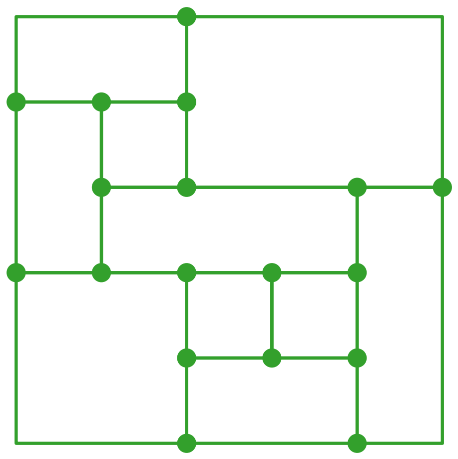
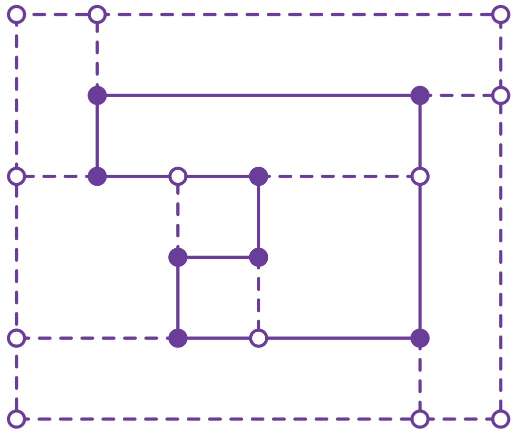
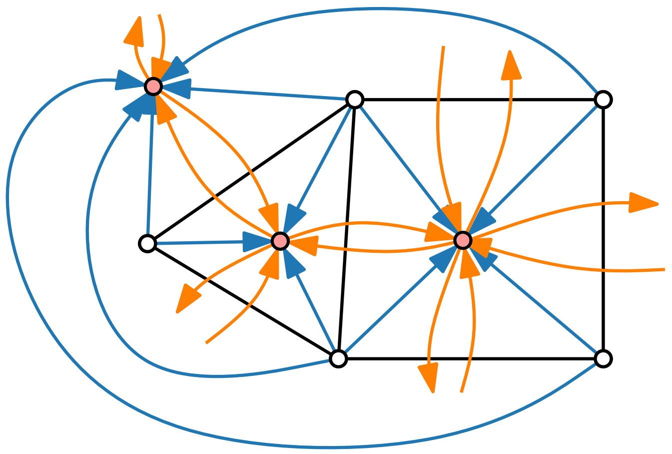
Visualization of Graphs





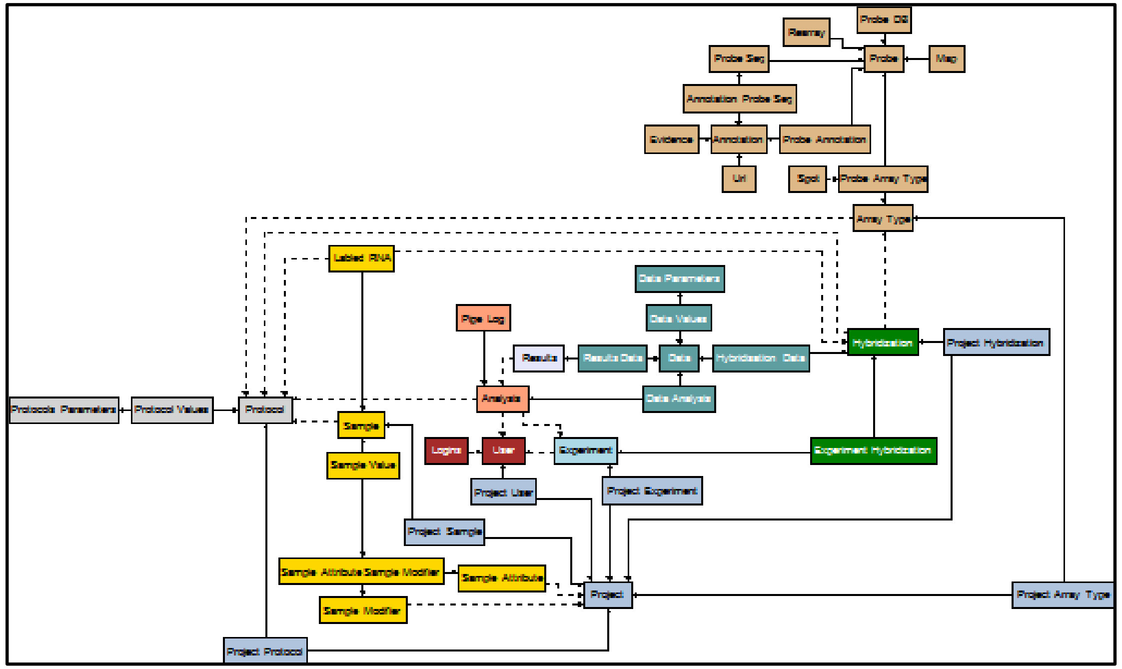
Lecture 6: Orthogonal Layouts



Alexander Wolff

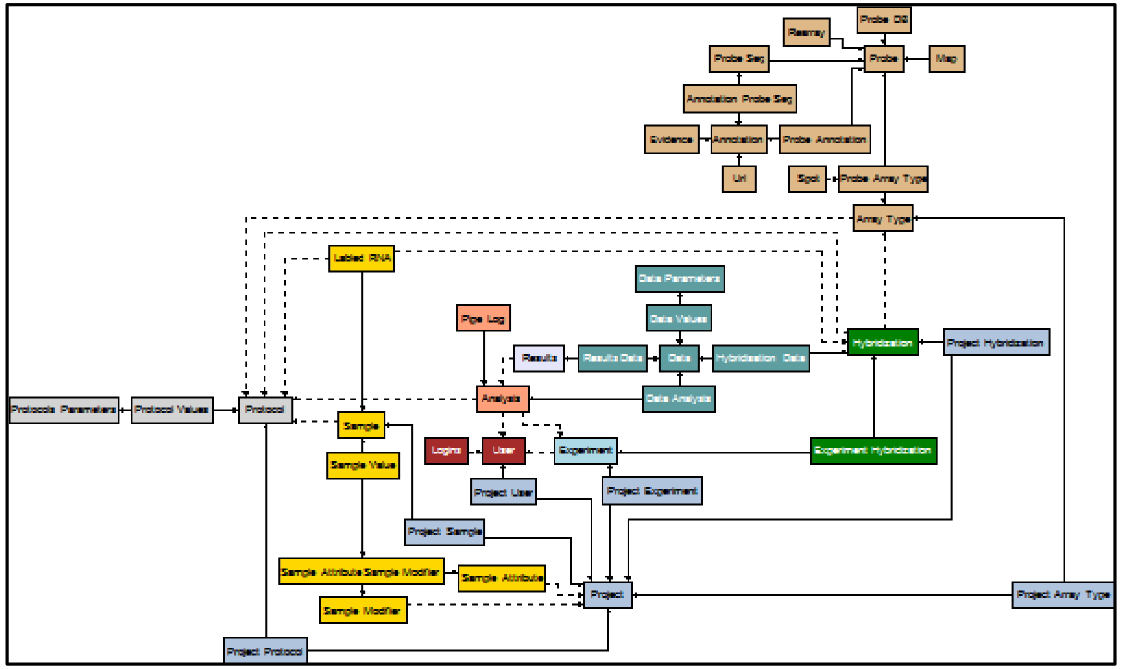
Summer term 2025

Orthogonal Layout – Applications



Entity-Relationship (ER) diagram in OGDF

Orthogonal Layout – Applications



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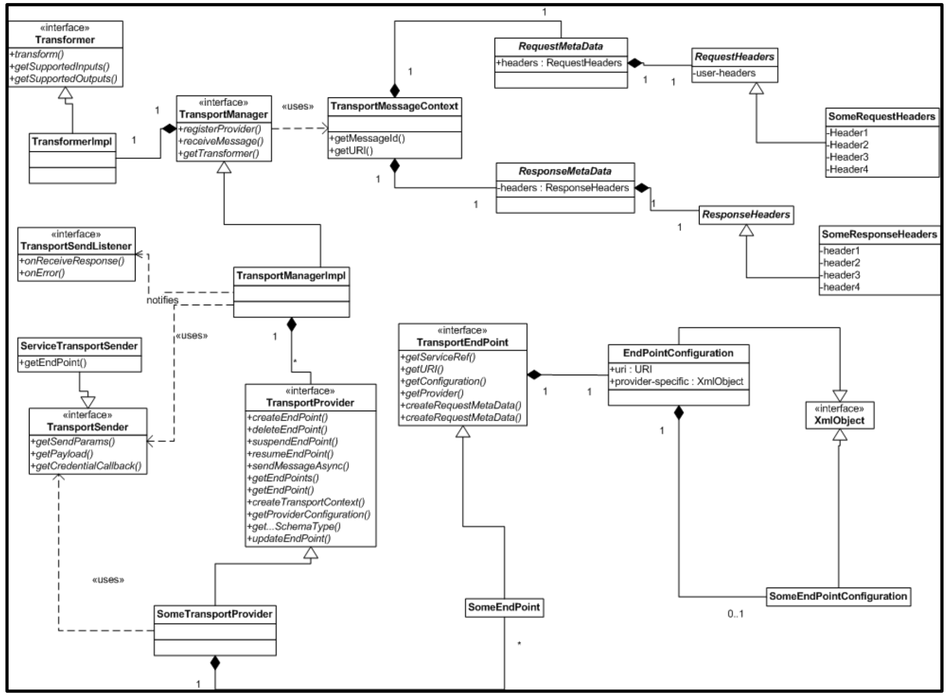
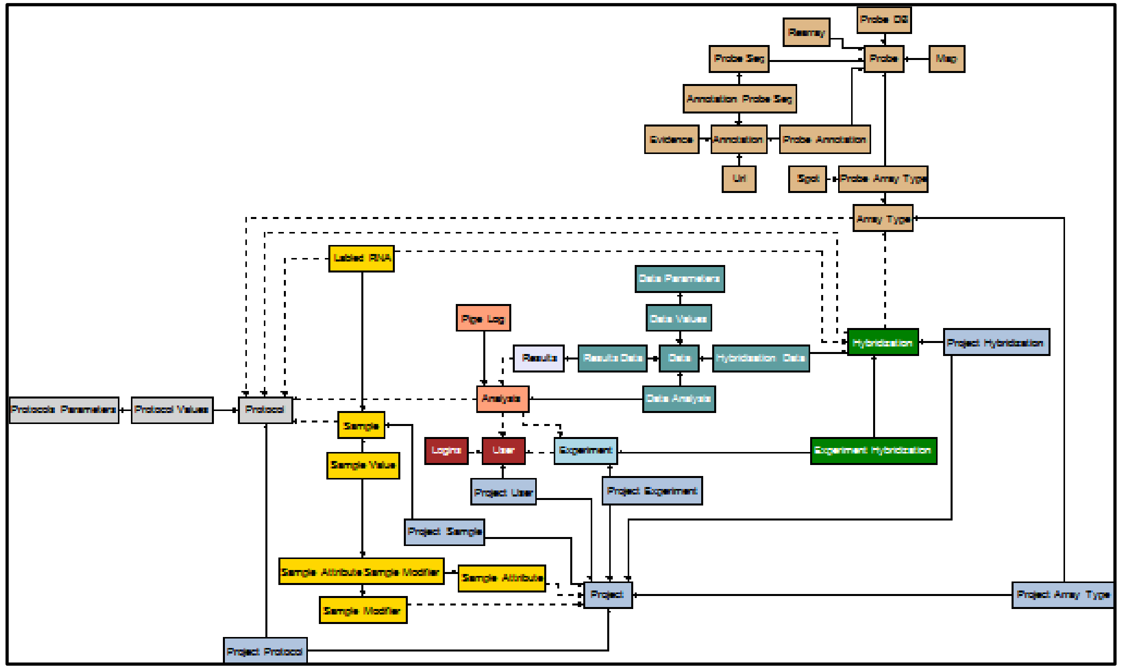


UML diagram by Oracle

Orthogonal Layout – Applications



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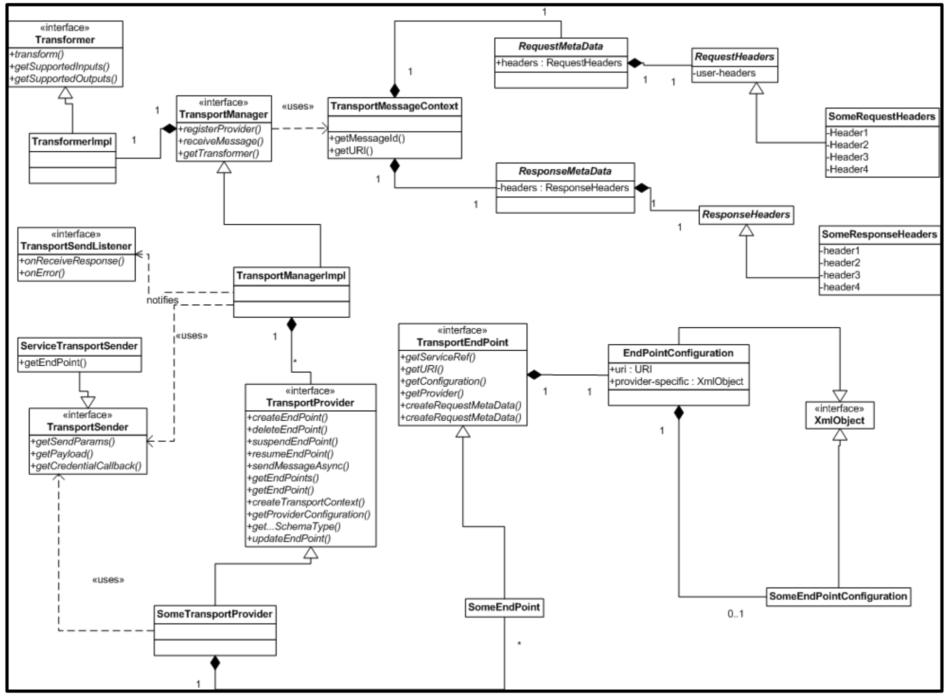

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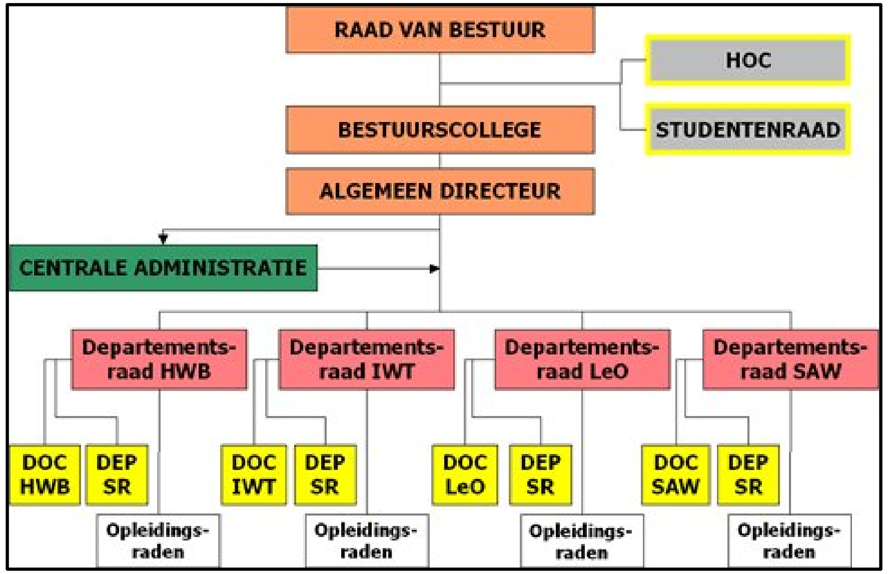
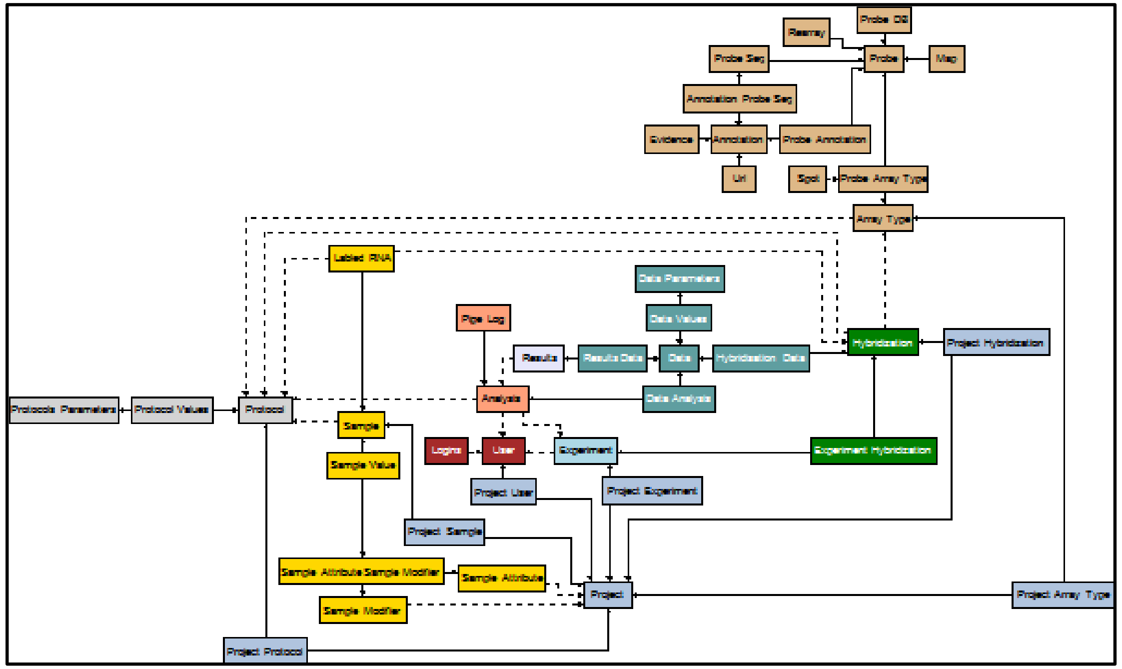


Organigram of HS Limburg

Orthogonal Layout – Applications



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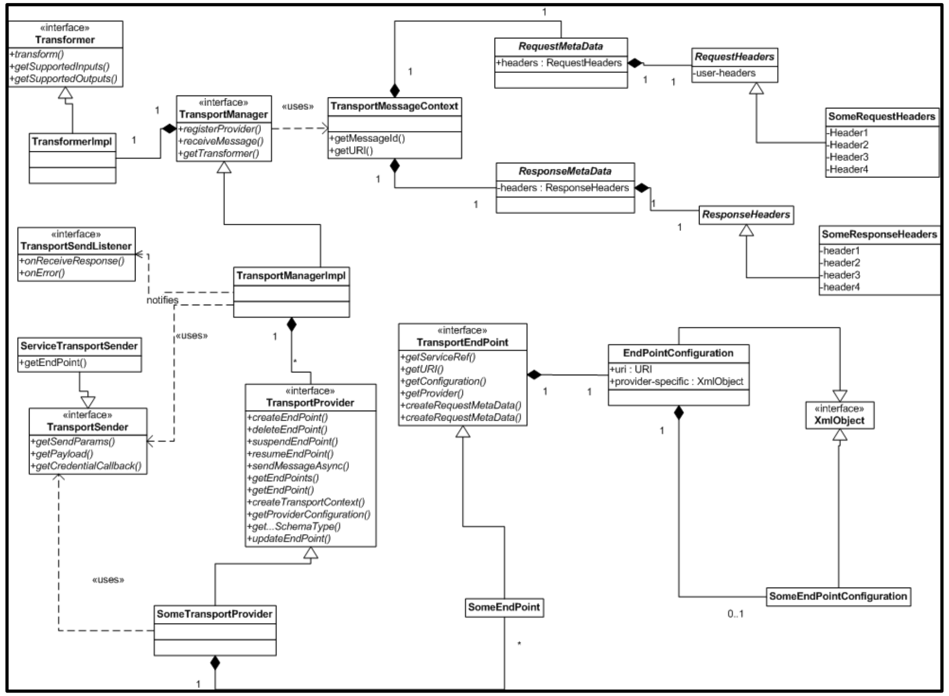

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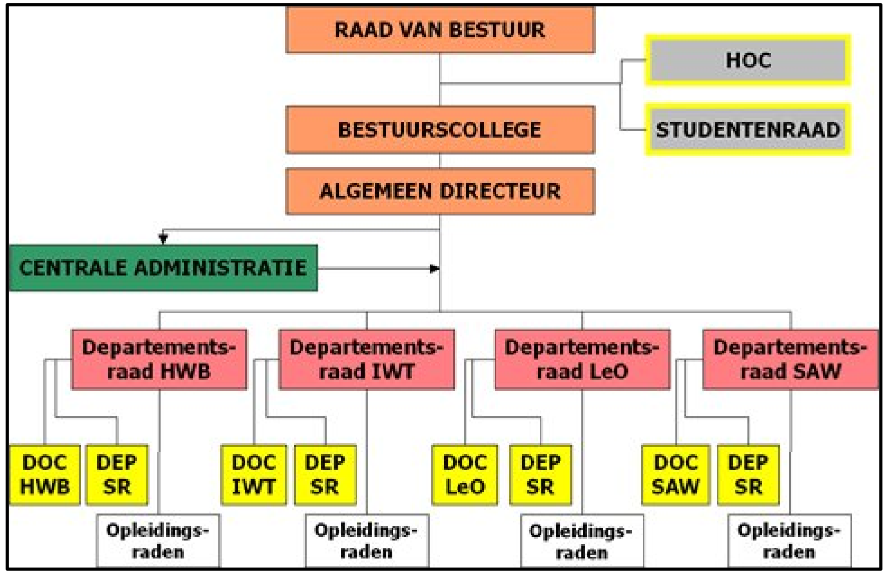

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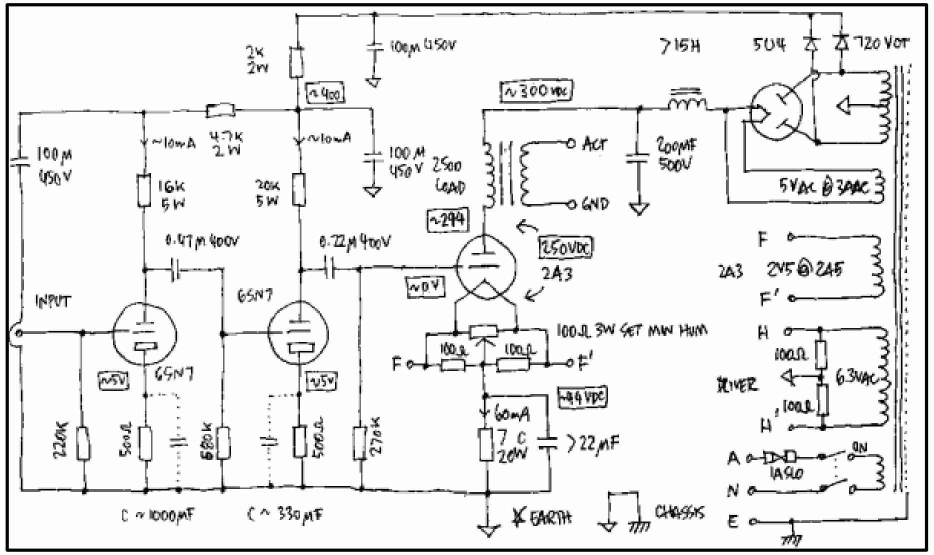
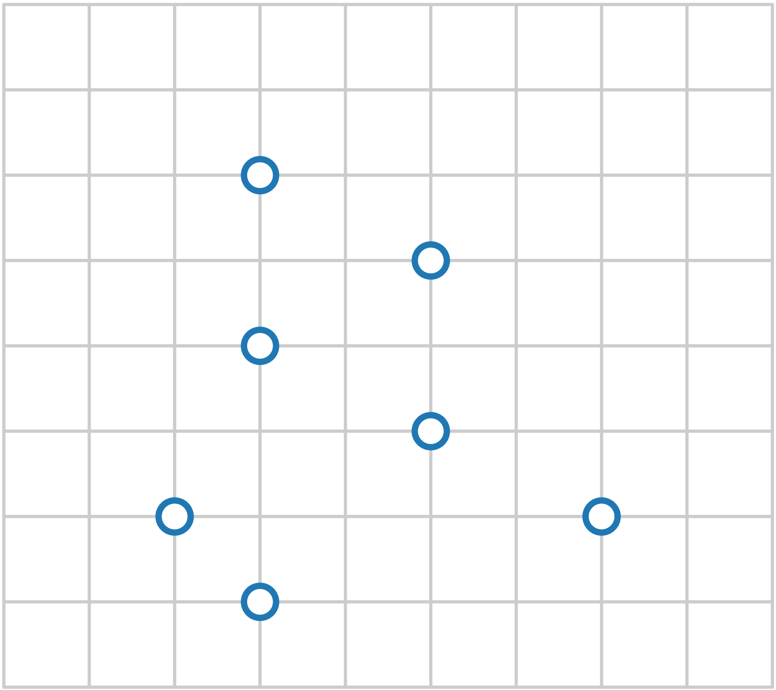


Circuit diagram by Jeff Atwood

Orthogonal Layout – Definition

Definition.
A drawing Γ of a graph G is called **orthogonal** if

Orthogonal Layout – Definition



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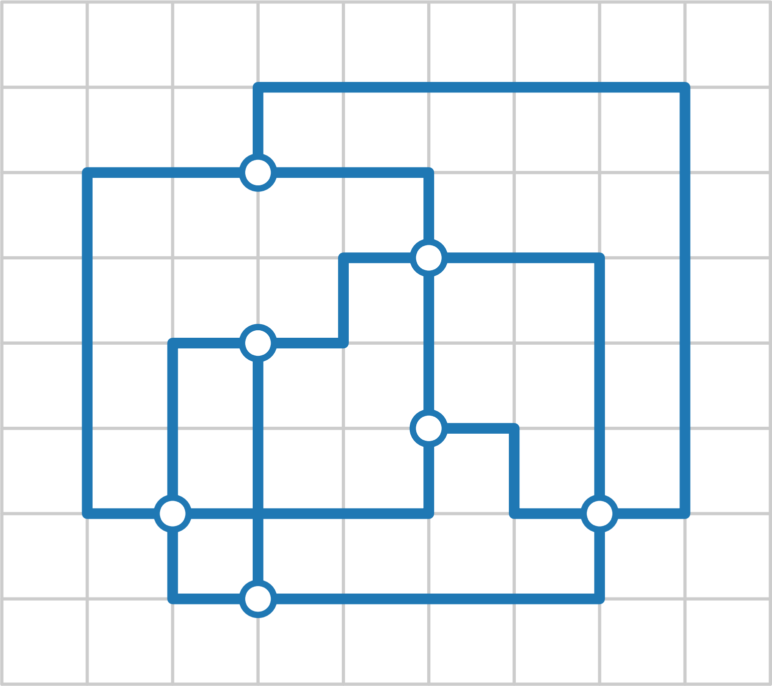

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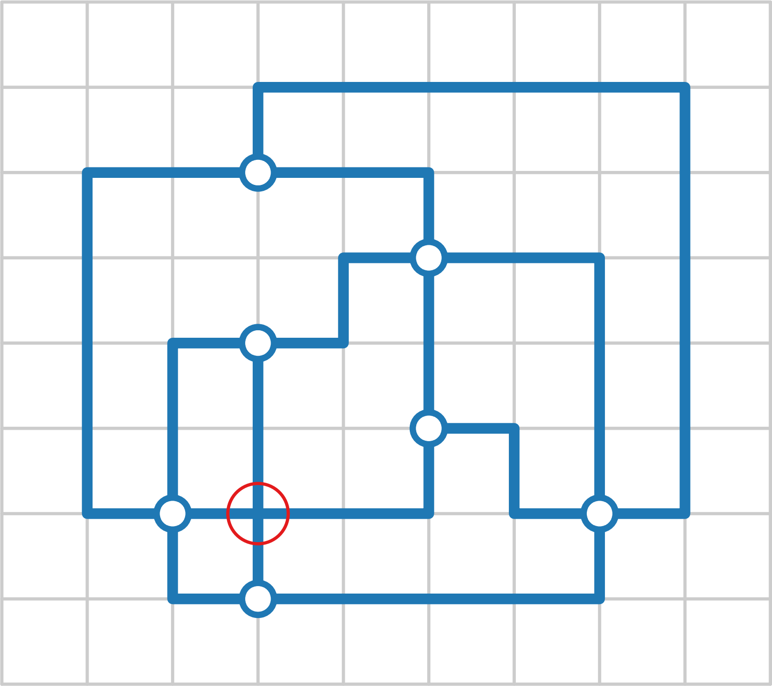

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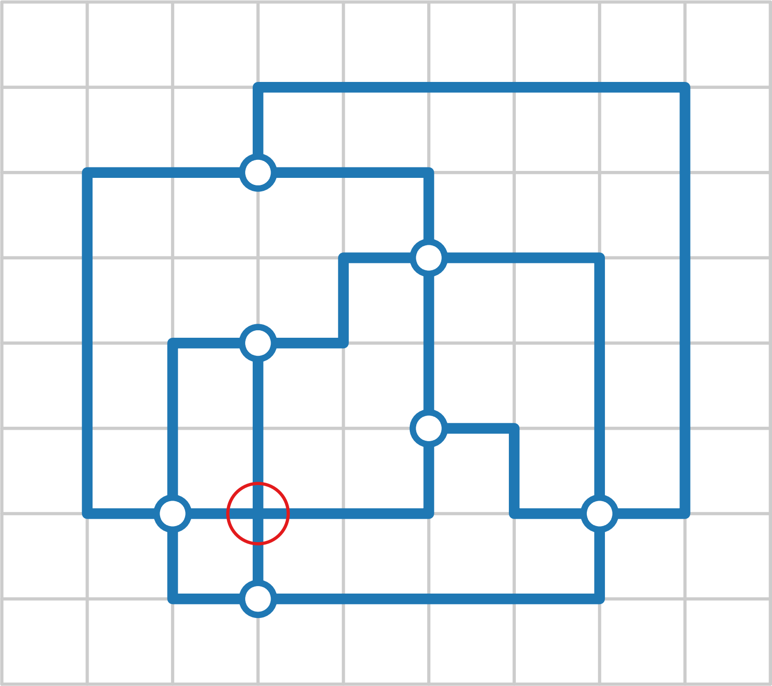

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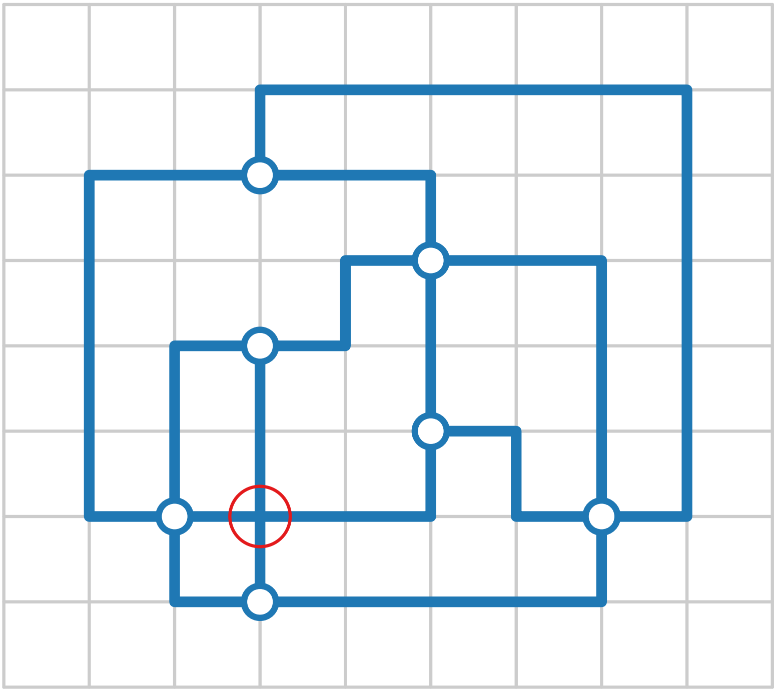

