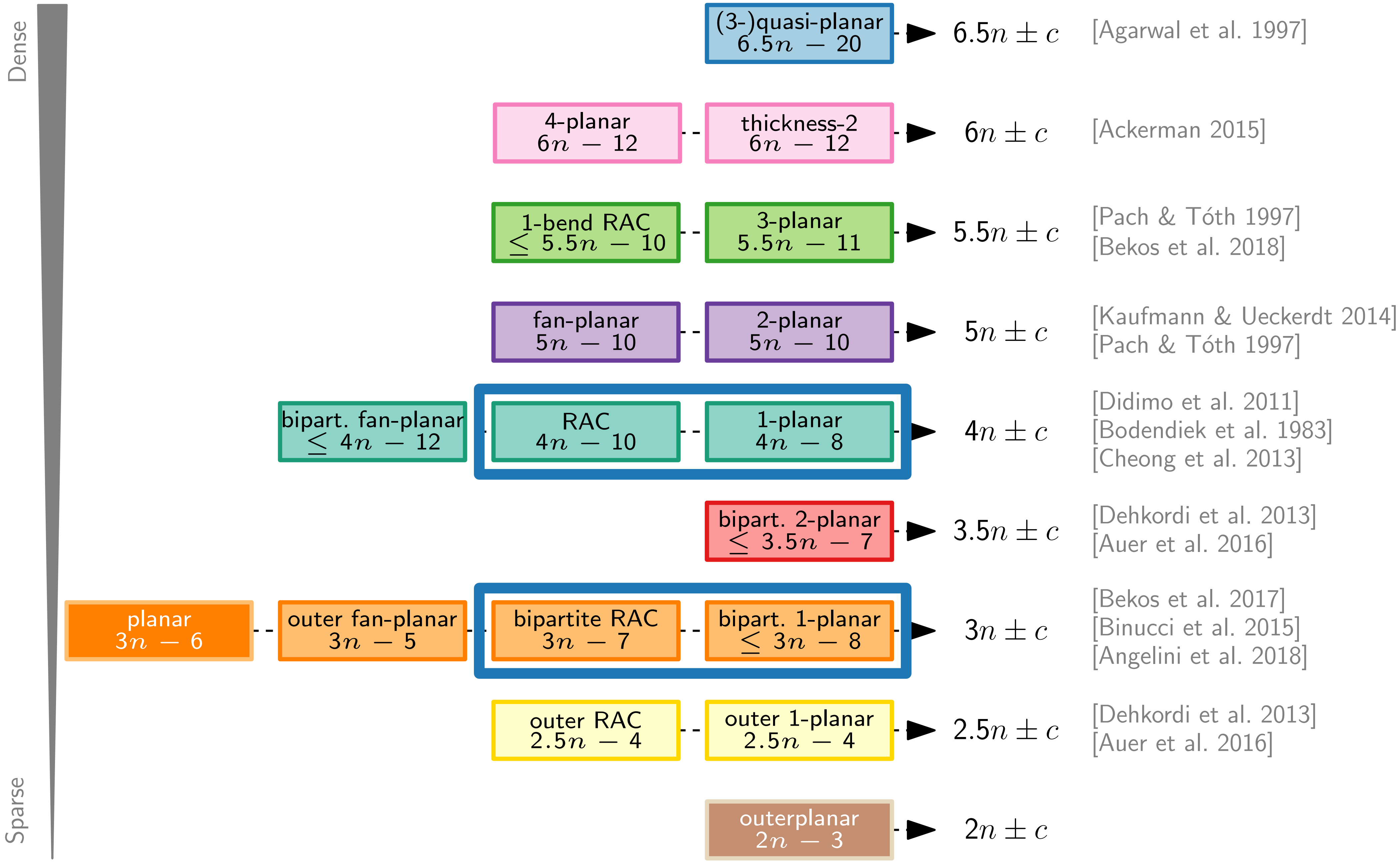

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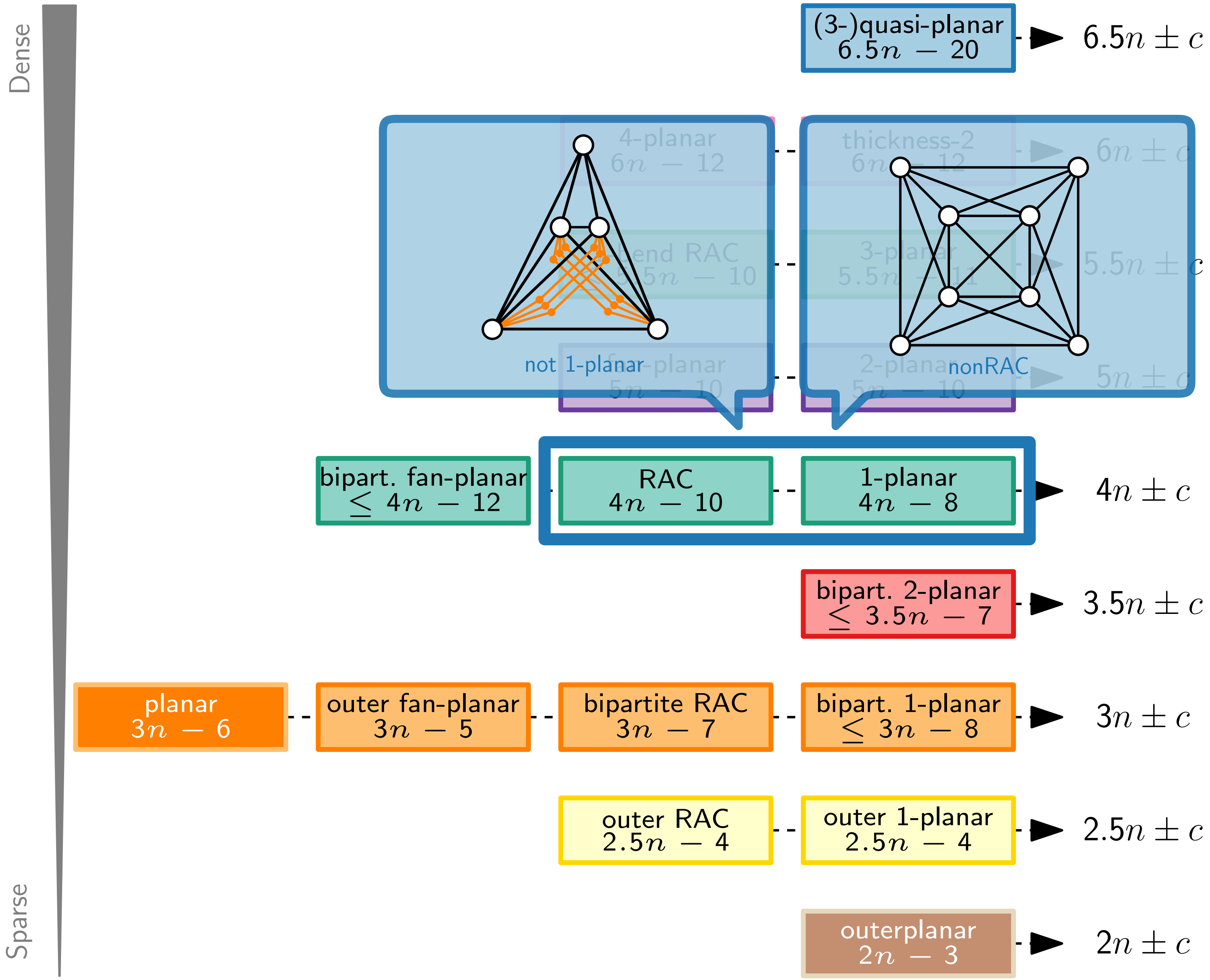

[Agarwal et al. 1997]

[Ackerman 2015]

[Pach & Tóth 1997]
[Bekos et al. 2018]

[Kaufmann & Ueckerdt 2014]
[Pach & Tóth 1997]

[Didimo et al. 2011]
[Bodendiek et al. 1983]
[Cheong et al. 2013]

[Dehkordi et al. 2013]
[Auer et al. 2016]

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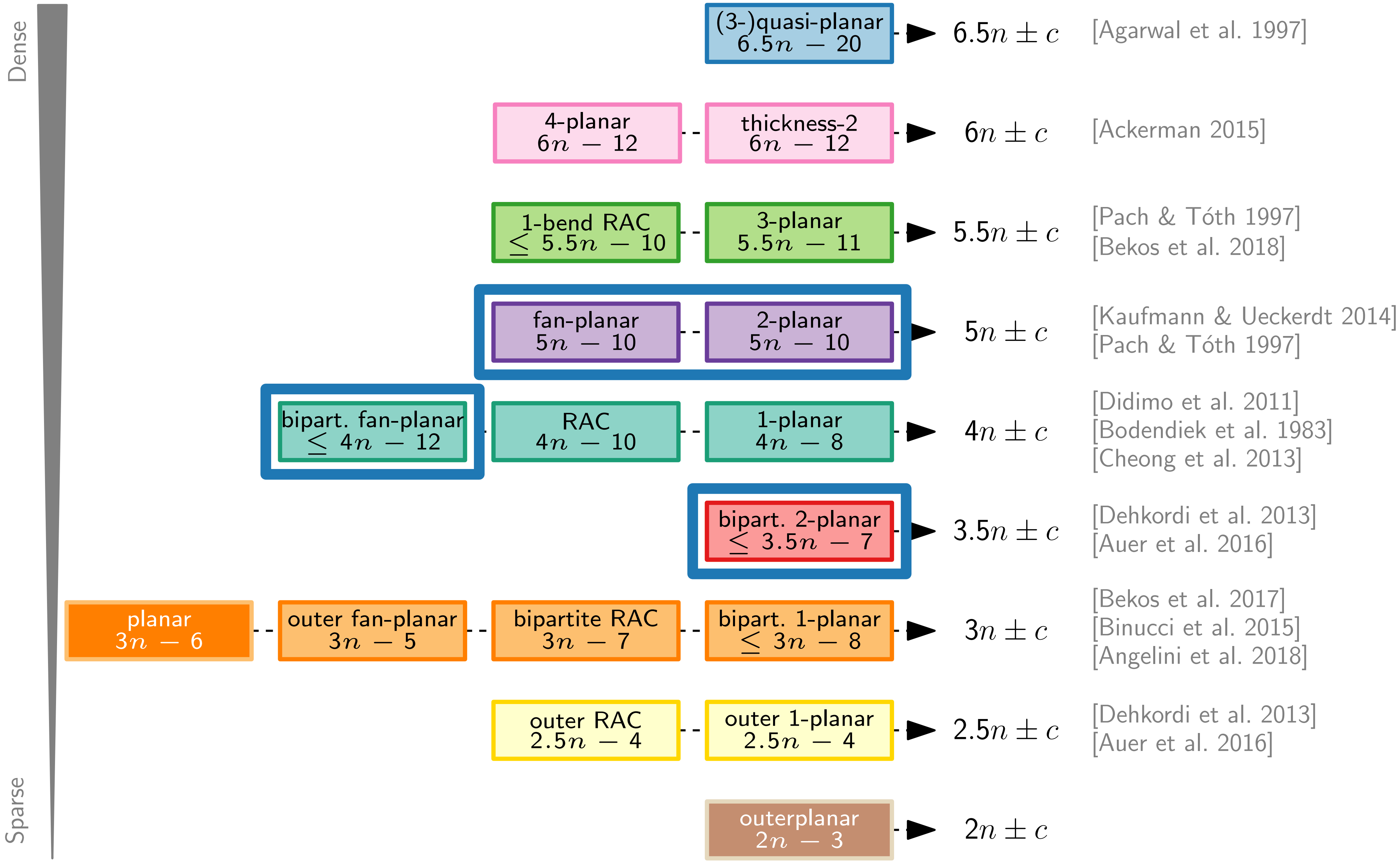
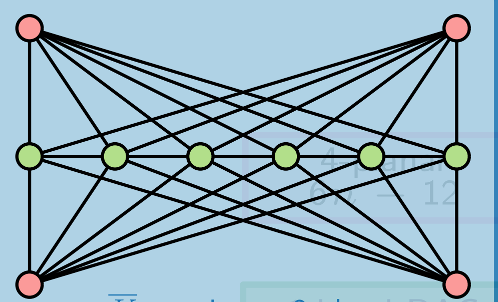
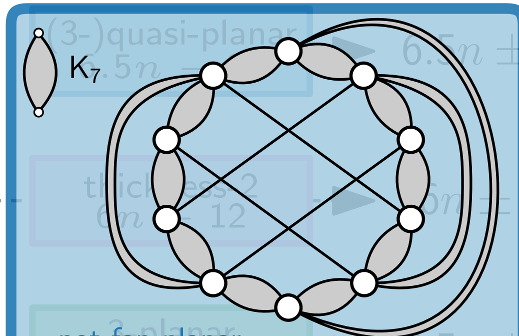


GD Beyond Planarity: a Hierarchy

Dense



$\overline{K}_{4,n-4}$ is not 2-planar

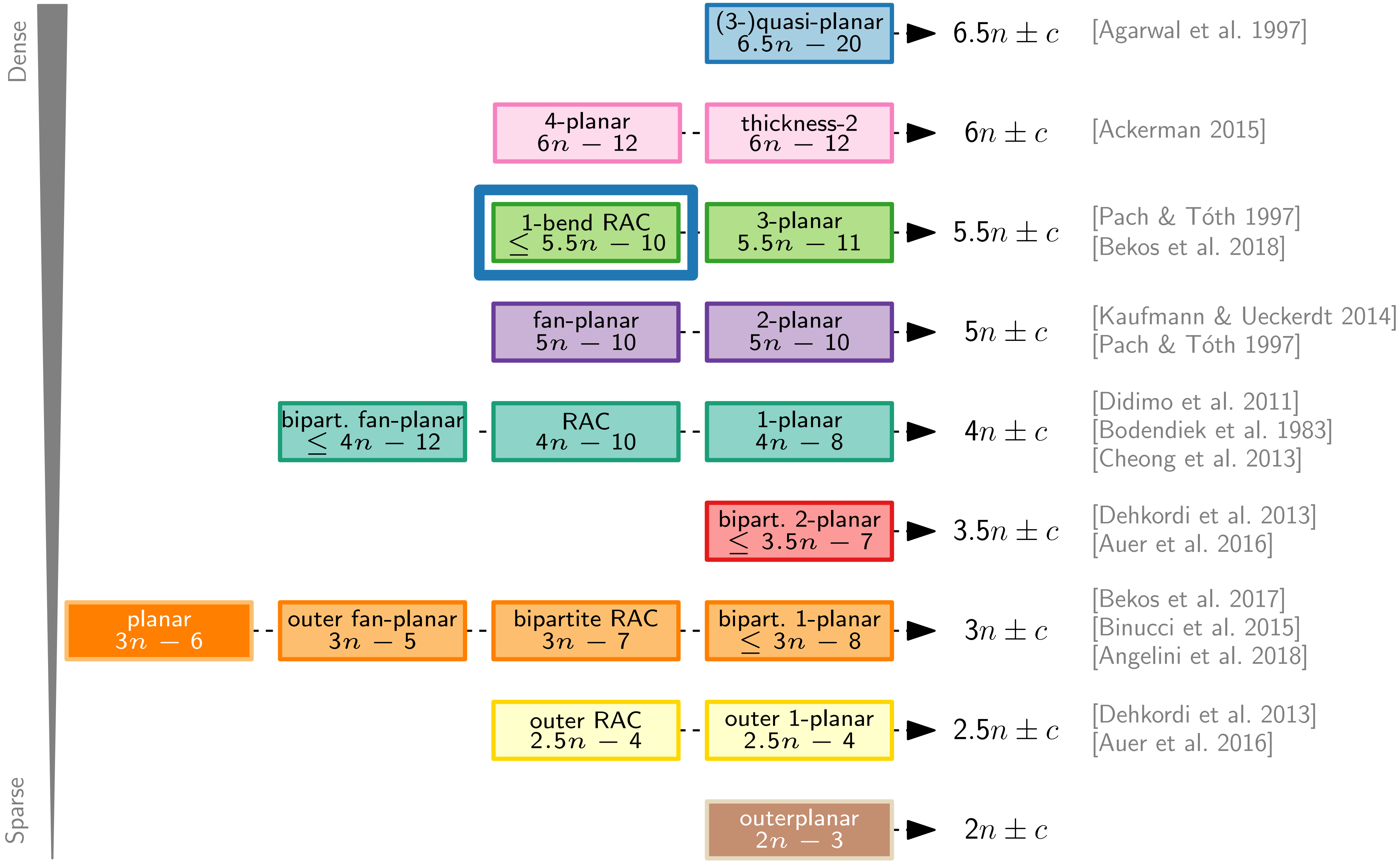


not fan-planar

				$6.5n \pm c$	[Agarwal et al. 1997]
				$6n \pm c$	[Ackerman 2015]
				$5.5n \pm c$	[Pach & Tóth 1997] [Bekos et al. 2018]
		fan-planar $5n - 10$	2-planar $5n - 10$	$5n \pm c$	[Kaufmann & Ueckerdt 2014] [Pach & Tóth 1997]
	bipart. fan-planar $\leq 4n - 12$	RAC $4n - 10$	1-planar $4n - 8$	$4n \pm c$	[Didimo et al. 2011] [Bodendiek et al. 1983] [Cheong et al. 2013]
			bipart. 2-planar $\leq 3.5n - 7$	$3.5n \pm c$	[Dehkordi et al. 2013] [Auer et al. 2016]
planar $3n - 6$	outer fan-planar $3n - 5$	bipartite RAC $3n - 7$	bipart. 1-planar $\leq 3n - 8$	$3n \pm c$	[Bekos et al. 2017] [Binucci et al. 2015] [Angelini et al. 2018]
		outer RAC $2.5n - 4$	outer 1-planar $2.5n - 4$	$2.5n \pm c$	[Dehkordi et al. 2013] [Auer et al. 2016]
			outerplanar $2n - 3$	$2n \pm c$	

Sparse

GD Beyond Planarity: a Hierarchy



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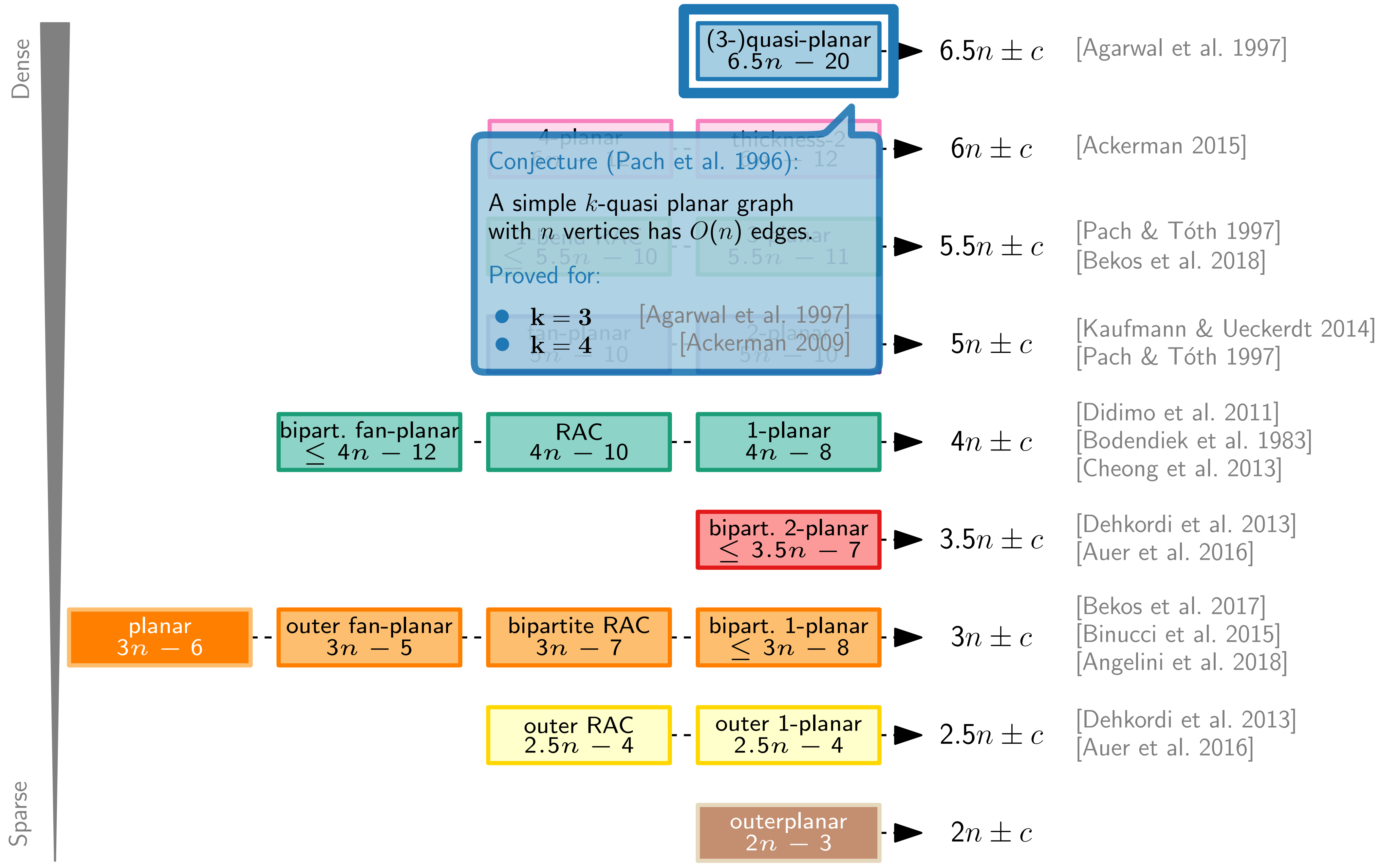

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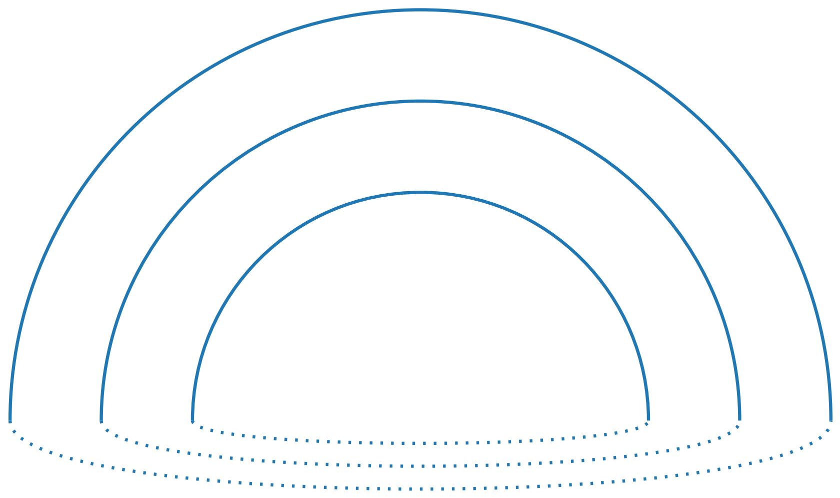

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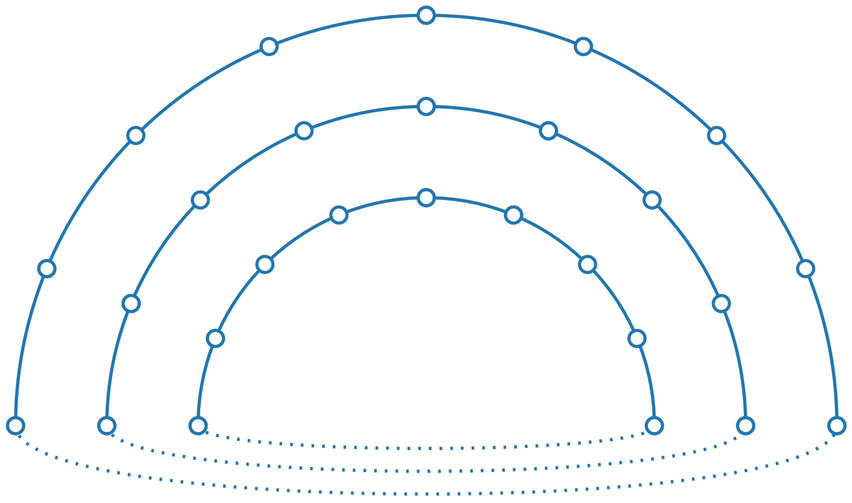

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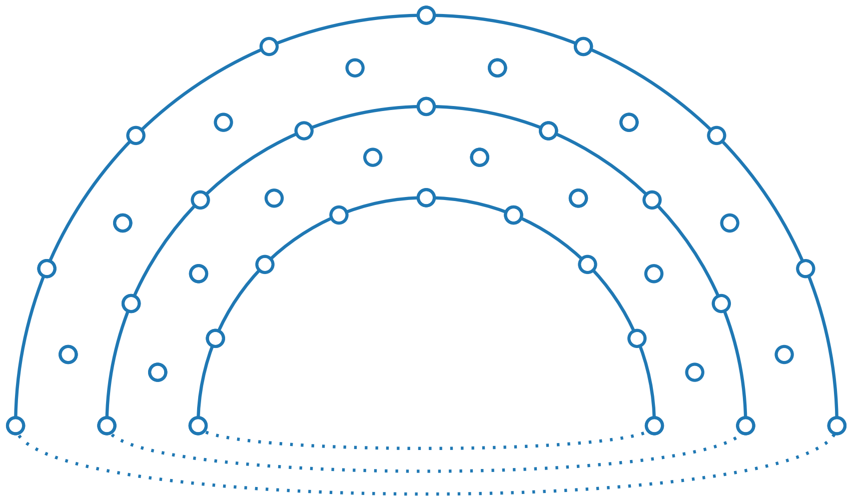

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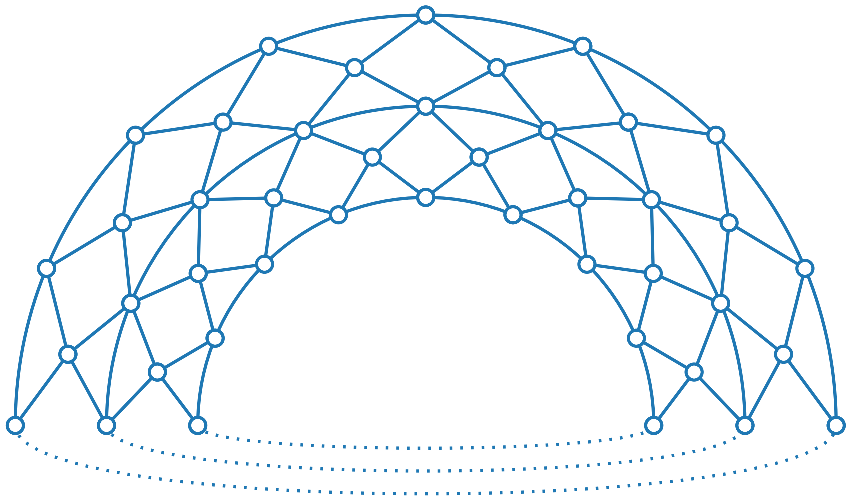

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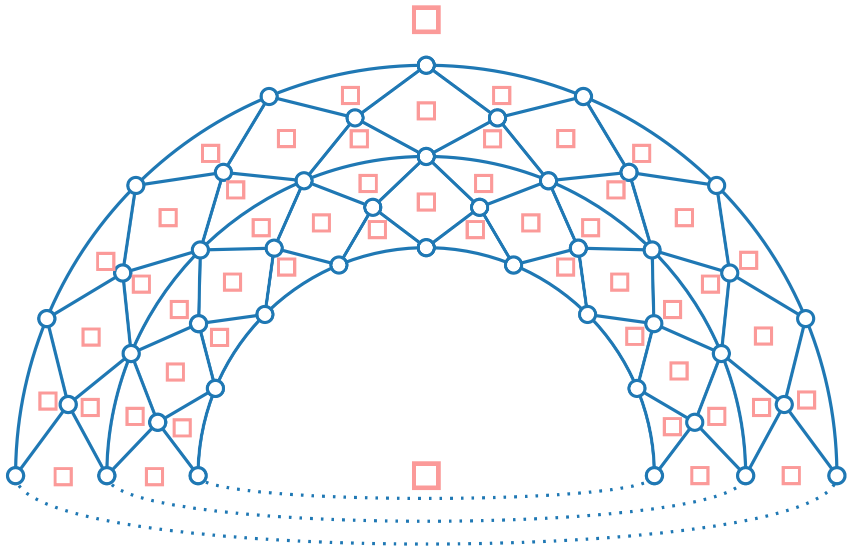

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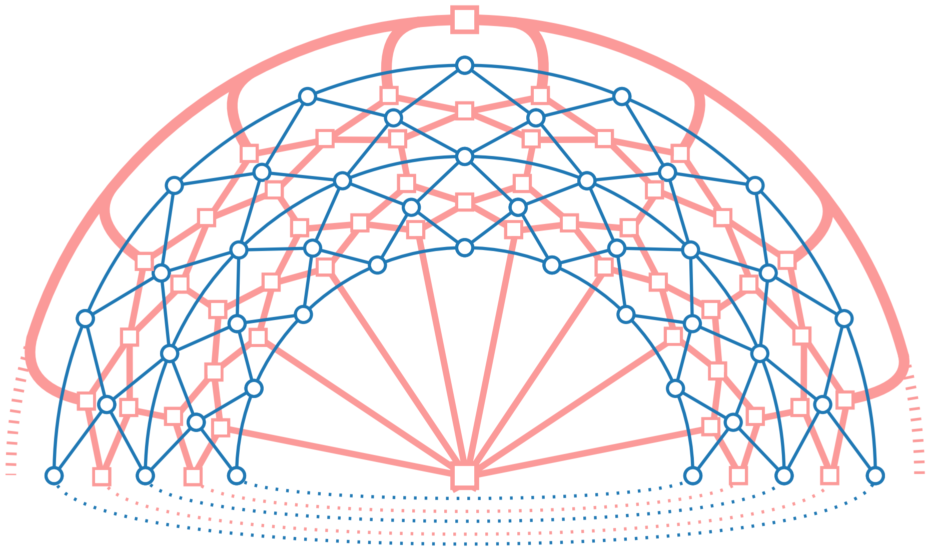

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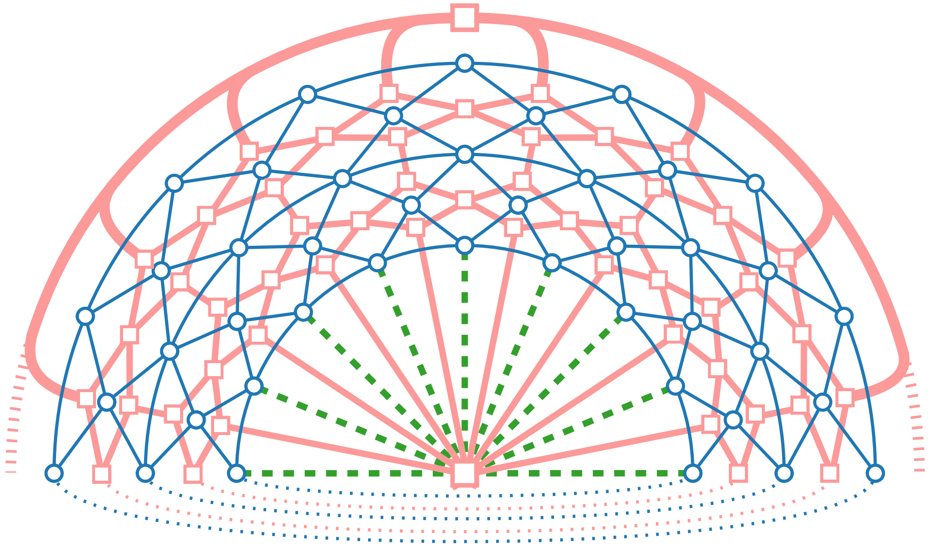

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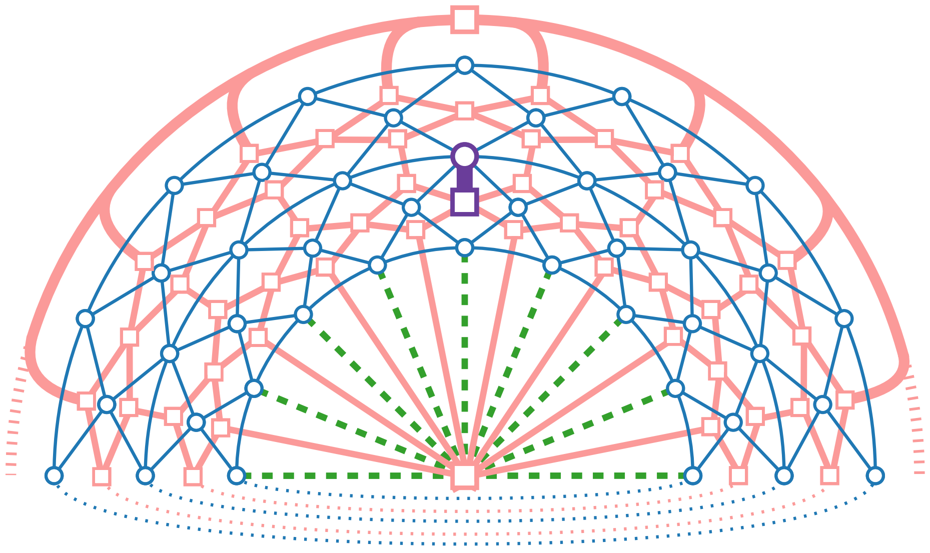

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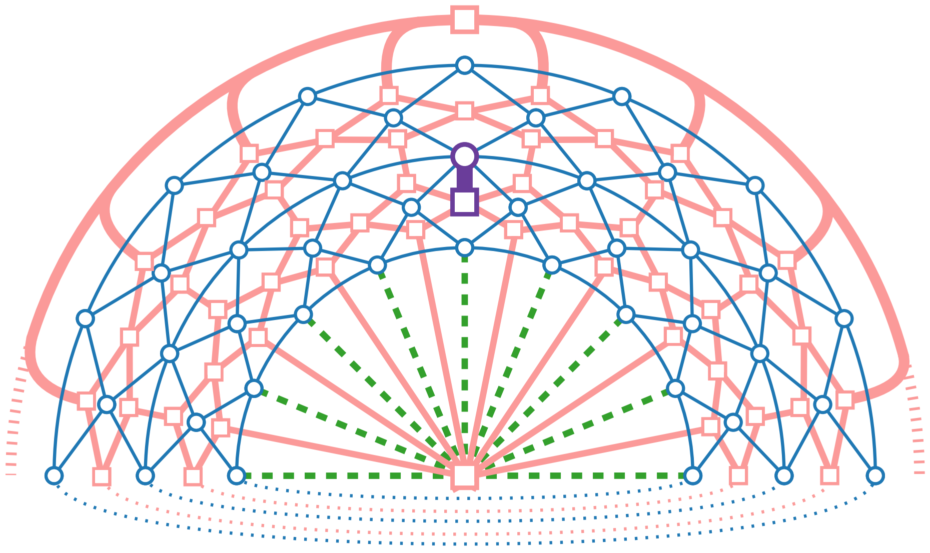

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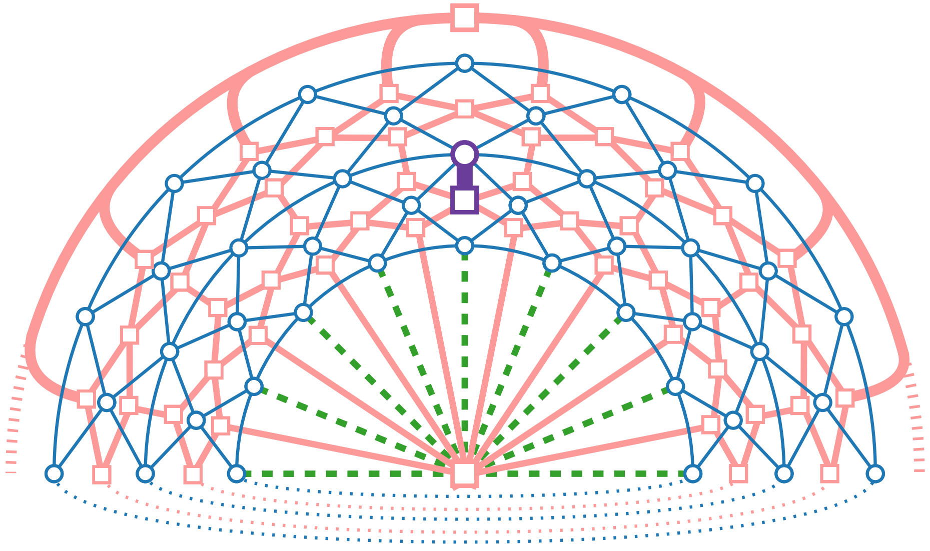

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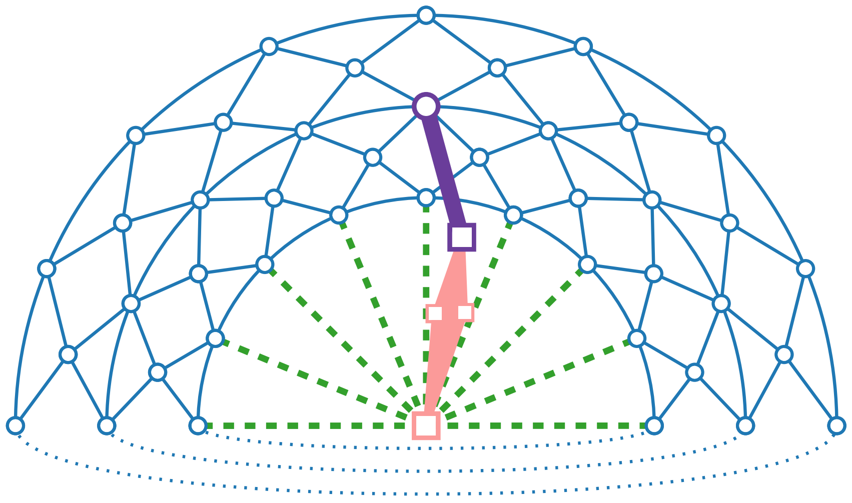

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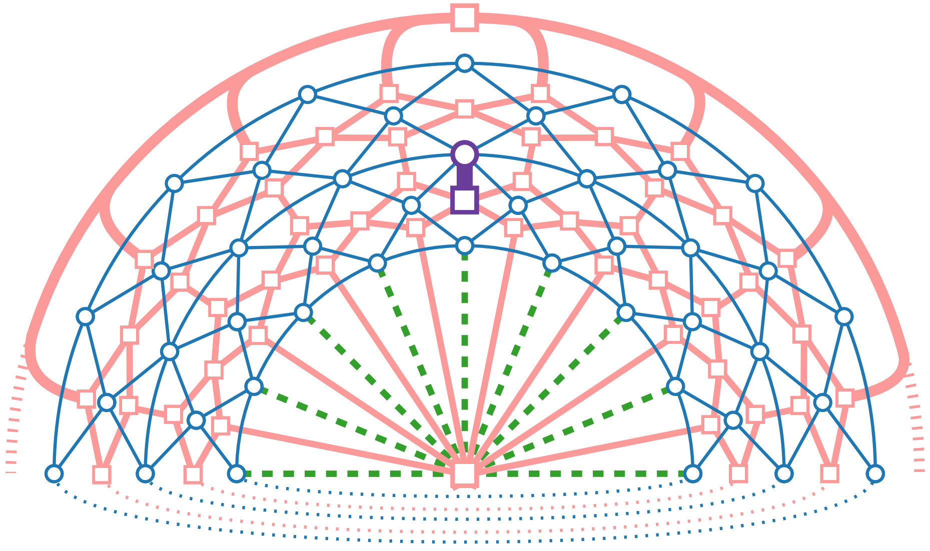

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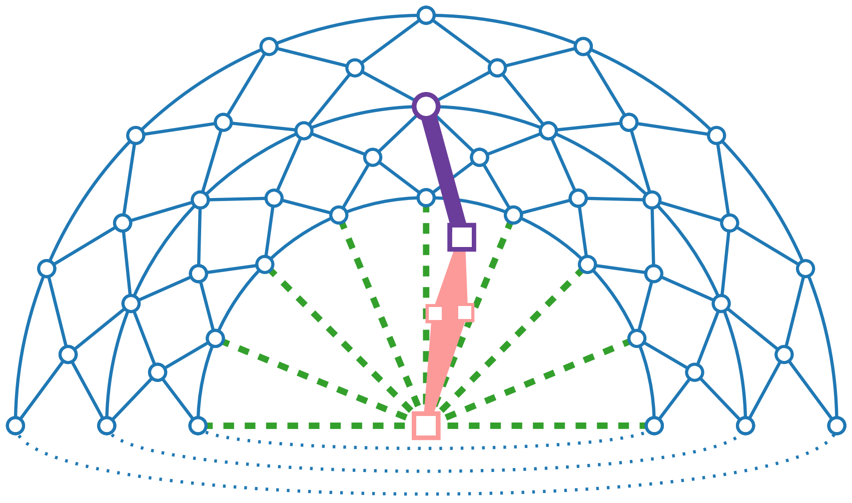

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Crossing ratio
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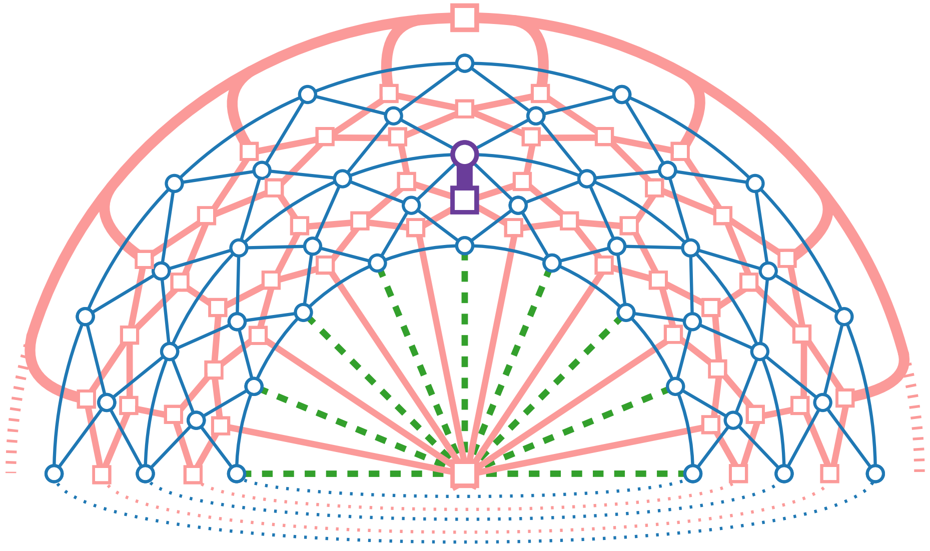

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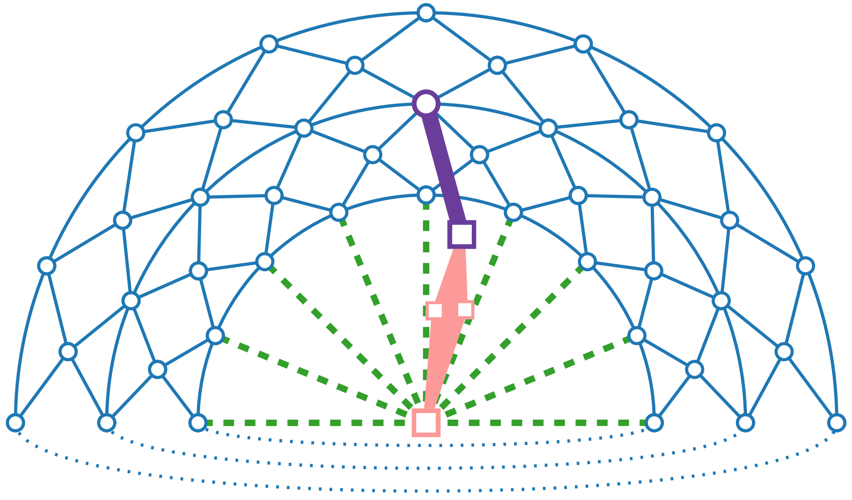

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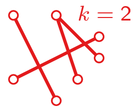
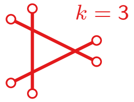
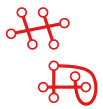
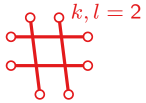
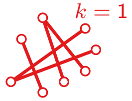
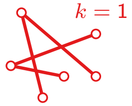
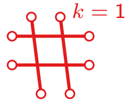
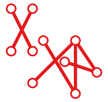
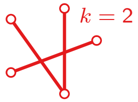
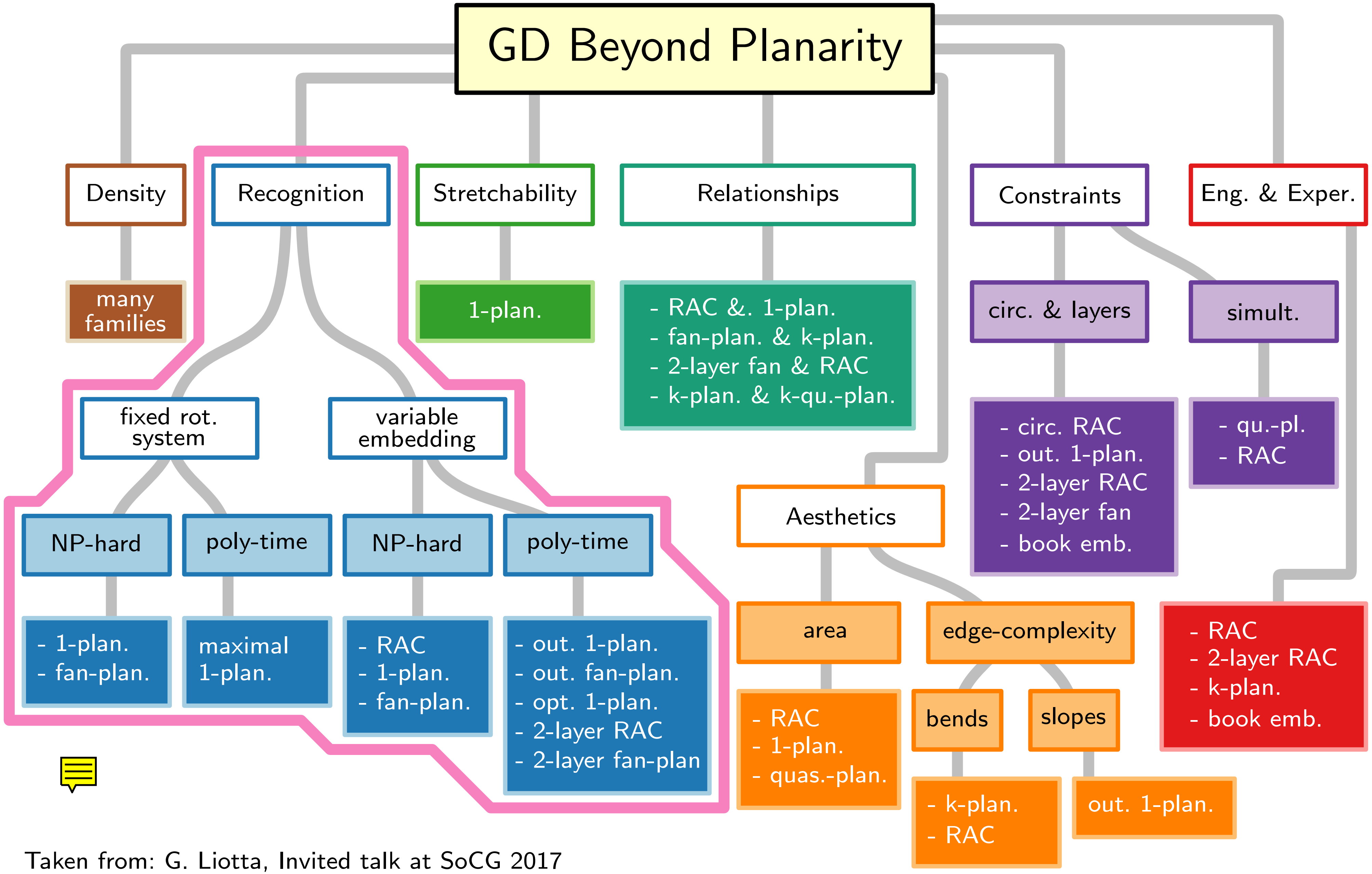
Crossing Ratios

Table from "Crossing Numbers of Beyond-Planar Graphs Revisited" [van Beusekom, Parada & Speckmann 2021]

Family	**Forbidden Configurations**	**Lower**	**Upper**
k-planar	An edge crossed more than k times 	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(k\sqrt{k}n)}$
k-quasi-planar	k pairwise crossing edges 	$\Omega(n/k^3)$	$f(k)n^2 \log^2 n$
Fan-planar	Two independent edges crossing a third or two adjacent edges crossing another edge from different "side" 	$\Omega(n)$	$O(n^2)$
(k,l)-grid-free	Set of k edges such that each edge crosses each edge from a set of l edges. 	$\Omega\left(\frac{n}{kl(k+l)}\right)$	$g(k,l)n^2$
k-gap-planar	More than k crossings mapped to an edge in an optimal mapping 	$\boldsymbol{\Omega(n/k^3)}$	$\boldsymbol{O(k\sqrt{k}n)}$
Skewness-k	Set of crossings not covered by at most k edges 	$\boldsymbol{\Omega(n/k)}$	$\boldsymbol{O(kn+k^2)}$
k-apex	Set of crossings not covered by at most k vertices 	$\Omega(n/k)$	$O(k^2n^2+k^4)$
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge 	$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$
k-fan-crossing-free	An edge that crosses k adjacent edges 	$\boldsymbol{\Omega(n^2/k^3)}$	$\boldsymbol{O(k^2n^2)}$
Straight-line RAC	Two edges crossing at an angle $< \frac{\pi}{2}$ 	$\boldsymbol{\Omega(n^2)}$	$\boldsymbol{O(n^2)}$

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

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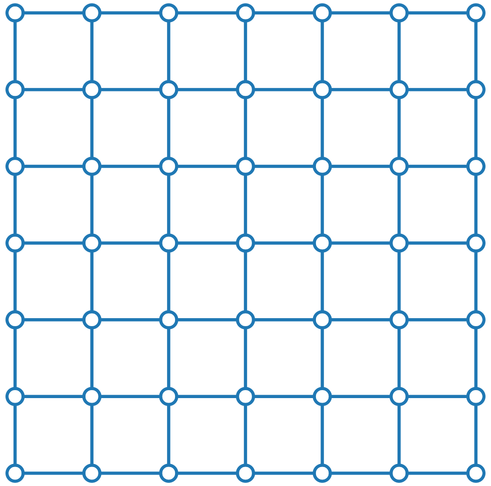

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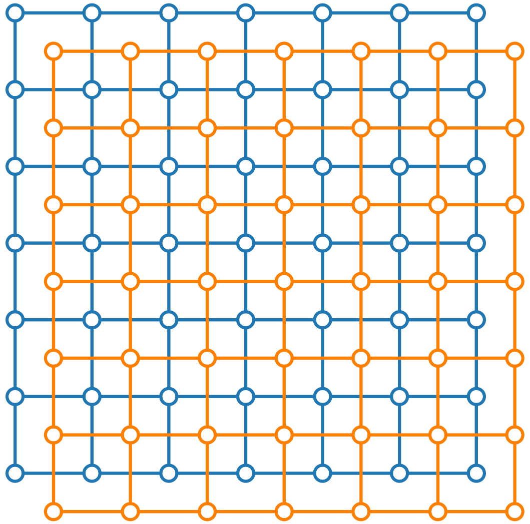

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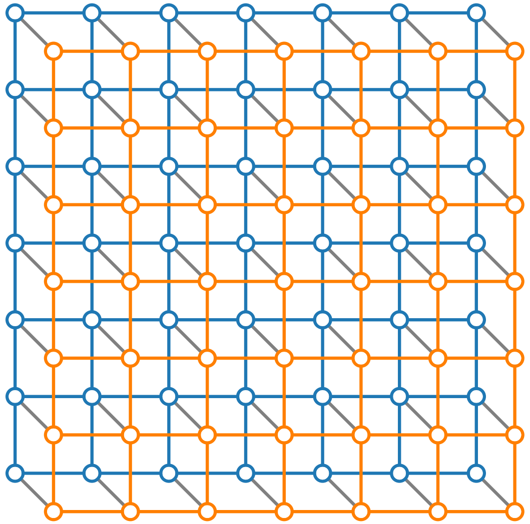

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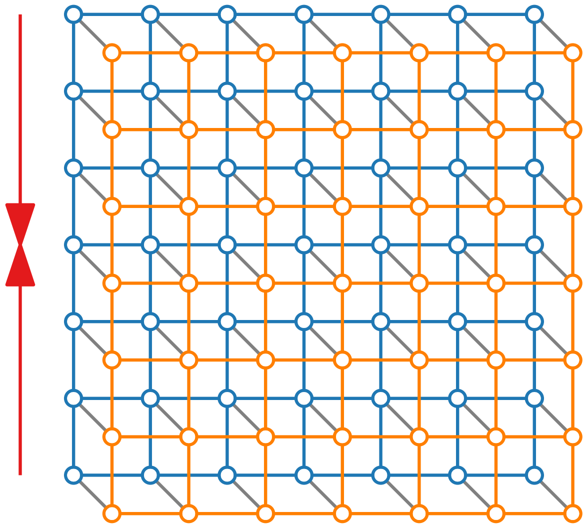

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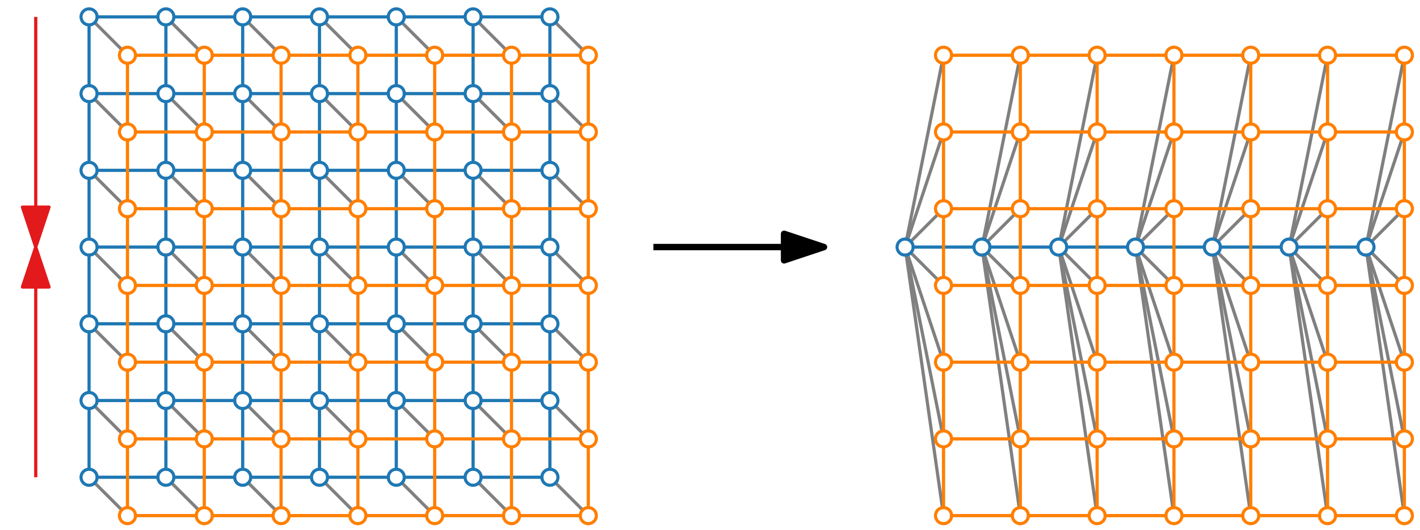

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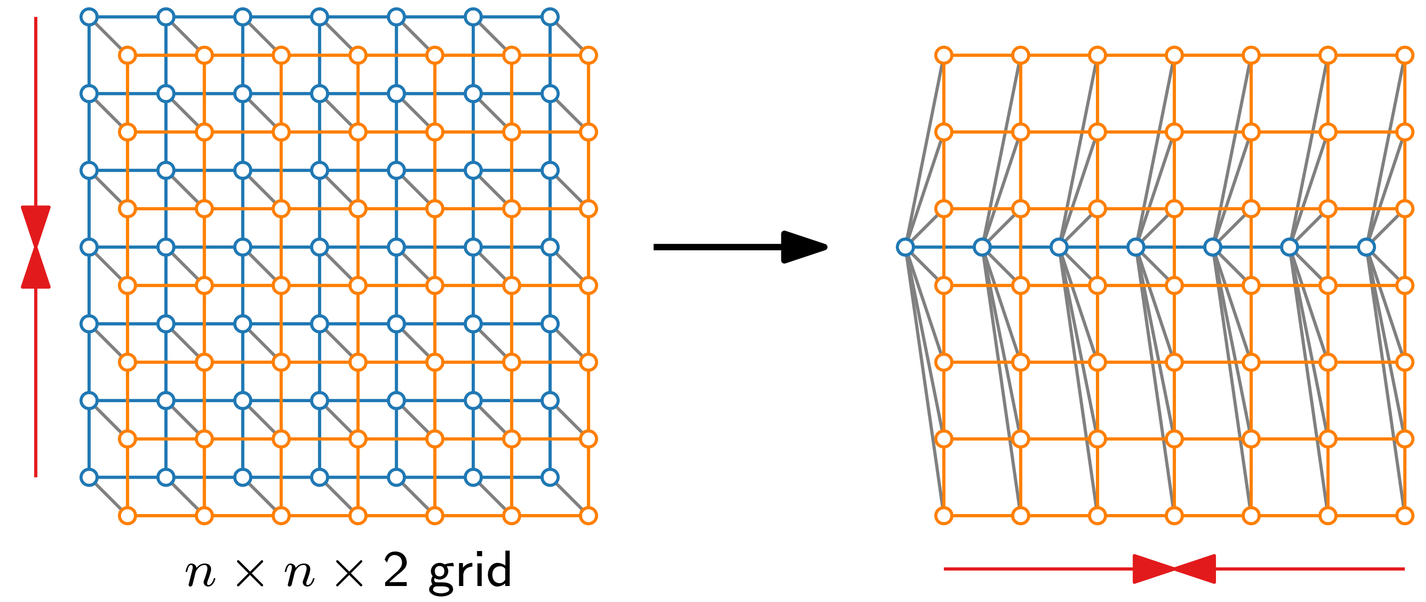

Minors of 1-Planar Graphs

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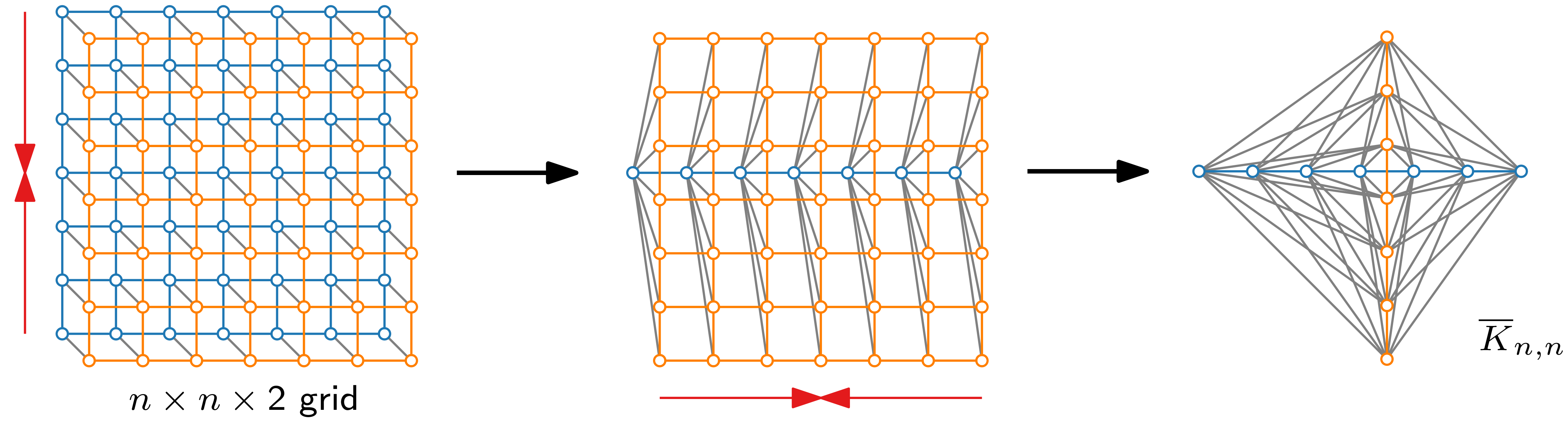

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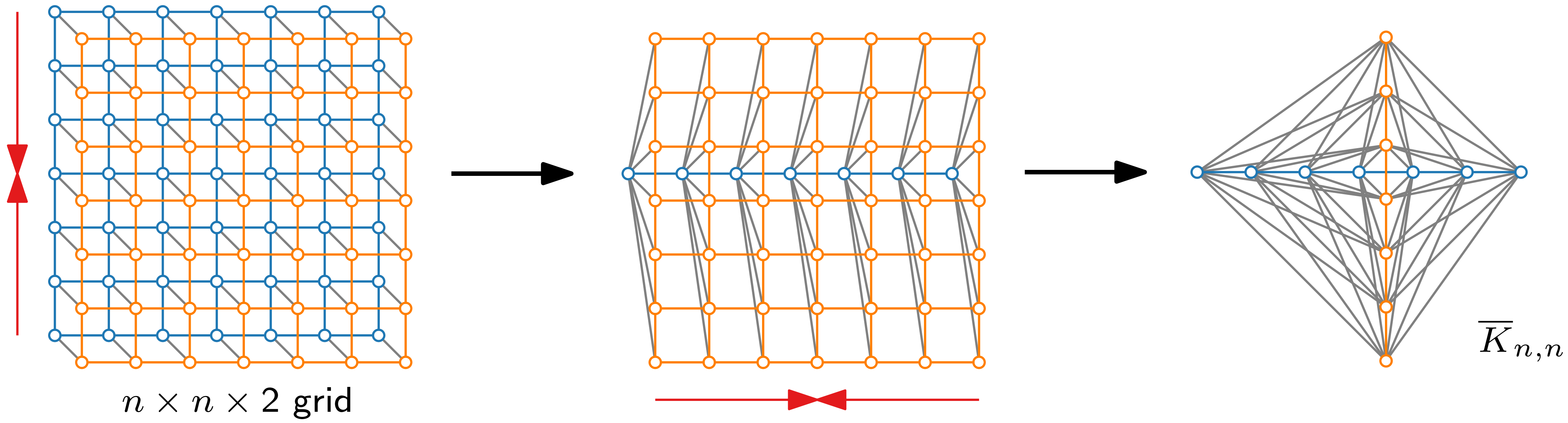

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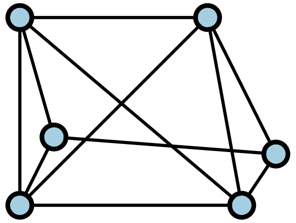

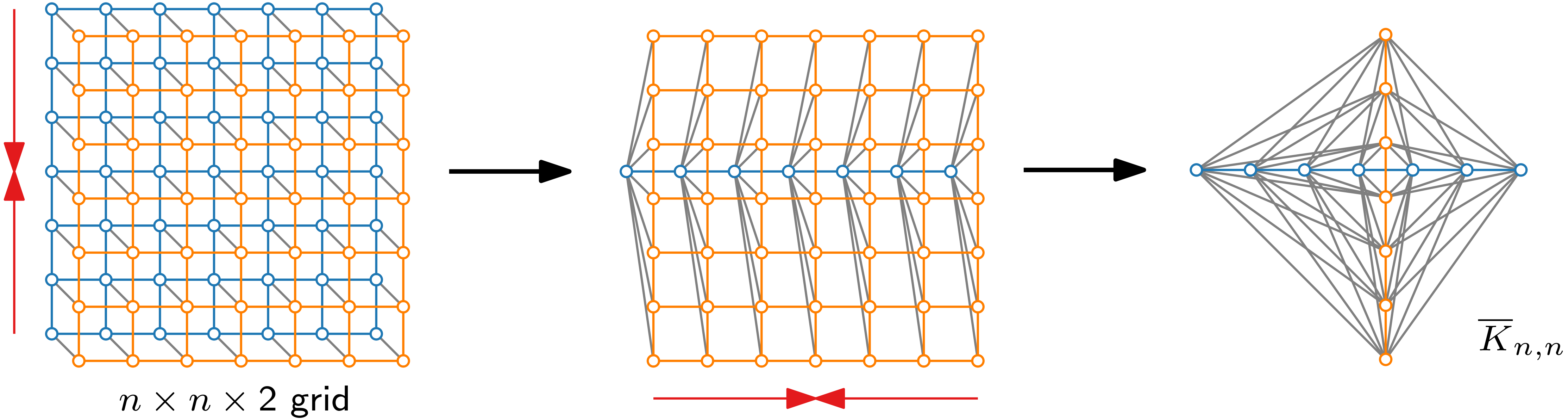

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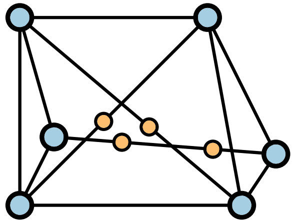

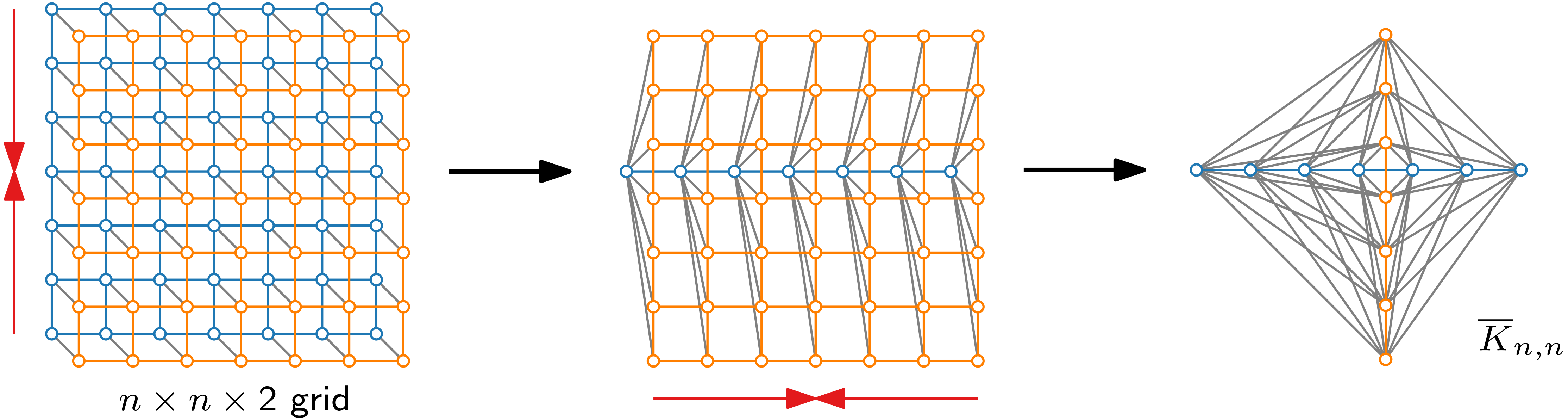

Theorem. [Korzhik & Mohar 2013]
For any n, there exist $\Omega(2^n)$ distinct n-vertex graphs that are not 1-planar but all their proper subgraphs are 1-planar.

