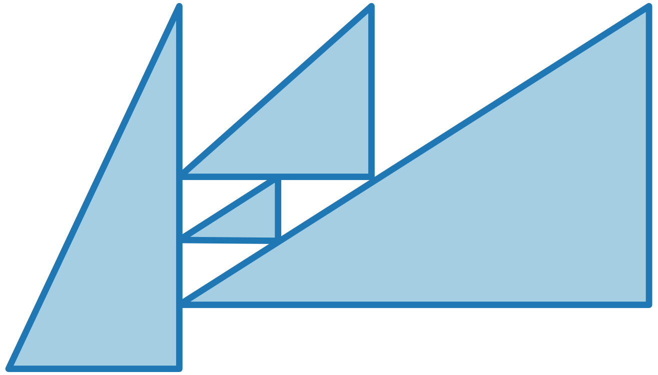
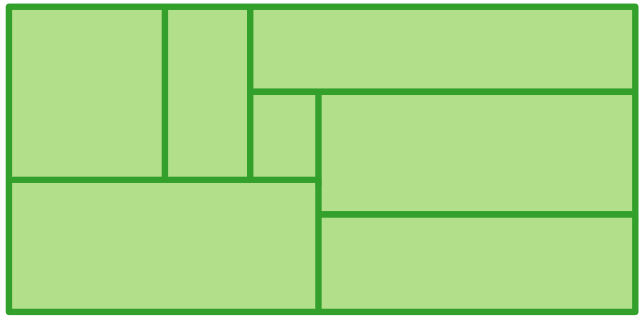
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 7:
Contact Representations of Planar Graphs: Triangle Contacts and Rectangular Duals



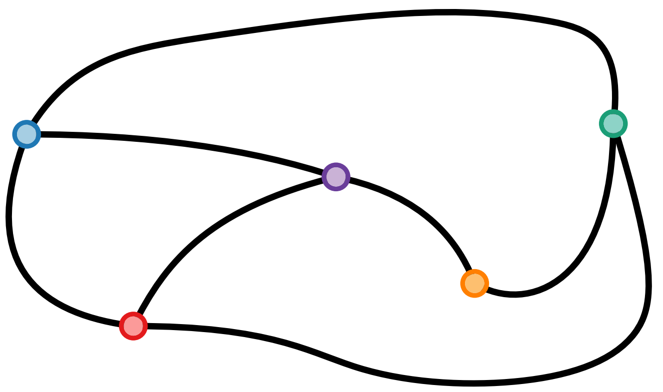


Alexander Wolff

Summer term 2025

Intersection Representation of Graphs

In an **intersection representation** of a graph,
– each vertex is represented by a set



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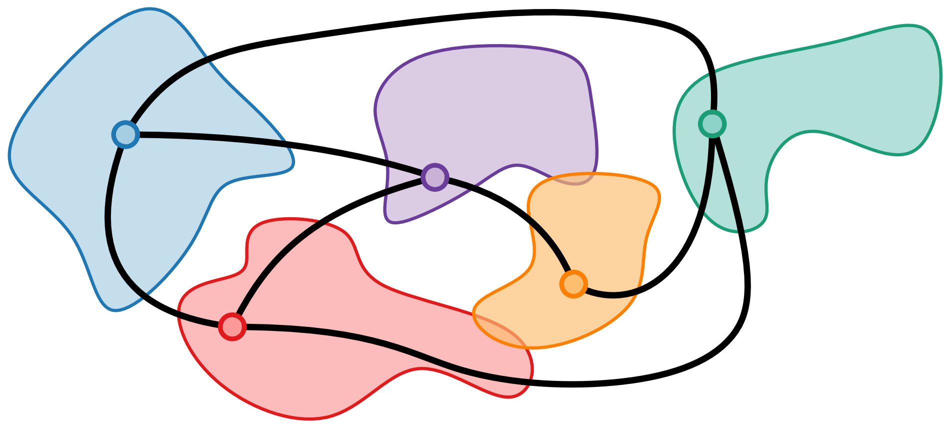

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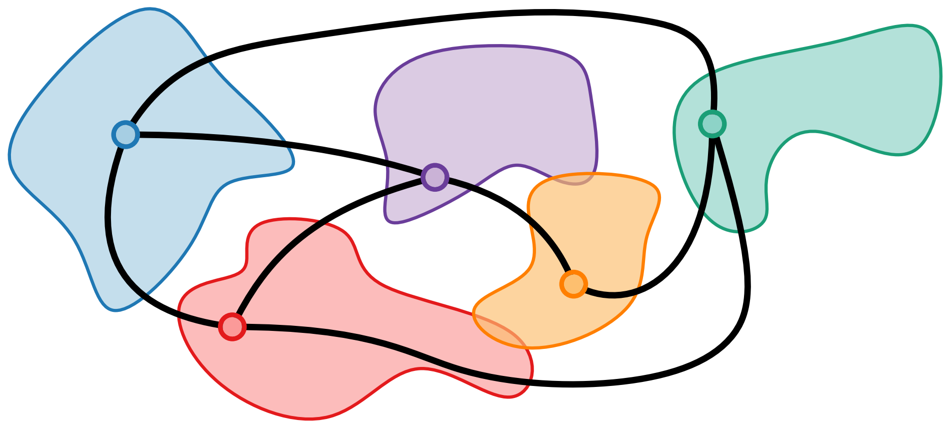

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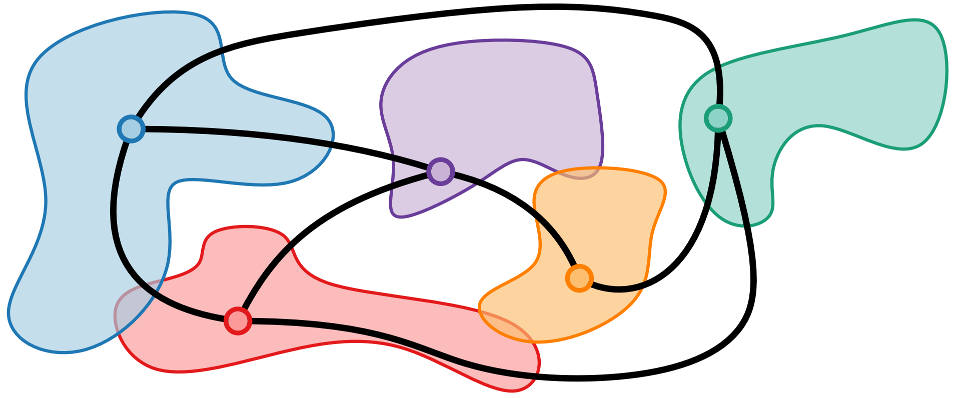

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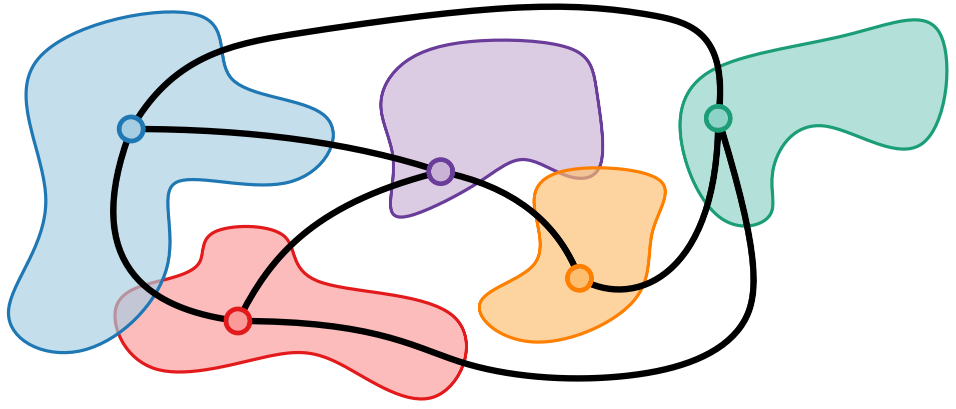

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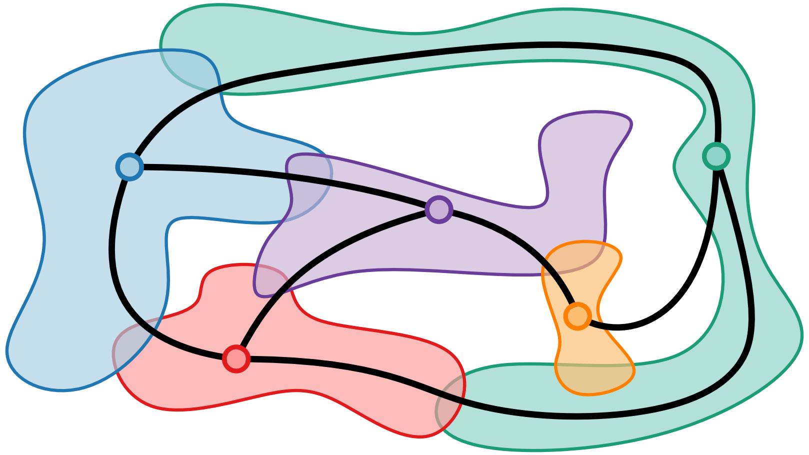
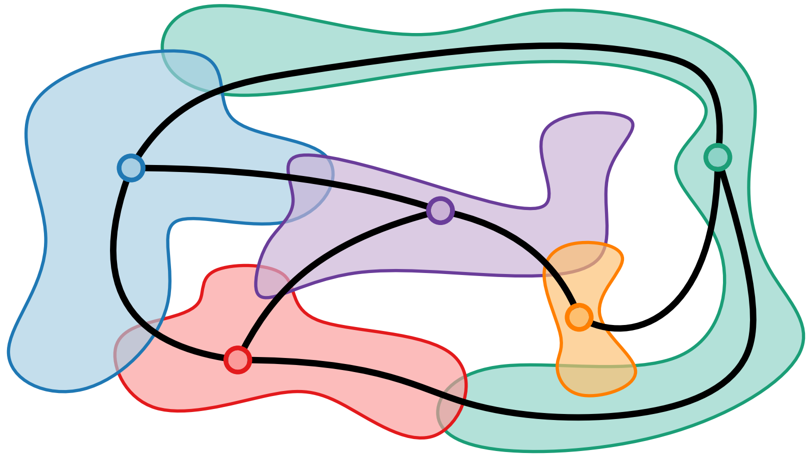


Intersection Representation of Graphs

In an **intersection representation** of a graph,
– each vertex is represented by a set
– such that two sets intersect $\Leftrightarrow$ the corresponding vertices are adjacent.

For a collection $\mathcal{S}$ of sets,
the **intersection graph** $G(\mathcal{S})$ of $\mathcal{S}$
has vertex set $\mathcal{S}$ and edge set
$\{\{S, S'\} \colon S, S' \in \mathcal{S}, S \neq S',$ and $S \cap S' \neq \emptyset\}$.



Contact Representation of Graphs

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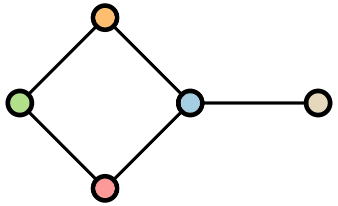

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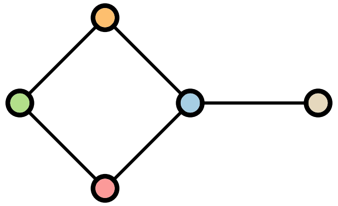

Represent each vertex v by a geometric object $S(v)$

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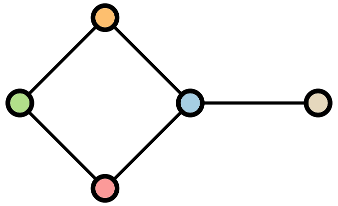

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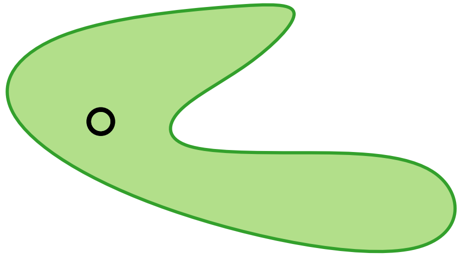

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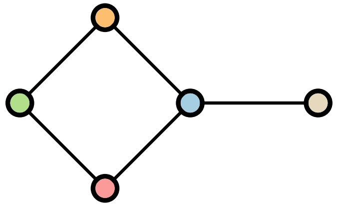

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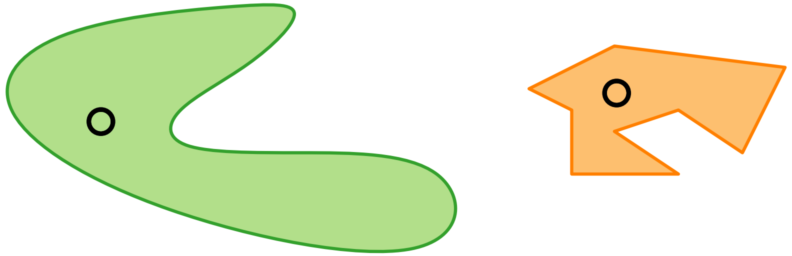

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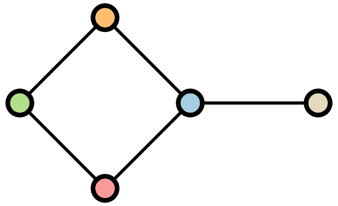

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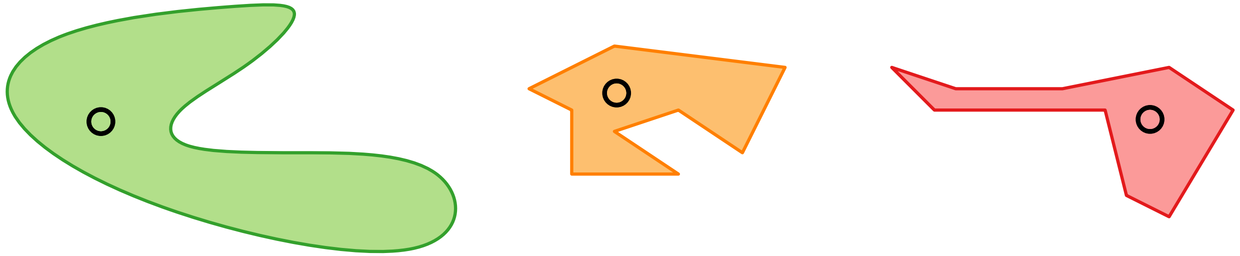

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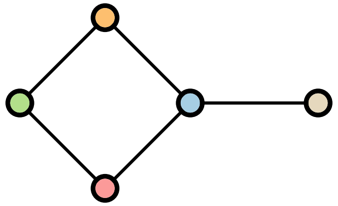

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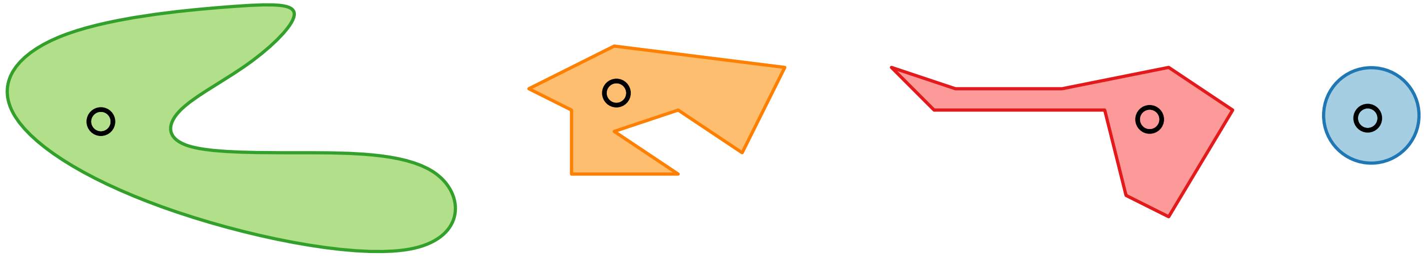

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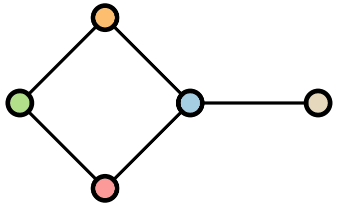

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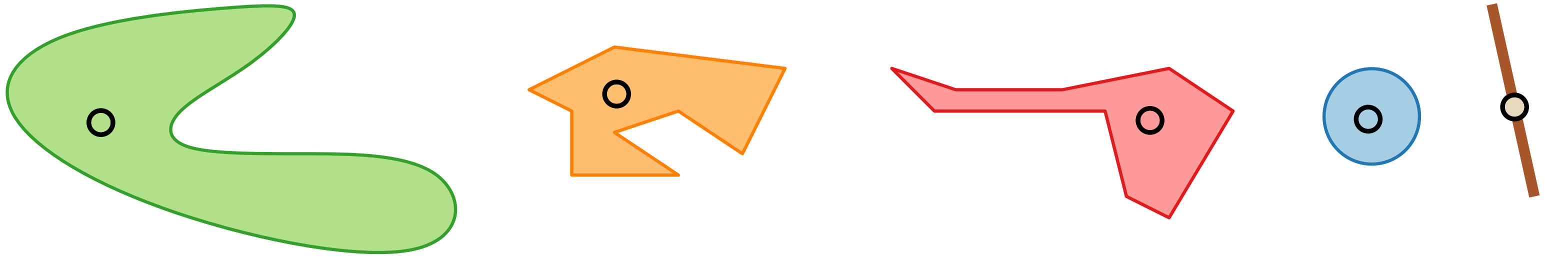

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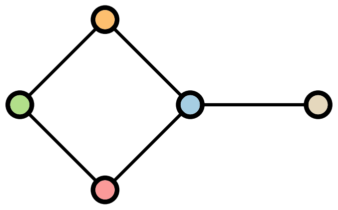

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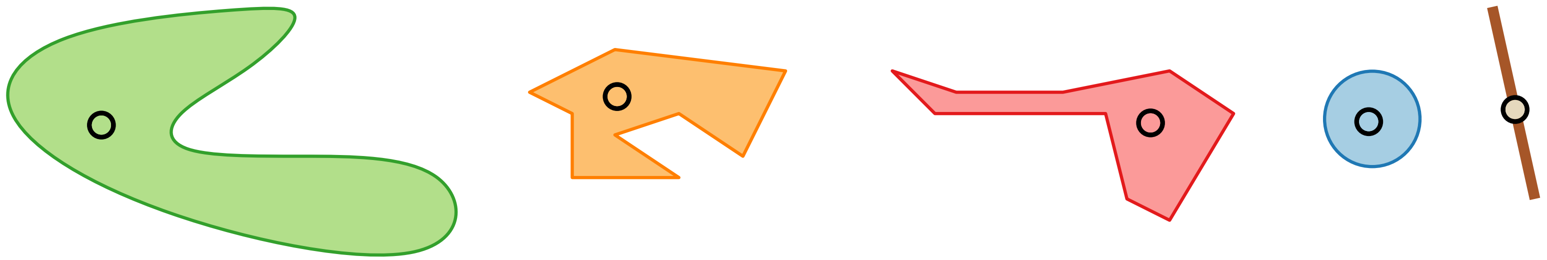

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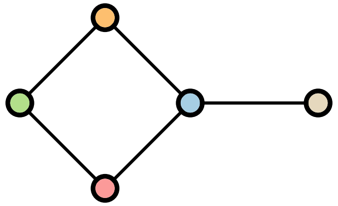

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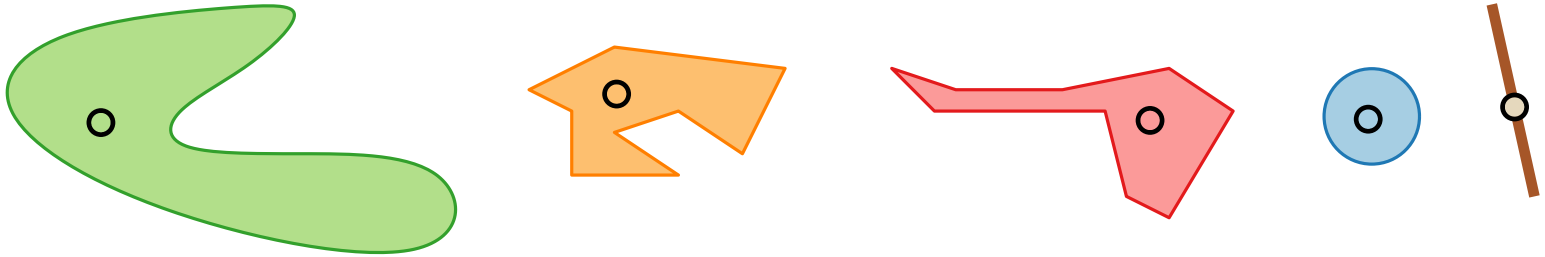

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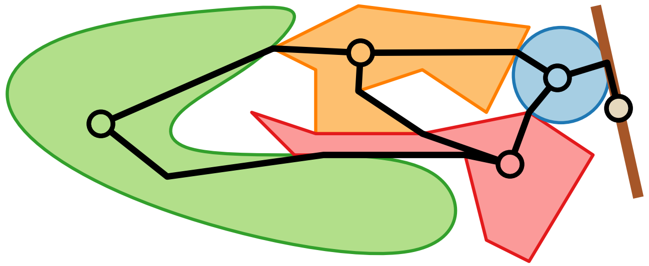

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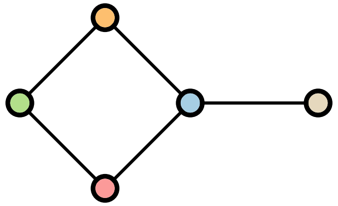

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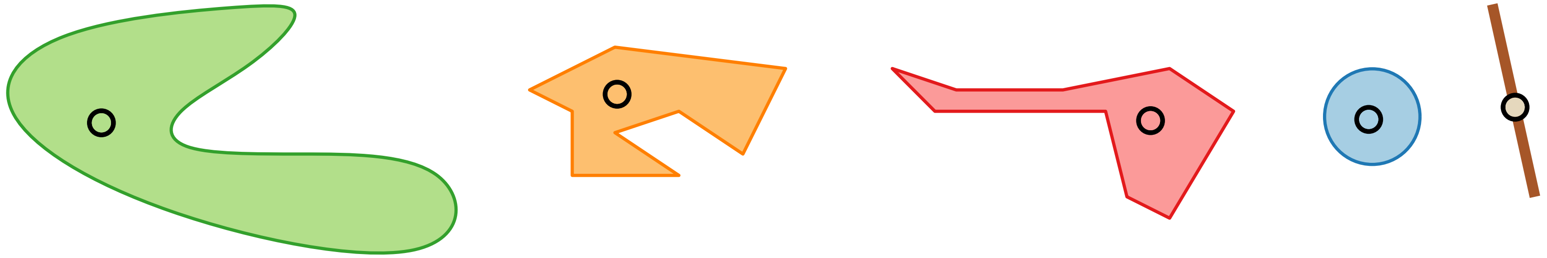

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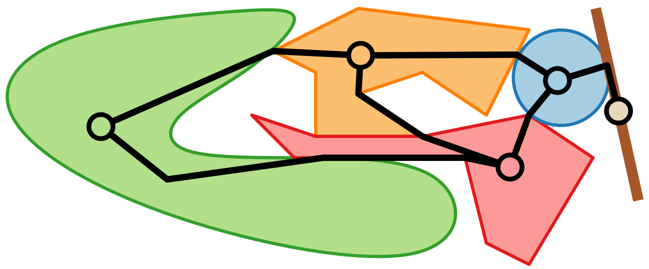

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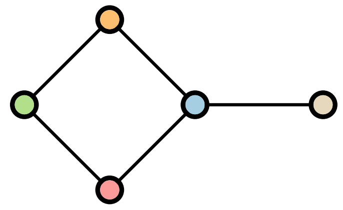

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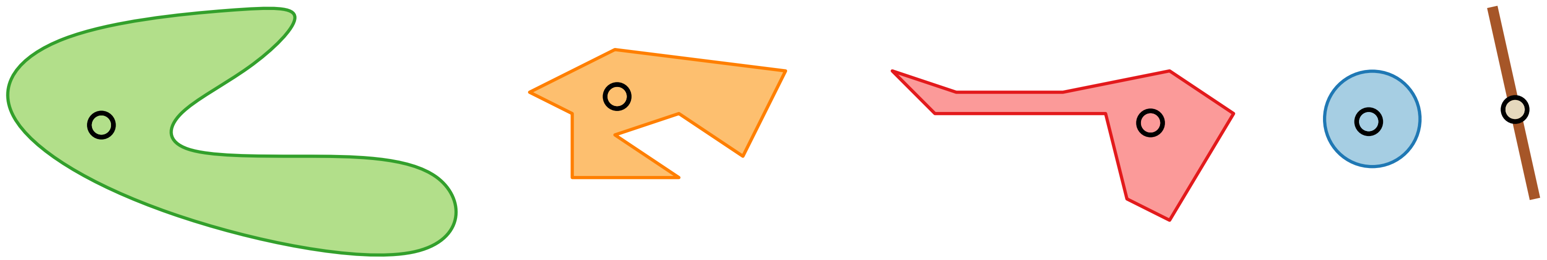

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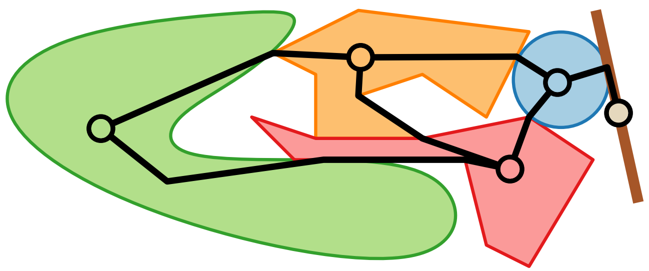

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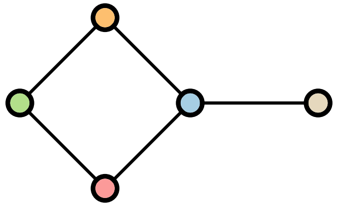

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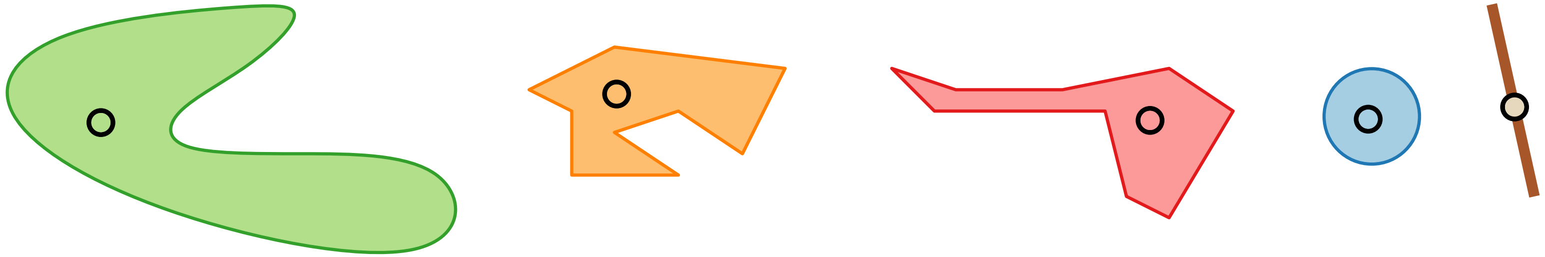

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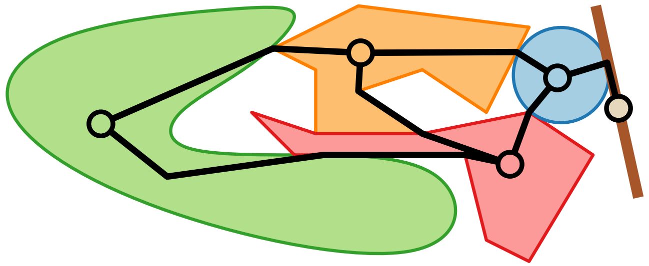

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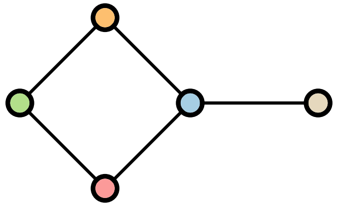

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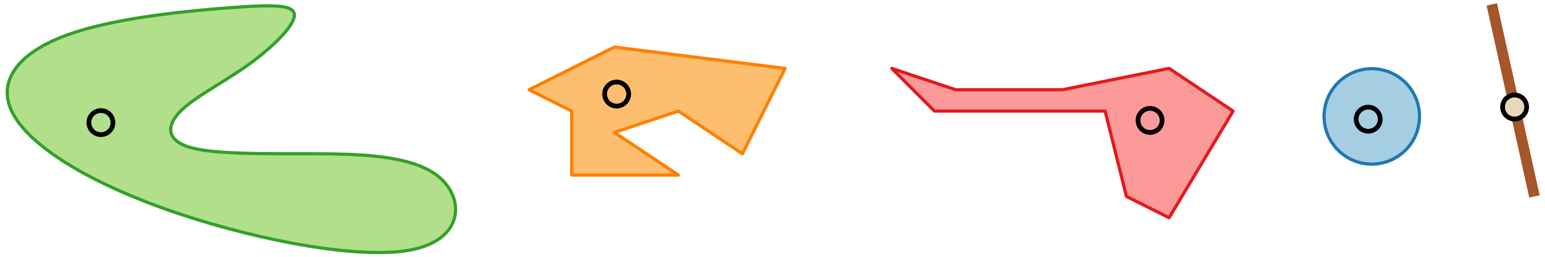

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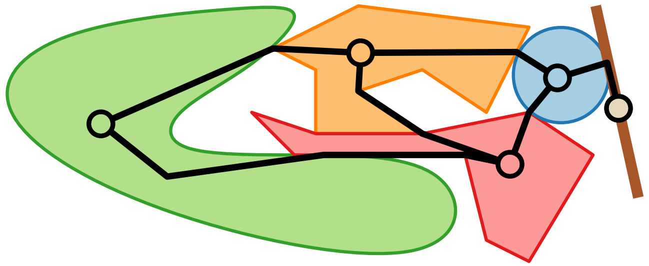

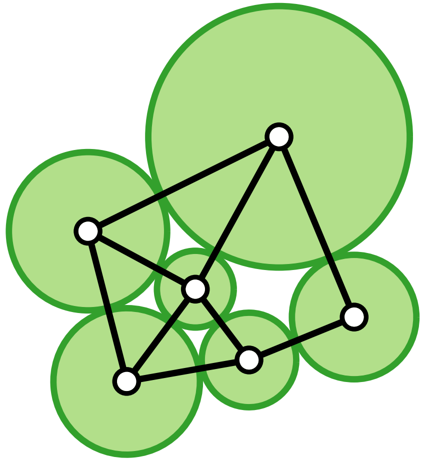

disks

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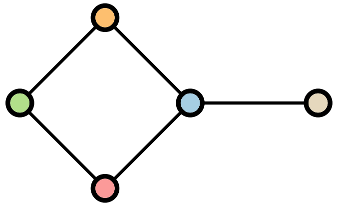

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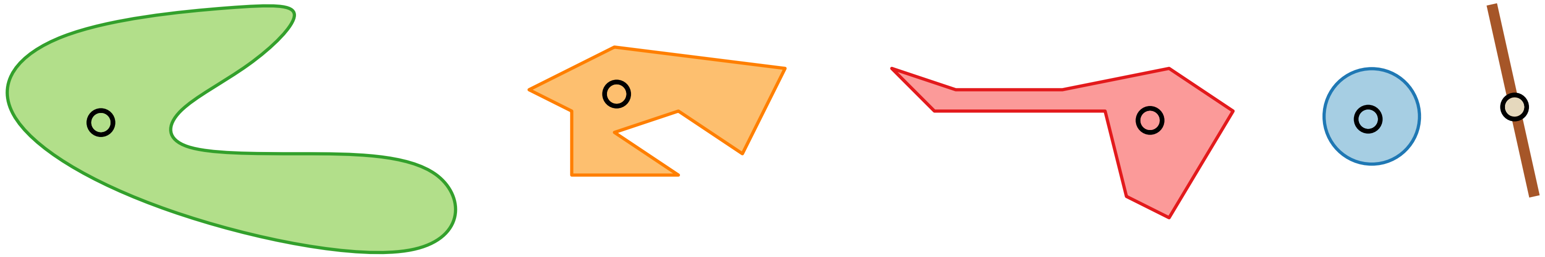

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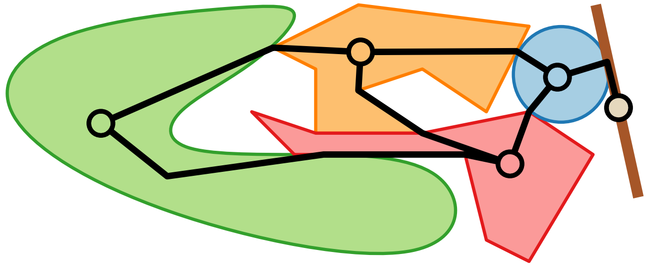

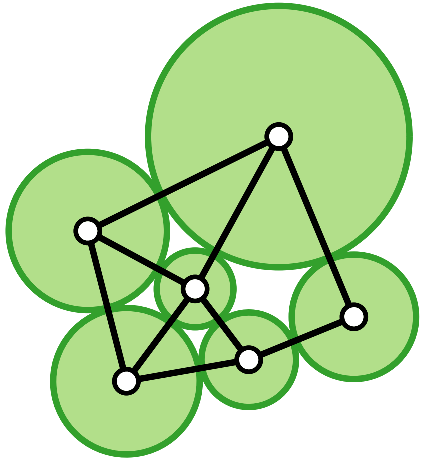

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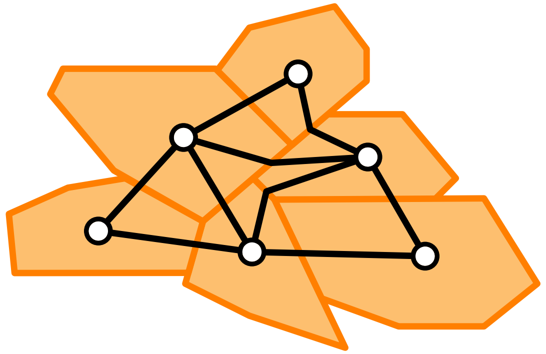

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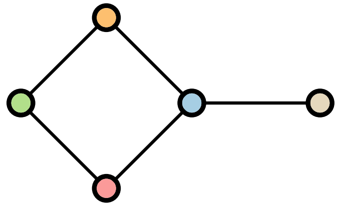

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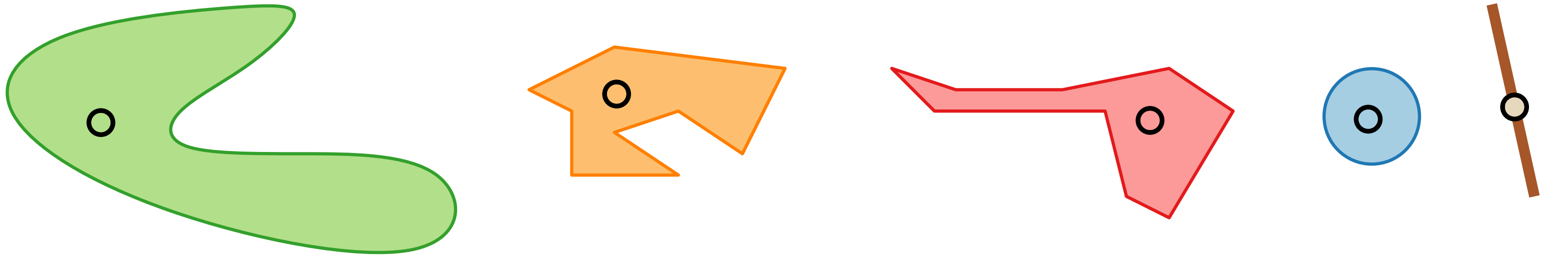

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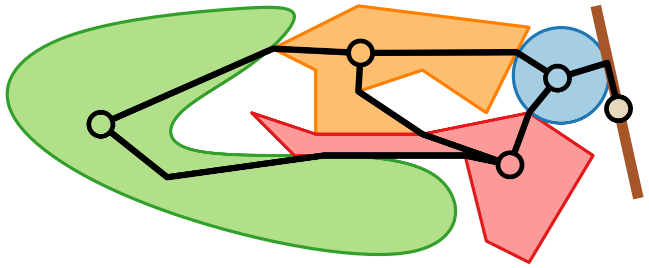

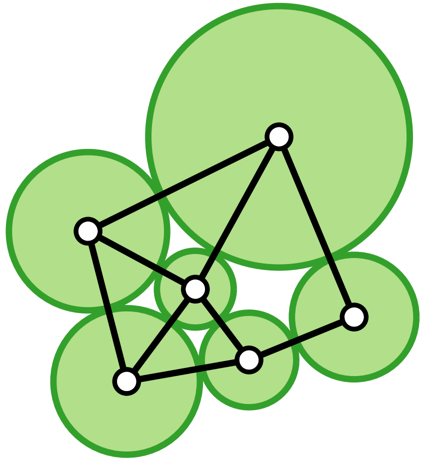

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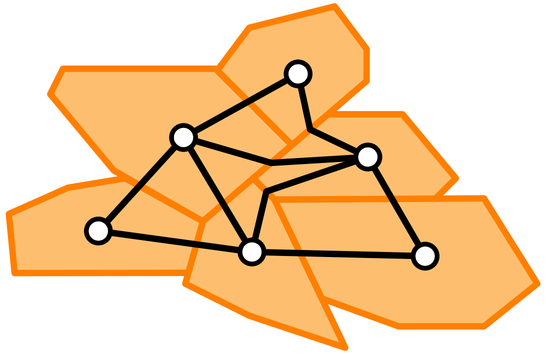

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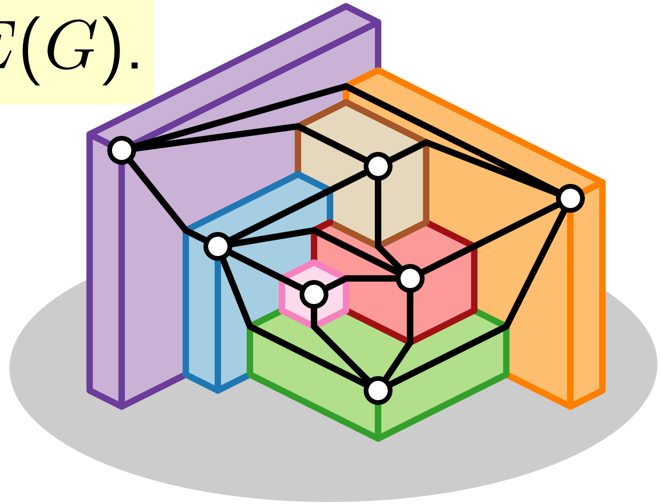

rectangular cuboids

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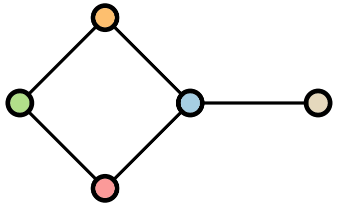

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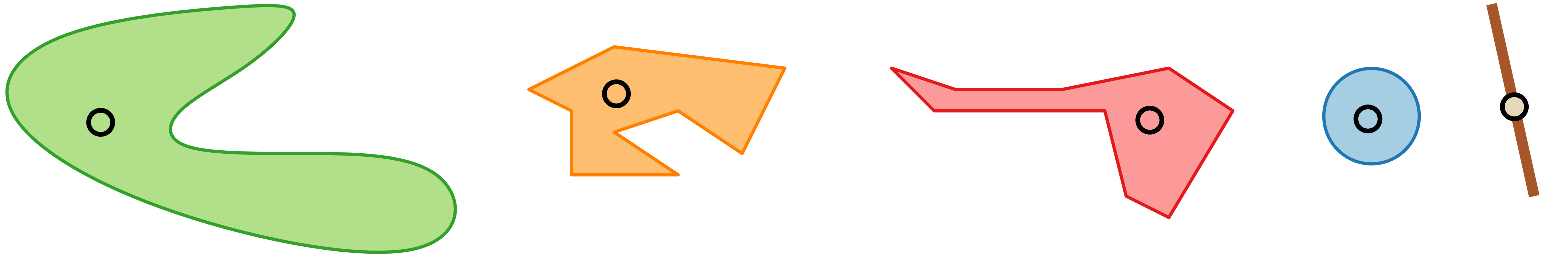

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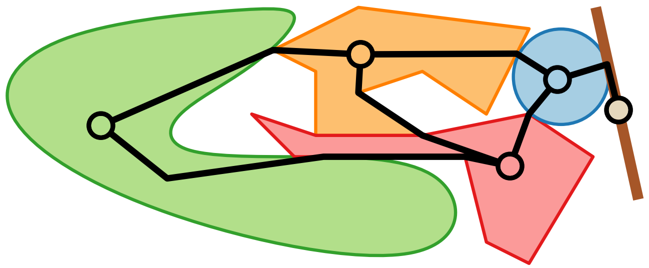
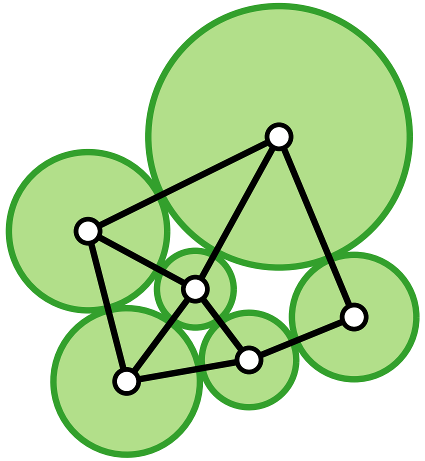


G is planar



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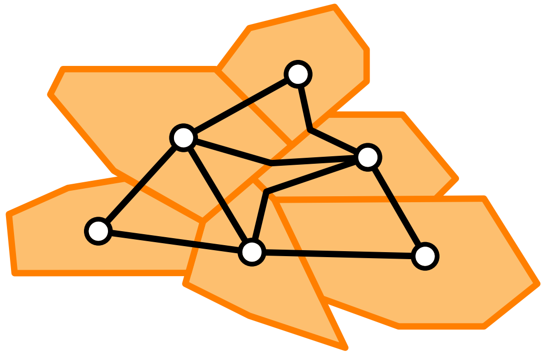

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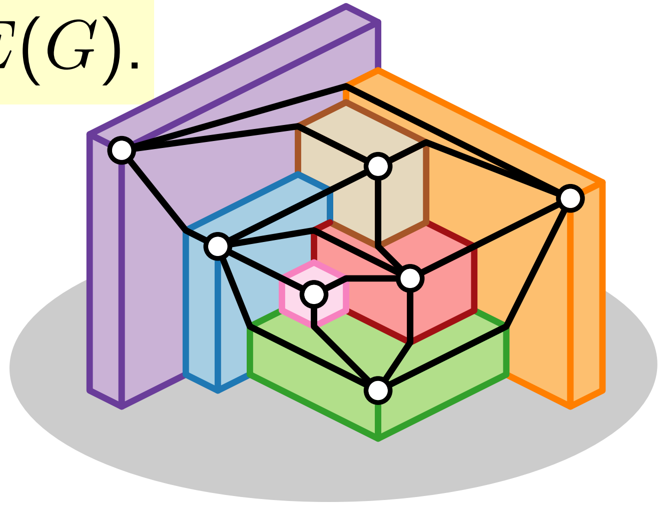

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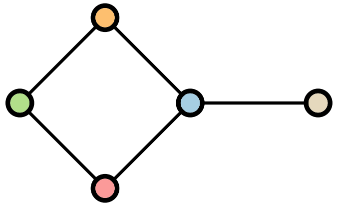

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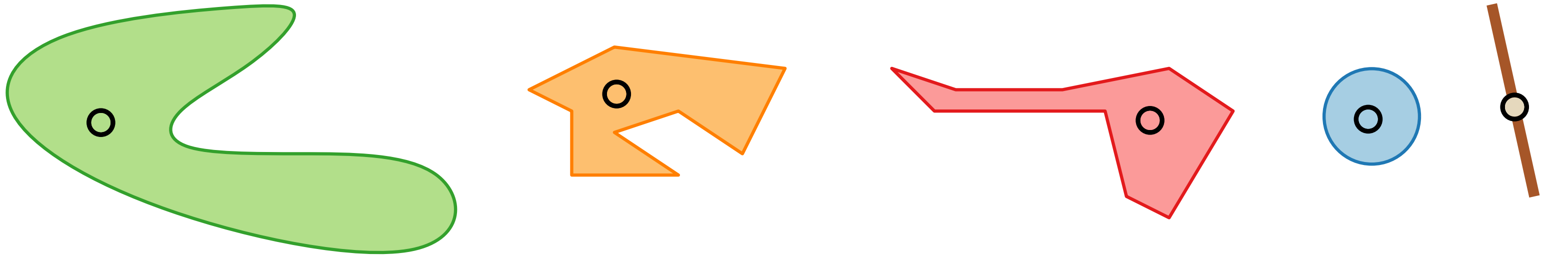

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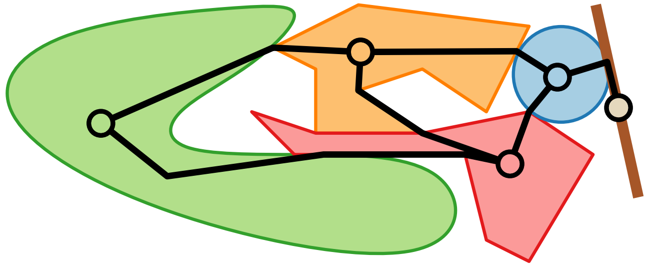
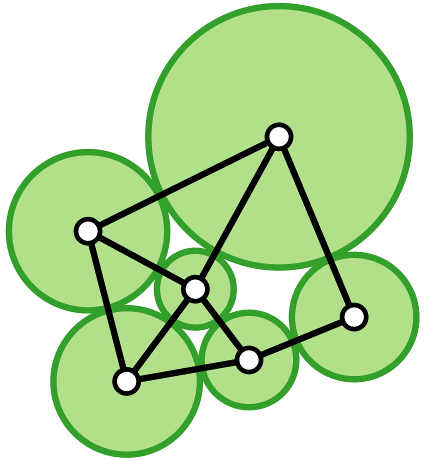
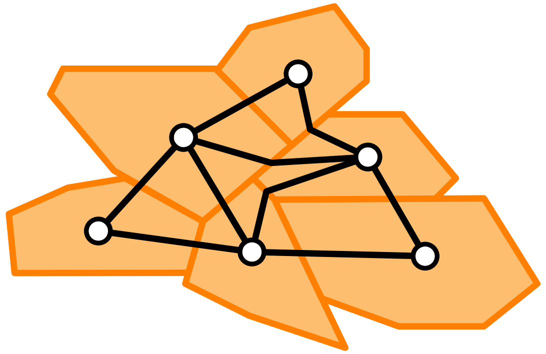




G is planar → disks [Koebe 1936]



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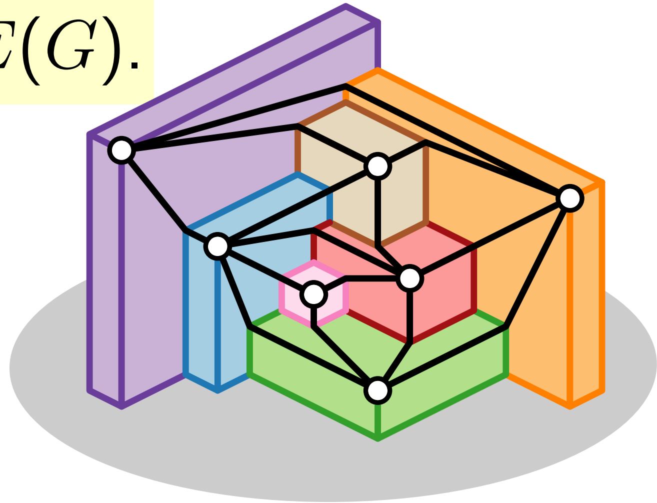

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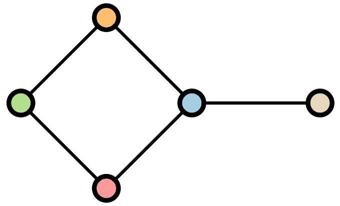

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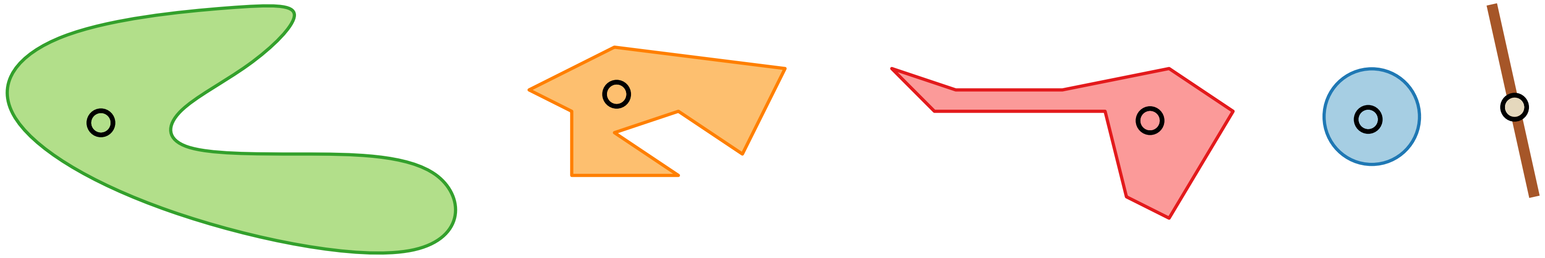
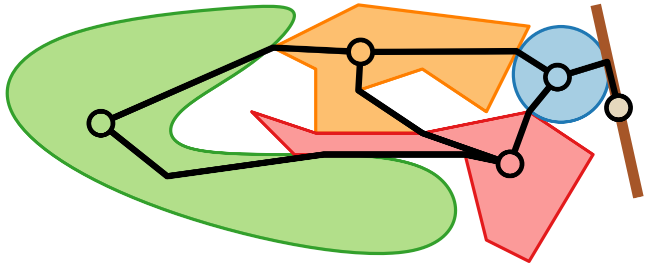
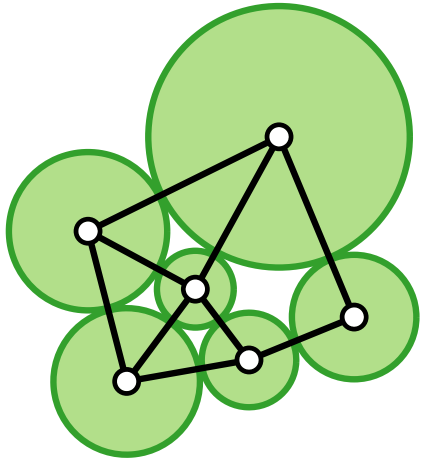
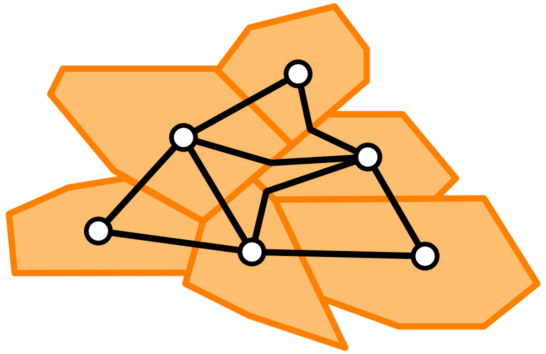
Represent each vertex v by a geometric object $S(v) \in \mathcal{S}$

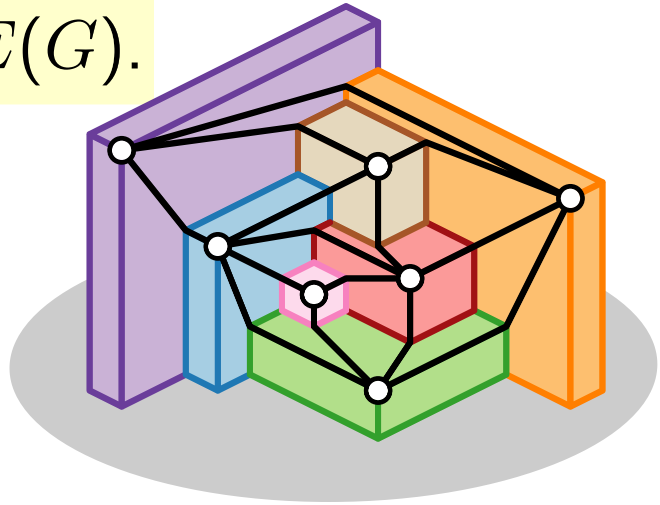


In an **$\mathcal{S}$-contact representation** of G, $S(u)$ and $S(v)$ *touch* iff $uv \in E(G)$.







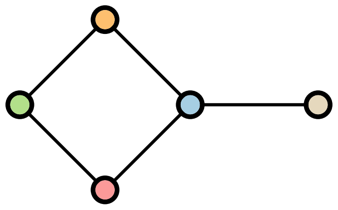


rectangular cuboids

G is planar → [Koebe 1936] → disks → polygons

Contact Representation of Graphs

Let G be a graph.



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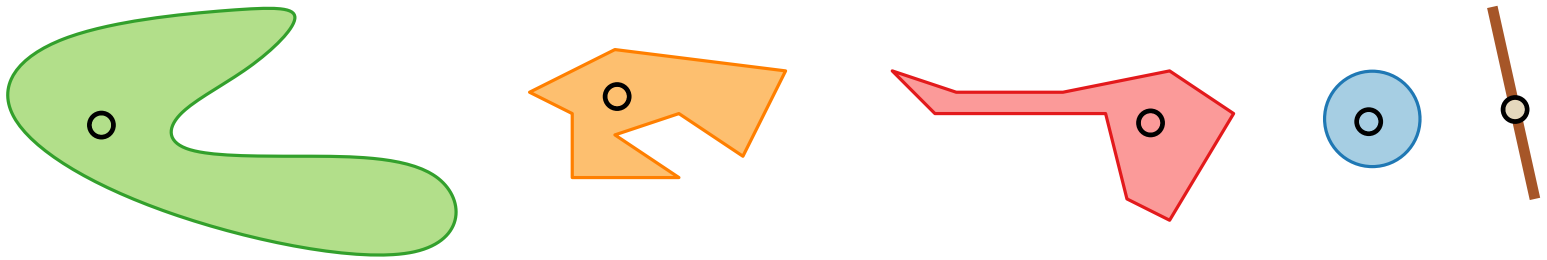

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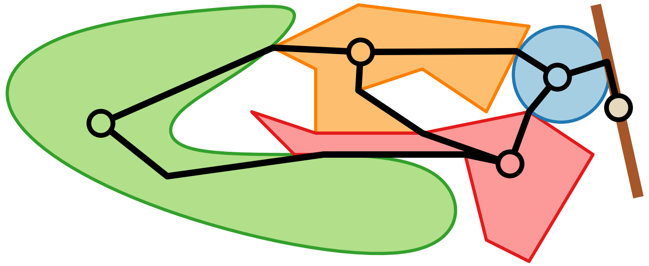

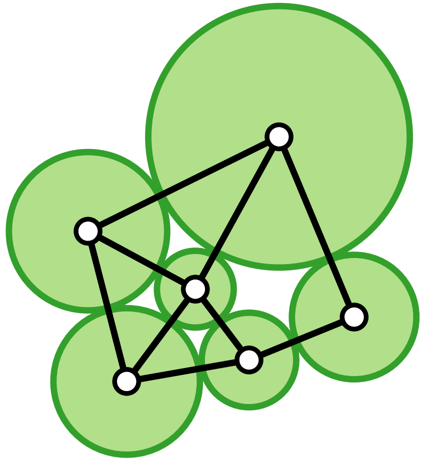

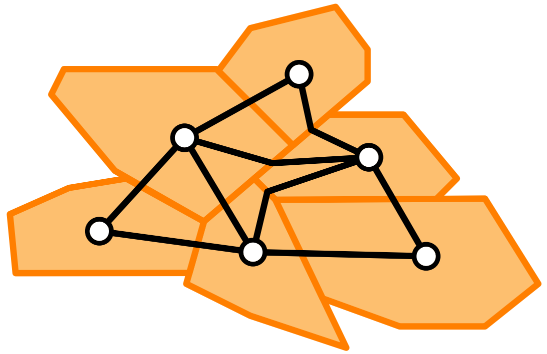

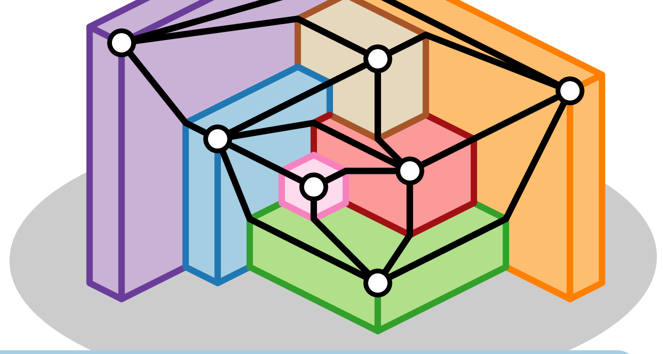

G is planar → disks → polygons

[Koebe 1936]

A contact representation is an intersection representation with interior-disjoint sets.

Contact Representation of Planar Graphs

Is the intersection graph of a contact representation always planar?

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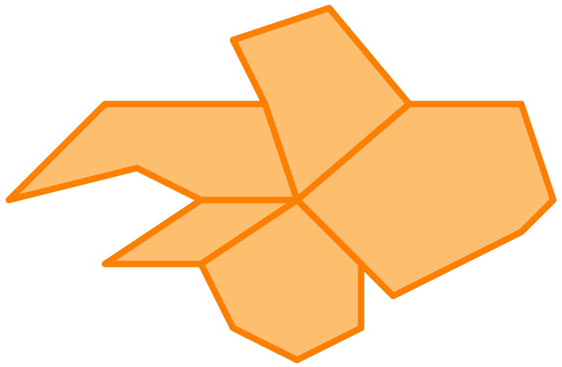

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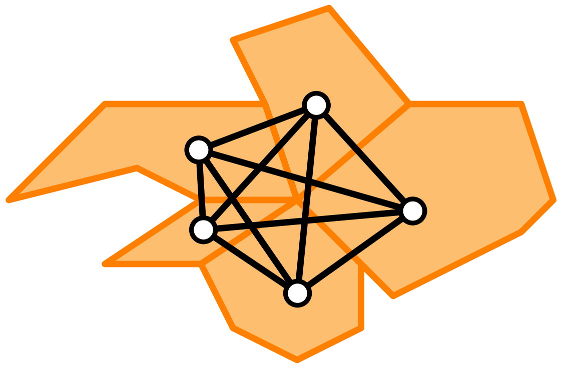

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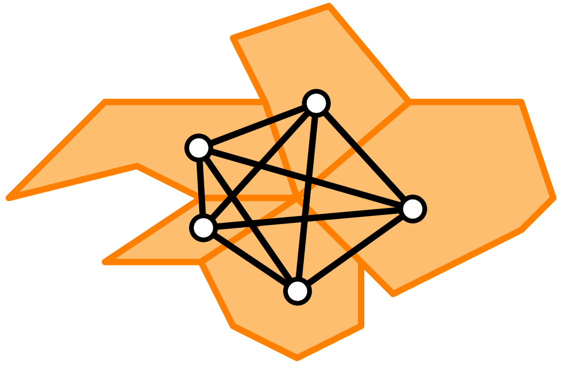

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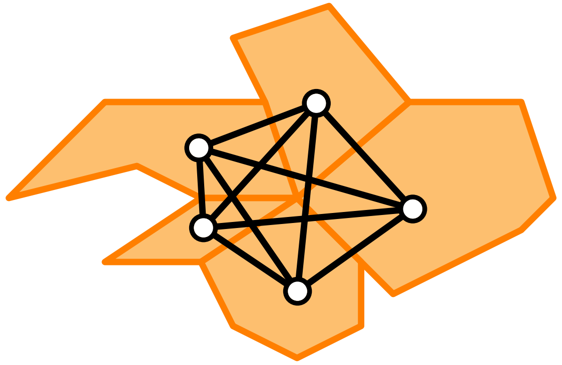

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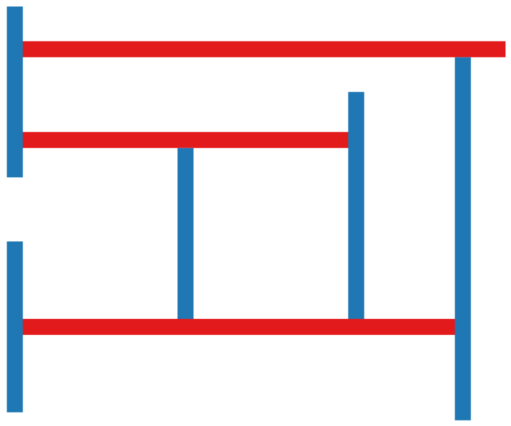

bipartite planar graphs

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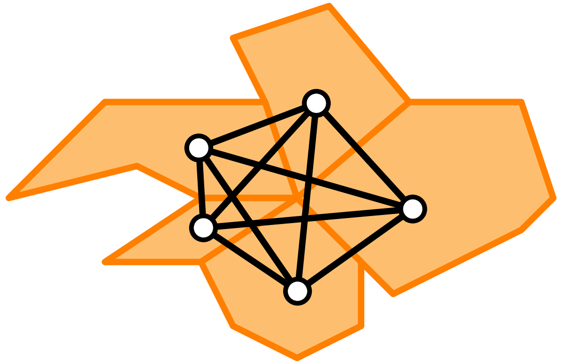

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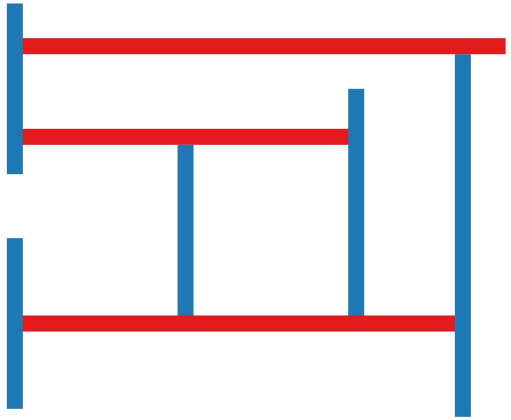

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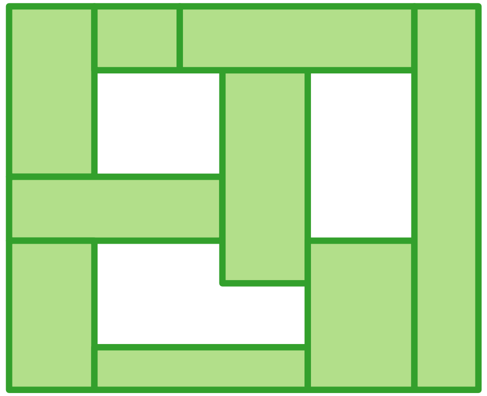

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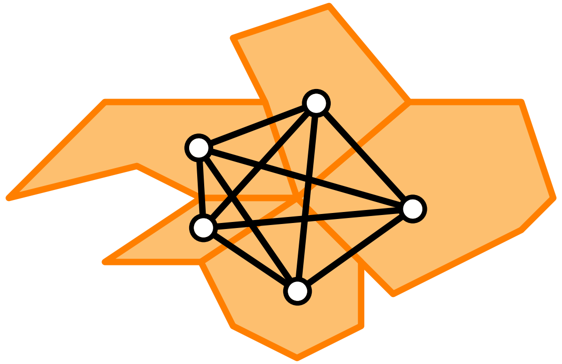

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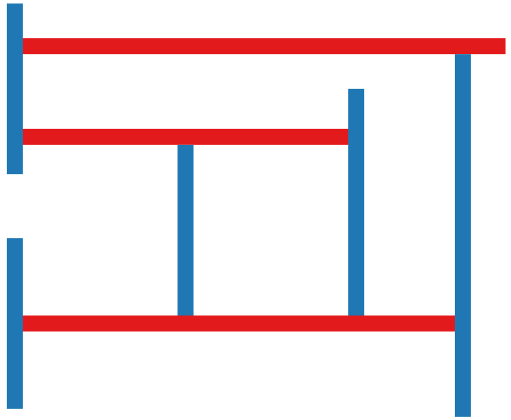

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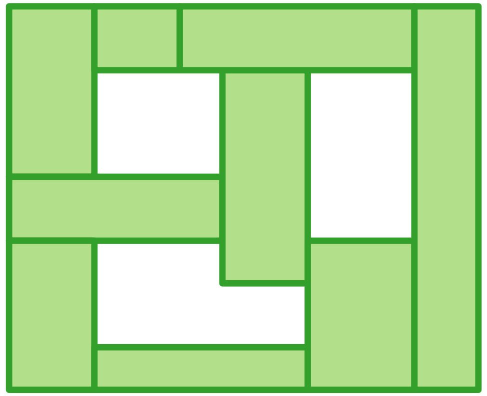

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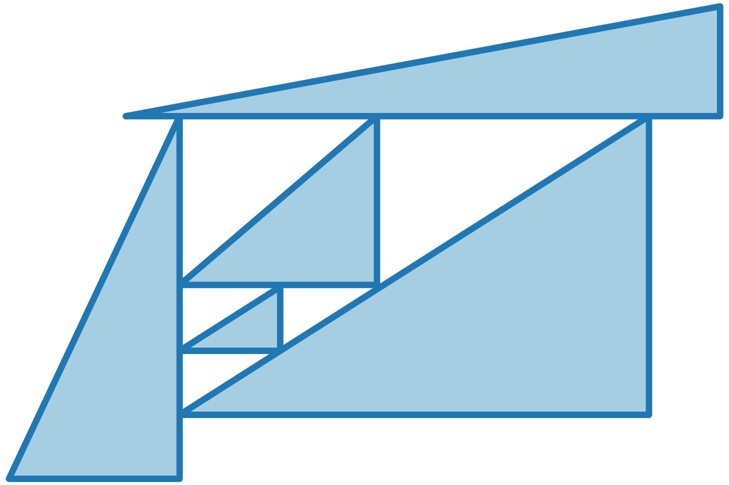

planar triangulations

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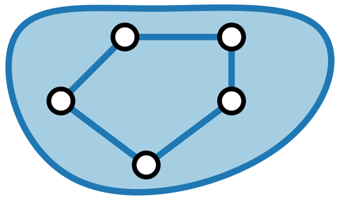

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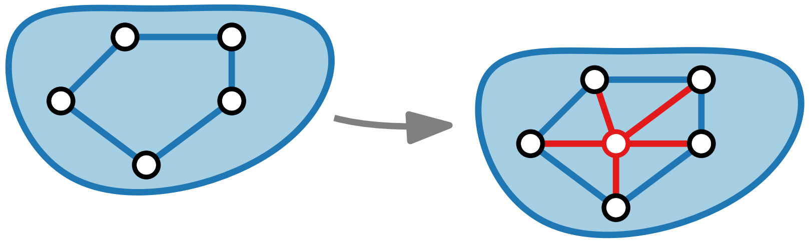

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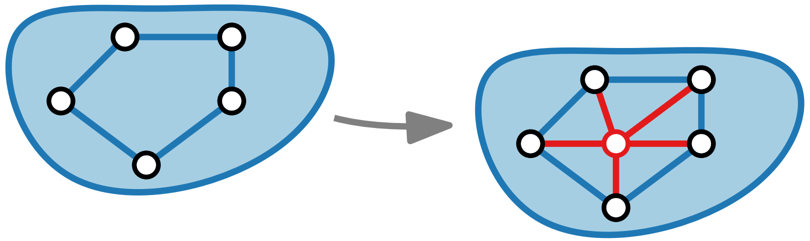

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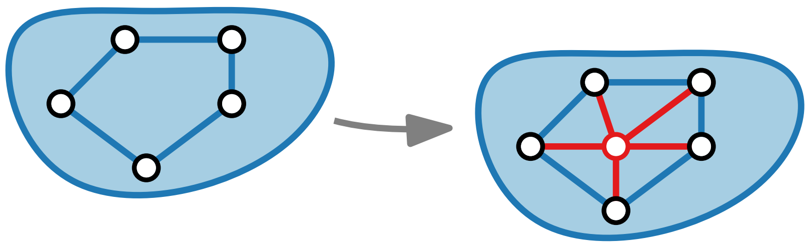

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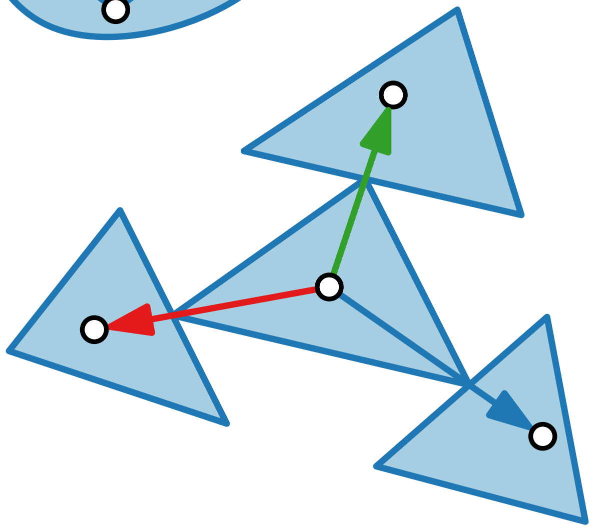

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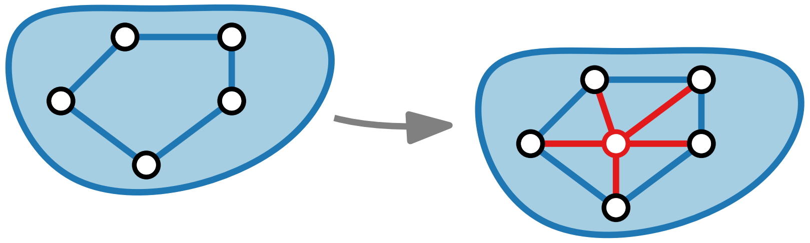

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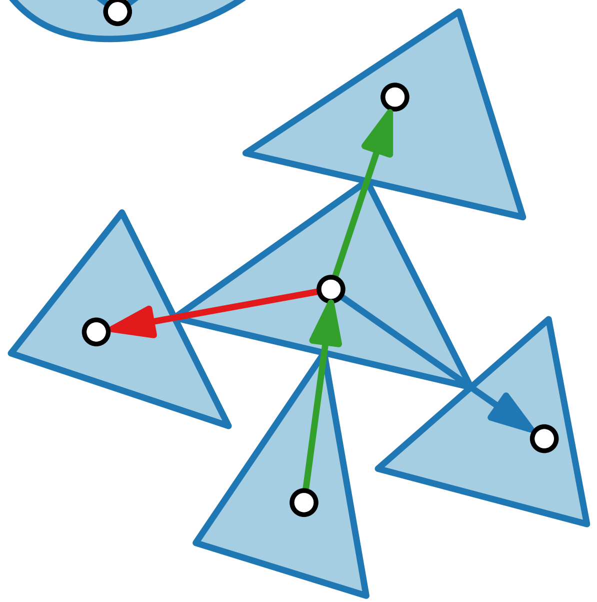

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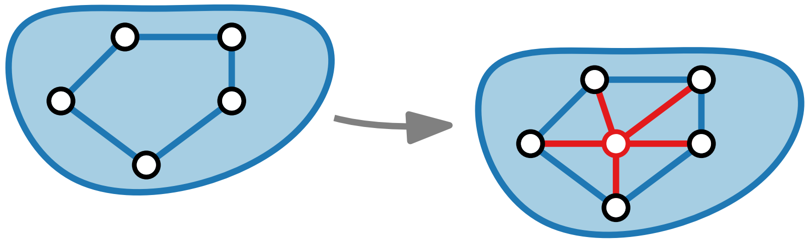

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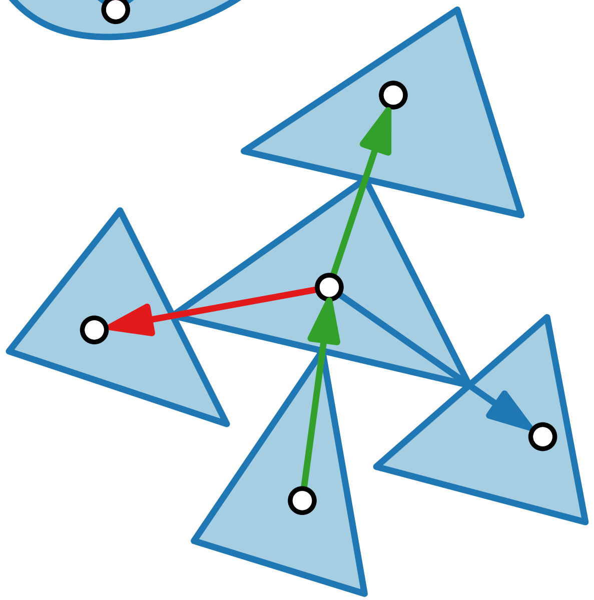

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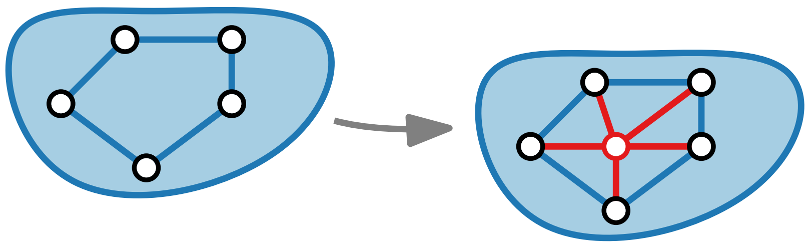
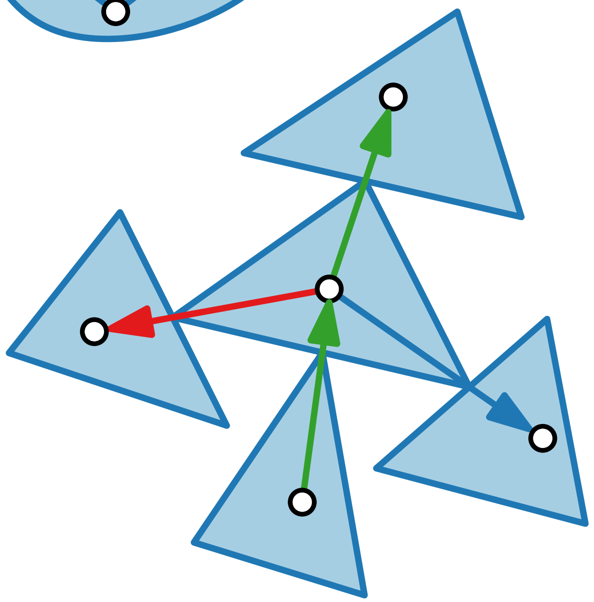
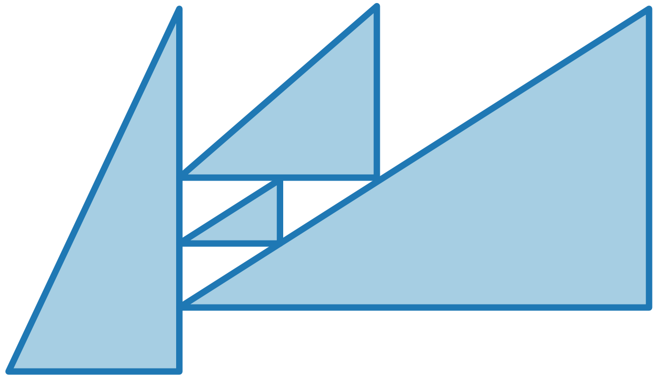


- Describe contact representation combinatorially.
  - Which objects touch each other in which way?
- Compute combinatorial description.
- Show that combinatorial description can be used to construct drawing.



This Lecture

Representation with right-angled triangles and corner contact:



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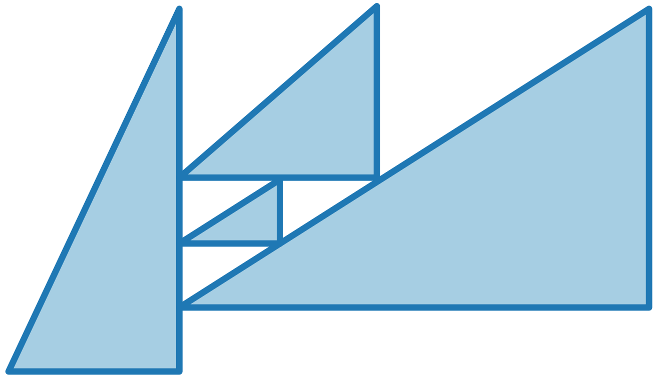

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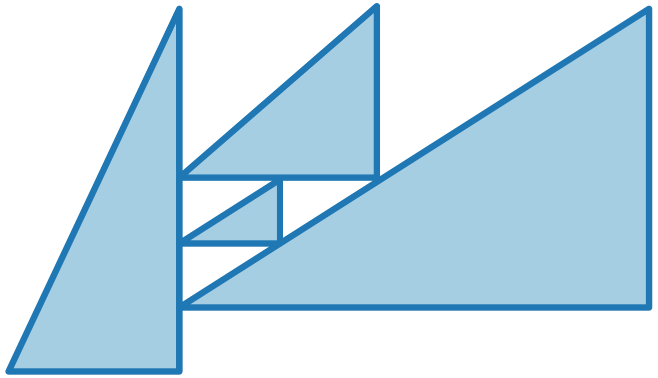

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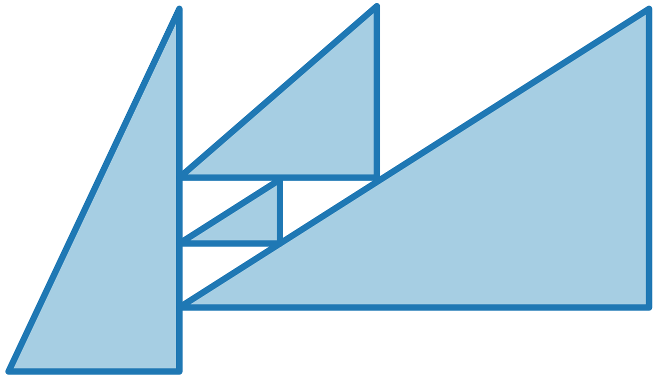

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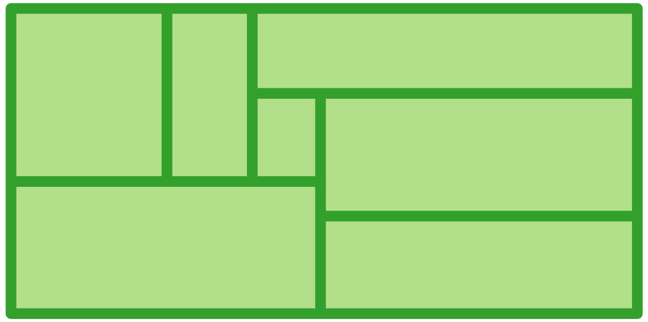

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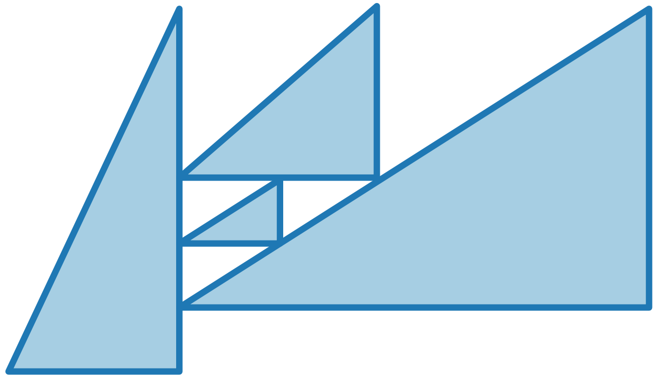

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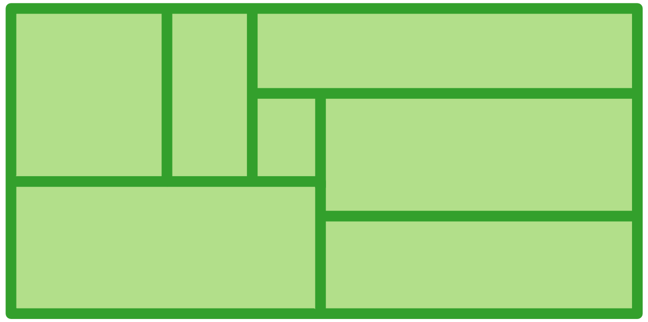

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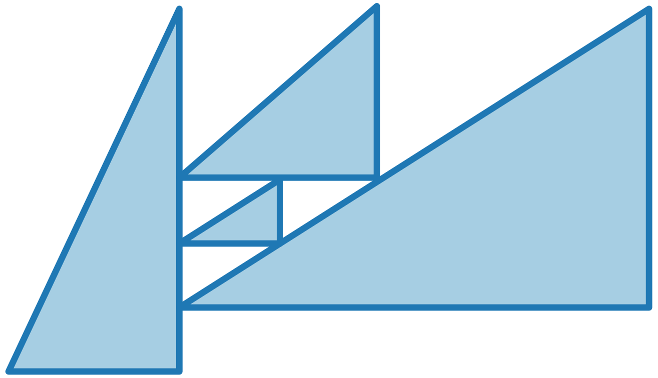

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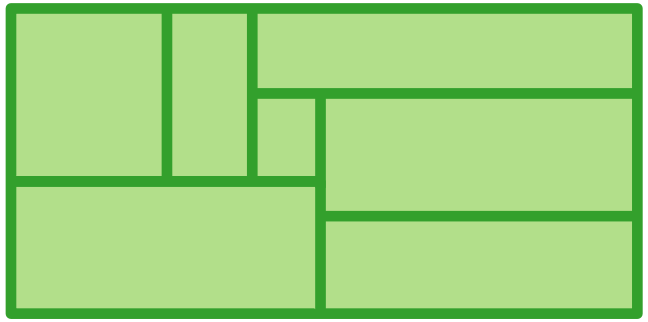

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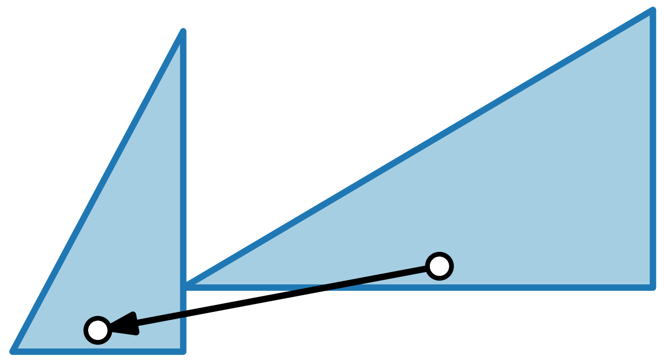

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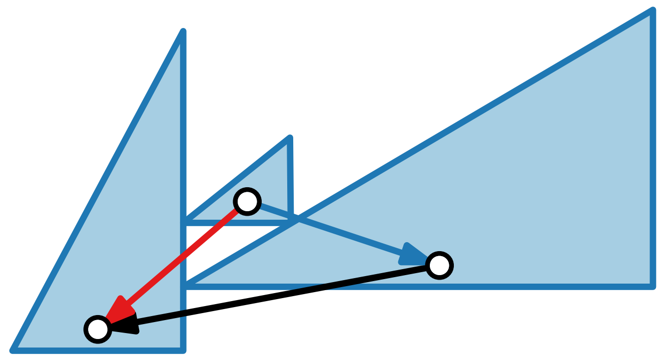

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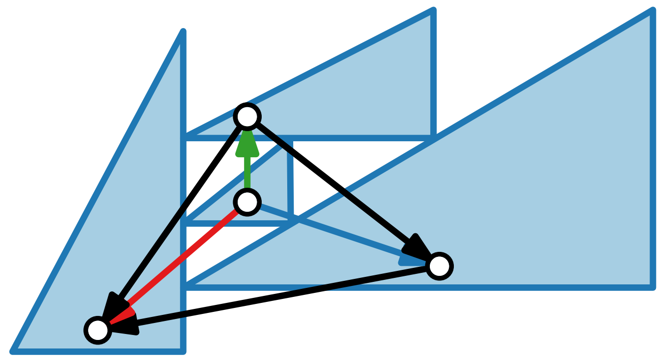

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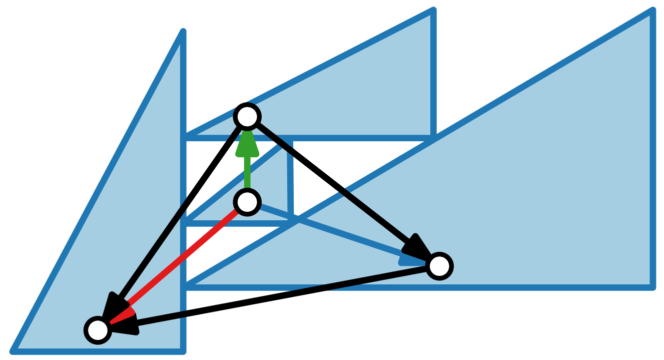

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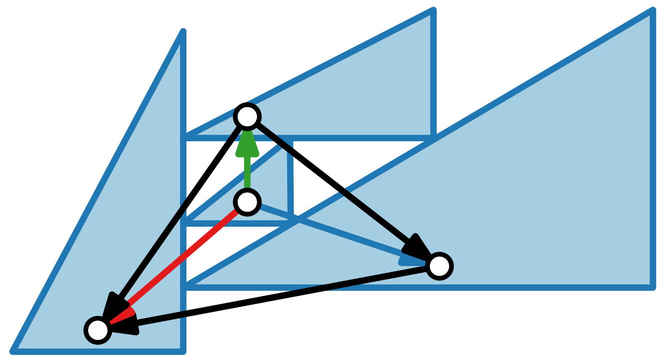

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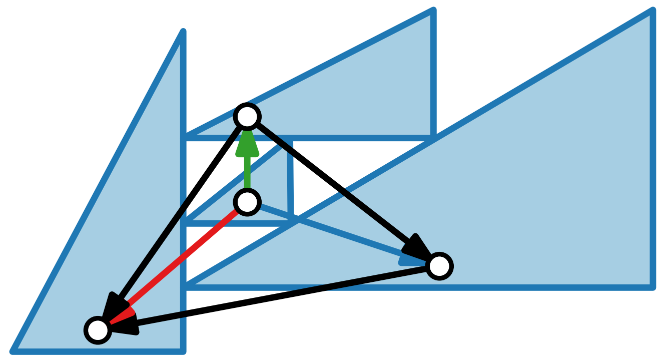
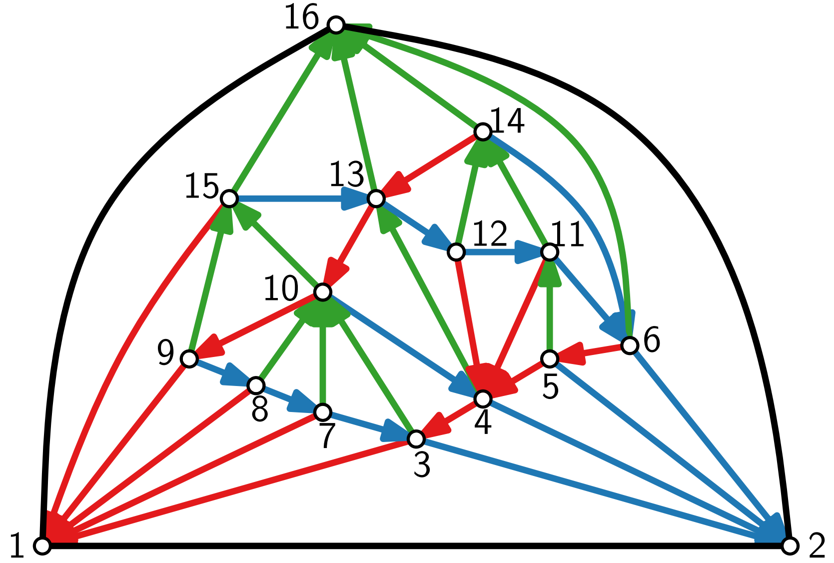


Detailed Idea.

- Place base of triangle at height equal to position in canonical order.
- Triangle tip is precisely at base of triangle corresponding to cover neighbor.
- Outgoing edges in Schnyder forest indicate corner contacts.