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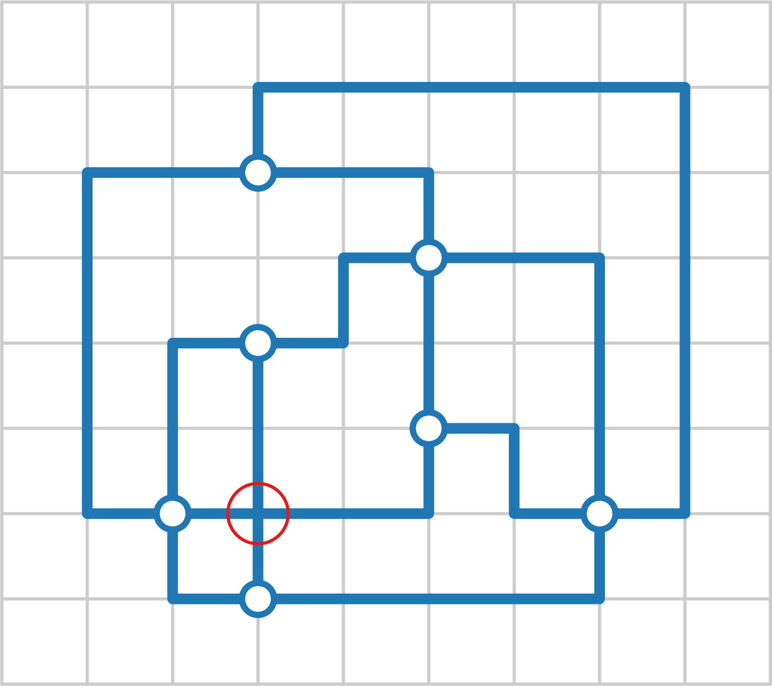

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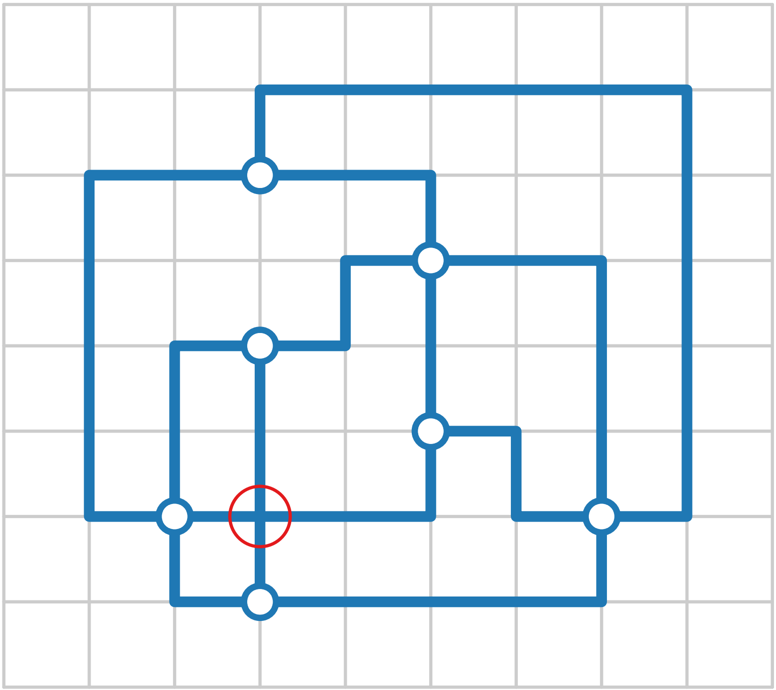

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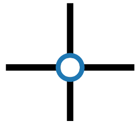
- Otherwise 

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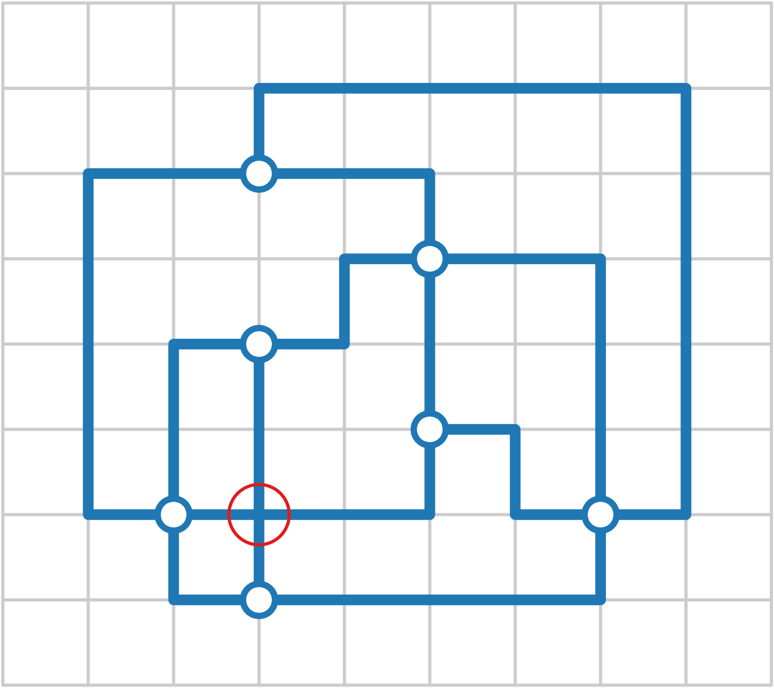

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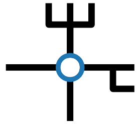
- Otherwise 

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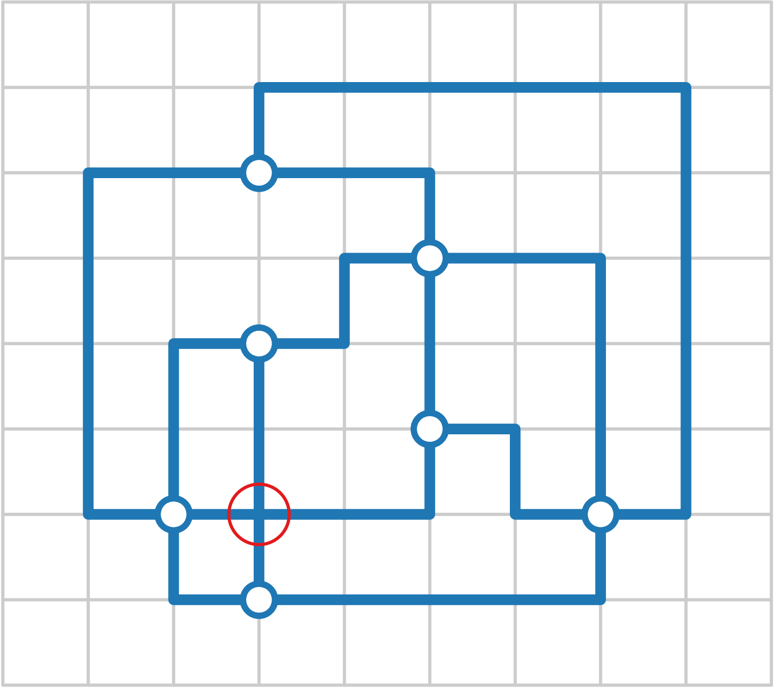

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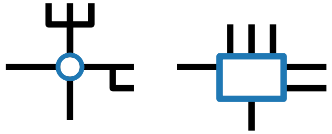

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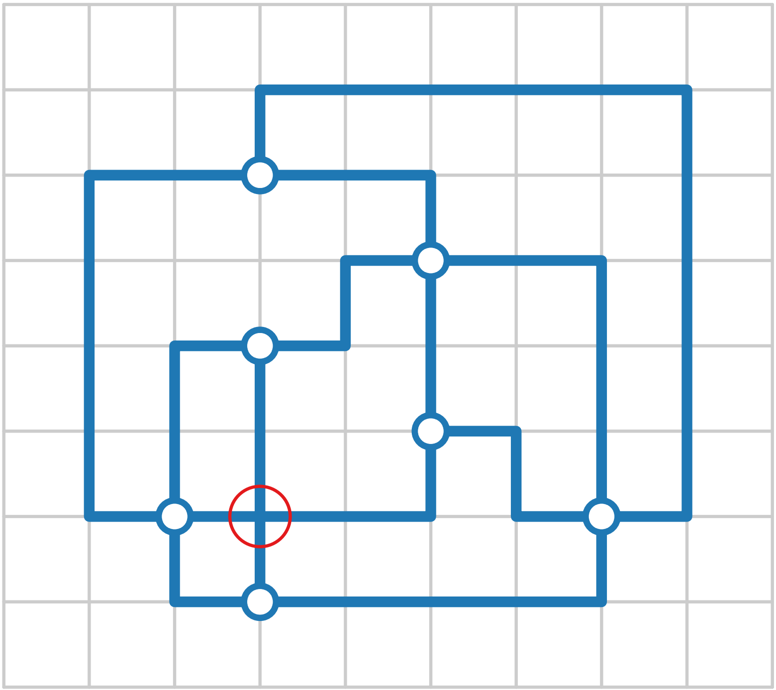

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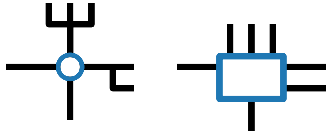
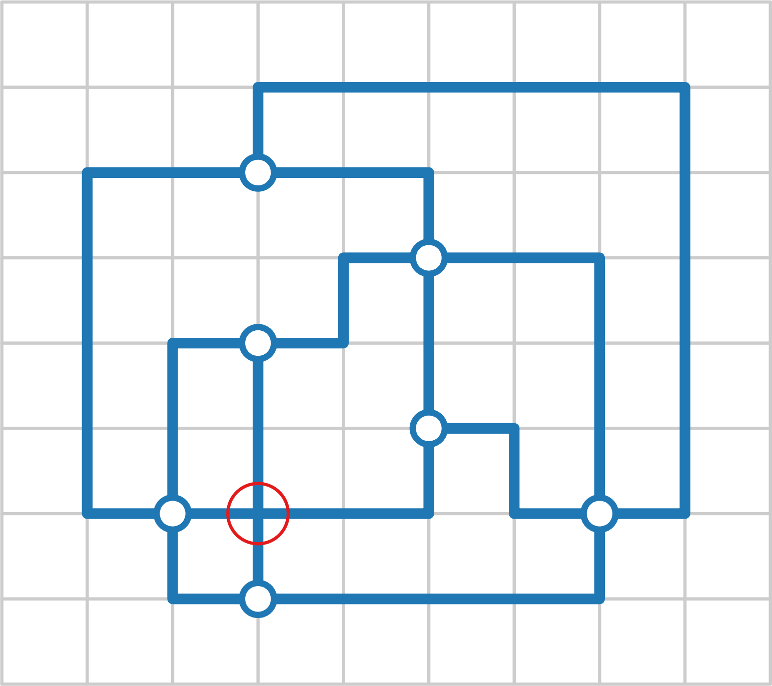


Planarization.

Orthogonal Layout – Definition



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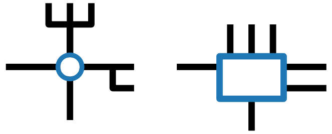
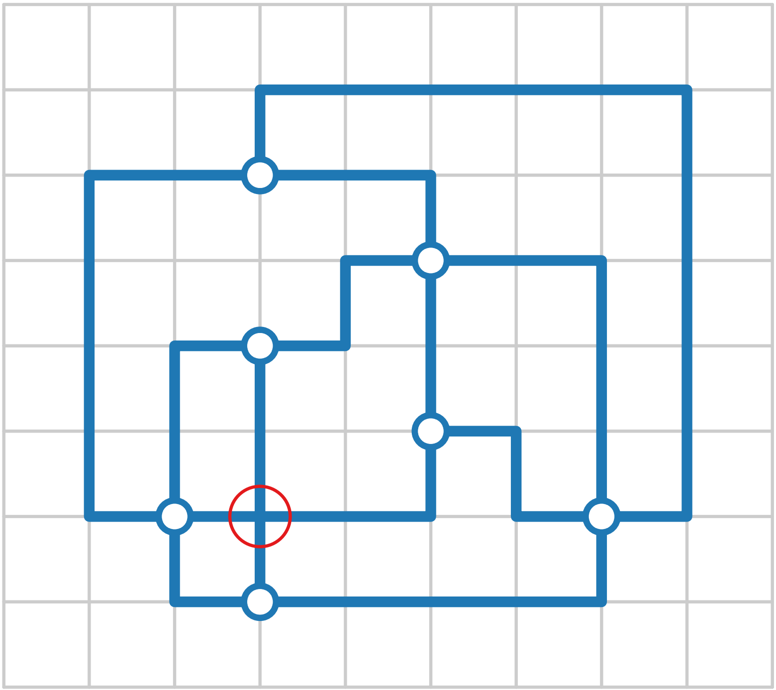


Planarization.
- Fix embedding

Orthogonal Layout – Definition



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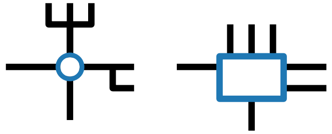
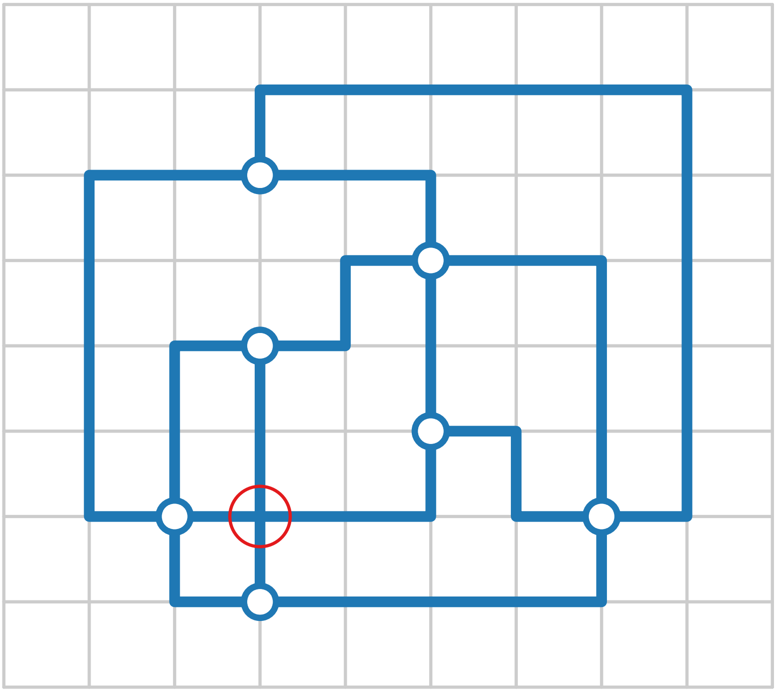


Planarization.
- Fix embedding
- Crossings become vertices

Orthogonal Layout – Definition



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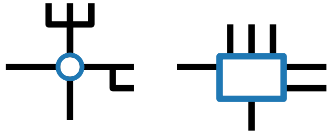
- Otherwise 

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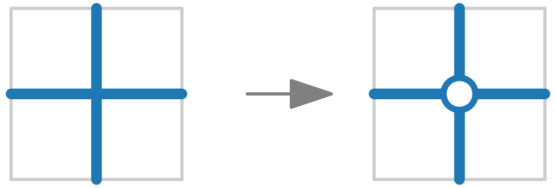

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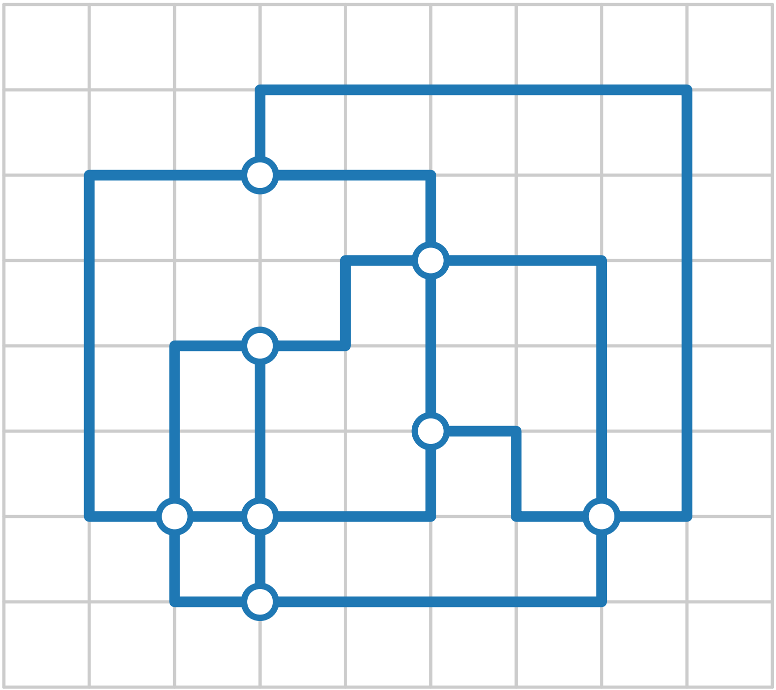

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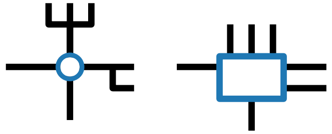

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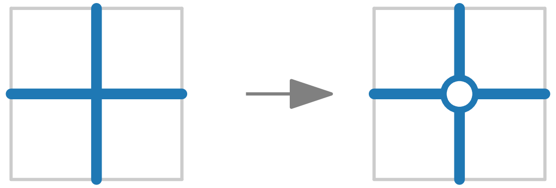

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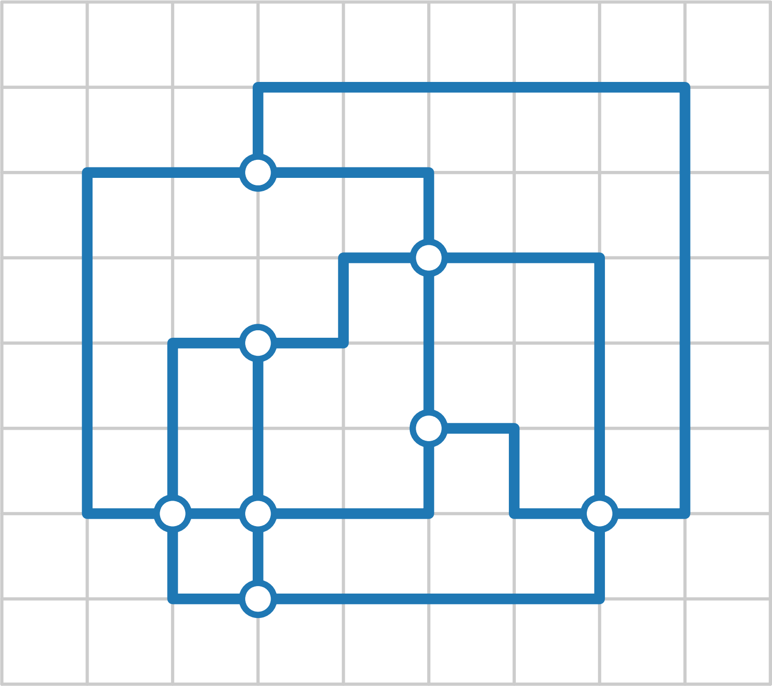

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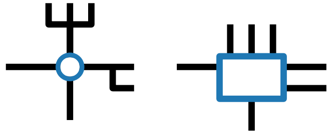

Planarization.
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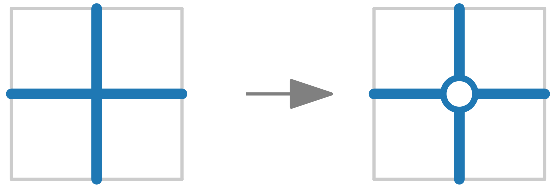
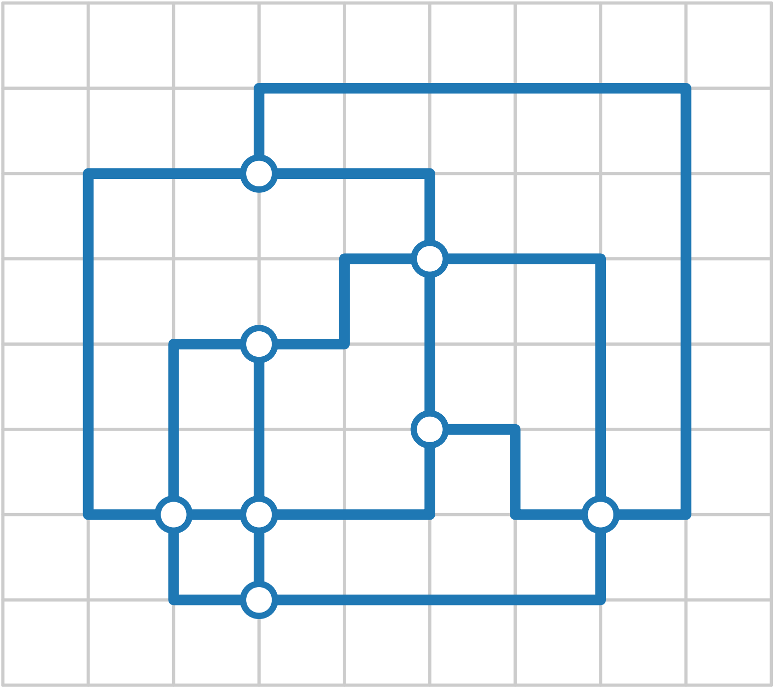


Aesthetic criteria to optimize.

Orthogonal Layout – Definition



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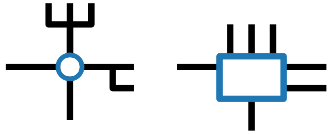

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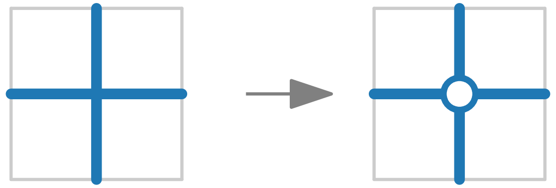
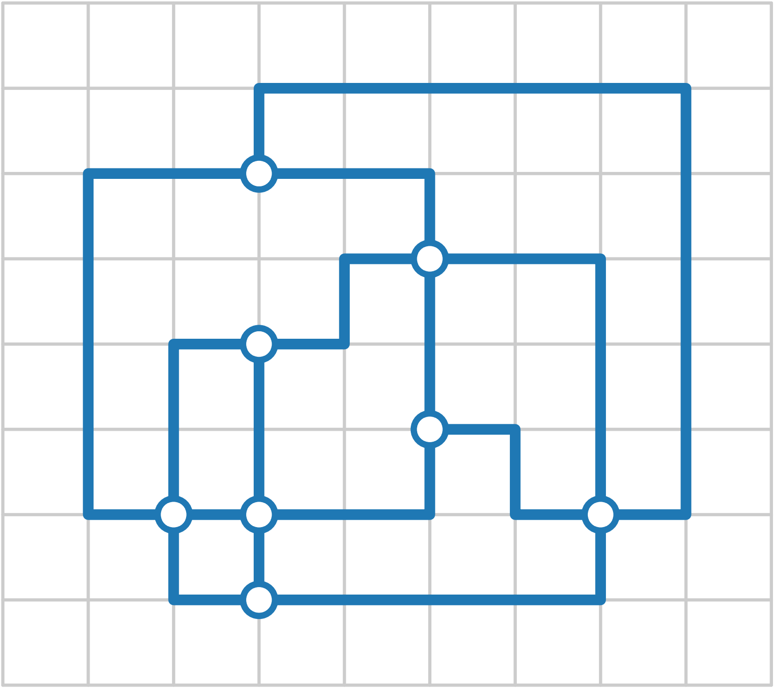


Aesthetic criteria to optimize.
- Number of bends

Orthogonal Layout – Definition



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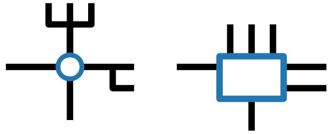

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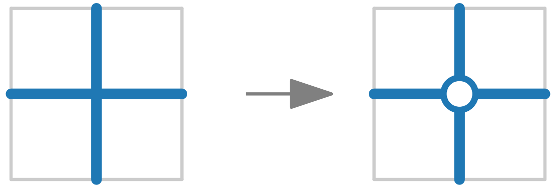
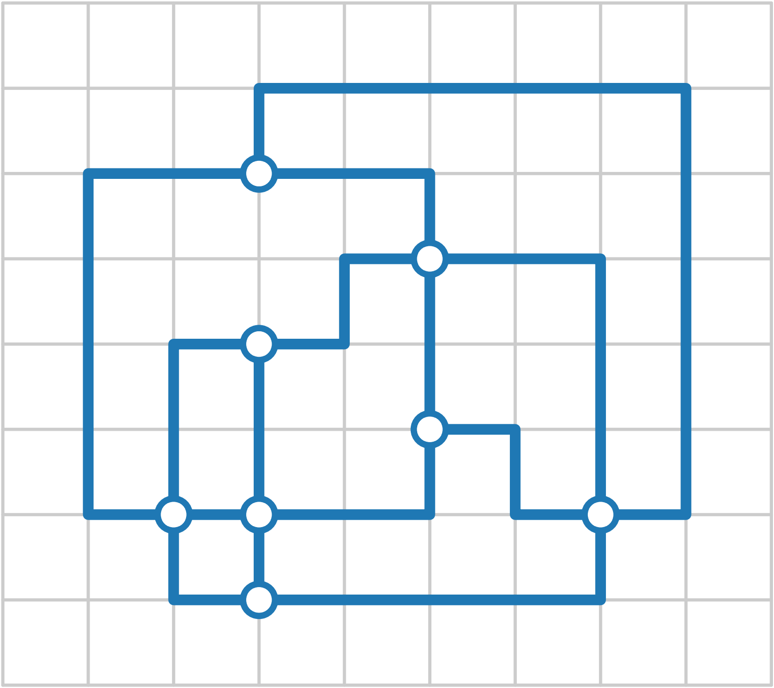


Aesthetic criteria to optimize.
- Number of bends
- Length of edges

Orthogonal Layout – Definition



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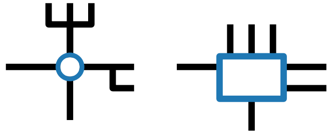

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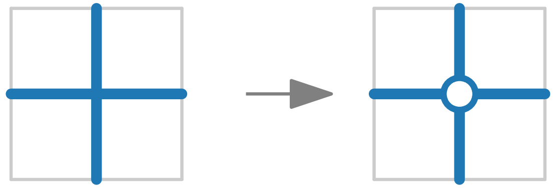
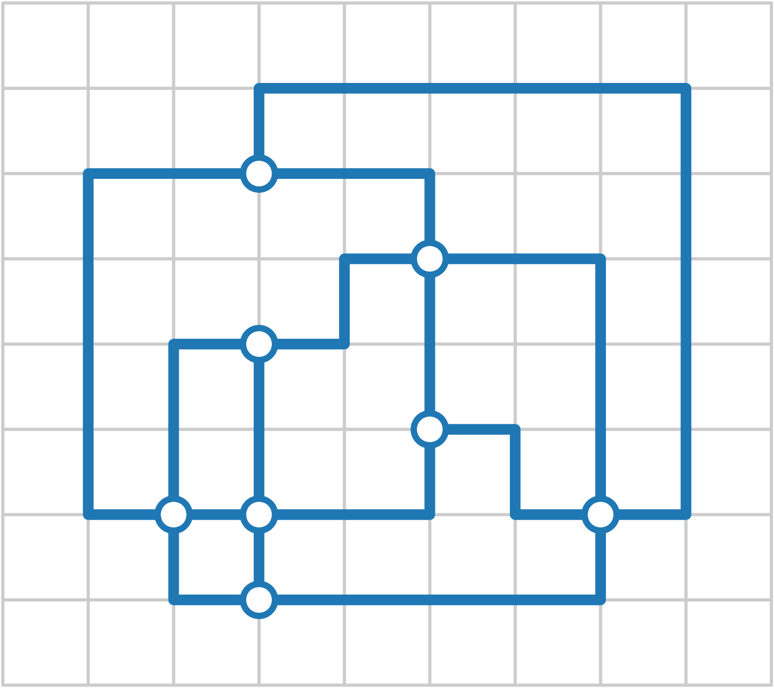


Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area

Orthogonal Layout – Definition



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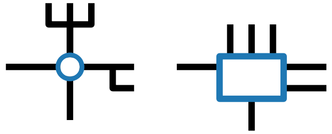

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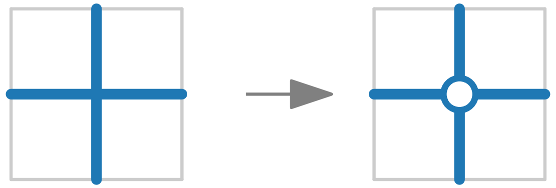

Aesthetic criteria to optimize.
- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

TOPOLOGY – SHAPE – METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

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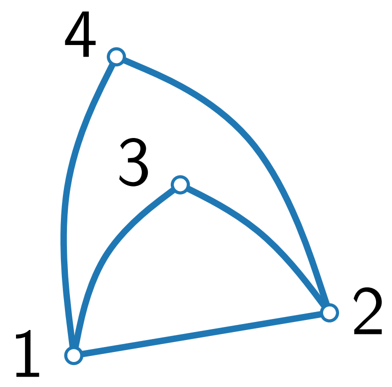
combinatorial
embedding/
planarization



TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

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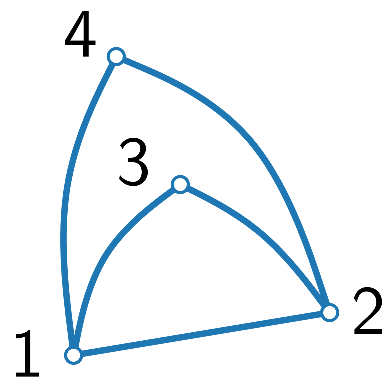

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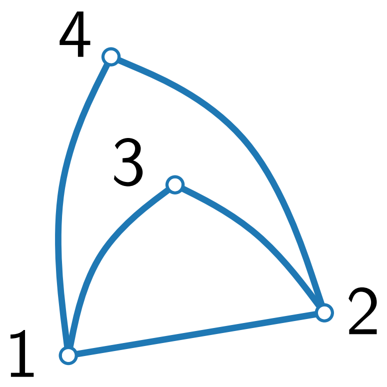

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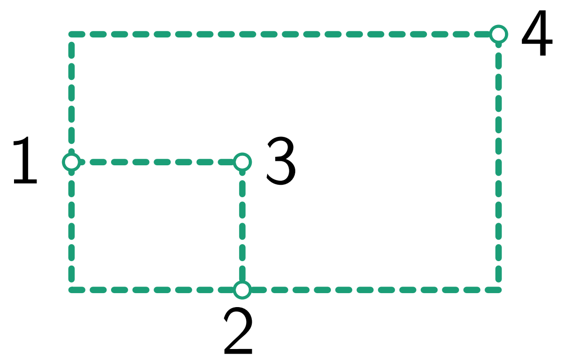

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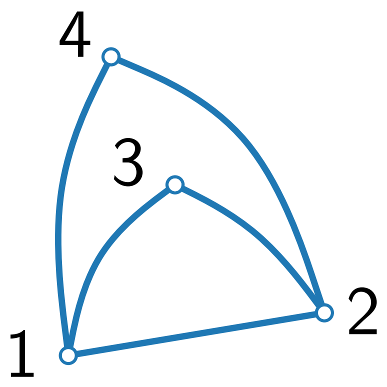

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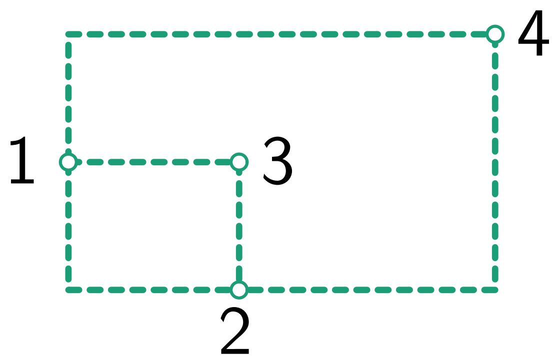

