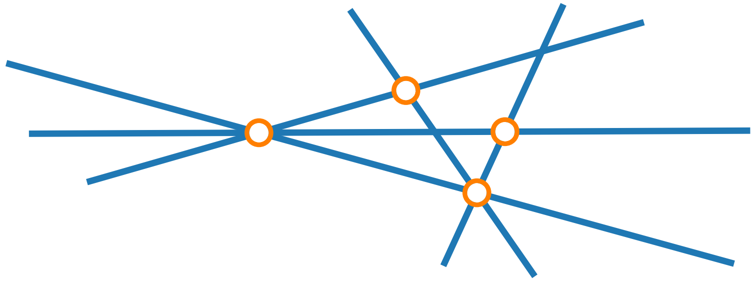
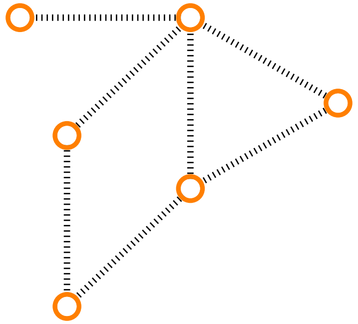
Visualization of Graphs

Lecture 9:

The Crossing Lemma

and Its Applications





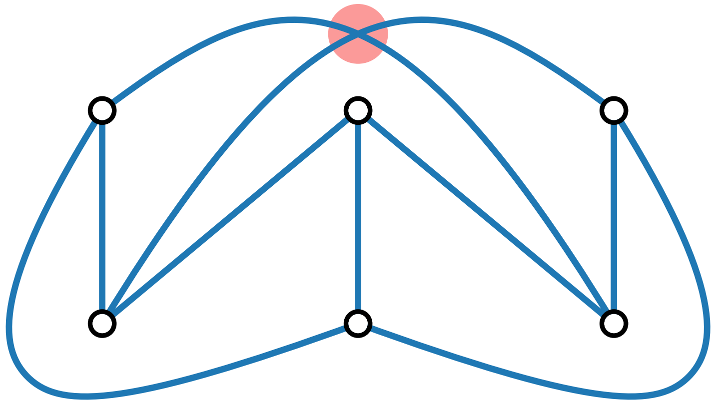
Alexander Wolff

Summer term 2025

Crossing Number and Topological Graphs

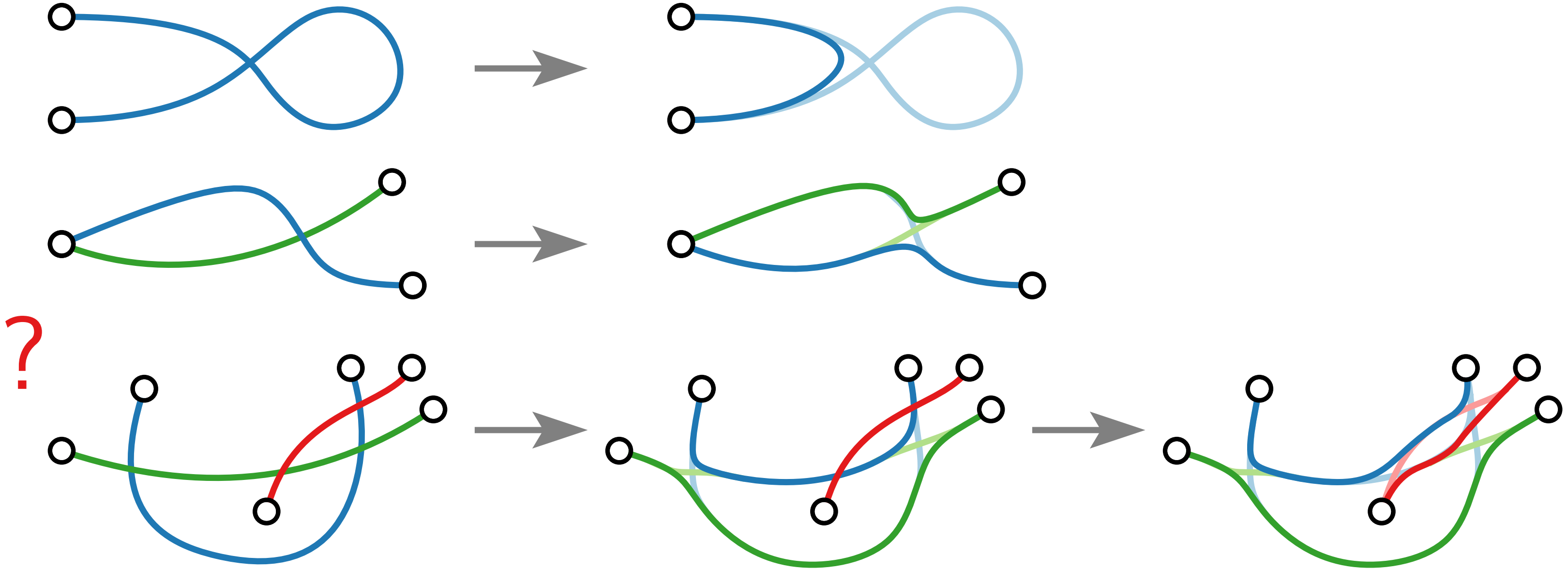
For a graph G, the **crossing number** $\mathrm{cr}(G)$ is the smallest number of pairwise edge crossings in a drawing of G (in the plane).