Triangle Contact Representation Example



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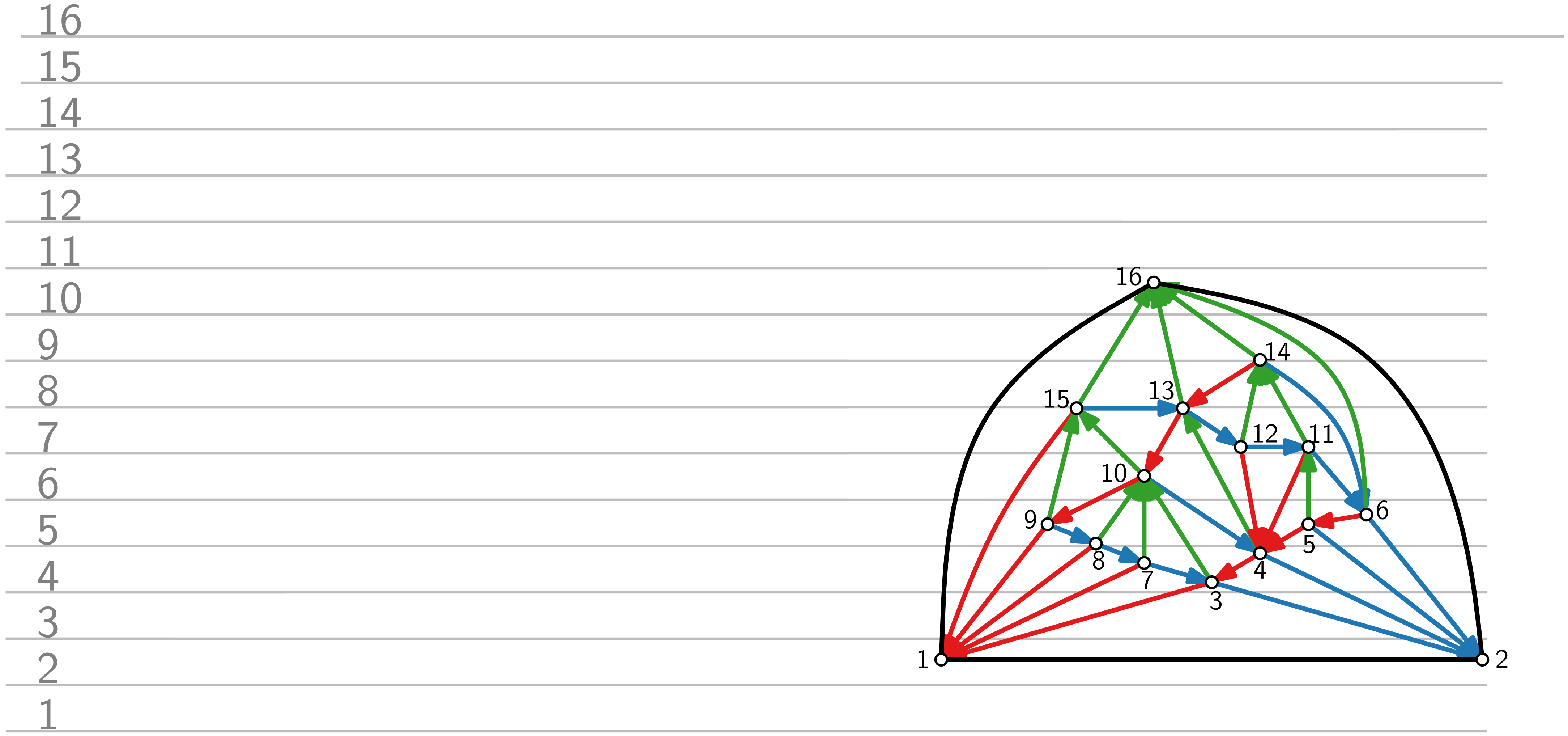

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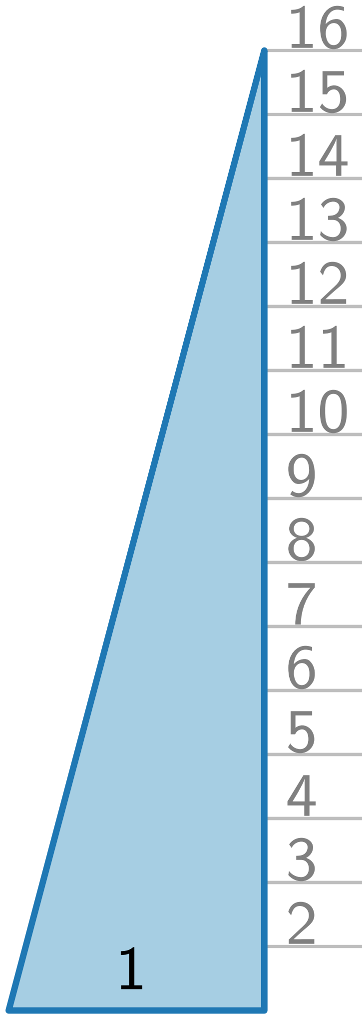

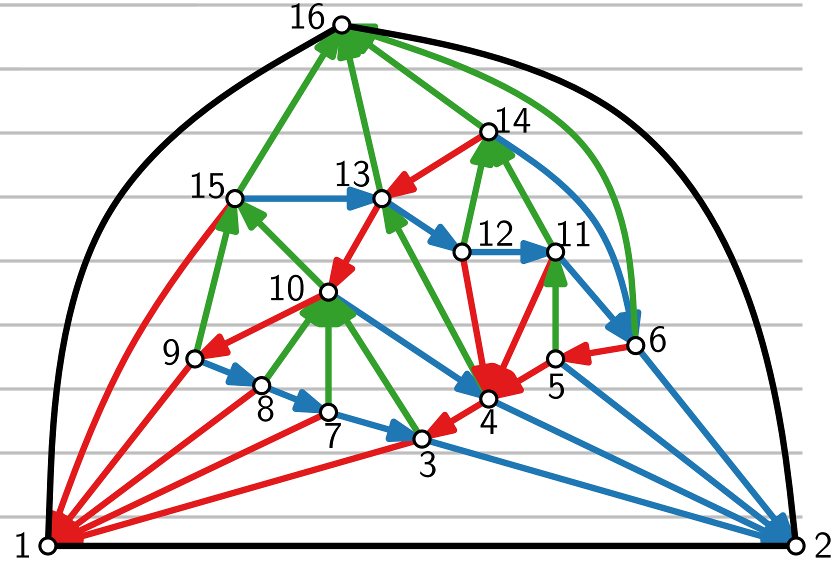

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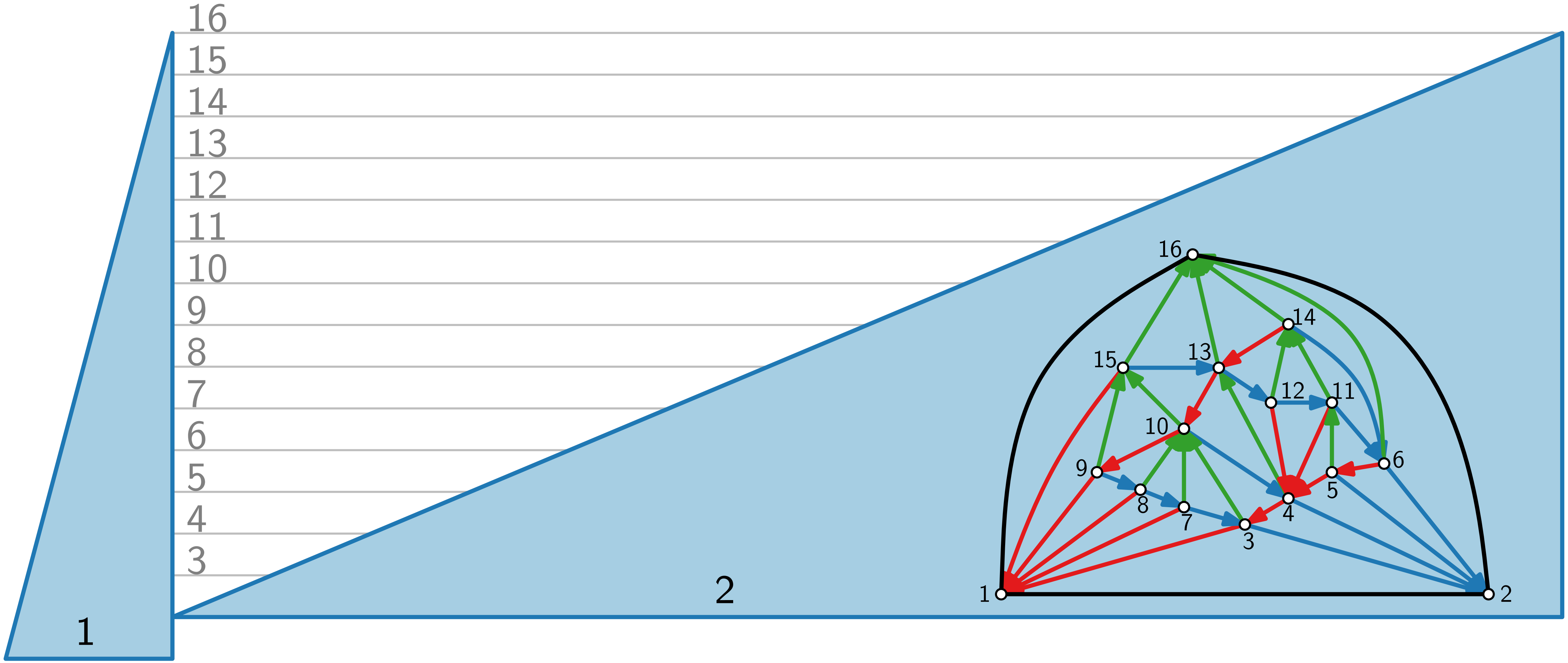

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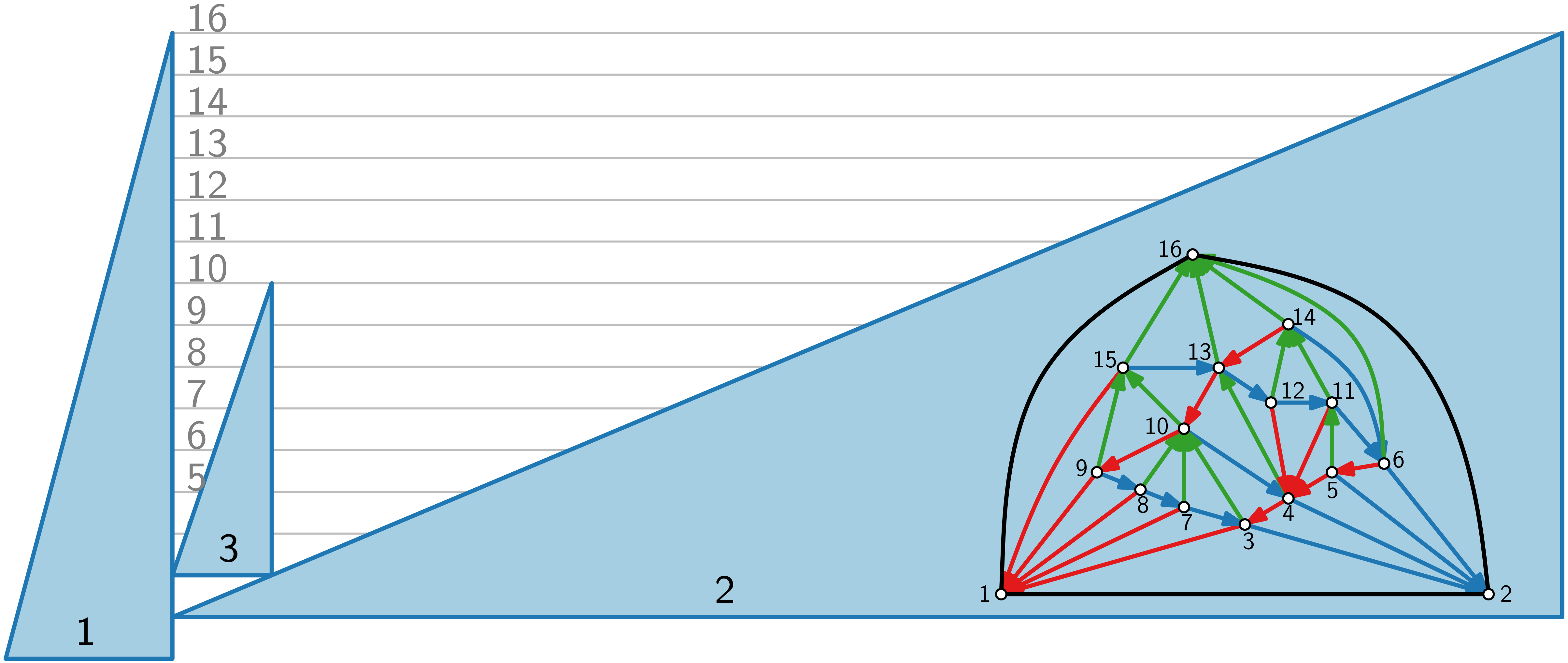

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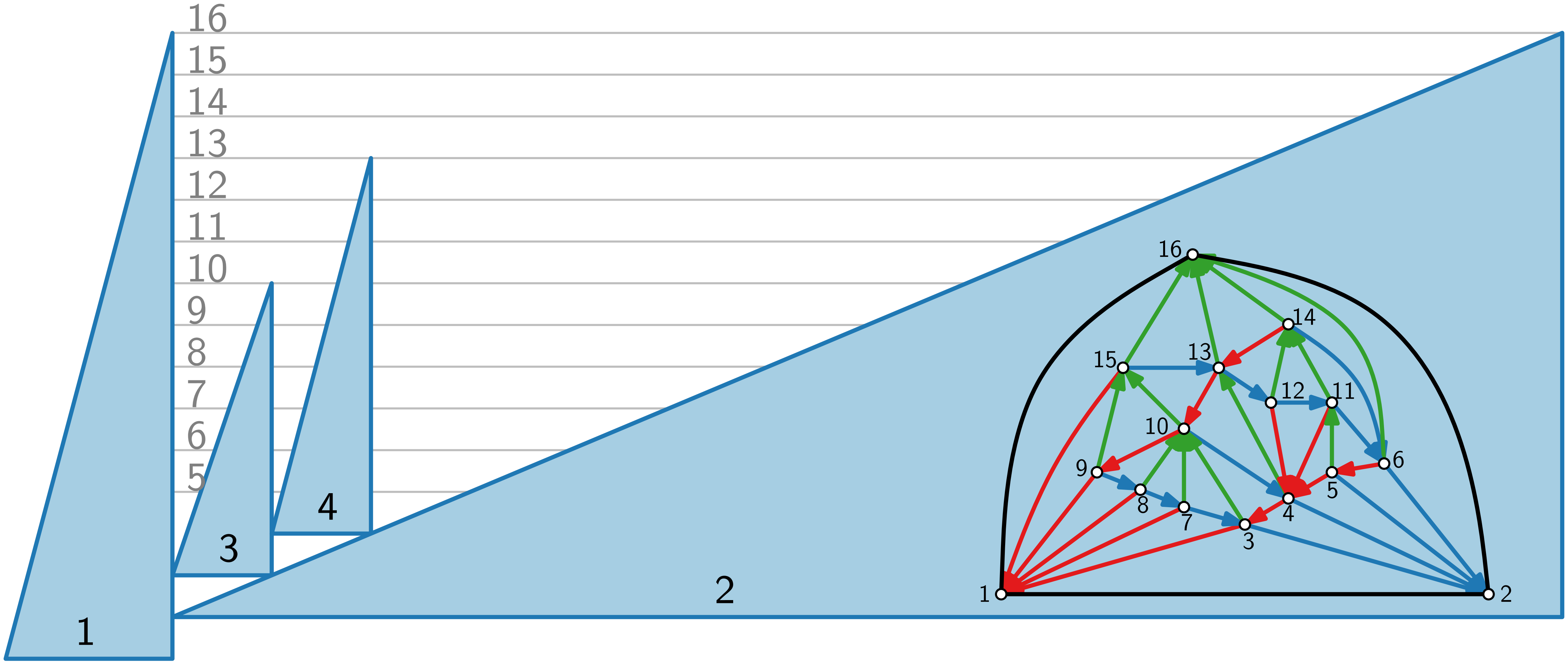

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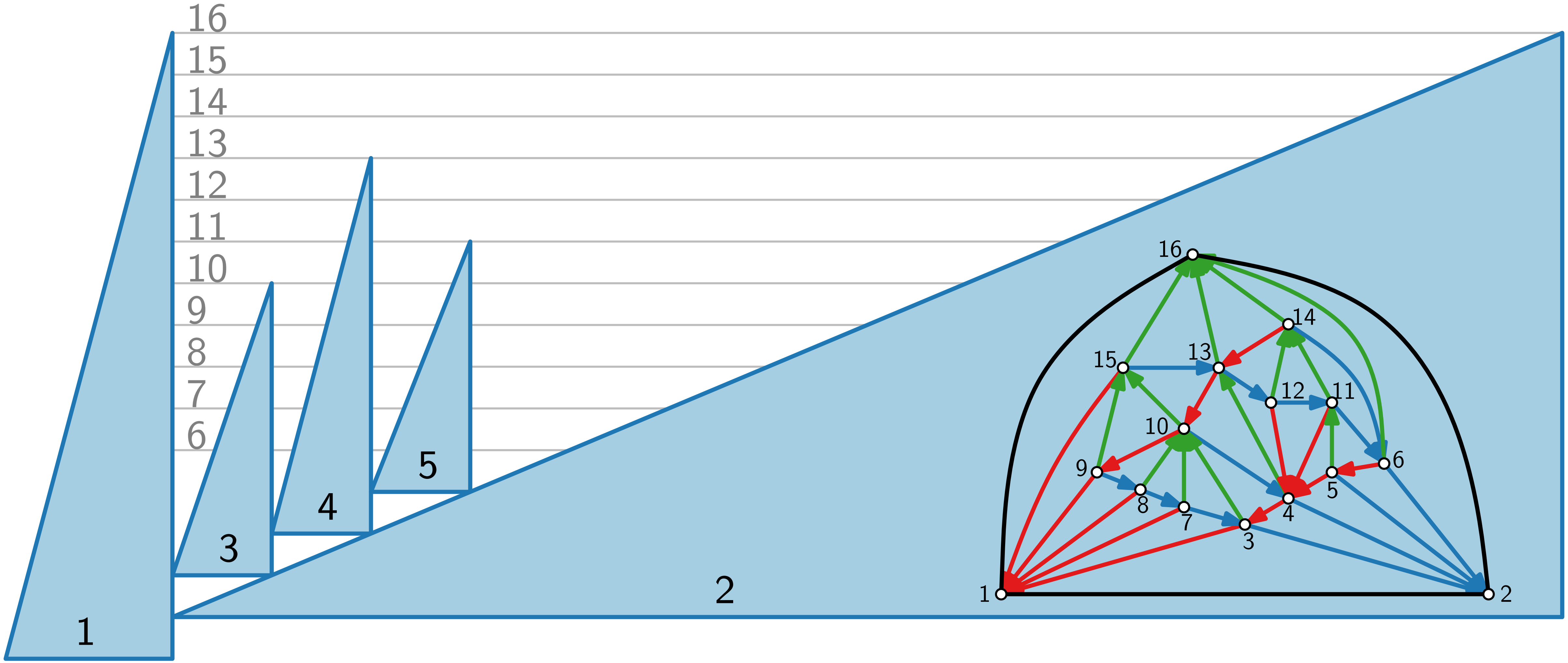

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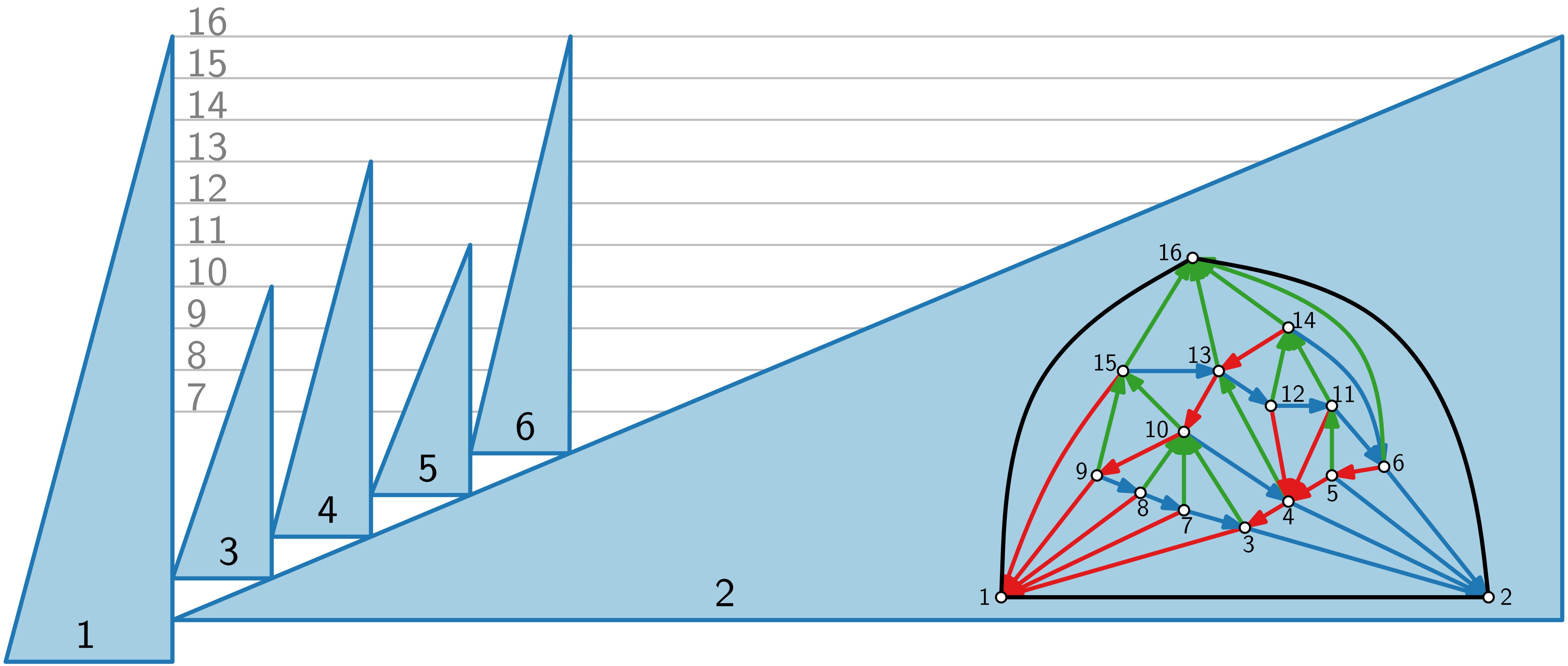

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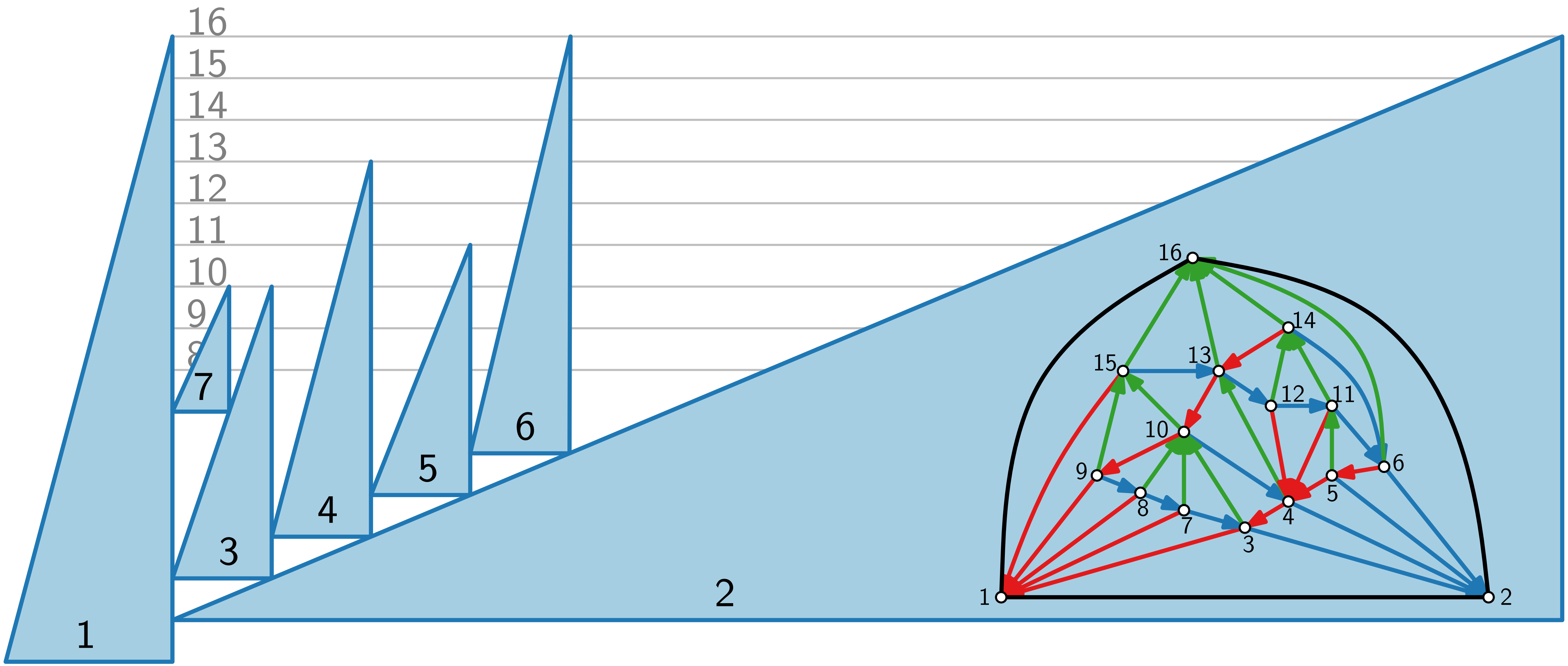
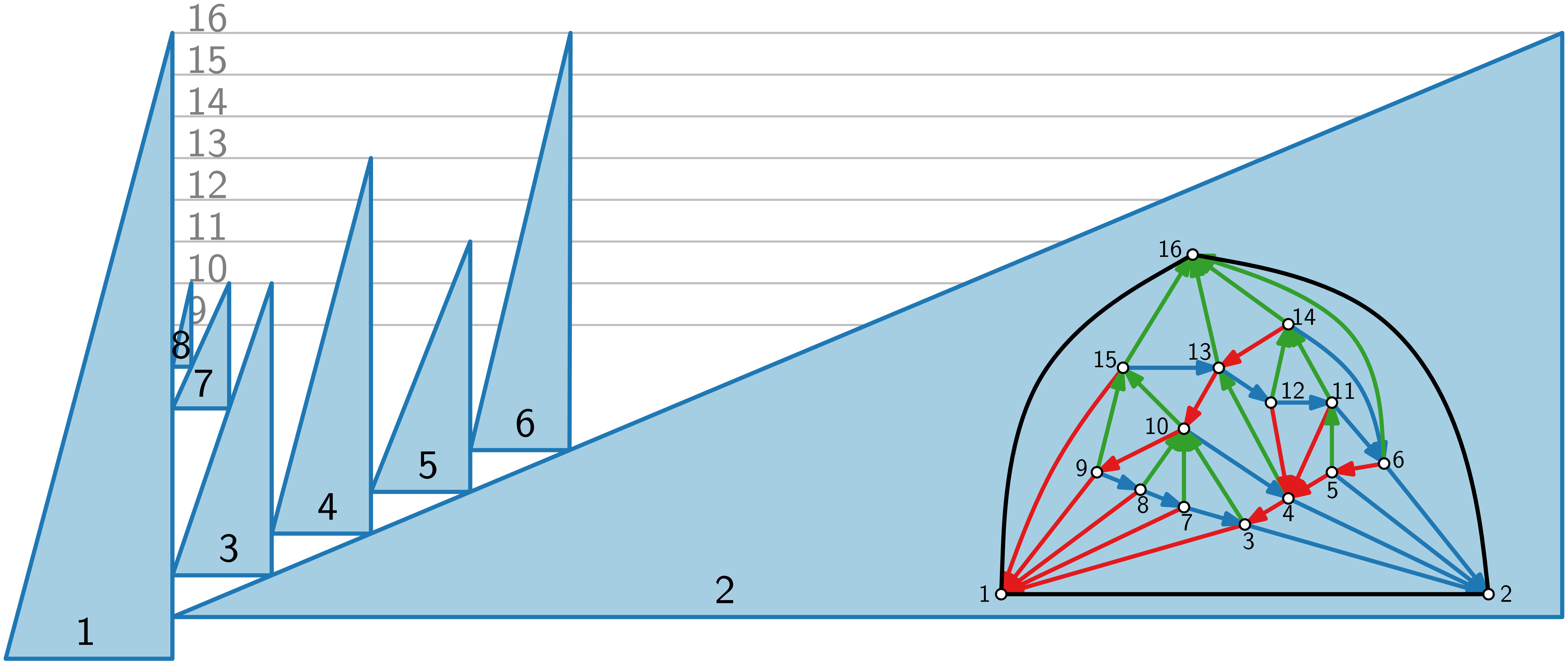


Triangle Contact Representation Example



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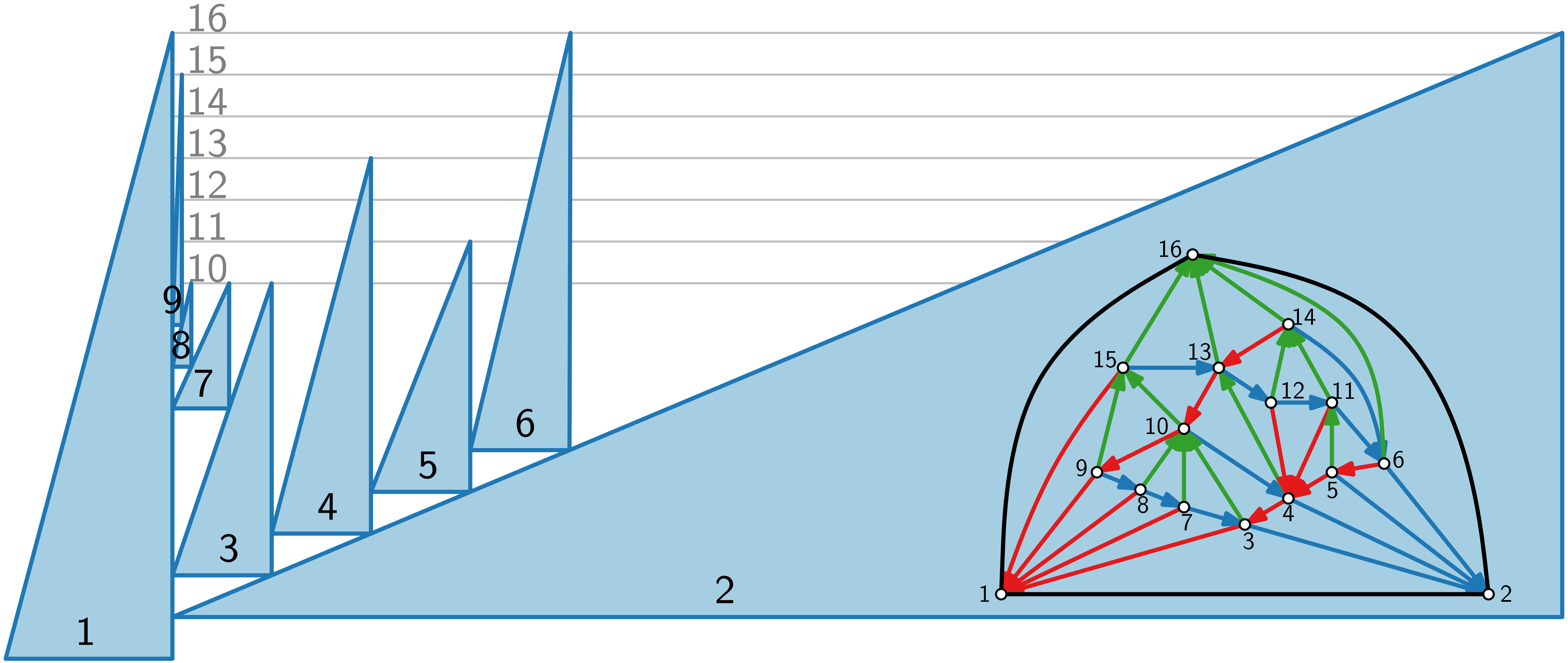

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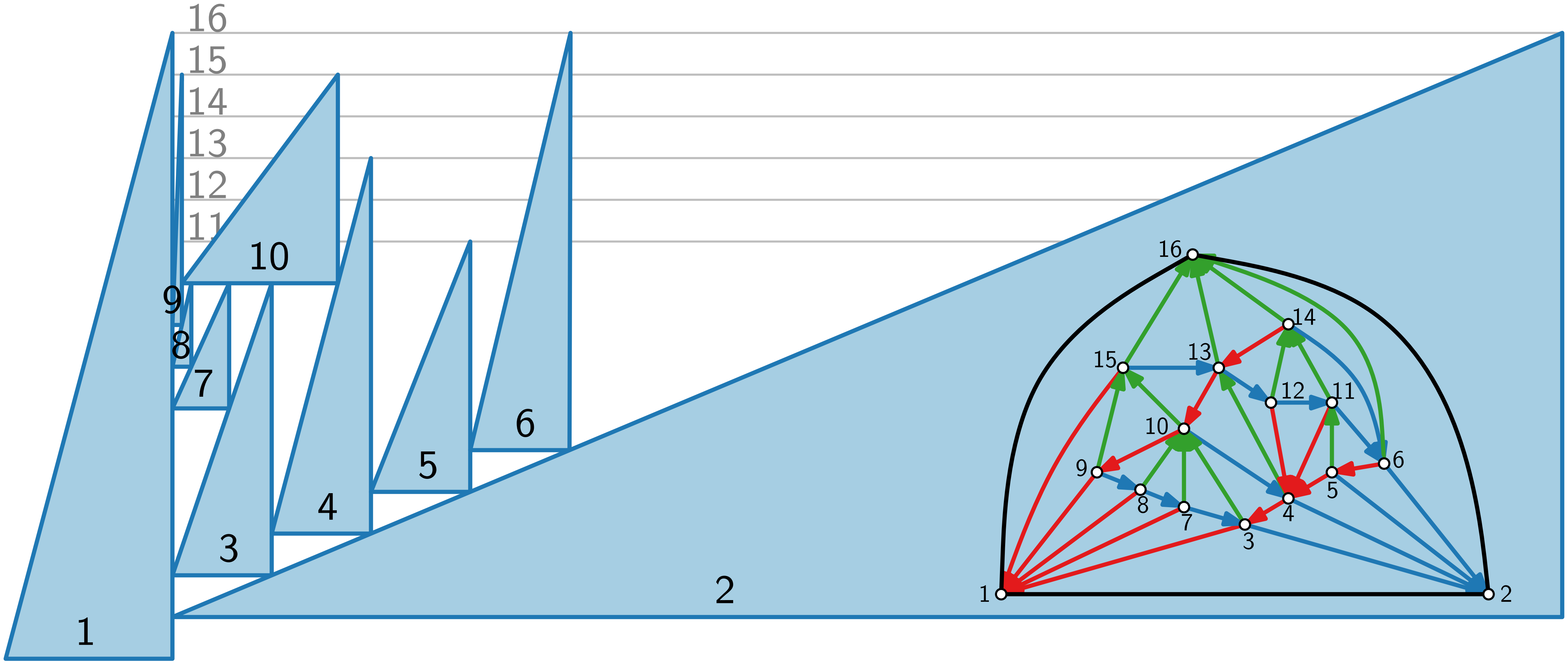

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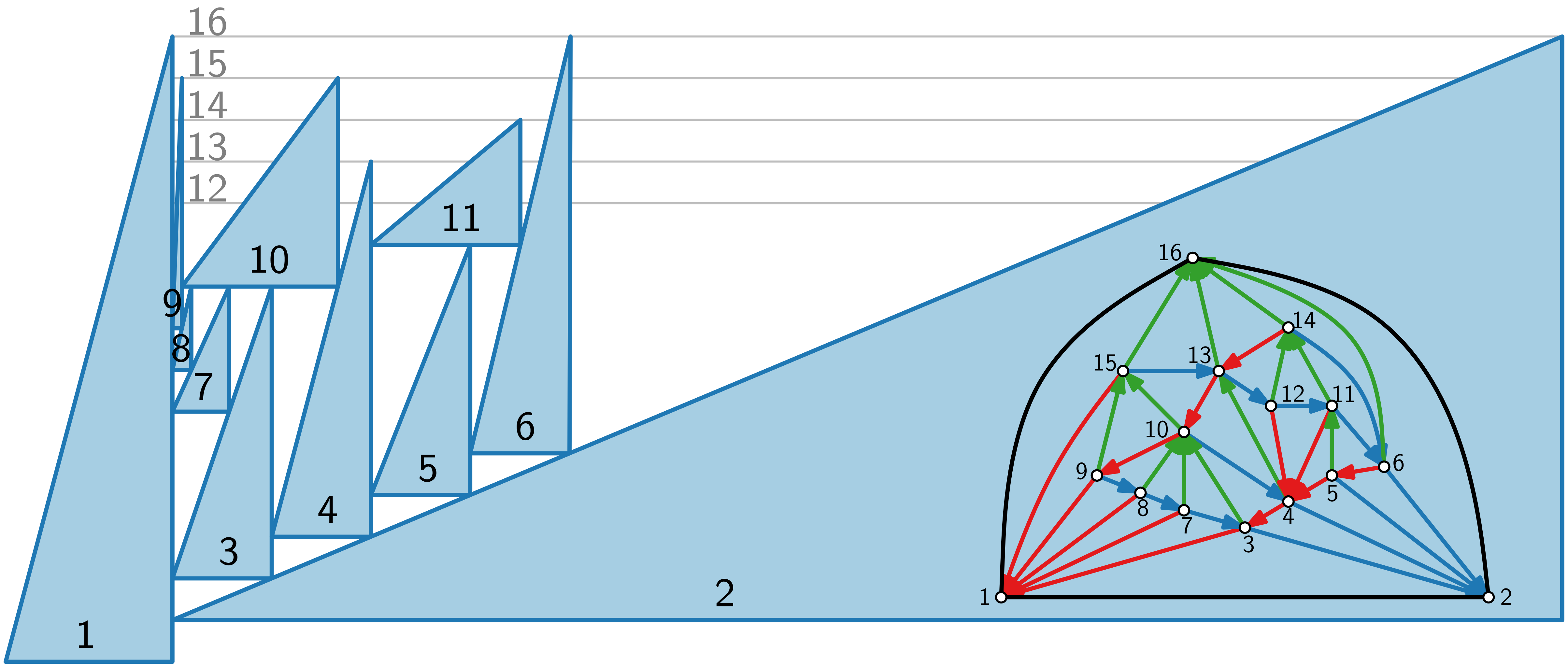

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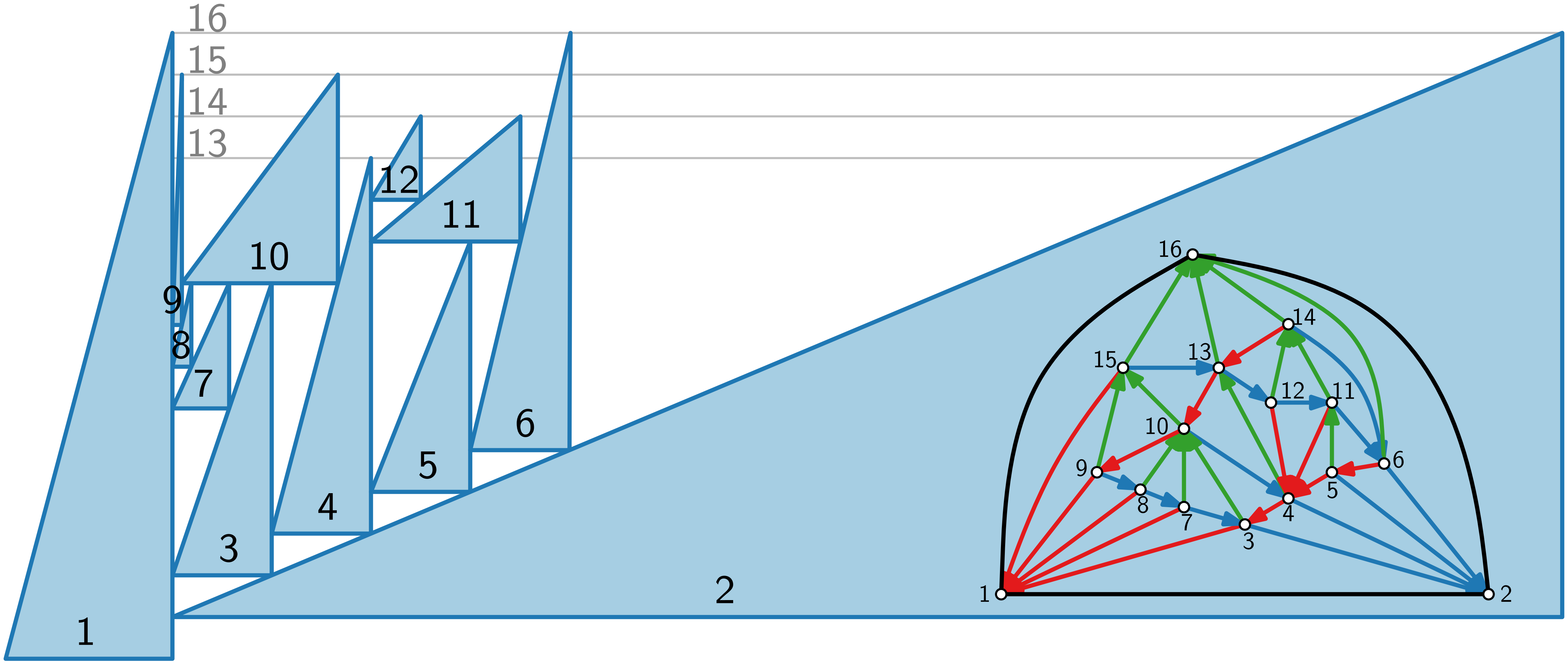

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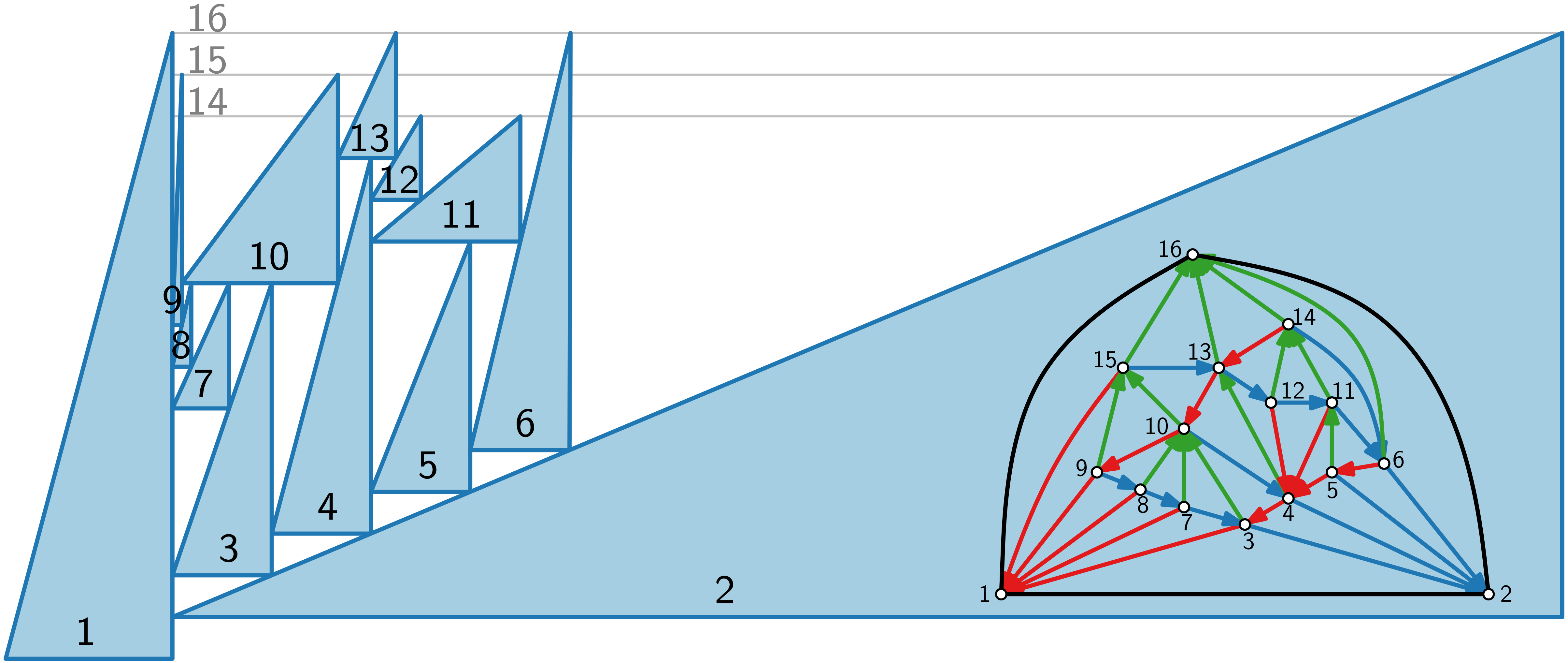

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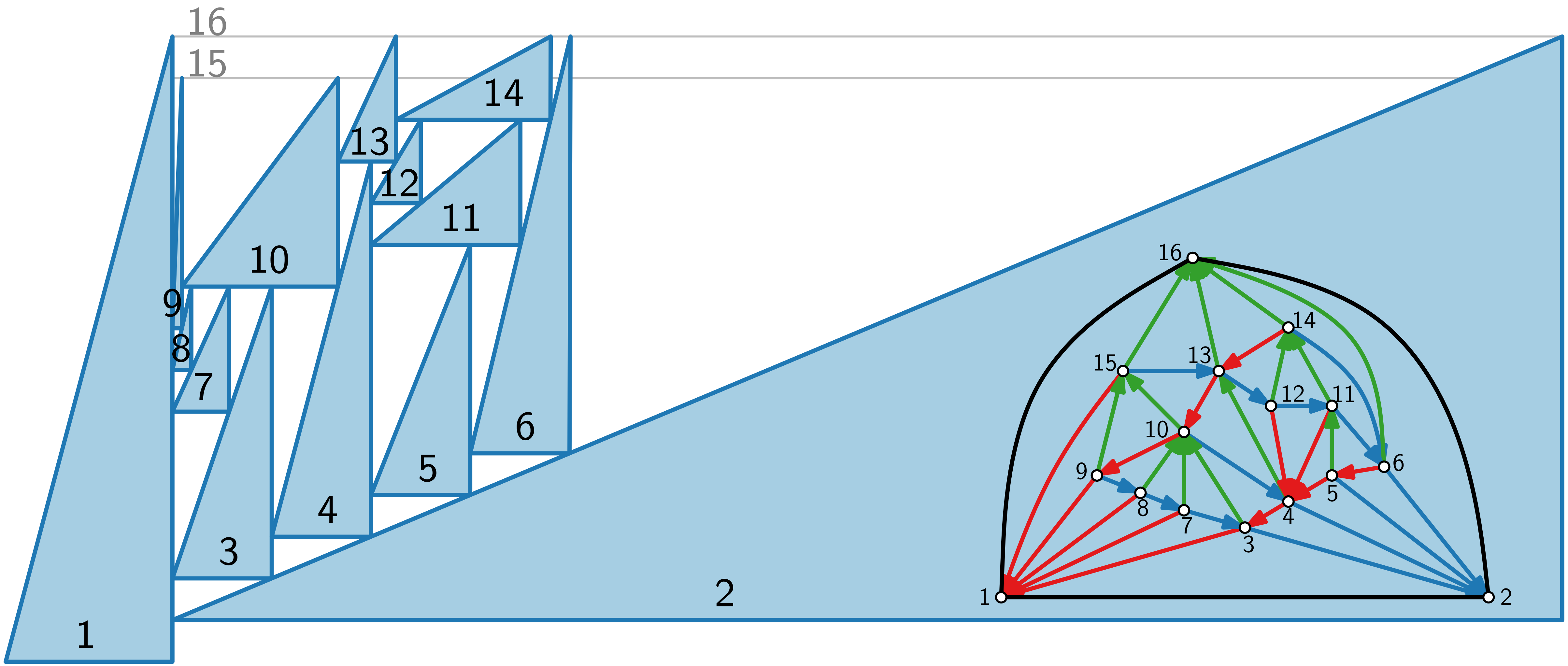

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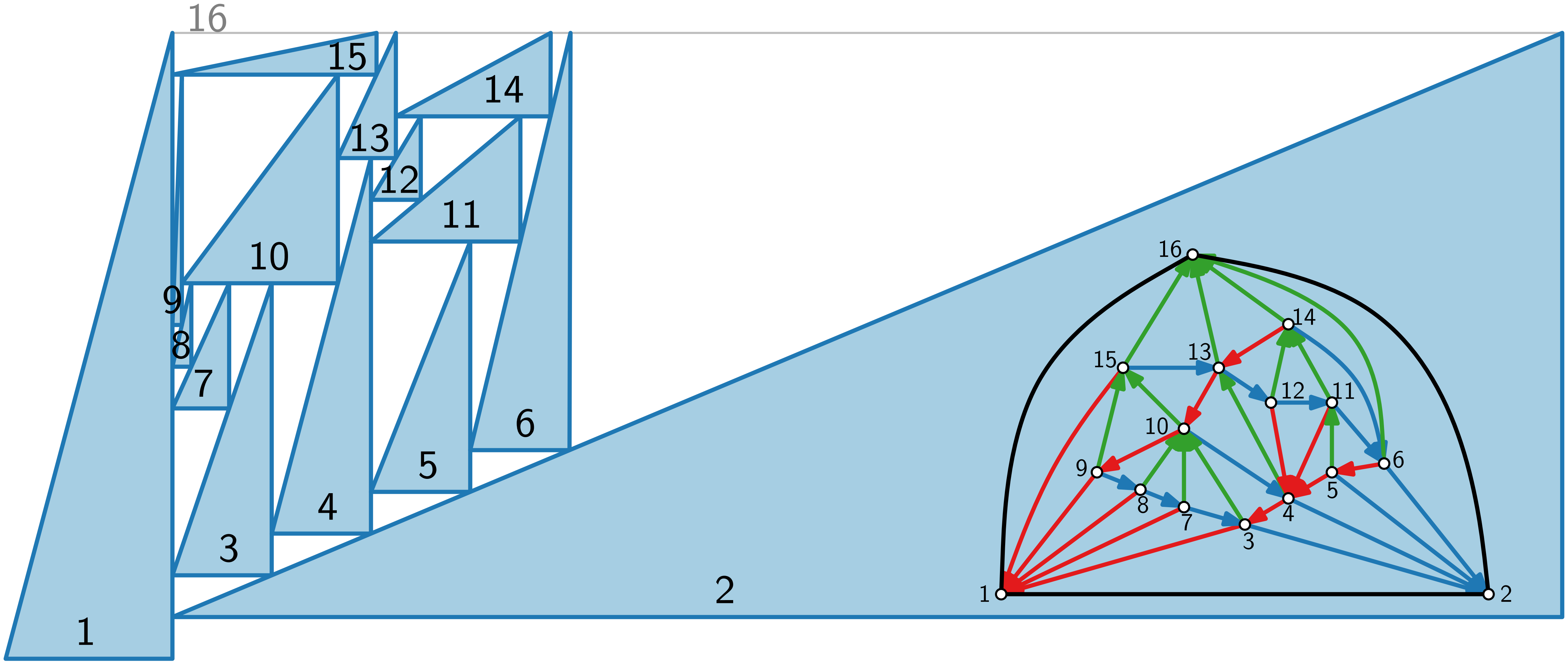

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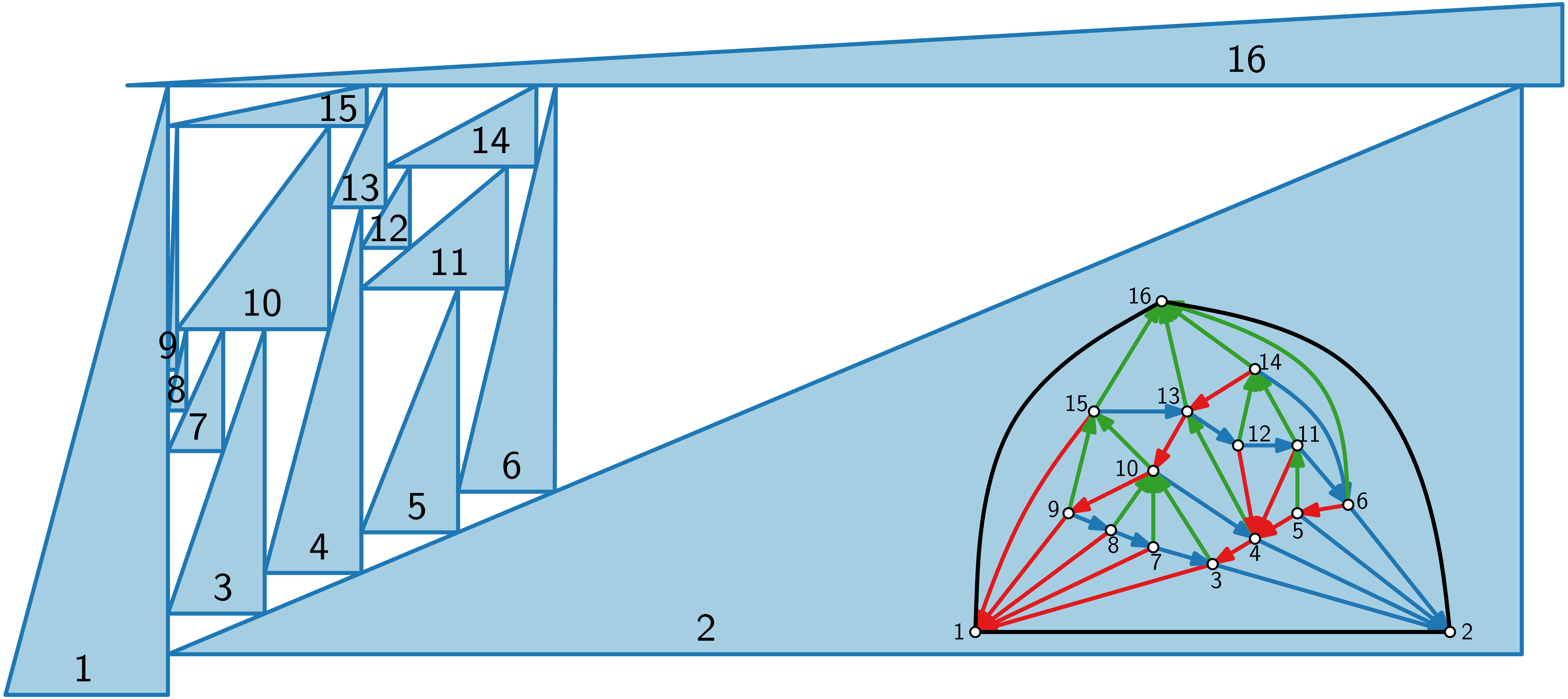
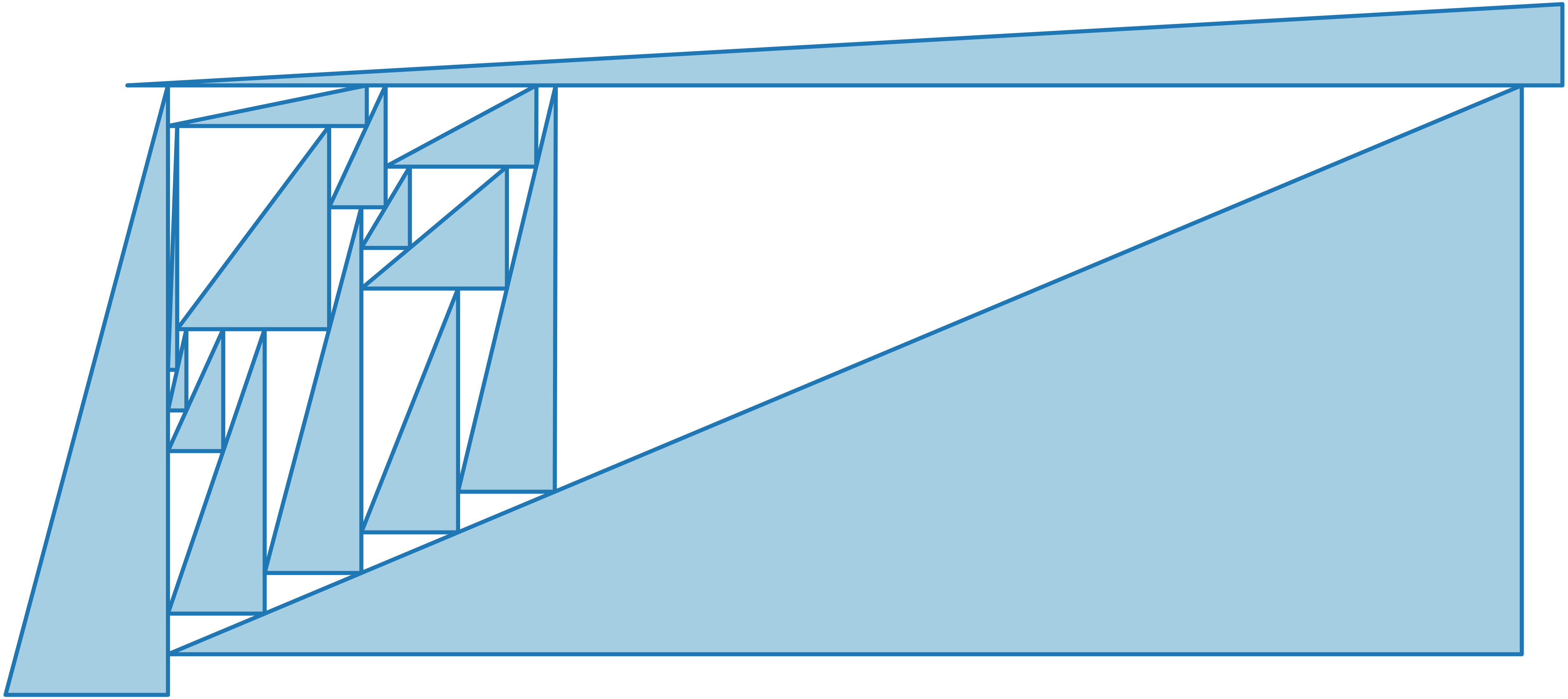


T-shape Contact Representation



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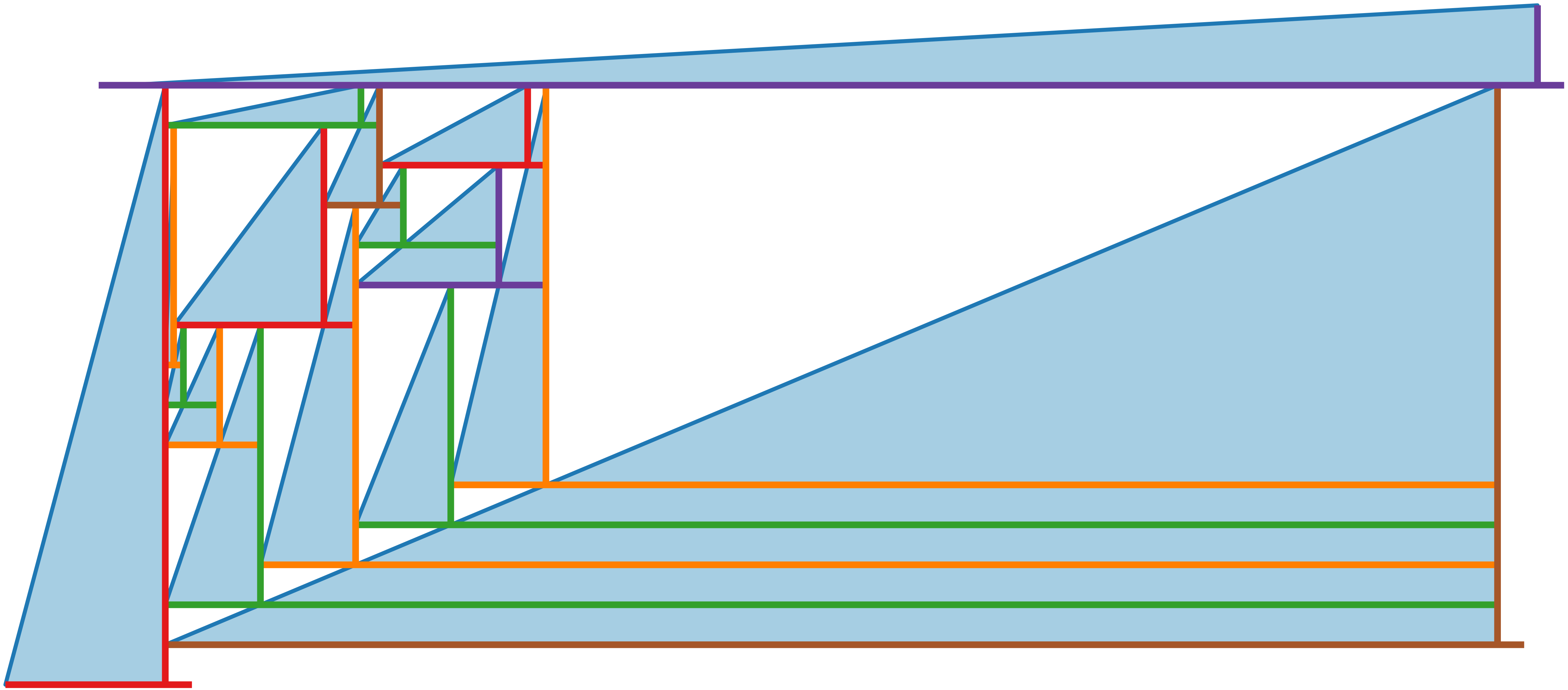

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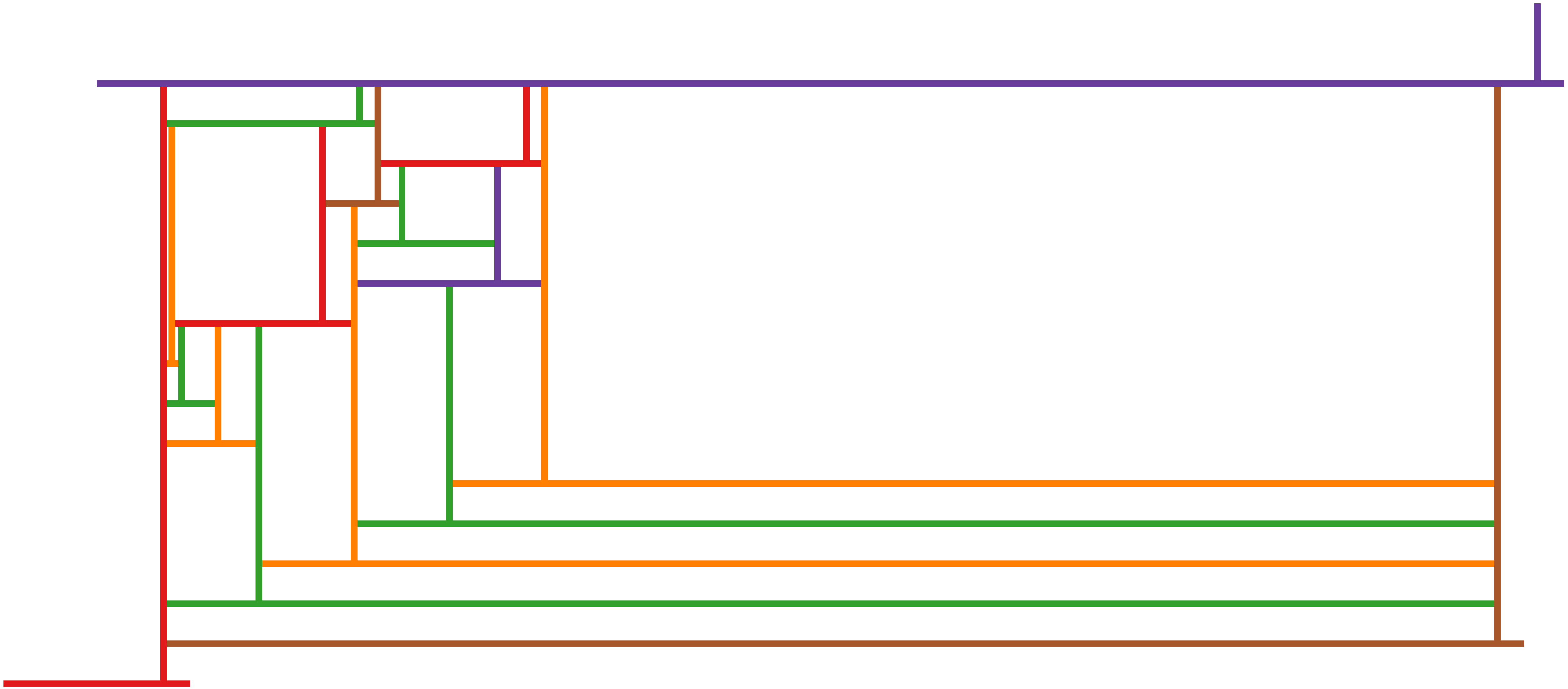

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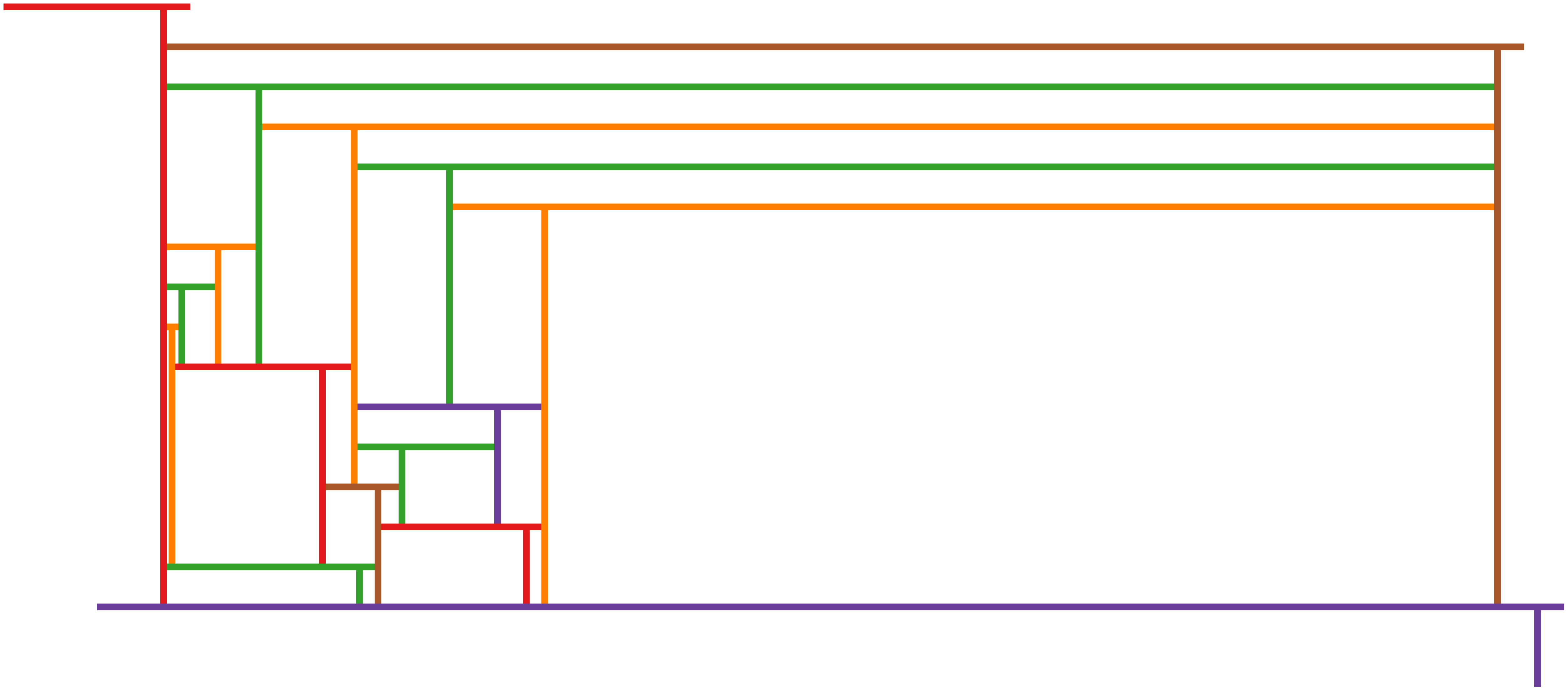
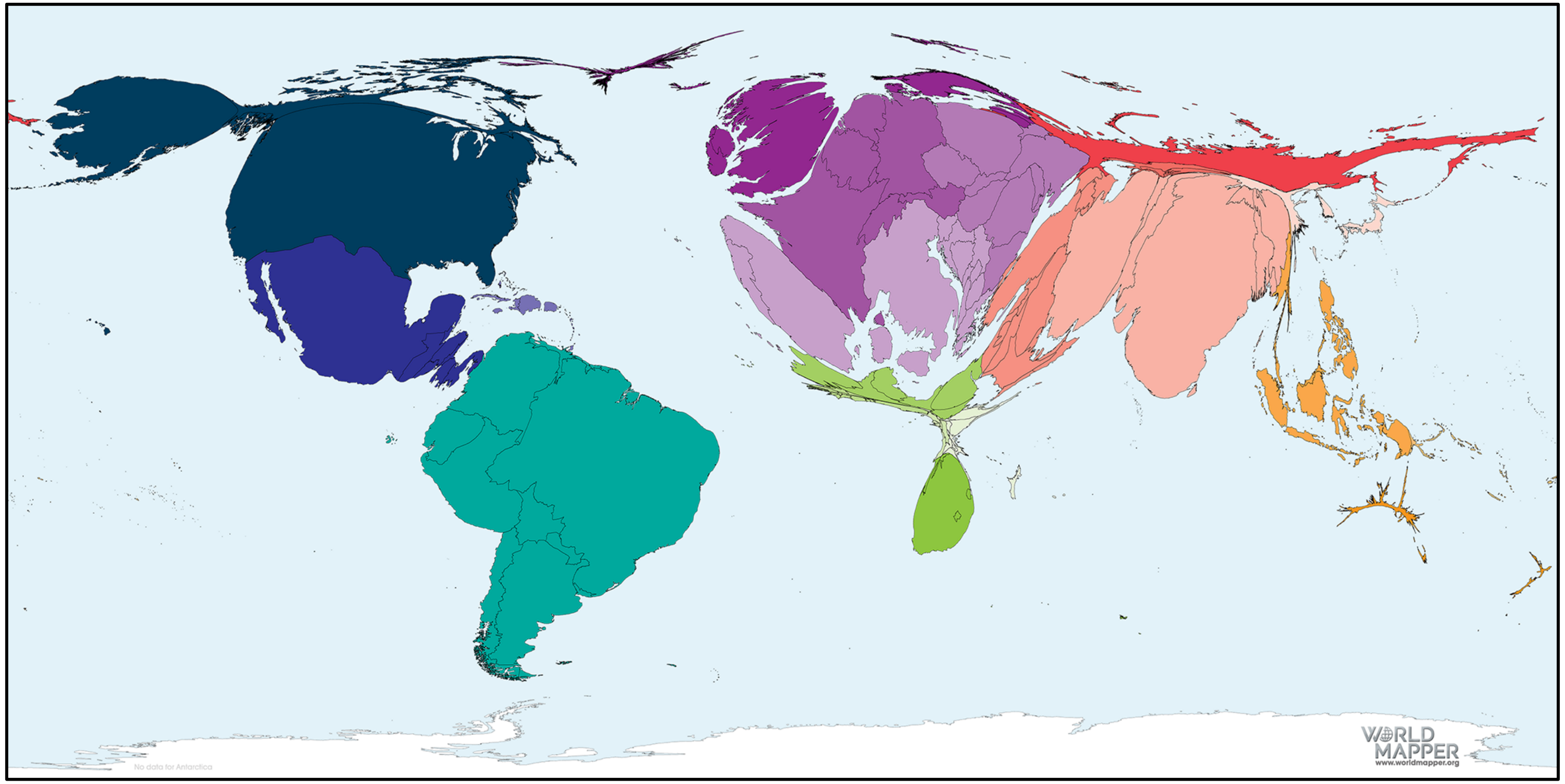


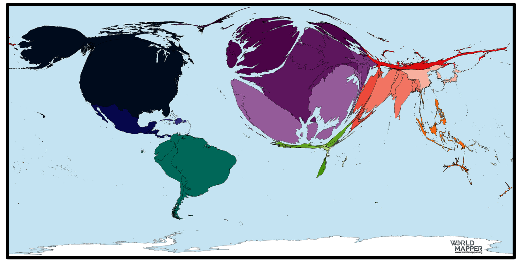
Cartograms

Cartograms



COVID19 reported deaths (January–December 2020)

Cartograms



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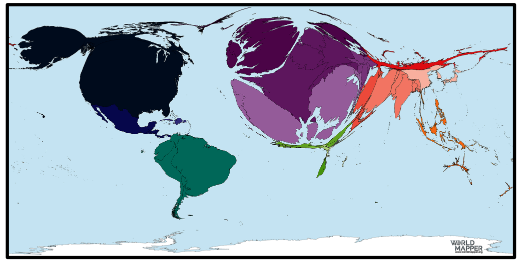

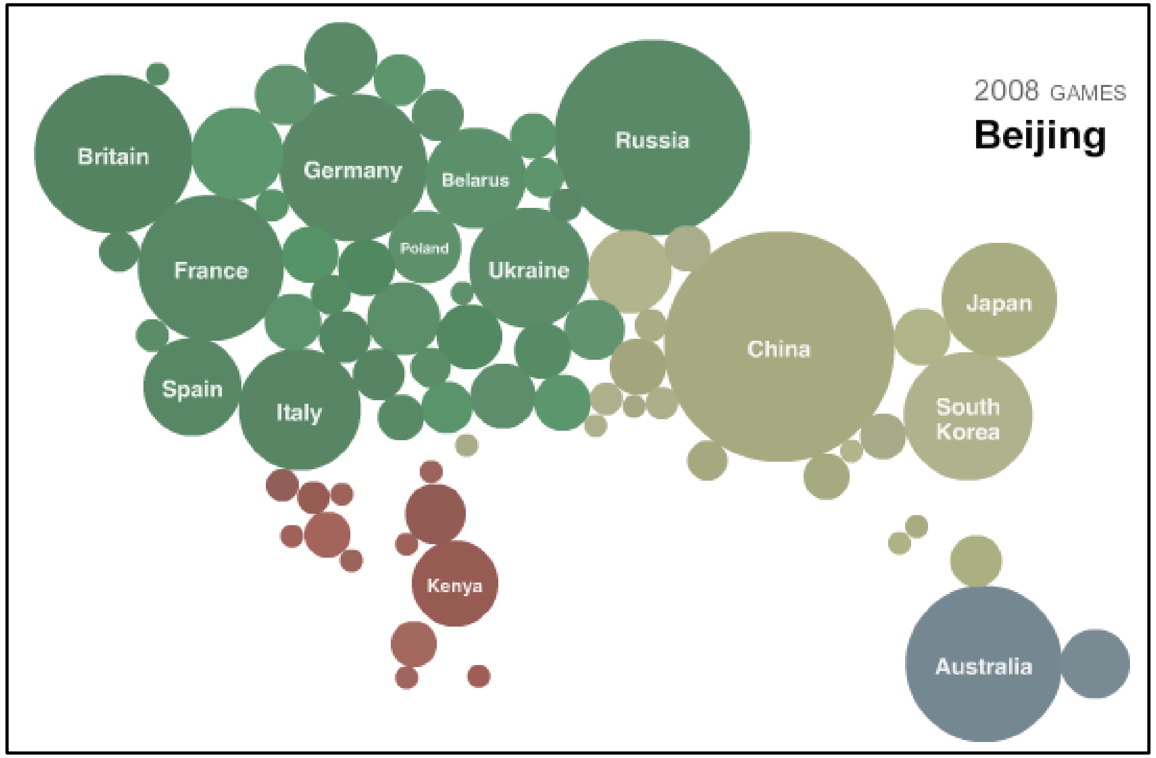
2008 GAMES
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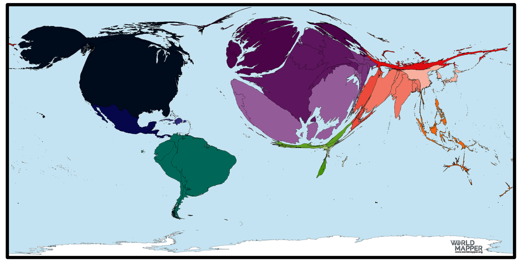
Germany
Belarus
Russia
France
Poland
Ukraine
China
Japan
Spain
Italy
South Korea
Kenya
Australia

Cartograms



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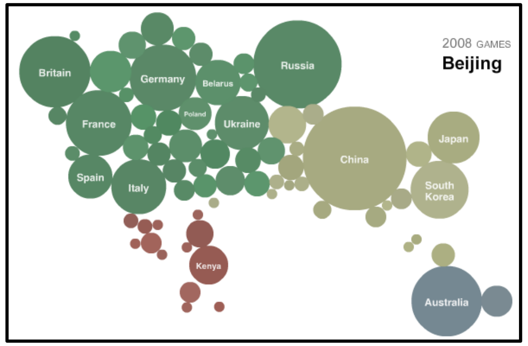

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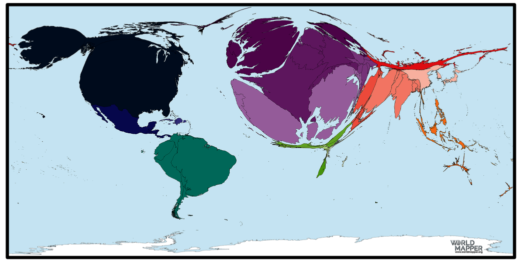

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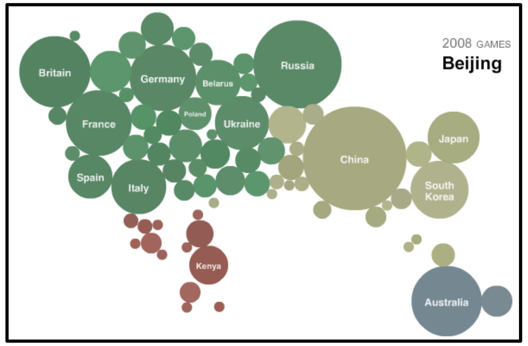

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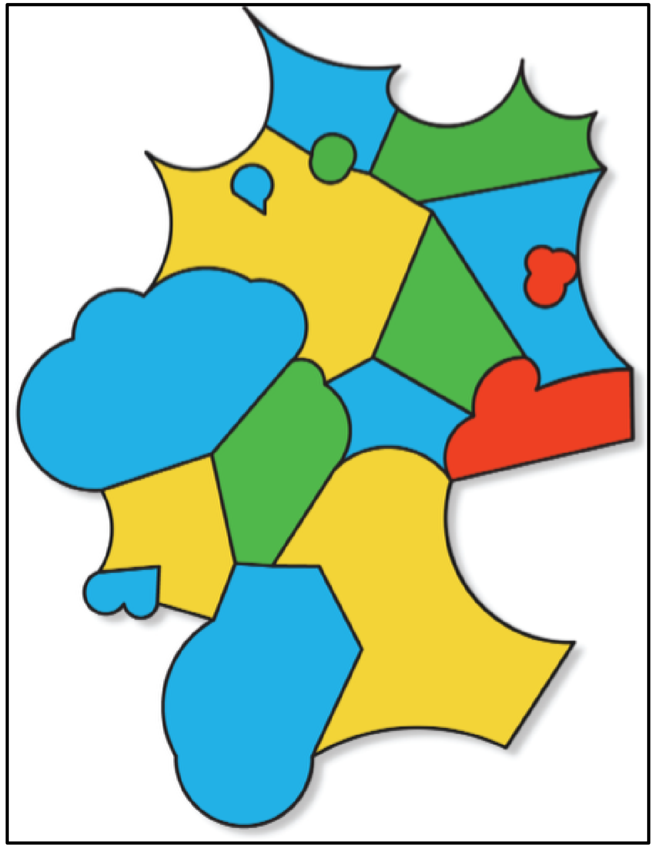

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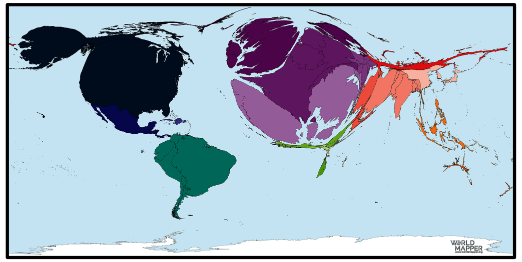

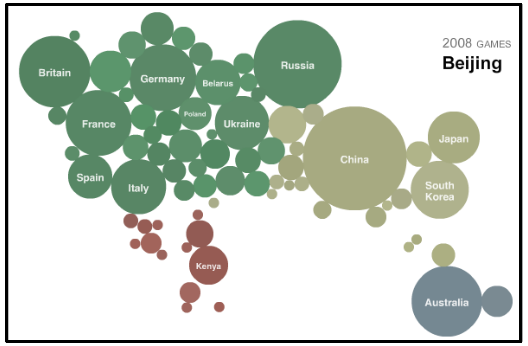
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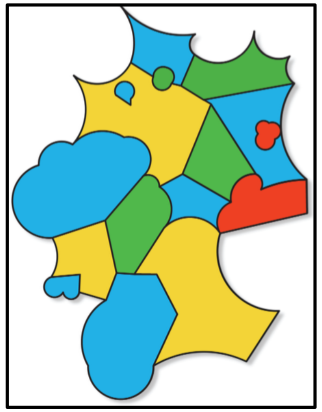

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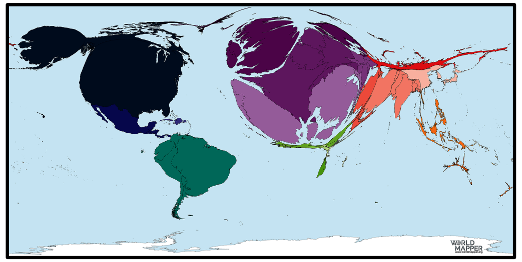

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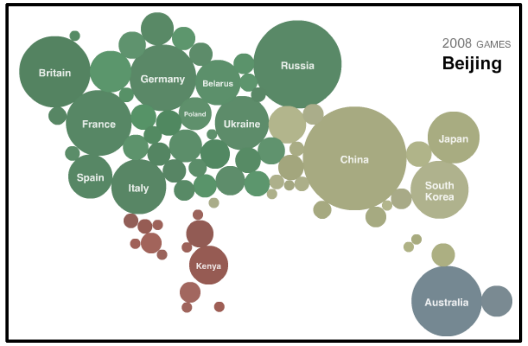

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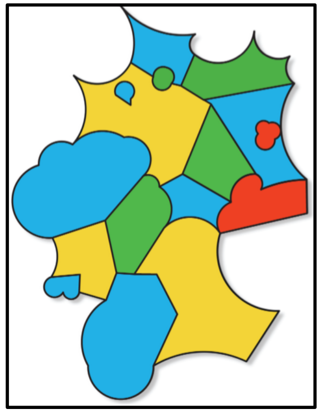

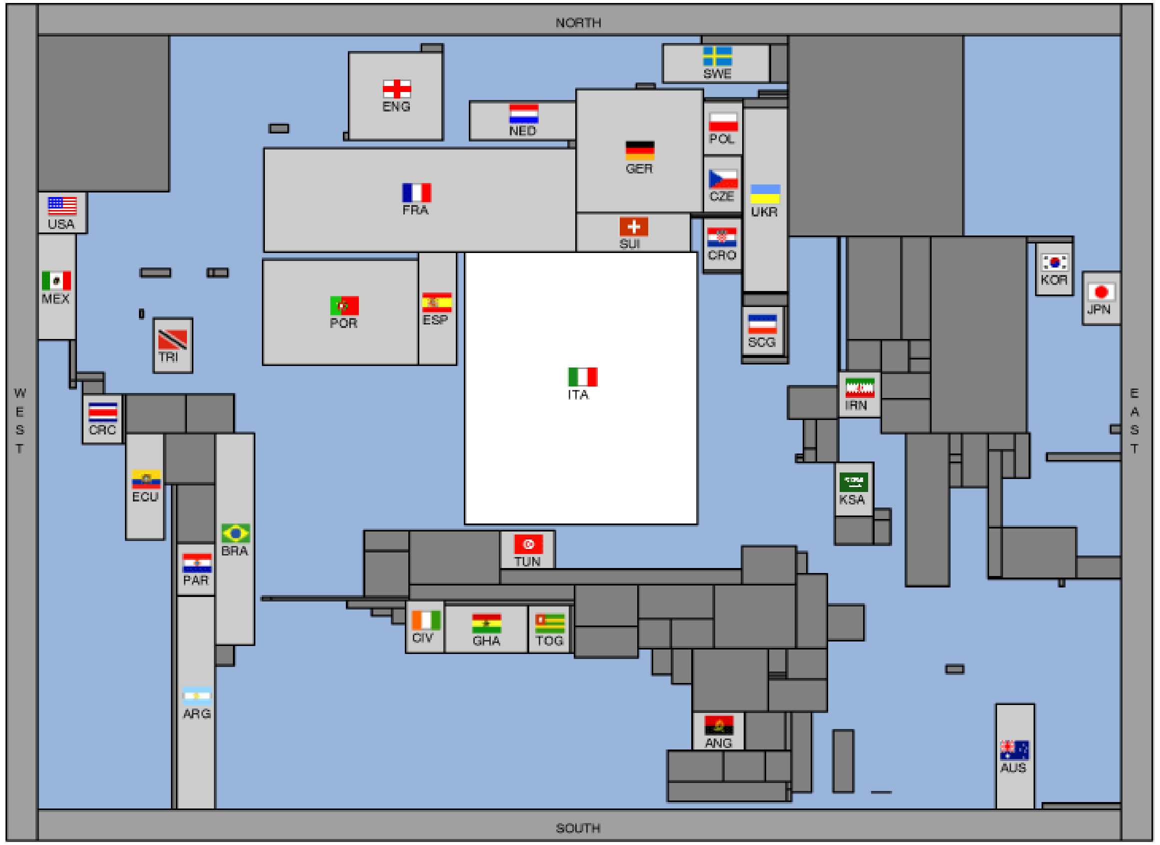

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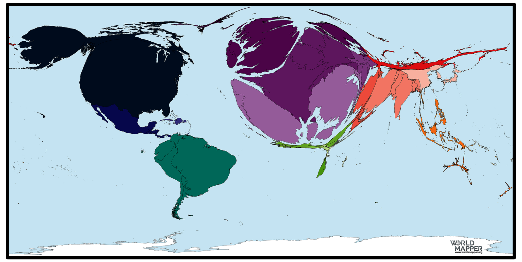

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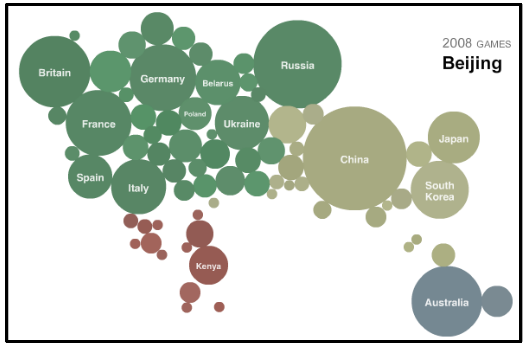

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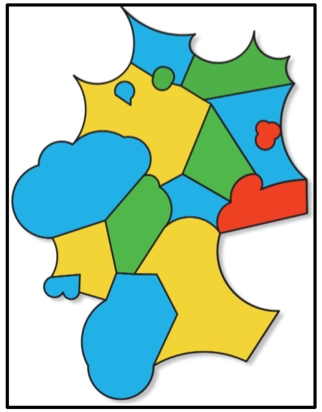

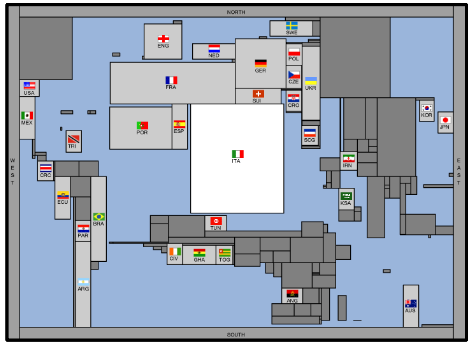

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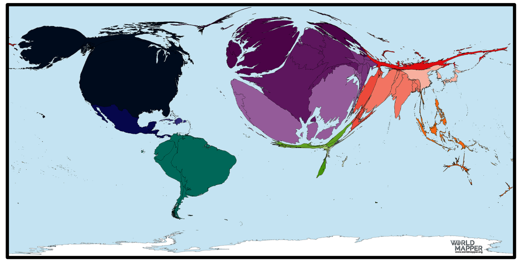

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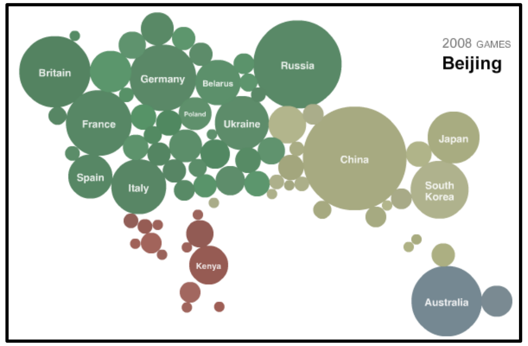

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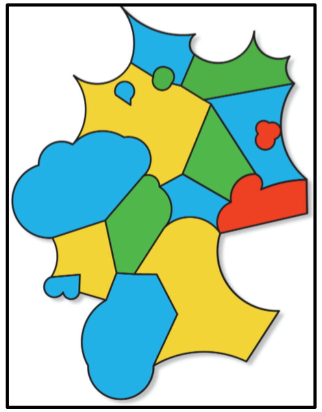

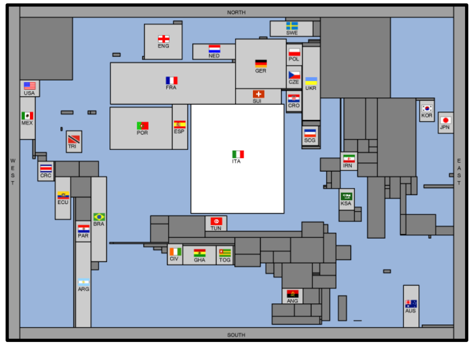

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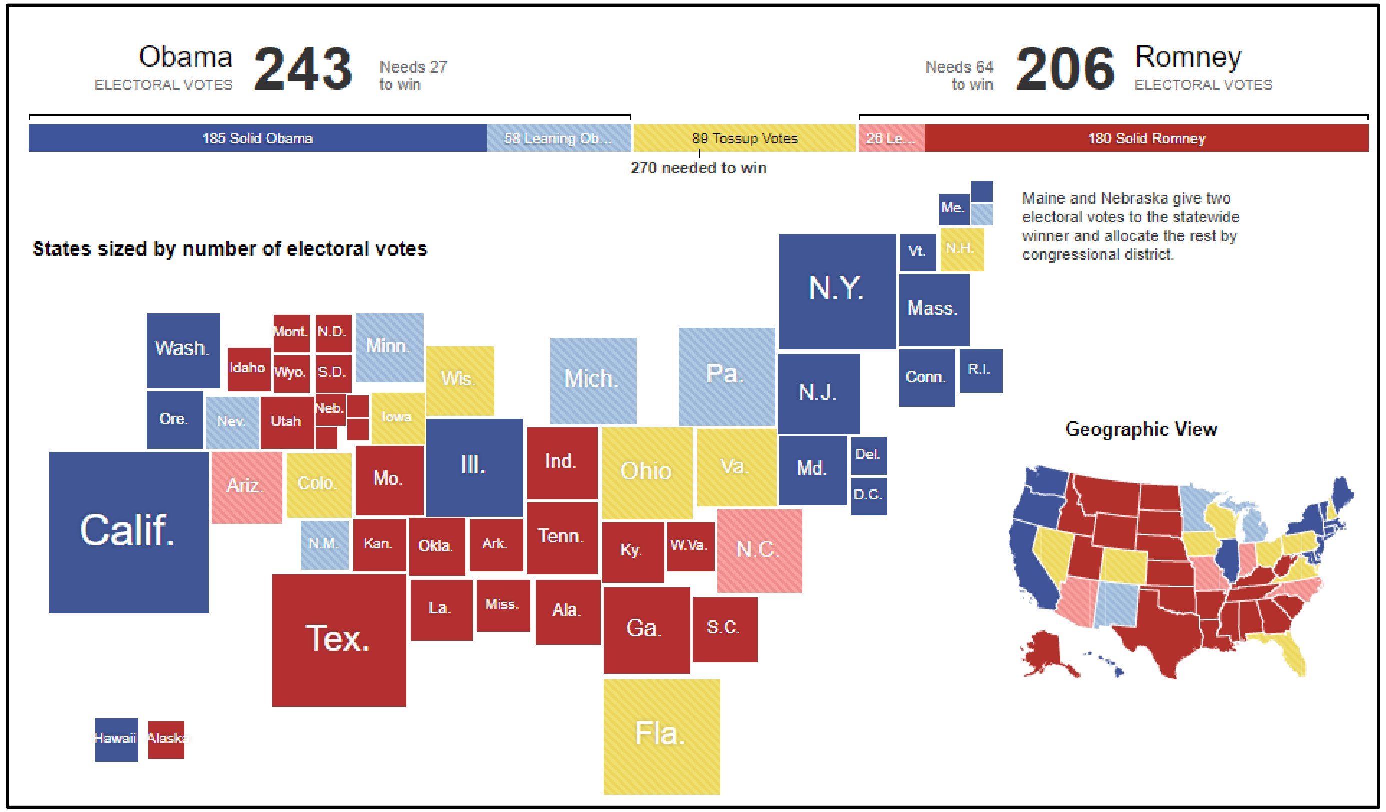

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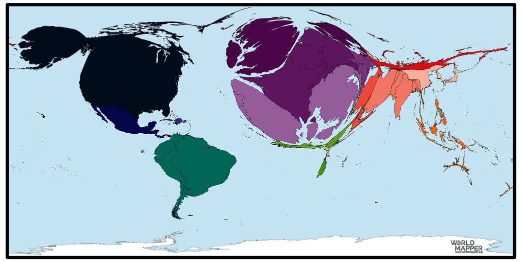

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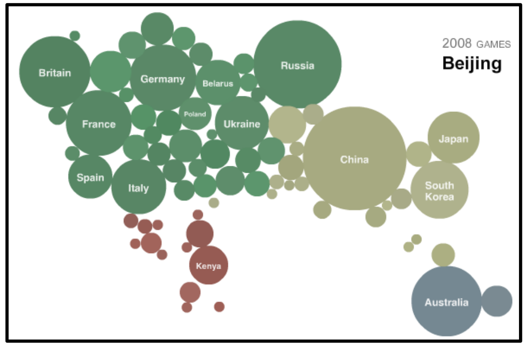

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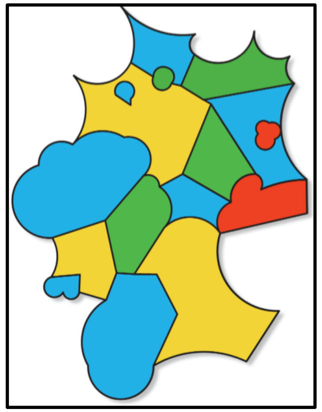

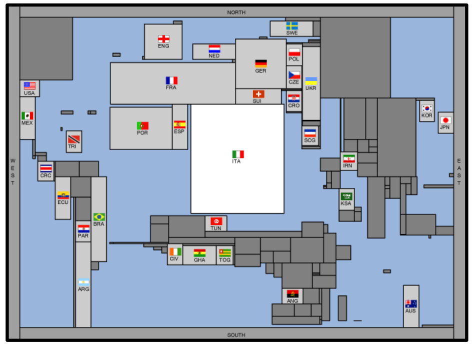

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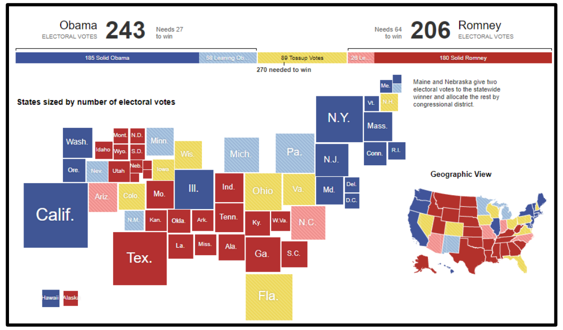

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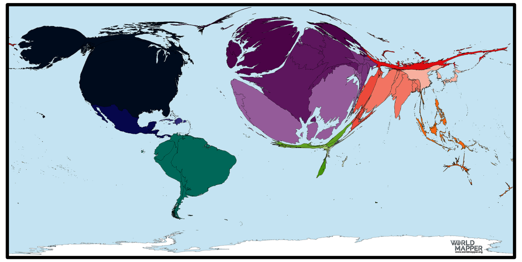

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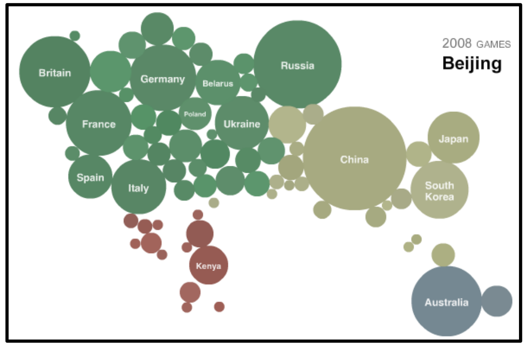

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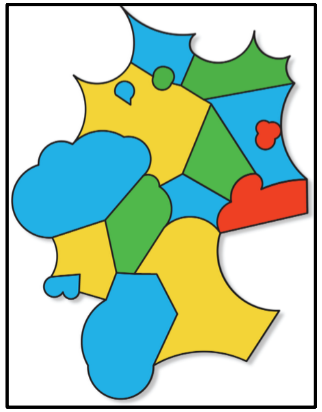

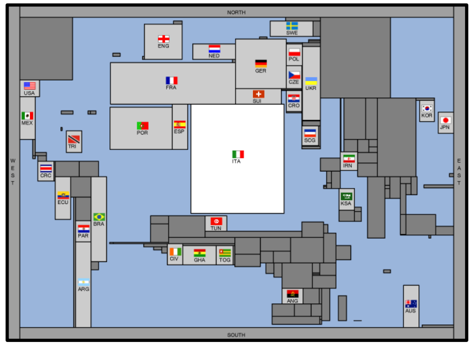

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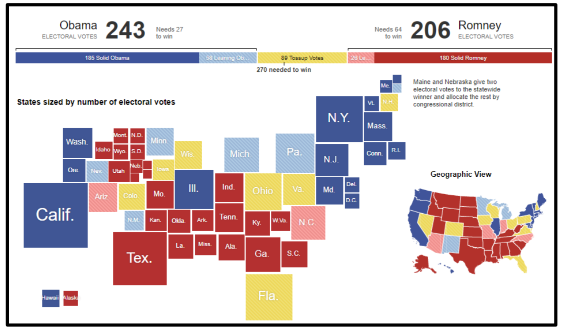

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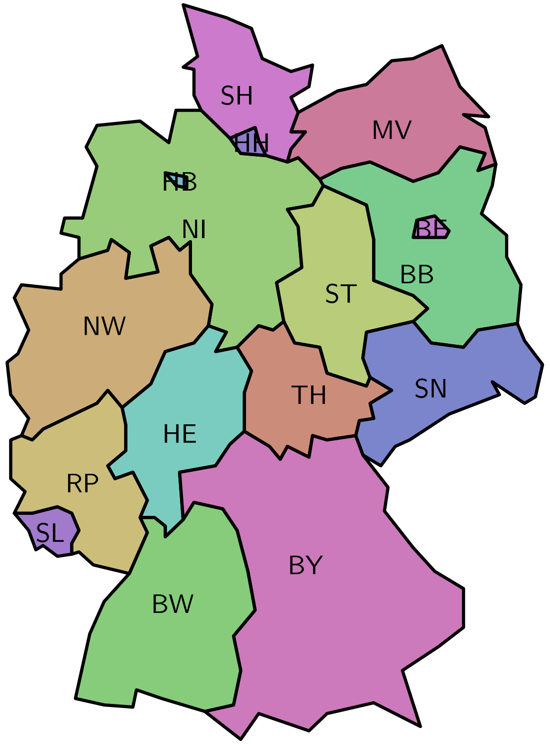

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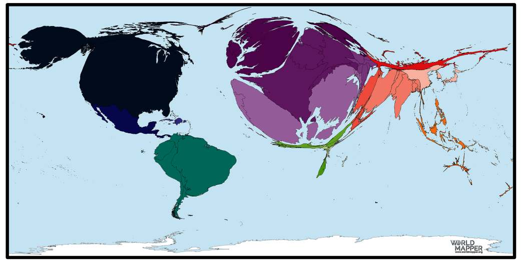

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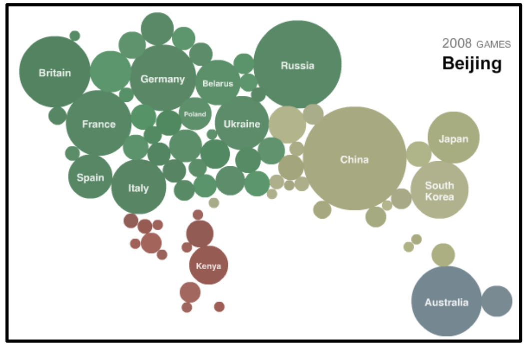

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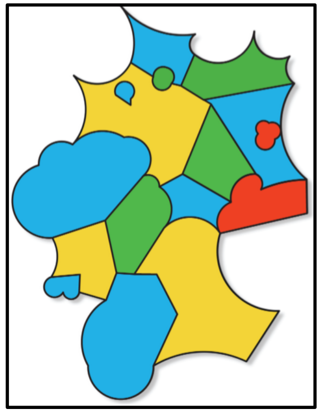

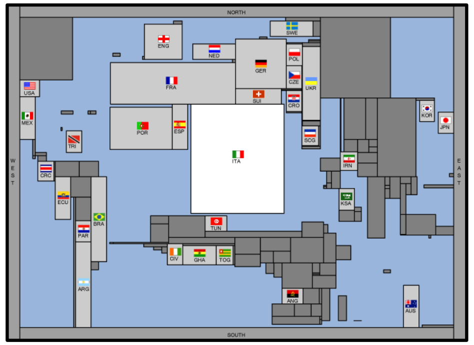

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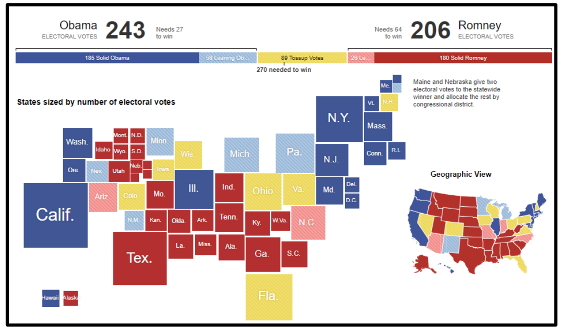

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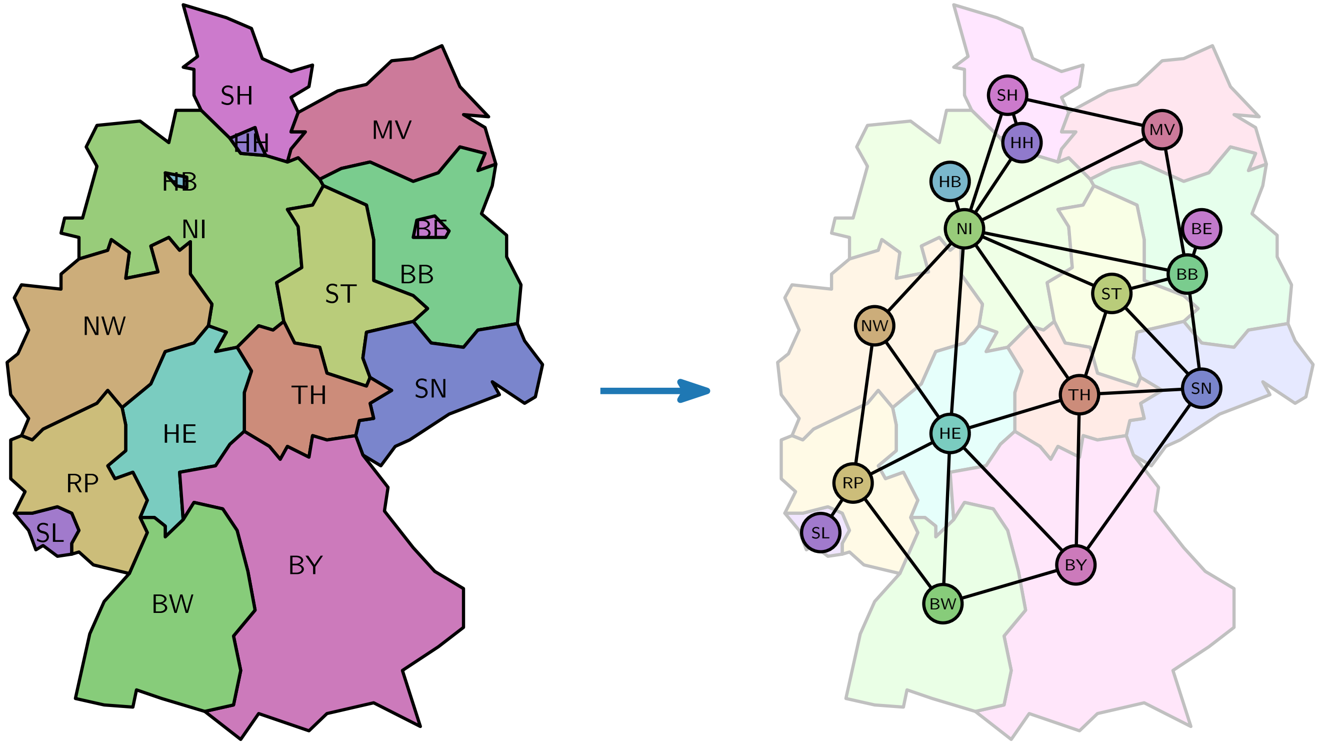

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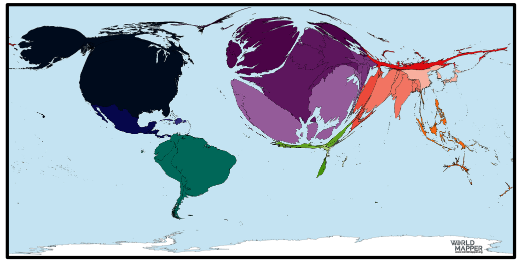

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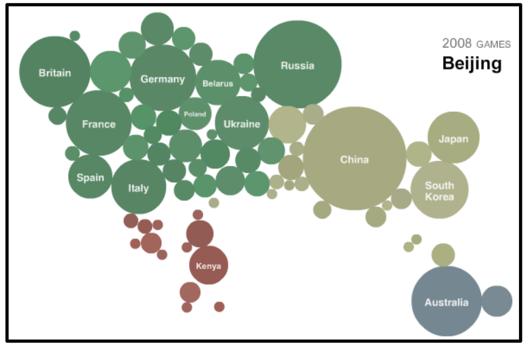

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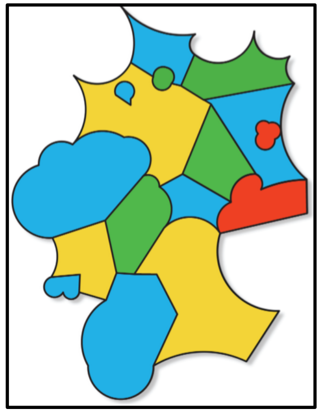

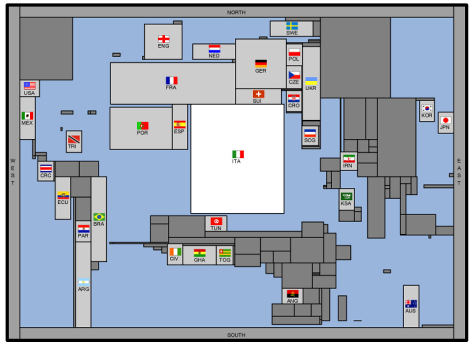

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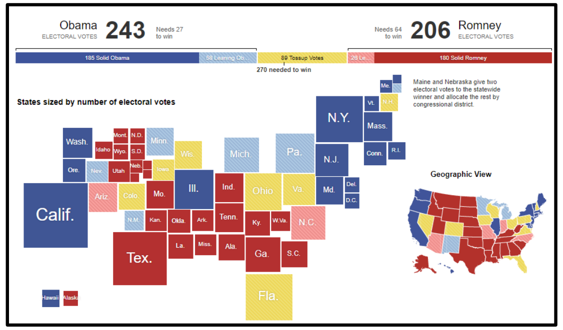