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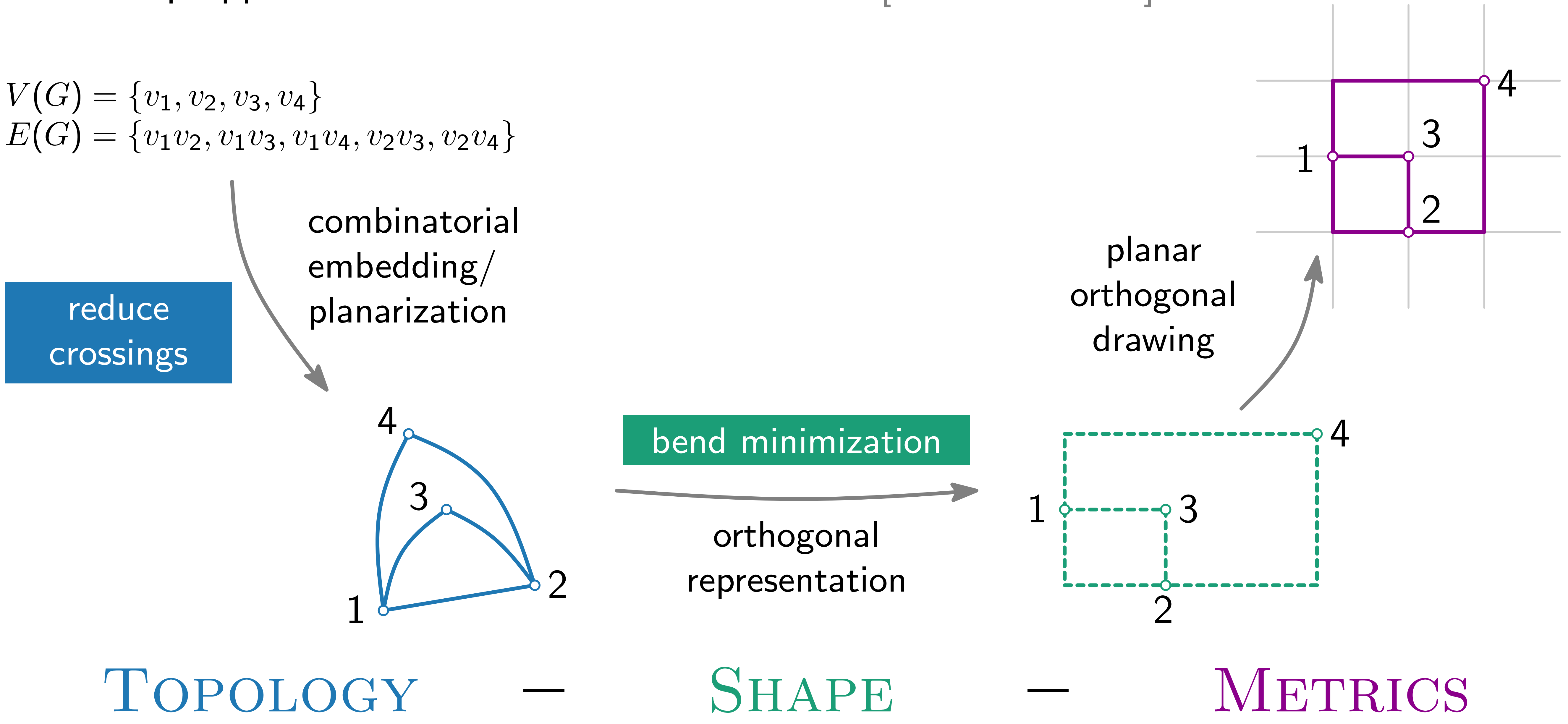

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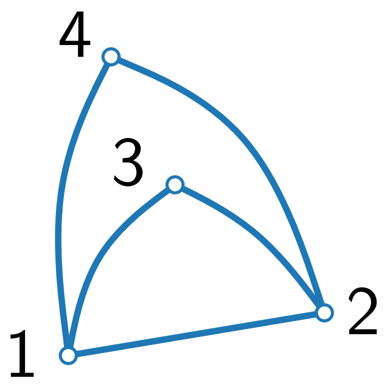

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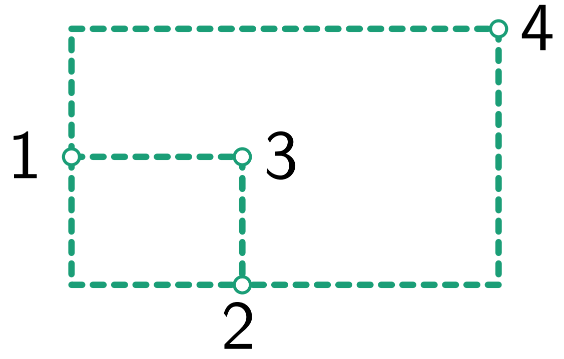
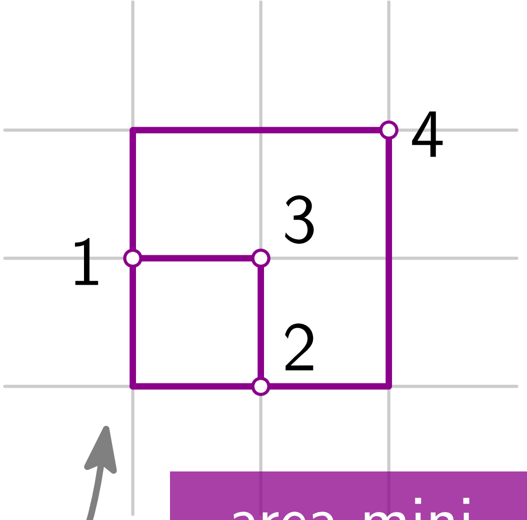


planar orthogonal drawing



area minimization

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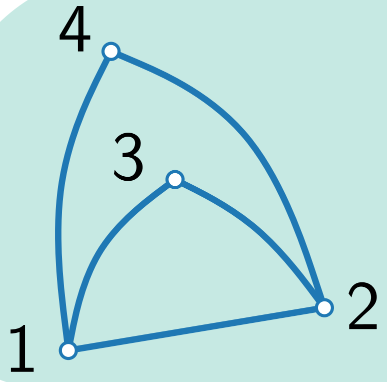

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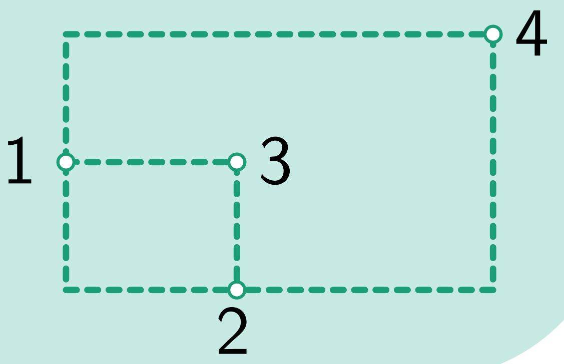

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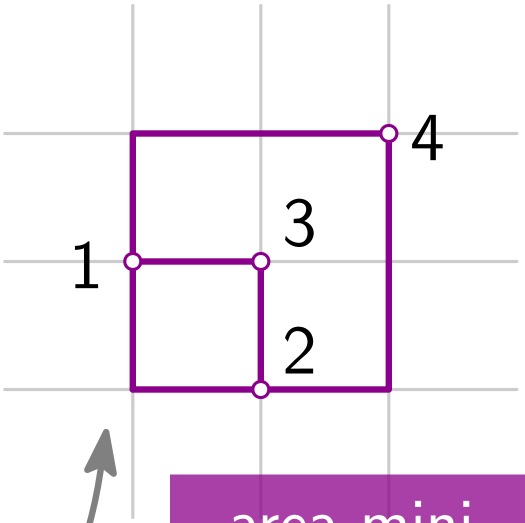

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TOPOLOGY — SHAPE — METRICS

Orthogonal Representation

Idea.
Describe orthogonal drawing combinatorially.

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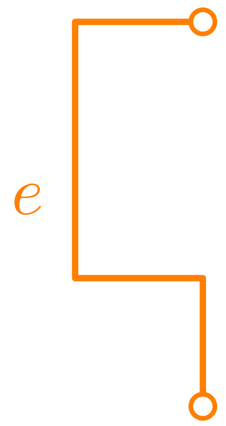

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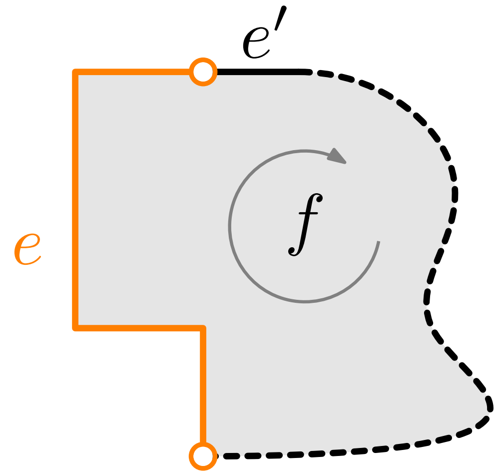

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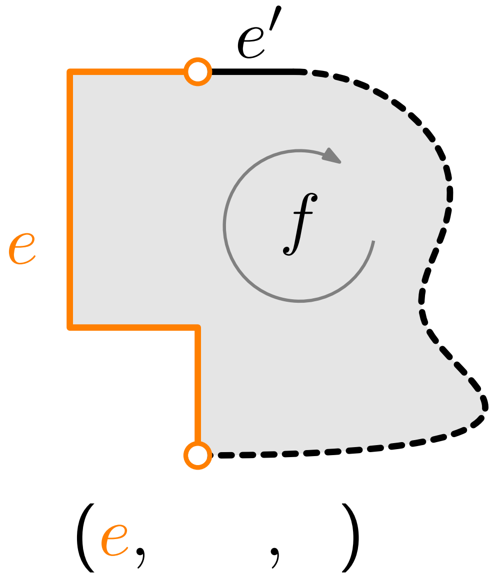

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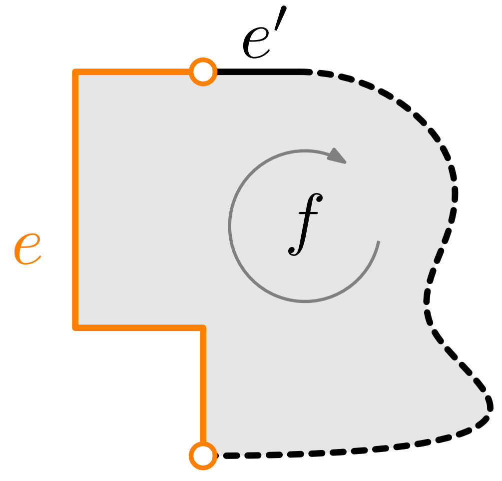

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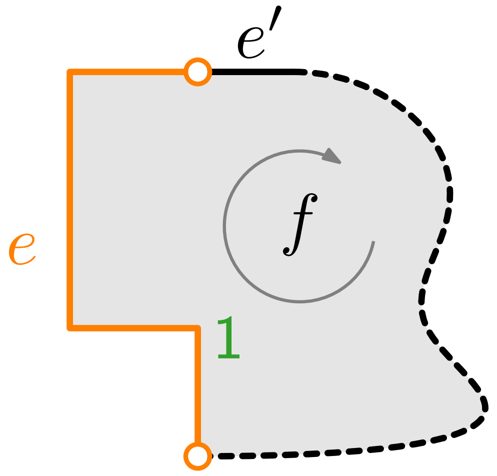

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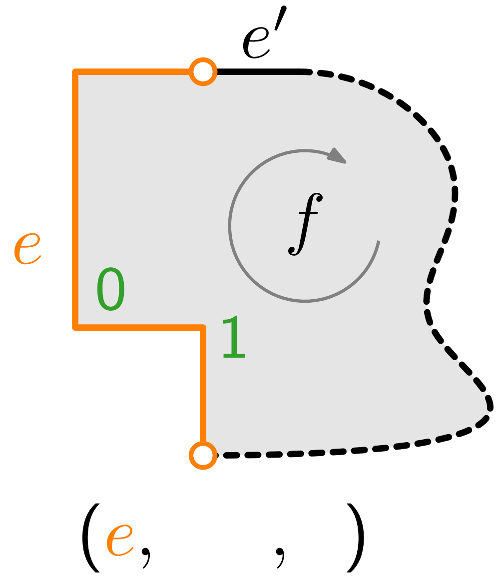

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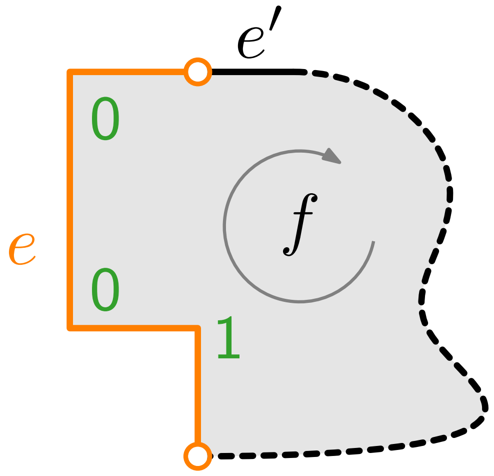

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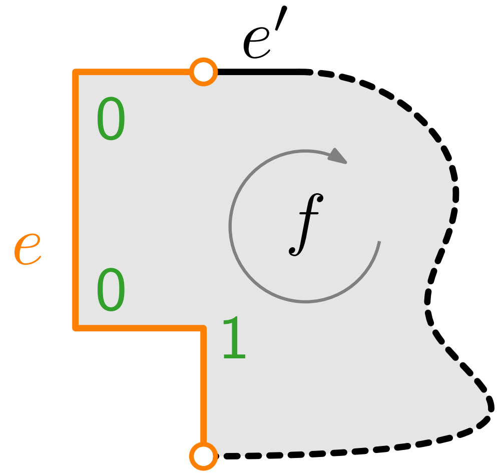

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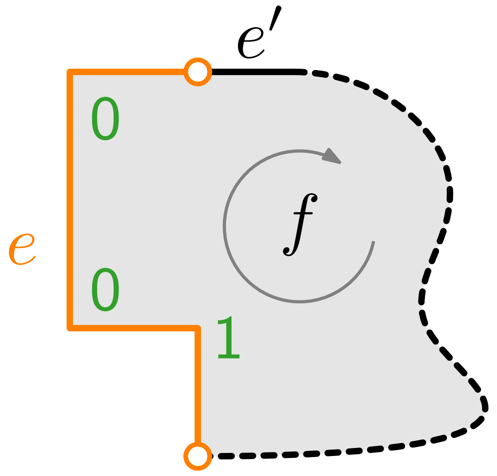

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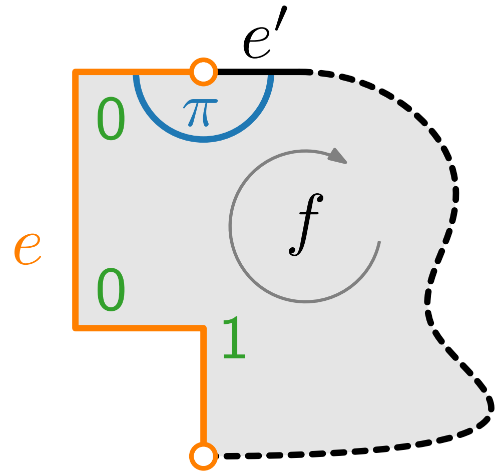

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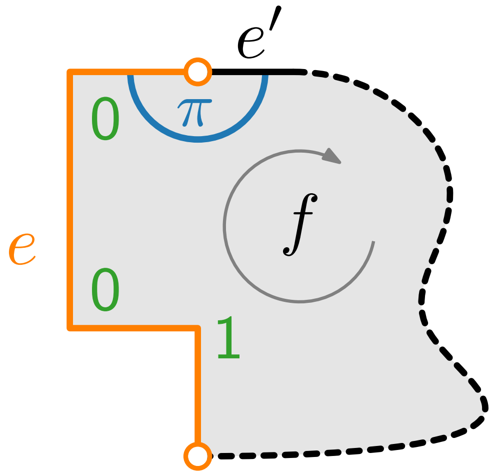

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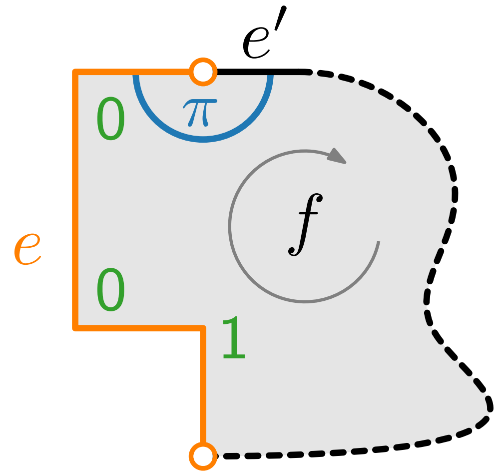

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- A **face representation** $H(f)$ of a face f is a clockwise ordered sequence $(e_1, \delta_1, \alpha_1), (e_2, \delta_2, \alpha_2), \dots, (e_{\deg(f)}, \delta_{\deg(f)}, \alpha_{\deg(f)})$ of edge descriptions w.r.t. f.

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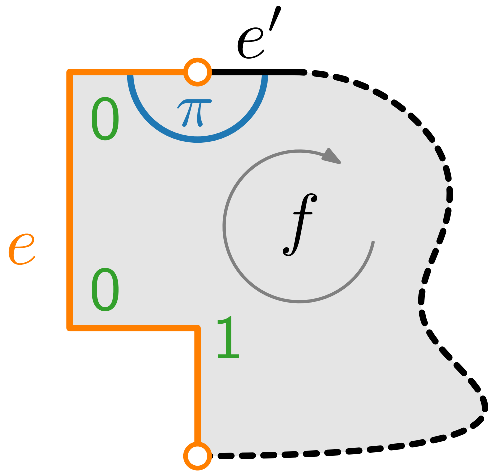

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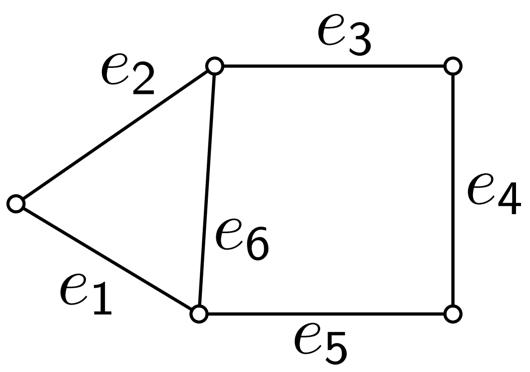
- An **orthogonal representation** $H(G)$ of G is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$

Orthogonal Representation – Example



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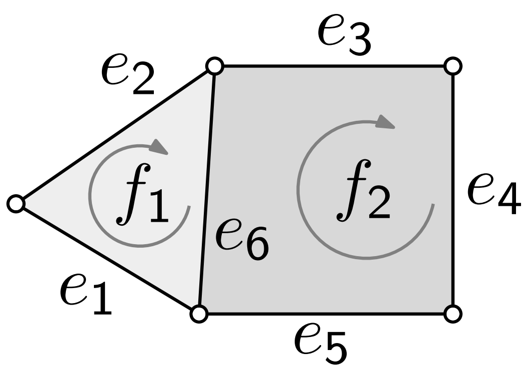

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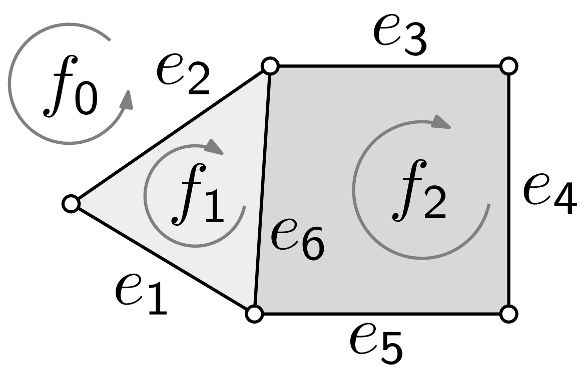

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

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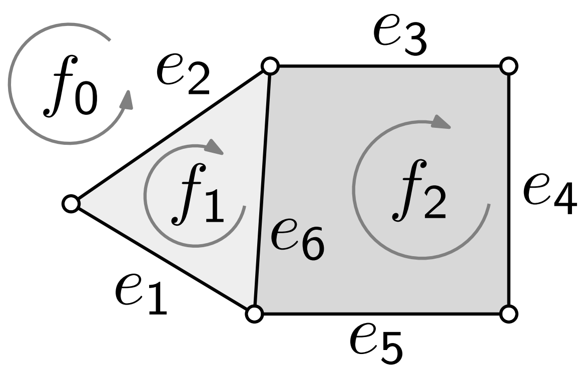

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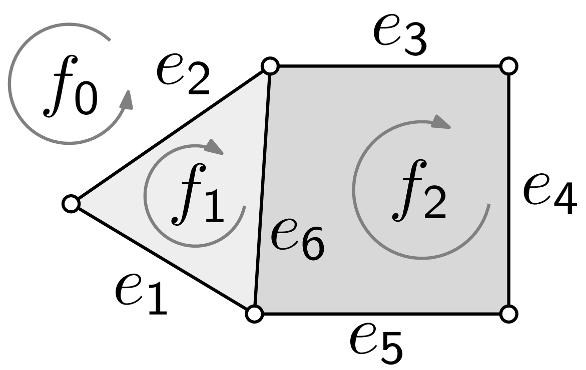

Combinatorial "drawing" of $H(G)$?

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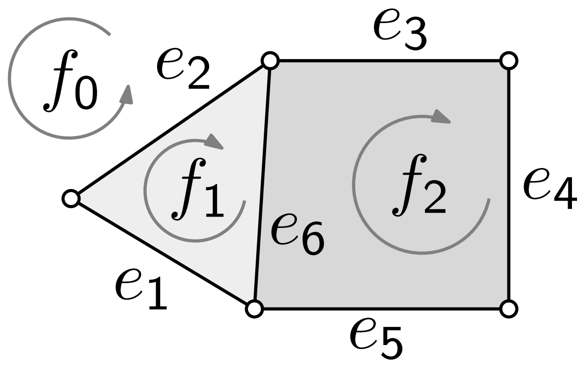

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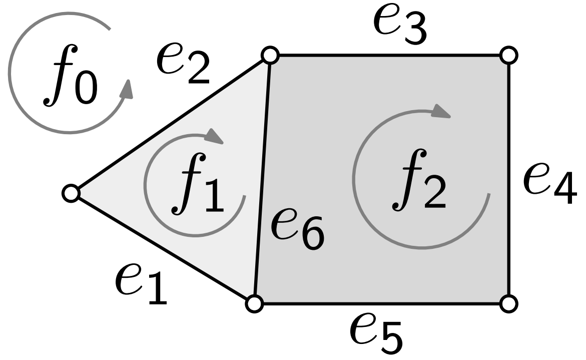

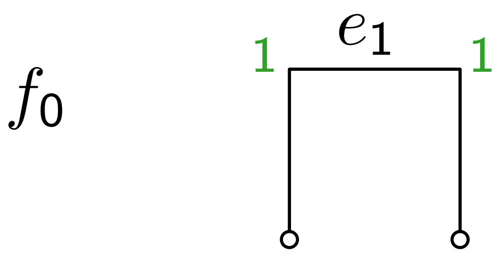

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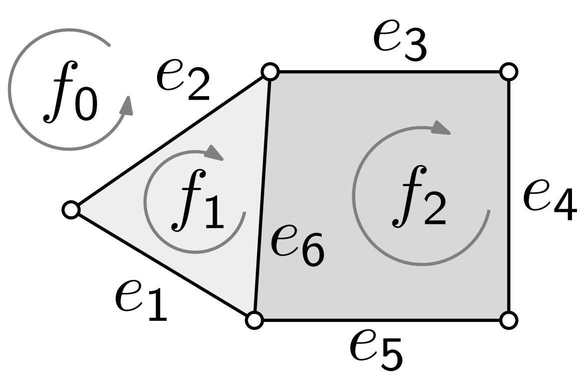

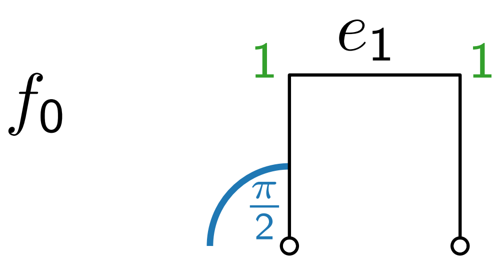

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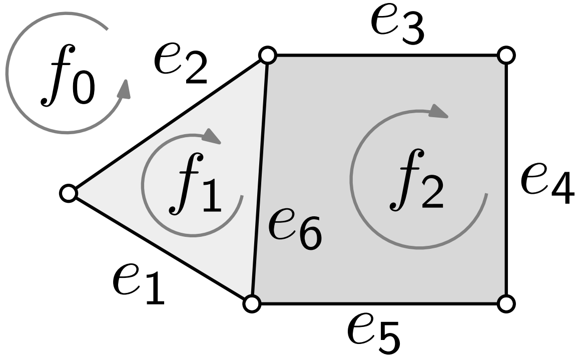

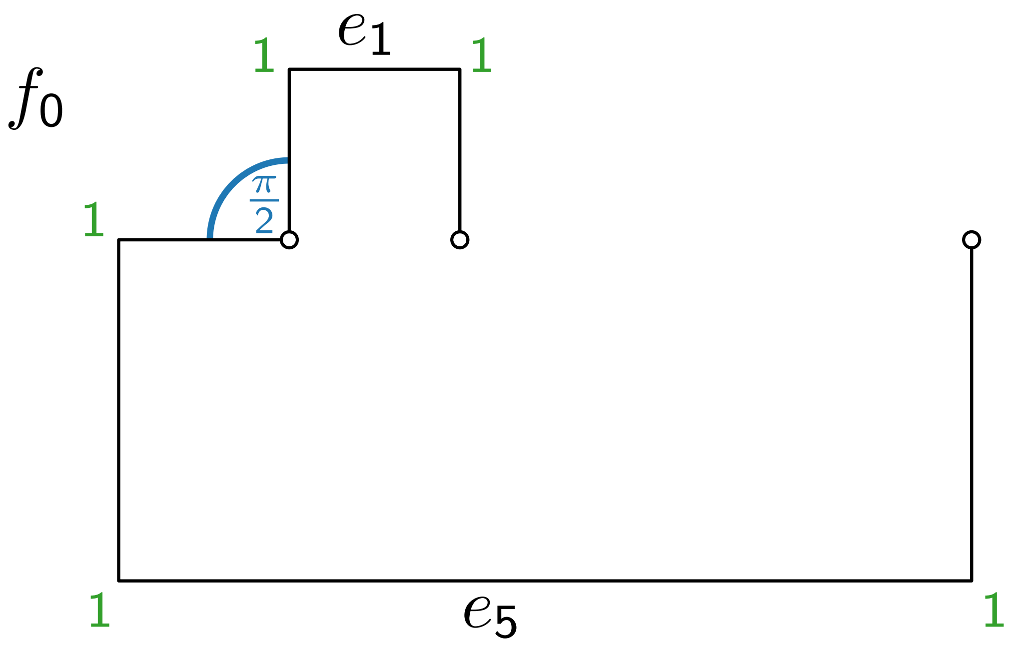

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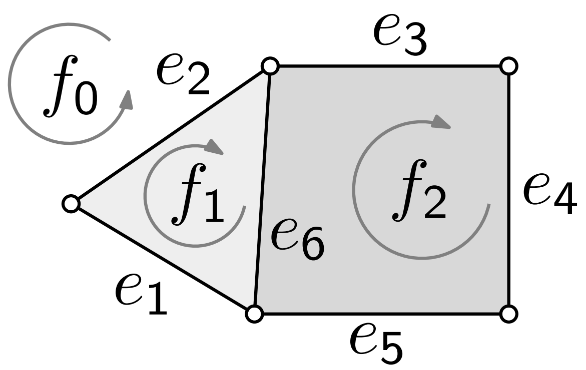
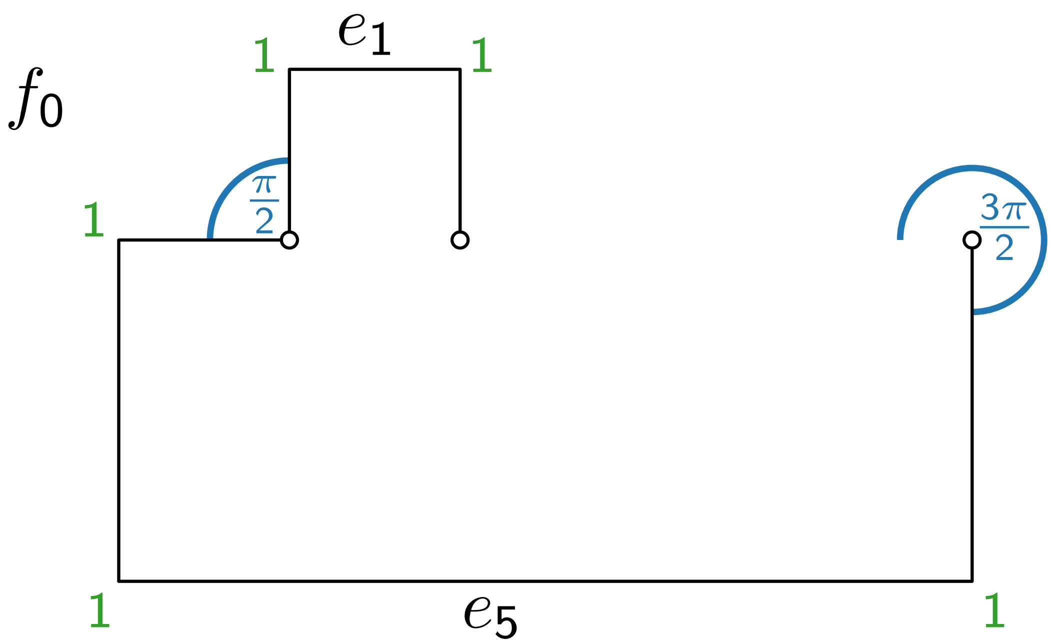


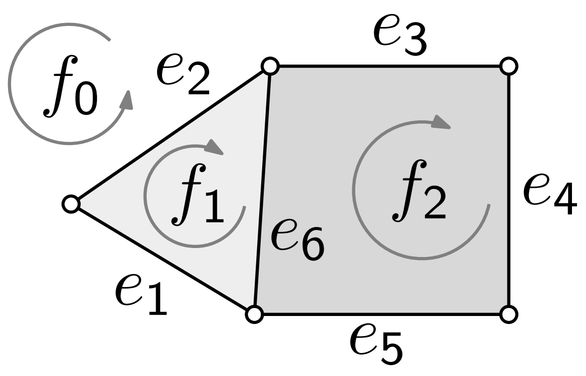


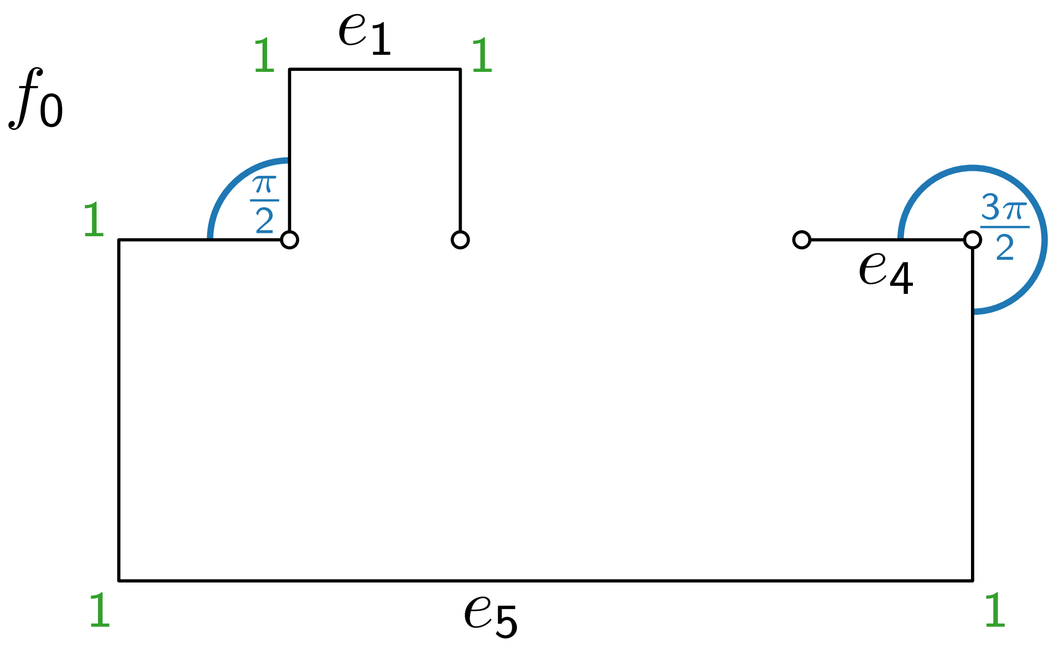
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$