Example.
$\mathrm{cr}(K_{3,3}) = 1$



In a crossing-minimal drawing of G

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once, ?
- and, w.l.o.g., at most two edges intersect at the same point.



crossings reduced; so, an iterative procedure terminates

Such a drawing is called a **topological drawing** of G.

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Proof sketch.

Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

Every non-planar graph has K_5 or $K_{3,3}$ as a minor, so there are two paths that cross an odd number of times.

Hence, there must be two edges on these paths that cross an odd number of times. □

Hanani–Tutte Theorem

Theorem. [Hanani '43, Tutte '70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

Corollary. ocr$(G) = 0 \Rightarrow$ pcr$(G) = 0 \Rightarrow$ cr$(G) = 0$

Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with ocr$(G) <$ cr$(G) \leq 10$

Theorem. [Pelsmajer, Schaefer & Štefankovič '07] [Pach & Tóth '00]
If Γ is a drawing of G and E_0 is the set of edges that cross any other edge an even number of times in Γ, then G can be drawn such that no edge in E_0 is involved in any crossings and no new pairs of edges cross.

The **pairwise crossing number** pcr(G) of G is the smallest number of pairs of edges that cross in a drawing of G.

By definition ocr$(G) \leq$ pcr$(G) \leq$ cr(G)

The **odd crossing number** ocr(G) of G is the smallest number of pairs of edges that cross oddly in a drawing of G.

Is ocr$(G) =$ cr(G)? No!

Is ocr$(G) =$ pcr(G)? No!

Is pcr$(G) =$ cr(G)? Open!

Note that, in the resulting drawing of G, an edge may cross some edges an odd number of times and some other edges an even number of times. So, no implications on ocr$(G) =$ pcr(G).

Theorem. [Pelsmajer, S. & Š.'08, Tóth'08]
There exist graphs where ocr$(G) <$ pcr(G).

