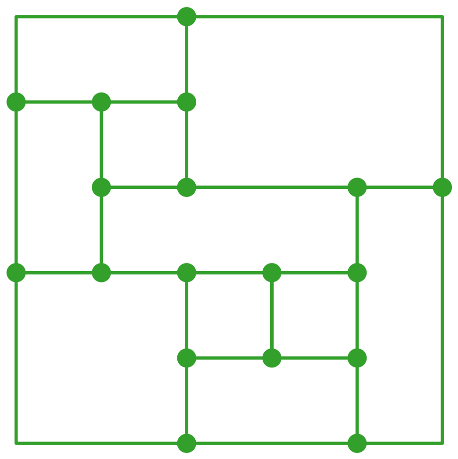
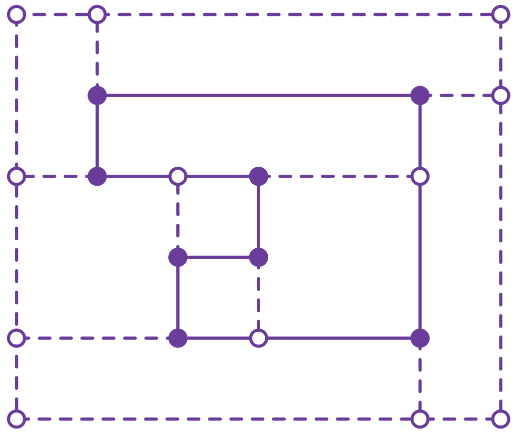
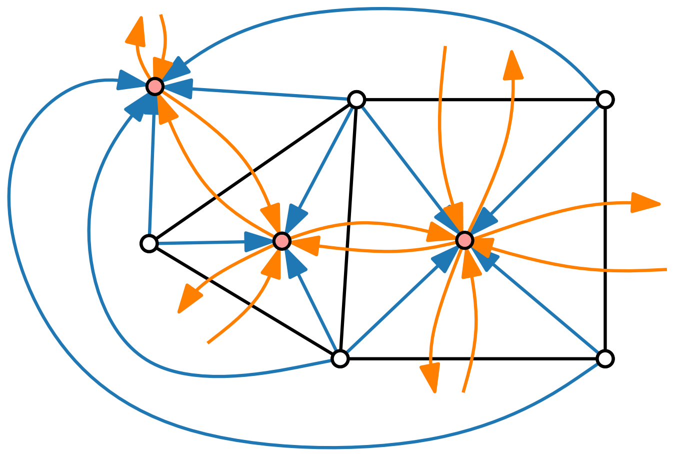
# Visualization of Graphs



## Lecture 6: Orthogonal Layouts

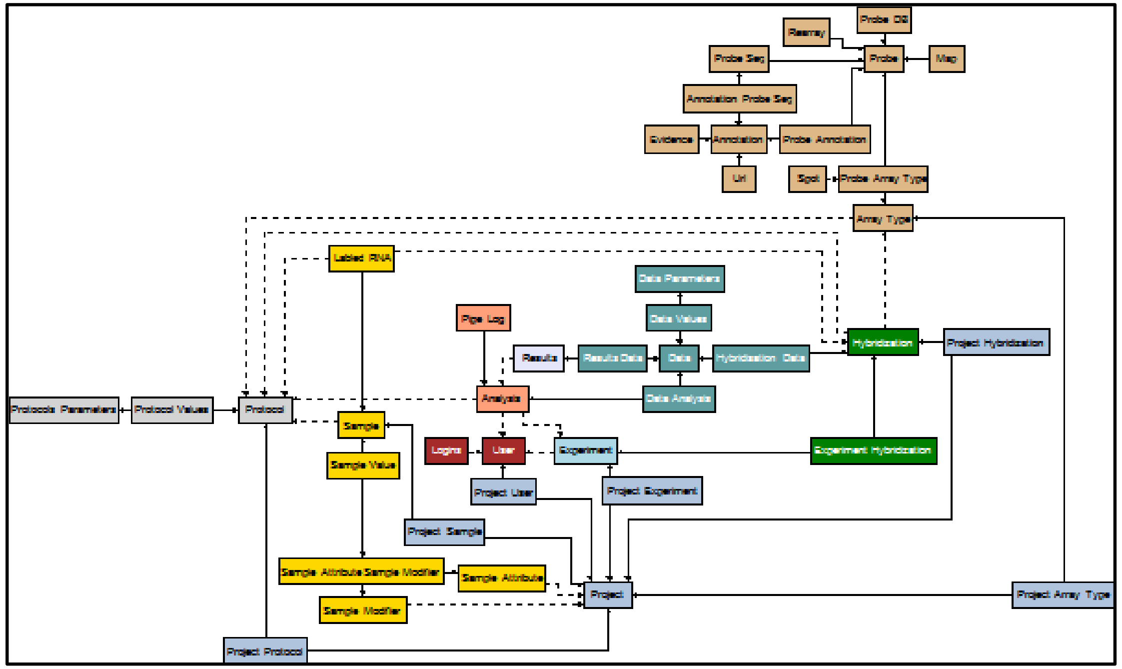




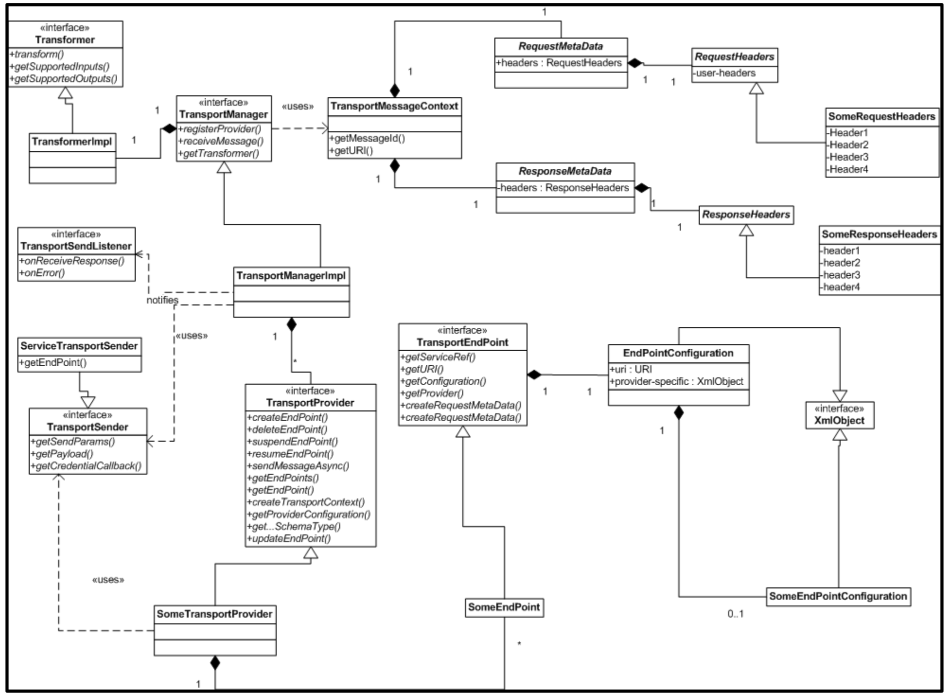
Alexander Wolff

Summer term 2025

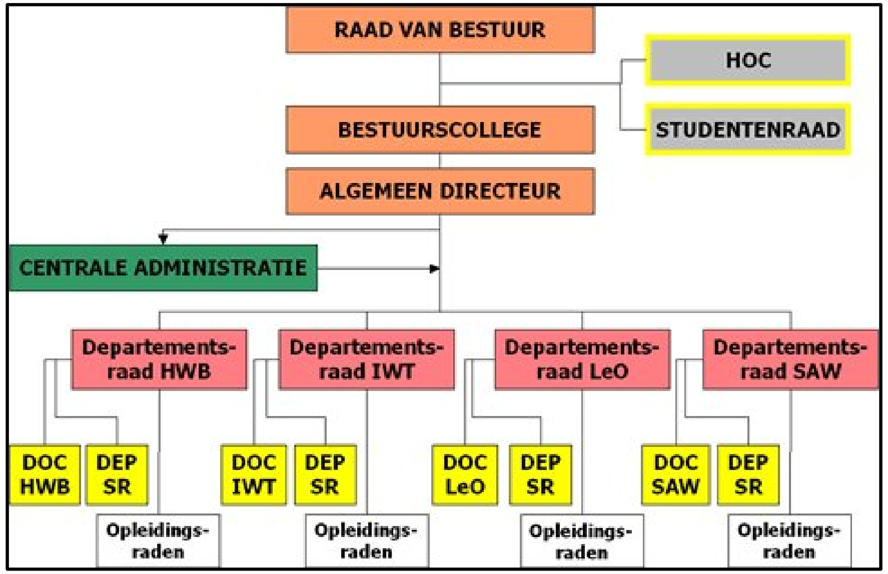
# Orthogonal Layout – Applications



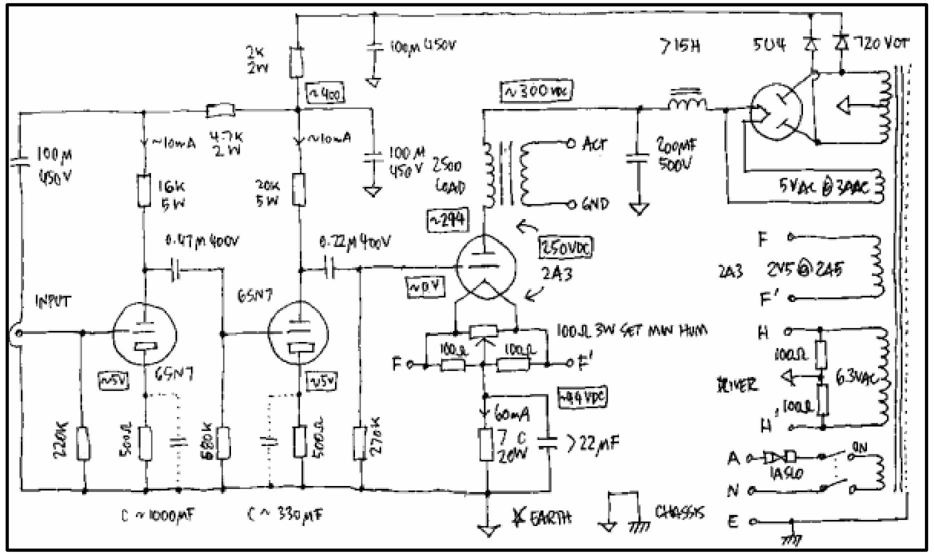
Entity-Relationship (ER) diagram in OGDF



UML diagram by Oracle

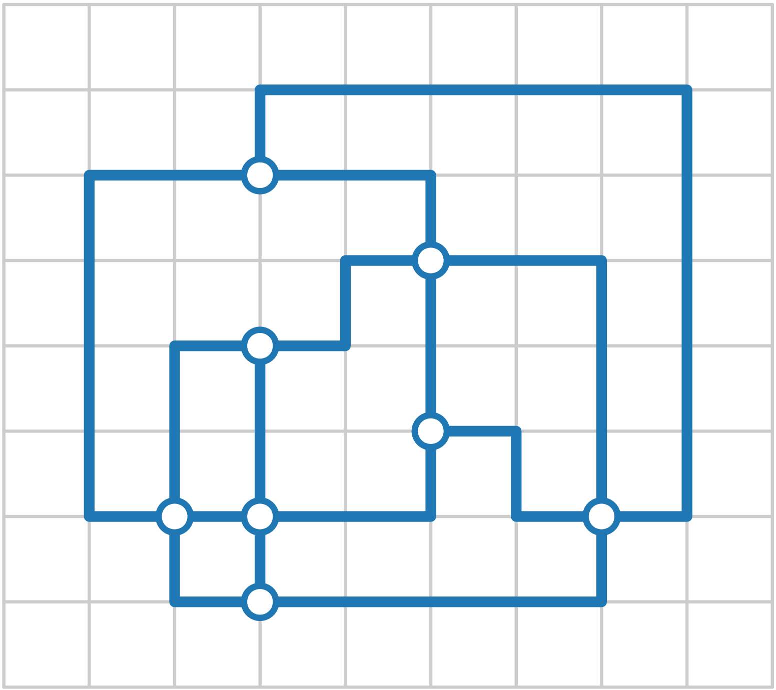



Organigram of HS Limburg



Circuit diagram by Jeff Atwood

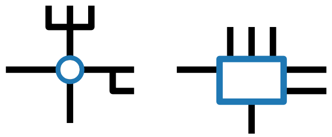
# Orthogonal Layout – Definition



**Definition.**
A drawing Γ of a graph $G$ is called **orthogonal** if
- vertices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical line segments of the grid, and
- pairs of edges are disjoint or cross orthogonally.

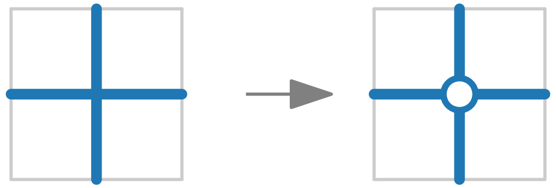
## Observations.