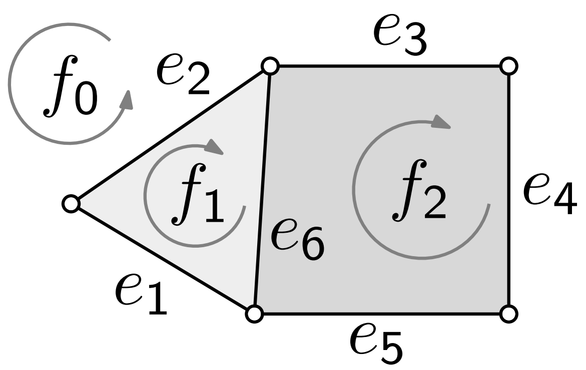


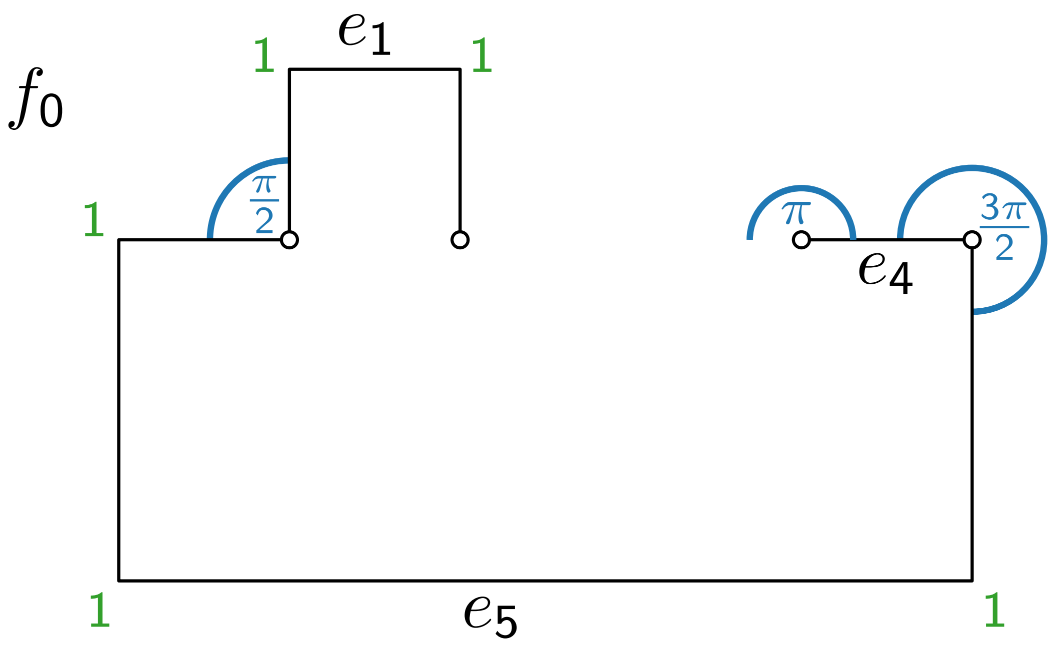
Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

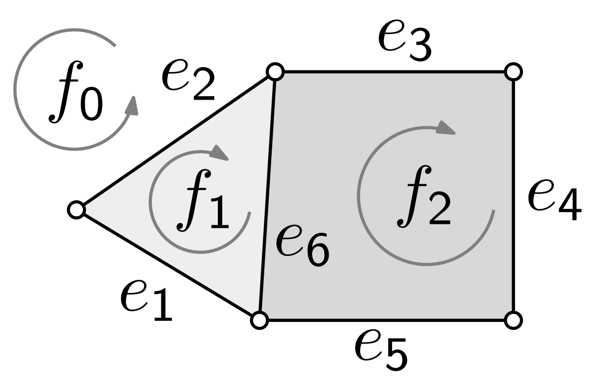
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

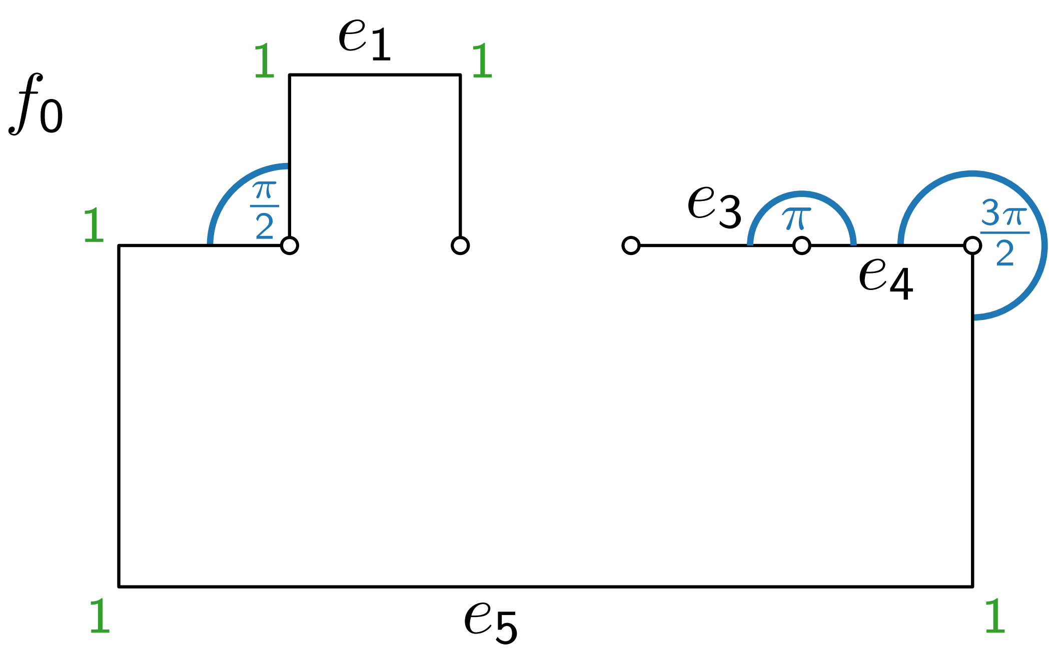




Orthogonal Representation – Example

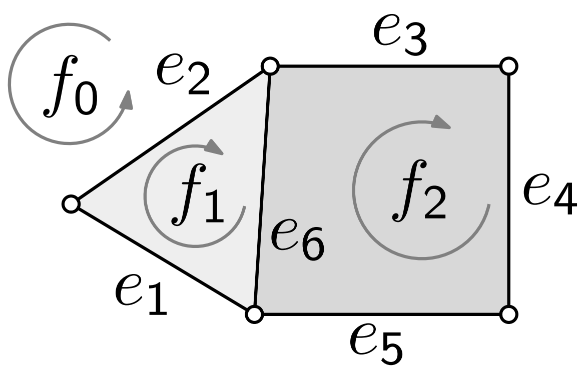
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

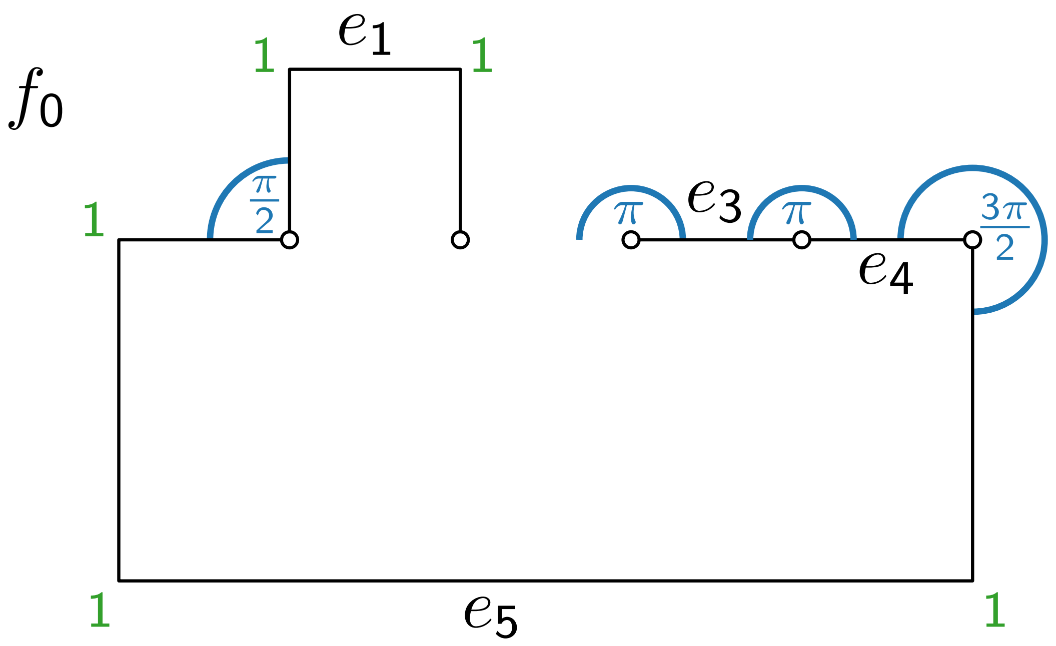




Orthogonal Representation – Example

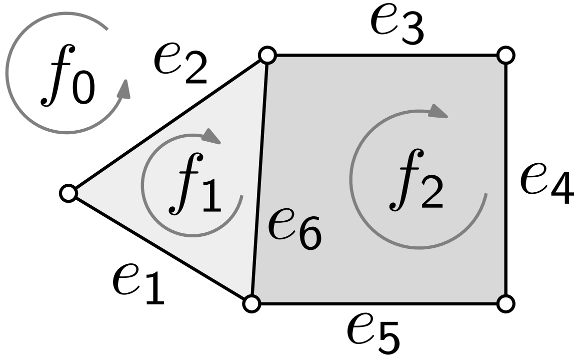
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

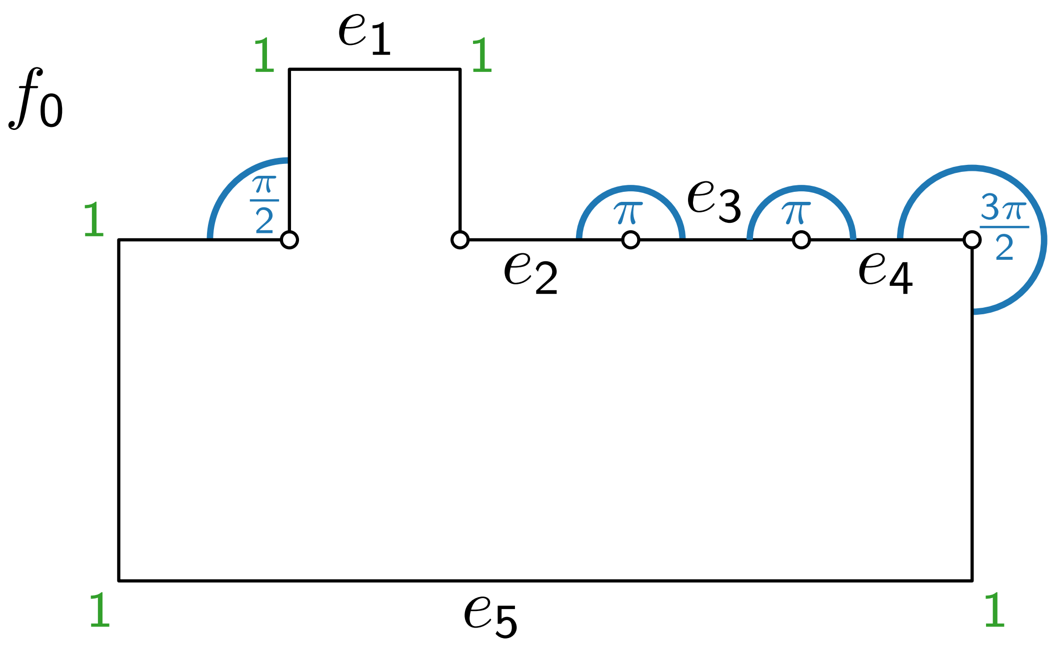




Orthogonal Representation – Example

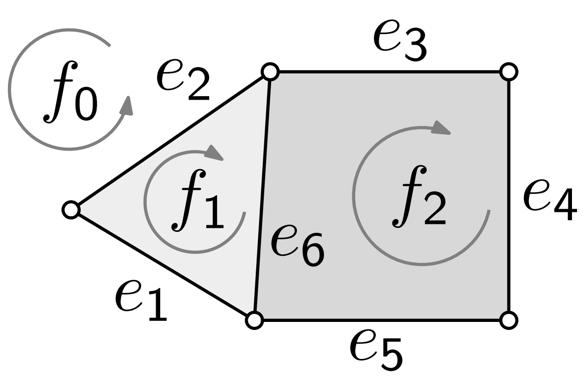
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

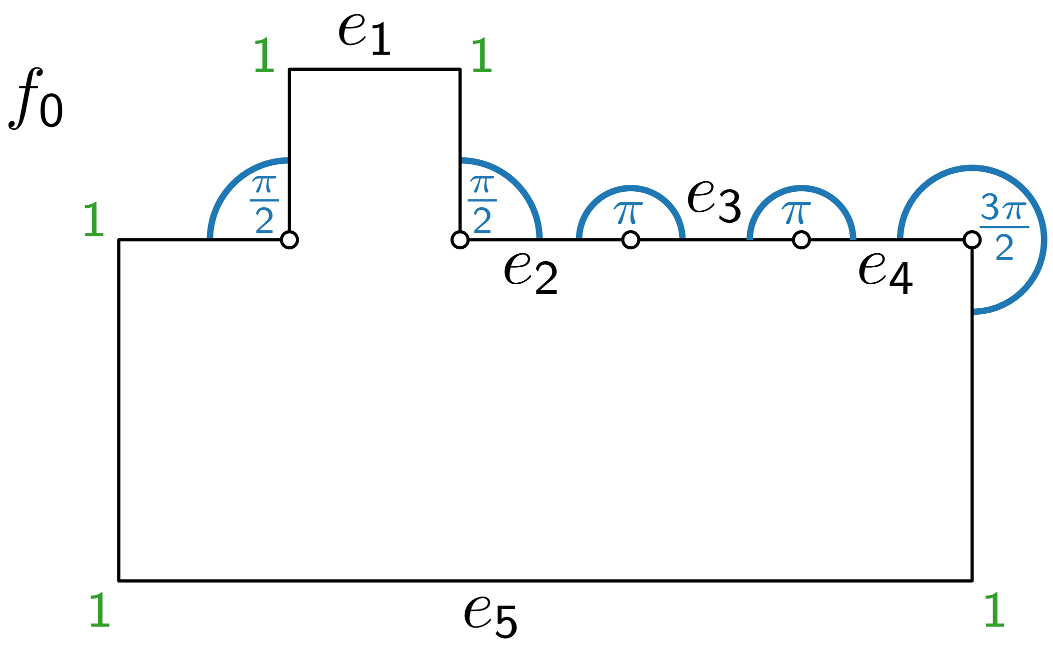




Orthogonal Representation – Example

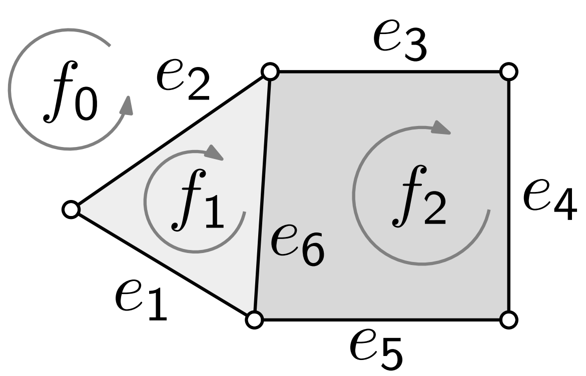
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

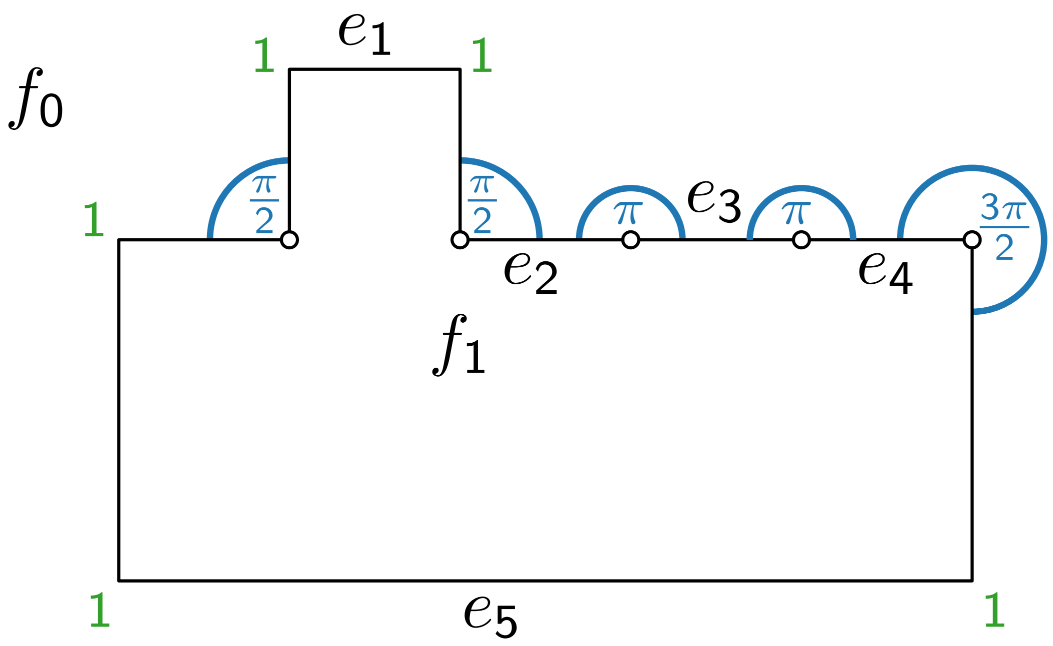




Orthogonal Representation – Example

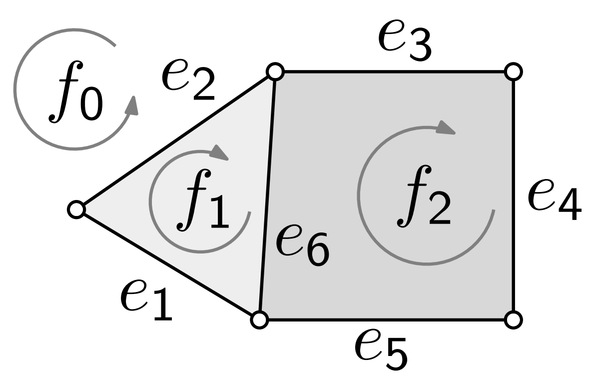
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

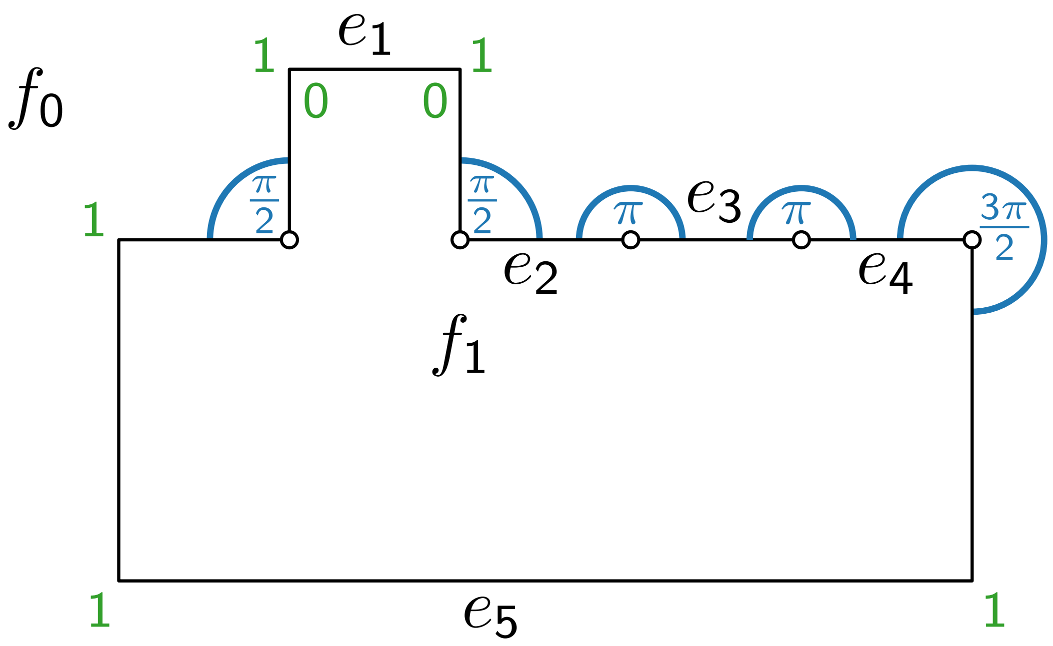




Orthogonal Representation – Example

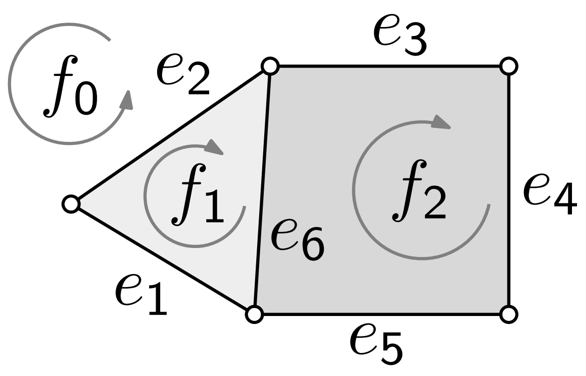
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

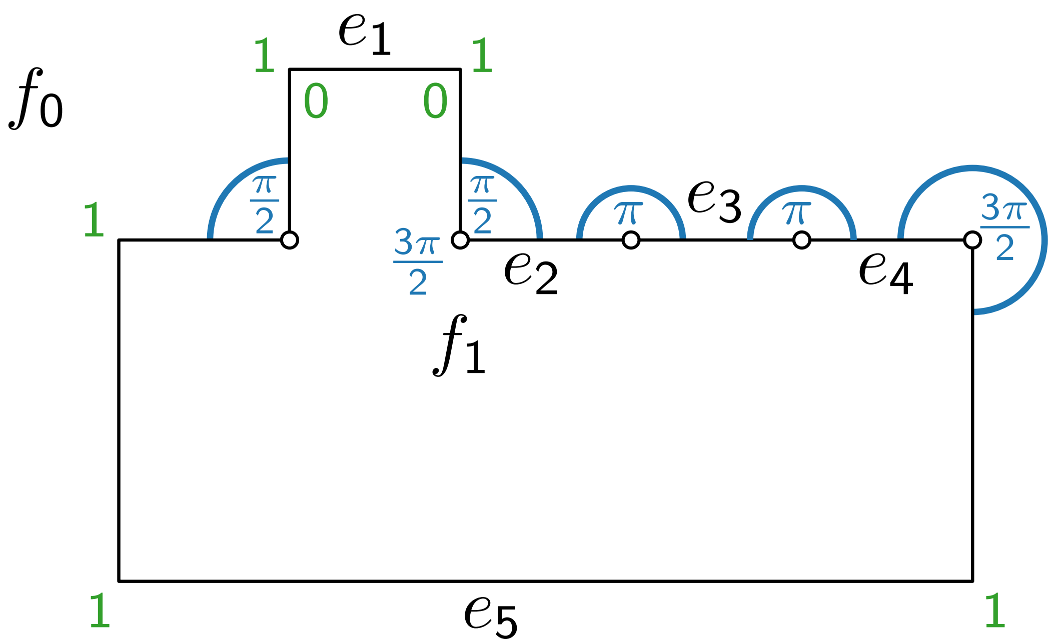




Orthogonal Representation – Example

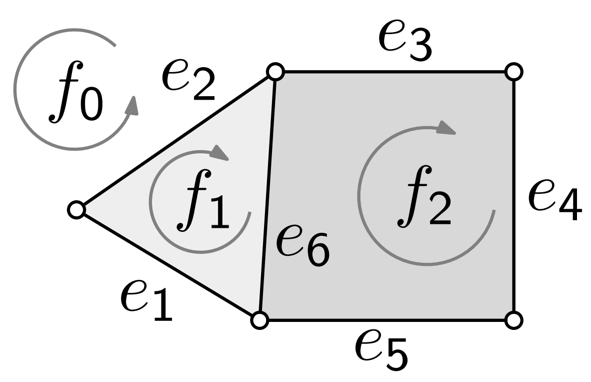
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

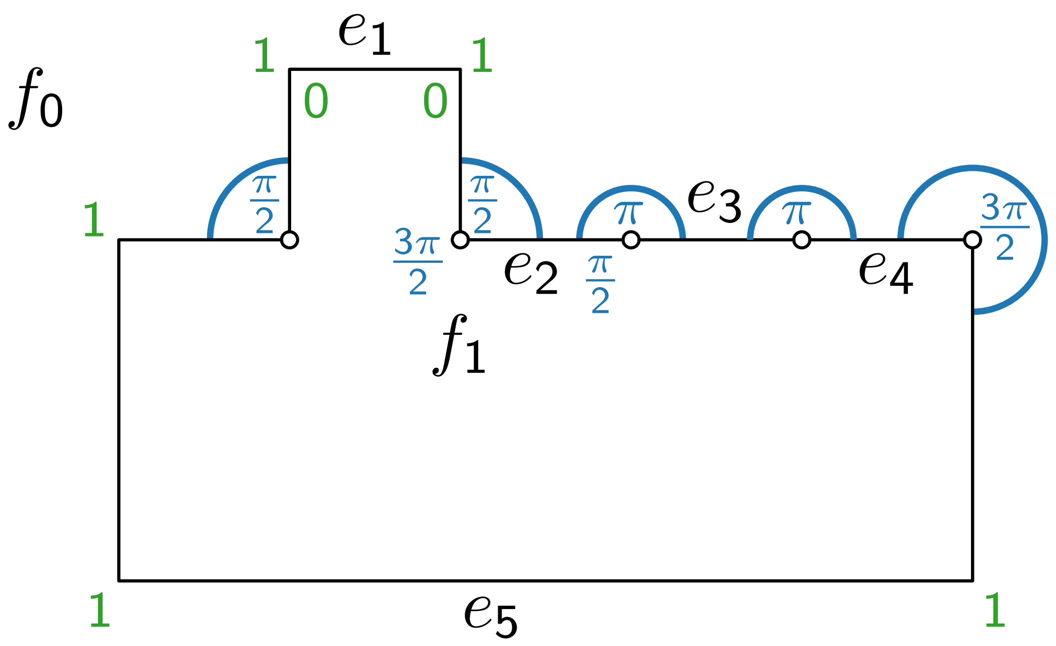




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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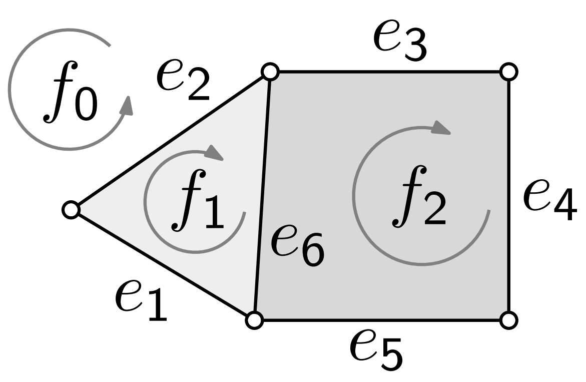
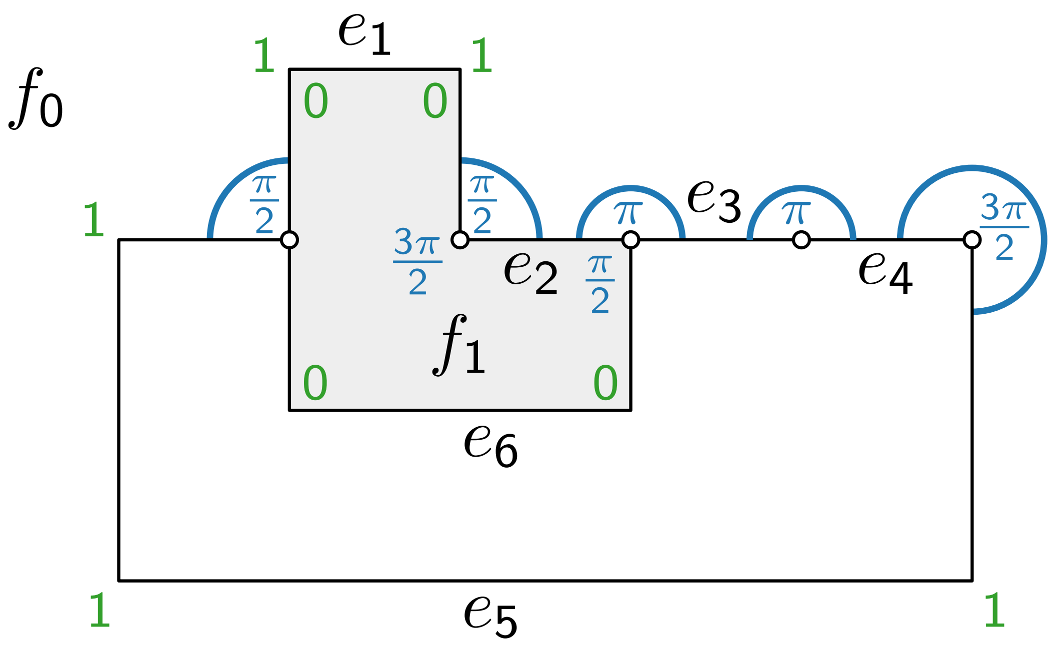
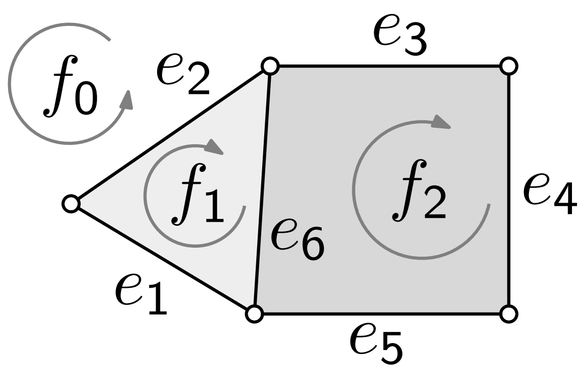
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