Computing the Crossing Number

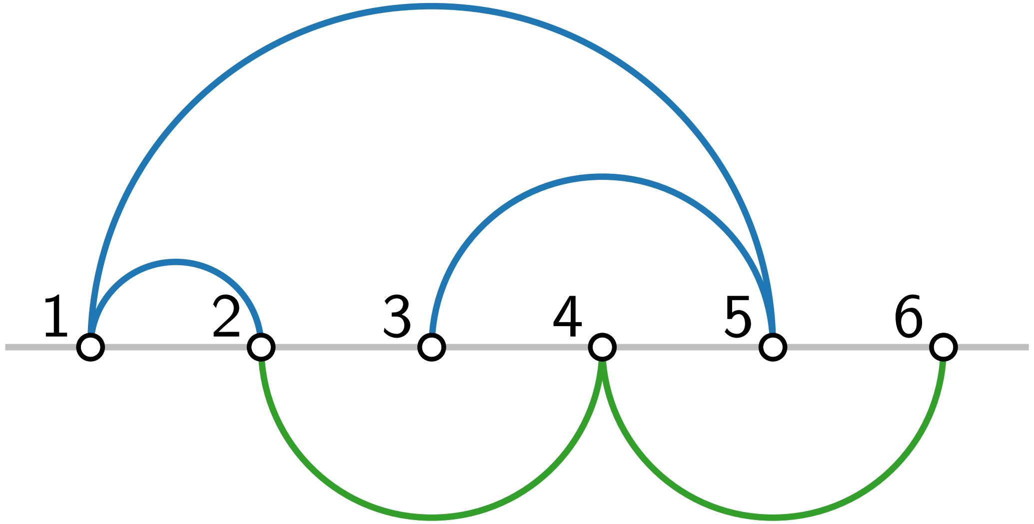
- Computing cr(G) is NP-hard. [Garey & Johnson '83]
  ... even if G is a planar graph plus one edge! [Cabello & Mohar '08]
- cr(G) often unknown, only conjectures exist
  (for K_n it is only known for up to ≈ 12 vertices)
- In practice, cr(G) is often not computed directly but rather drawings of G are optimized with
  - force-based methods,
  - multidimensional scaling,
  - heuristics, . . .

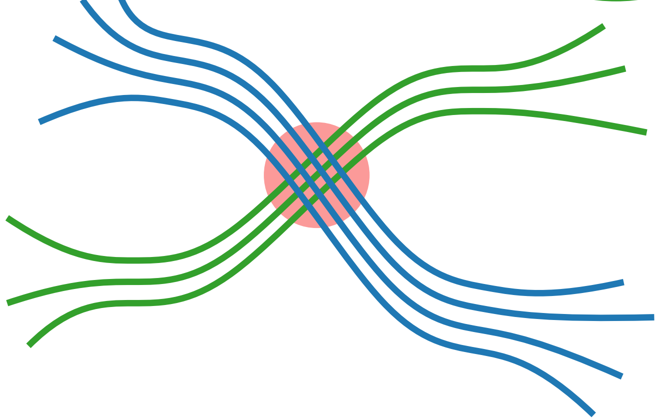
For exact computations,
check out `http://crossings.uos.de`!

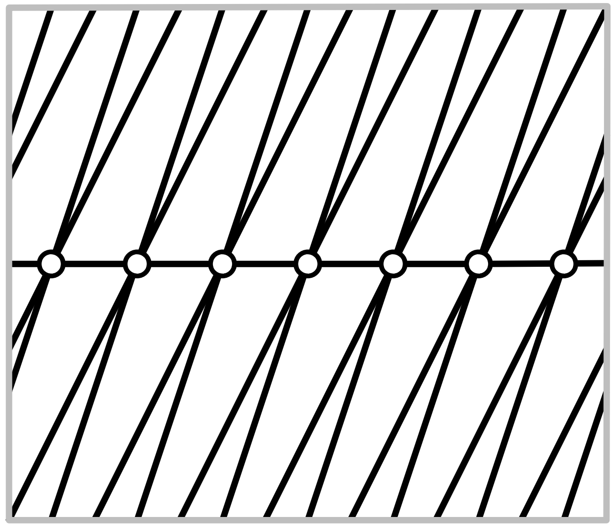
- cr(G) is a measure of how far G is from being planar.
- For planarization, where we replace crossings with dummy vertices, also only heuristic approaches are known.

Other Crossing Numbers

- Schaefer [Sch20] has a survey on many variants of crossing numbers (including precise definitions).
- One-sided crossing minimization (see lecture 8)
- Fixed linear crossing number
- Book embeddings (vertices on a line, edges assigned to few "pages" where edges do not cross)
- Crossings of edge bundles
- On other surfaces, such as donuts
- Weighted crossings
- Crossing minimization is NP-hard for most variants.