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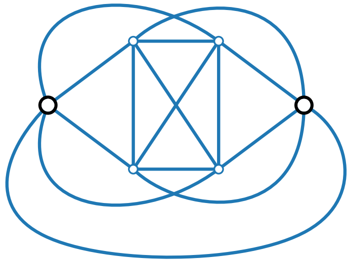

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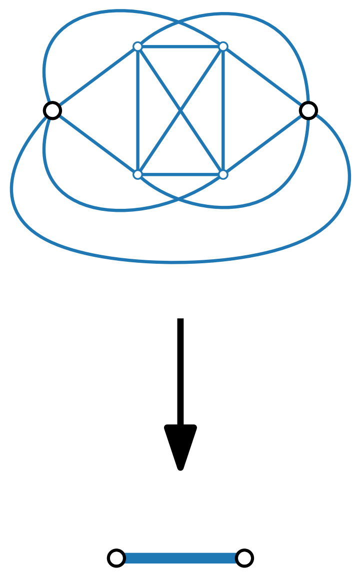

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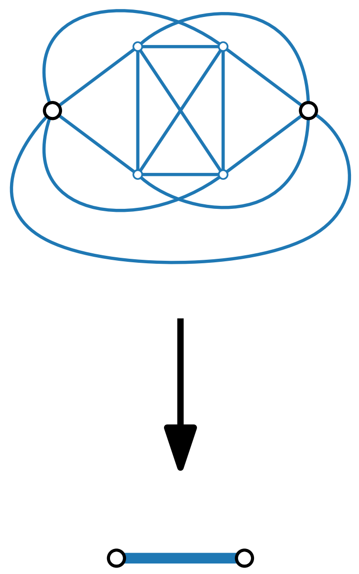

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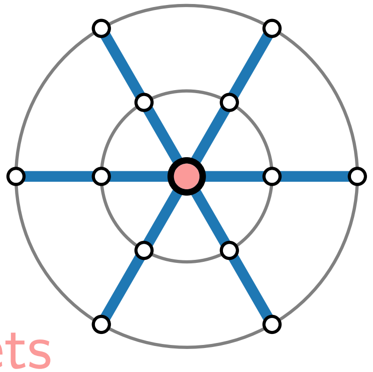

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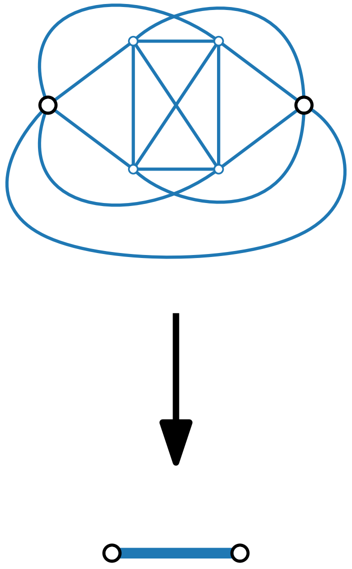

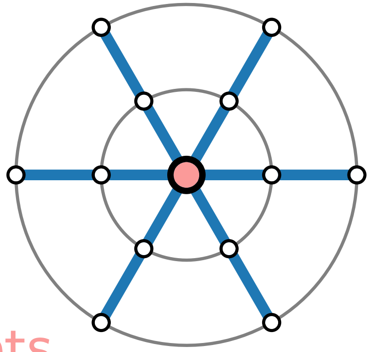
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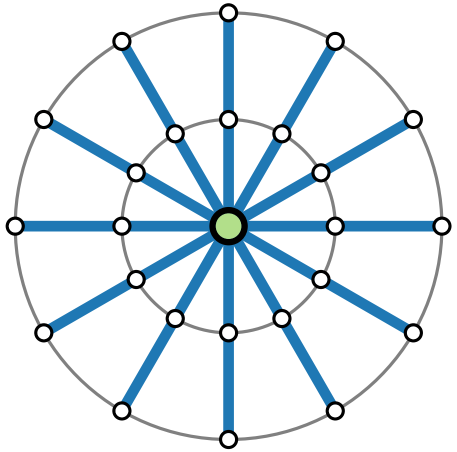
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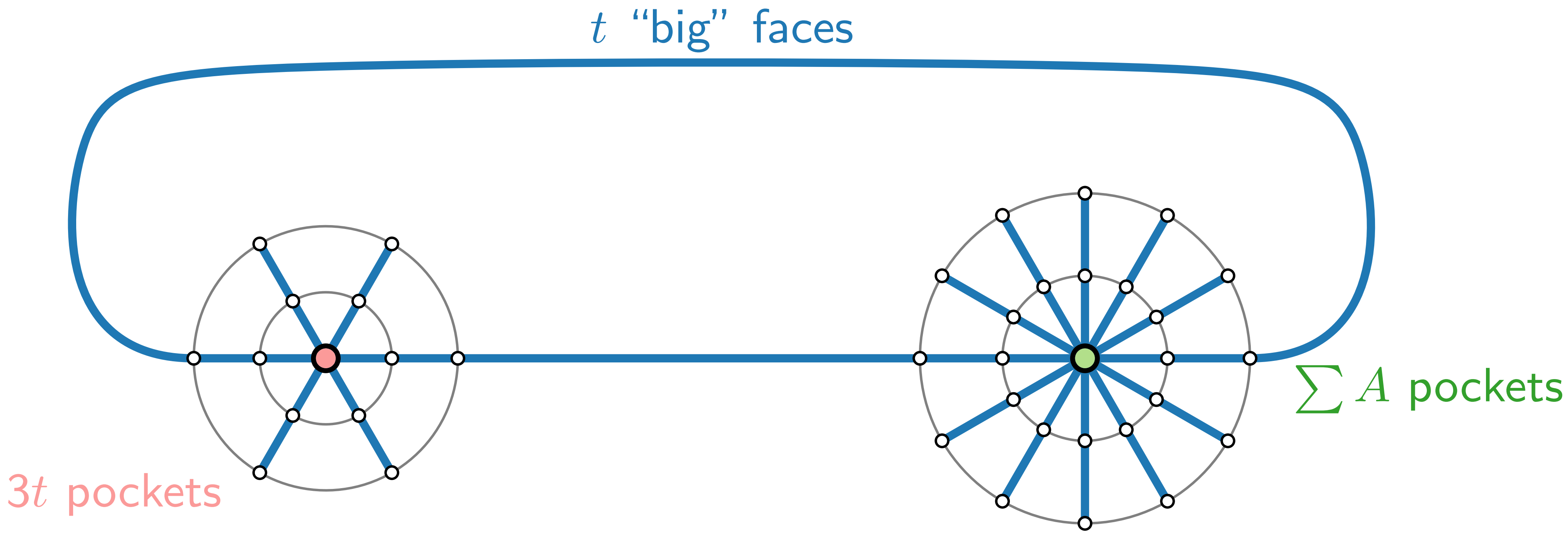

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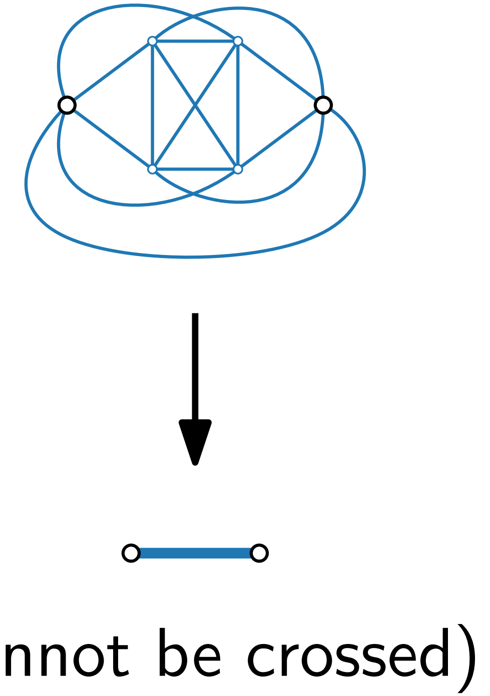

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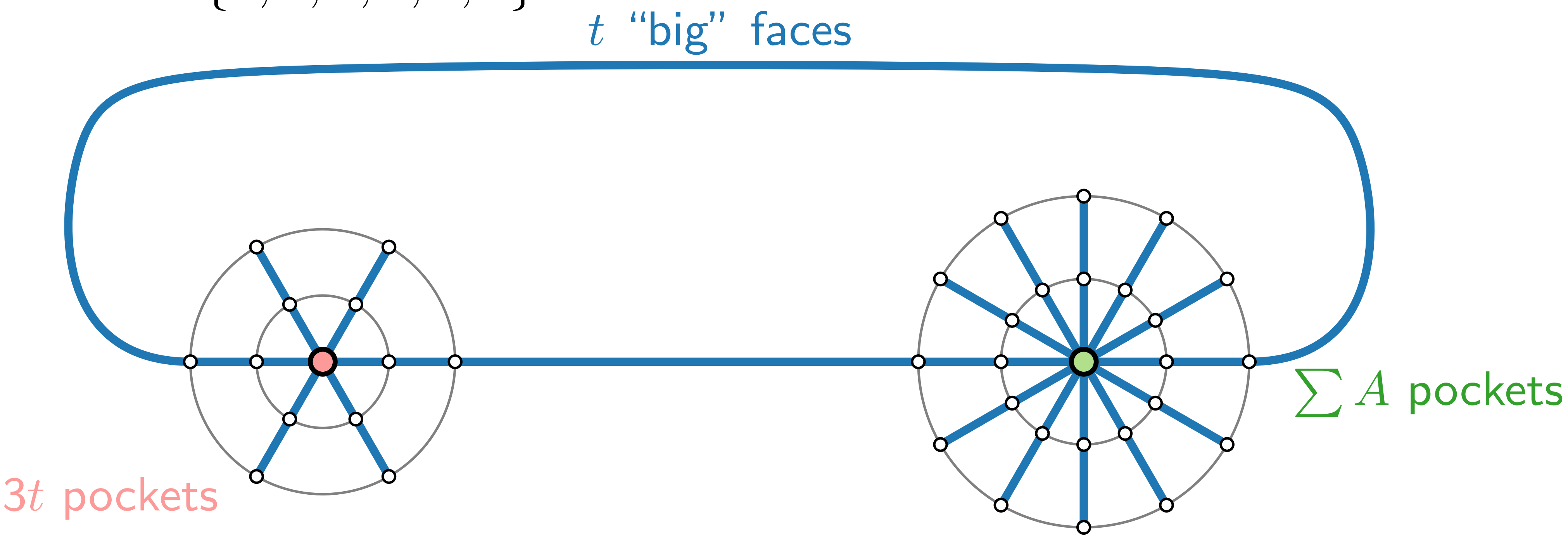

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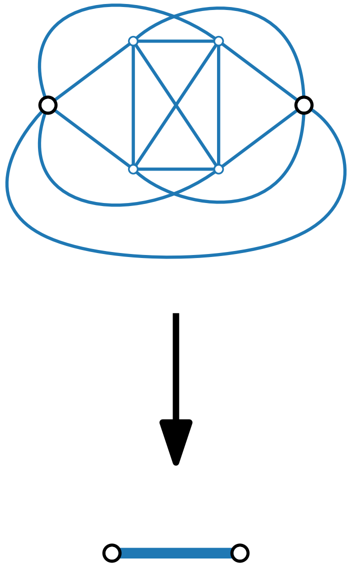

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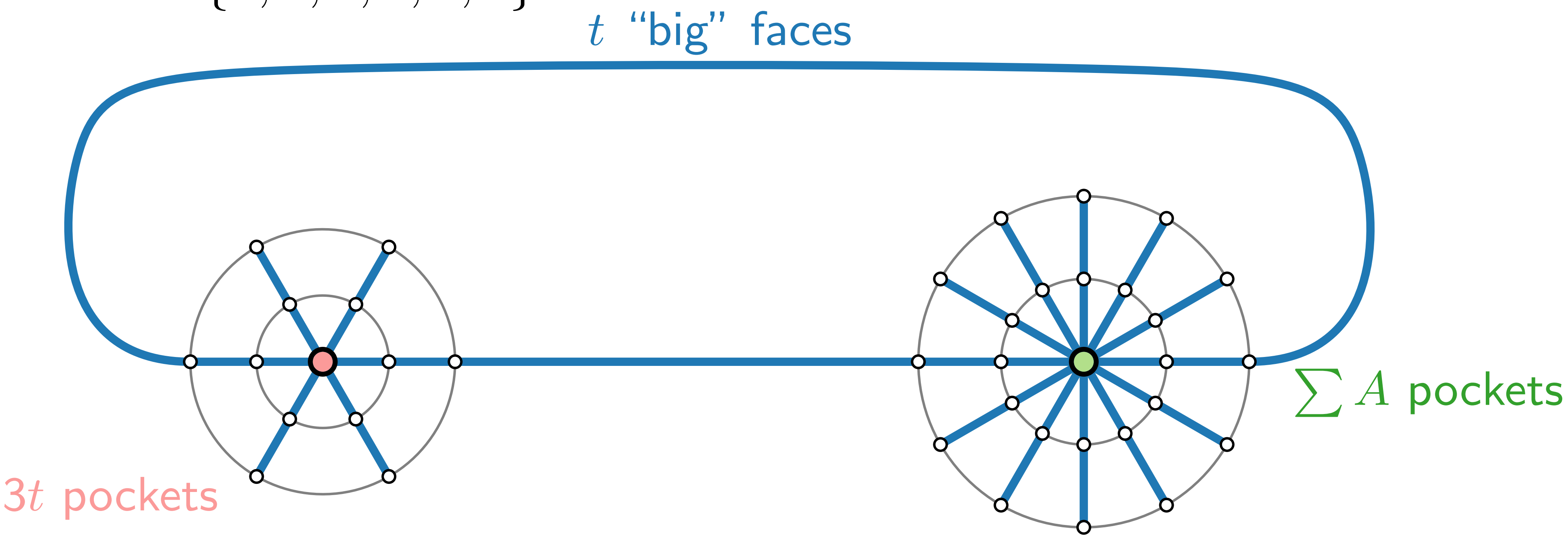

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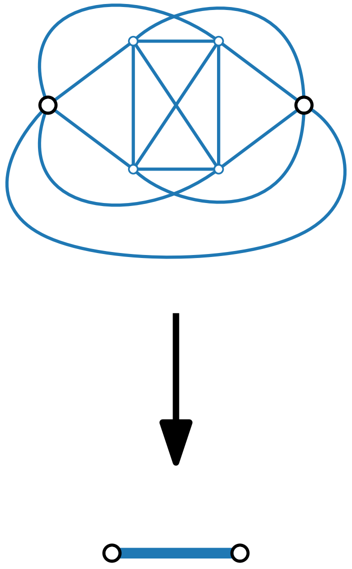

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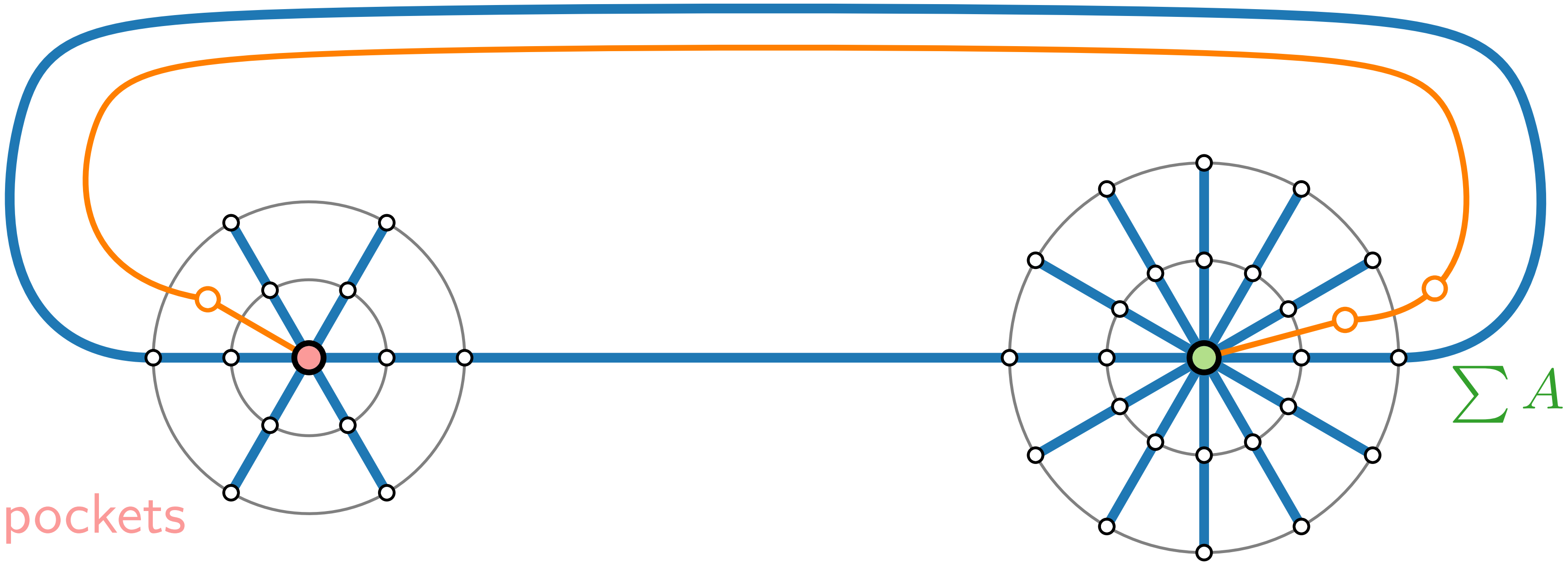

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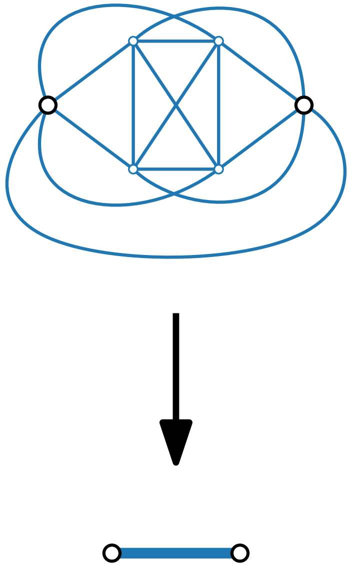

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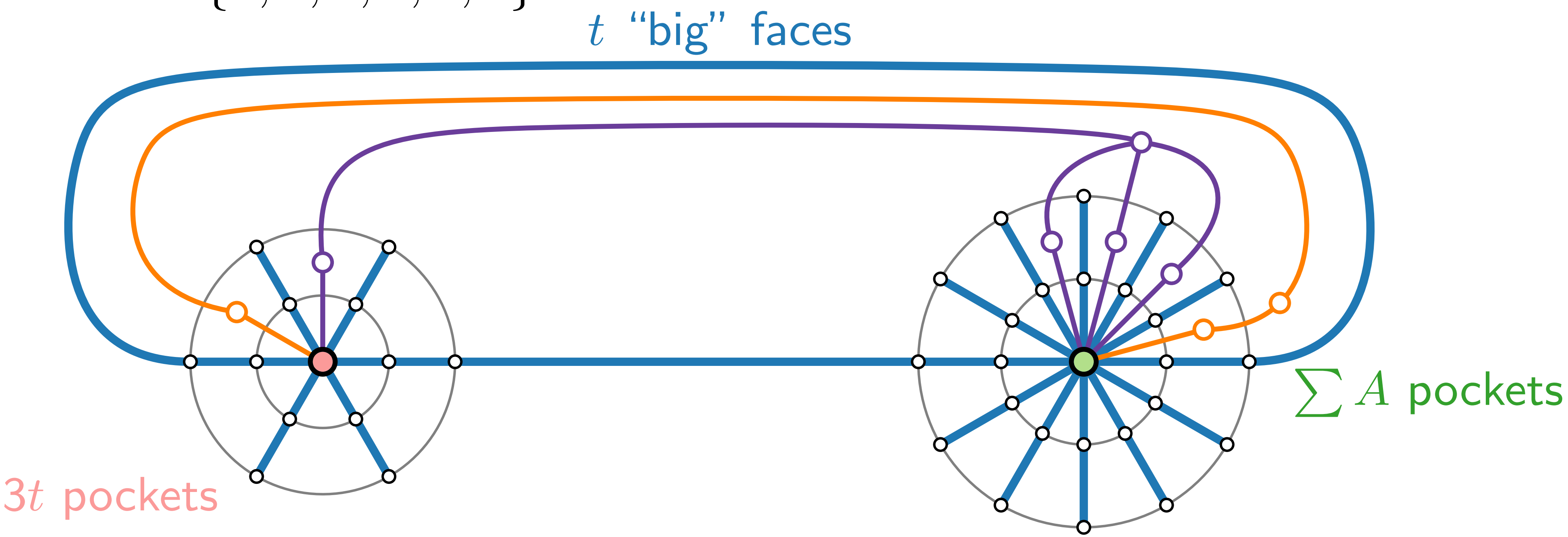

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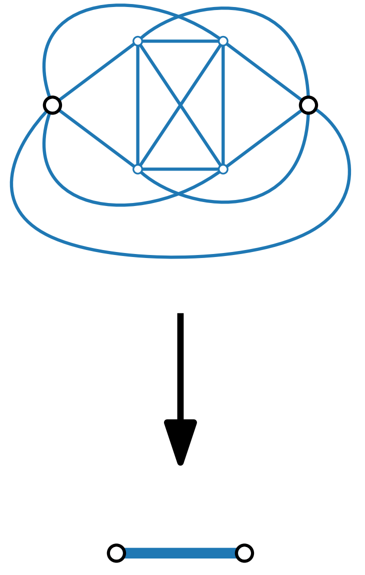

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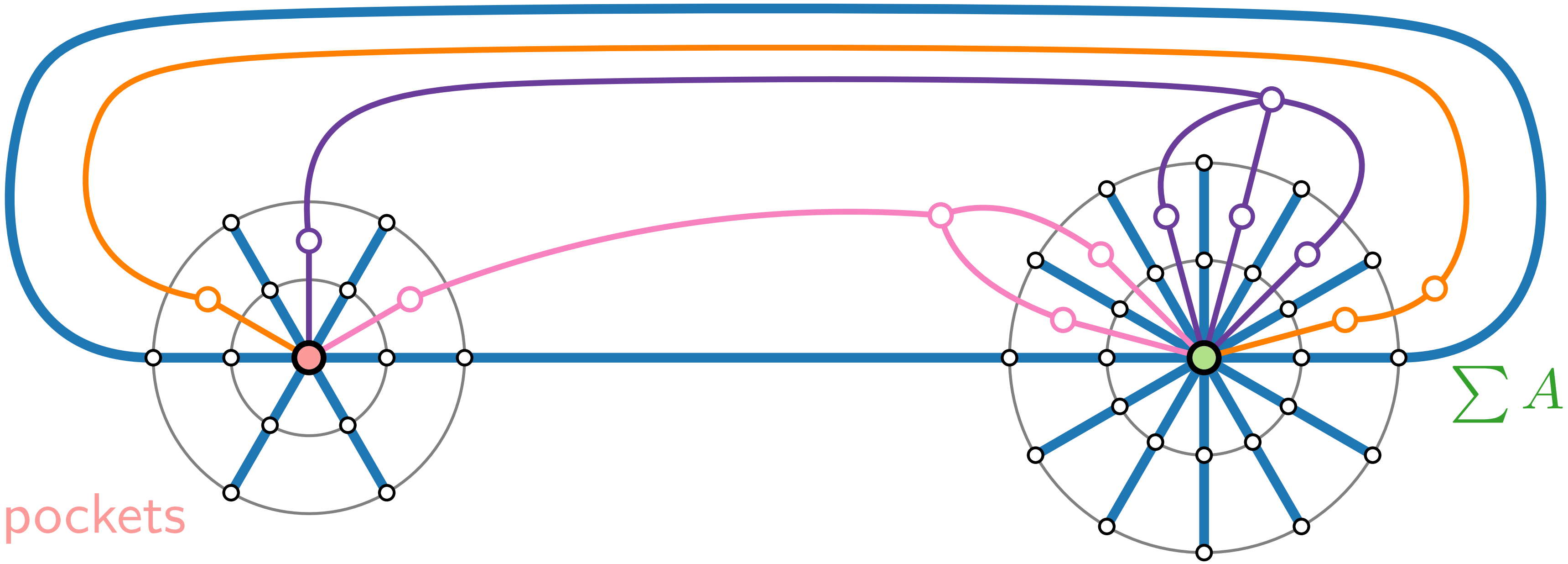

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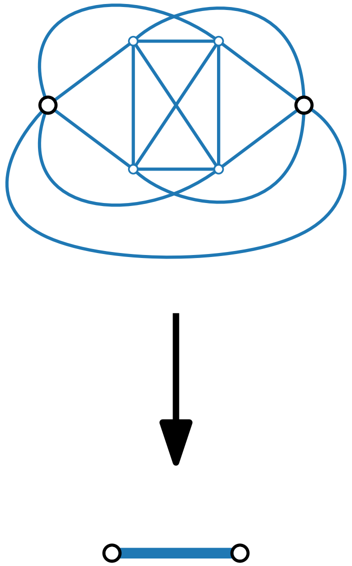

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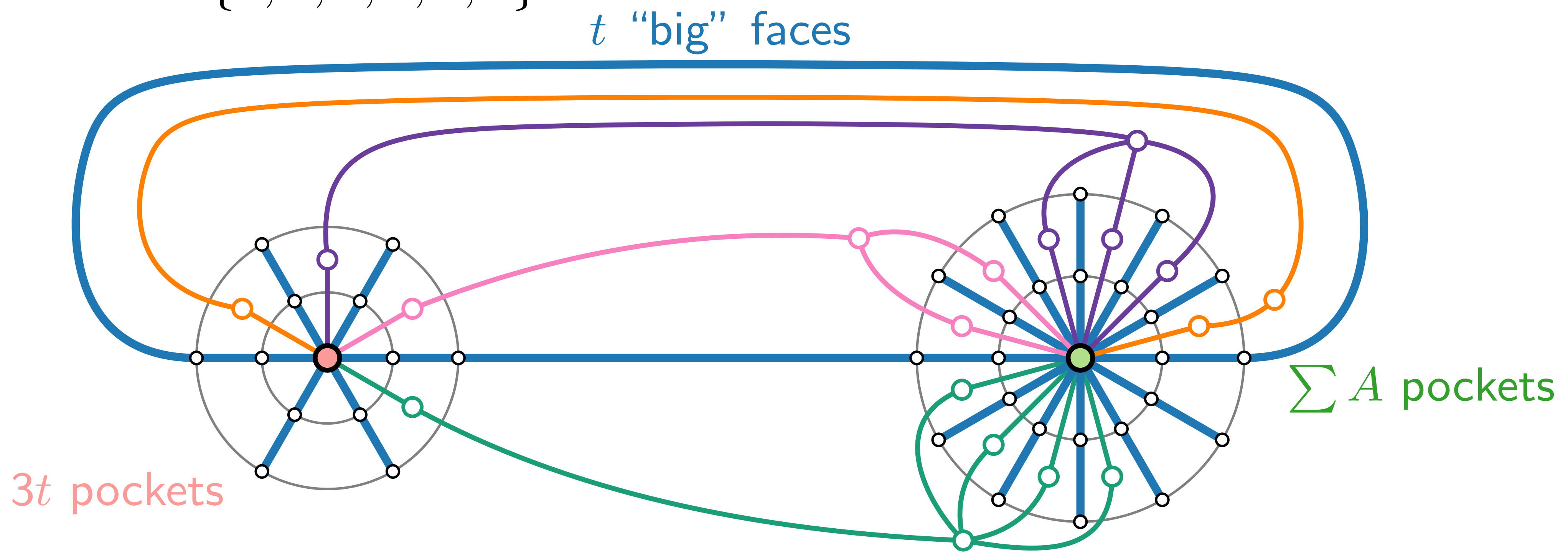

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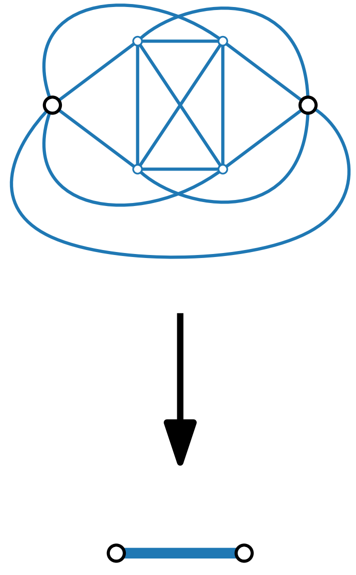

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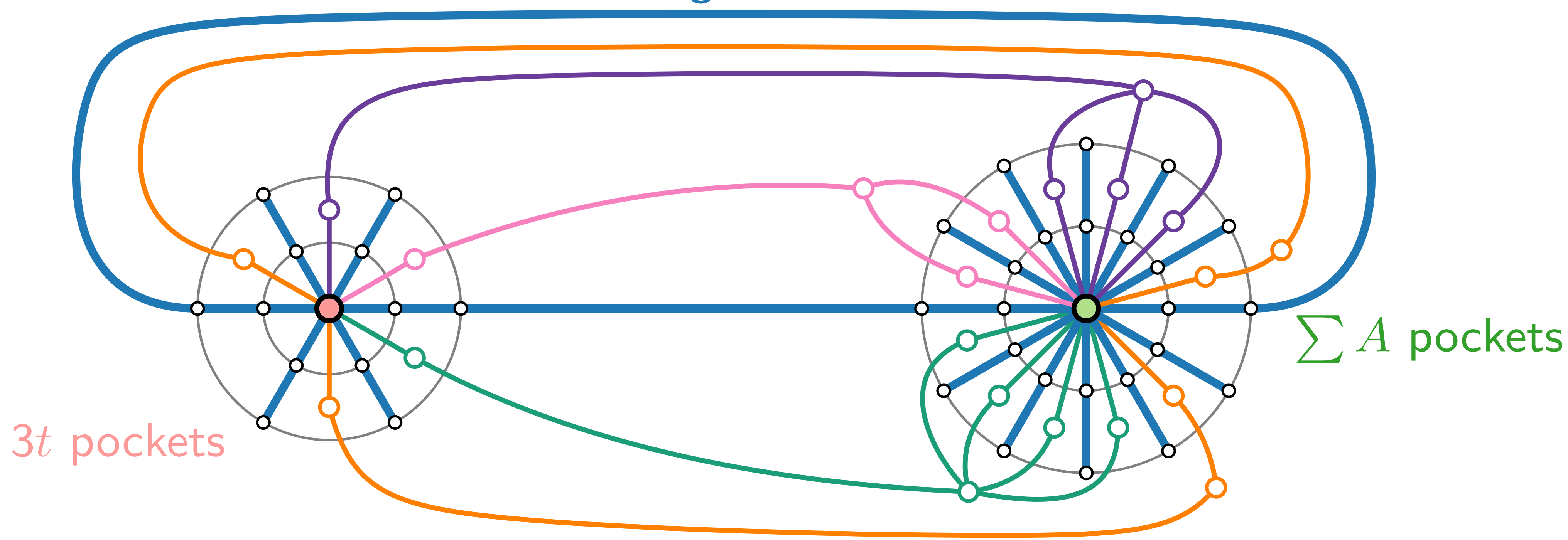

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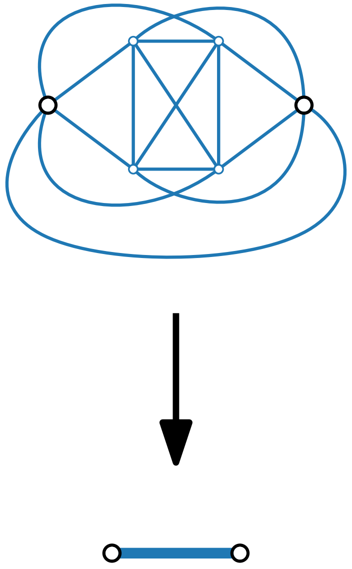

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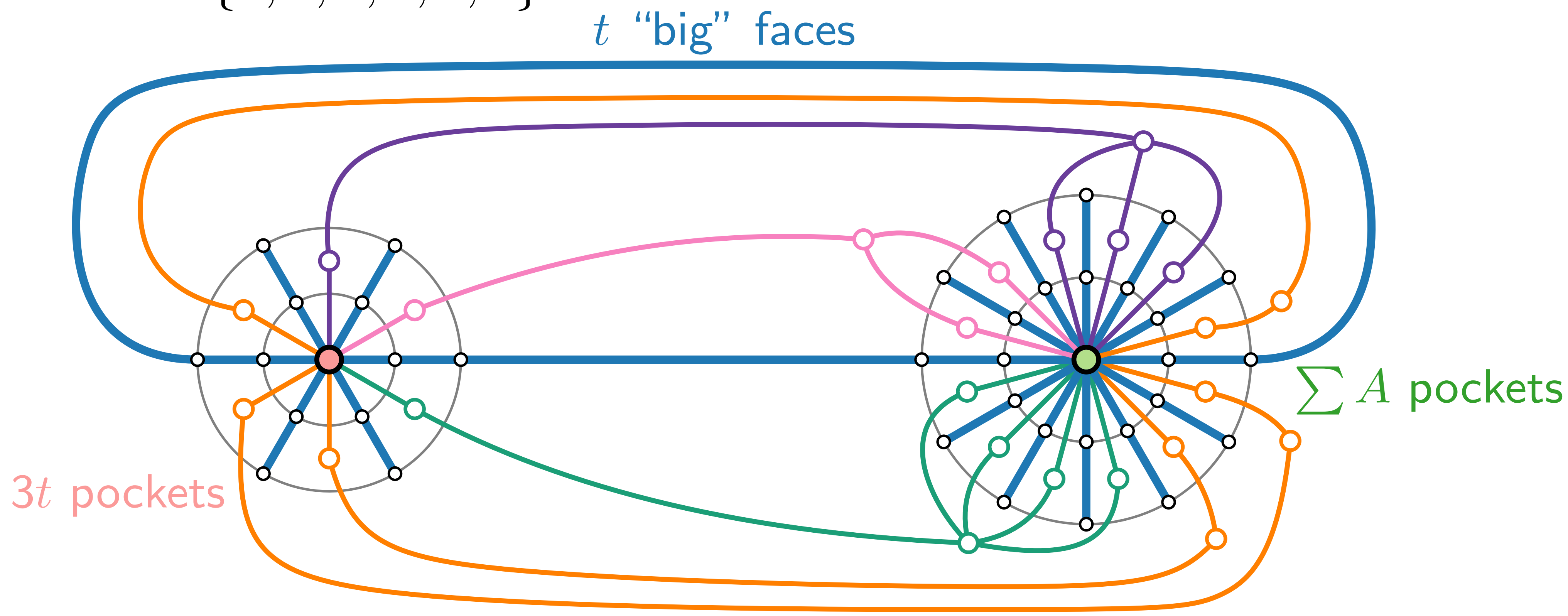

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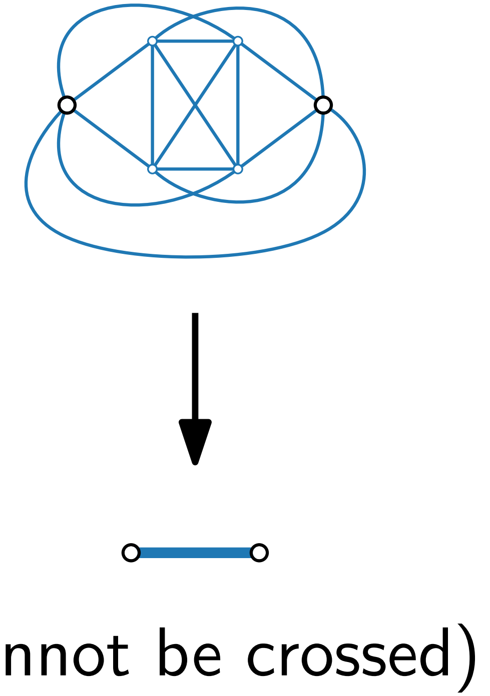

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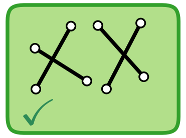

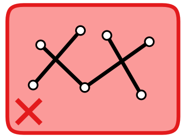

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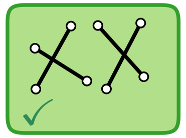

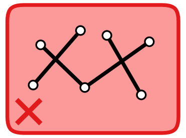

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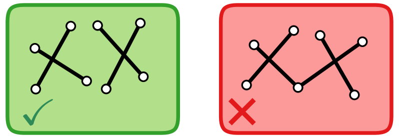

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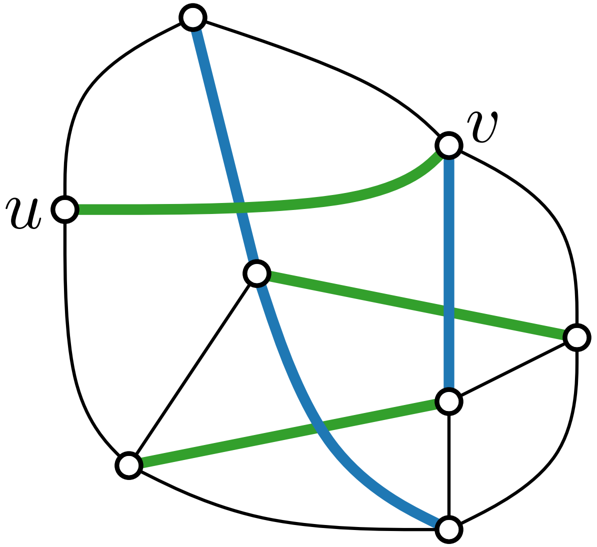

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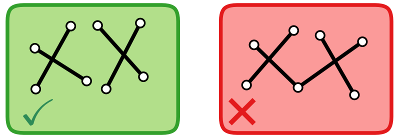

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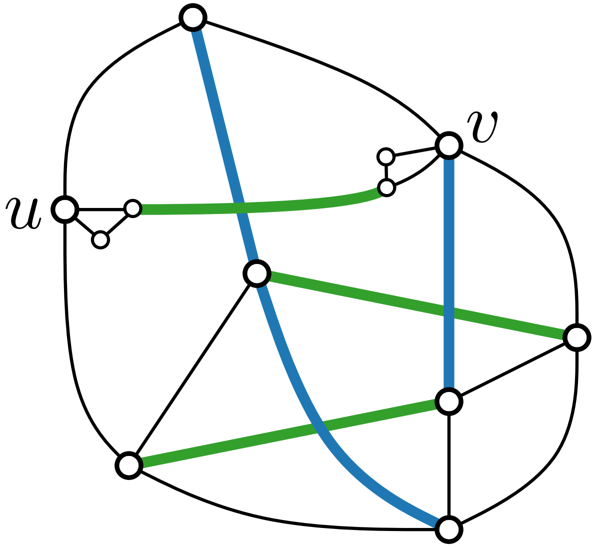

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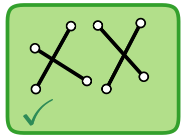 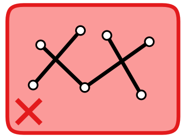

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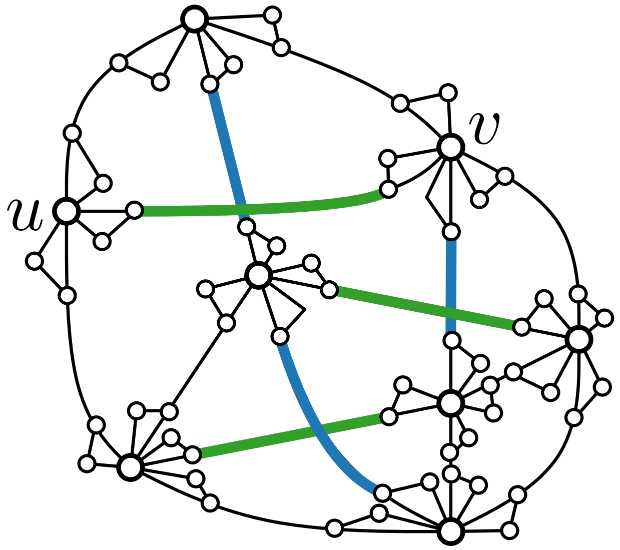

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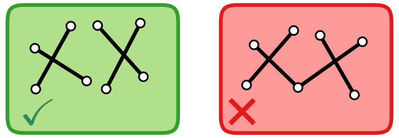

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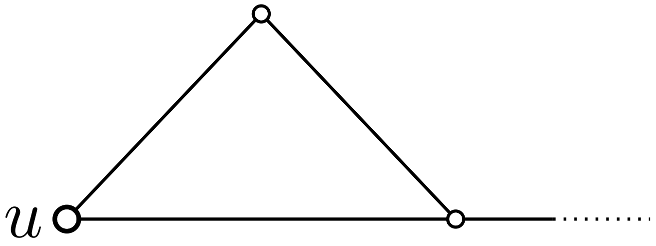
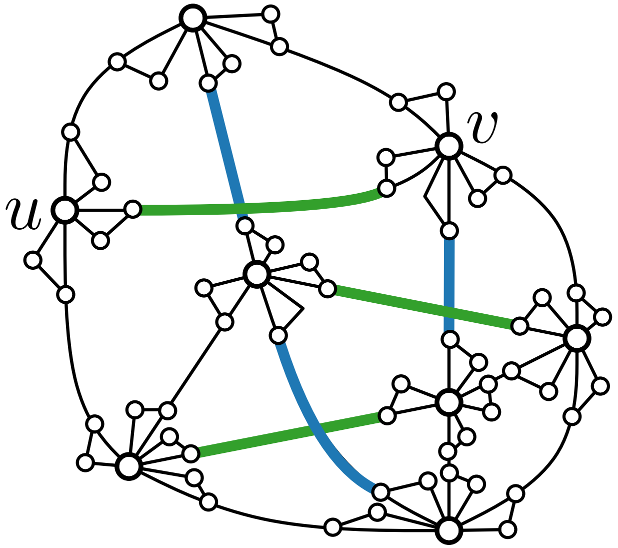
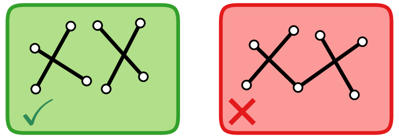




Recognition of IC-Planar Graphs

Theorem. [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.



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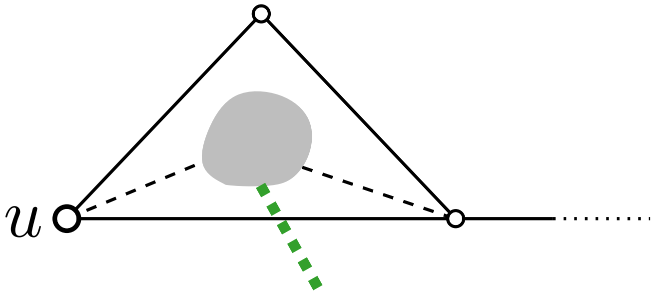

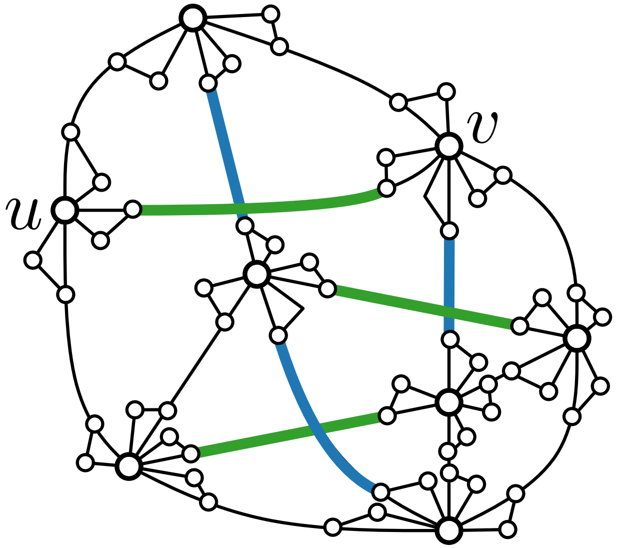

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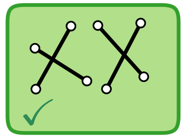

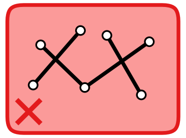

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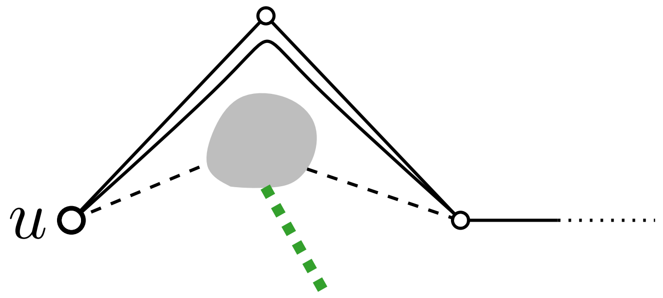

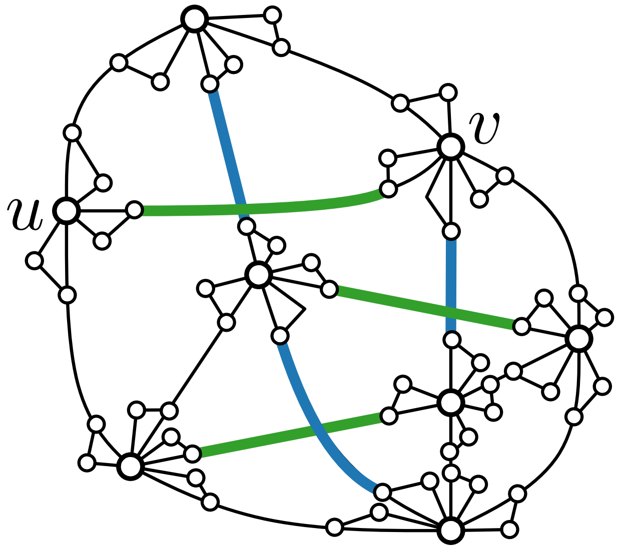

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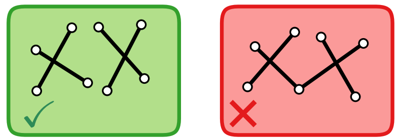

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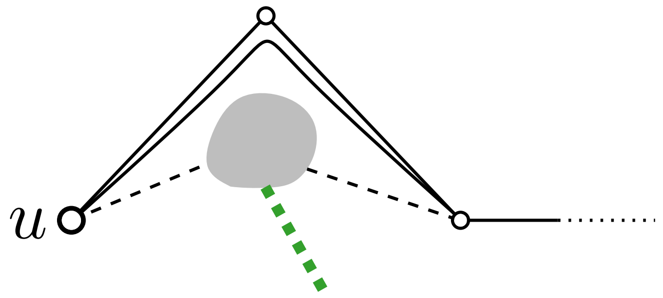

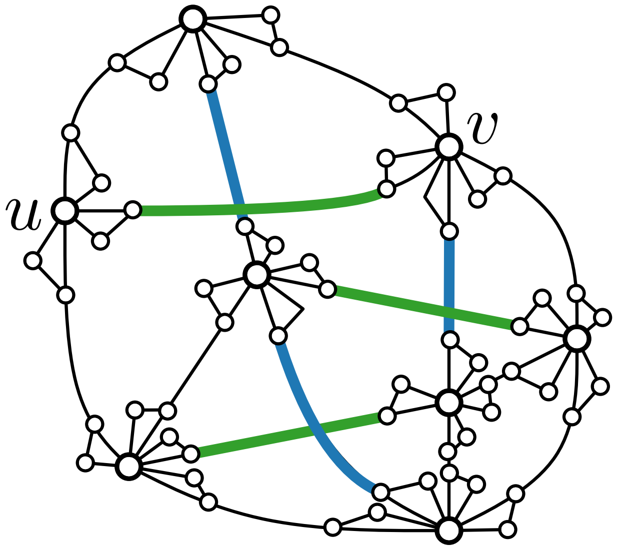

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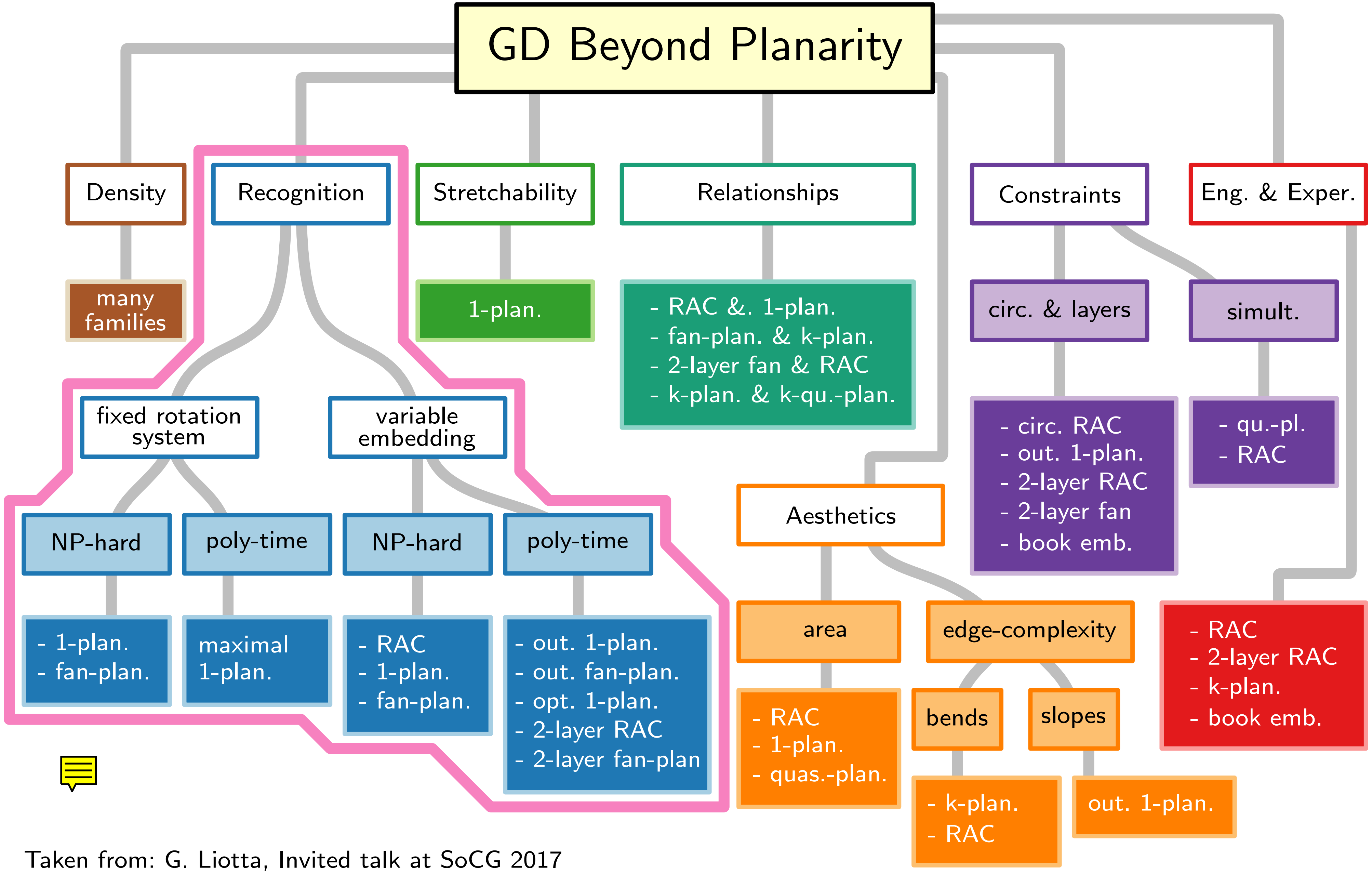
GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

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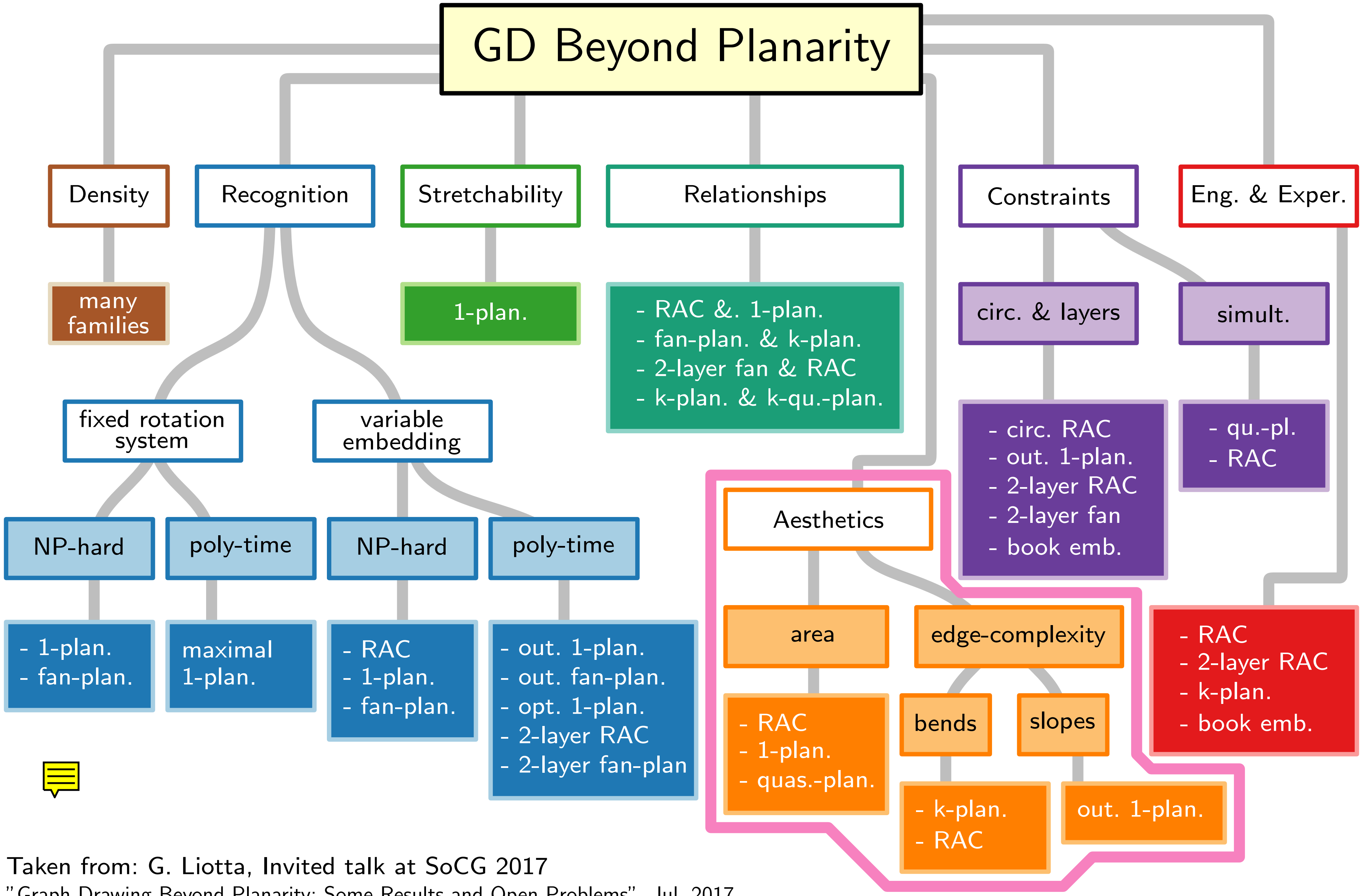

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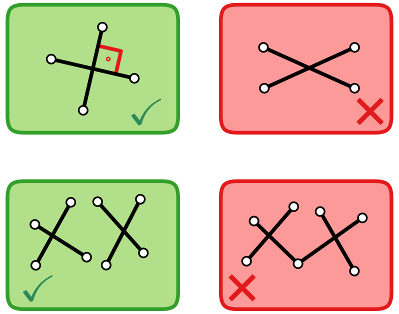
Area of Straight-Line RAC Drawings



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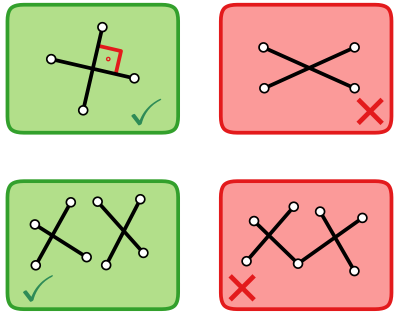

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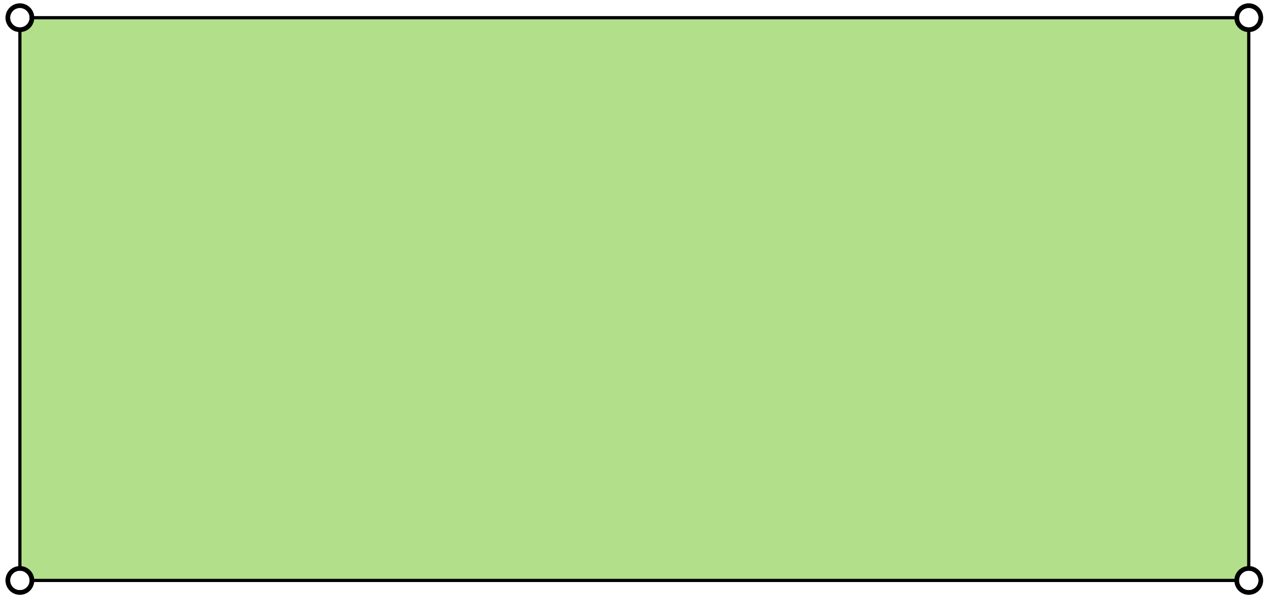

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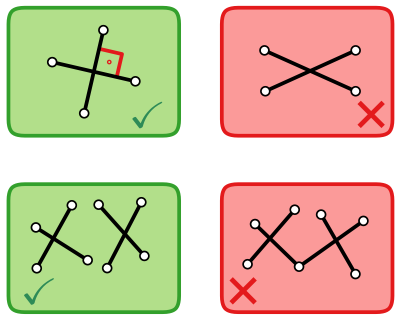

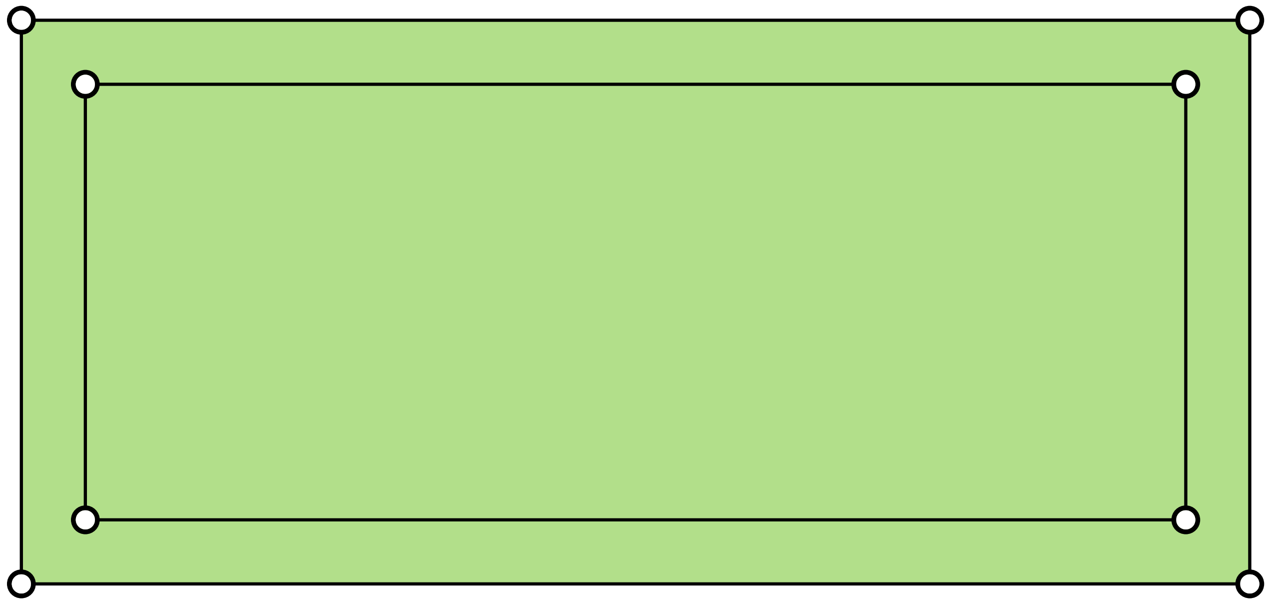

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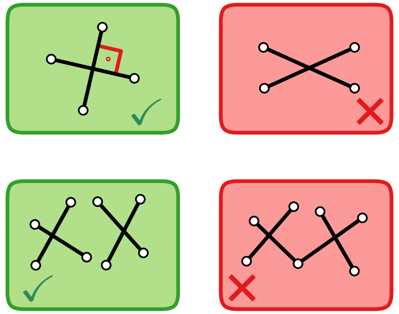