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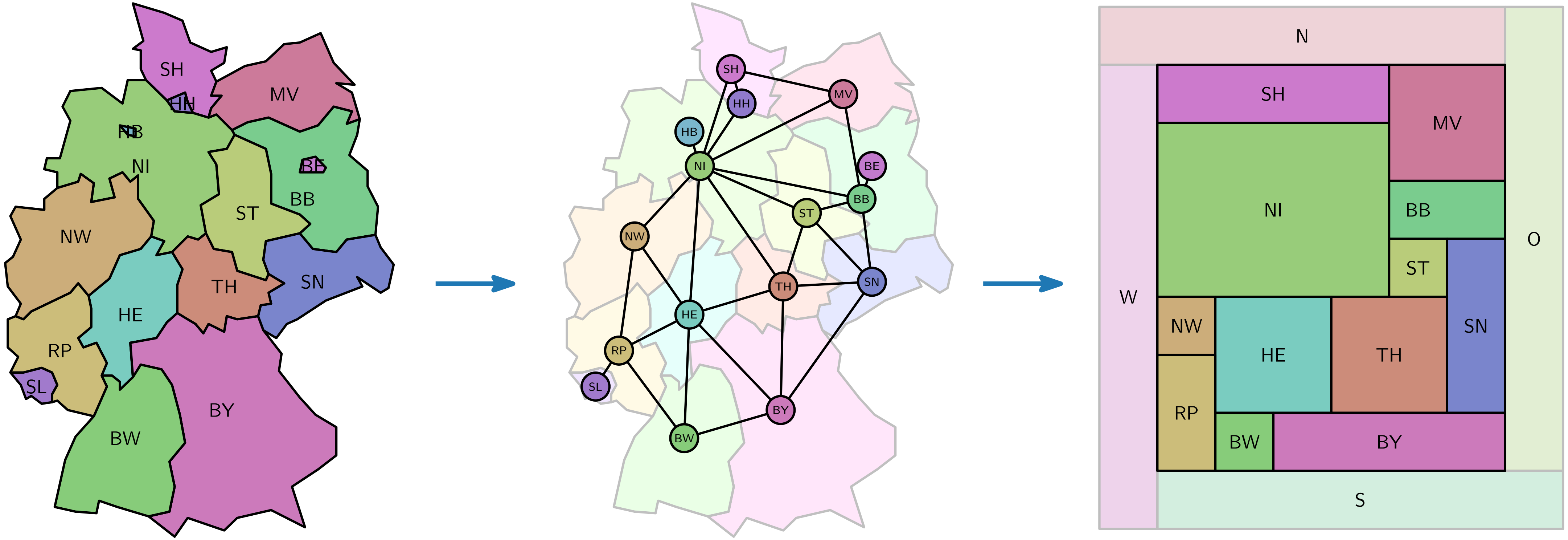

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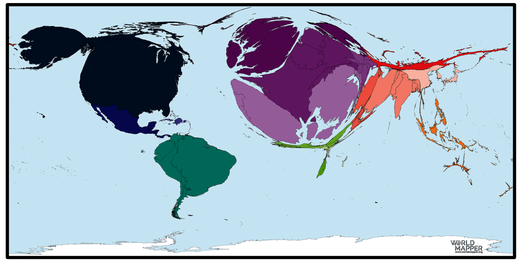

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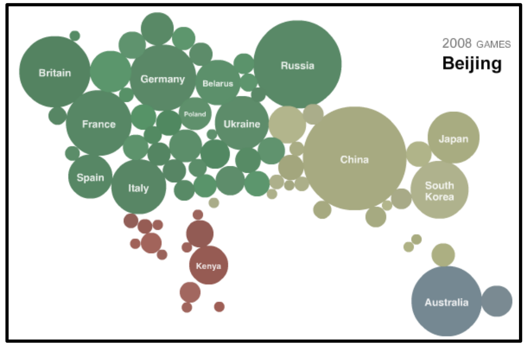

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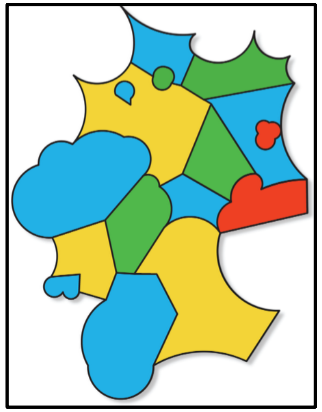

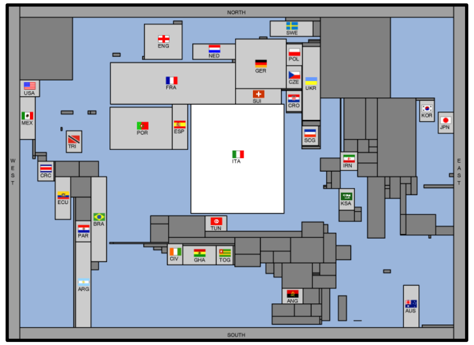

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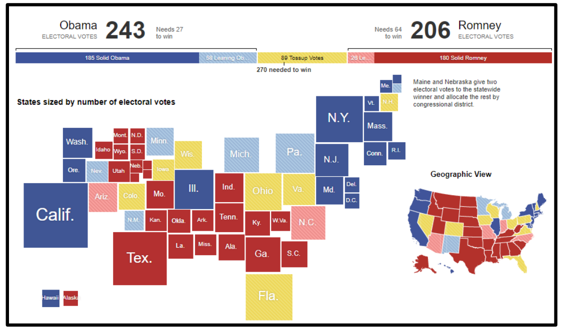

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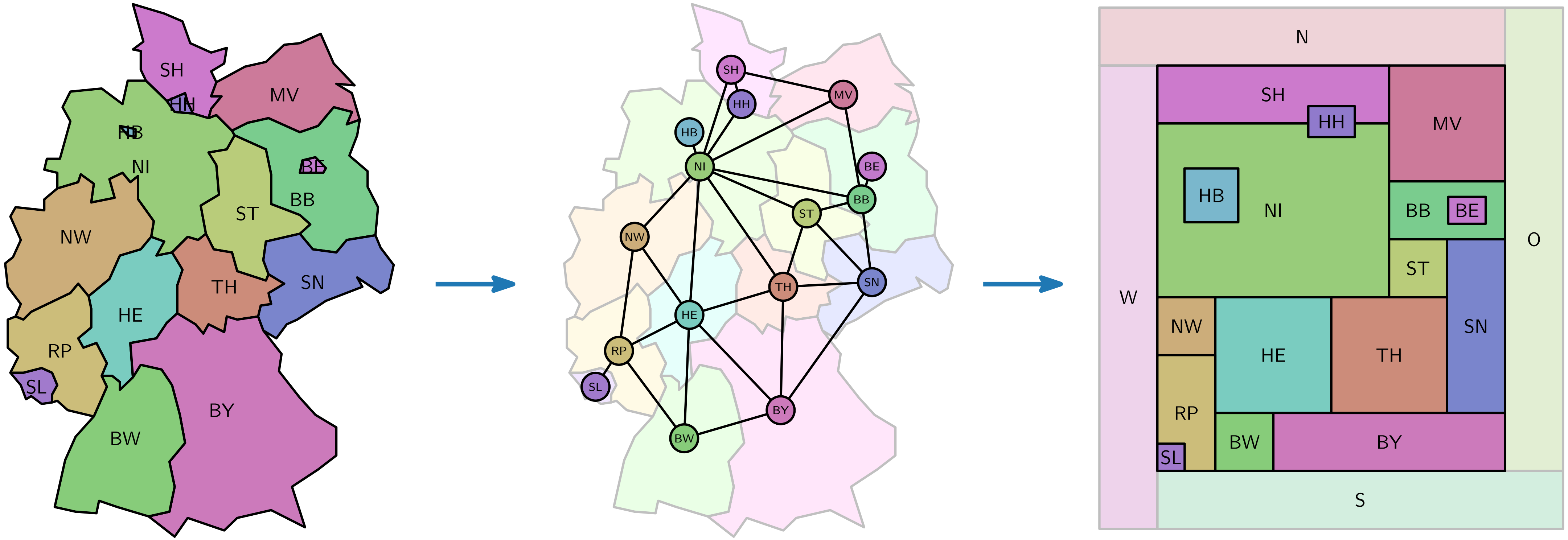
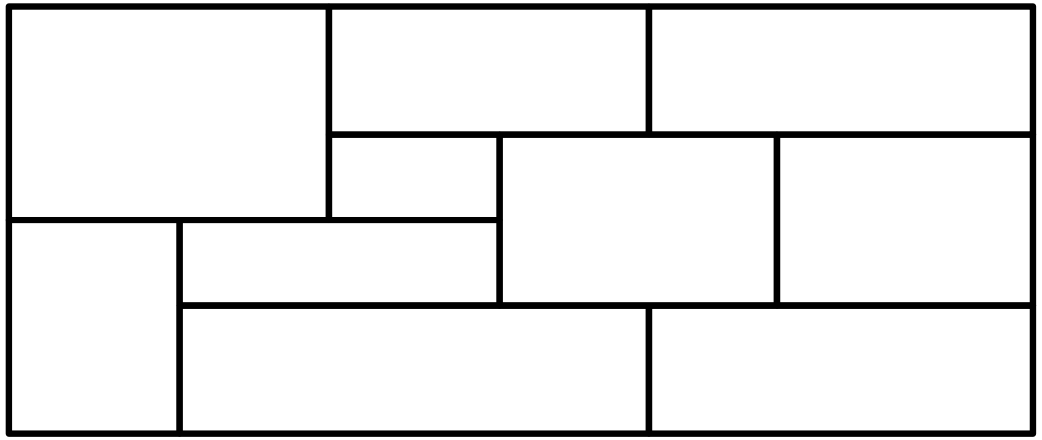


Rectangular Dual

Rectangular Dual



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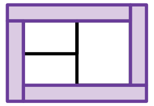
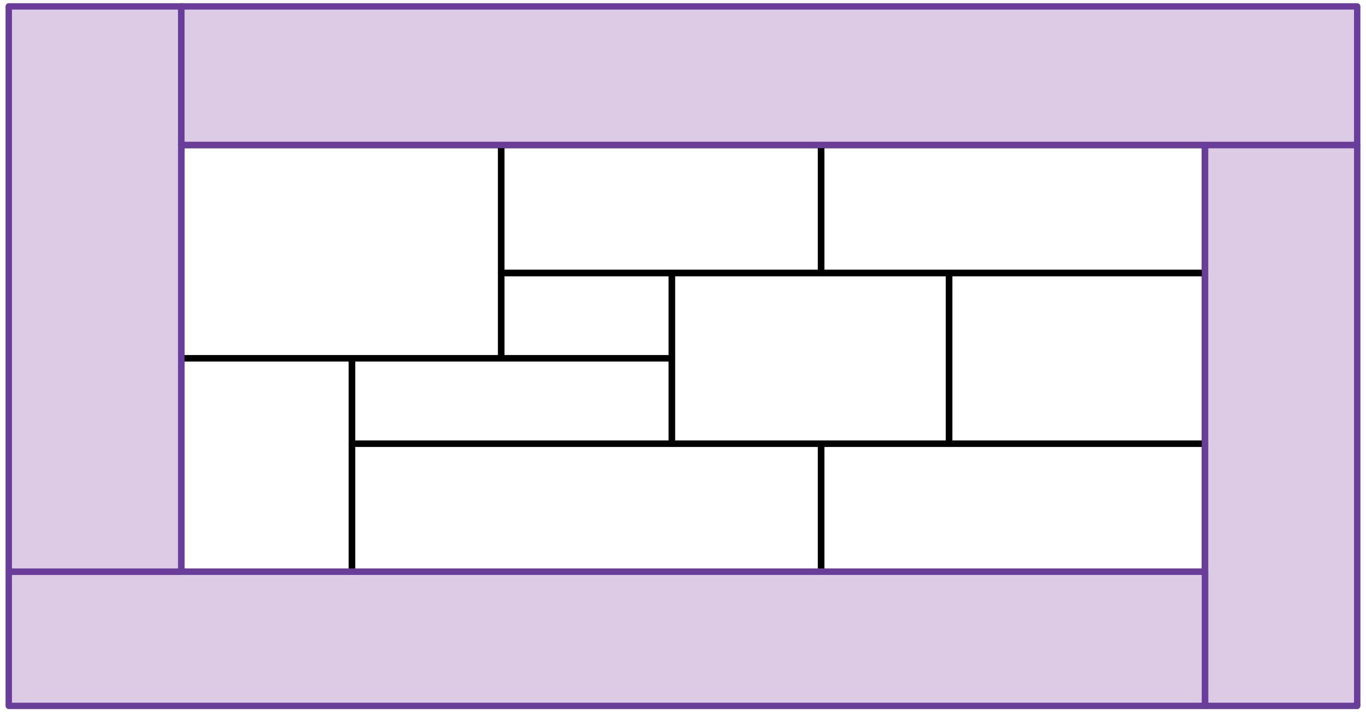


Rectangular Dual $\mathcal{R}$



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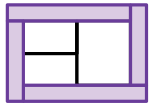
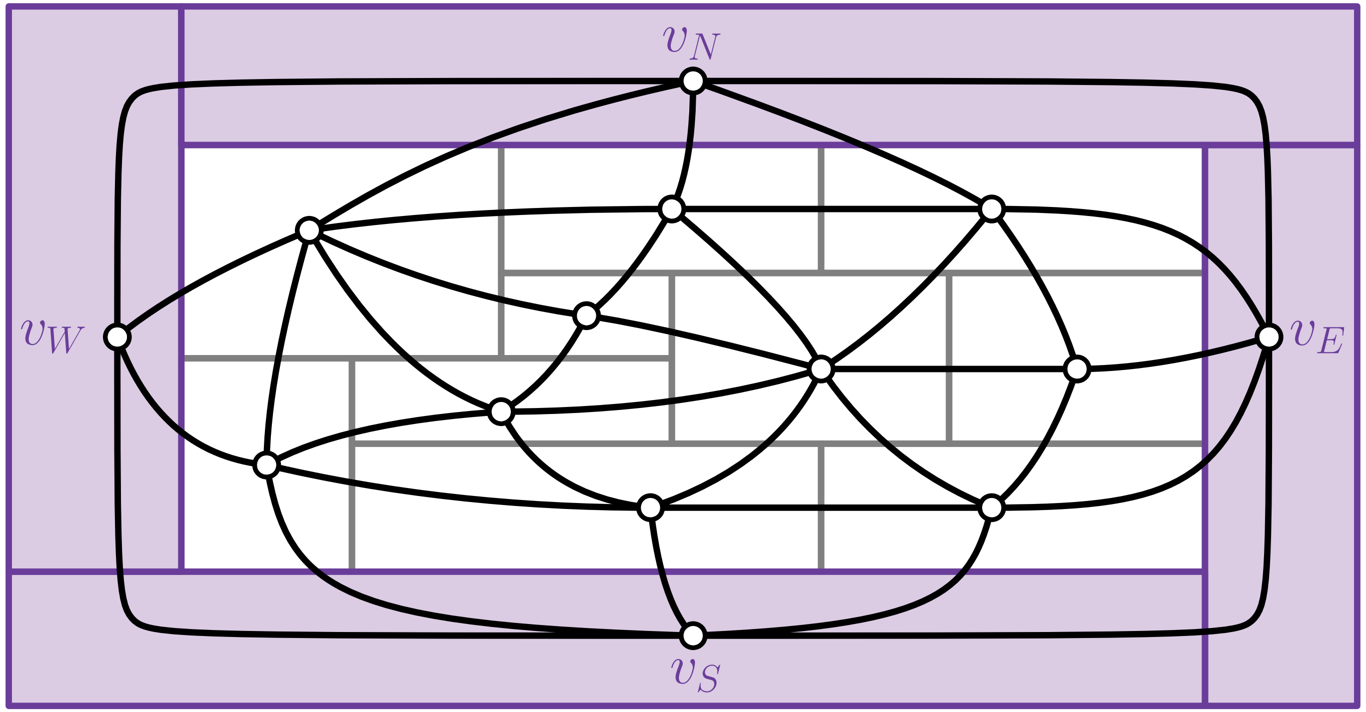
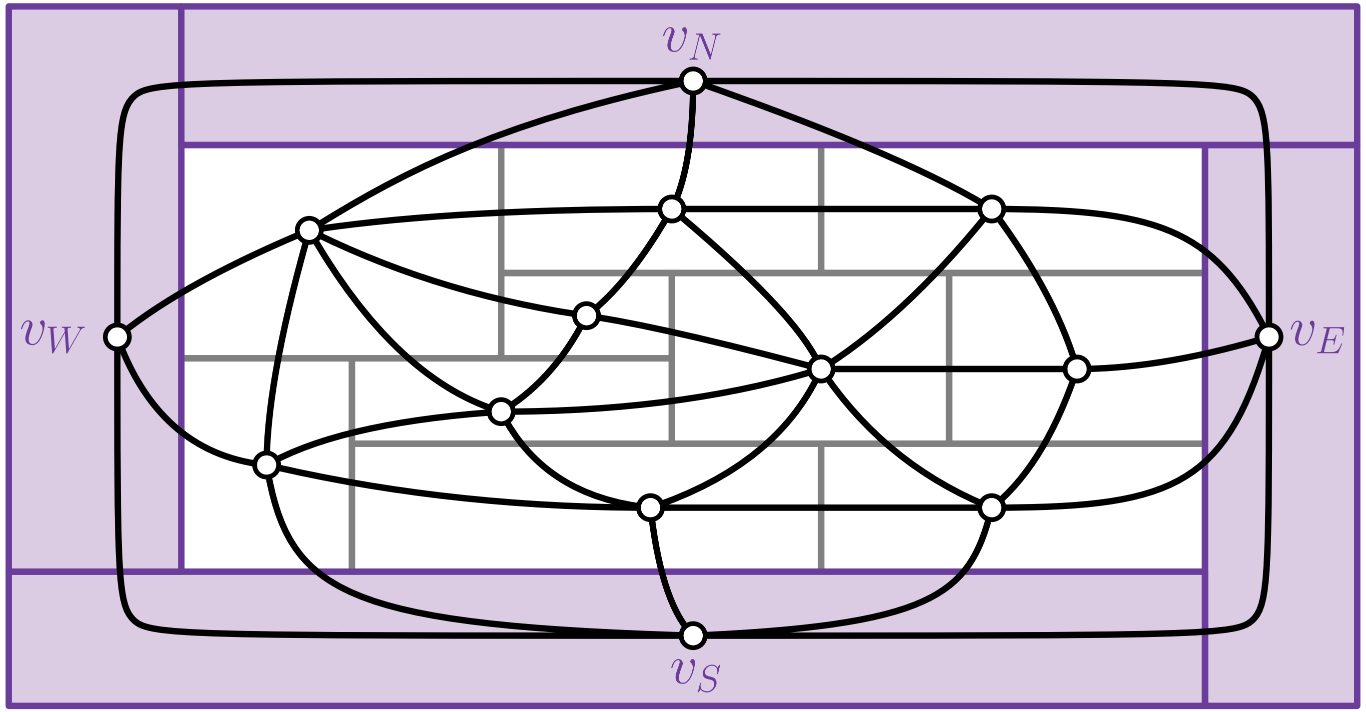
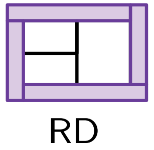


Rectangular Dual $\mathcal{R}$



Rectangular Dual

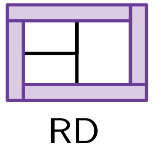




Rectangular Dual $\mathcal{R}$

A **rectangular dual** of a graph G is a contact representation with axis-aligned rectangles s.t.

Rectangular Dual



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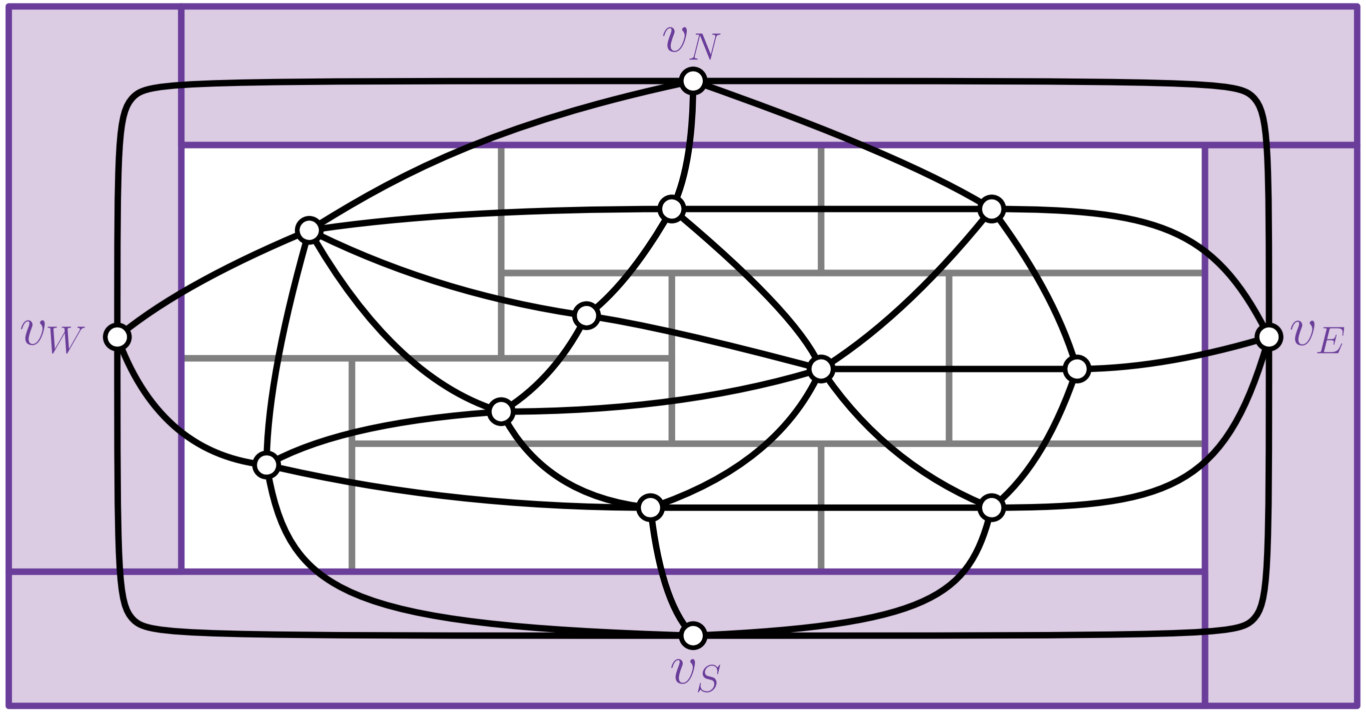

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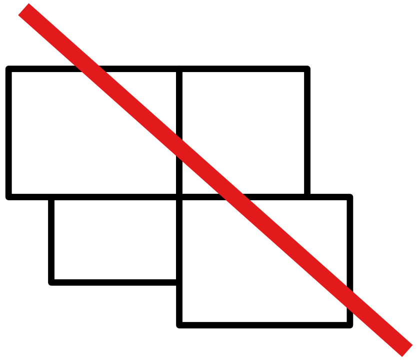

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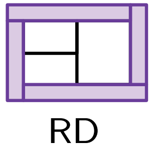

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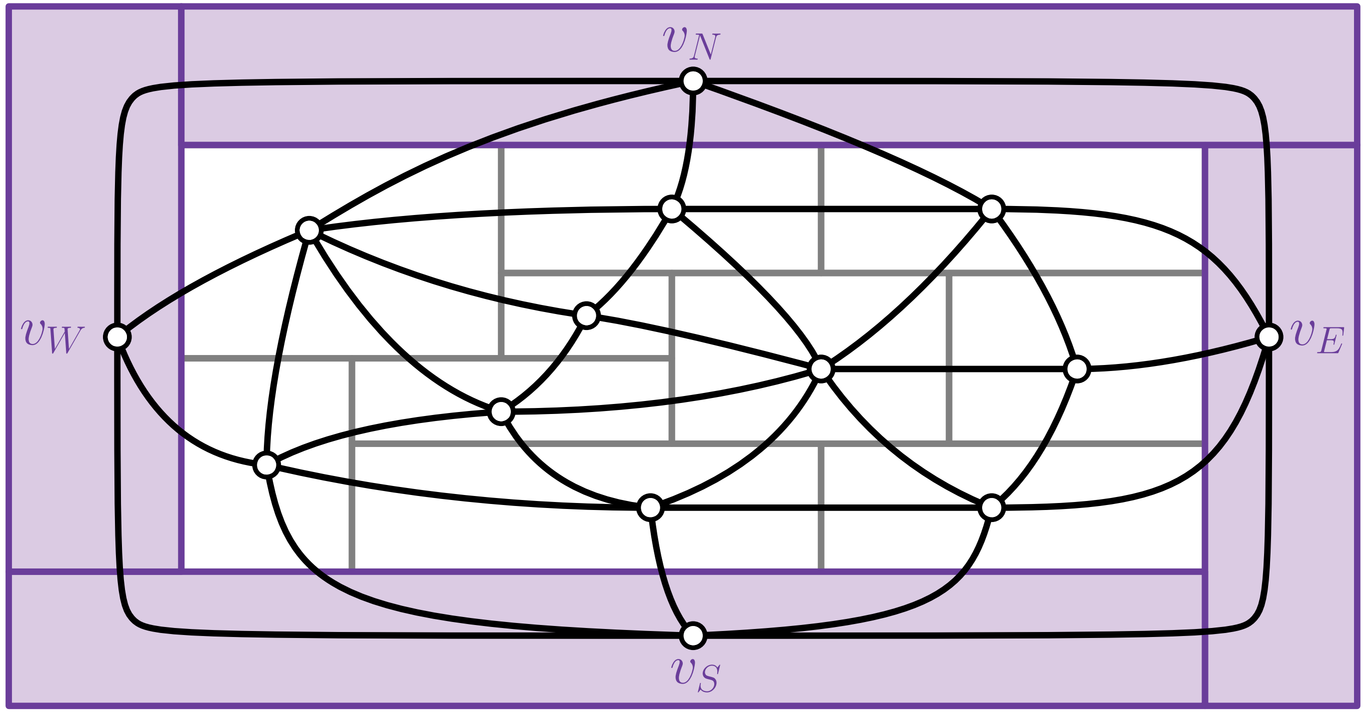

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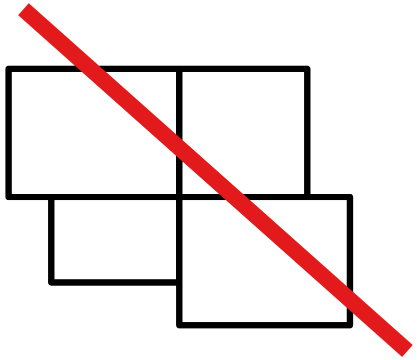
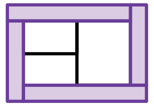
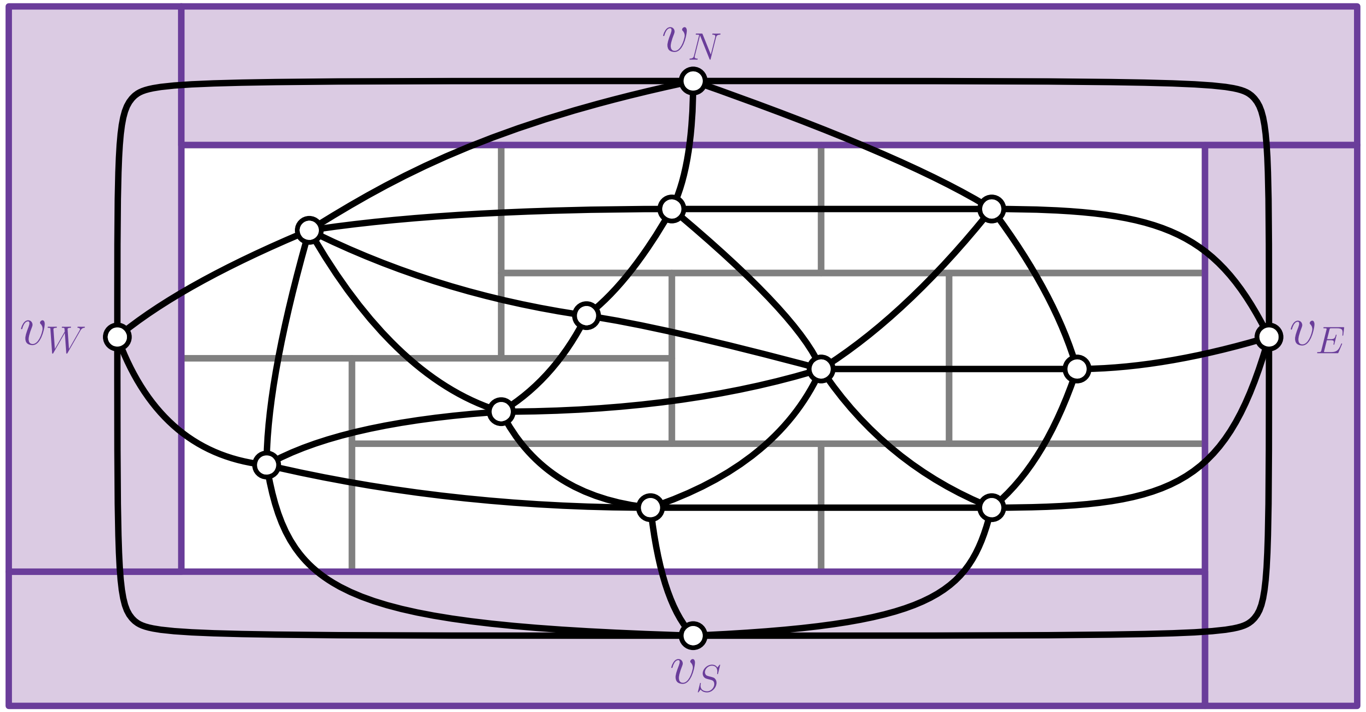


Rectangular Dual


RD

Rectangular Dual $\mathcal{R}$



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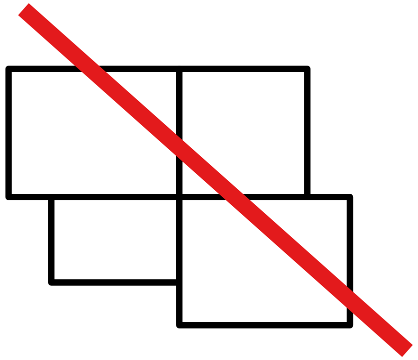

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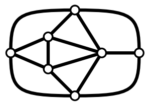
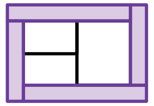
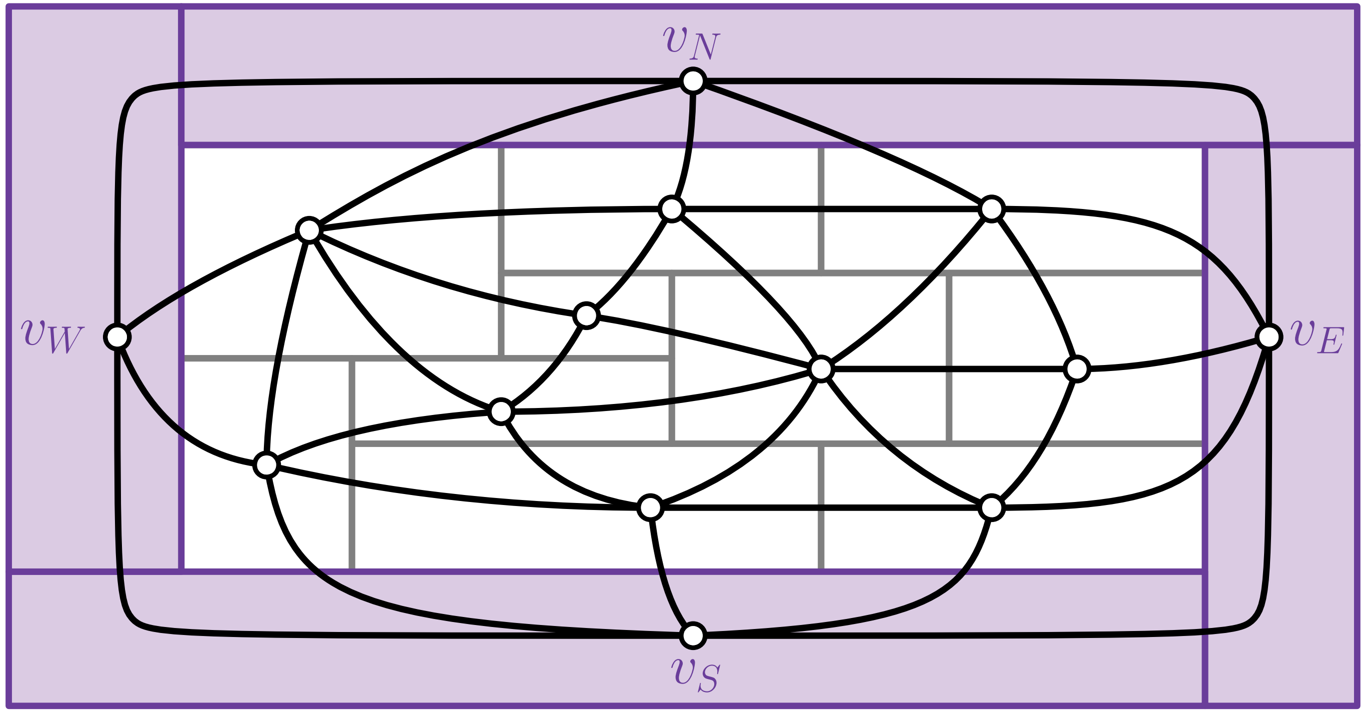


Properly Triangulated Planar Graph G



Rectangular Dual $\mathcal{R}$



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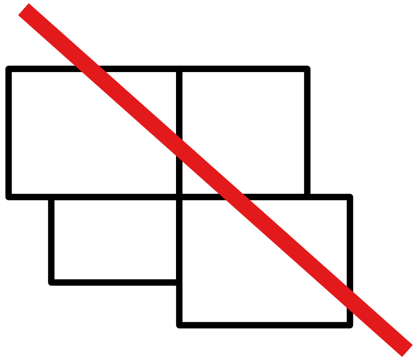

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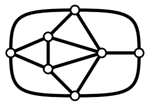

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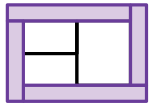
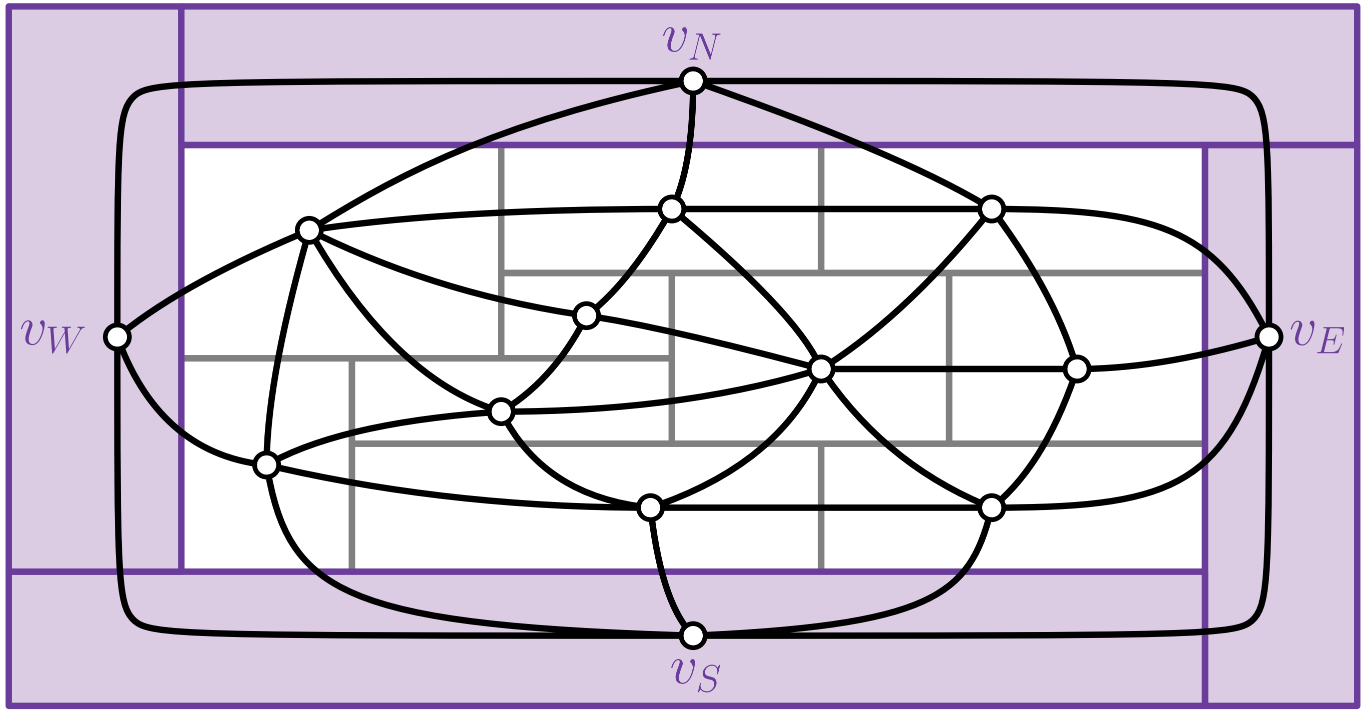


Rectangular Dual $\mathcal{R}$



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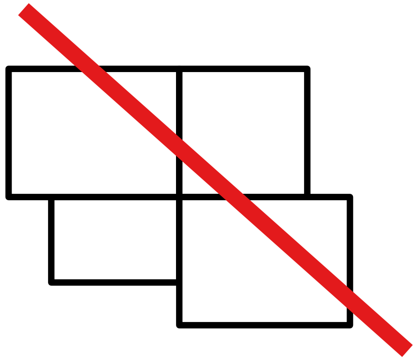

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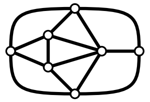

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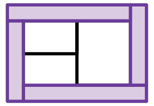
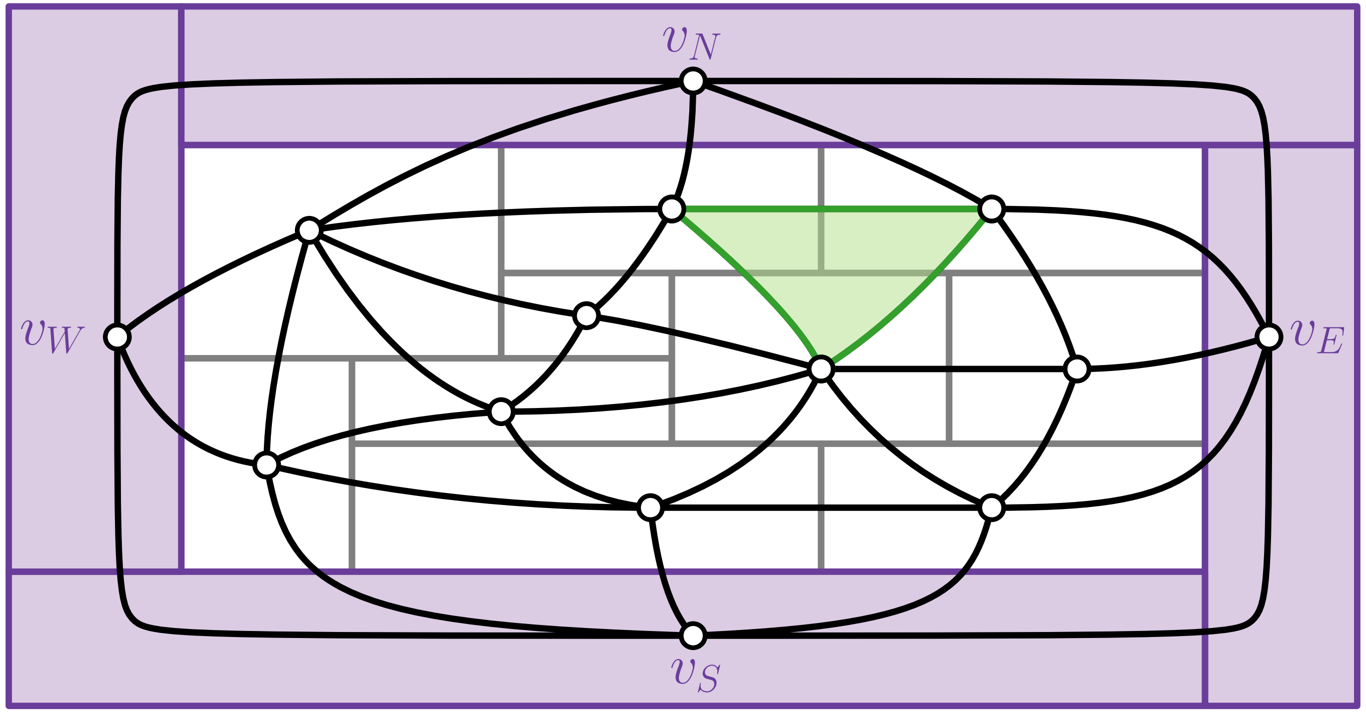


Rectangular Dual $\mathcal{R}$



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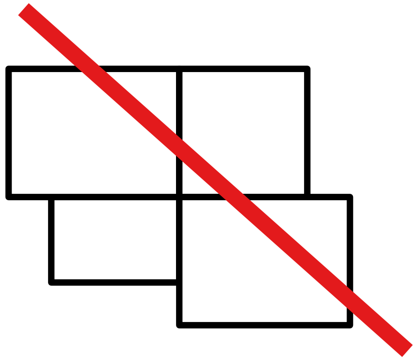

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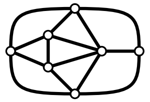
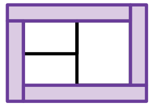


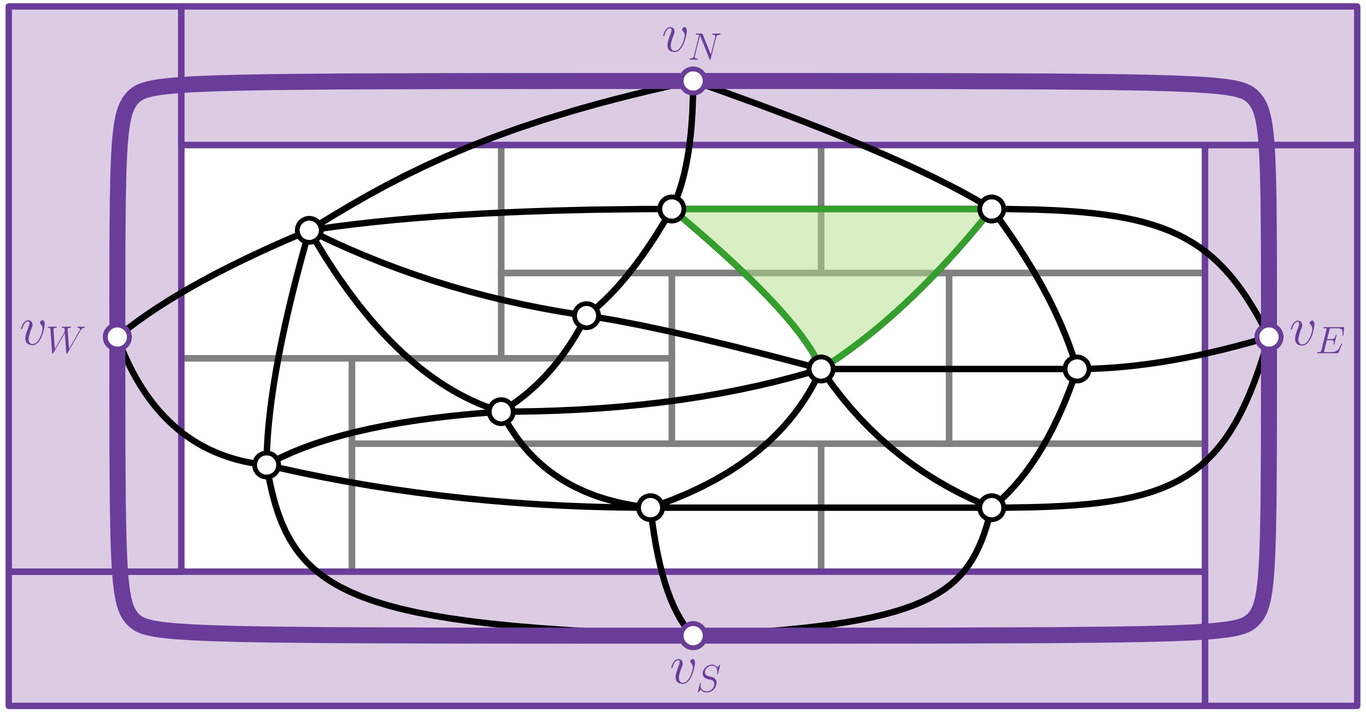
PTP

Properly Triangulated Planar Graph G



RD

Rectangular Dual $\mathcal{R}$



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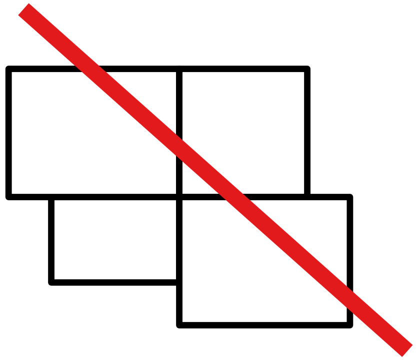

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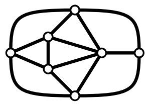
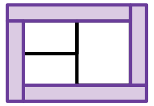
Rectangular Dual

Exactly four vertices on the outer face.



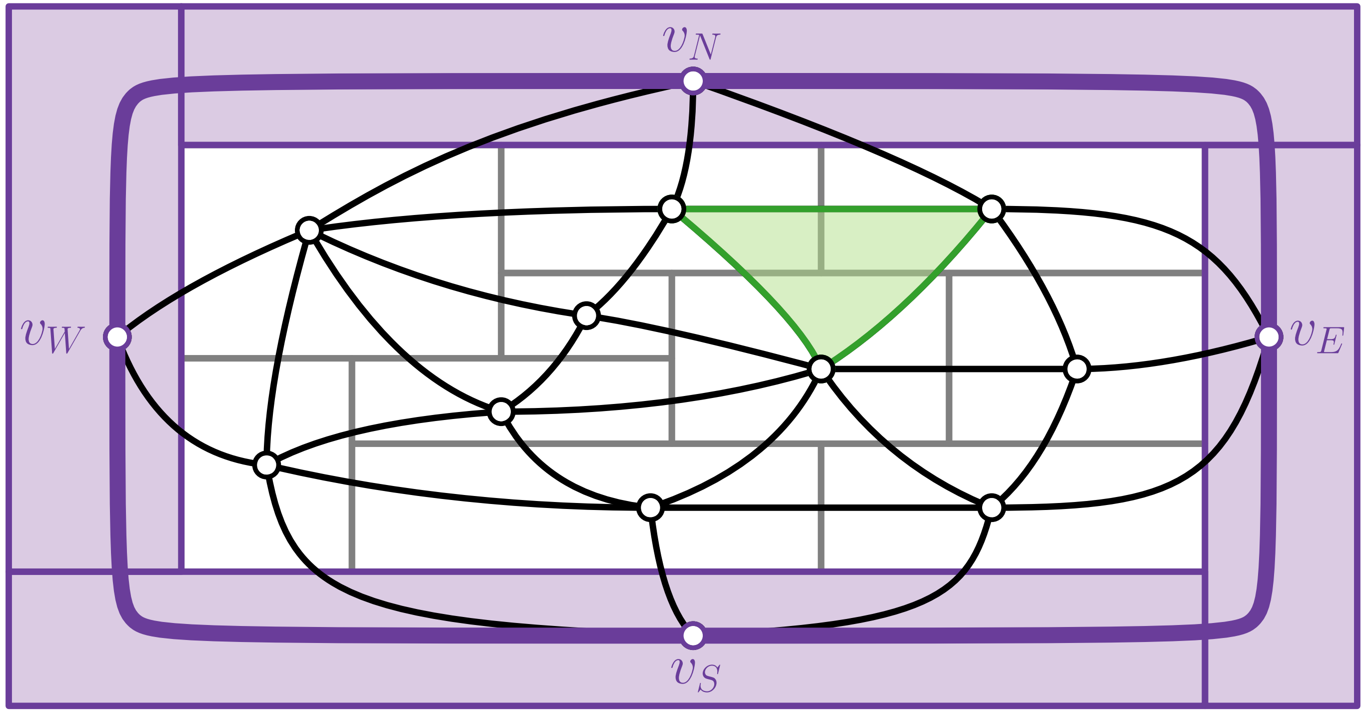
PTP

Properly Triangulated Planar Graph G



RD

Rectangular Dual $\mathcal{R}$



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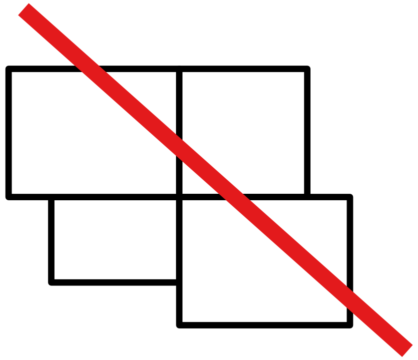

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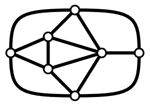
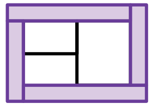


PTP

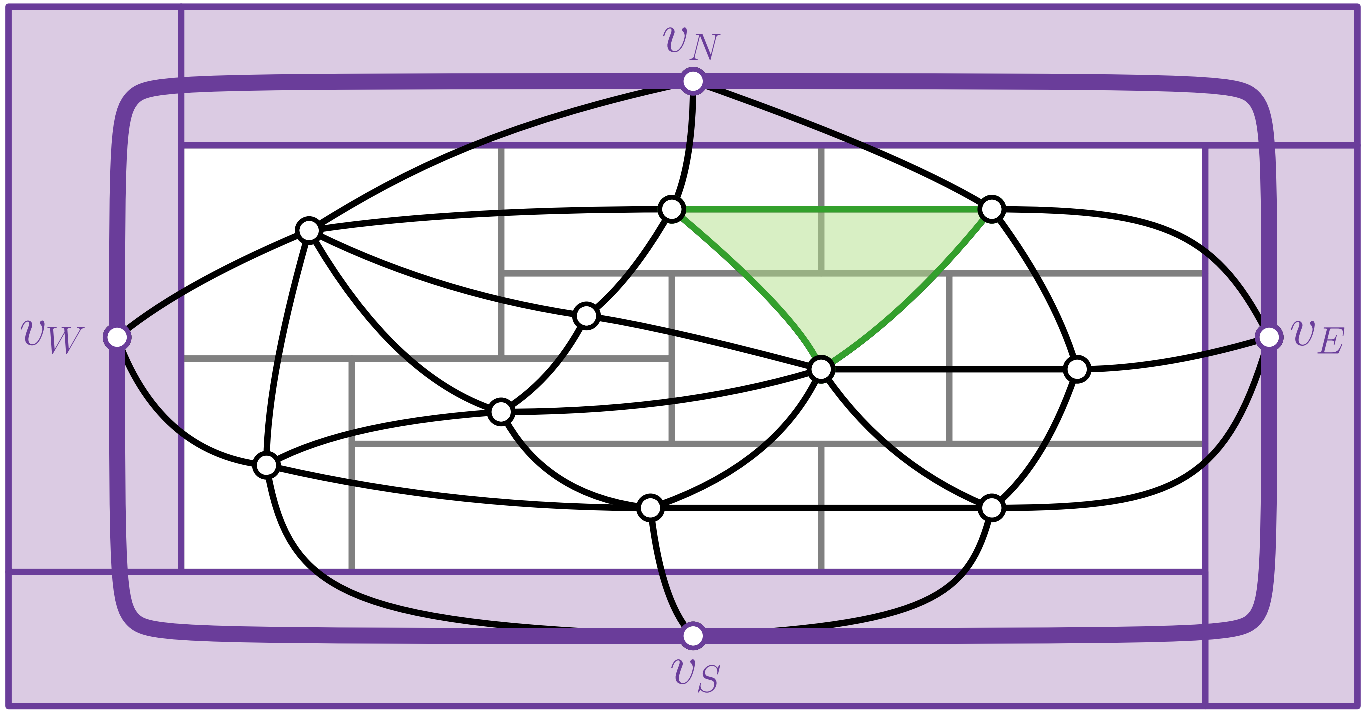
Properly Triangulated Planar Graph G



RD

Rectangular Dual $\mathcal{R}$

Exactly four vertices on the outer face.



No separating triangle!

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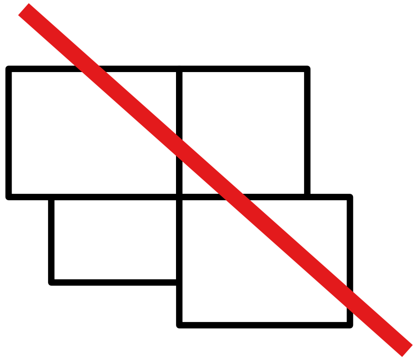

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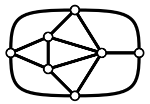
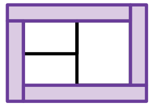


PTP

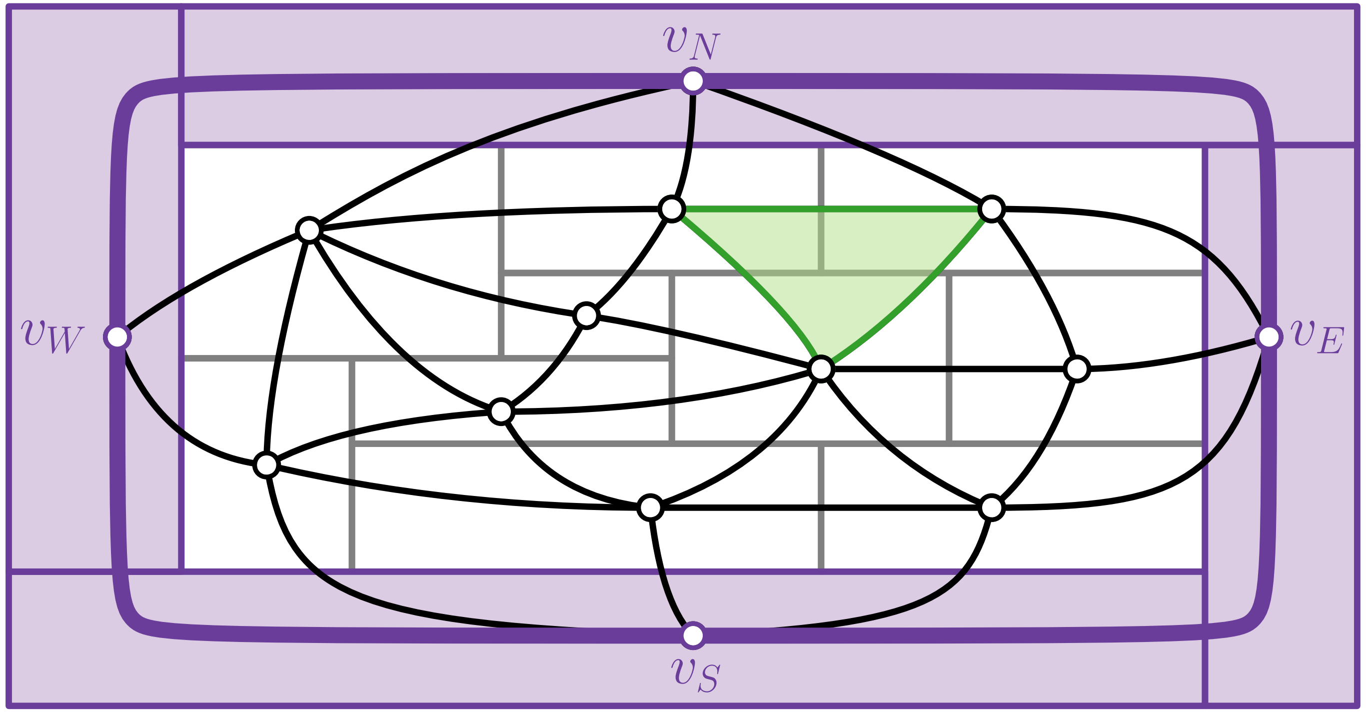
Properly Triangulated Planar Graph G

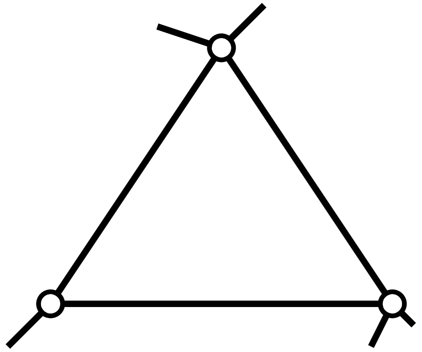


RD

Rectangular Dual $\mathcal{R}$

Exactly four vertices on the outer face.





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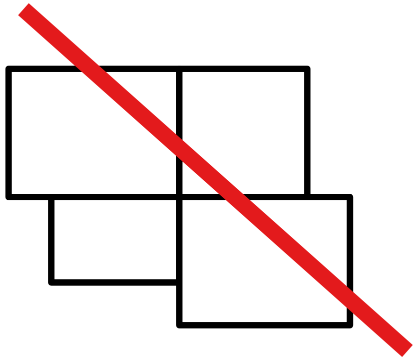

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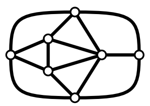
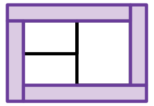
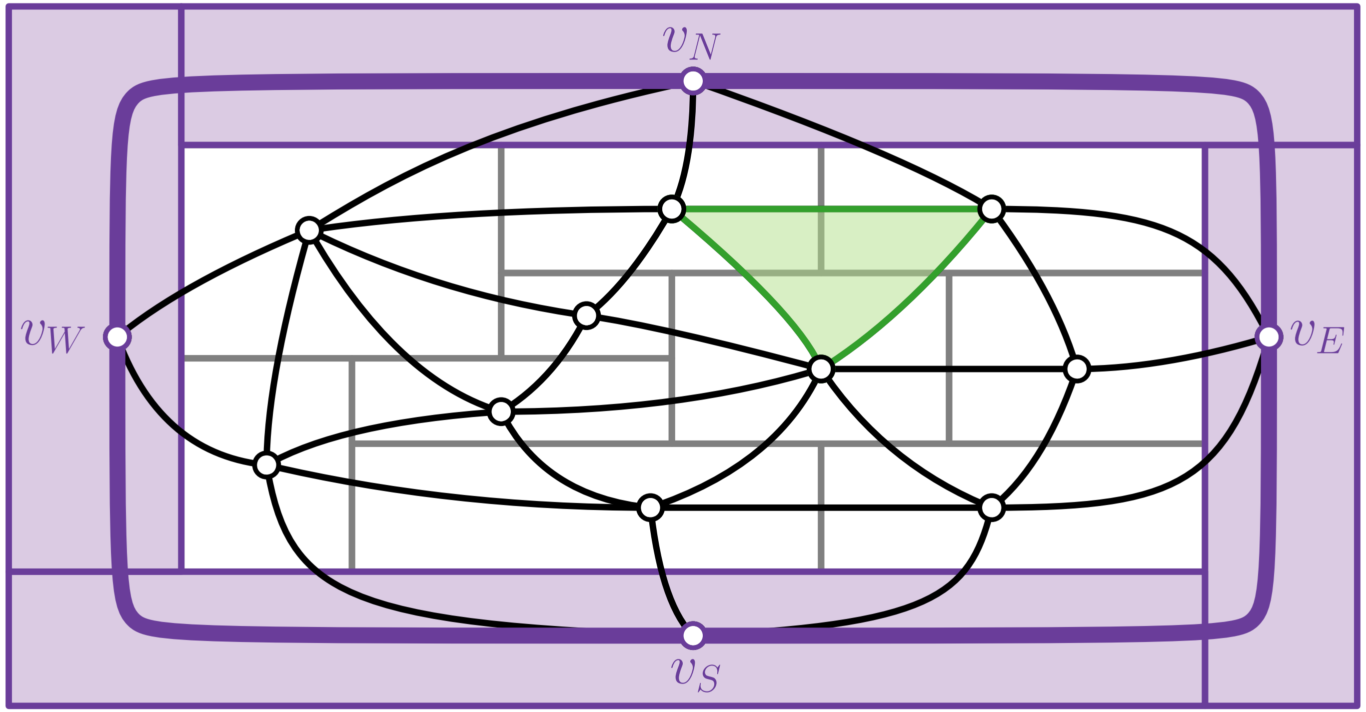

PTP

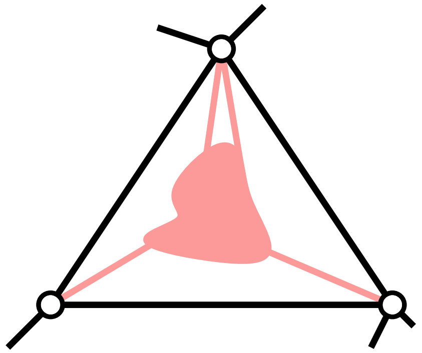
Properly Triangulated Planar Graph G


RD

Rectangular Dual $\mathcal{R}$

Exactly four vertices on the outer face.





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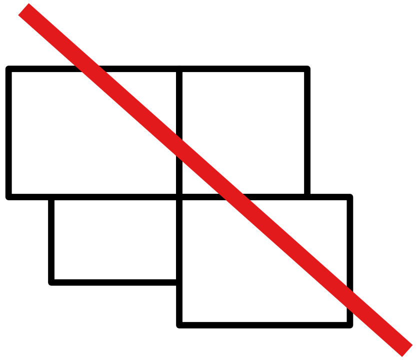

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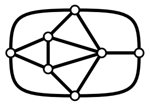
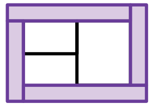


PTP

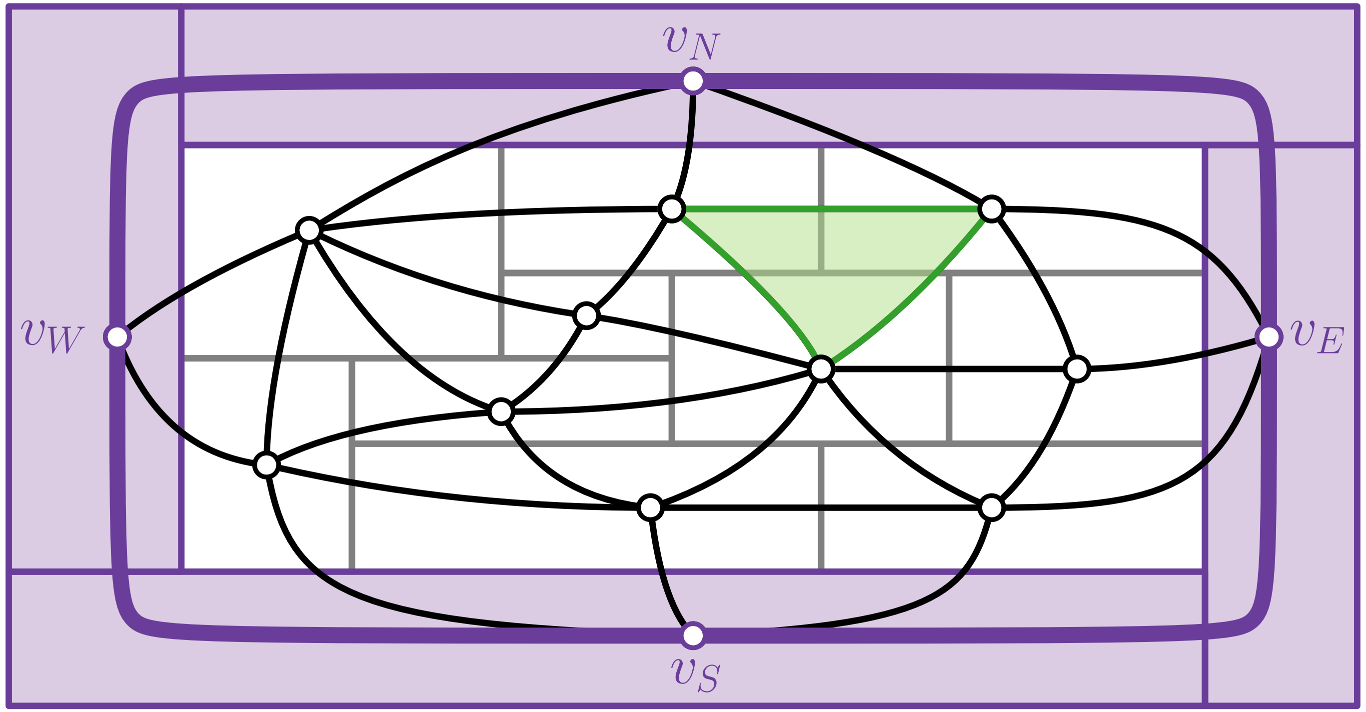
Properly Triangulated Planar Graph G

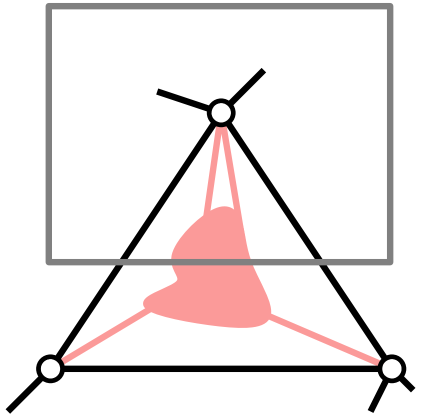


RD

Rectangular Dual $\mathcal{R}$

Exactly four vertices on the outer face.





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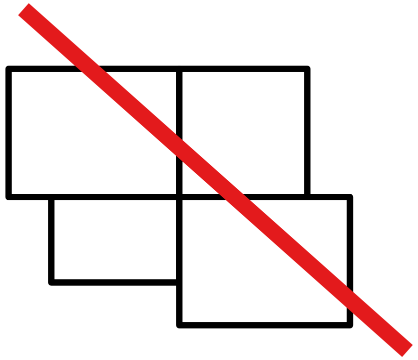

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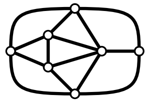
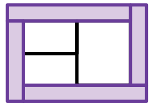


Properly Triangulated Planar Graph G



Rectangular Dual $\mathcal{R}$

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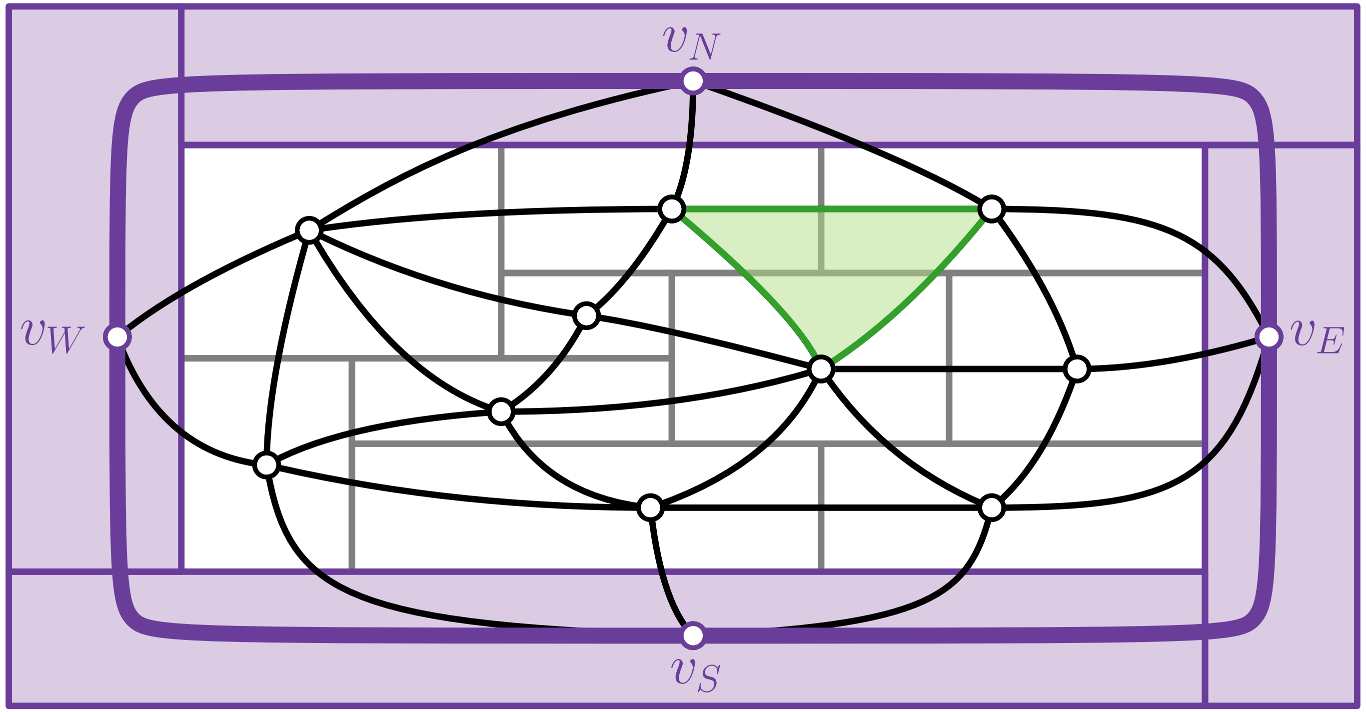

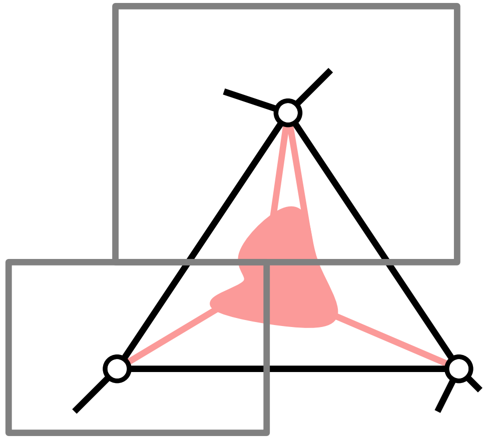

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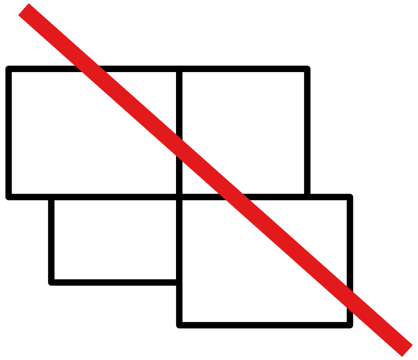

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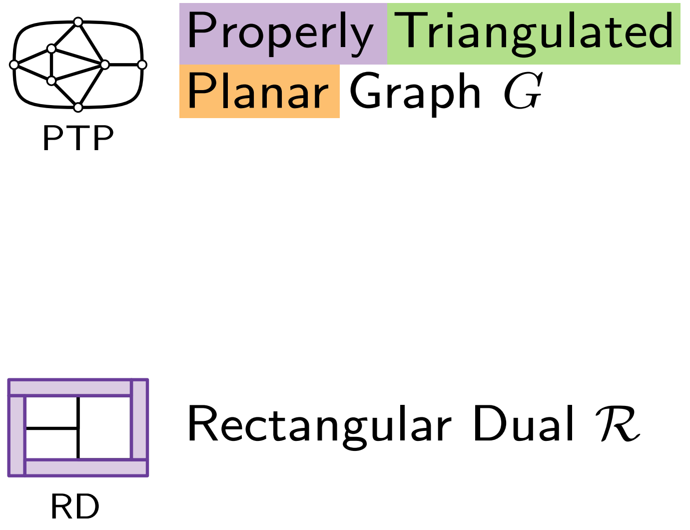

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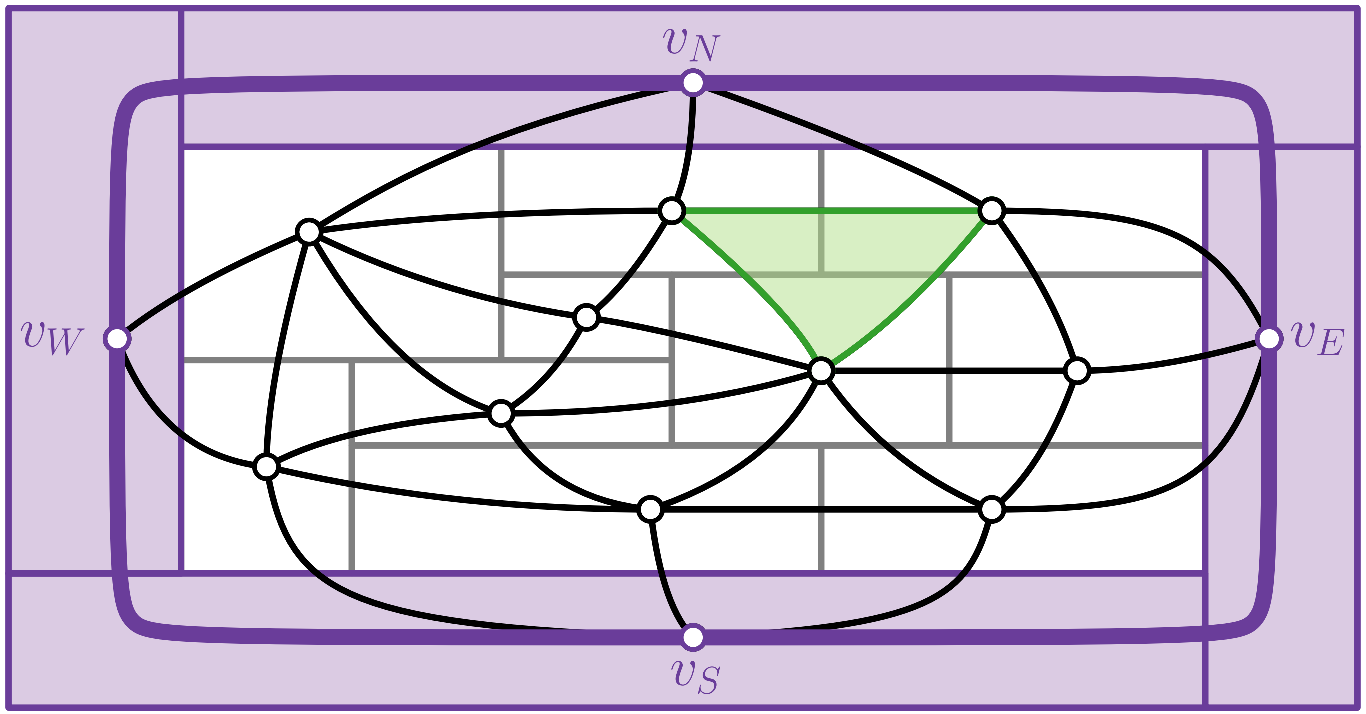

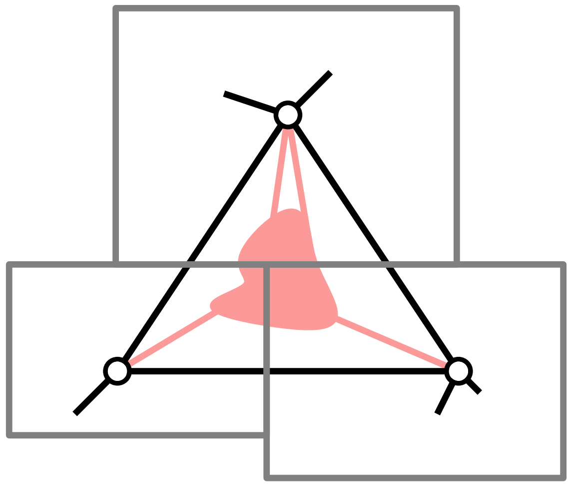

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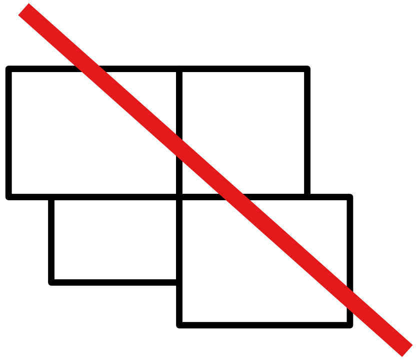

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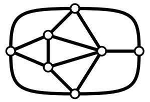
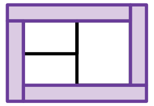
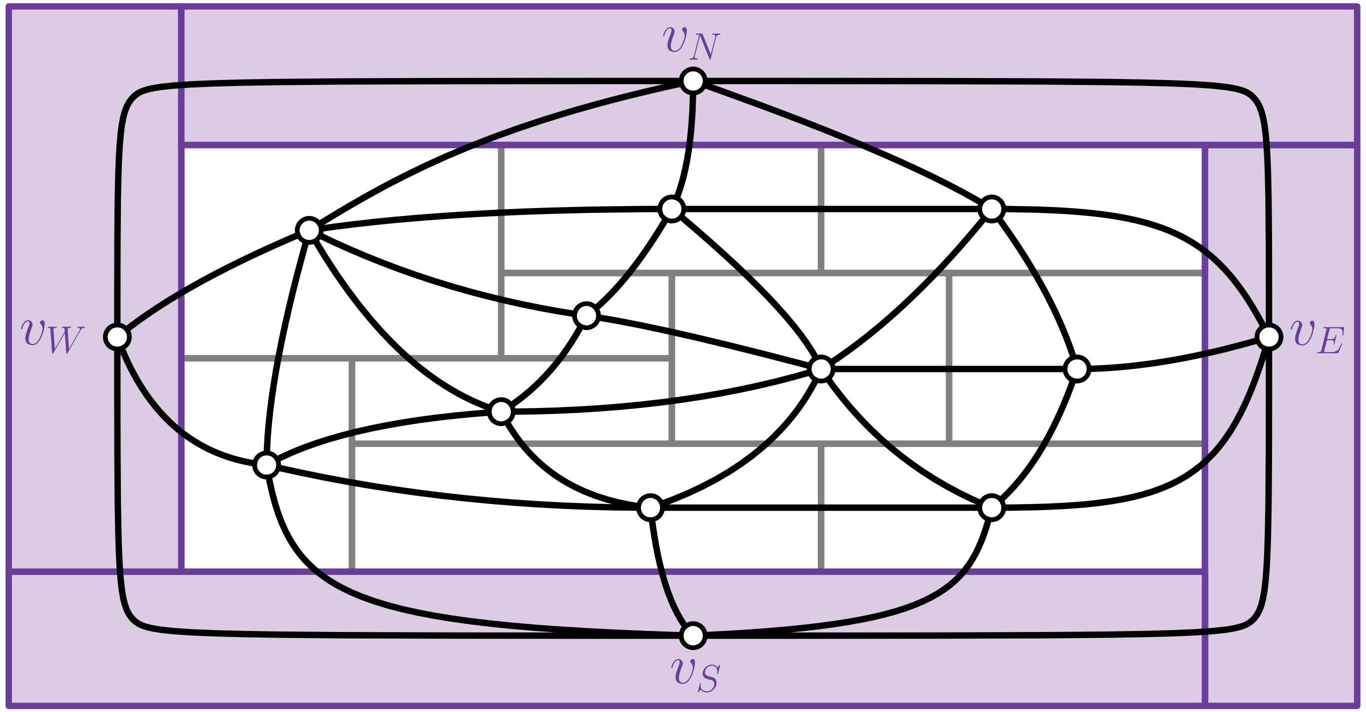
Regular Edge Labeling



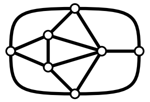
Properly Triangulated Planar Graph G



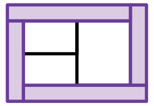
Rectangular Dual $\mathcal{R}$



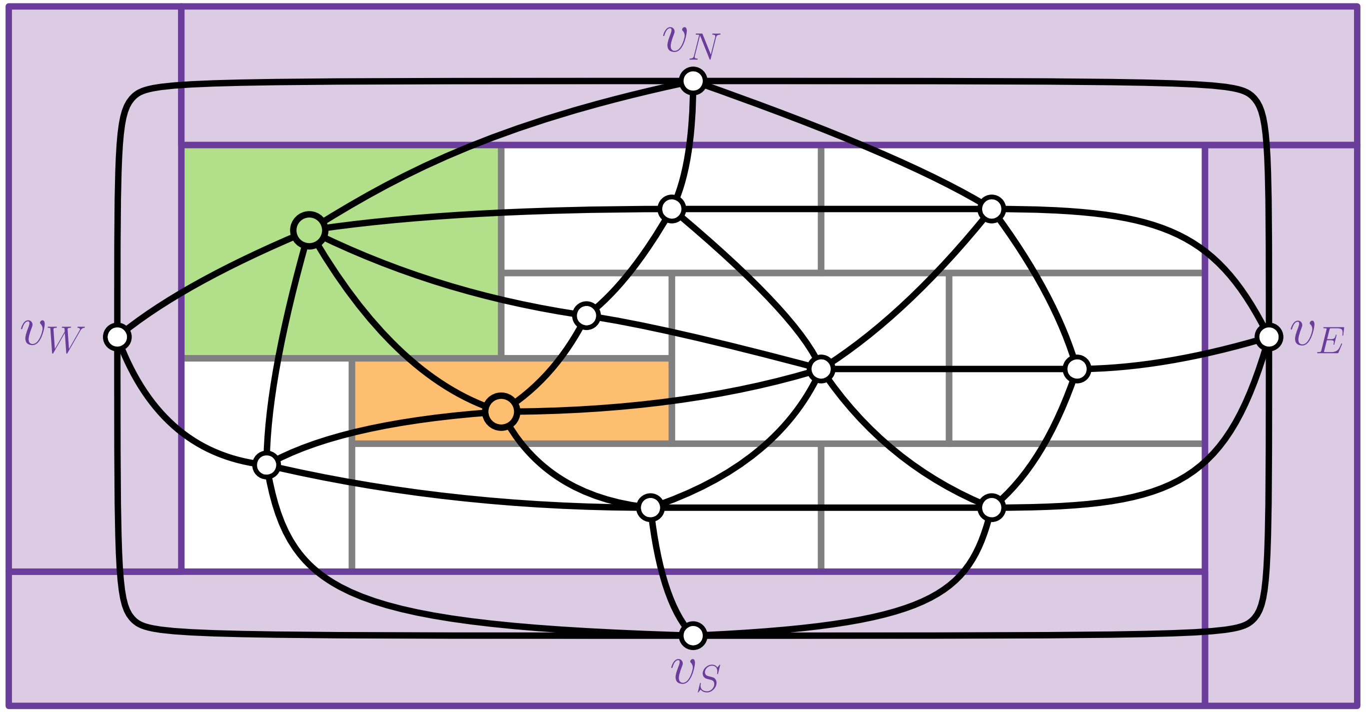
Regular Edge Labeling



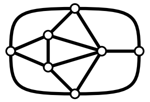
Properly Triangulated Planar Graph G



Rectangular Dual $\mathcal{R}$



Regular Edge Labeling



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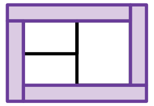
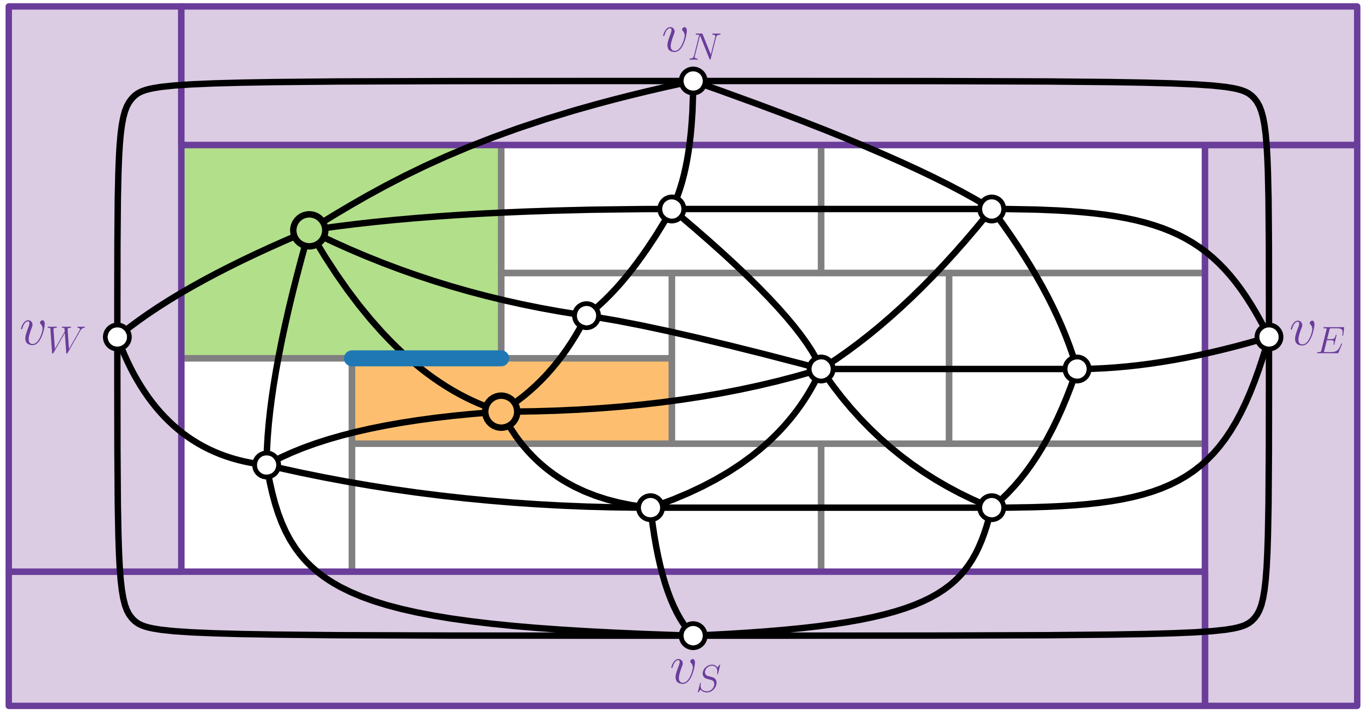
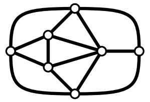


Rectangular Dual $\mathcal{R}$



Regular Edge Labeling



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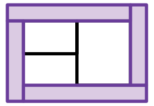
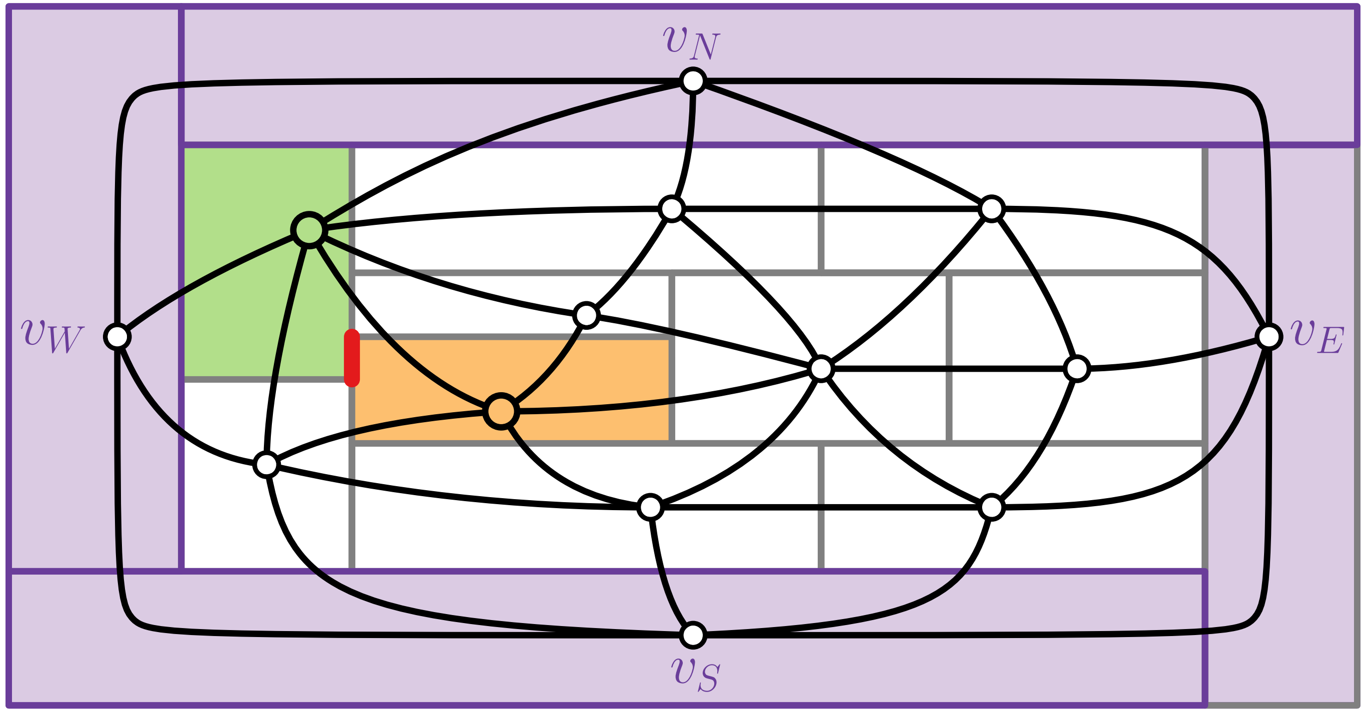
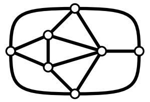


Rectangular Dual $\mathcal{R}$



Regular Edge Labeling



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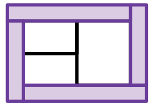
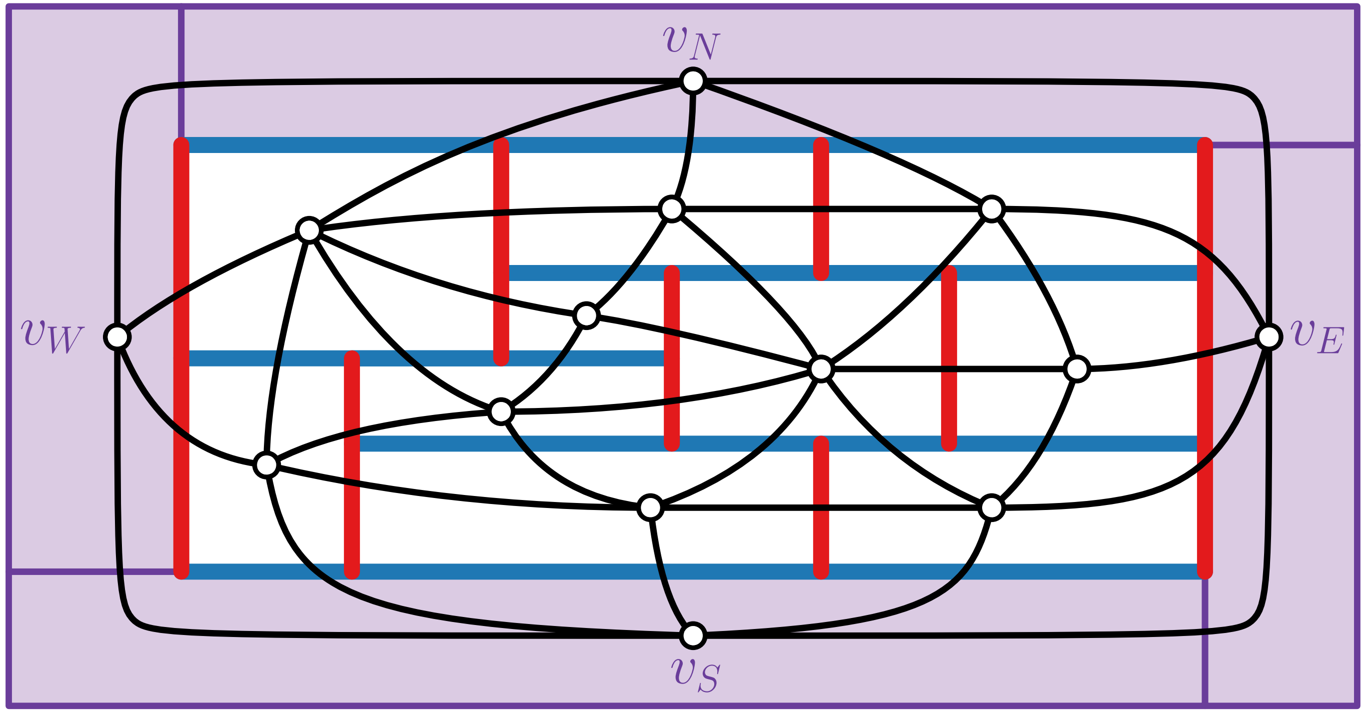


Rectangular Dual $\mathcal{R}$



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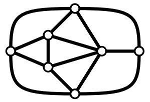

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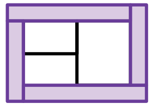
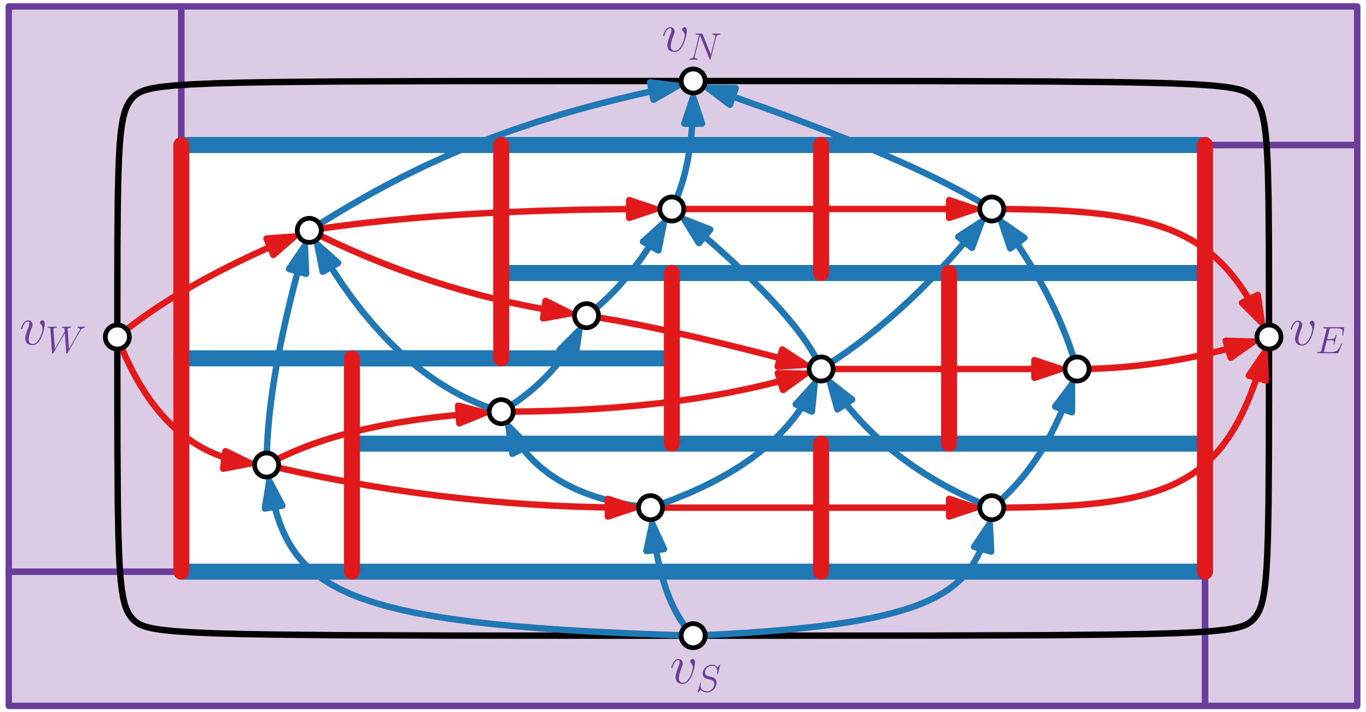
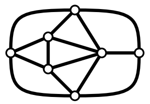


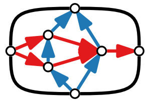
Rectangular Dual $\mathcal{R}$



Regular Edge Labeling



Properly Triangulated Planar Graph G



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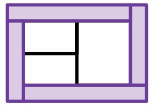
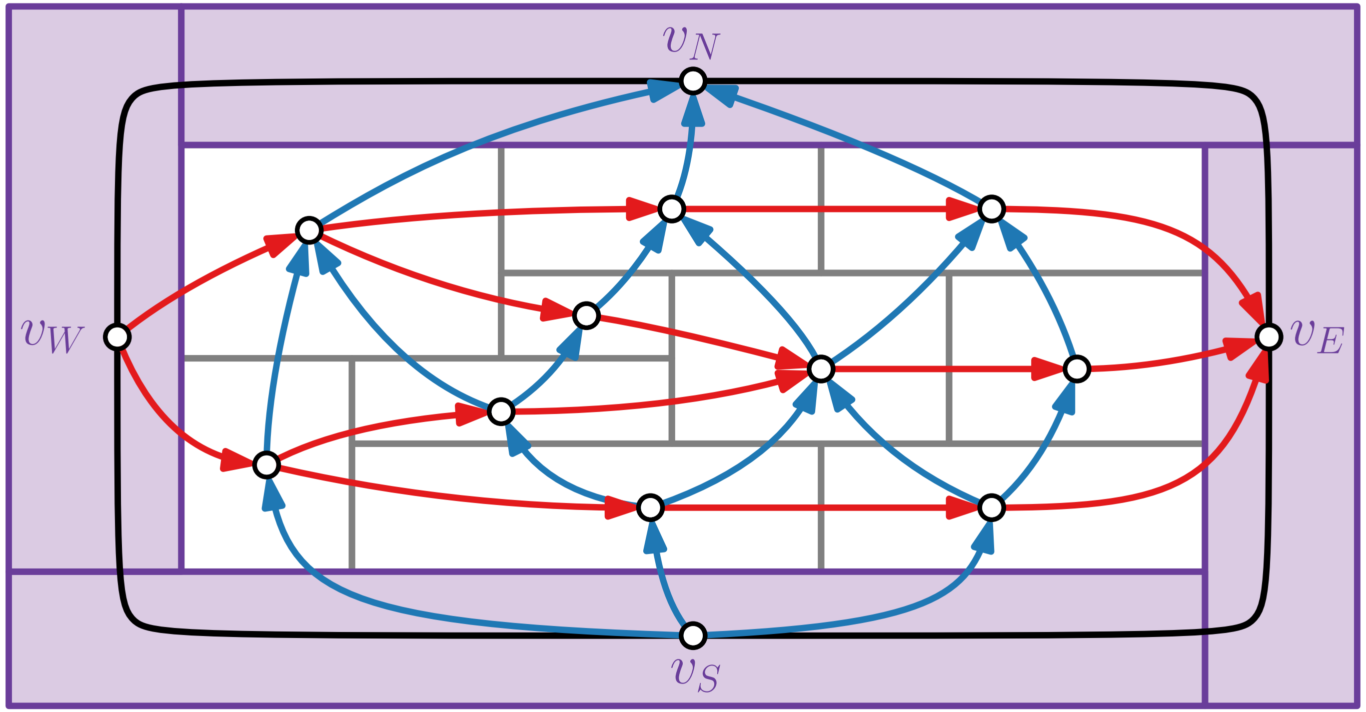


Rectangular Dual $\mathcal{R}$



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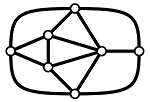
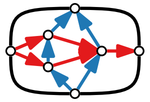