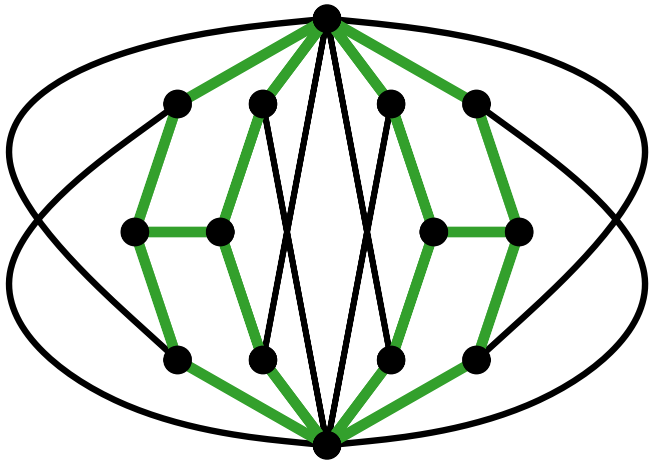


K_7 on the torus

Rectilinear Crossing Number

Definition.
For a graph G, the **rectilinear (straight-line) crossing number** $\overline{\text{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of G.

Separation.
$\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.

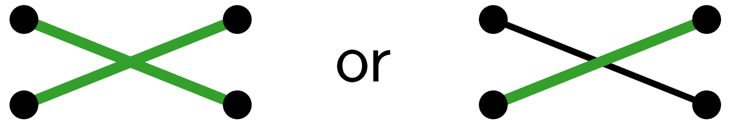


G_1

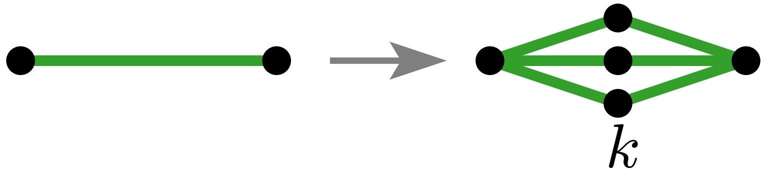
Even more . . .

Lemma 1. [Bienstock, Dean '93]
For every $k \geq 4$, there exists a graph G_k with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$.

- Each straight-line drawing of G_1 has at least one crossing of the following types:



- From G_1 to G_k do



Bounds for Complete Graphs

Theorem. **Conjecture.** [Guy '60]

$$\mathrm{cr}(K_n) \underset{=}{\cancel{\le}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is tight for $n \le 12$.

complete bipartite graph with $m \times n$ edges

Theorem. **Conjecture.** [Zarankiewicz '54, Urbaník '55]

$$\mathrm{cr}(K_{m,n}) \underset{=}{\cancel{\le}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{m-1}{2}\right\rceil$$

Pál Turán
*1910 – 1976
Budapest, Hungary

Turán's brick factory problem (1944)

© TruckinTim

Bounds for Complete Graphs

Theorem. **Conjecture.** [Guy '60]

$$\mathrm{cr}(K_n) \underset{=}{\cancel{\leq}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is tight for $n \leq 12$.

complete bipartite graph with $m \times n$ edges

Theorem. **Conjecture.** [Zarankiewicz '54, Urbaník '55]

$$\mathrm{cr}(K_{m,n}) \underset{=}{\cancel{\leq}} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{m}{2}\right\rceil\left\lceil\frac{m-1}{2}\right\rceil$$

Theorem. [Lovász et al. '04, Aichholzer et al. '06]

$$\left(\frac{3}{8}+\varepsilon\right)\binom{n}{4} + O(n^3) < \overline{\mathrm{cr}}(K_n) < 0.3807\binom{n}{4} + O(n^3)$$

Exact numbers are known for $n \leq 27$.

Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`

First Lower Bounds on cr(G)

Lemma 2.
For a graph G with n vertices and m edges,

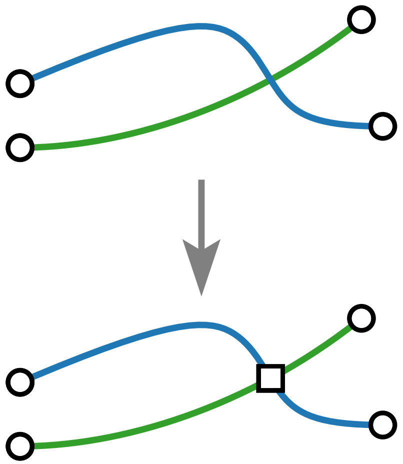
$$\text{cr}(G) \geq m - 3n + 6.$$

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

Proof.

- Consider a drawing of G with cr(G) crossings.
- Obtain a graph H by turning crossings into dummy vertices.
- H has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- H is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6. \qquad \square$$



First Lower Bounds on cr(G)

Lemma 3.
For a non-planar graph G with n vertices and m edges,

$$\text{cr}(G) \geq r \cdot \binom{\lfloor m/r \rfloor}{2} \in \Omega\left(\frac{m^2}{n}\right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of G.

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

Proof sketch.

- Take $\lfloor m/r \rfloor$ edge-disjoint subgraphs $G_1, G_2, \ldots, G_{\lfloor m/r \rfloor}$ of G with (at least) r edges.
- In the best case, they are all planar.
- For every $i < j$, any edge of G_j induces at least one crossings with G_i. (Otherwise, we could add an edge to G_i and obtain a planar subgraph of G with $r + 1$ edges.)
- So, for each of the $\binom{\lfloor m/r \rfloor}{2}$ pairs of subgraphs, there are at least r crossings.

The Crossing Lemma

- In 1973 Erdős and Guy conjectured that $\text{cr}(G) \in \Omega(m^3/n^2)$.

- In 1982 Leighton and, independently, Ajtai, Chávtal, Newborn, and Szemerédi showed that

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

- Bound is asymptotically tight.

- Result stayed hardly known until Székely demonstrated its usefulness (in 1997).

- We go through the proof of Chazelle, Sharir, and Welzl (see "THE BOOK").

- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman [CGTA 2019].

The Crossing Lemma

Crossing Lemma.
For a graph G with n vertices and m edges, $m \geq 4n$,

$$\text{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

Proof.

- Consider a crossing-minimal drawing of G.
- Let p be a number in $(0, 1]$.
- Keep every vertex of G independently with probability p.
- $G_p =$ remaining graph (with drawing Γ_p).
- Let n_p, m_p, X_p be the random variables counting the numbers of vertices / edges / crossings of Γ_p, resp.
- By Lemma 2, $\text{cr}(G_p) - m_p + 3n_p \geq 6$.

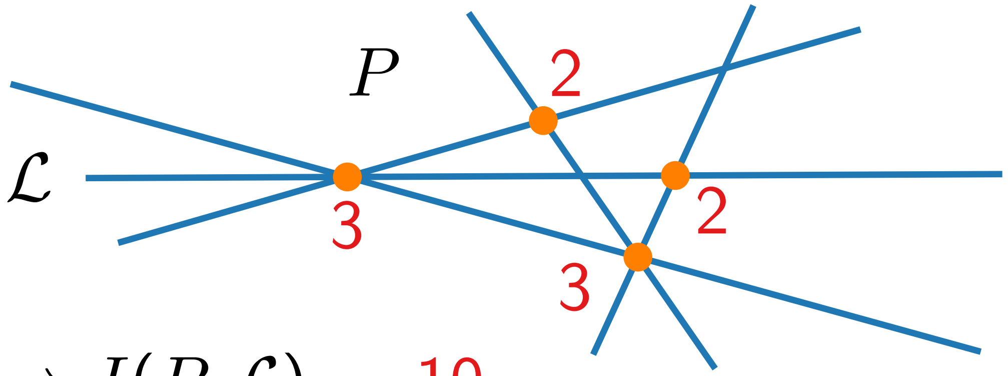
$\text{cr}(G) \geq m - 3n + 6$

$\Rightarrow \text{E}[X_p - m_p + 3n_p] \geq 0.$

- $\text{E}[n_p] = pn$ and $\text{E}[m_p] = p^2 m$
- $\text{E}[X_p] = p^4 \text{cr}(G)$
- $0 \leq \text{E}[X_p] - \text{E}[m_p] + 3\text{E}[n_p]$
  $= p^4 \text{cr}(G) - p^2 m + 3pn$
- $\text{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64}\frac{m^3}{n^2}$ □

