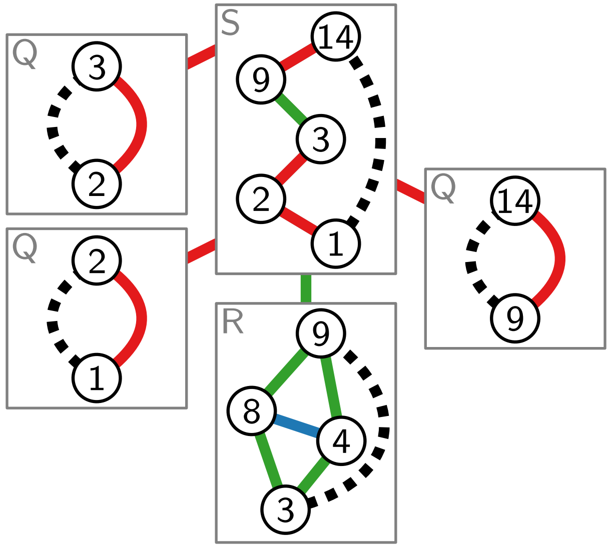
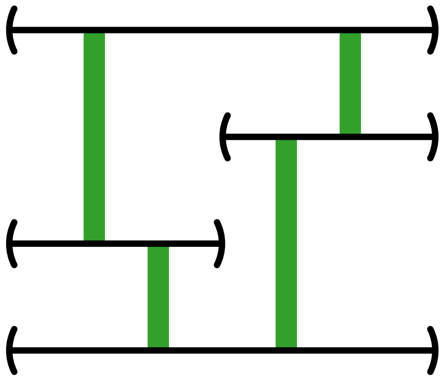
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 10:
Partial Visibility Representation Extension



Alexander Wolff



Summer semester 2025

Partial Representation Extension Problem

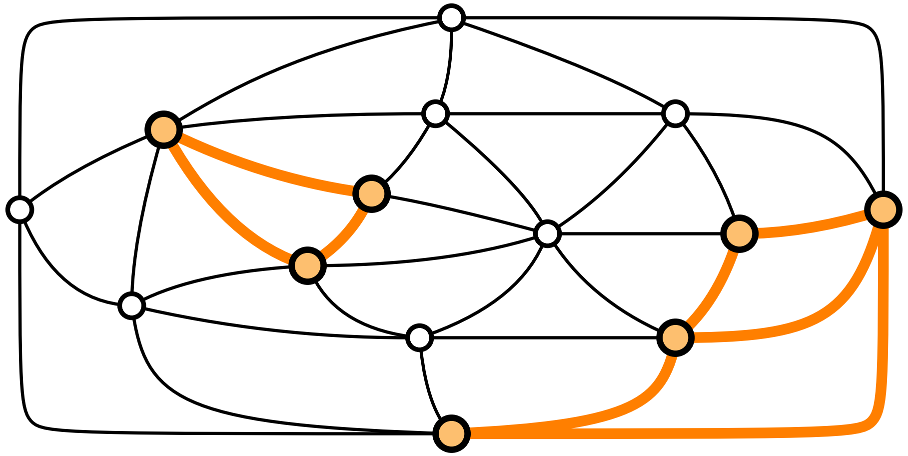
Let G be a graph.

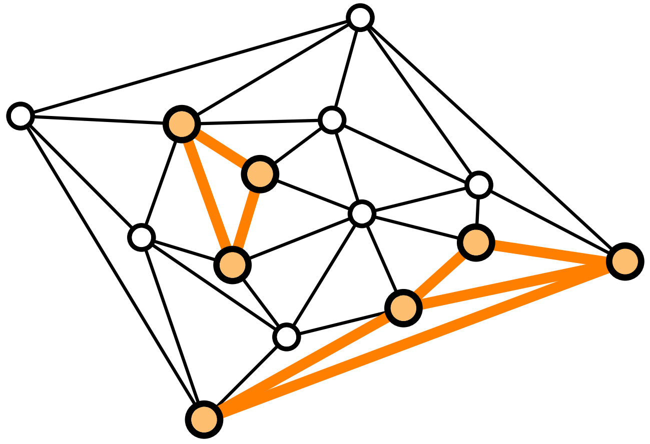
Let $V' \subseteq V(G)$ and $H = G[V']$

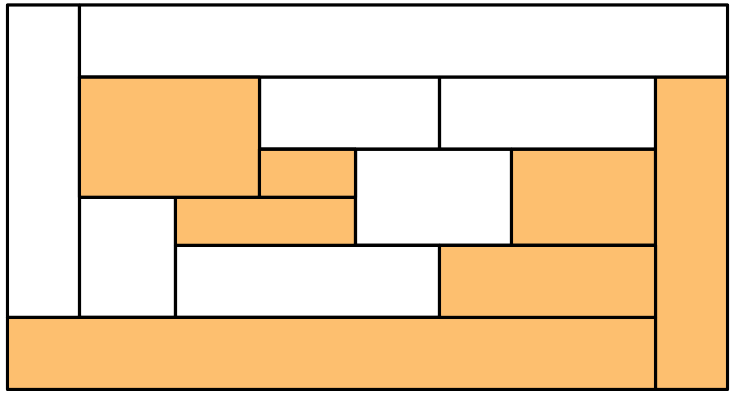
induced subgraph of G w.r.t. V': V' and all edges among V'

Let Γ_H be a representation of H.

Find a representation Γ_G of G that *extends* Γ_H.

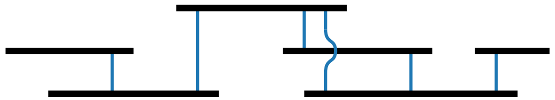




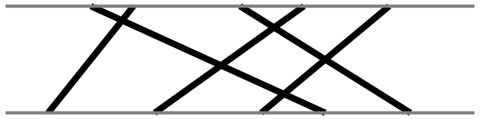


Polytime for:

- (unit) interval graphs



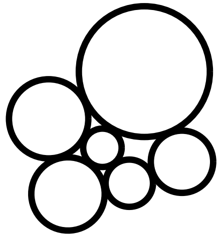
- permutation graphs

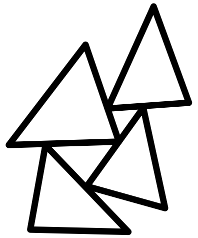


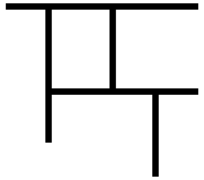
- circle graphs

NP-hard for:

- planar straight-line drawings
- contacts of
  - disks
  - triangles
  - orthogonal segments

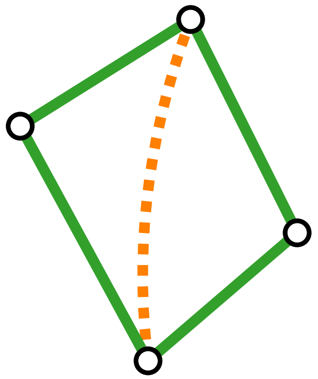


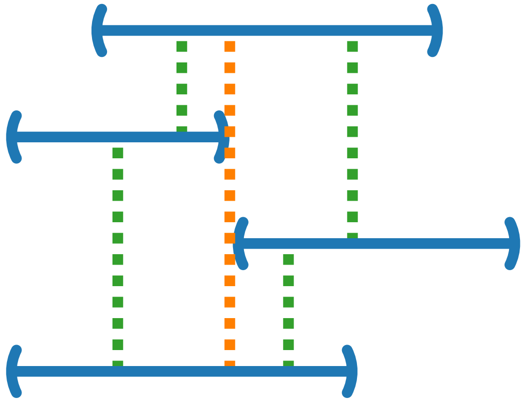




Bar Visibility Representation

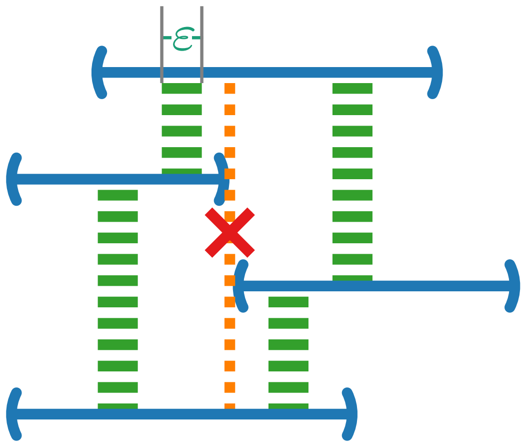
- Vertices correspond to horizontal (open) line segments called **bars**.
- **Edges** correspond to unobstructed vertical lines of sight.
- What about unobstructed **0-width** vertical lines of sight? Do all visibilities induce edges?

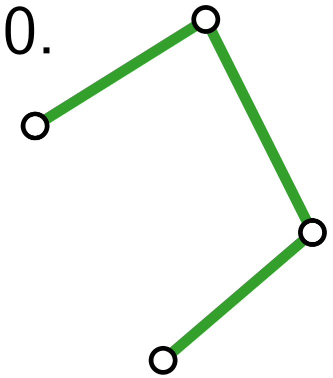




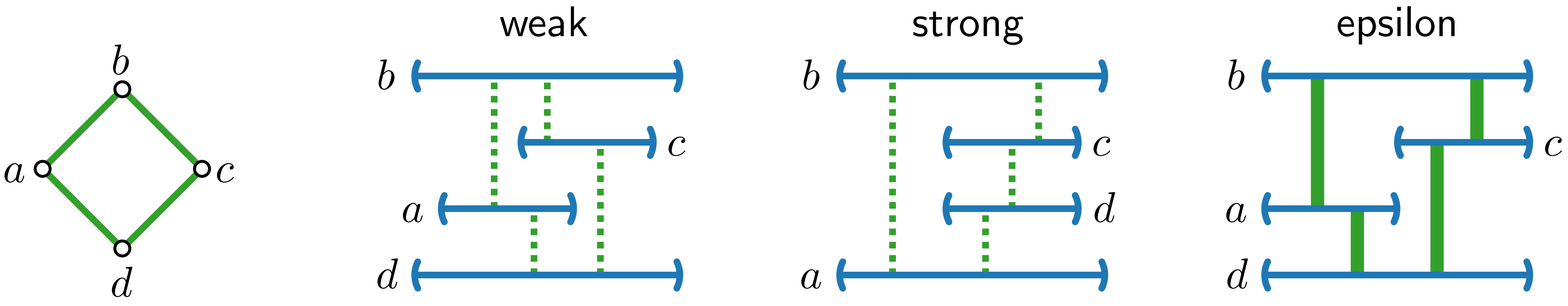
Models.

- **Strong:**
  Edge $uv \Leftrightarrow$ unobstructed **0-width** vertical lines of sight.
- **Epsilon:**
  Edge $uv \Leftrightarrow$ ε-wide vertical lines of sight for some $\varepsilon > 0$.
- **Weak:**
  Edge $uv \Rightarrow$ unobstructed vertical lines of sight exists, i.e., any subset of *visible* pairs





Problems

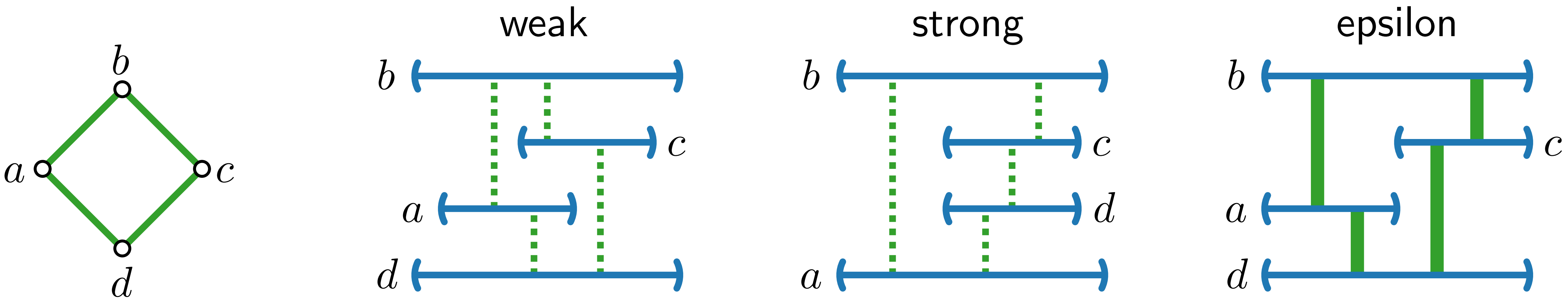


Recognition Problem.
Given a graph G, **decide** whether there exists a weak/strong/ε-bar visibility representation ψ of G.