- Edges lie on a grid ⇒ **bends** lie on grid points
- Max. degree of each vertex is at most 4
- Otherwise



## Planarization.

- Fix embedding
- Crossings become vertices



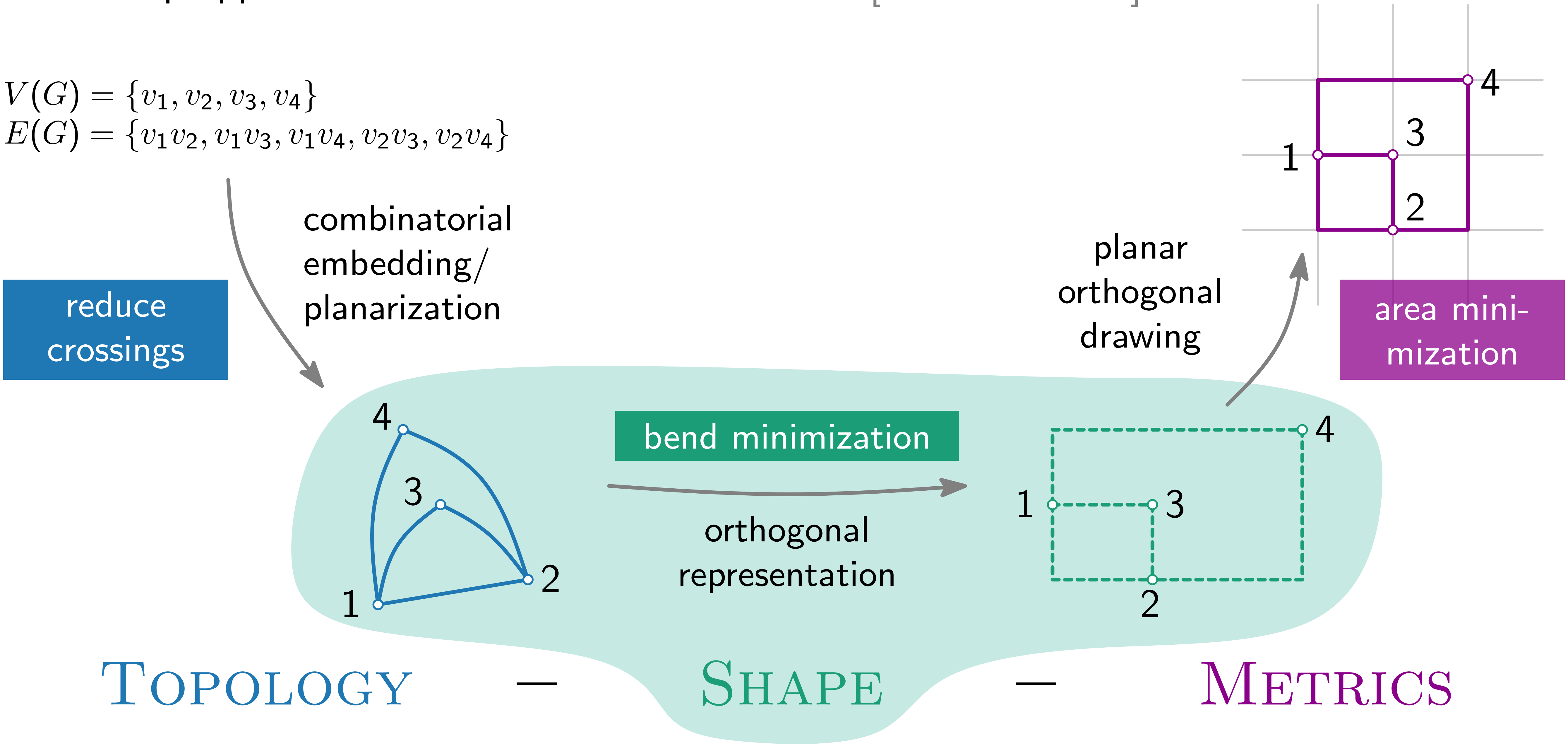
## Aesthetic criteria to optimize.

- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

# Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

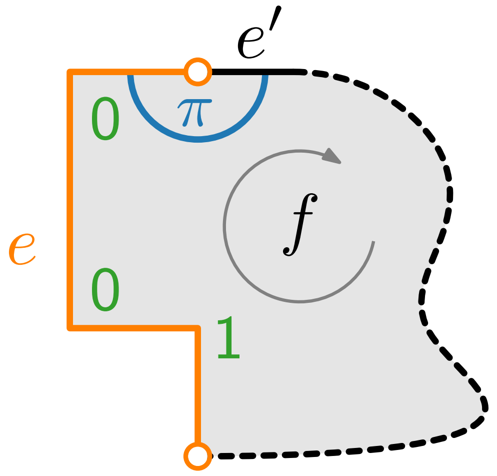


# Orthogonal Representation

**Idea.**
Describe orthogonal drawing combinatorially.

**Definitions.**
Let $G$ be a plane graph with set $F$ of faces and outer face $f_0 \in F$.

- Let $e$ be an edge with the face $f$ to the right.
  An **edge description** of $e$ w.r.t. $f$ is a triple $(e, \delta, \alpha)$ where
  - $\delta \in \{0, 1\}^*$ (where 0 = right bend, 1 = left bend)
  - $\alpha$ is angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between $e$ and next edge $e'$



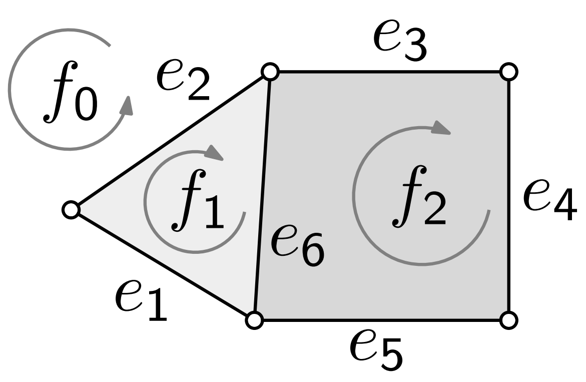

$(e, 100, \pi)$

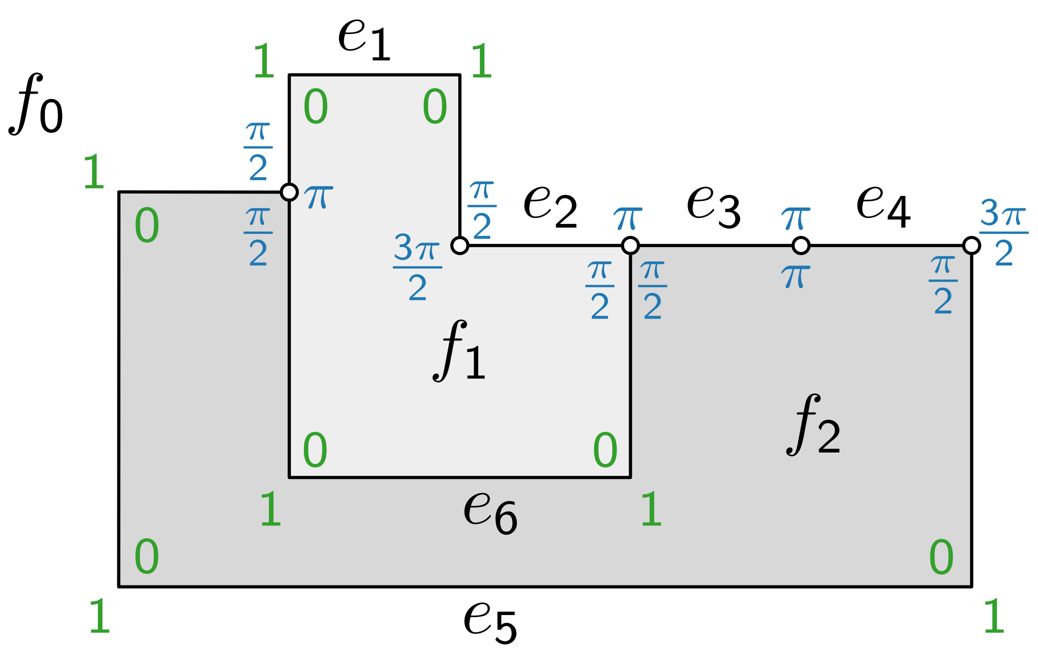
- A **face representation** $H(f)$ of a face $f$ is a clockwise ordered sequence $(e_1, \delta_1, \alpha_1), (e_2, \delta_2, \alpha_2), \ldots, (e_{\deg(f)}, \delta_{\deg(f)}, \alpha_{\deg(f)})$ of edge descriptions w.r.t. $f$.
- An **orthogonal representation** $H(G)$ of $G$ is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$

# Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$




Coordinates are not fixed yet!

# Correctness of an Orthogonal Representation

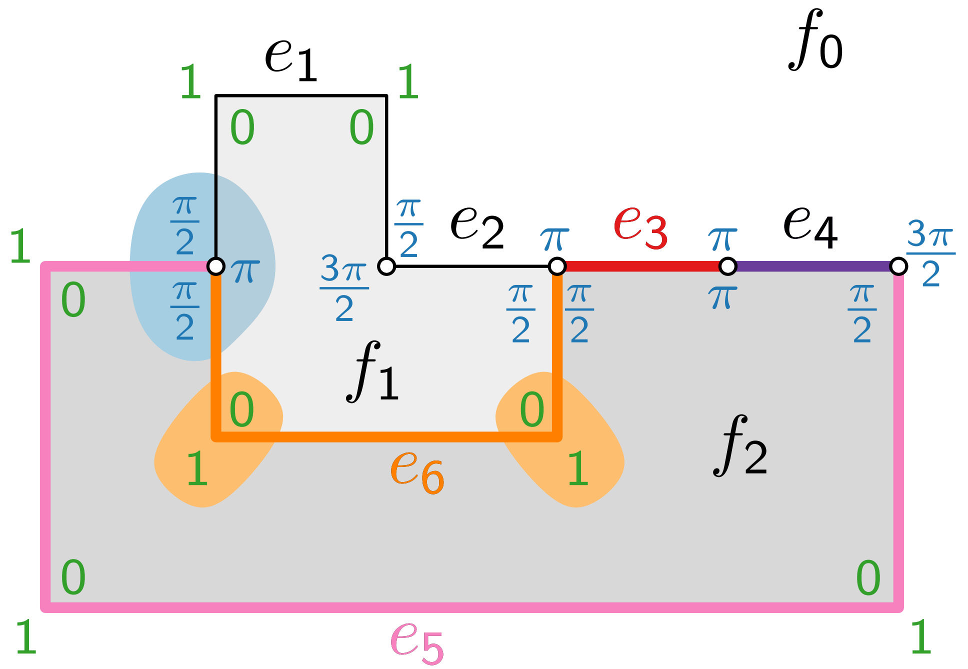
(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$ with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$, the sequence $\delta_1$ is like $\delta_2$, but reversed and inverted.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in $\delta$, and let $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 - \alpha/\frac{\pi}{2} + 2$.
For each **face** $f$, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$, the sum of incident angles is $2\pi$.




$$C((e_3, \emptyset, \pi)) = 0 - 0 - 2 + 2 = 0$$

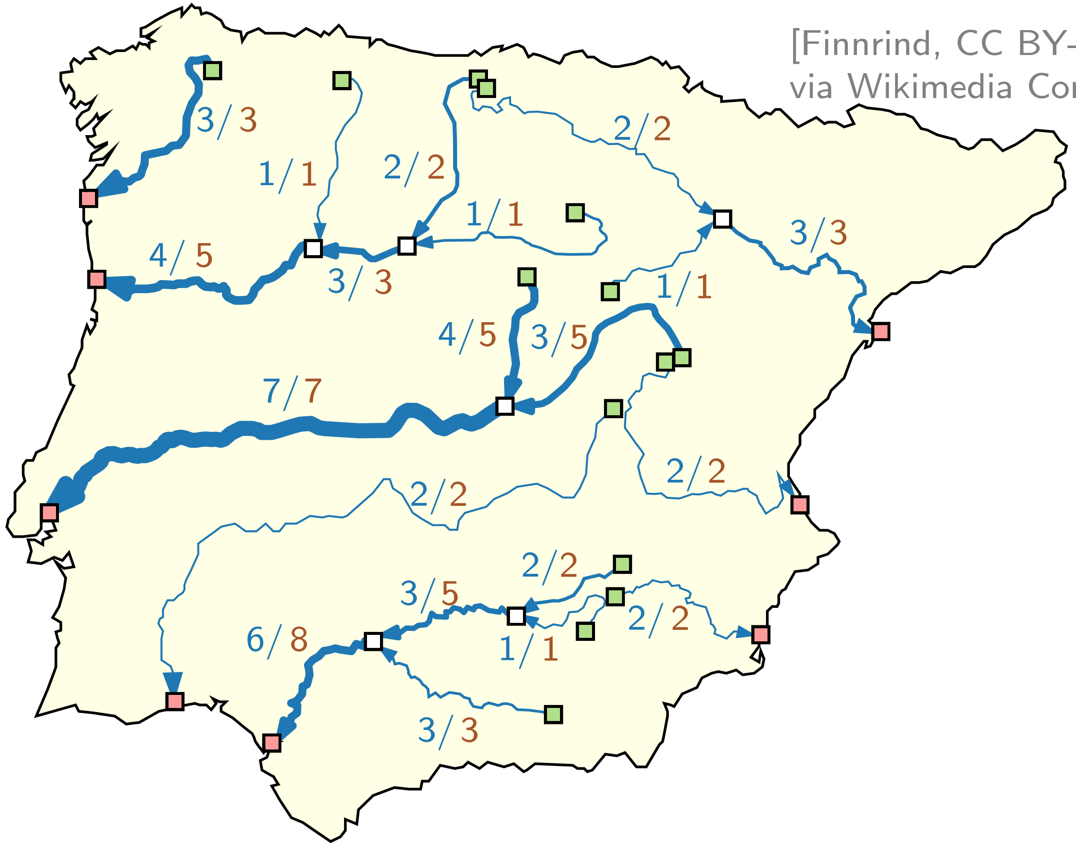
$$C((e_4, \emptyset, \tfrac{\pi}{2})) = 0 - 0 - 1 + 2 = 1$$

$$C((e_5, 000, \tfrac{\pi}{2})) = 3 - 0 - 1 + 2 = 4$$

$$C((e_6, 11, \tfrac{\pi}{2})) = 0 - 2 - 1 + 2 = -1$$

$$\sum_{r \in H(f_2)} C(r) = +4$$

# Reminder: $s$–$t$ Flow Networks






**Flow network** $(G; S, T; u)$ with

- directed graph $G$
- *sources* $S \subseteq V(G)$, *sinks* $T \subseteq V(G)$
- edge *capacity* $u \colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