Application 1: Point–Line Incidences

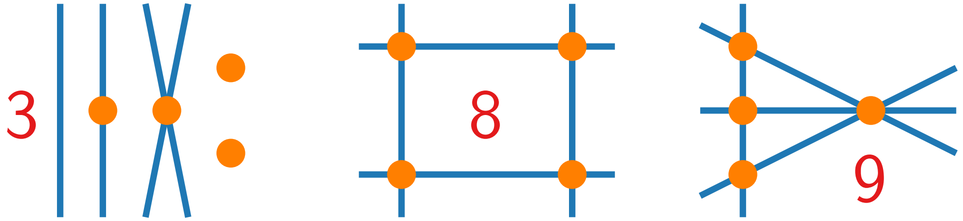
For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P, \mathcal{L}) = \#$ point–line incidences in $(P, \mathcal{L})$.



$\Rightarrow I(P, \mathcal{L}) = 10$

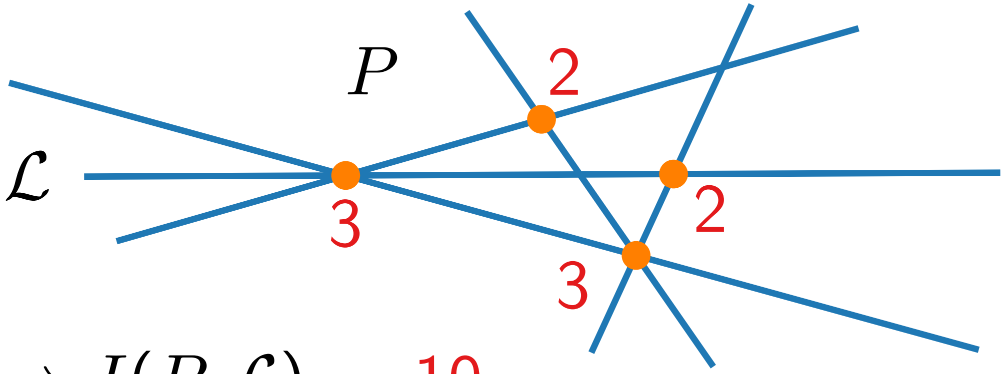
Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n, k) \leq 2.7 n^{2/3} k^{2/3} + 6n + 2k.$

- Define $I(n, k) = \max\limits_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.
- For example: $I(4, 4) = 9$



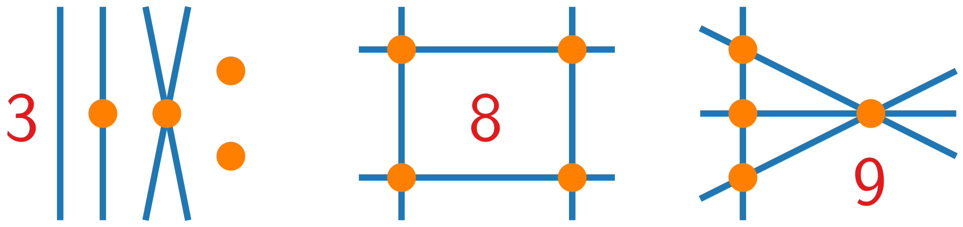
Application 1: Point–Line Incidences

For a set $P \subset \mathbb{R}^2$ of points and a set $\mathcal{L}$ of lines, let $I(P,\mathcal{L}) = \#$ point–line incidences in $(P,\mathcal{L})$.