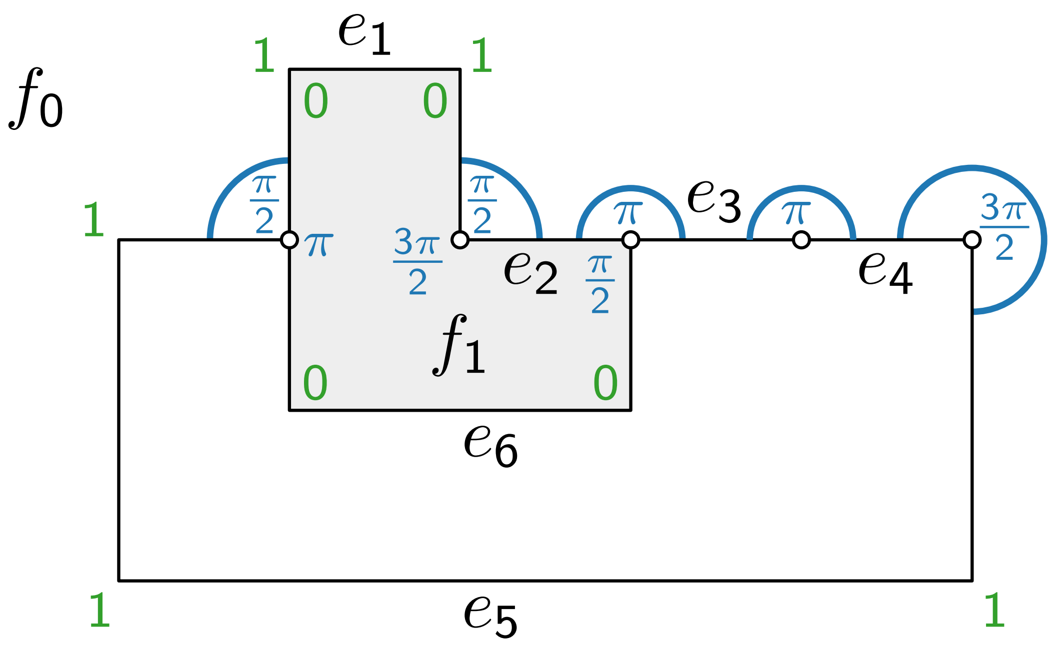




Orthogonal Representation – Example

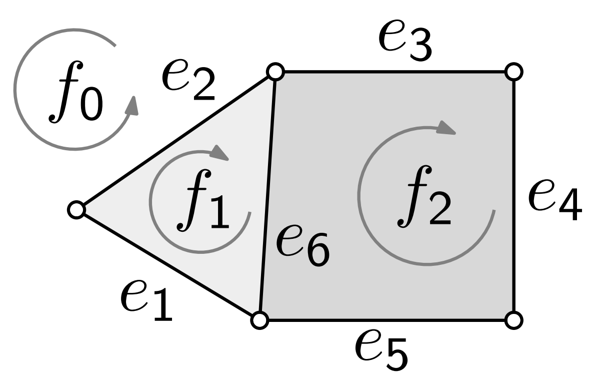
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

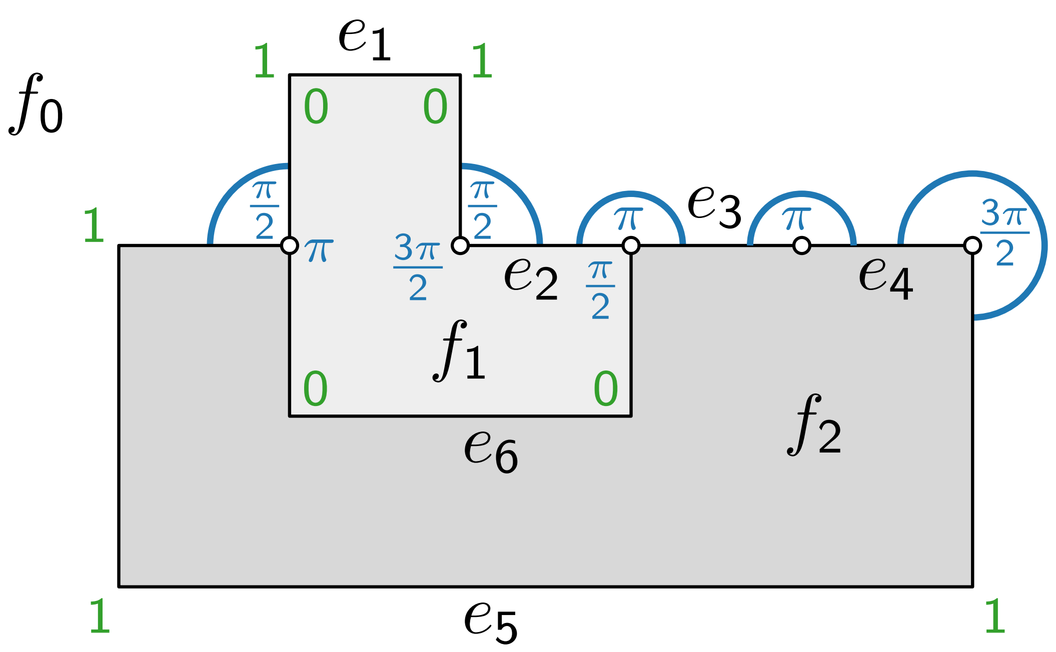




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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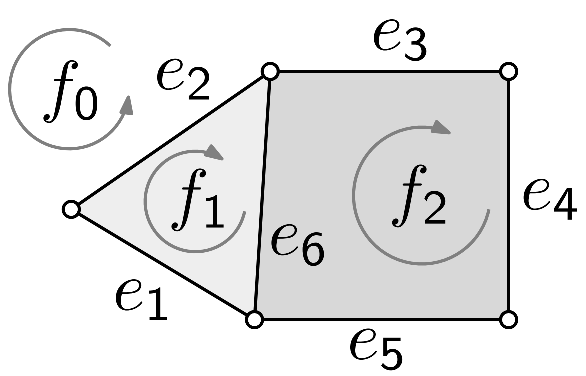

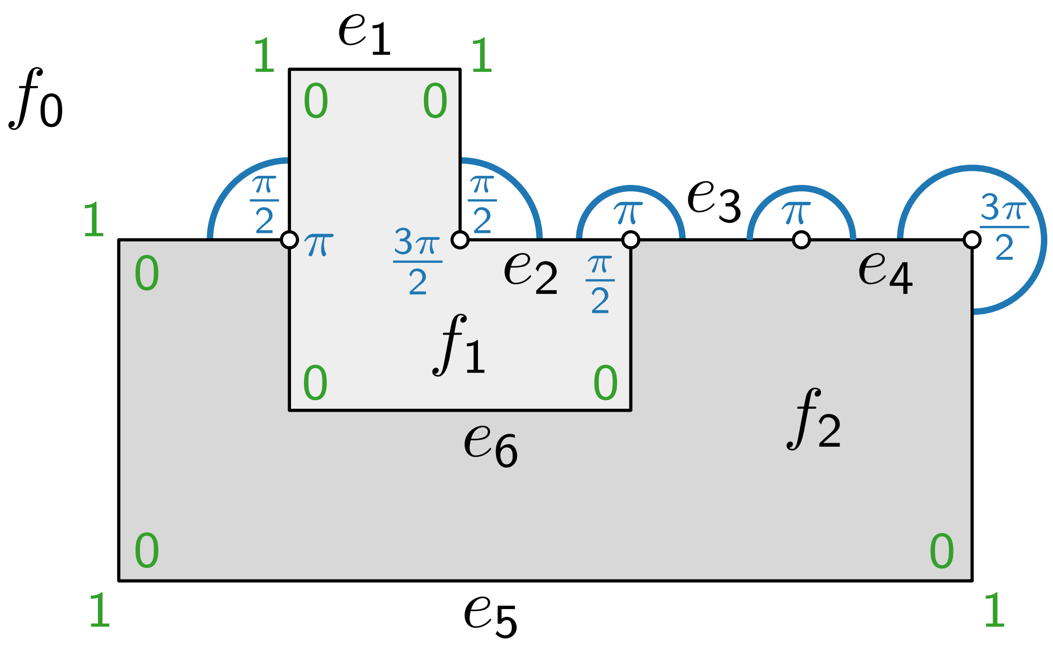

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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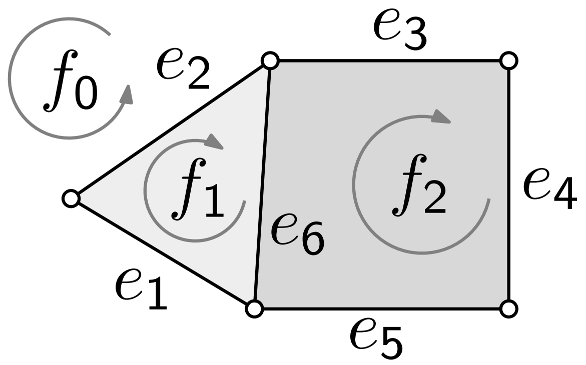

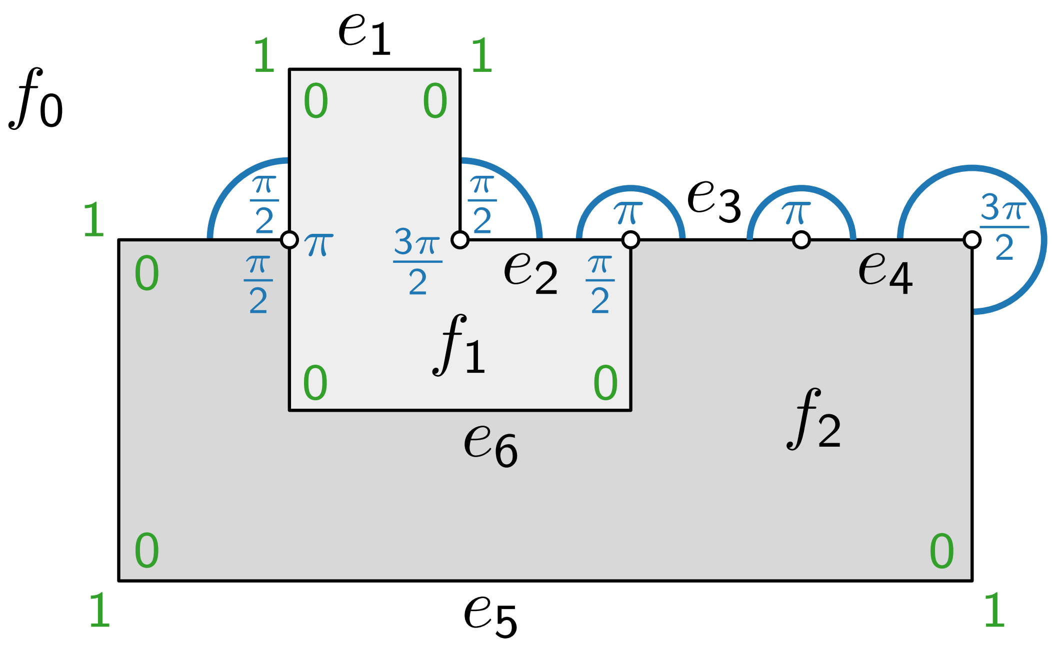

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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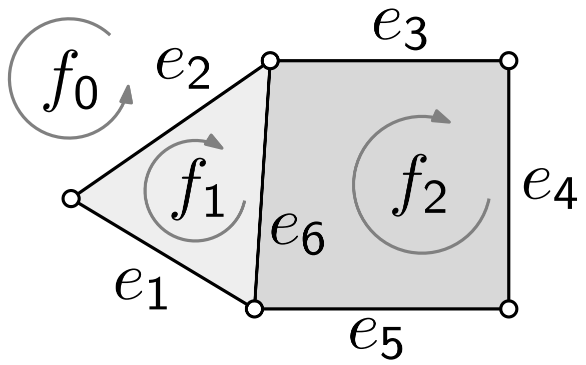

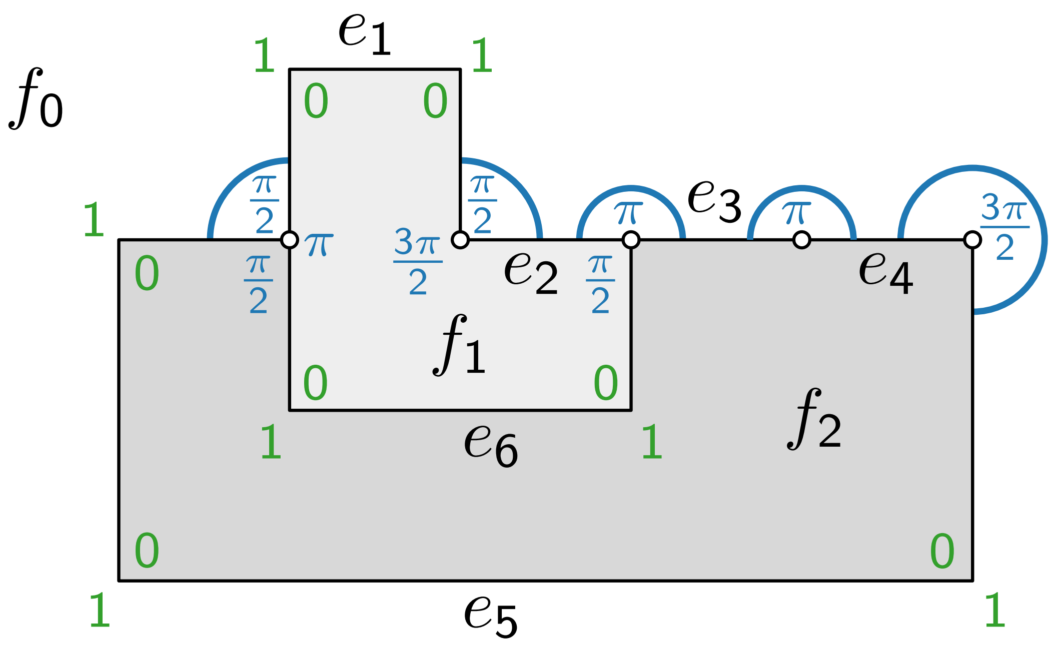

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$

$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

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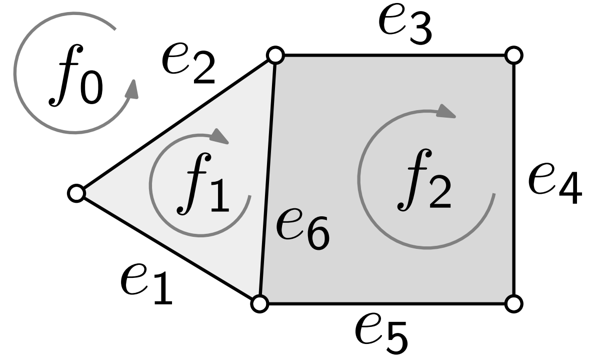

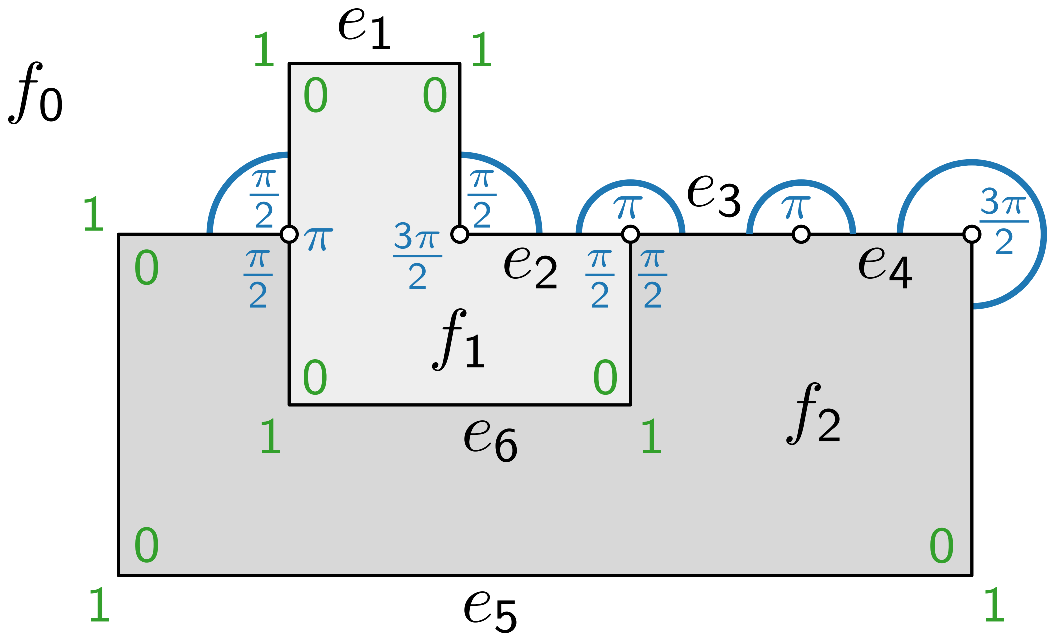

Orthogonal Representation – Example

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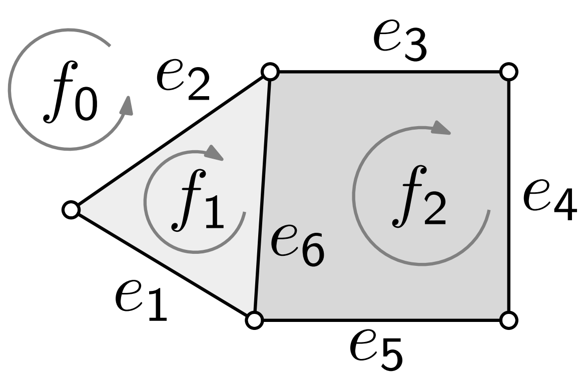
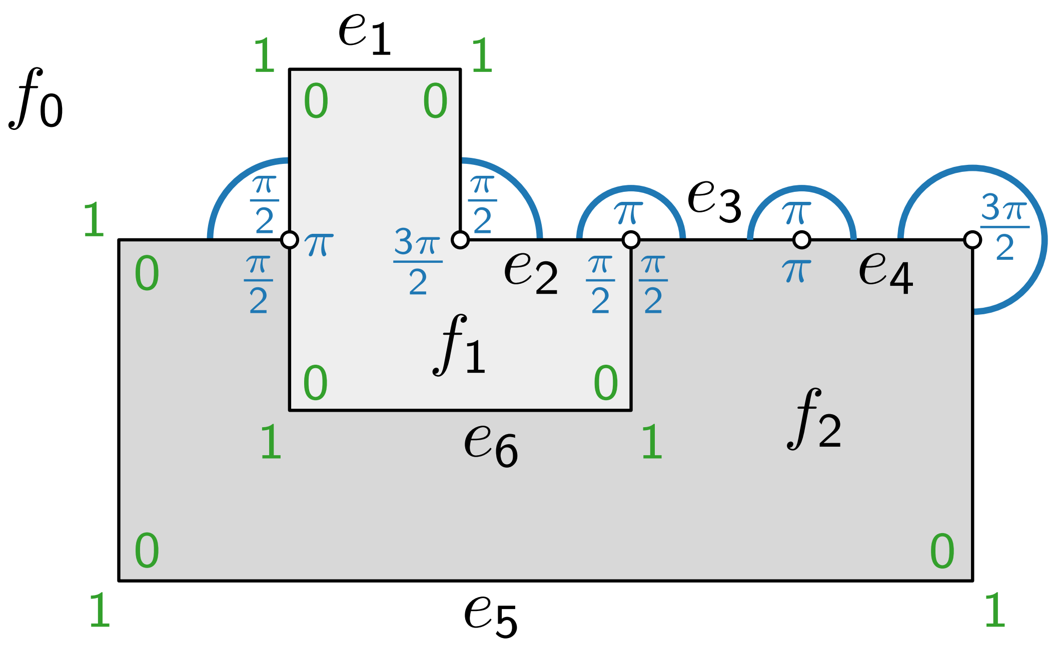
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$

$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

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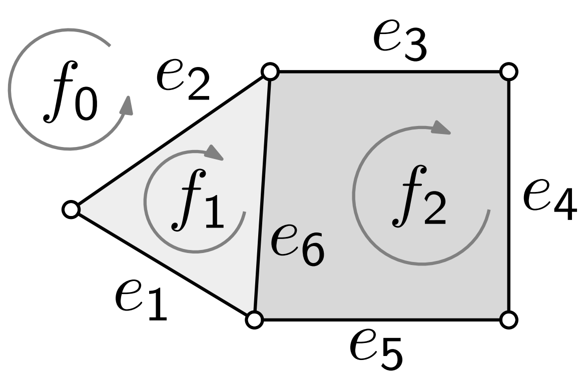

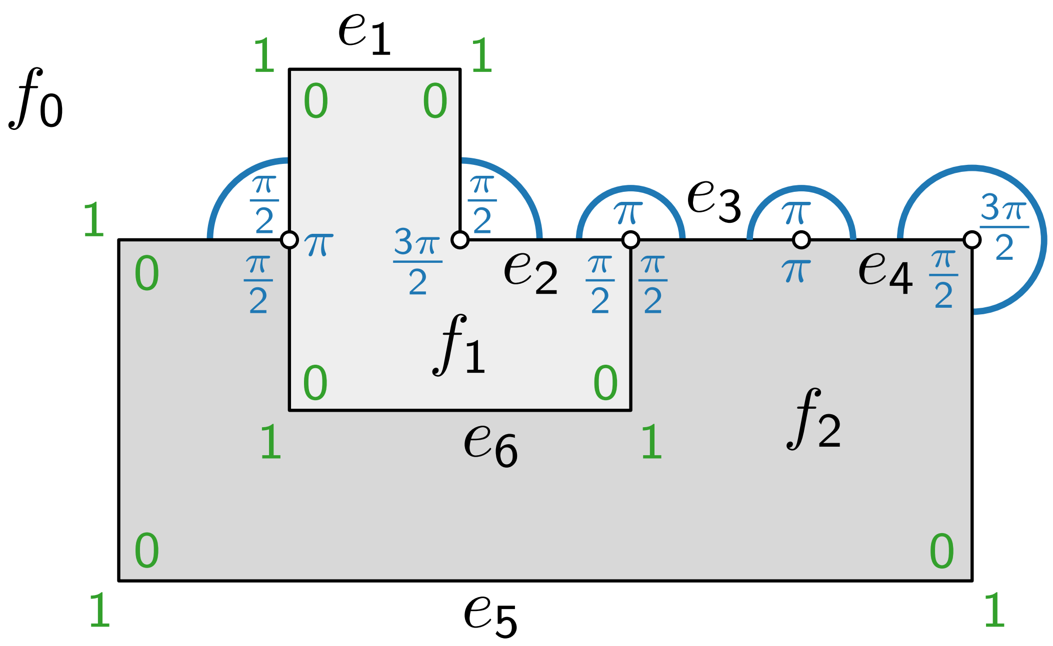

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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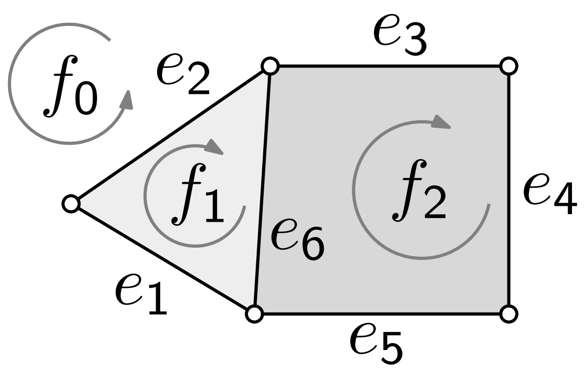

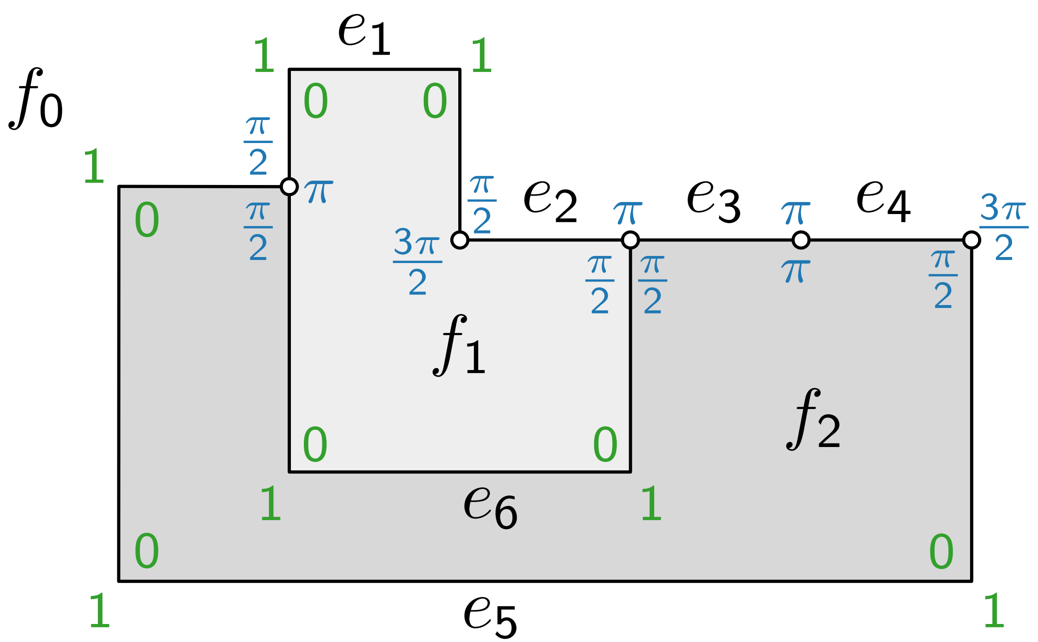

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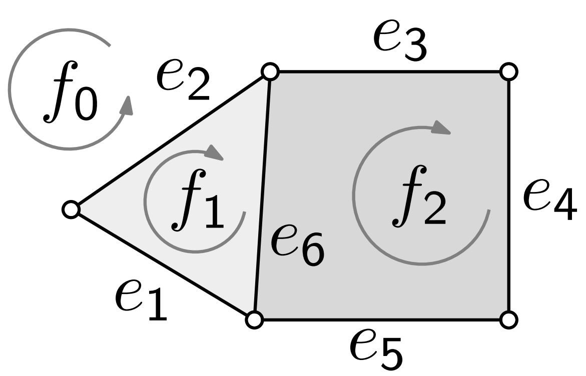
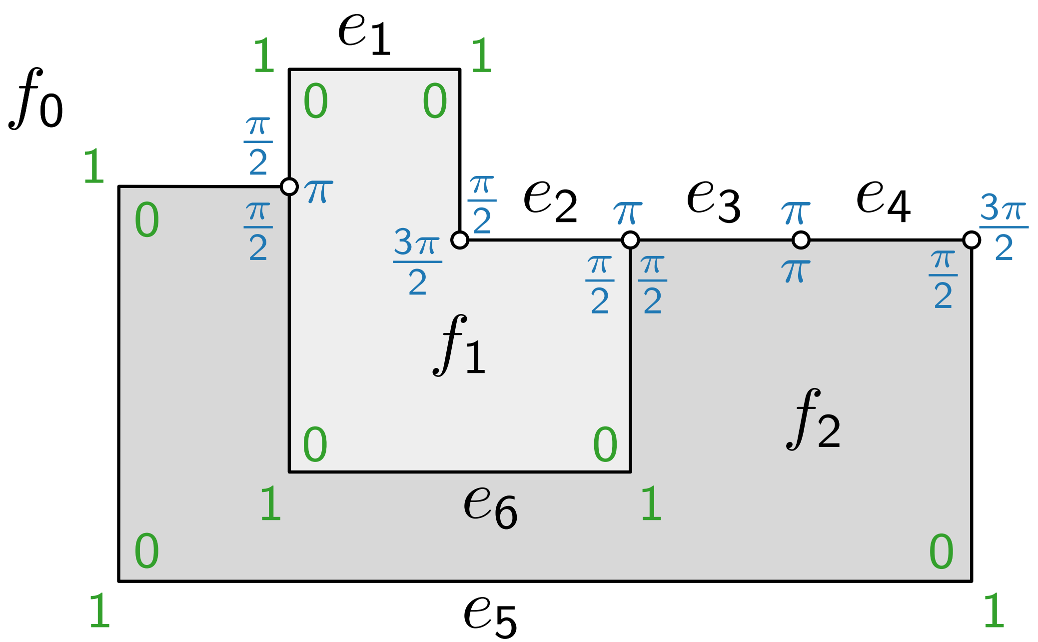
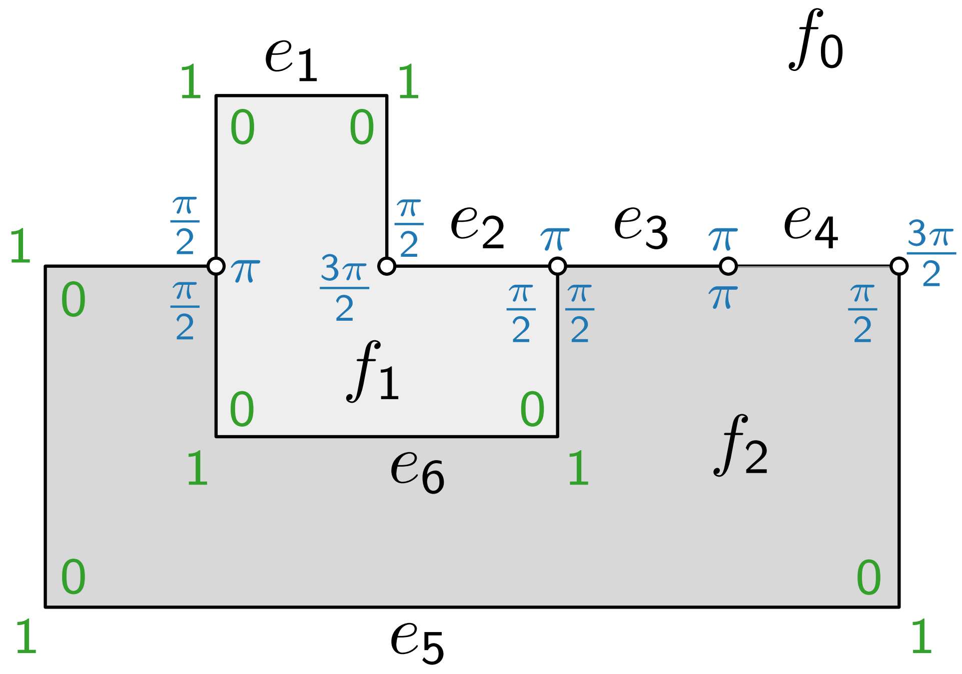




Coordinates are not fixed yet!

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.