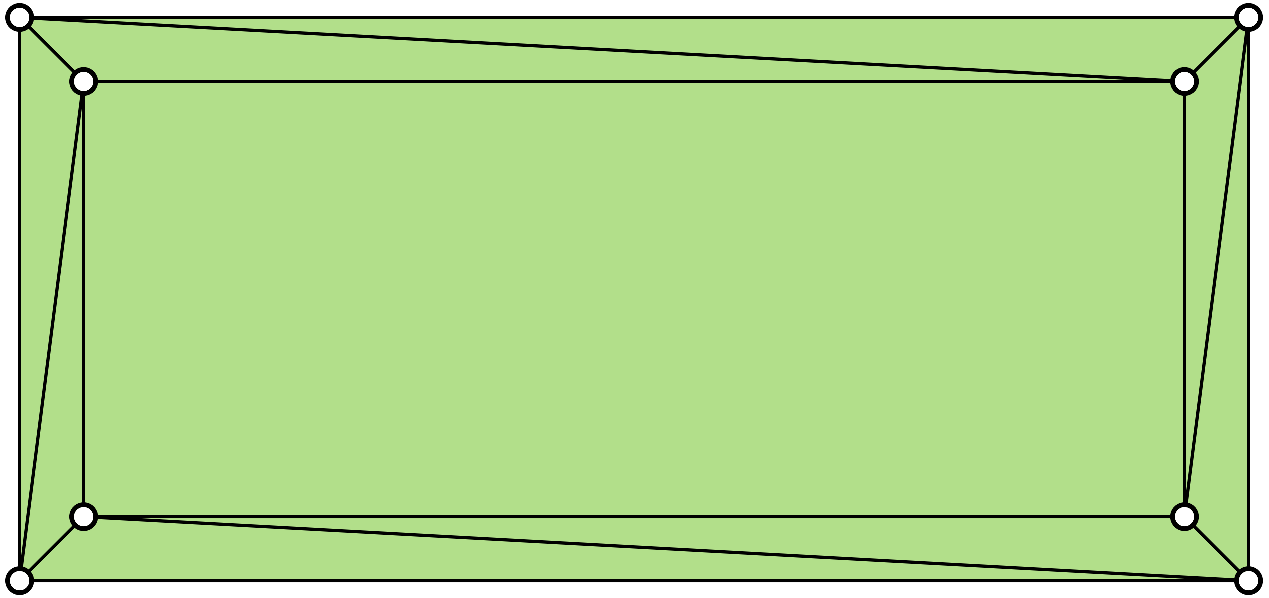

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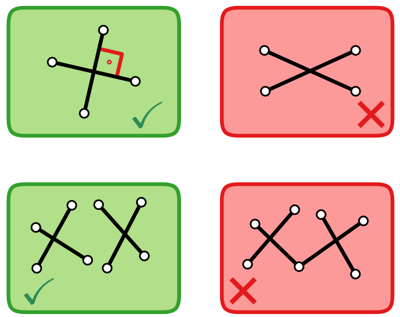

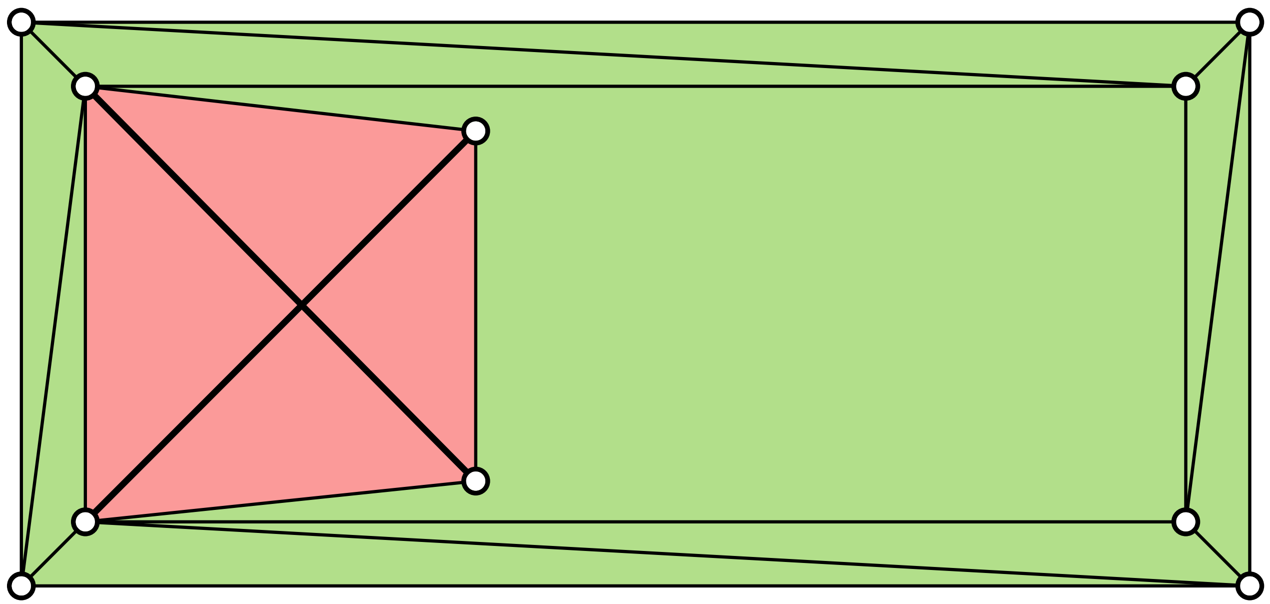

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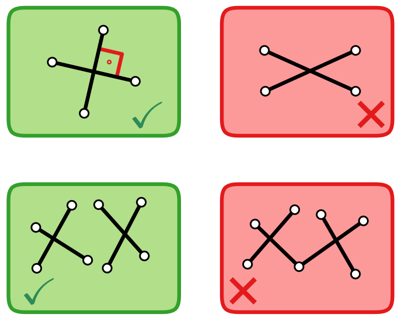

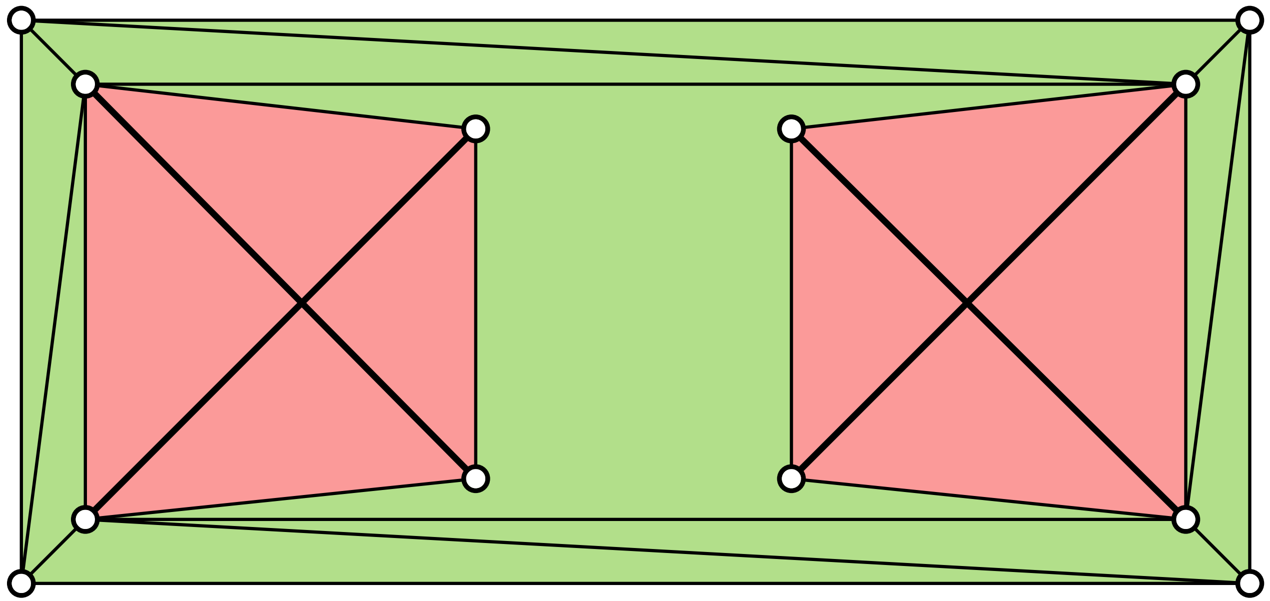

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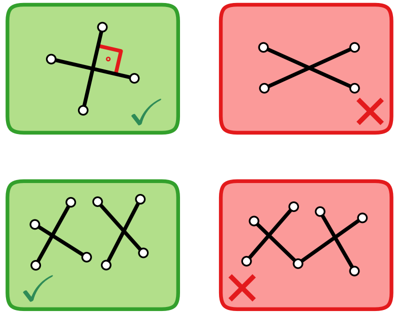

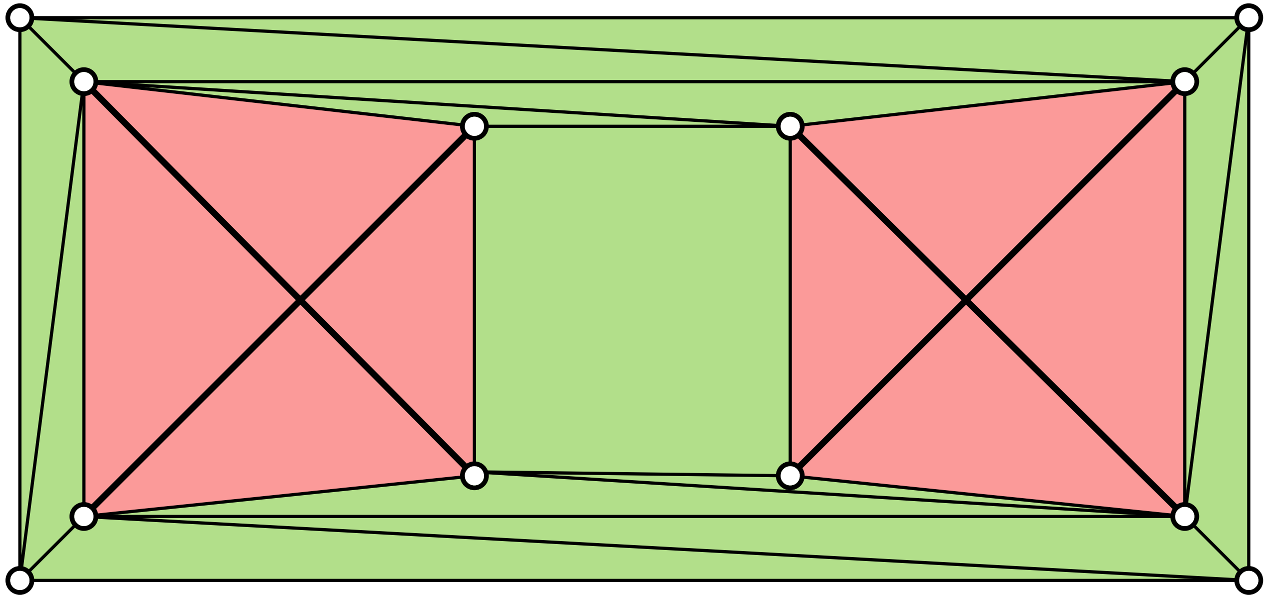

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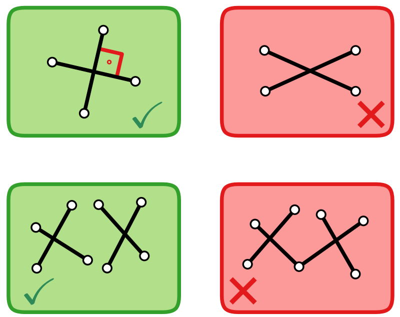

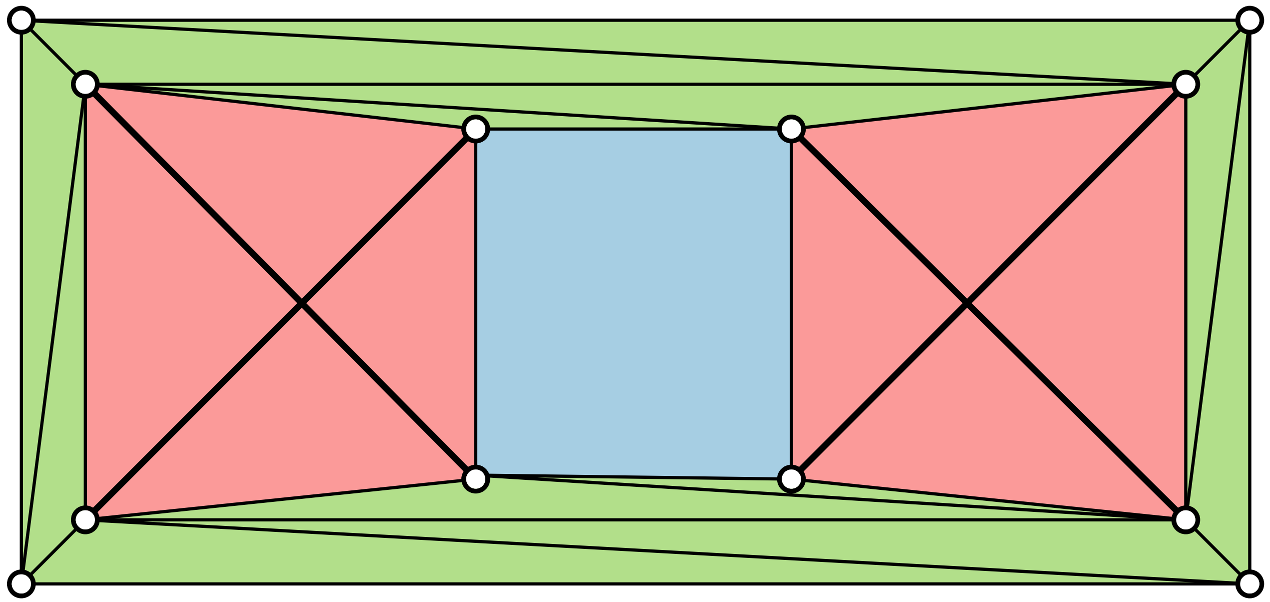

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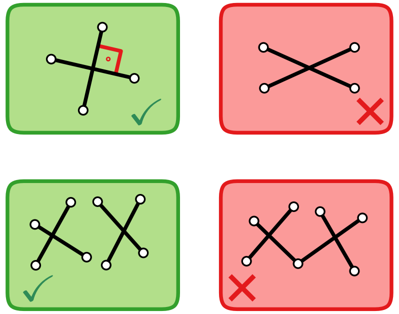

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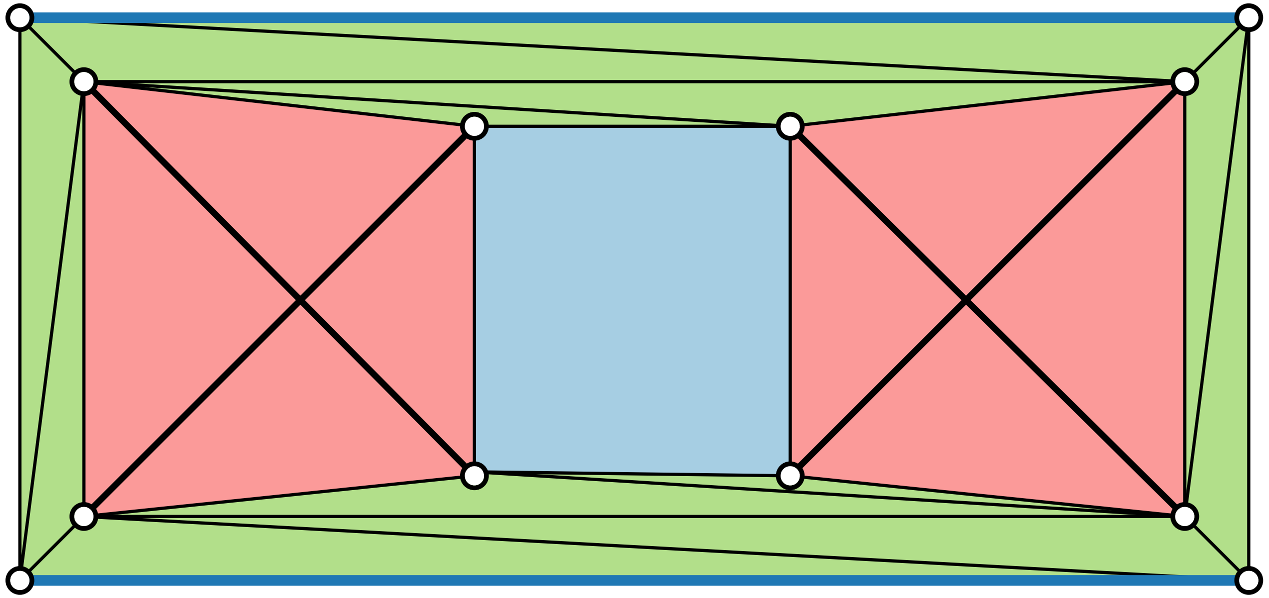

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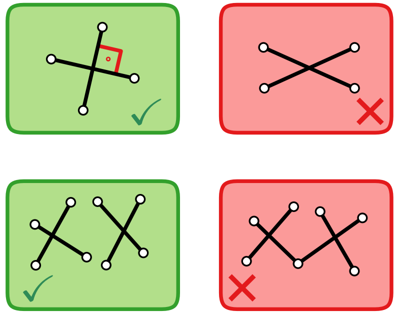

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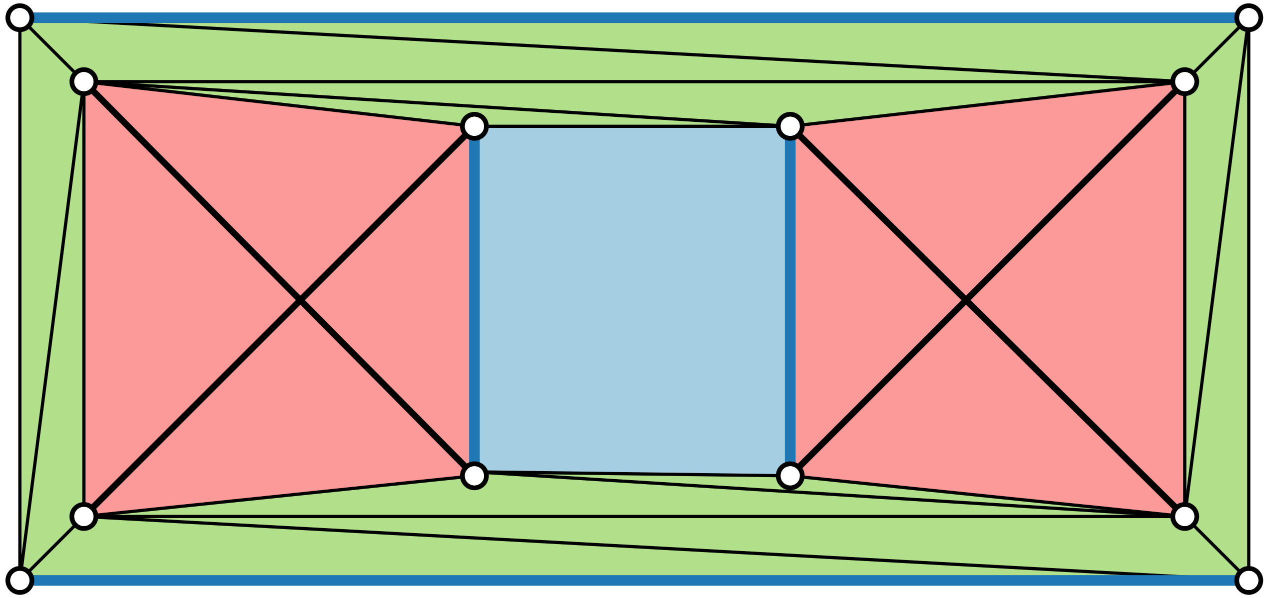

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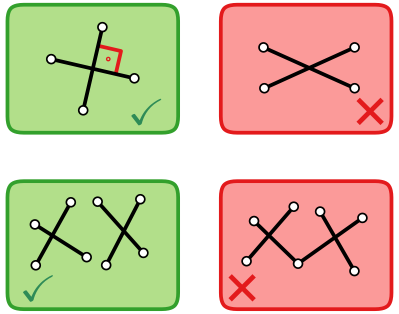

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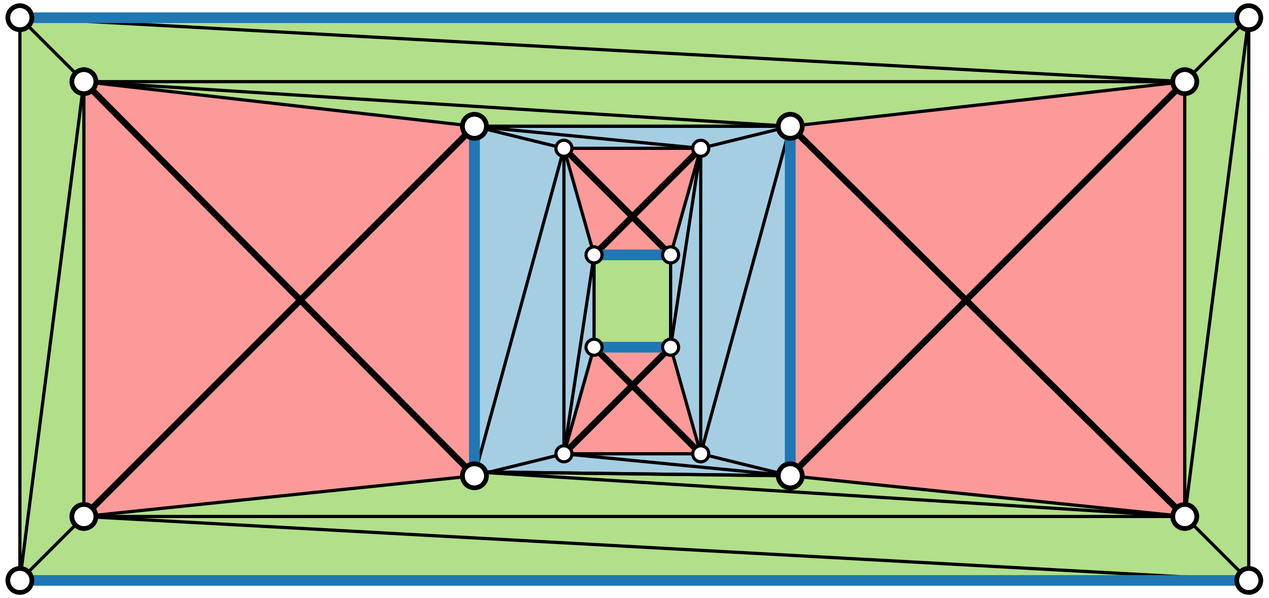

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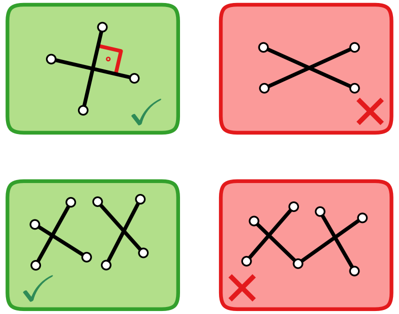

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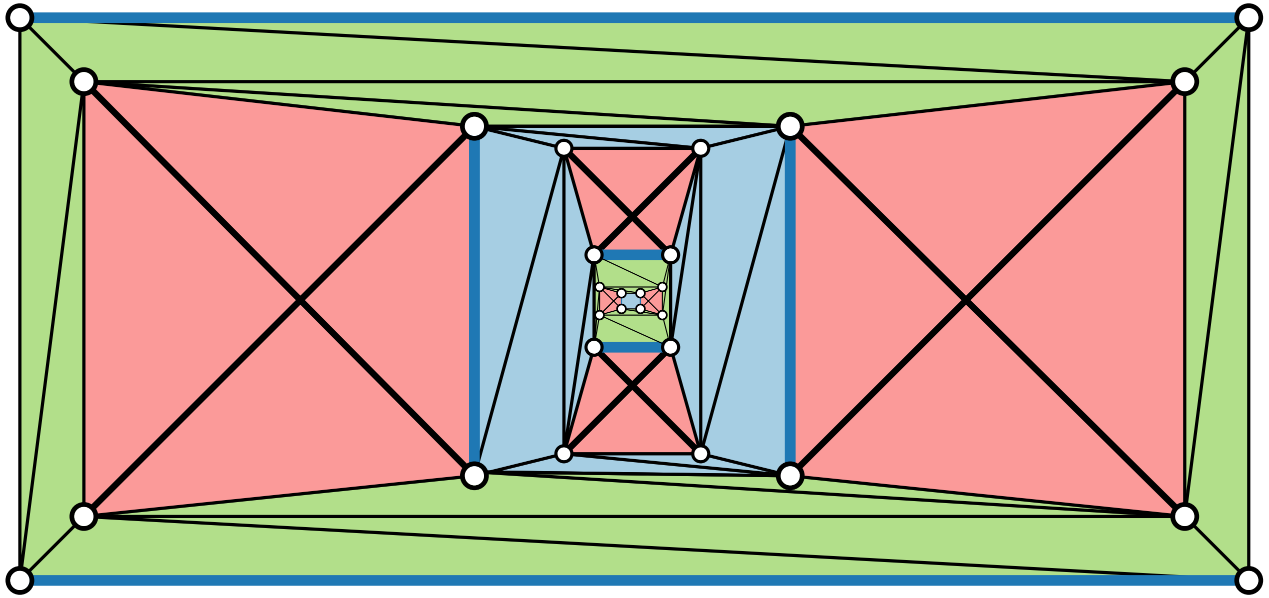

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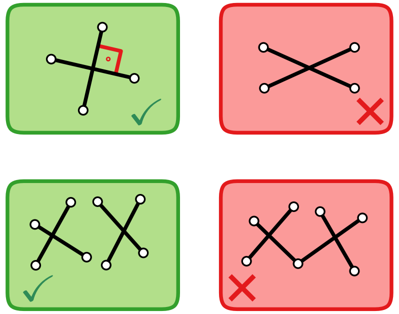

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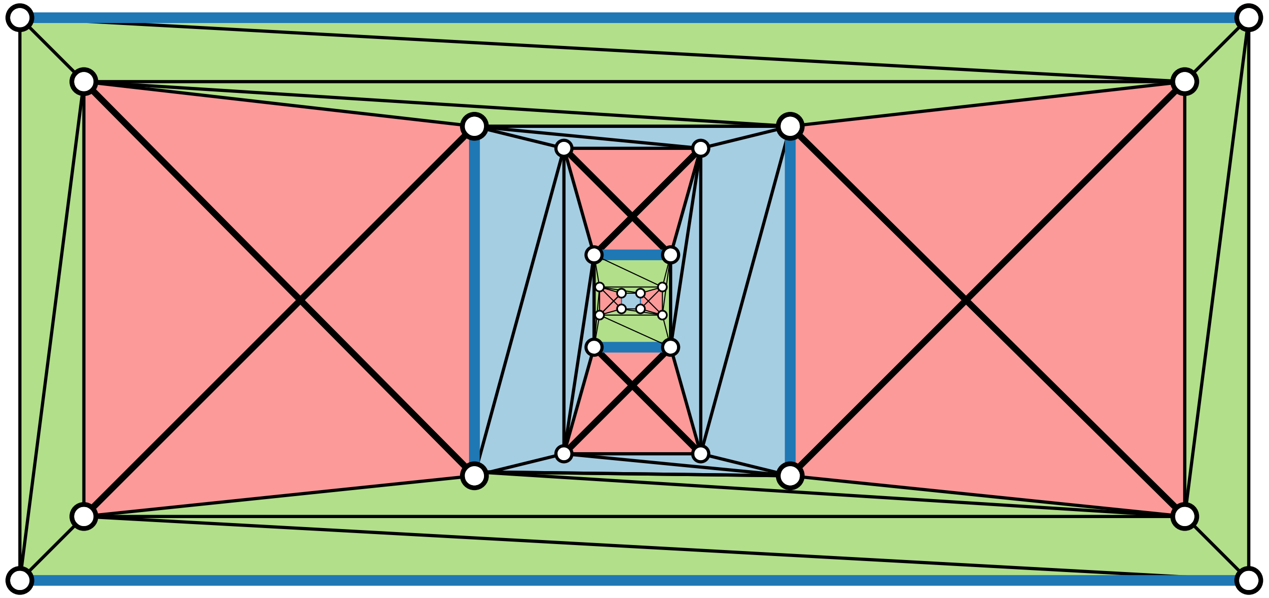

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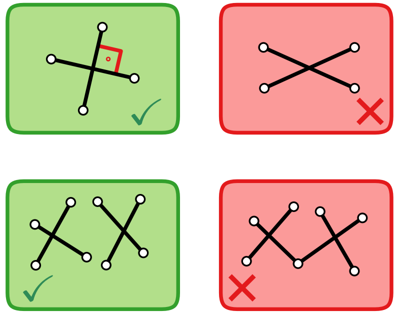

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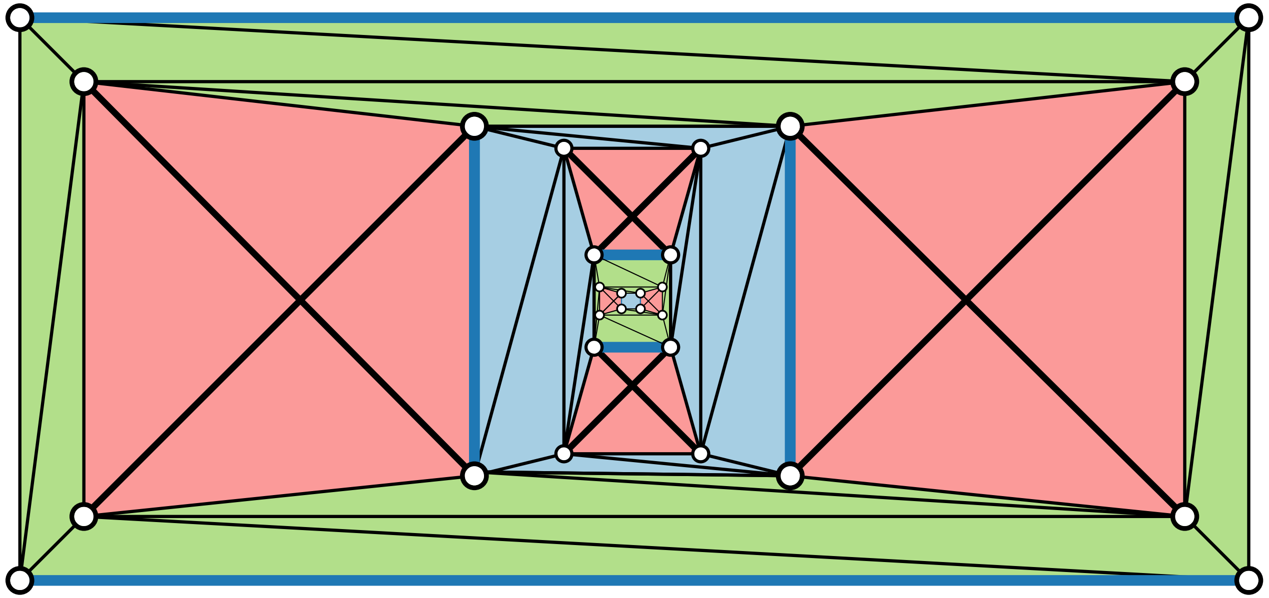

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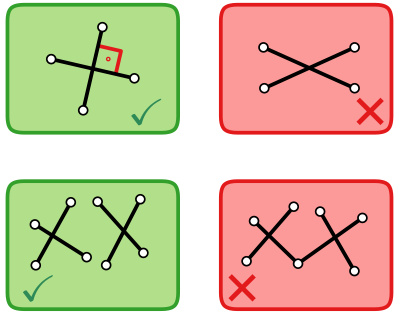

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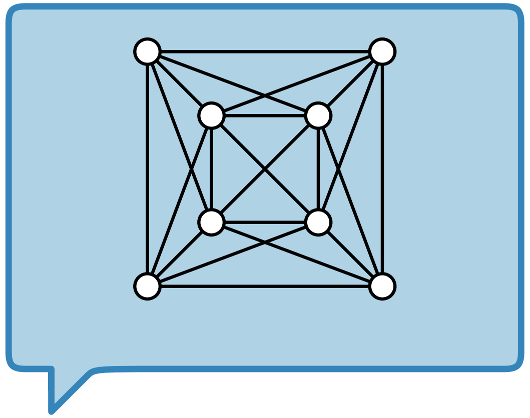

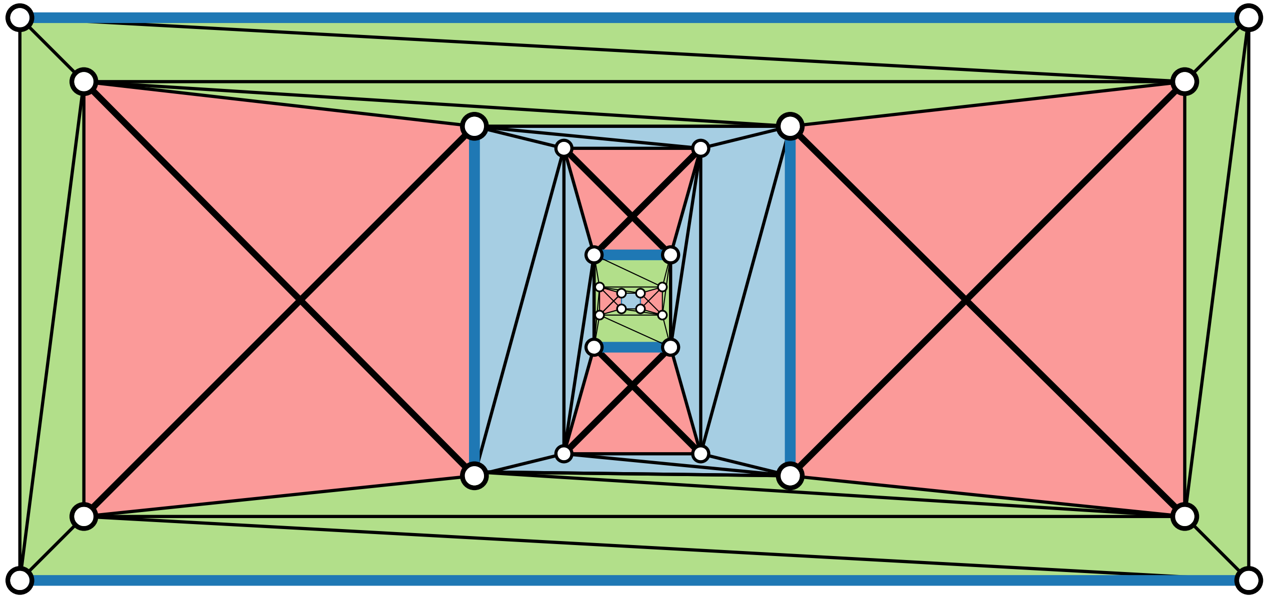

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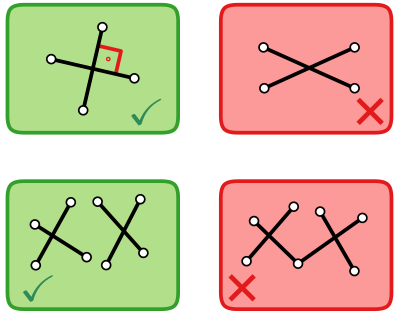

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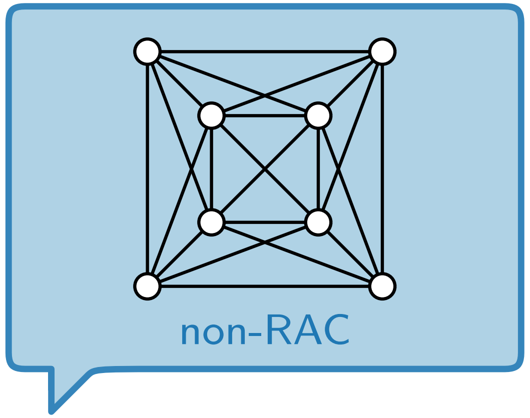

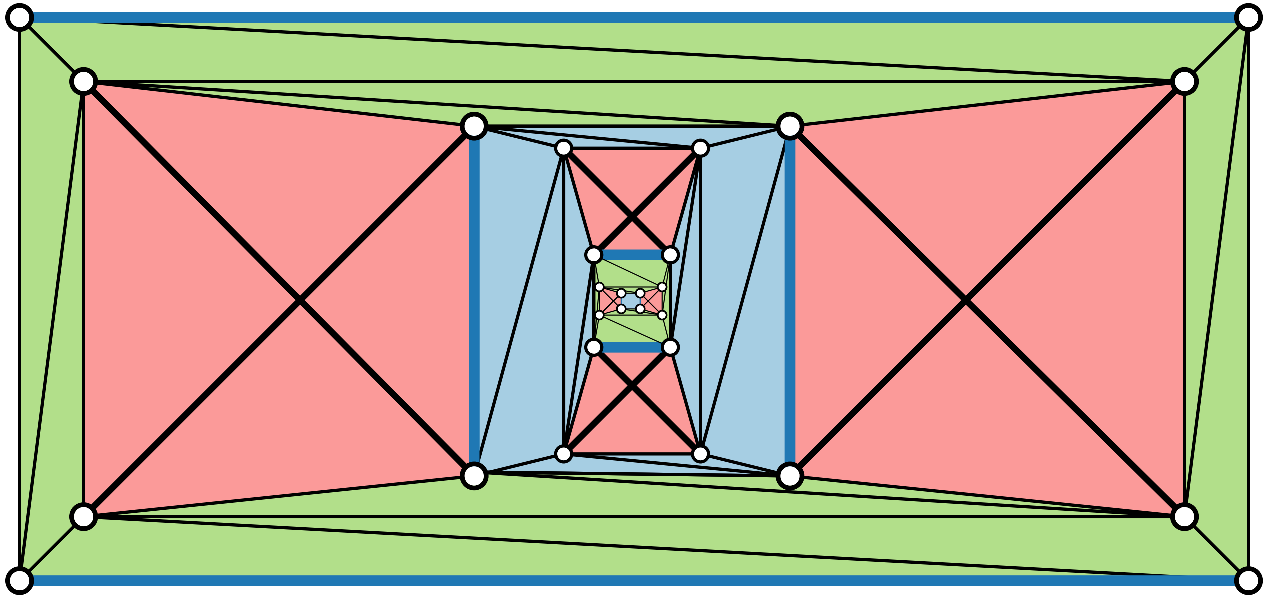

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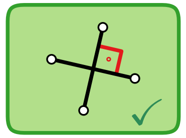

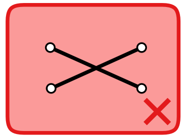

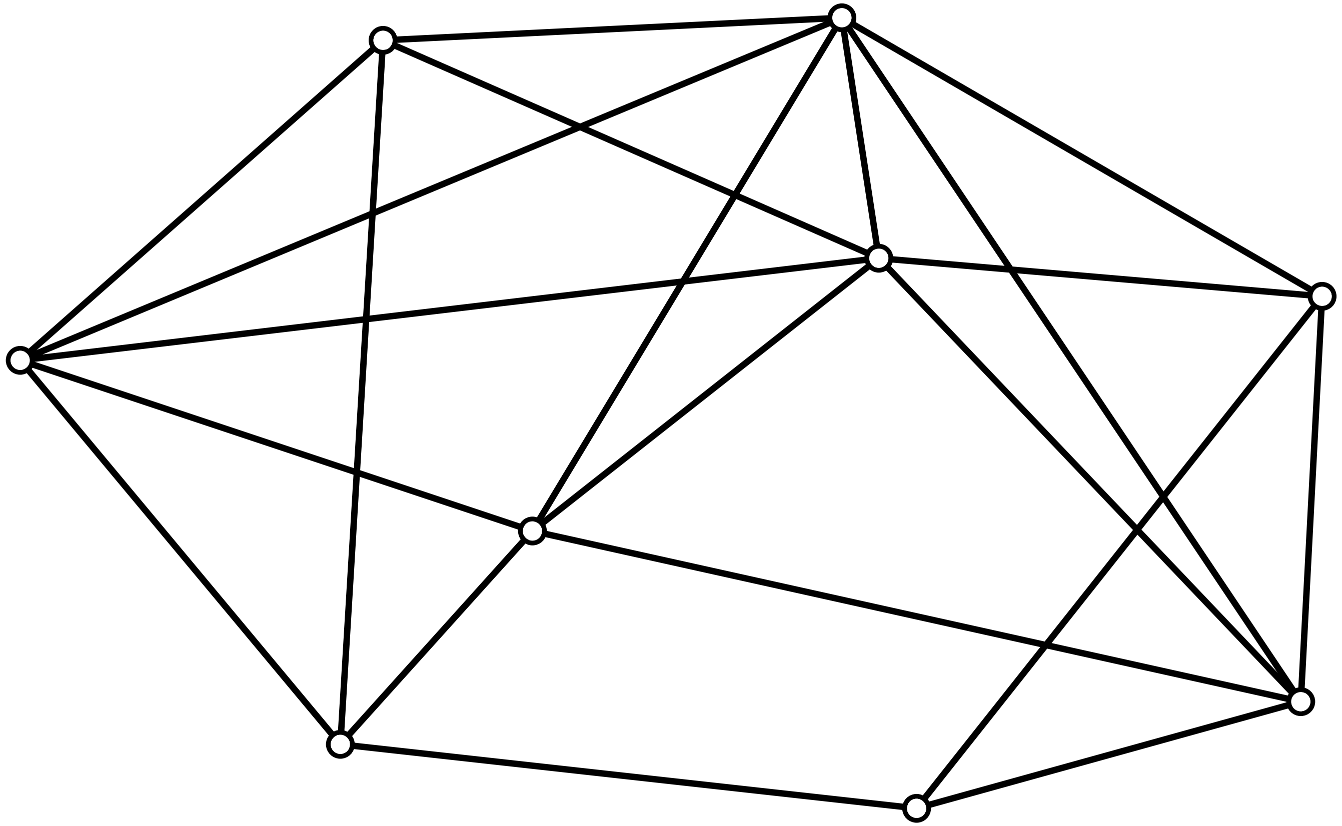

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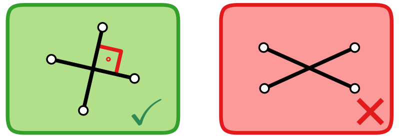

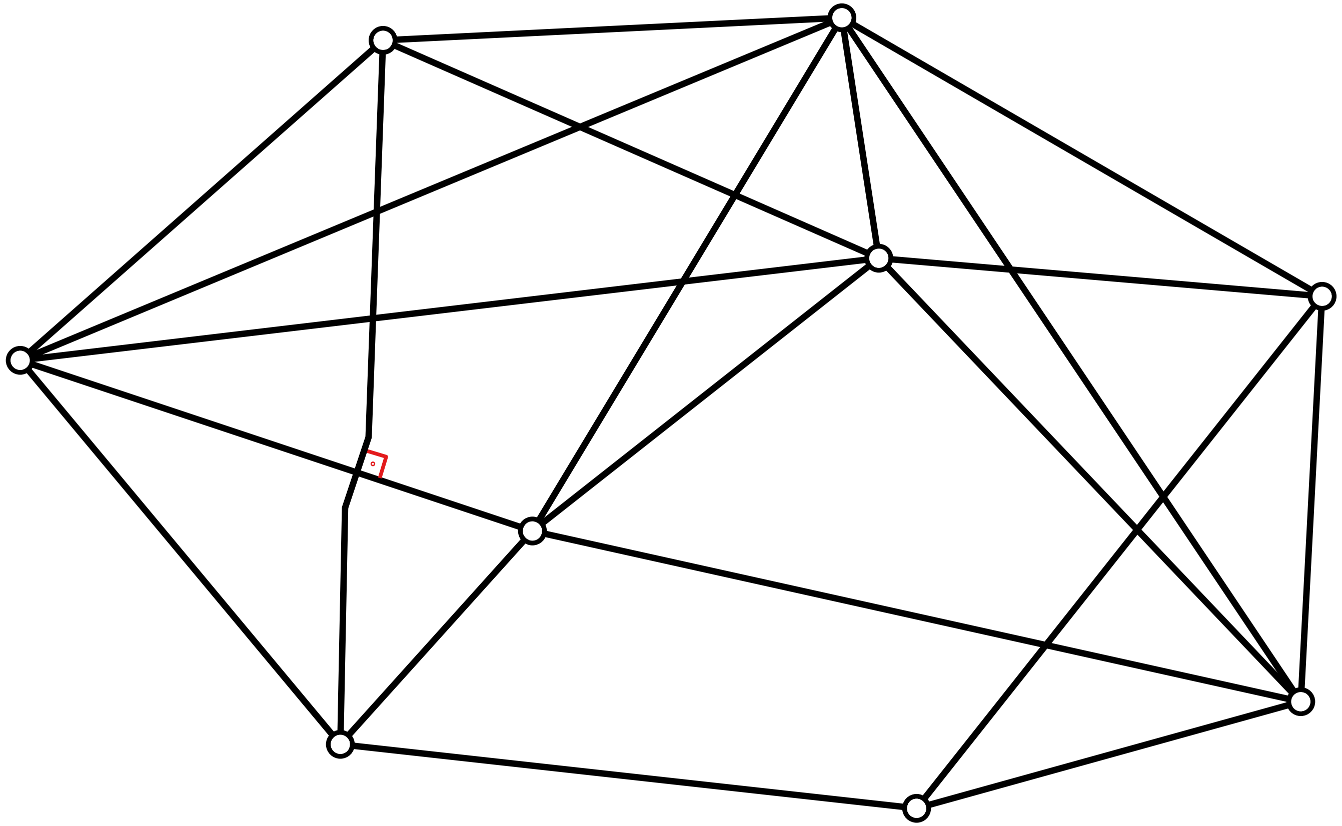

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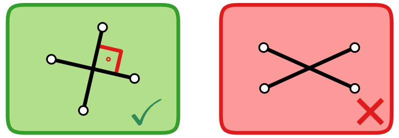

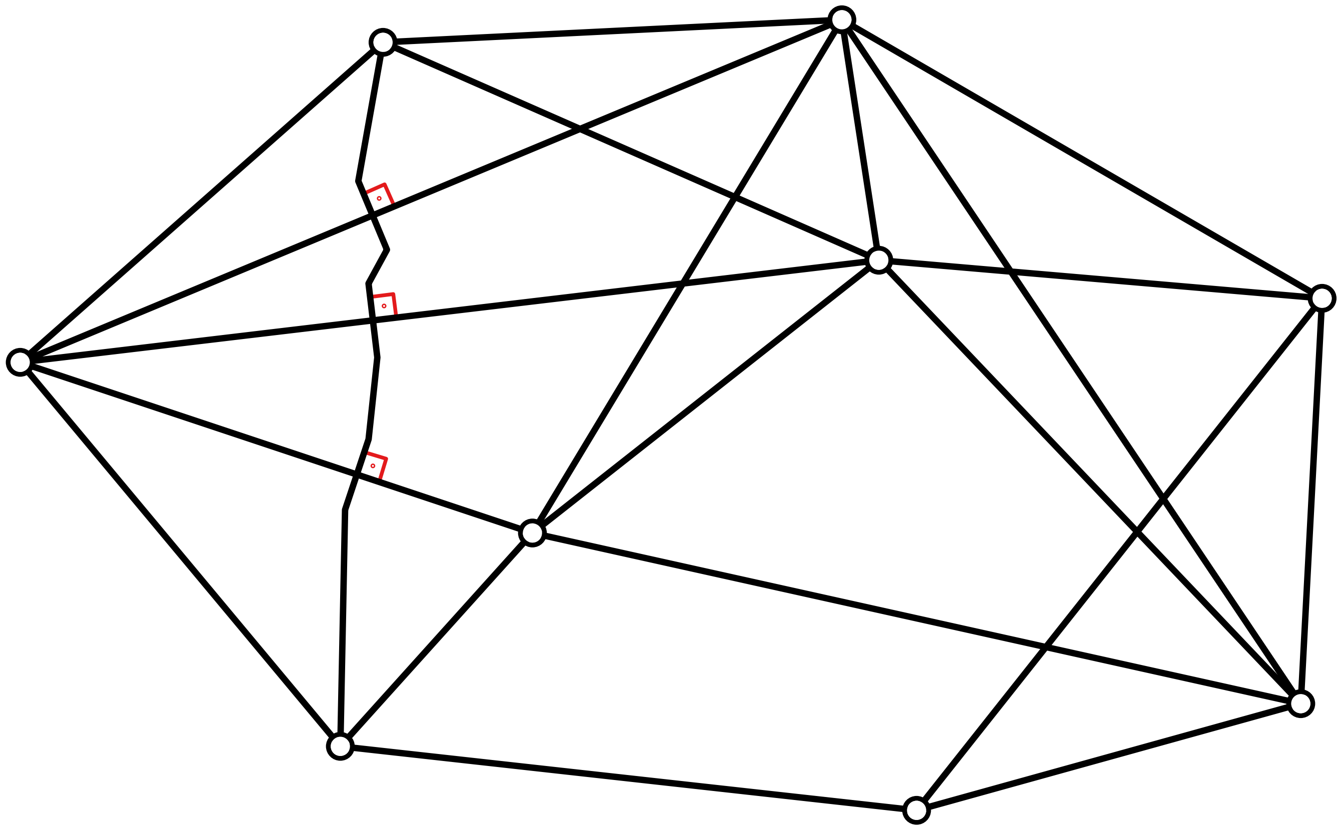

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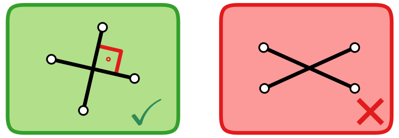

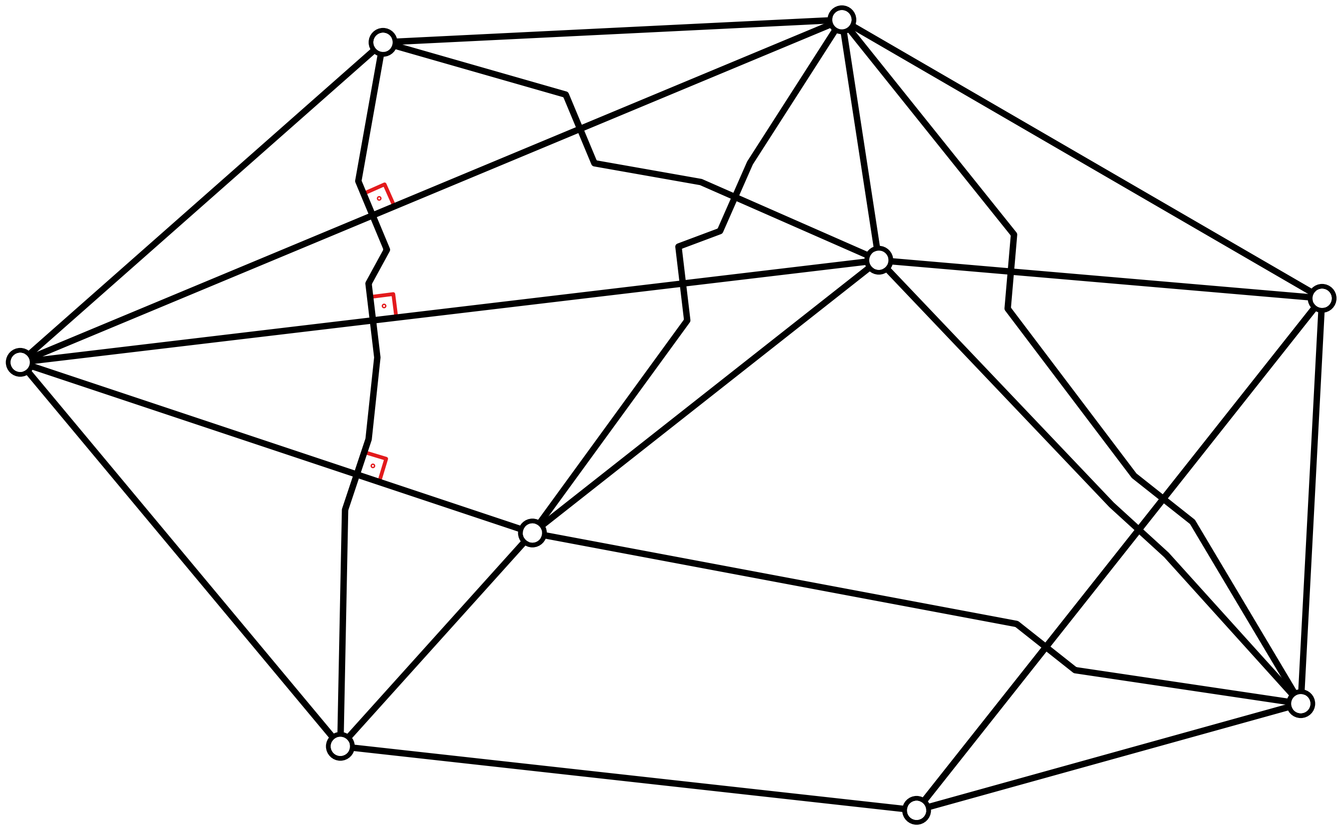

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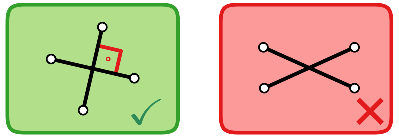
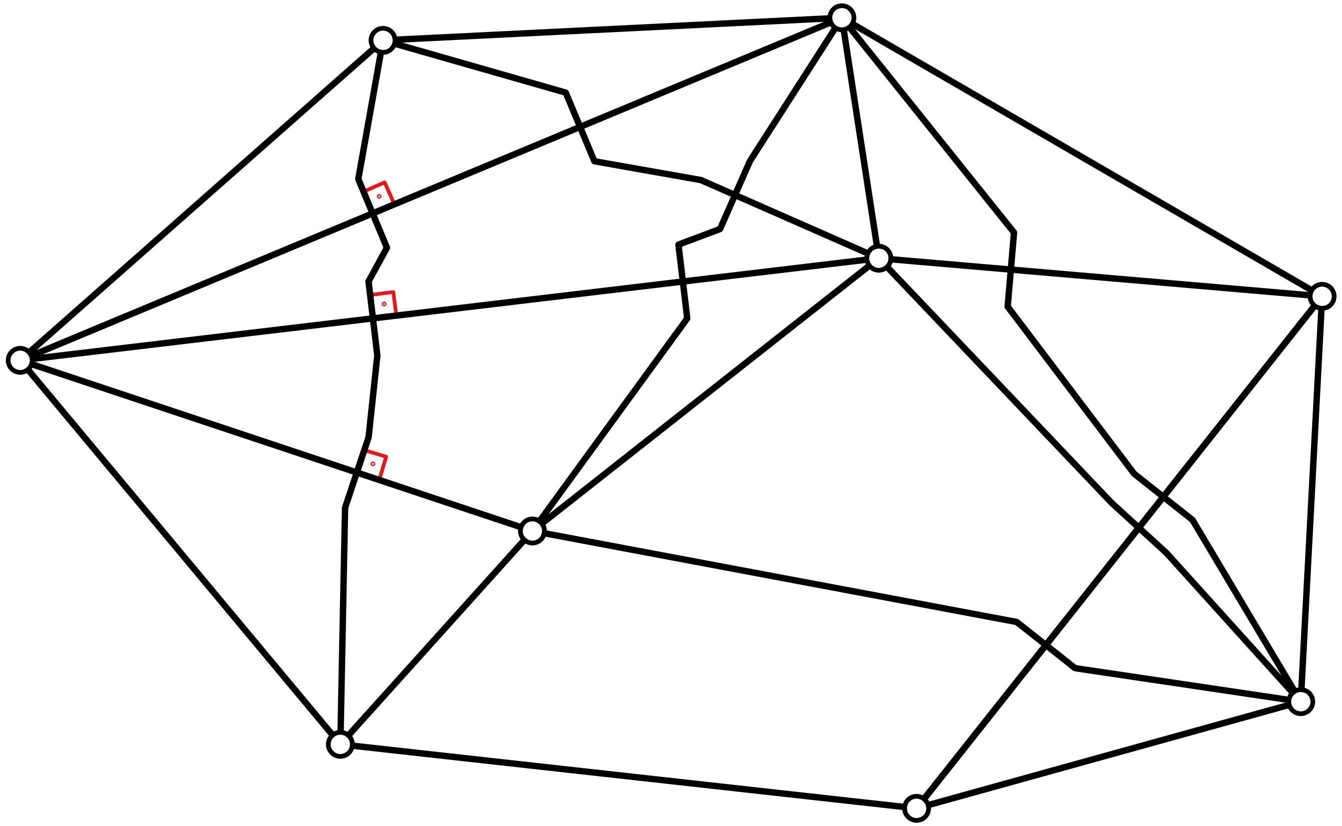
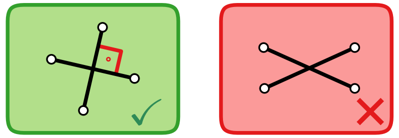
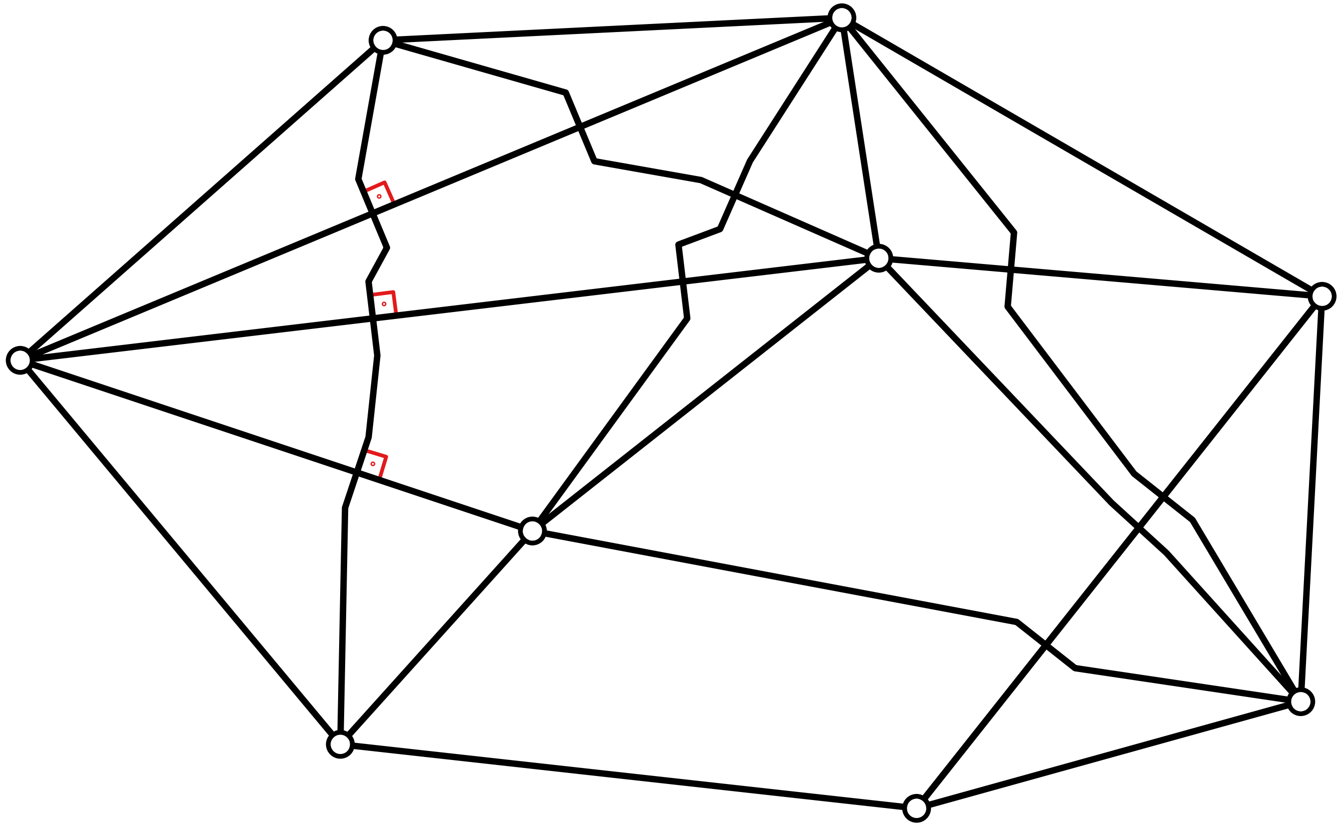




Every graph admits a RAC drawing . . .

RAC Drawings With Enough Bends





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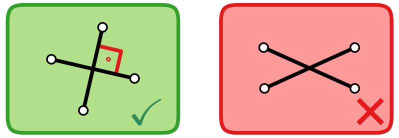

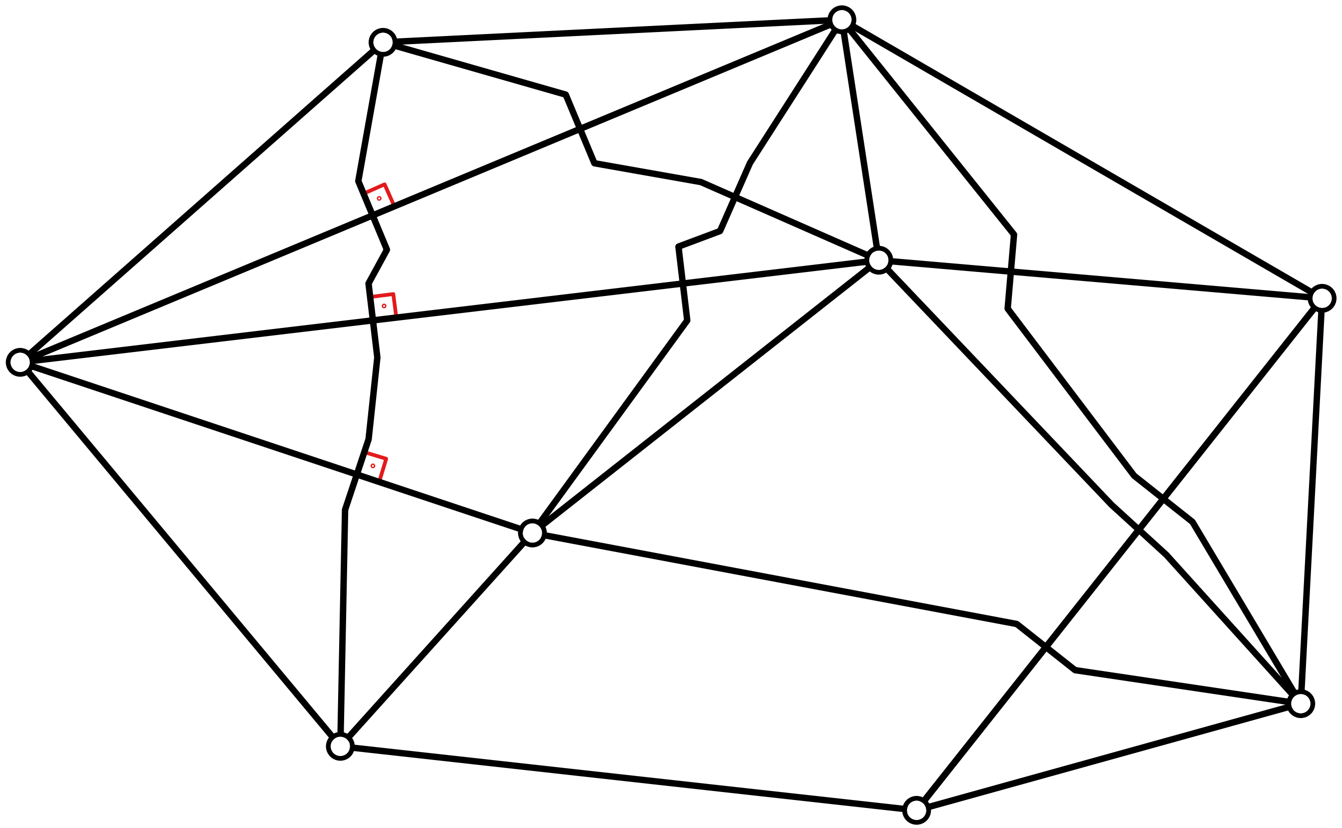

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How many do we need – in total or per edge?