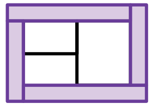
PTP

Properly Triangulated Planar Graph G



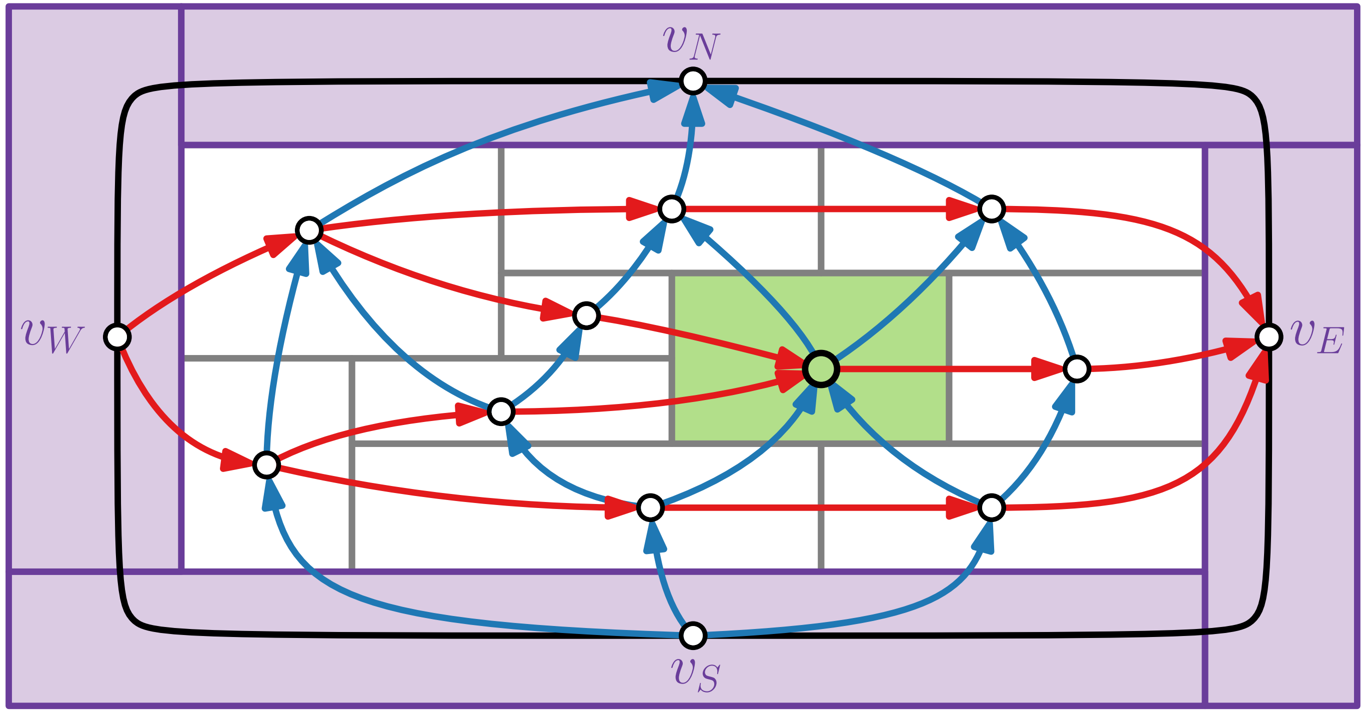
REL

Regular Edge Labeling



RD

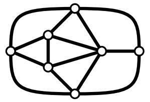
Rectangular Dual $\mathcal{R}$



Properties:

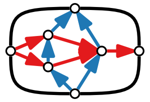
Regular Edge Labeling

Properties:



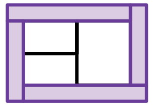
PTP

Properly Triangulated Planar Graph G



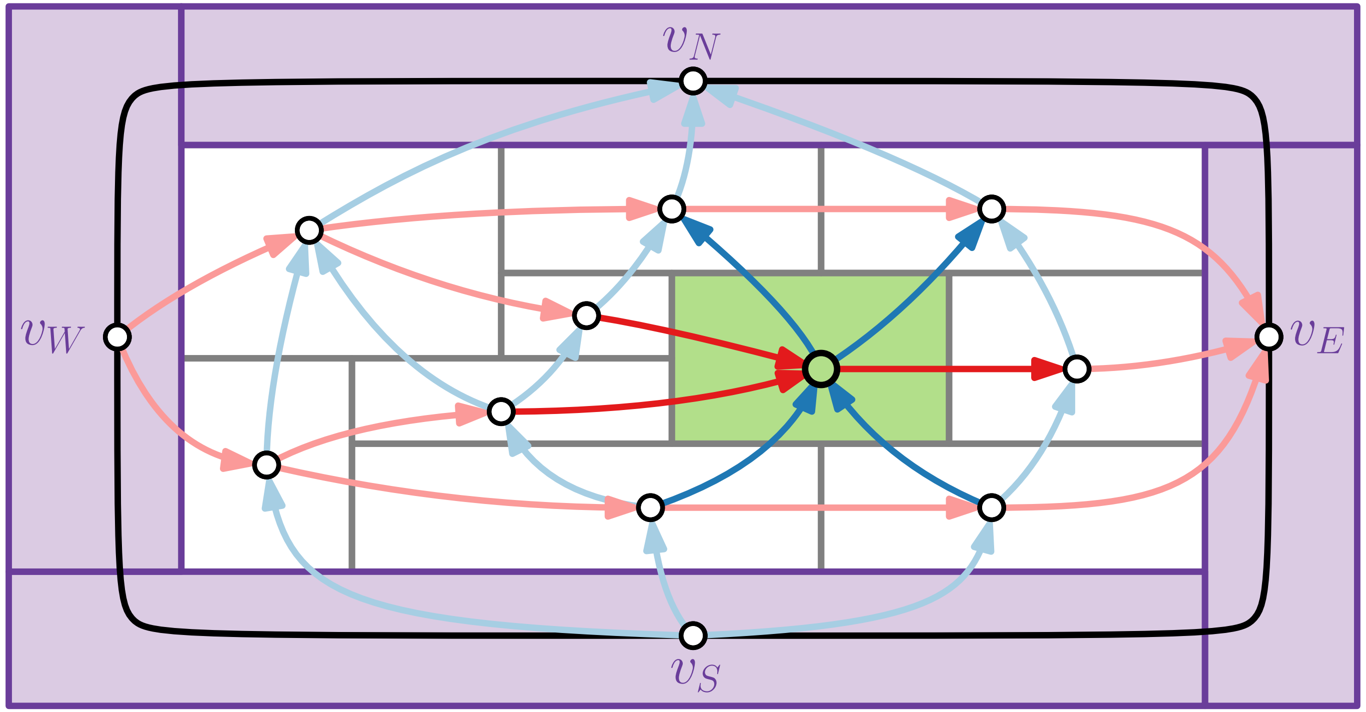
REL

Regular Edge Labeling

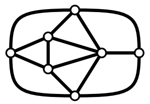


RD

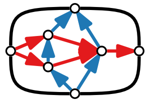
Rectangular Dual $\mathcal{R}$



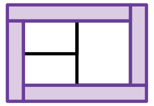
Regular Edge Labeling


PTP

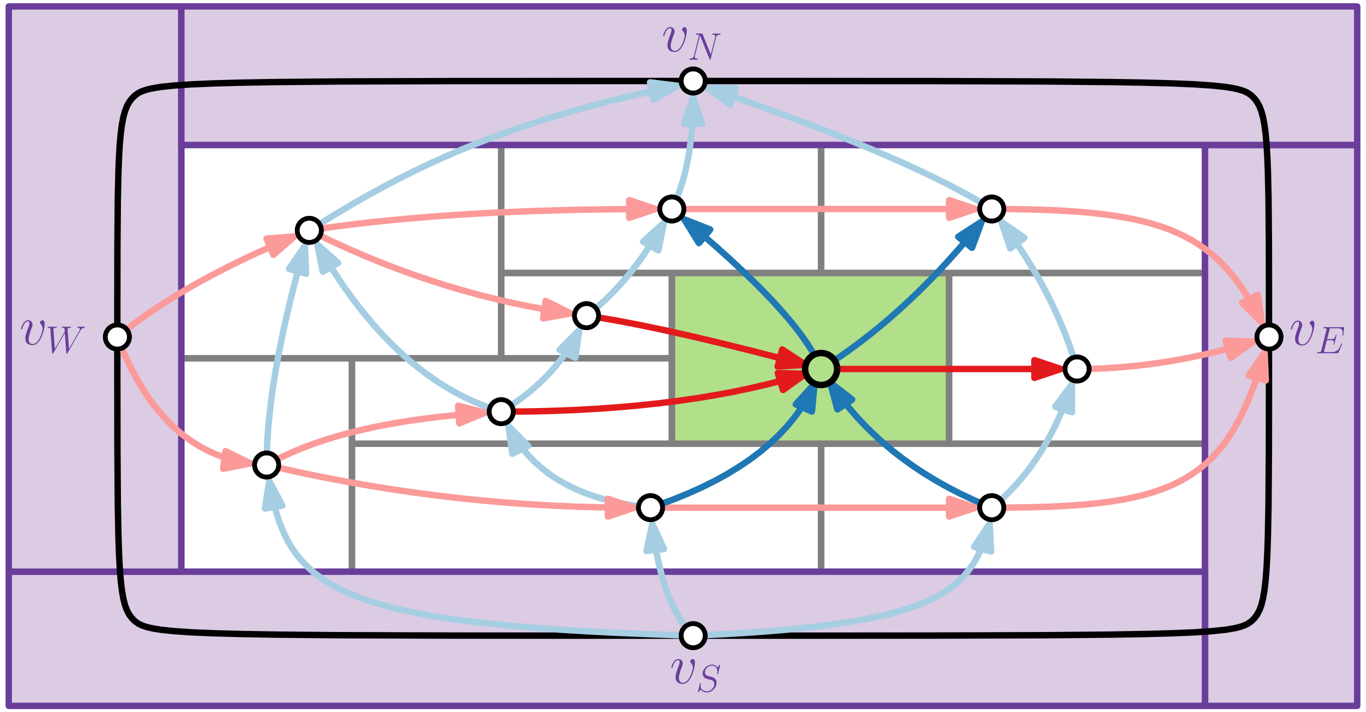
Properly Triangulated Planar Graph G


REL

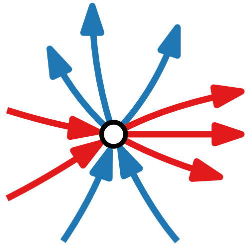
Regular Edge Labeling


RD

Rectangular Dual $\mathcal{R}$

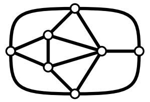


Properties:

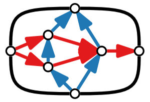


for every inner vertex

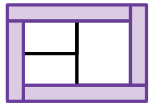
Regular Edge Labeling


Properly Triangulated Planar Graph G

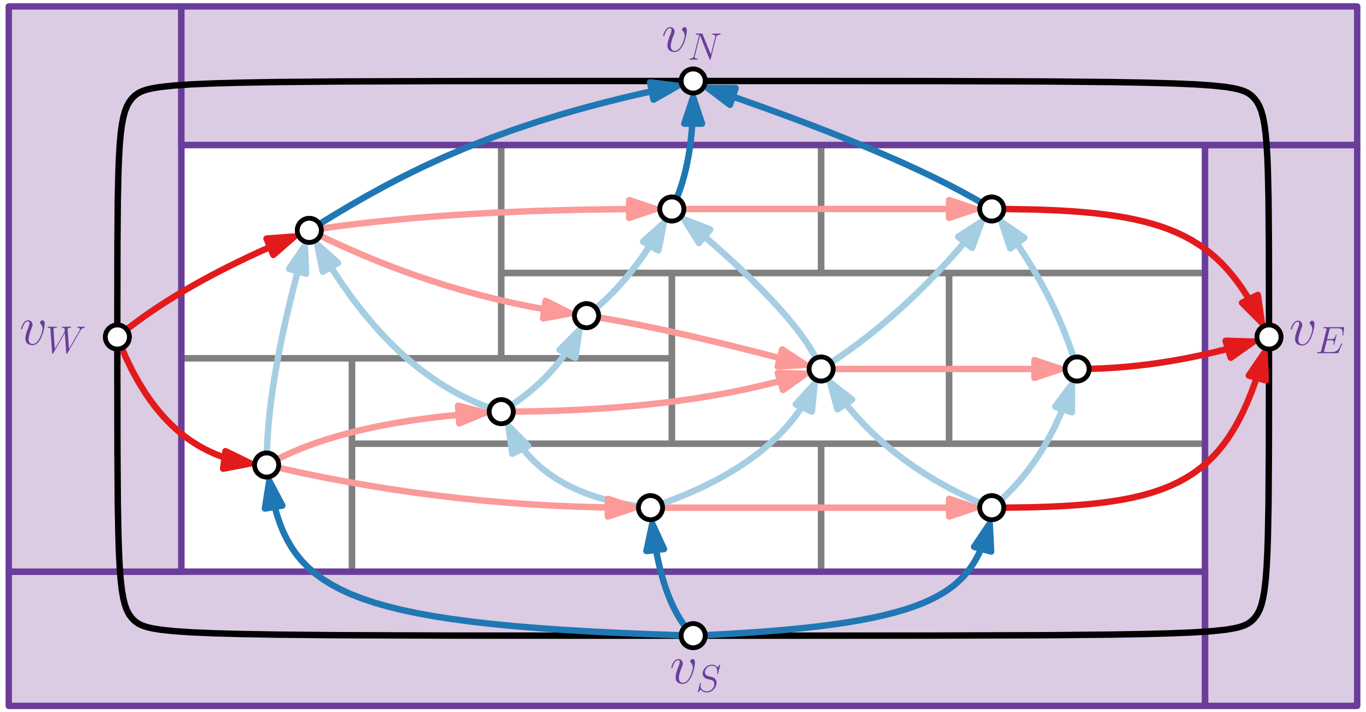
PTP


Regular Edge Labeling

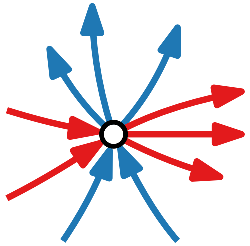
REL


Rectangular Dual $\mathcal{R}$

RD

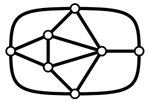


Properties:

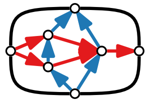


for every inner vertex

Regular Edge Labeling


PTP

Properly Triangulated Planar Graph G


REL

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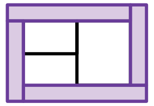
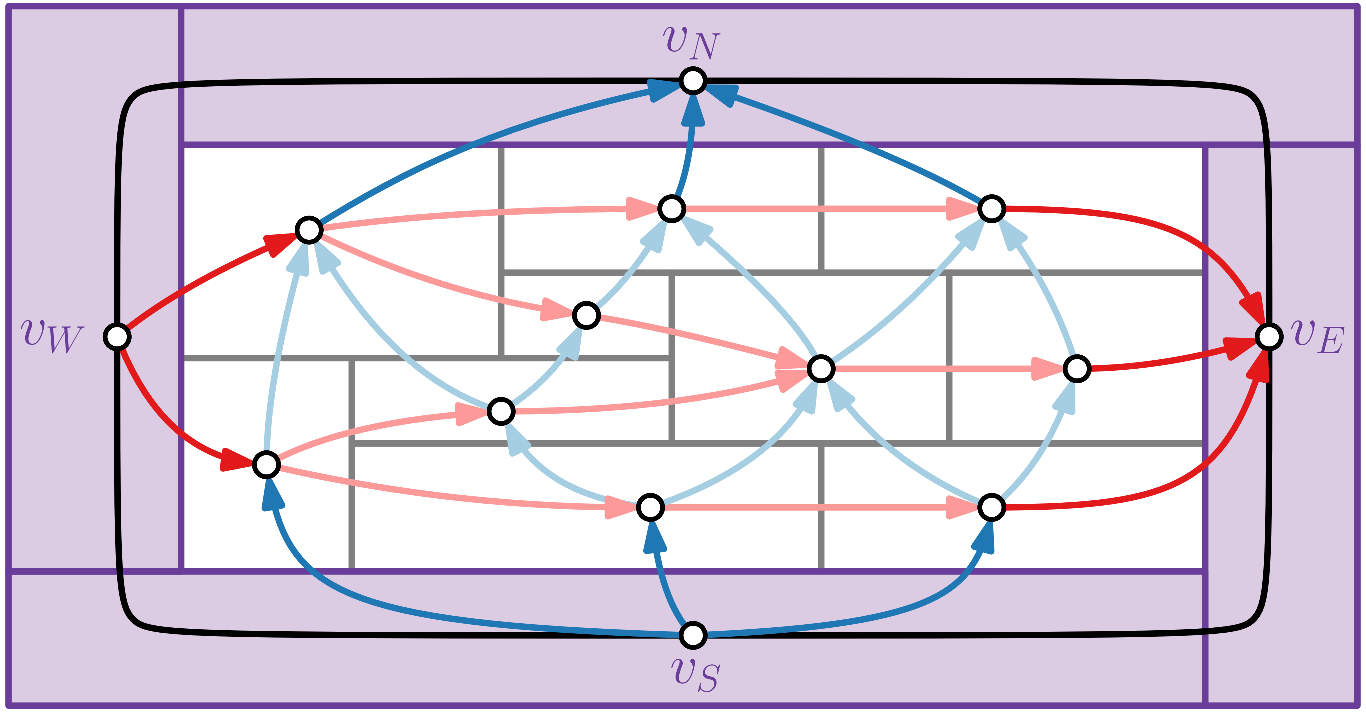

RD

Rectangular Dual $\mathcal{R}$



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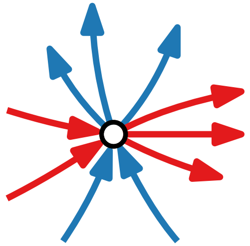

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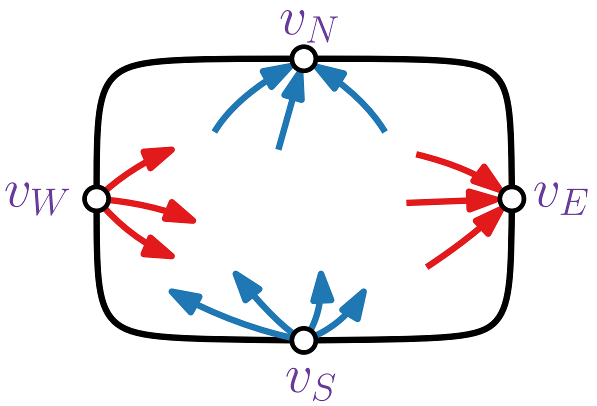
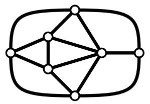
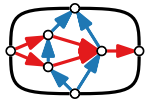


for four outer vertices

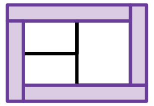
Regular Edge Labeling



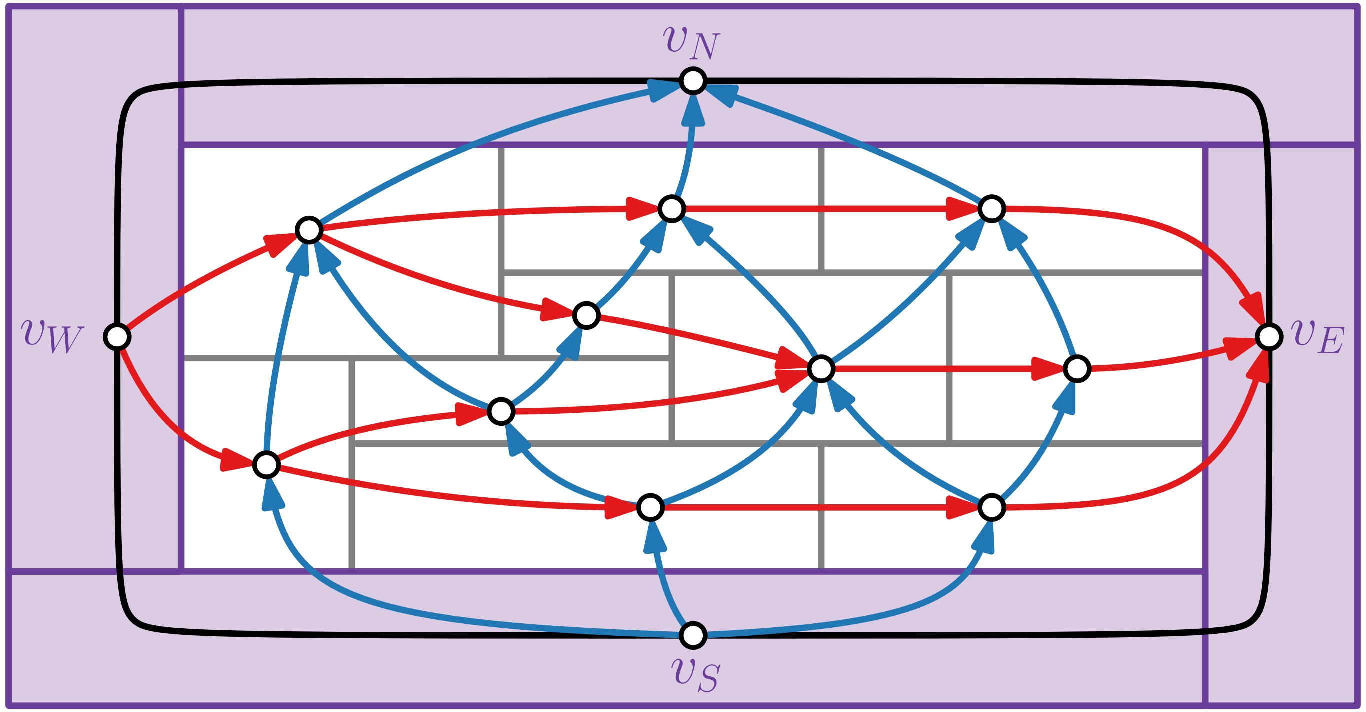
Properly Triangulated Planar Graph G



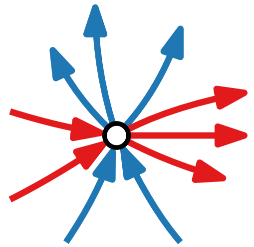
Regular Edge Labeling



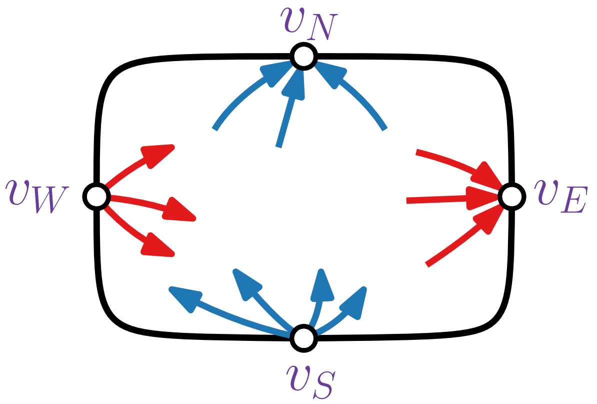
Rectangular Dual $\mathcal{R}$



Properties:

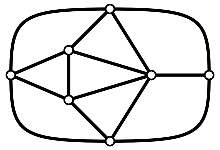


for every inner vertex



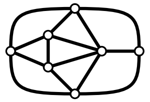
for four outer vertices

[Kant, He '94]:

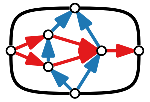


PTP

Regular Edge Labeling


PTP

Properly Triangulated Planar Graph G


REL

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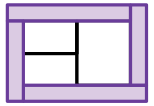
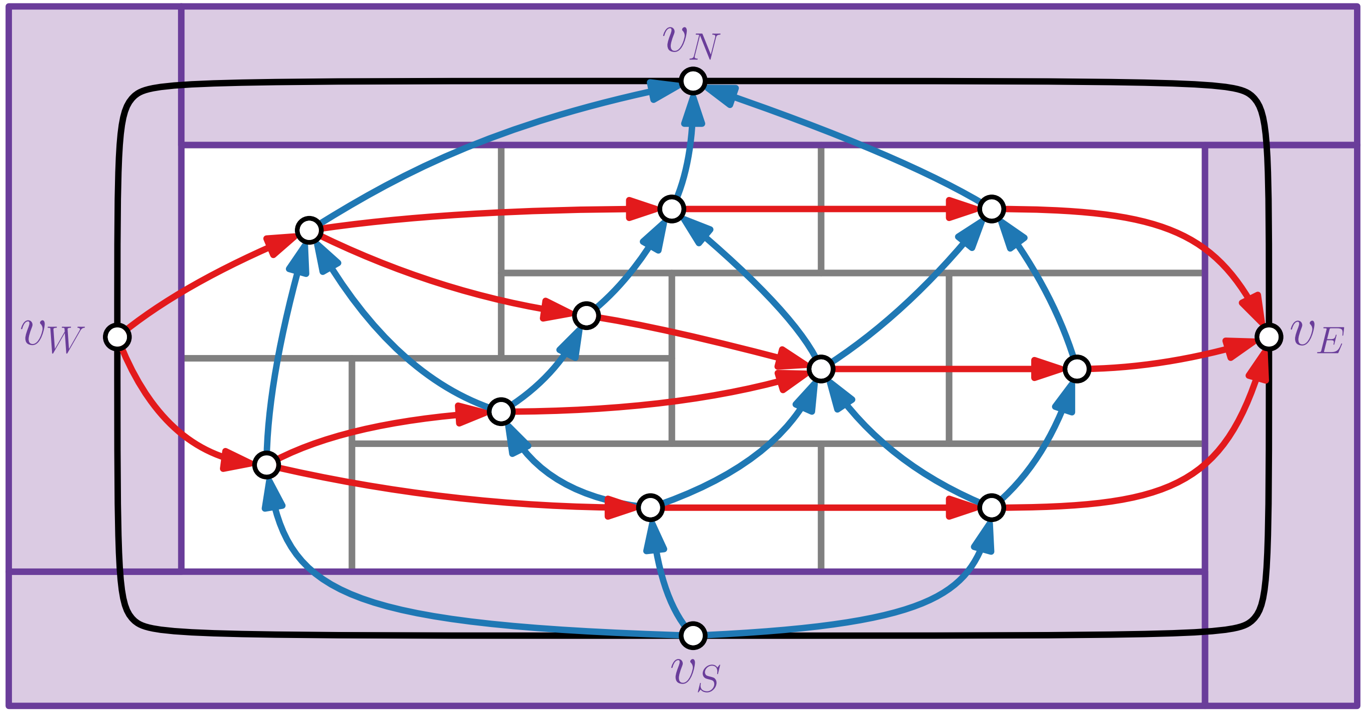
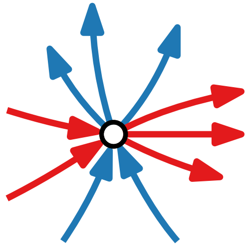

RD

Rectangular Dual $\mathcal{R}$



Properties:



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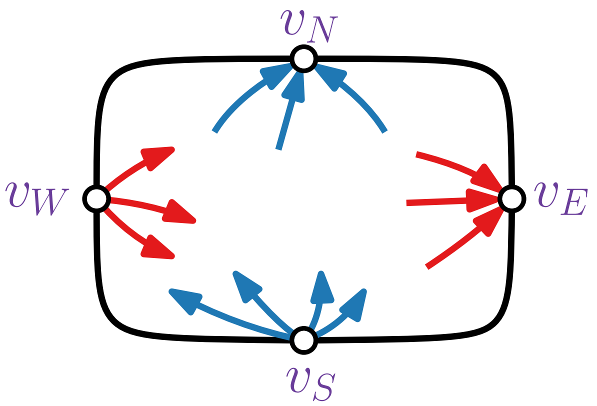

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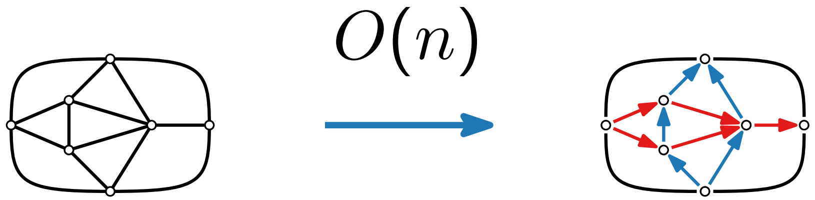
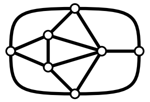
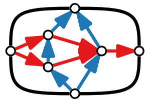


PTP REL

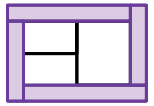
Regular Edge Labeling


PTP

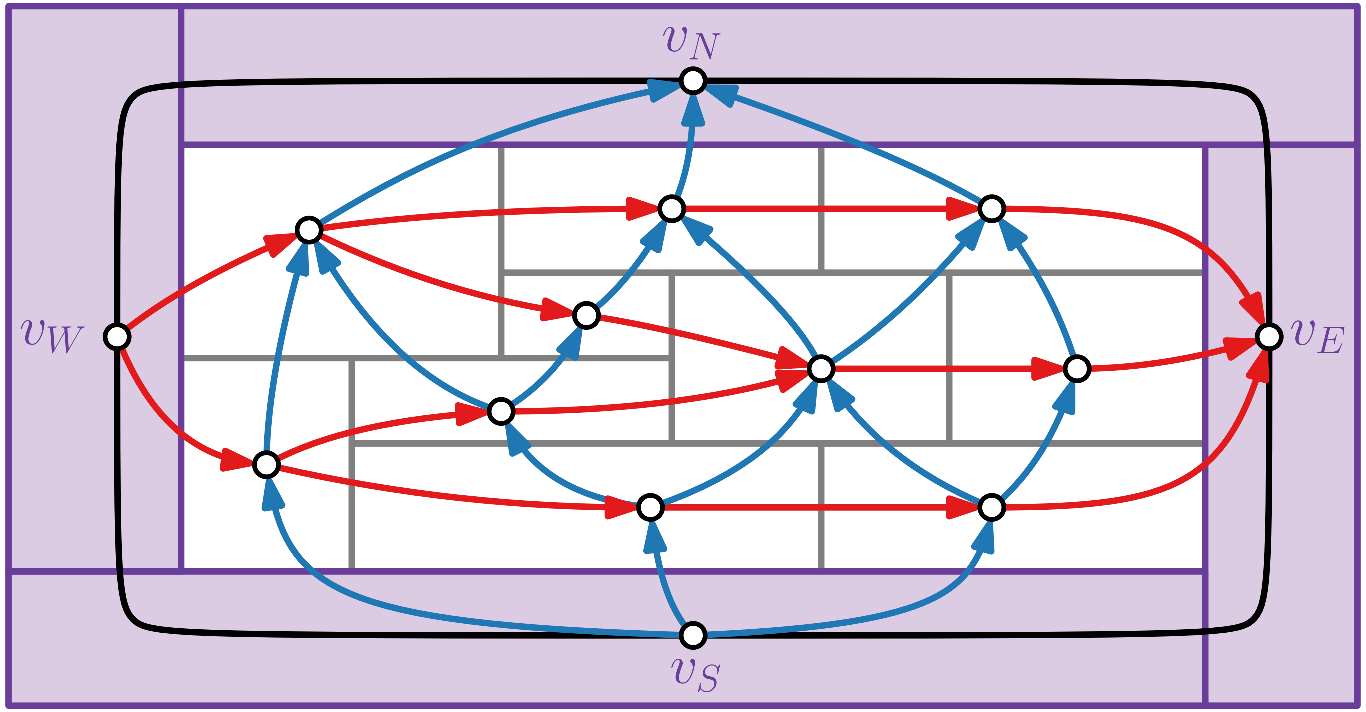
Properly Triangulated Planar Graph G


REL

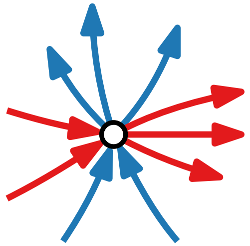
Regular Edge Labeling


RD

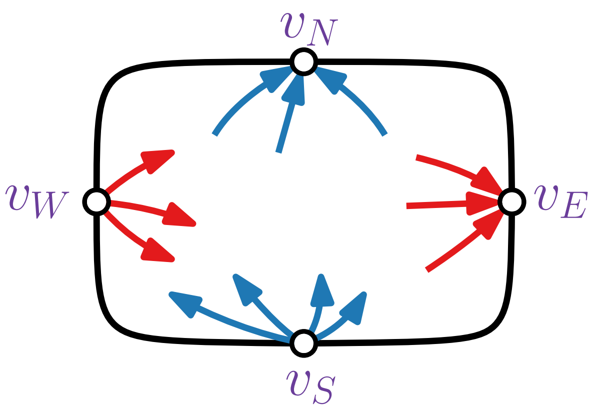
Rectangular Dual $\mathcal{R}$



Properties:



for every inner vertex



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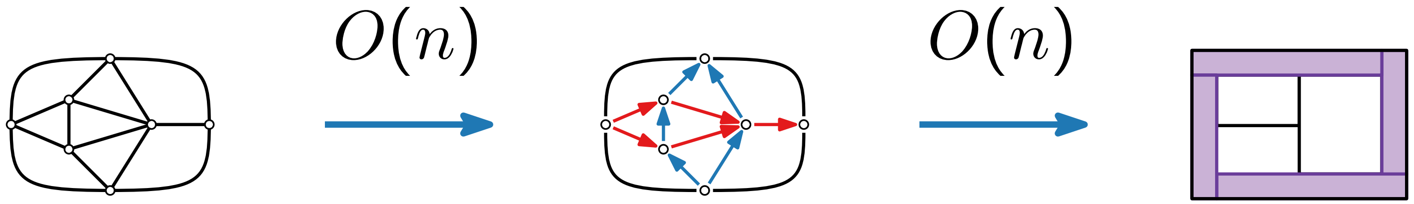

PTP REL RD

Refined Canonical Order

Theorem.
Let G be a PTP graph that is embedded in its unique planar embedding with counter-clockwise outer face $\langle v_W, v_S, v_E, v_N \rangle$.

Refined Canonical Order

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- The subgraph G_{k-1} induced by $v_1, \dots, v_{k-1}$ is biconnected

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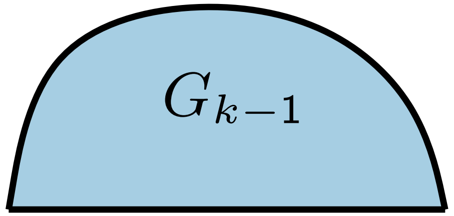

Refined Canonical Order

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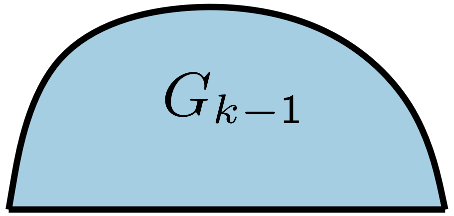

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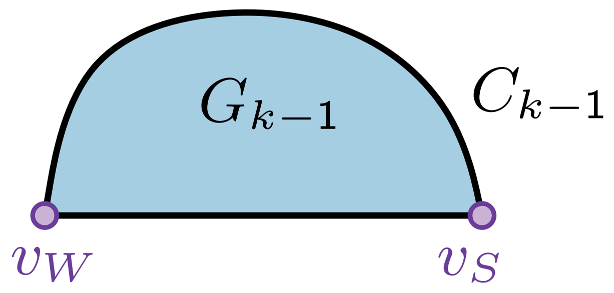

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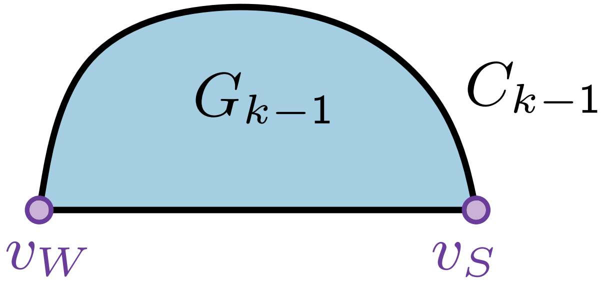

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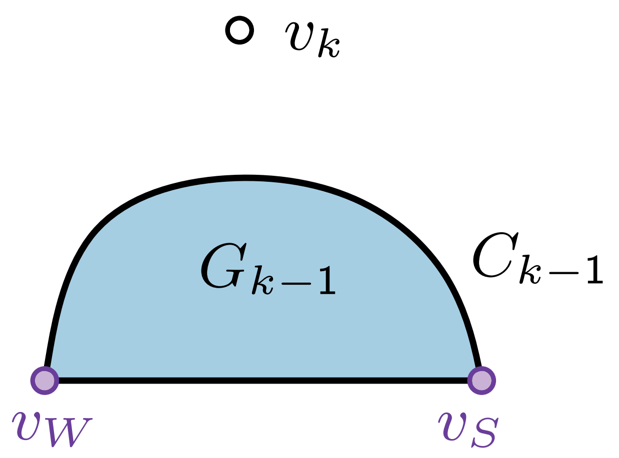

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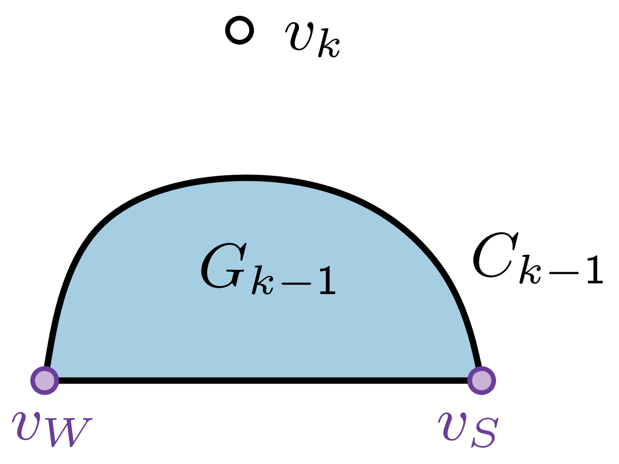

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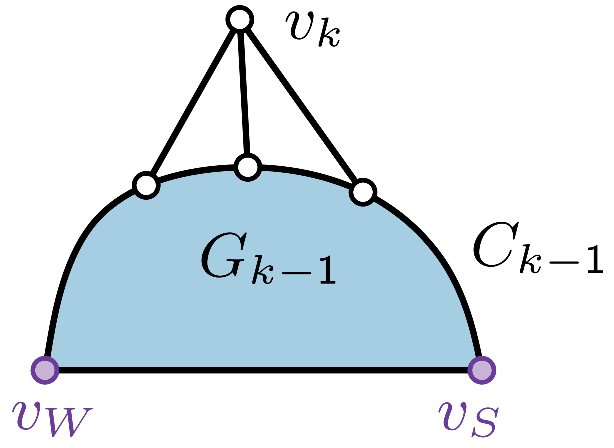

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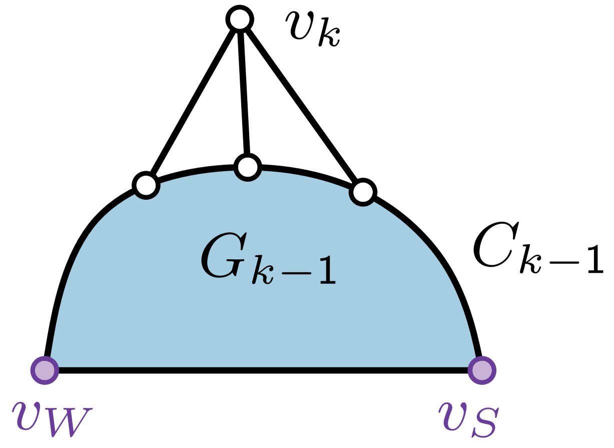

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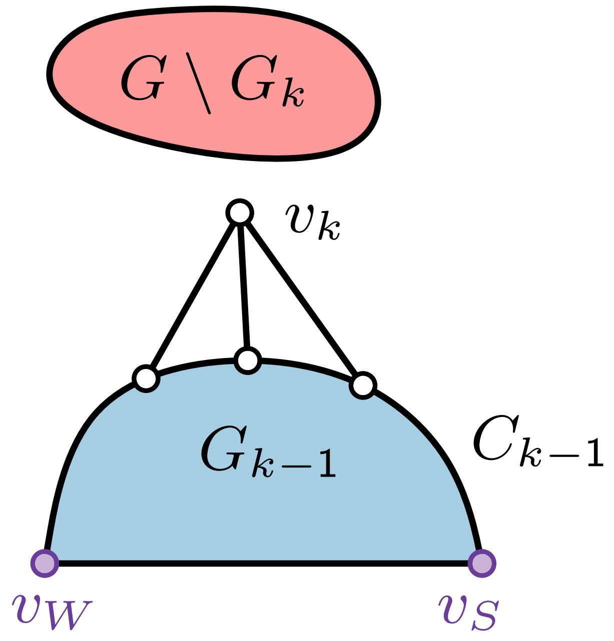

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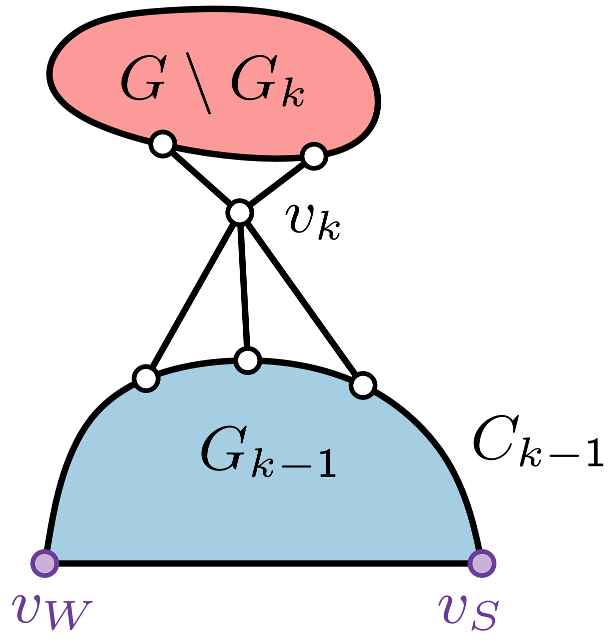

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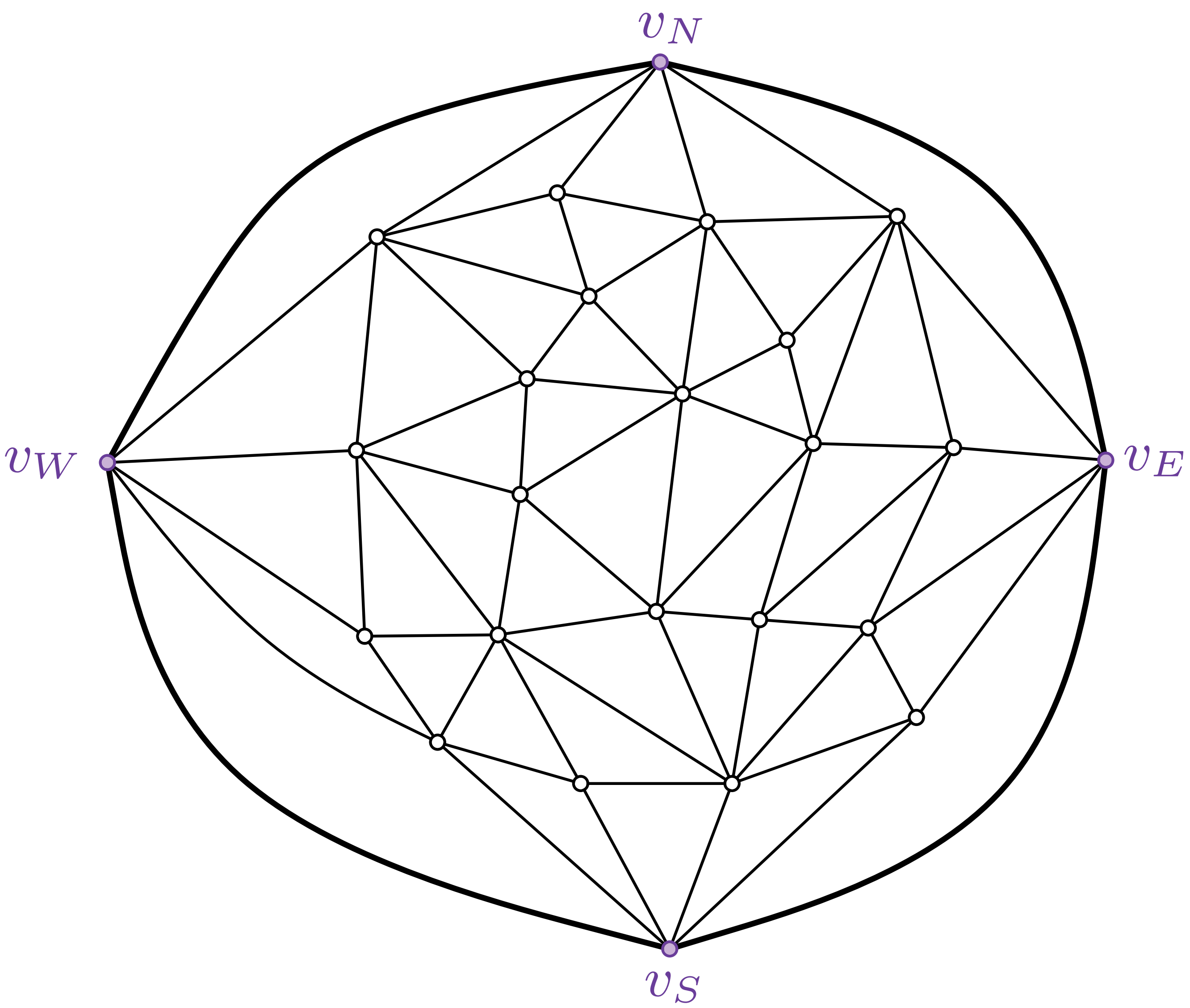

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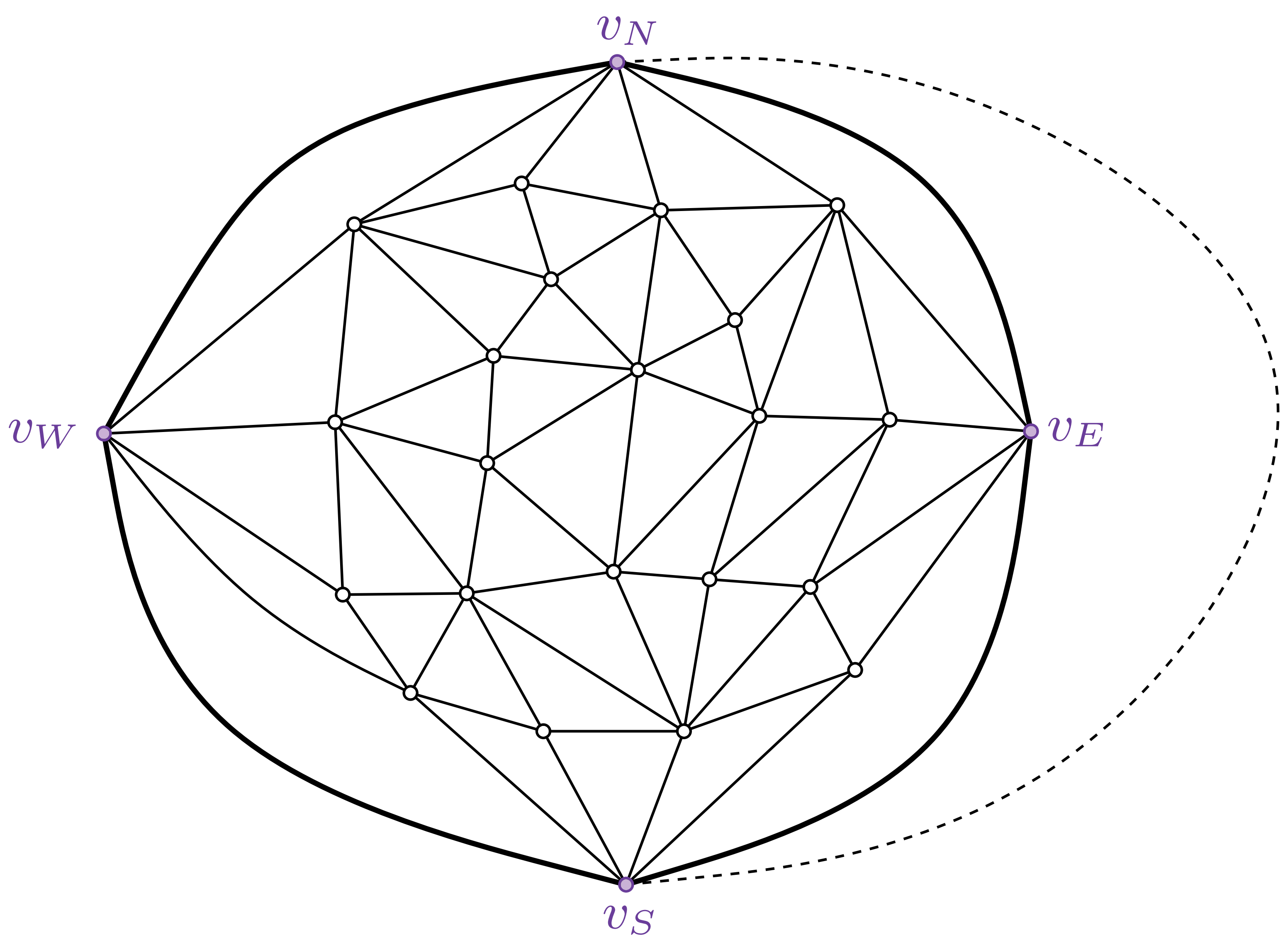

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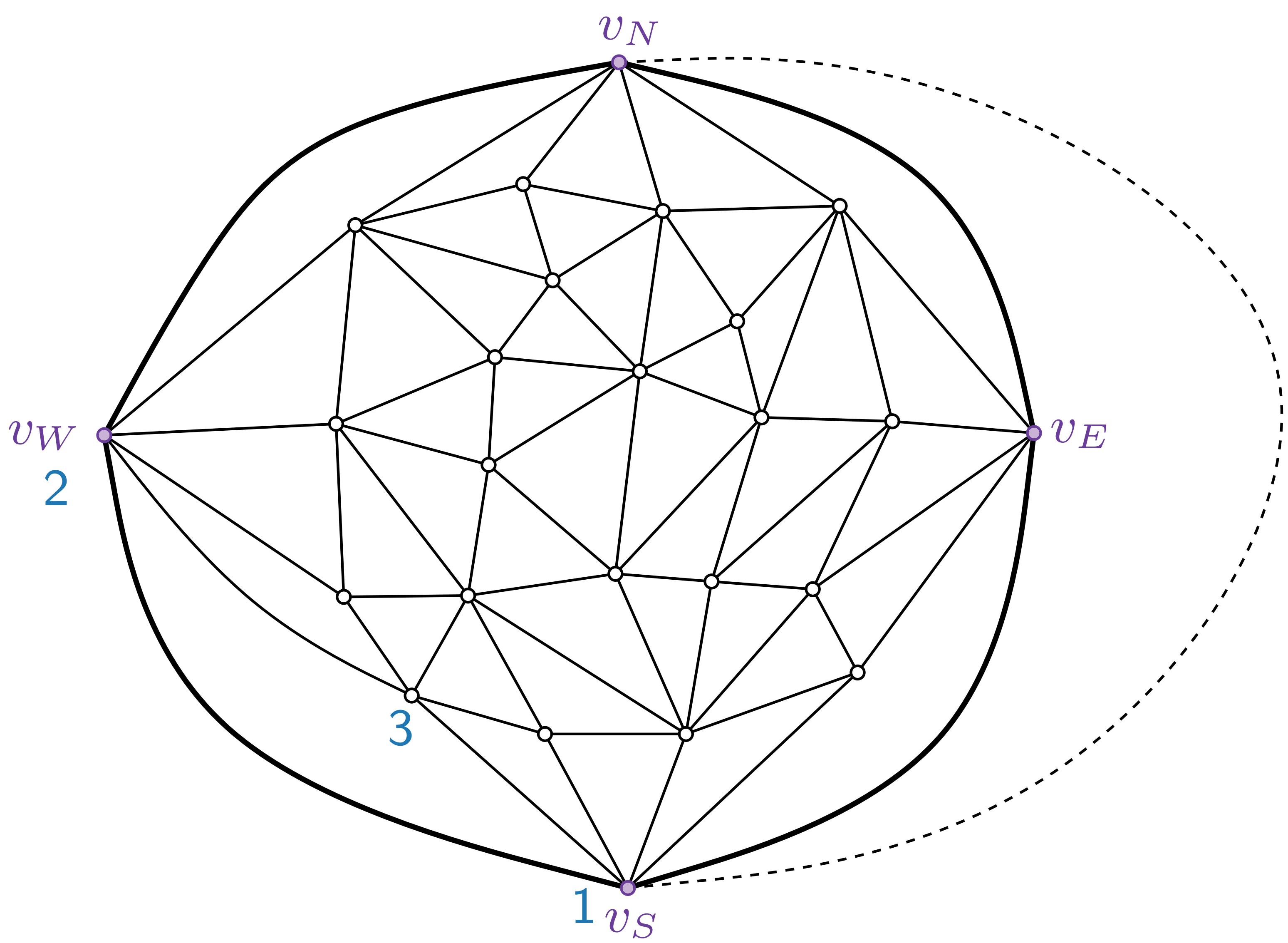

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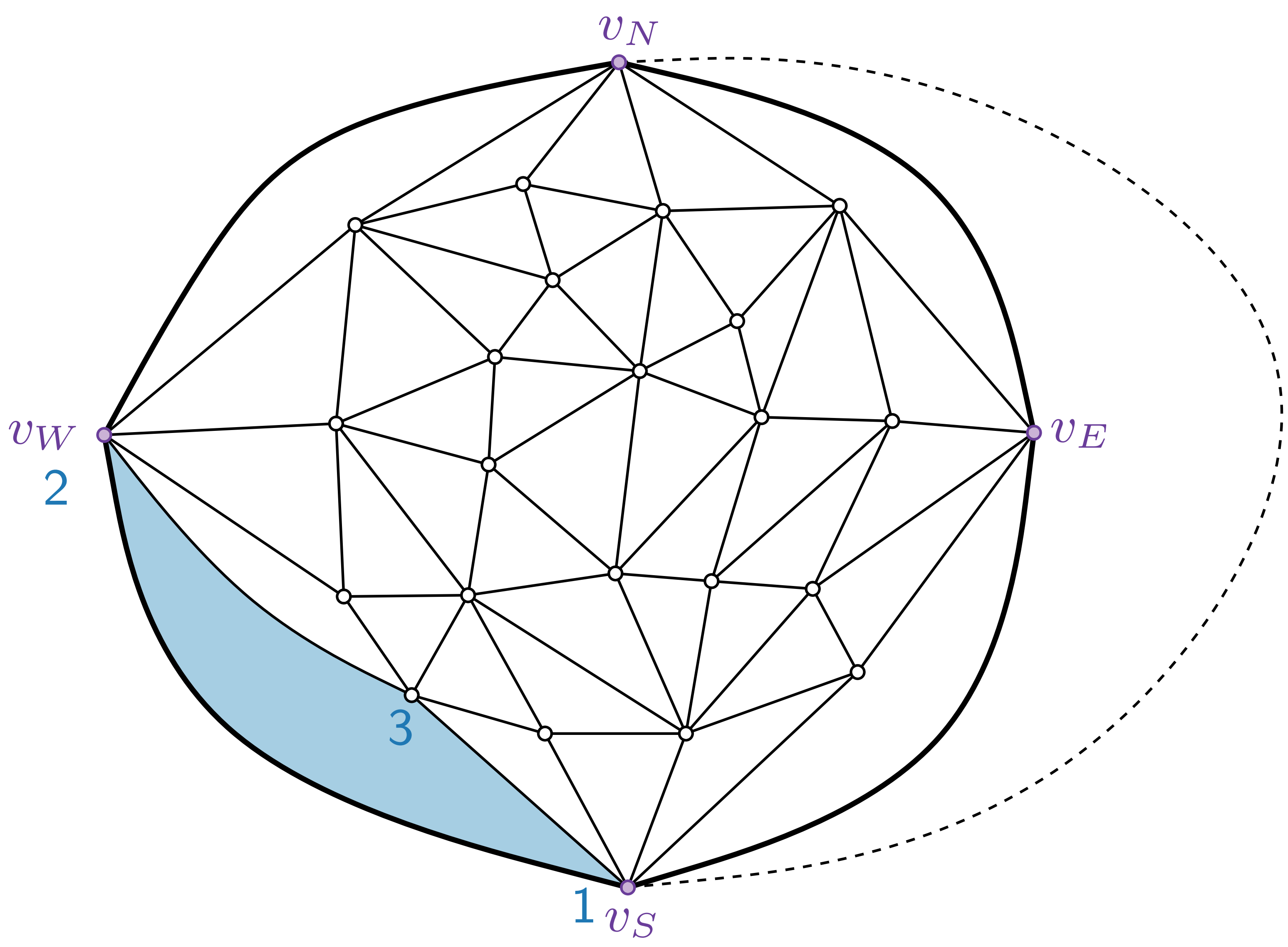

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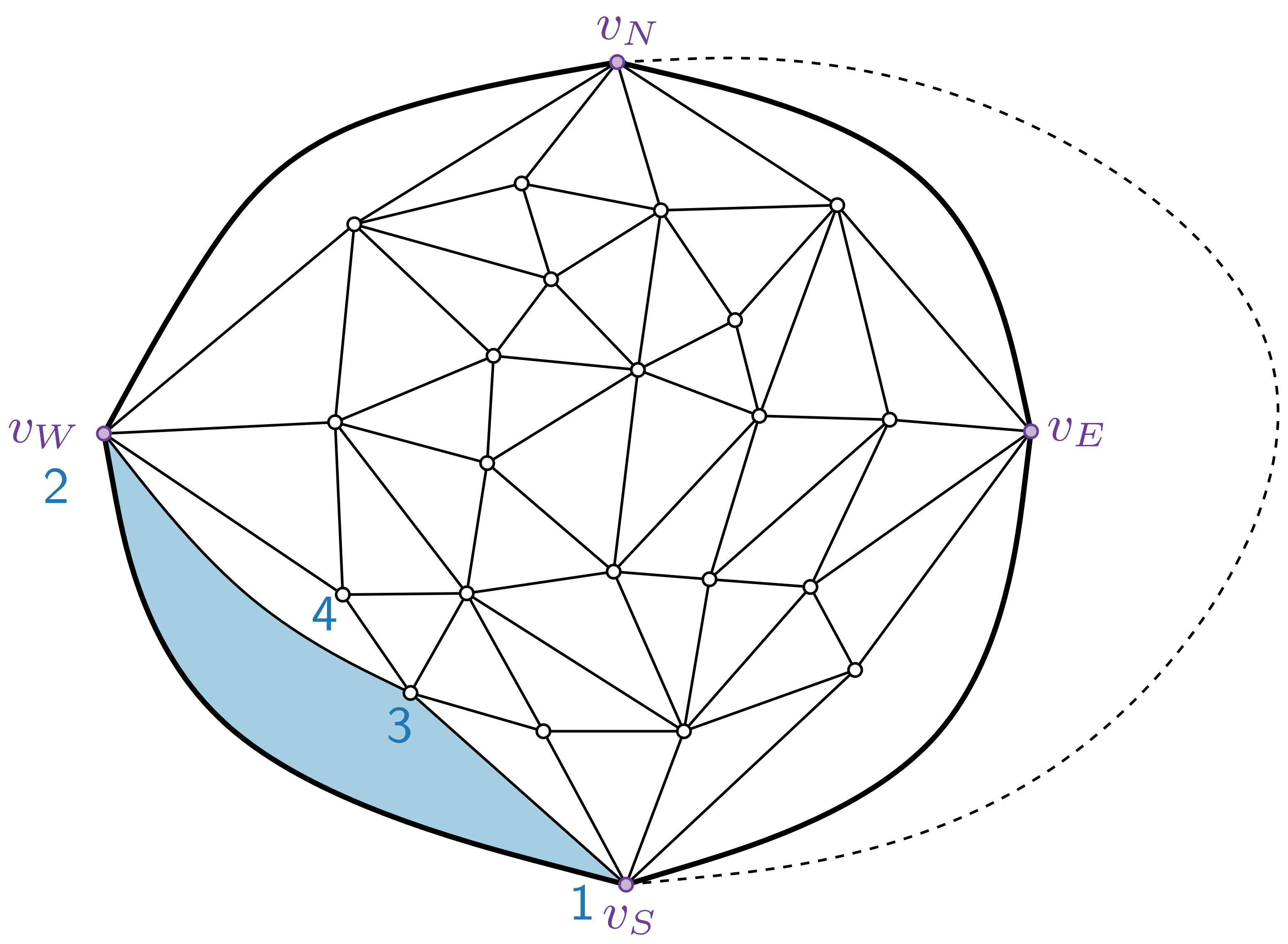

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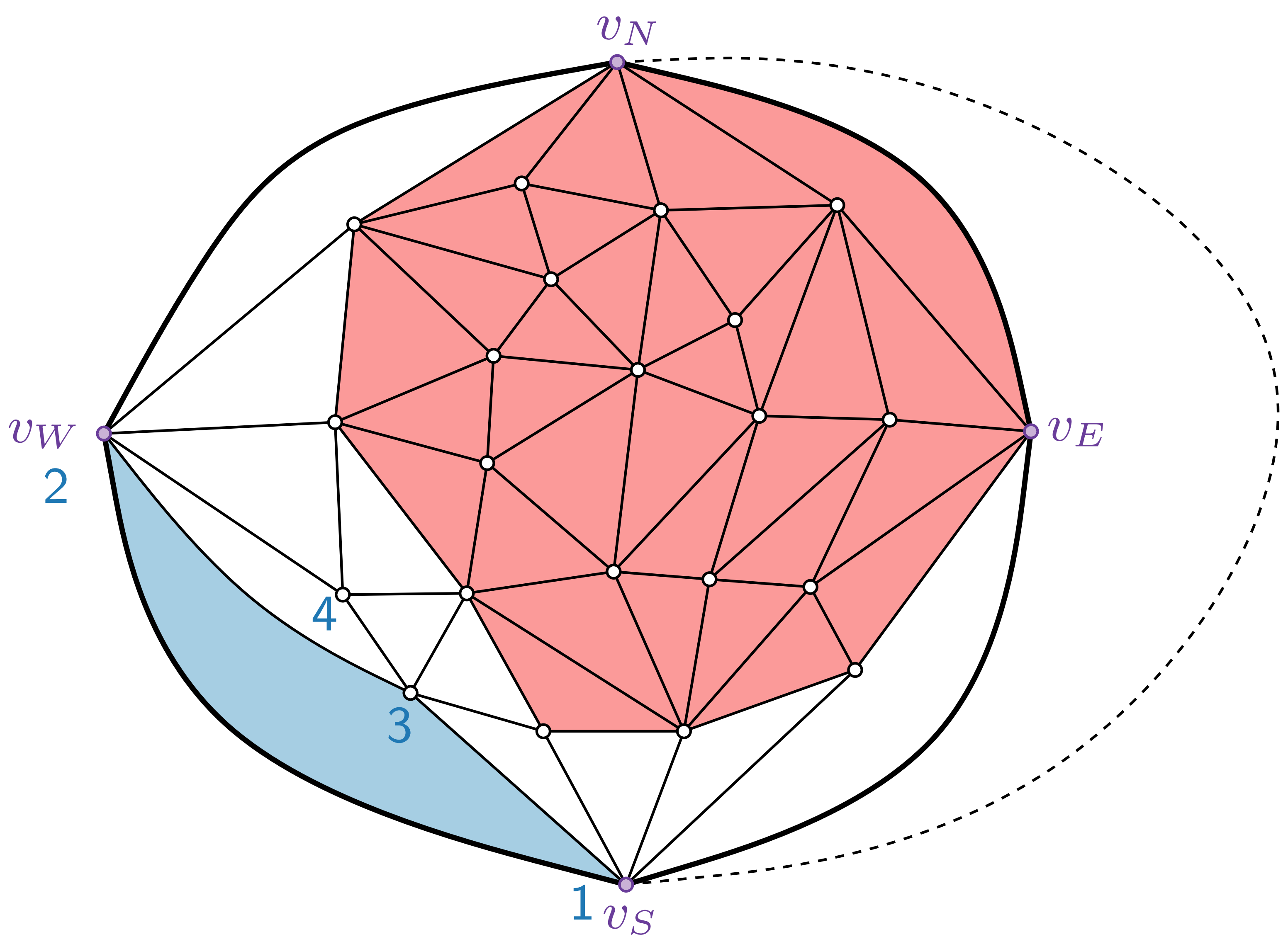

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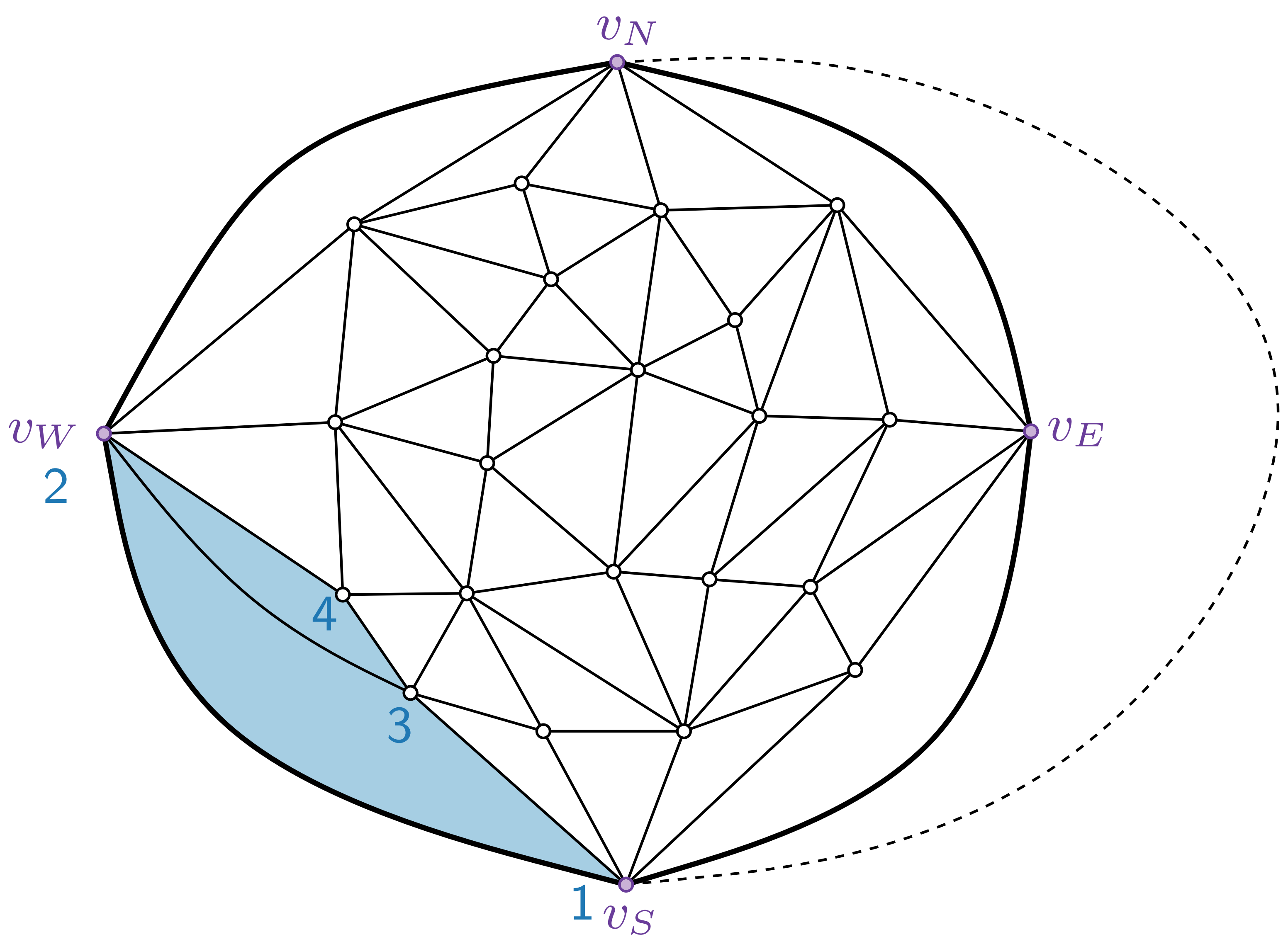

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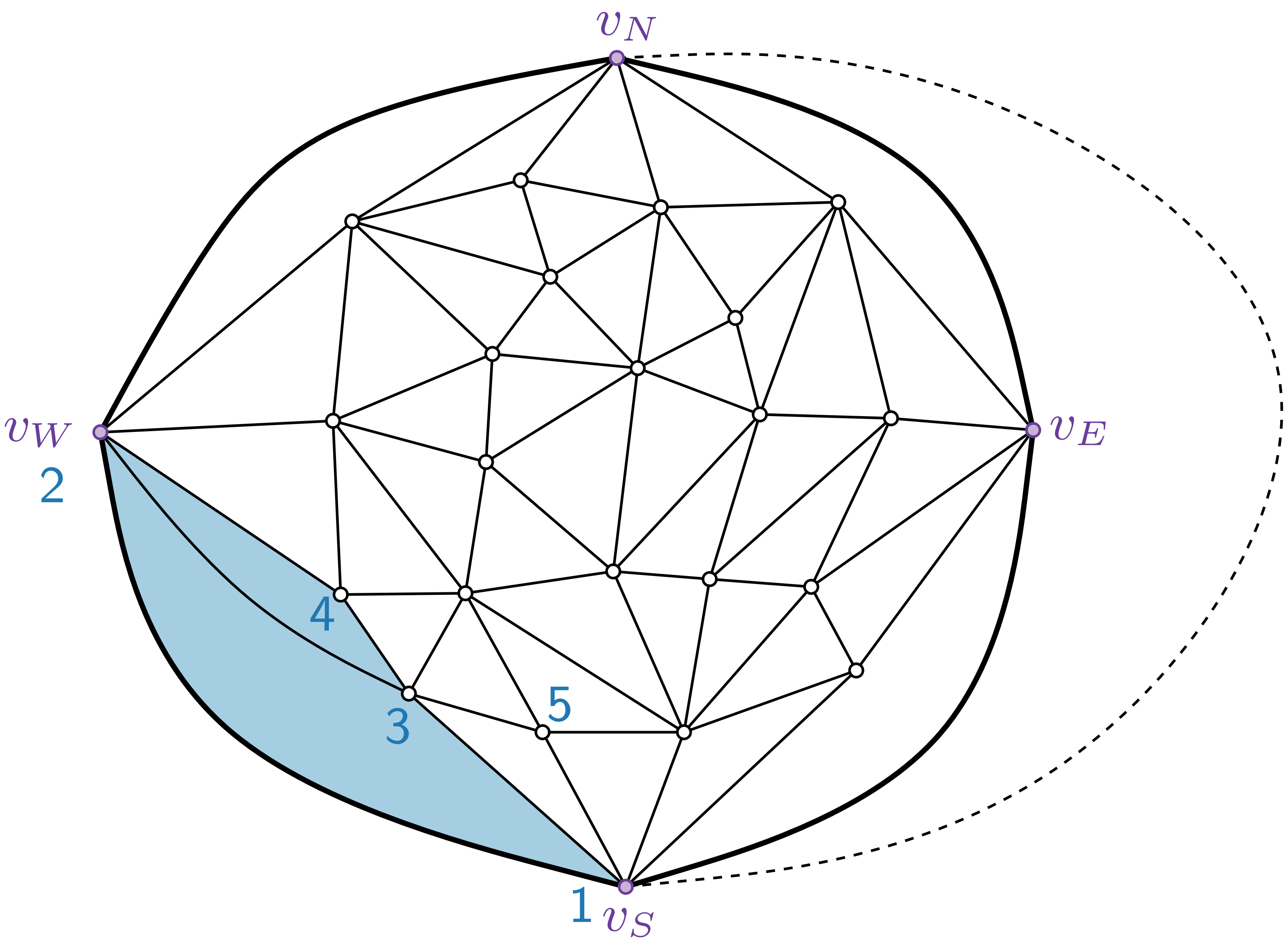

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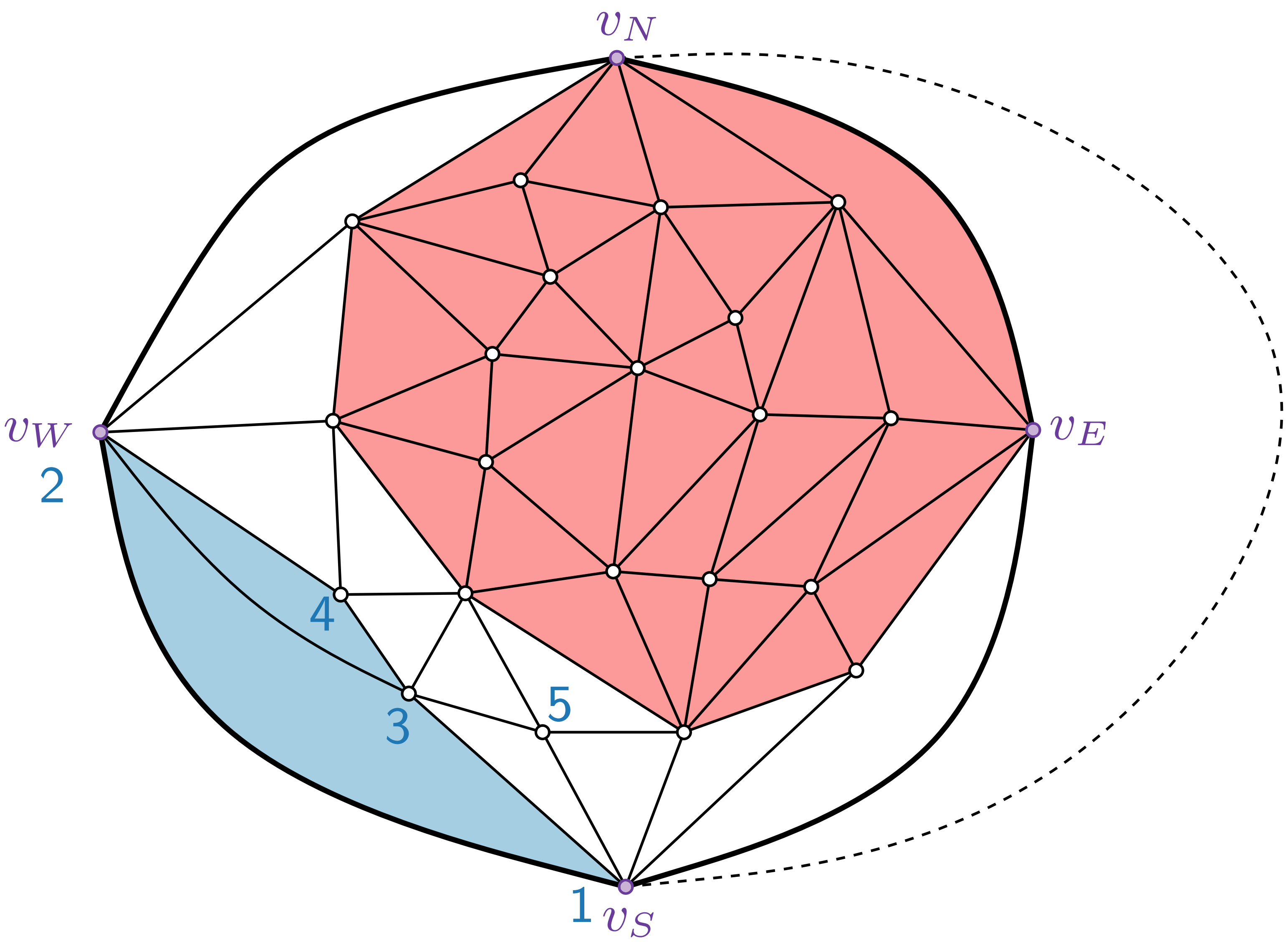

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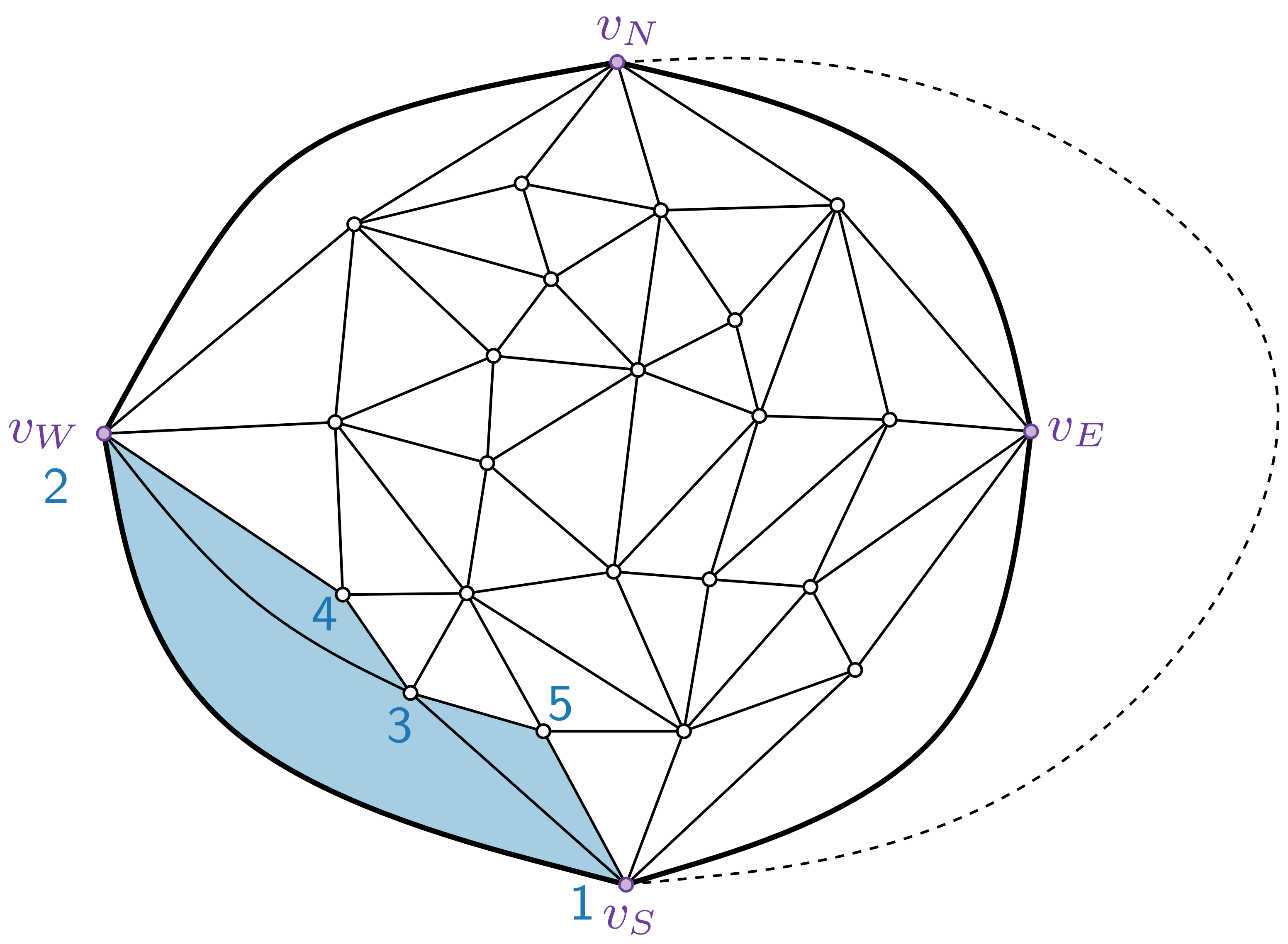

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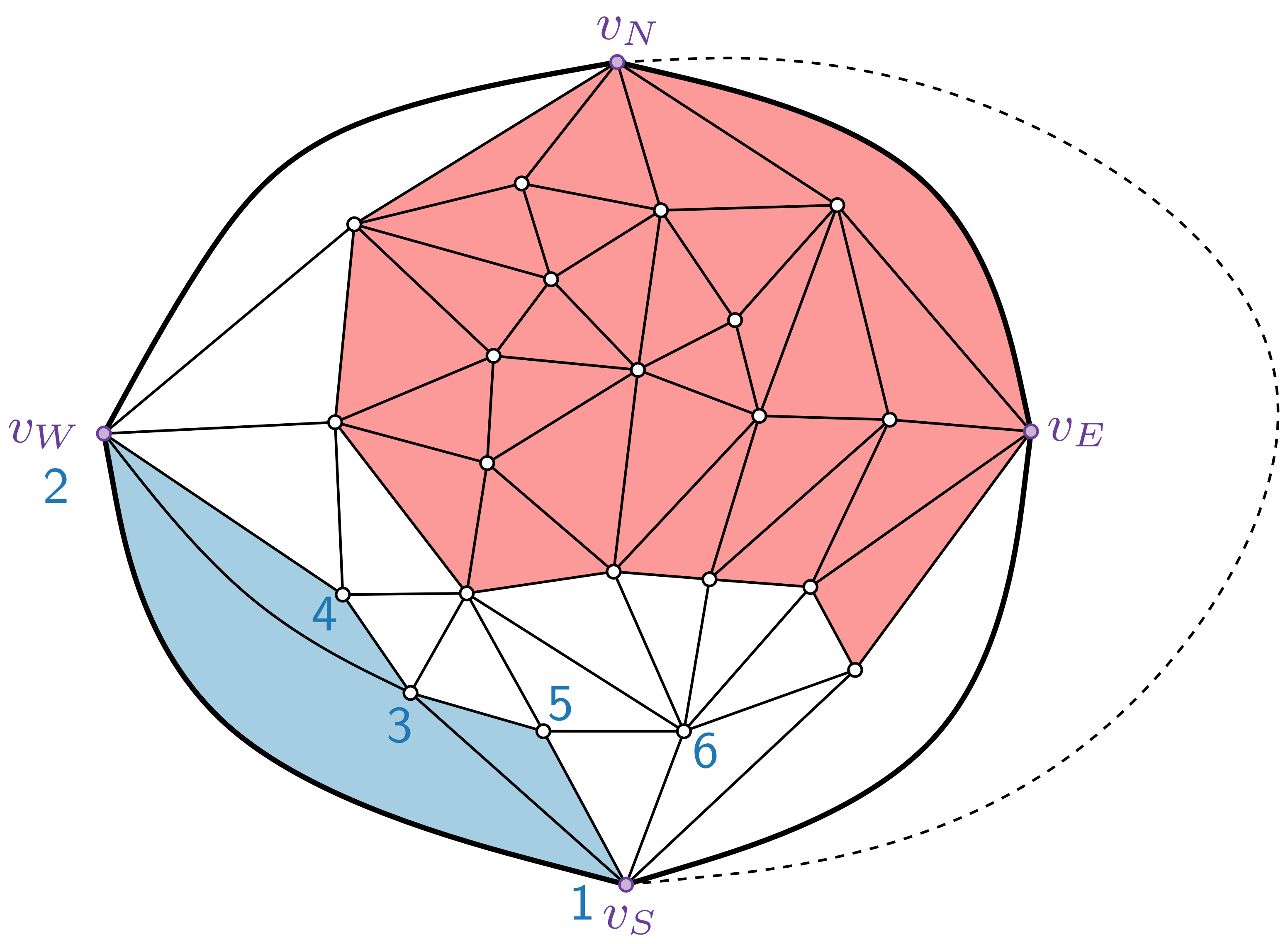

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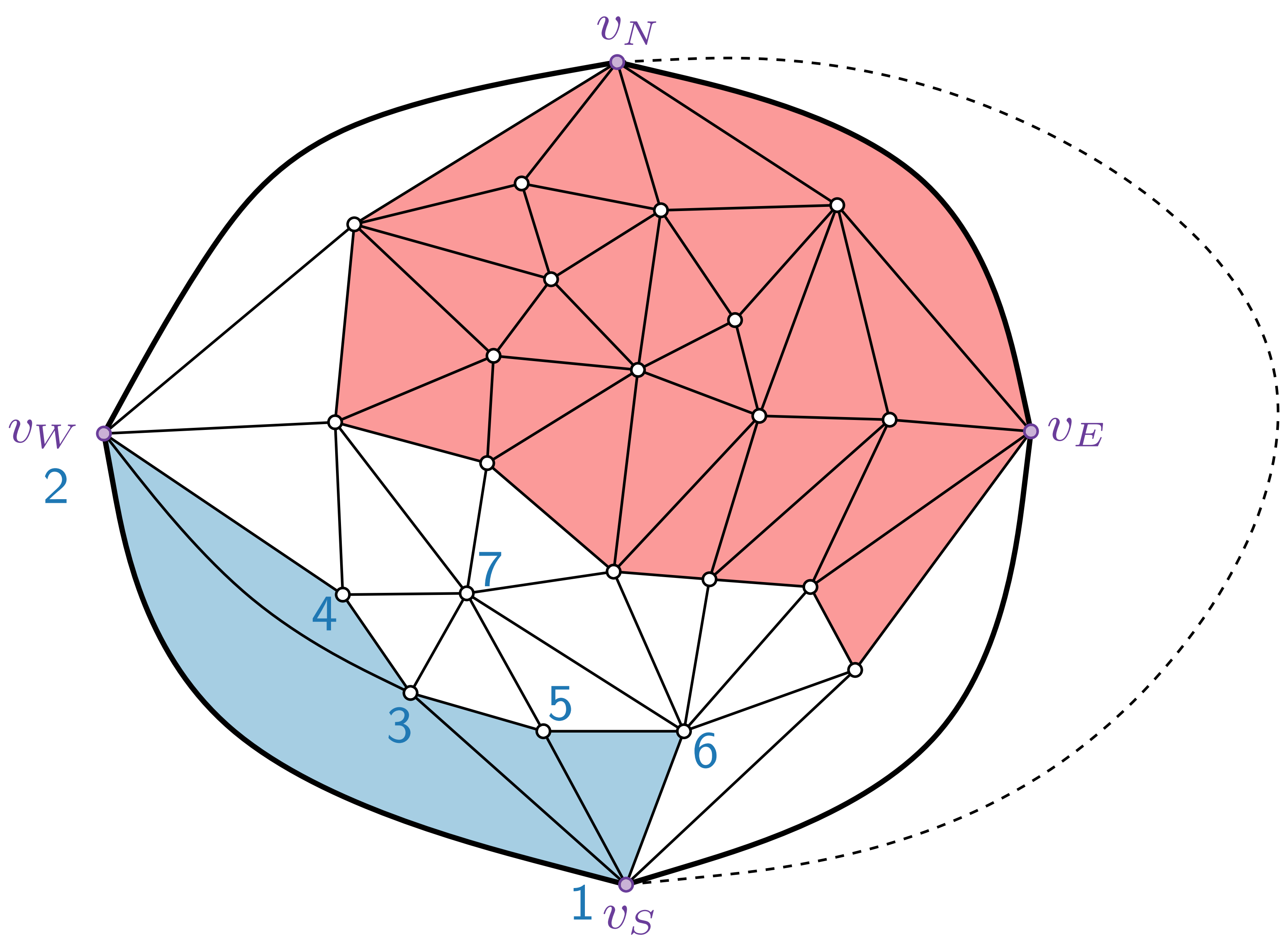

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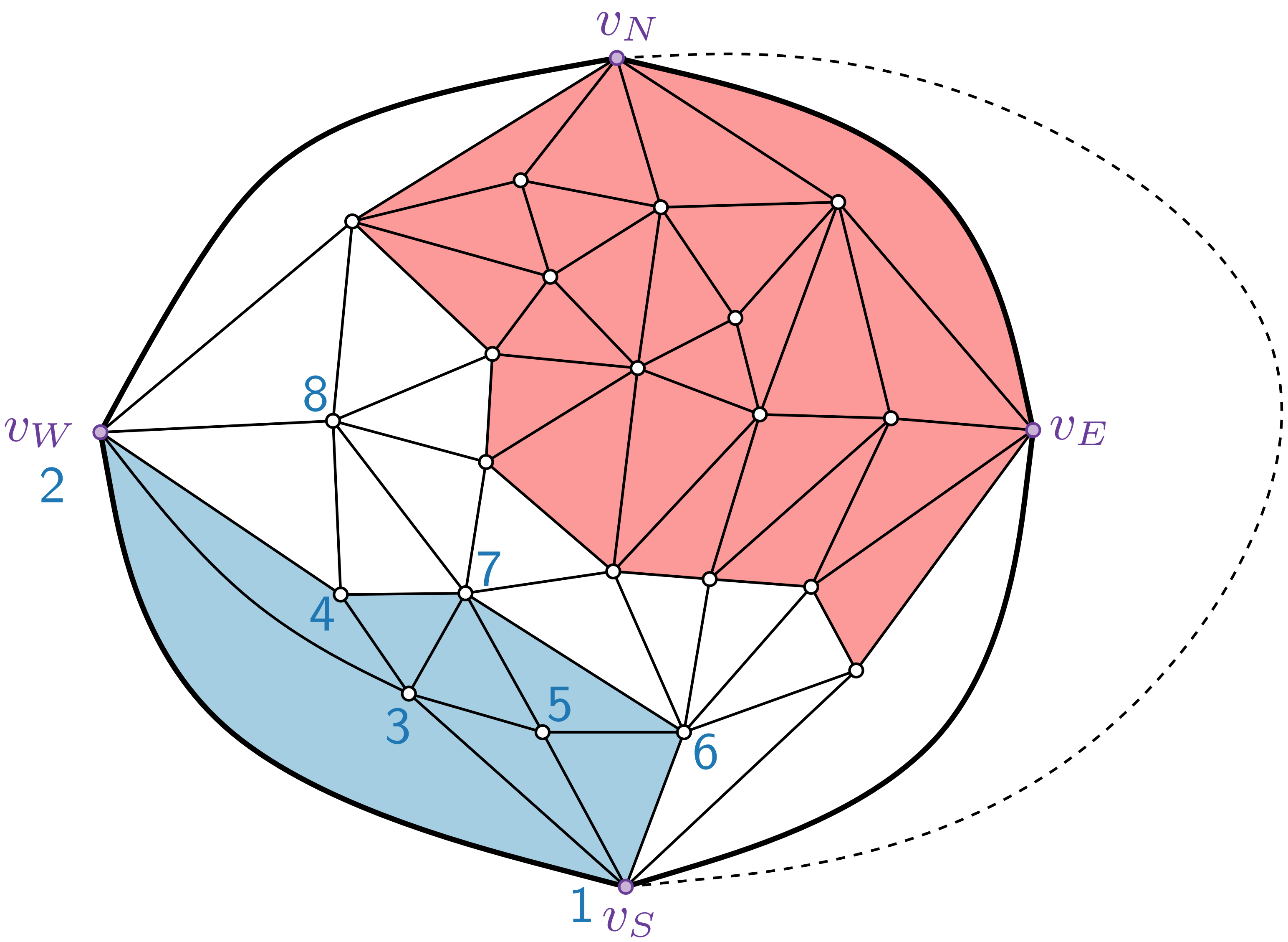

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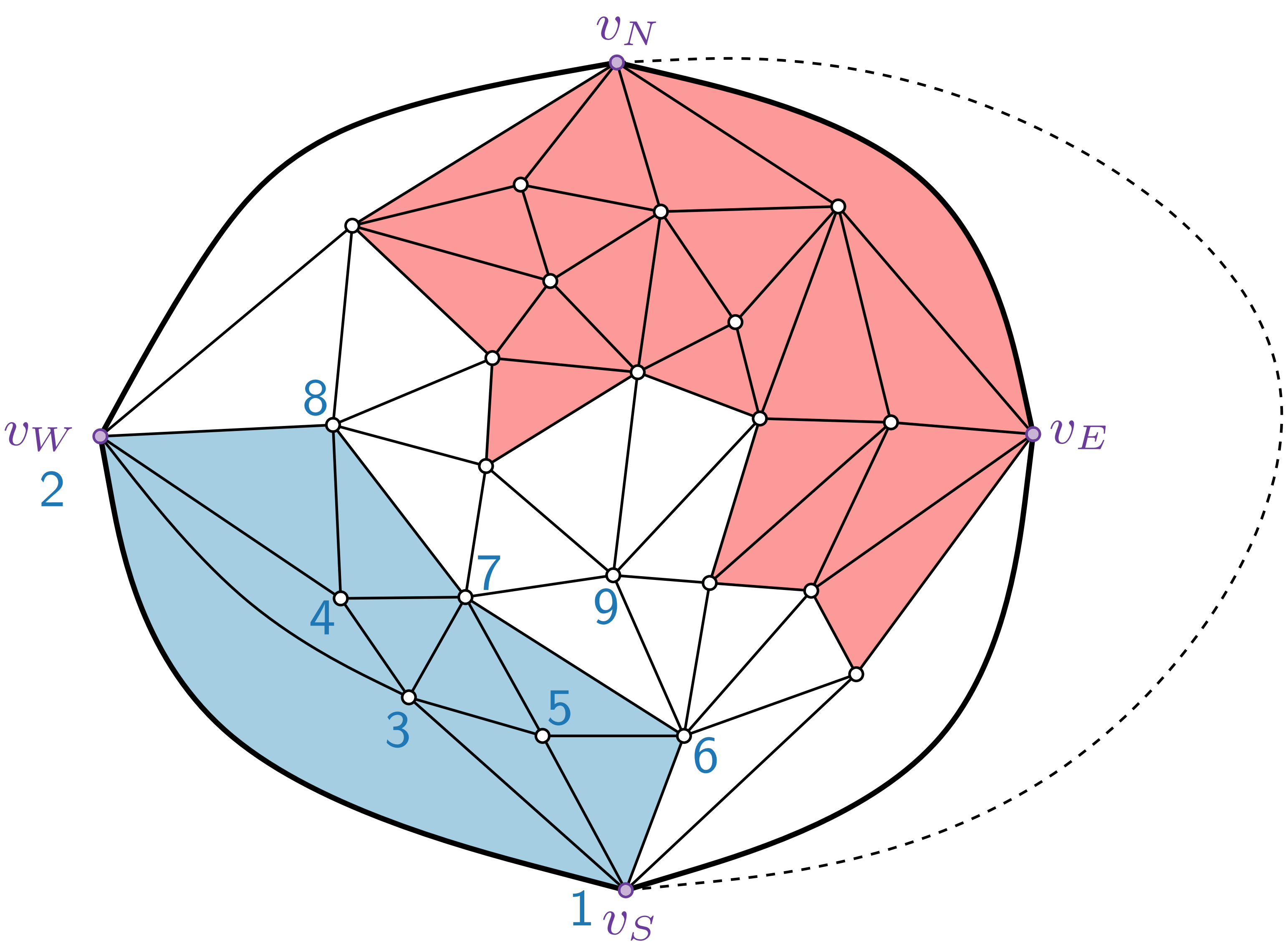

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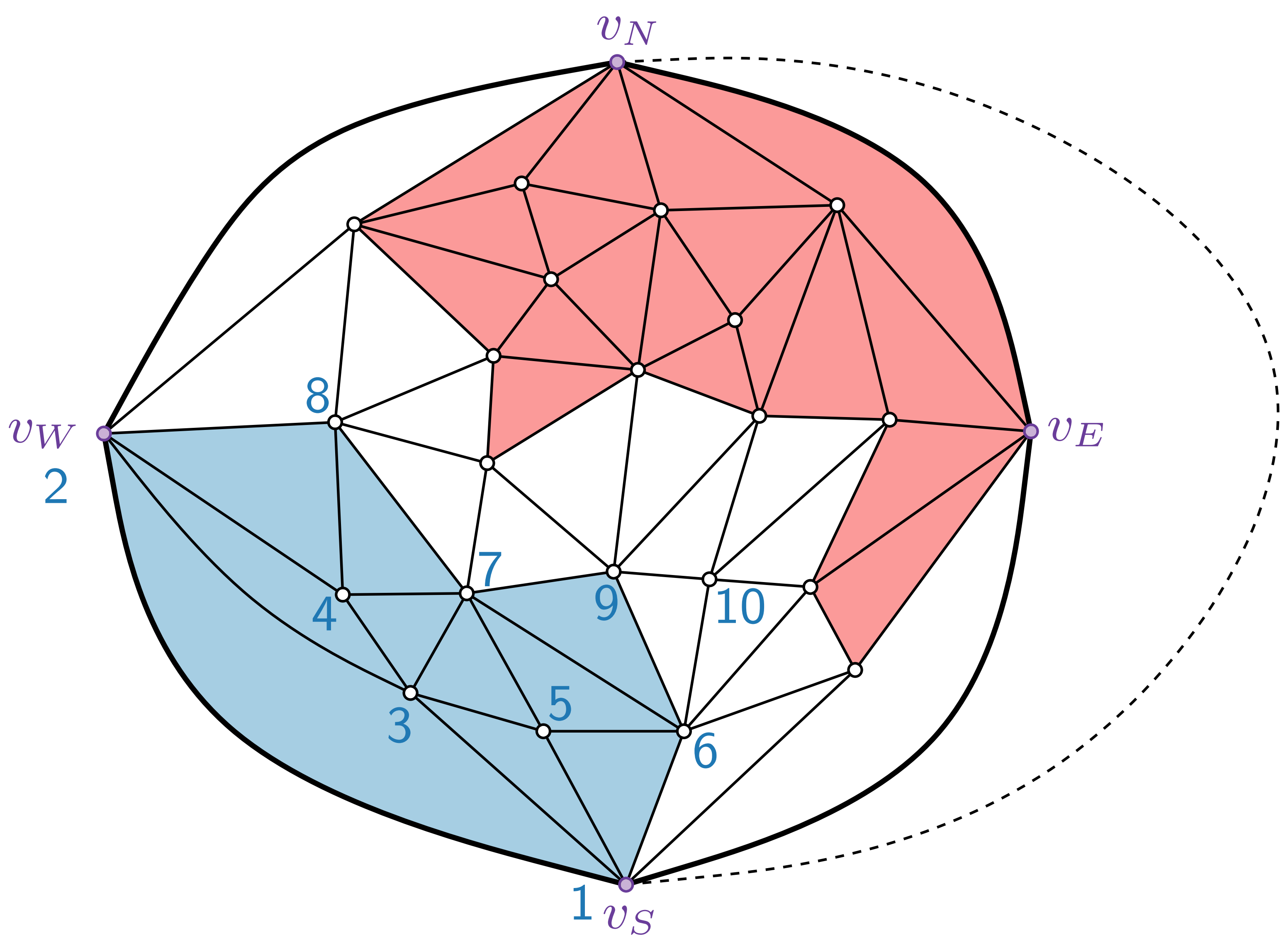

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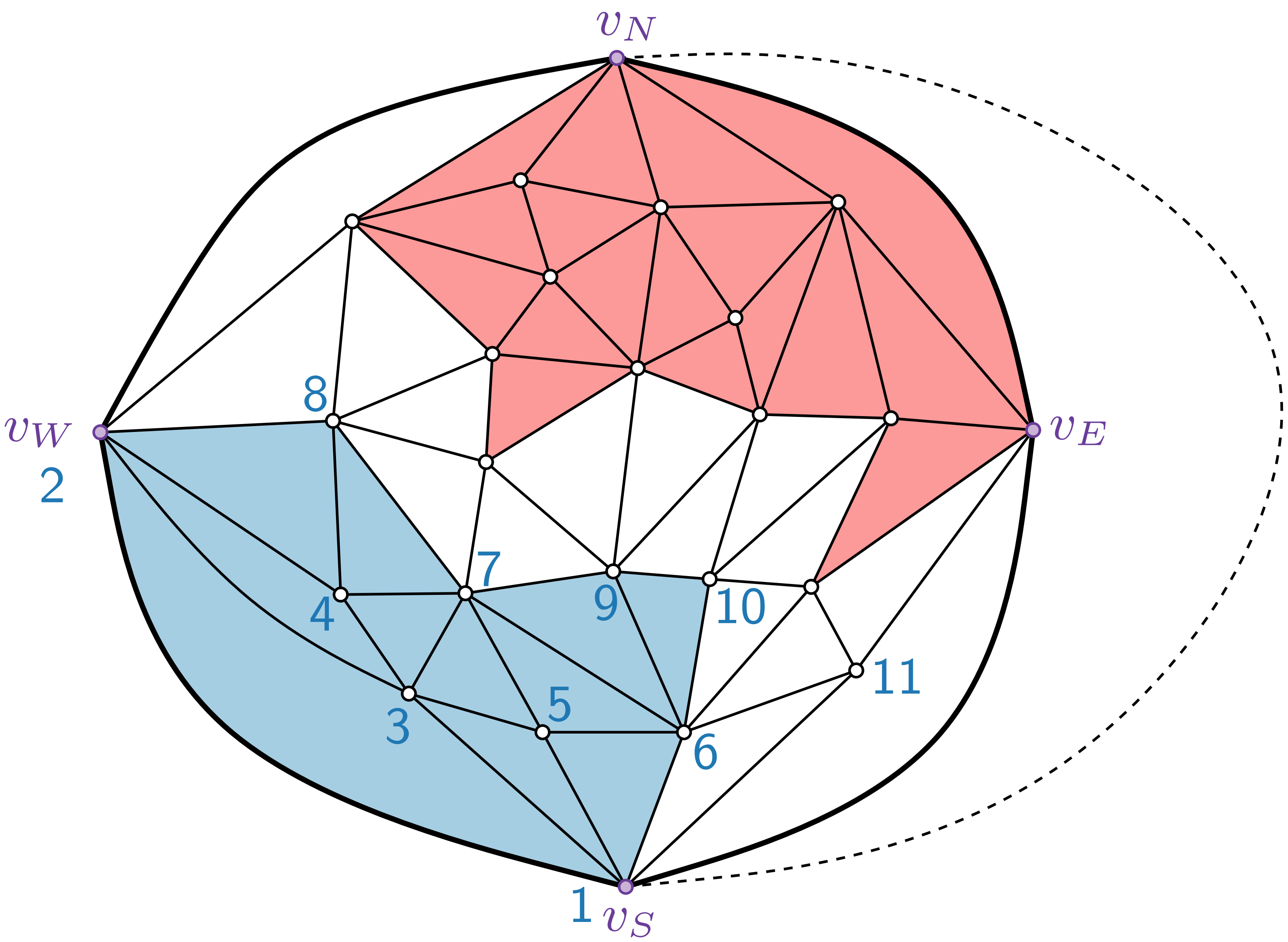