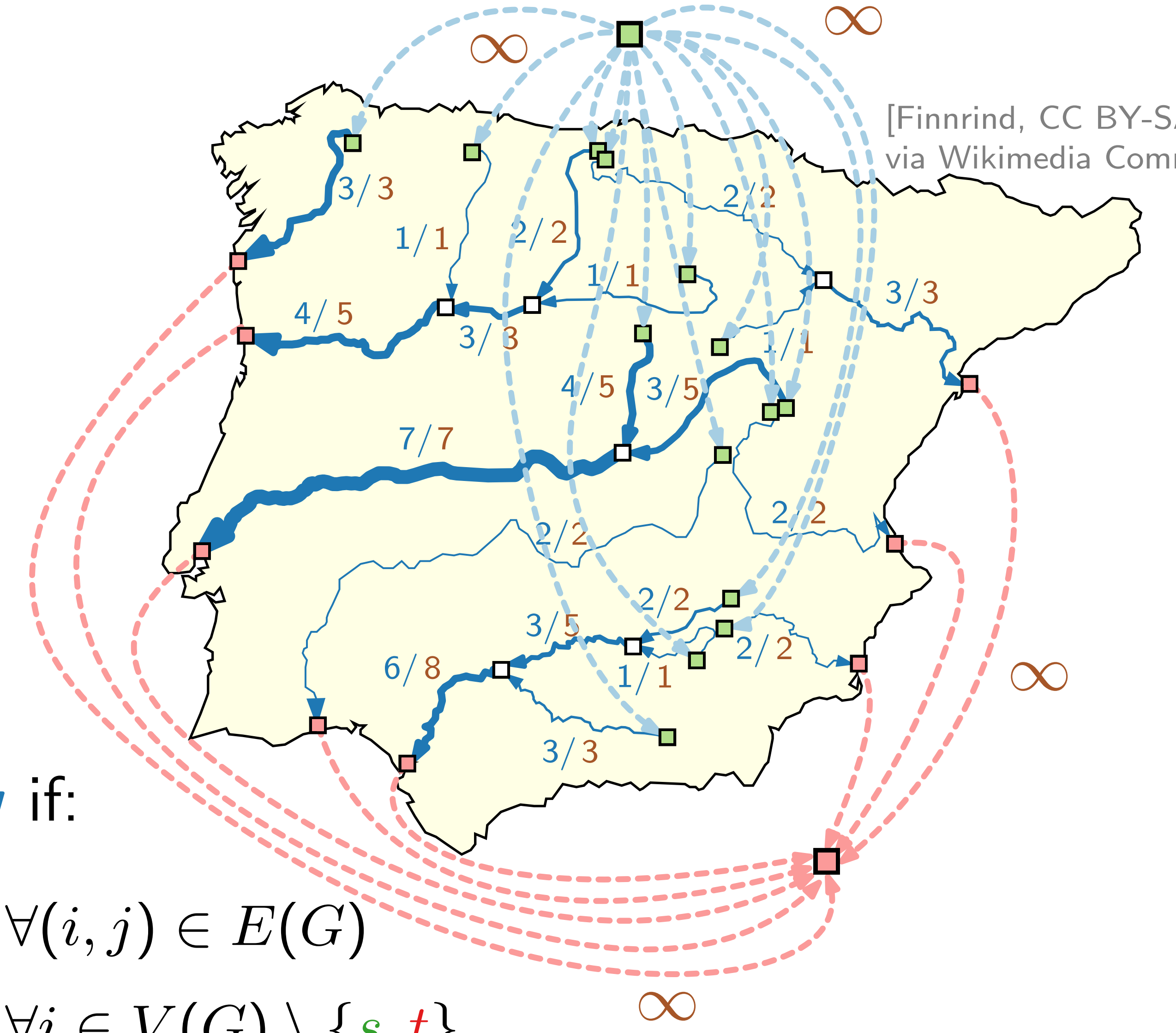
A function $X \colon E(G) \to \mathbb{R}_0^+$ is called $S$–$T$ **flow** if:

$$0 \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E(G)$$

$$\sum_{(i,j)\in E(G)} X(i,j) - \sum_{(j,i)\in E(G)} X(j,i) = 0 \qquad \forall i \in V(G) \setminus (S \cup T)$$

A **maximum** $S$–$T$ flow is an $S$–$T$ flow where $\sum_{(i,j)\in E(G), i\in S} X(i,j) - \sum_{(j,i)\in E(G), i \in S} X(j,i)$ is maximized.

# Reminder: $s$–$t$ Flow Networks



[Finnrind, CC BY-SA 3.0, via Wikimedia Commons]

**Flow network** $(G; s, t; u)$ with

- directed graph $G$
- *source* $s \in V(G)$, *sink* $t \in V(G)$
- edge *capacity* $u\colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

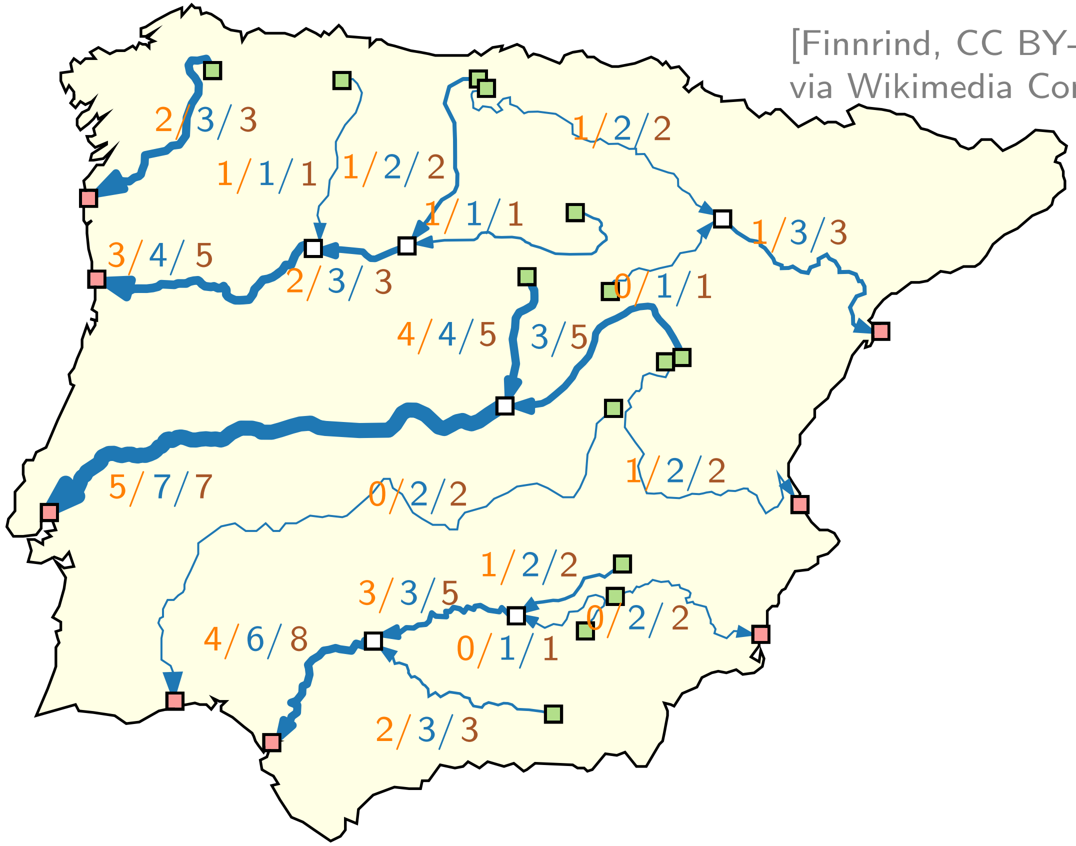
A function $X\colon E(G) \to \mathbb{R}_0^+$ is called $s$–$t$ **flow** if:

$$0 \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E(G)$$

$$\sum_{(i,j)\in E(G)} X(i,j) - \sum_{(j,i)\in E(G)} X(j,i) = 0 \qquad \forall i \in V(G) \setminus \{s,t\}$$

A **maximum** $s$–$t$ flow is an $s$–$t$ flow where $\sum_{(s,j)\in E(G)} X(s,j) - \sum_{(j,s)\in E(G)} X(j,s)$ is maximized.

# General Flow Network






**Flow network** $(G; S, T; \ell; u)$ with

- directed graph $G$
- *sources* $S \subseteq V(G)$, *sinks* $T \subseteq V(G)$
- edge *lower bound* $\ell \colon E(G) \to \mathbb{R}_0^+$
- edge *capacity* $u \colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

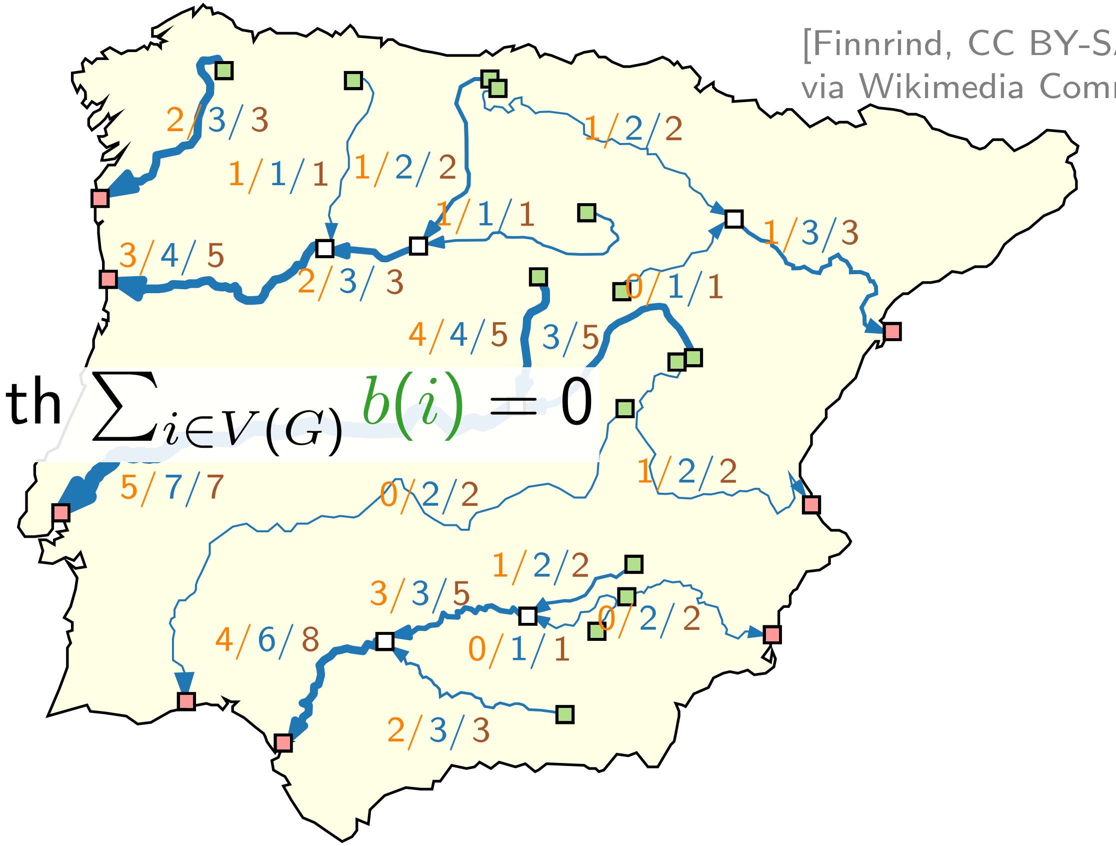
A function $X \colon E(G) \to \mathbb{R}_0^+$ is called $S$–$T$ **flow** if:

$$\ell(i,j) \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E(G)$$

$$\sum_{(i,j)\in E(G)} X(i,j) - \sum_{(j,i)\in E(G)} X(j,i) = 0 \qquad \forall i \in V(G) \setminus (S \cup T)$$

A **maximum** $S$–$T$ flow is an $S$–$T$ flow where $\sum_{(i,j)\in E(G), i \in S} X(i,j) - \sum_{(j,i)\in E(G), i\in S} X(j,i)$ is maximized.

# General Flow Network






**Flow network** $(G; b; \ell; u)$ with

- directed graph $G$
- node *production/consumption* $b\colon V(G) \to \mathbb{R}$ with $\sum_{i \in V(G)} b(i) = 0$
- edge *lower bound* $\ell\colon E(G) \to \mathbb{R}_0^+$
- edge *capacity* $u\colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

A function $X\colon E(G) \to \mathbb{R}_0^+$ is called **valid flow** if:

$$\ell(i,j) \le X(i,j) \le u(i,j) \qquad \forall (i,j) \in E(G)$$

$$\sum_{(i,j) \in E(G)} X(i,j) - \sum_{(j,i) \in E(G)} X(j,i) = b(i) \qquad \forall i \in V(G)$$

- *Cost function*: $\text{cost}\colon E(G) \to \mathbb{R}_0^+$ and $\text{cost}(X) := \sum_{(i,j) \in E(G)} \text{cost}(i,j) \cdot X(i,j)$

$X$ is a **minimum-cost flow** if $X$ is a valid flow that minimizes $\text{cost}(X)$.

# General Flow Network – Algorithms

$n$: #vertices
$m$: #edges

**Polynomial Algorithms**

| # | Due to | Year | Running Time |
|---|---|---|---|
| 1 | Edmonds and Karp | 1972 | O((n + m') log U S(n, m, nC)) |
| 2 | Rock | 1980 | O((n + m') log U S(n, m, nC)) |
| 3 | Rock | 1980 | O(n log C M(n, m, U)) |
| 4 | Bland and Jensen | 1985 | O(m log C M(n, m, U)) |
| 5 | Goldberg and Tarjan | 1987 | O(nm log ($n^2$/m) log (nC)) |
| 6 | Goldberg and Tarjan | 1988 | O(nm log n log (nC)) |
| 7 | Ahuja, Goldberg, Orlin and Tarjan | 1988 | O(nm log log U log (nC)) |

**Strongly Polynomial Algorithms**

| # | Due to | Year | Running Time |
|---|---|---|---|
| 1 | Tardos | 1985 | O($m^4$) |
| 2 | Orlin | 1984 | O($(n + m')^2$ log n S(n, m)) |
| 3 | Fujishige | 1986 | O($(n + m')^2$ log n S(n, m)) |
| 4 | Galil and Tardos | 1986 | O($n^2$ log n S(n, m)) |
| 5 | Goldberg and Tarjan | 1987 | O($nm^2$ log n log($n^2$/m)) |
| 6 | Goldberg and Tarjan | 1988 | O($nm^2$ $\log^2$ n) |
| 7 | Orlin (this paper) | 1988 | O((n + m') log n S(n, m)) |

| | | | |
|---|---|---|---|
| S(n, m) | = | O( m + n log n) | Fredman and Tarjan [1984] |
| S(n, m, C) | = | O( Min (m + n$\sqrt{\log C}$), (m log log C)) | Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977] |
| M(n, m) | = | O(min (nm + $n^{2+\varepsilon}$, nm log n) where ε is any fixed constant. | King, Rao, and Tarjan [1991] |
| M(n, m, U) | = | O(nm log ($\frac{n}{m}\sqrt{\log U}$ + 2)) | Ahuja, Orlin and Tarjan [1989] |

[Orlin 1991]

**Theorem.** [Orlin 1991]
The minimum-cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

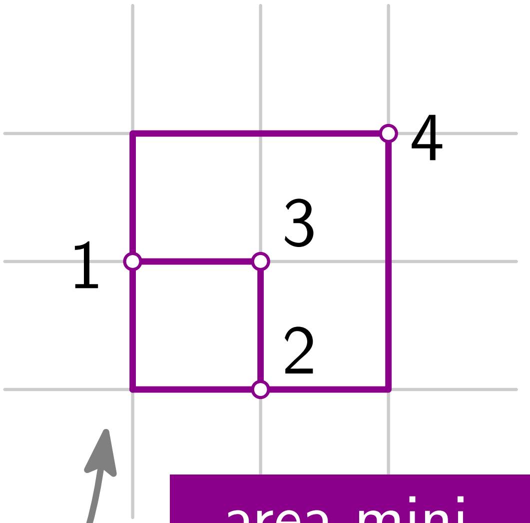
**Theorem.** [Cornelsen & Karrenbauer 2011]
The minimum-cost flow problem for planar graphs with bounded costs and face sizes can be solved in $O(n^{3/2})$ time.