Construction Problem.
Given a graph G, **construct** a weak/strong/ε-bar visibility representation ψ of G – if one exists.

Partial Representation Extension Problem.
Given a graph G and a **set of bars ψ' of $V' \subseteq V(G)$**, **decide** whether there exists a weak/strong/ε-bar visibility representation ψ of G **where $\psi|_{V'} = \psi'$** (and **construct** ψ if a representation exists).

Background



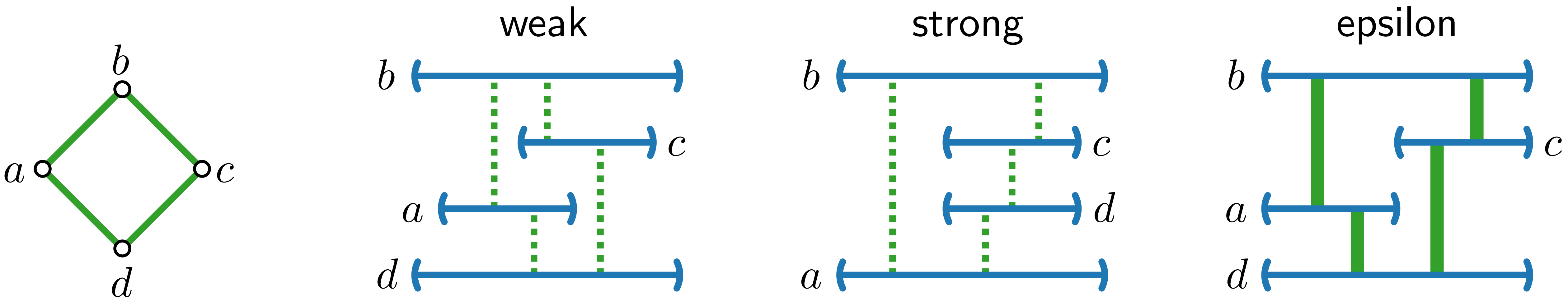
Weak Bar Visibility.

- Exactly all planar graphs [Tamassia & Tollis '86; Wismath '85]
- Linear-time recognition and construction [T&T '86]
- Representation extension is NP-complete [Chaplick et al. '14]

Strong Bar Visibility.

- NP-complete to recognize [Andreae '92]

Background

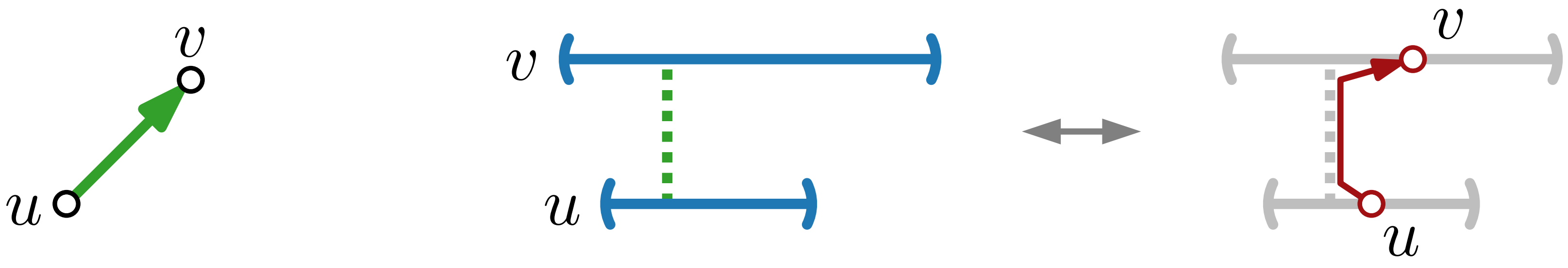


ε-Bar Visibility.

- Exactly all planar graphs that can be embedded with all **cut vertices** on the outerface [T&T '86, Wismath '85]
- Linear-time recognition and construction [T&T '86]
- Representation extension? This Lecture!

Bar Visibility Representation of Digraphs

- Instead of an undirected graph, we are given a directed graph G.
- The task is to construct a weak/strong/ε-bar visibility representation of G such that . . .
- . . . for each directed edge uv, the bar representing u is below the bar representing v.



Weak Bar Visibility.

- NP-complete for directed (acyclic planar) graphs!
- This is because upward planarity testing is NP-complete. [Garg & Tamassia '01]

Strong/ε-Bar Visibility.

- Open for directed graphs!

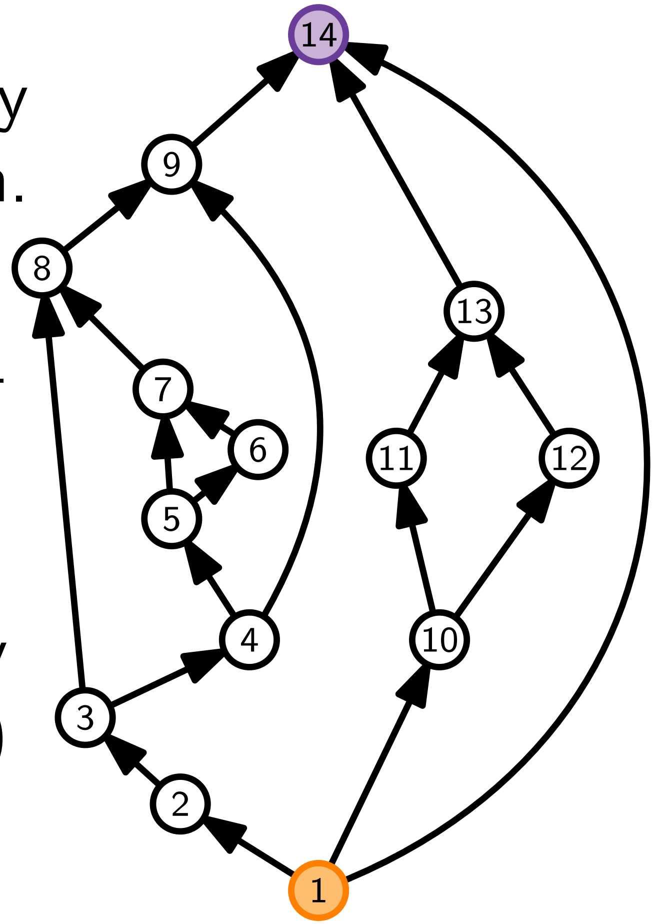
Next, we consider ε-bar visibility representations of specific directed graphs ($\rightarrow$ st-graphs)

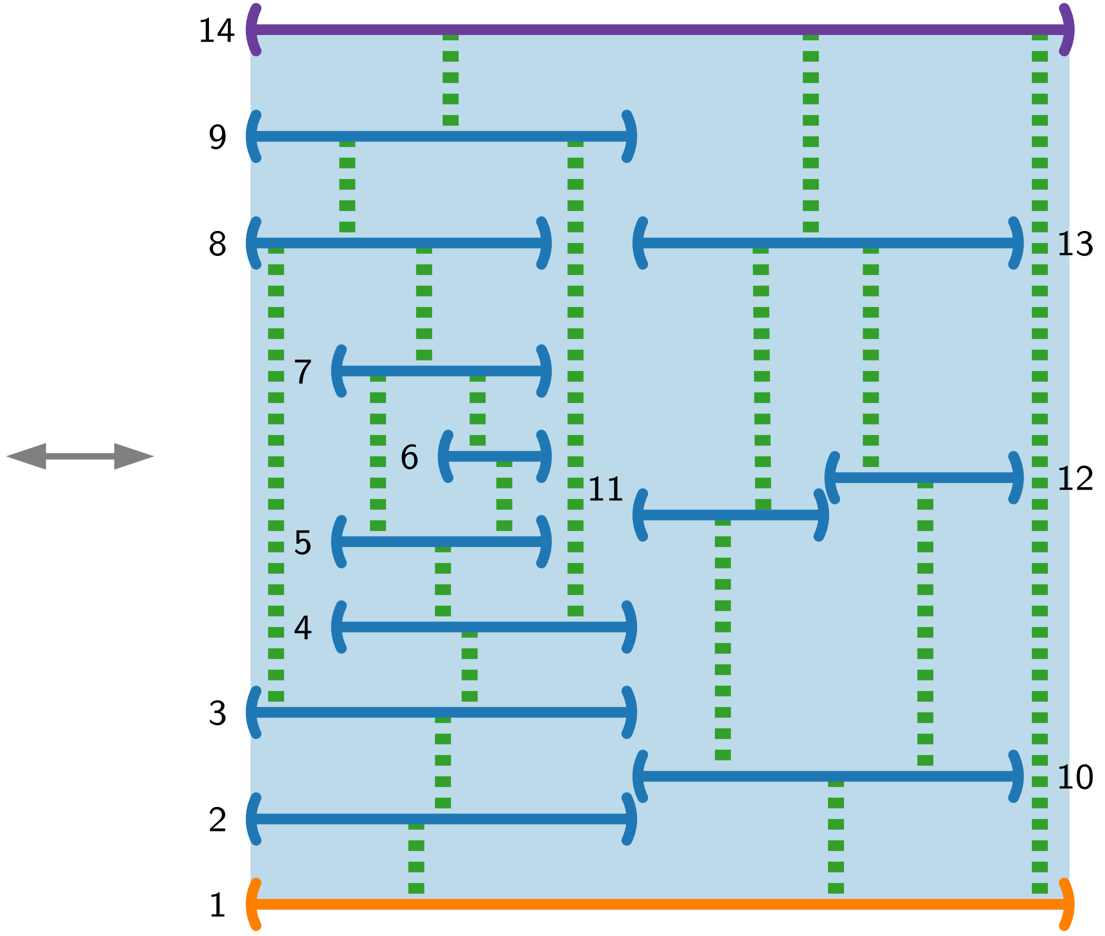
ε-Bar Visibility and st-Graphs

Recall that an **st-graph** is a planar acylic digraph G with exactly one source s and one sink t where s and t occur on the outer face of an embedding of G.

Observation.
st-orientations correspond to ε-bar visibility representations.

- ε-bar visibility testing is easily done via st-graph recognition.
- Strong bar visibility recognition. . . open!
- In a **rectangular** bar visibility representation $\psi(s)$ and $\psi(t)$ span an enclosing rectangle.





Results and Outline

[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]

Theorem 1.
Rectangular ε-bar visibility representation extension can be solved in $\mathcal{O}(n \log^2 n)$ time for st-graphs.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2.
ε-bar visibility representation extension is NP-complete.