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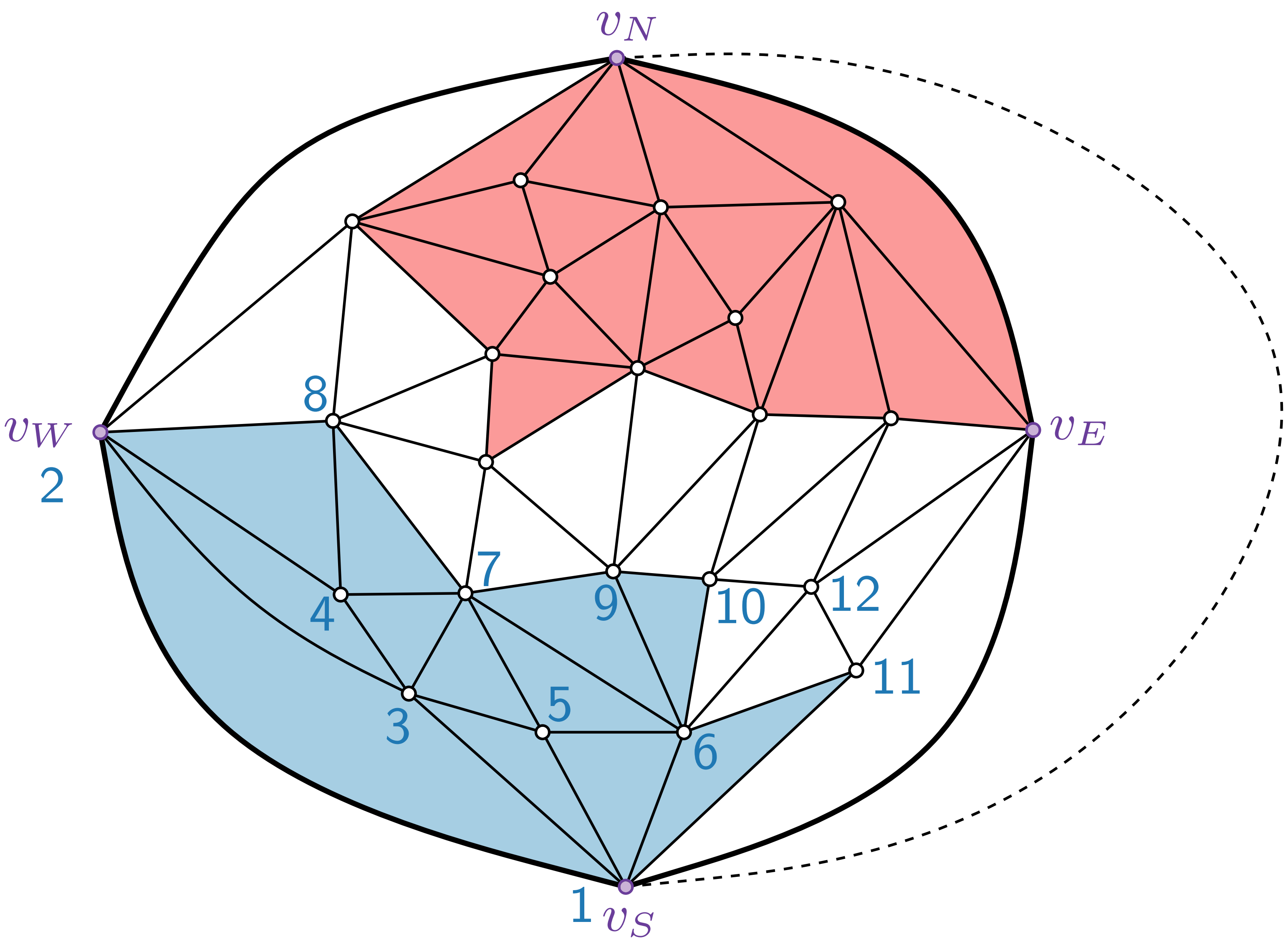

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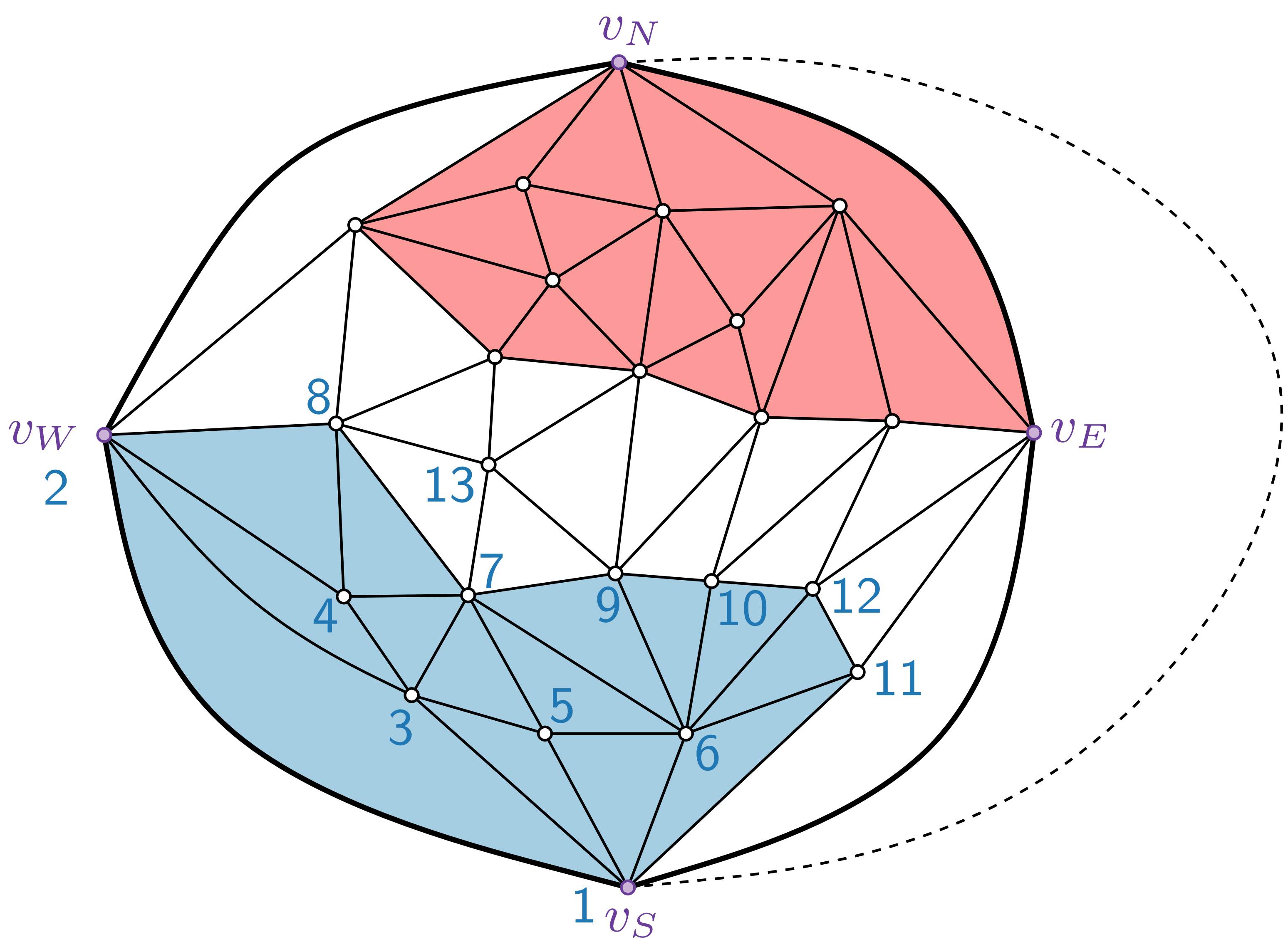

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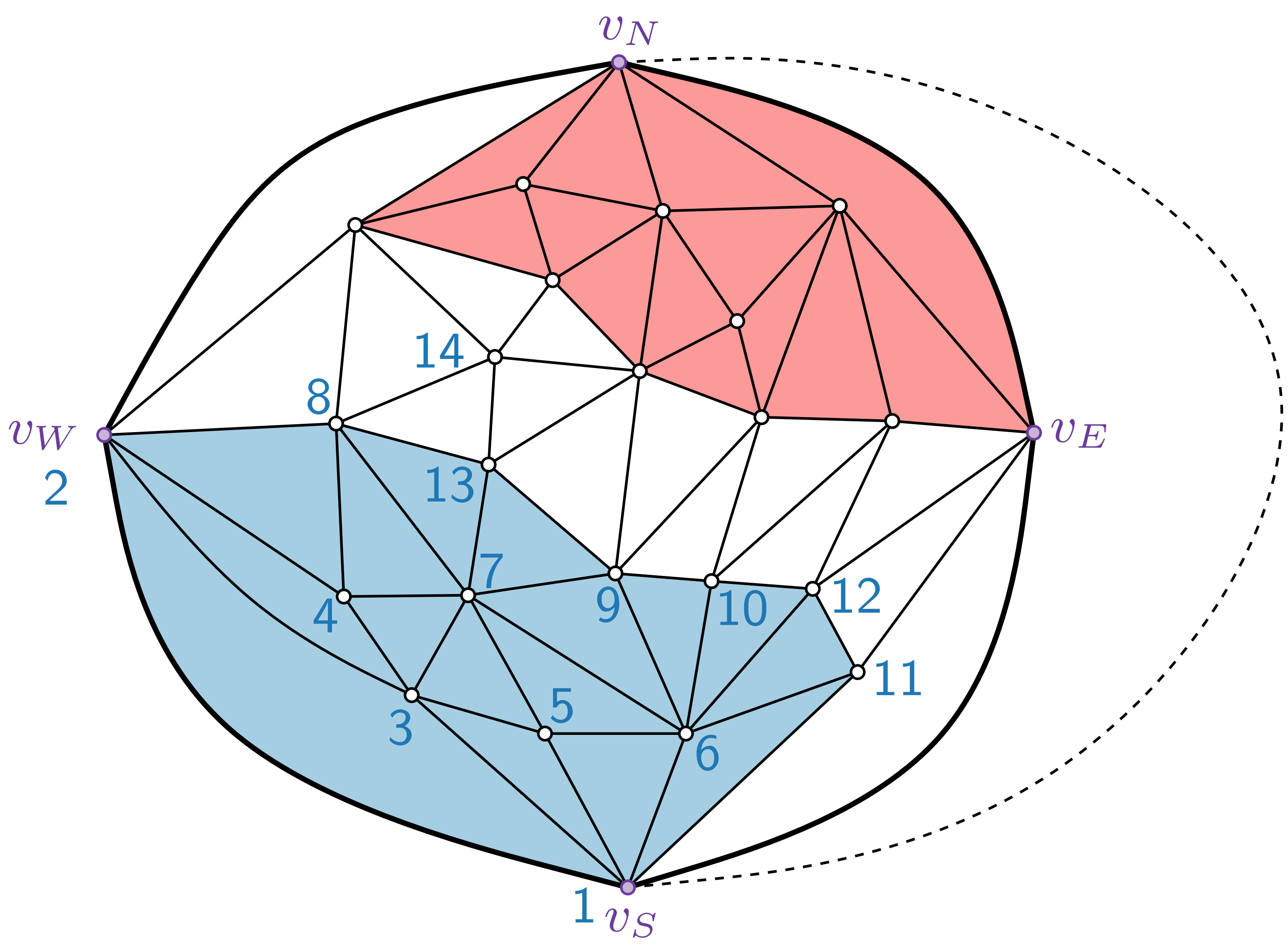

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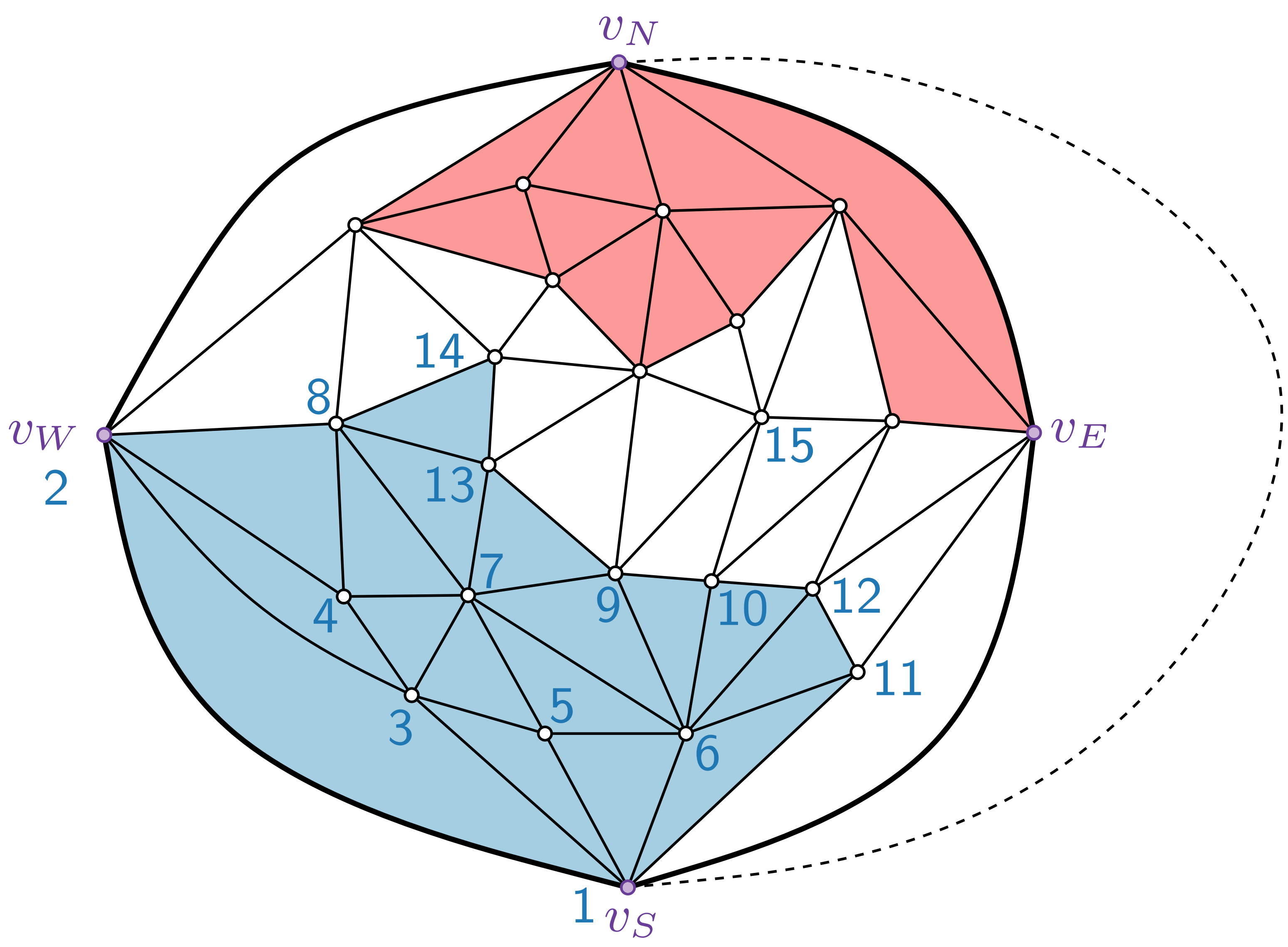

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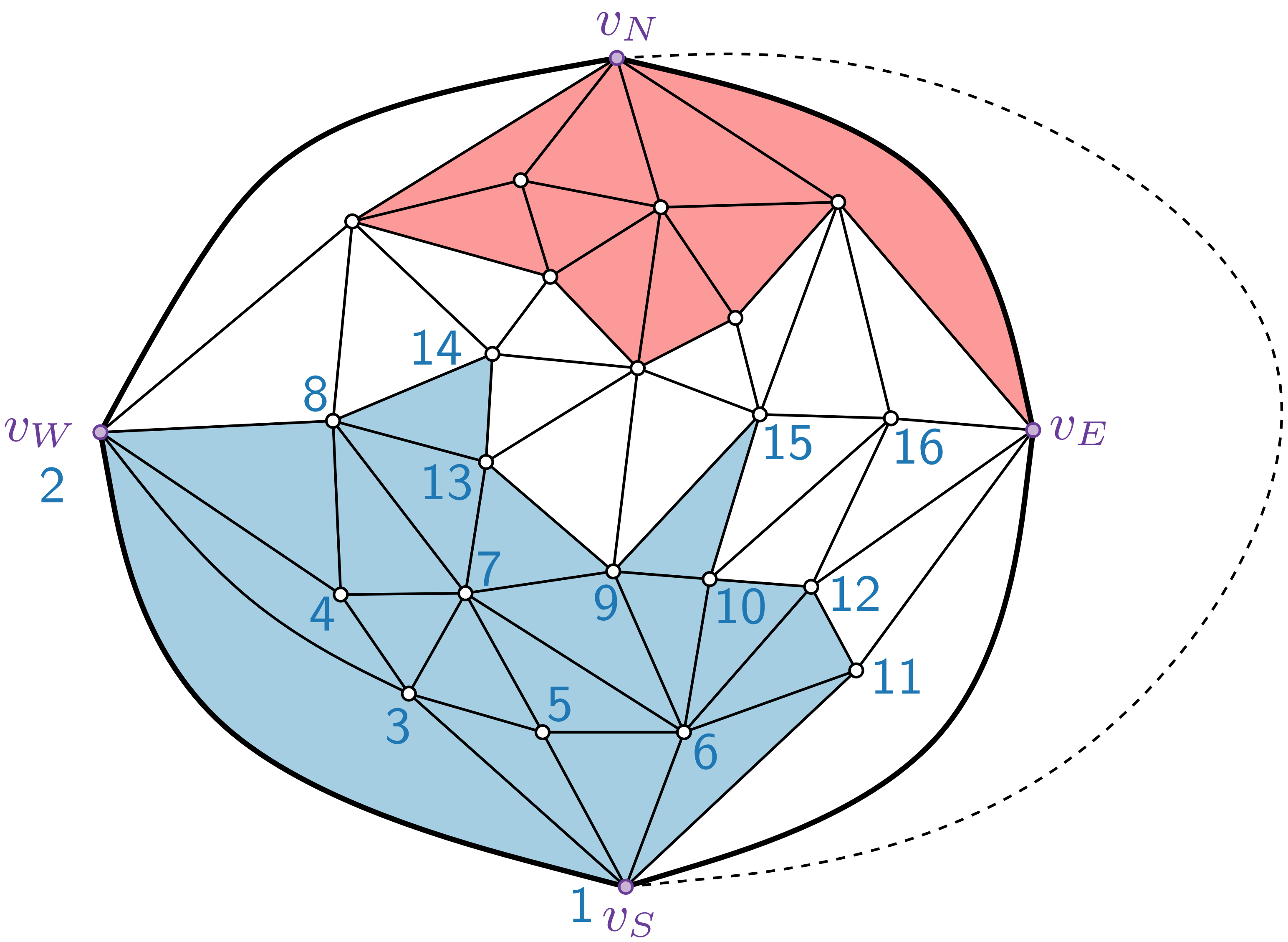

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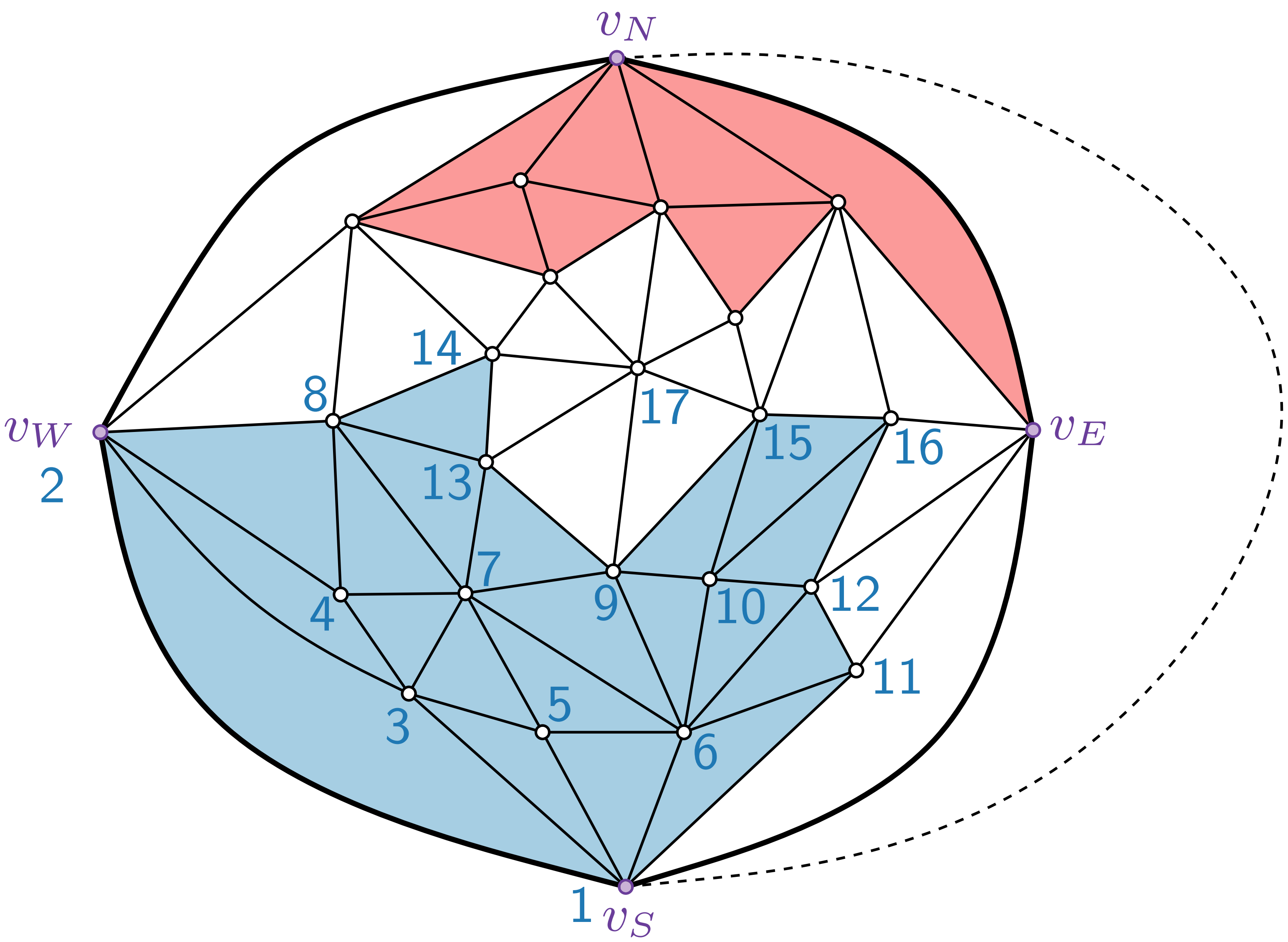

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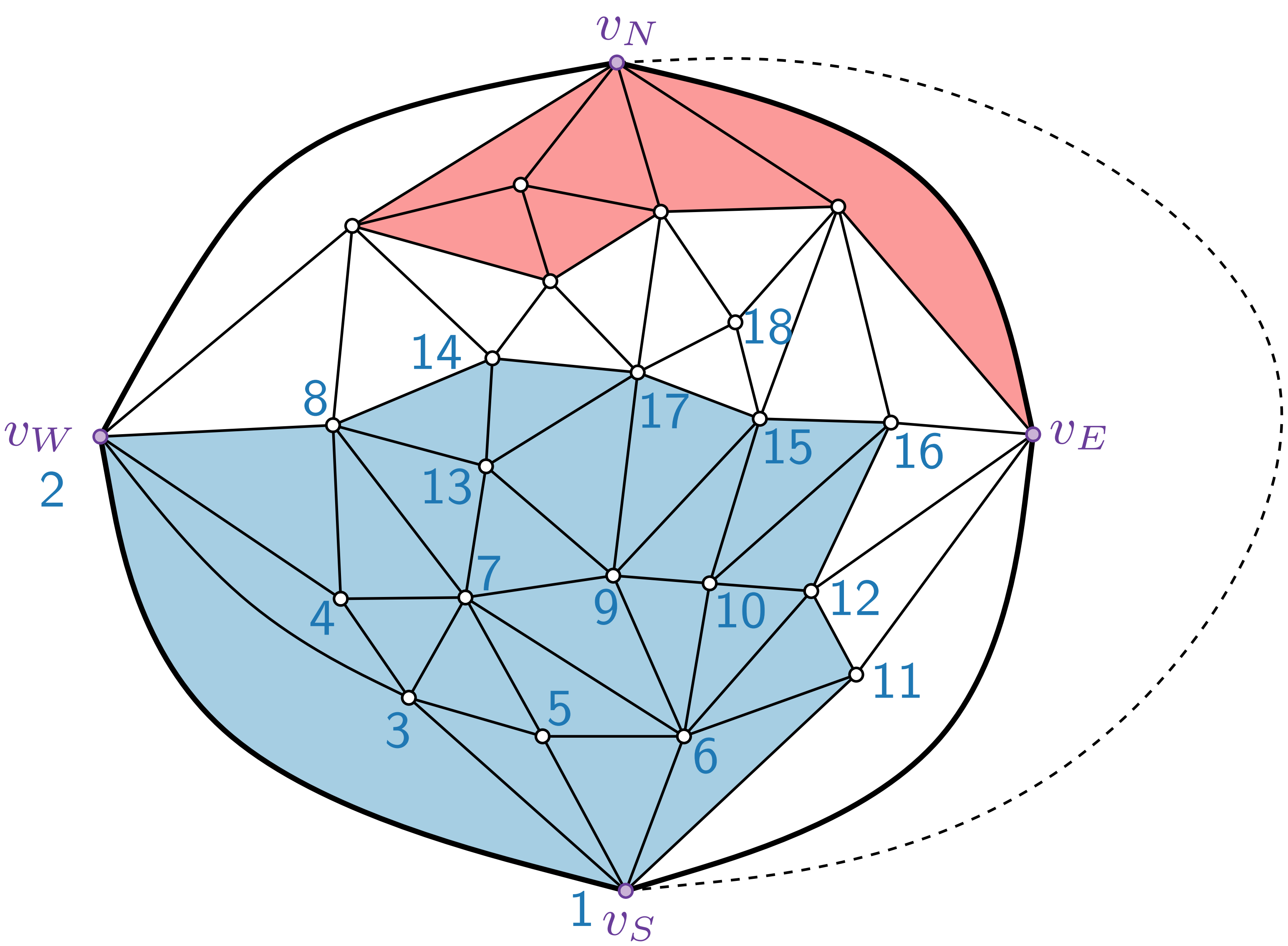

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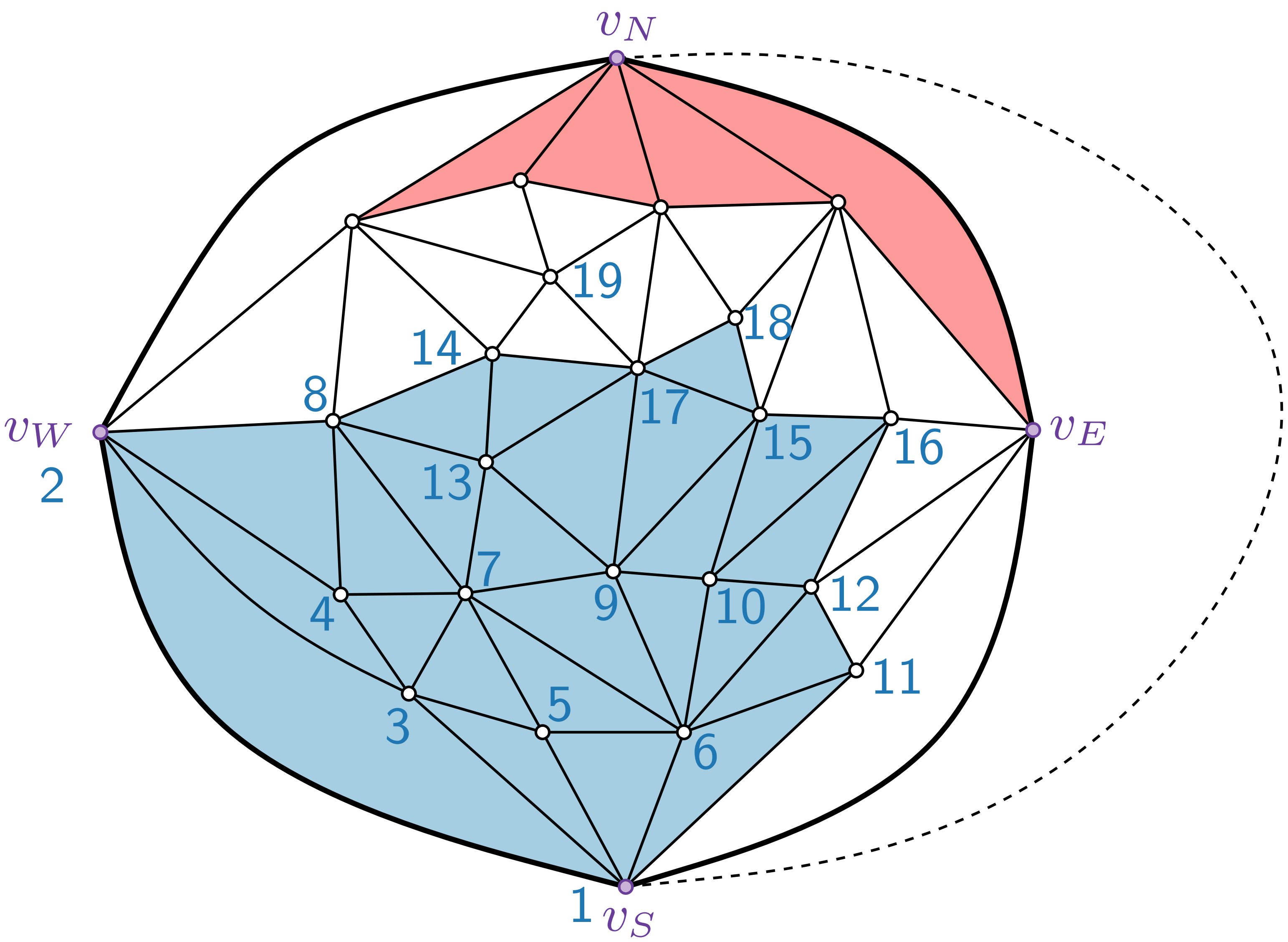

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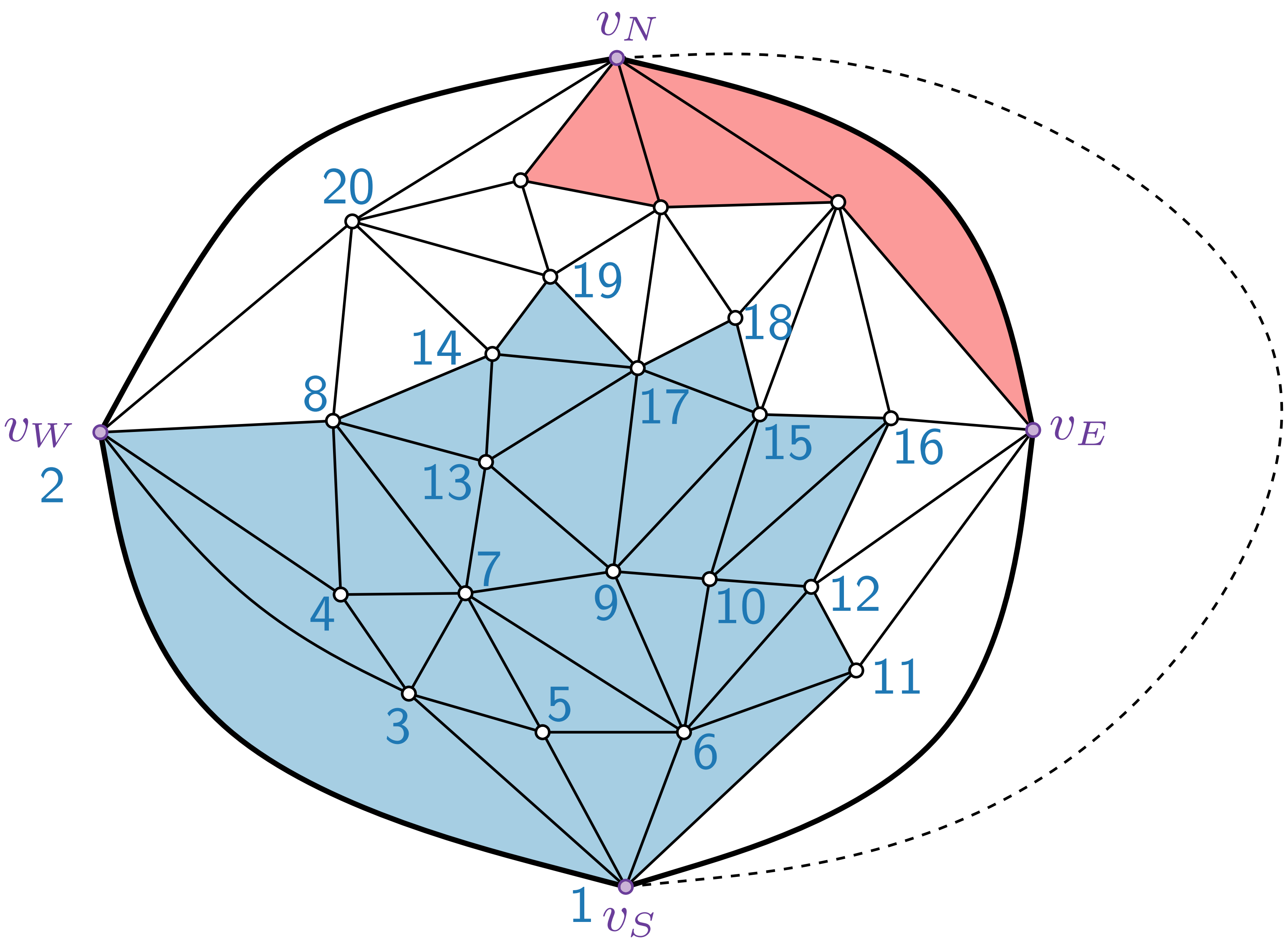

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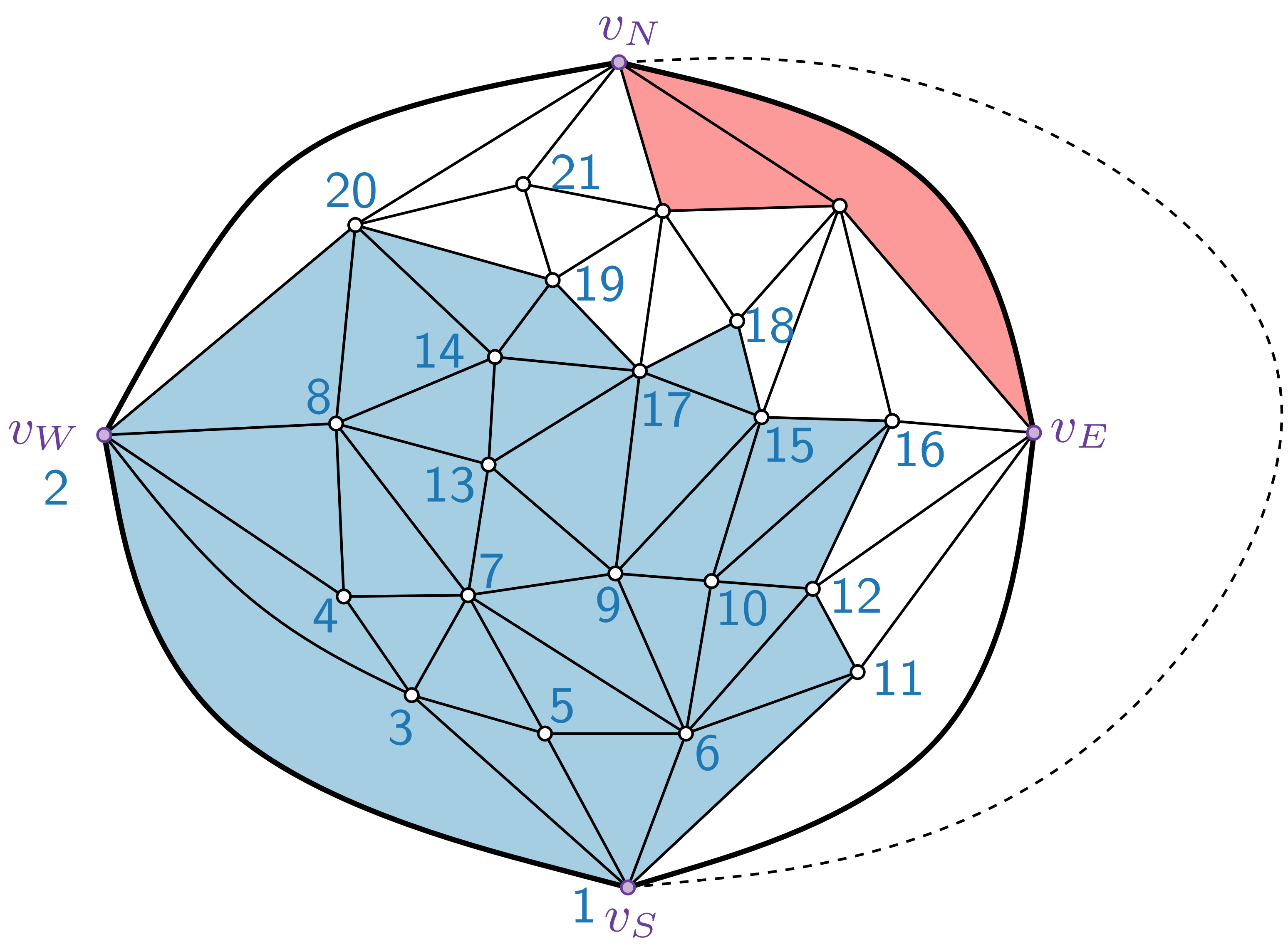

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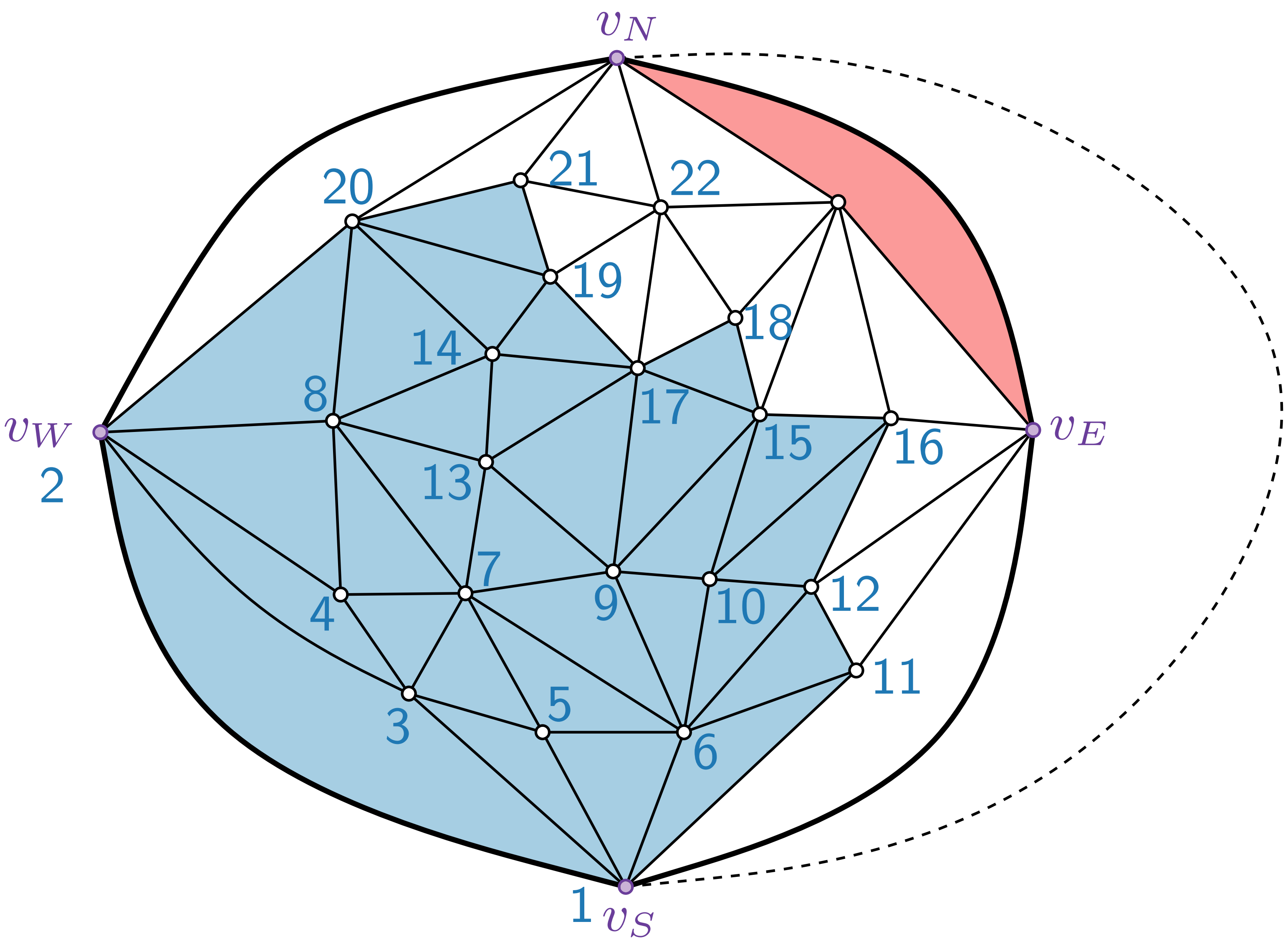

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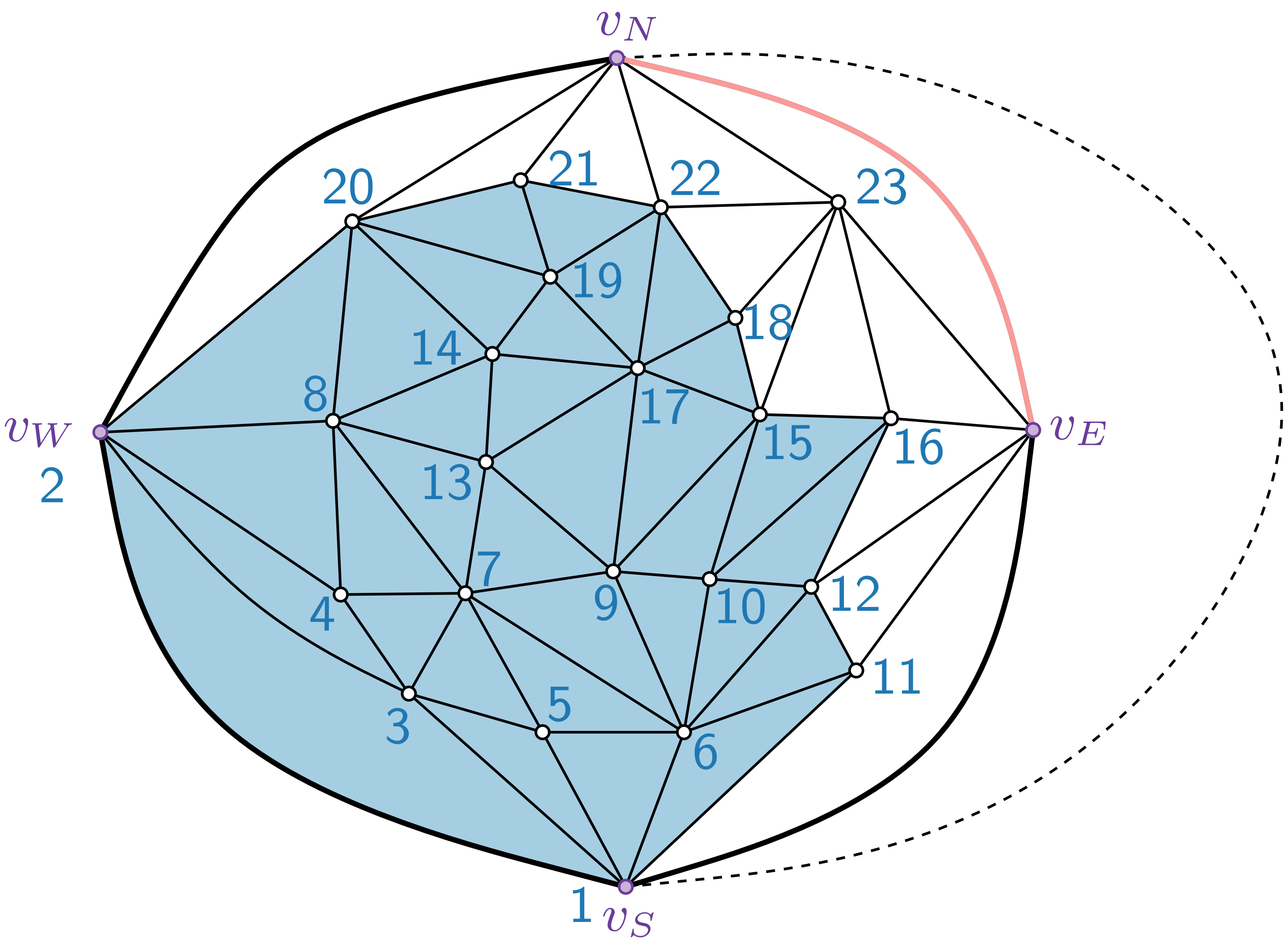

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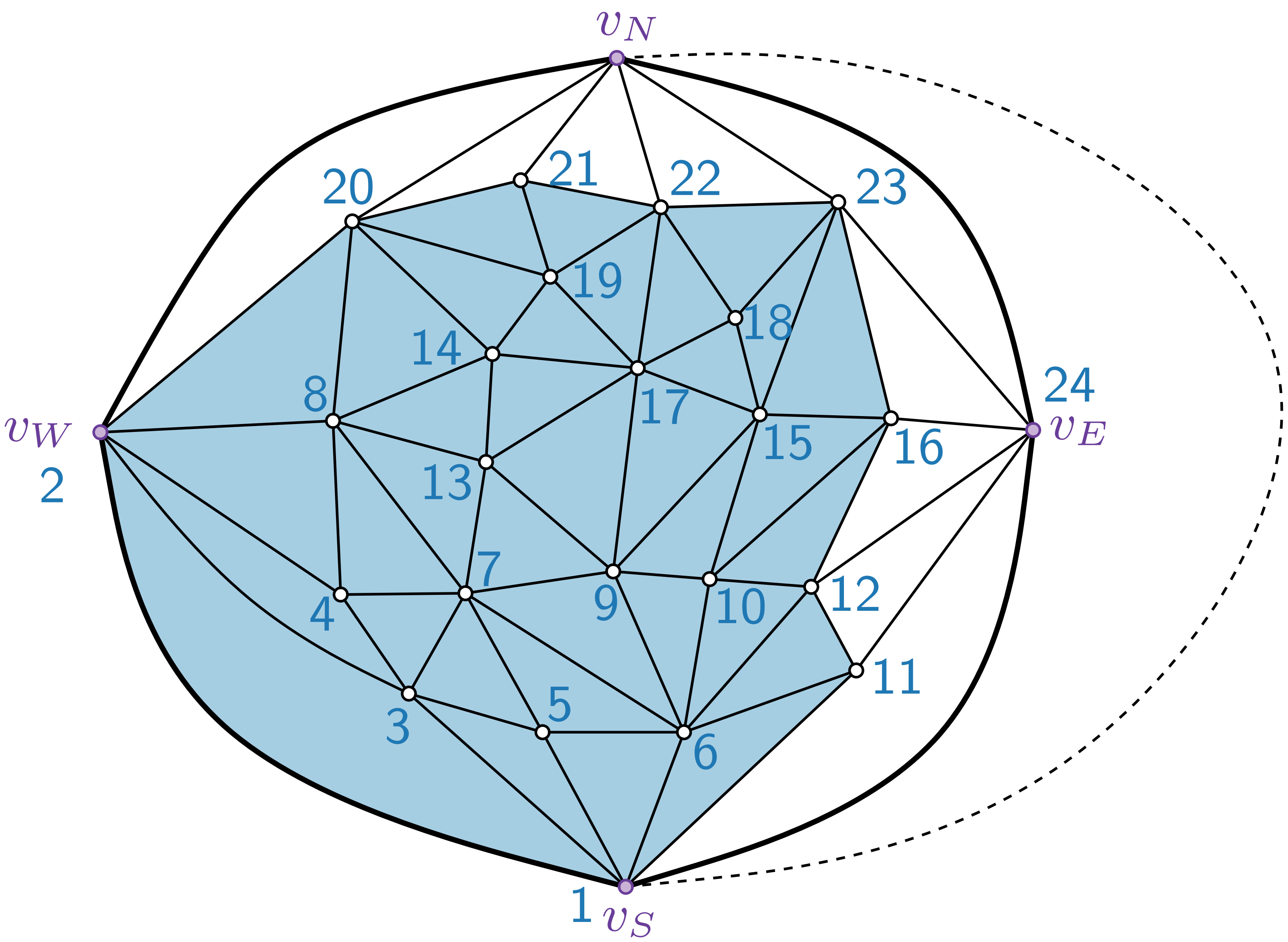

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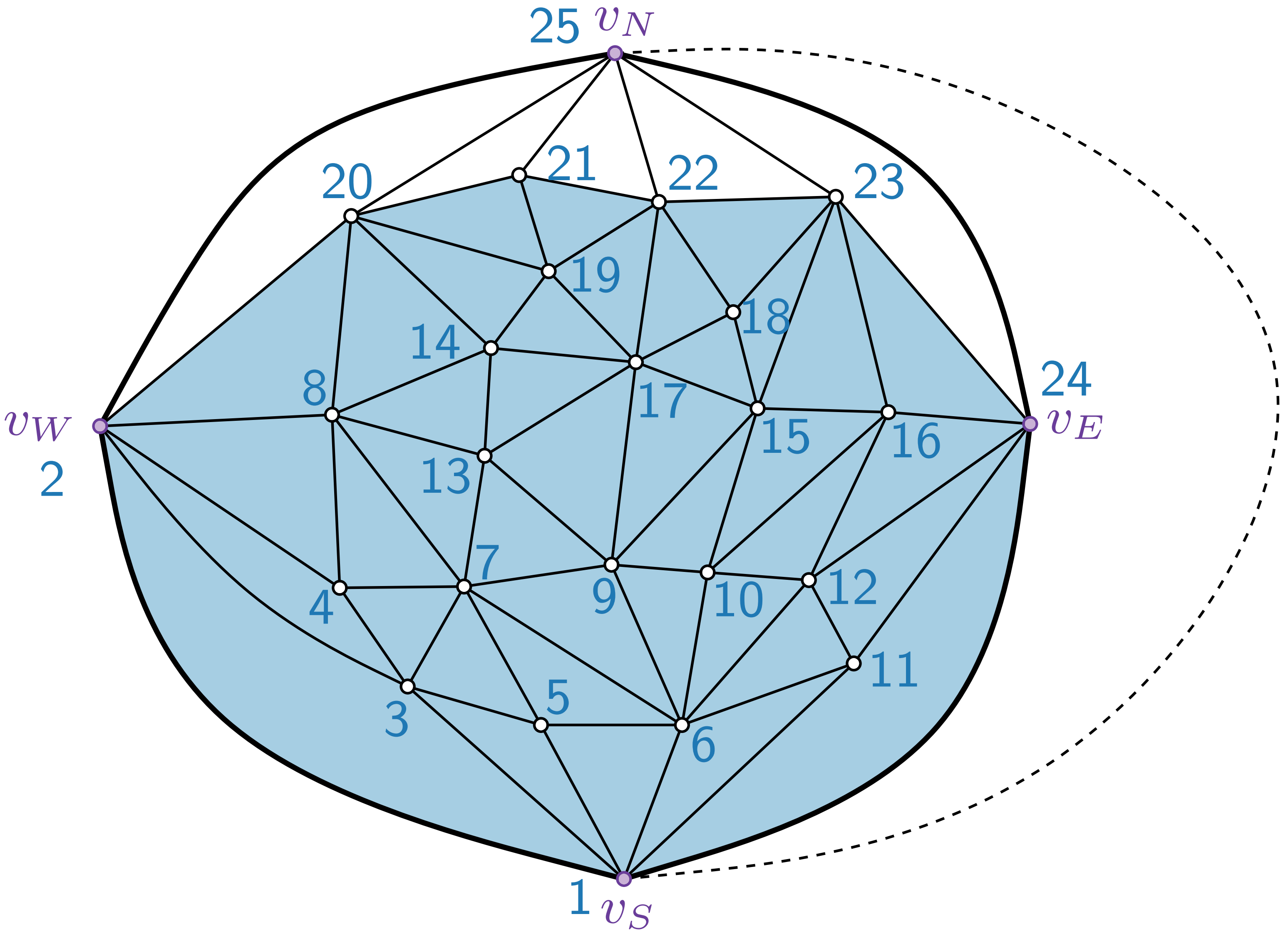

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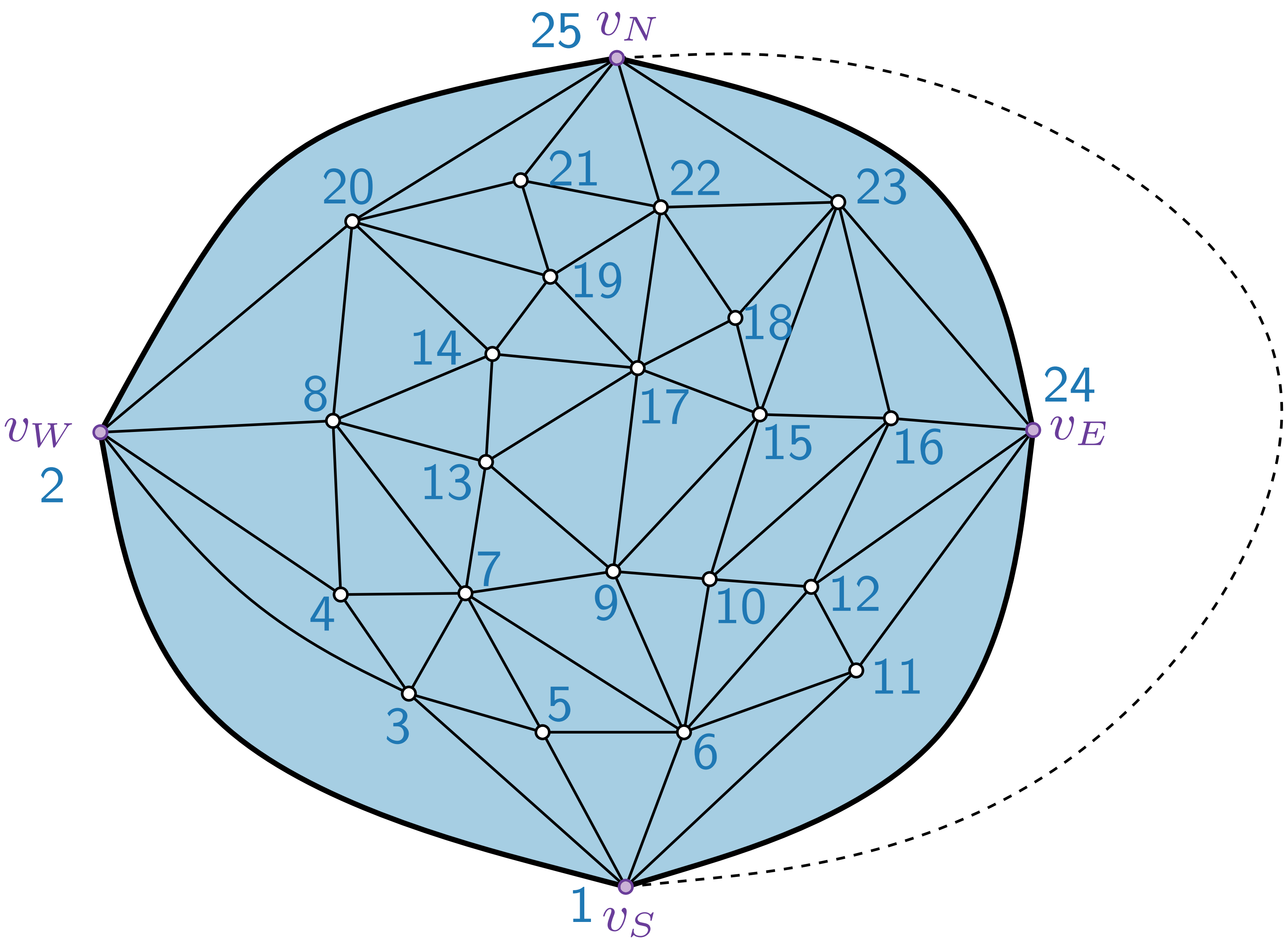

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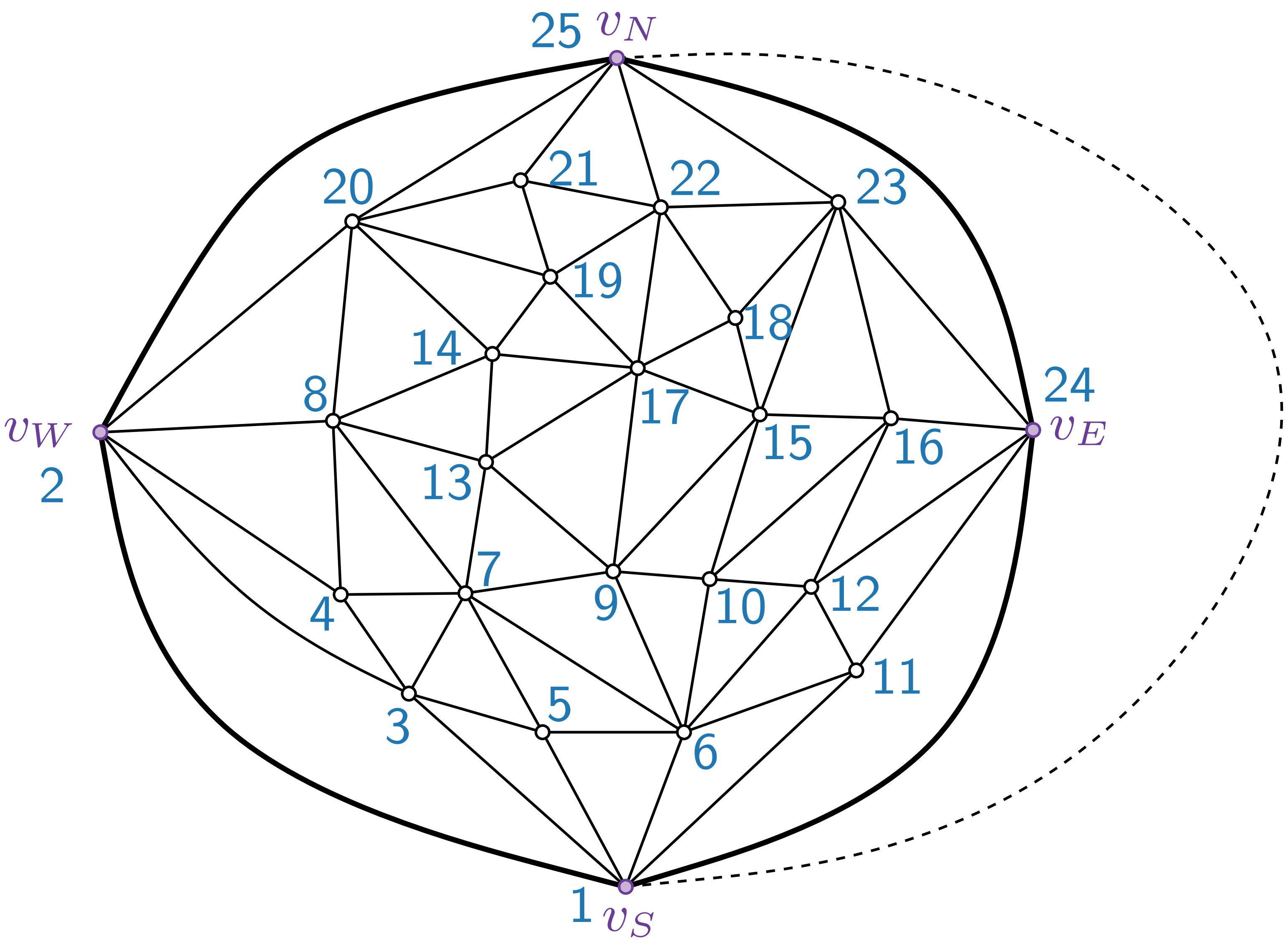

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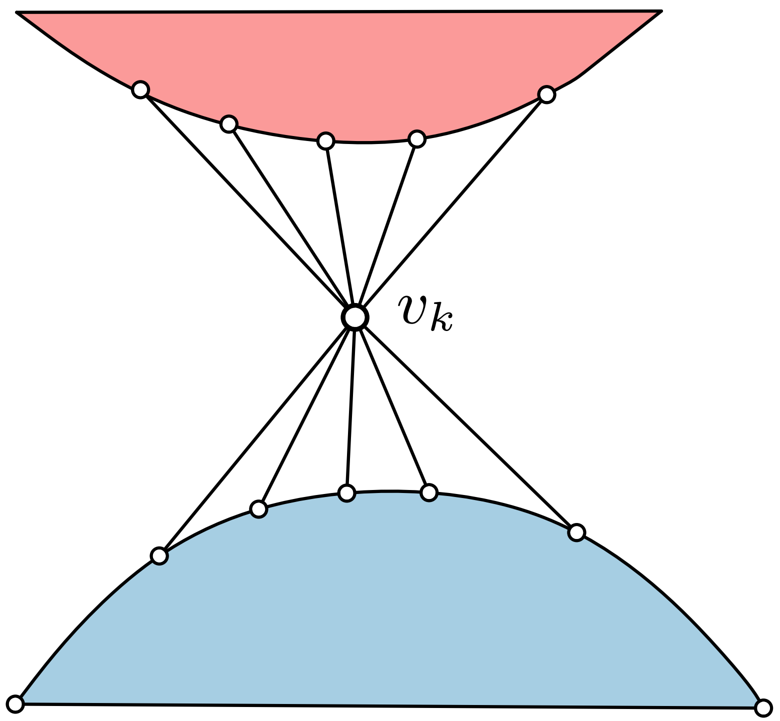

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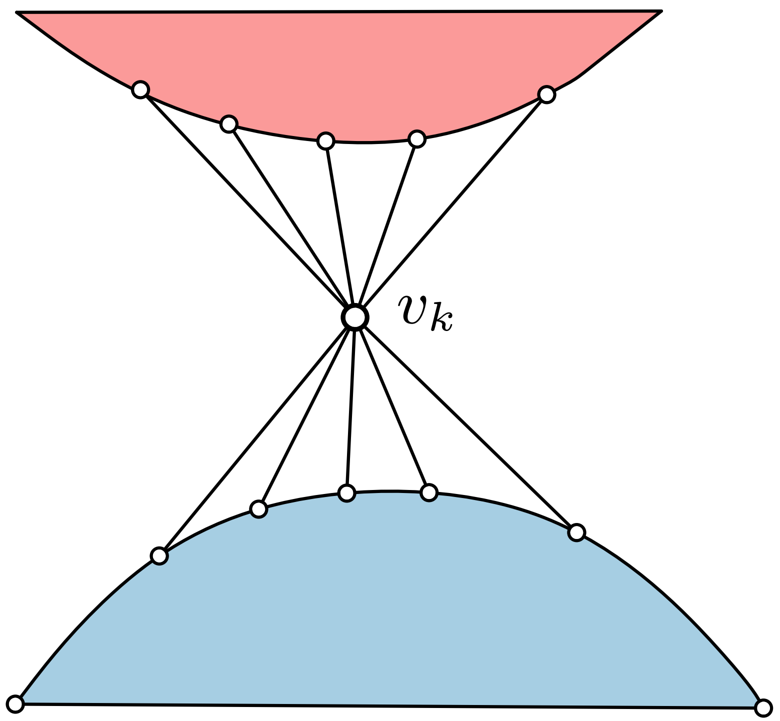

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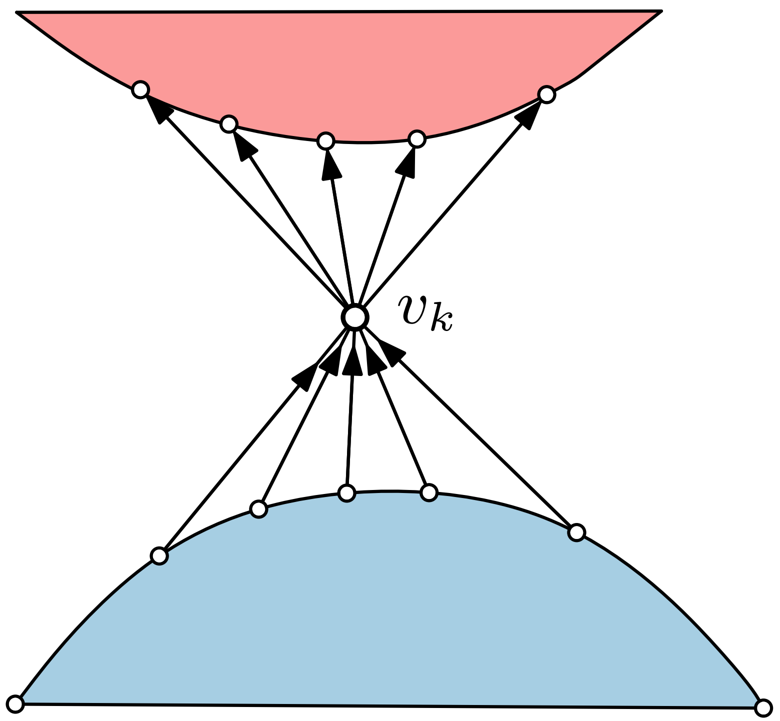

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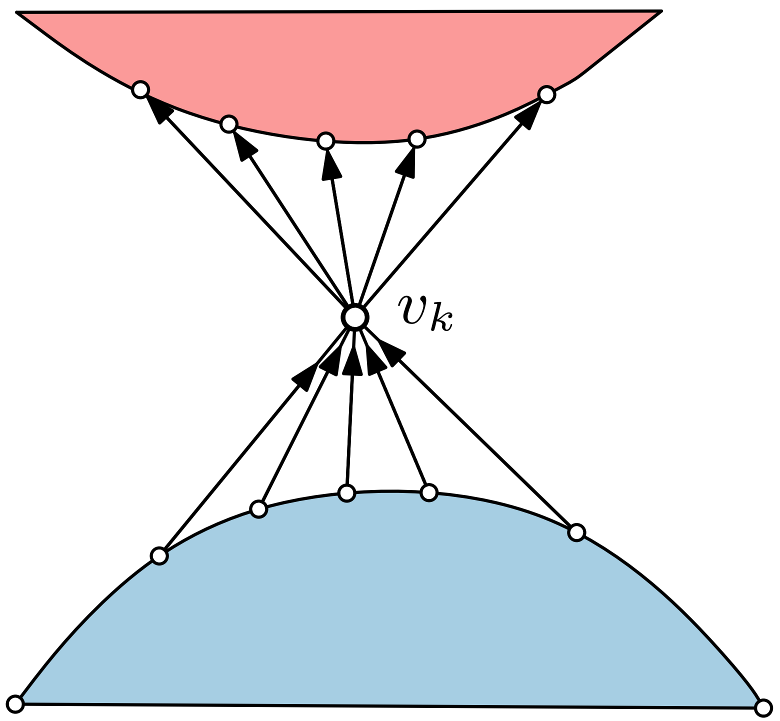

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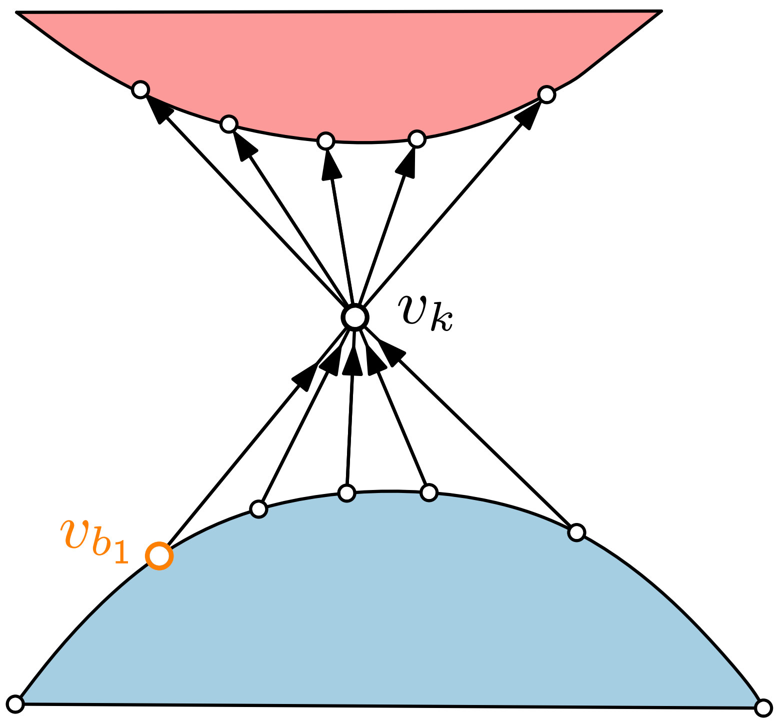

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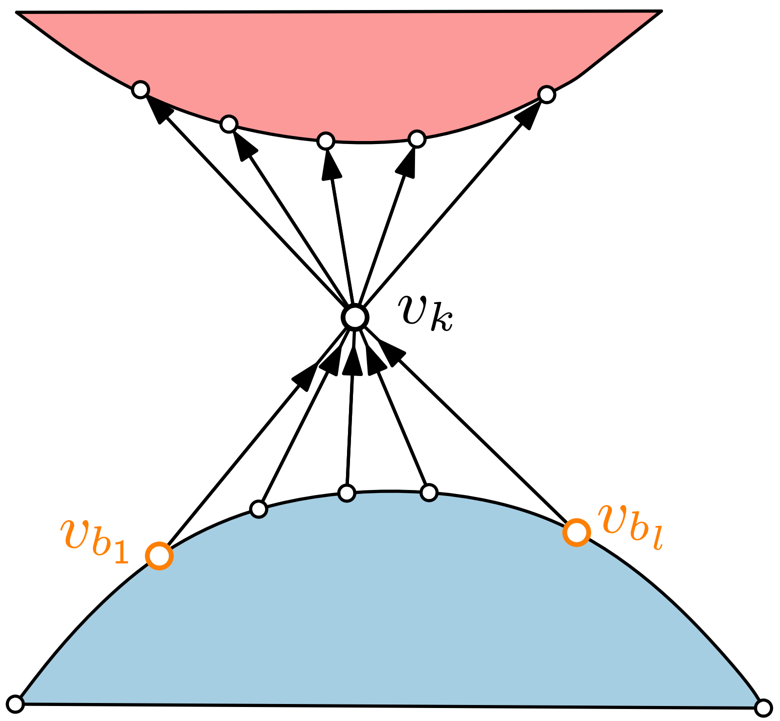

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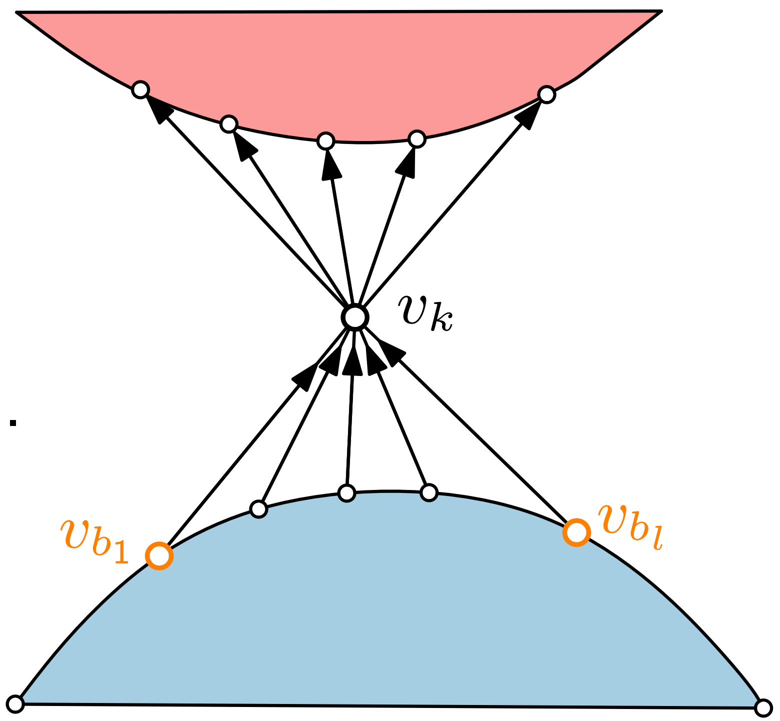

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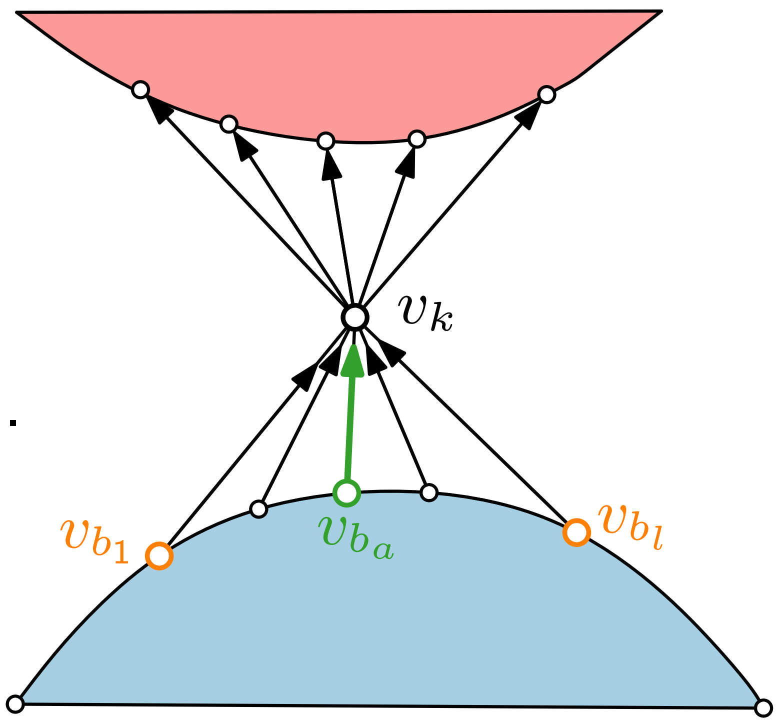

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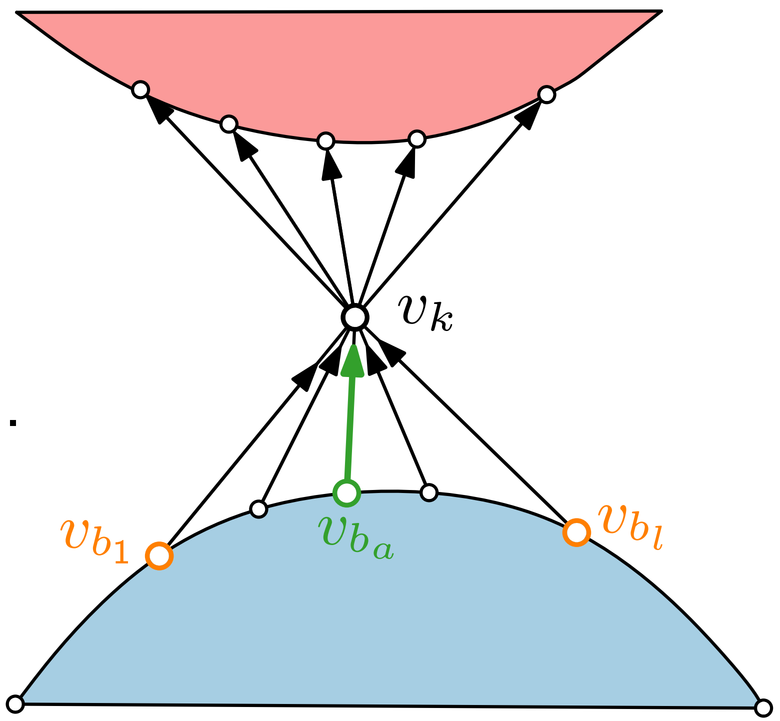

Refined Canonical Order → REL

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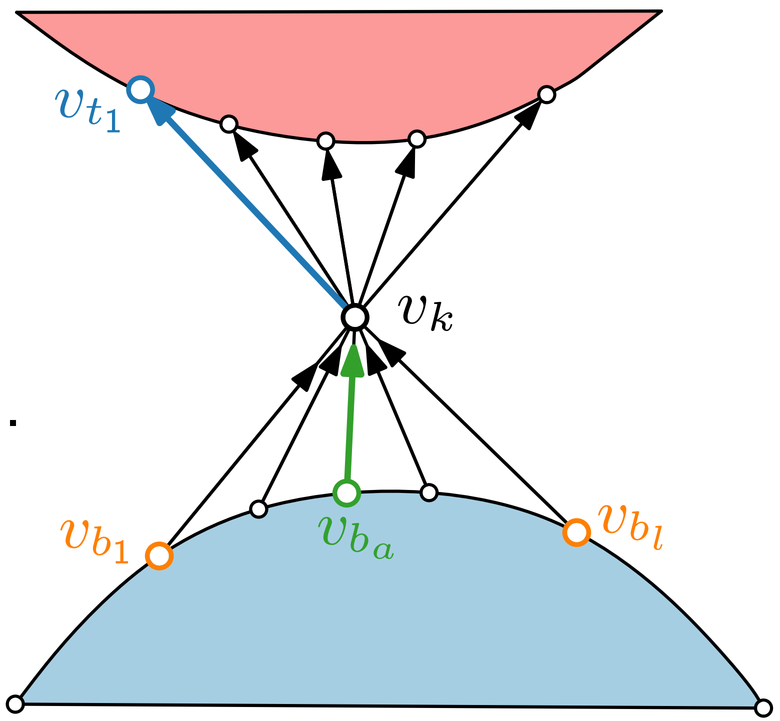

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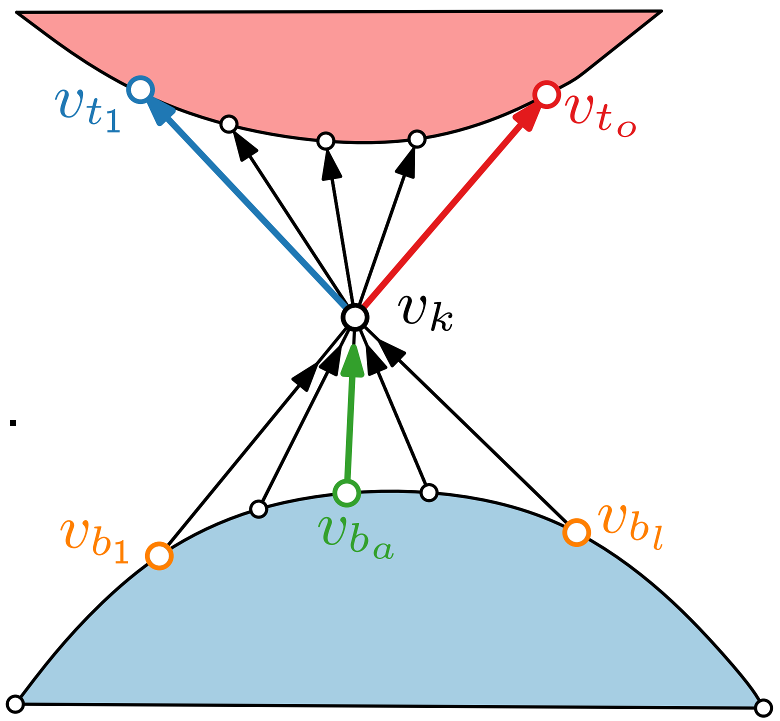

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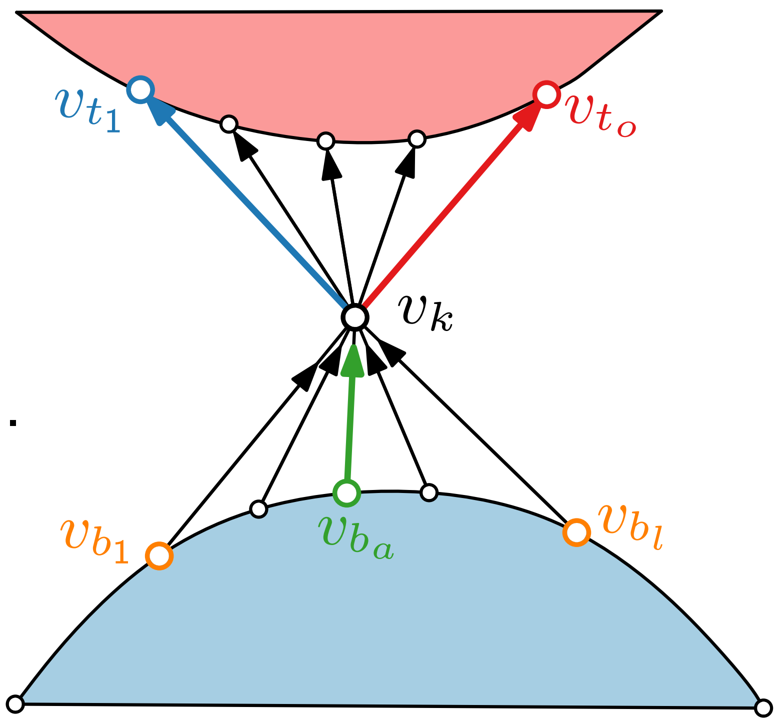

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A left edge or right edge cannot be a base edge.

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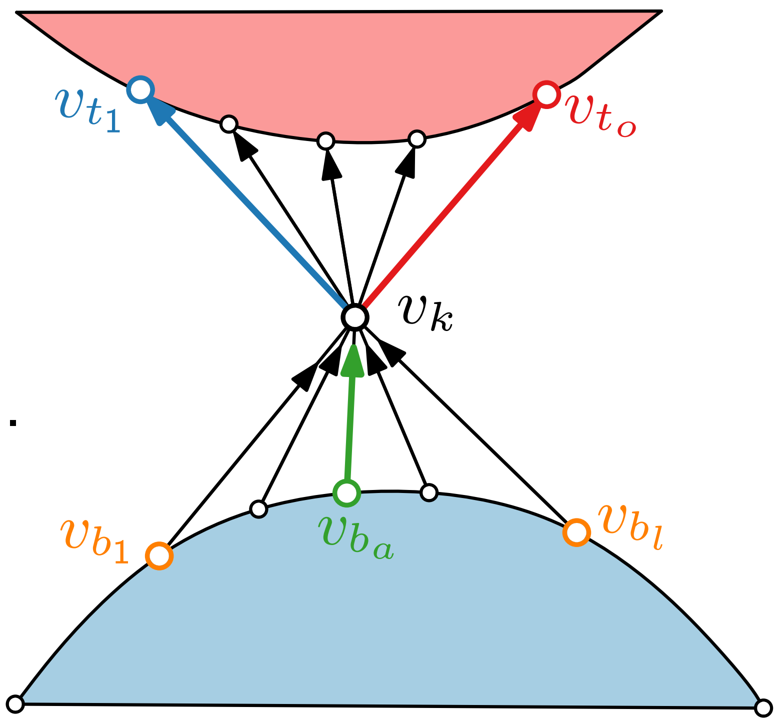

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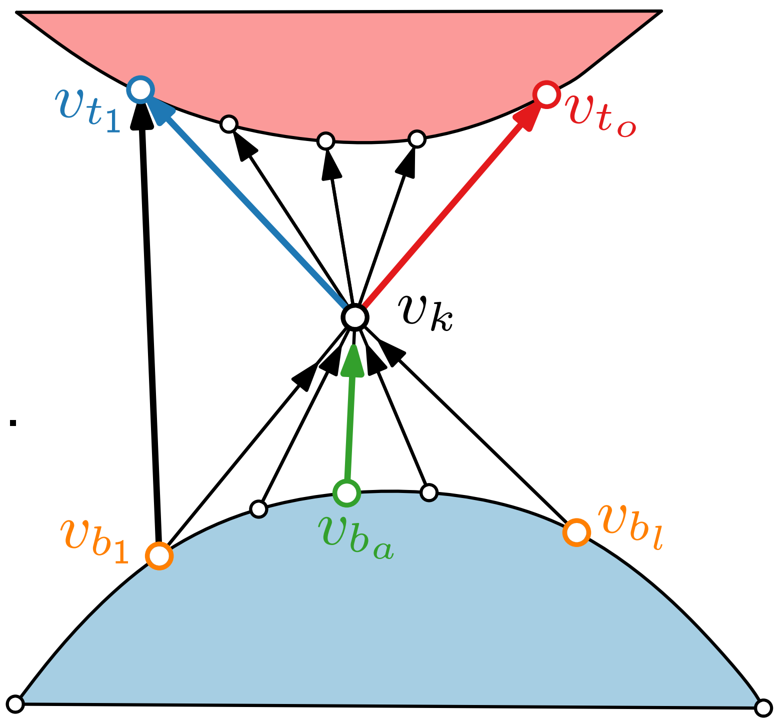

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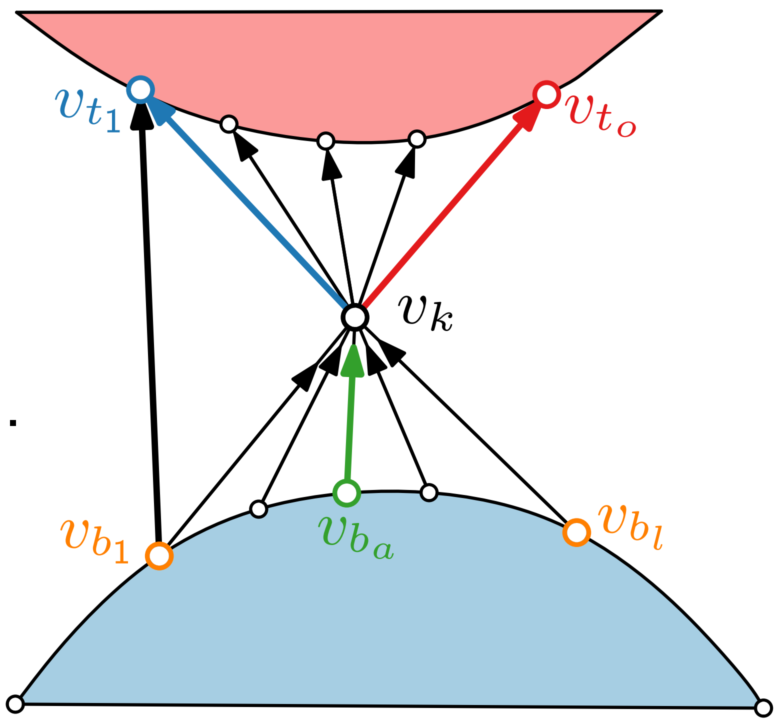

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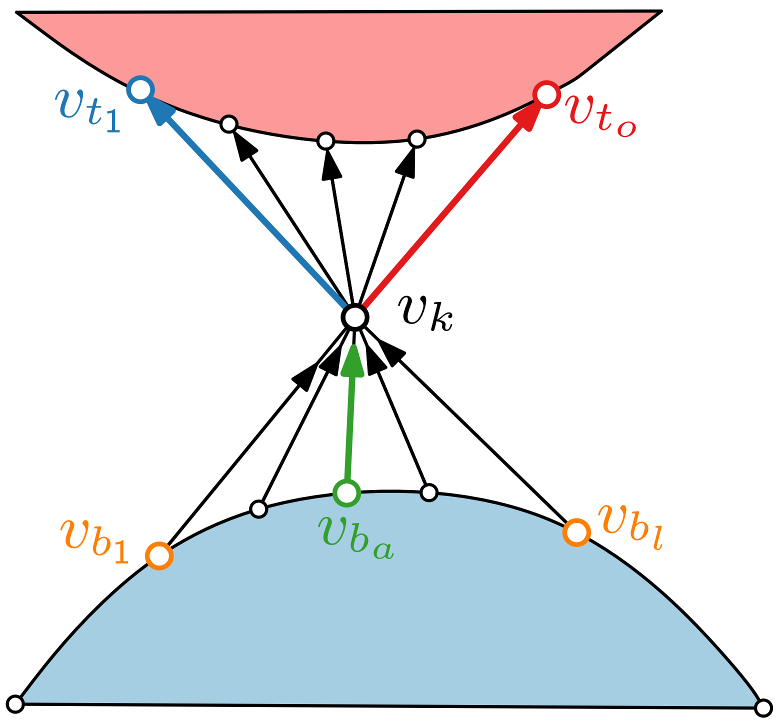
Contradiction since $k > b_1$.

Refined Canonical Order → REL

Lemma 2.
Every edge is either a left edge, a right edge or a base edge.



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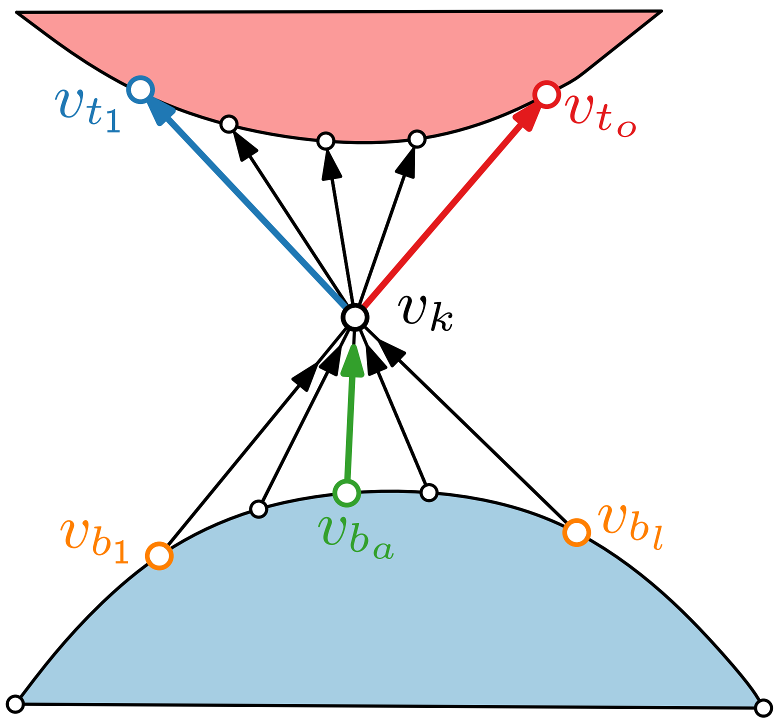

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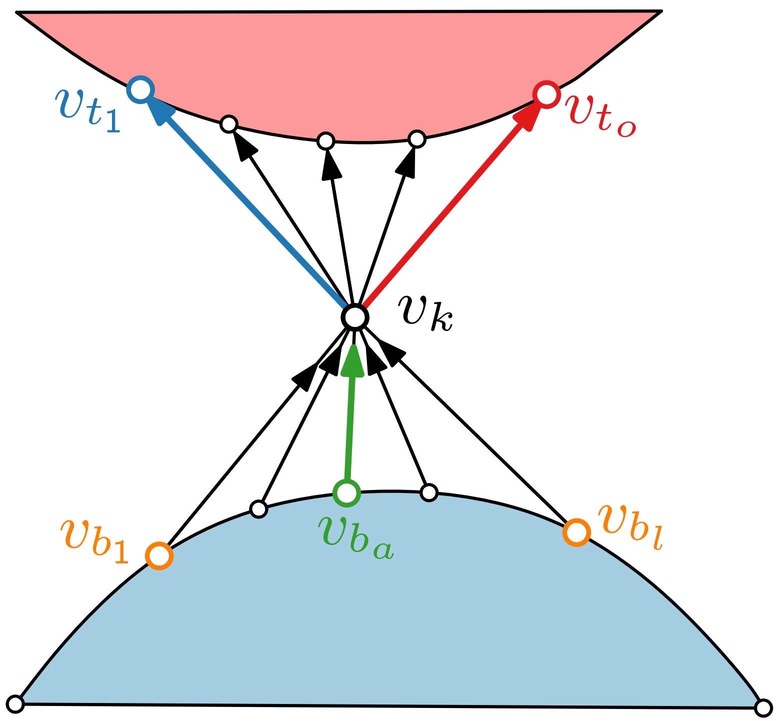

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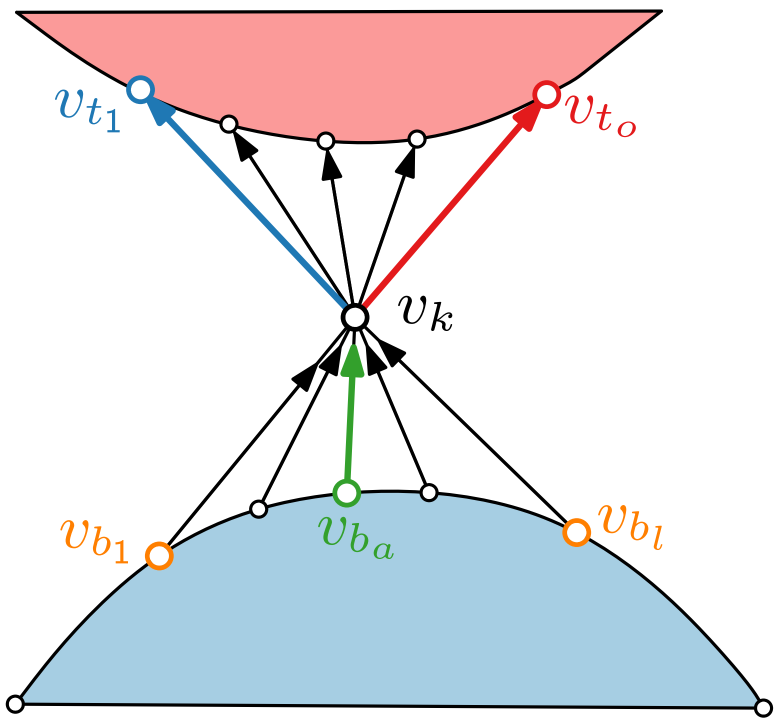

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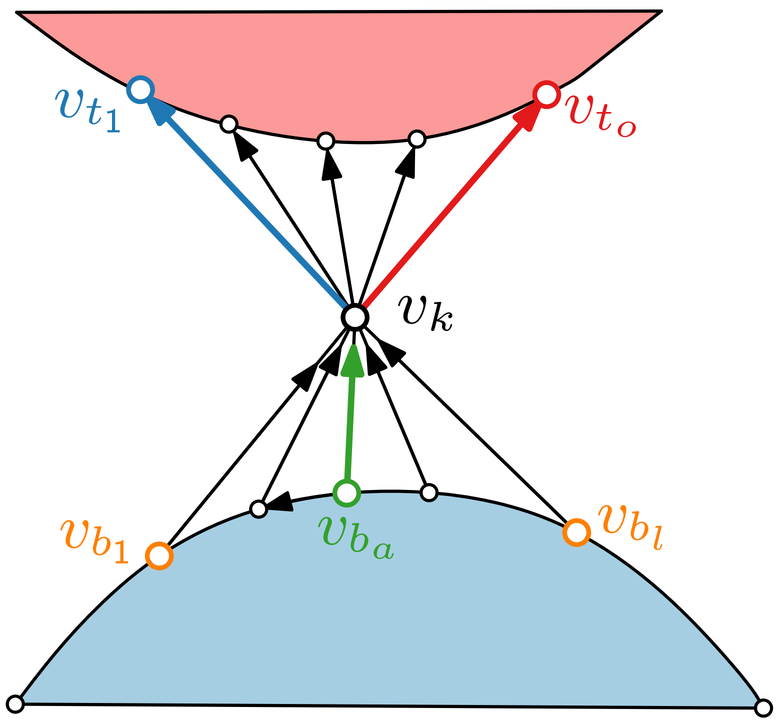

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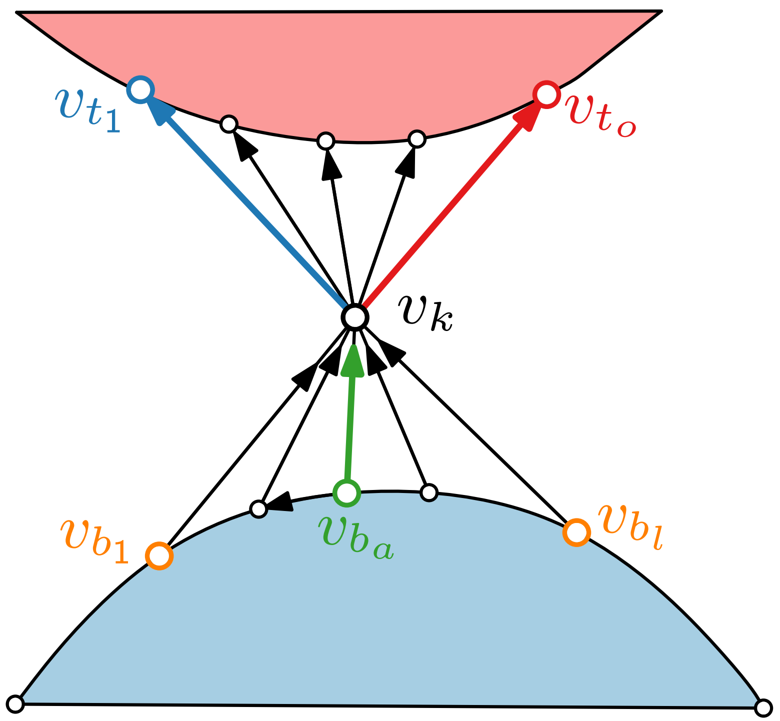

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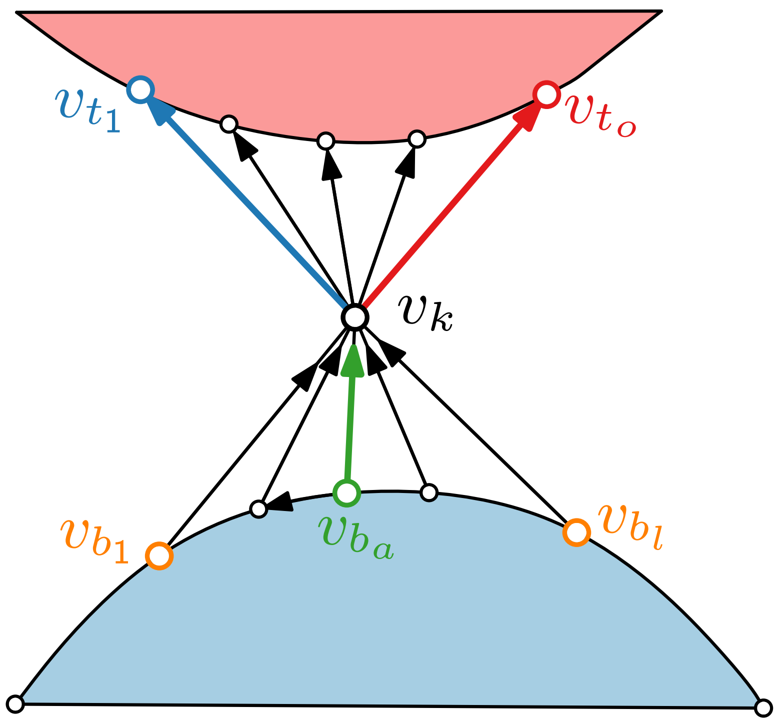

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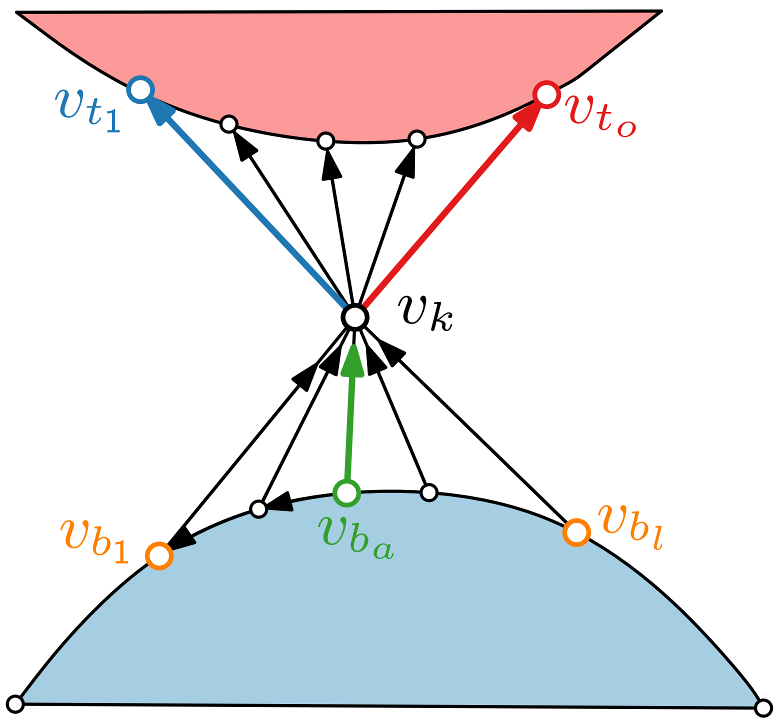

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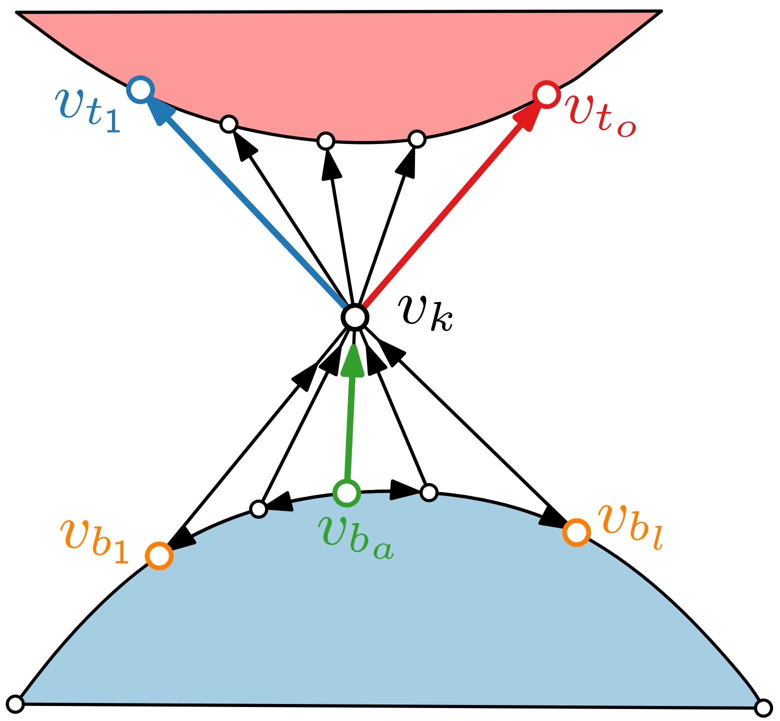

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- Analogously, v_{b_i} is the left point of $v_{b_{i+1}}$ for $i \geq a$.

