Lehrstuhl für Informatik I
– Algorithmen und Komplexität –
Universität Würzburg

Würzburg, July 4, 2025

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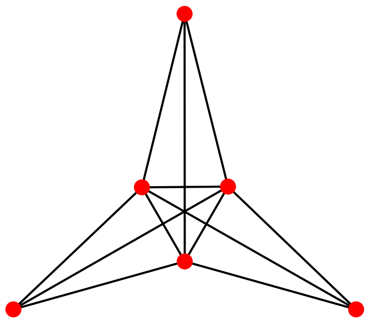
Exercise Sheet #9

Graph Visualization (SS 2025)

Exercise 1 – Finding a crossing number

What is the crossing number of the graph below and why? **3 Points**



Exercise 2 – Adding an edge with a minimum number of new crossings

Suppose that you are given a planar drawing Γ of a biconnected graph G where each edge is represented by a polygonal line (i.e., a sequence of line segments) with a constant number of bends. Let u and v be a pair of non-adjacent vertices of G.

Devise an efficient algorithm that adds the edge uv to the existing drawing causing as few crossings as possible. The new edge should also be drawn as a polygonal path, but the number of bends is not restricted. Show the correctness of your algorithm. Can you come up with a linear-time implementation? **7 Points**

Exercise 3 – Fixed linear crossing number

In the lecture we talked about the problem *fixed linear crossing number*, which is the crossing number for a fixed linear layout: For a graph G with given vertex numbering $V(G) = \{v_1, v_2, \ldots, v_n\}$, vertex v_i has position $(i, 0)$ and every edge is drawn as a semi-circle. The only decision when drawing an edge is, therefore, whether it is drawn in the halfplane above or below the x-axis. Given a graph G with numbered vertices and an integer k, it is NP-hard to decide whether a fixed linear layout with at most k crossings exists.

a) Devise an algorithm that decides for a graph with given vertex numbering whether a fixed linear layout with zero crossings exists. Can you implement your algorithm such that it runs in linear time? **6 Points**

b) Show that, when restricting the input graphs to matchings (i.e., all vertices have degree 1), the decision problem for the fixed linear crossing number problem is still NP-hard. **4 Points**

Exercise 4 – Tightness of the asymptotic lower bound on the crossing number

To show that the bound $\mathrm{cr}(G) \in \Omega(m^3/n^2)$ from the lecture is asymptotically tight, find a family of graphs that contains, for every combination of n and m that makes sense, a graph G with n vertices and m edges that admits a drawing with $O(m^3/n^2)$ crossings. **4 Extra points**

Note that the family of complete graphs is *not* a solution since, for every n, there is just one graph in the family, and it has $m \in \Theta(n^2)$ many edges. In other words, all graphs of the family have the same density. You should find a graph family for any (integer) edge density $k = m/n$.

This assignment is due at the beginning of the next lecture, that is, on July 11 at 10:15 am. Please submit your solutions via WueCampus. The questions can be asked in the tutorial session on July 9 at 16:00 and the solutions will be discussed two weeks after that on July 16.