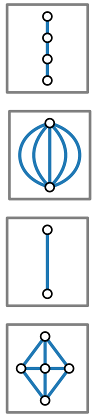
- Reduction from Planar Monotone 3-SAT

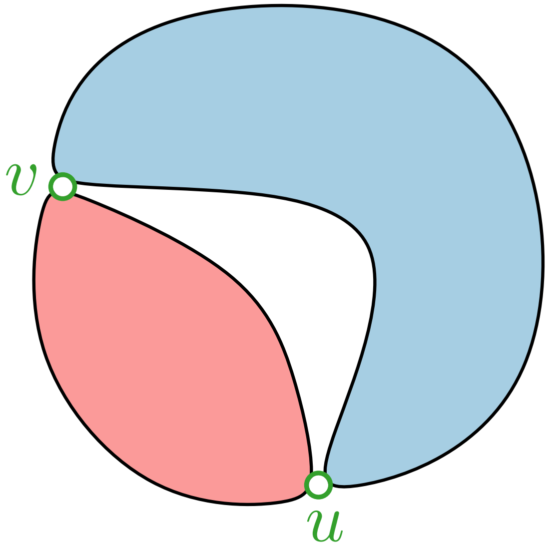
Theorem 3.
ε-bar visibility representation extension is NP-complete even for (series-parallel) st-graphs when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

- Reduction from 3-Partition

SPQR-Tree

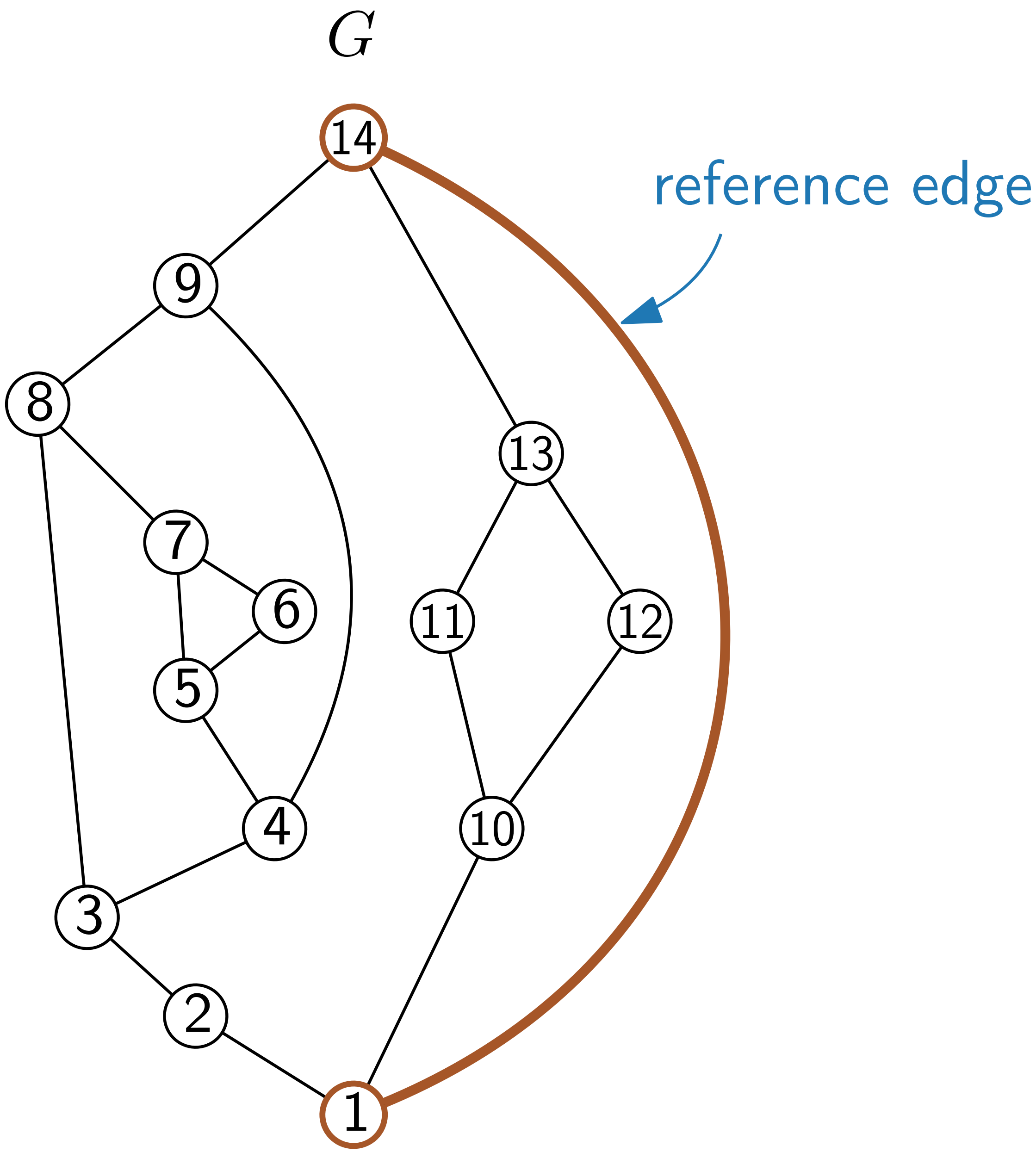
- An **SPQR-tree** T is a decomposition of a planar graph G by **separation pairs**.
- The nodes of T are of four types:
  - **S**-nodes represent a series composition
  - **P**-nodes represent a parallel composition
  - **Q**-nodes represent a single edge
  - **R**-nodes represent 3-connected (*rigid*) subgraphs

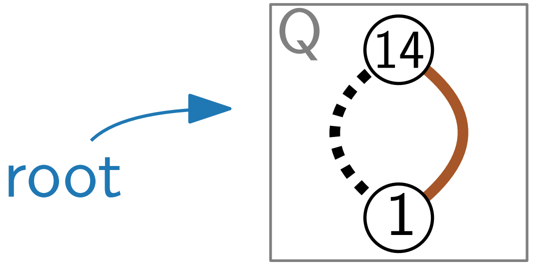




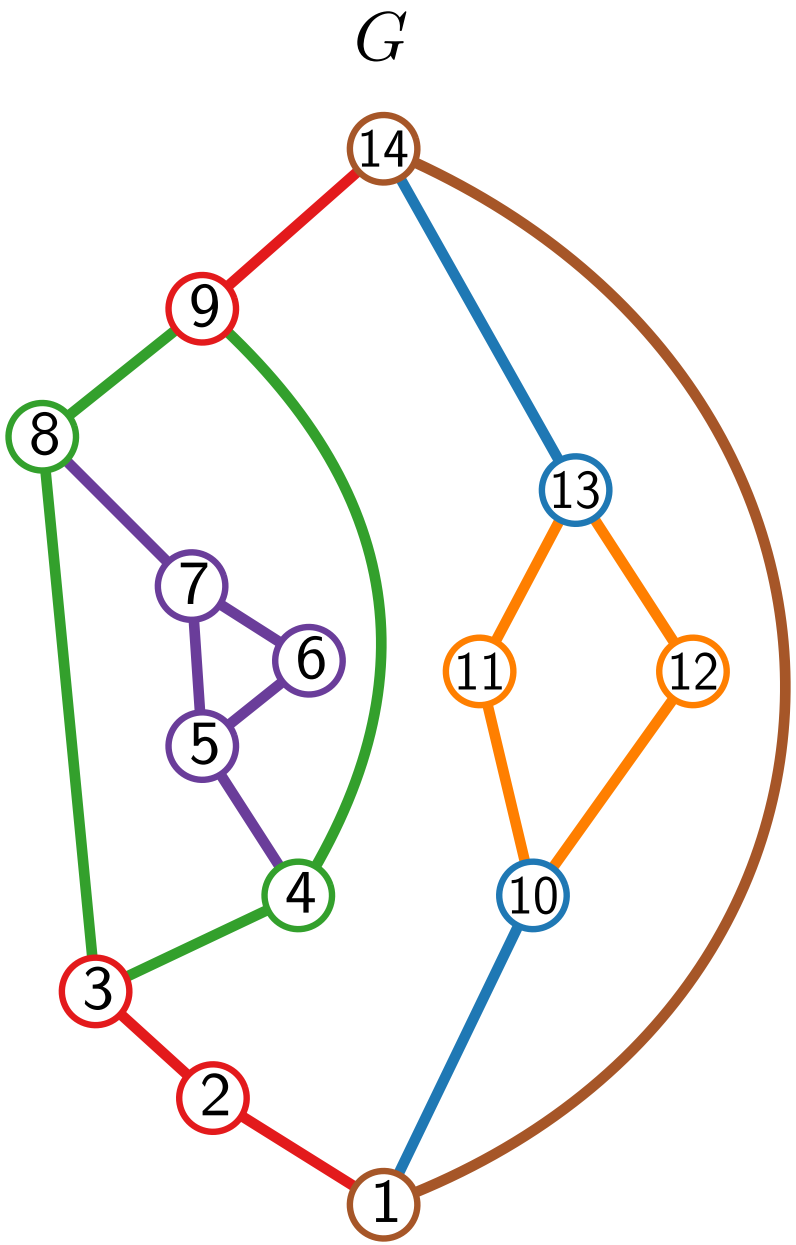
- A decomposition tree of a series-parallel graph is an SPQR-tree without **R**-nodes.
- T represents all planar embeddings of G.
- T can be computed in time linear in the size of G. [Gutwenger, Mutzel '01]