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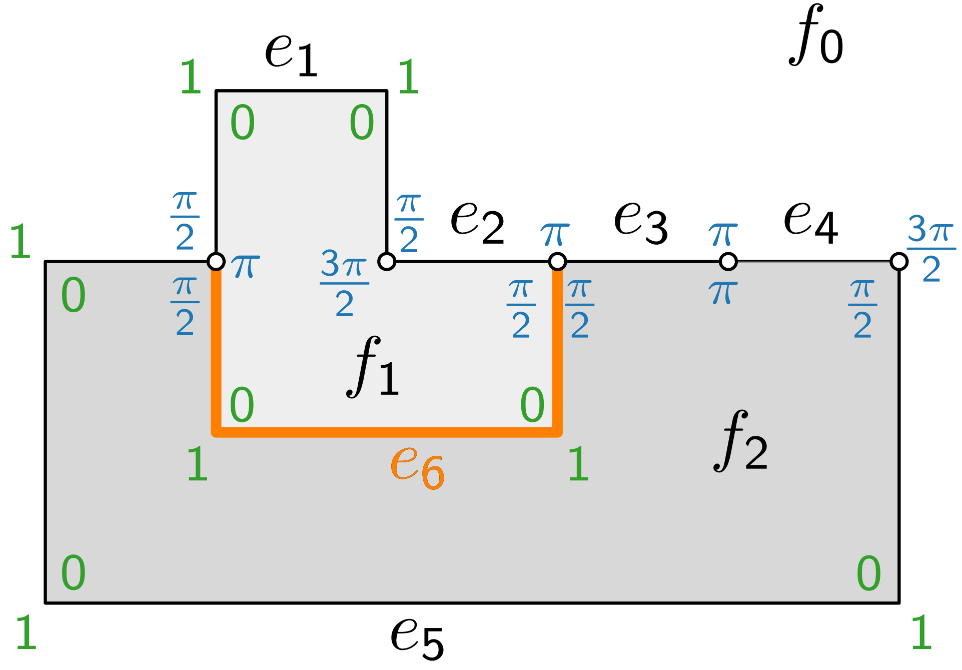
(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$



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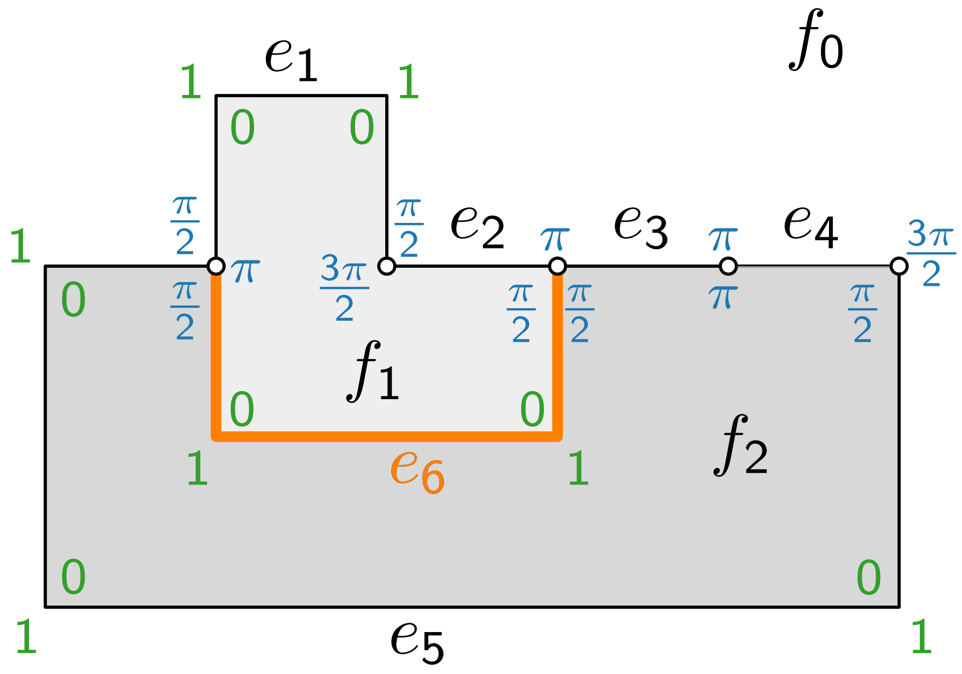

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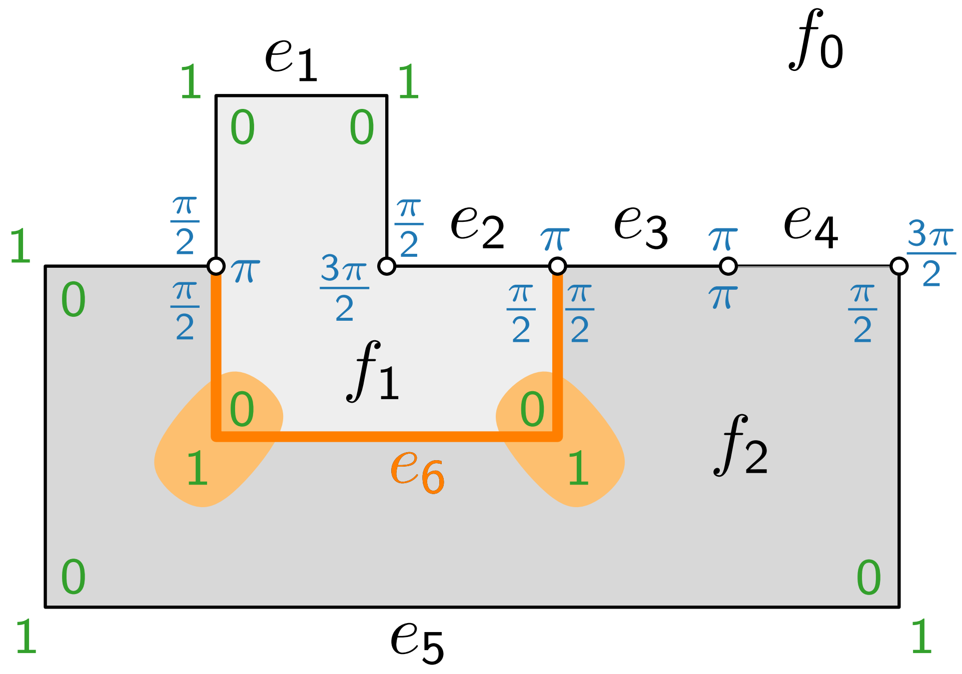

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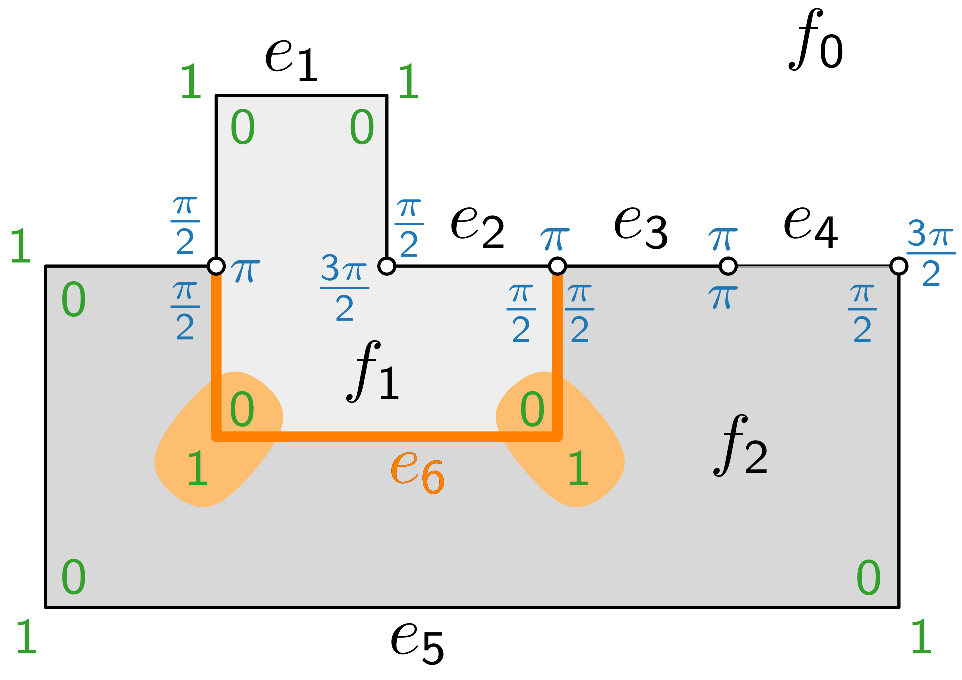

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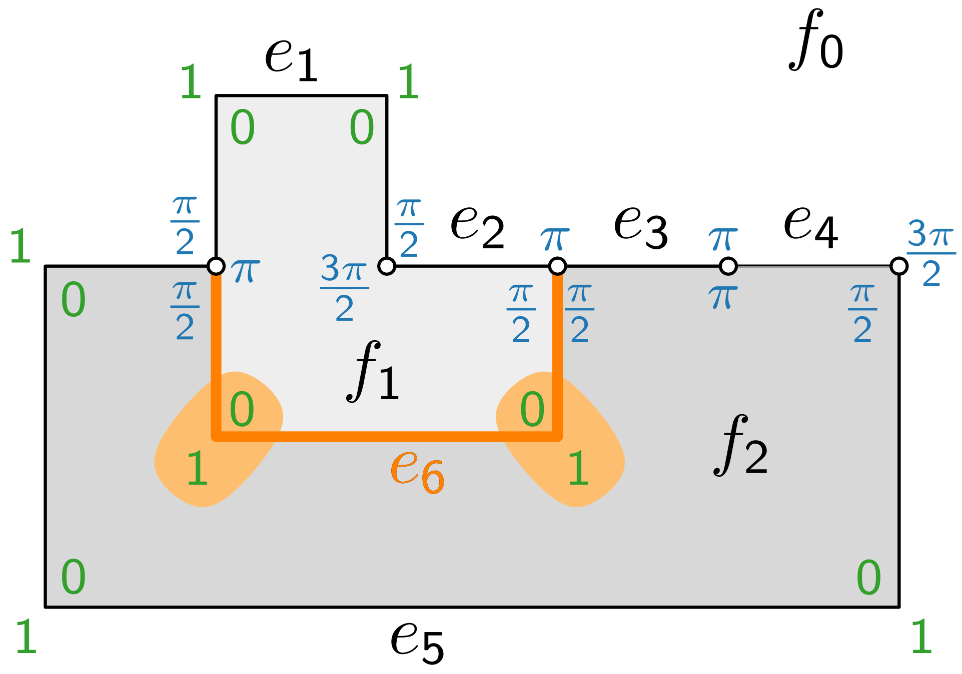

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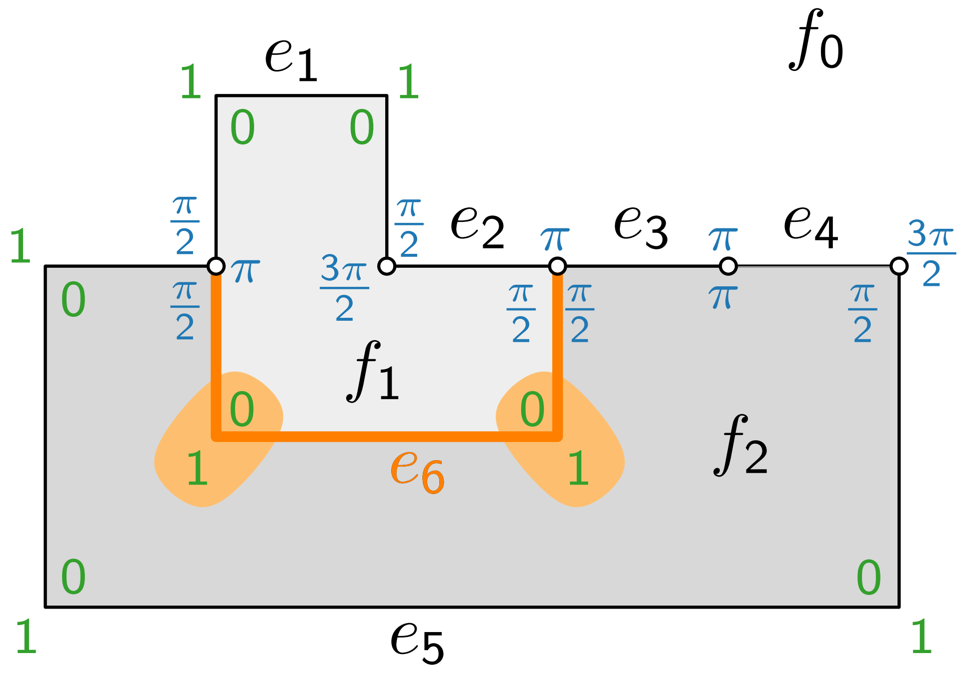

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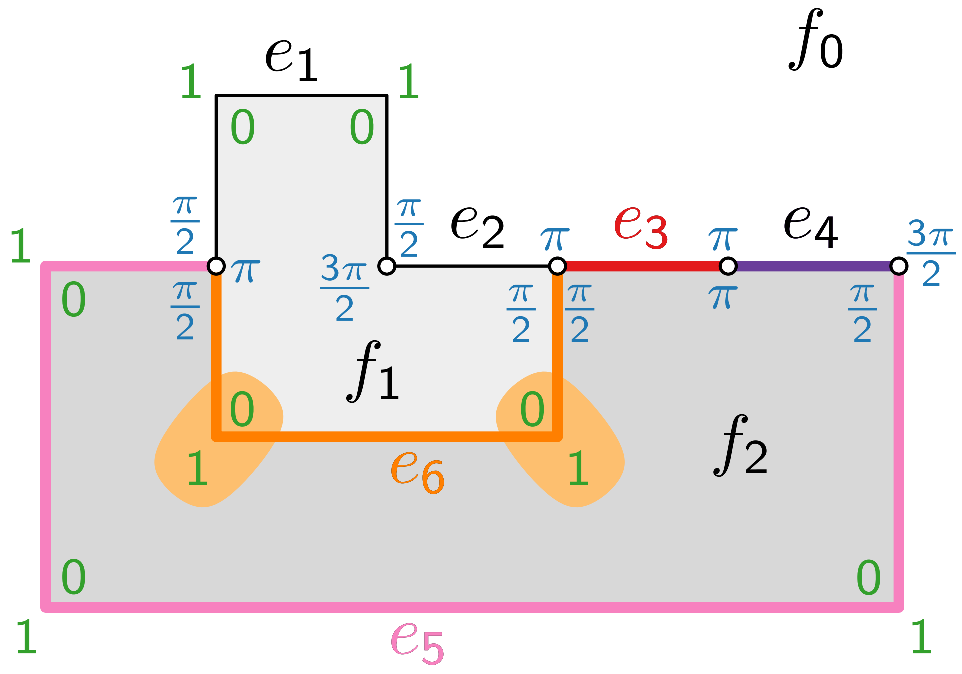

$C((e_3, \emptyset, \pi)) = \quad - \quad - \quad + 2 =$

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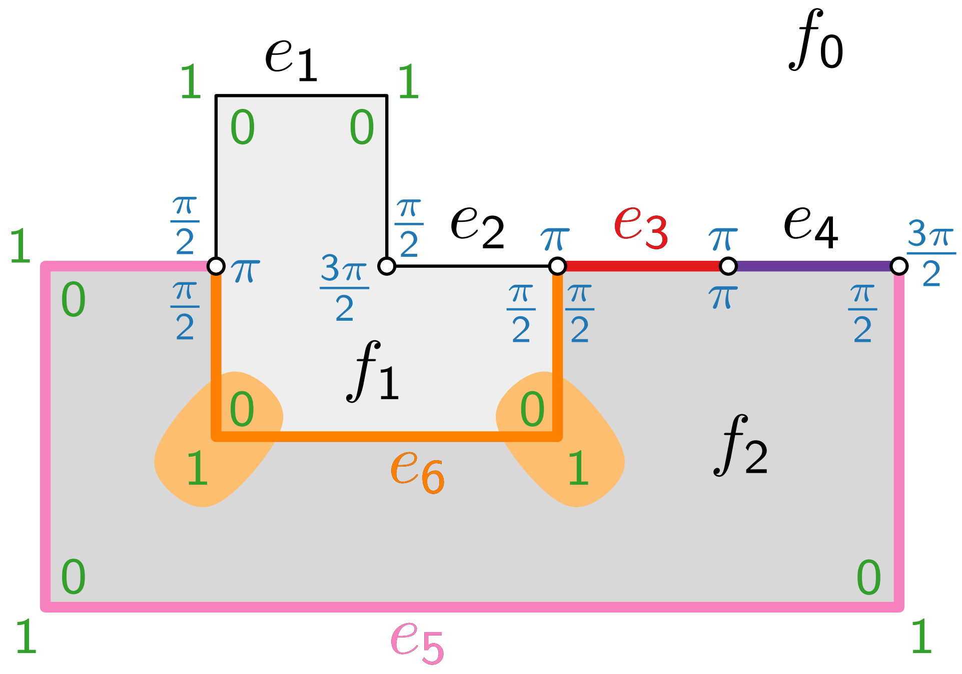

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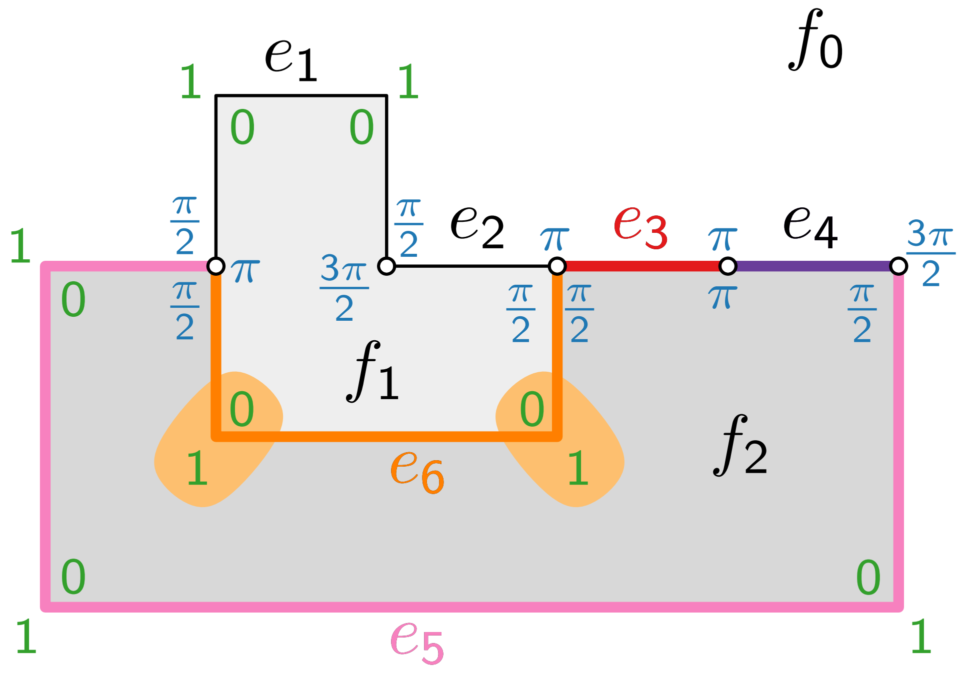

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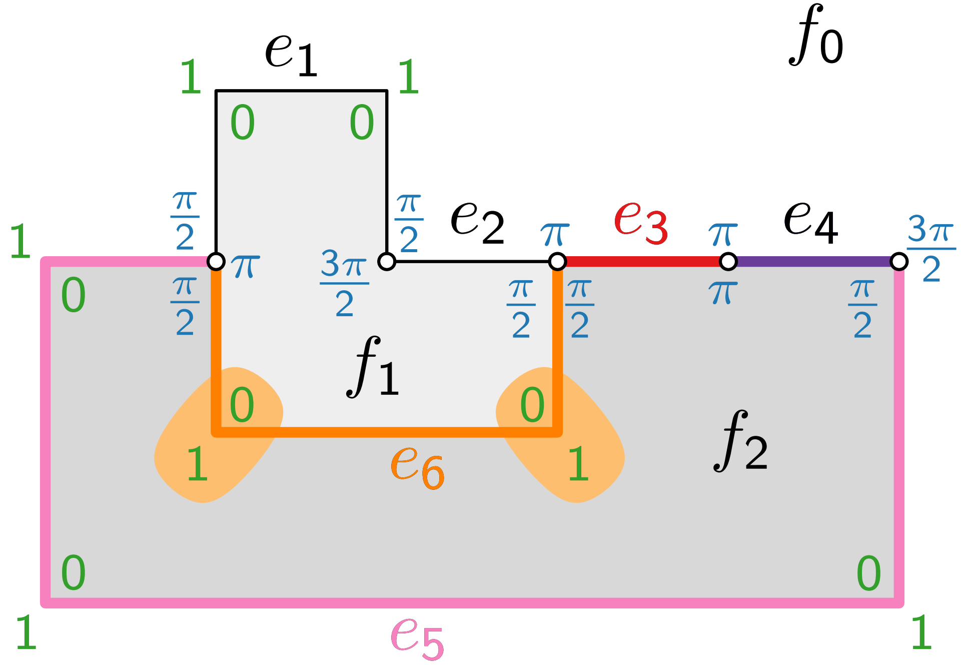

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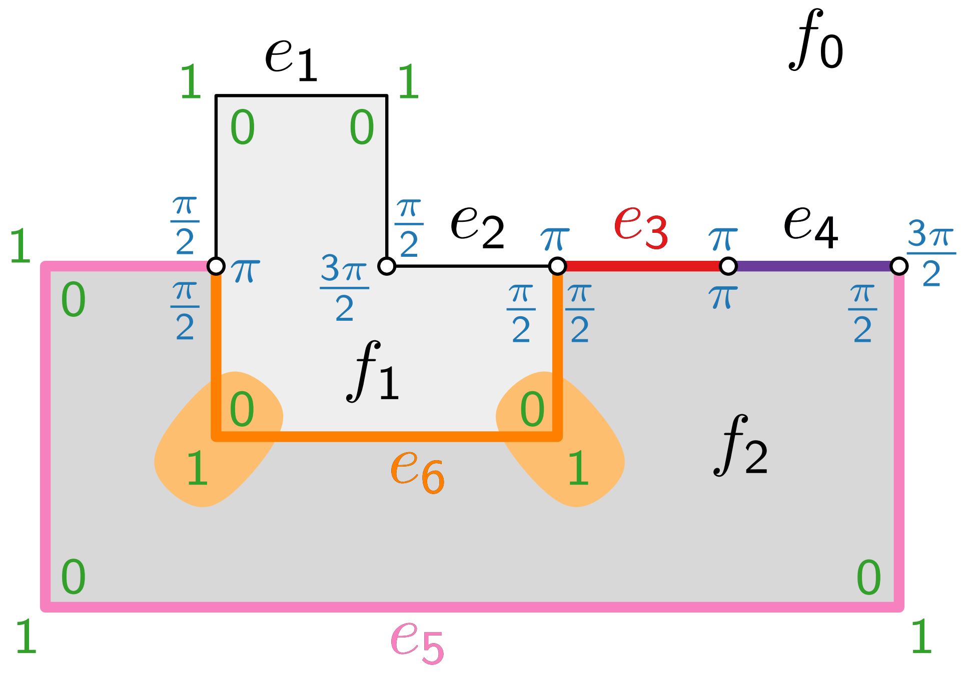

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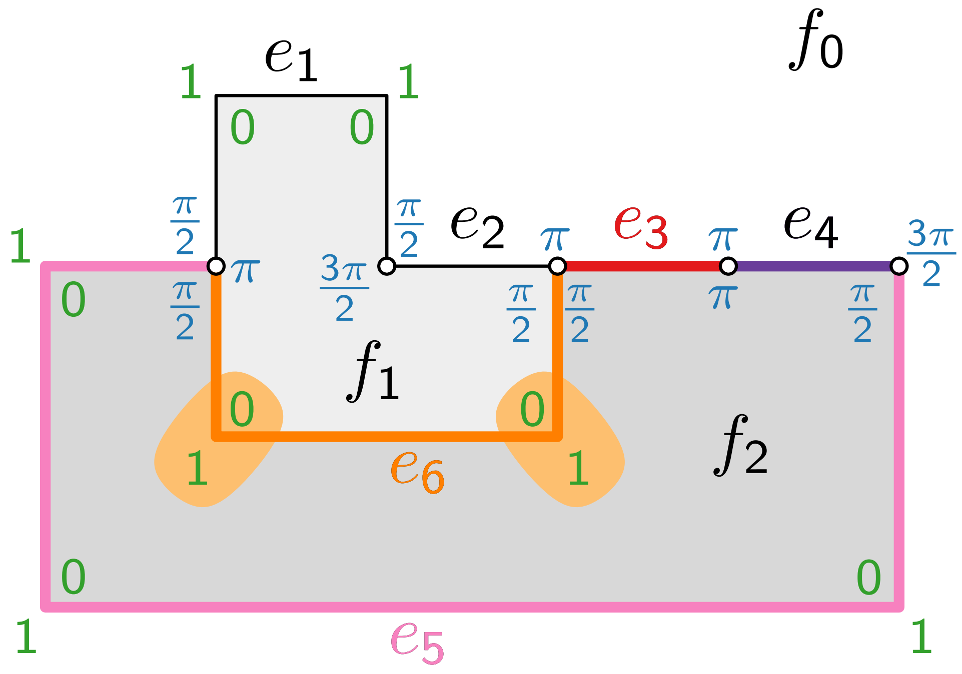

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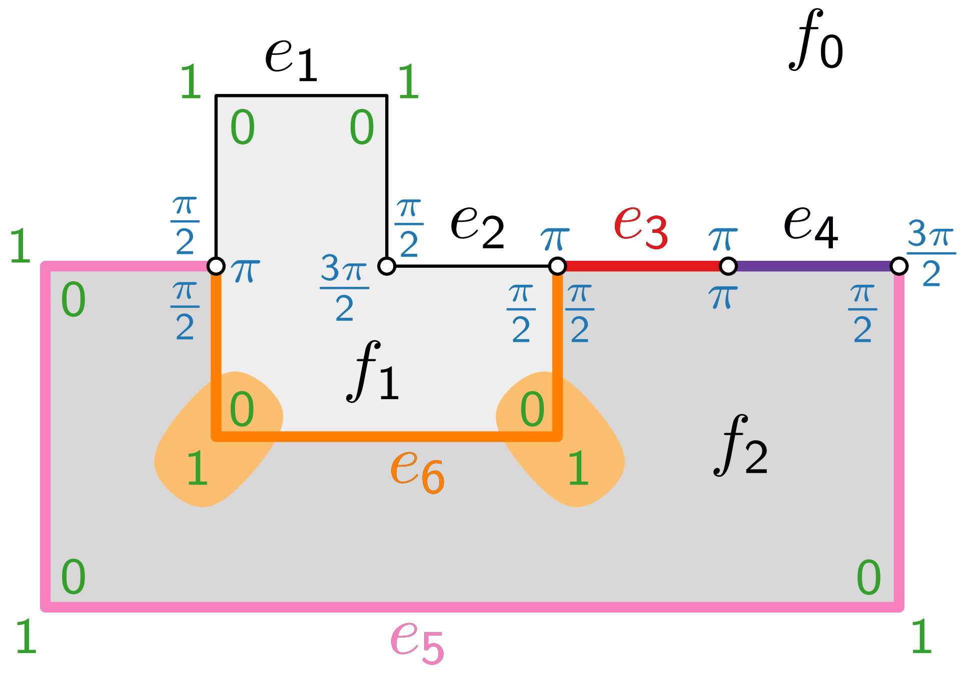

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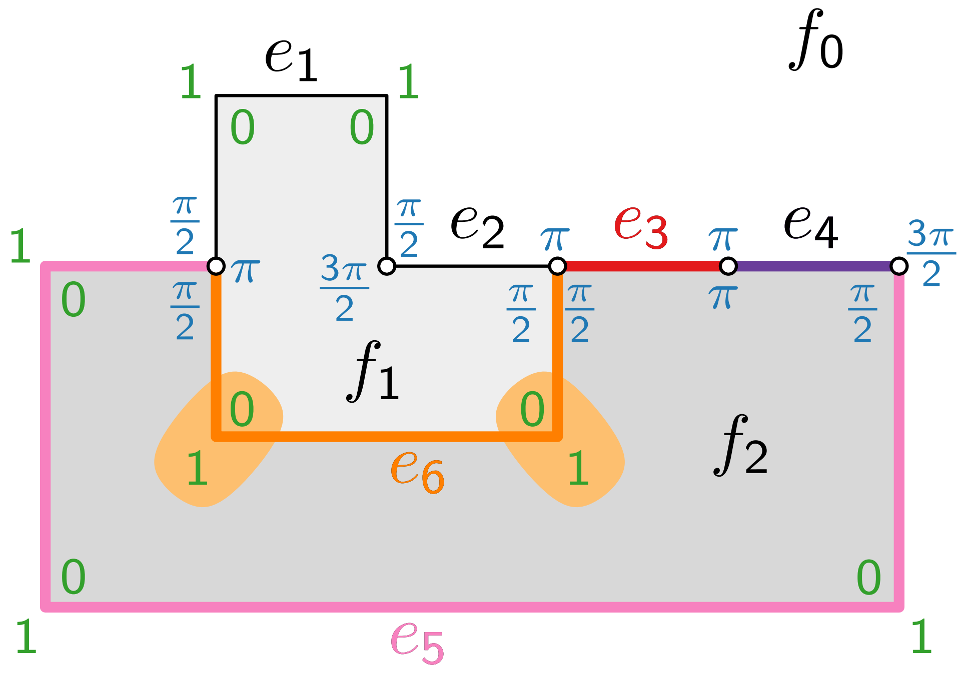

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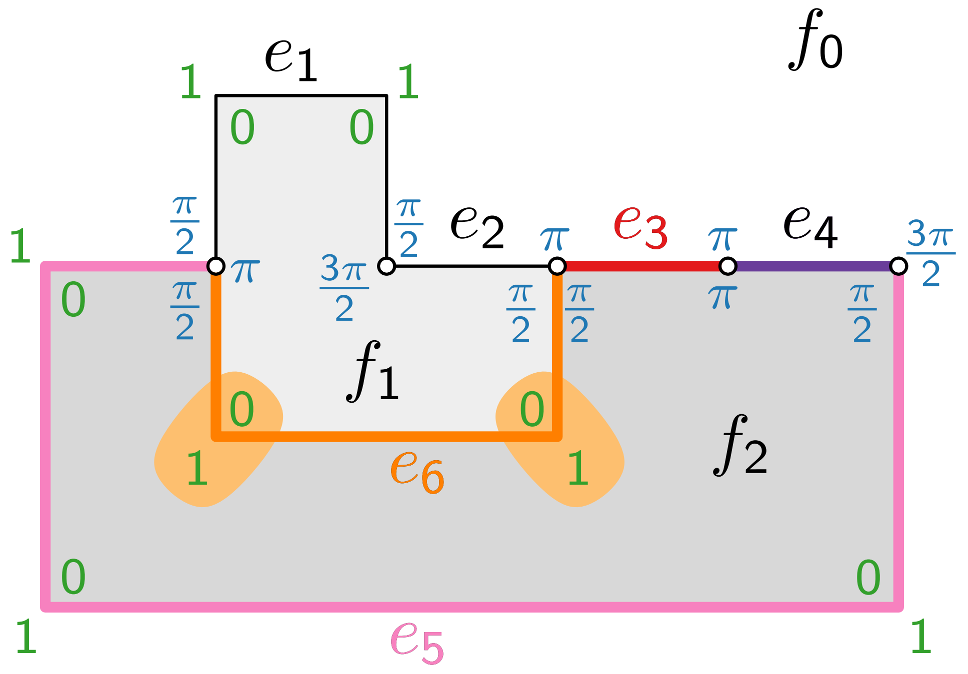

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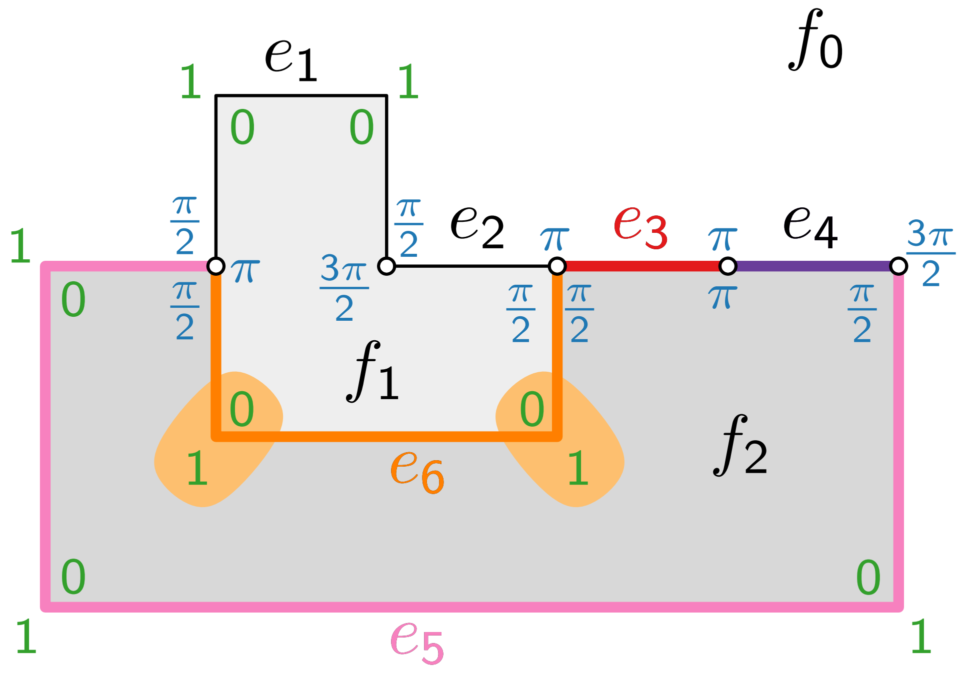

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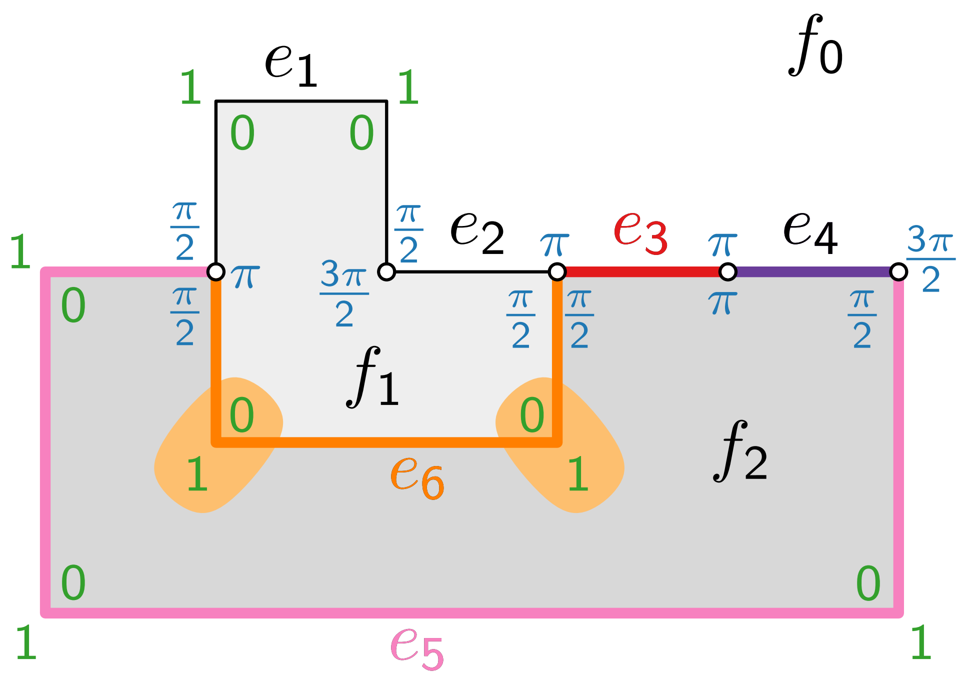