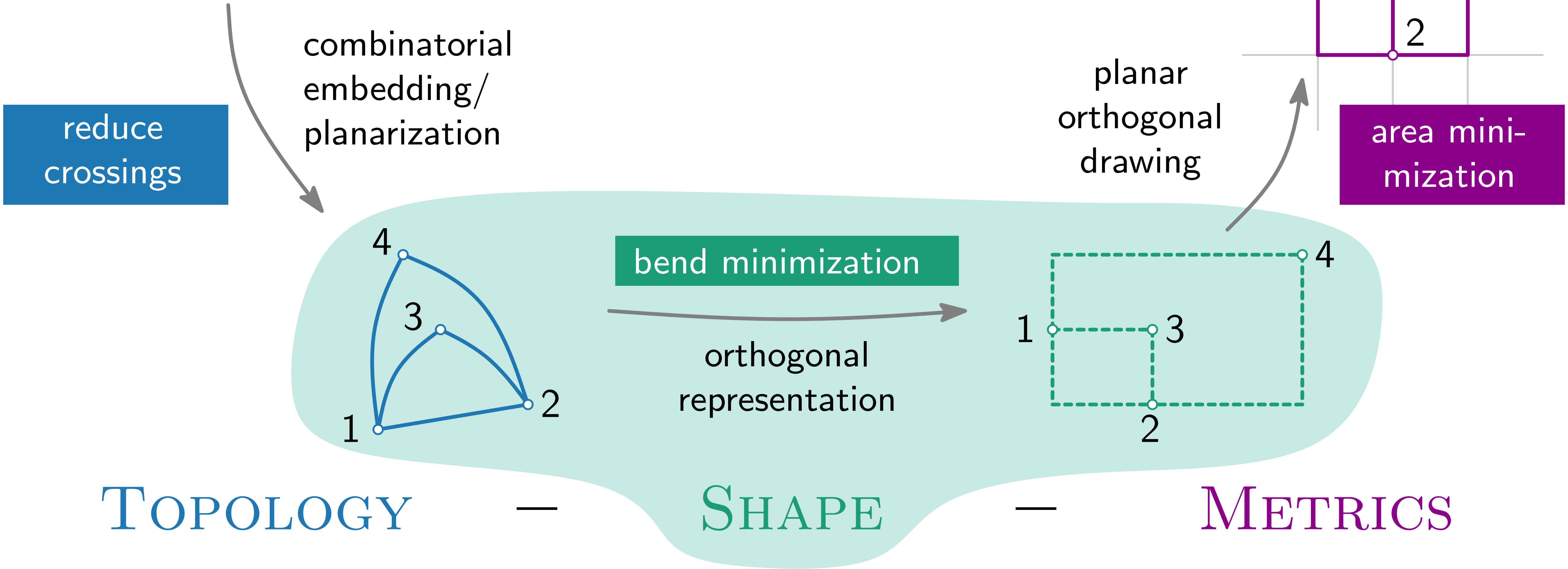
**Theorem.** [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
The minimum-cost flow problem with integral vertex demands, edge capacities, and edge costs can be solved in $O(m^{1+o(1)} \log U \log C)$ time, where $U$ is the maximum capacity and $C$ are the maximum costs.

# Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$



# Bend Minimization with Given Embedding

**Geometric orthogonal bend minimization.**

Given:
- Plane graph $G$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

Compare with the following variant:

**Combinatorial orthogonal bend minimization.**

Given:
- Plane graph $G$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

# Bend Minimization with Given Embedding

How to solve the combinatorial orthogonal bend minimization problem?

**Idea.**
Formulate as a network-flow problem:

- a unit of flow $= \measuredangle \frac{\pi}{2}$
- vertices $\xrightarrow{\measuredangle}$ faces ($\# \measuredangle \frac{\pi}{2}$ per face)
- faces $\xrightarrow{\measuredangle}$ neighboring faces ($\#$ bends toward the neighbor)

**Combinatorial orthogonal bend minimization.**

Given:
- Plane graph $G$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

# Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$, the sequence $\delta_1$ is reversed and inverted copy of $\delta_2$.

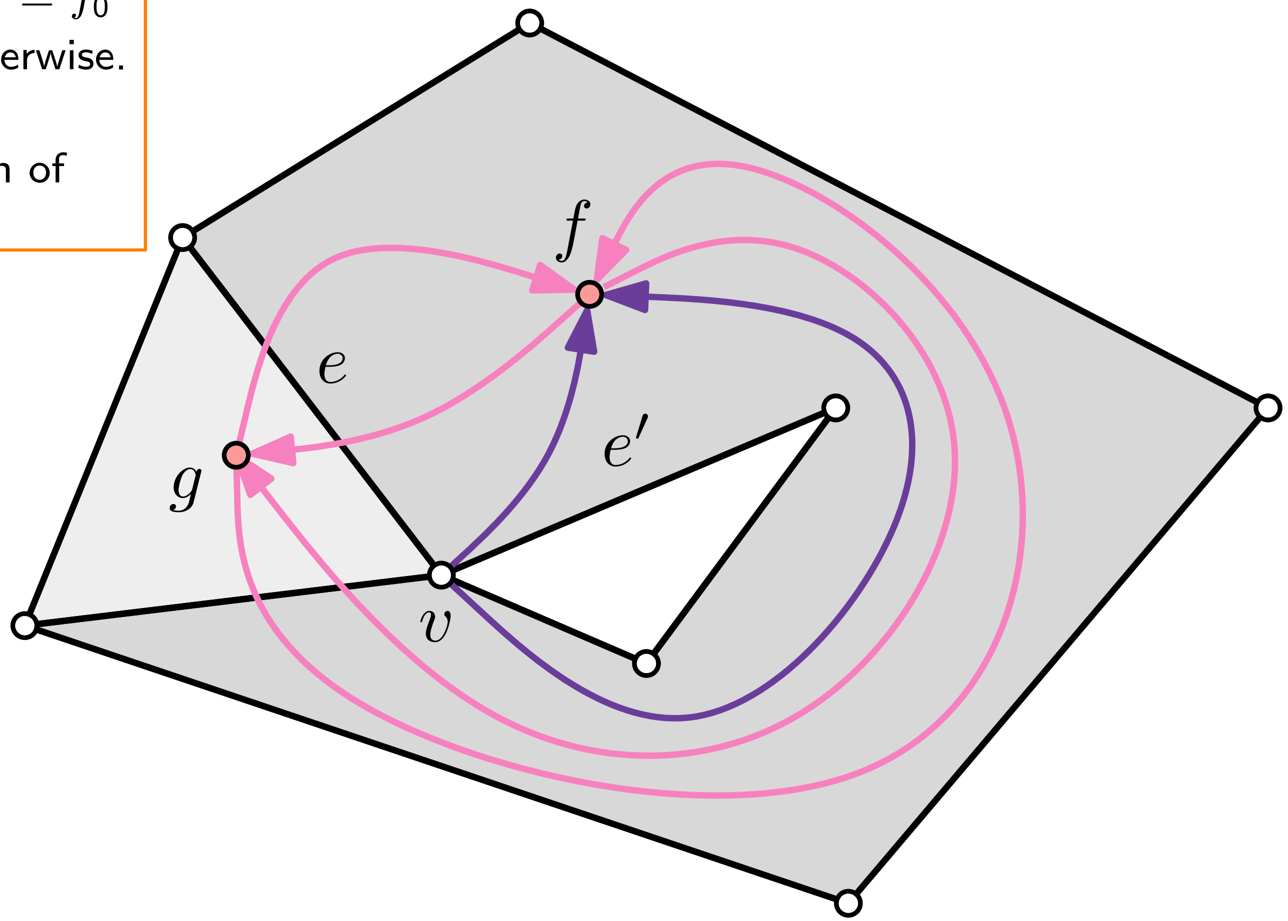
(H3) For each **face** $f$, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$, the sum of incident angles is $2\pi$.

Define flow network $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$:

- $E' = \{(v, f)_{ee'} \in V(G) \times F \mid v$ between edges $e, e'$ of $\partial f\} \cup$ $\{(f, g)_e \in F \times F \mid f, g$ have common edge $e\}$




*Directed multigraph!*

# Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to $F$, $f_0$.

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Define flow network $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$:

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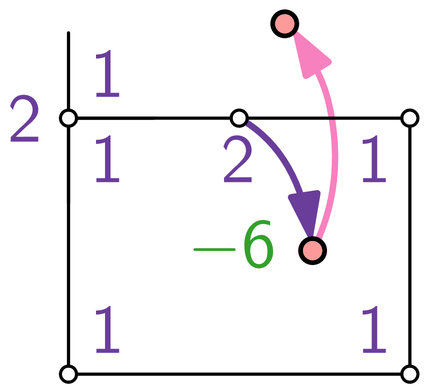
- $b(v) = 4 \quad \forall v \in V(G)$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

$\Rightarrow \sum_{w \in V(G) \cup F} b(w) = 0$ (Euler)

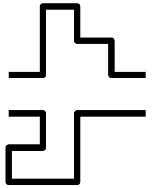



$\forall (v, f) \in E', v \in V(G), f \in F$ $\quad \ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$

$\text{cost}(v, f) = 0$

$\forall (f, g) \in E', f, g \in F$ $\quad \ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$

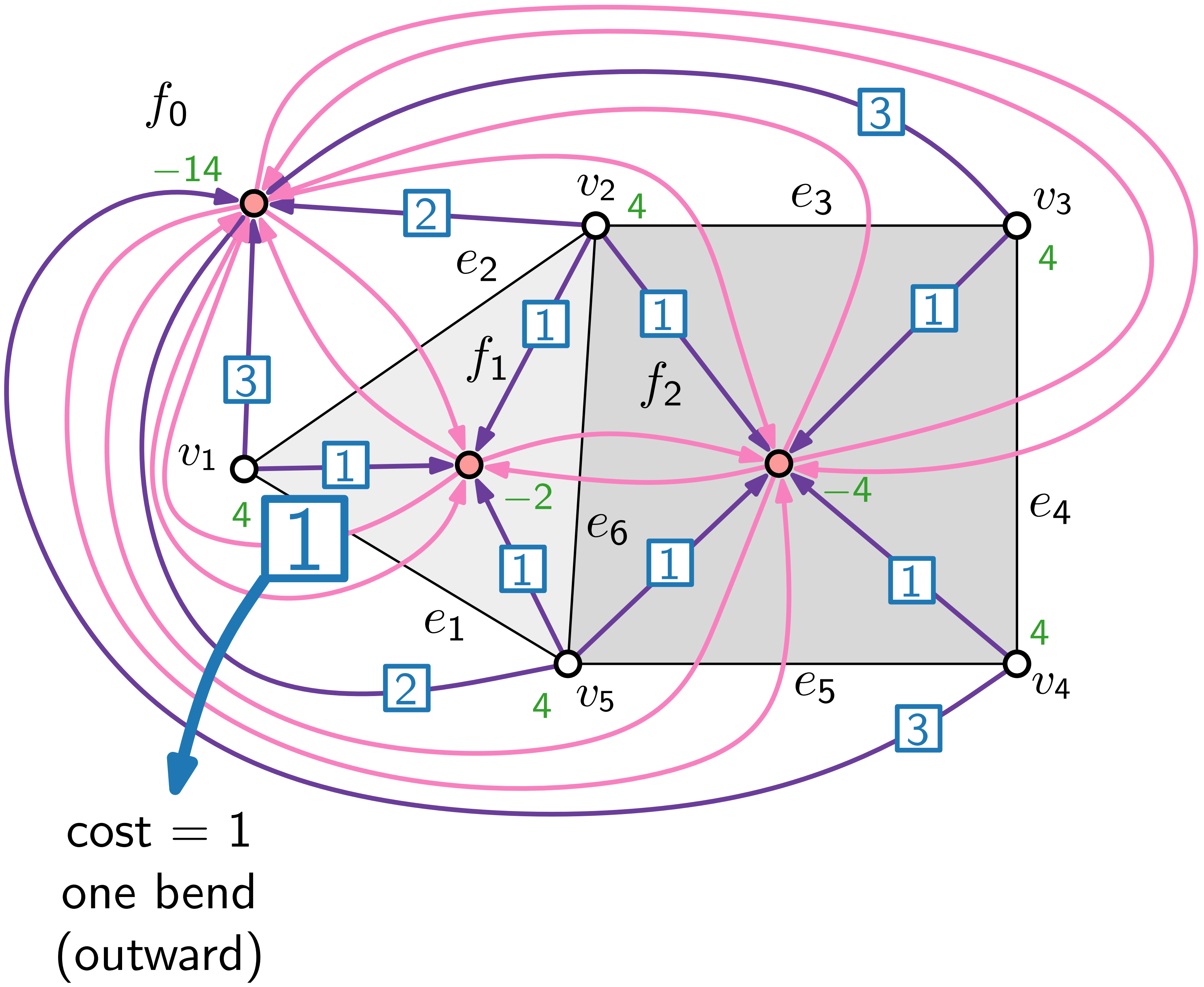
$\text{cost}(f, g) = 1$



We model only the *number* of bends. Why is it enough?