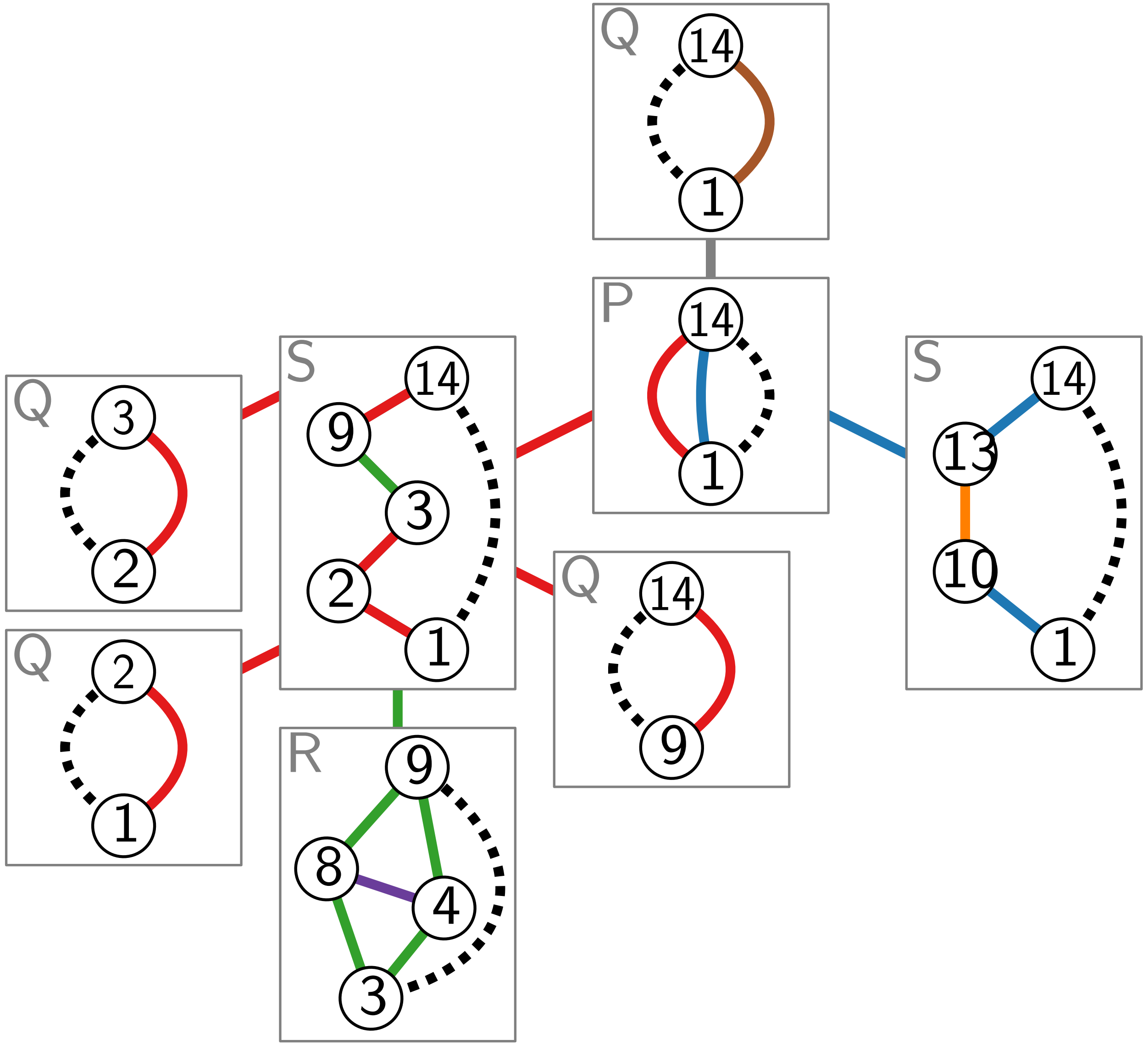
SPQR-Tree – Example



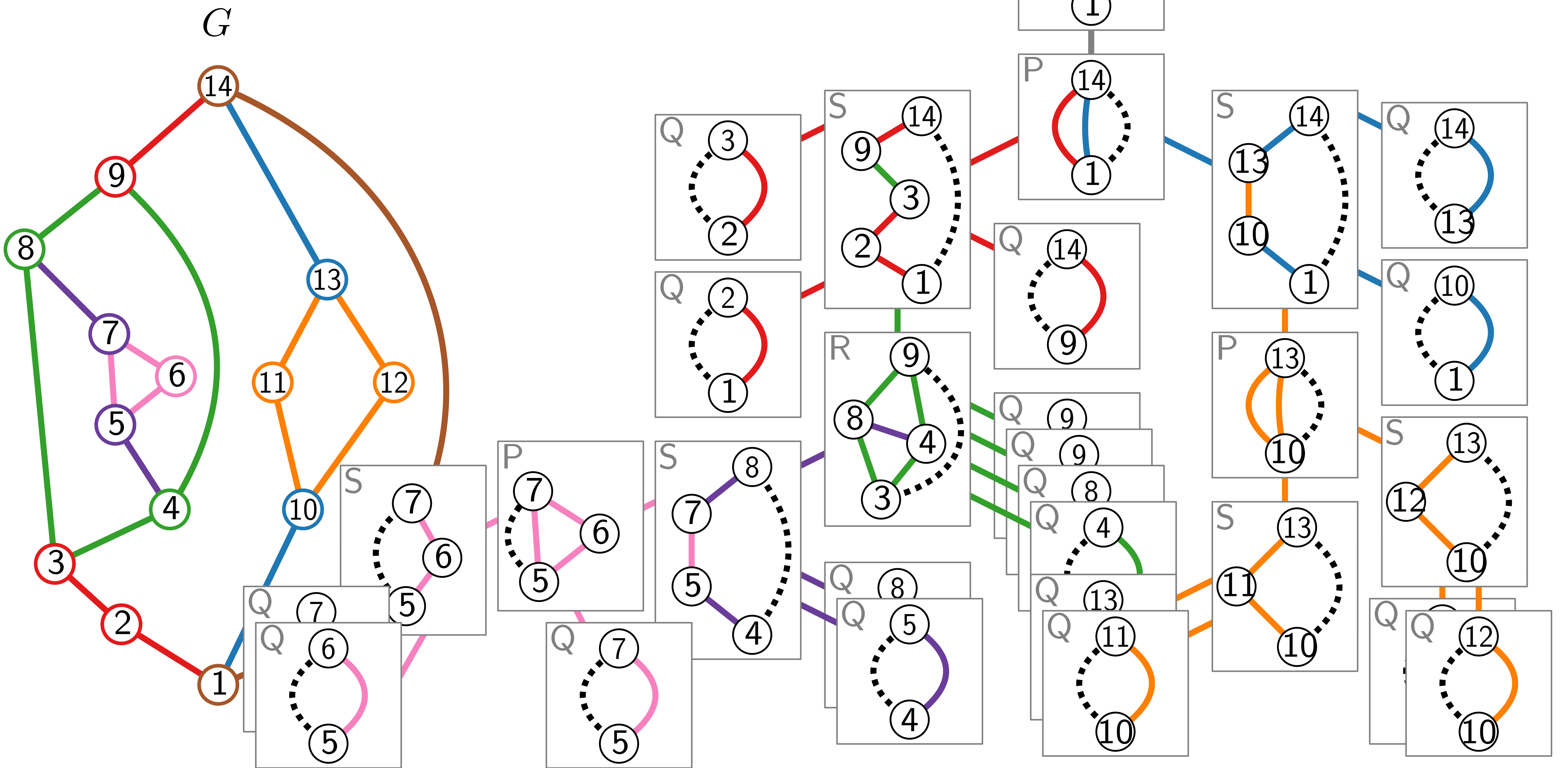


SPQR-Tree – Example



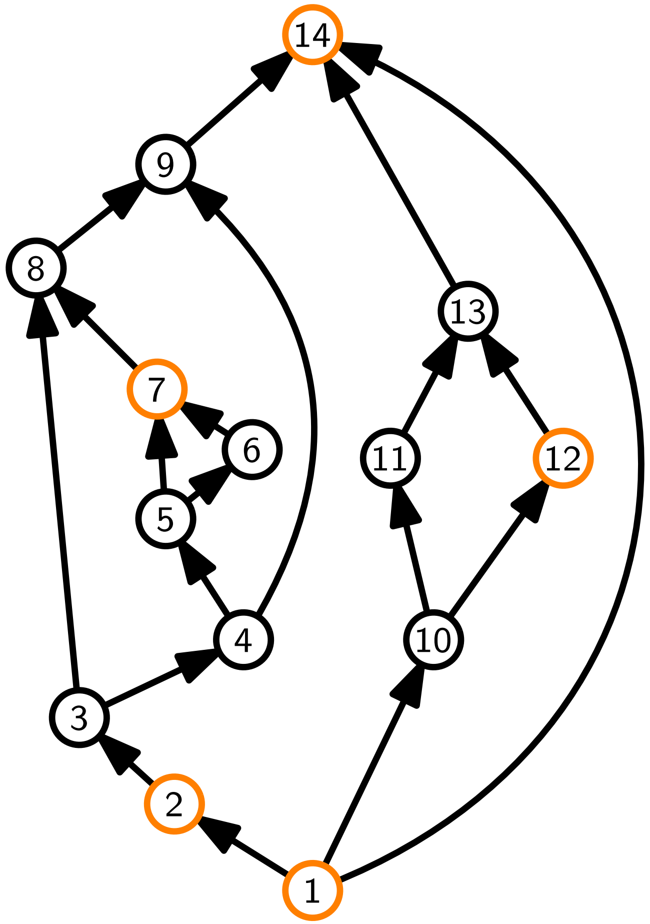


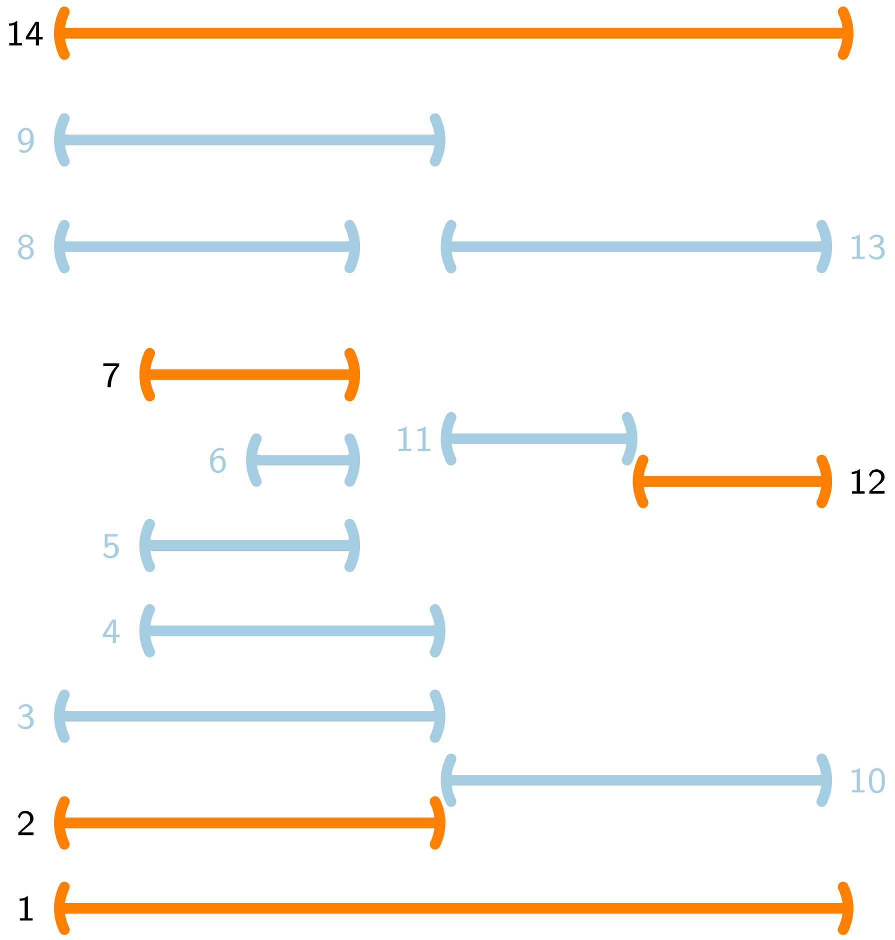
SPQR-Tree – Example



Representation Extension for st-Graphs

Theorem 1'.
Rectangular ε-bar visibility representation extension can be solved in $\mathcal{O}(n^2)$ time for st-graphs.





- Simplify problem via assumption regarding y-coordinates
- Exploit connection between SPQR-trees and rectangle tiling
- Solve problems for **S**-, **P**-, and **R**-nodes
- Dynamic program via structure of SPQR-tree

y-Coordinate Invariant

- Let G be an st-graph, and let ψ' be a representation of $V' \subseteq V(G)$.
- Let $y\colon V(G) \to \mathbb{R}$ such that
  - for each $v \in V'$, $y(v) =$ the y-coordinate of $\psi'(v)$.
  - for each edge (u, v), $y(u) < y(v)$.

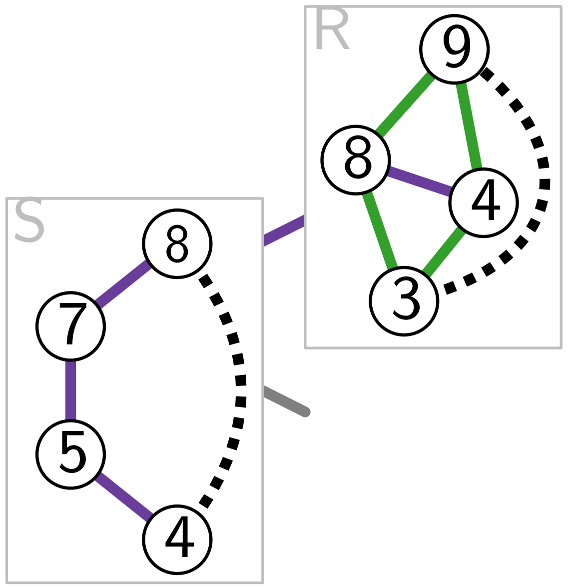
Lemma 1.
G has a representation extending ψ' $\Leftrightarrow$
G has a representation extending ψ'
where the y-coordinates of the bars are as in y.

We can now assume that all y-coordinates are given!