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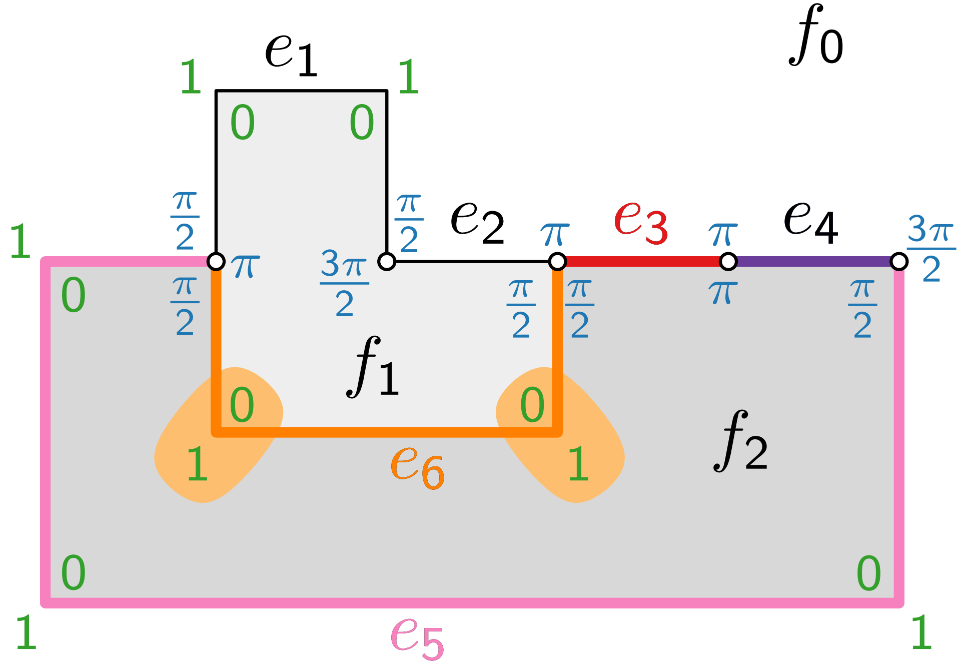

$$C((e_3, \emptyset, \pi)) = 0 - 0 - 2 + 2 = 0$$

$$C((e_4, \emptyset, \tfrac{\pi}{2})) = 0 - 0 - 1 + 2 = 1$$

$$C((e_5, 000, \tfrac{\pi}{2})) = 3 - 0 - 1 + 2 = 4$$

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Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u,v), \delta_1, \alpha_1) \in H(f)$ and $((v,u), \delta_2, \alpha_2) \in H(g)$, the sequence δ_1 is like δ_2, but reversed and inverted.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ, and let $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 - \alpha / \frac{\pi}{2} + 2$.
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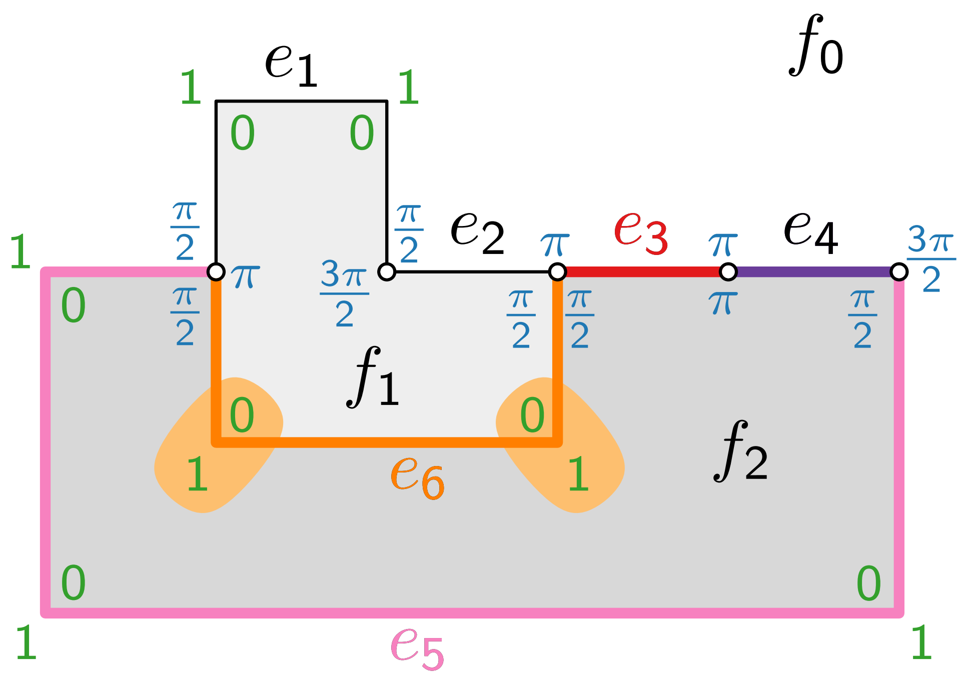

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$$\sum_{r \in H(f_2)} C(r) =$$

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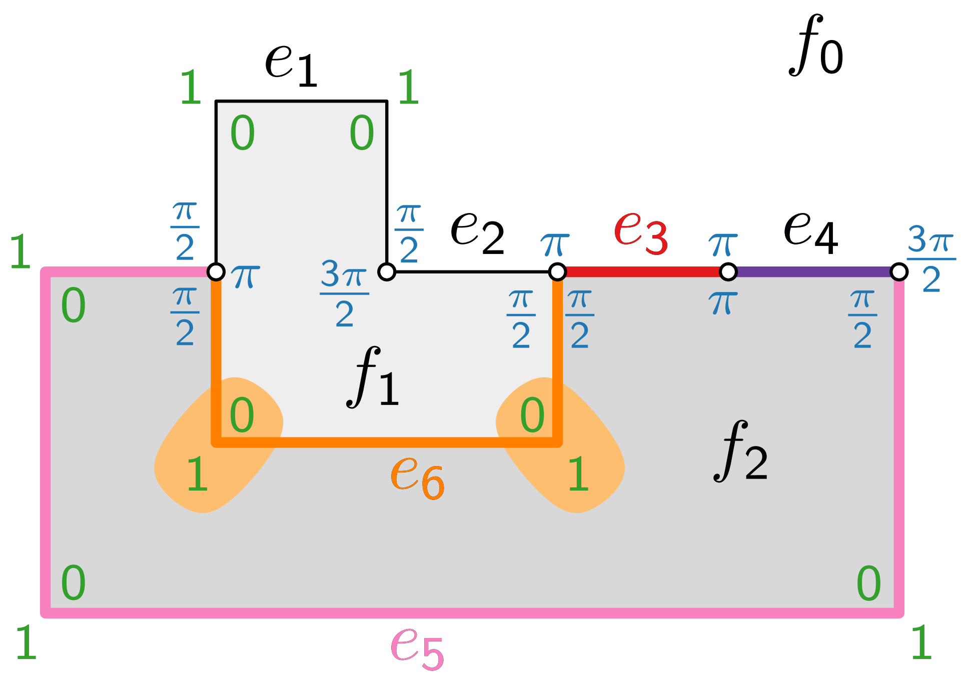

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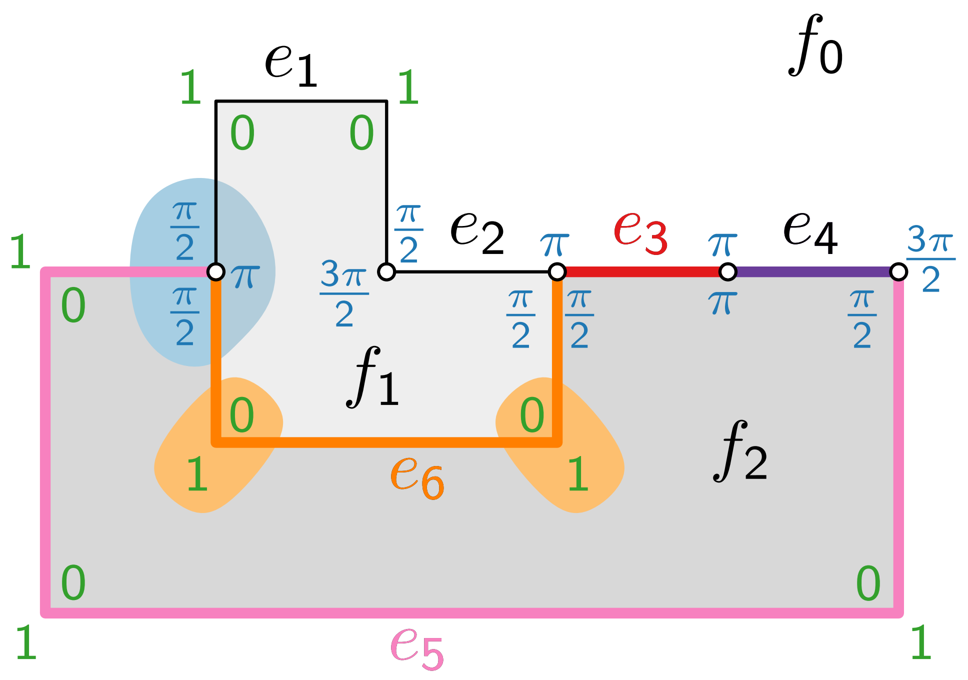
(H4) For each **vertex** v, the sum of incident angles is 2π.



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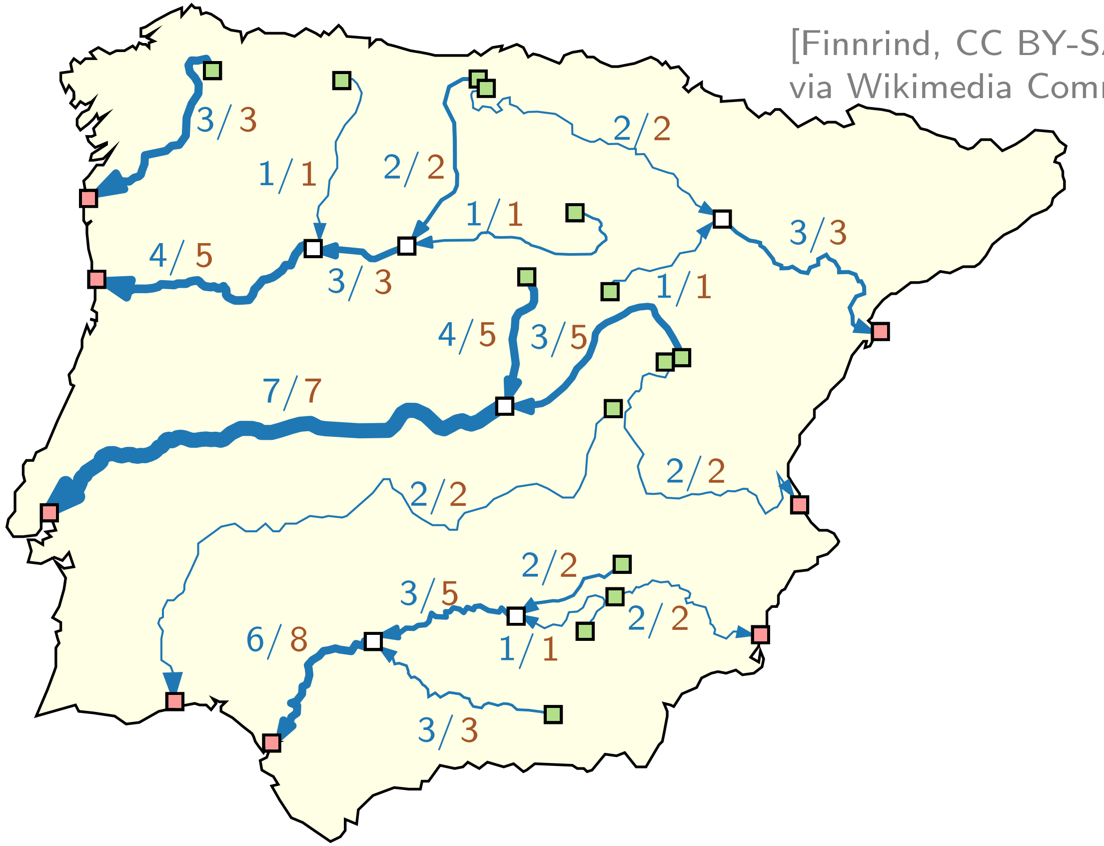
Reminder: s–t Flow Networks



Flow network $(G; S, T; u)$ with

- directed graph G
- *sources* $S \subseteq V(G)$, *sinks* $T \subseteq V(G)$
- edge *capacity* $u \colon E(G) \to \mathbb{R}_0^+ \cup \{\infty\}$

A function $X \colon E(G) \to \mathbb{R}_0^+$ is called S–T **flow** if:

$$0 \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E(G)$$

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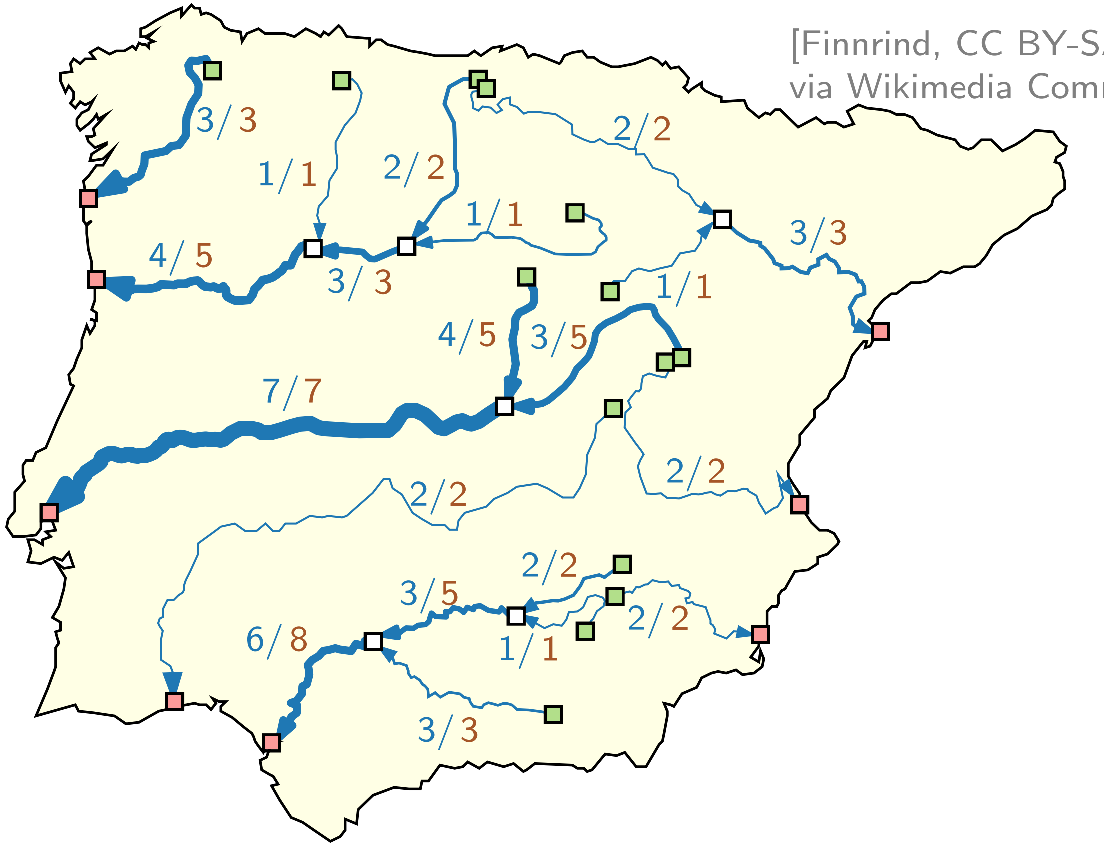

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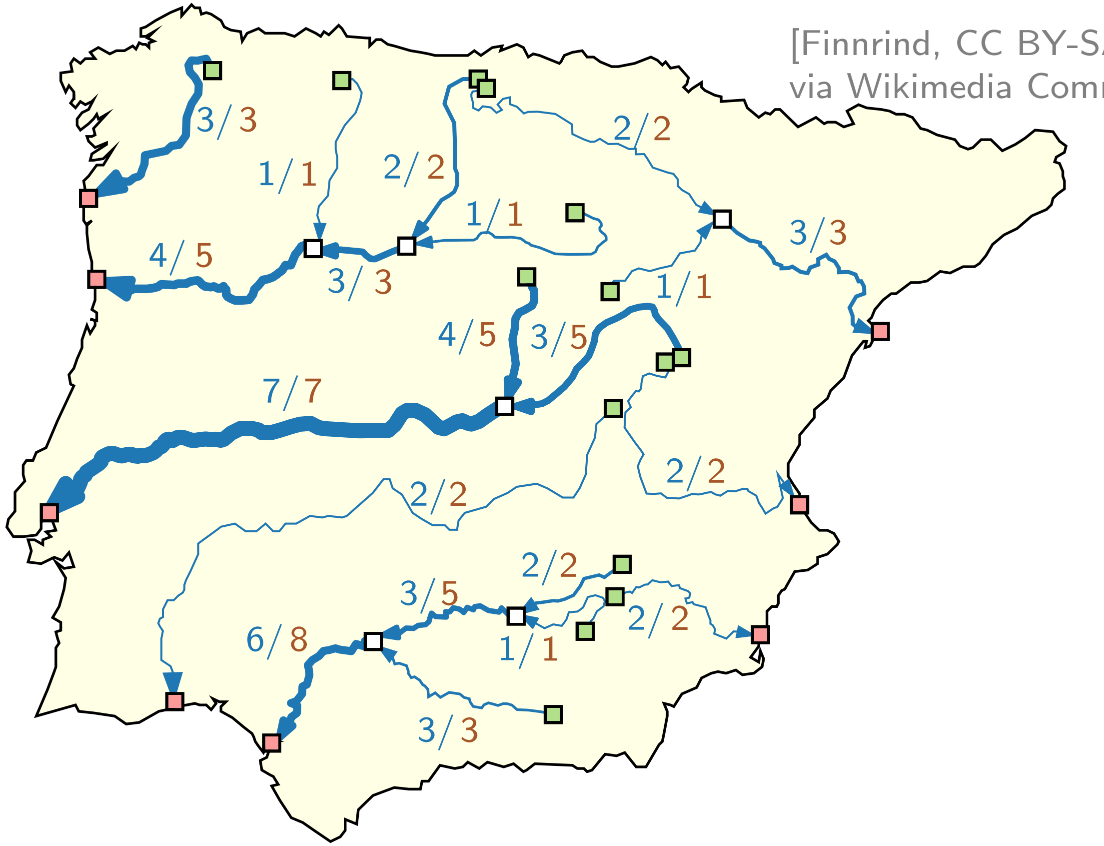

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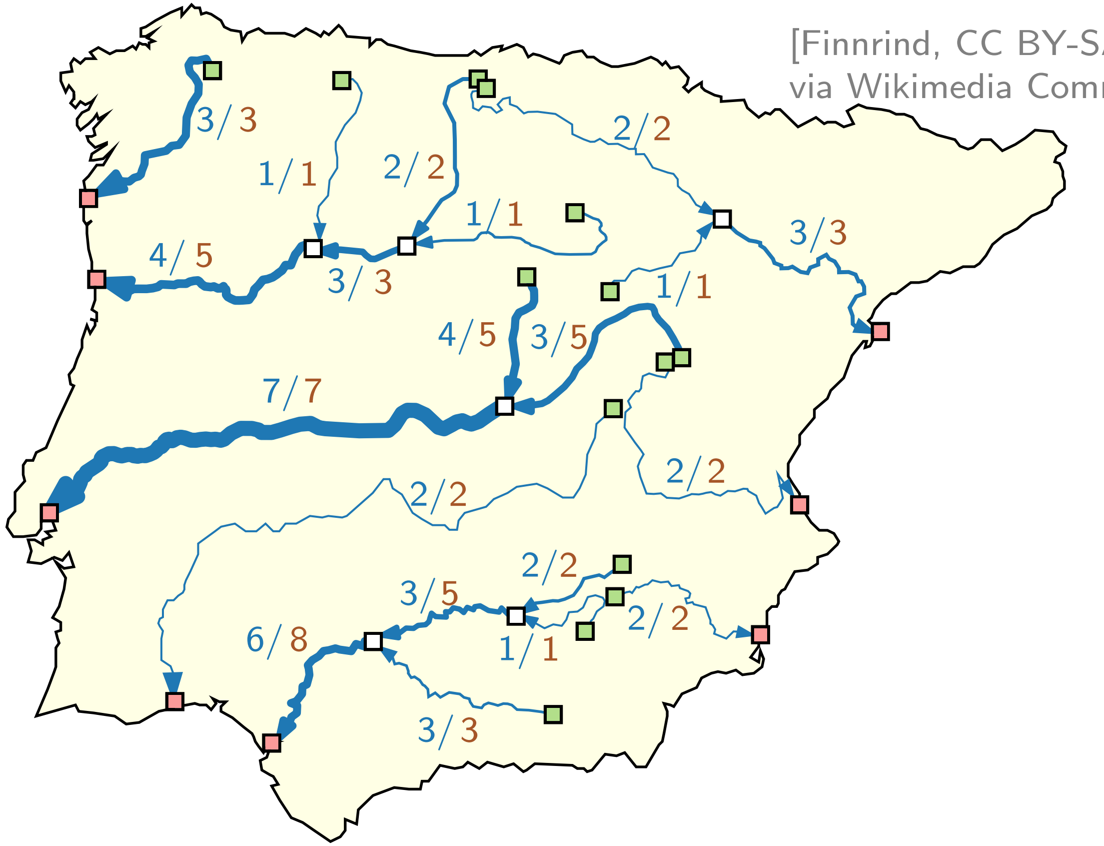

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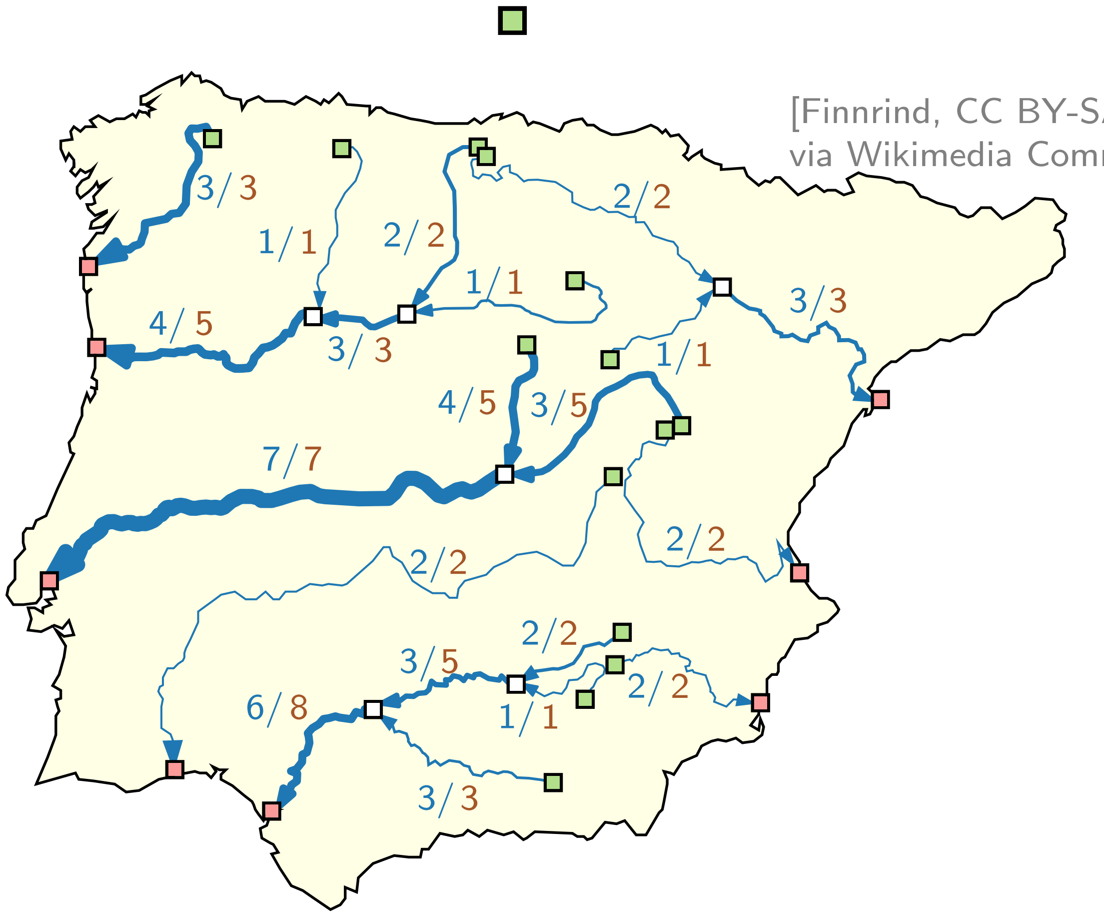

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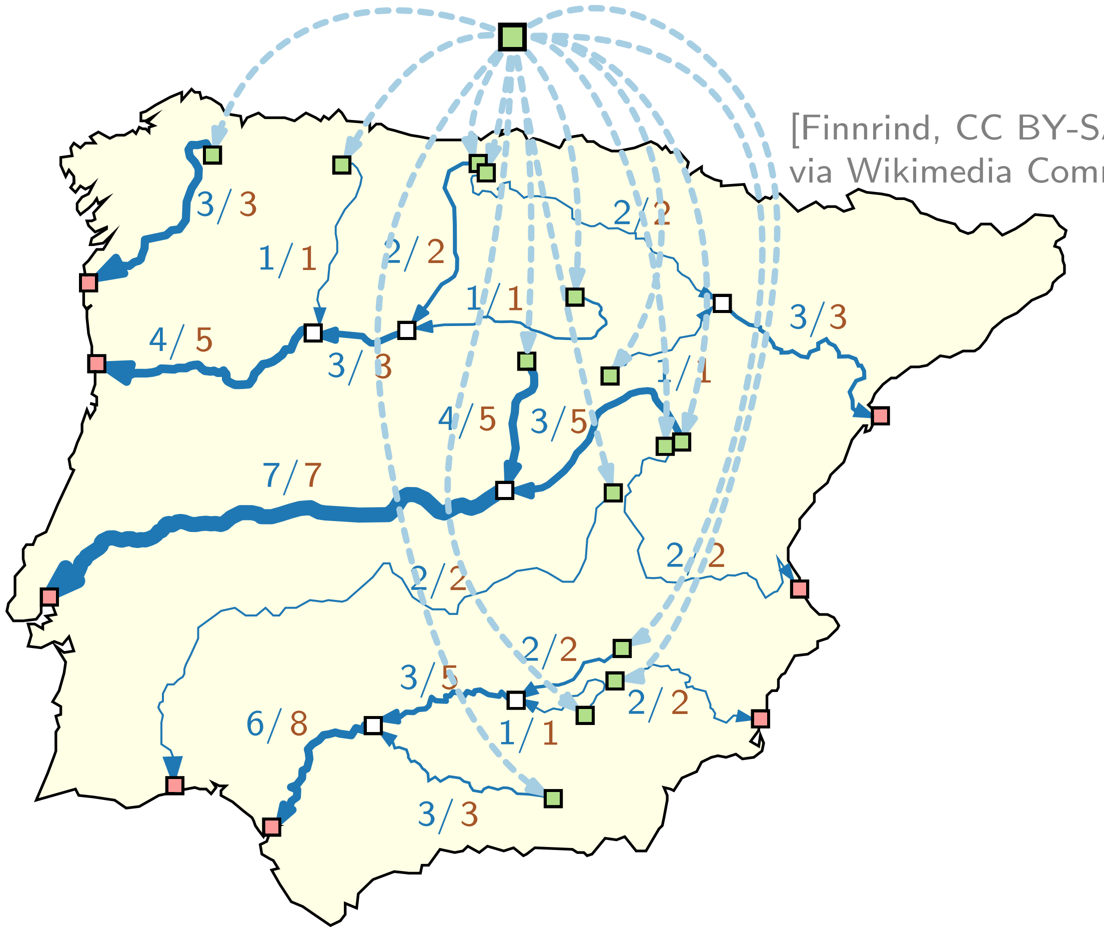

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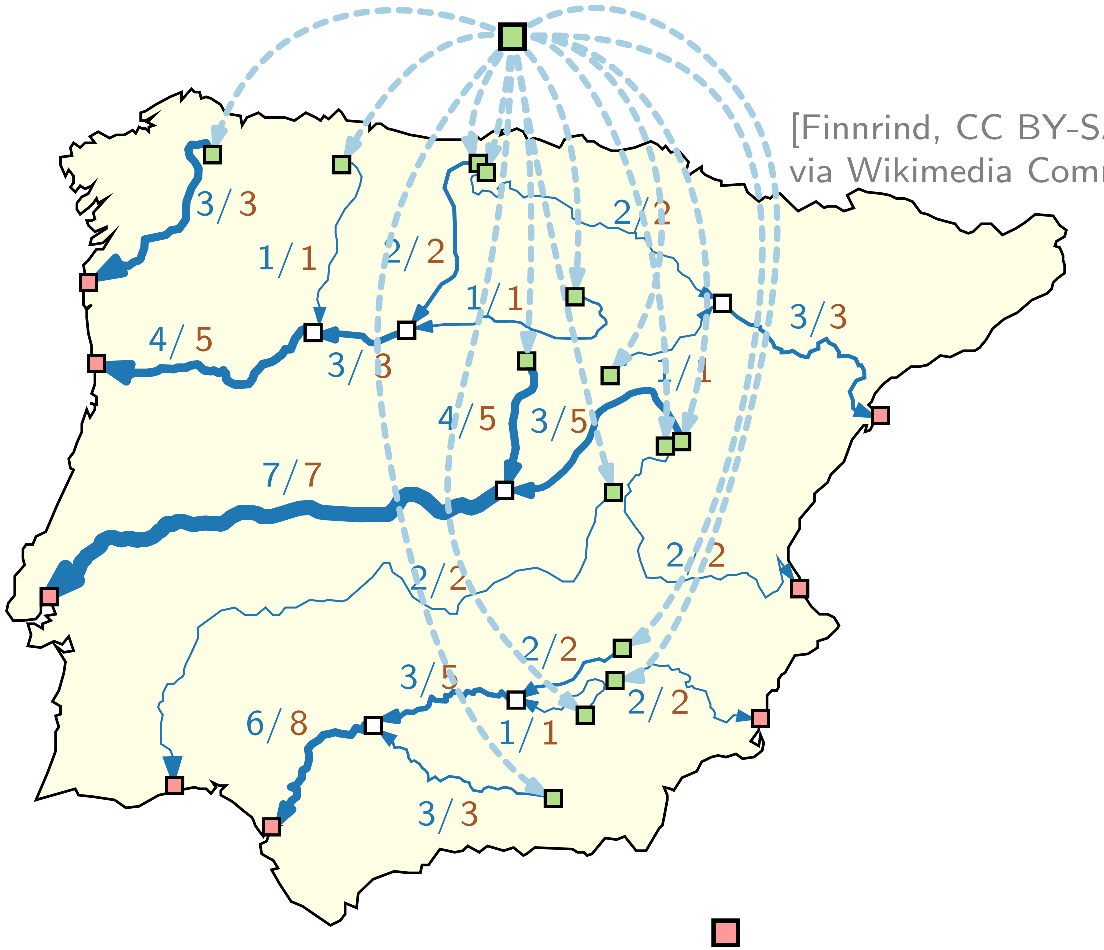