→ *Exercise!*

# Flow Network Example




Legend

$V(G)$ ○

$F$ ●

$\ell/u/\text{cost}$

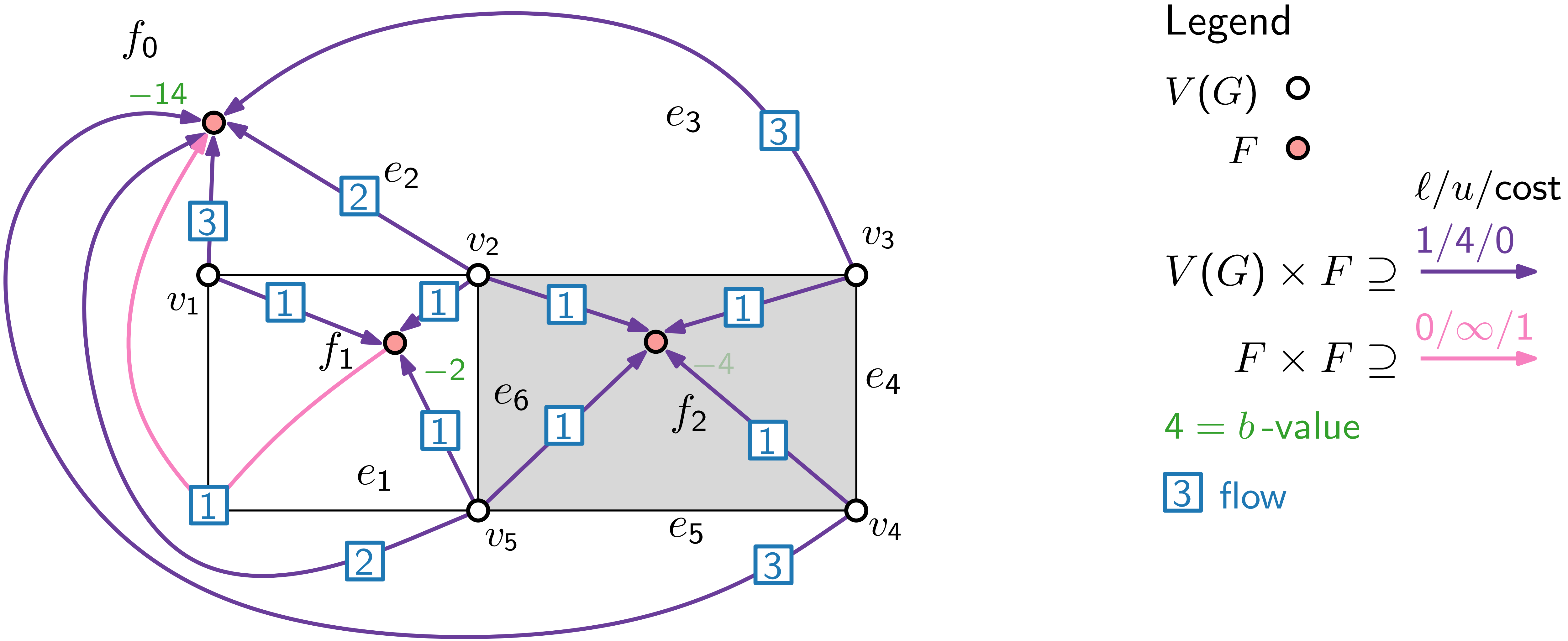
$V(G) \times F \supseteq$ 1/4/0

$F \times F \supseteq$ 0/∞/1

4 = $b$-value

3 flow

# Flow Network Example



# Bend Minimization – Result

**Theorem.** [Tamassia '87]
A plane graph $(G, F, f_0)$ has a valid orthogonal representation $H(G)$ with $k$ bends. $\Leftrightarrow$
The flow network $N(G)$ has a valid flow $X$ with cost $k$.

(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$, sequence $\delta_1$ is reversed and inverted $\delta_2$.

(H3) For each **face** $f$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$ the sum of incident angles is $2\pi$.

**Proof.**

"$\Leftarrow$": Given a valid flow $X$ in $N(G)$ of cost $k$, construct an orthogonal representation $H(G)$ with $k$ bends.

- Transform from flow to orthogonal description.
- Show properties (H1)–(H4).

(H1) H(G) matches $F, f_0$ ✓

(H2) Bend order inverted and reversed on opposite sides ✓

(H3) Angle sum of $f = \pm 4$ ✓ $\rightarrow$ *Exercise.*

(H4) Total angle at each vertex $= 2\pi$ ✓

# Bend Minimization – Result

**Theorem.** [Tamassia '87]
A plane graph $(G, F, f_0)$ has a valid orthogonal representation $H(G)$ with $k$ bends. $\Leftrightarrow$
The flow network $N(G)$ has a valid flow $X$ with cost $k$.

- $b(v) = 4 \quad \forall v \in V(G)$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$
- $\ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$
  $\text{cost}(v, f) = 0$
  $\ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$
  $\text{cost}(f, g) = 1$

**Proof.**

"$\Rightarrow$": Given an orthogonal representation $H(G)$ with $k$ bends, construct a valid flow $X$ in $N(G)$ of cost $k$.

- Define flow $X \colon E' \to \mathbb{R}_0^+$.
- Show that $X$ is a valid flow and has cost $k$.

(N1) $X(vf) = 1/2/3/4$ ✓

(N2) $X((fg)_e) = |\delta|_0$, where $(e, \delta, x)$ describes edge $e$ in $H(f)$ ✓

(N3) capacities, deficit/demand coverage ✓

(N4) cost $= k$ ✓

# Bend Minimization – Remarks

- The theorem implies that the combinatorial orthogonal bend minimization problem for plane graphs can be solved using an algorithm for min-cost flow.

**Theorem.** [Garg & Tamassia 1996]
The min-cost flow problem for planar graphs with bounded costs and vertex degrees can be solved in $O(n^{7/4}\sqrt{\log n})$ time.

**Theorem.** [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
The minimum-cost flow problem with integral vertex demands, edge capacities & costs can be solved in $O(m^{1+o(1)} \log U \log C)$ time, where $U$ is max. capacity and $C$ are max. costs.

$m \in O(n)$ for planar graphs

$C \in \{0, 1\}$

$U \in O(n)$ because $2n + 4$ bends in total are always sufficient [Storer 1984]

Further, $\log n = n^{\log_n \log n} = n^{\log \log n / \log n} \in n^{o(1)}$ since $\lim_{n \to \infty} \frac{\log \log n}{\log n} = 0$

**Corollary.**
The combinatorial orthogonal bend minimization problem can be solved in $O(n^{1+o(1)})$ time.

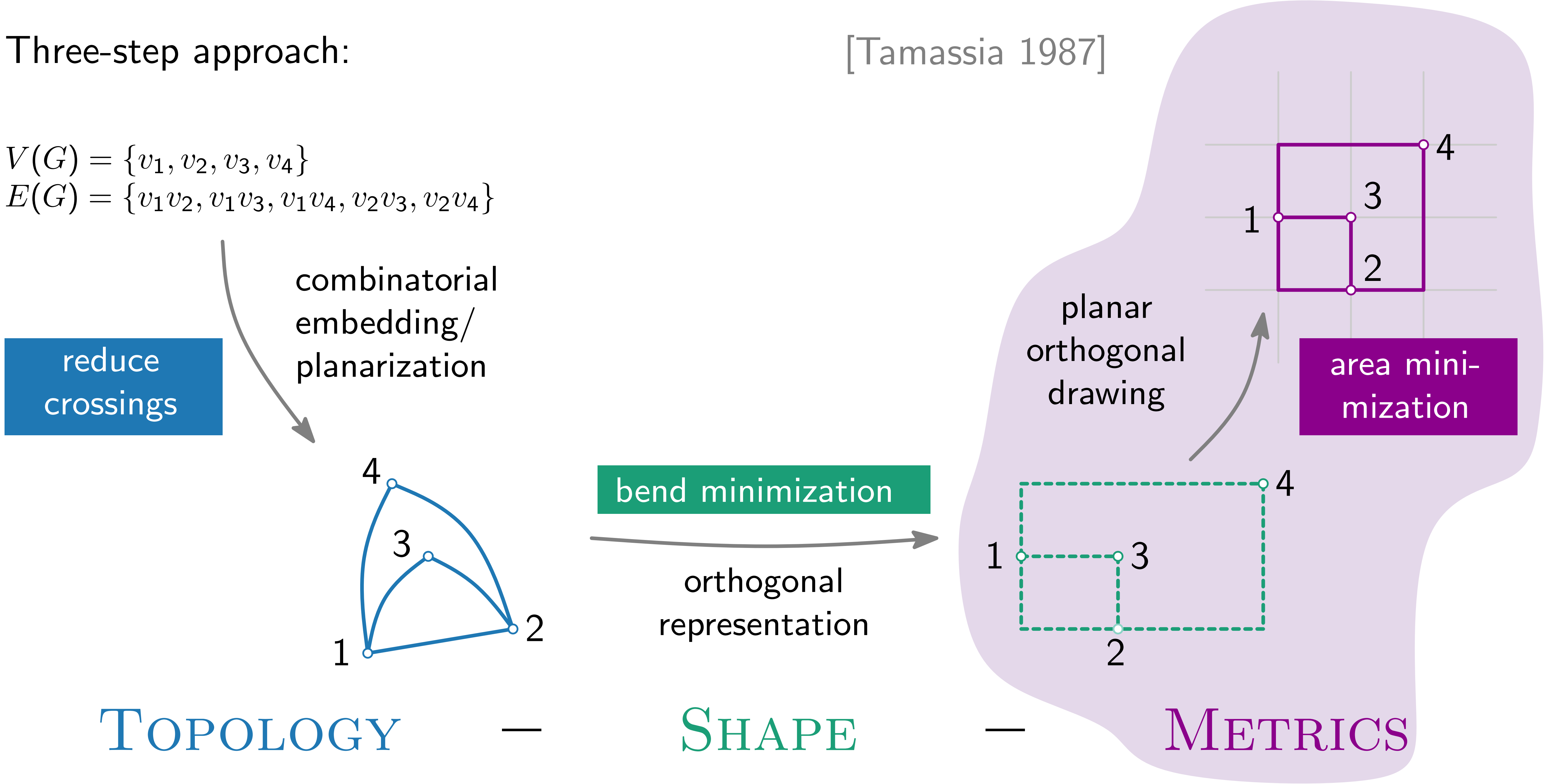
**Theorem.** [Garg & Tamassia 2001]
Bend minimization without given combinatorial embedding is NP-hard.

# Topology – Shape – Metrics

Three-step approach:

[Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



# Compaction

**Compaction problem.**

Given:
- Plane graph $G$ with maximum degree 4
- Orthogonal representation $H(G)$

Find: Compact orthogonal layout of $G$ that realizes $H(G)$

**Special case.**

All faces are rectangles.

This guarantees:
- minimum total edge length
- minimum area

**Properties.**

- bends only on the outer face
- opposite sides of a face have the same length

**Idea.**

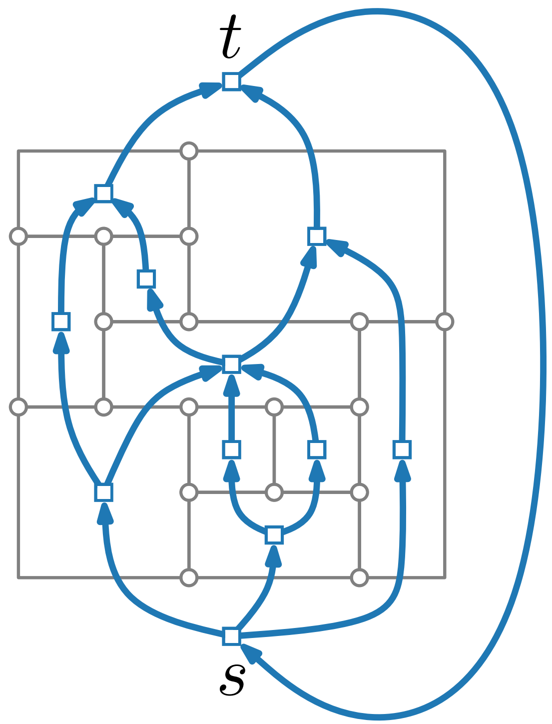
- Formulate flow network for horizontal/vertical compaction

# Flow Network for Edge-Length Assignment

**Definition.**

Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

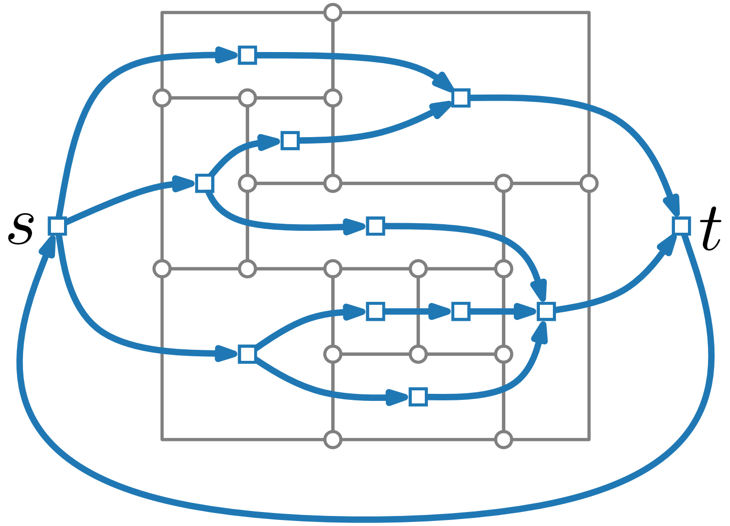
- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and $f$ lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$



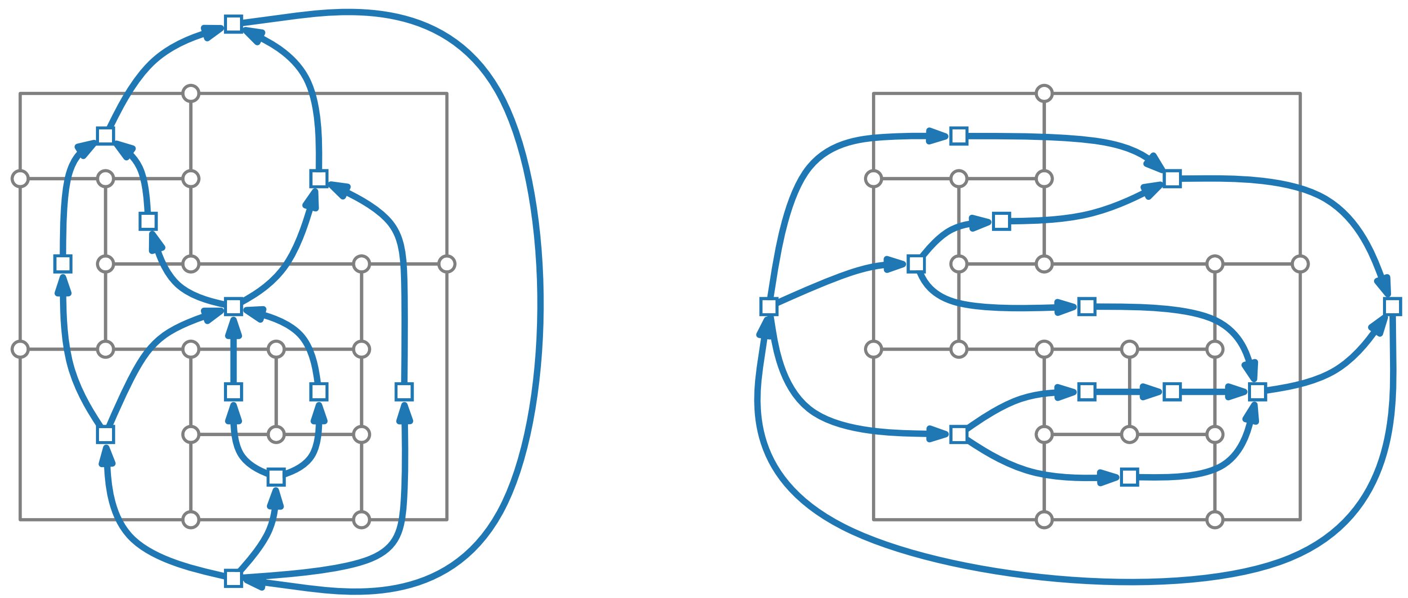
# Flow Network for Edge-Length Assignment