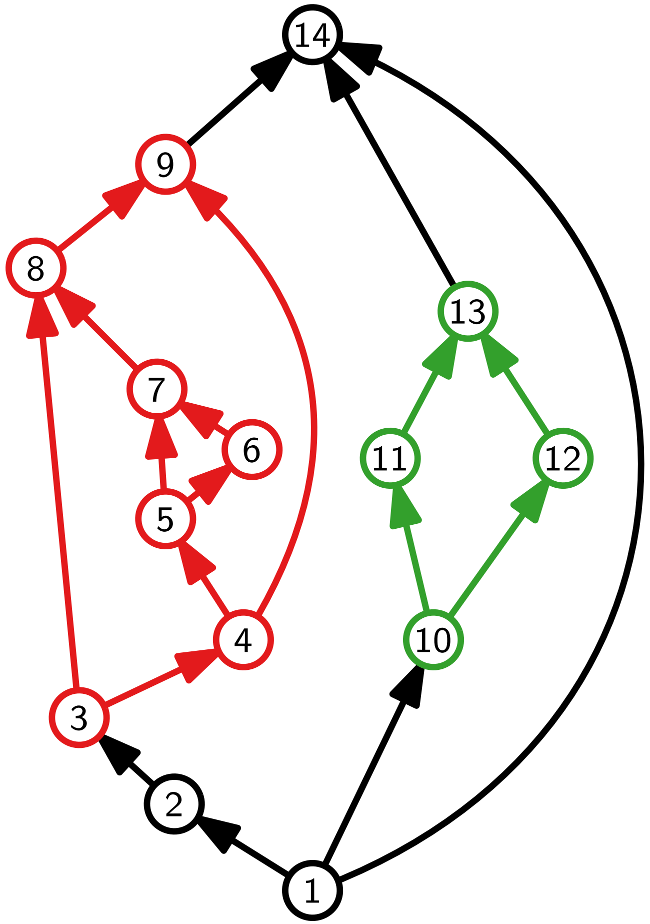
Proof idea. The relative positions of **adjacent** bars must match the order given by y.
So, we can adjust the y-coordinates of any solution to be as in y by sweeping from bottom to top.

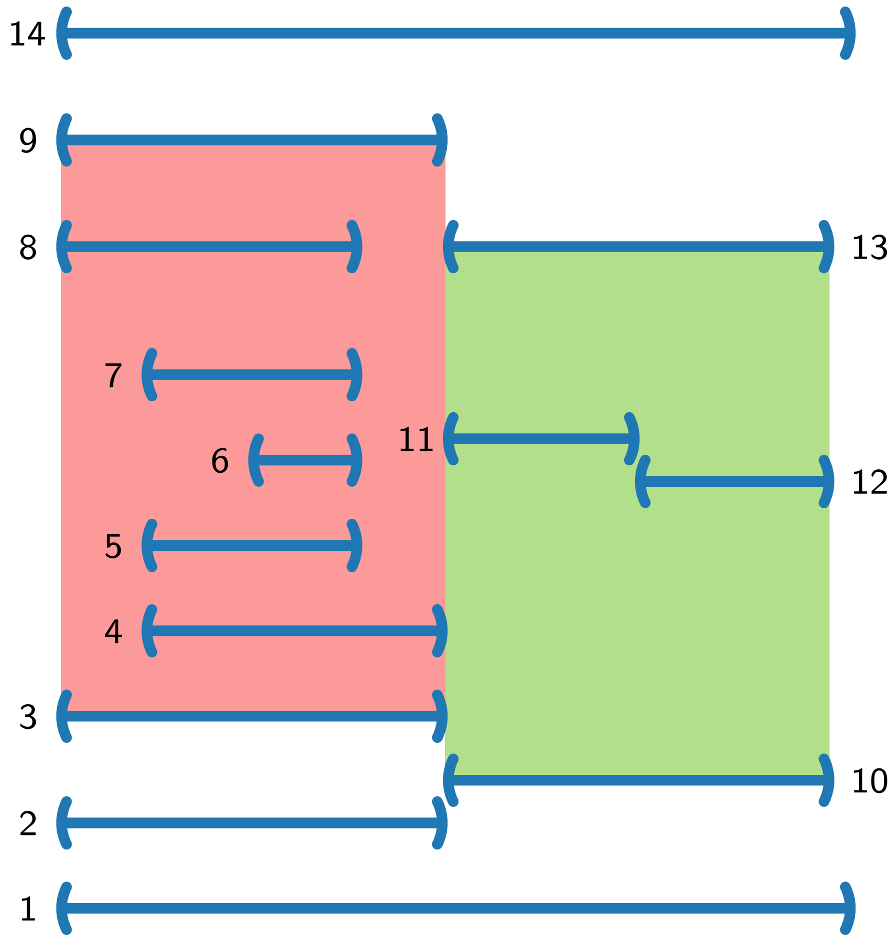
But Why Do SPQR-Trees Help?

Lemma 2.
The SPQR-tree of an st-graph G induces a recursive **tiling** of any ε-bar visibility representation of G.



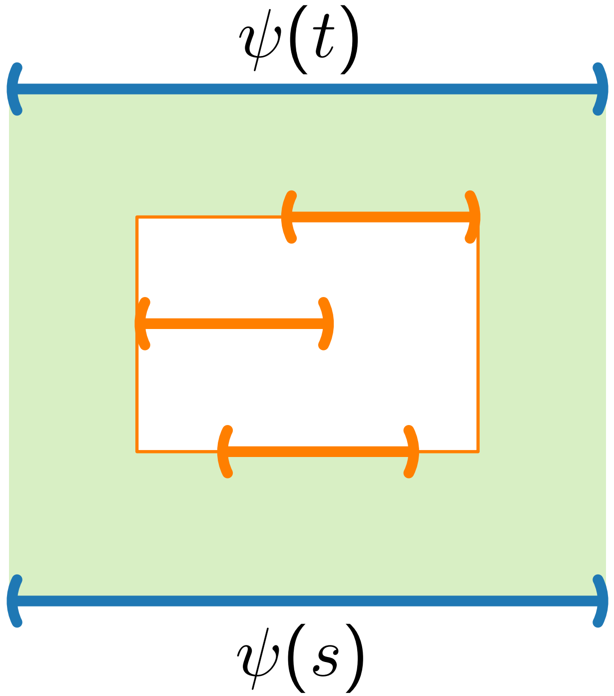
Solve tiles bottom-up.





Tiles

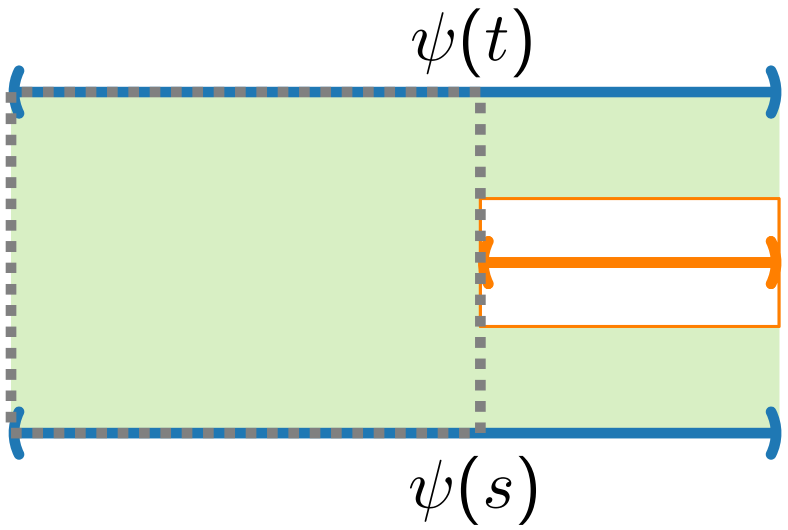
Convention. **Orange** bars are from the given partial representation.



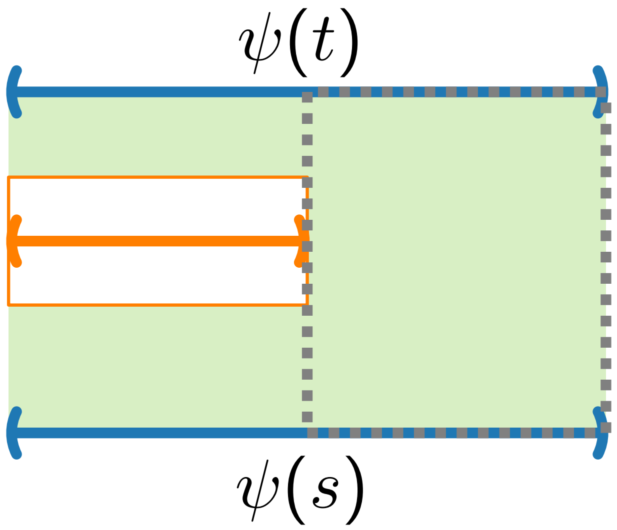
Observation.
The bounding box (tile) of any solution ψ **contains** the bounding box of the partial representation.

How many different **types** of tiles are there?