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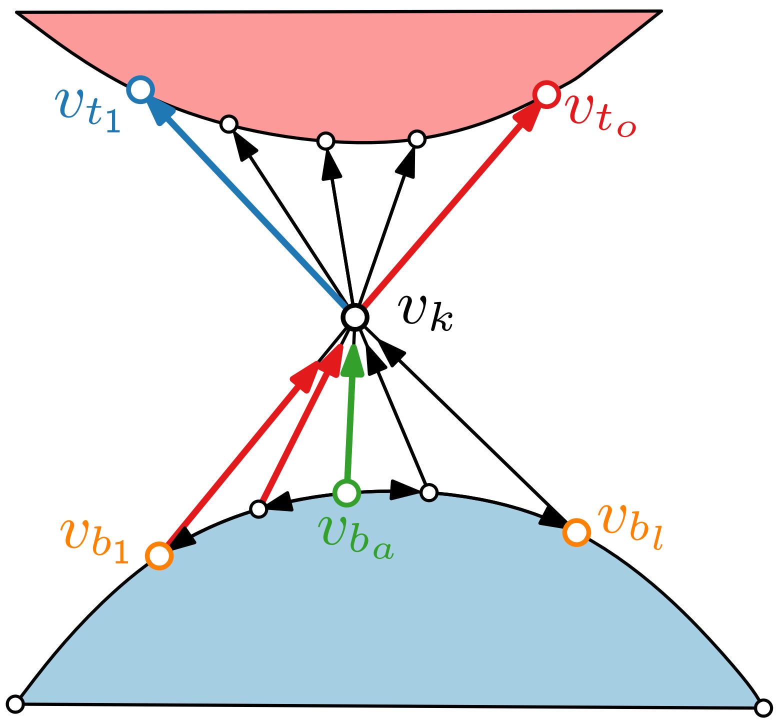

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  - For $1 \leq i < a-1$, it is $v_{b_{i-1}}$. Thus, v_{b_i} is the right point of $v_{b_{i-1}}$.
- Analogously, v_{b_i} is the left point of $v_{b_{i+1}}$ for $i \geq a$.
- Edges (v_{b_i}, v_k), $1 \leq i < a-1$, are right edges.

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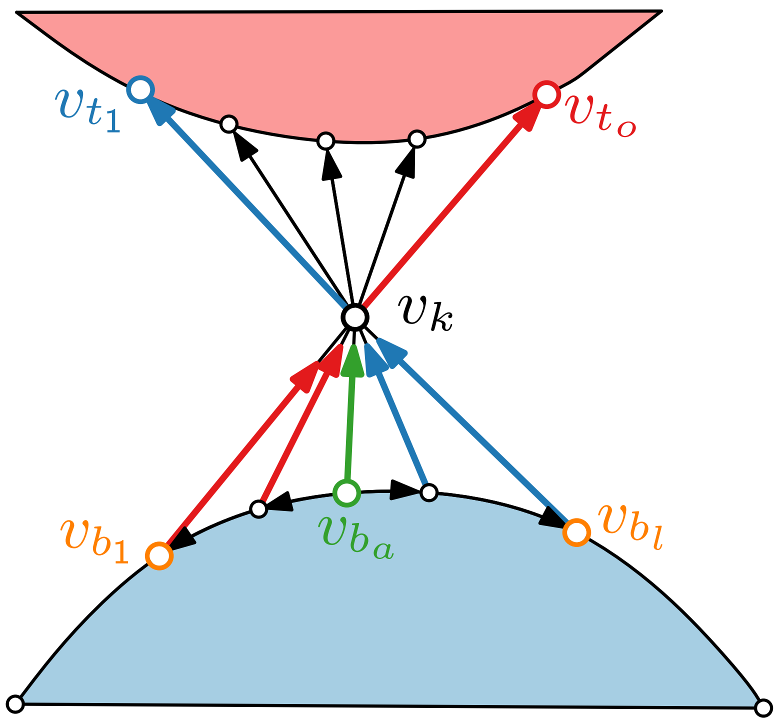

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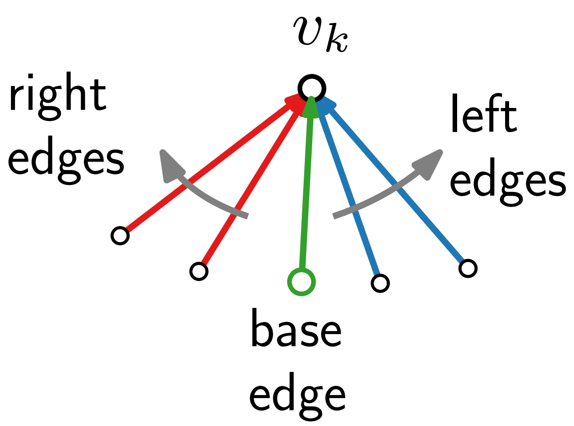
- Similarly, (v_{b_i}, v_k), for $a+1 \leq i \leq l$, are left edges.

Refined Canonical Order → REL



Refined Canonical Order → REL

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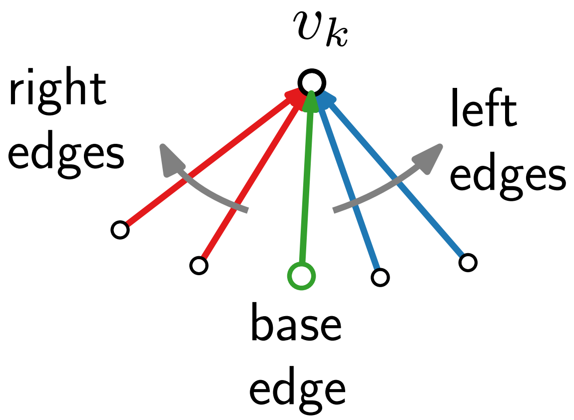

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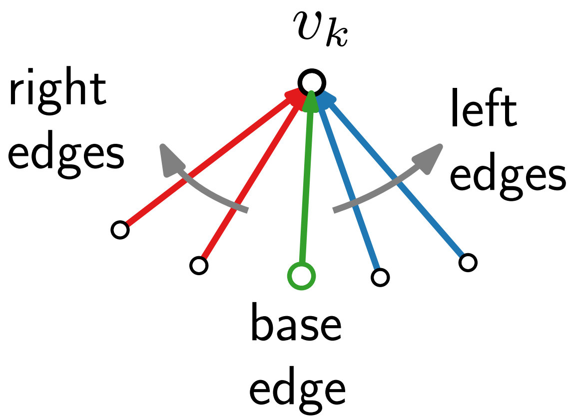

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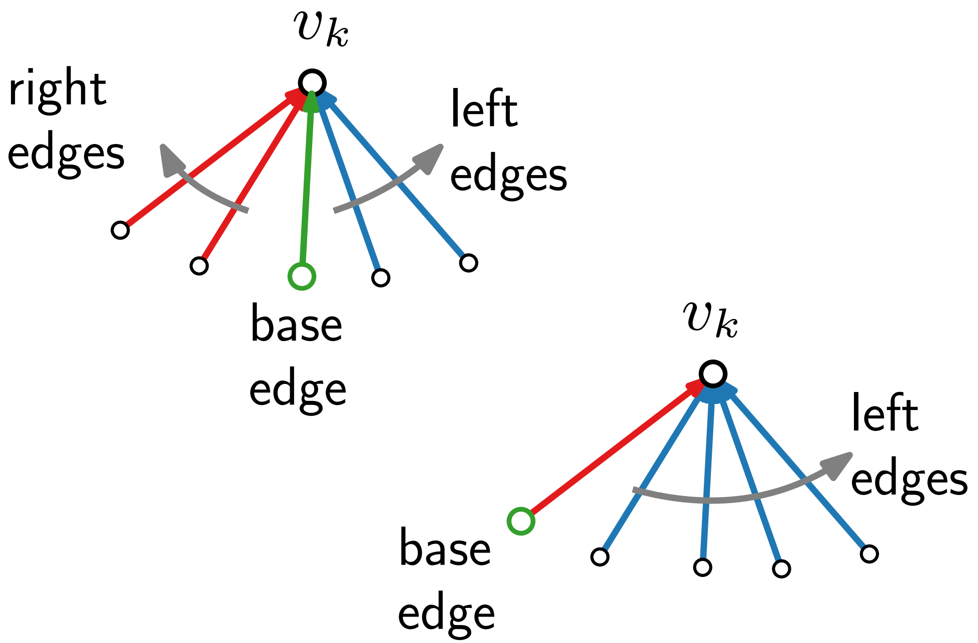

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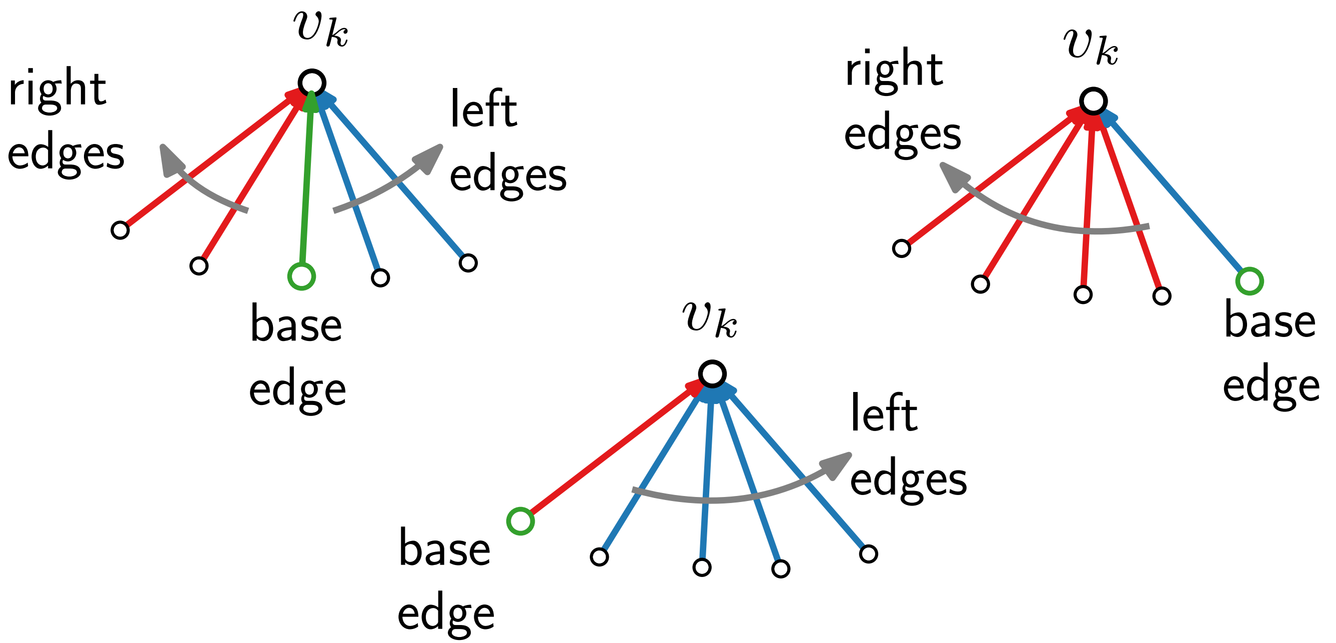

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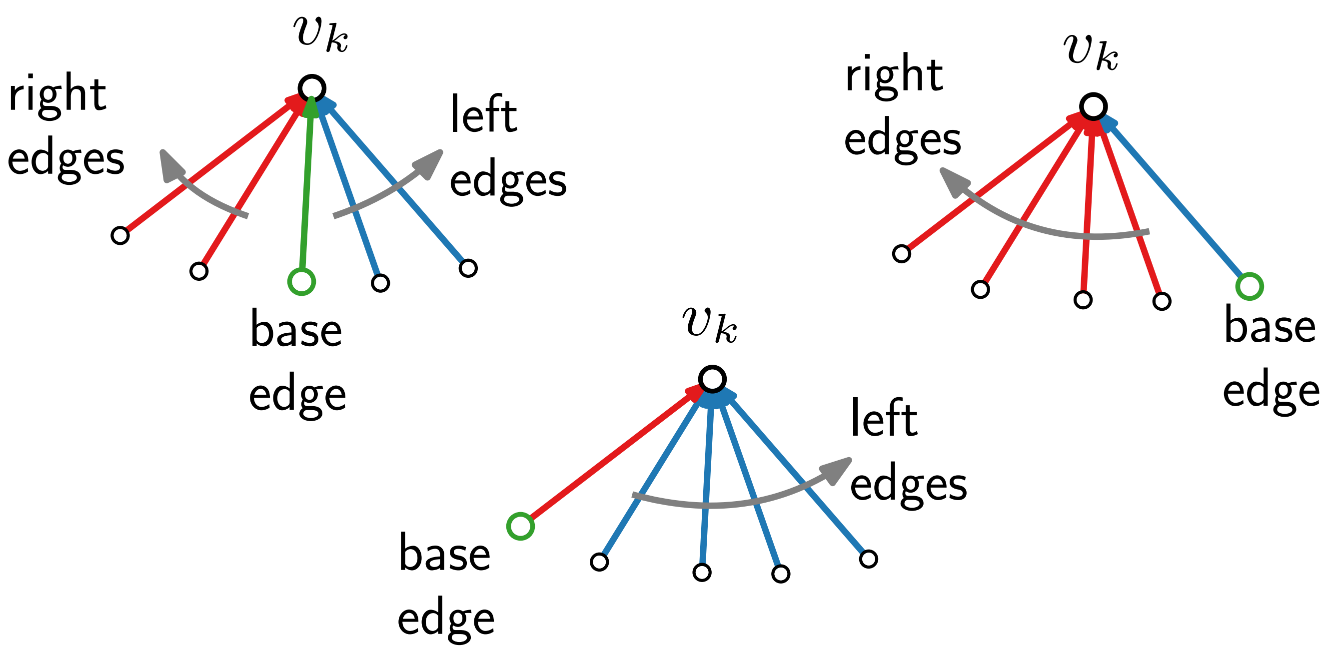

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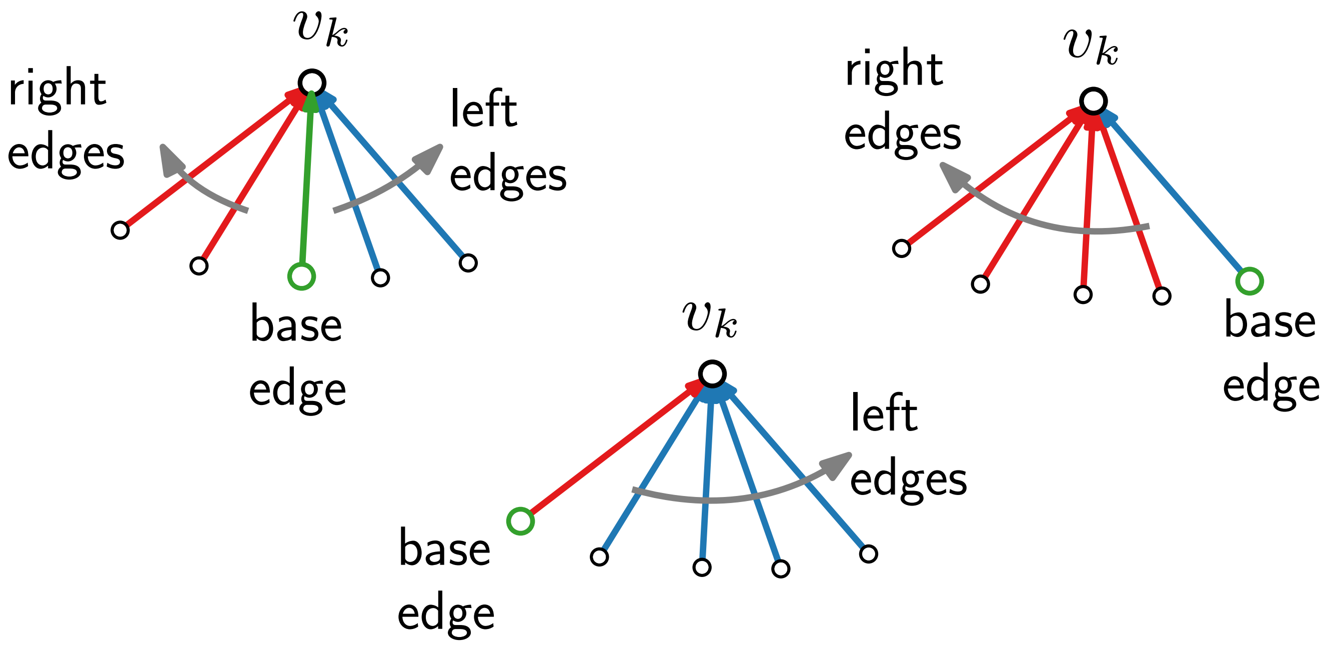

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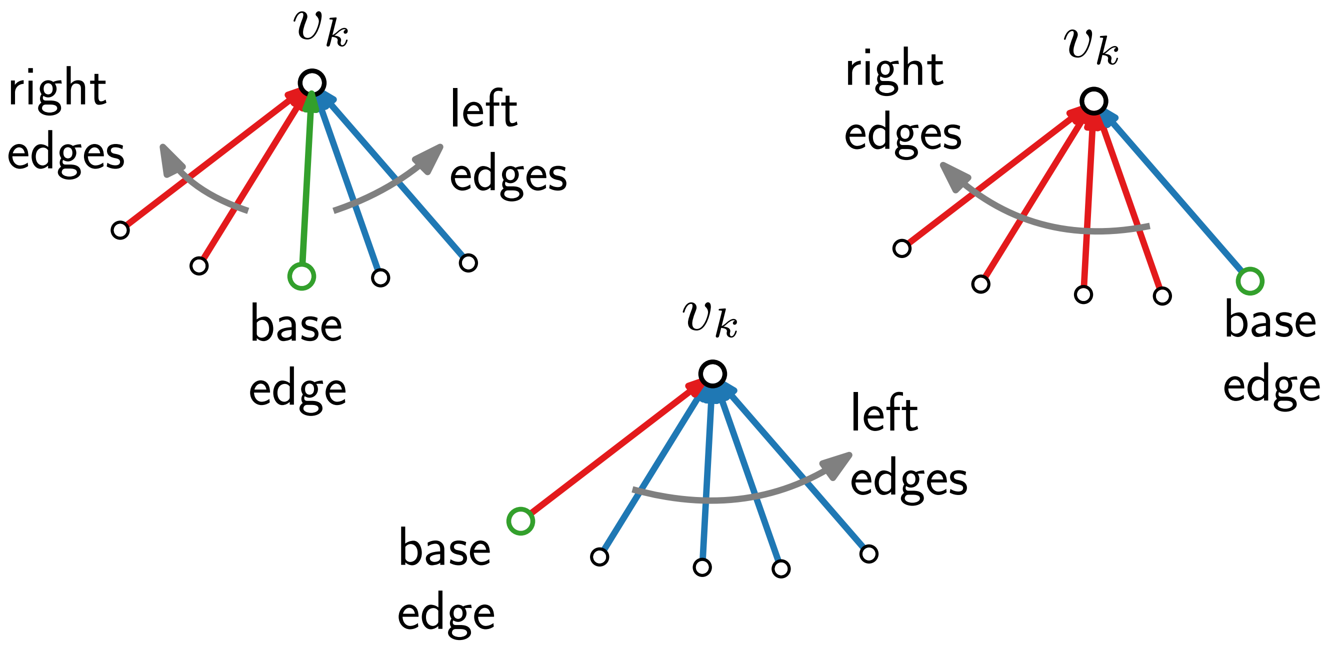

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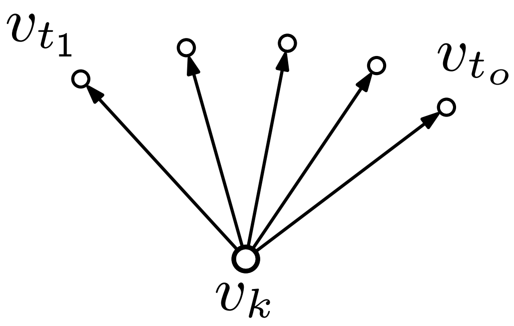
$t_o \geq 2$



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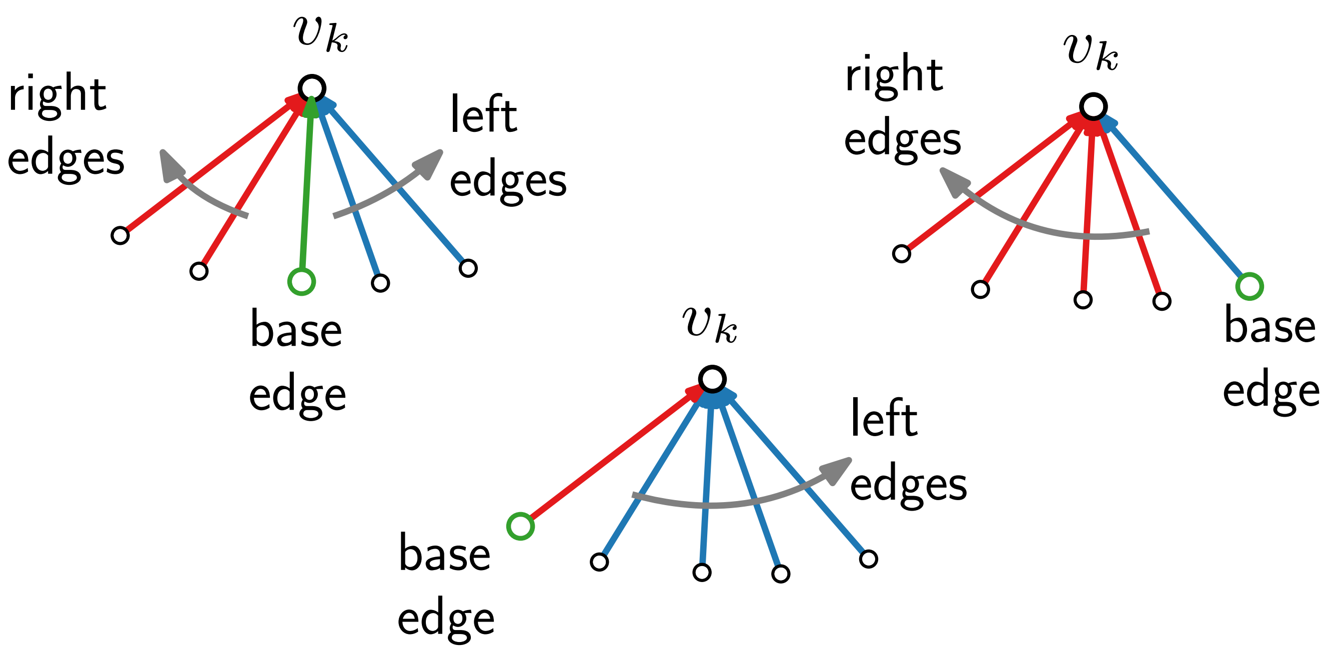

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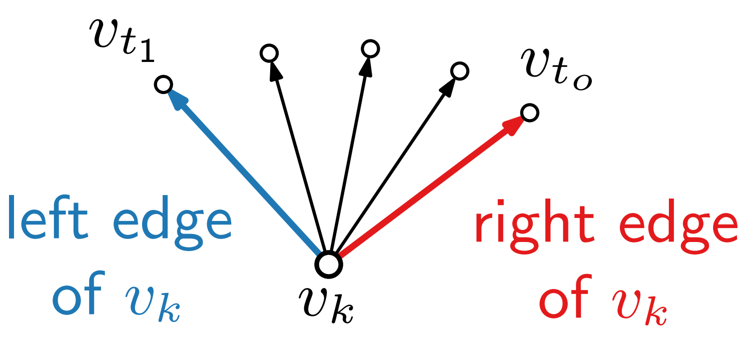

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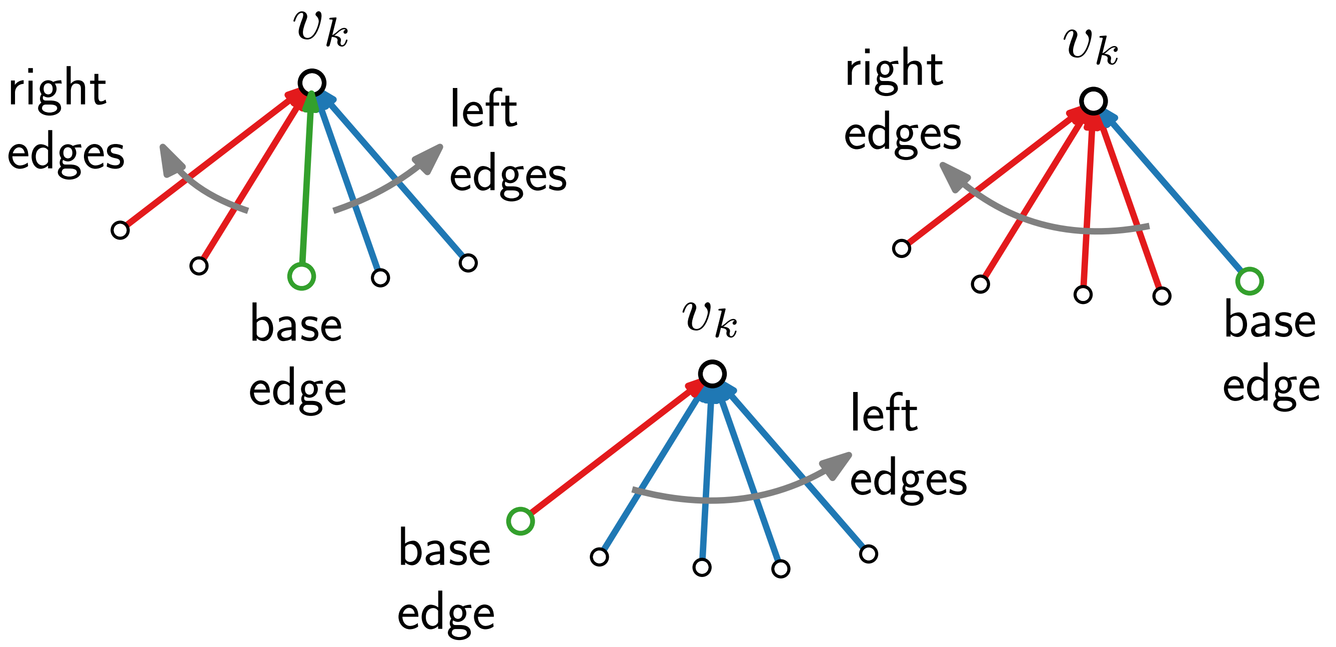

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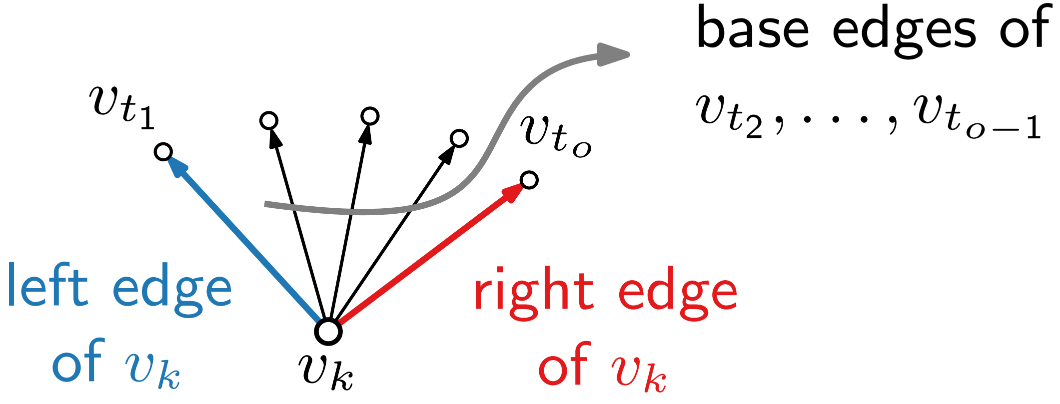

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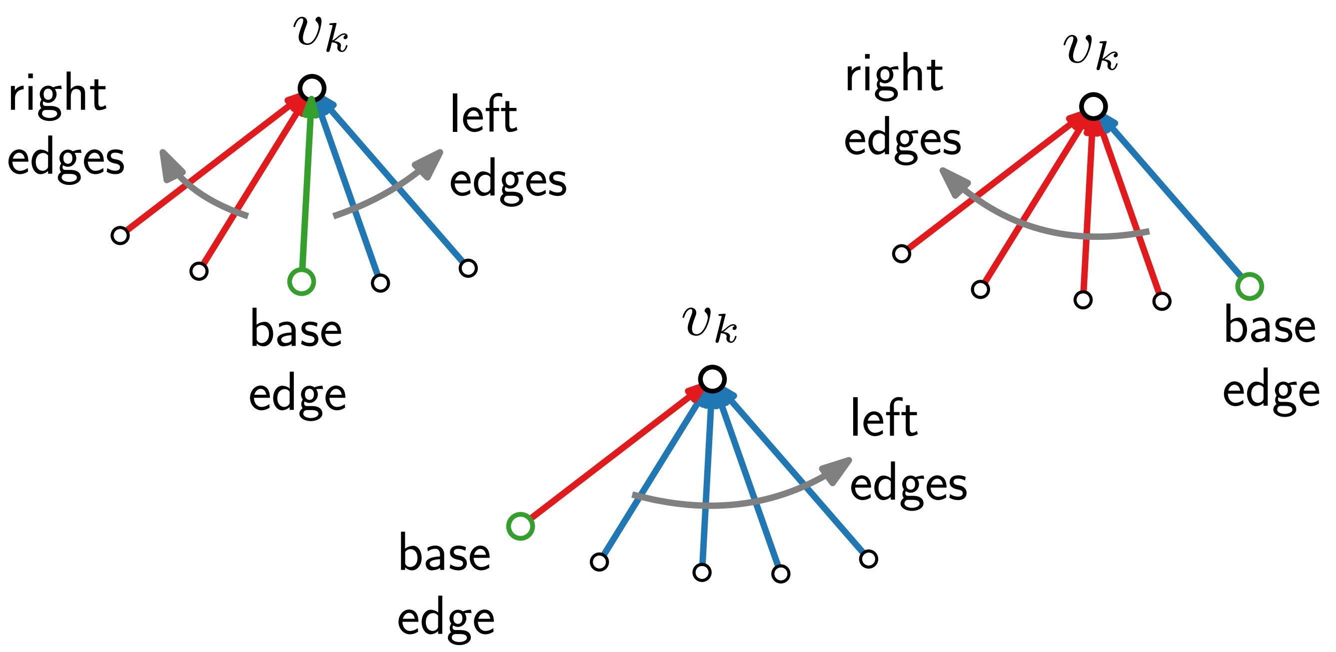

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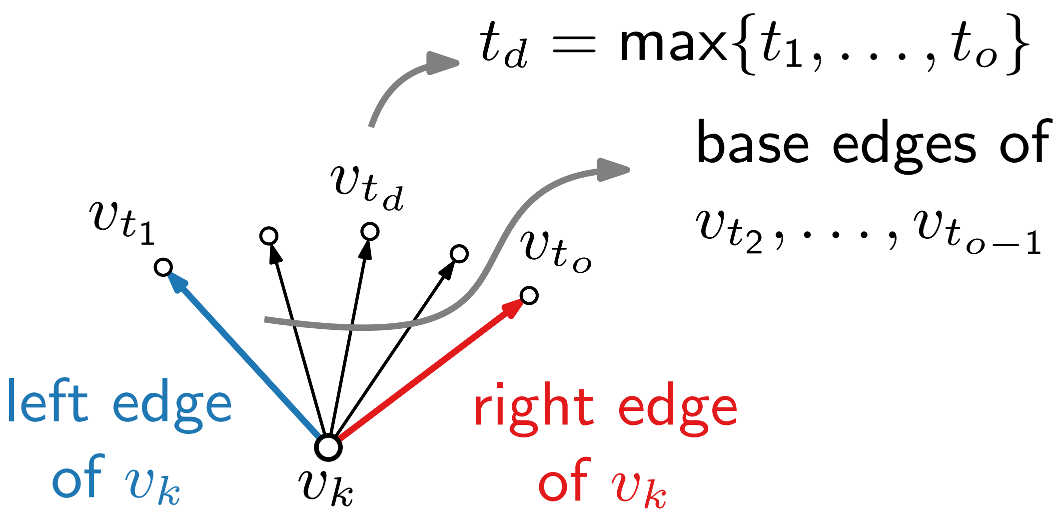

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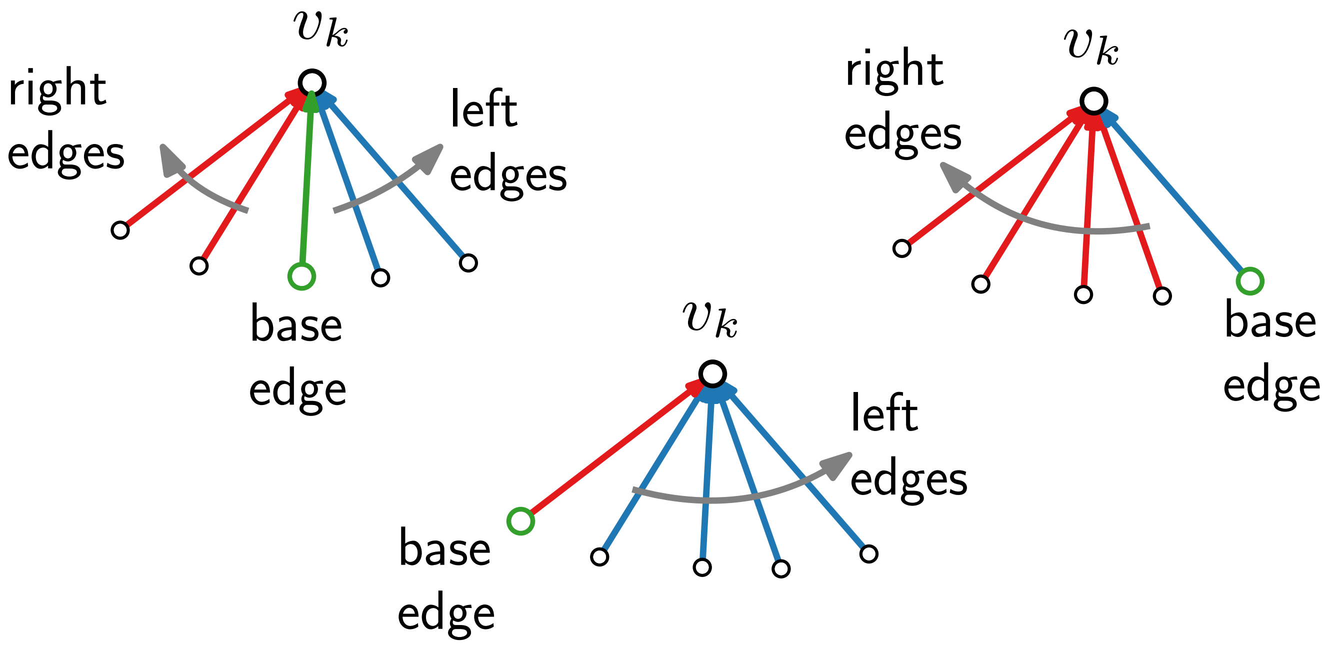

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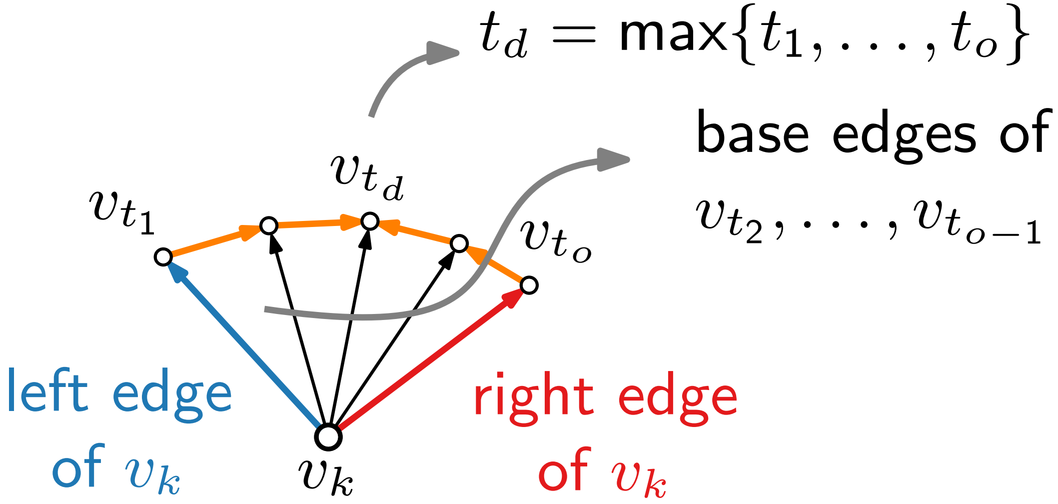

- $t_1 < t_2 < \ldots < t_d$ and $t_d > t_{d+1} > \ldots > t_o$

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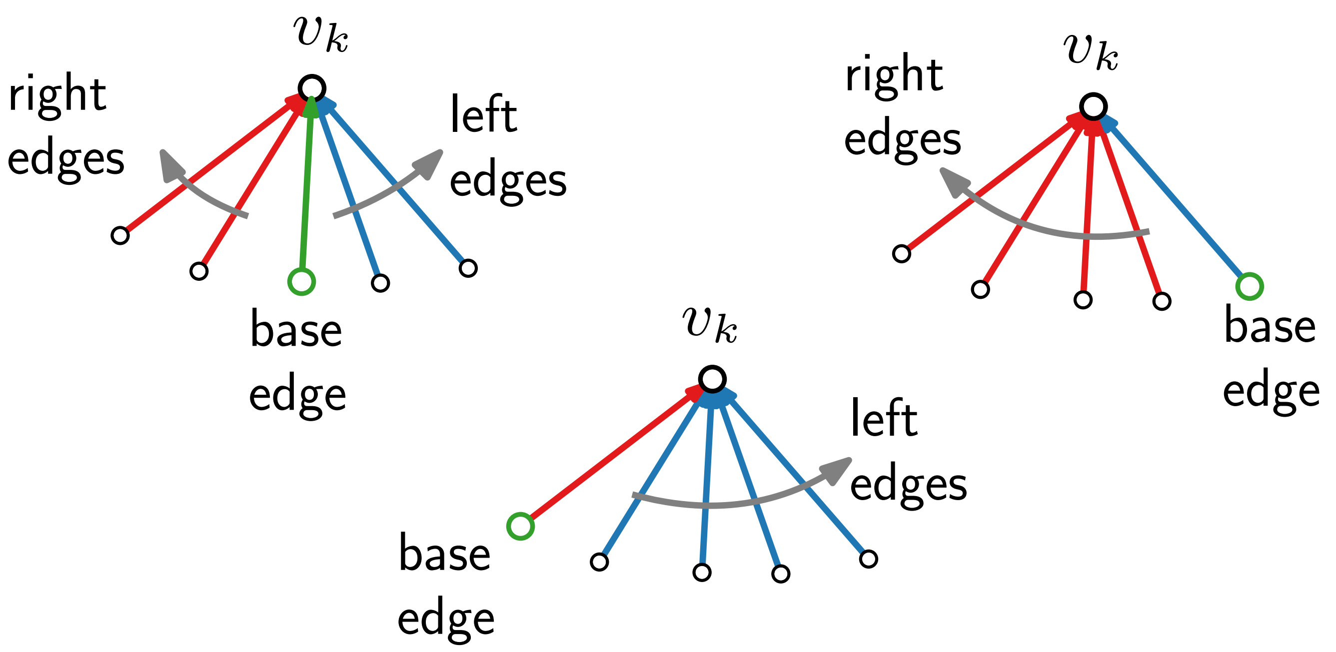
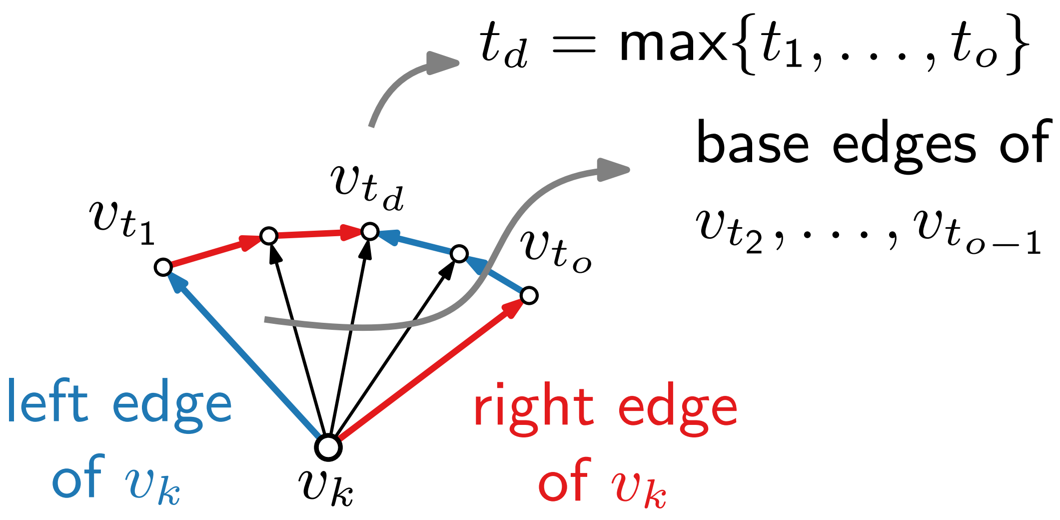


Let T_r be the red edges and T_b the blue edges.

Lemma 3.
$\{T_r, T_b\}$ is a regular edge labeling.

Proof.

$t_o \geq 2$



- $t_1 < t_2 < \ldots < t_d$ and $t_d > t_{d+1} > \ldots > t_o$

Refined Canonical Order → REL

Coloring.

- Color right (left) edges in red (blue).
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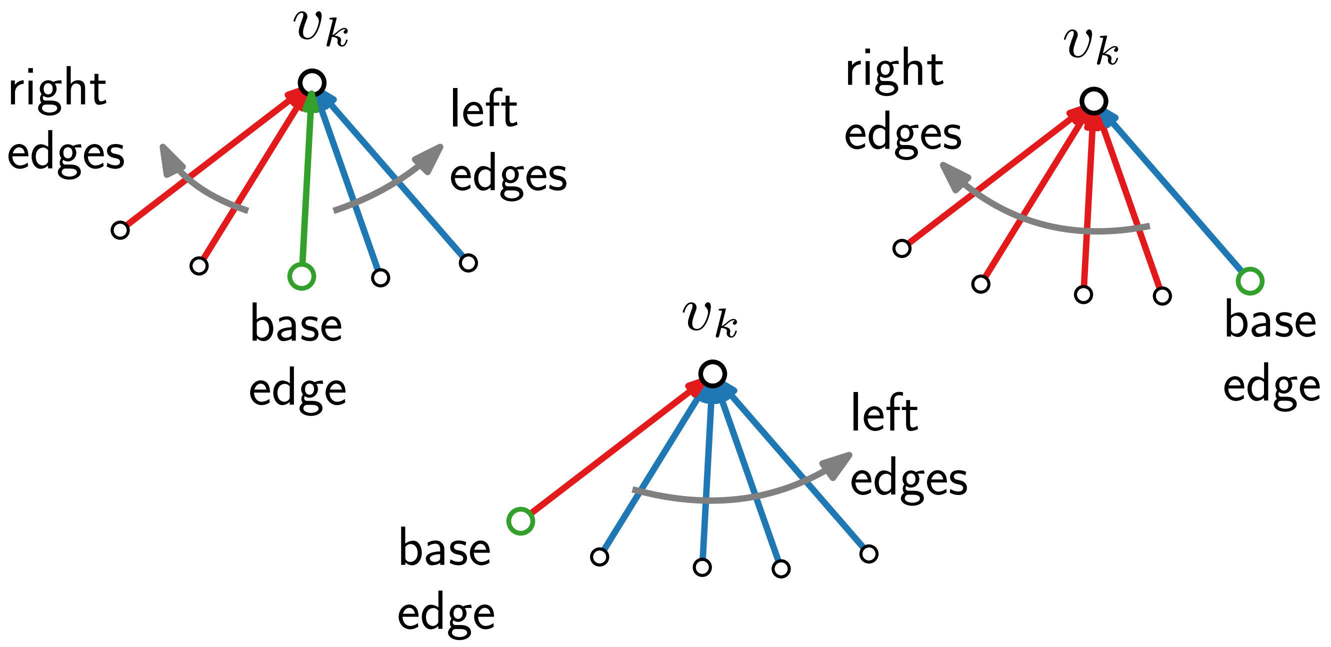

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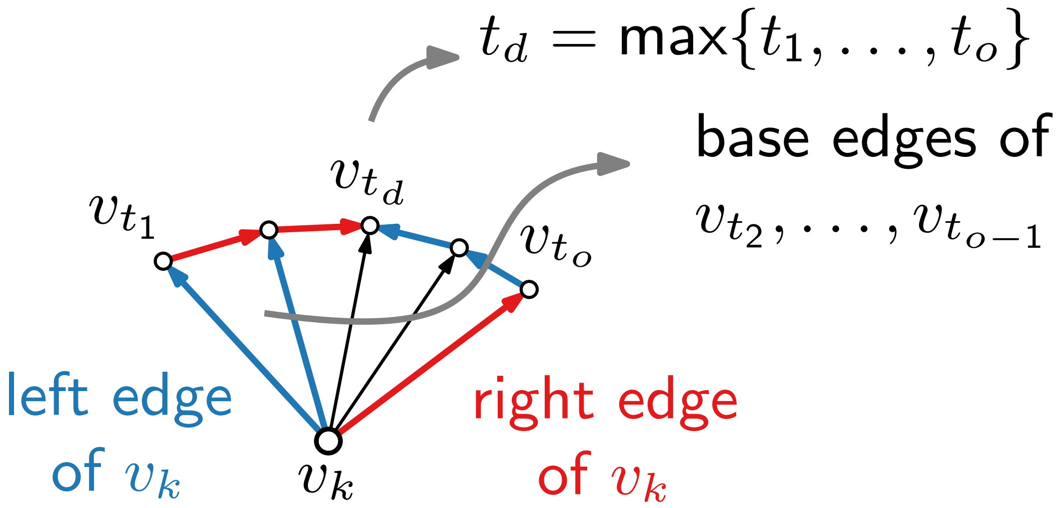

- $t_1 < t_2 < \ldots < t_d$ and $t_d > t_{d+1} > \ldots > t_o$
- (v_k, v_{t_i}), $2 \leq i \leq d-1$ are blue

Refined Canonical Order → REL

Coloring.

- Color right (left) edges in red (blue).
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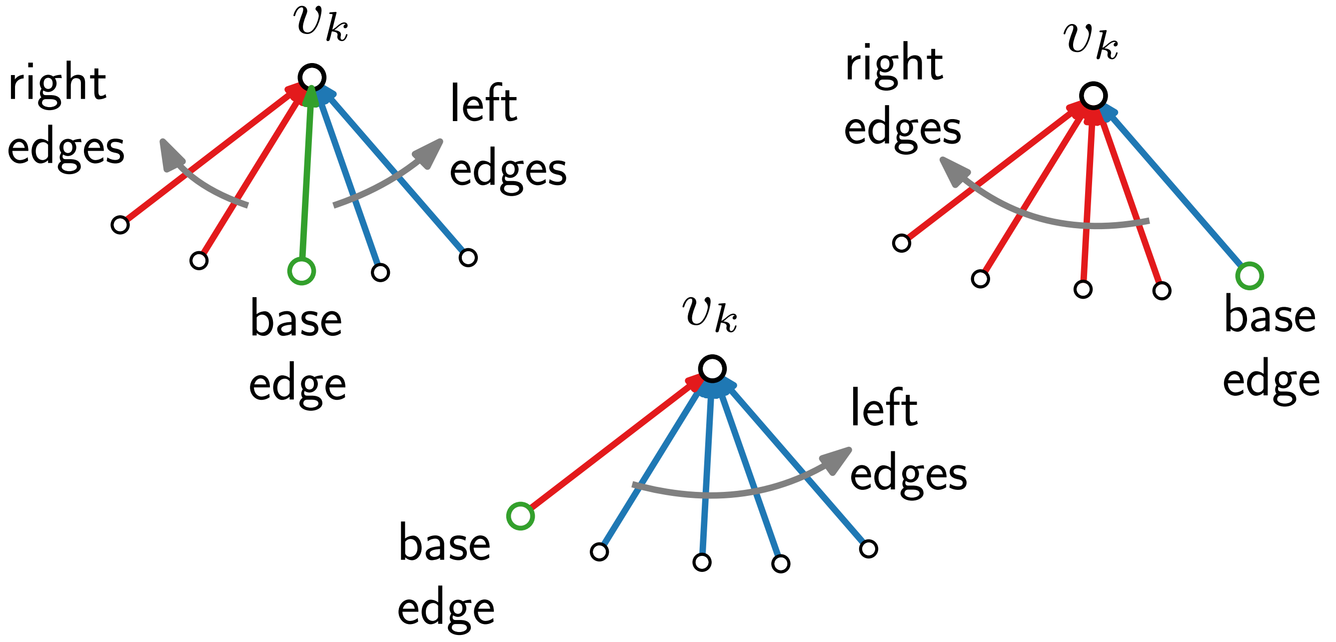

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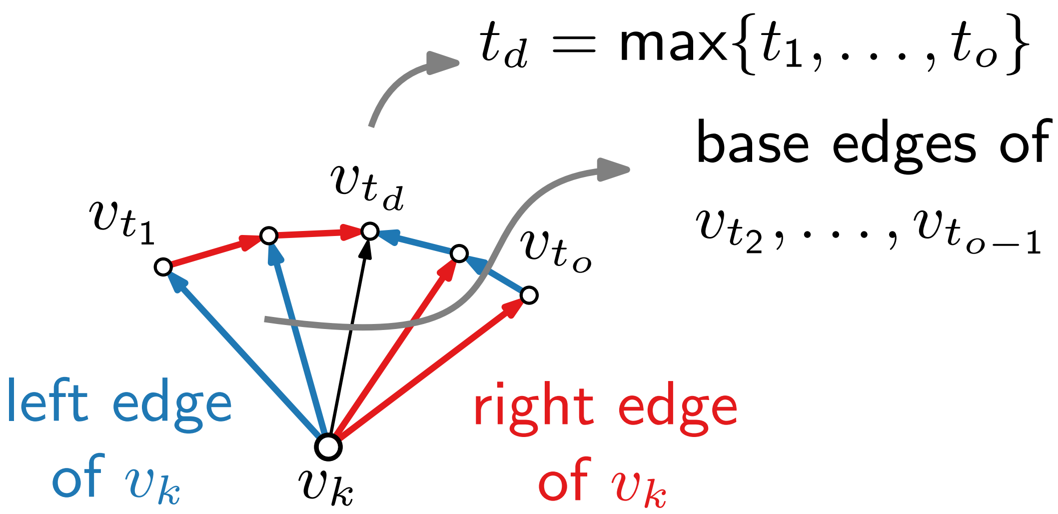

- $t_1 < t_2 < \ldots < t_d$ and $t_d > t_{d+1} > \ldots > t_o$
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- (v_k, v_{t_i}), $d+1 \leq i \leq o-1$ are red

Refined Canonical Order → REL

Coloring.

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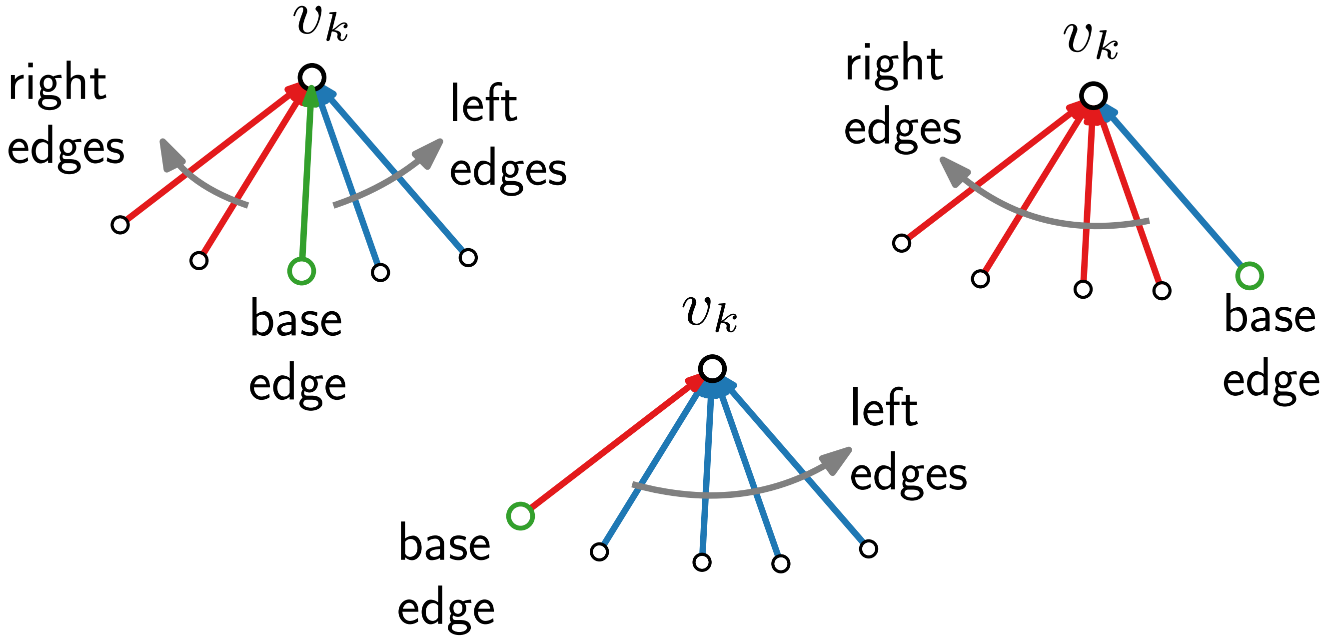
- Color a base edge (v_{b_i}, v_k) red if $i = 1$ and blue if $i = l$ and otherwise arbitrarily.



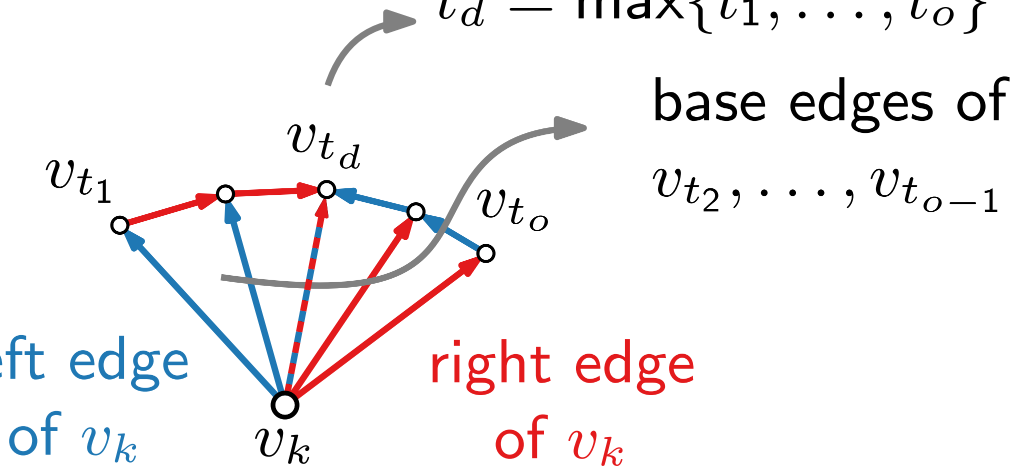
Let T_r be the red edges and T_b the blue edges.

Lemma 3.
$\{T_r, T_b\}$ is a regular edge labeling.

Proof.

$t_o \geq 2$

$t_d = \max\{t_1, \ldots, t_o\}$



- $t_1 < t_2 < \ldots < t_d$ and $t_d > t_{d+1} > \ldots > t_o$
- (v_k, v_{t_i}), $2 \leq i \leq d-1$ are blue
- (v_k, v_{t_i}), $d+1 \leq i \leq o-1$ are red
- (v_k, v_{t_d}) is either red or blue

Refined Canonical Order → REL

Coloring.

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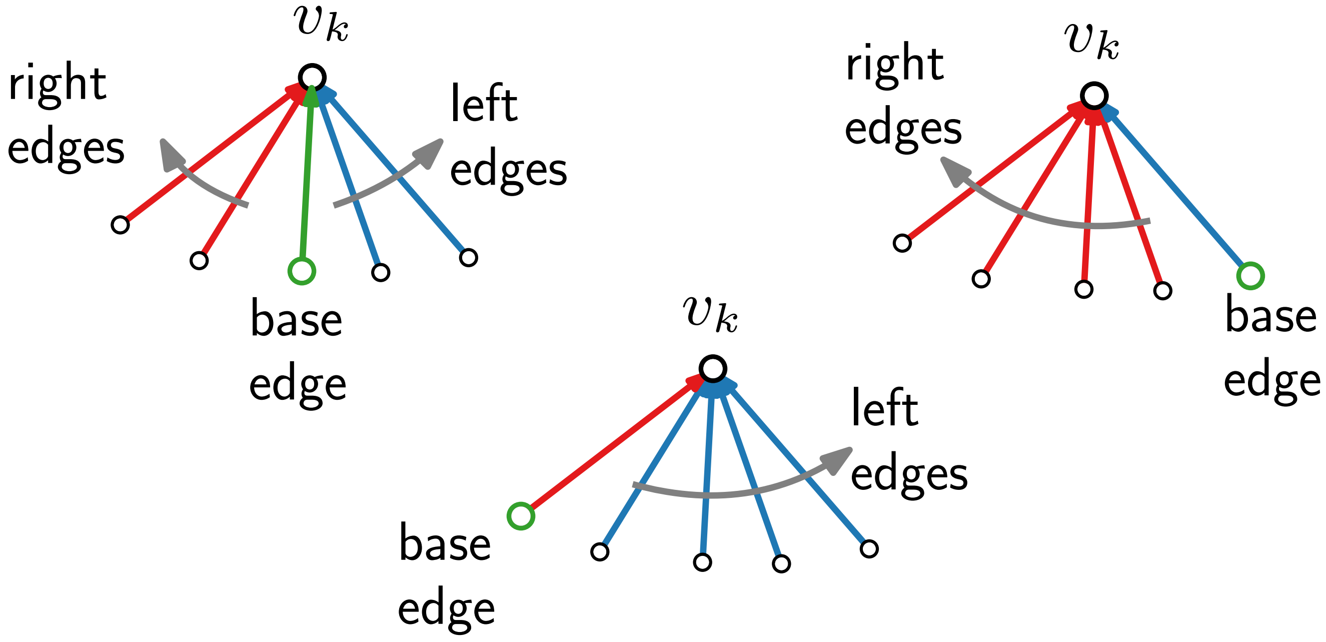

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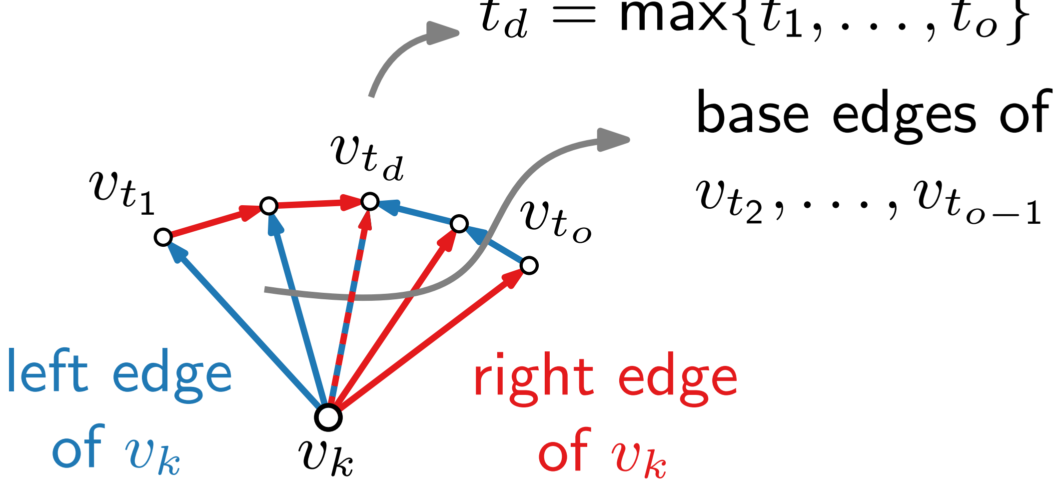

- $t_1 < t_2 < \dots < t_d$ and $t_d > t_{d+1} > \dots > t_o$
- (v_k, v_{t_i}), $2 \leq i \leq d-1$ are blue
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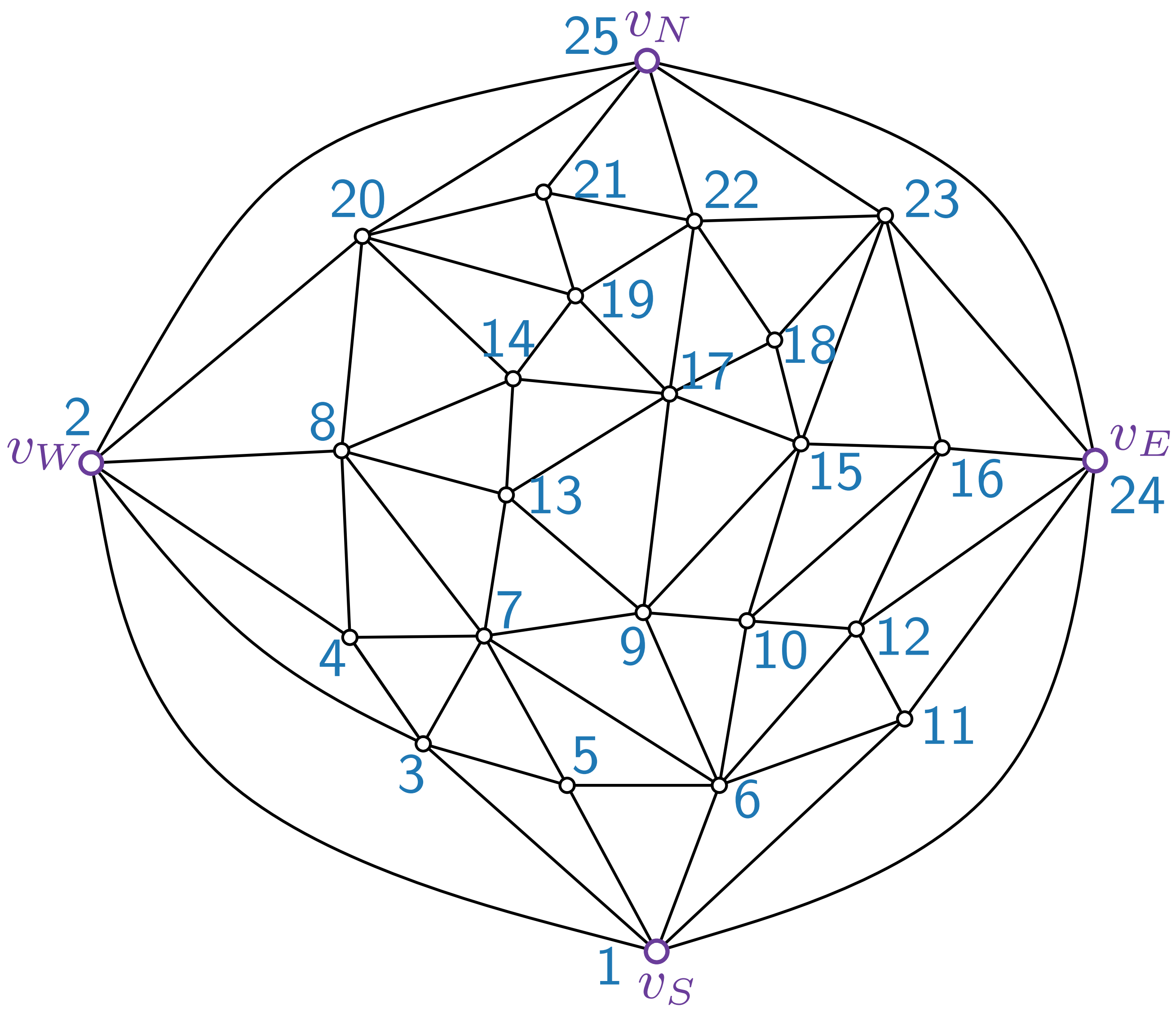
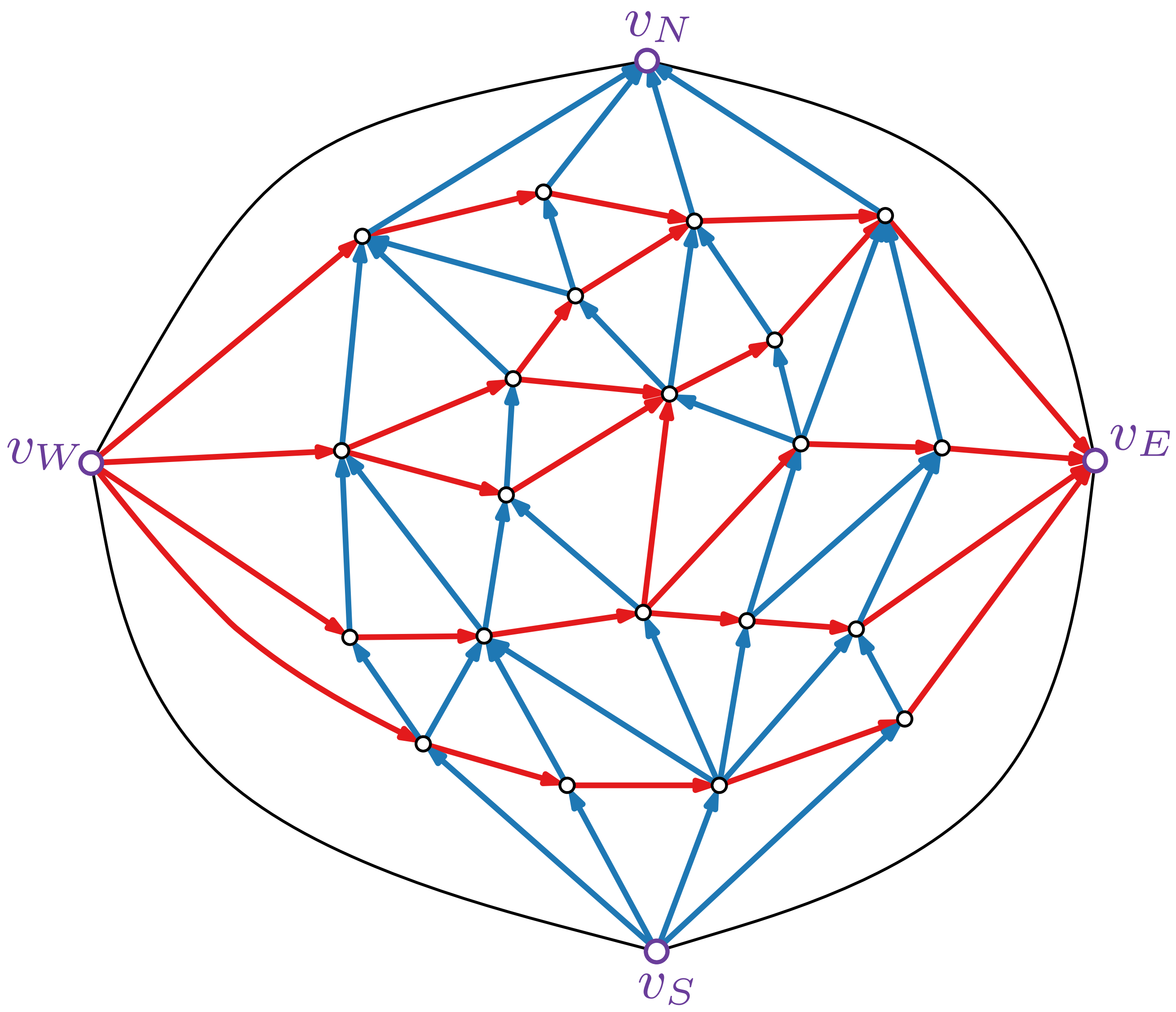
- (v_k, v_{t_d}) is either red or blue

$\Rightarrow$ Circular order of outgoing edges at v_k correct.

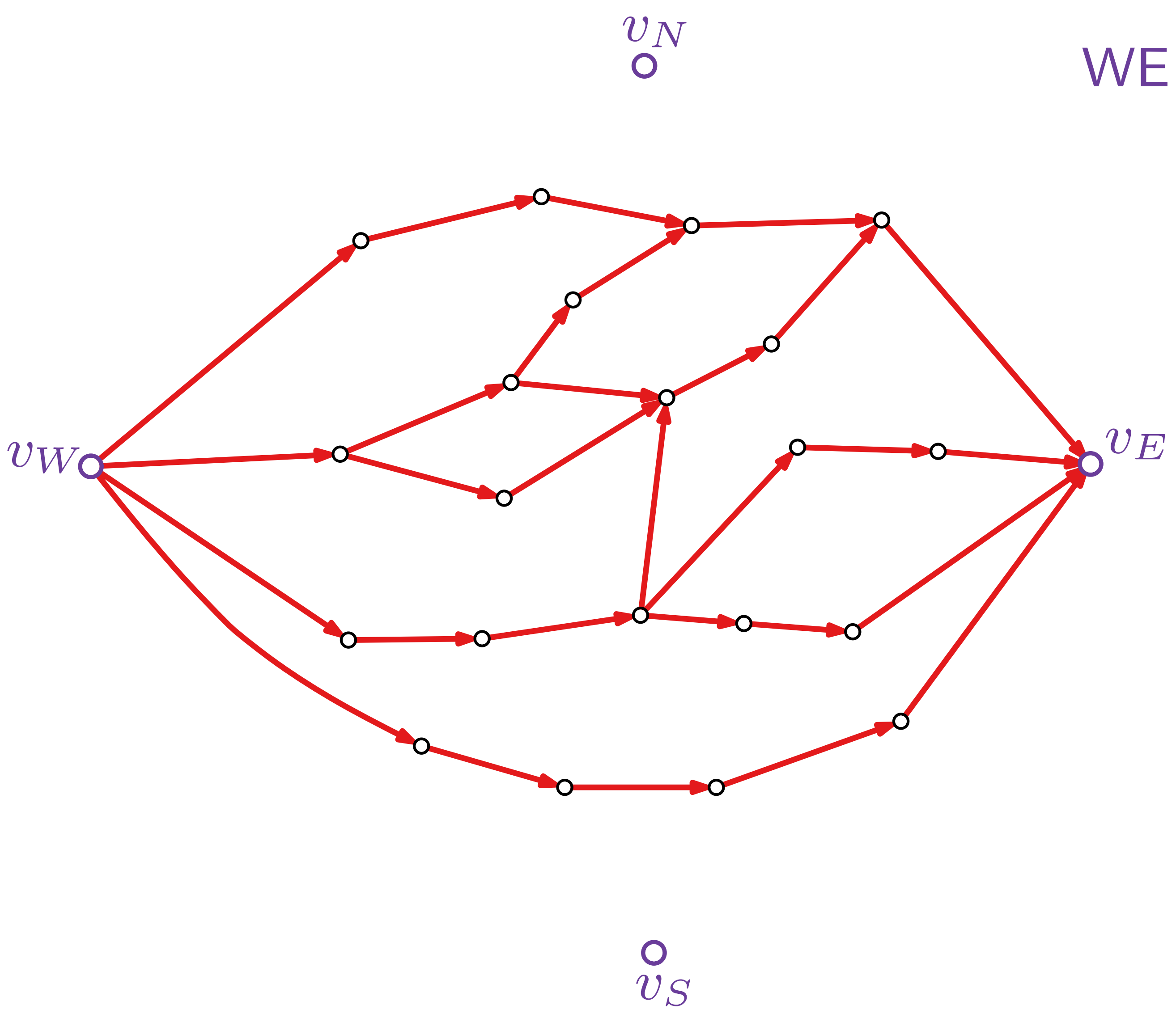
From REL to st-Digraphs to Coordinates



From REL to st-Digraphs to Coordinates

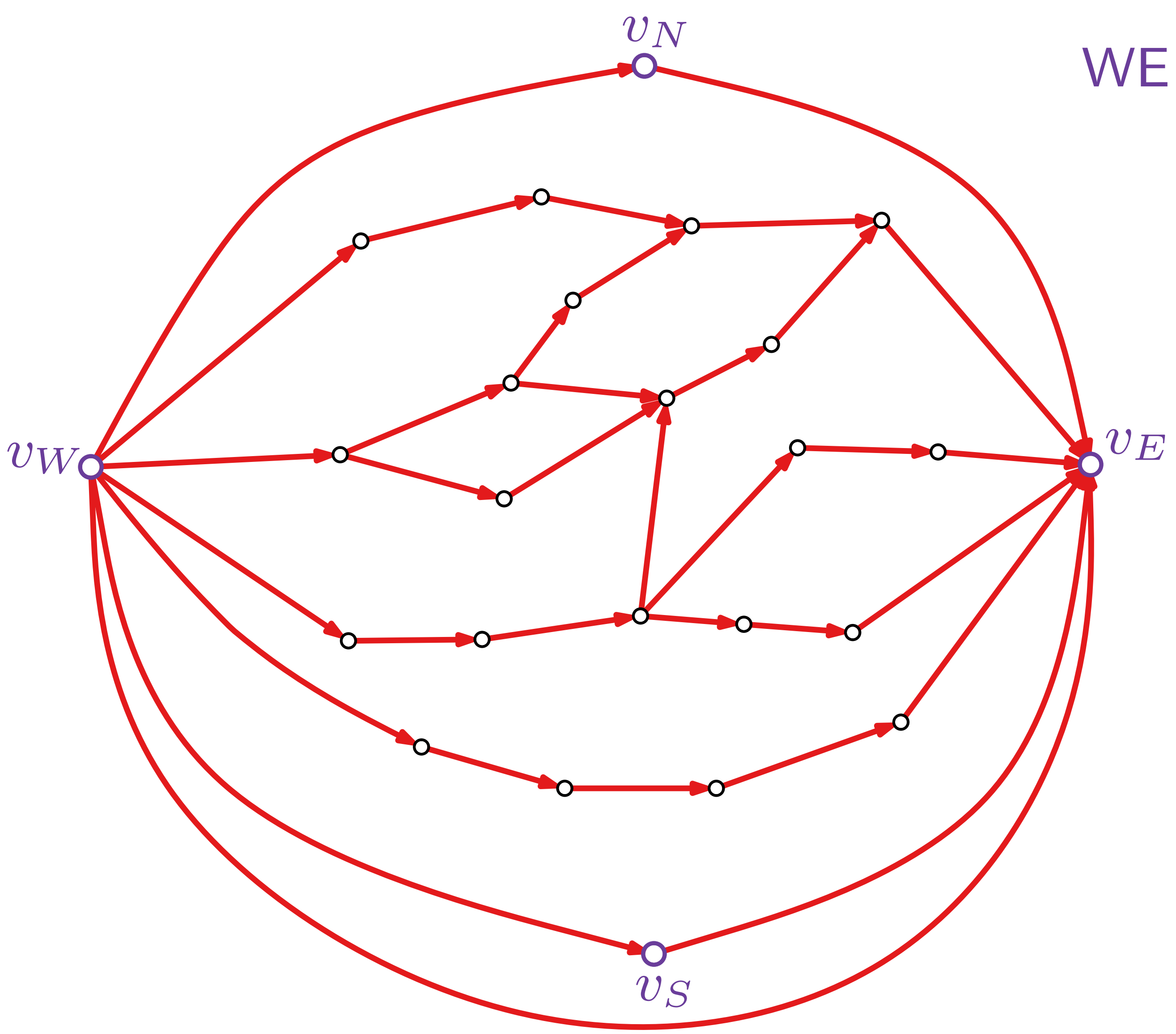


From REL to st-Digraphs to Coordinates



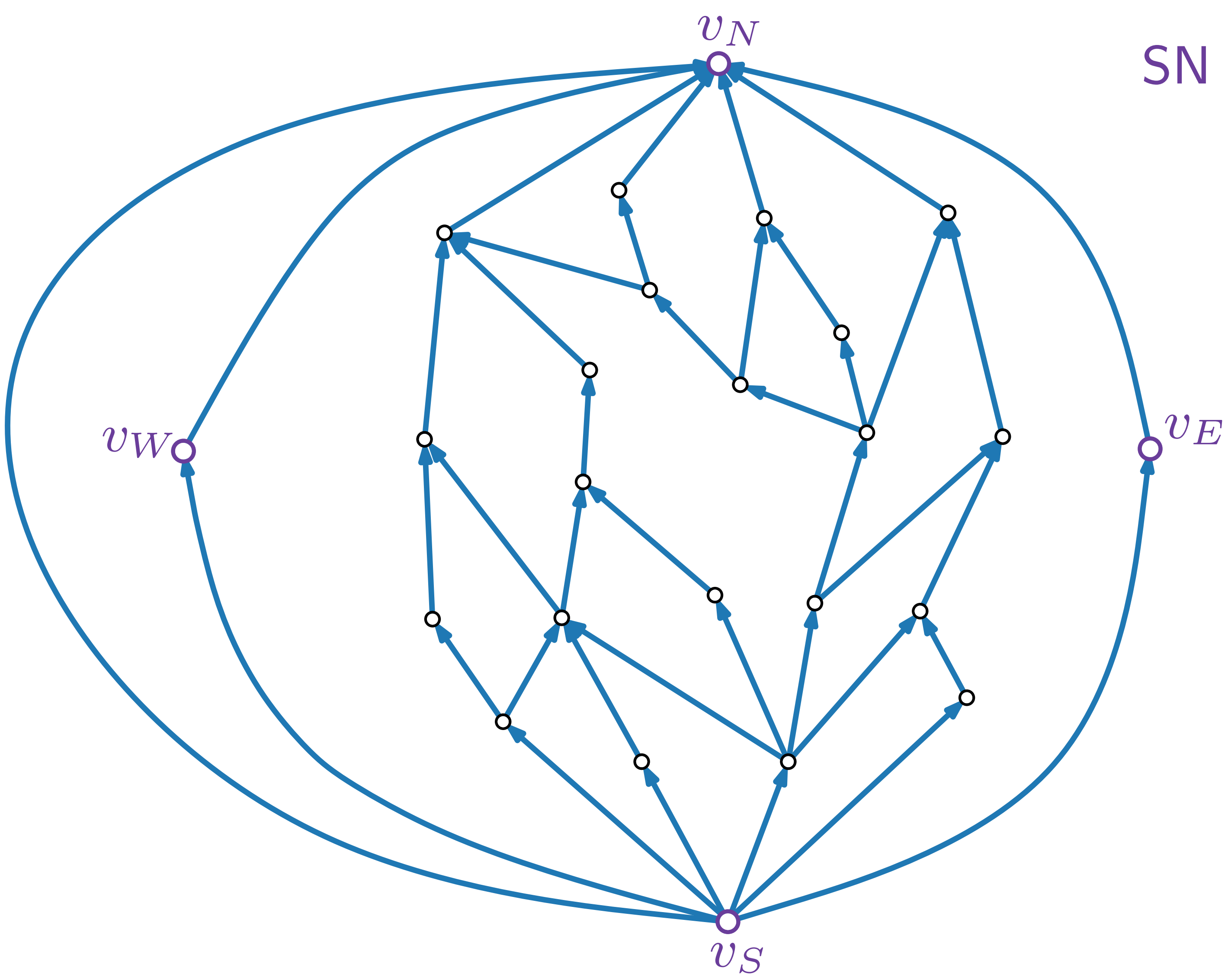
WE network G_{hor}

From REL to st-Digraphs to Coordinates



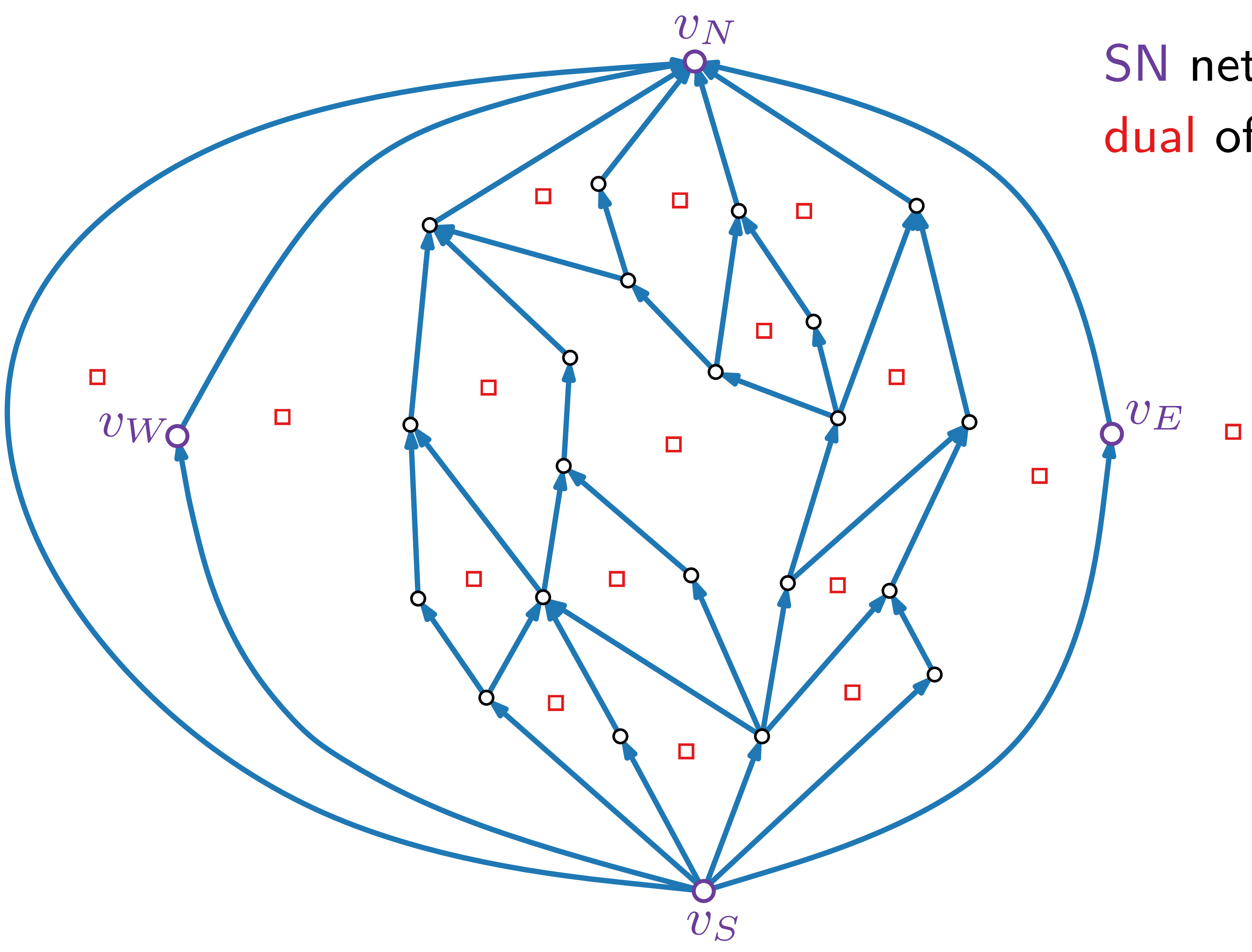
WE network G_{hor}

From REL to st-Digraphs to Coordinates



SN network G_{ver}

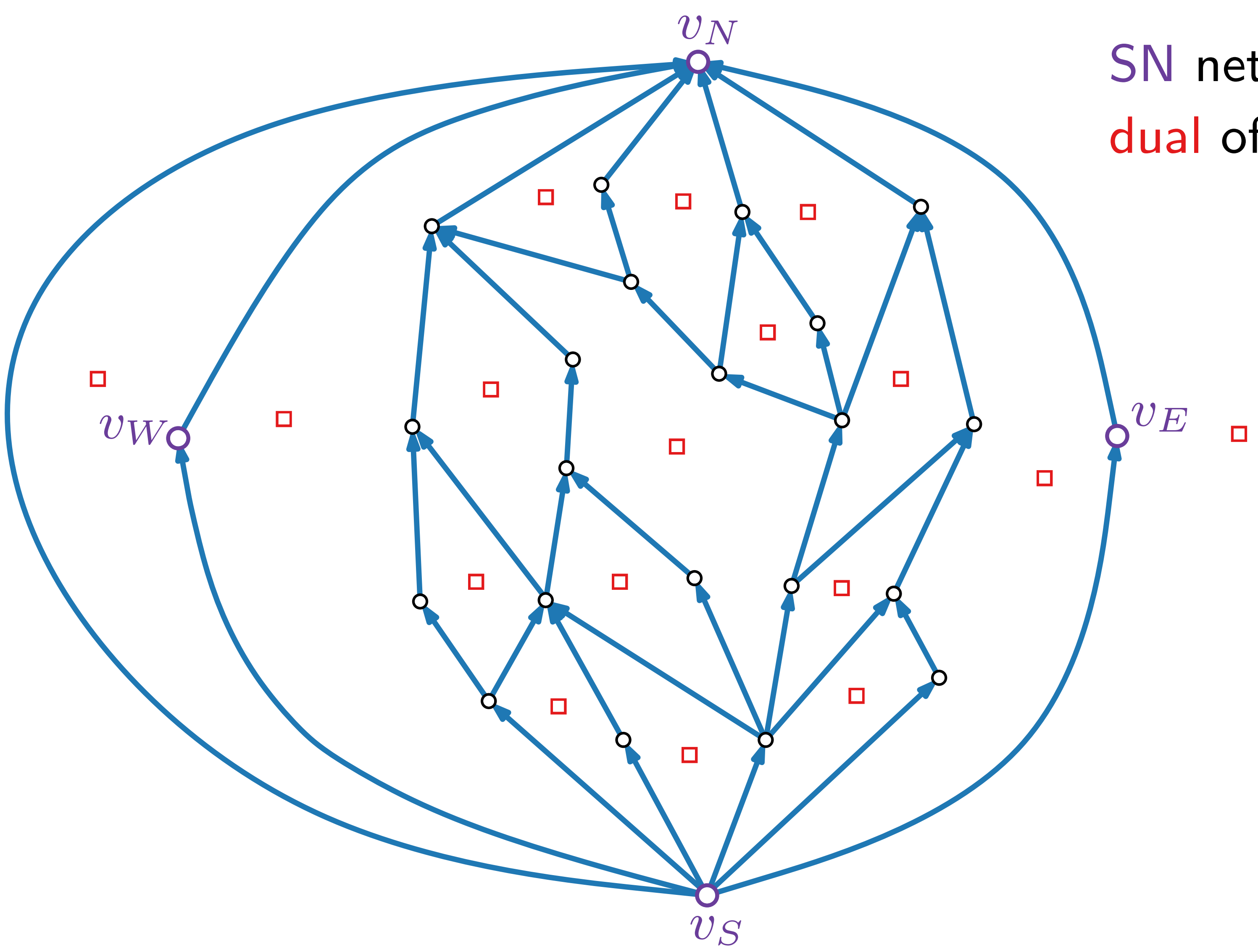
From REL to st-Digraphs to Coordinates



SN network G_{ver}

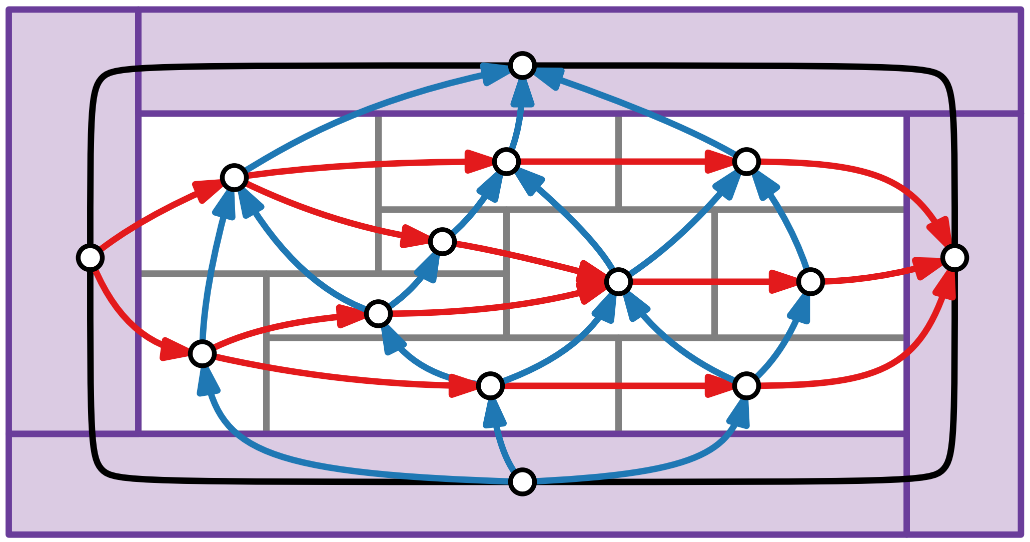
dual of G_{ver}

From REL to st-Digraphs to Coordinates



SN network G_{ver}

dual of G_{ver}



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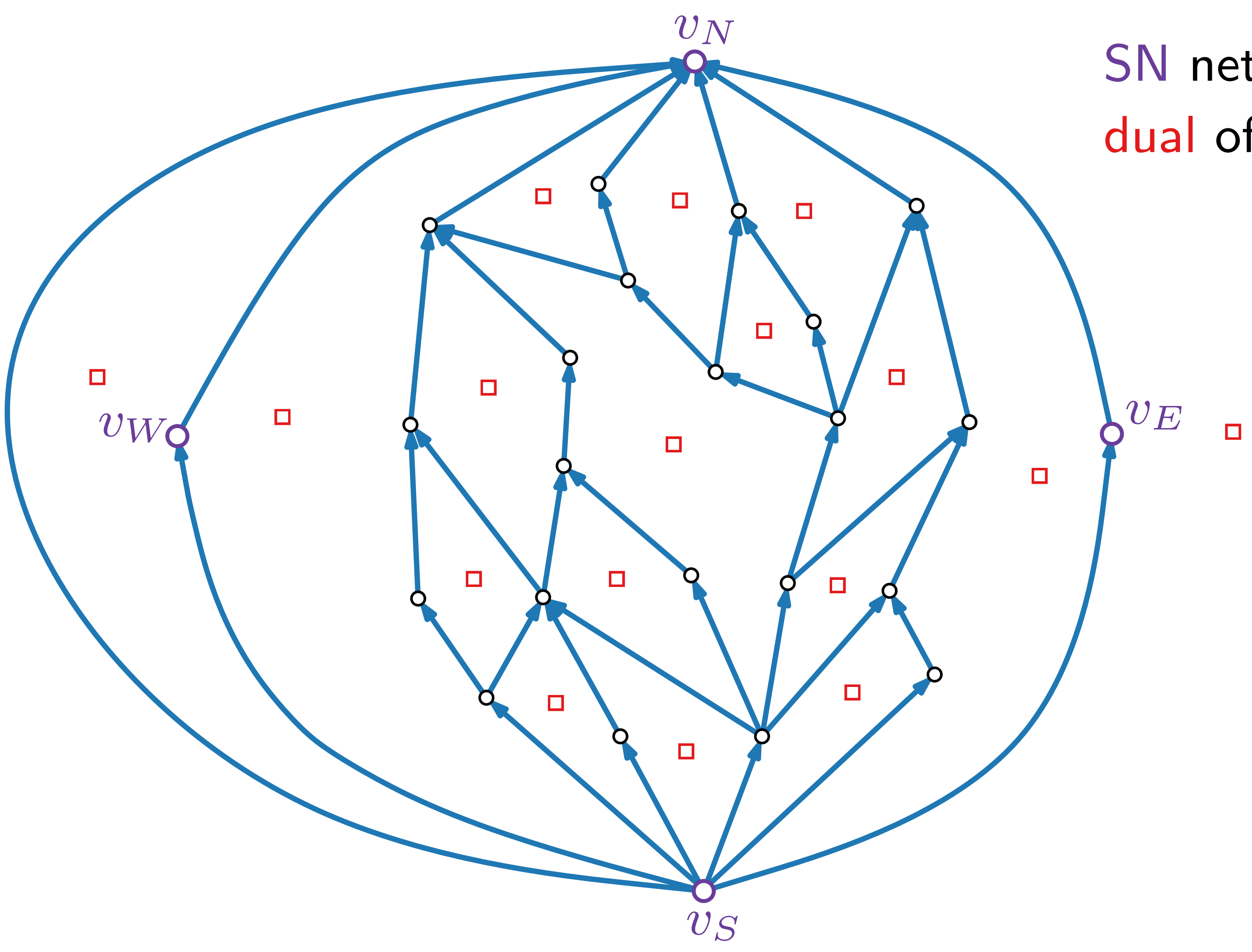

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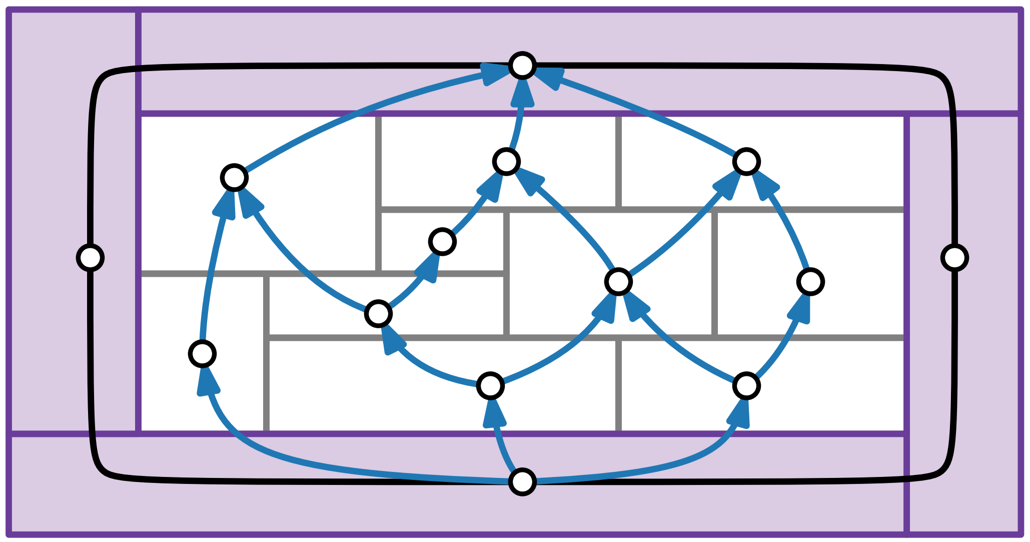