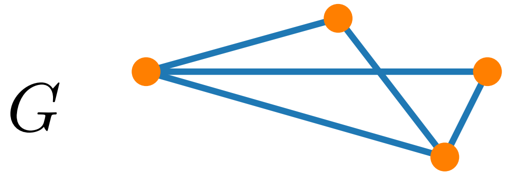
$\Rightarrow I(P,\mathcal{L}) =$ 10

- Define $I(n,k) = \max_{|P|=n, |\mathcal{L}|=k} I(P,\mathcal{L})$.
- For example: $I(4,4) =$ 9



Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n,k) \leq c(n^{2/3}k^{2/3} + n + k)$.

Proof.



- cr$(G) \leq k^2/2$
- #(points on a line ℓ) $-$ 1 $=$ #(edges on ℓ)
  $\Rightarrow I(n,k) - k = m$ (sum up over $\mathcal{L}$ in an "optimal" instance)
- If $m \leq 4n$, then $I(n,k) - k = m \leq 4n$.
  $\Rightarrow I(n,k) \leq 4n + k \leq c(n + k + n^{2/3}k^{2/3})$
- Otherwise, employ the Crossing Lemma:

$$\frac{m^3}{64n^2} \leq \text{cr}(G) \leq k^2/2 \quad \Rightarrow \frac{(I(n,k)-k)^3}{64n^2} \leq k^2/2$$

$$\Leftrightarrow I(n,k) \leq c(n^{2/3}k^{2/3} + k)$$

$$\leq c(n^{2/3}k^{2/3} + k + n).$$

☐

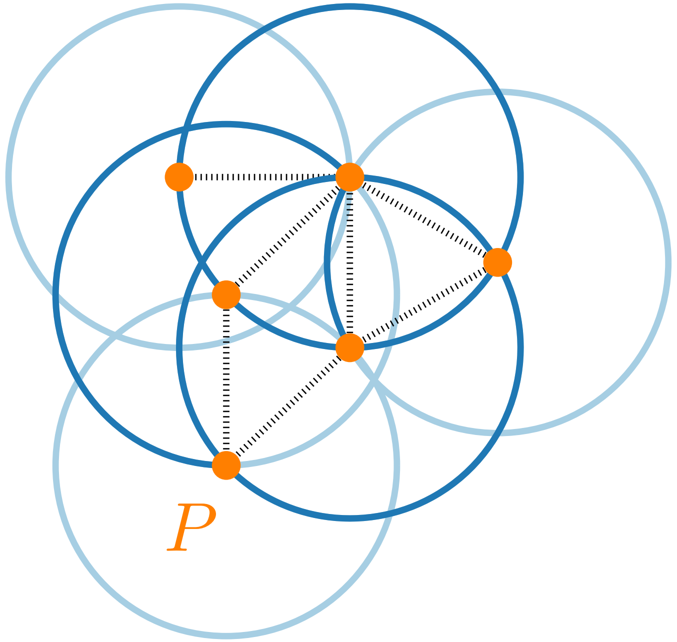
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P) =$ number of pairs in P at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

Theorem 2.
[Spencer, Szemerédi, Trotter '84, Székely '97]
$U(n) < 6.7n^{4/3}$

Proof sketch.



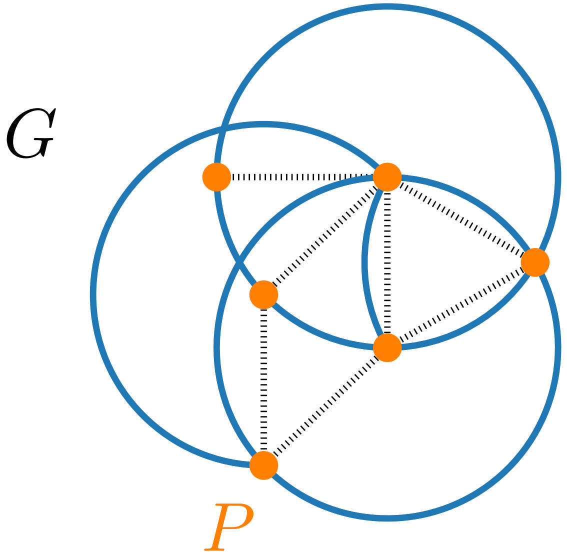
Application 2: Unit Distances

For a set $P \subset \mathbb{R}^2$ of points, define

- $U(P) =$ number of pairs in P at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