Types of Tiles



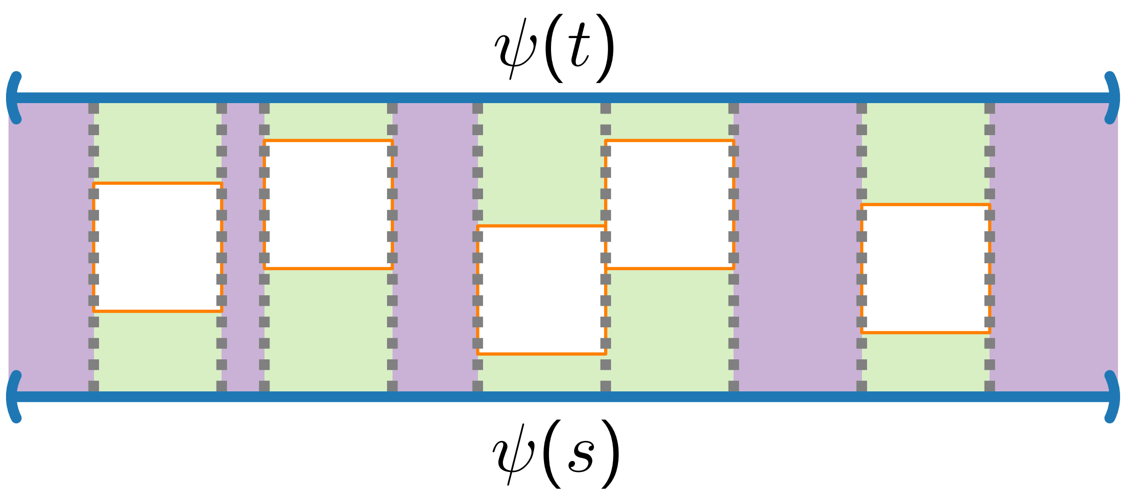
- Right **F**ixed
- Left **L**oose

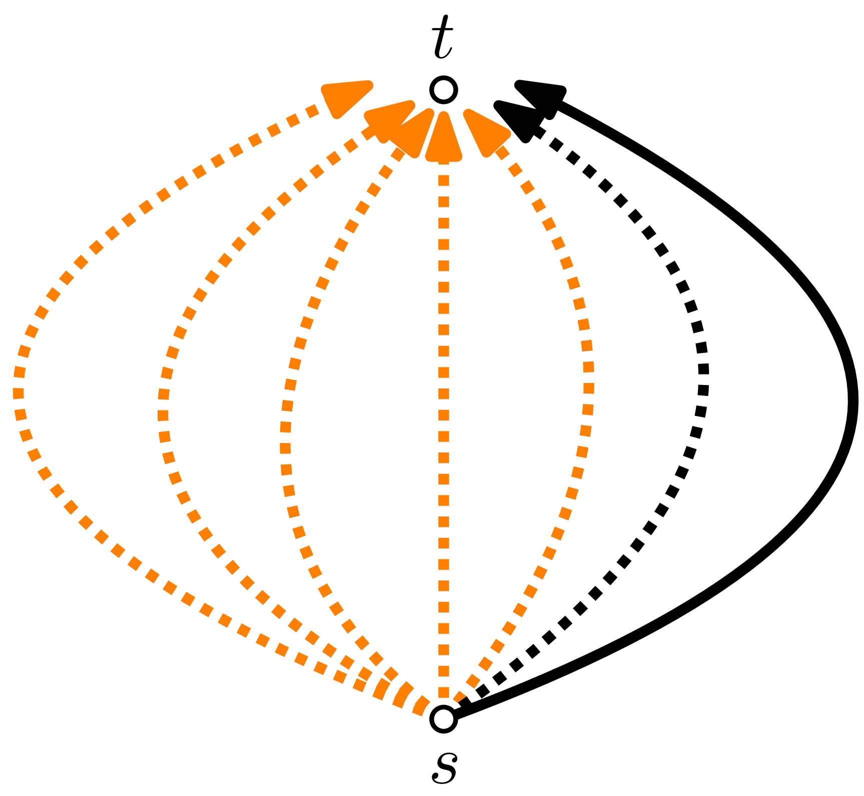


- Left **F**ixed
- Right **L**oose

Four different types: **FF, FL, LF, LL**

P-Nodes

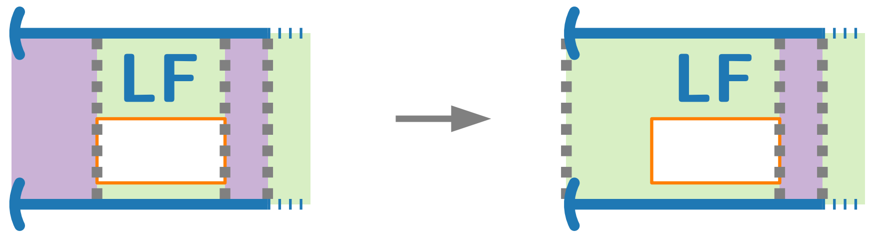




- Children of **P**-node with prescribed bars occur in given left-to-right order
- But there might be some gaps. . .

Idea.

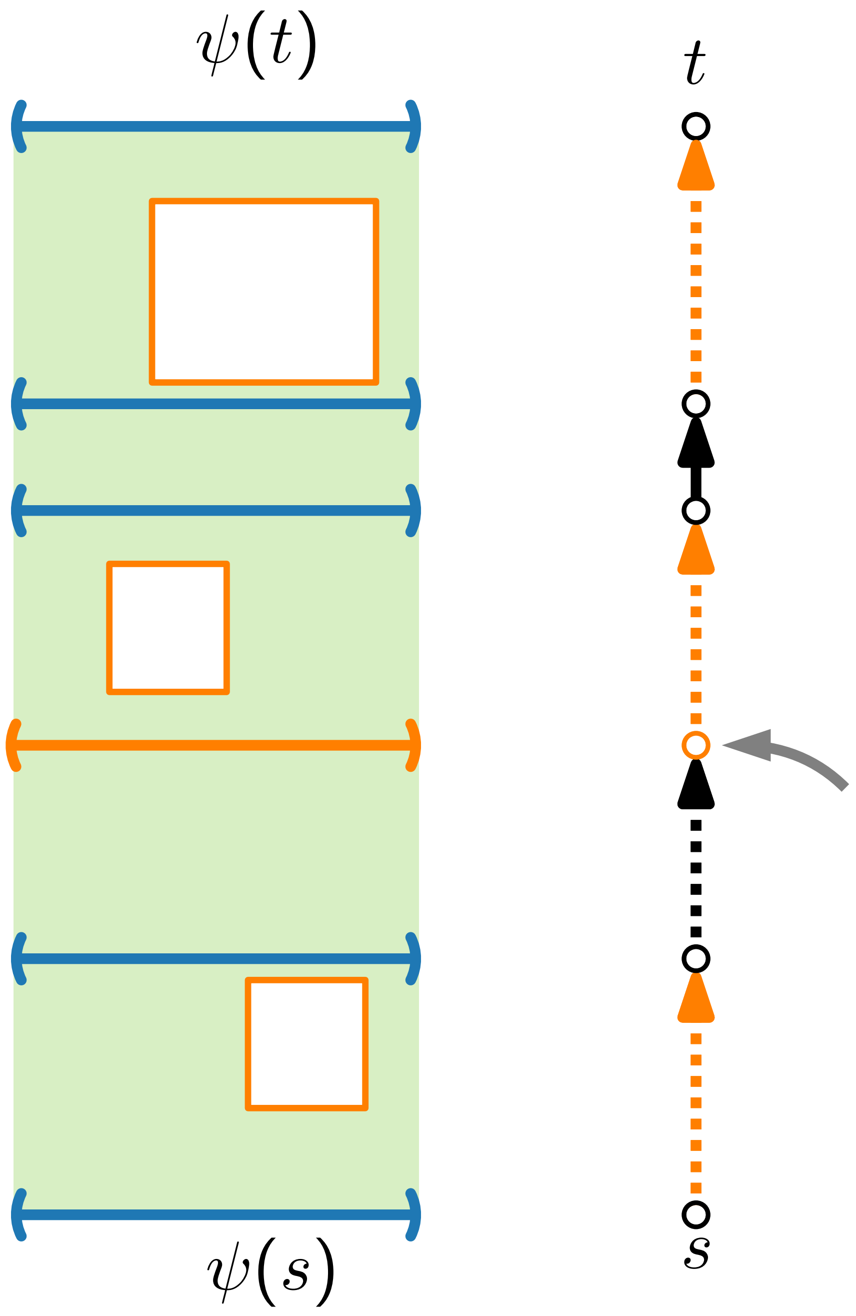
Greedily *fill* the gaps by preferring to "stretch" the children with prescribed bars.



Outcome.

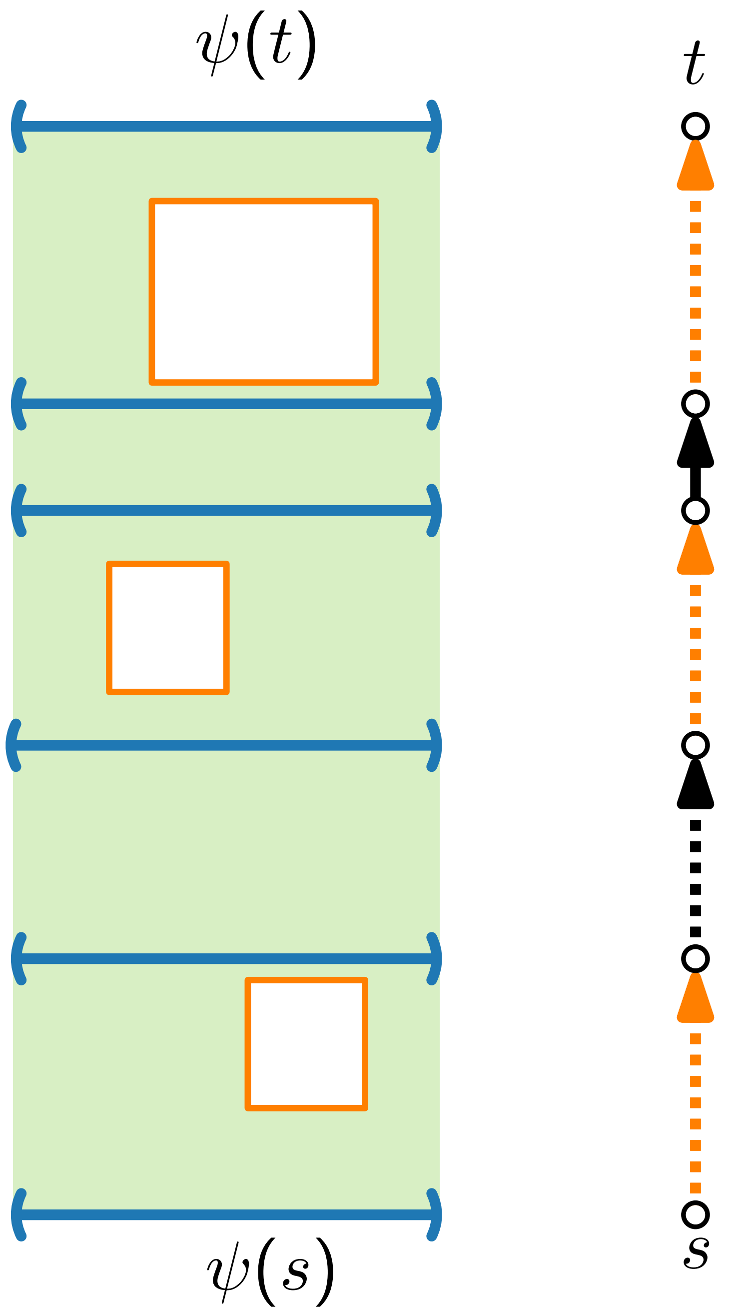
After processing, we must know the valid types for the corresponding subgraphs.

S-Nodes



Here we have a chance to make all (**LL, FL, LF, FF**) types.

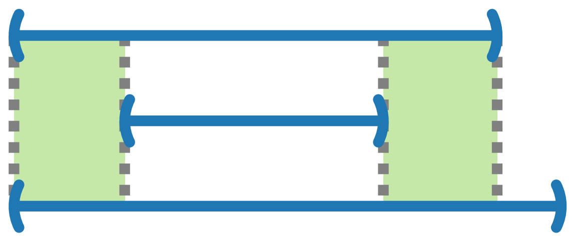
This fixed vertex means we can only make a **F**ixed-**F**ixed representation!

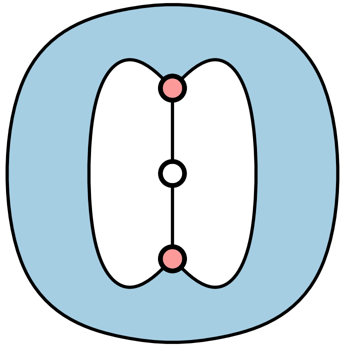


R-Nodes with 2-SAT Formulation

- For each child (edge) e:
  - Find all types of $\{$**FF**, **FL**, **LF**, **LL**$\}$ that admit a drawing.
  - Use two variables (l_e and r_e) to encode the type of its tile (**F** $= 0$).
  - Add *consistency clauses*: e.g., $\neg(\neg r_e \wedge \neg l_f)$ $\rightarrow O(n^2)$ many.