**Definition.**
Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and $f$ lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$



# Compaction – Result



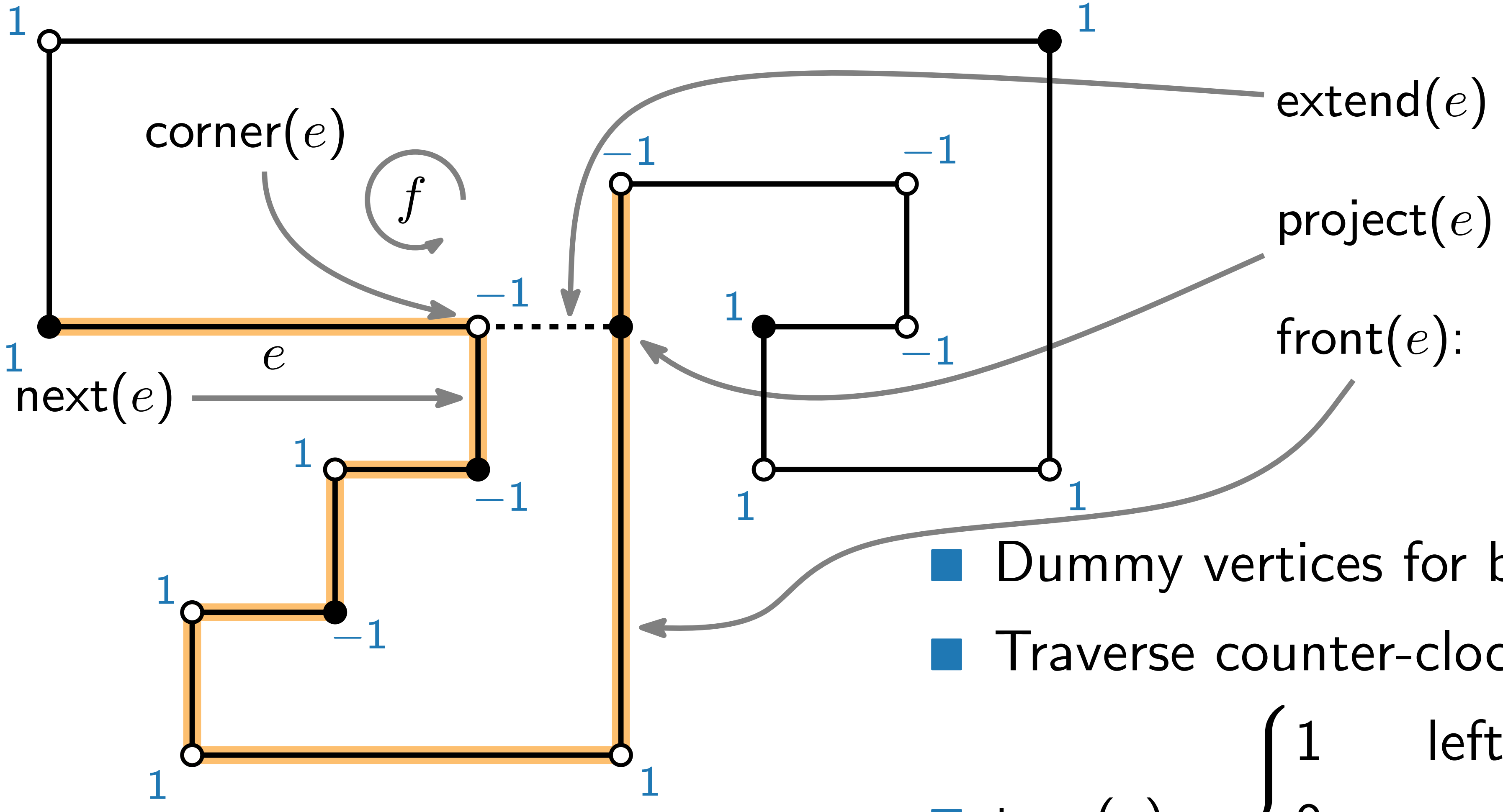
What if not all faces are rectangular?

**Theorem.**
A valid flow for $N_{\text{hor}}$ and $N_{\text{ver}}$ exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

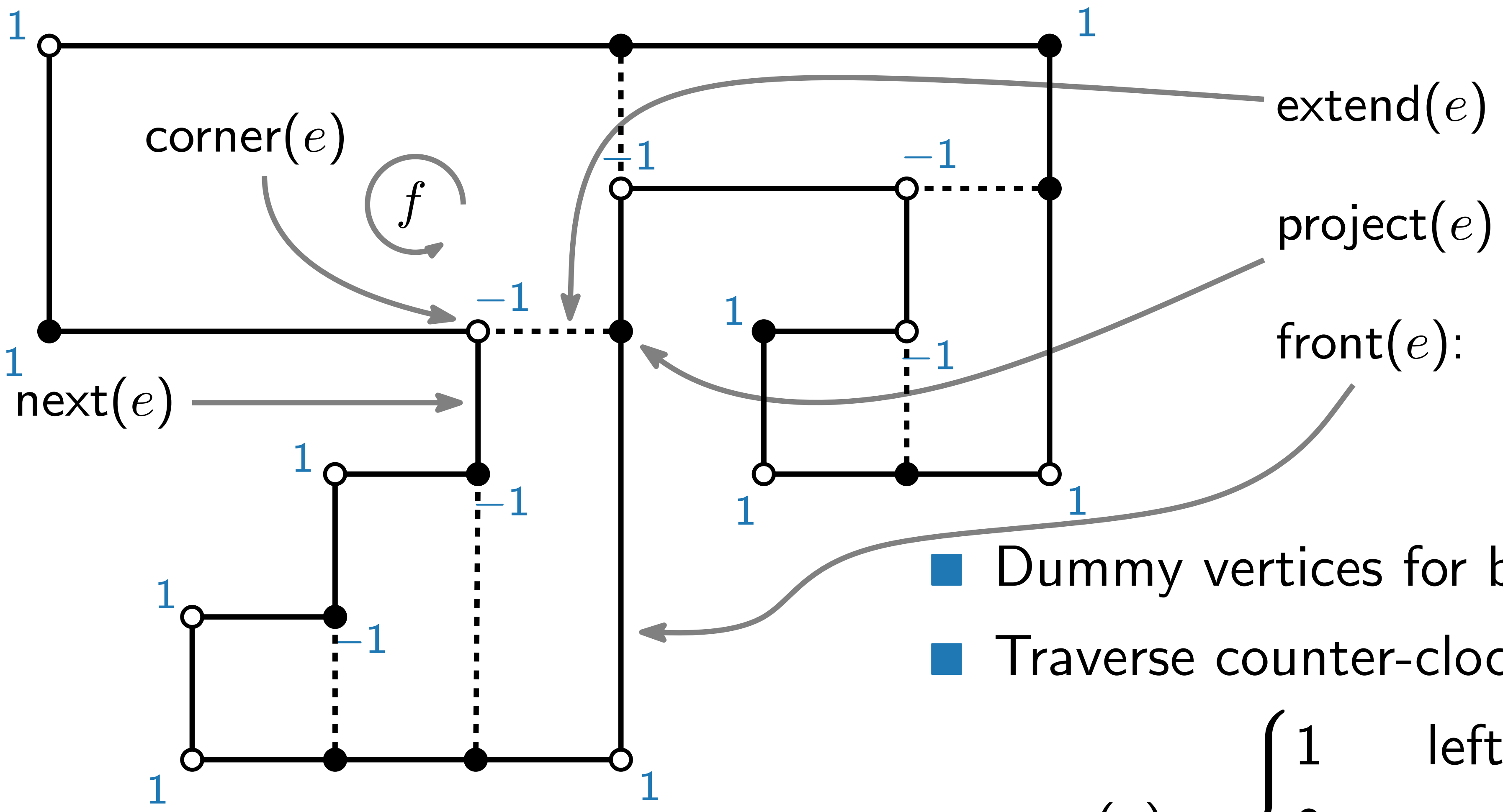
# Refinement of $G$ and $H(G)$ – Inner Face




front($e$): First edge $e$ after $e$ such that the angle sum between $e$ and $e'$ is positive.

- Dummy vertices for bends
- Traverse counter-clockwise
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

# Refinement of $G$ and $H(G)$ – Inner Face

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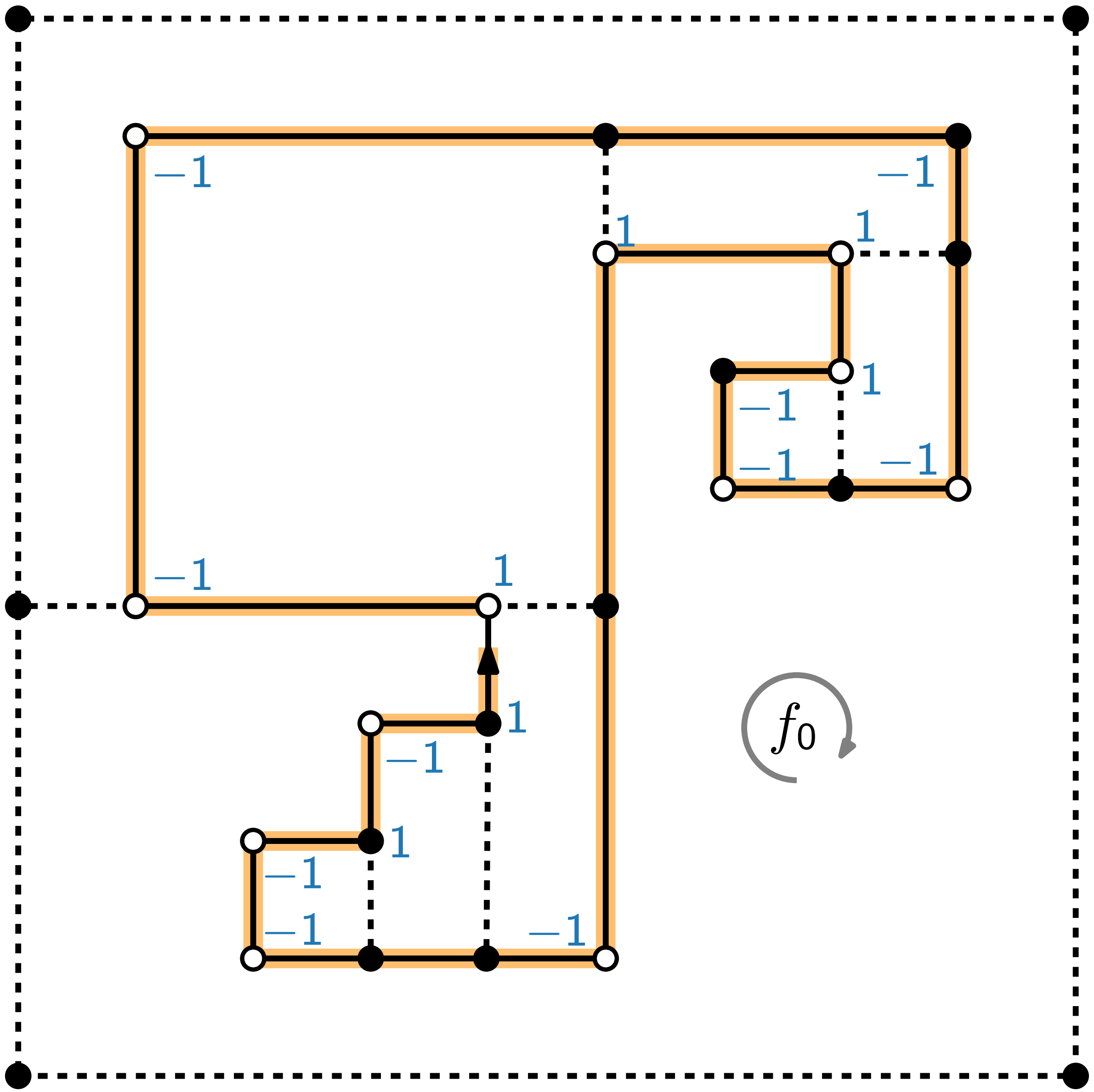
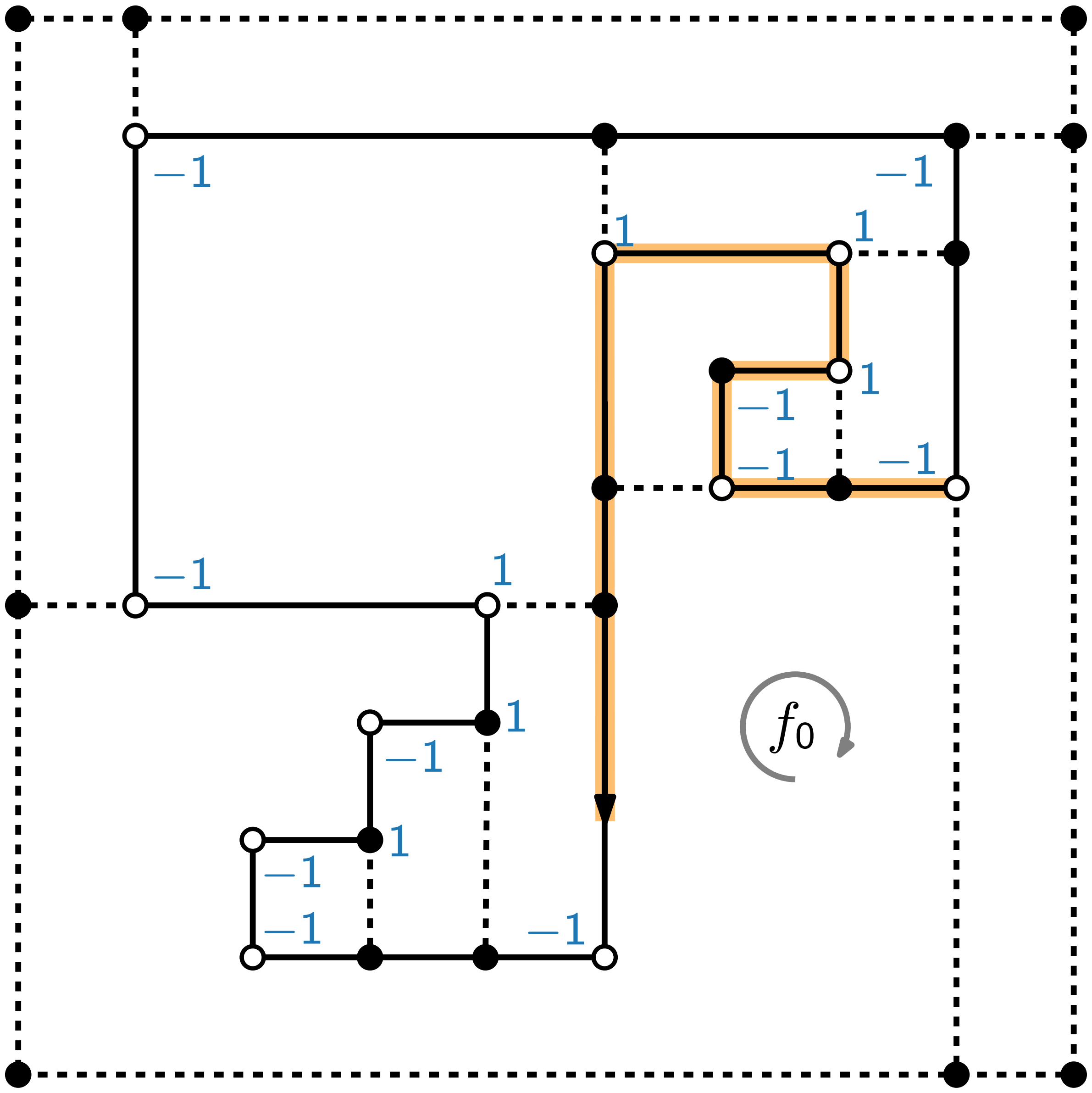
# Refinement of $G$ and $H(G)$ – Outer Face