3-Bend RAC Drawings

Theorem. [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most three bends.

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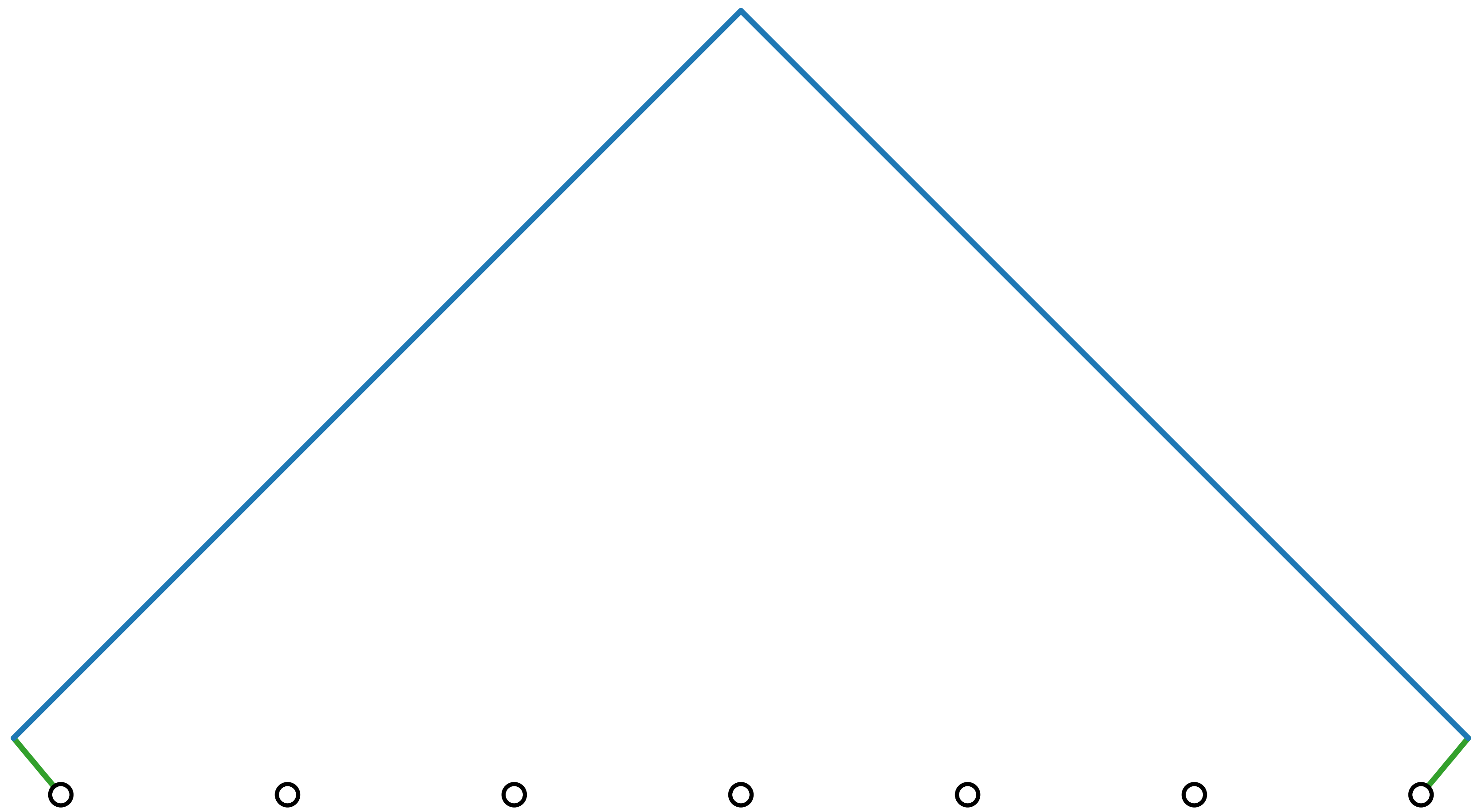

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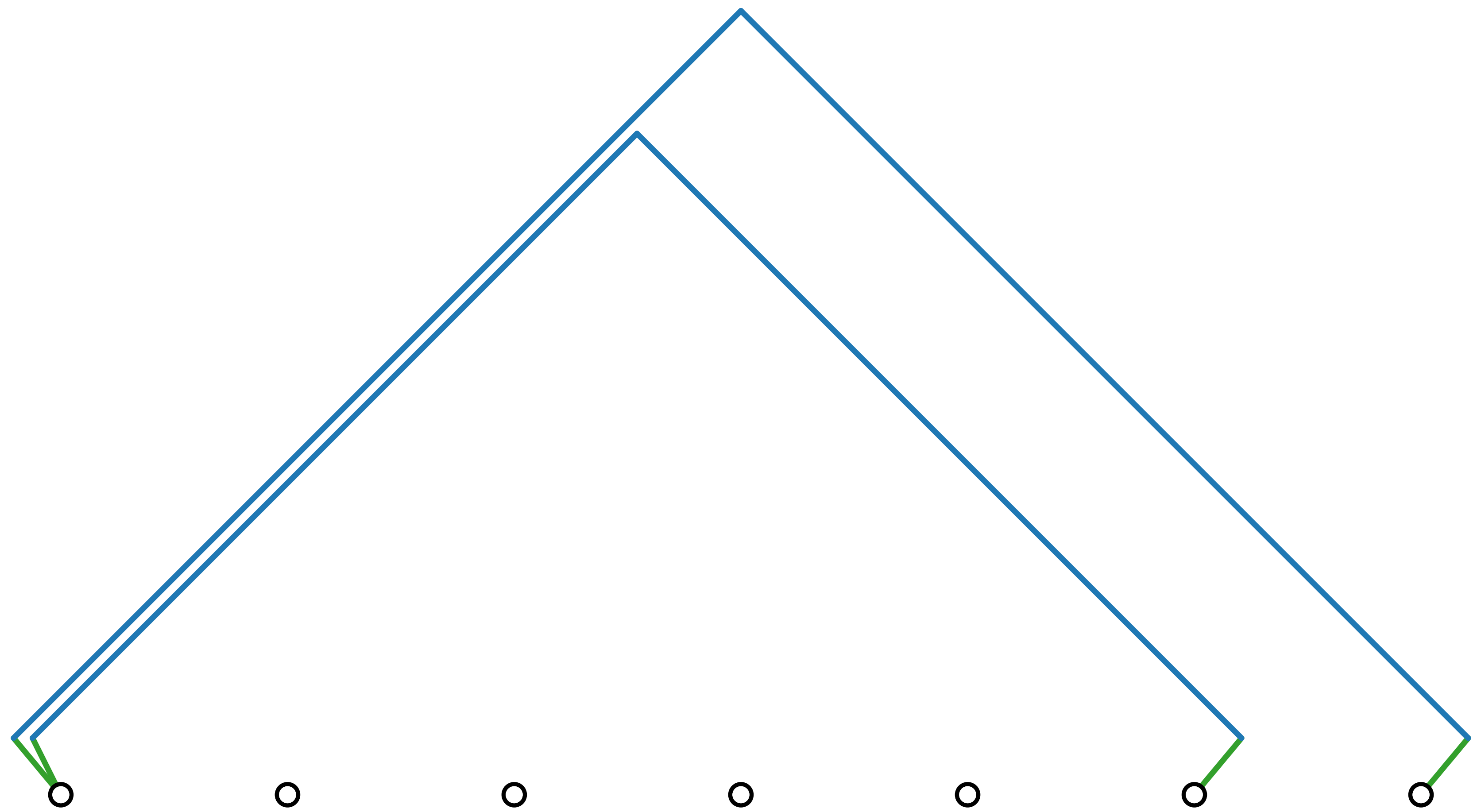

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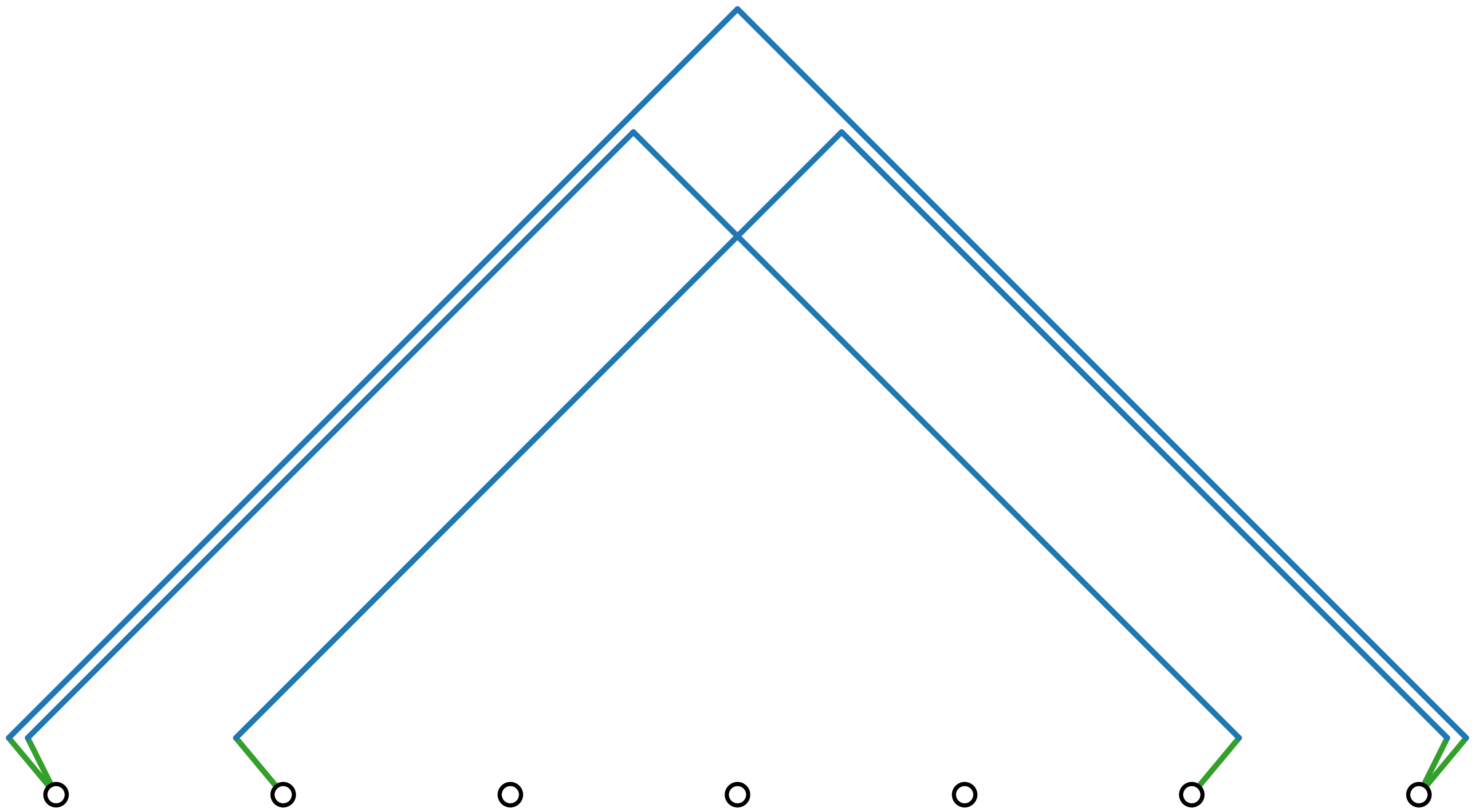

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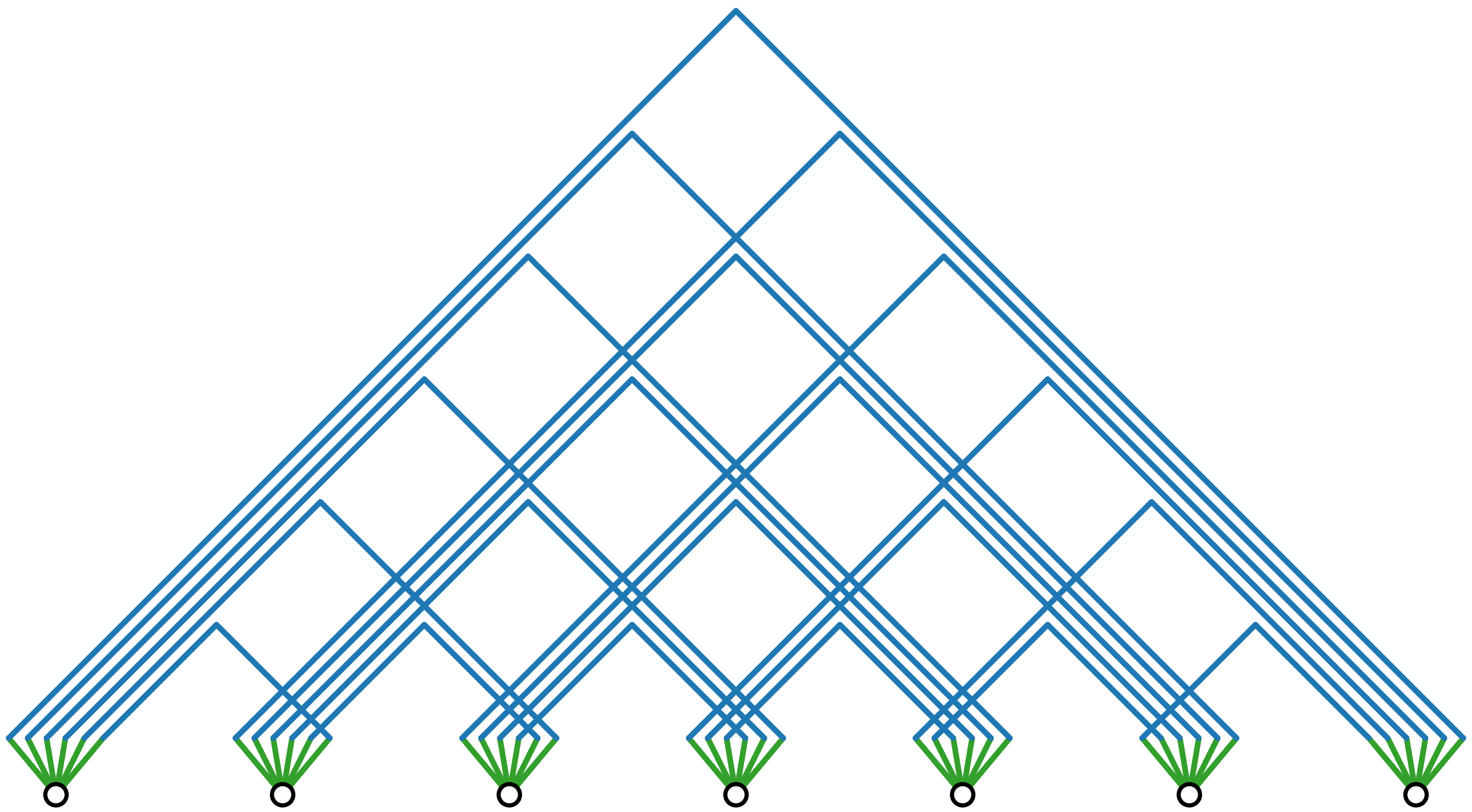

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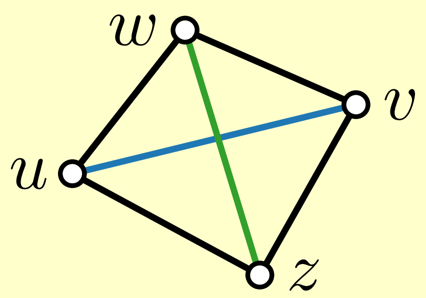

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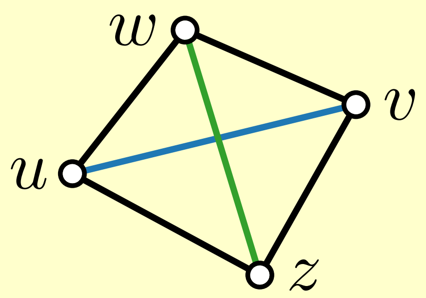

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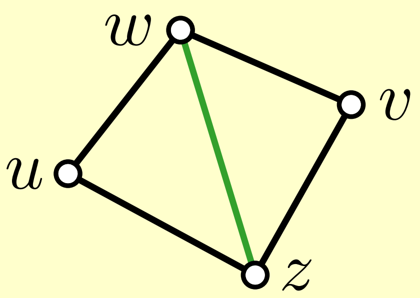

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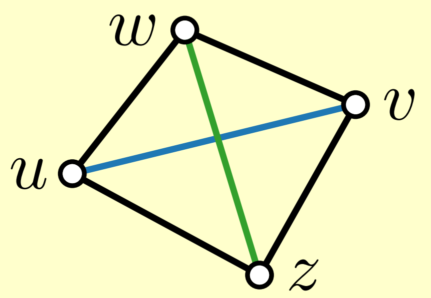

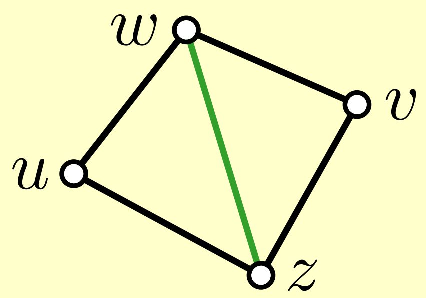

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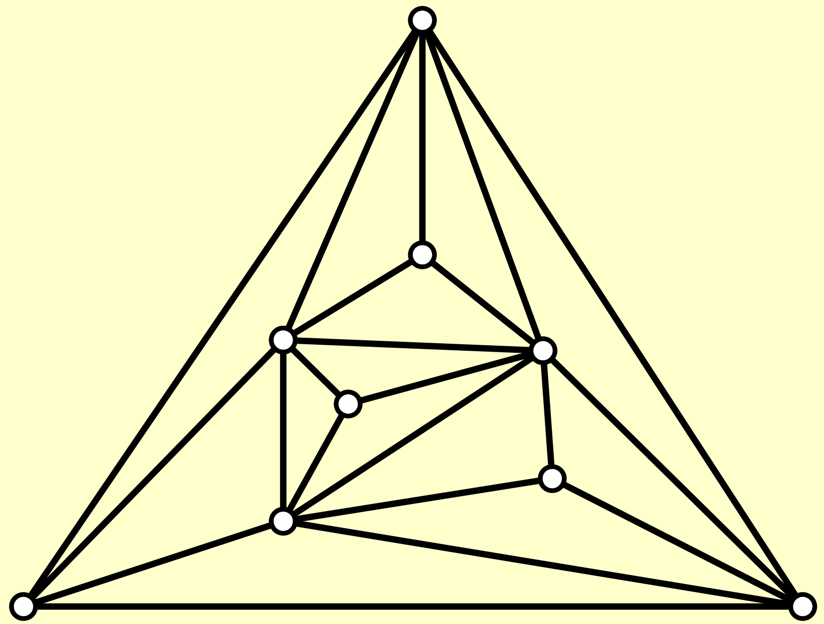

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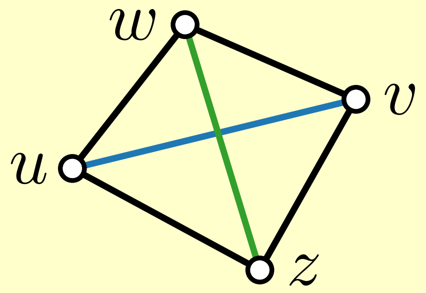

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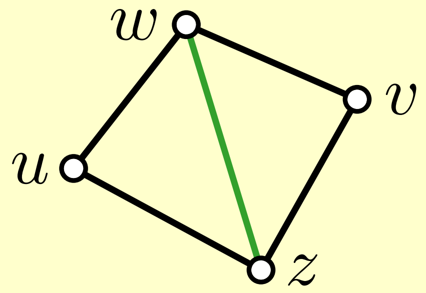

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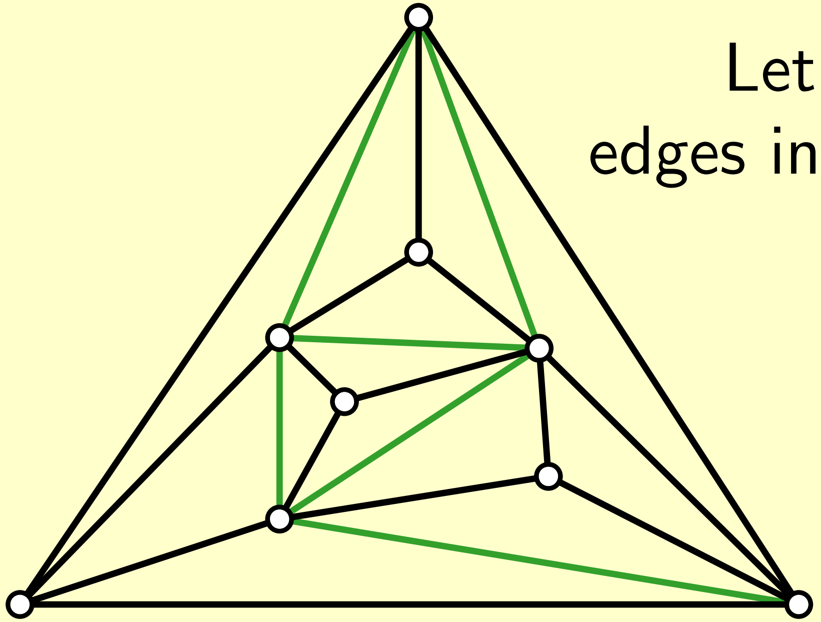

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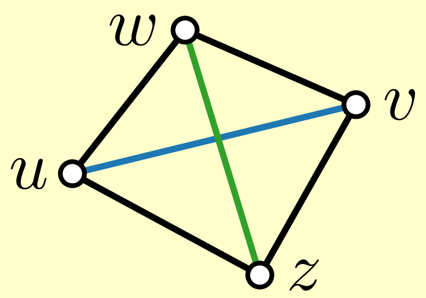

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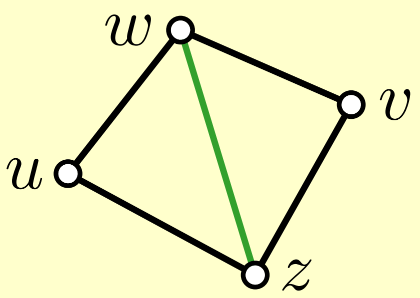

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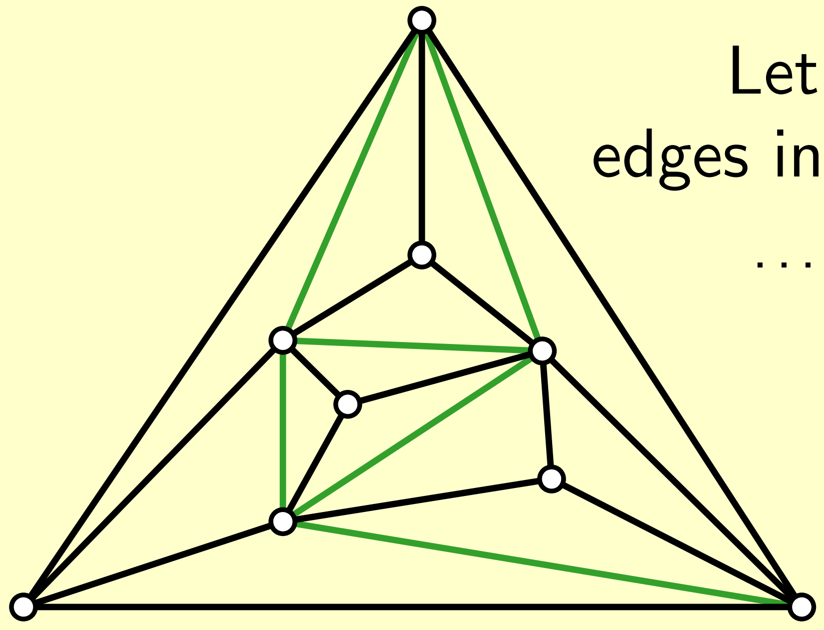

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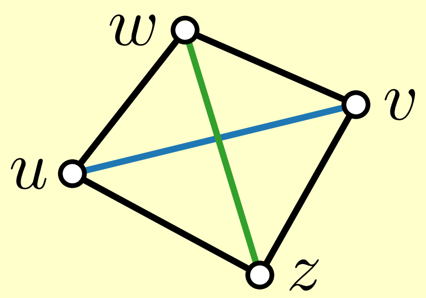

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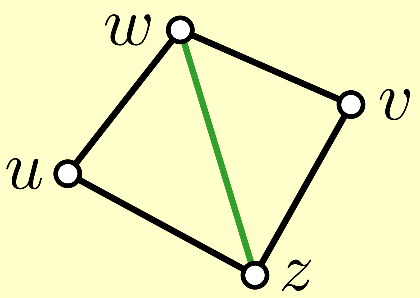

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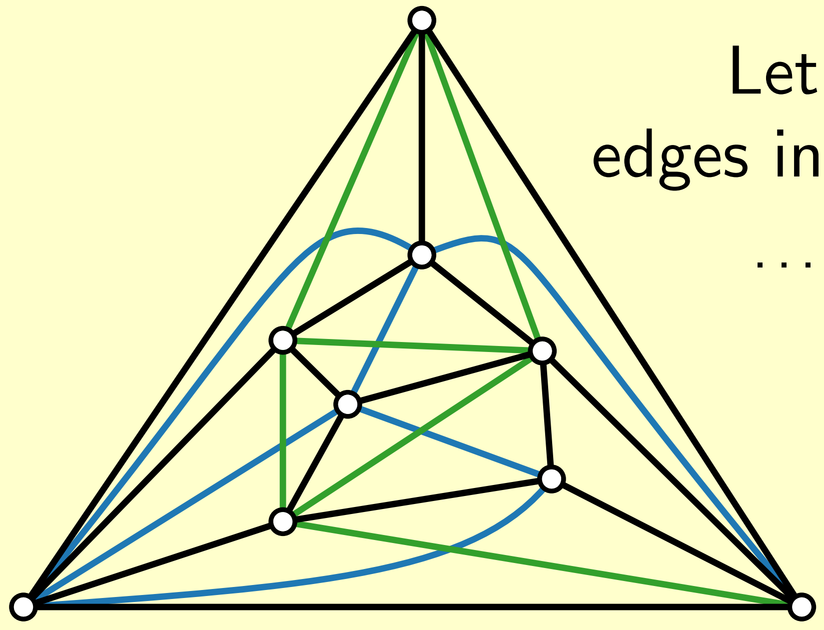

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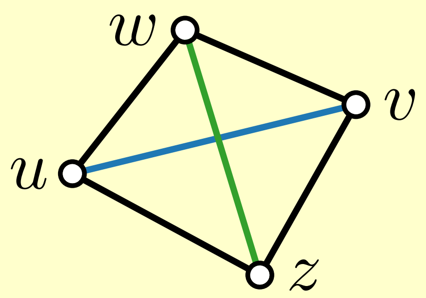

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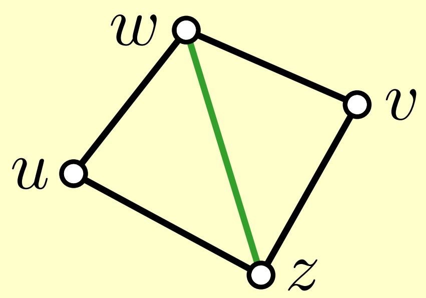

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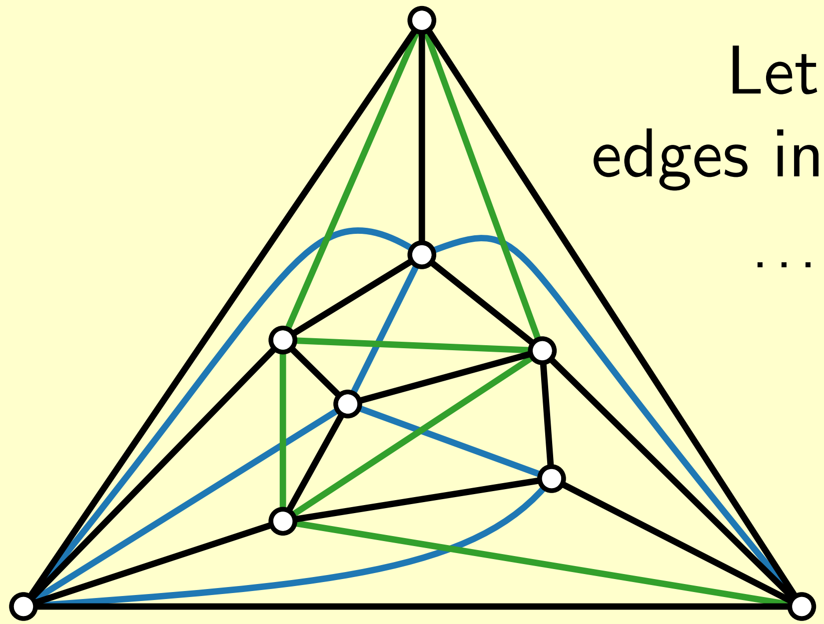

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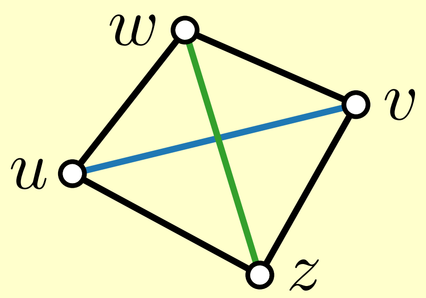

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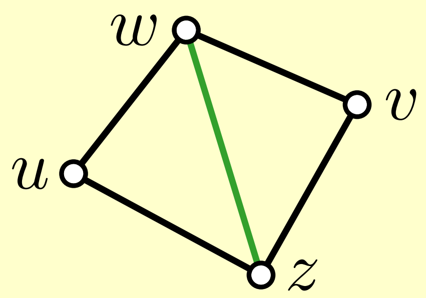

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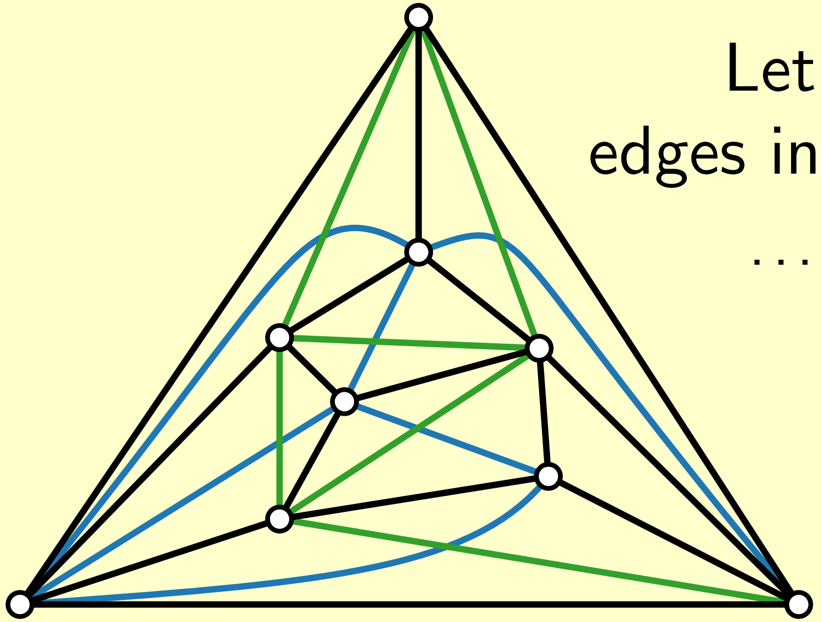

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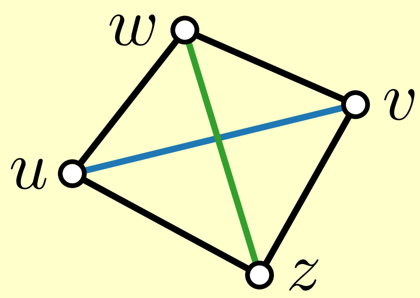

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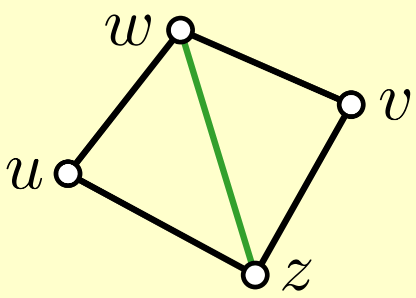

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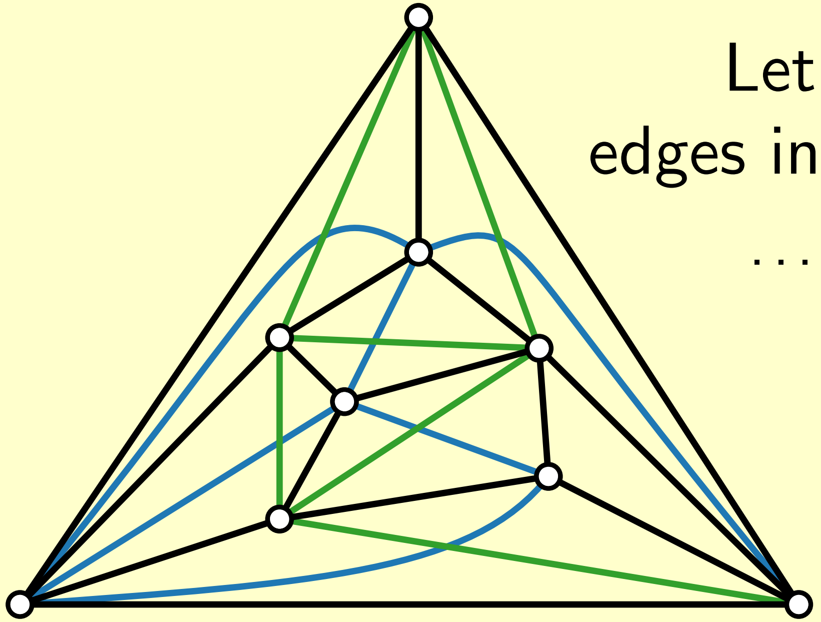

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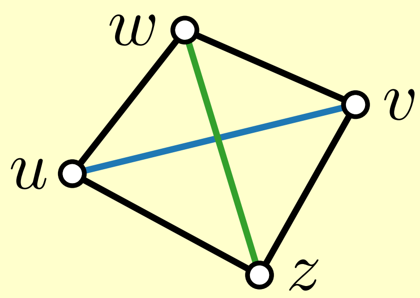

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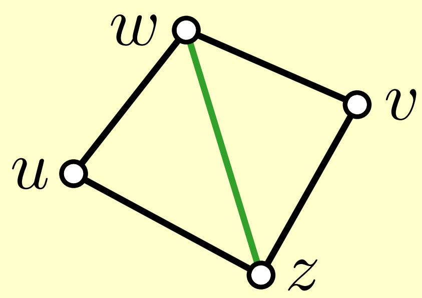

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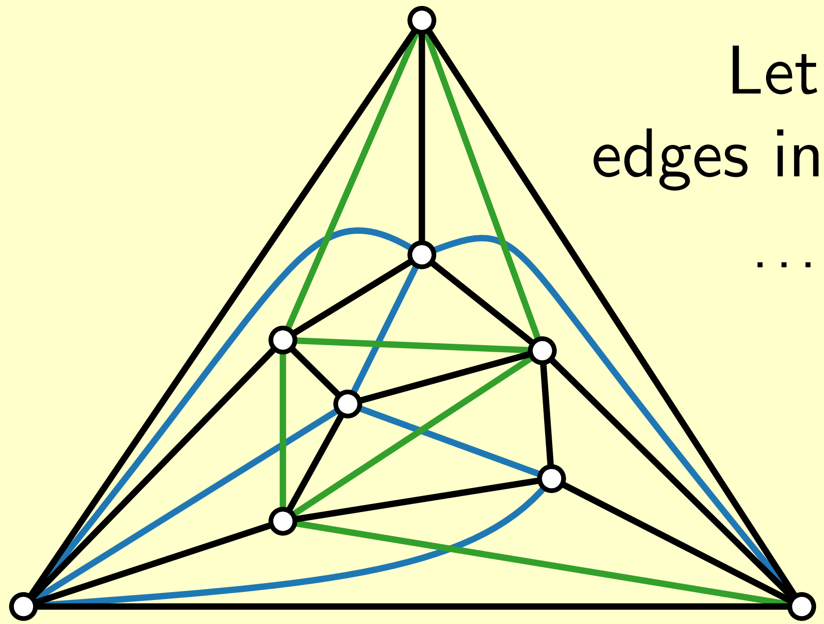

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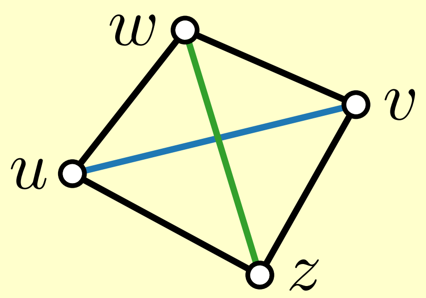

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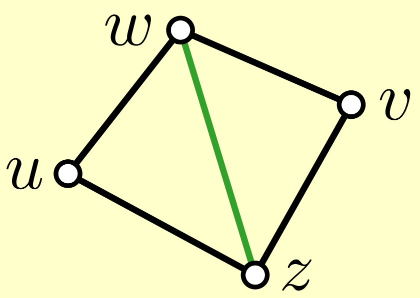

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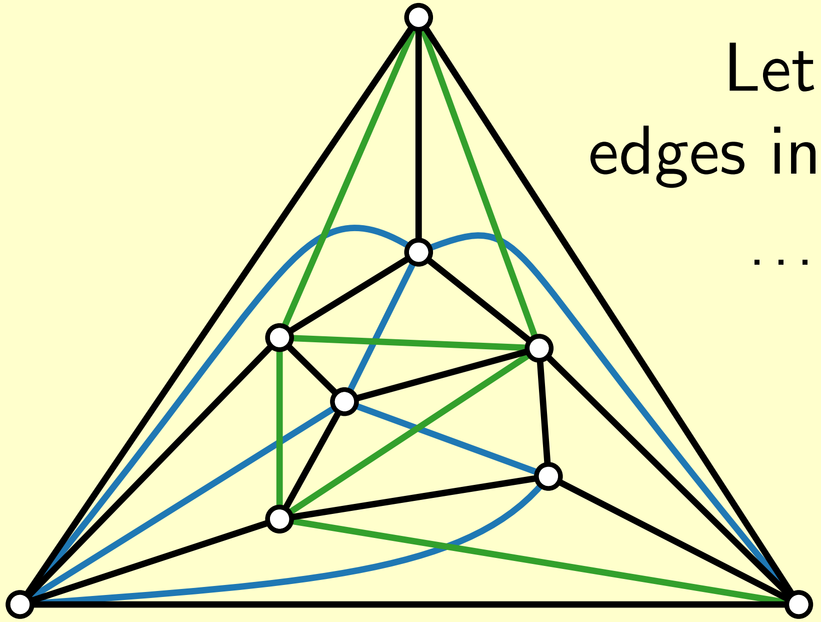

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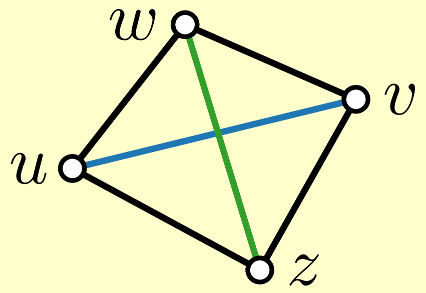

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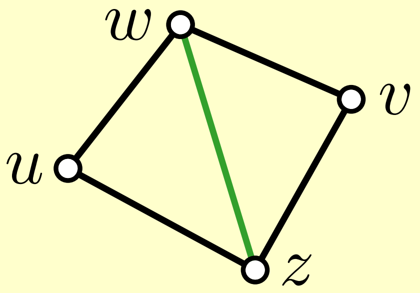

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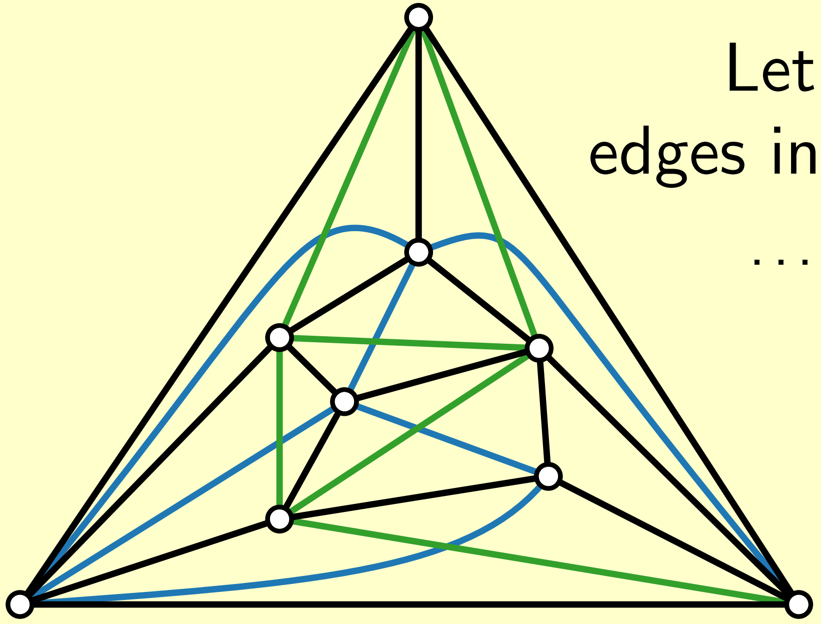

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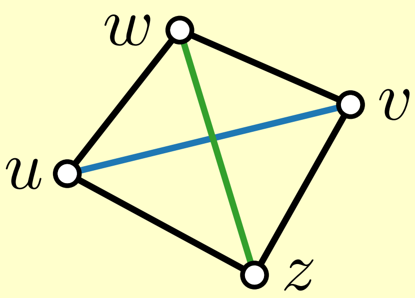

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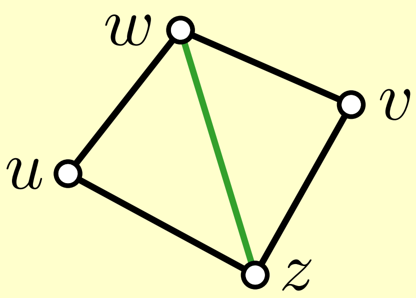

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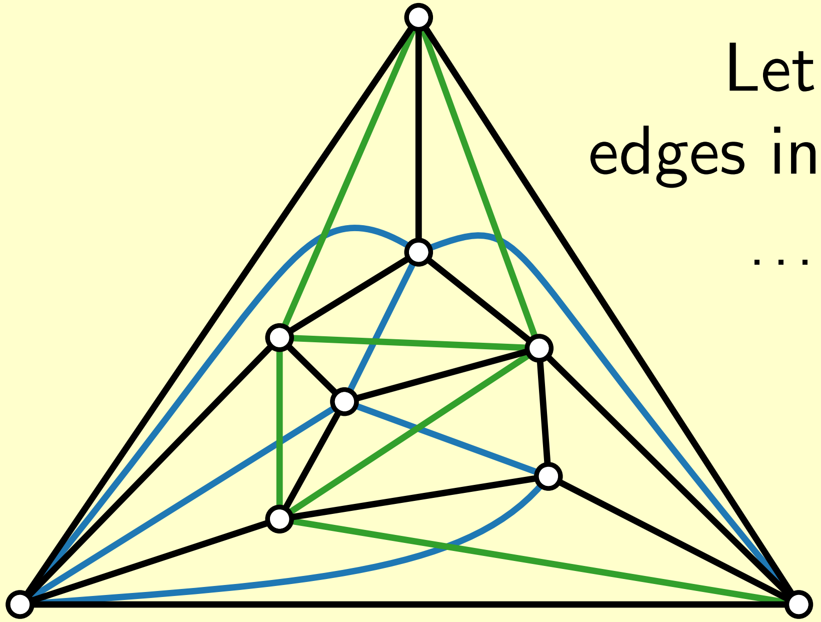

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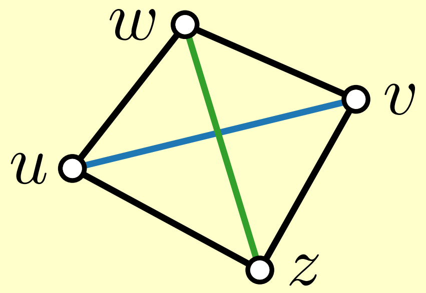

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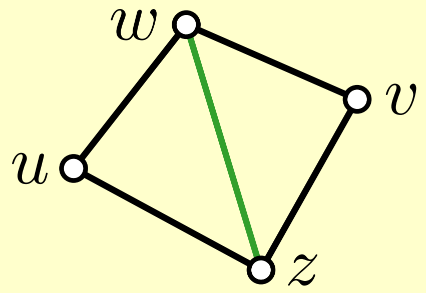

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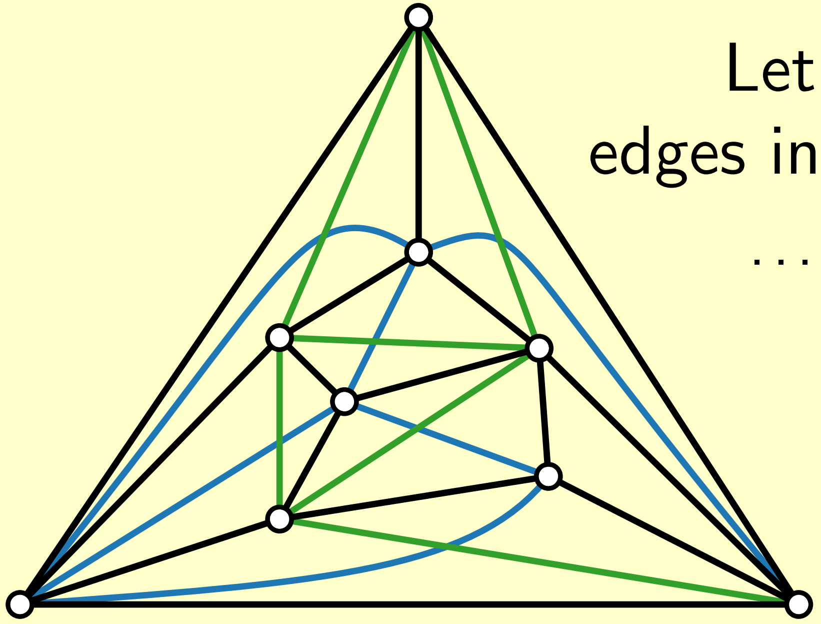

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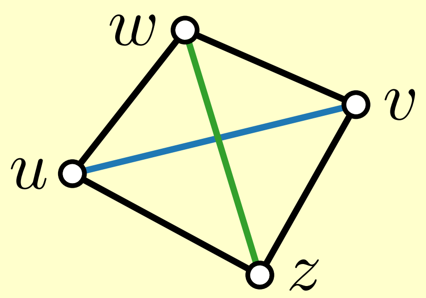

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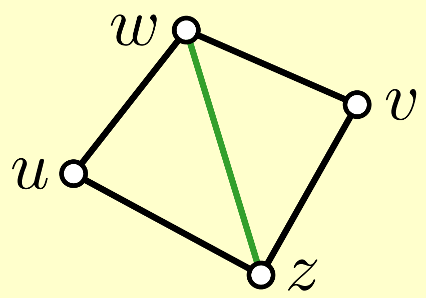

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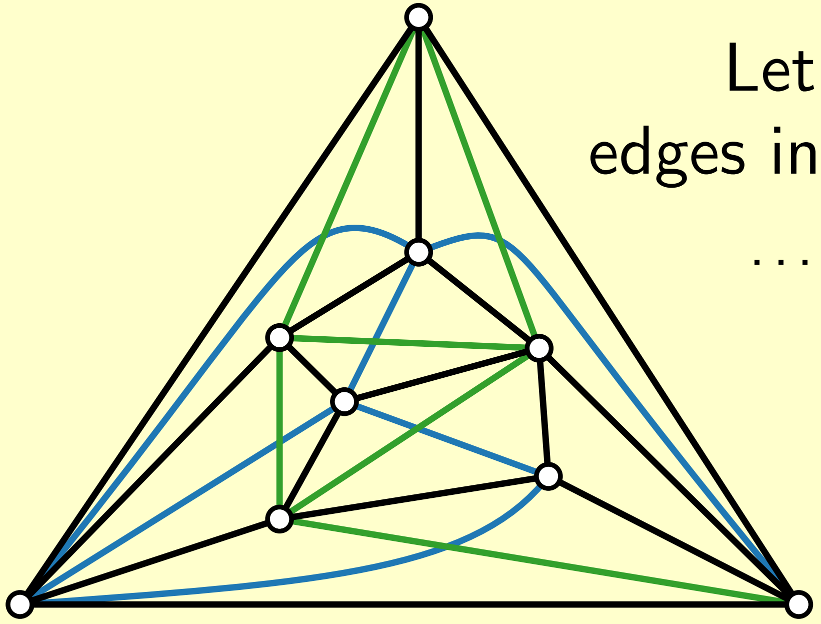

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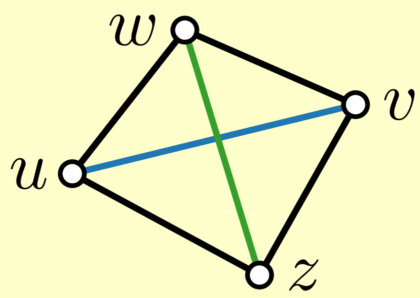

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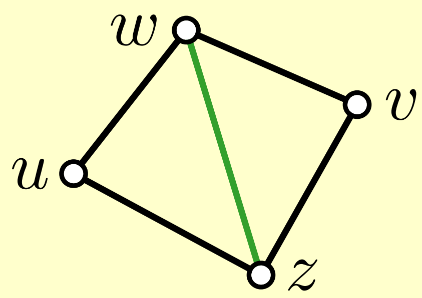

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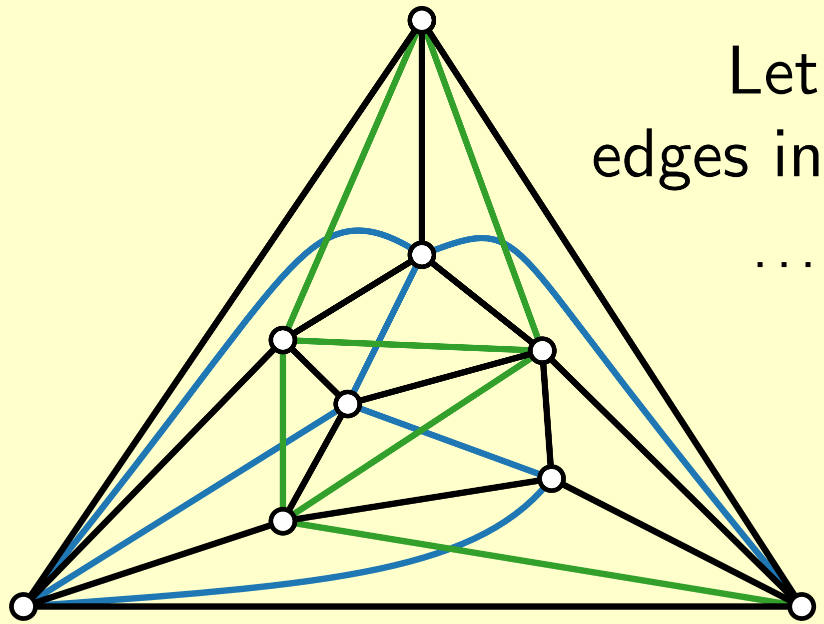

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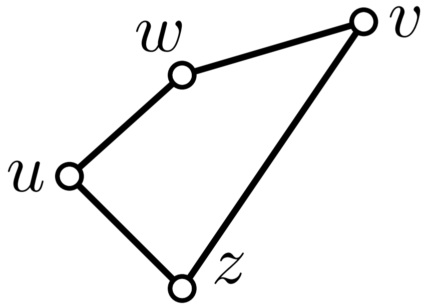

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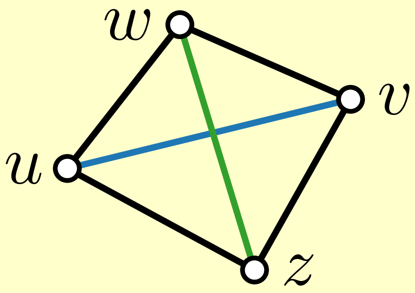

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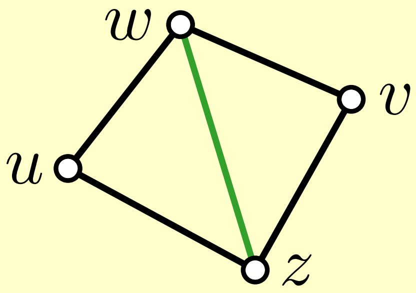

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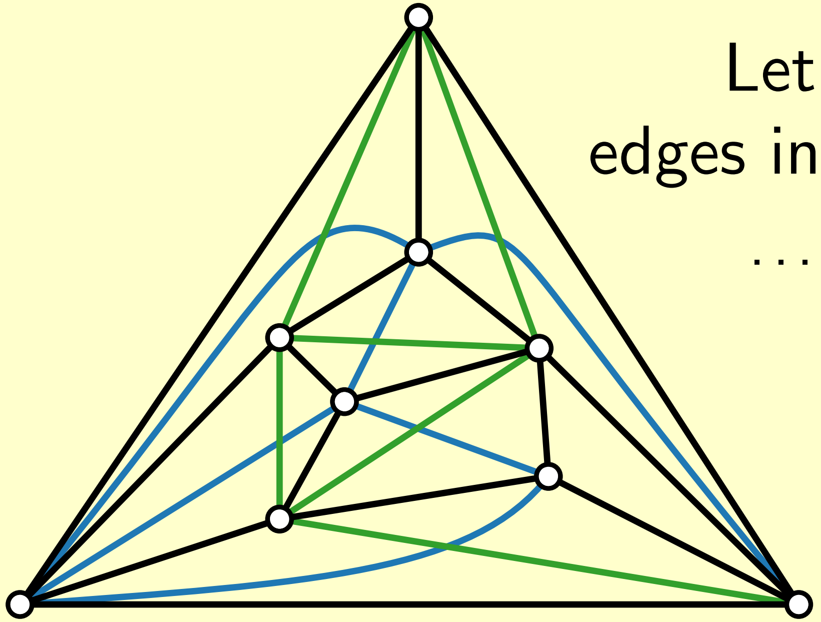

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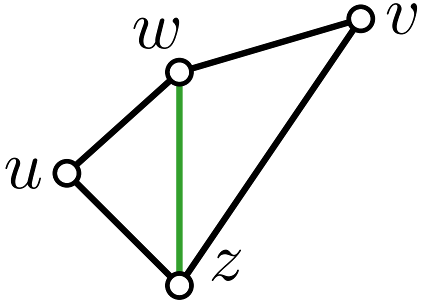

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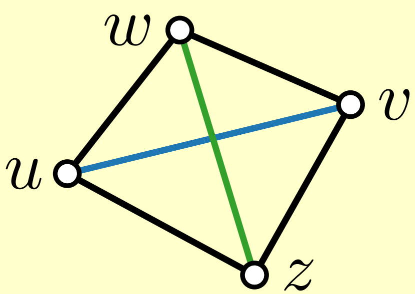

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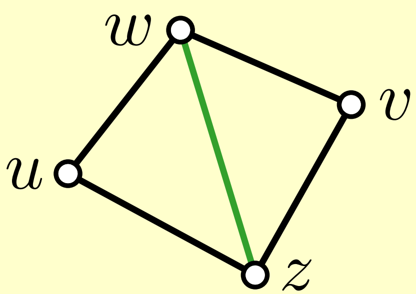

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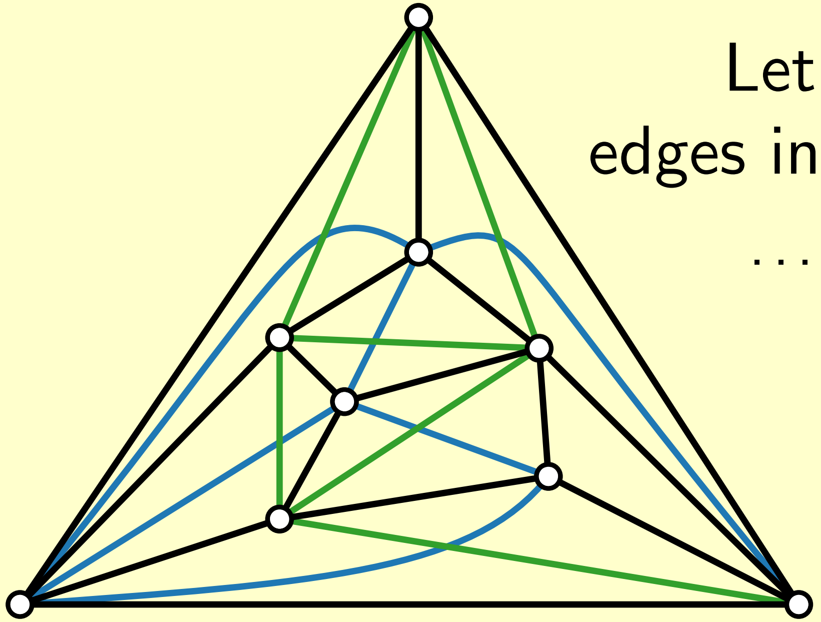

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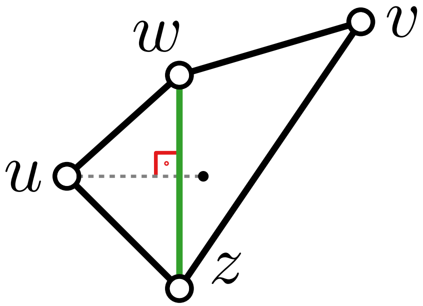

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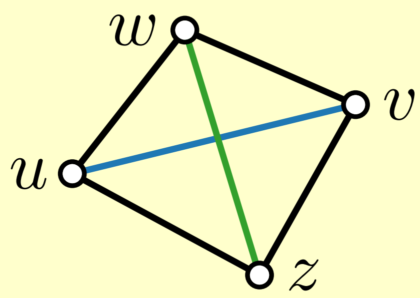

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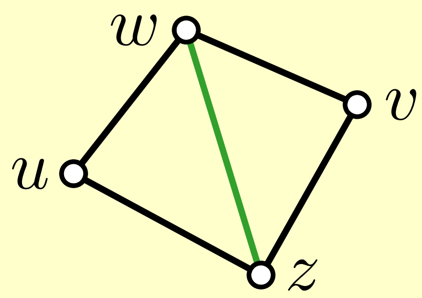

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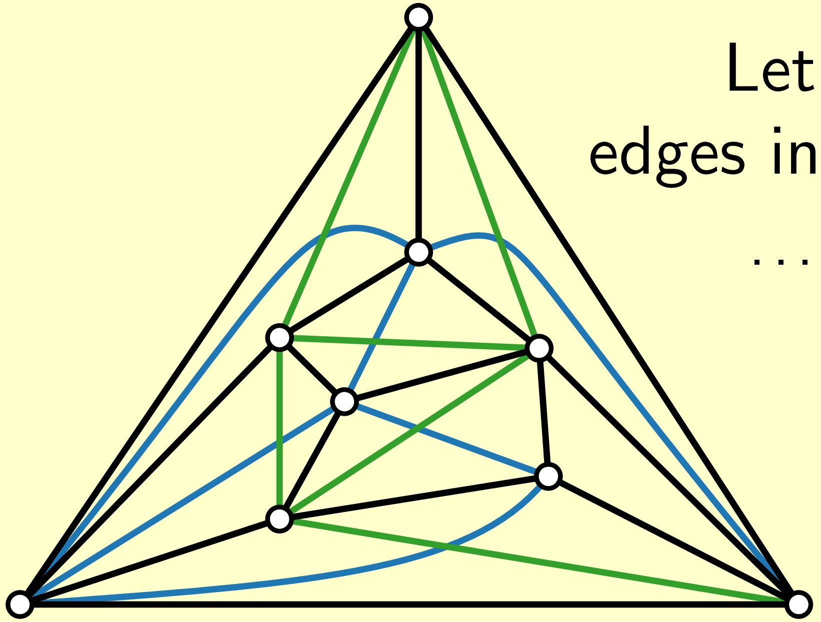

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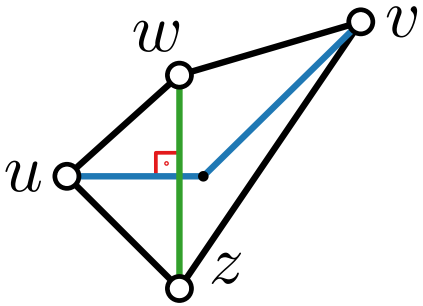

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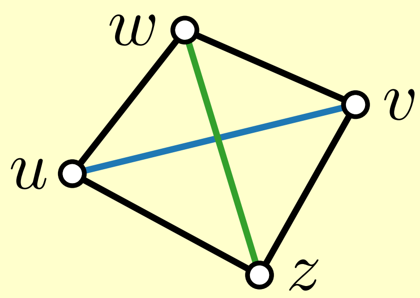

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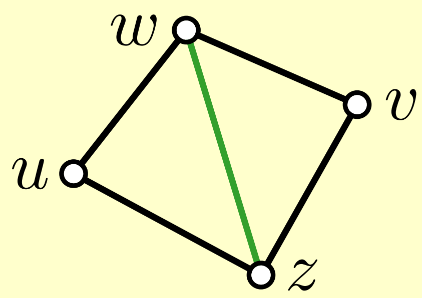

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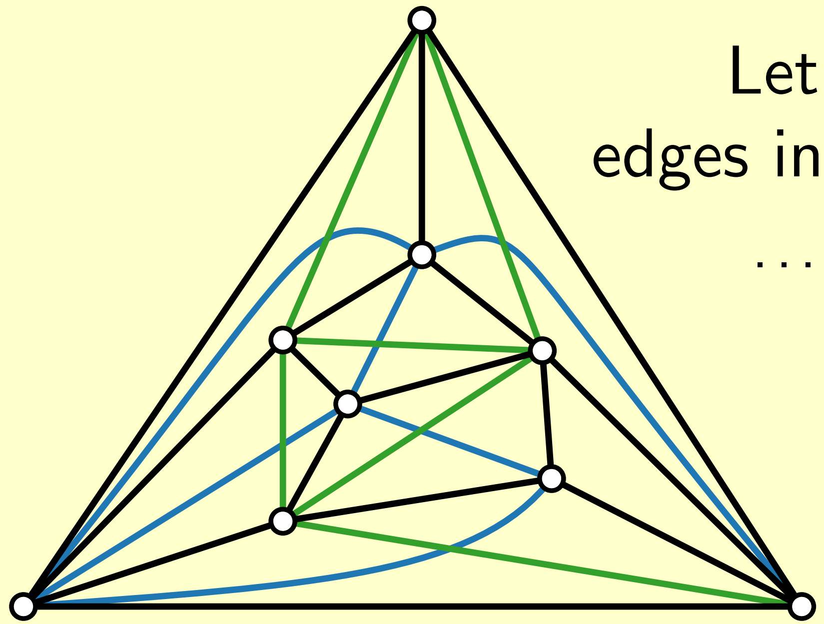