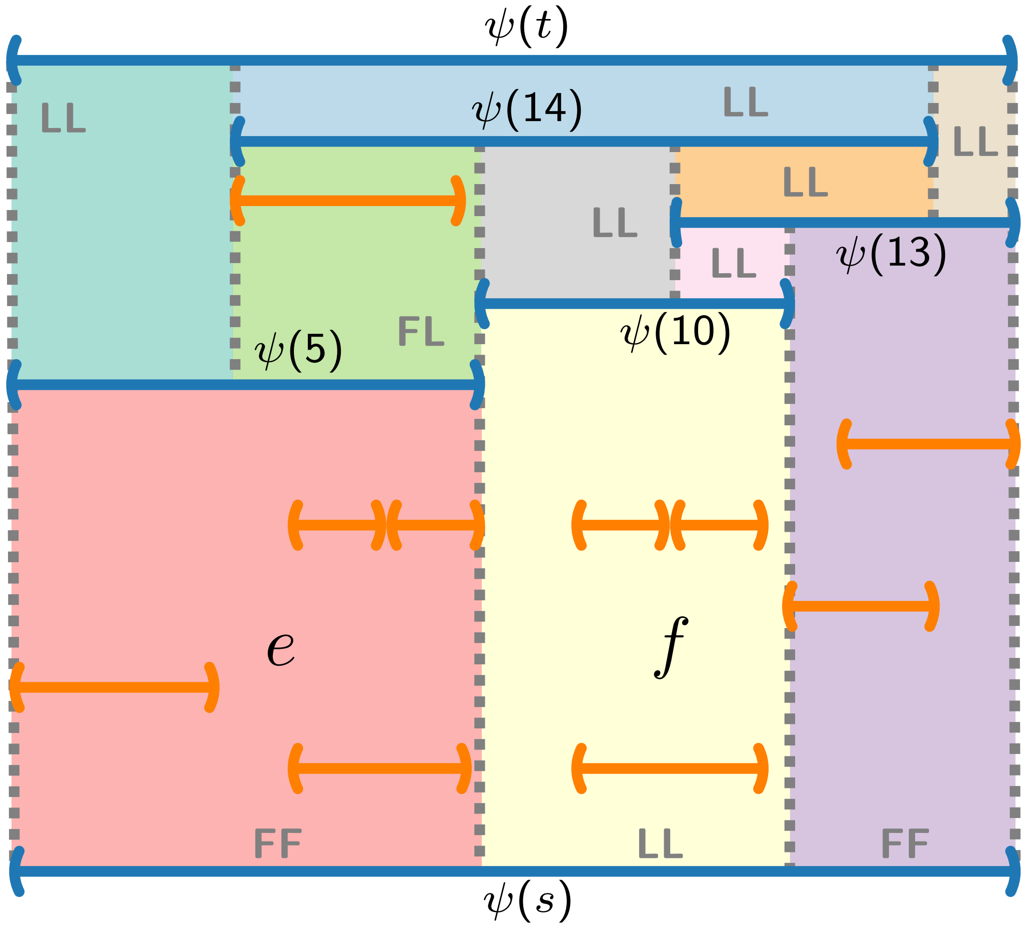
  (can be reduced to $O(n \log^2 n)$)

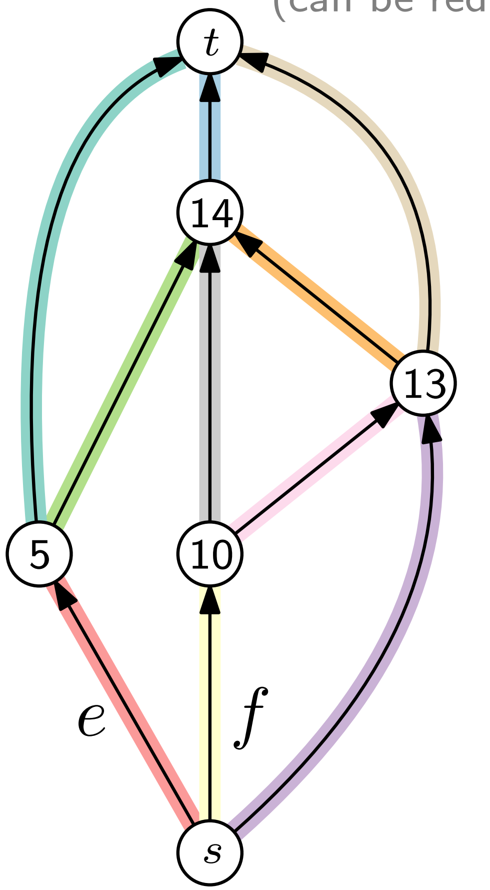




Separation pair!

($\nexists$ in **R**-component.)





- Finding a satisfying assignment of a 2-SAT formula can be done in linear time!

  $\Rightarrow O(n^2)$ time in total
  or $O(n \log^2 n)$

Results and Outline

[Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta ’18]

Theorem 1.
Rectangular ε-bar visibility representation extension can be solved in $\mathcal{O}(n \log^2 n)$ time *for st-graphs*.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2.
ε-bar visibility representation extension is NP-complete.

- Reduction from PLANAR MONOTONE 3-SAT

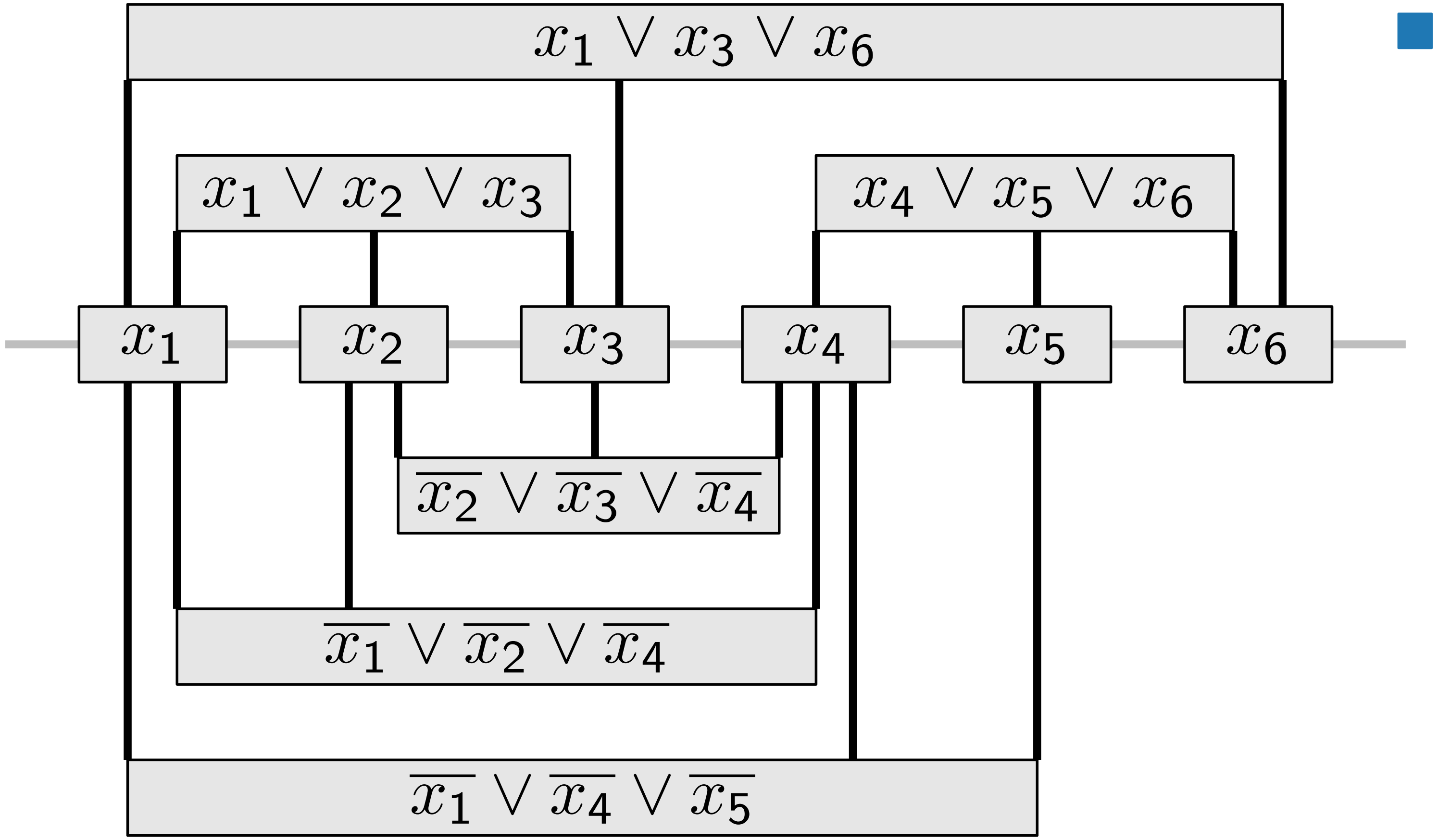
Theorem 3.
ε-bar visibility representation extension is NP-complete even for (series-parallel) st-graphs when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

- Reduction from 3-PARTITION

NP-Hardness of RepExt in the General Case

Theorem 2.
ε-Bar visibility representation extension is NP-complete.

- Membership in NP?
- NP-hard: Reduction from Planar Monotone 3-SAT

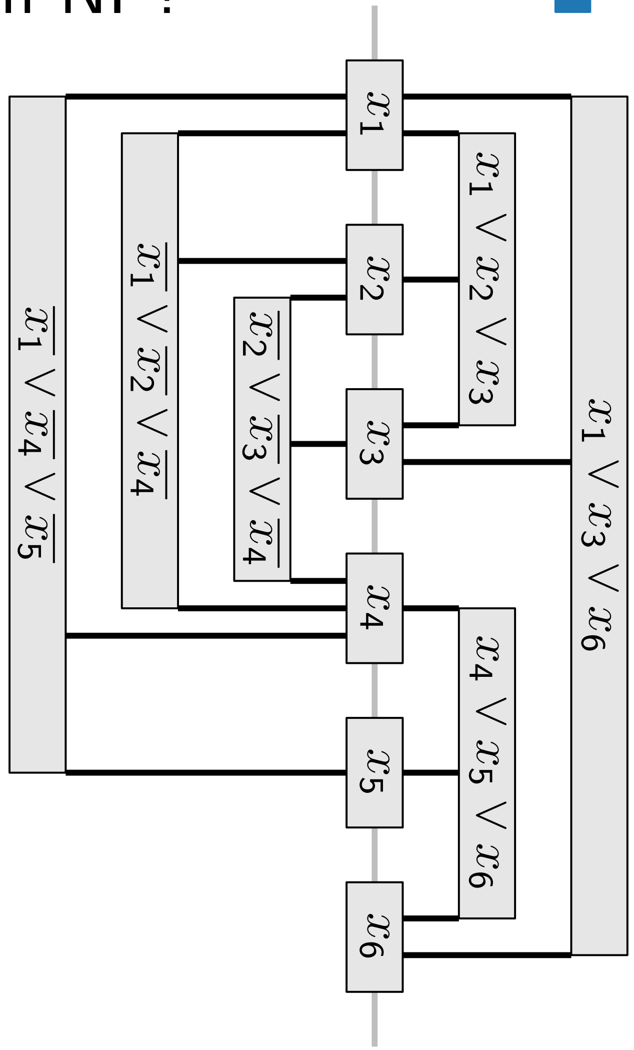


- NP-complete
  [de Berg & Khosravi ’10]

NP-Hardness of RepExt in the General Case

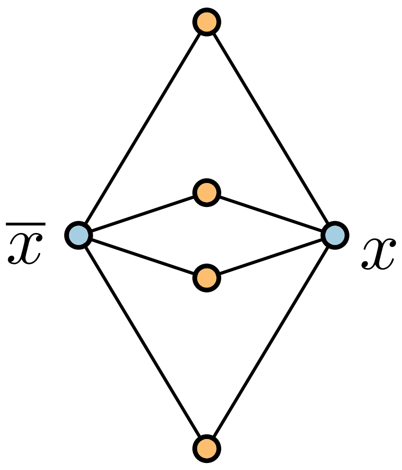
Theorem 2.
ε-Bar visibility representation extension is NP-complete.