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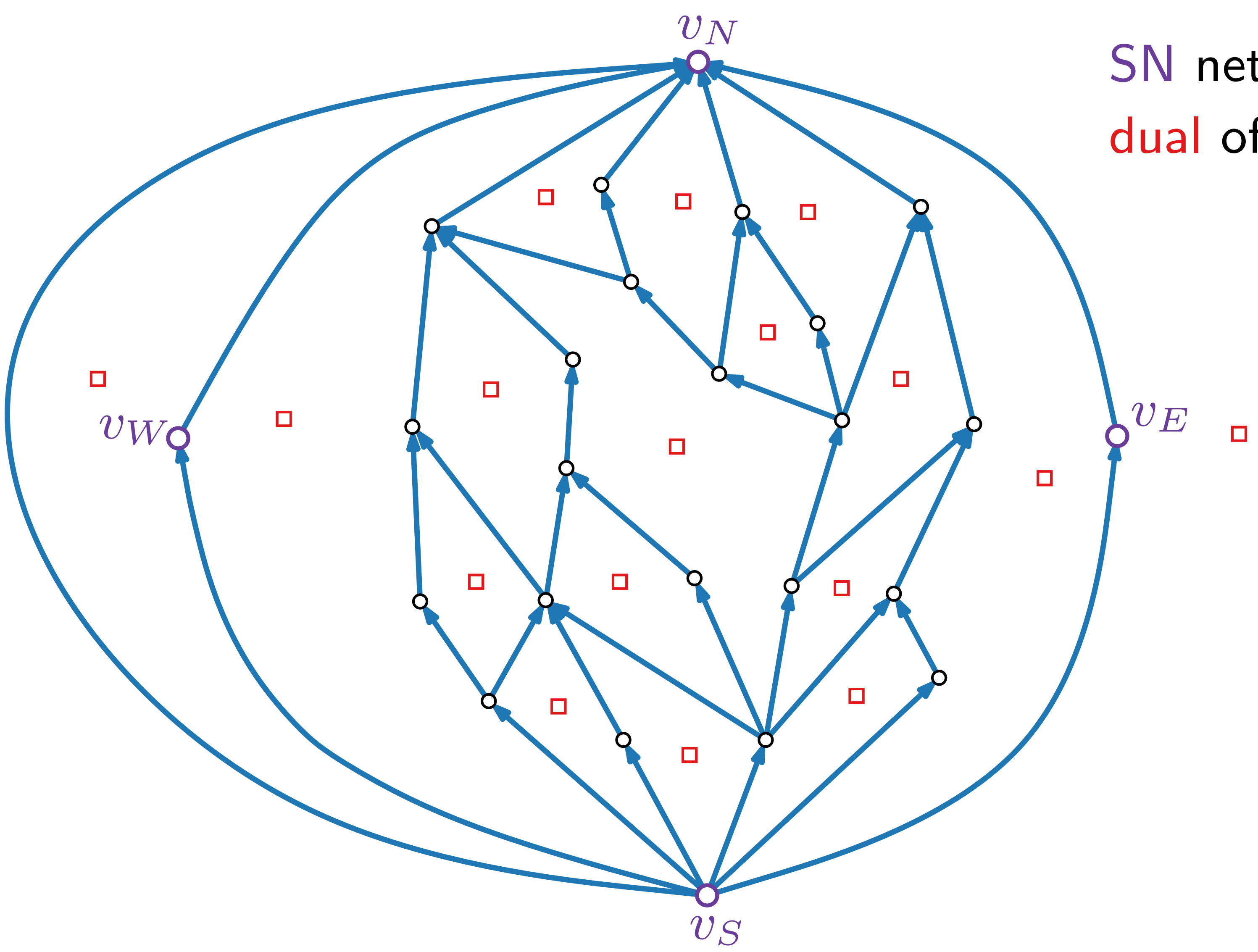

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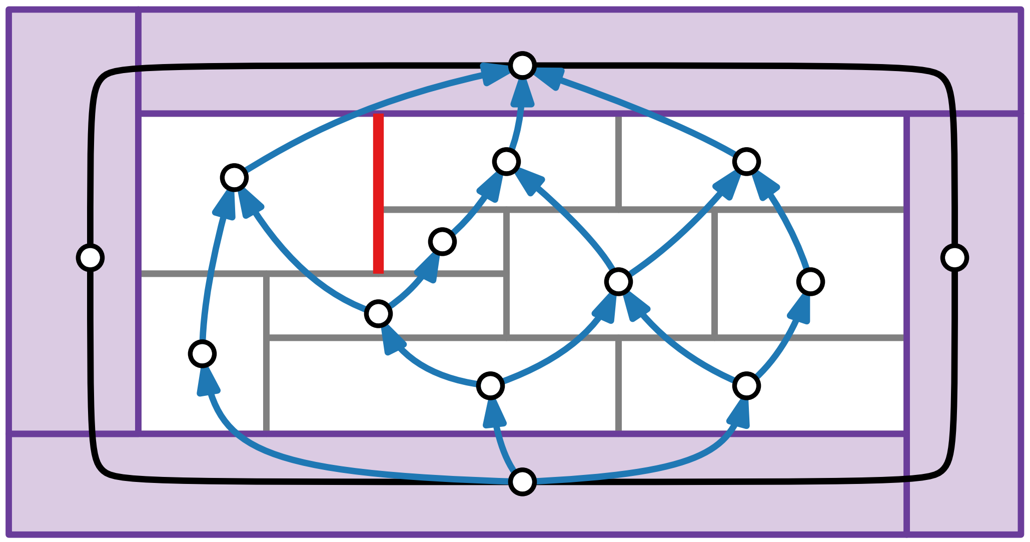
dual of G_{ver}



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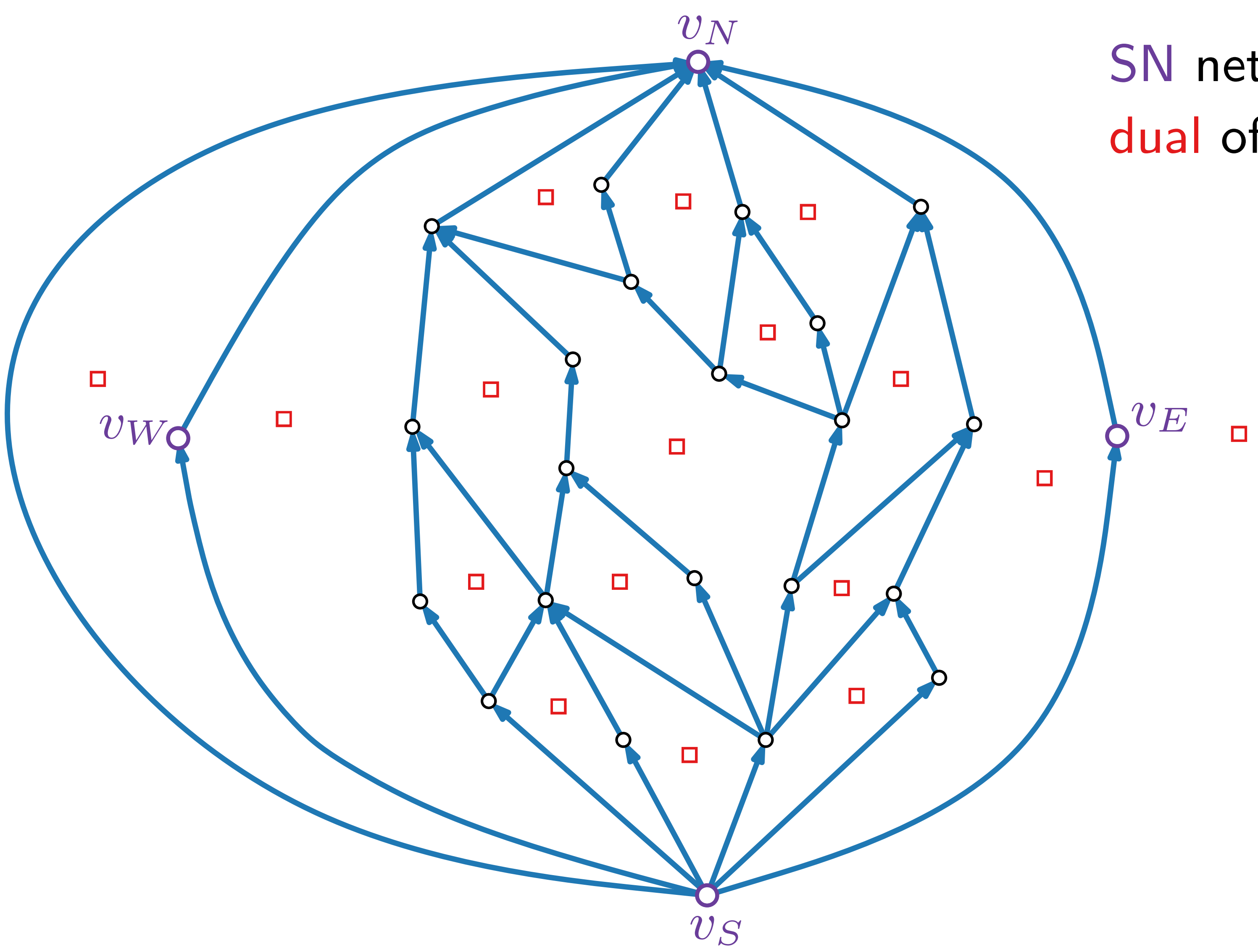

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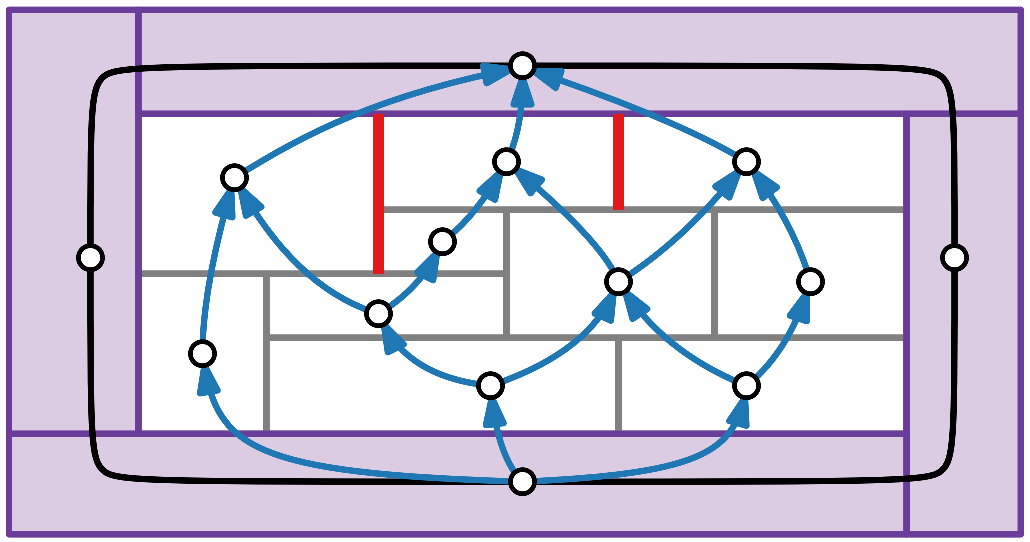

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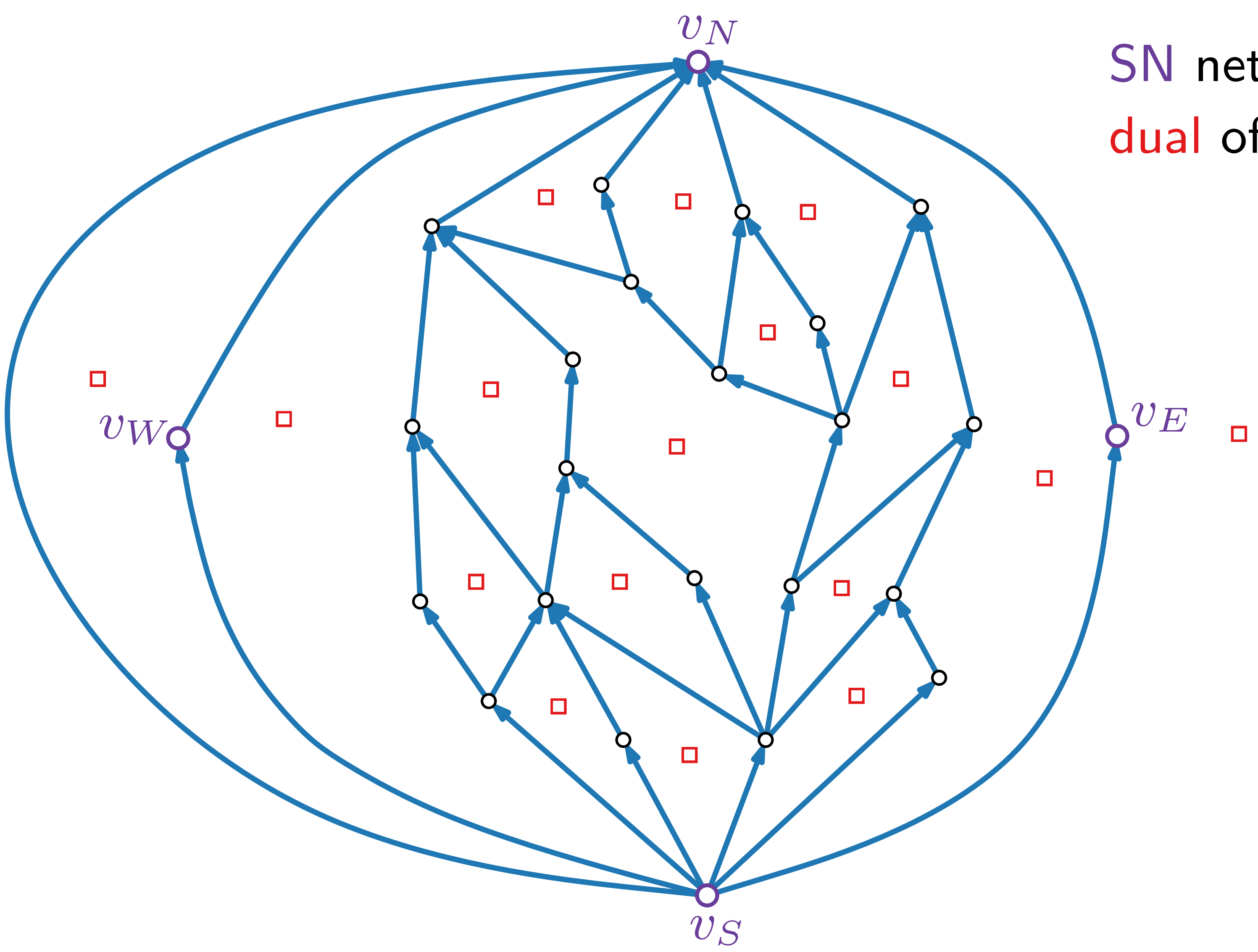

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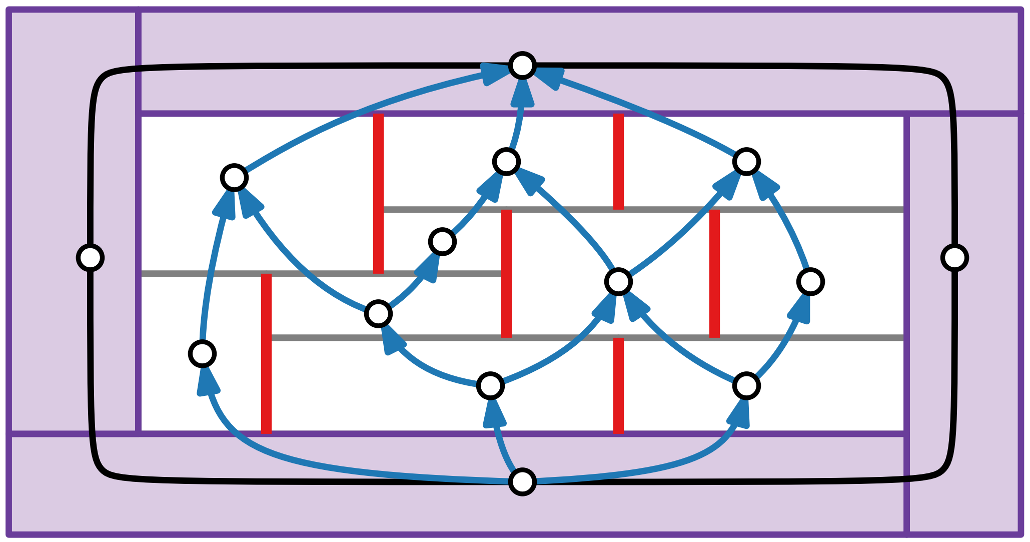

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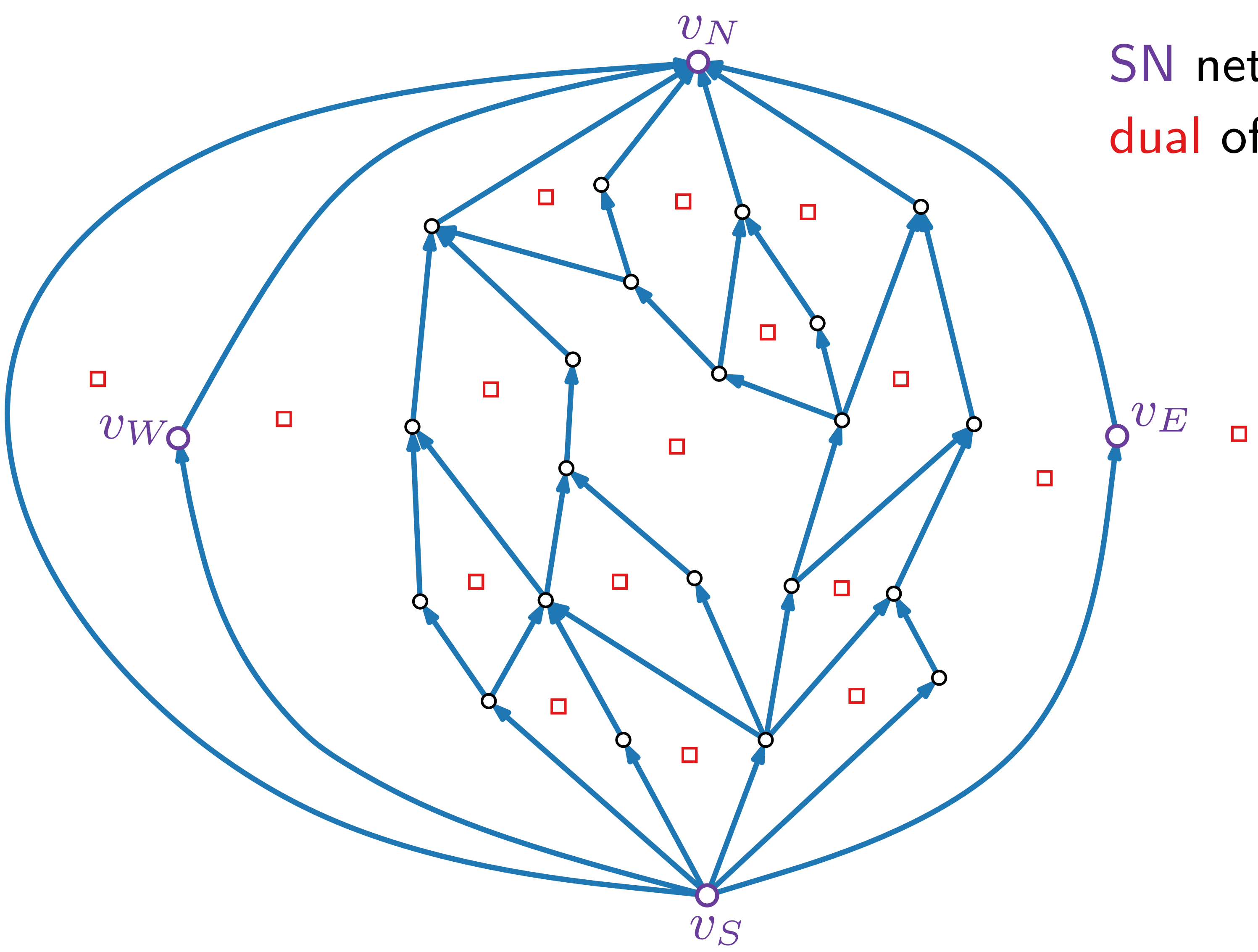

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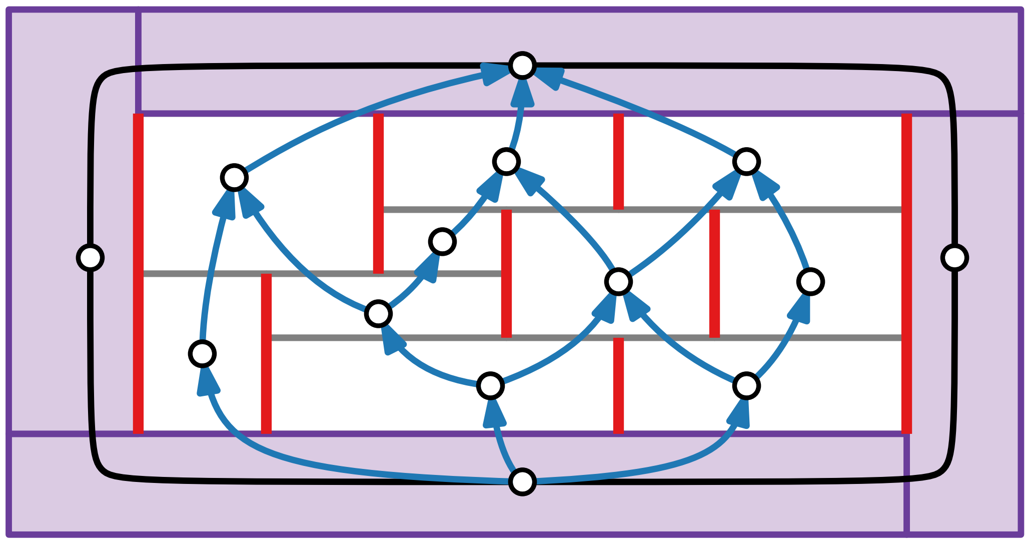

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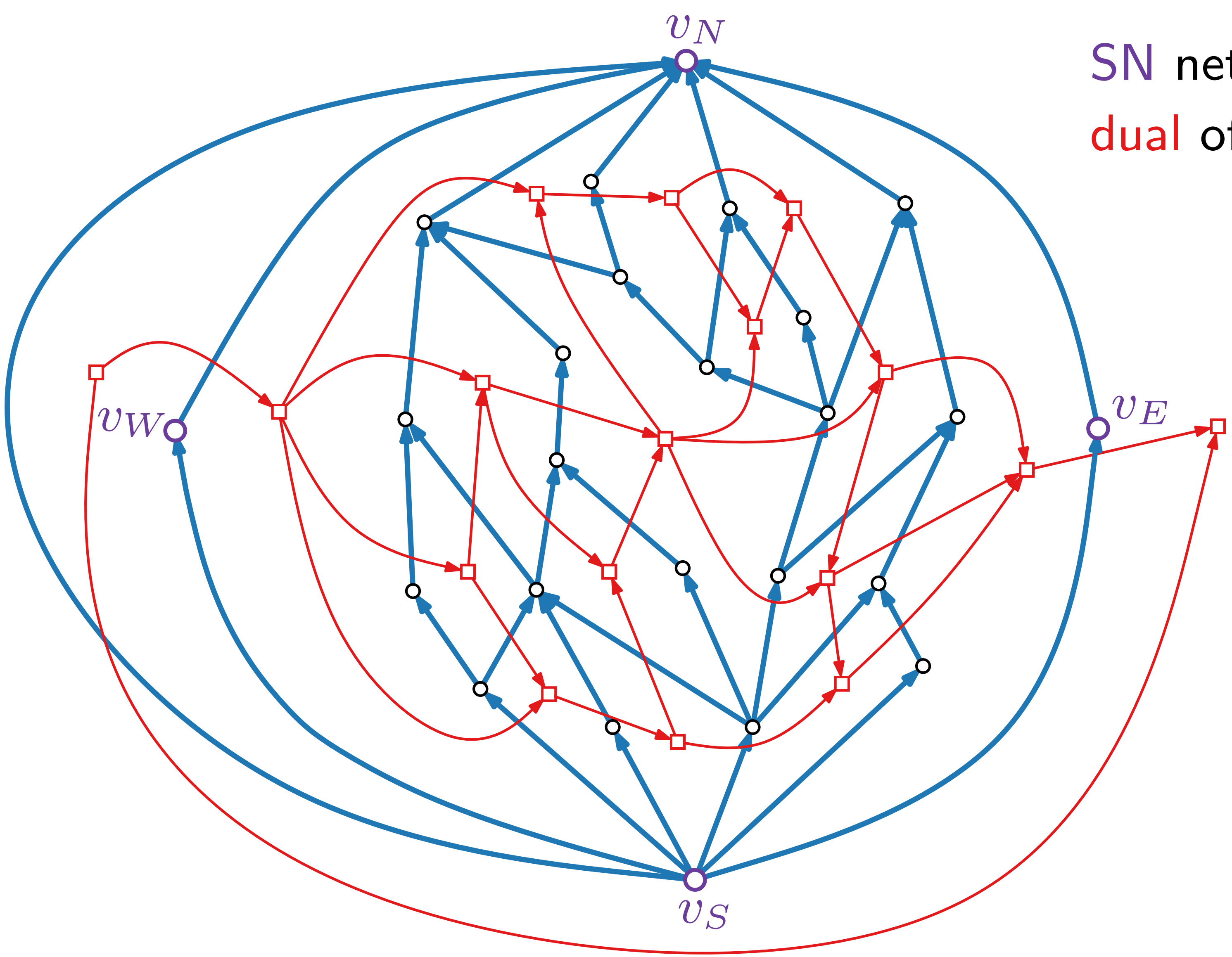
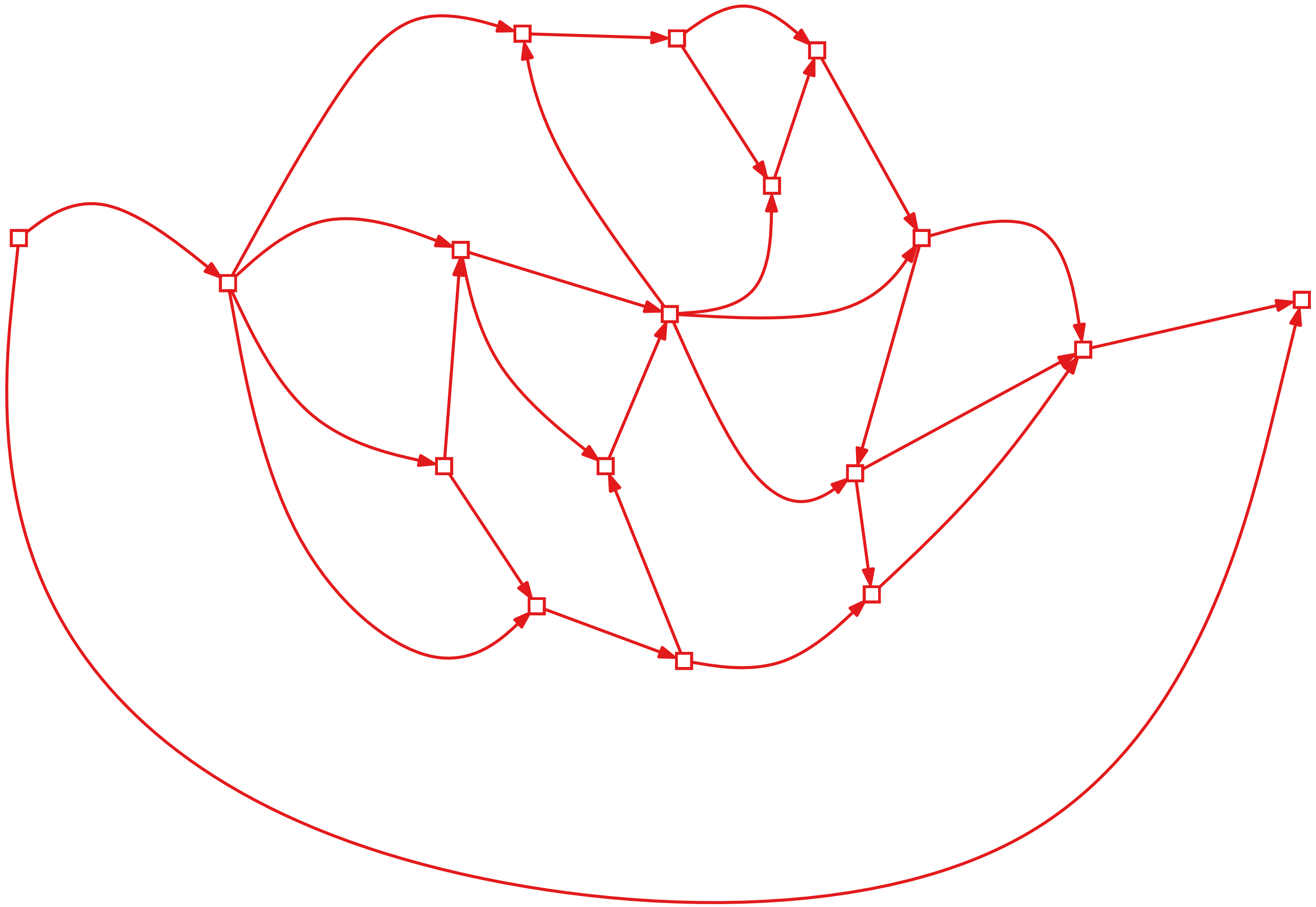


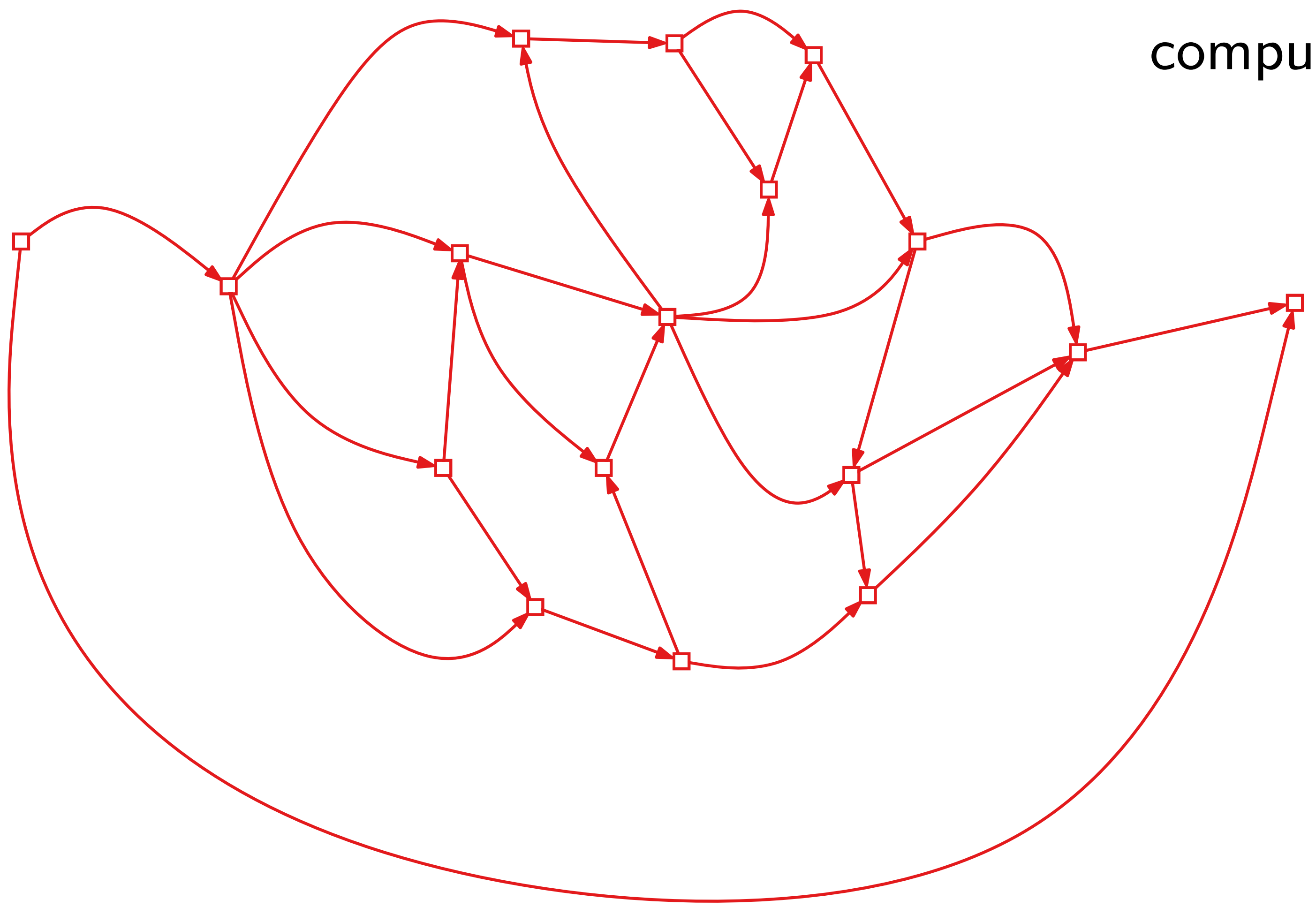
SN network G_{ver}
dual of G_{ver}

From REL to st-Digraphs to Coordinates

dual of G_{ver}



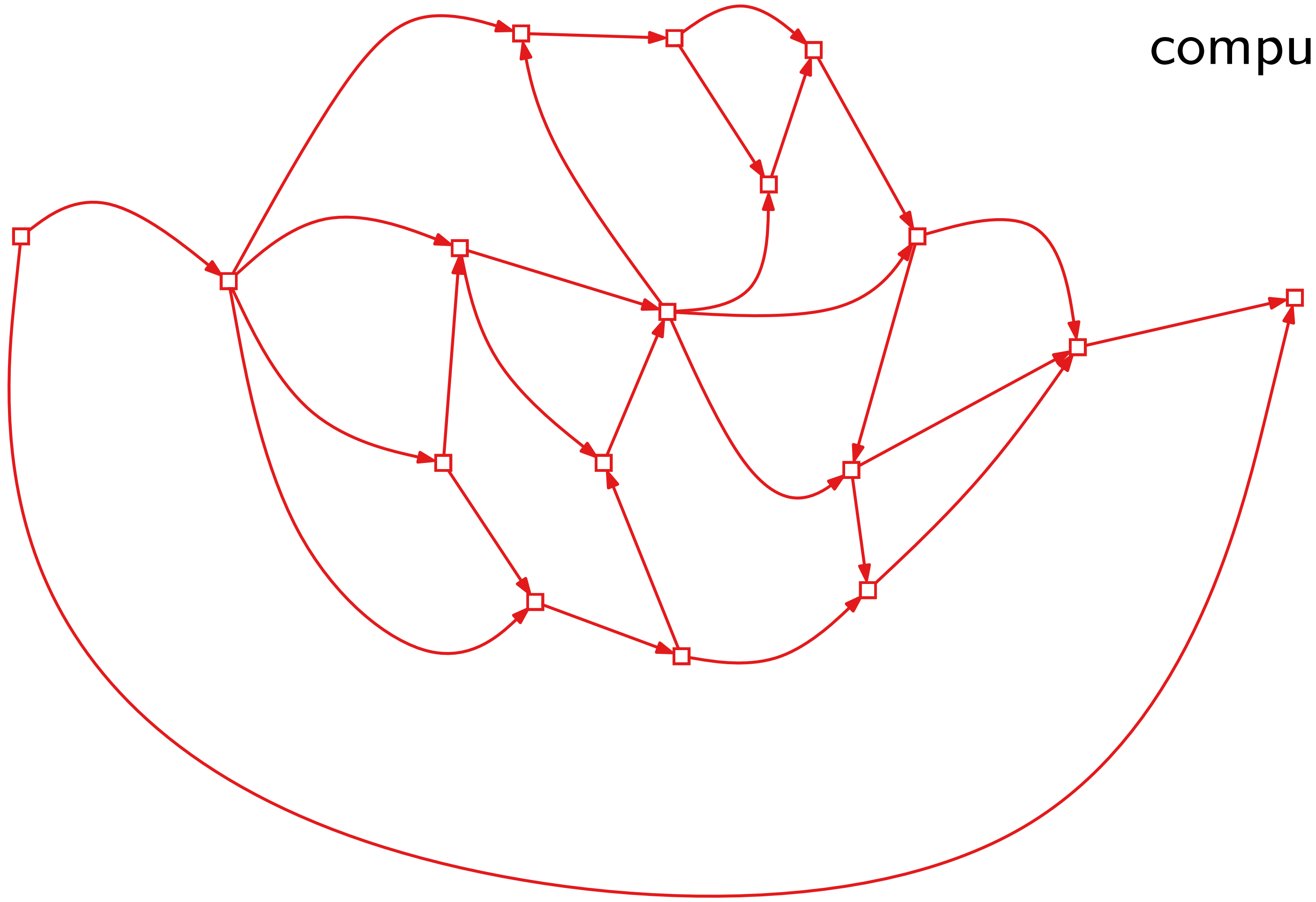
From REL to st-Digraphs to Coordinates



dual of G_{ver}

compute topological order

From REL to st-Digraphs to Coordinates

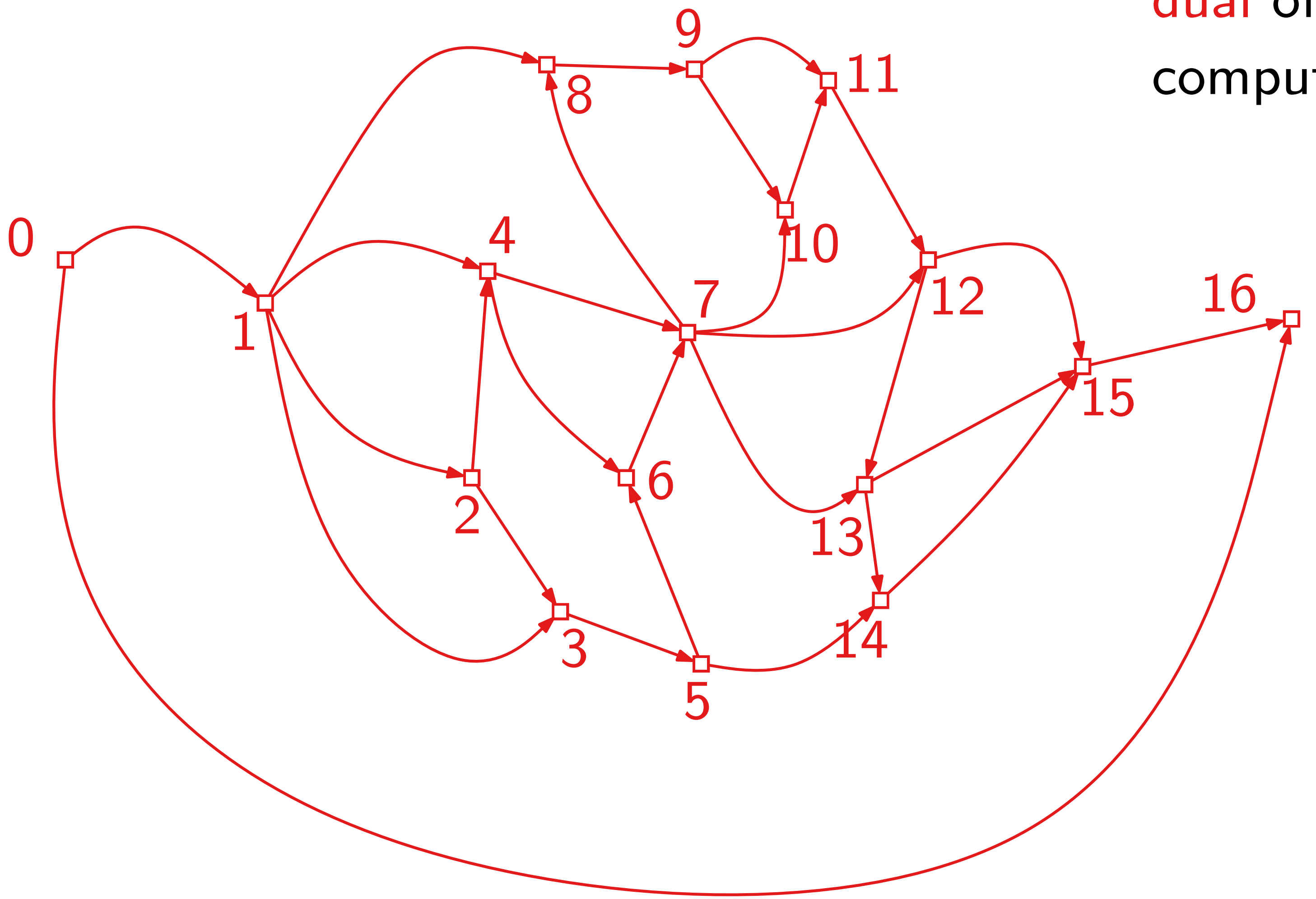


dual of G_{ver}

compute topological order

i.e., linear order of all vertices such that all directed edges point in the same direction

From REL to st-Digraphs to Coordinates



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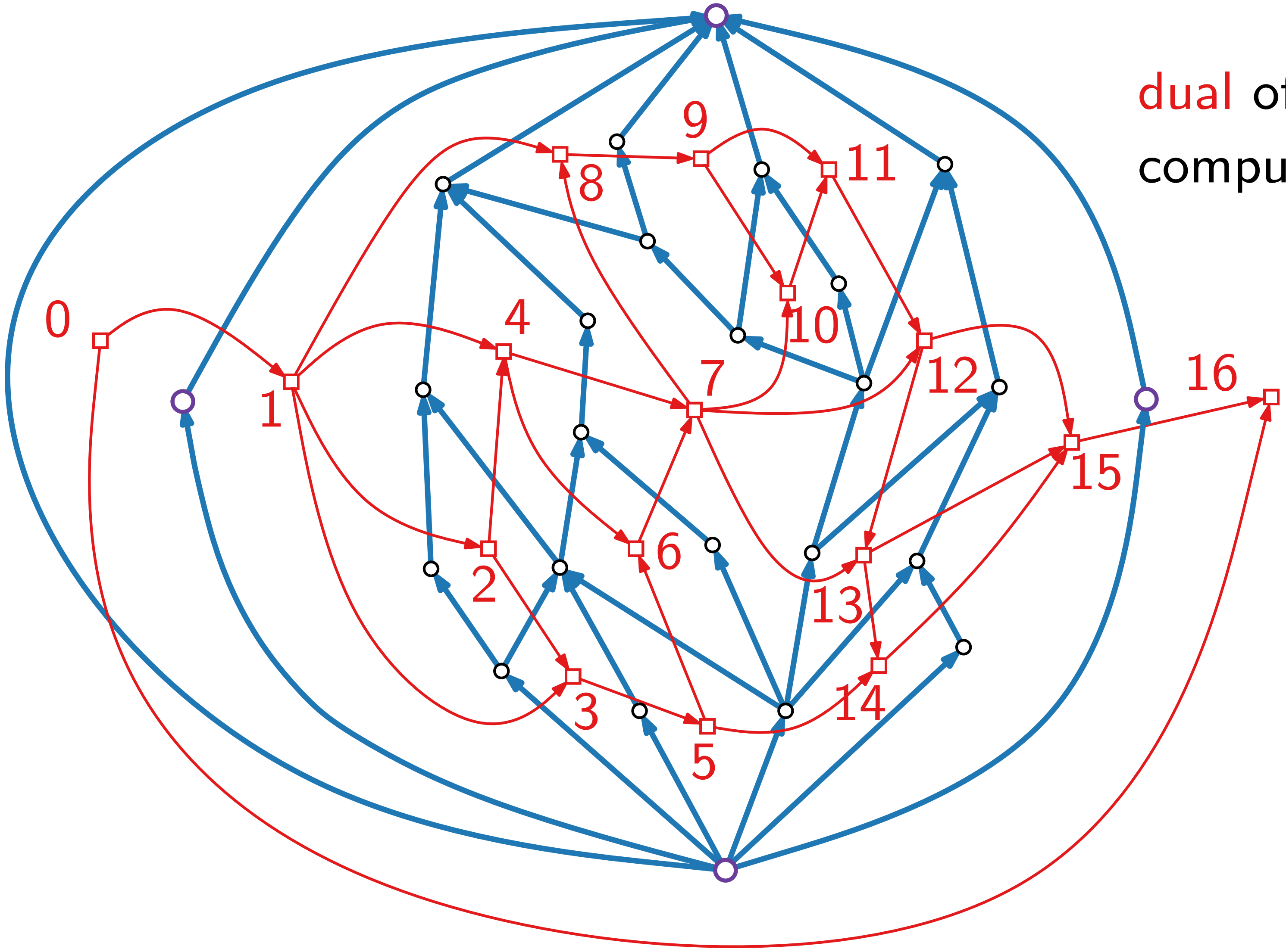

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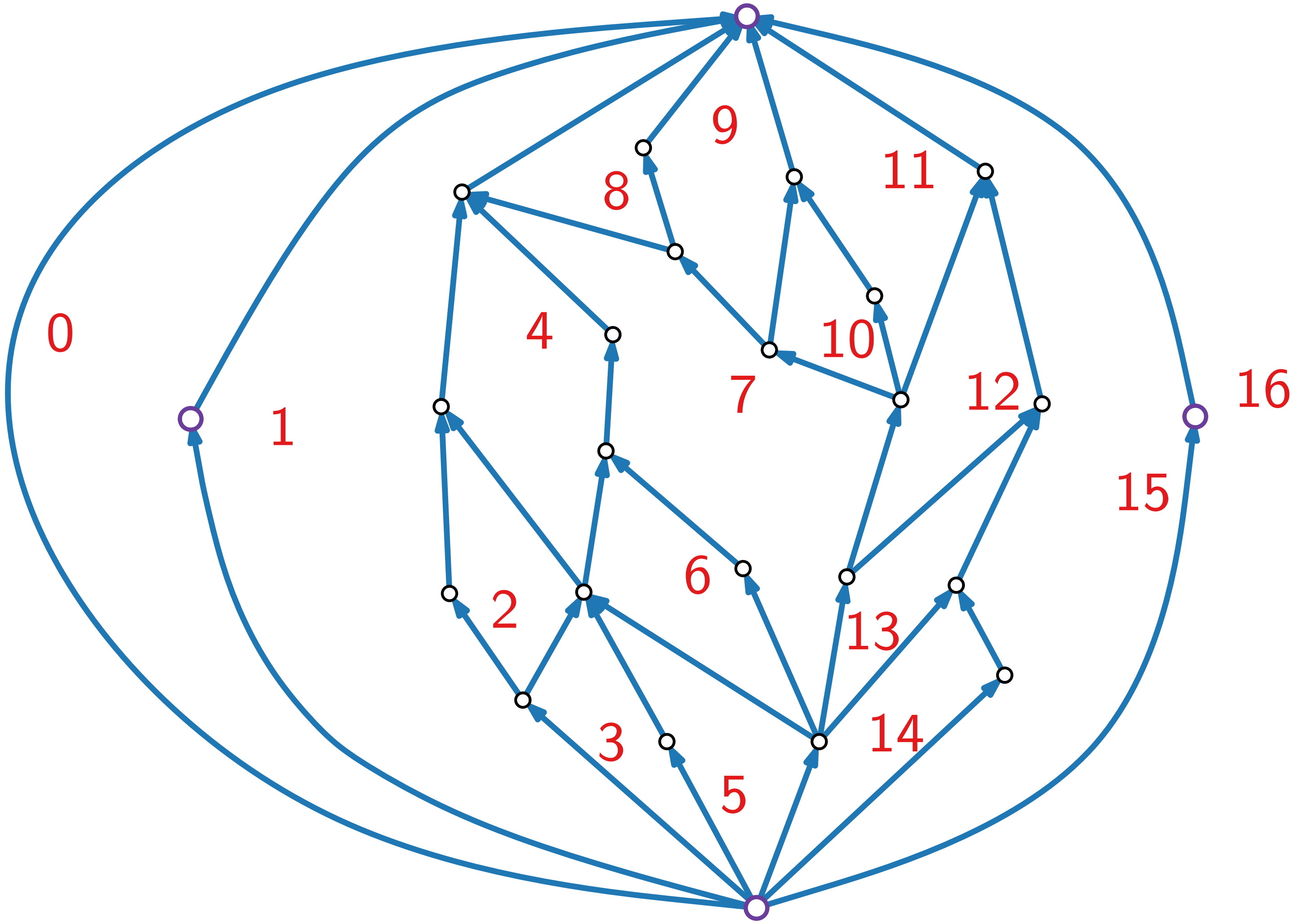

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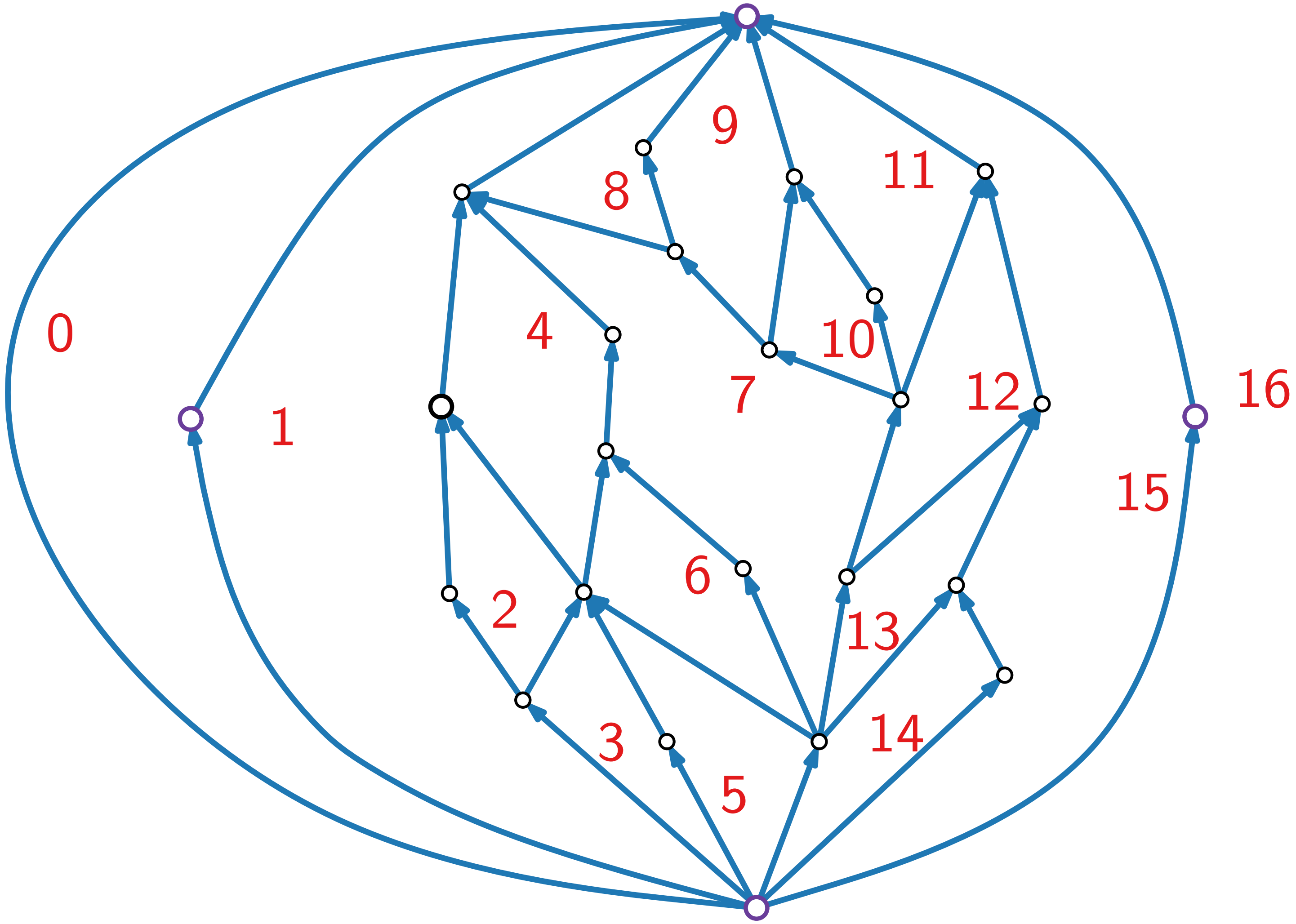

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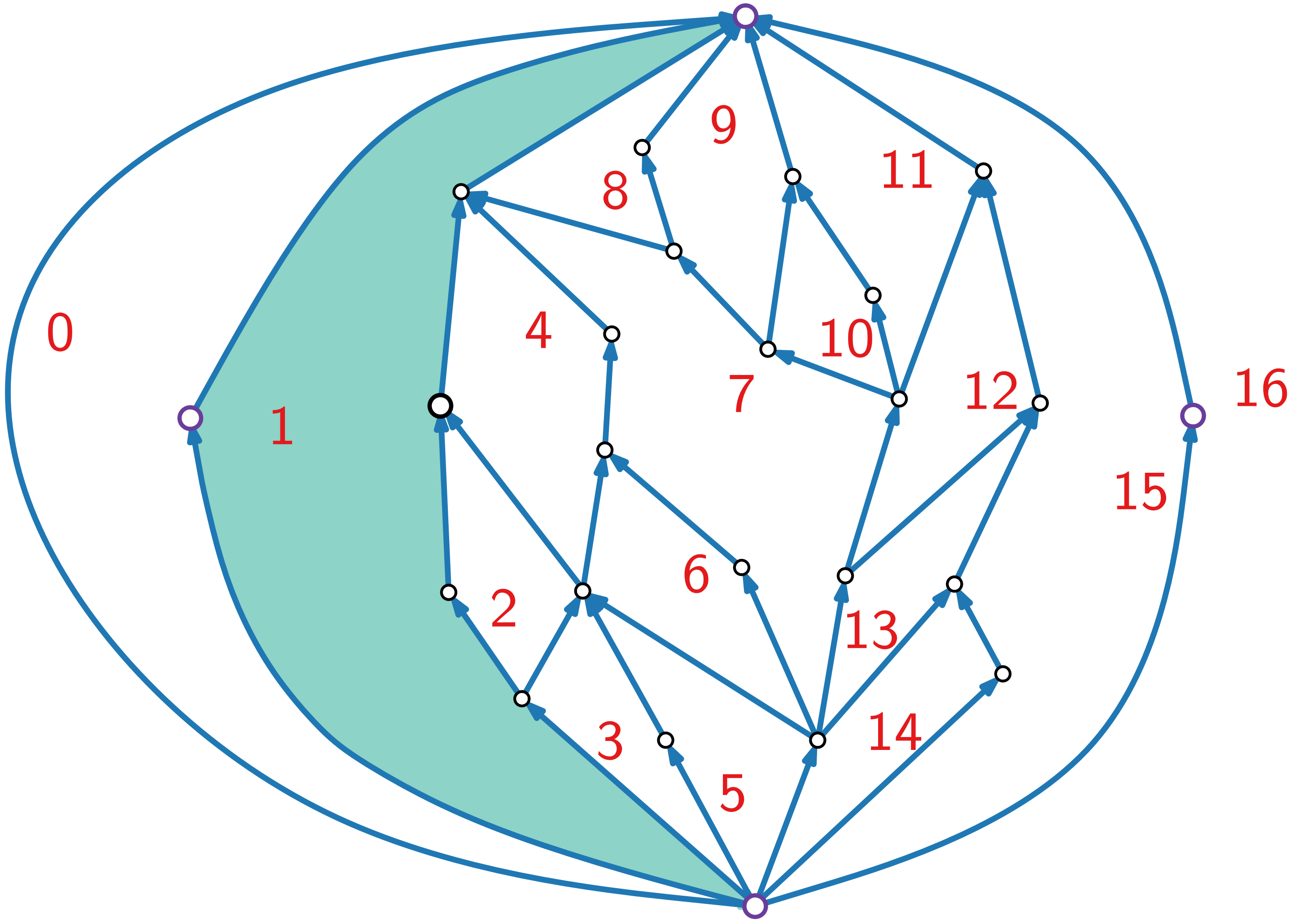

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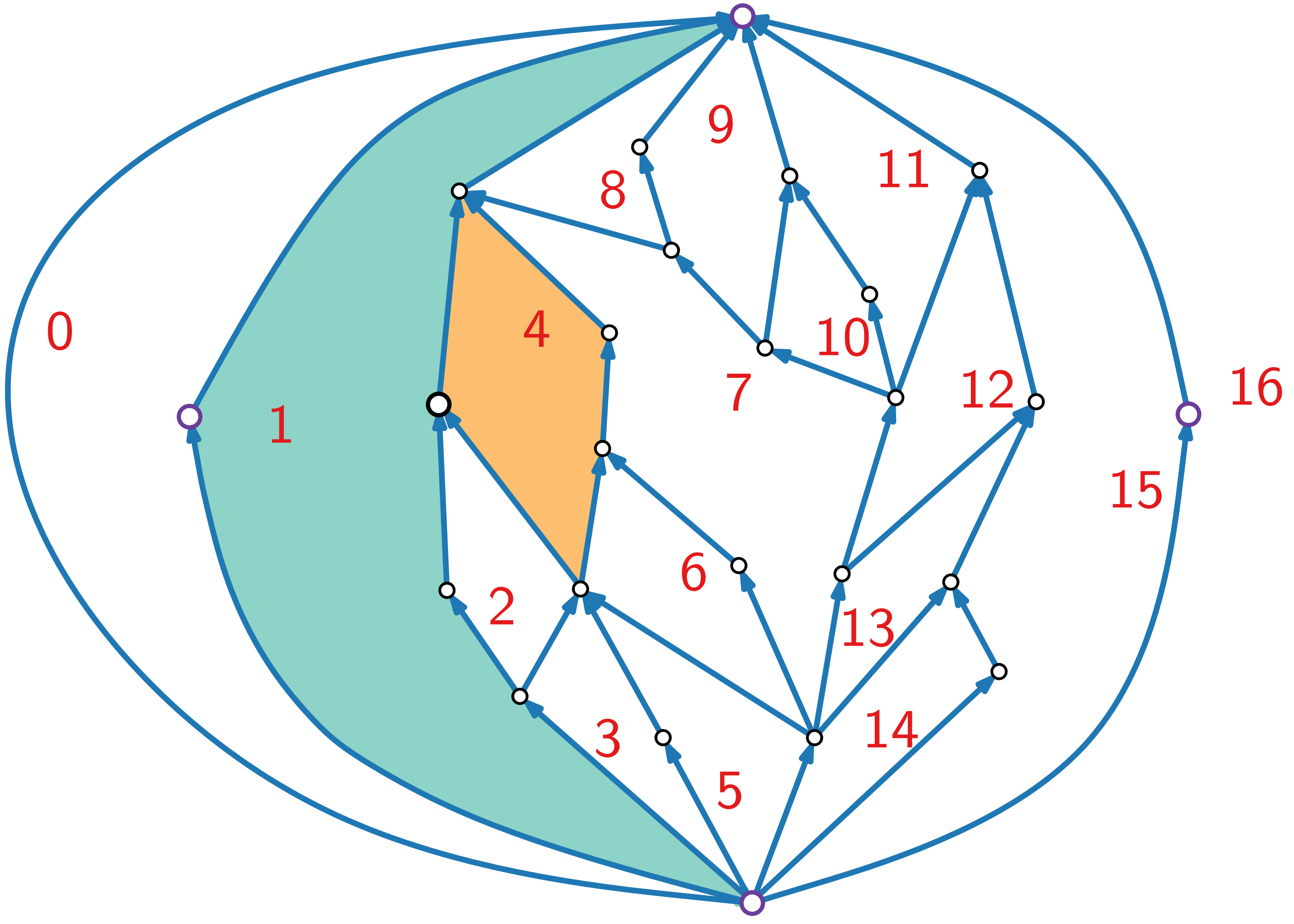

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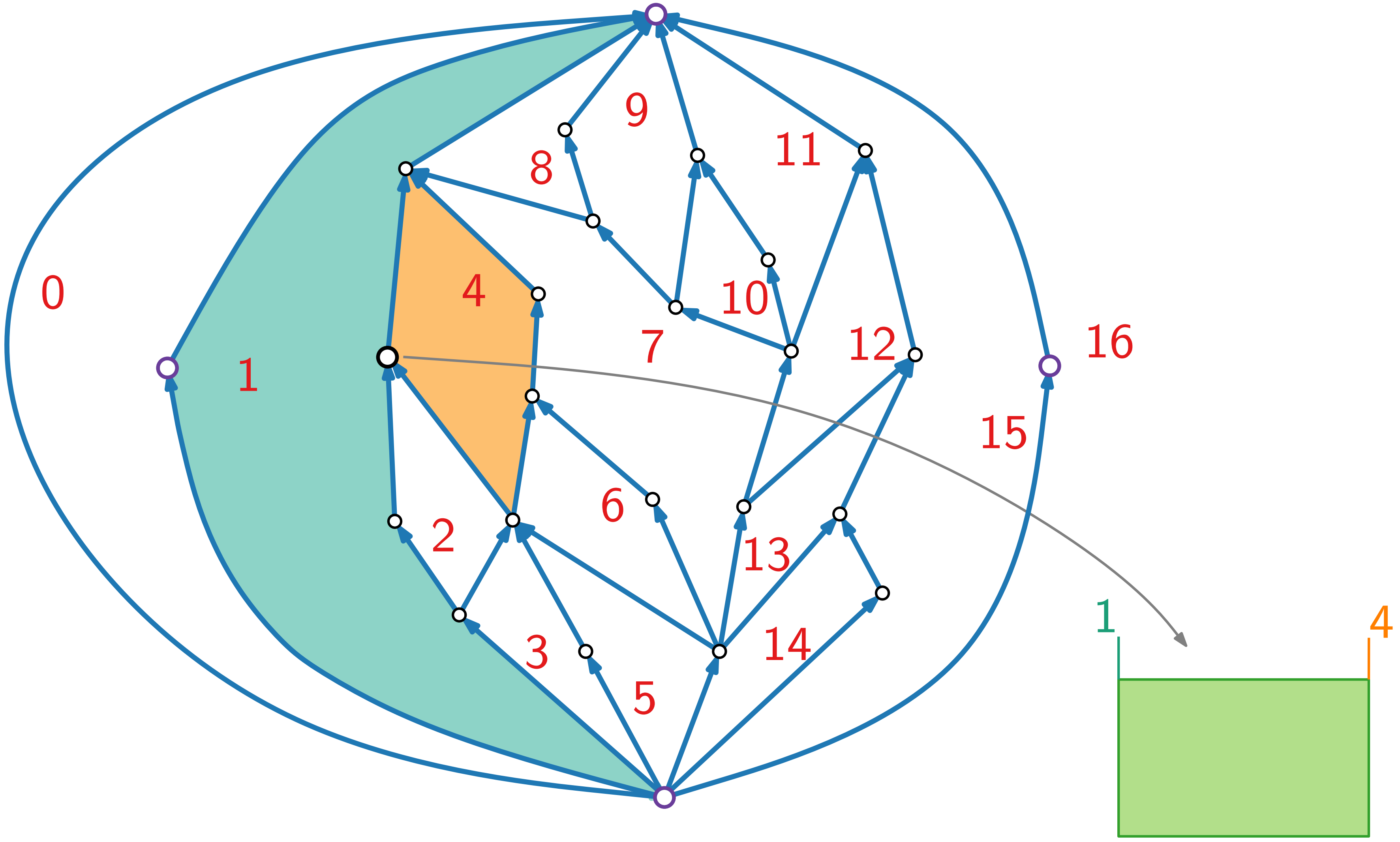

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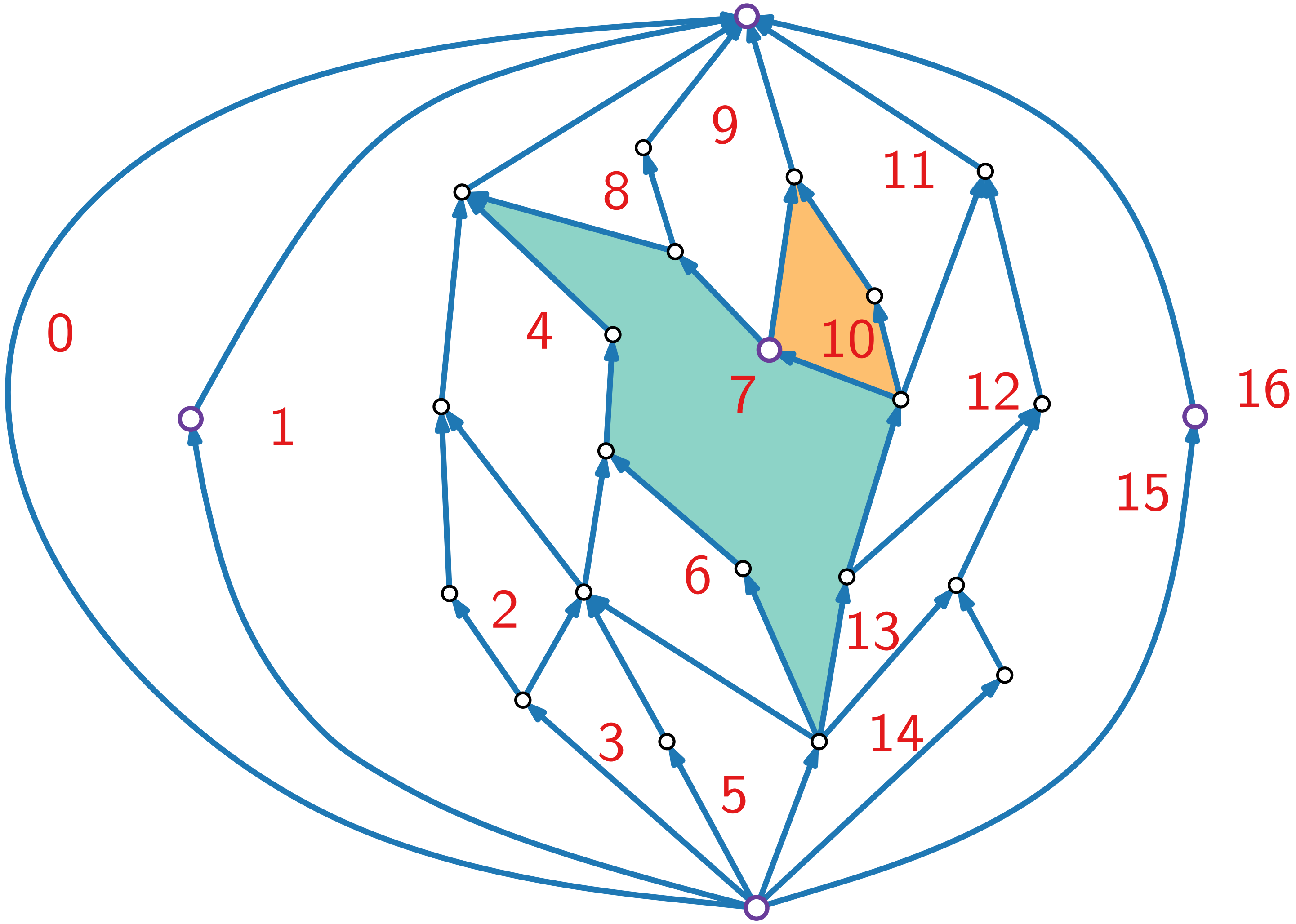

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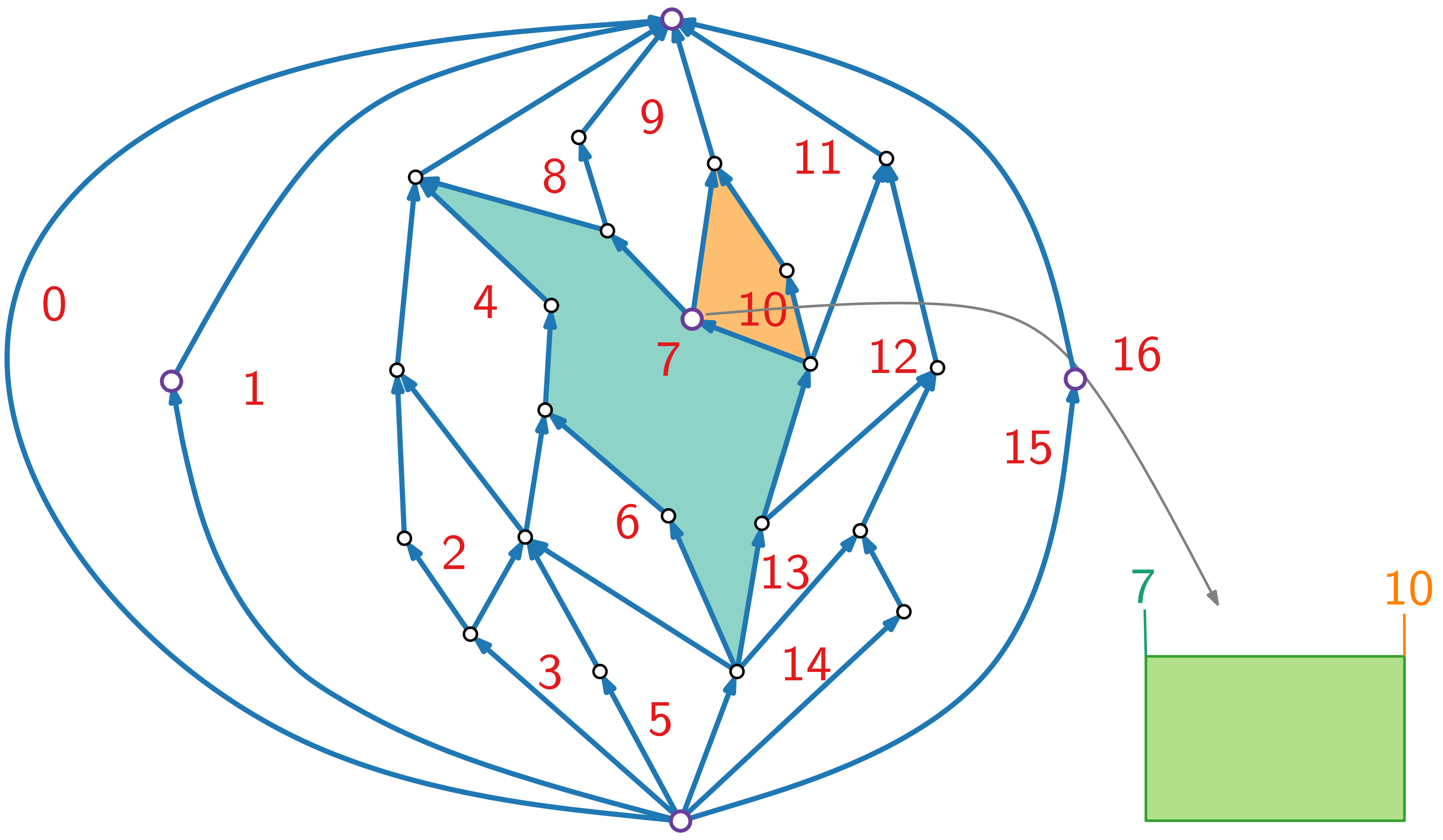

Rectangular Dual Algorithm

For a PTP graph G:

Rectangular Dual Algorithm

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- Find a REL $\{T_r, T_b\}$ of G.

Rectangular Dual Algorithm

For a PTP graph G:

- Find a REL $\{T_r, T_b\}$ of G.
- Construct a SN network G_{ver} of G (consists of T_b plus outer edges).

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  Set $x_1(v) = f_{\text{ver}}(g)$ and $x_2(v) = f_{\text{ver}}(h)$.

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- Define $x_1(v_N) = 0, x_1(v_S) = 1$ and $x_2(v_N) = \max f_{\text{ver}} - 1, x_2(v_S) = \max f_{\text{ver}}$.

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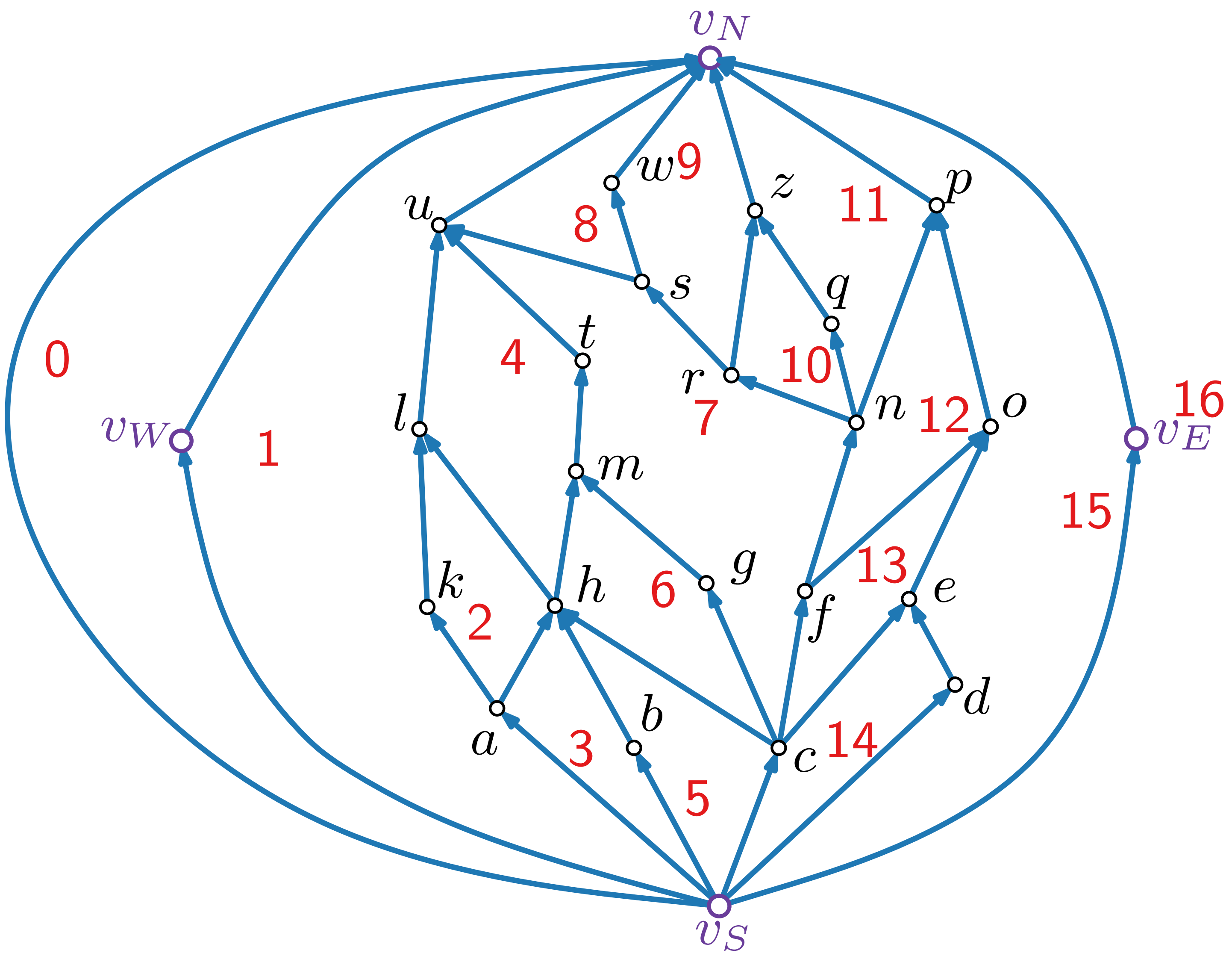
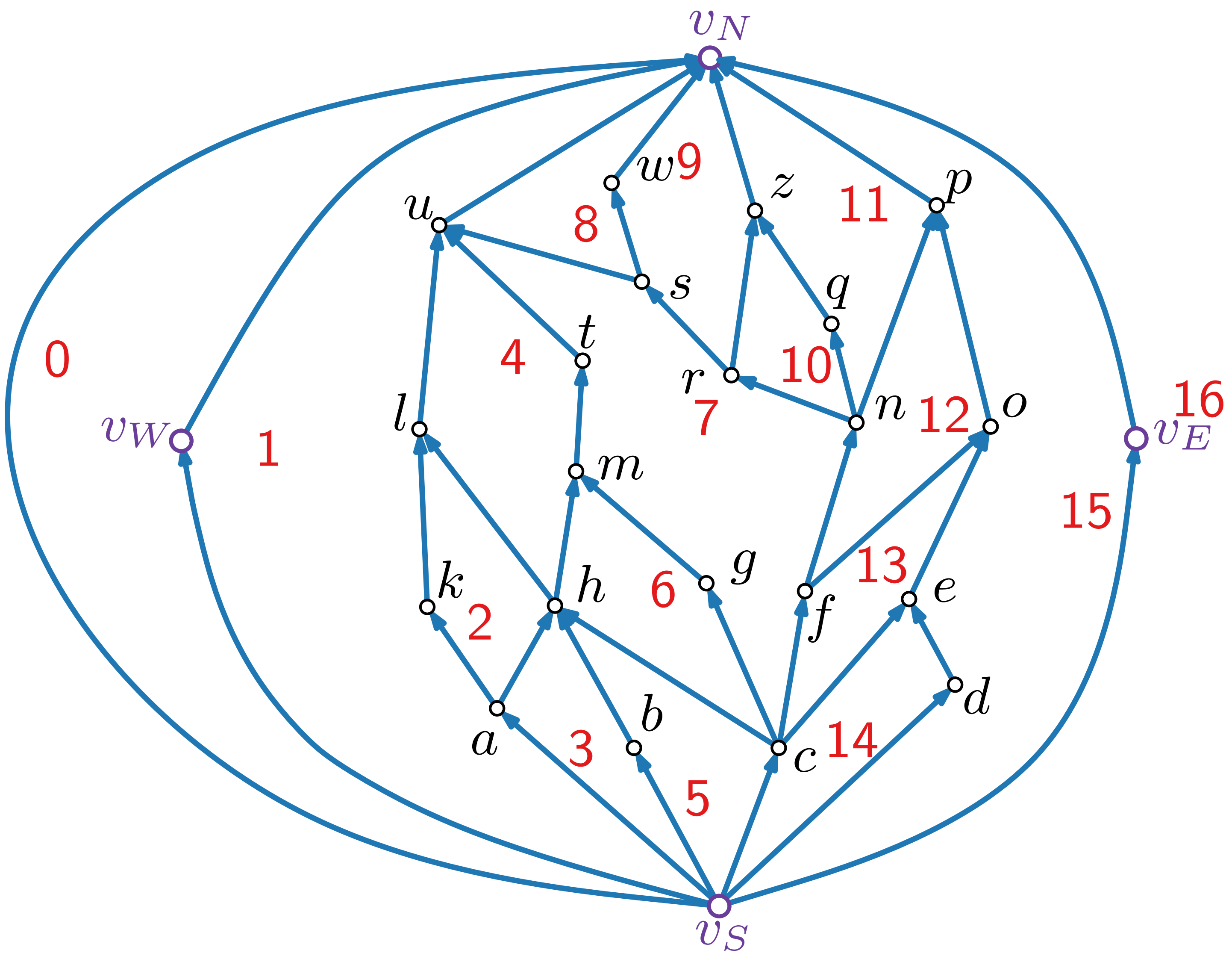
- For each vertex v of G, let $R(v) = [x_1(v), x_2(v)] \times [y_1(v), y_2(v)]$.

Reading off Coordinates to Get Rectangular Dual

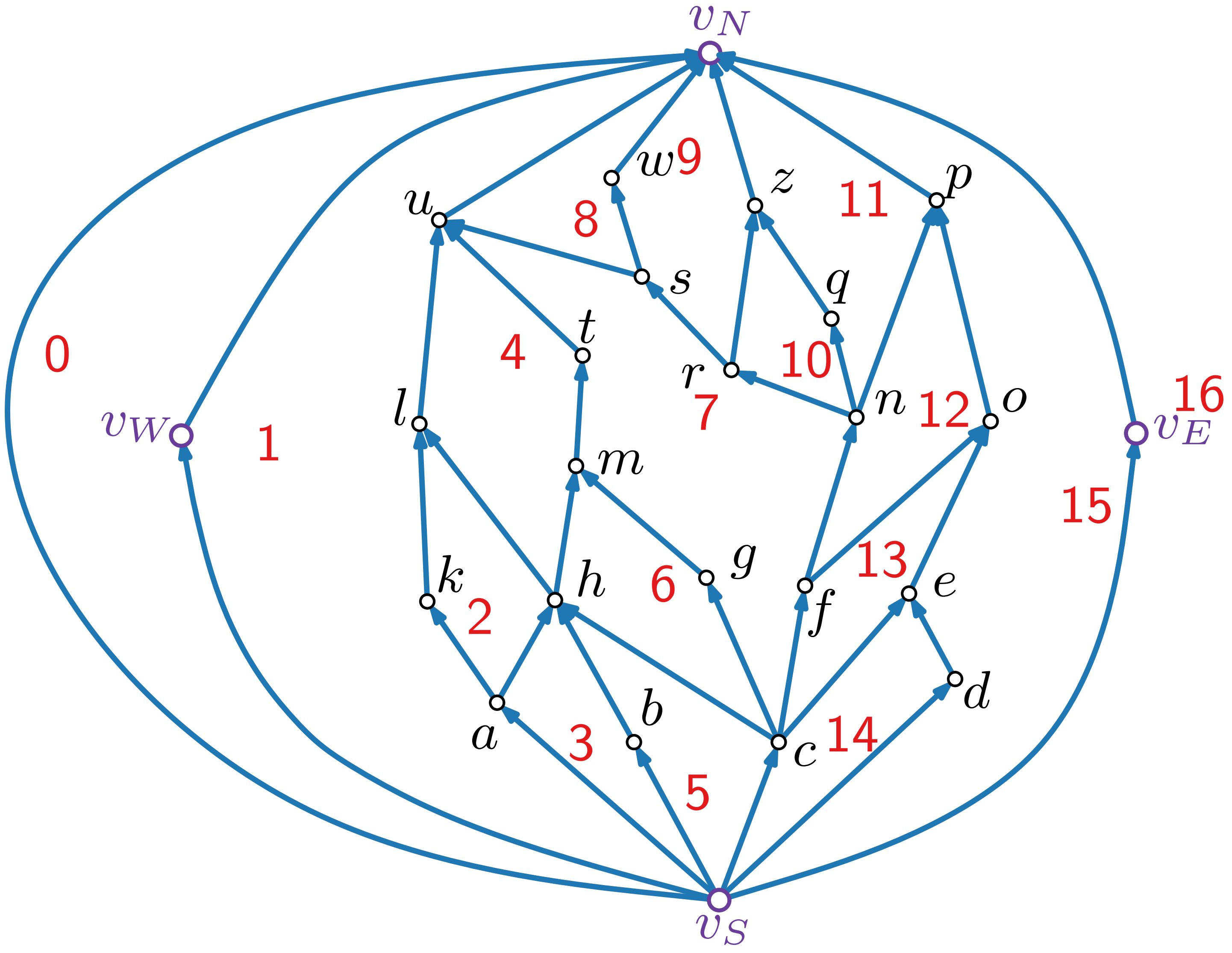


Reading off Coordinates to Get Rectangular Dual

$x_1(v_N) = 0,\ x_2(v_N) = 15$

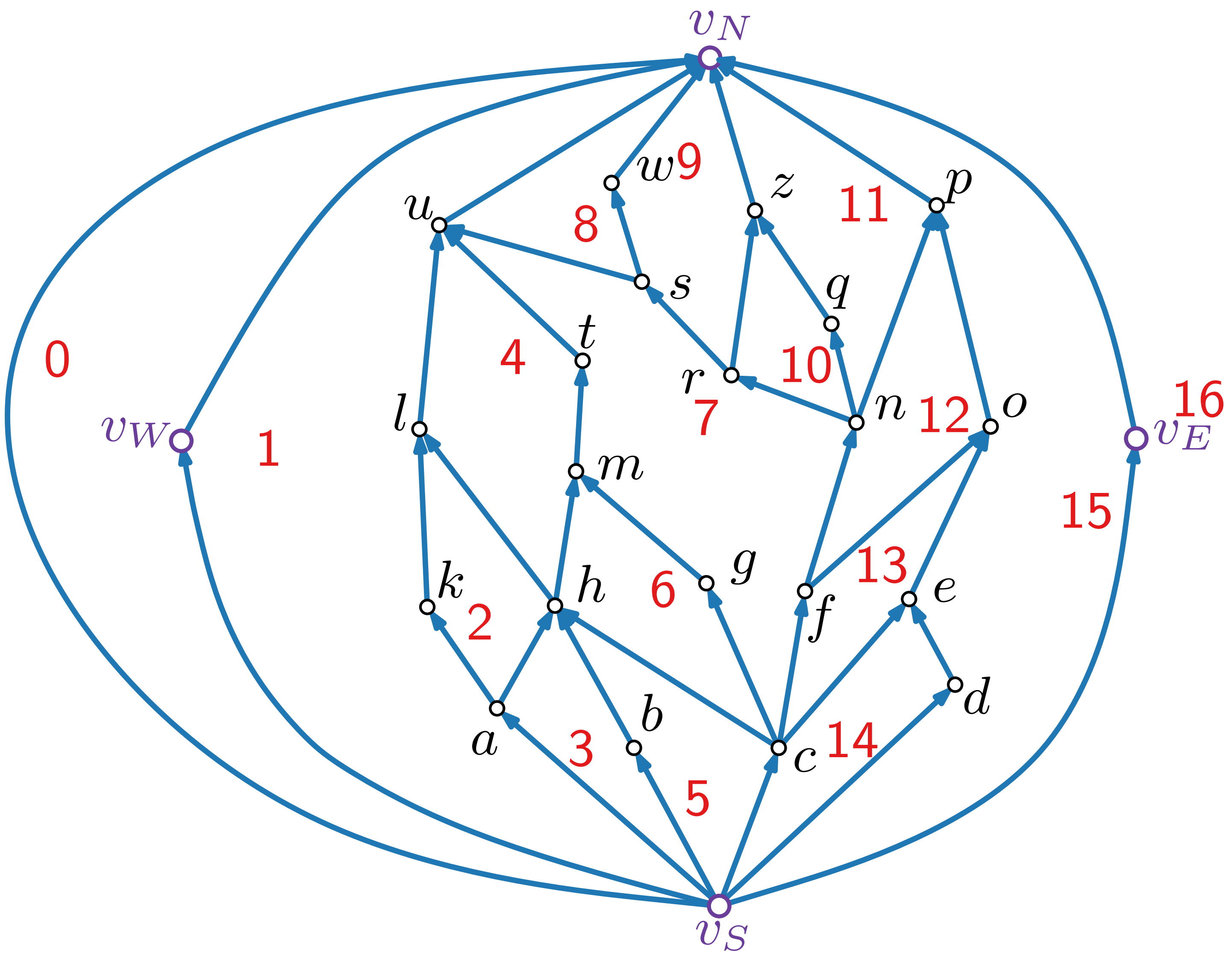


Reading off Coordinates to Get Rectangular Dual



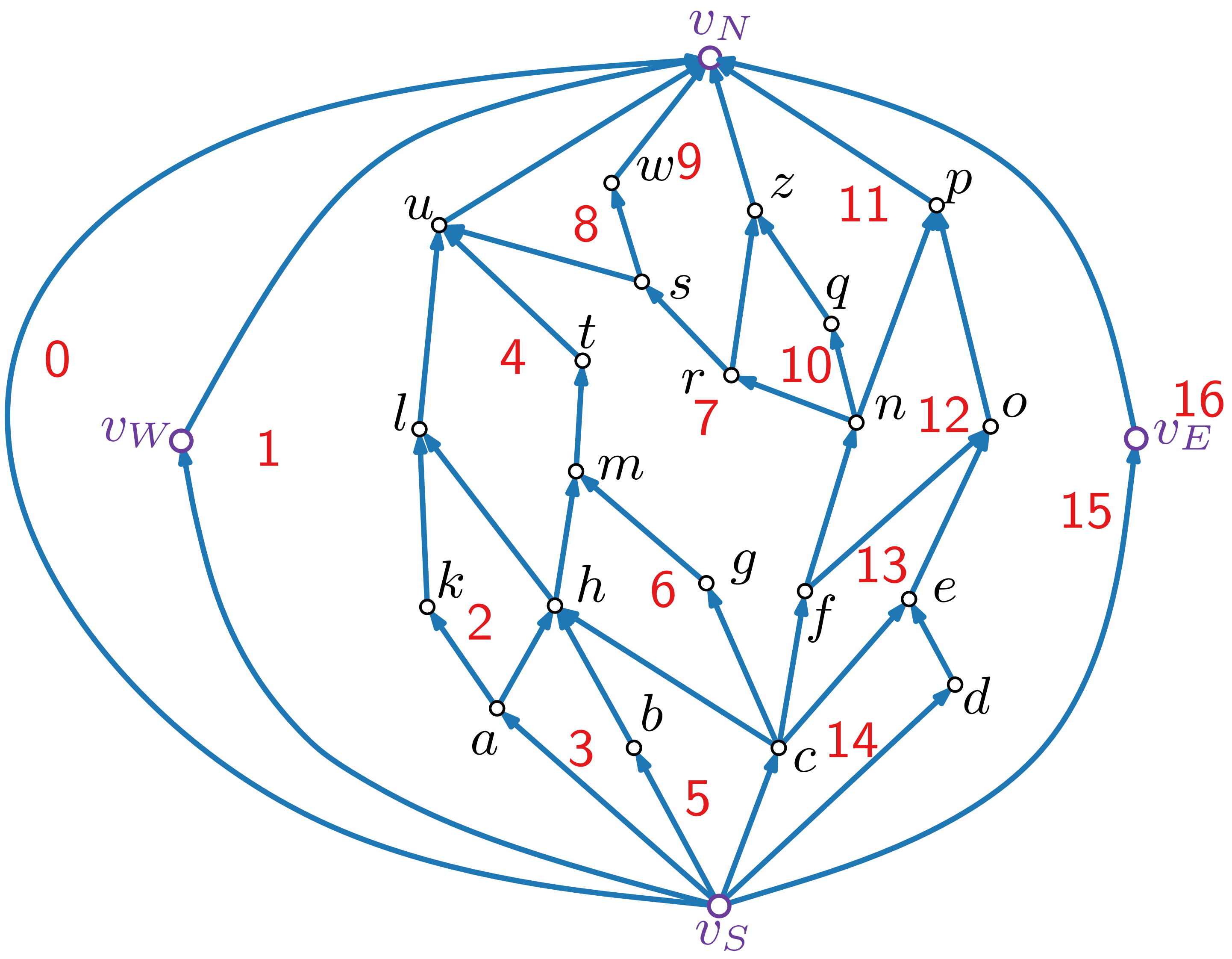
$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$

Reading off Coordinates to Get Rectangular Dual



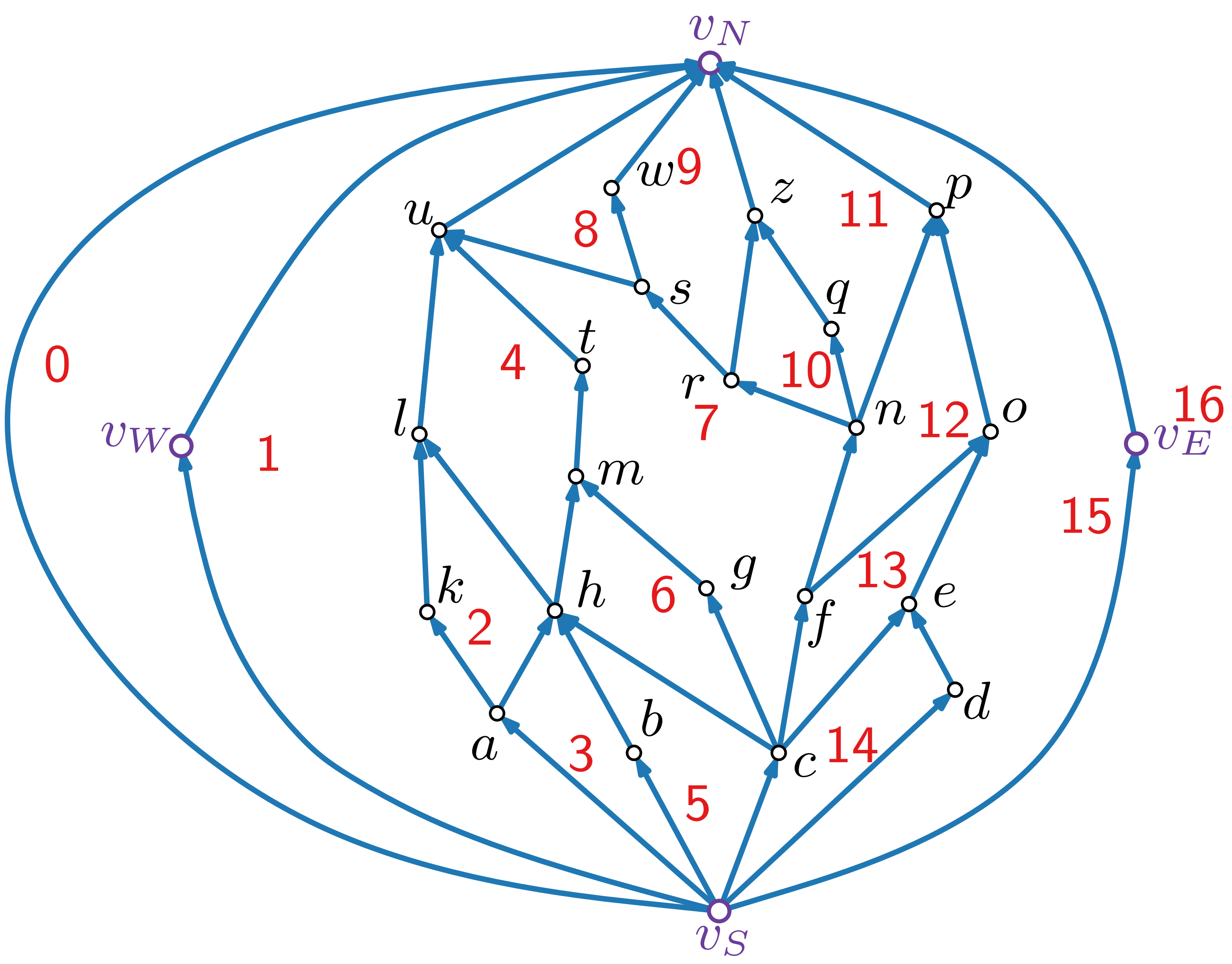
$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
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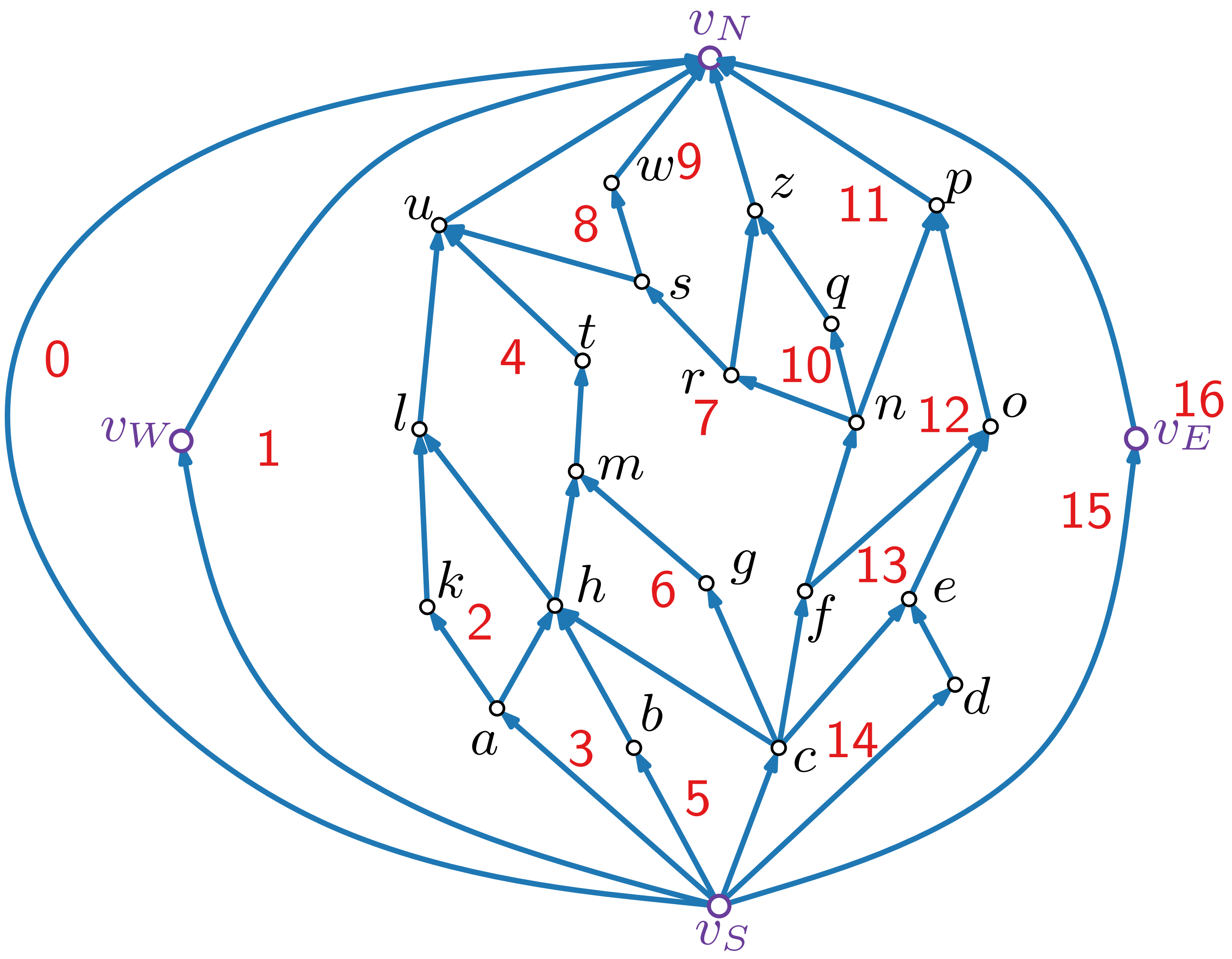
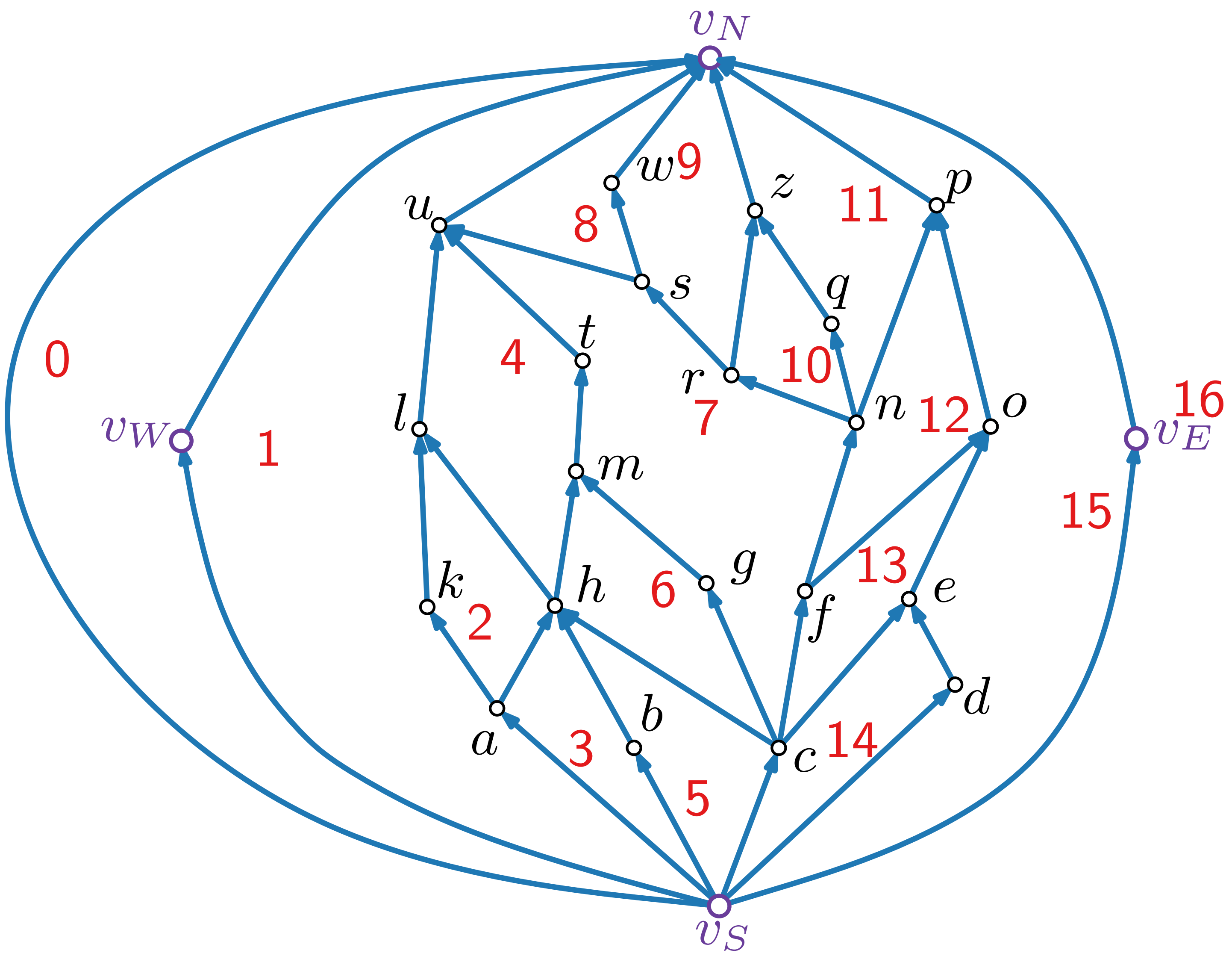
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$