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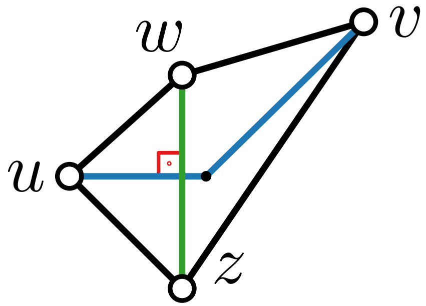

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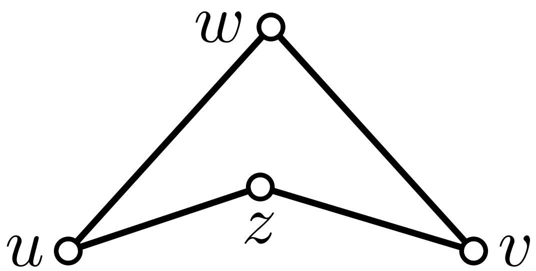

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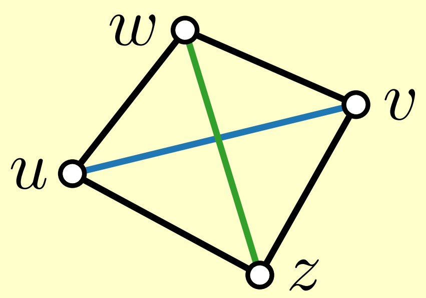

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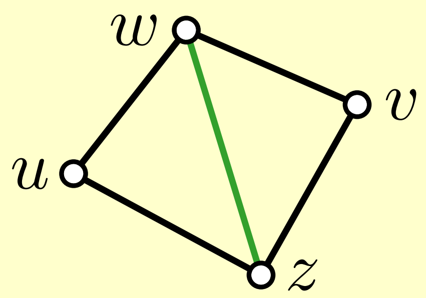

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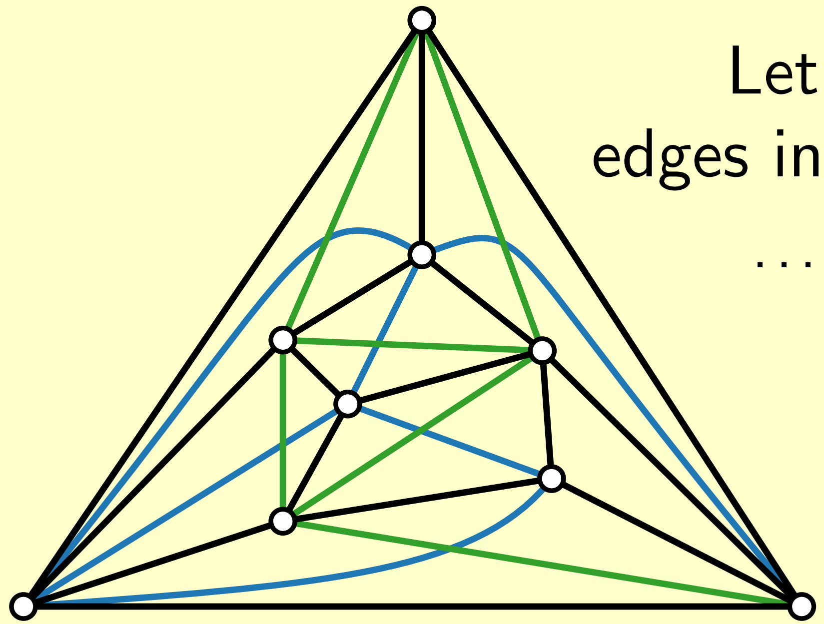

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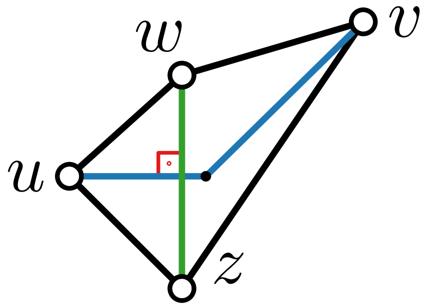

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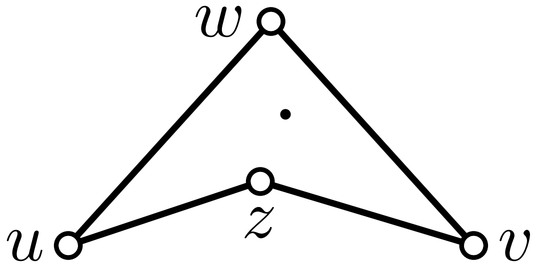

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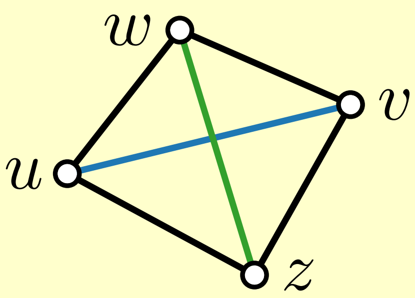

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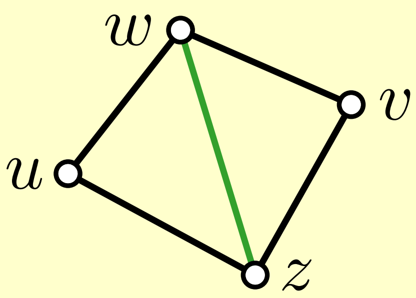

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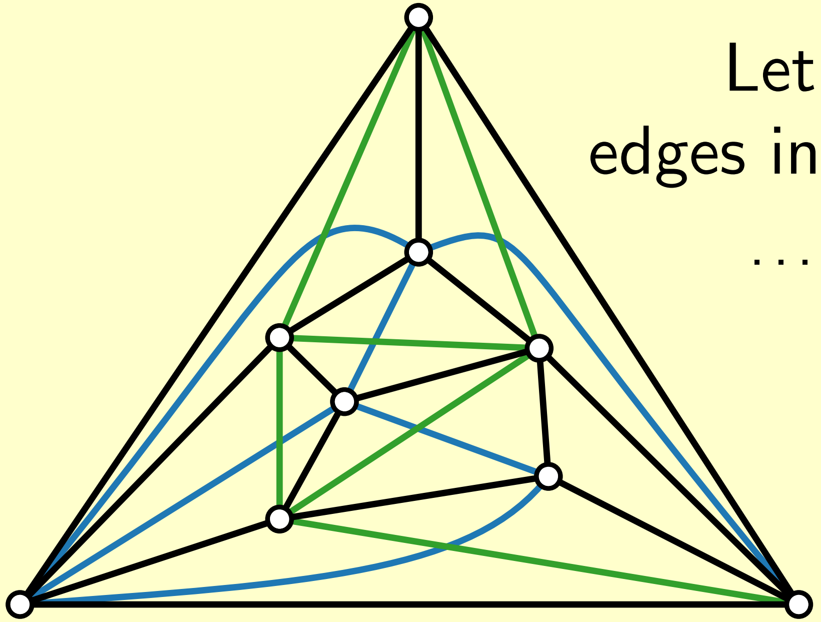

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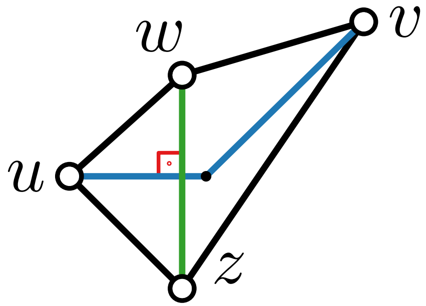

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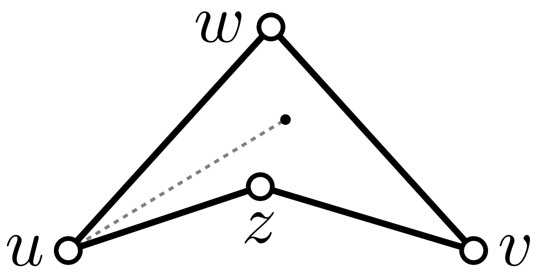

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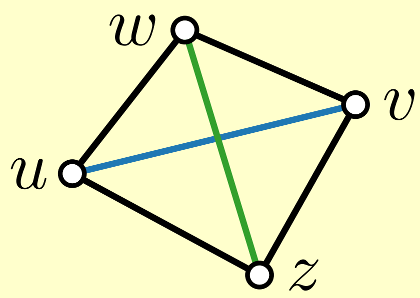

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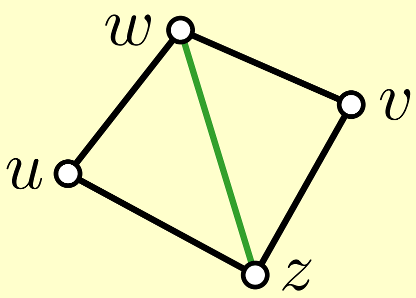

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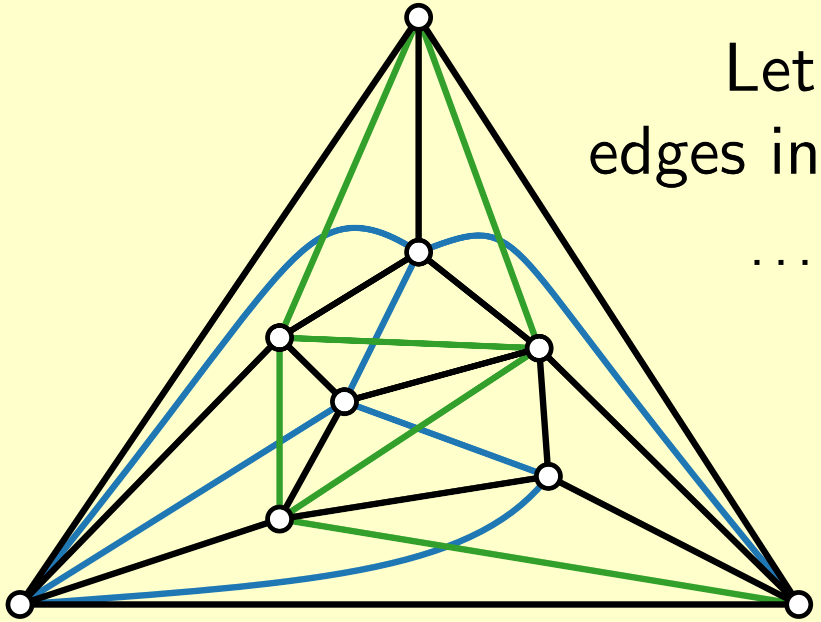

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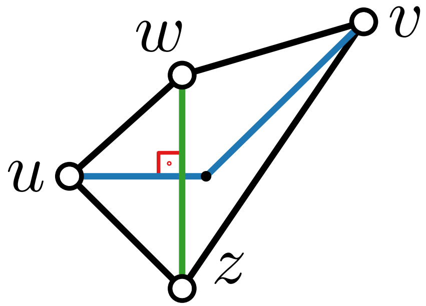

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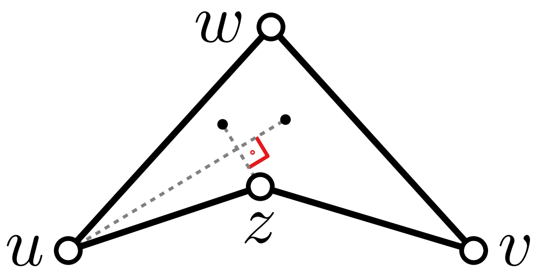

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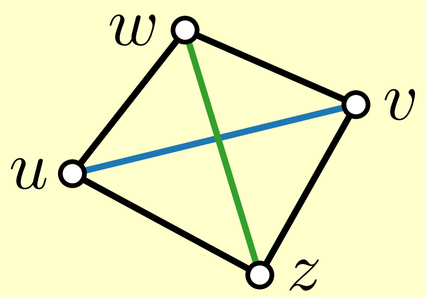

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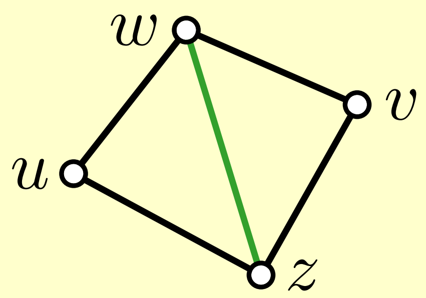

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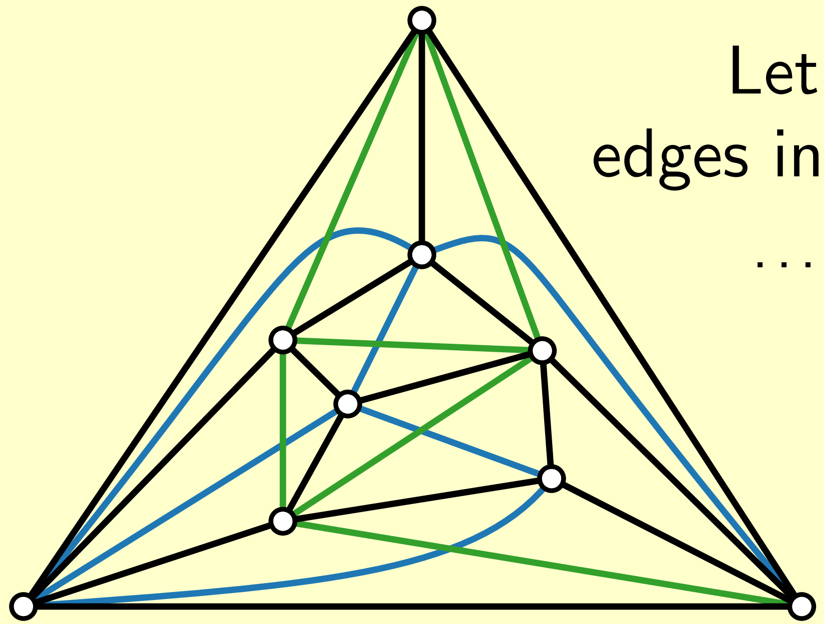

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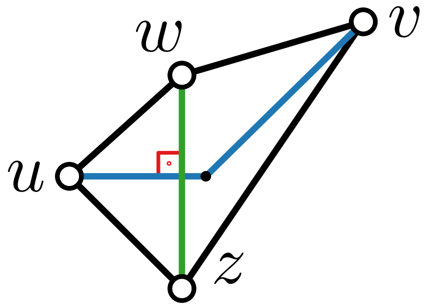

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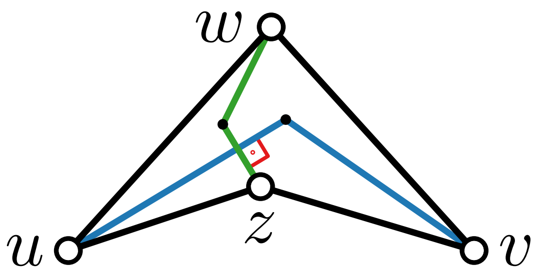

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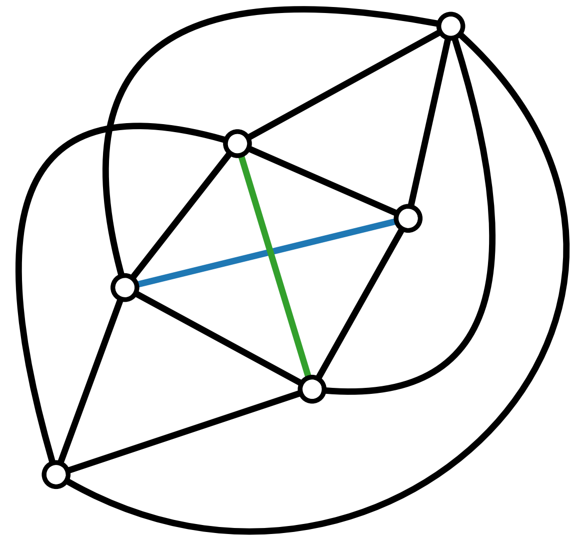

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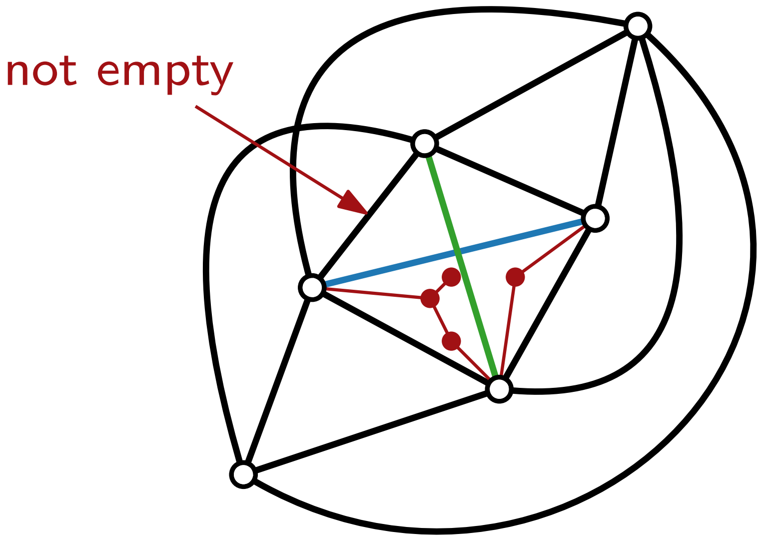

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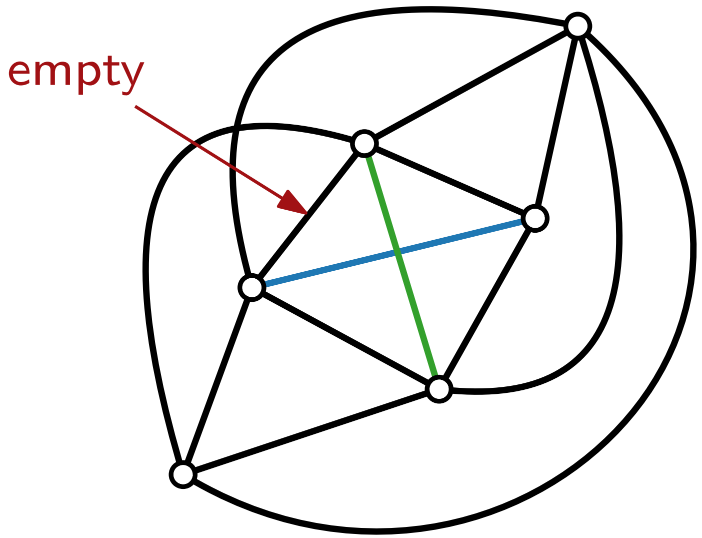

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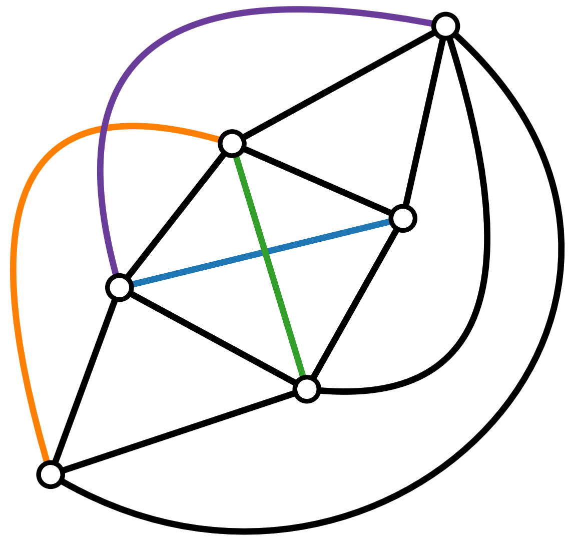

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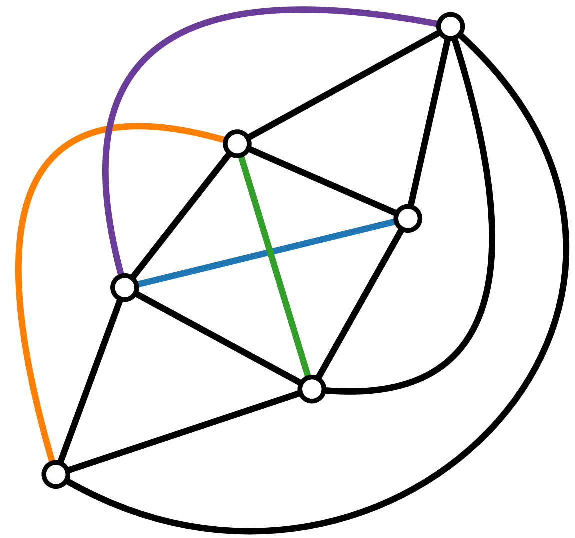
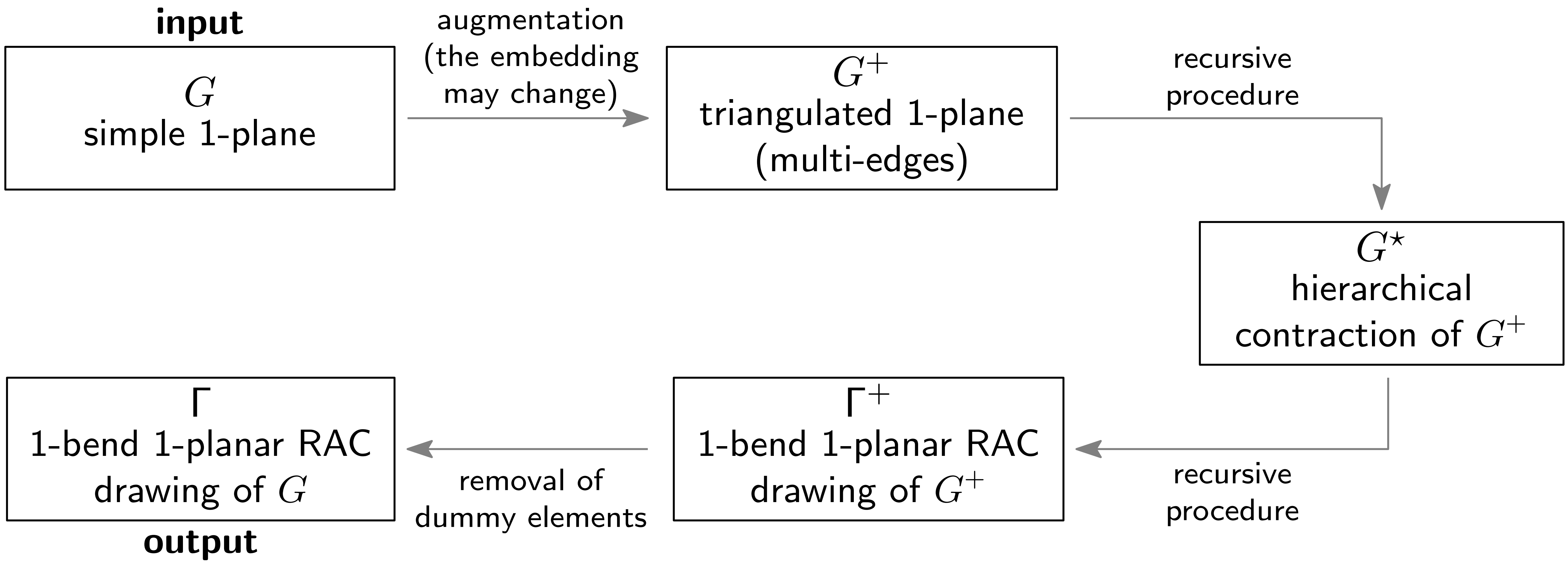


Theorem. [Chiba, Yamanouchi & Nishizeki 1984]
For every 2-connected plane graph G with outer face C_k and every convex k-gon P, there is a strictly convex planar straight-line drawing of G whose outer face coincides with P. Such a drawing can be computed in linear time.

Algorithm Outline



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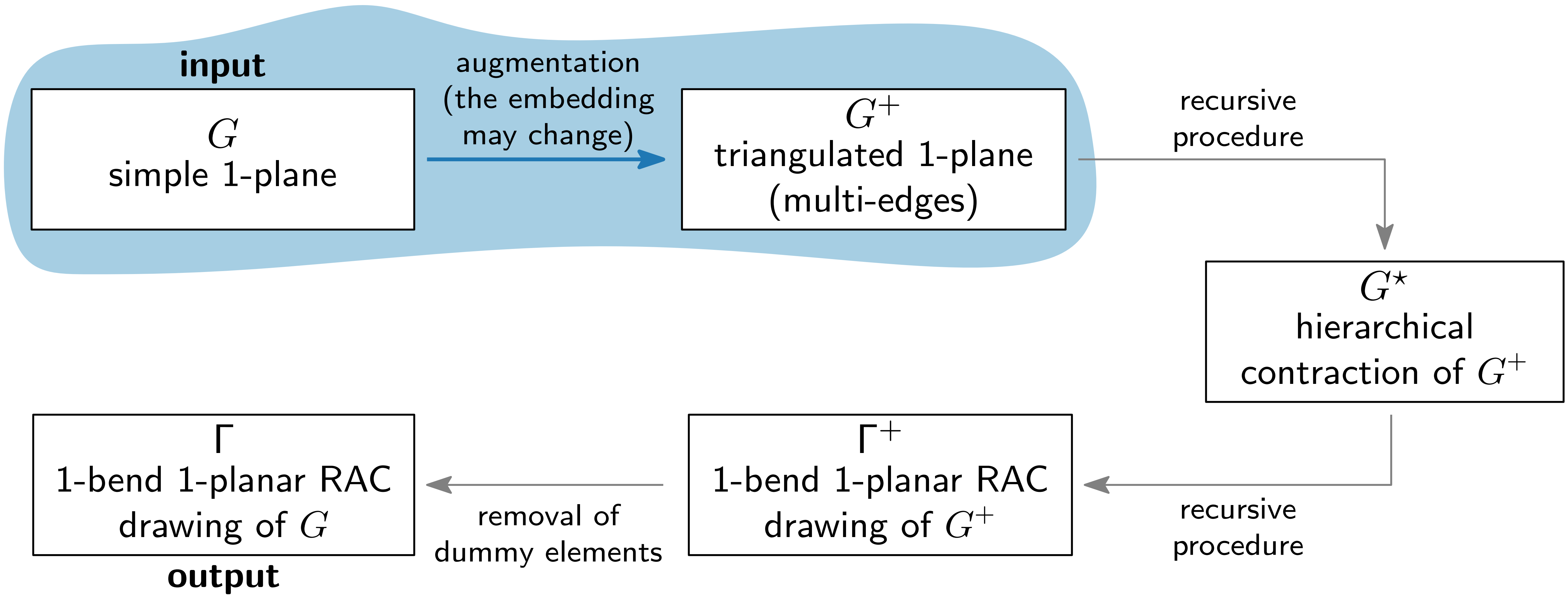

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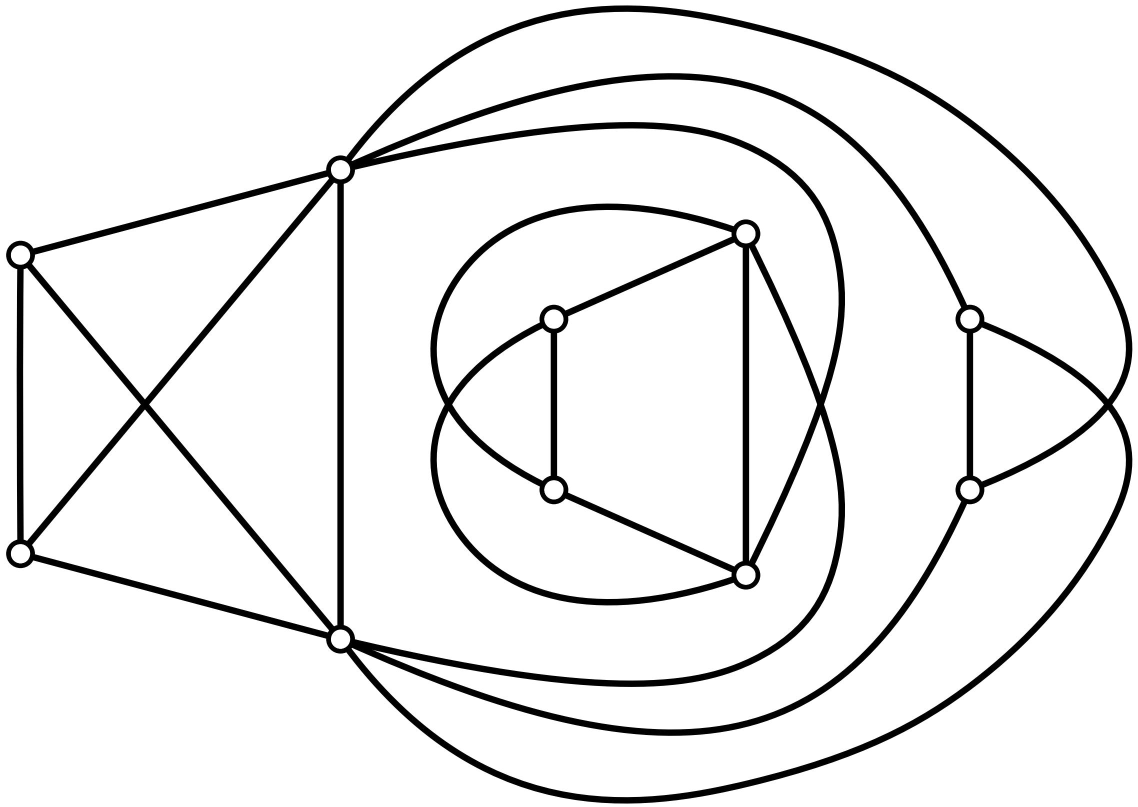

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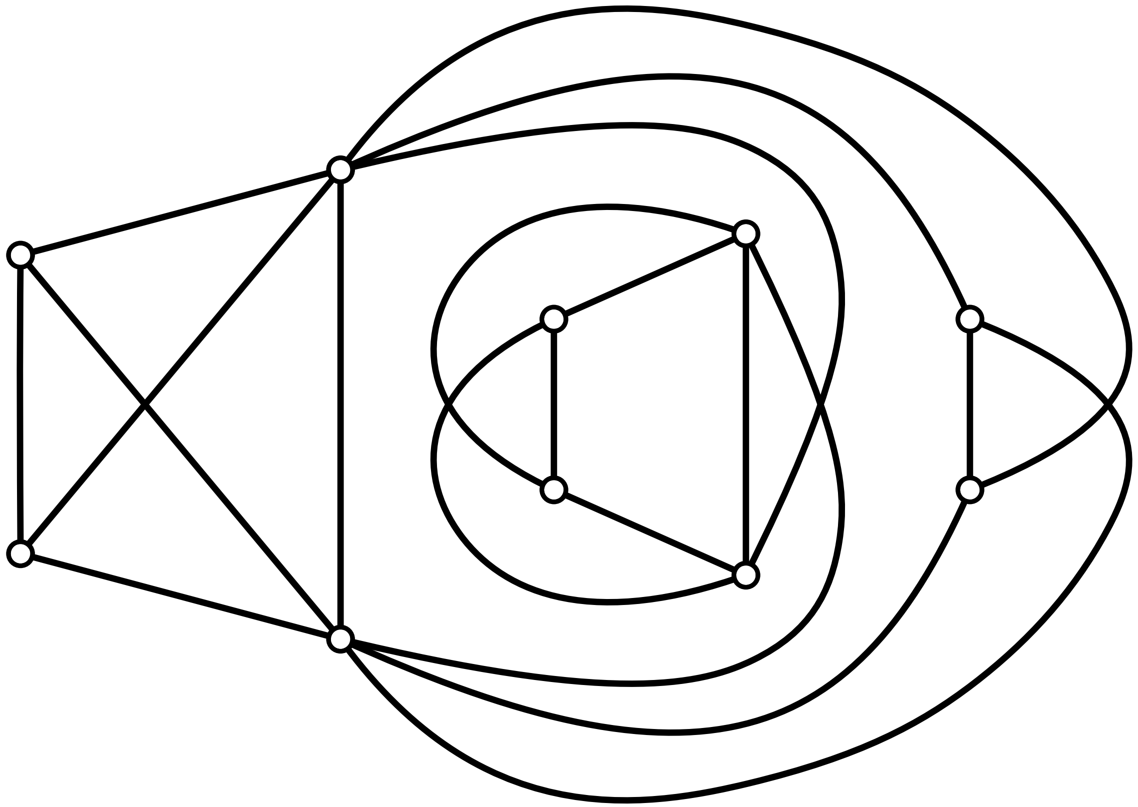

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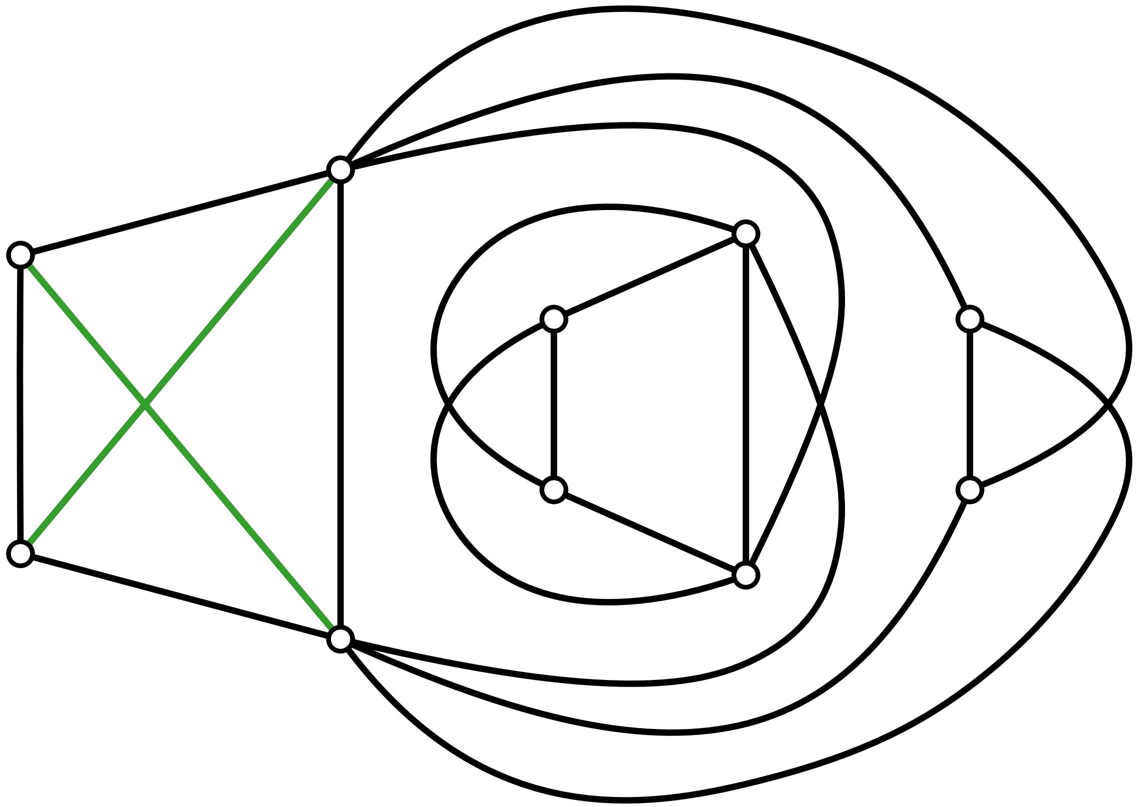

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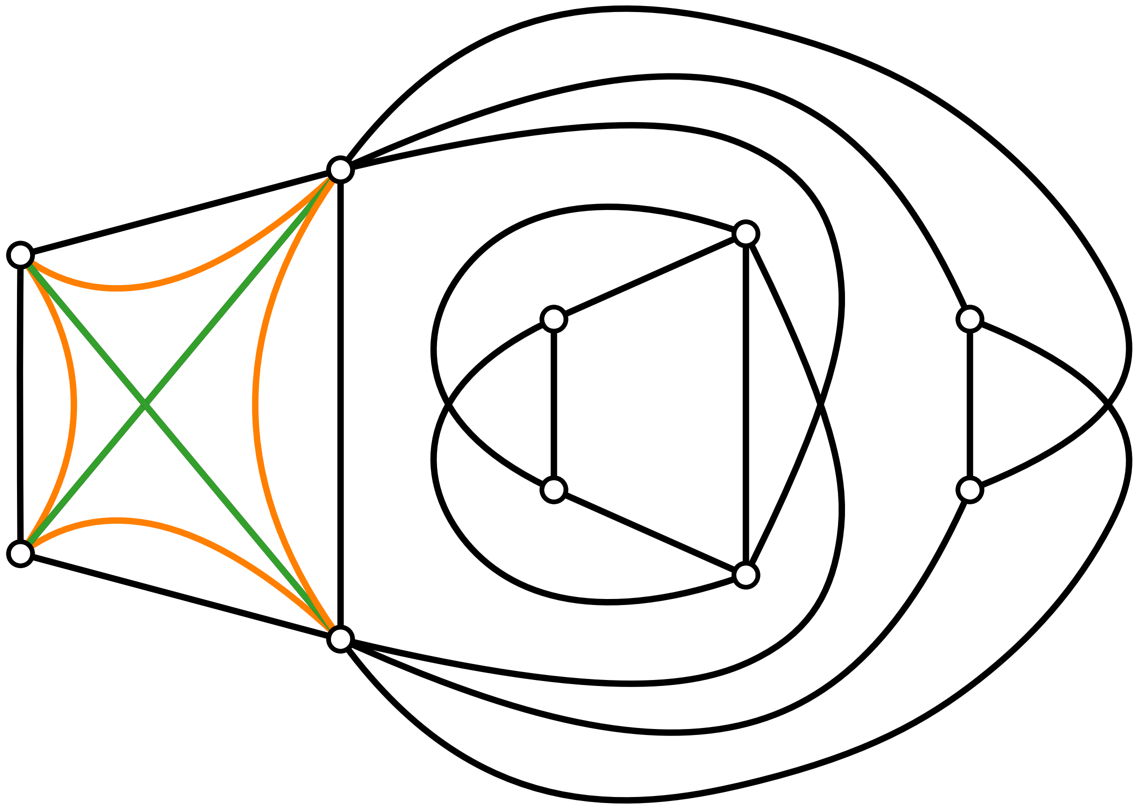

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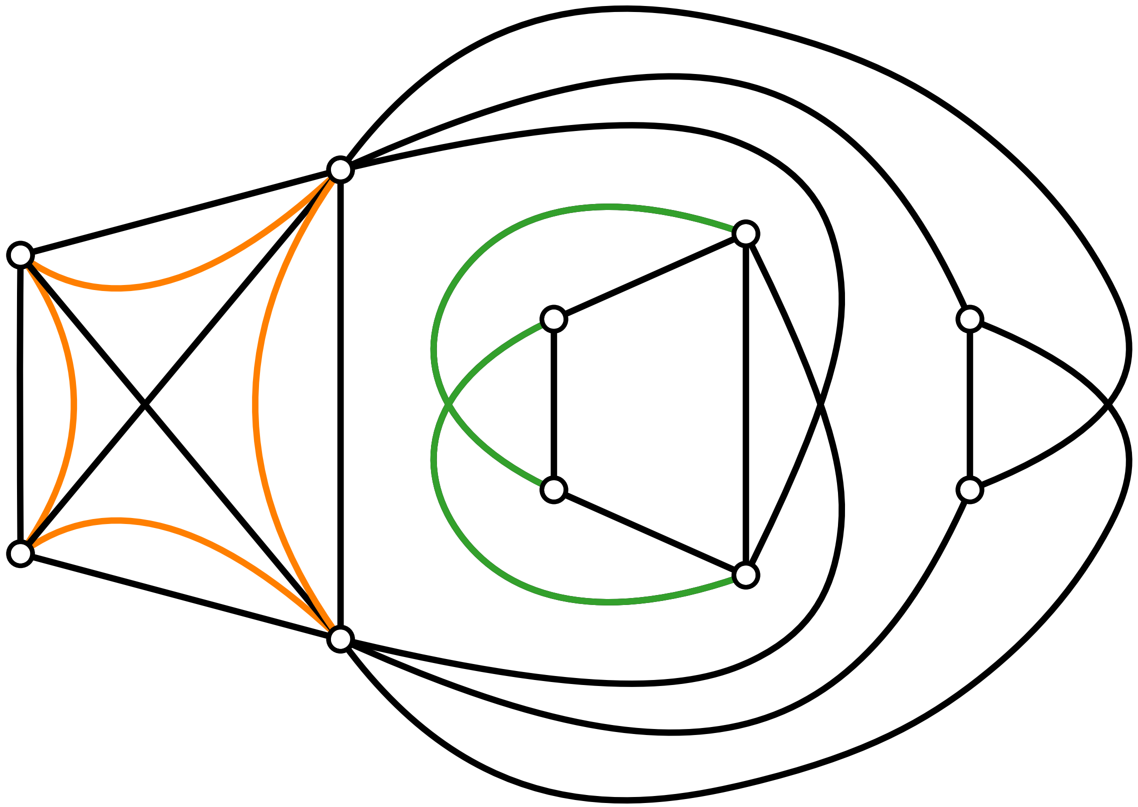

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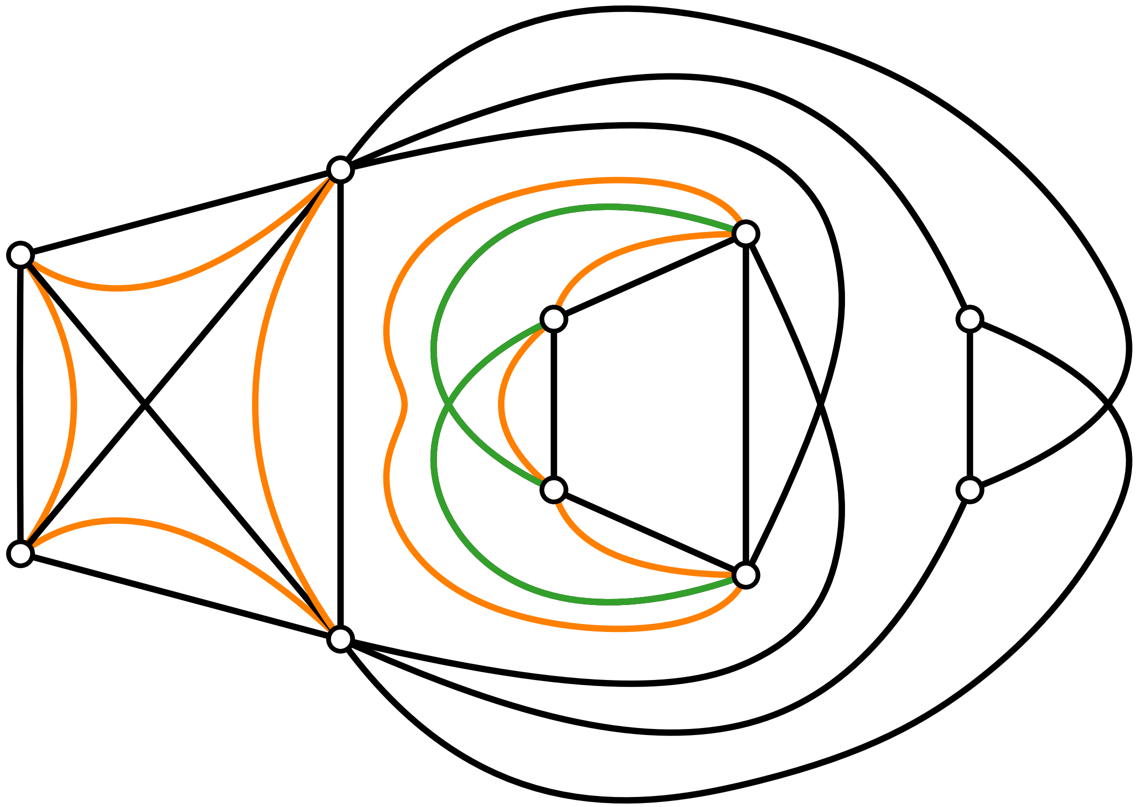

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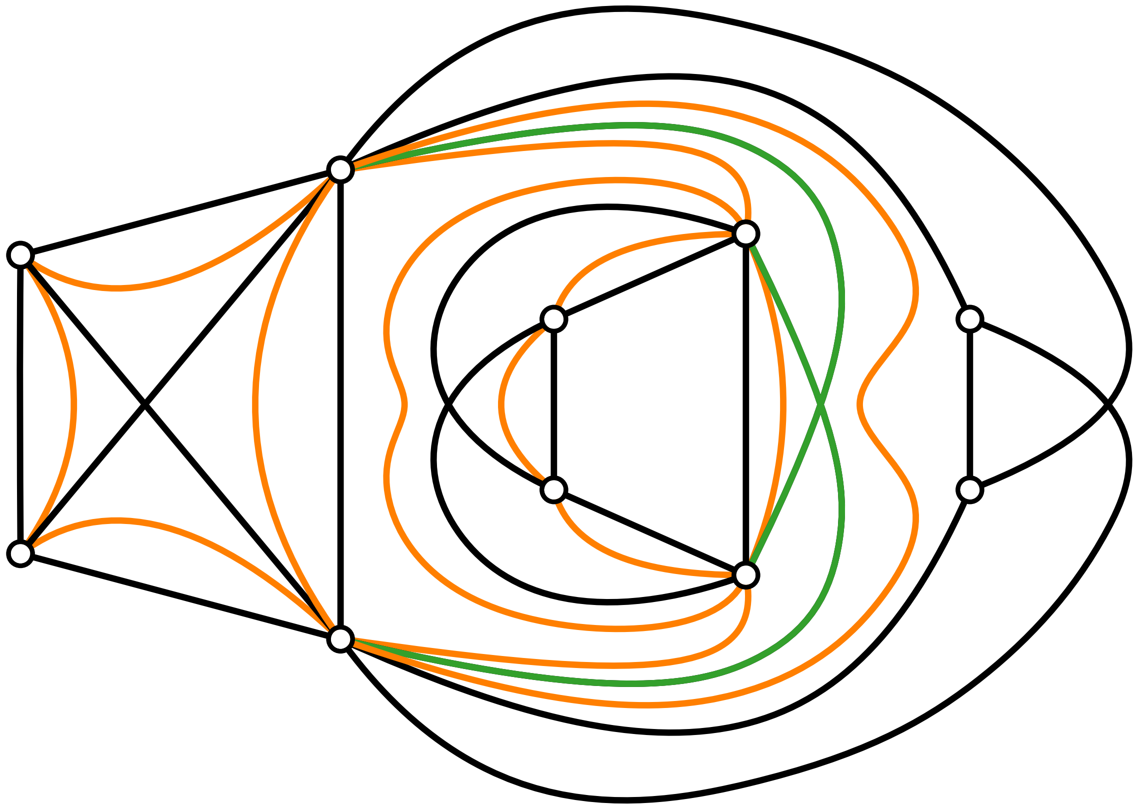

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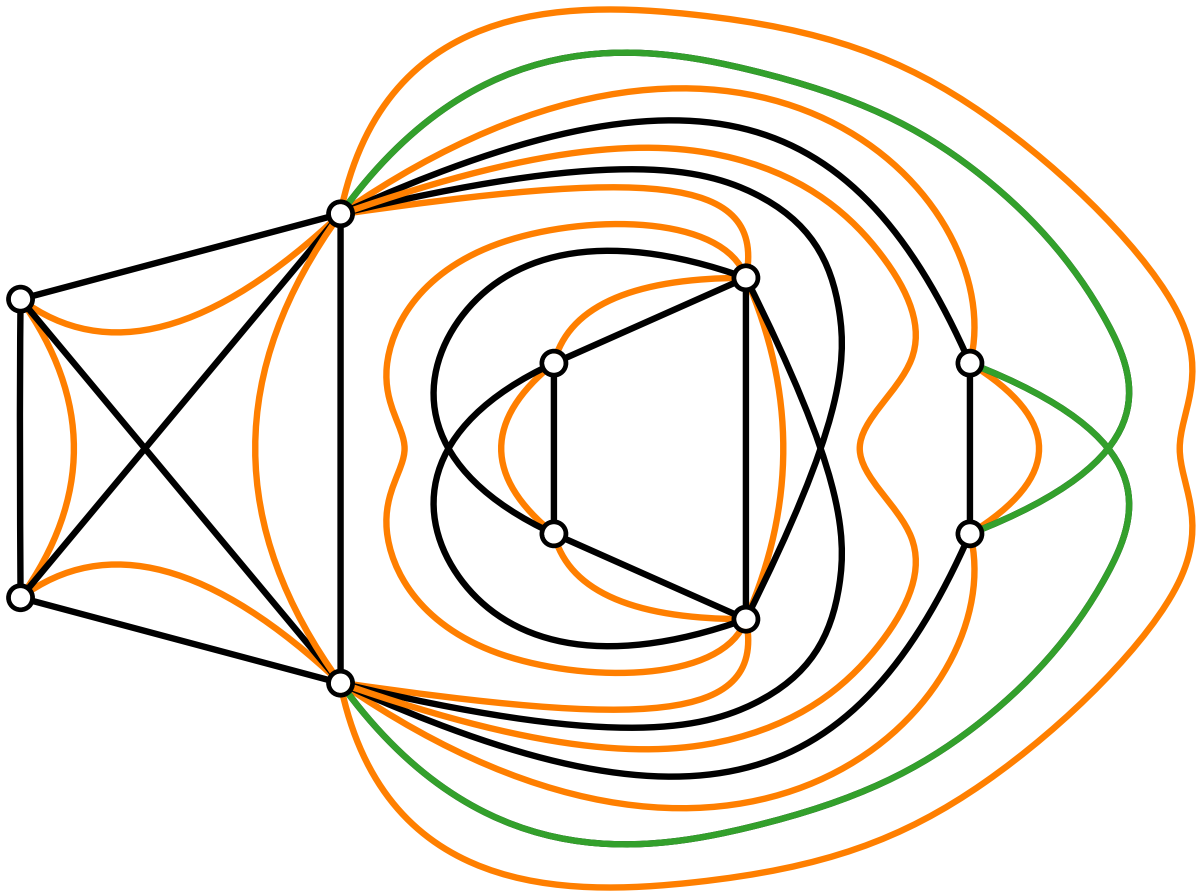

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2. Remove those multiple edges that belong to G.

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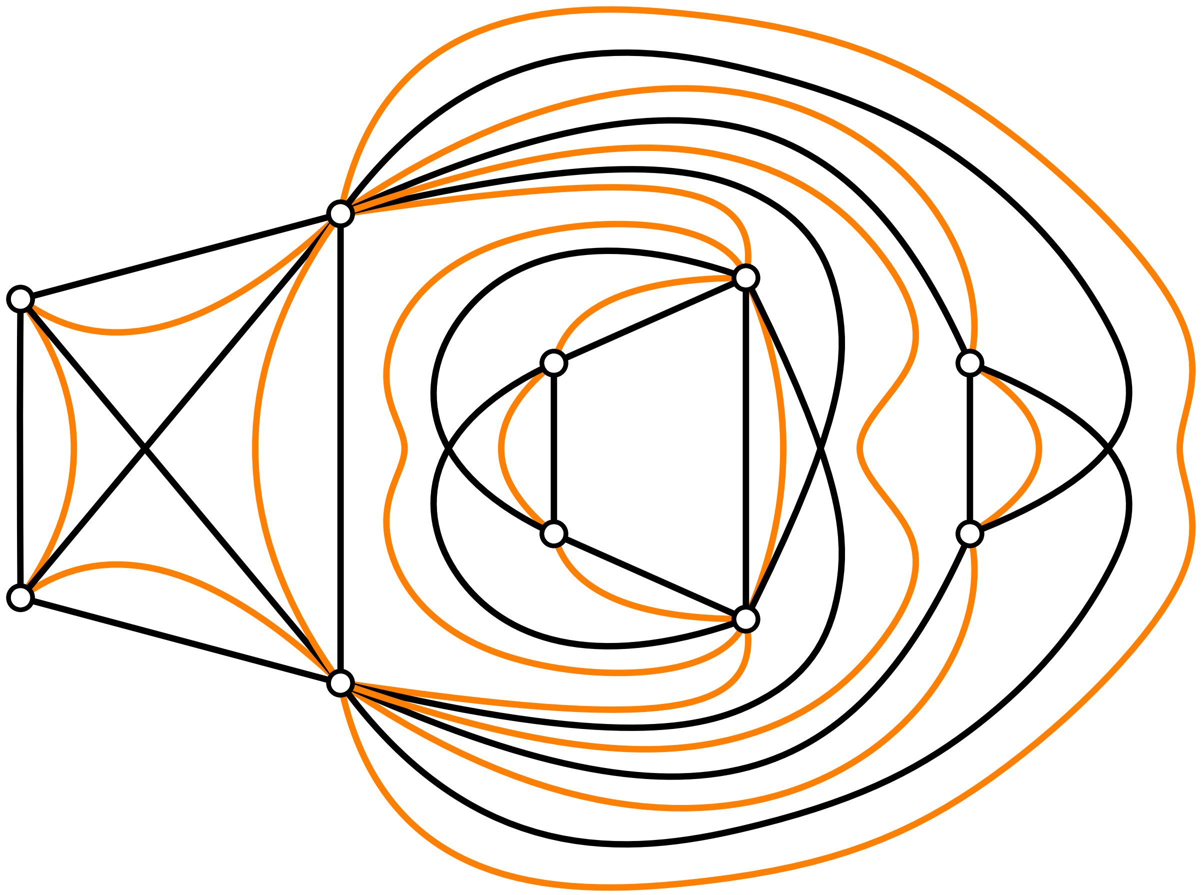

Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

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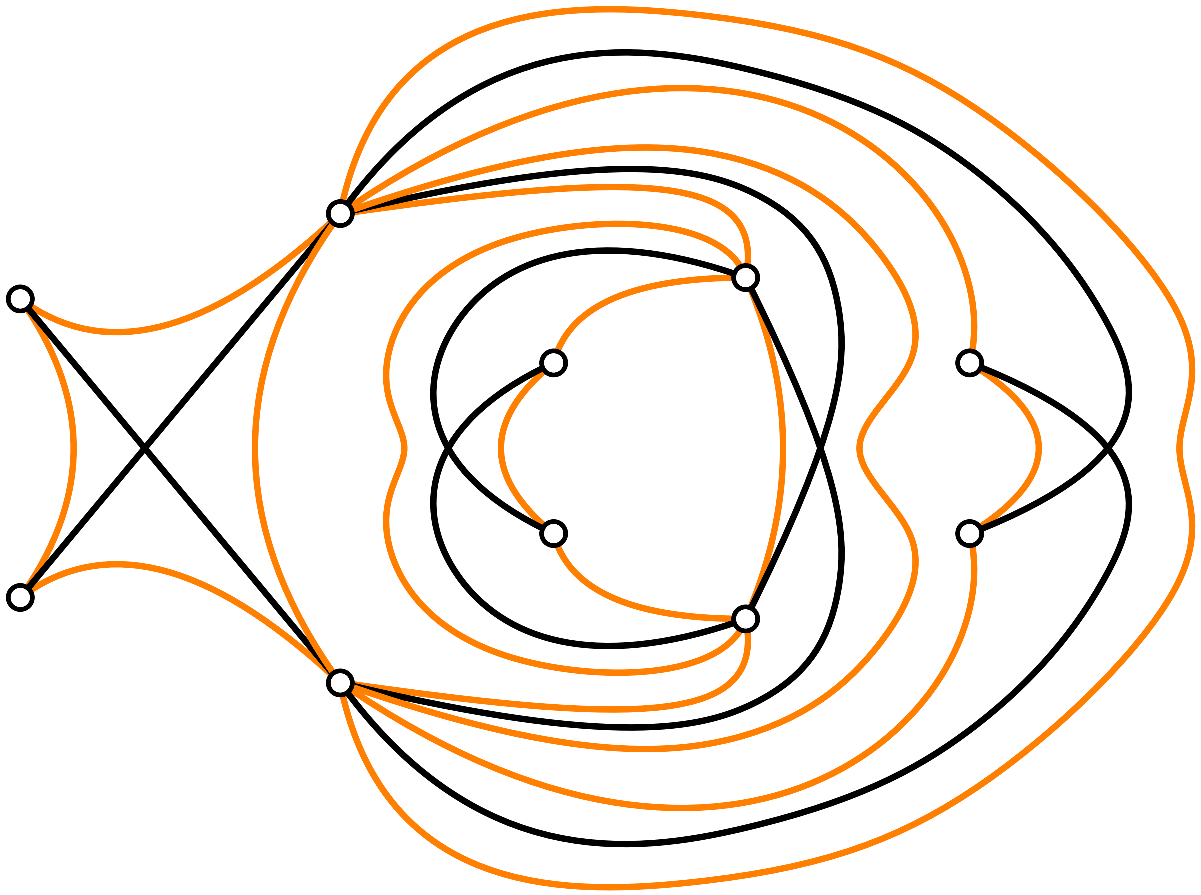
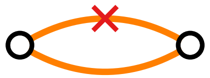


Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to G.

3. Remove one (multiple) edge from each face of degree two (if any).



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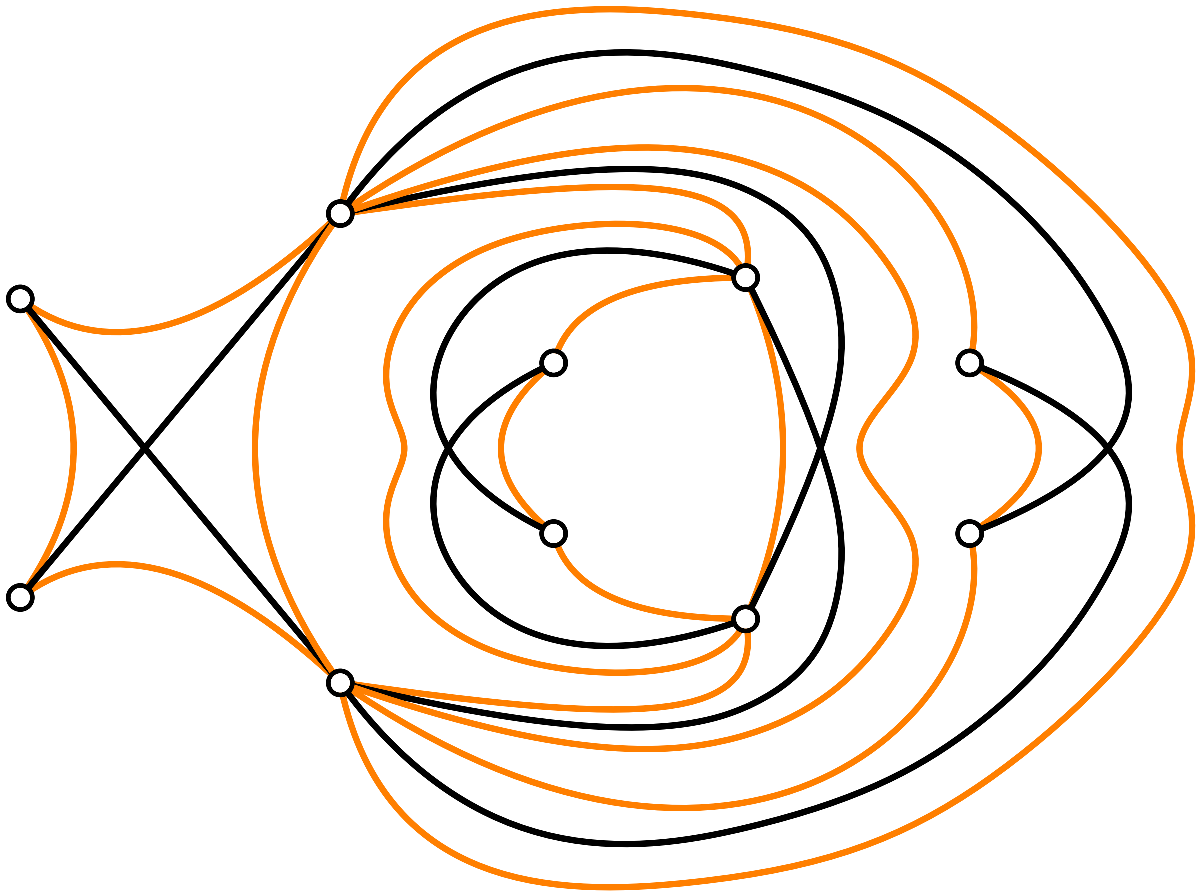

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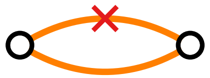

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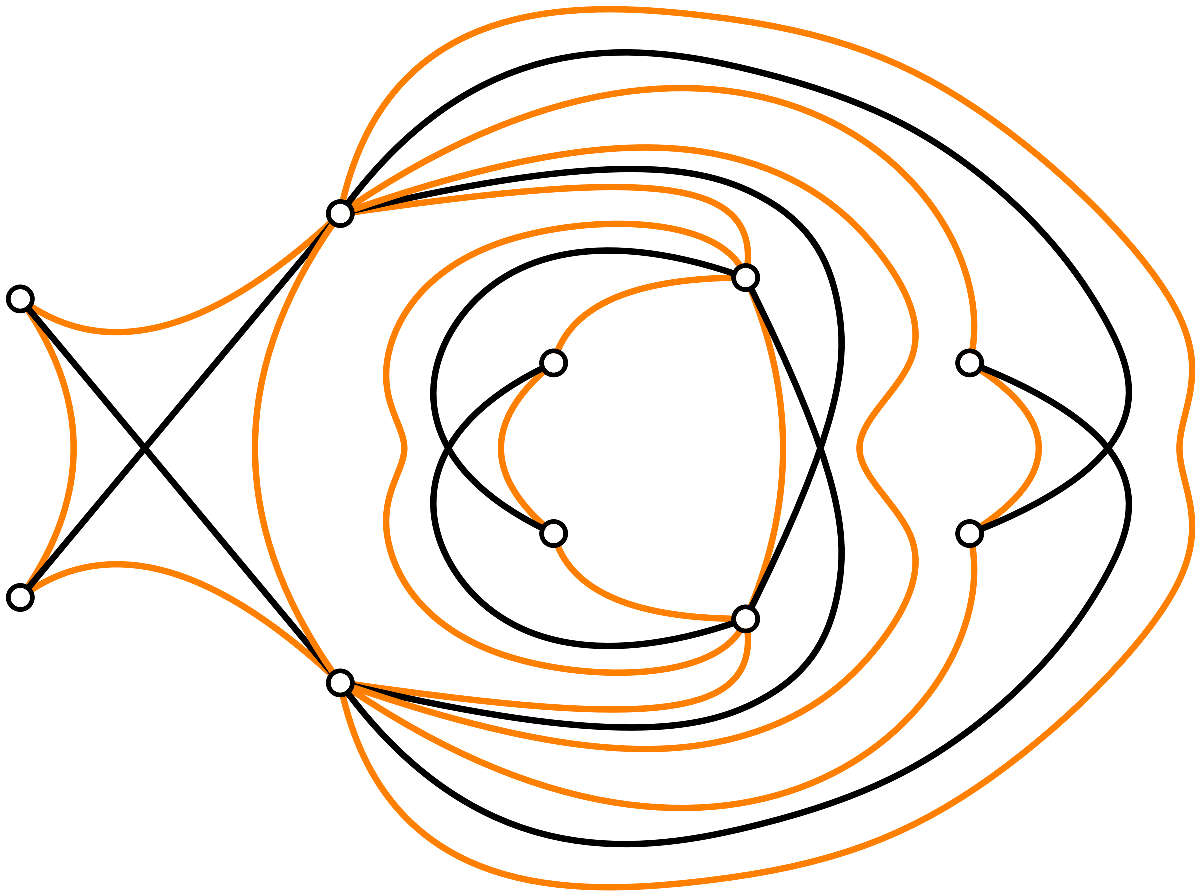

Note that we can still have parallel (orange) edges

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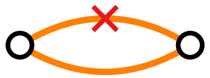
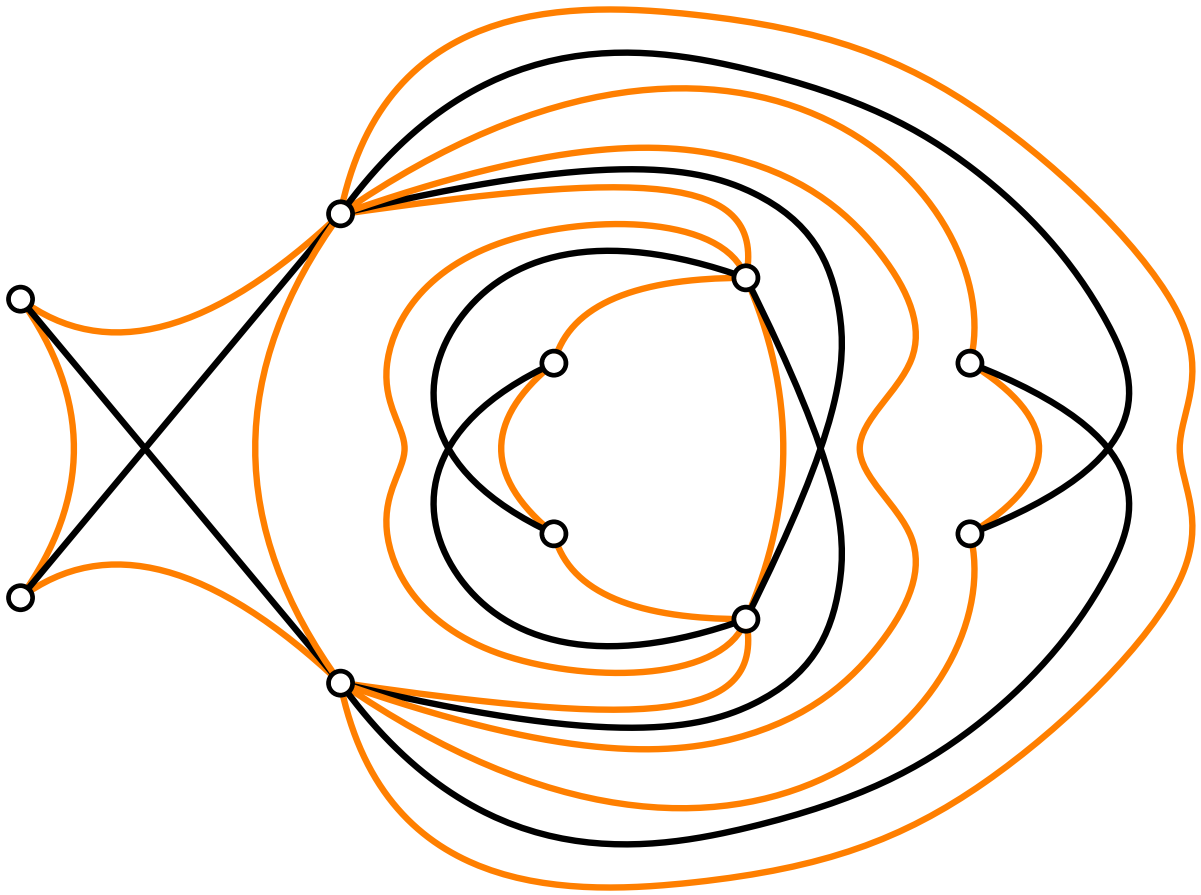


4. Triangulate faces of degree > 3 by inserting a star inside them.

G: simple 1-plane graph



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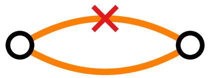
3. Remove one (multiple) edge from each face of degree two (if any). 