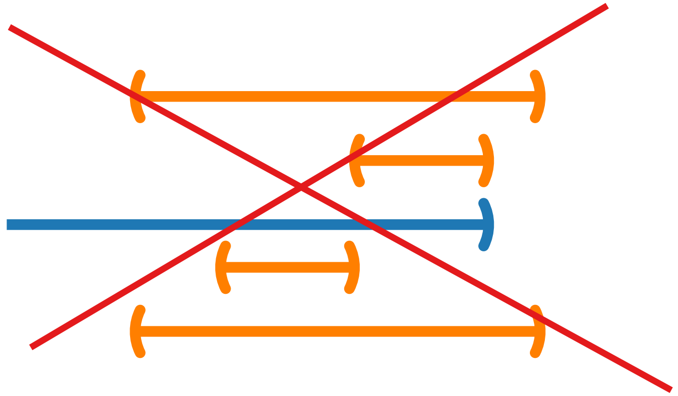
- Membership in NP?
- NP-hard: Reduction from Planar Monotone 3-SAT



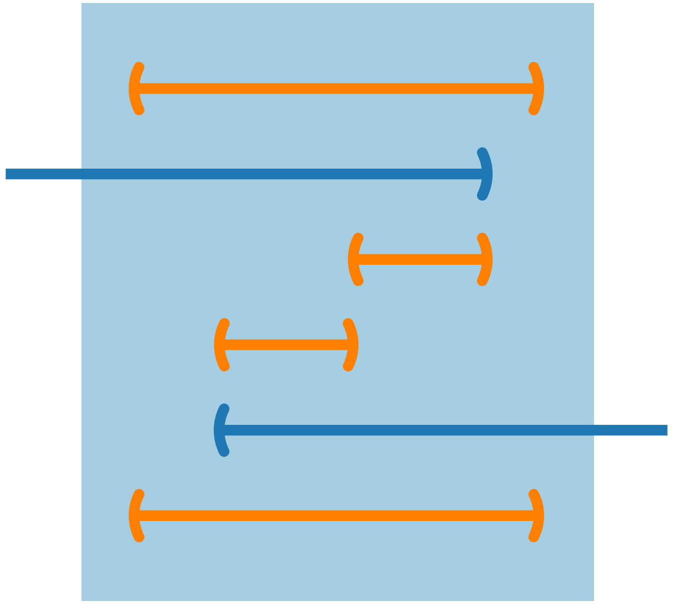
- NP-complete
  [de Berg & Khosravi '10]

Variable Gadget

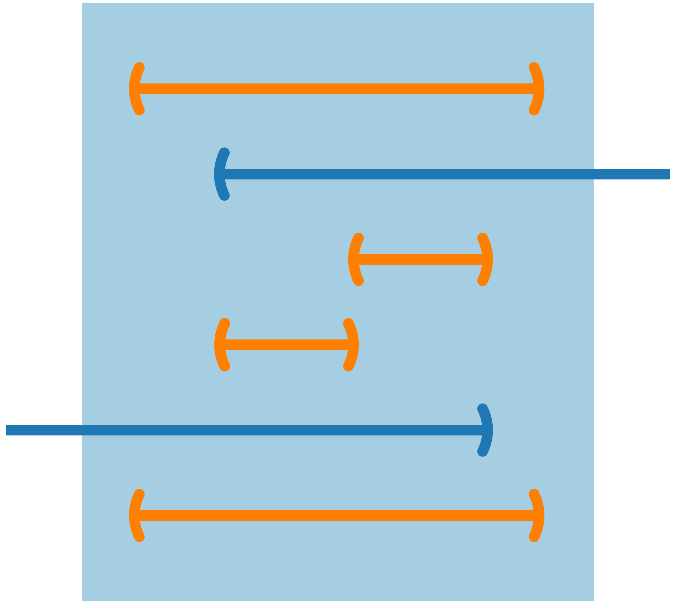




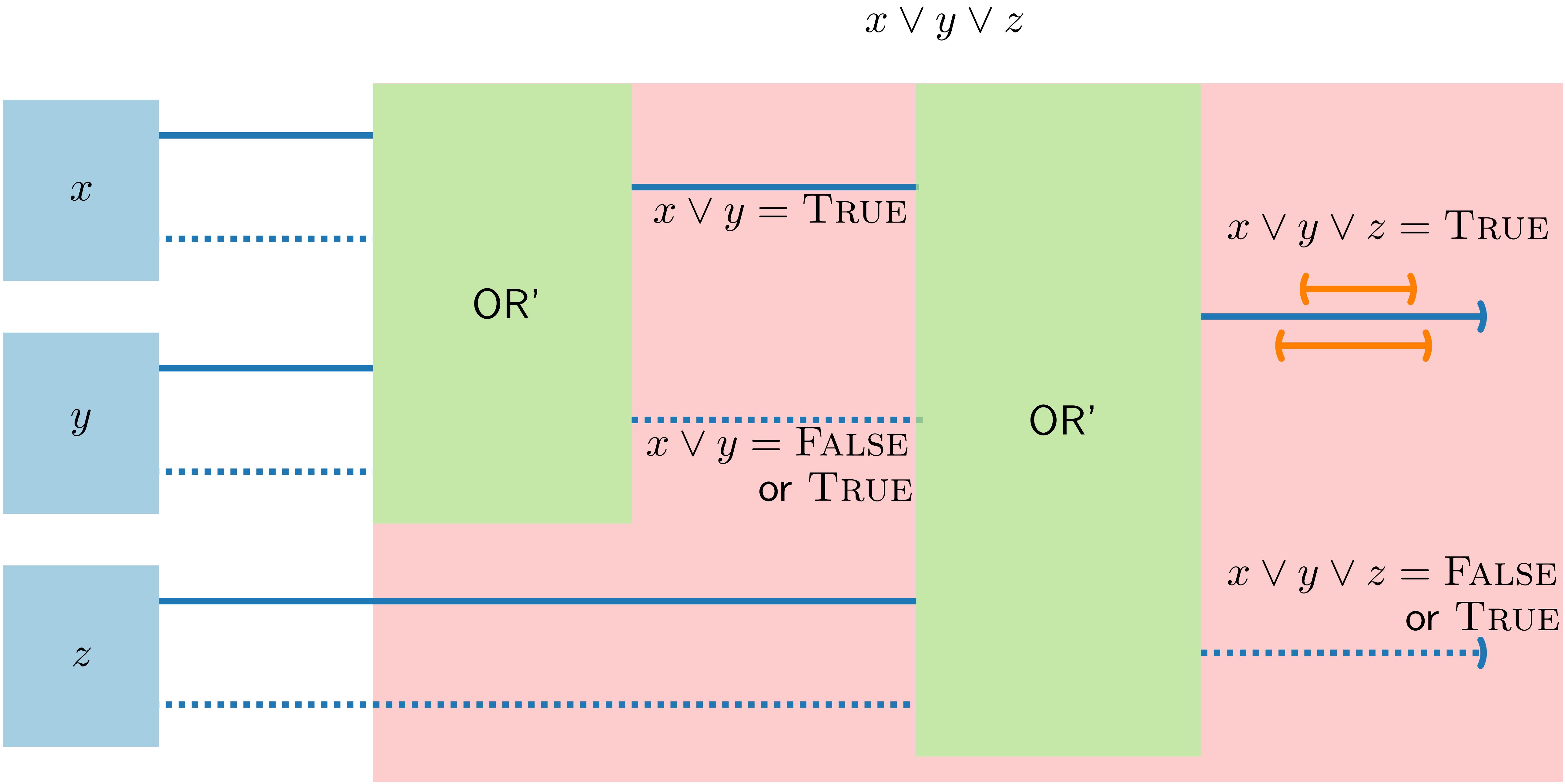
$x = \textsc{False}$



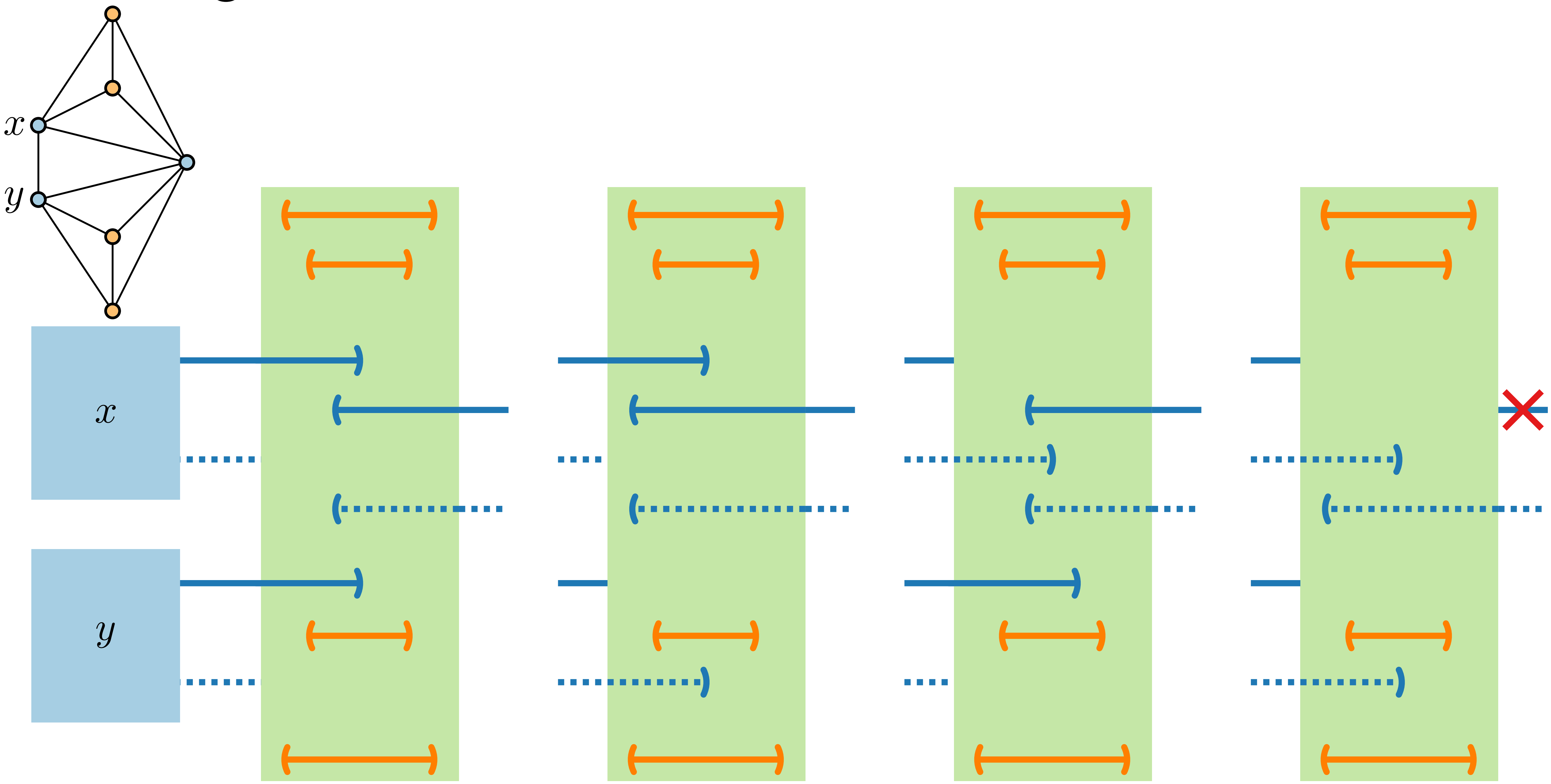
$x = \textsc{True}$



Clause Gadget



OR' Gadget



Discussion

- *Rectangular* ε-bar visibility representation extension can be solved in $O(n \log^2 n)$ time *for st-graphs*.
- ε-bar visibility representation extension is NP-complete.
- ε-bar visibility representation extension is NP-complete for (series-parallel) st-graphs when restricted to the *integer grid* (or if any fixed $\varepsilon > 0$ is specified).

Open Problems:

- Can ~~*rectangular*~~ ε-bar visibility representation extension be solved in polynomial time for st-graphs? For DAGs?
- Can *strong* bar visibility recognition / representation extension be solved in polynomial time for st-graphs?

Literature

Main source:

- [Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18]
  The Partial Visibility Representation Extension Problem

Referenced papers:

- [Tamassia, Tollis '86] Algorithms for visibility representations of planar graphs
- [Wismath '85] Characterizing bar line-of-sight graphs
- [Chaplick, Dorbec, Kratochvíl, Montassier, Stacho '14]
  Contact representations of planar graphs: Extending a partial representation is hard
- [Andreae '92] Some results on visibility graphs
- [Garg, Tamassia '01]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Gutwenger, Mutzel '01] A Linear Time Implementation of SPQR-Trees
- [de Berg, Khosravi '10] Optimal Binary Space Partitions in the Plane