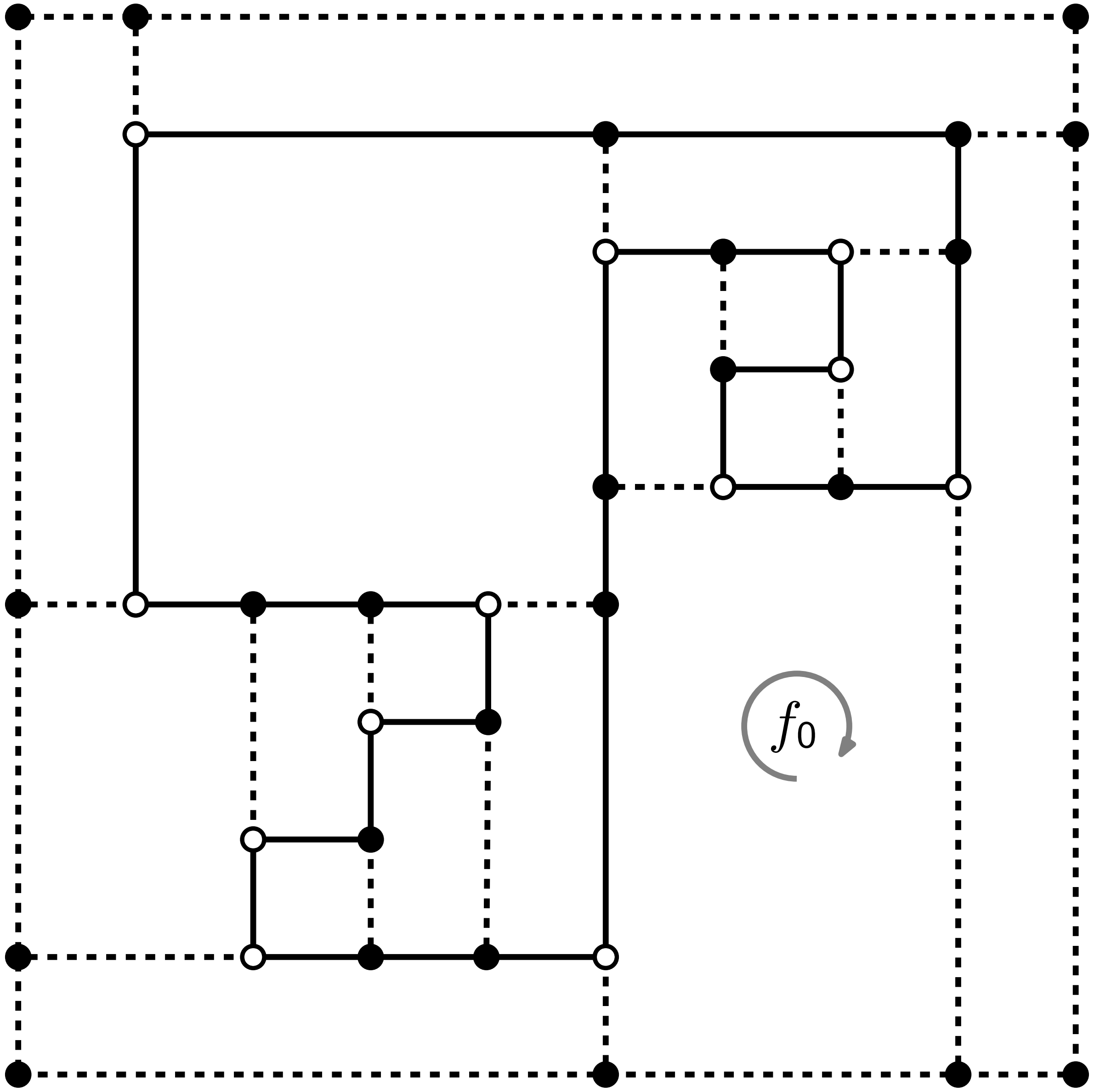

- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face



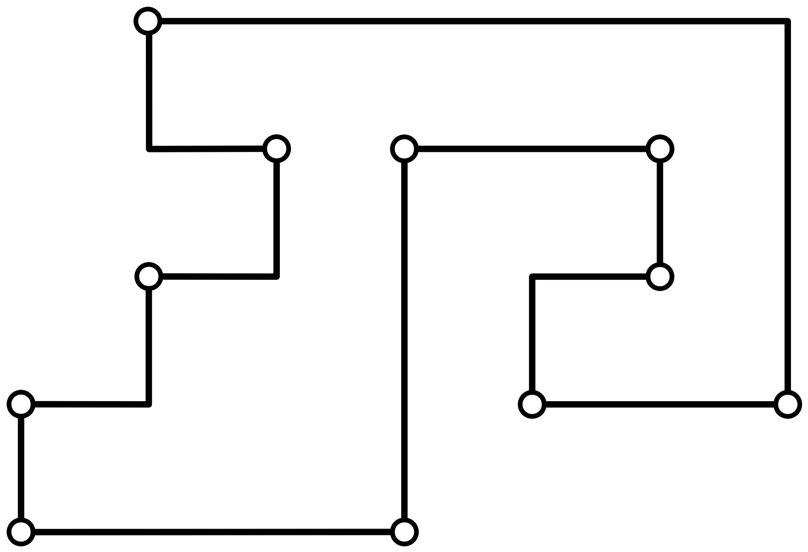

- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face




- Add an outer rectangle
- Traverse clockwise

# Refinement of $G$ and $H(G)$ – Outer Face



Area minimized? No!

# vertices

# bends

But we get bound $O((n + b)^2)$ on the area.

**Theorem.** [Patrignani 2001]
Compaction for a given orthogonal representation is NP-hard in general.

**Theorem.** [EFKSSW 2022]
Compaction is NP-hard even for orthogonal representations of *cycles*.

# Compaction is NP-hard

Polynomial-time reduction from the NP-complete satisfiability problem ($\mathrm{SAT}$).

In an instance of the $\mathrm{SAT}$ problem we have:

- set of $n$ Boolean variables $X = \{x_1, x_2, \dots, x_n\}$
- $m$ clauses $C_1, C_2, \dots, C_m$, where each clause is a disjunction of literals from $X$, e.g., $C_1 = x_1 \vee \neg x_2 \vee x_3$

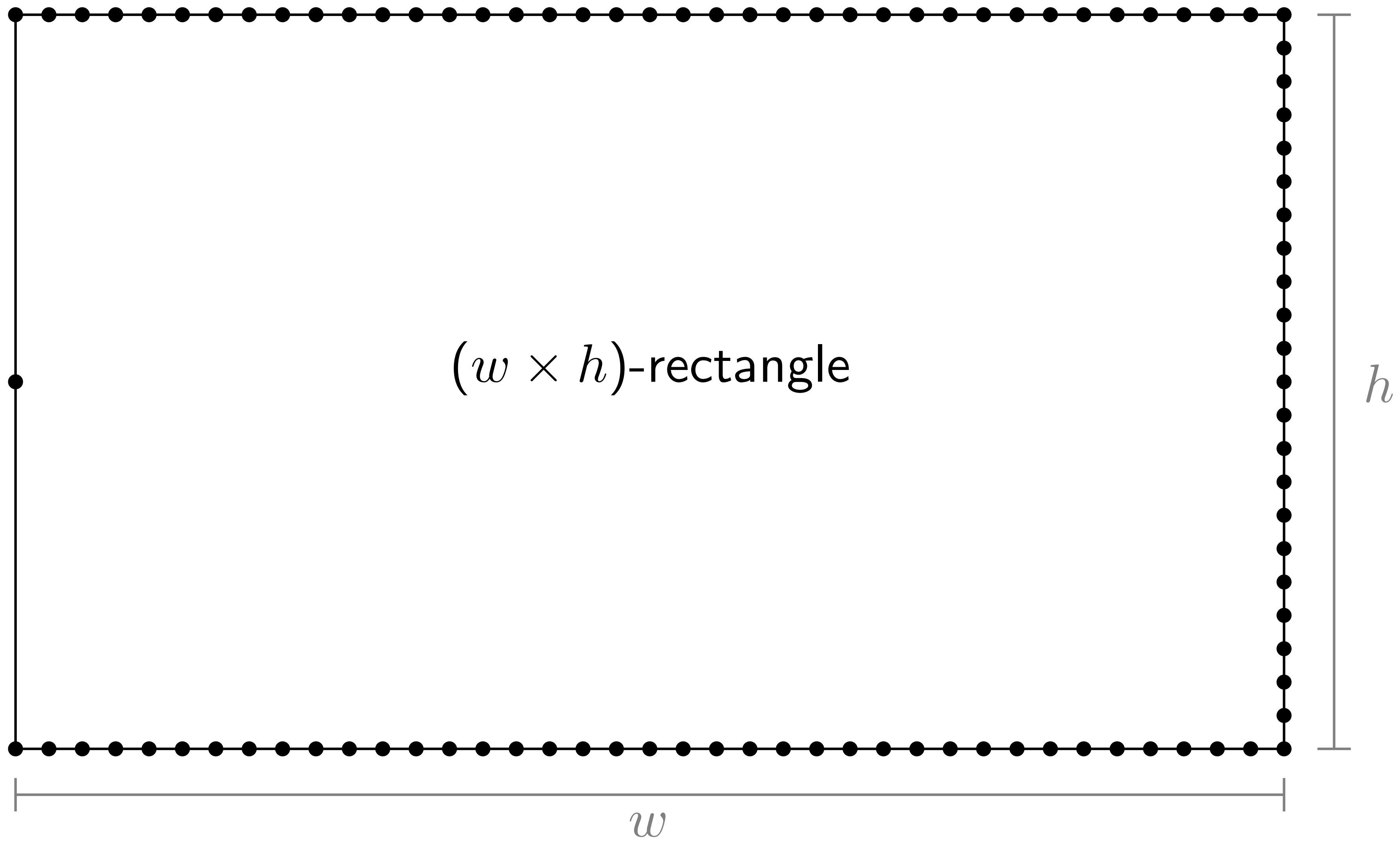
  a literal is a variable $x$ or a negated variable $\neg x$
- Boolean formula $\Phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$

Question: Is there an assignment of truth values to the variables in $X$ such that $\Phi$ is true?

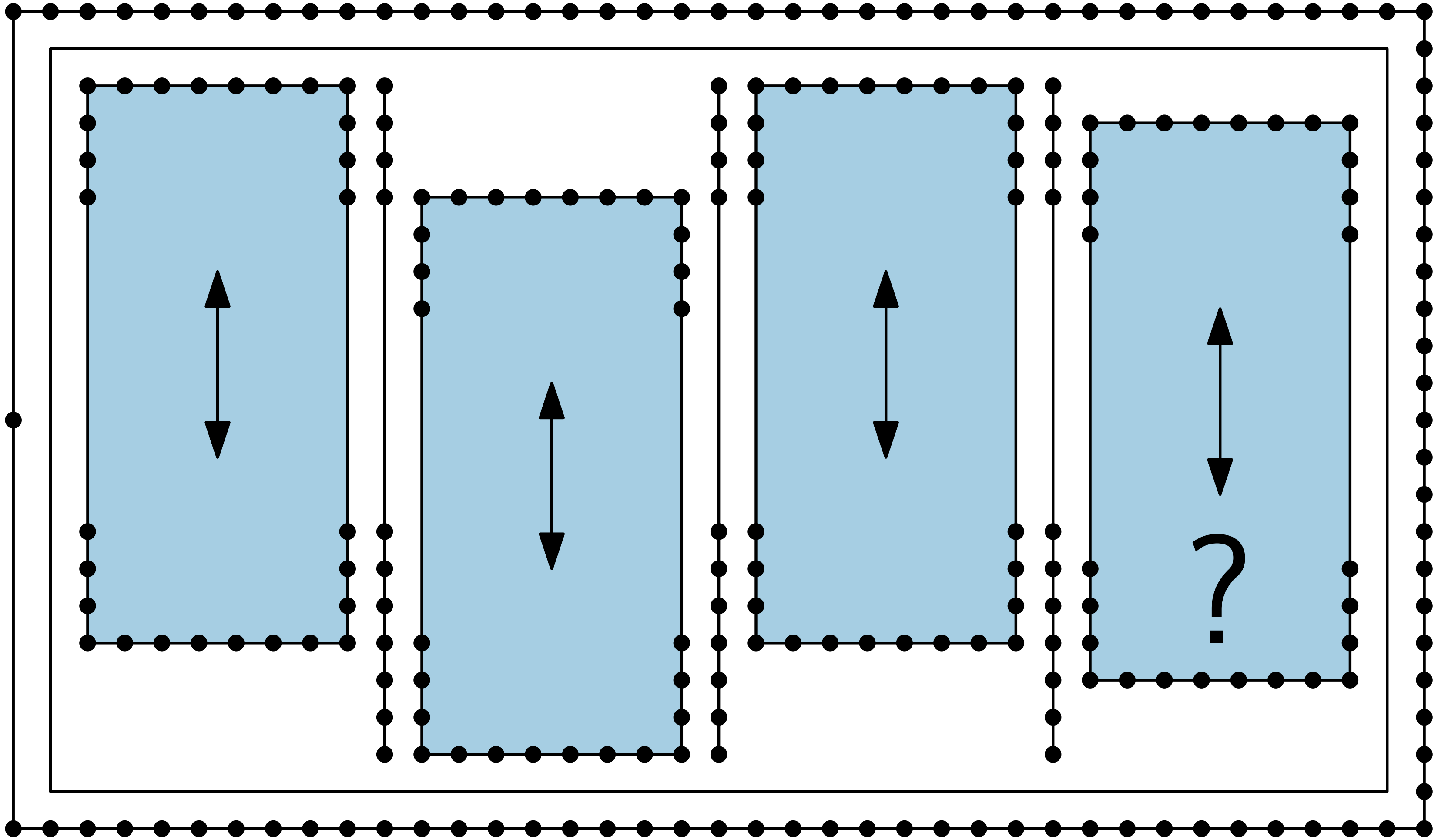
Idea of the reduction:

- Given $\mathrm{SAT}$ instance $\Phi \Rightarrow$ construct a plane graph $G$ and a orthogonal description $H(G)$
- $\Phi$ is satisfiable $\Leftrightarrow$ $G$ can be drawn w.r.t. $H(G)$ in area $K$ for some specific number $K$

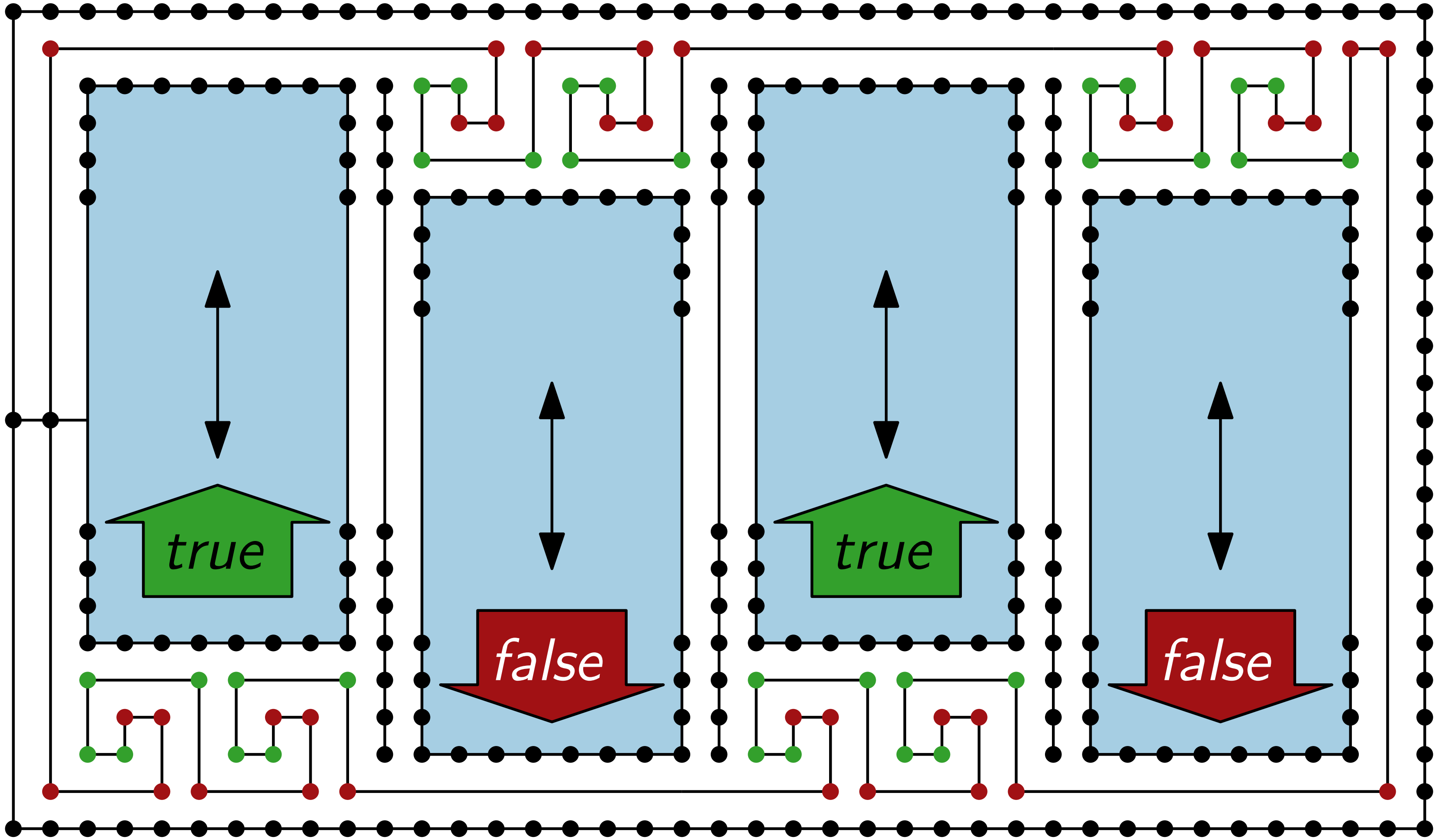
# Boundary, Belt, and “Piston” Gadget



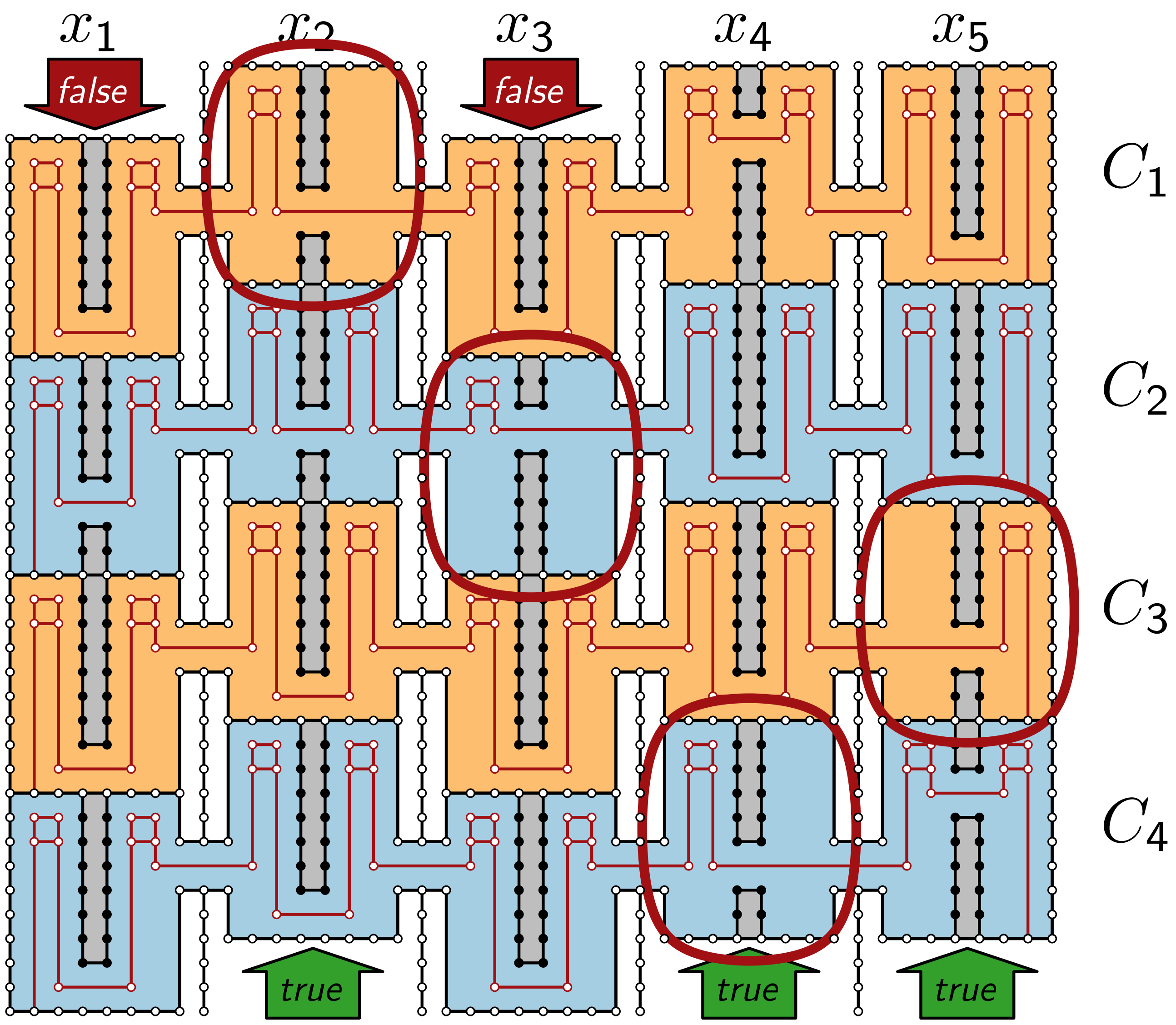
# Boundary, Belt, and “Piston” Gadget



# Boundary, Belt, and “Piston” Gadget

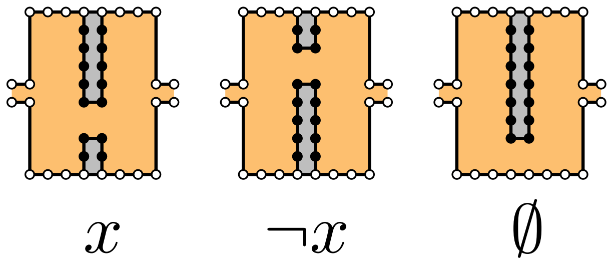


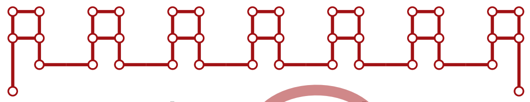
# Clause Gadgets




Example:

$C_1 = x_2 \vee \neg x_4$
$C_2 = x_1 \vee x_2 \vee \neg x_3$
$C_3 = x_5$
$C_4 = x_4 \vee \neg x_5$

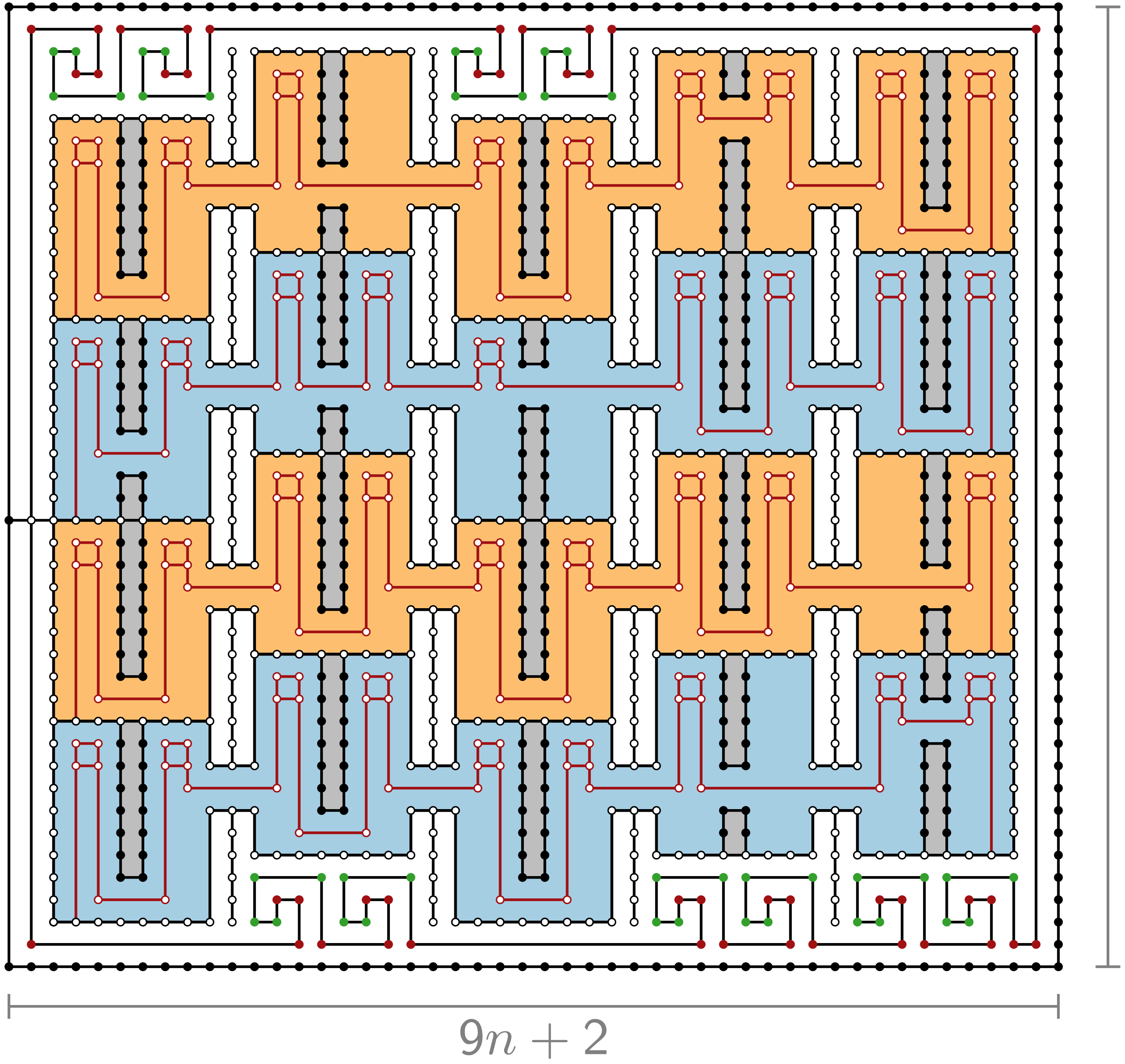


insert $(2n - 1)$-chain
through each clause

$\rightarrow$ for every clause, there needs to be $\geq 1$ "gap of a literal" to be on the same height as the "tunnel" to the next literal

# Complete Reduction




Pick
$K = (9n+2) \times (9m+7)$

Then:

$G$ under $H(G)$ has an orthogonal drawing in area $K$

$\Leftrightarrow$

$\Phi$ satisfiable

$\square$

# Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] “On embedding a graph in the grid with the minmum number of bends”
  Original paper on flow for bend minimization.
- [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
  “A Deterministic Almost-Linear Time Algorithm for Minimum-Cost Flow”
  State-of-the-art algorithm for solving the minimum-cost flow problem
  (published recently in the proceedings of the FOCS 2023 conference).
- [Patrignani 2001] “On the complexity of orthogonal compaction”
  NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.
- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]
  “Minimum rectilinear polygons for given angle sequences”: Compacting cycles is NP-hard.
- [Antić, Liotta, Masařík Ortali, Pfretzschner, Stumpf, Wolff, Zink 2025]
  “Unbent Collections of Orthogonal Drawings”: It is NP-hard to find two drawings such that each edge is straight in one and the total number of bends is minimum.