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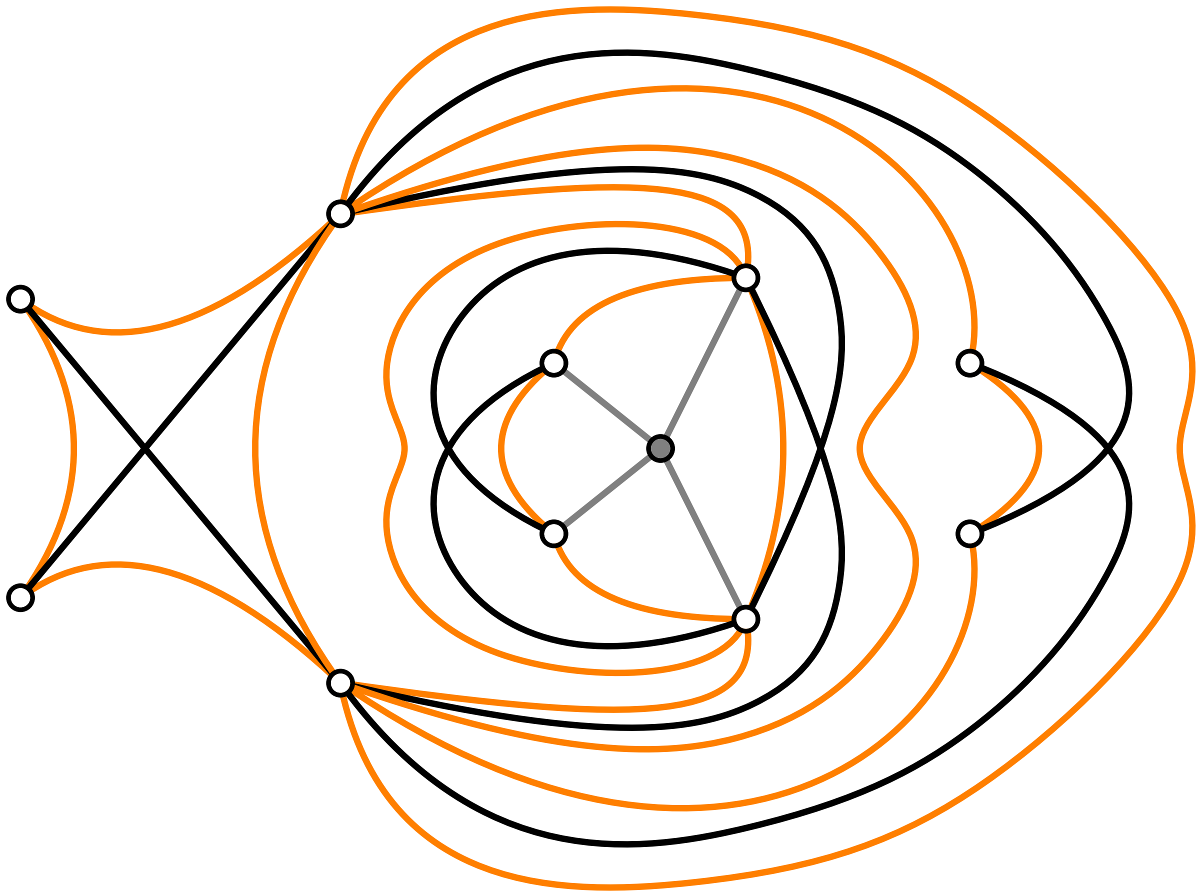

Note that we can still have parallel (orange) edges

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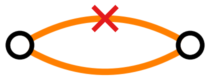
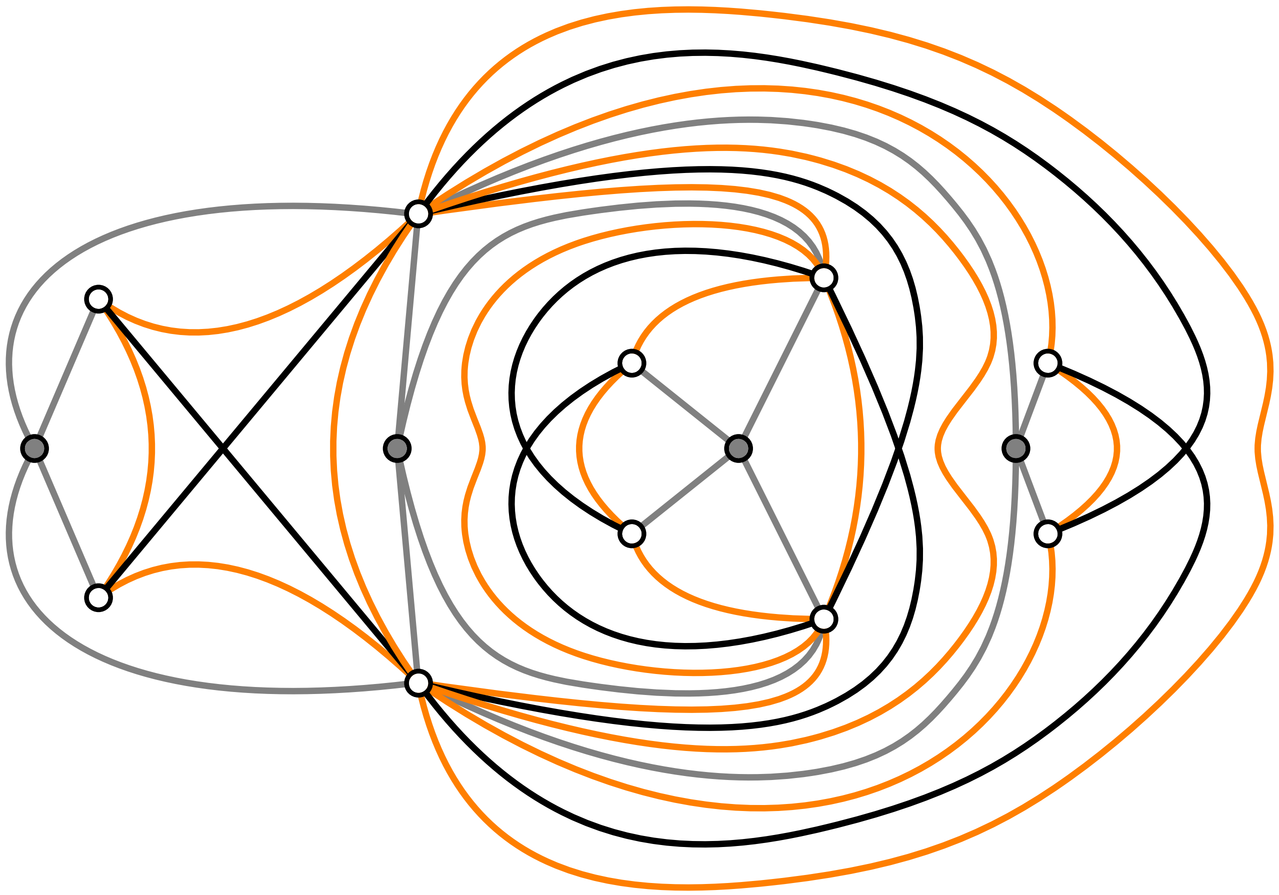


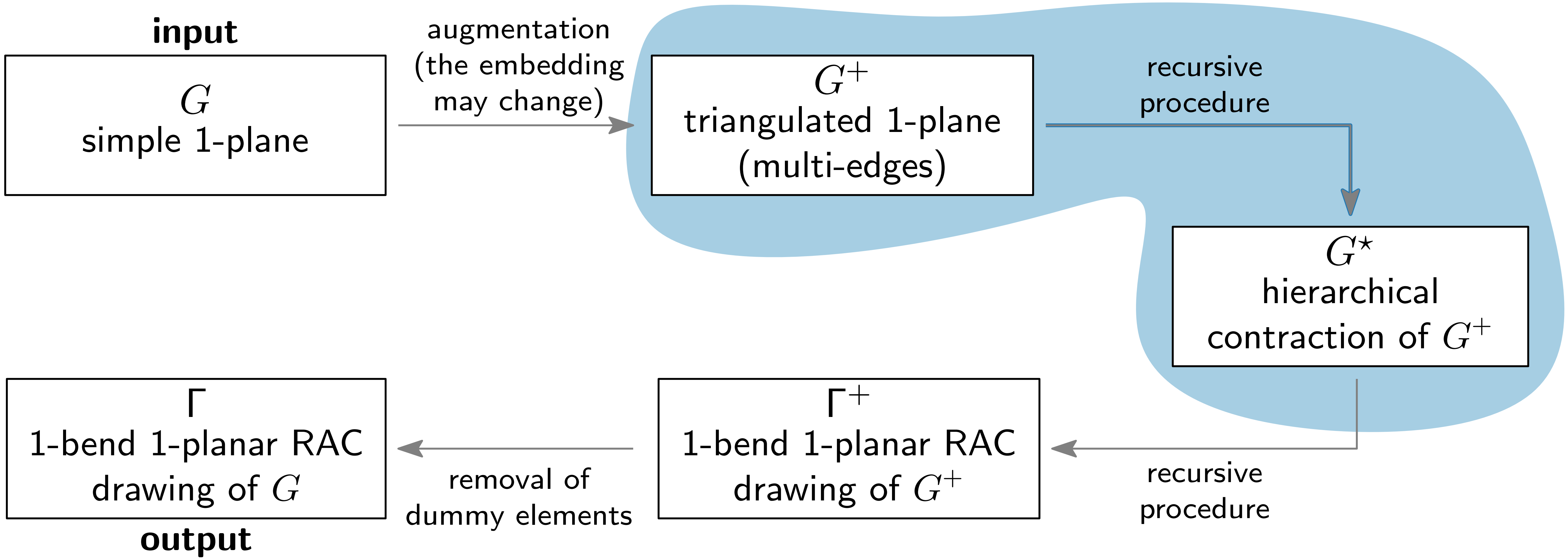
4. Triangulate faces of degree > 3 by inserting a star inside them.

G: simple 1-plane graph → G^+: triangulated 1-plane (possibly with multi-edges)



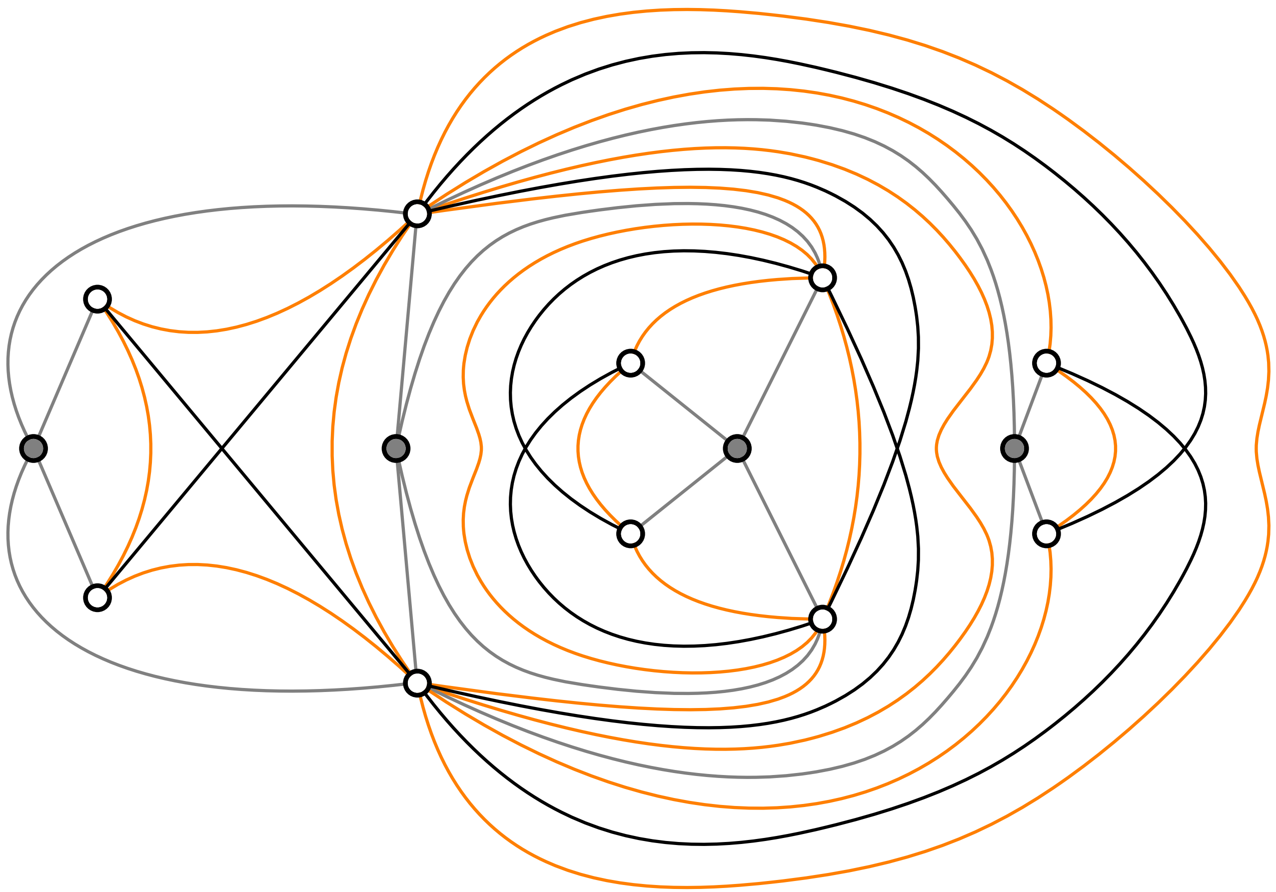
Note that we can still have parallel (orange) edges

Algorithm Outline



Algoritm Step 2: Hierarchical Contractions

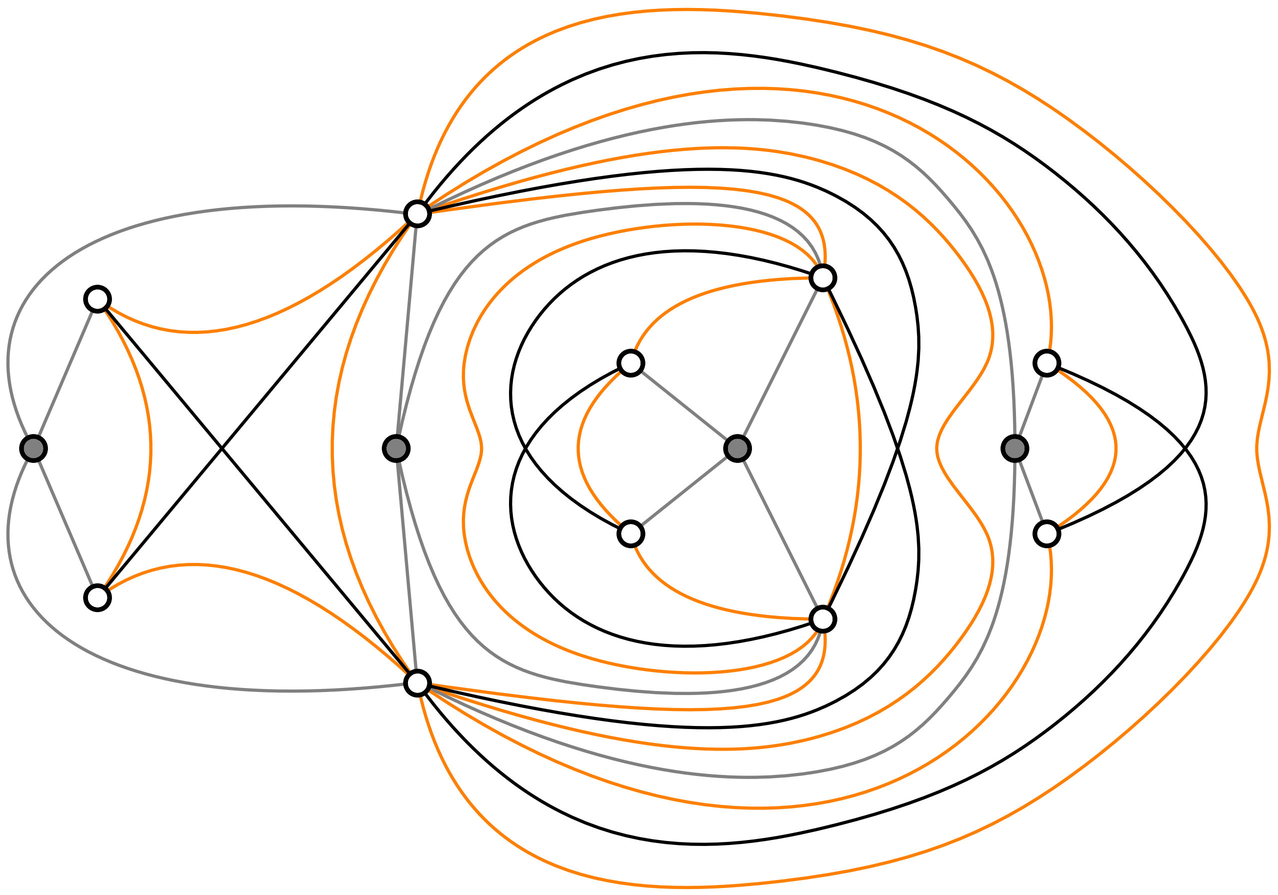
G^+
triangulated 1-plane
(multi-edges)



Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

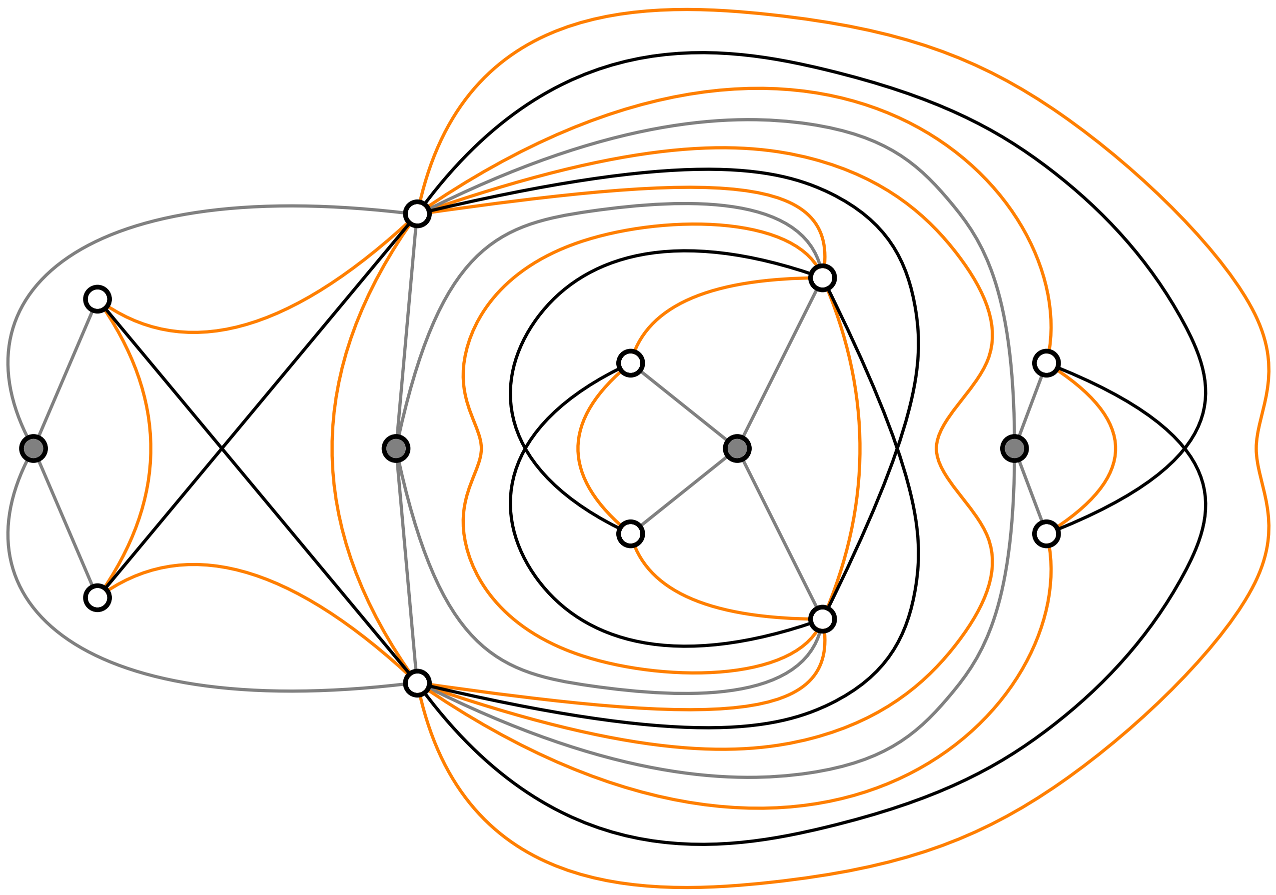
- triangular faces



Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

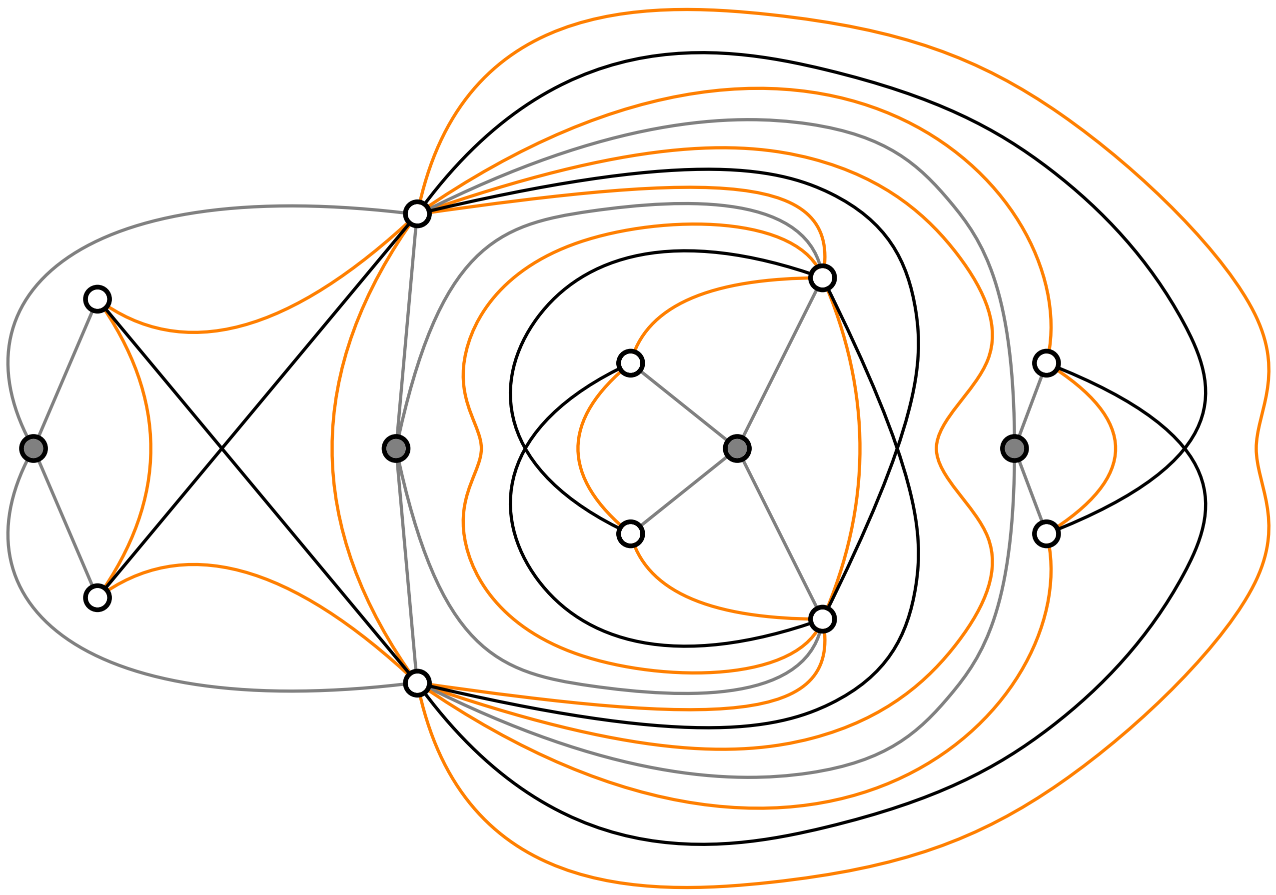
- triangular faces
- multiple edges never crossed



Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



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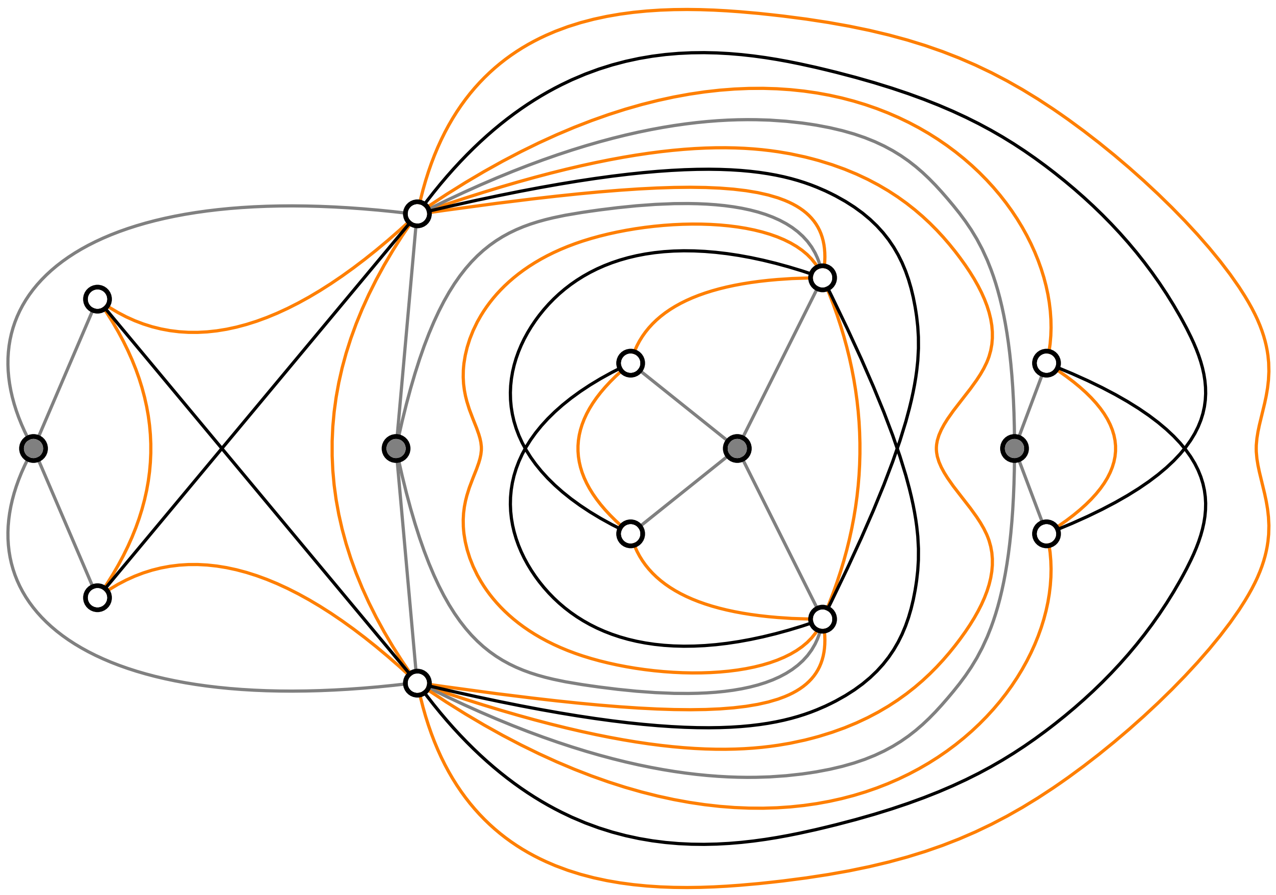
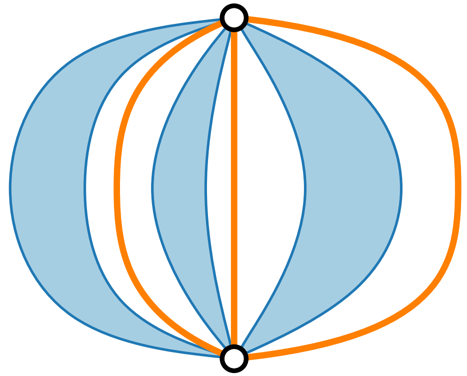


$\Rightarrow$



structure of each separation pair

Algoritm Step 2: Hierarchical Contractions

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(multi-edges)

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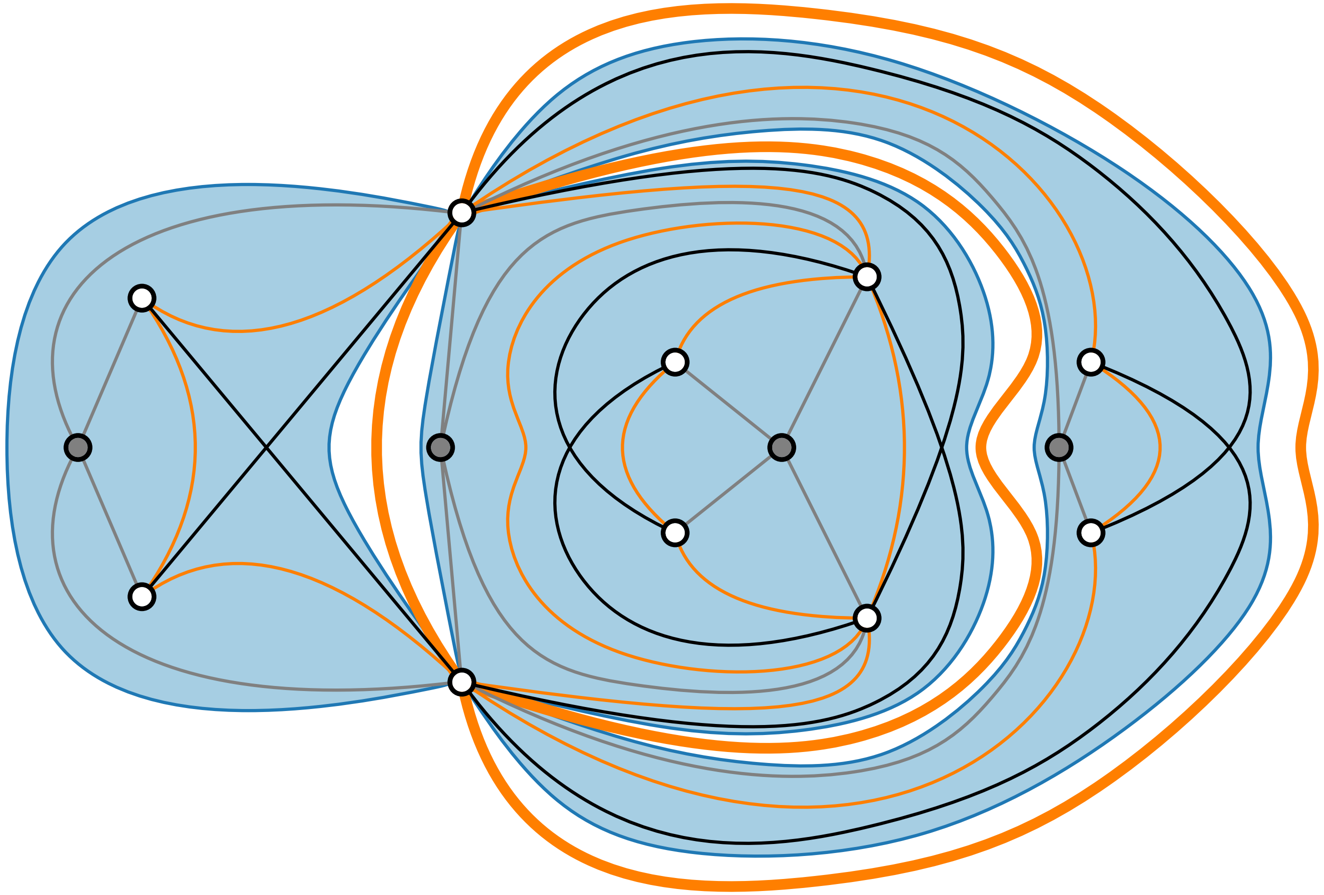

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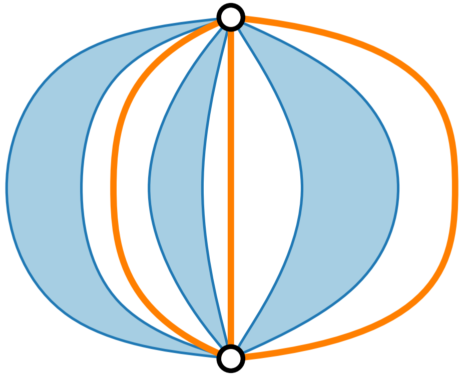

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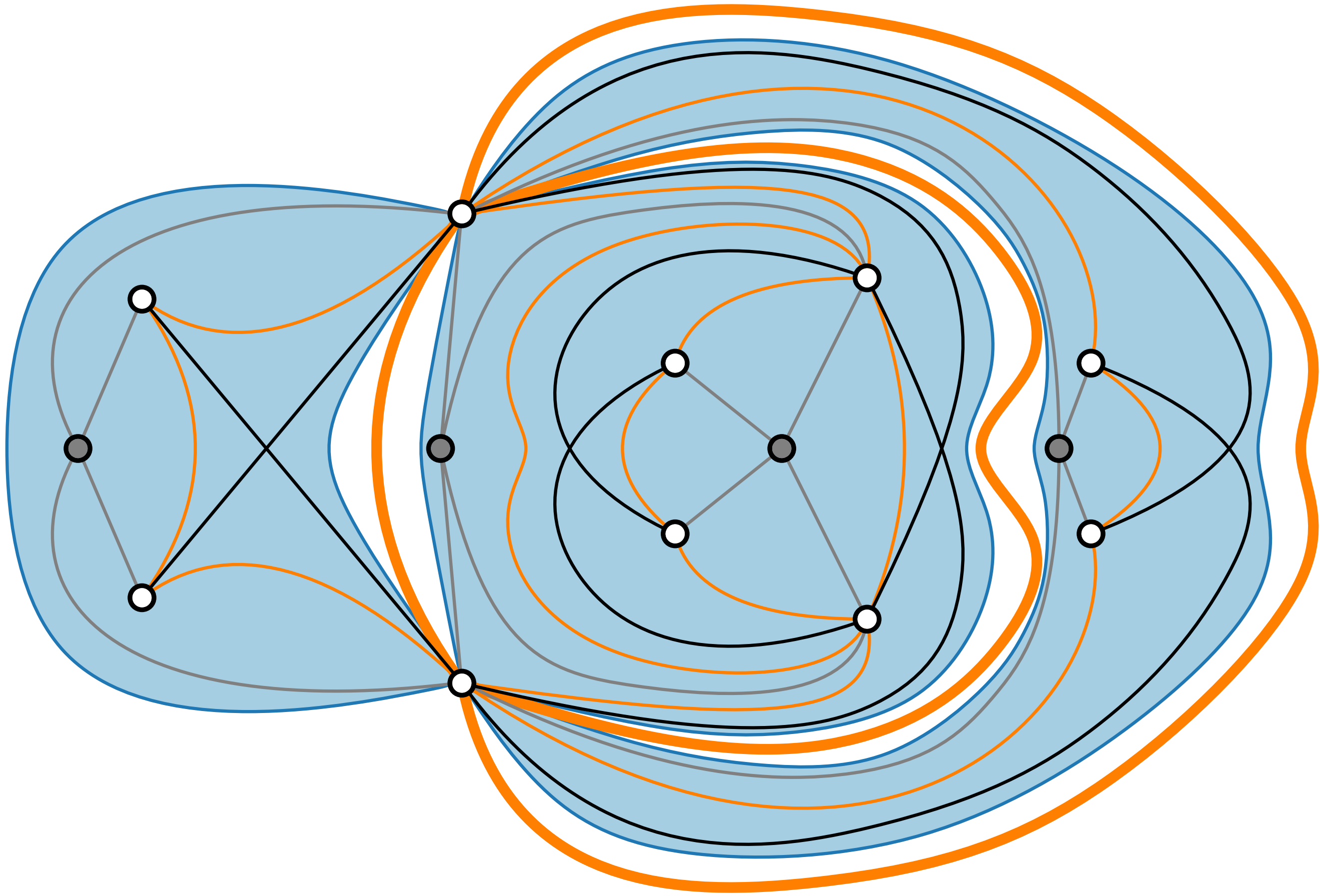

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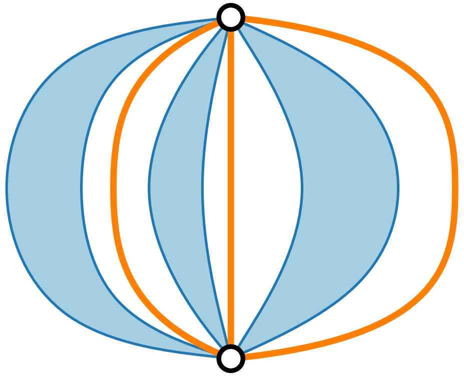

structure of each separation pair

Contract all inner components of each separation pair into a thick edge.

Algoritm Step 2: Hierarchical Contractions

G^+
triangulated 1-plane
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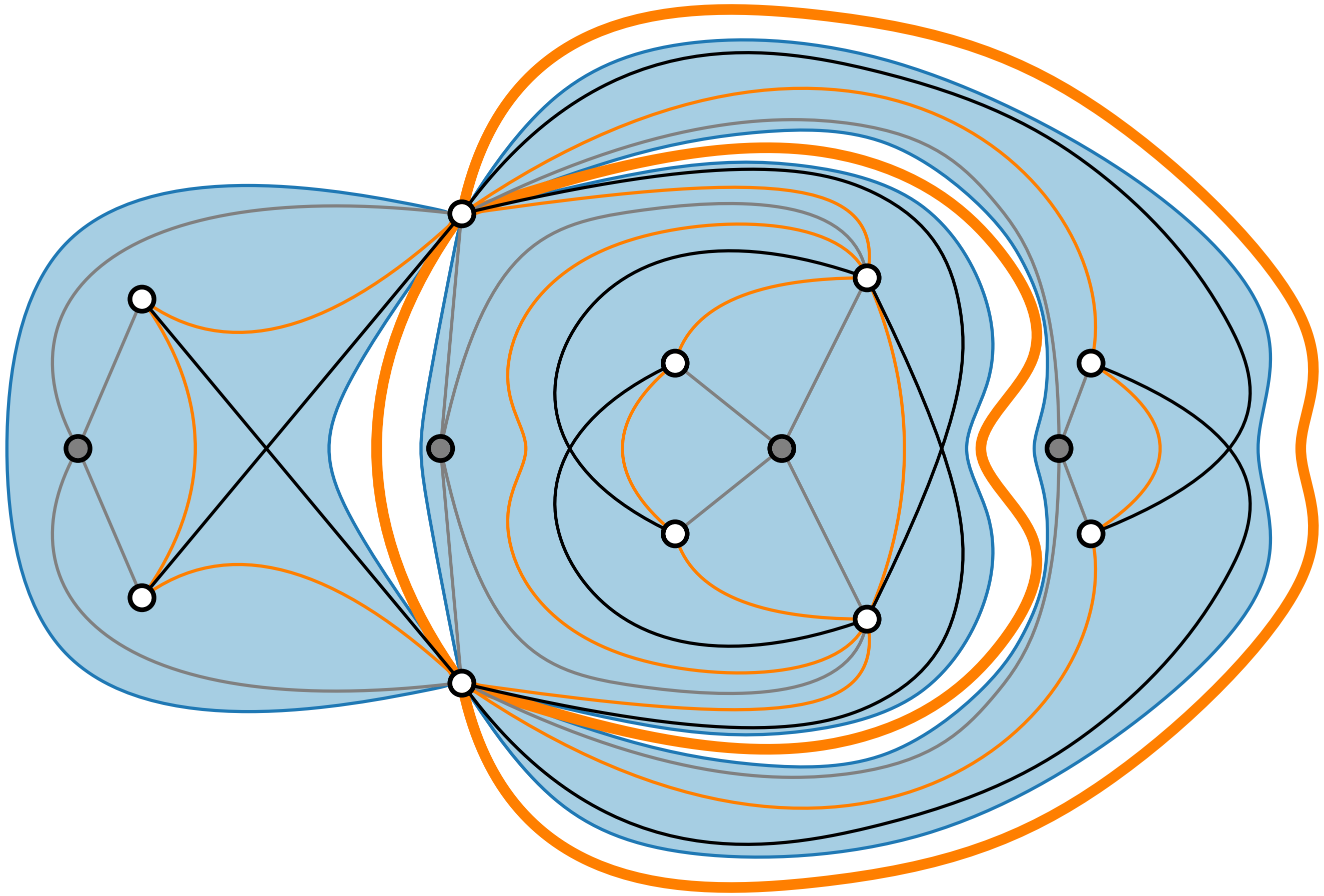
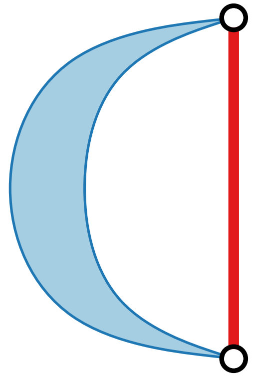


⇒



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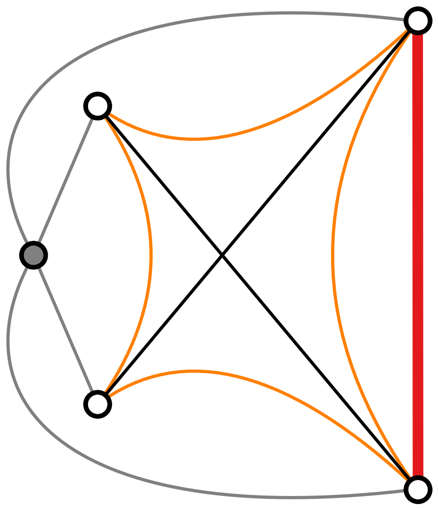
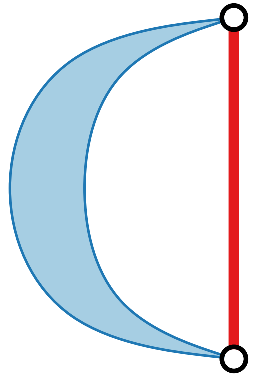


⇒



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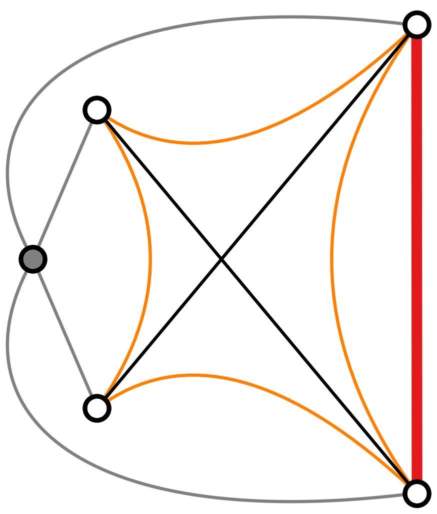

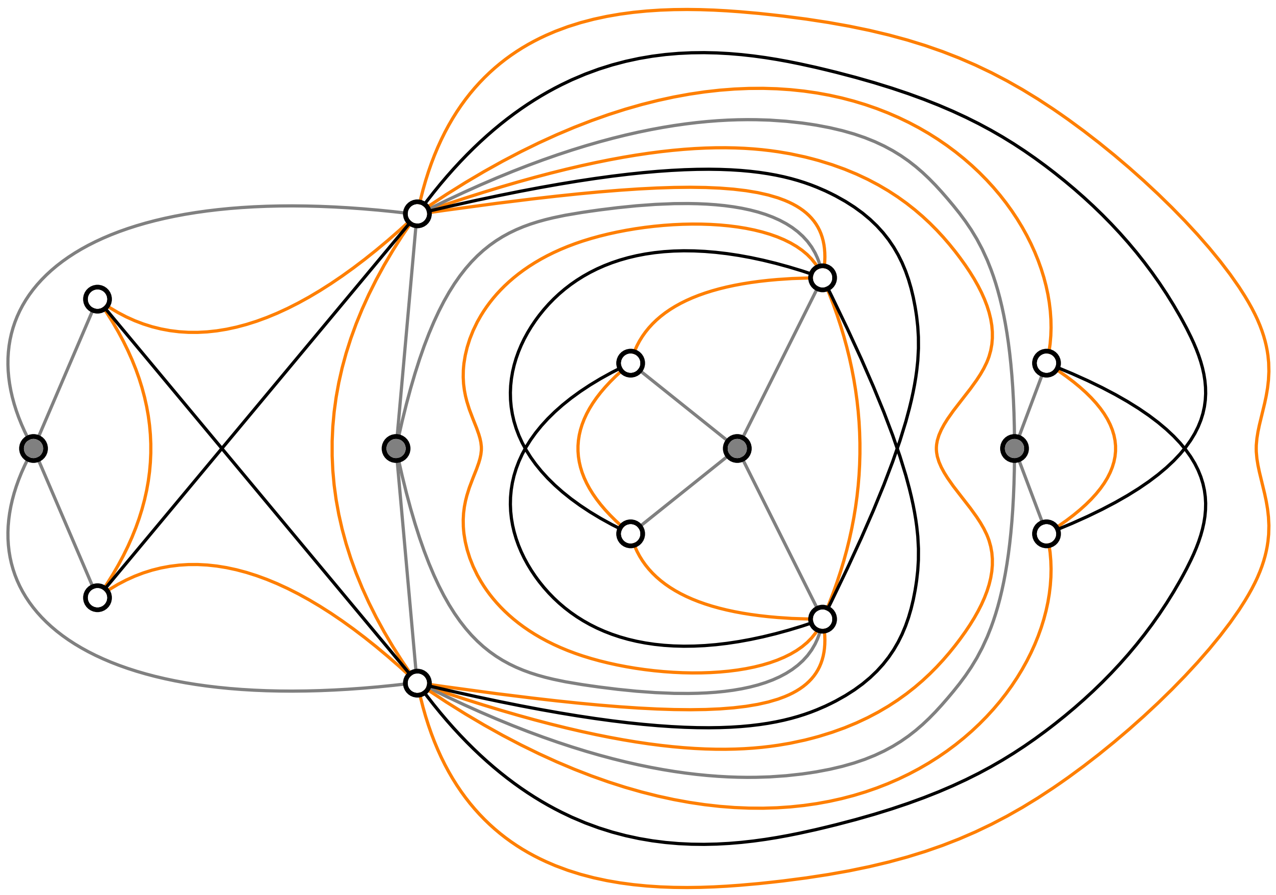

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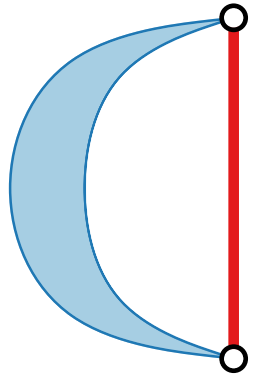

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Algoritm Step 2: Hierarchical Contractions

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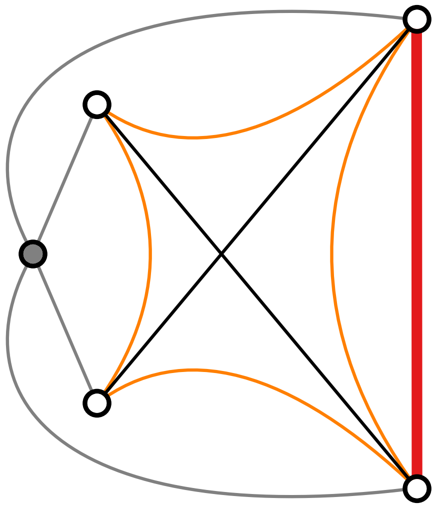

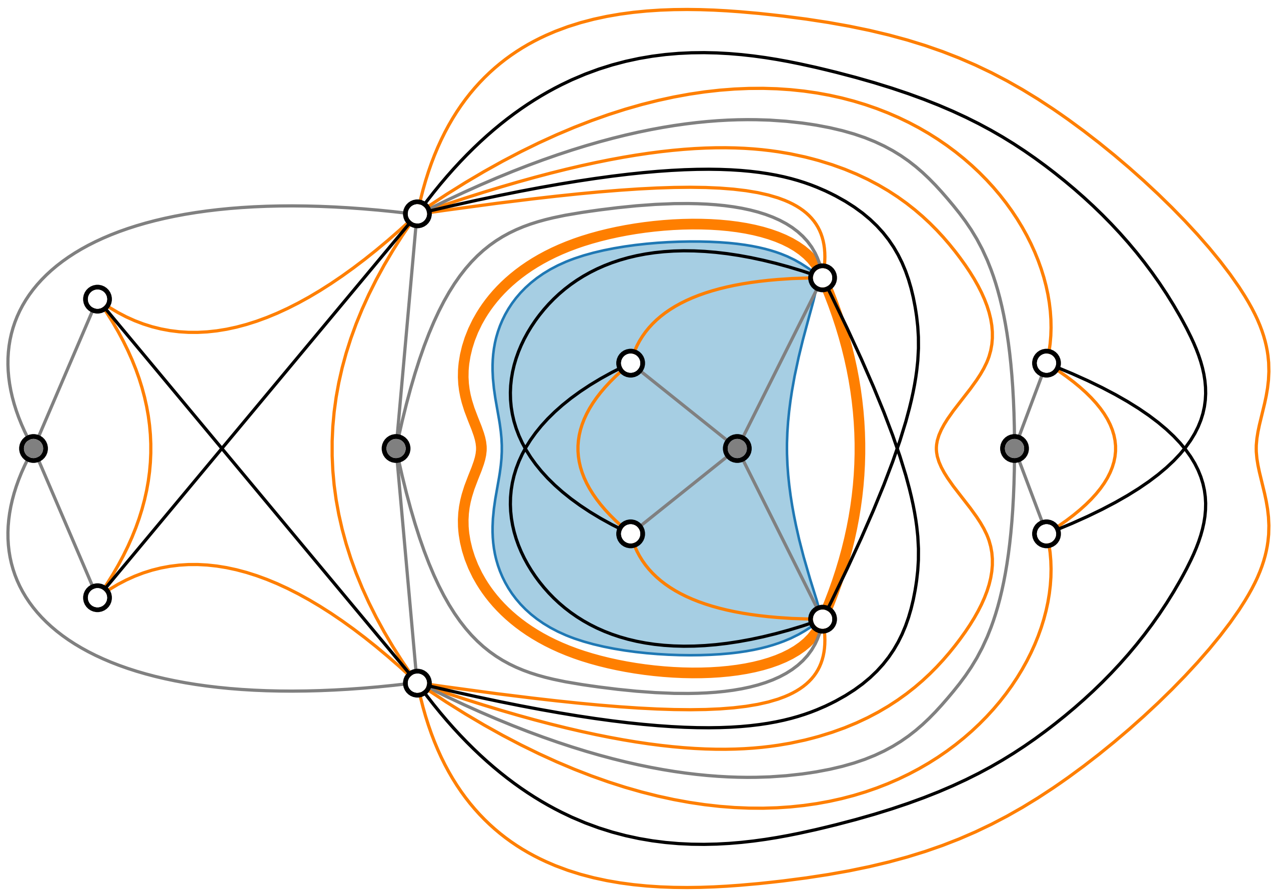

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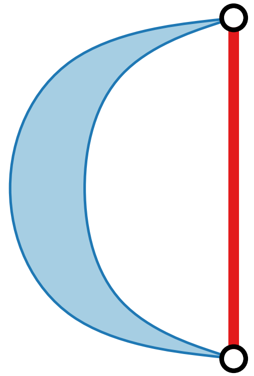

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Algoritm Step 2: Hierarchical Contractions

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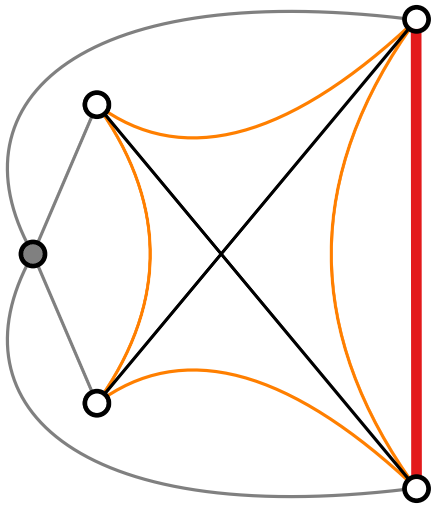

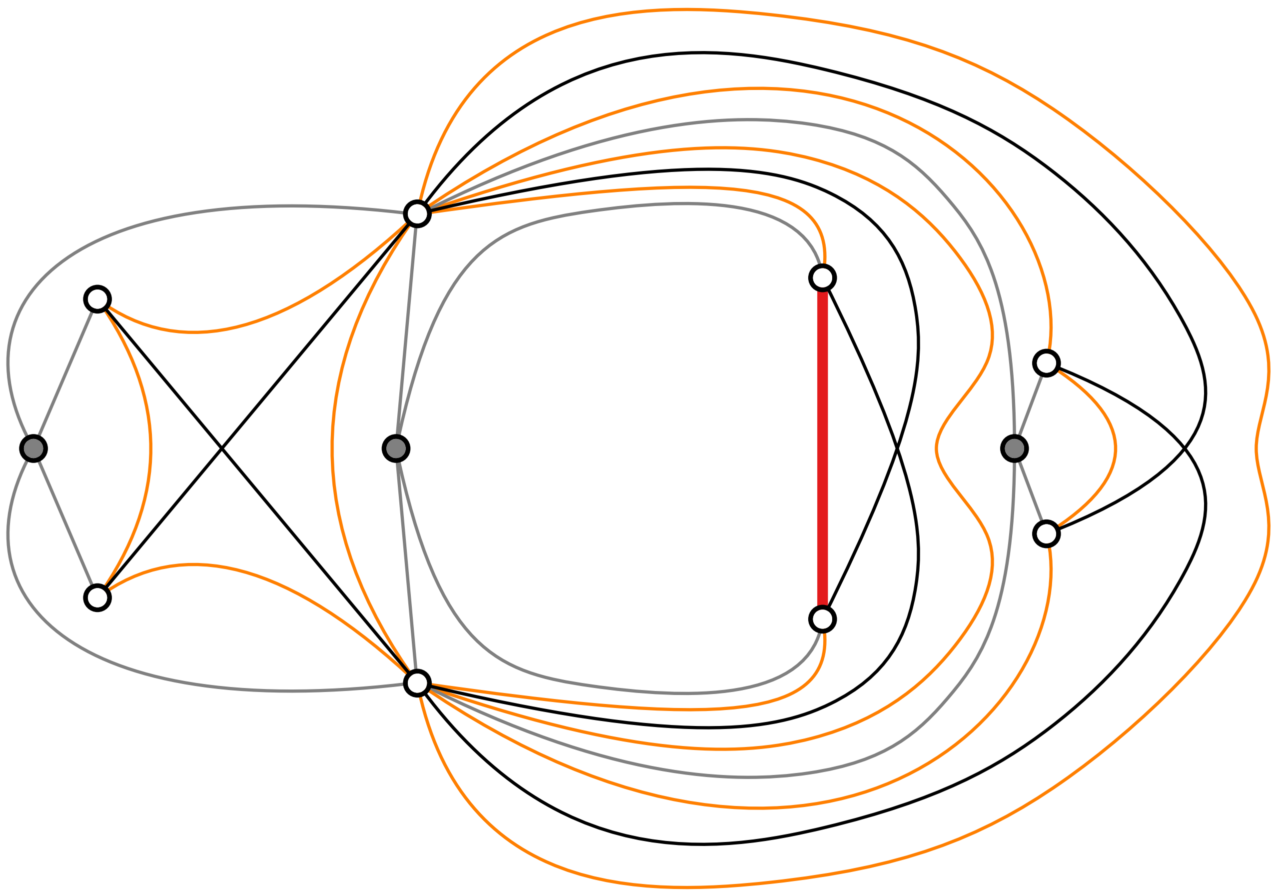

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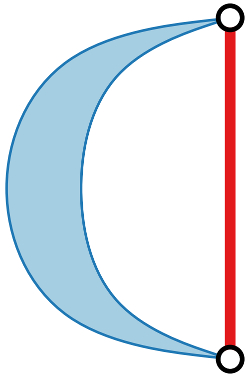

structure of each separation pair

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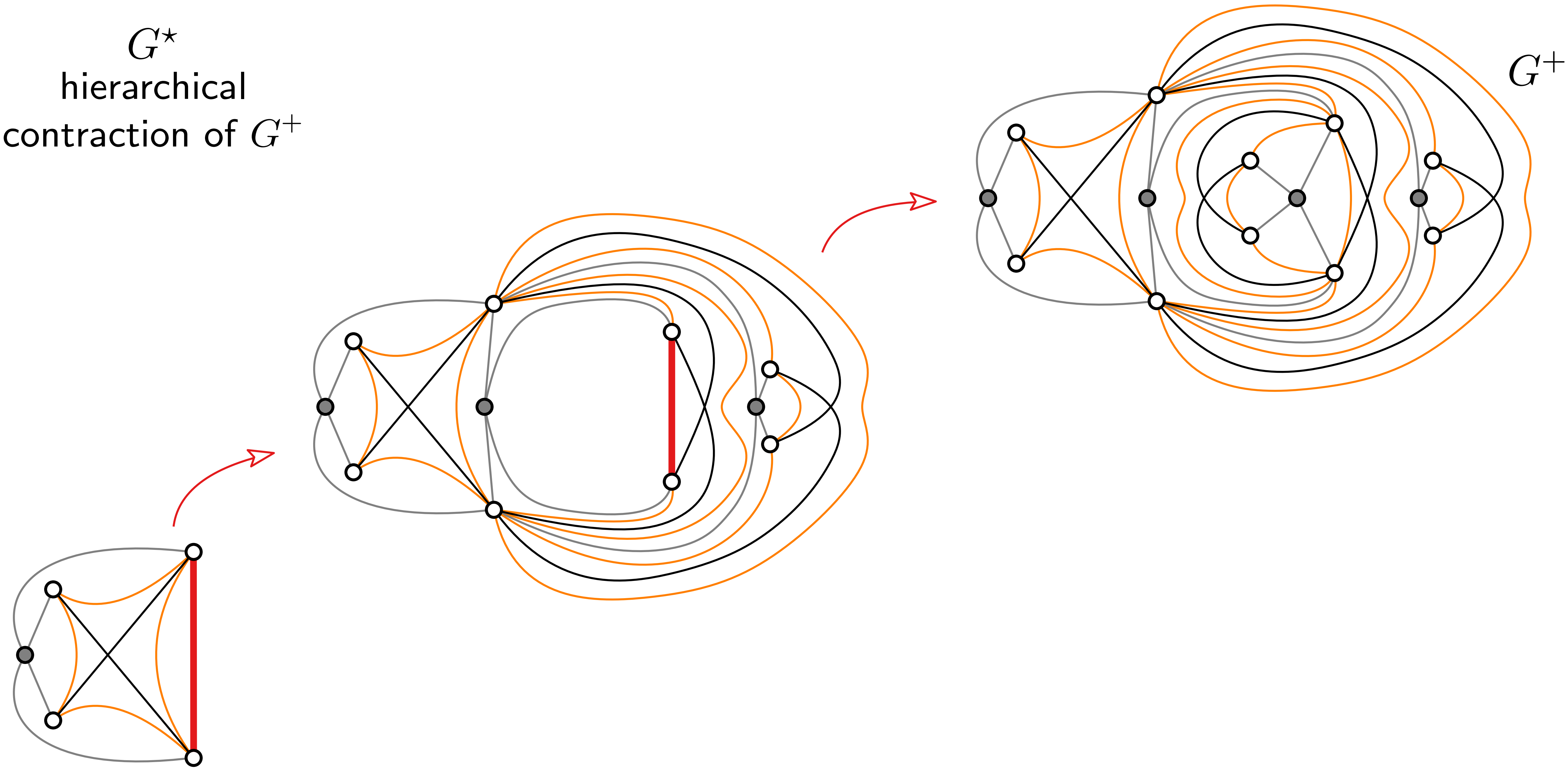
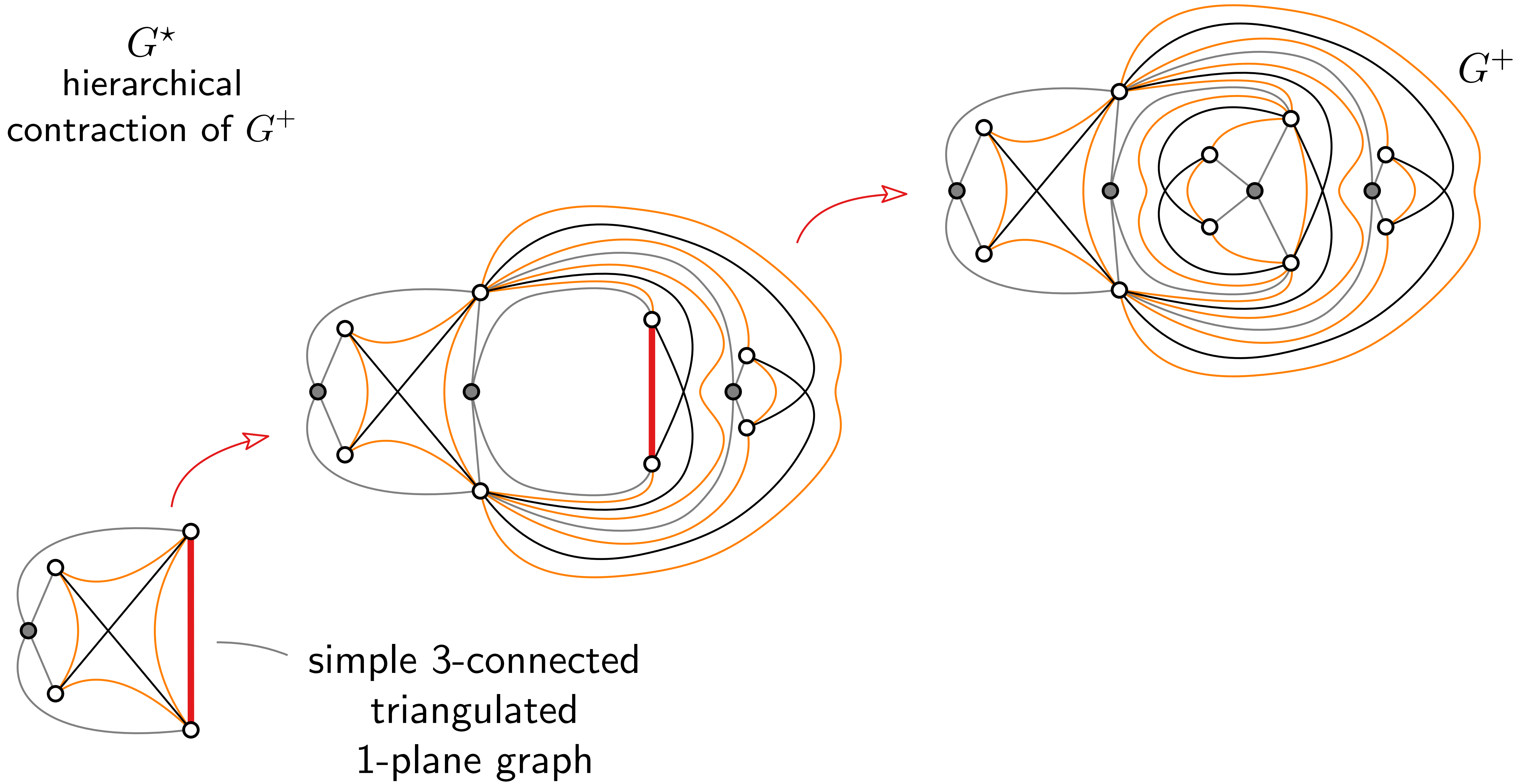
Algoritm Step 2: Hierarchical Contractions