$x_1(v_S) = 1,\ x_2(v_S) = 16$

$x_1(v_W) = 0, x_2(v_W) = 1$

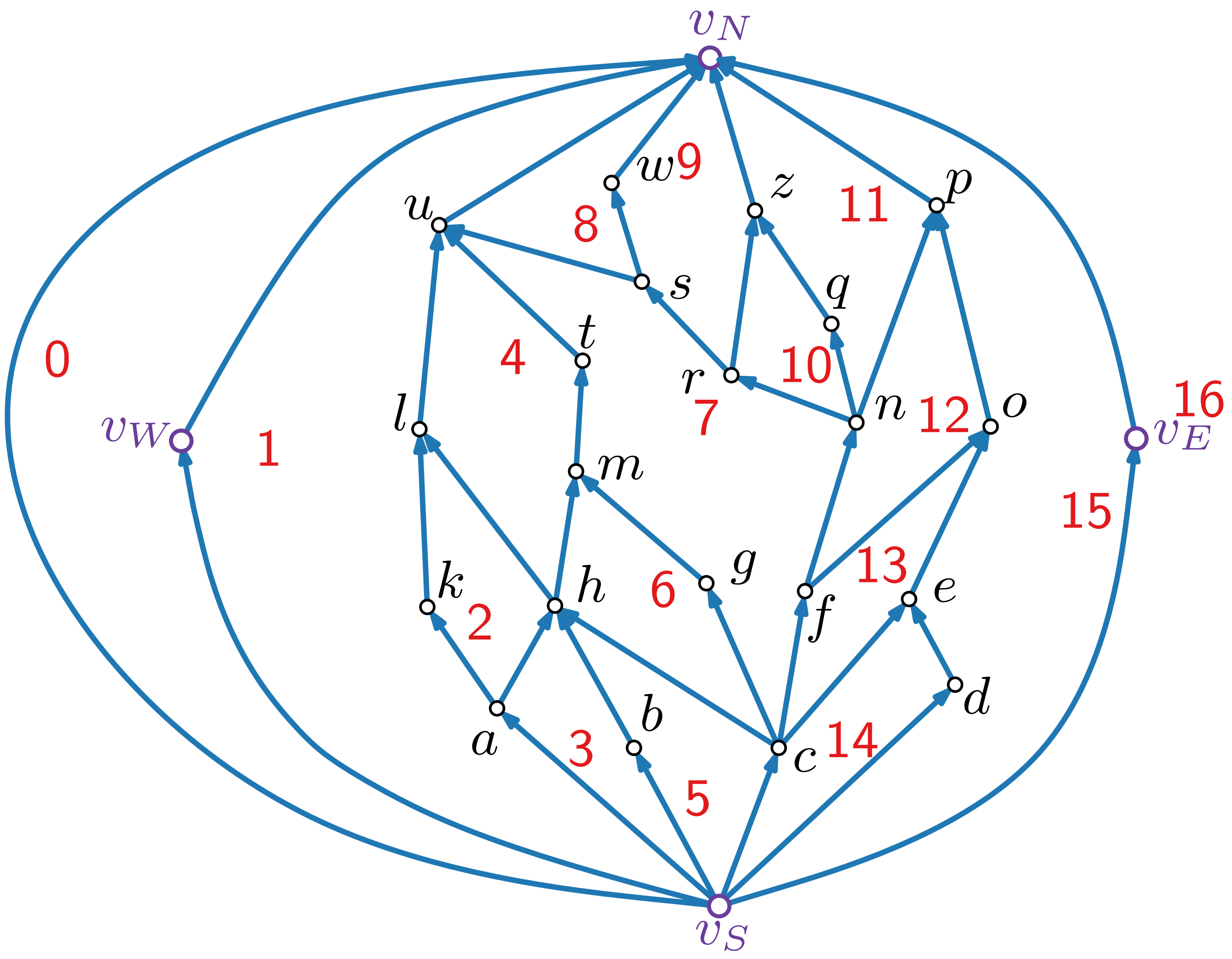
$x_1(v_E) = 15,\ x_2(v_E) = 16$

$x_1(a) = 1,\ x_2(a) = 3$

$x_1(b) = 3,\ x_2(b) = 5$

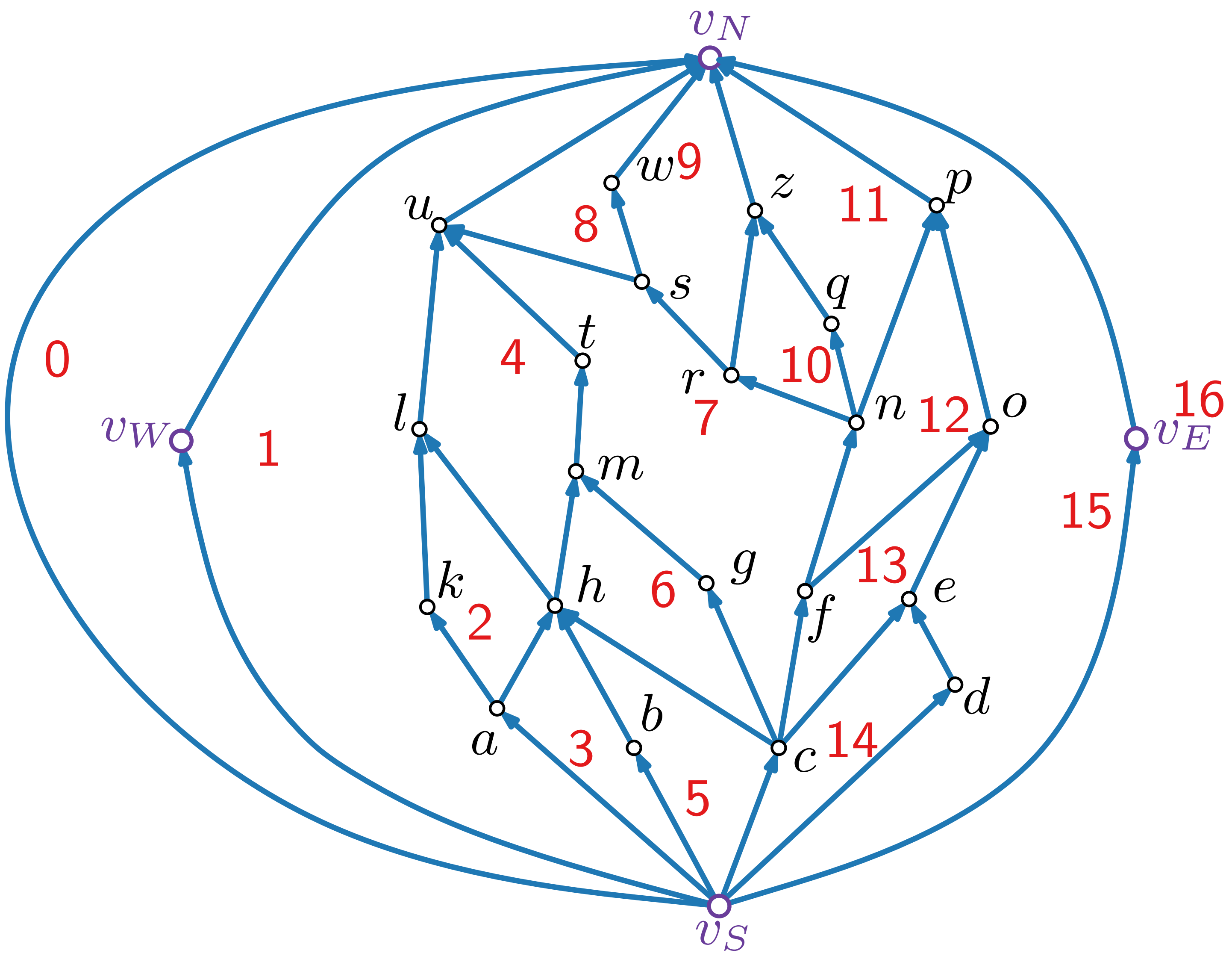
$x_1(c) = 5,\ x_2(c) = 14$

Reading off Coordinates to Get Rectangular Dual



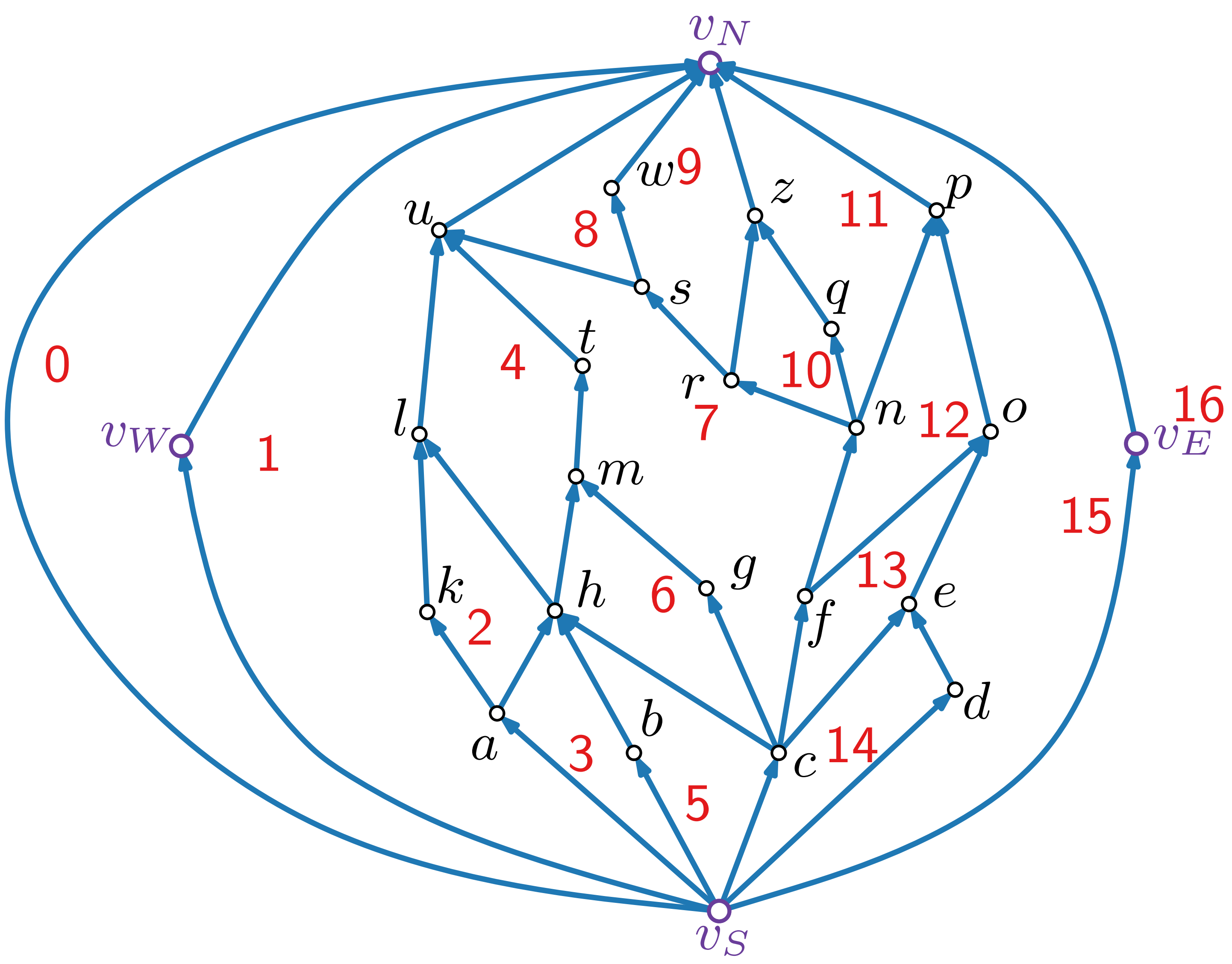
$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$

$x_1(v_S) = 1,\ x_2(v_S) = 16$

$x_1(v_W) = 0, x_2(v_W) = 1$

$x_1(v_E) = 15,\ x_2(v_E) = 16$

$x_1(a) = 1,\ x_2(a) = 3$

$x_1(b) = 3,\ x_2(b) = 5$

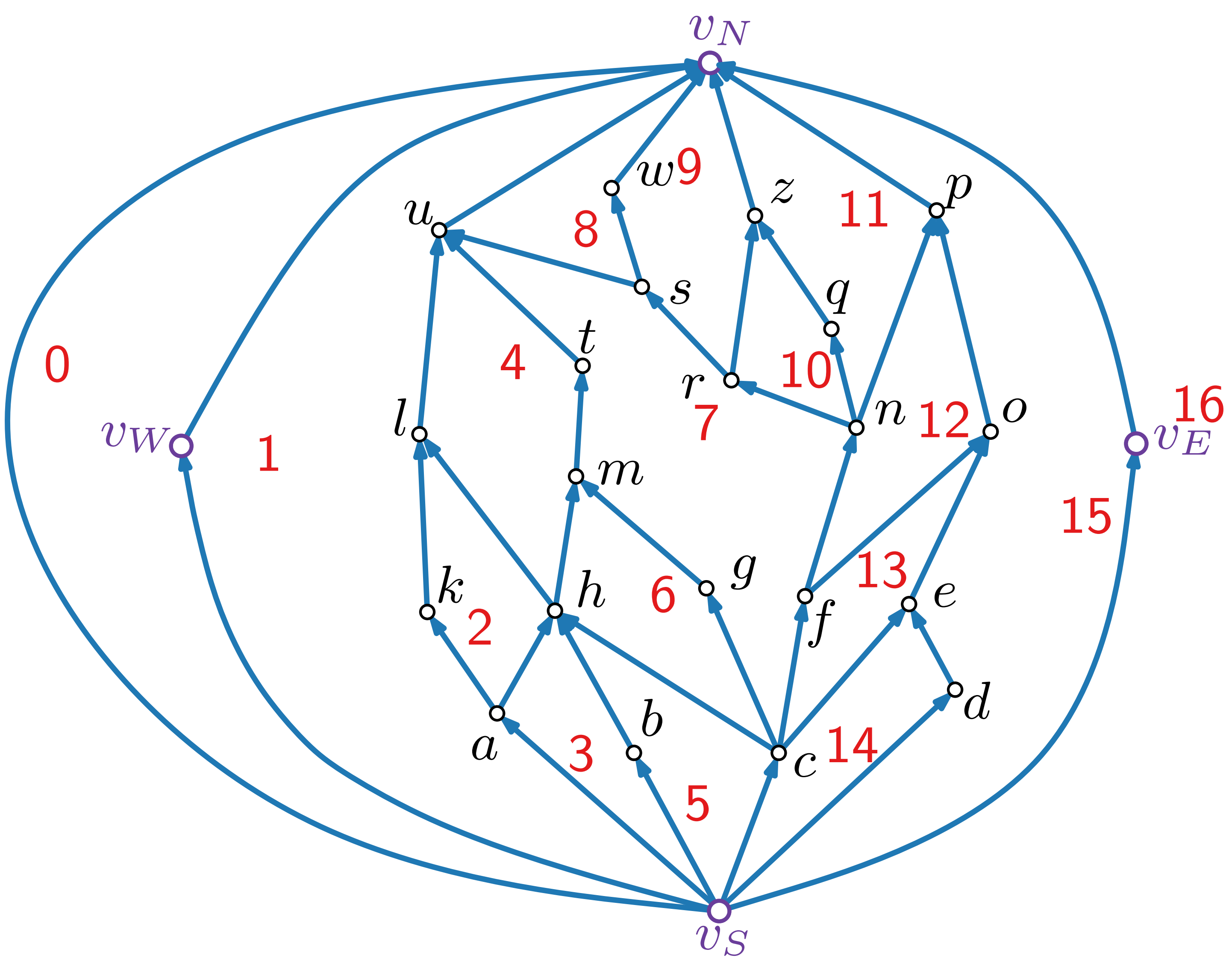
$x_1(c) = 5,\ x_2(c) = 14$

$x_1(d) = 14,\ x_2(d) = 15$

$x_1(e) = 13,\ x_2(e) = 15$

...

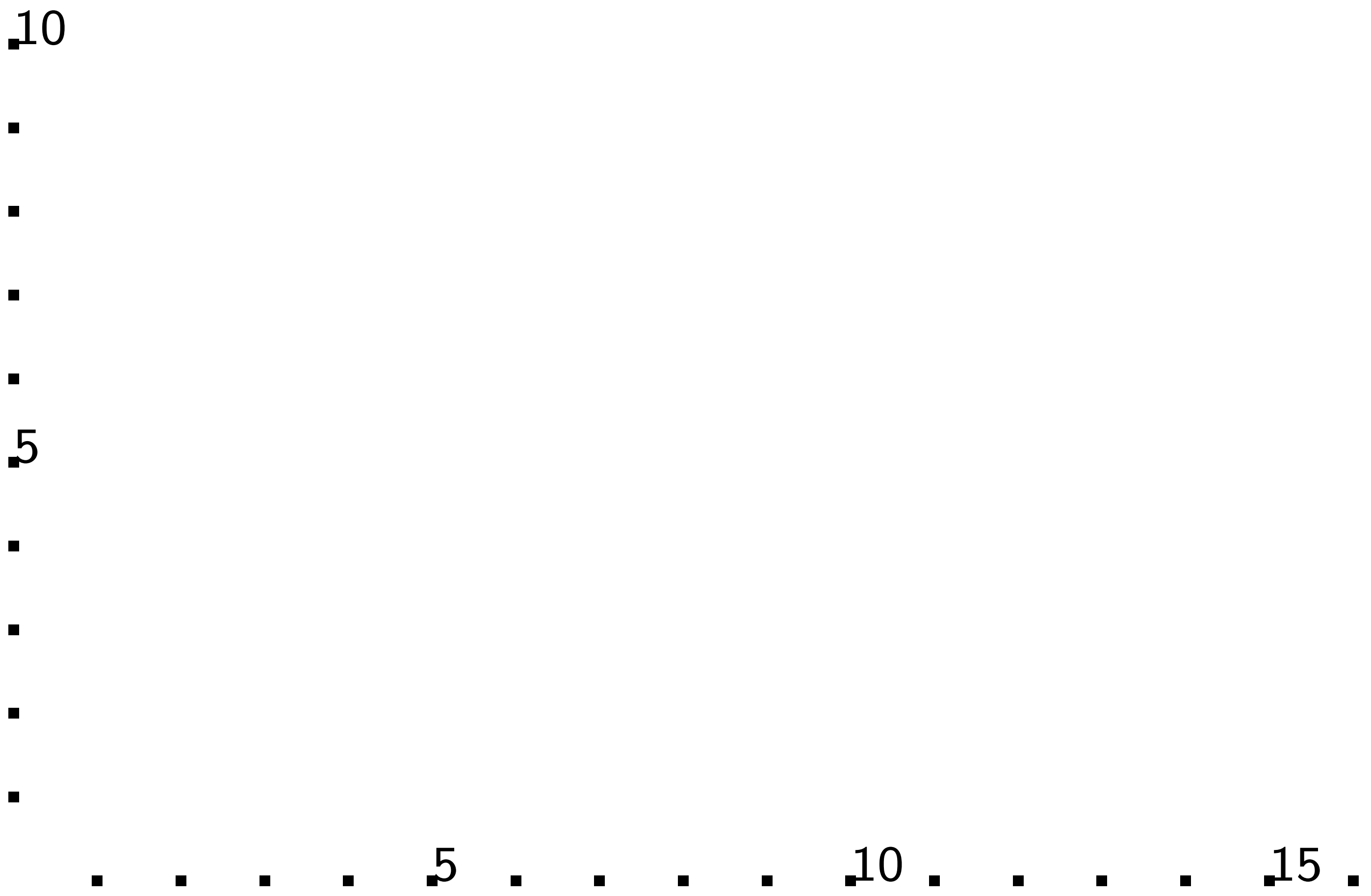
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\dots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\dots$

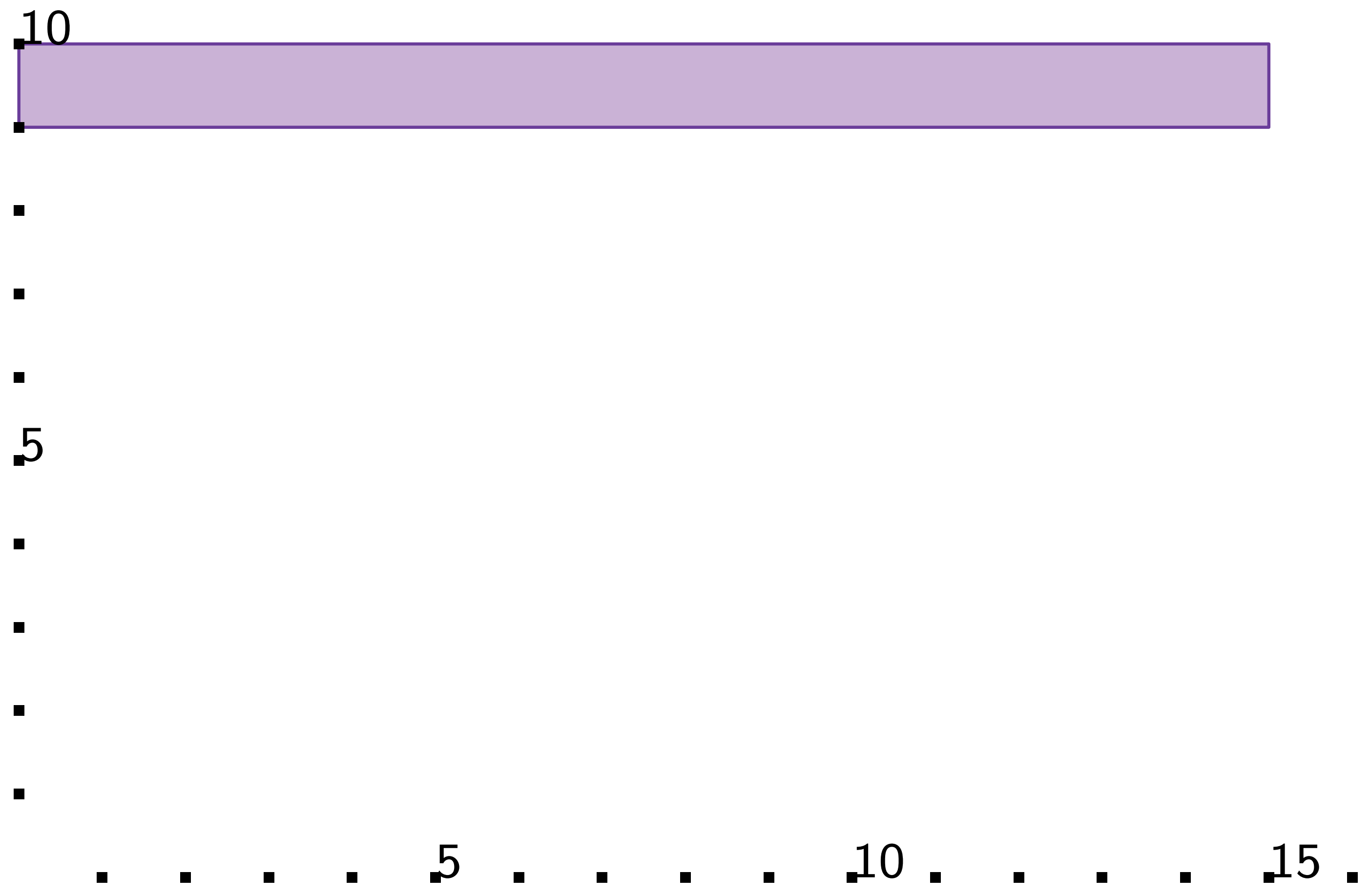
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

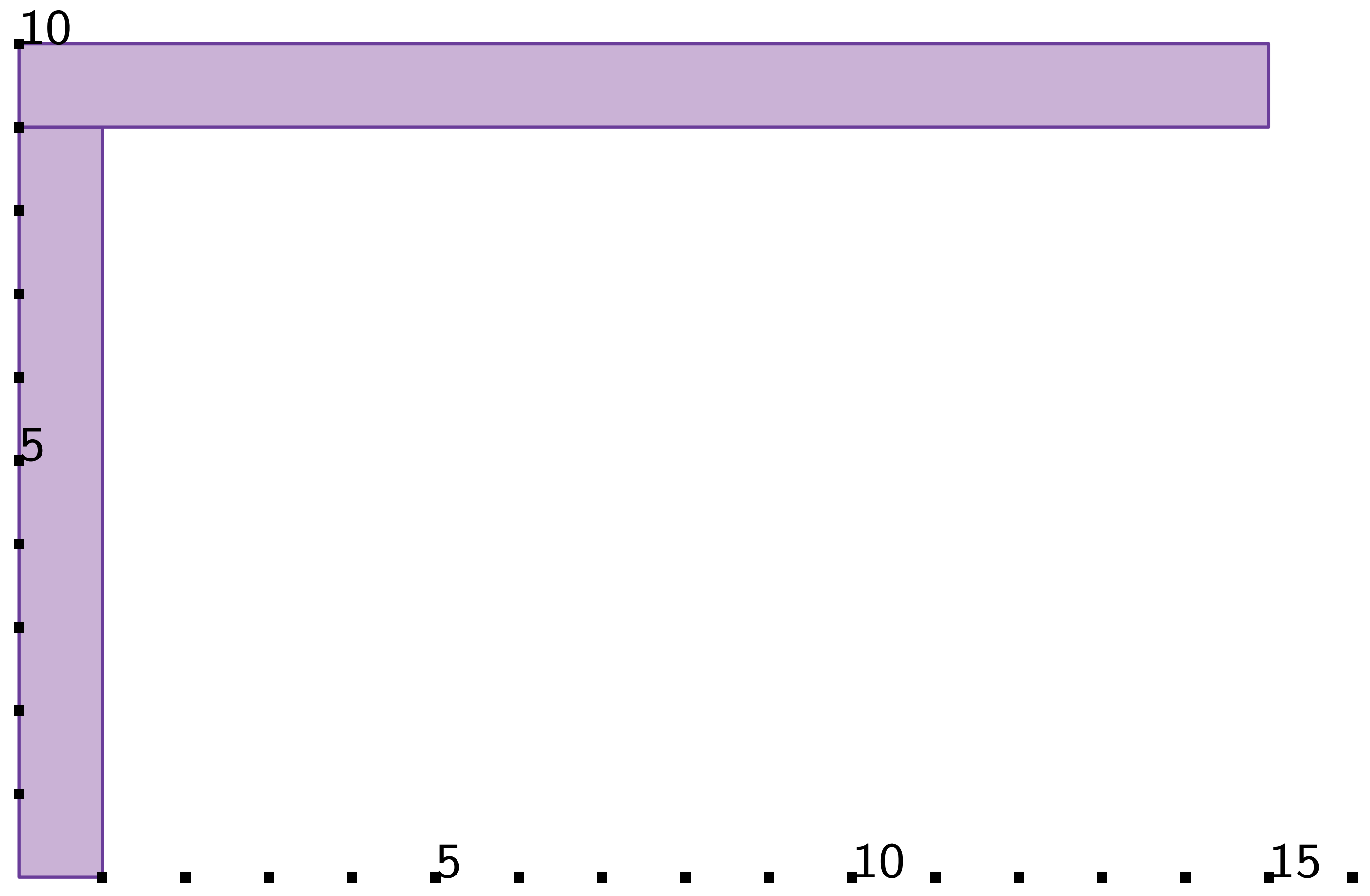
Reading off Coordinates to Get Rectangular Dual



$$x_1(v_N) = 0,\ x_2(v_N) = 15$$
$$x_1(v_S) = 1,\ x_2(v_S) = 16$$
$$x_1(v_W) = 0, x_2(v_W) = 1$$
$$x_1(v_E) = 15,\ x_2(v_E) = 16$$
$$x_1(a) = 1,\ x_2(a) = 3$$
$$x_1(b) = 3,\ x_2(b) = 5$$
$$x_1(c) = 5,\ x_2(c) = 14$$
$$x_1(d) = 14,\ x_2(d) = 15$$
$$x_1(e) = 13,\ x_2(e) = 15$$
$$\ldots$$

$$y_1(v_W) = 0, y_2(v_W) = 9$$
$$y_1(v_E) = 1,\ y_2(v_E) = 10$$
$$y_1(v_N) = 9,\ y_2(v_N) = 10$$
$$y_1(v_S) = 0,\ y_2(v_S) = 1$$
$$y_1(a) = 1,\ y_2(a) = 2$$
$$y_1(b) = 1,\ y_2(b) = 2$$
$$\ldots$$

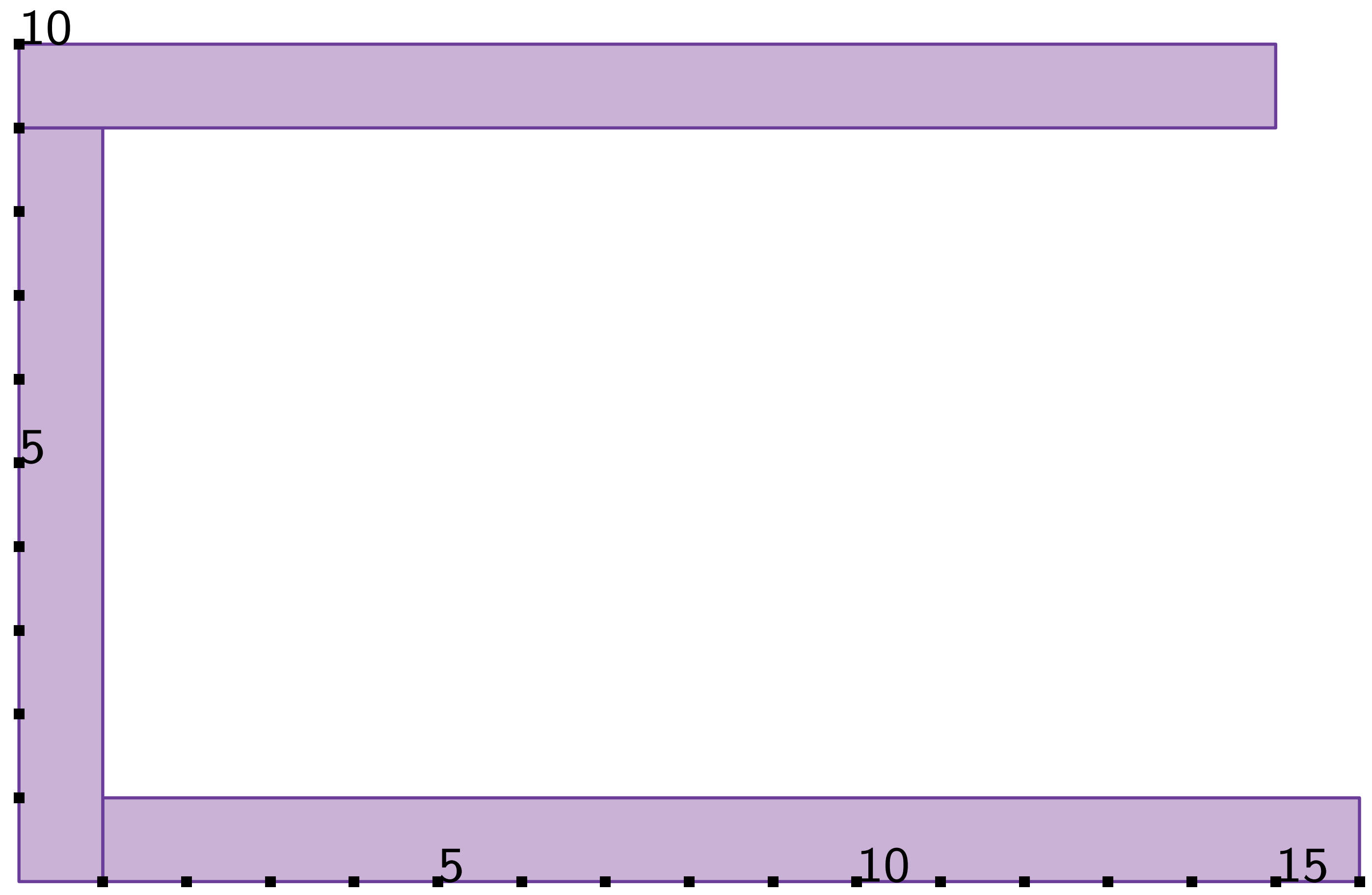
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

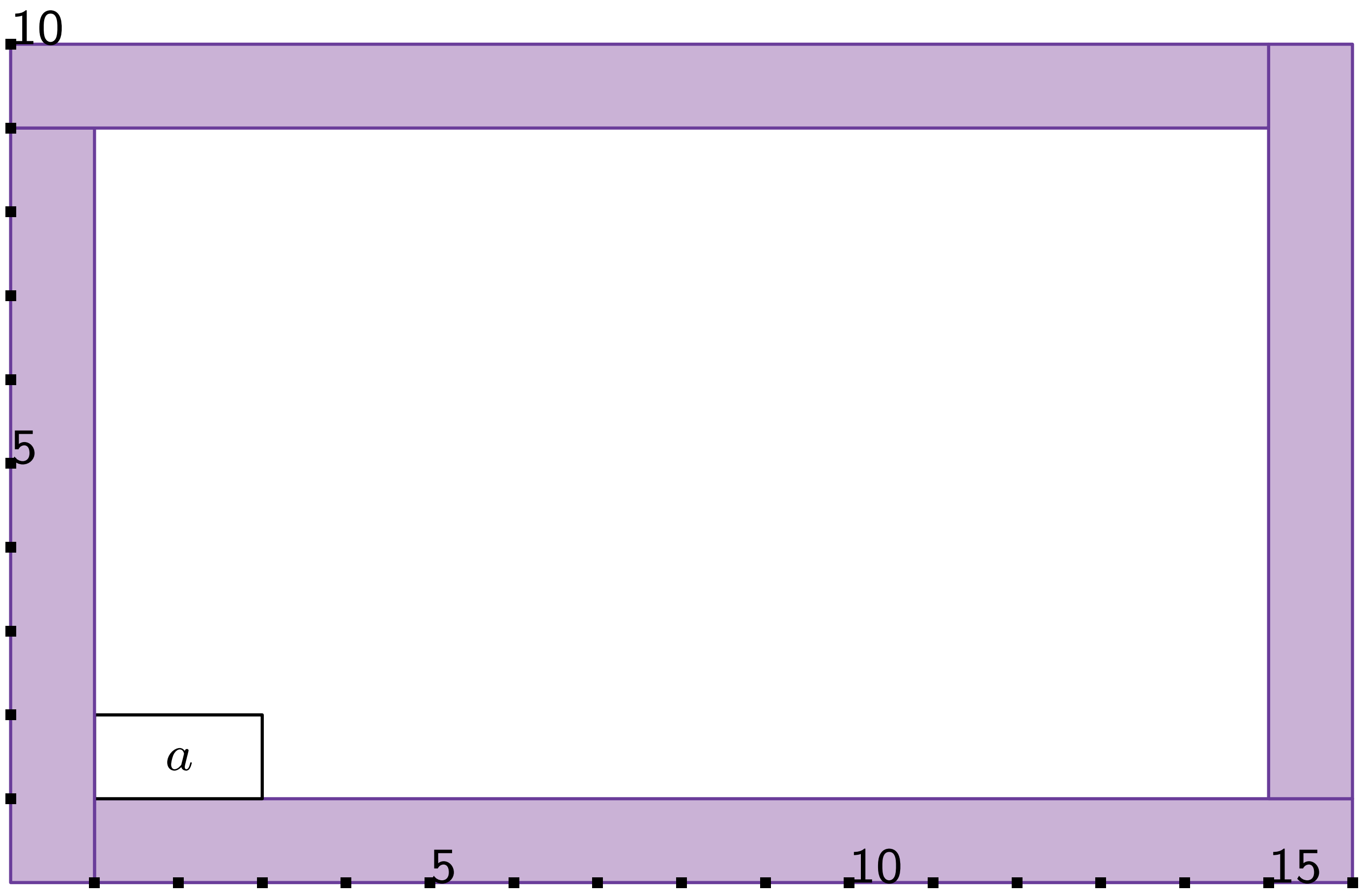
$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

Reading off Coordinates to Get Rectangular Dual

$x_1(v_N) = 0, \; x_2(v_N) = 15$
$x_1(v_S) = 1, \; x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15, \; x_2(v_E) = 16$
$x_1(a) = 1, \; x_2(a) = 3$
$x_1(b) = 3, \; x_2(b) = 5$
$x_1(c) = 5, \; x_2(c) = 14$
$x_1(d) = 14, \; x_2(d) = 15$
$x_1(e) = 13, \; x_2(e) = 15$
...

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1, \; y_2(v_E) = 10$
$y_1(v_N) = 9, \; y_2(v_N) = 10$
$y_1(v_S) = 0, \; y_2(v_S) = 1$
$y_1(a) = 1, \; y_2(a) = 2$
$y_1(b) = 1, \; y_2(b) = 2$
...

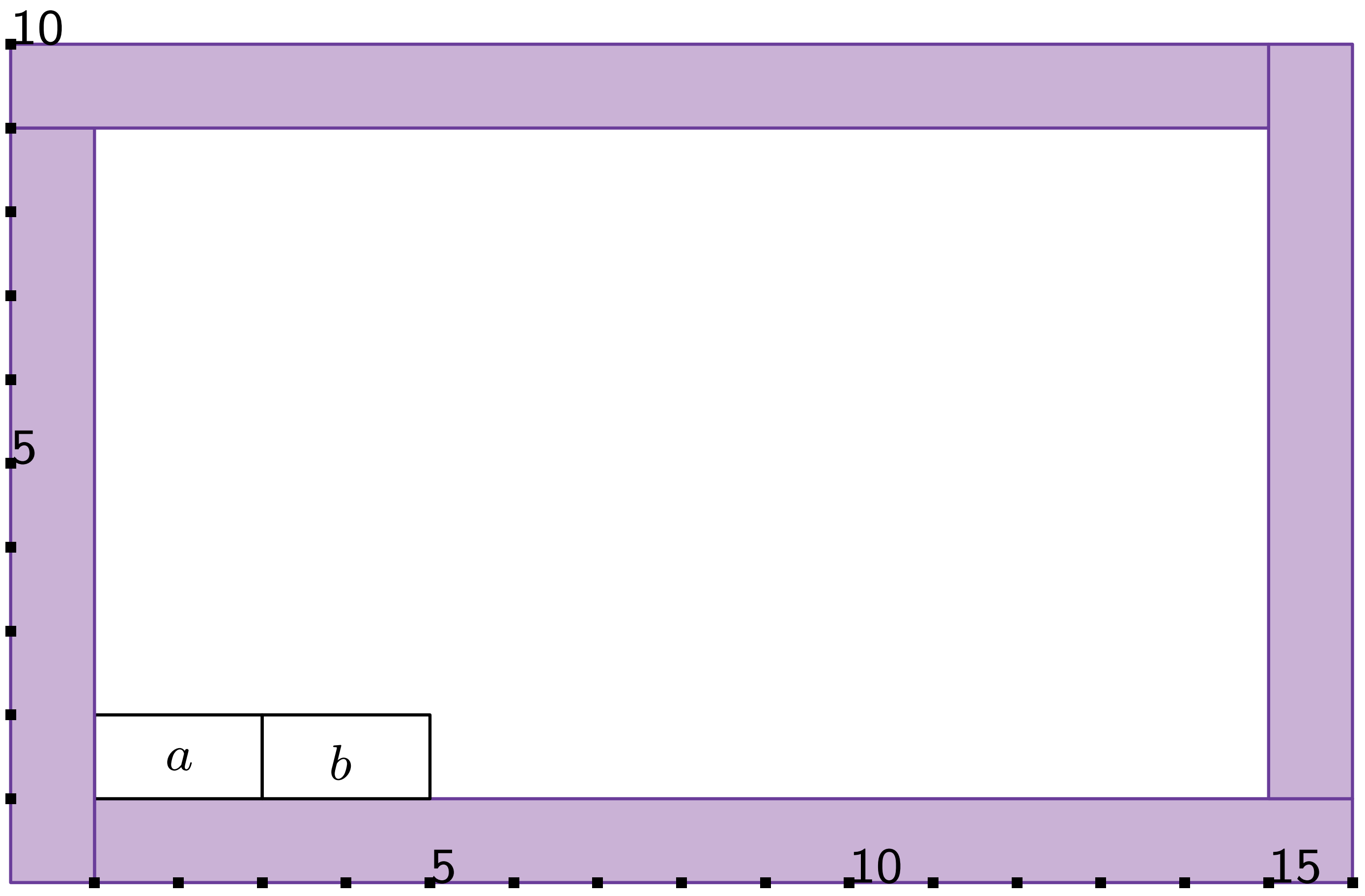
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

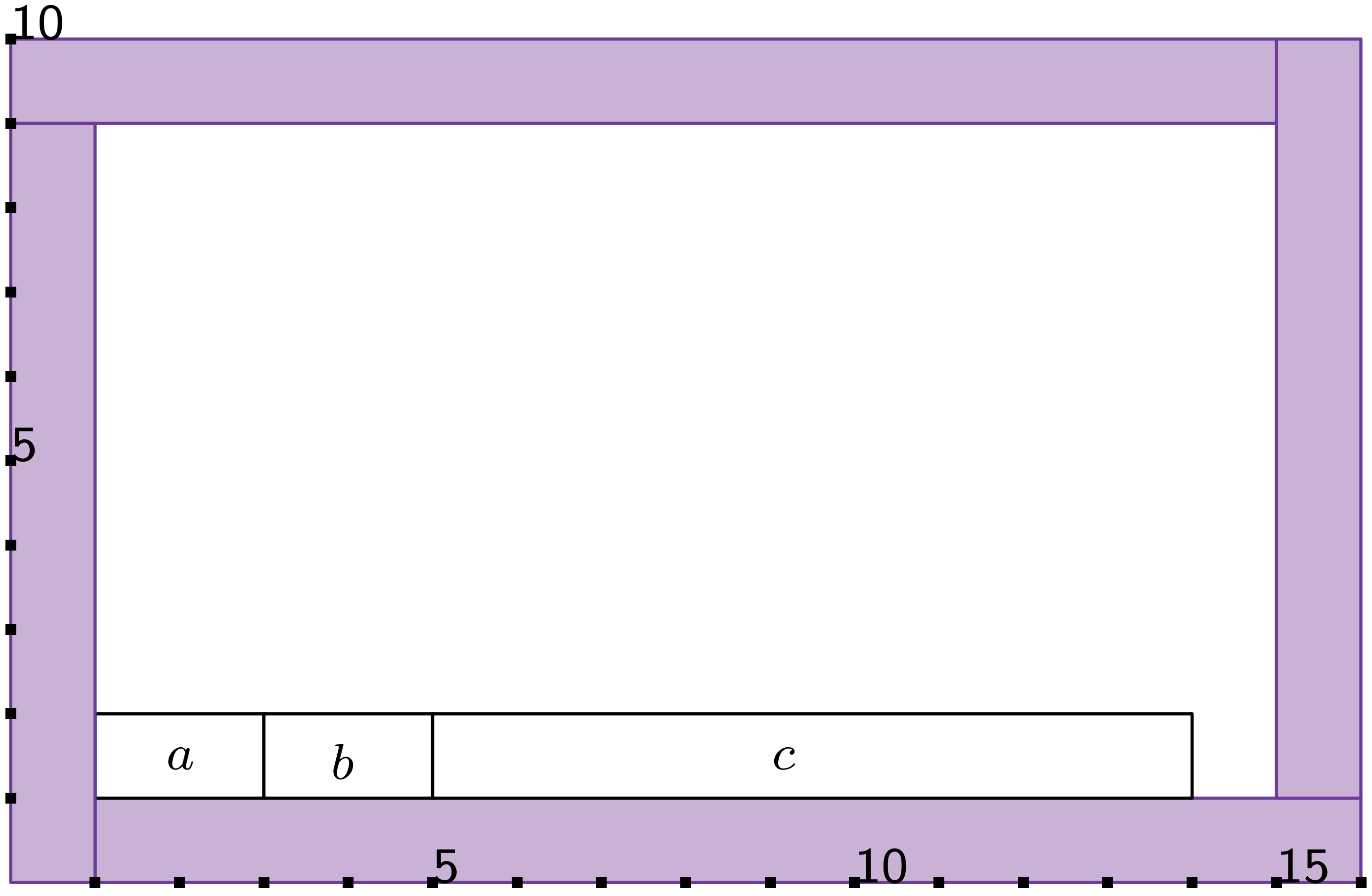
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

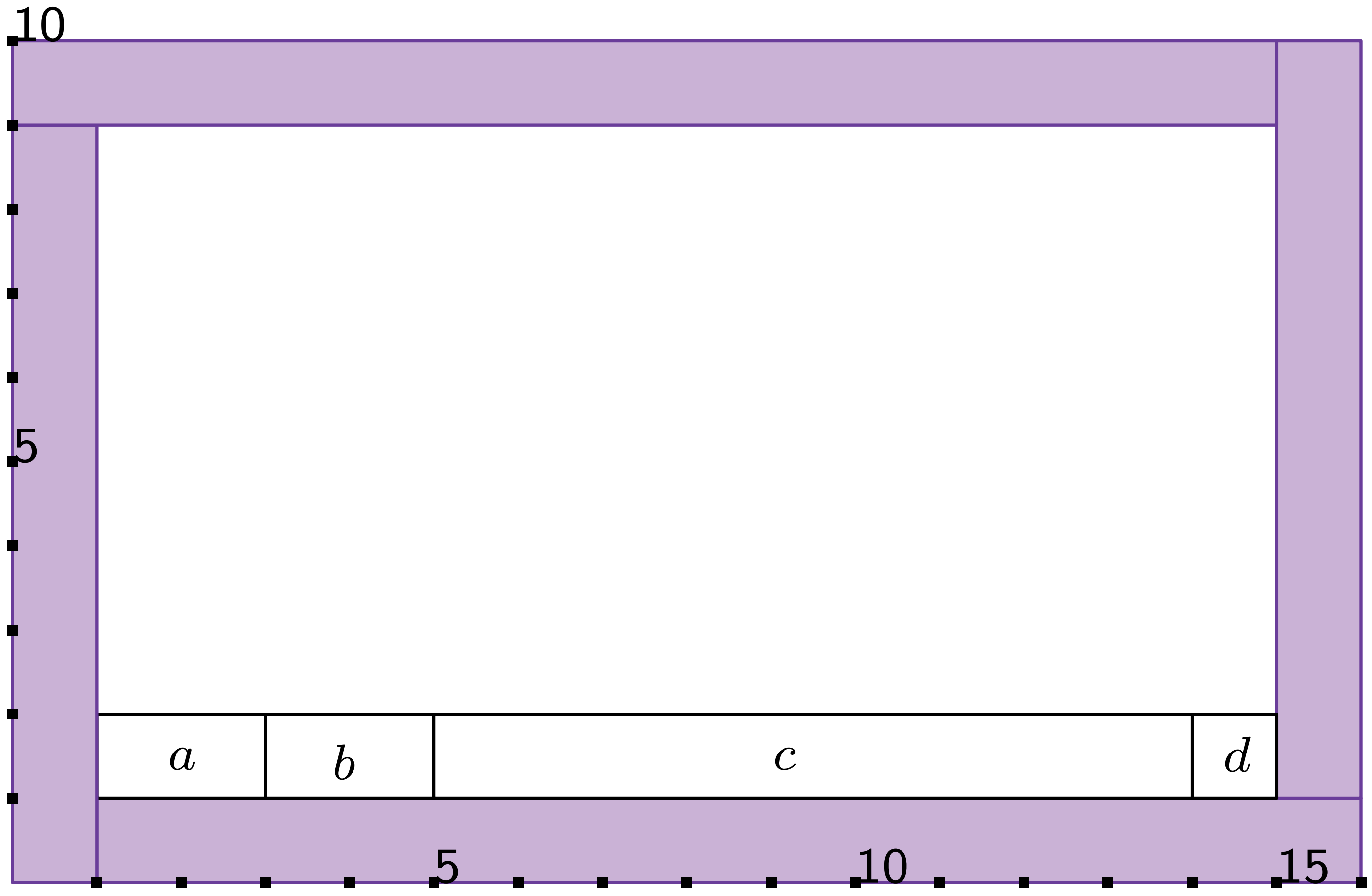
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

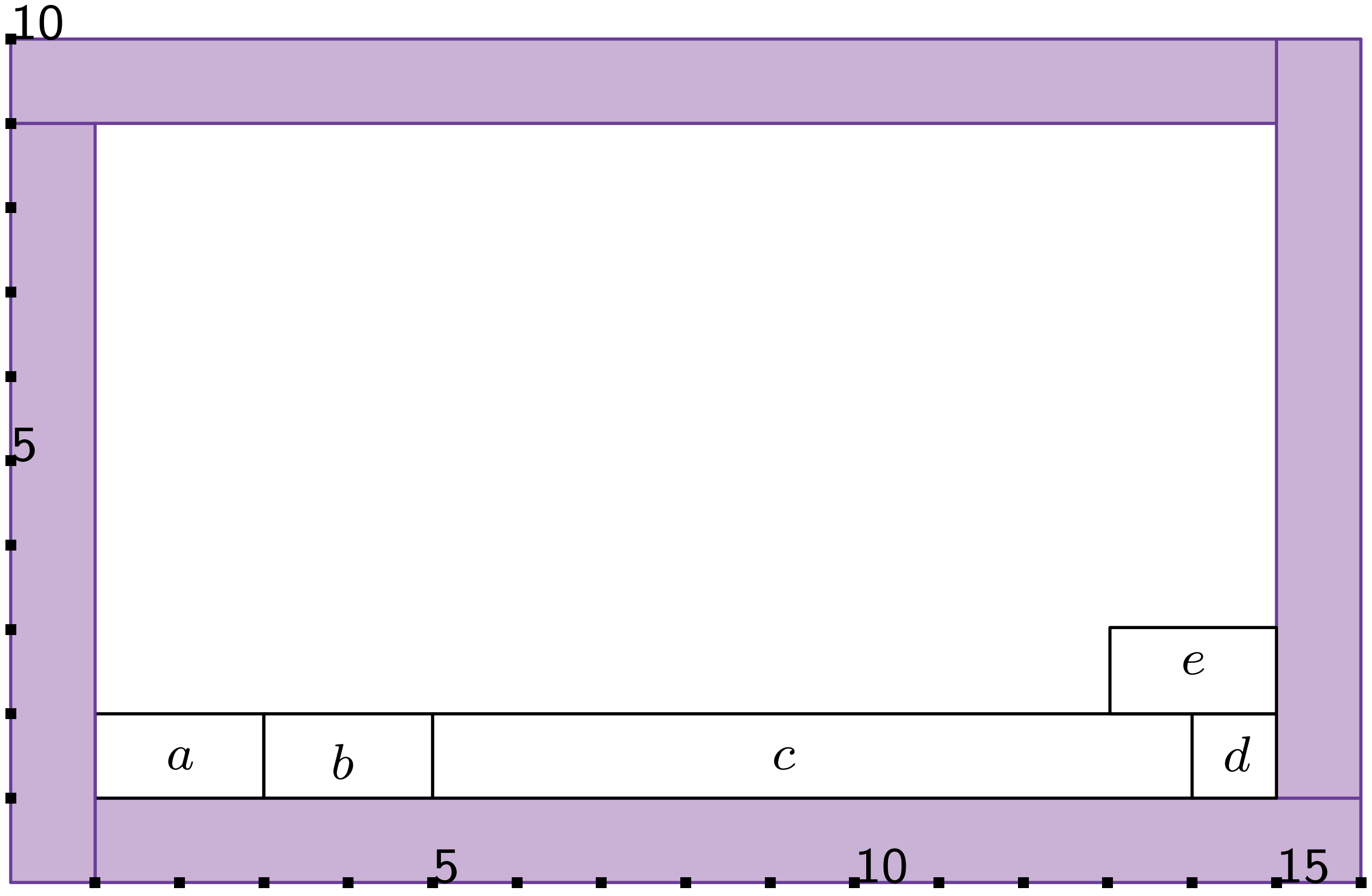
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

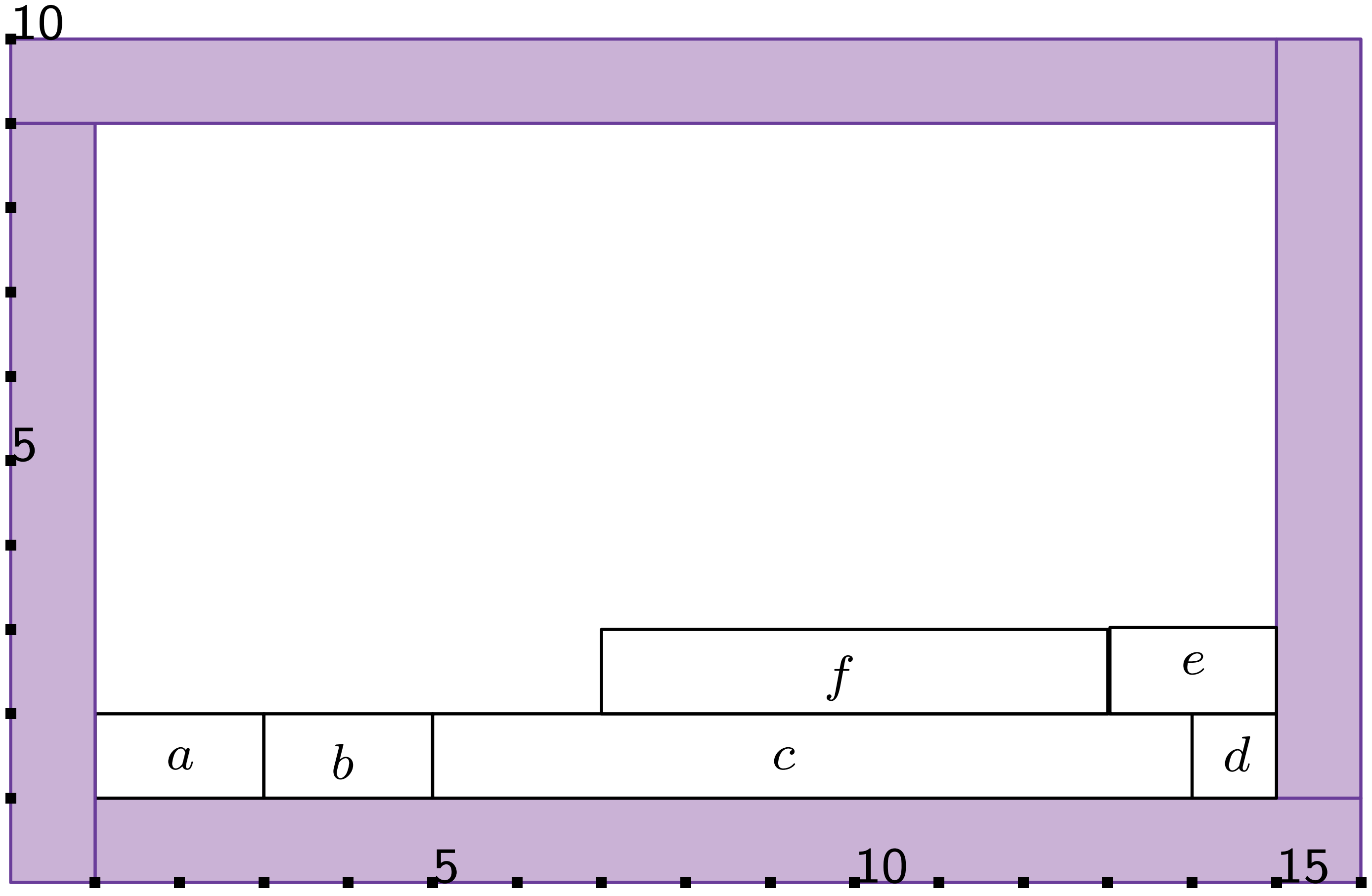
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\dots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\dots$

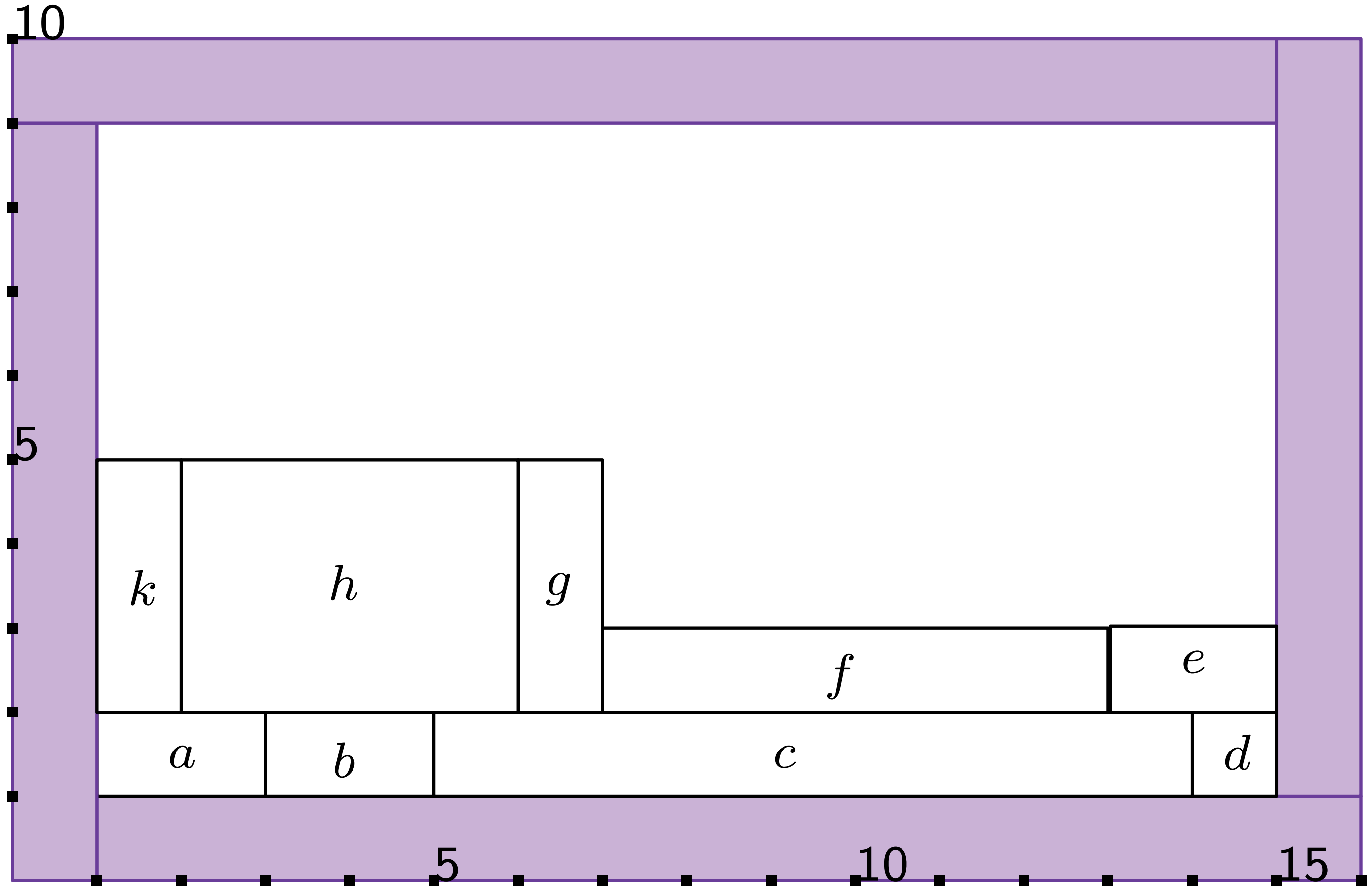
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\ldots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\ldots$

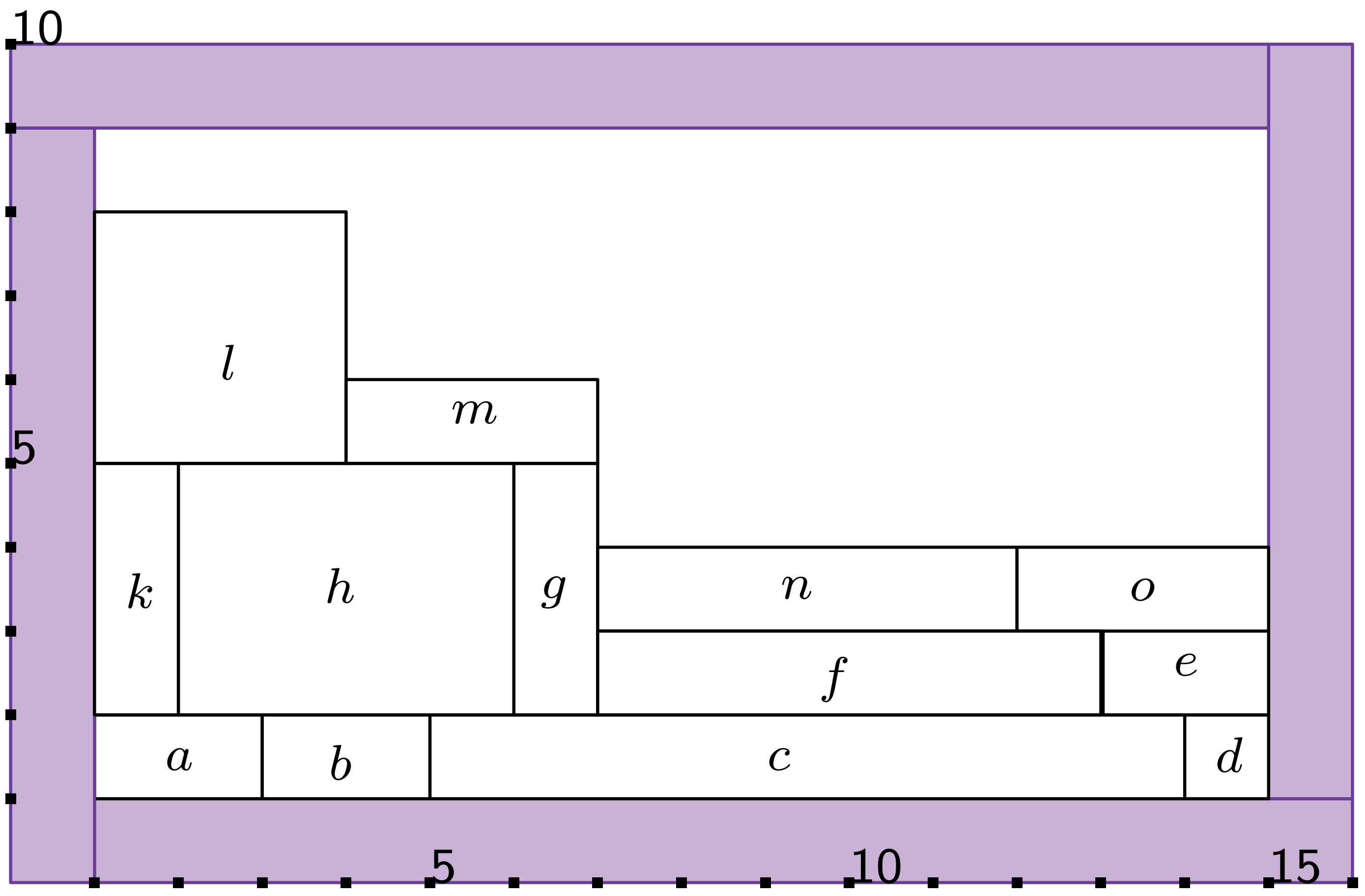
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
$\dots$

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$\dots$

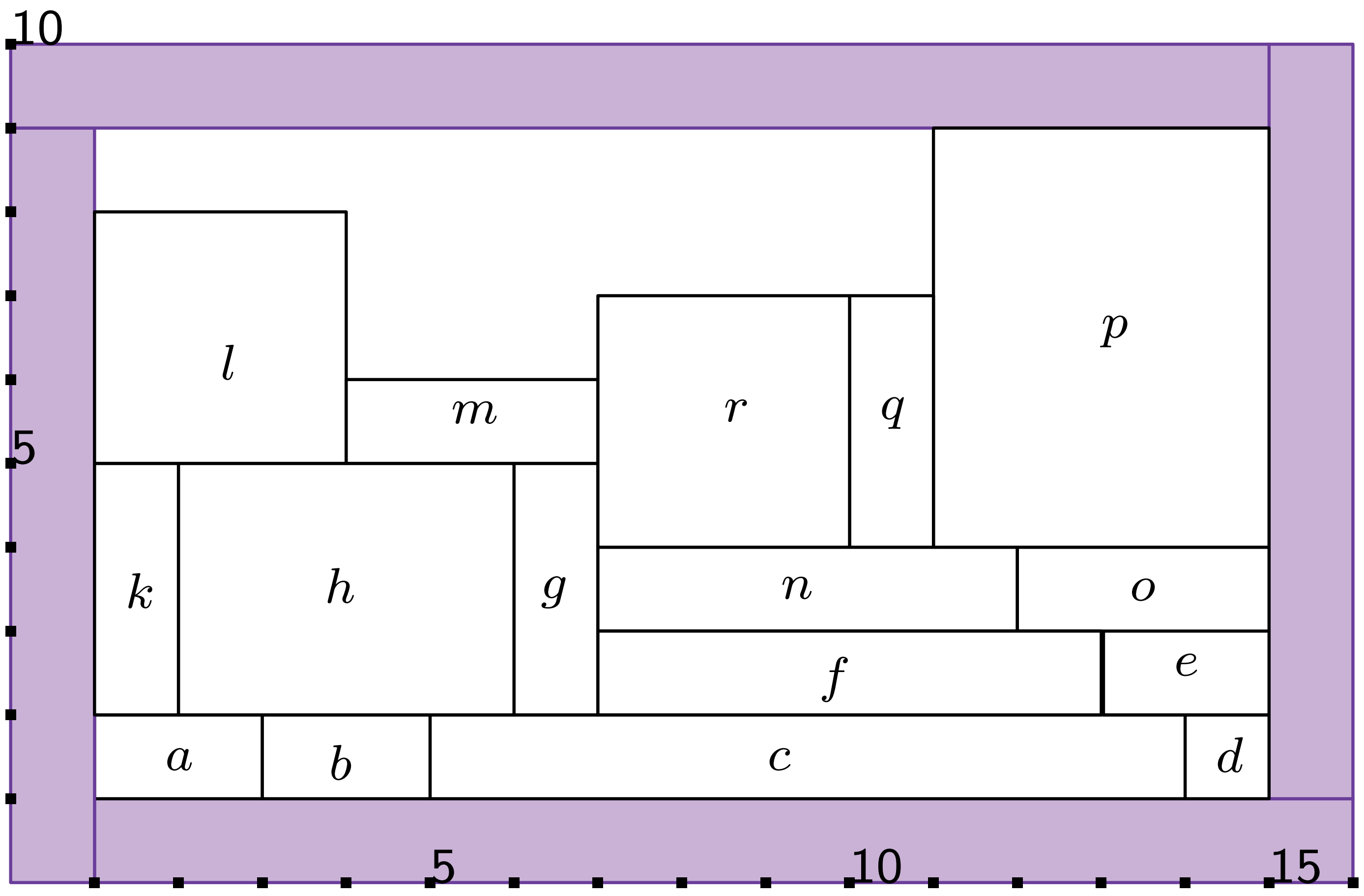
Reading off Coordinates to Get Rectangular Dual



$x_1(v_N) = 0,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$
...

$y_1(v_W) = 0, y_2(v_W) = 9$
$y_1(v_E) = 1,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
...

Reading off Coordinates to Get Rectangular Dual



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$x_1(v_S) = 1,\ x_2(v_S) = 16$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
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$\dots$

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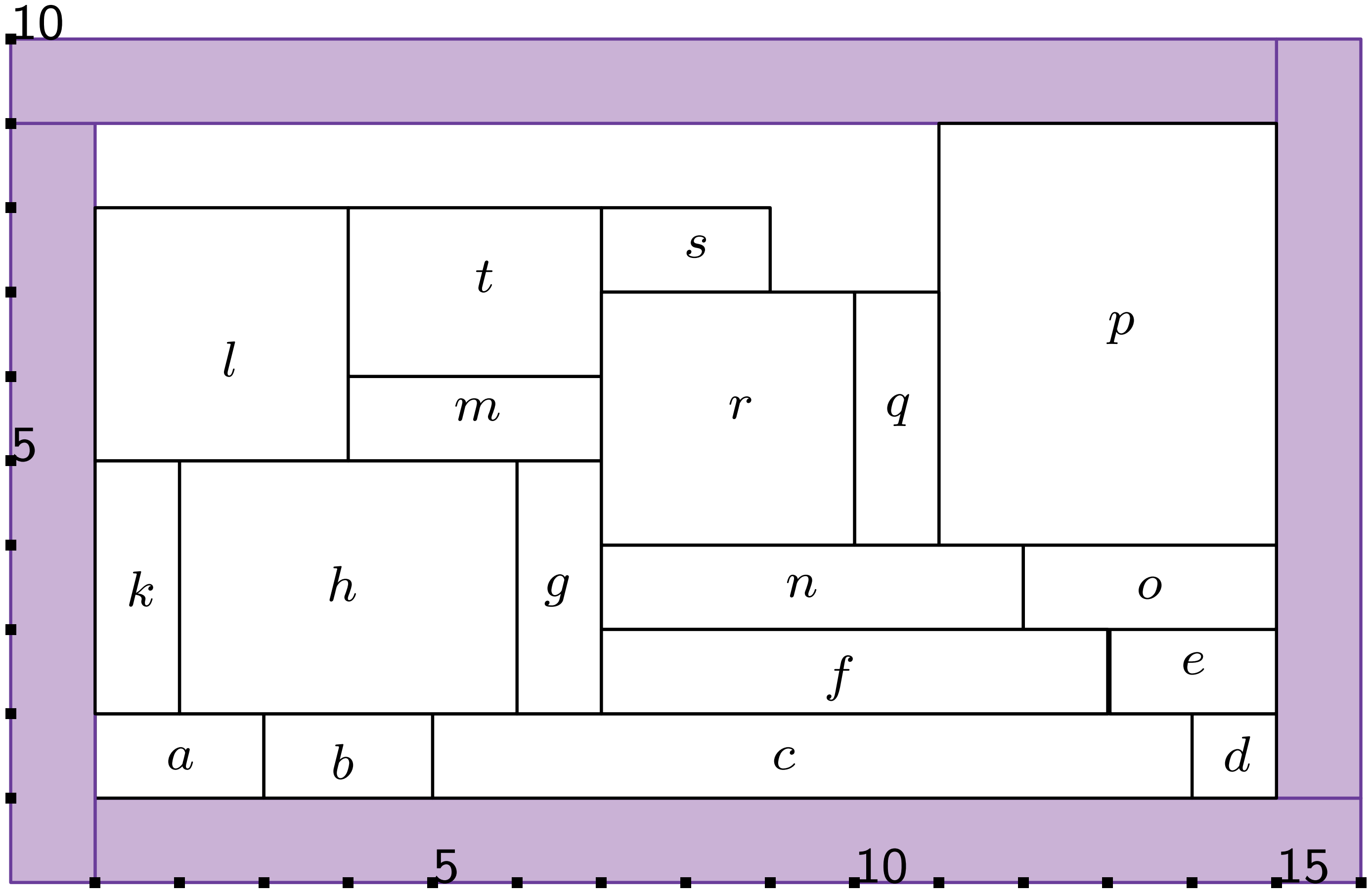

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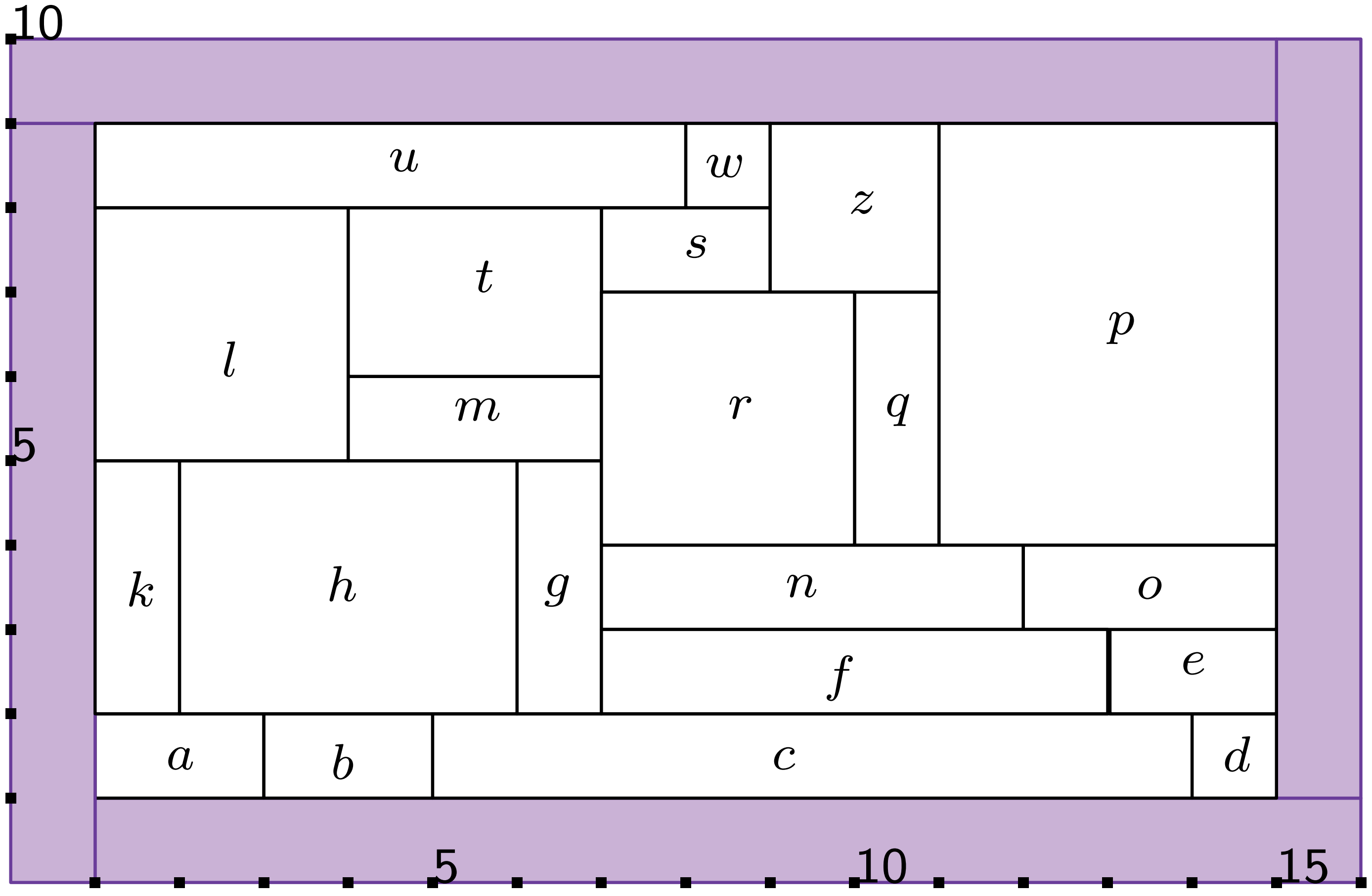

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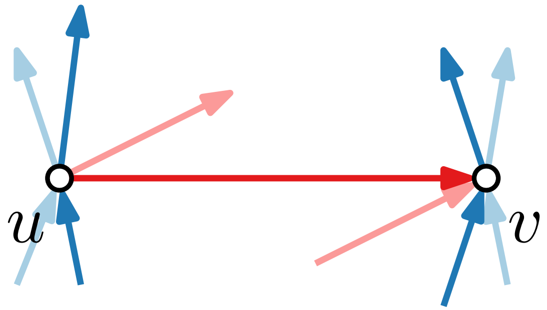

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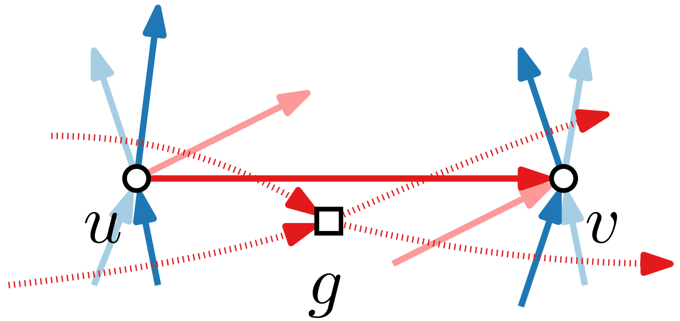

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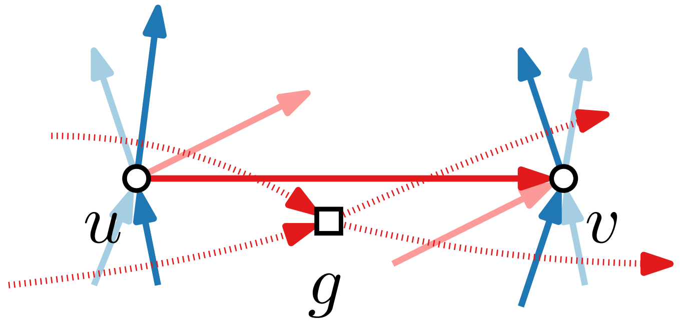

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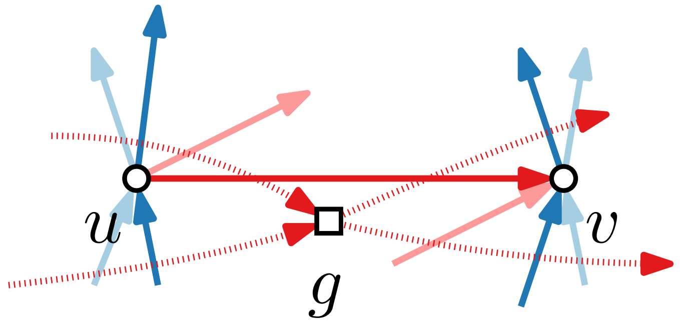

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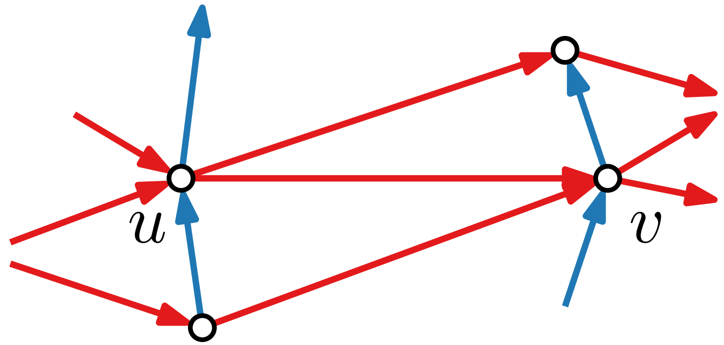

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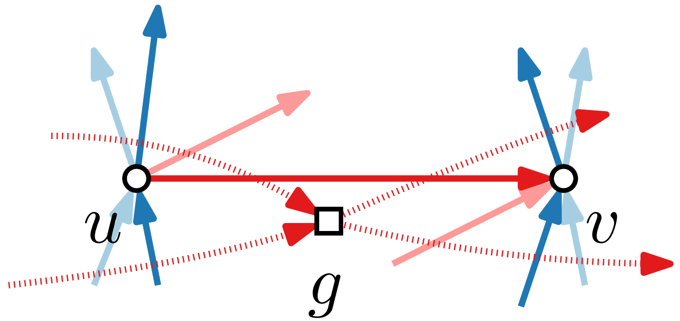

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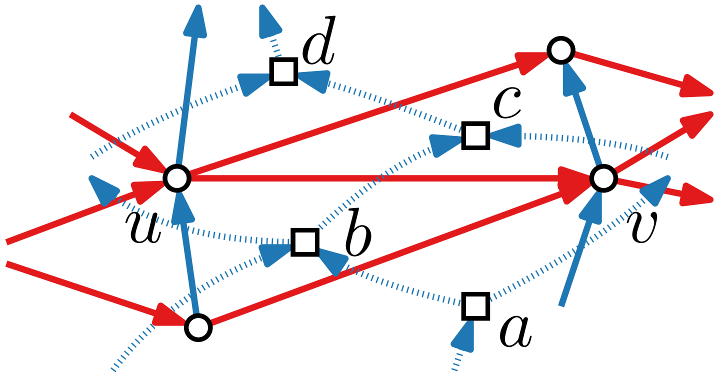

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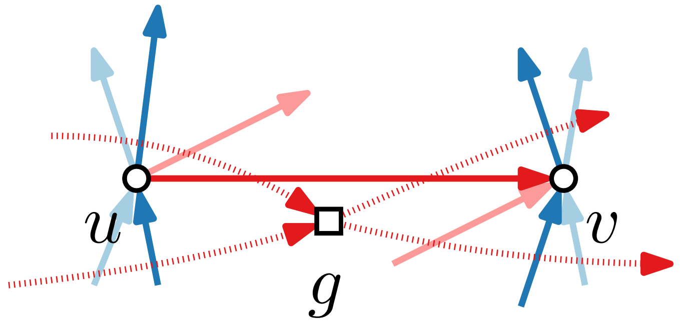

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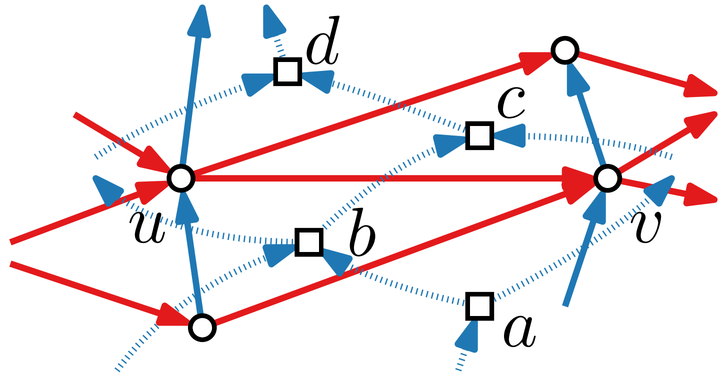

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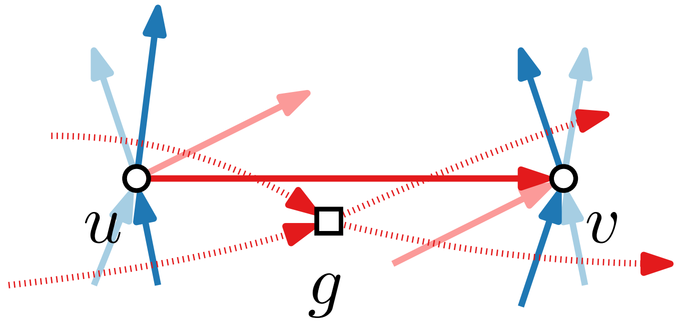

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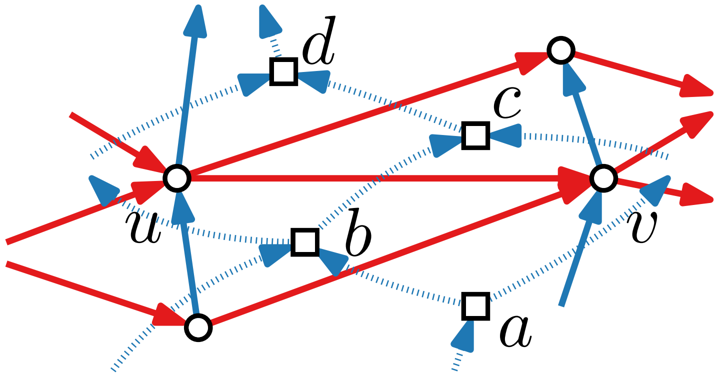

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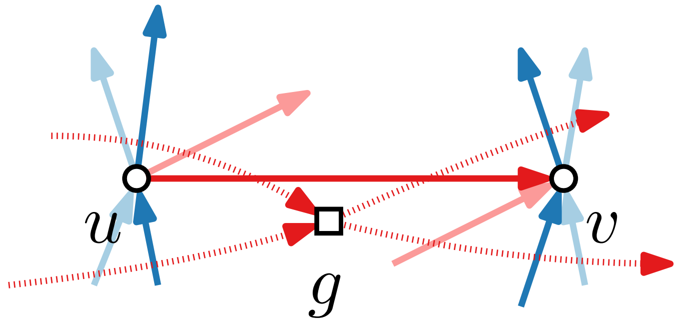

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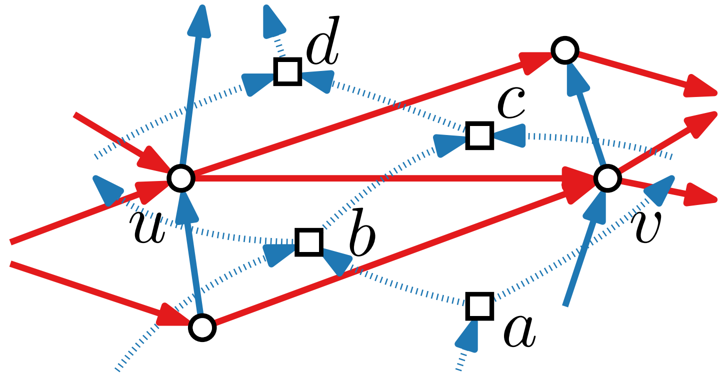

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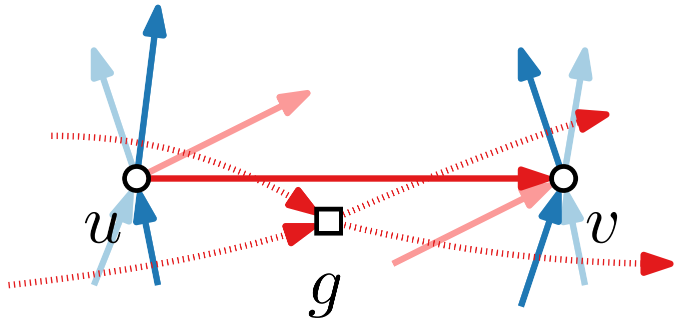

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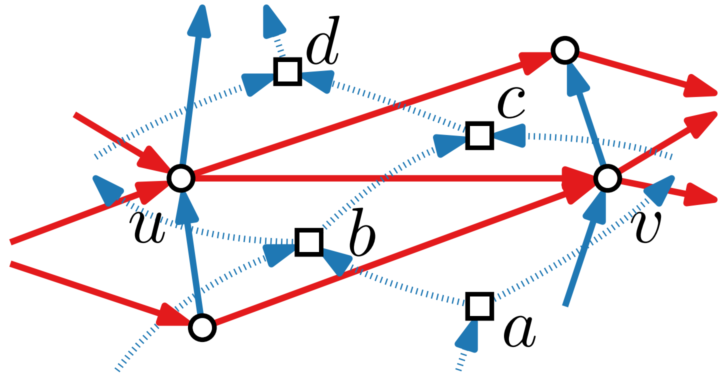

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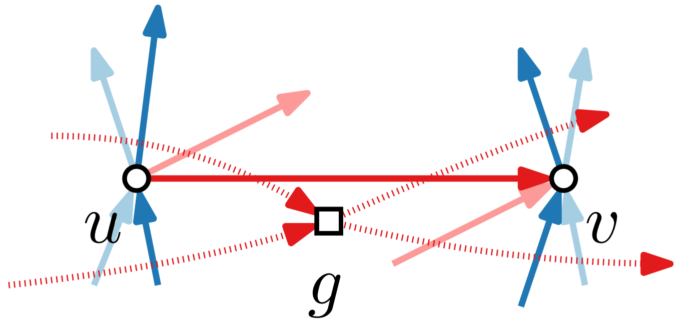

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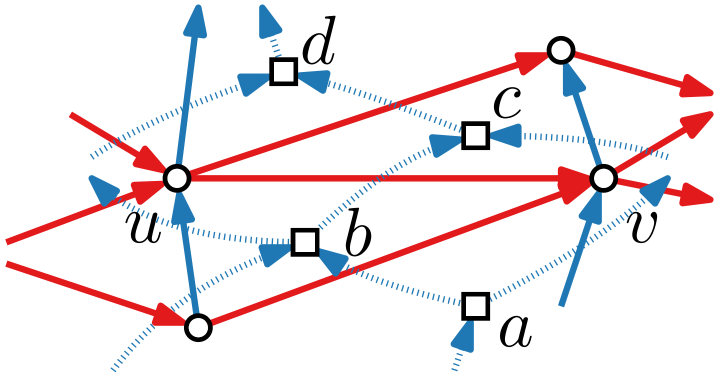

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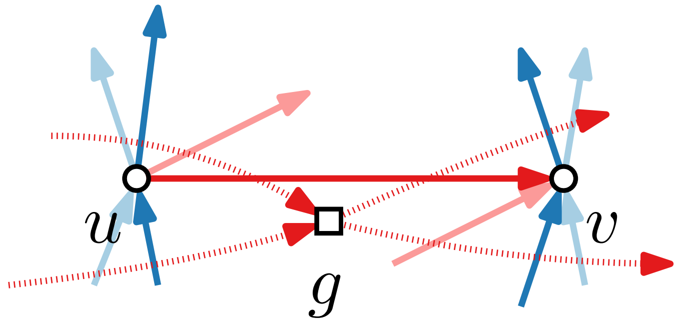

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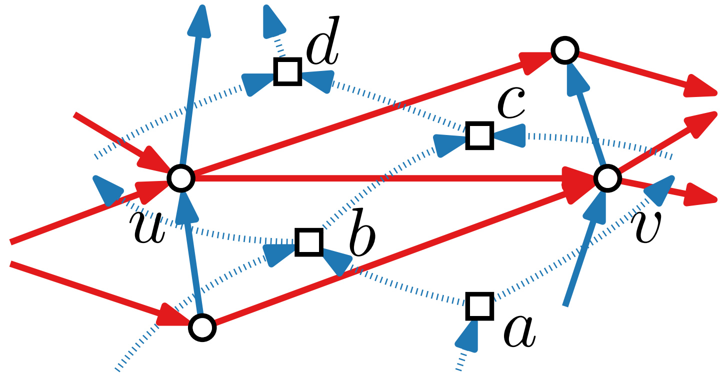

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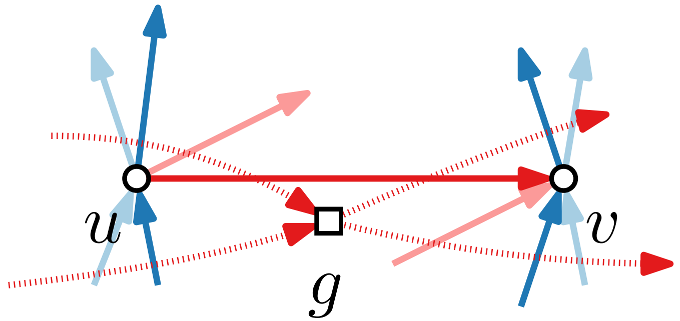

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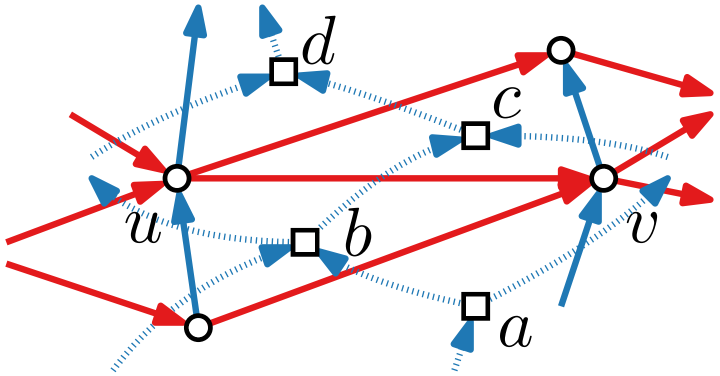

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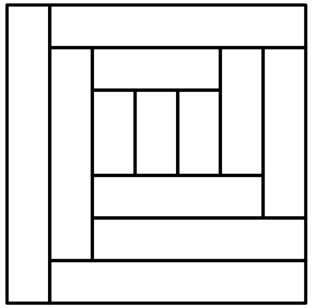

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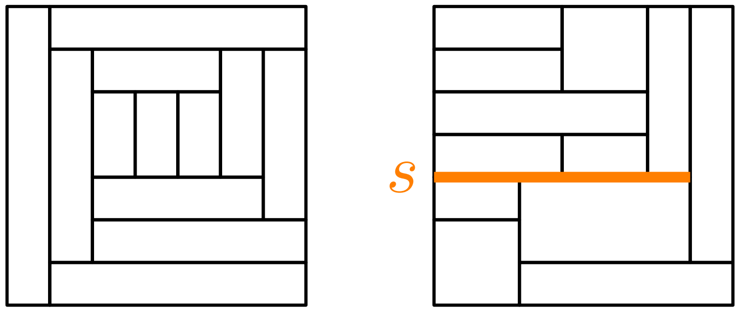

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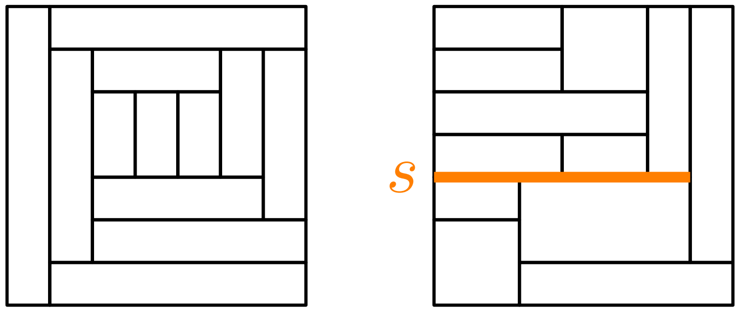

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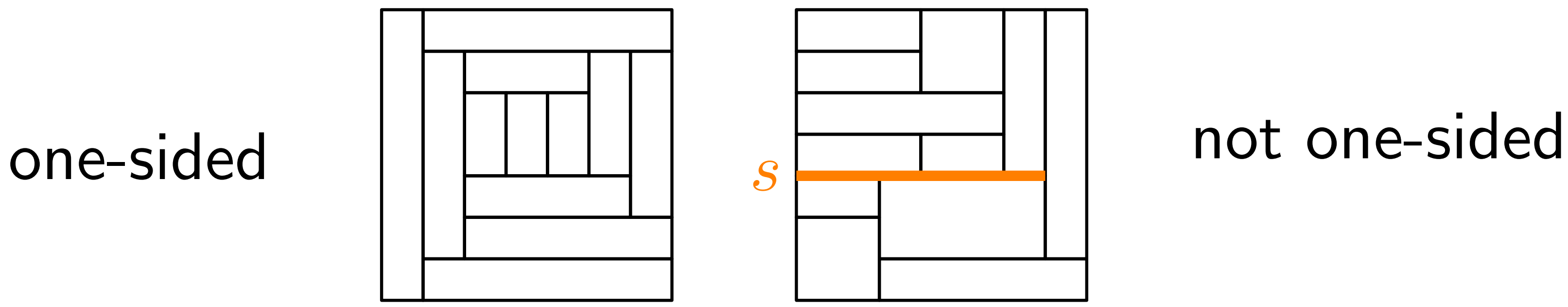

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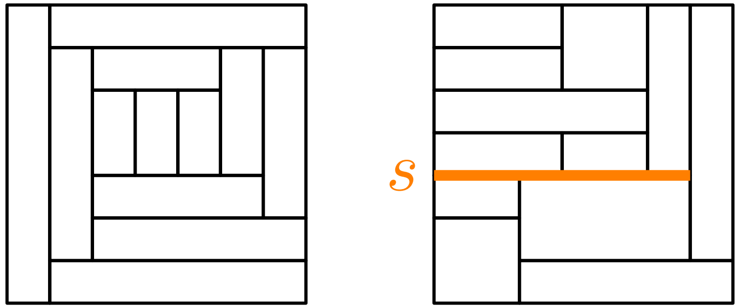

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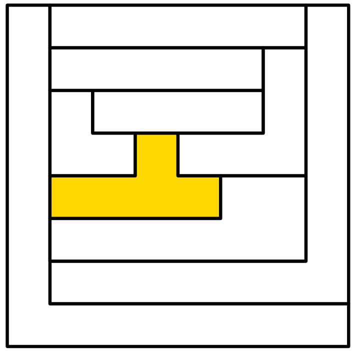

Literature

Construction of triangle contact representations based on

- [de Fraysseix, Ossona de Mendez, Rosenstiehl '94] On Triangle Contact Graphs

Construction of rectangular dual based on

- [He '93] On Finding the Rectangular Duals of Planar Triangulated Graphs
- [Kant, He '94] Two algorithms for finding rectangular duals of planar graphs

and originally from

- [Koźmiński, Kinnen '85] Rectangular Duals of Planar Graphs