Theorem 2.
[Spencer, Szemerédi, Trotter '84, Székely '97]
$U(n) < 6.7n^{4/3}$

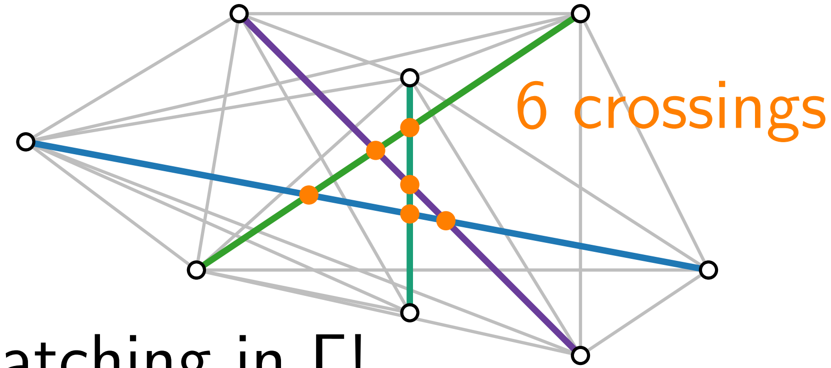
Proof sketch.



- $U(P) \le c''m$ some constant number of edges in G
- $\mathrm{cr}(G) \le 2\binom{n}{2} \le n^2$ (Circles intersect each other at most twice.)
- $n^2 \ge \mathrm{cr}(G) \ge \frac{m^3}{64n^2} \ge$ by the Crossing Lemma.

Application 3: Expected Number of Crossings in a Matching

Given set of n points (in general position, n even) –
what is the average number of crossings in a perfect matching?



Point set spans drawing Γ of K_n.
We will analyze the number of crossings in a **random** perfect matching in Γ!

Number of crossings in Γ $\geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$ [Lovász et al. '04, Aichholzer et al. '06]
Number of edges in K_n: $\binom{n}{2}$
Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3\binom{n}{4}$

Pick two random edges e_1 and e_2.
$\Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \overline{\text{cr}}(K_n)/\text{pot}(K_n) > \frac{1}{8}$.

Pick random perfect matching M; it has $n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges.
By linearity of expectation,
the expected number of crossings in M is $> \frac{1}{8}\binom{n/2}{2} = \frac{1}{64}n(n-2) \in \Omega(n^2)$. □

Literature

- [Aigner, Ziegler] Proofs from THE BOOK [https://doi.org/10.1007/978-3-662-57265-8]
- [Schaefer ’20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao’s blog post “The crossing number inequality” from 2007
- [Hanani ’43] Über wesentlich unplättbare Kurven im dreidimensionalen Raume
- [Tutte ’70] Toward a theory of crossing numbers
- [Pach & Tóth ’00] Which crossing number is it anyway?
- [Pelsmajer, Schaefer & Štefankovič ’07] Removing even crossings
- [Pelsmajer, Schaefer & Štefankovič ’08] Odd Crossing Number and Crossing Number Are Not the Same
- [Tóth ’08] Note on the Pair-crossing Number and the Odd-crossing Number
- [Garey, Johnson ’83] Crossing number is NP-complete
- [Bienstock, Dean ’93] Bounds for rectilinear crossing numbers
- [Lovász et al. ’04] Towards a theory of geometric graphs
- [Aichholzer et al. ’06] On the Crossing Number of Complete Graphs
- [Székely ’97] Crossing Numbers and Hard Erdős Problems in Discrete Geometry
- Documentary/Biography “N Is a Number: A Portrait of Paul Erdős”
- Exact computations of crossing numbers: `http://crossings.uos.de`