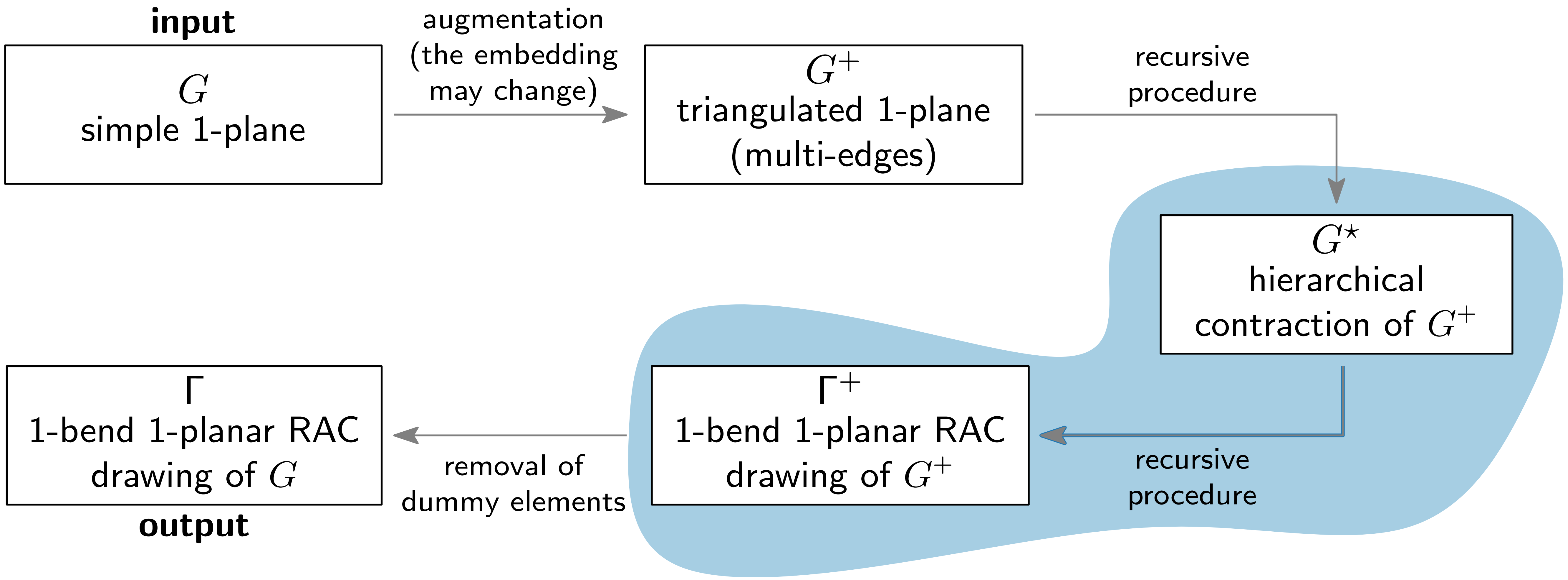
$G^\star$
hierarchical
contraction of G^+



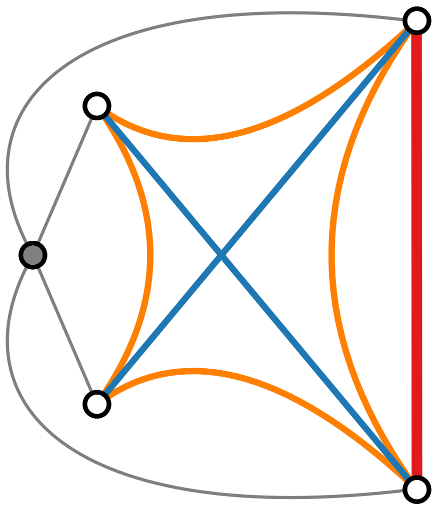
Algoritm Step 2: Hierarchical Contractions



Algorithm Outline



Algorithm Step 3: Drawing Procedure



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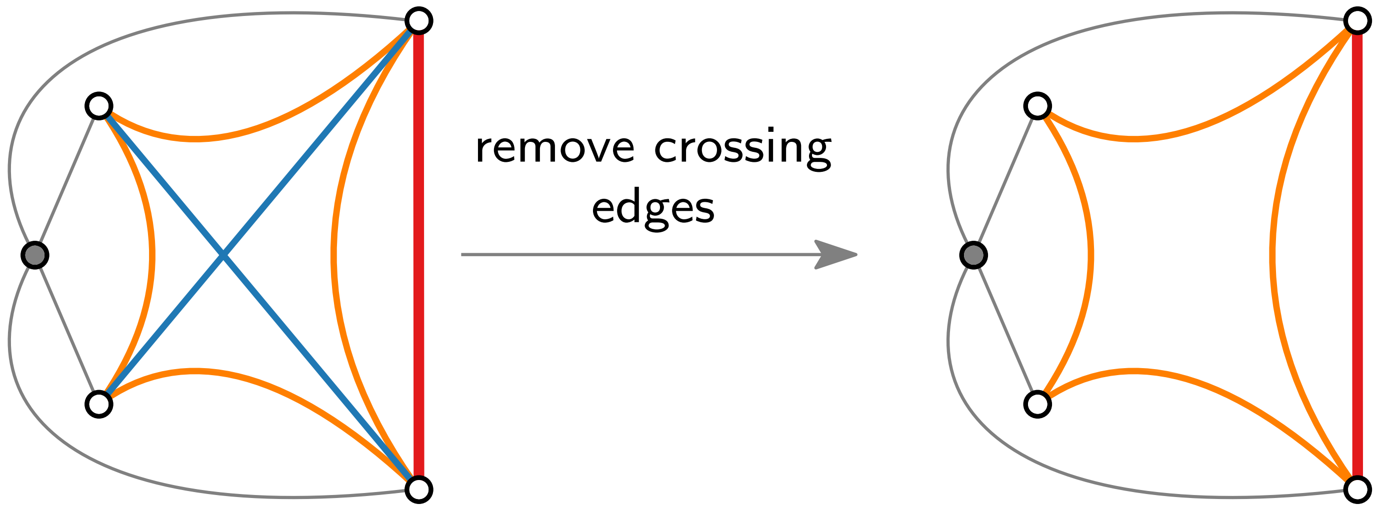

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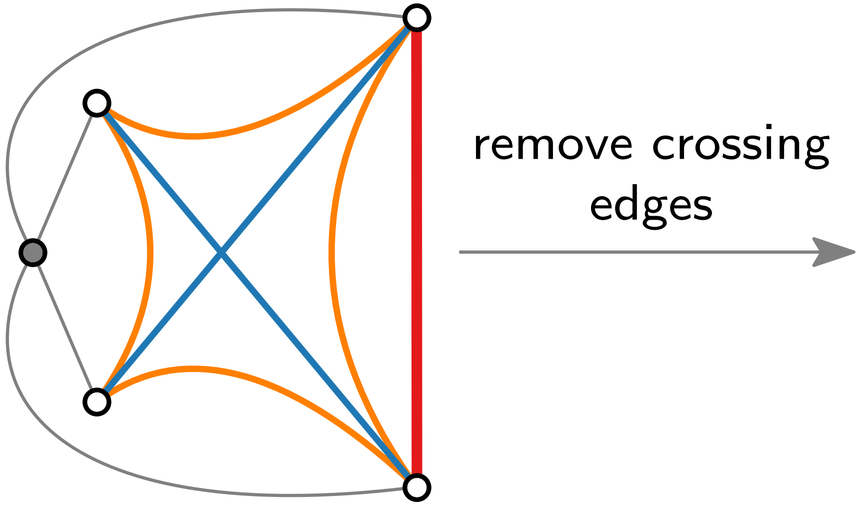
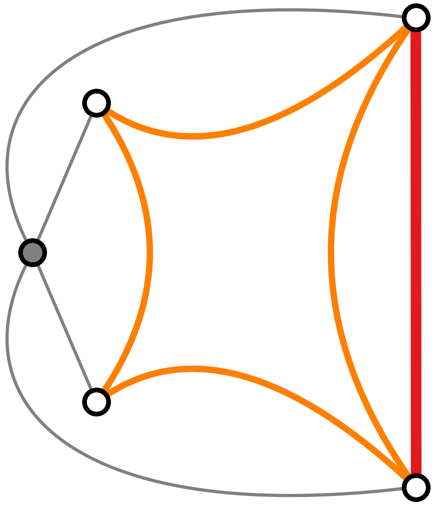
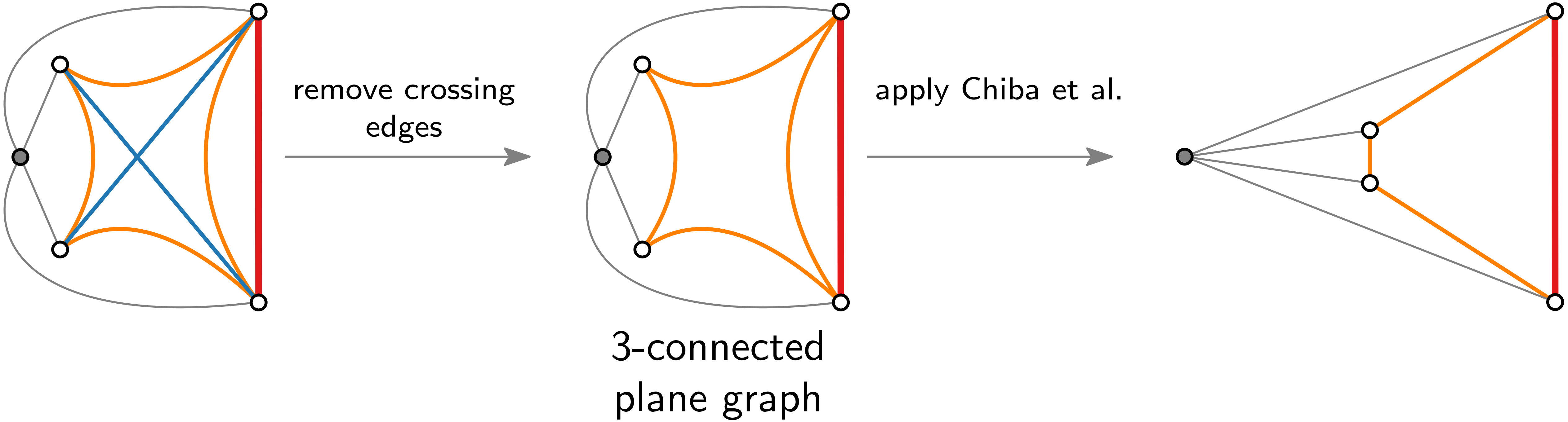


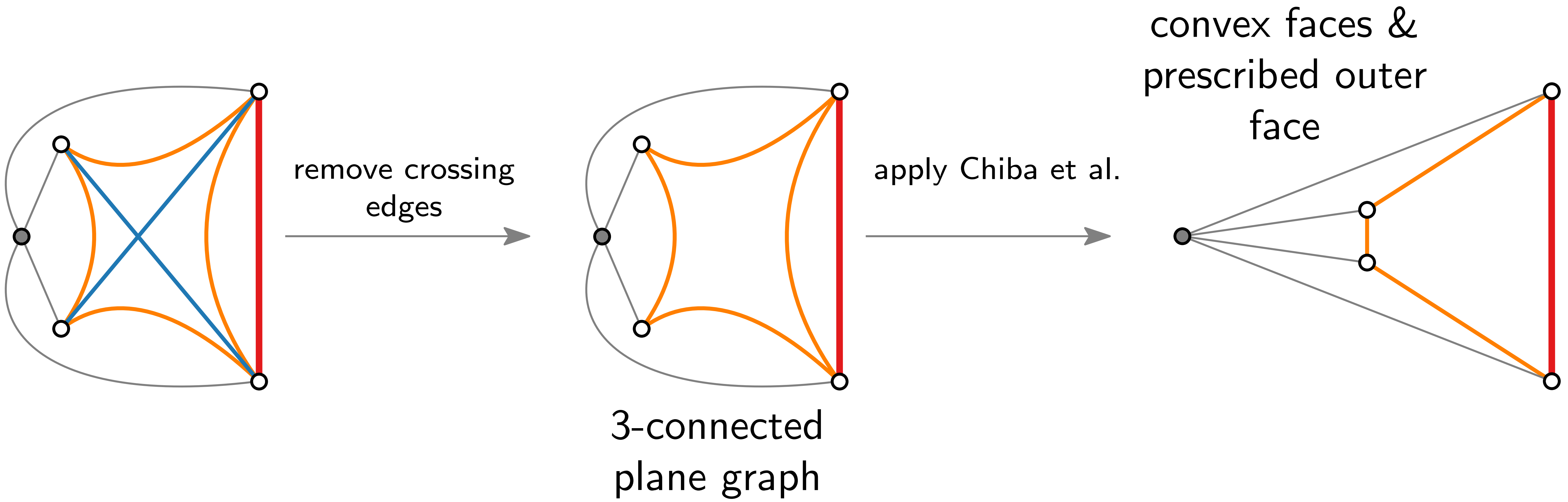


3-connected
plane graph

Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure



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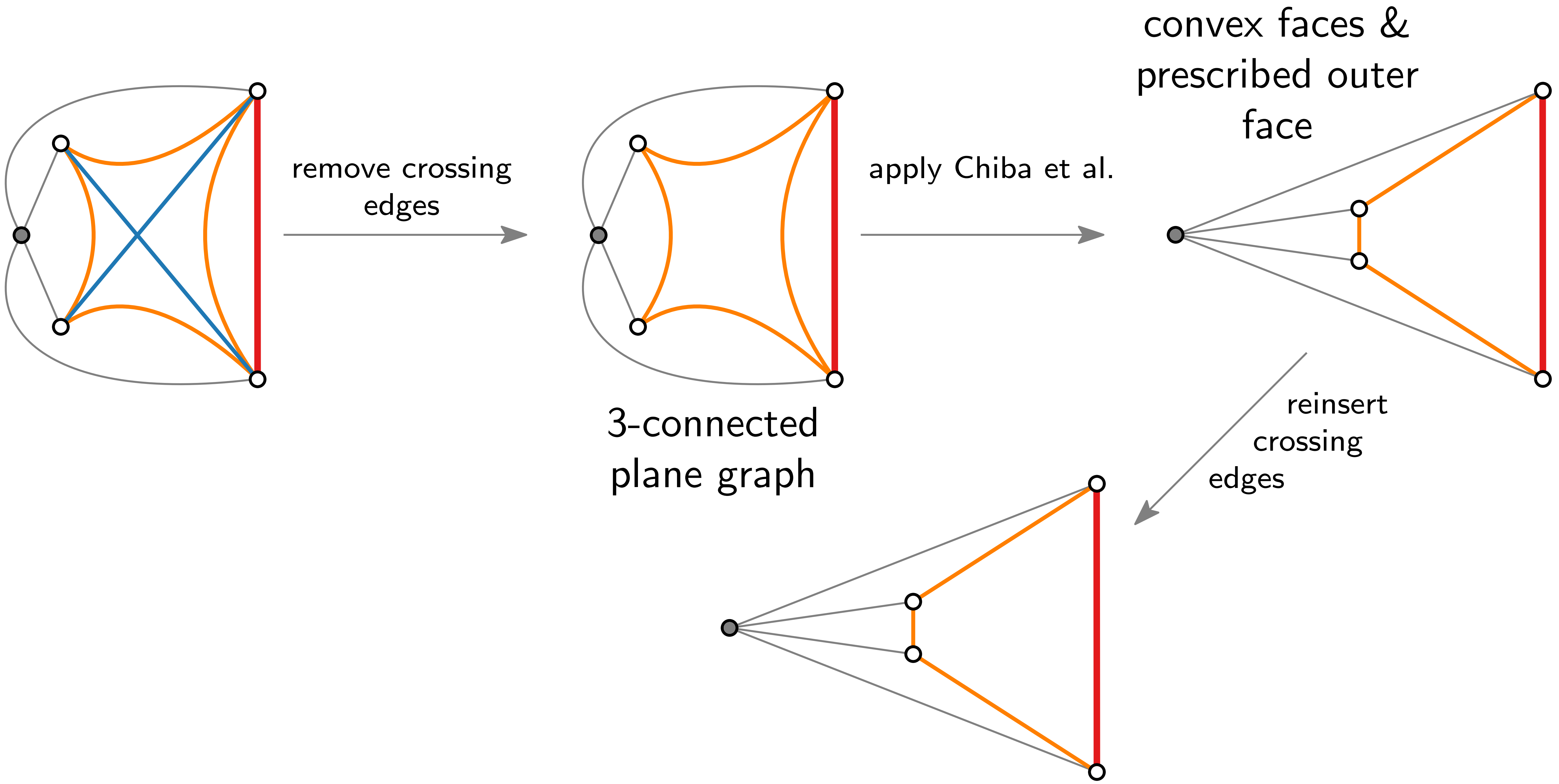
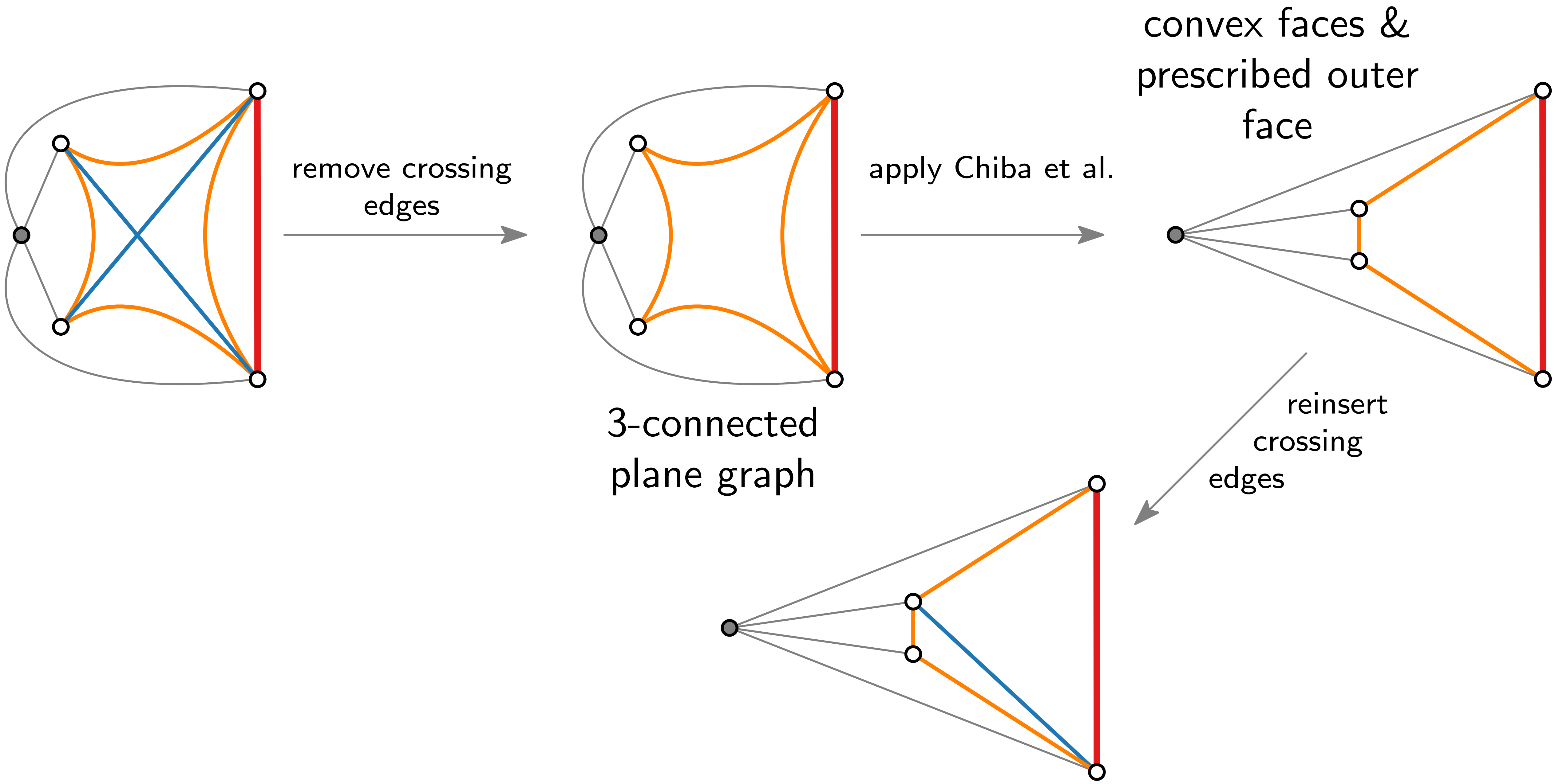
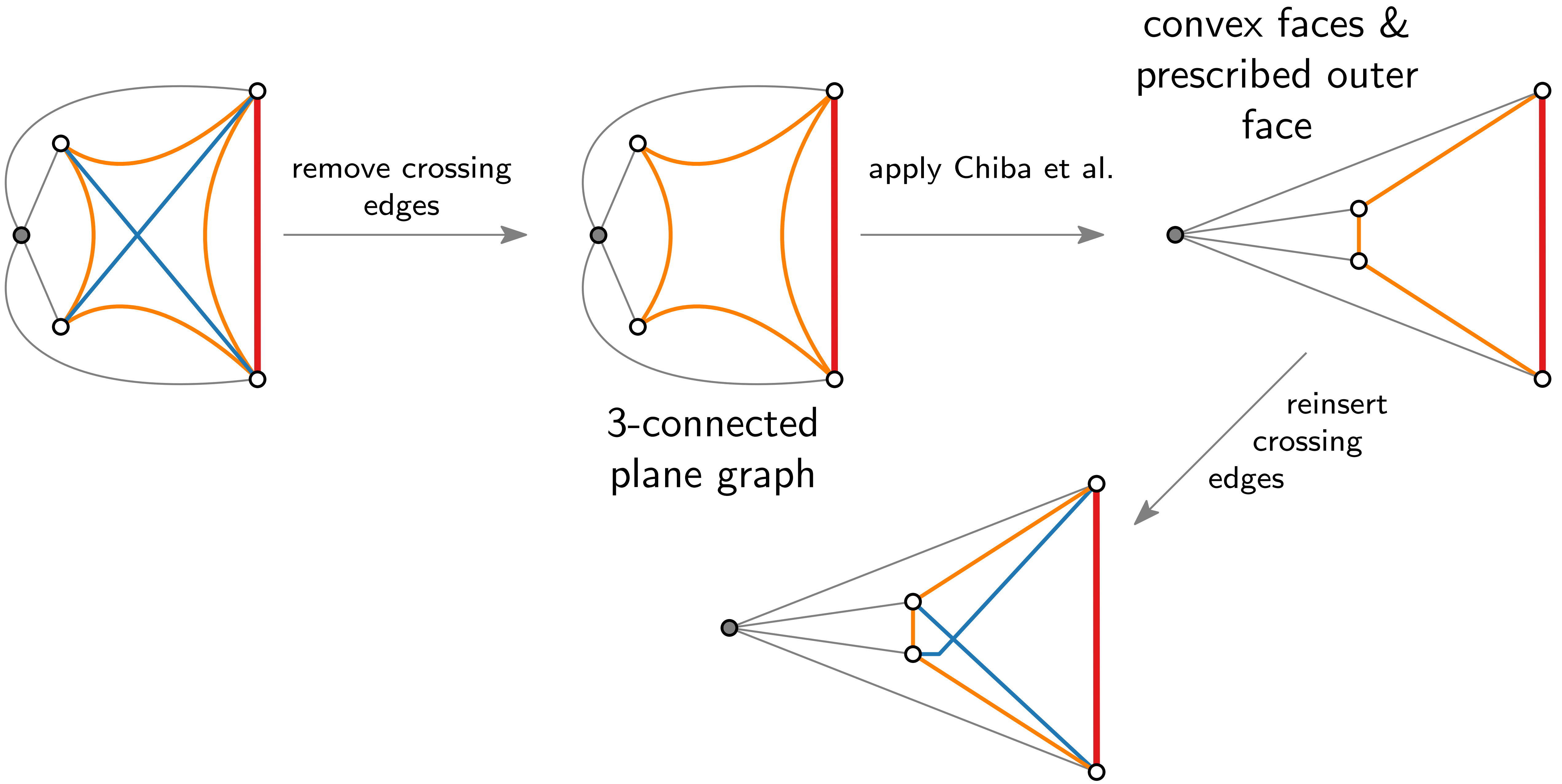


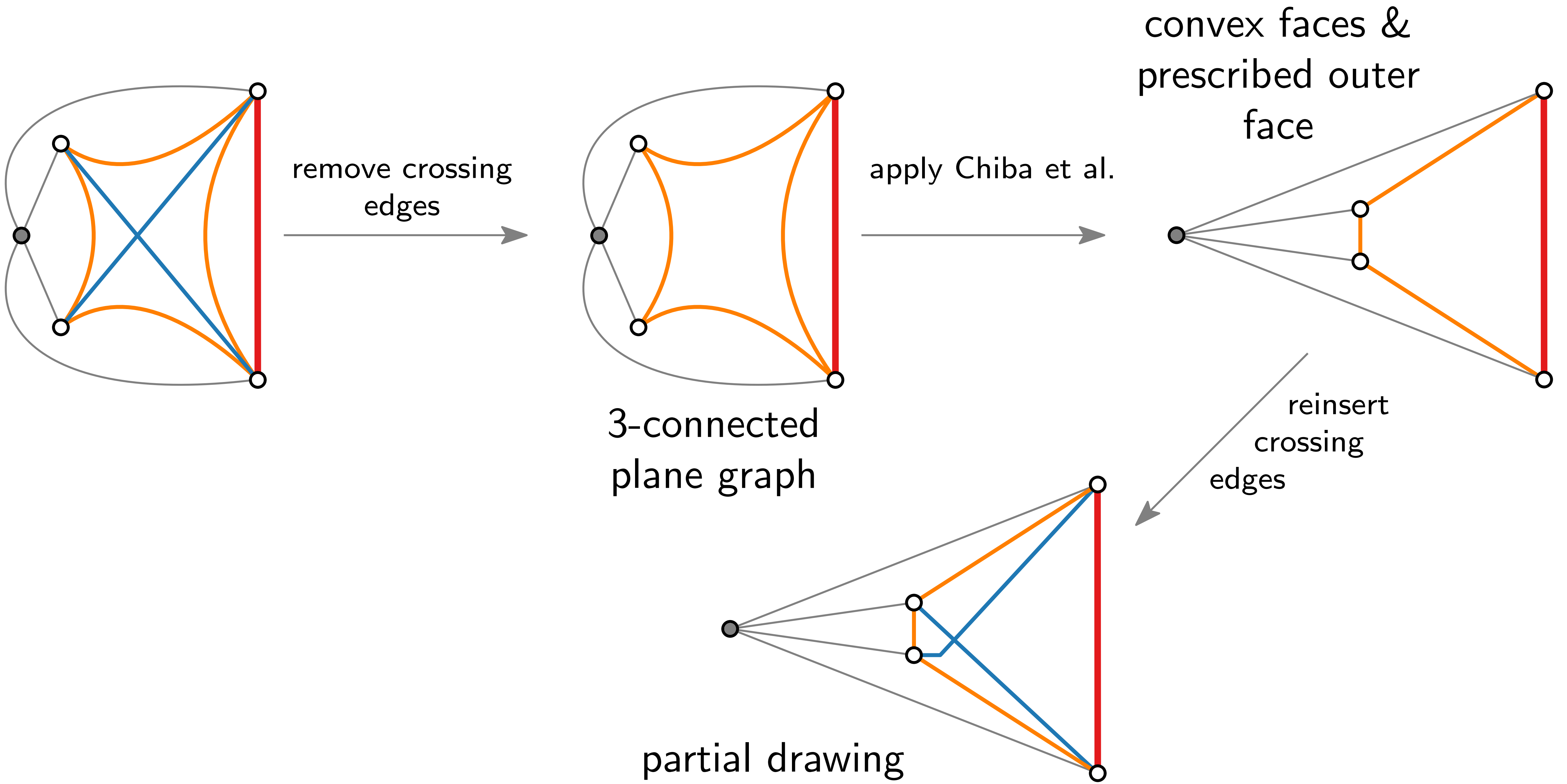
Algorithm Step 3: Drawing Procedure



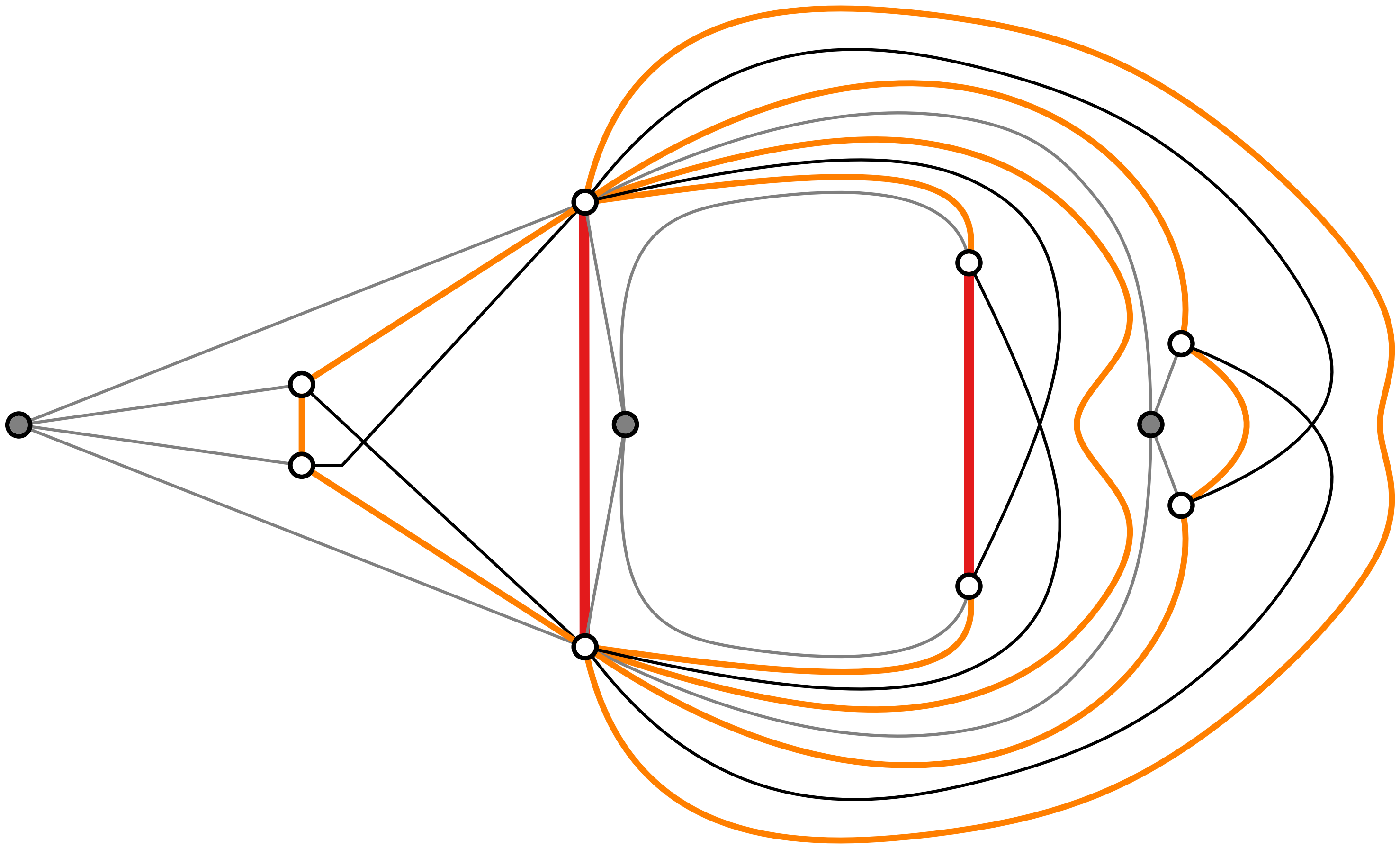
Algorithm Step 3: Drawing Procedure



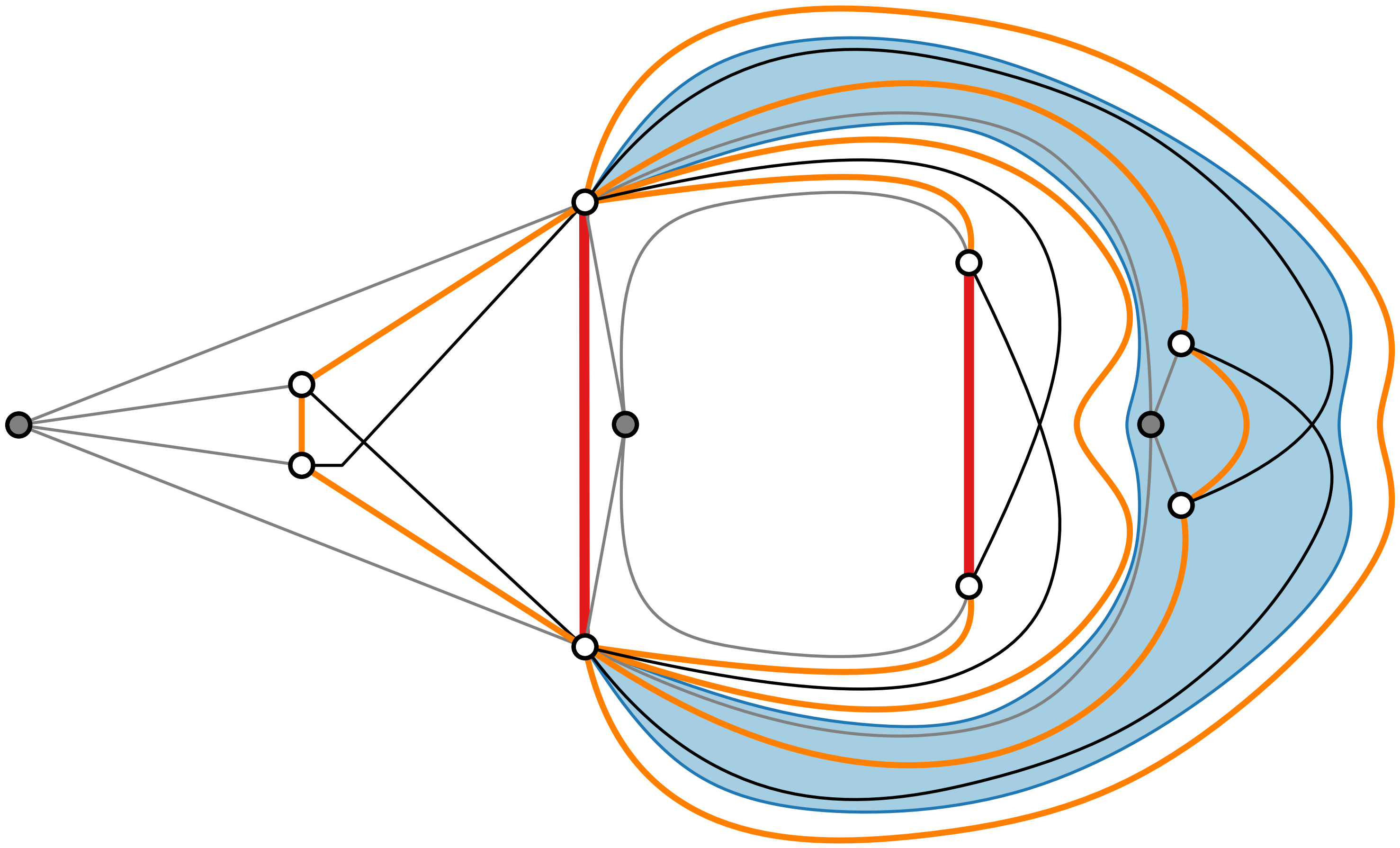
Algorithm Step 3: Drawing Procedure



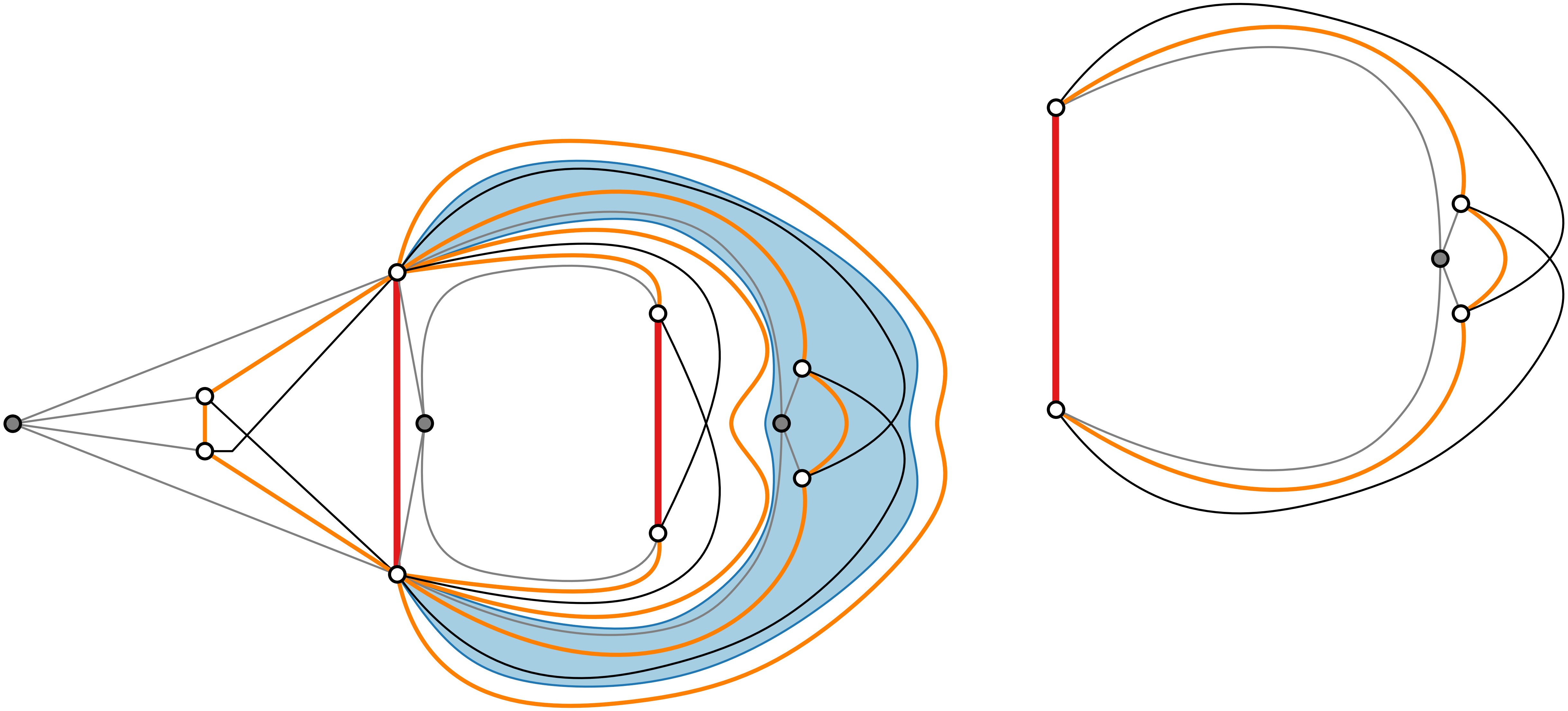
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure



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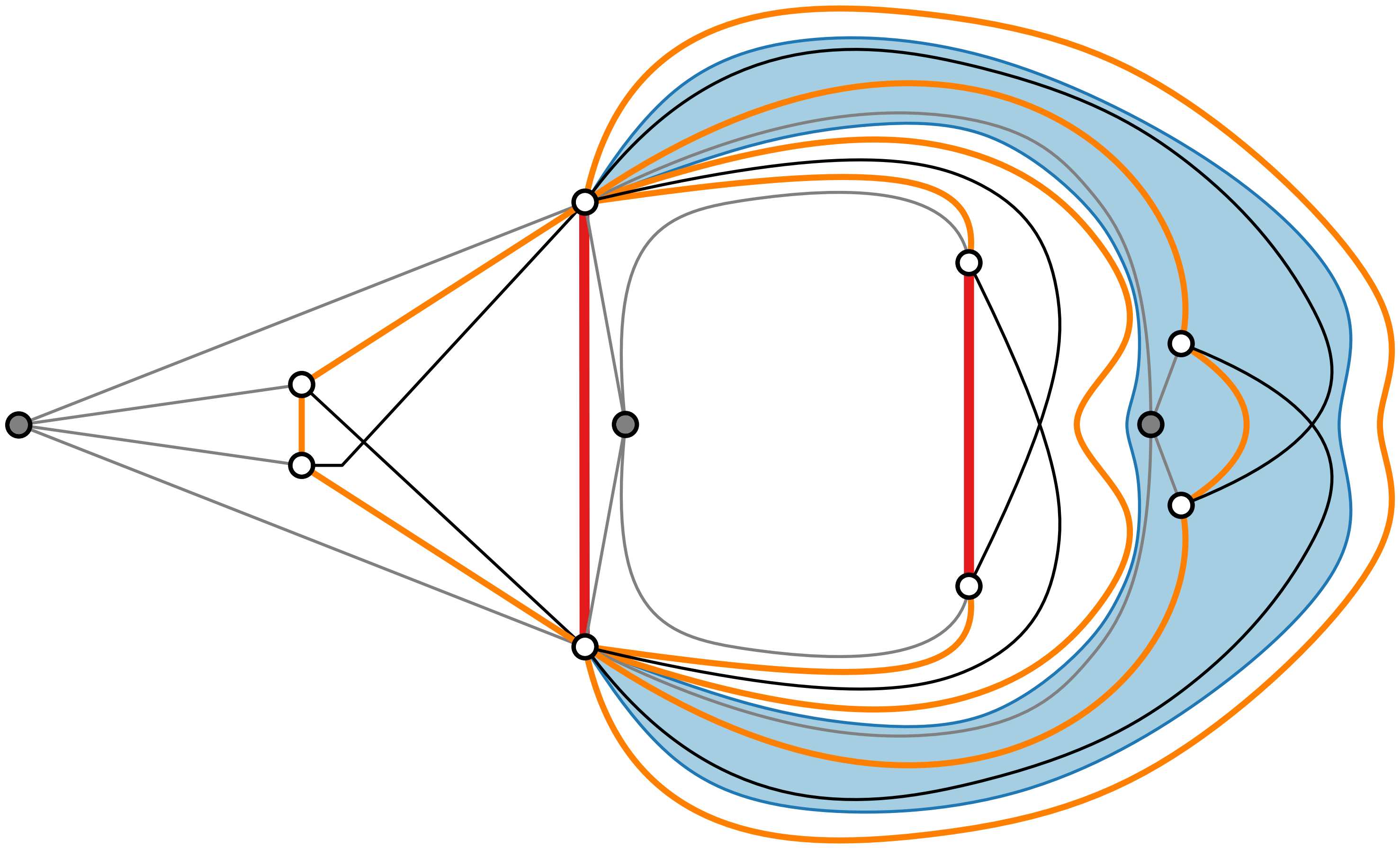
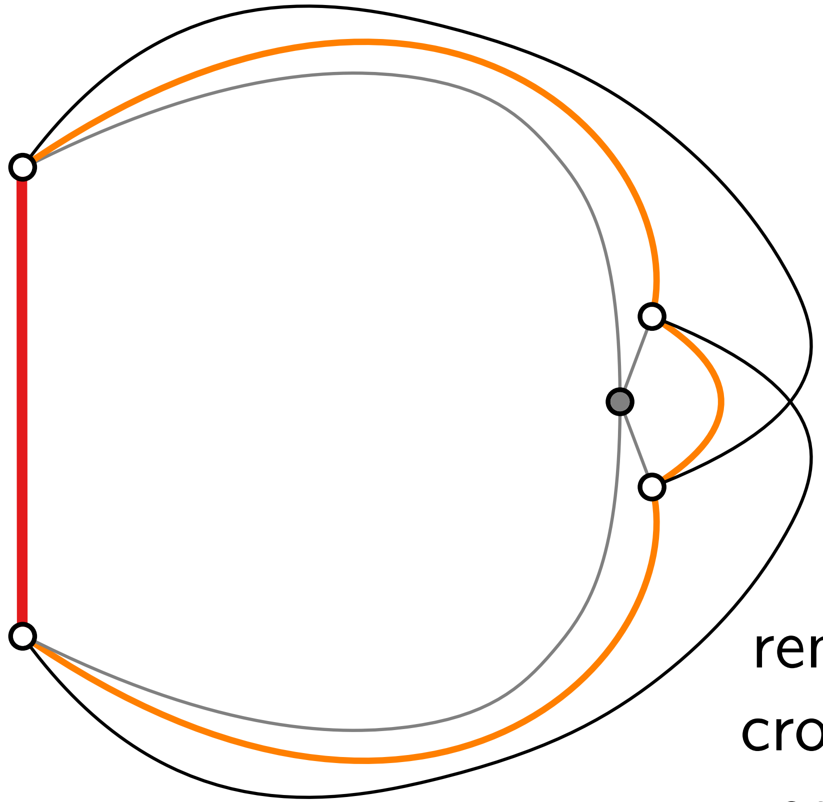
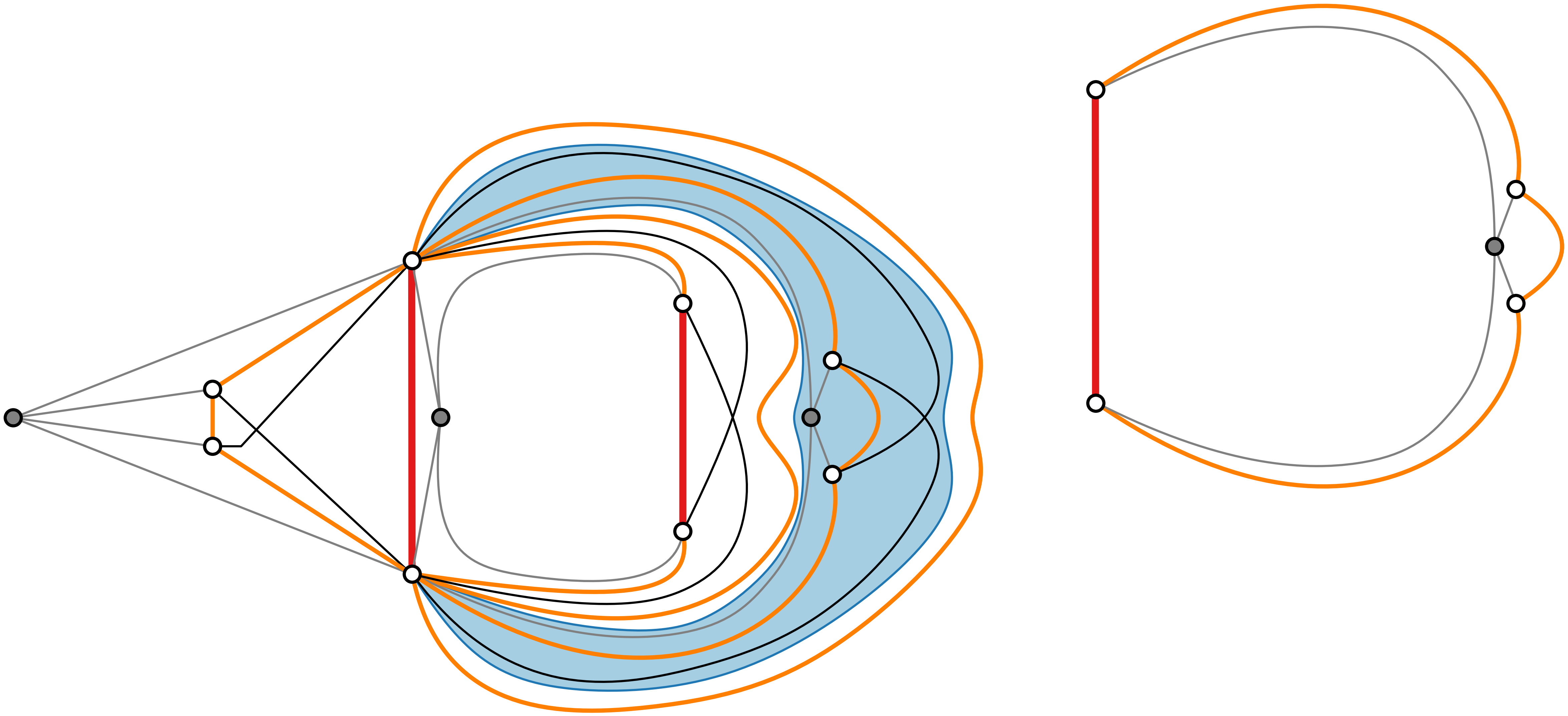




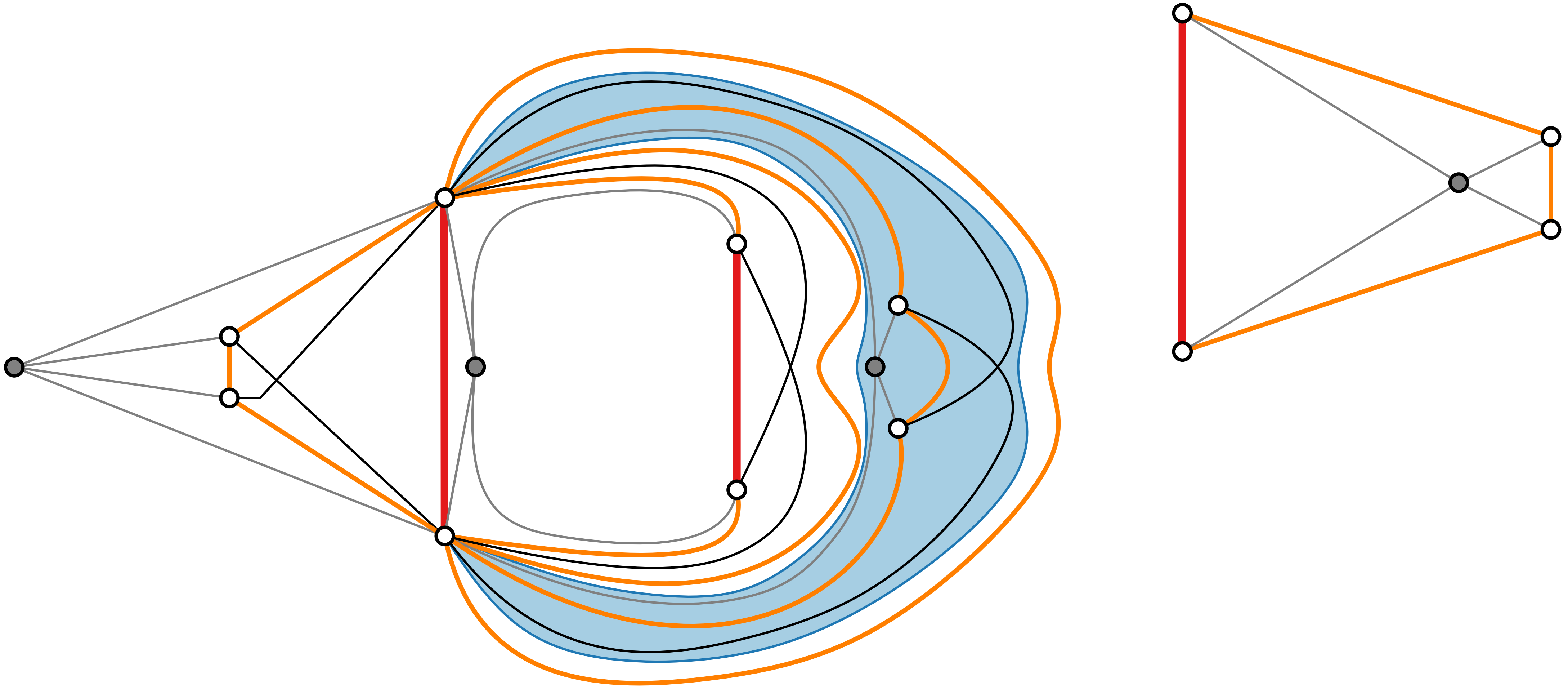
remove crossing edges

Algorithm Step 3: Drawing Procedure



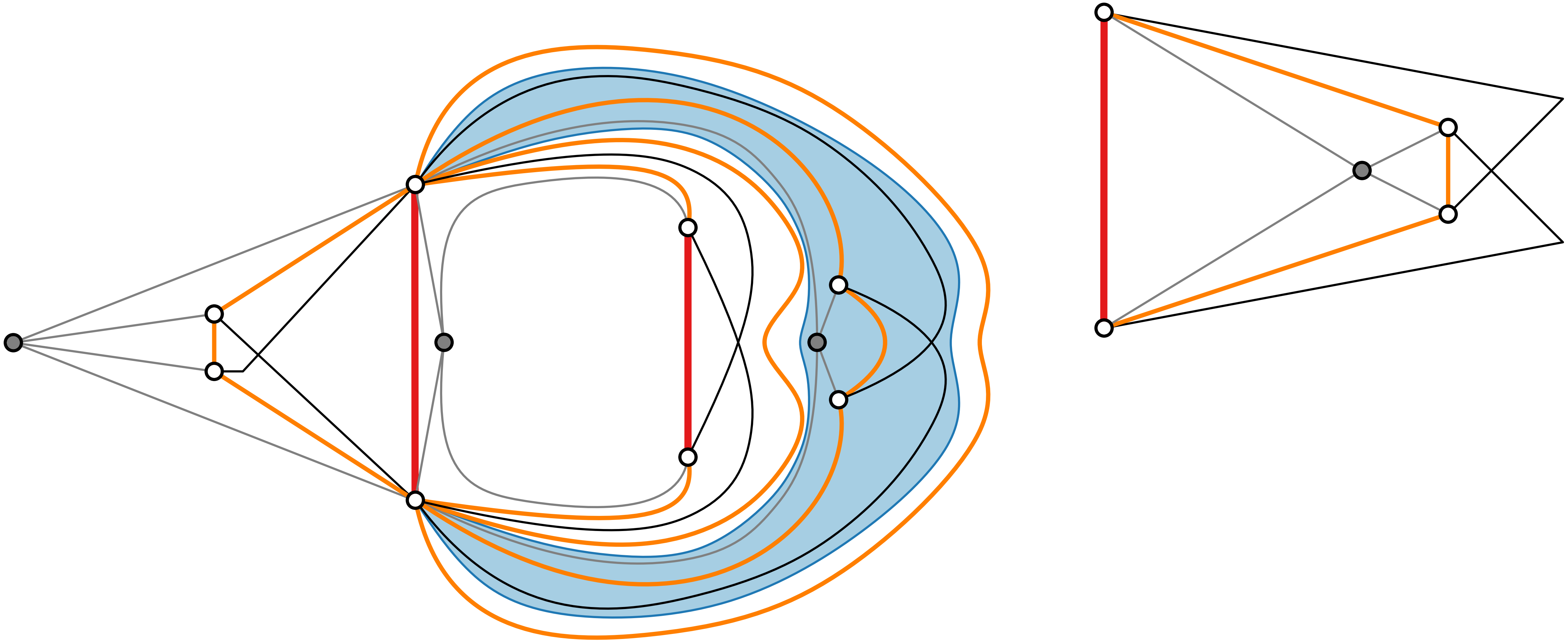
remove crossing edges

Algorithm Step 3: Drawing Procedure



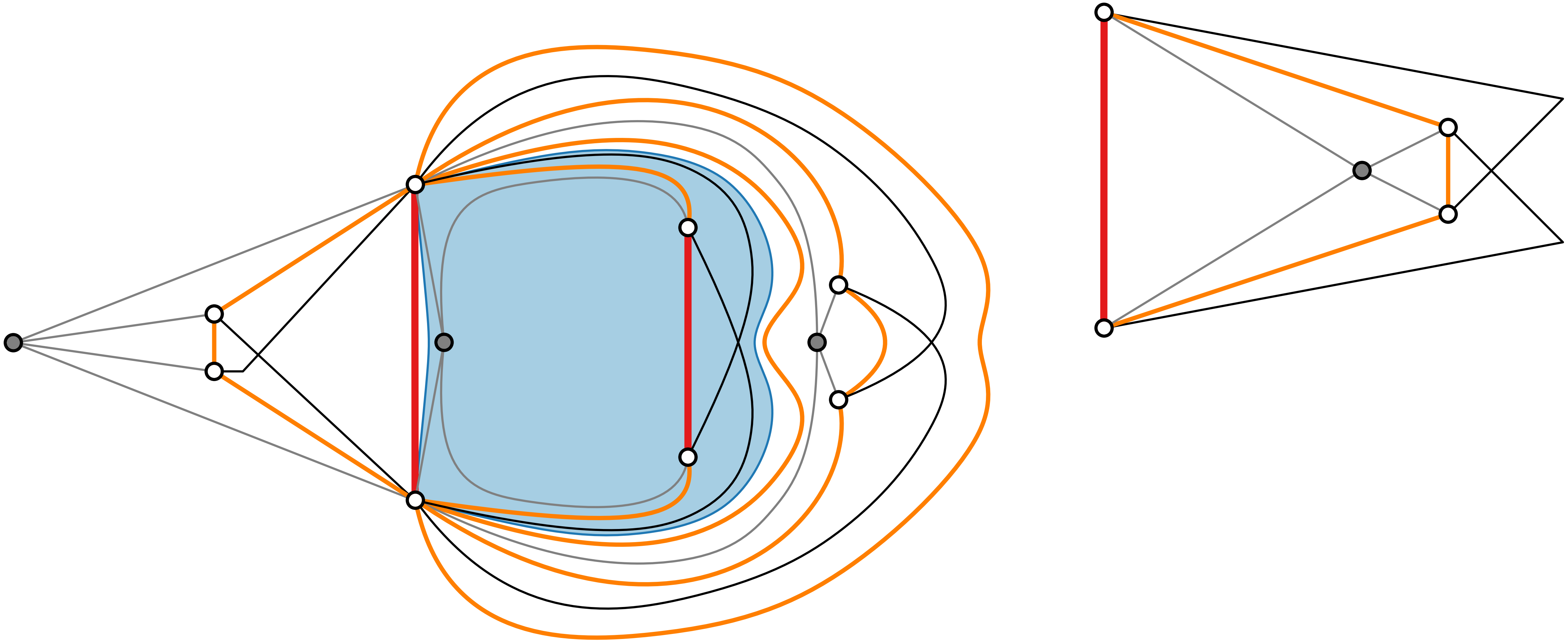
apply Chiba et al.

Algorithm Step 3: Drawing Procedure

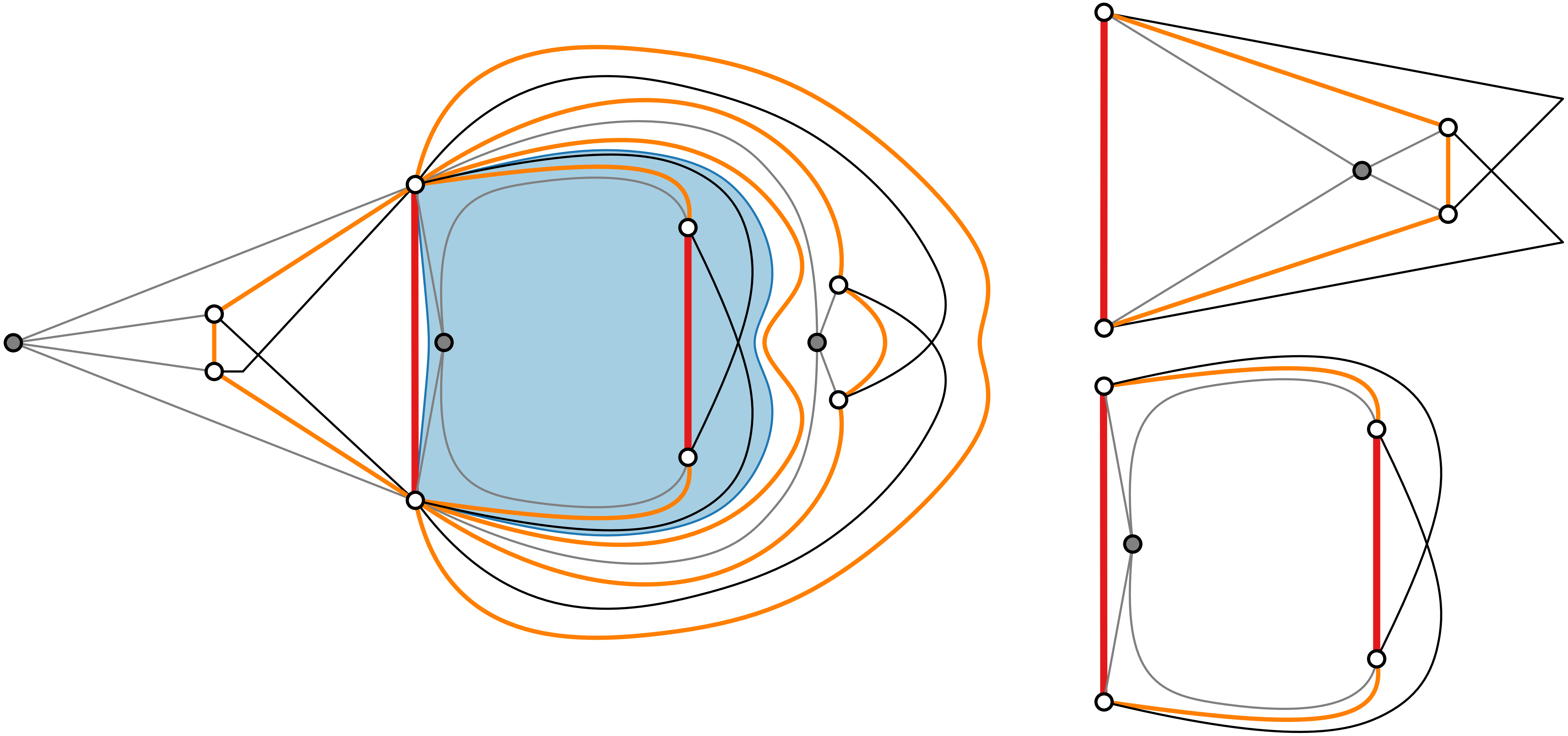


reinsert
crossing
edges

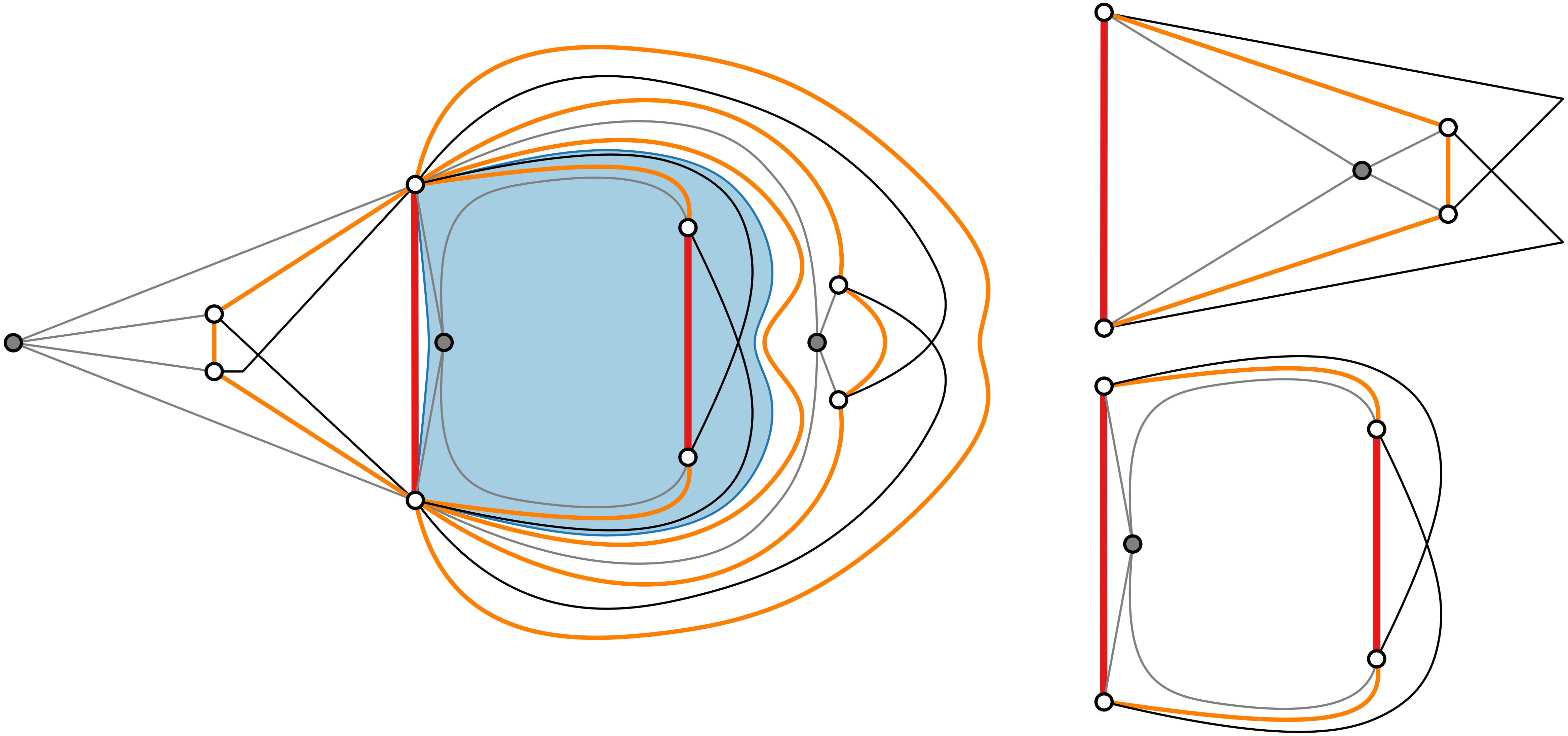
Algorithm Step 3: Drawing Procedure



Algorithm Step 3: Drawing Procedure

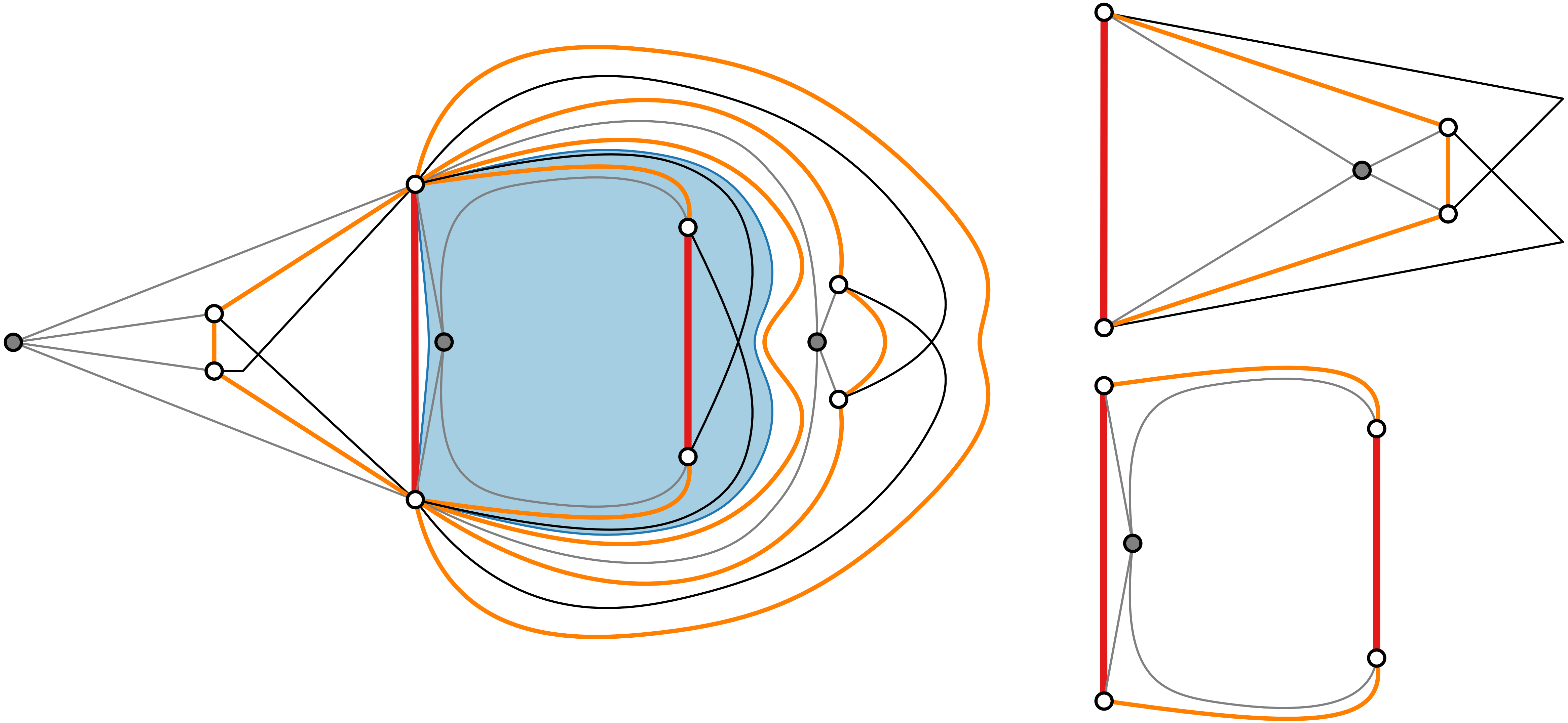


Algorithm Step 3: Drawing Procedure



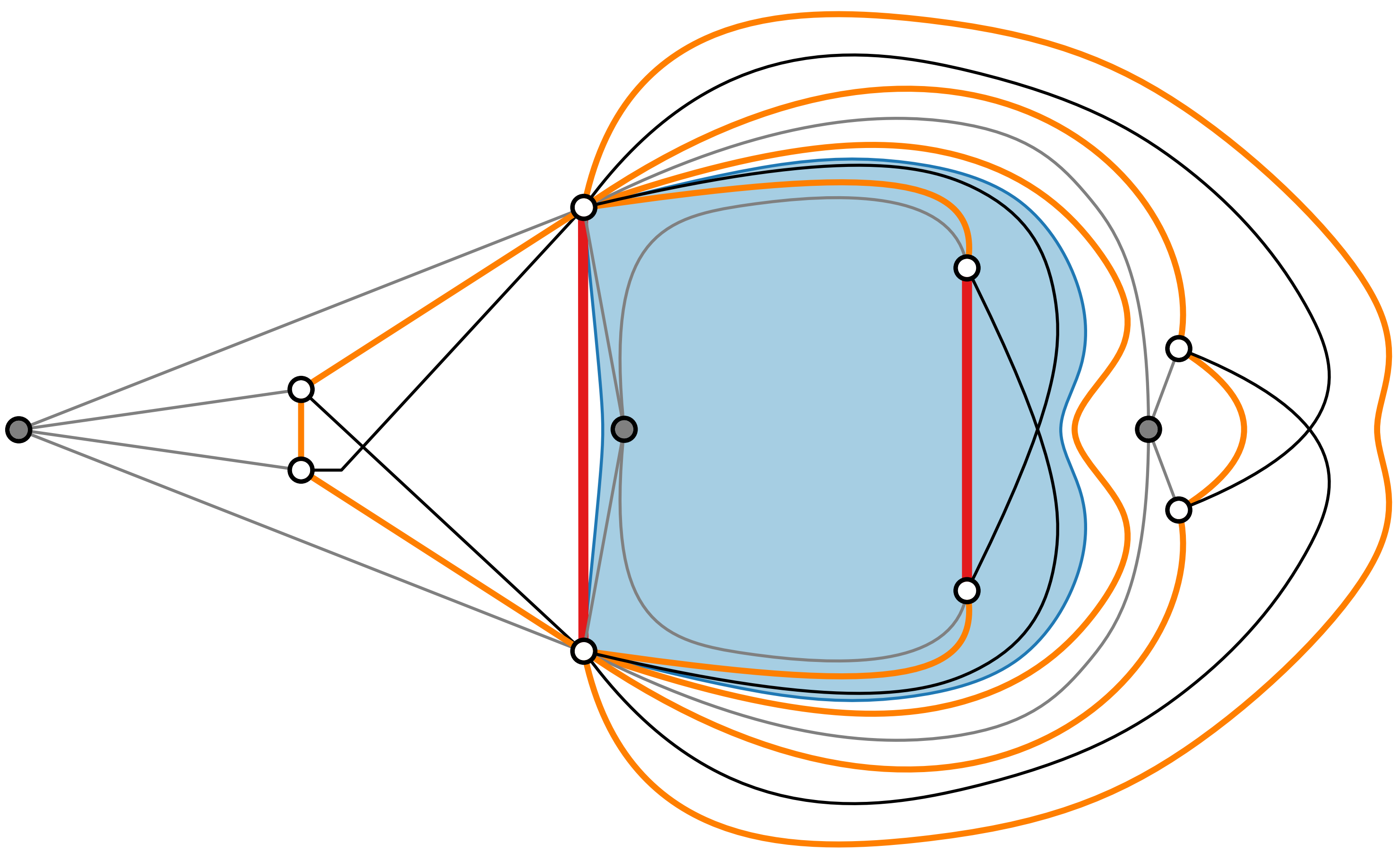
remove crossing edges

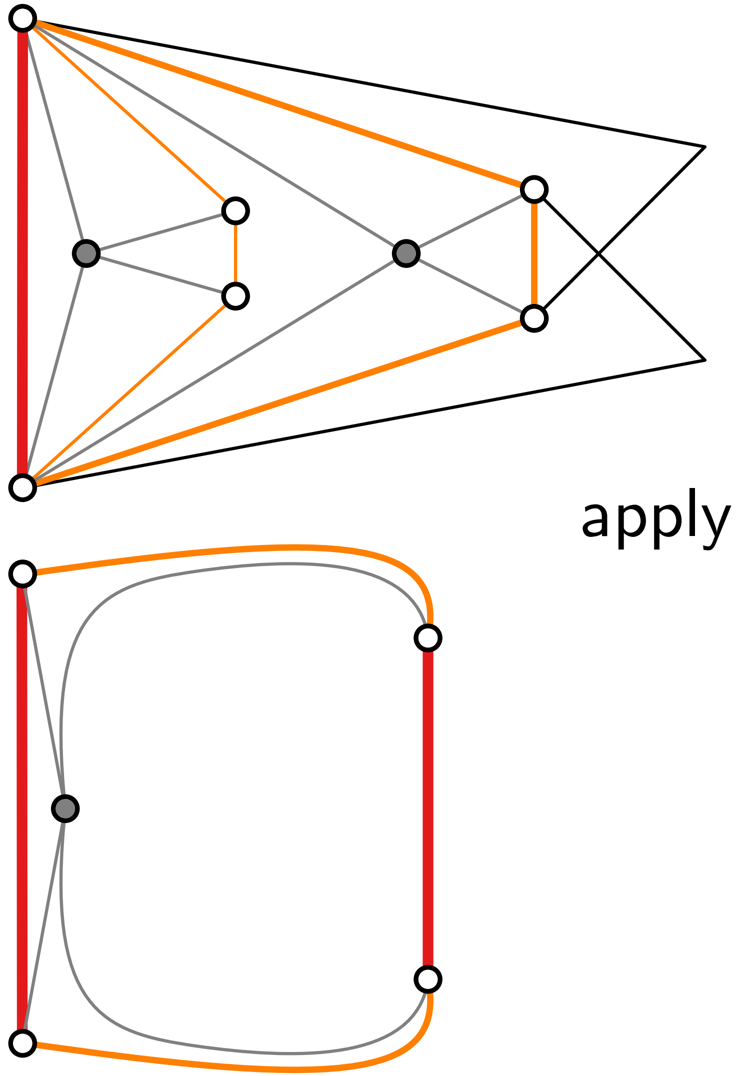
Algorithm Step 3: Drawing Procedure



remove crossing edges

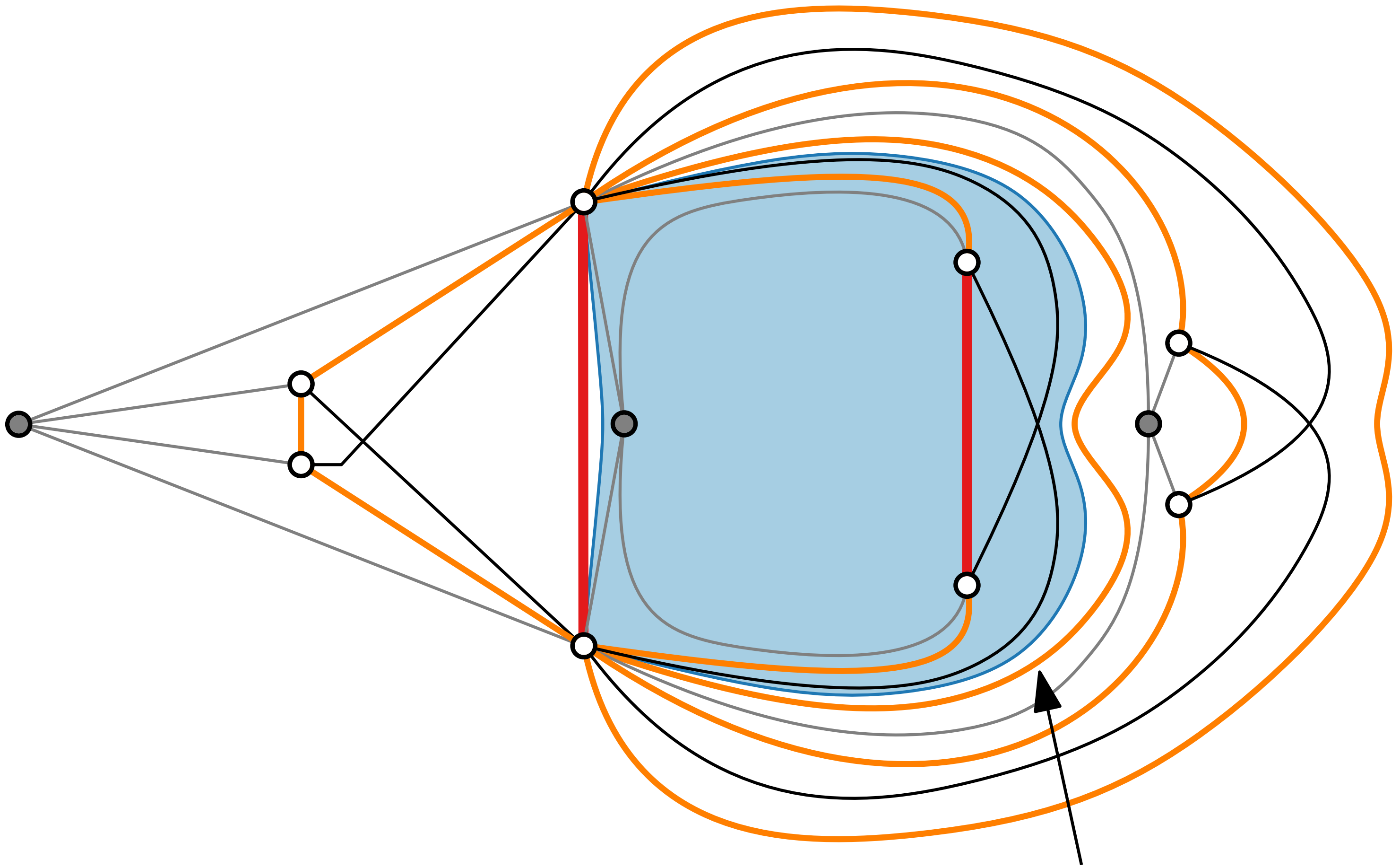
Algorithm Step 3: Drawing Procedure



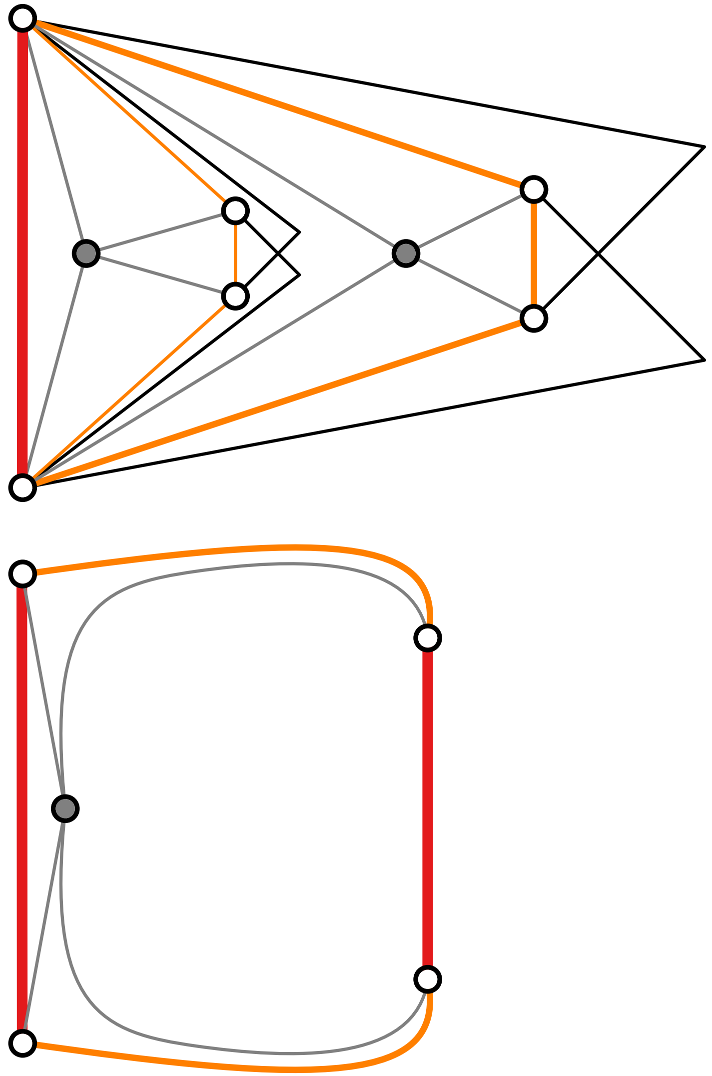


apply Chiba et al.

Algorithm Step 3: Drawing Procedure

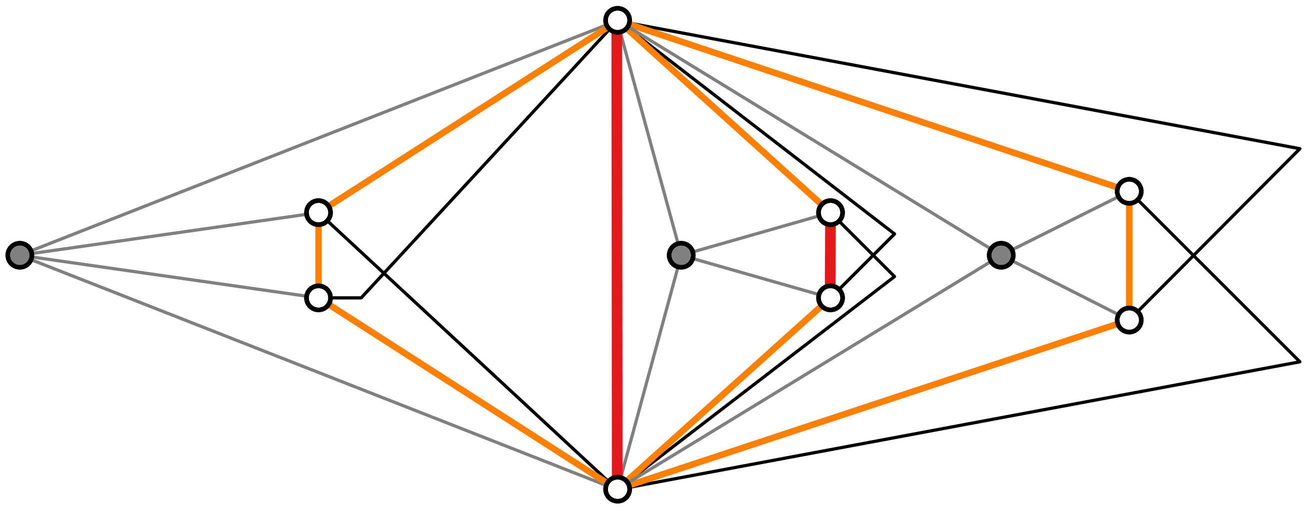


This kite edge, which is a multi-edge, is not drawn.

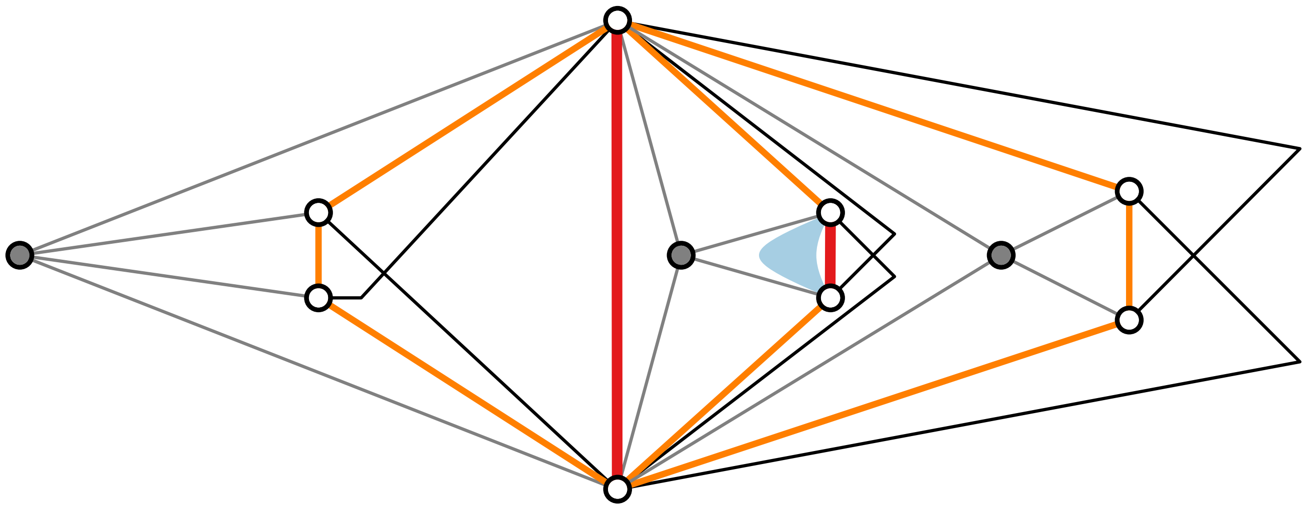


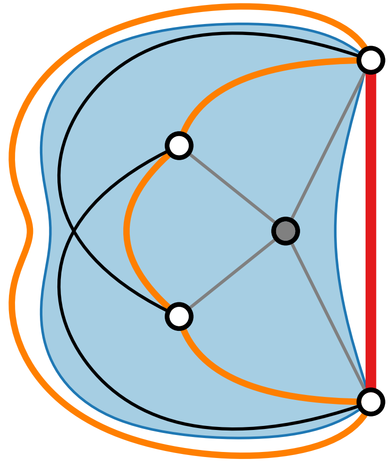
reinsert crossing edges

Algorithm Step 3: Drawing Procedure

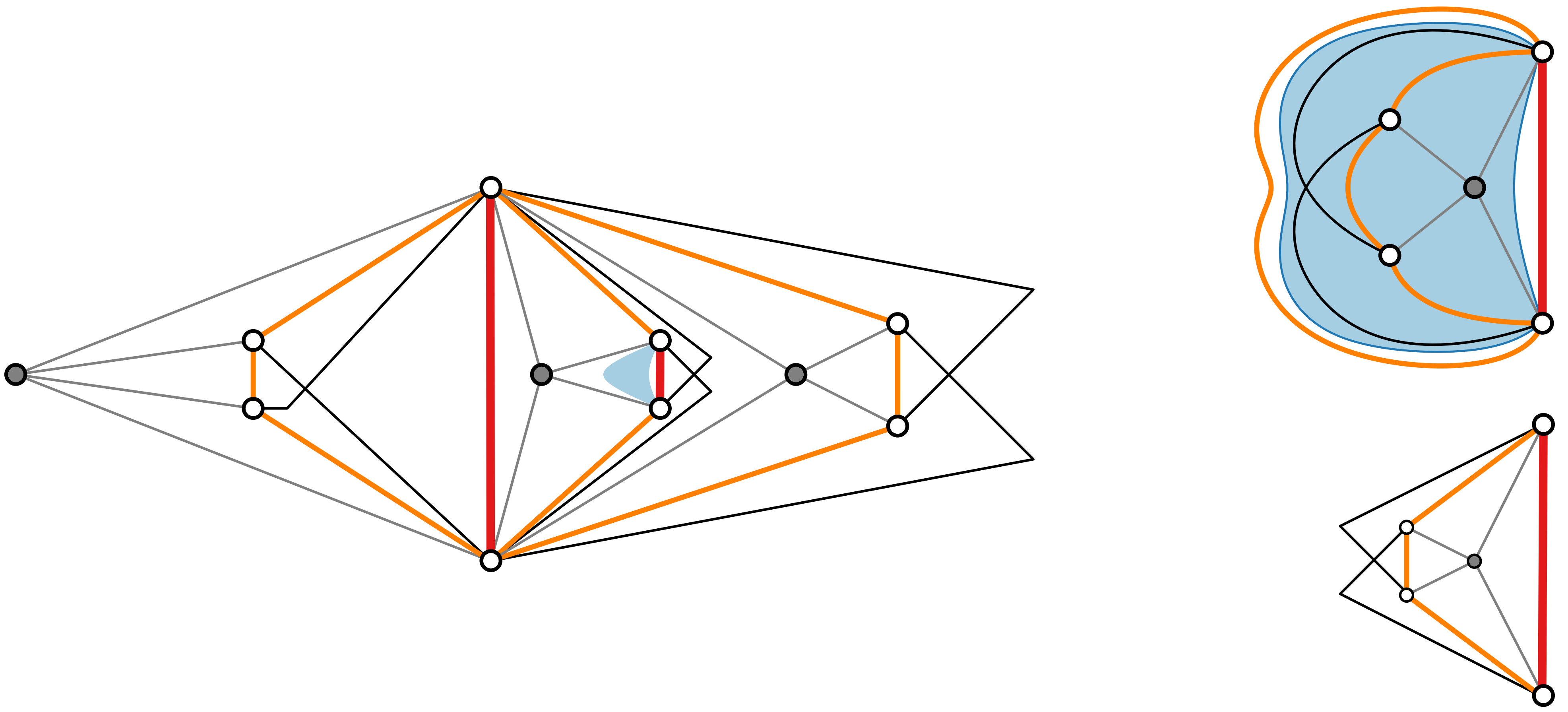


Algorithm Step 3: Drawing Procedure

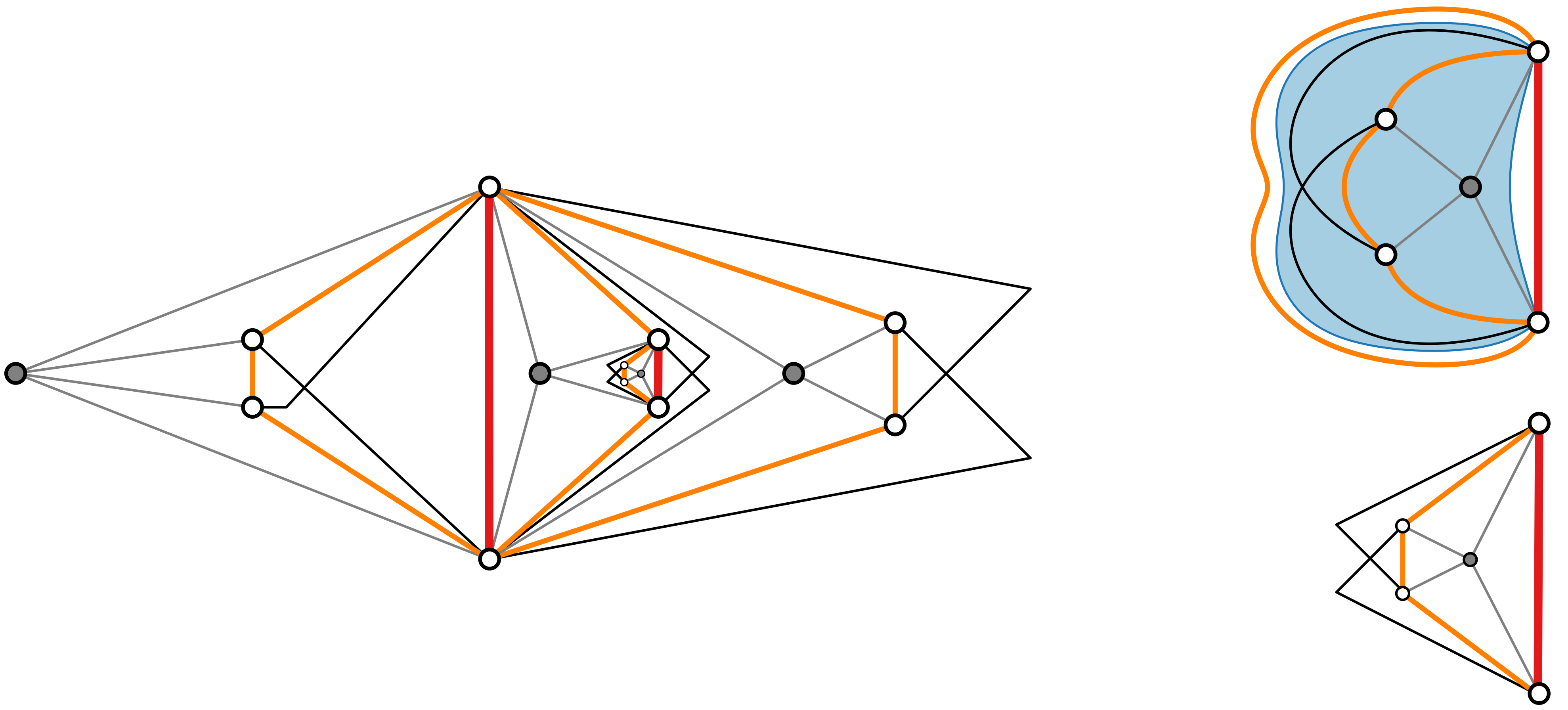




Algorithm Step 3: Drawing Procedure

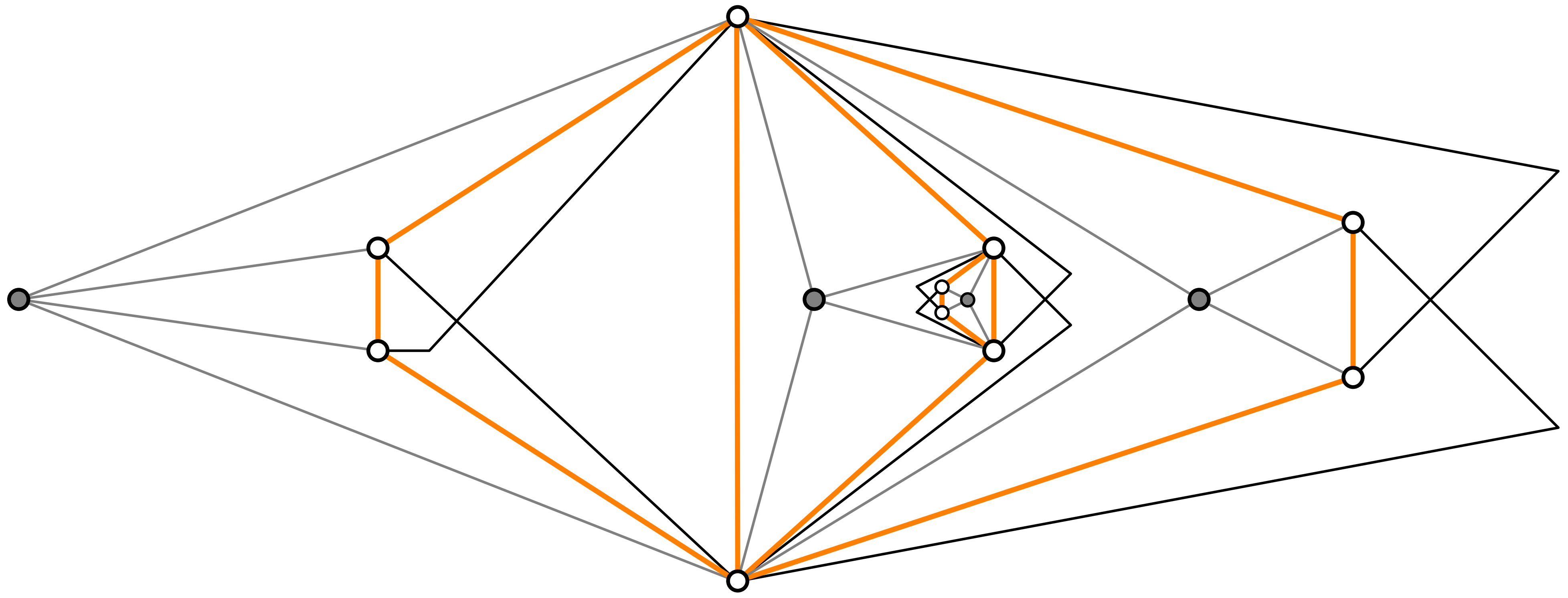


Algorithm Step 3: Drawing Procedure

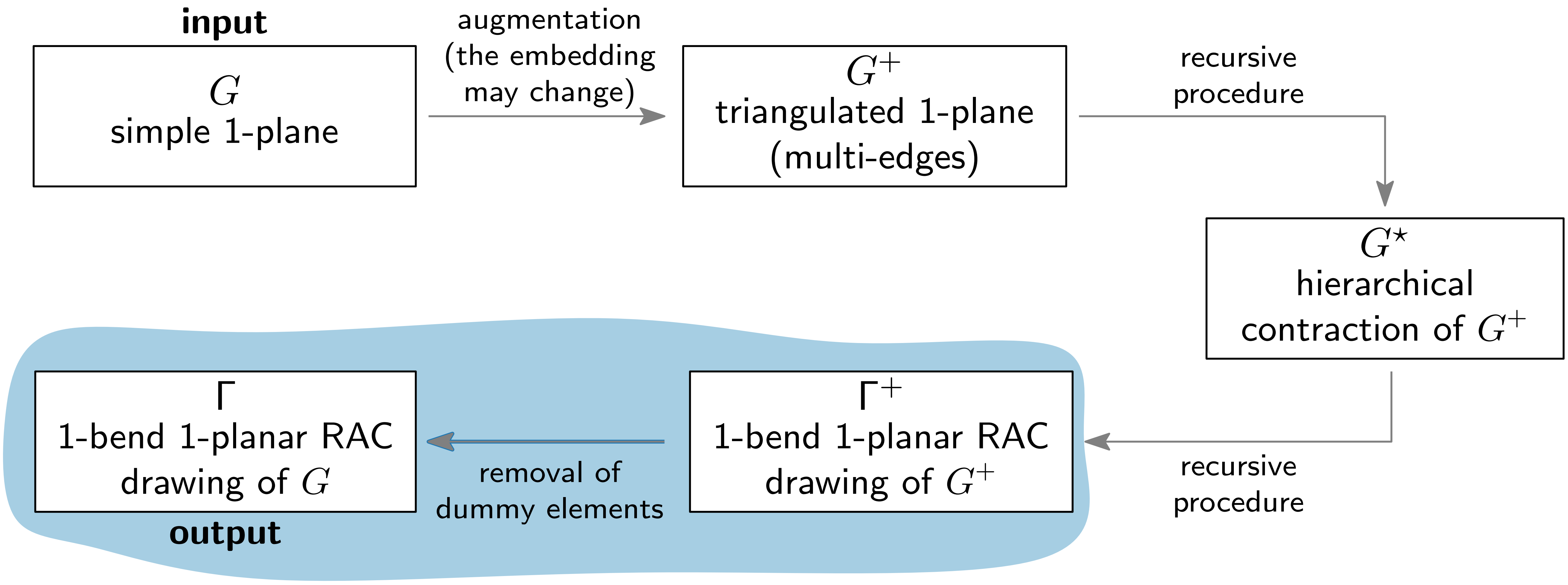


Algorithm Step 3: Drawing Procedure

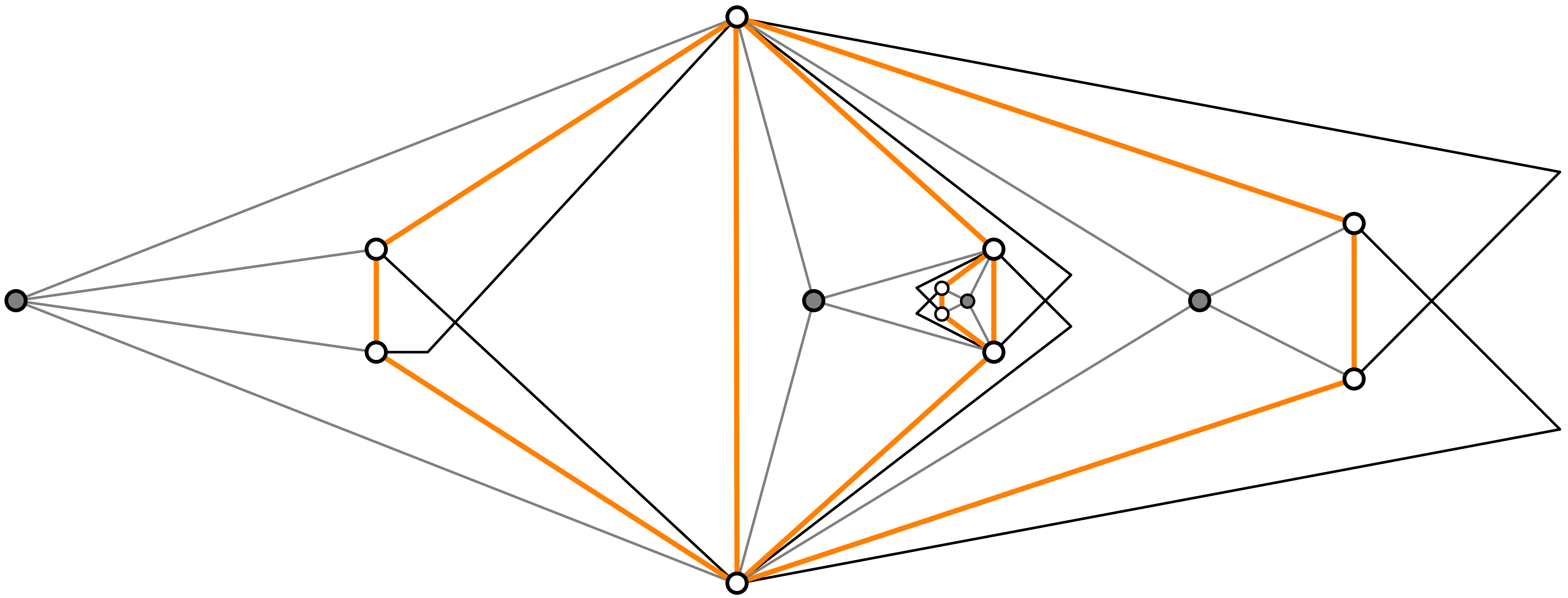
Γ^+: 1-bend 1-planar RAC drawing of G^+



Algorithm Outline

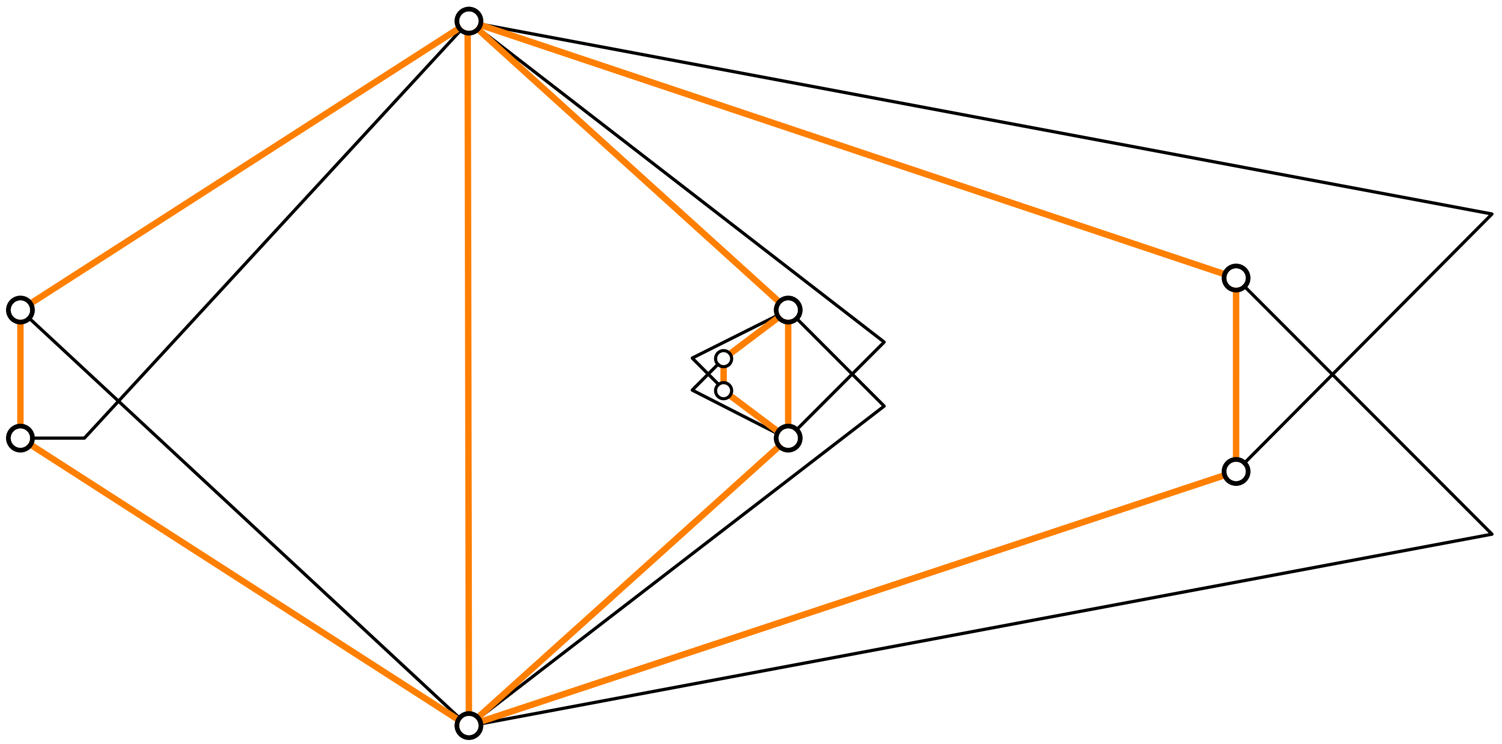


Algorithm Step 4: Removal of Dummy Vertices



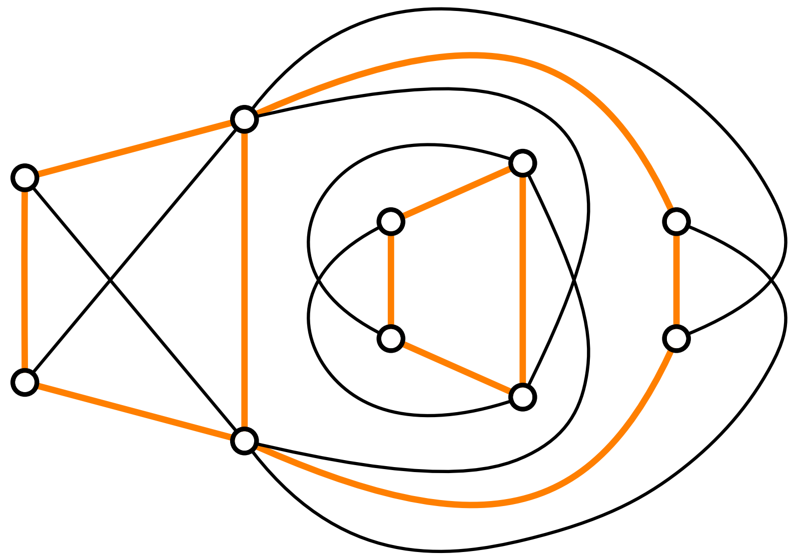
Algorithm Step 4: Removal of Dummy Vertices

Γ: 1-bend 1-planar RAC drawing of G



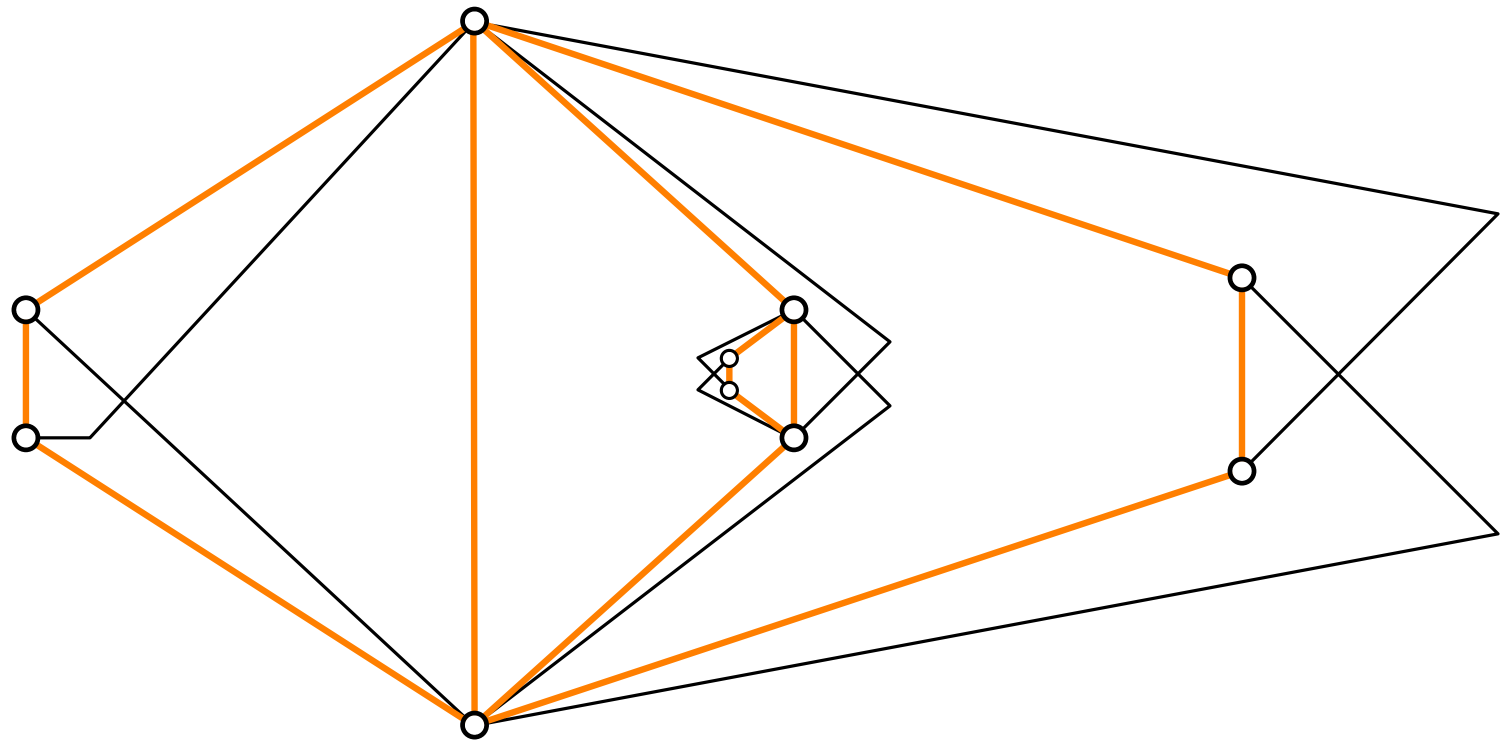
Algorithm Step 4: Removal of Dummy Vertices

G: simple 1-plane graph



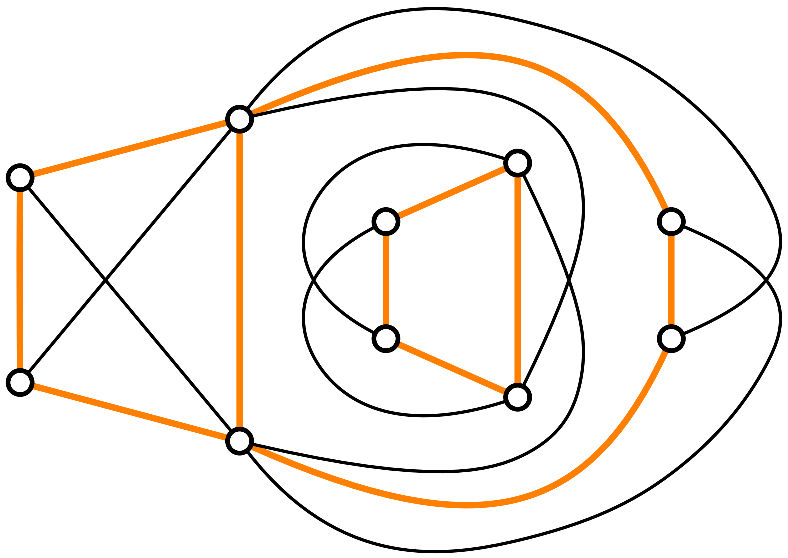
Γ: 1-bend 1-planar RAC drawing of G

(embedding may differ)



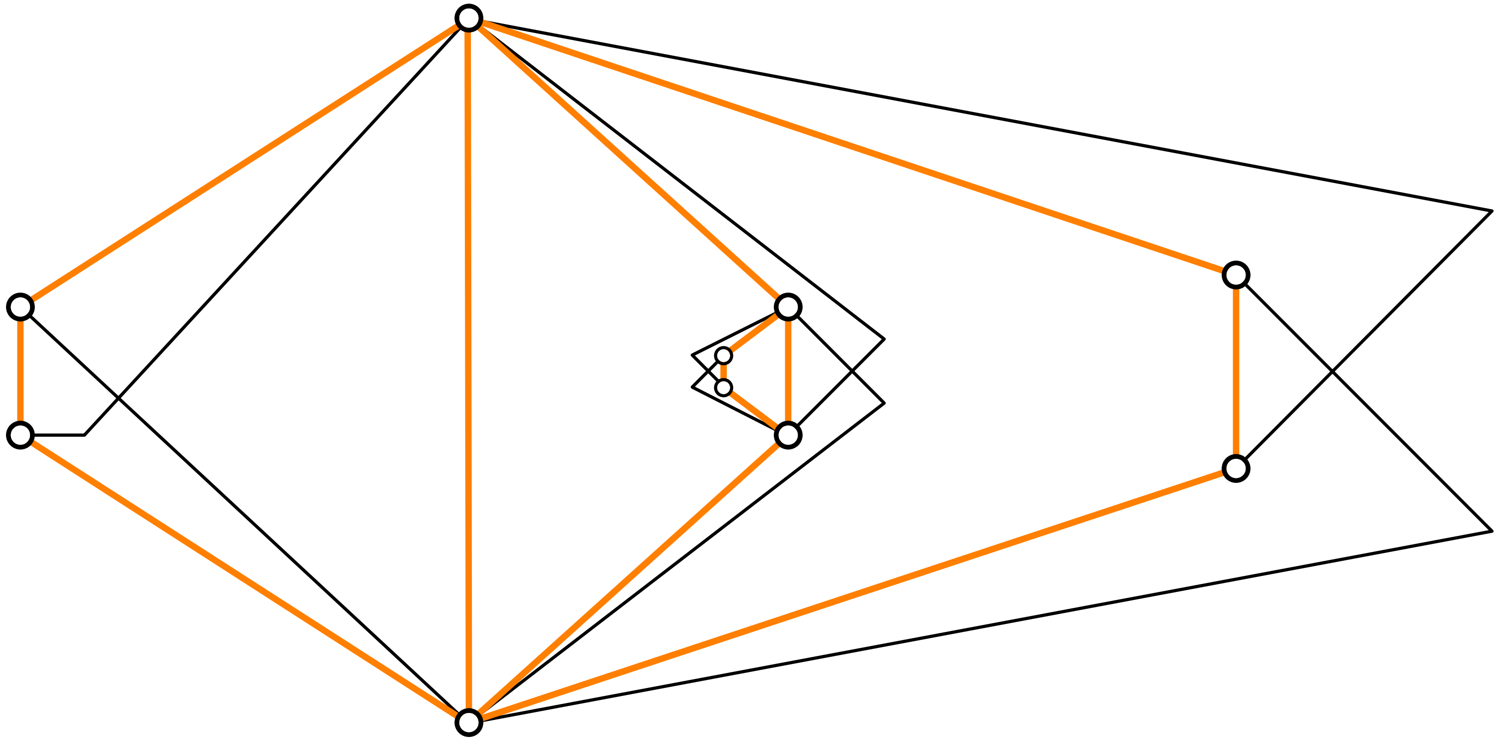
Algorithm Step 4: Removal of Dummy Vertices

G: simple 1-plane graph



Γ: 1-bend 1-planar RAC drawing of G

(embedding may differ)

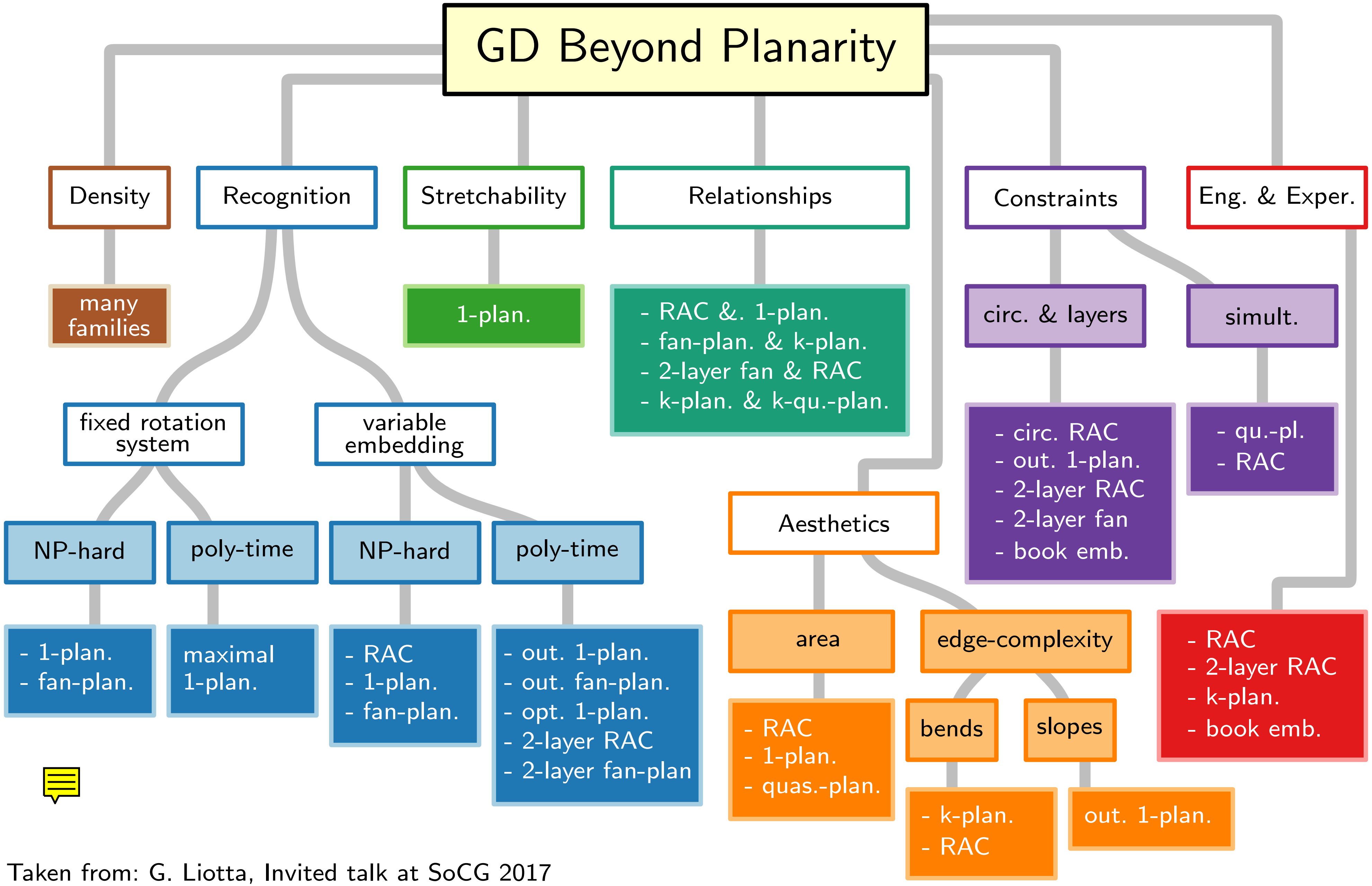


Remark.

By modifying the algorithm slightly, the given input embedding can be preserved.

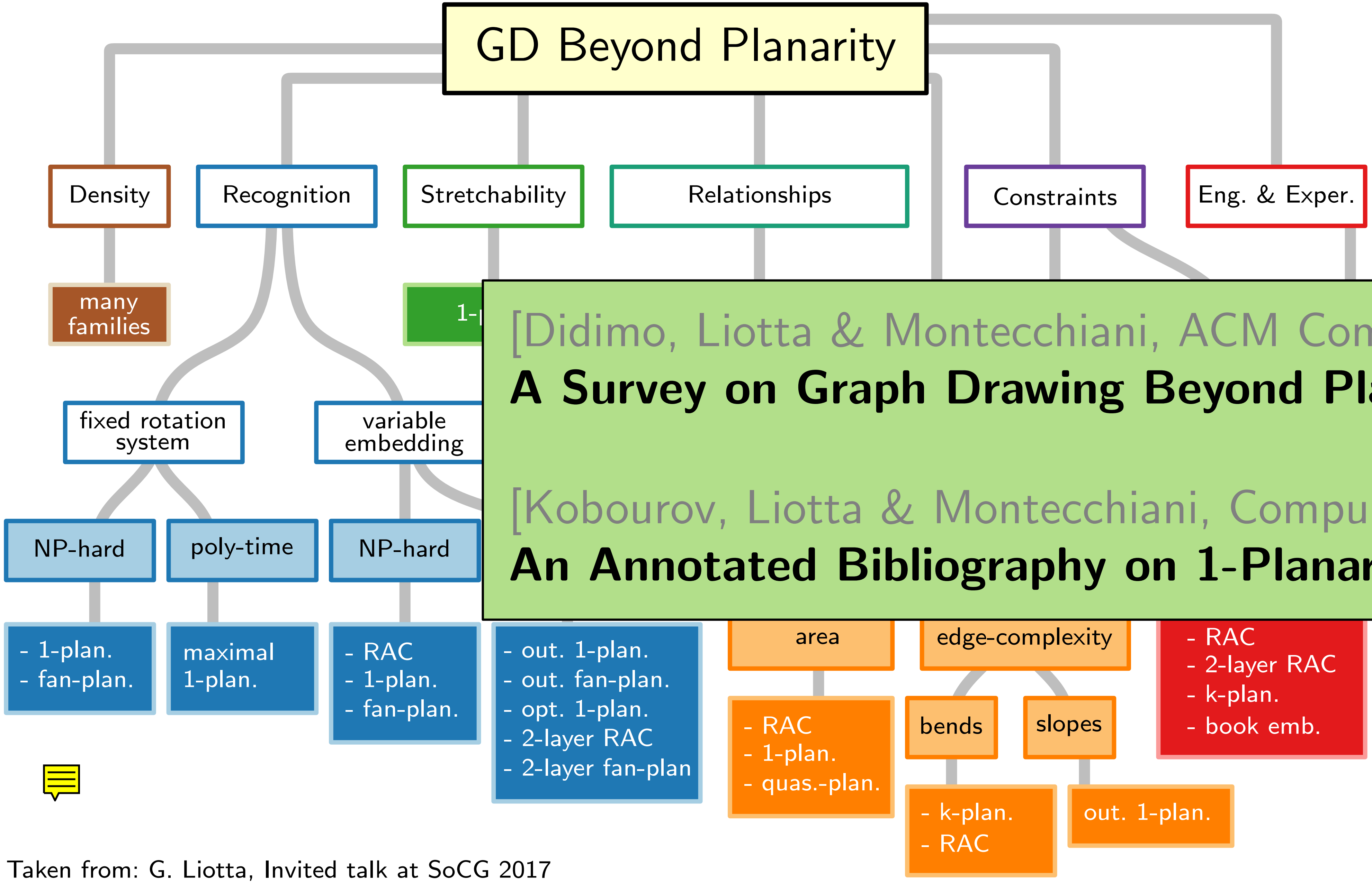
[Chaplick, Lipp, Wolff, Zink 2019]

GD Beyond Planarity: a Taxonomy



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

GD Beyond Planarity: a Taxonomy



[Didimo, Liotta & Montecchiani, ACM Comput. Surv. 2019]
A Survey on Graph Drawing Beyond Planarity.

[Kobourov, Liotta & Montecchiani, Compu. Sci. Review 2017]
An Annotated Bibliography on 1-Planarity.

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Hong and Tokuyama, editors '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchani, Valtr '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angelini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
- [Didimo, Eades, Liotta '17] Drawing graphs with right angle crossings
- [Bekos et al. '17] On RAC drawings of 1-planar graphs