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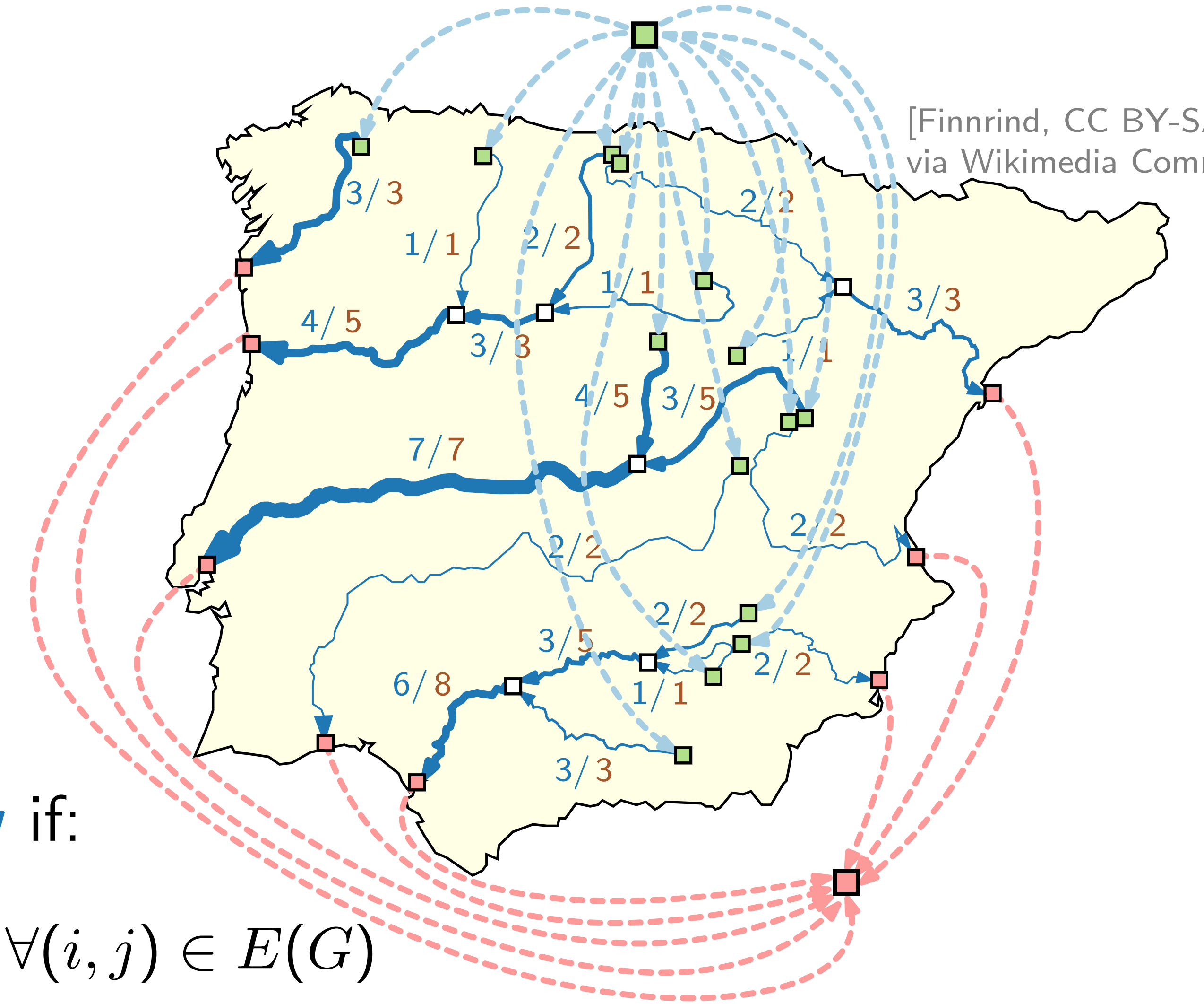

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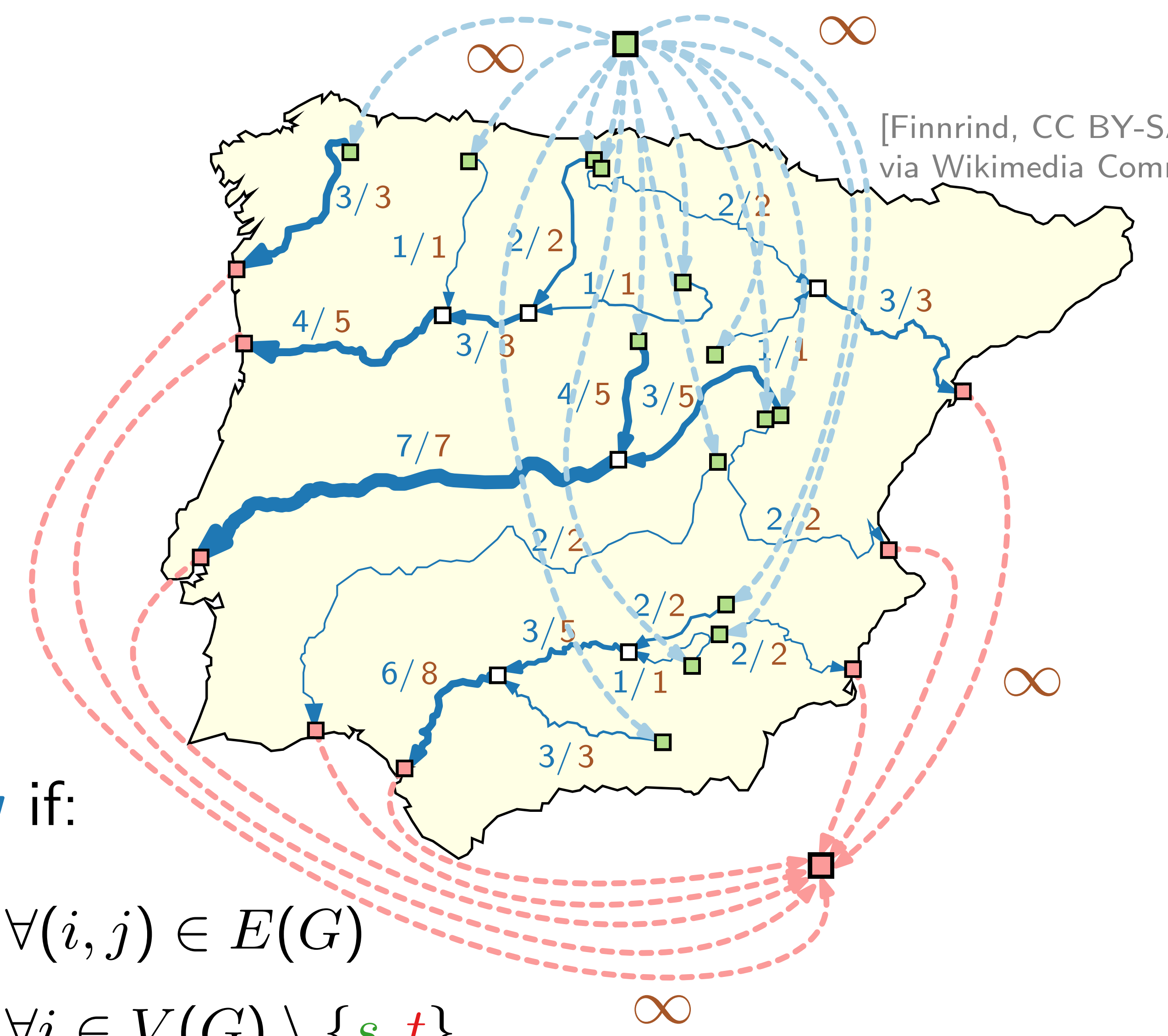

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General Flow Network

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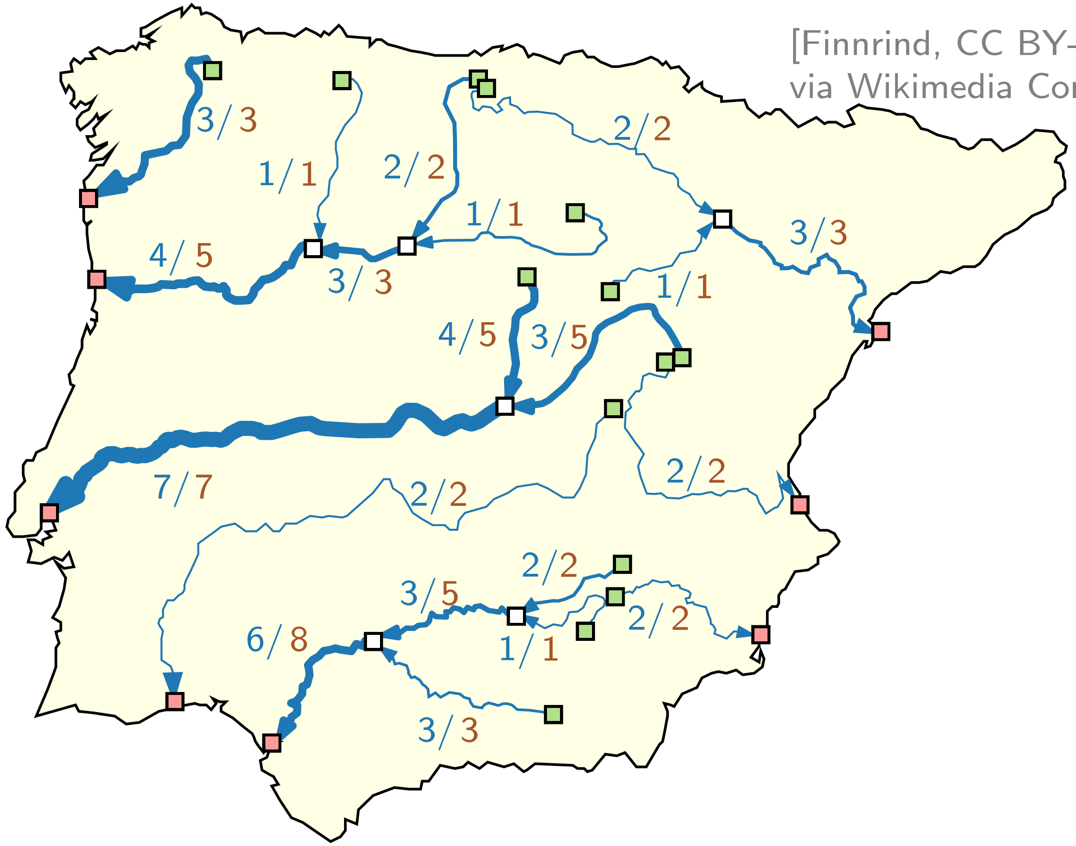

A function $X\colon E(G) \to \mathbb{R}_0^+$ is called S–T **flow** if:

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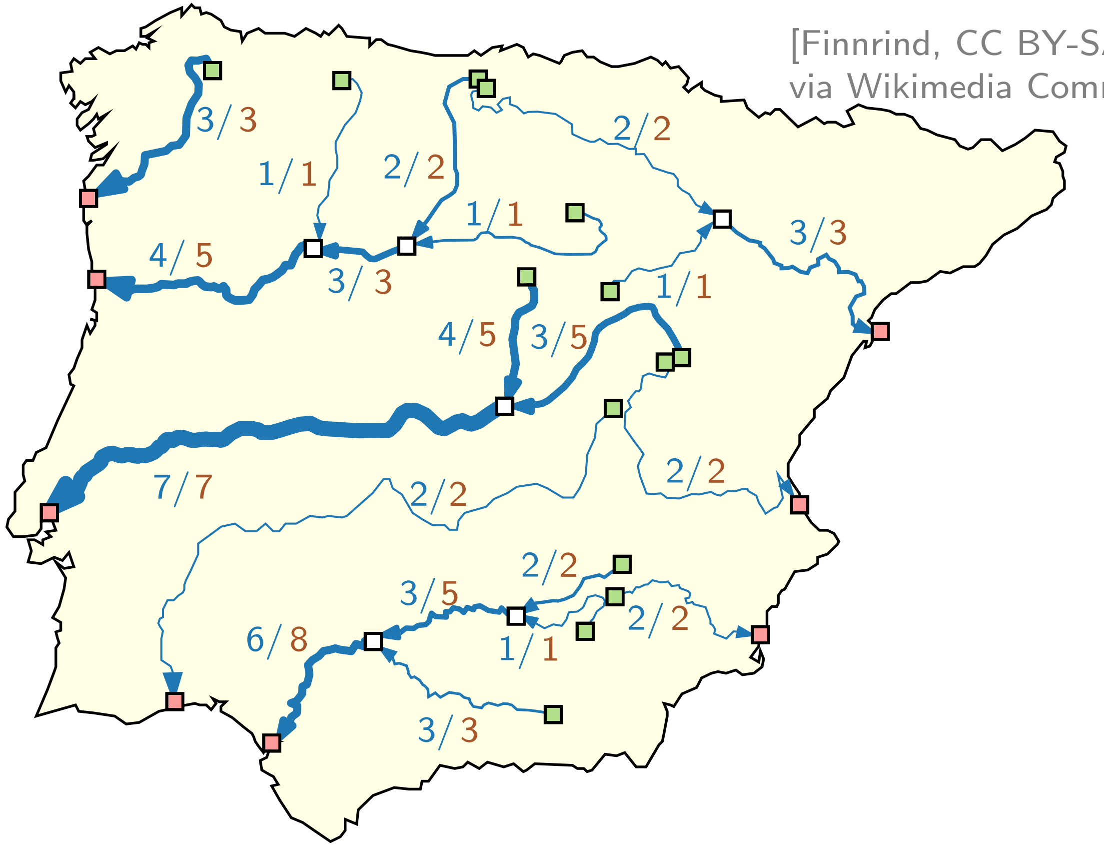
General Flow Network



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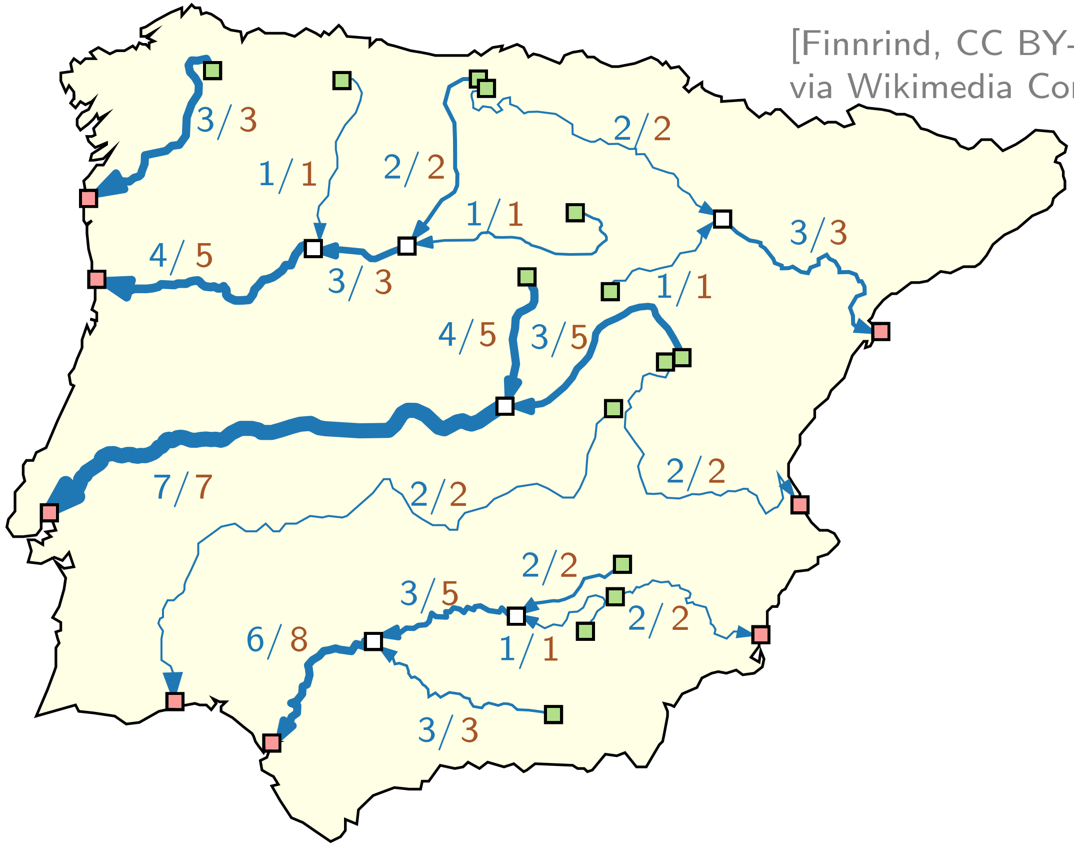

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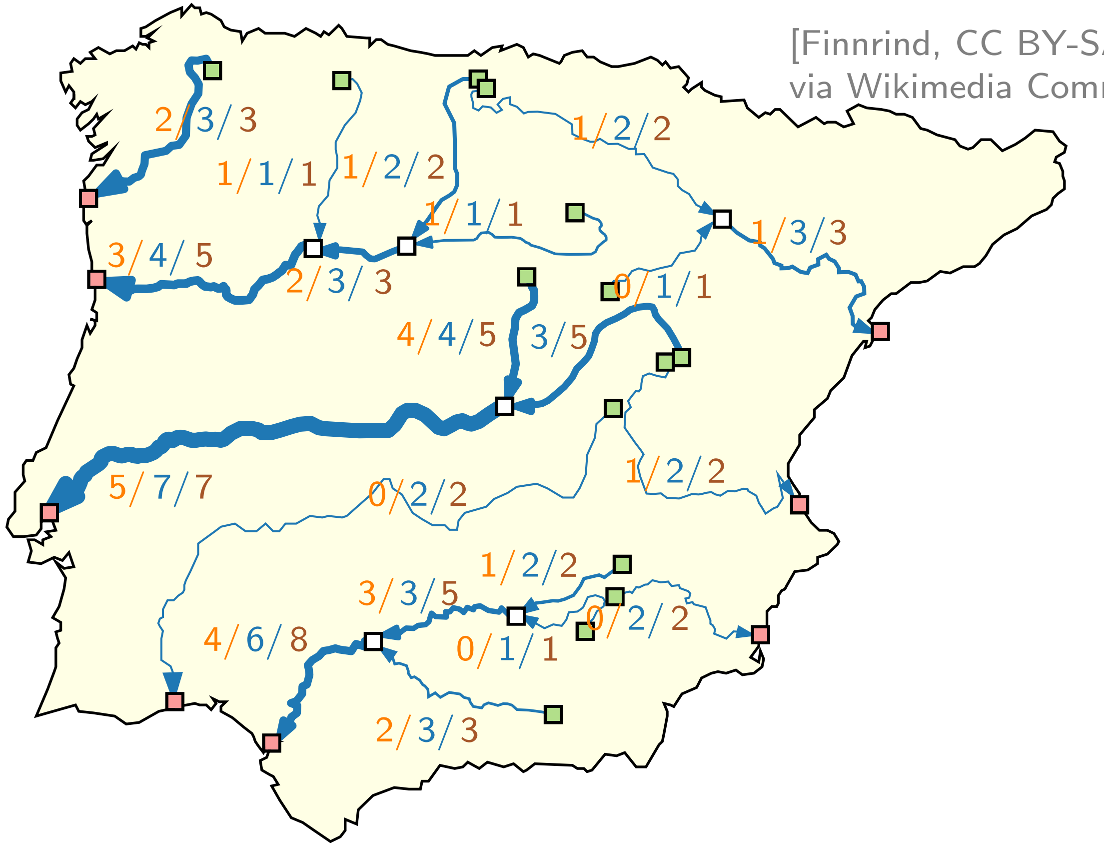
General Flow Network



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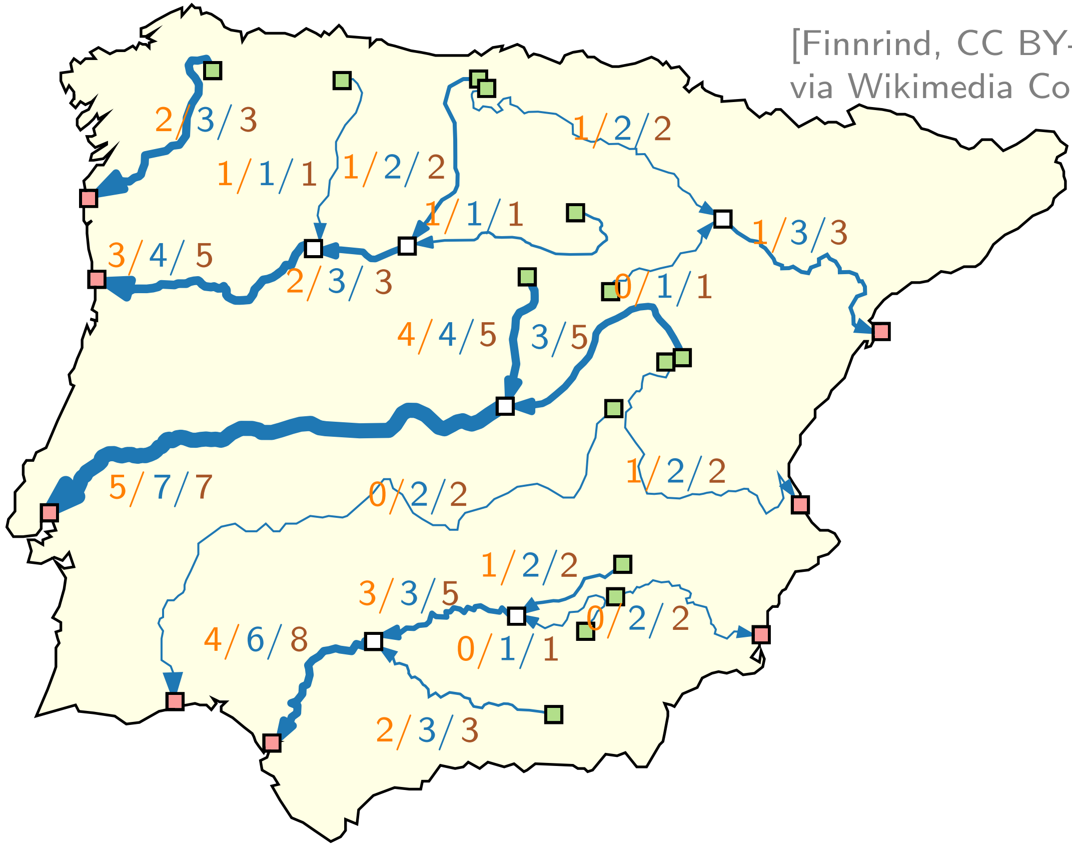
General Flow Network



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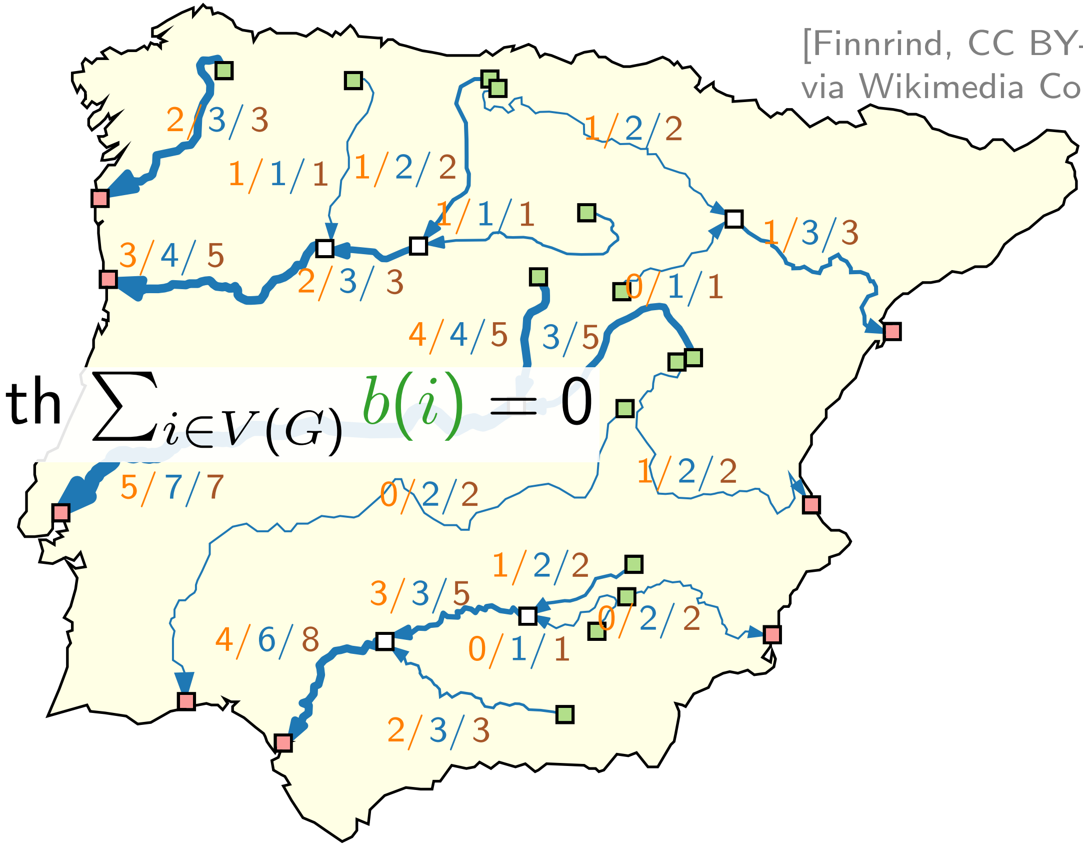

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Flow network $(G; b; \ell; u)$ with

- directed graph G
- node *production/consumption* $b\colon V(G) \to \mathbb{R}$ with $\sum_{i \in V(G)} b(i) = 0$
- edge *lower bound* $\ell\colon E(G) \to \mathbb{R}_0^+$
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A function $X\colon E(G) \to \mathbb{R}_0^+$ is called S–T **flow** if:

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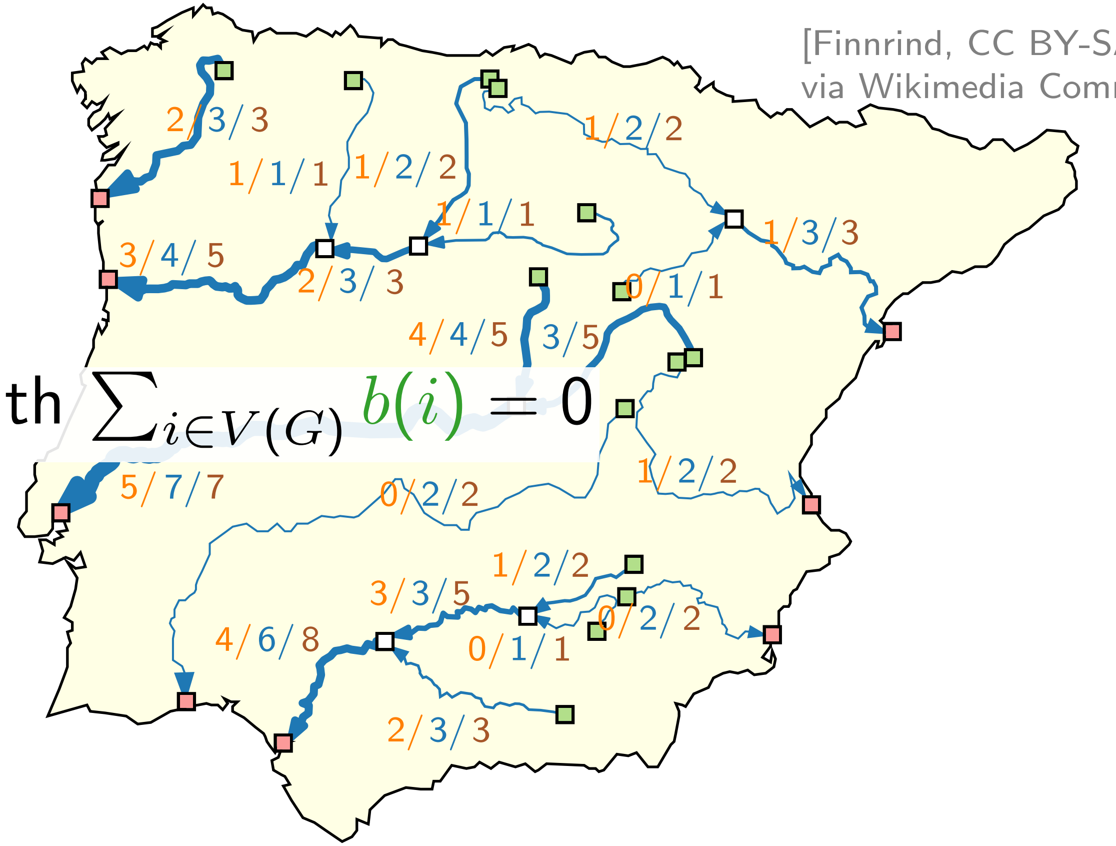
General Flow Network



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A function $X\colon E(G) \to \mathbb{R}_0^+$ is called **valid flow** if:

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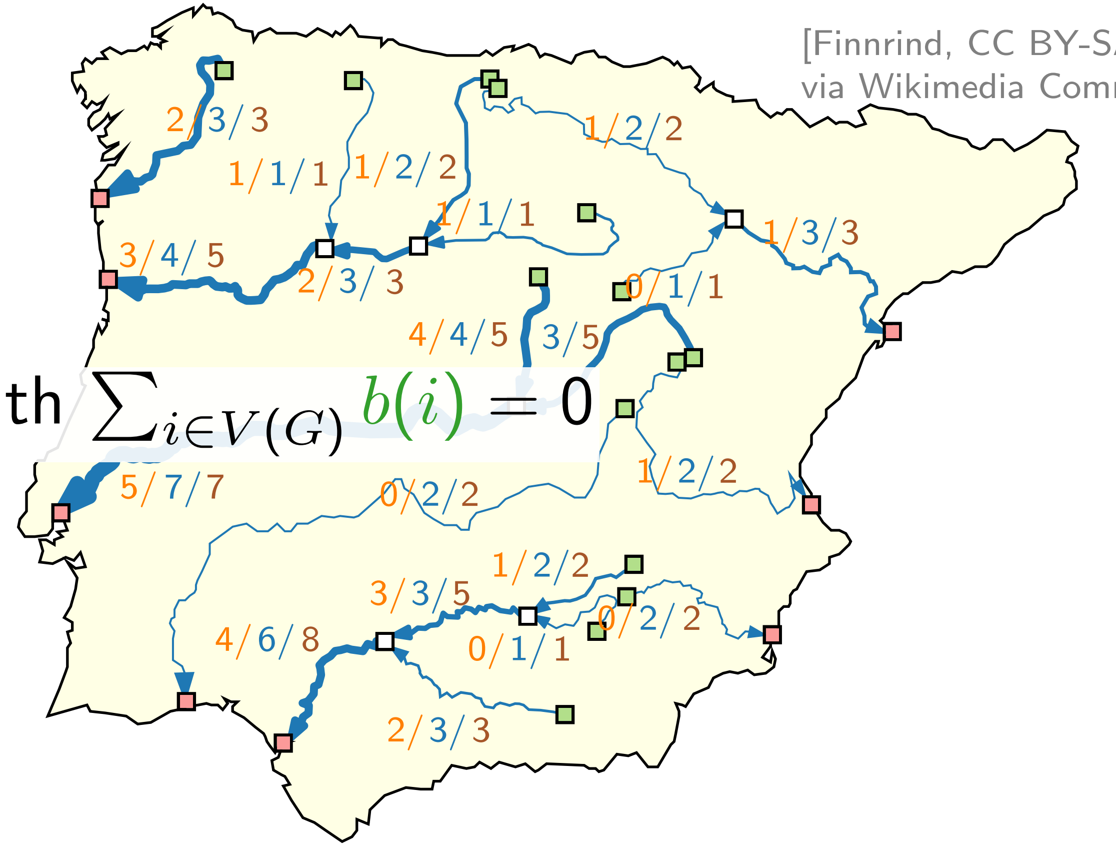

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- node *production/consumption* $b\colon V(G) \to \mathbb{R}$ with $\sum_{i \in V(G)} b(i) = 0$
- edge *lower bound* $\ell\colon E(G) \to \mathbb{R}_0^+$
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A function $X\colon E(G) \to \mathbb{R}_0^+$ is called **valid flow** if:

$$\ell(i,j) \le X(i,j) \le u(i,j) \qquad \forall (i,j) \in E(G)$$

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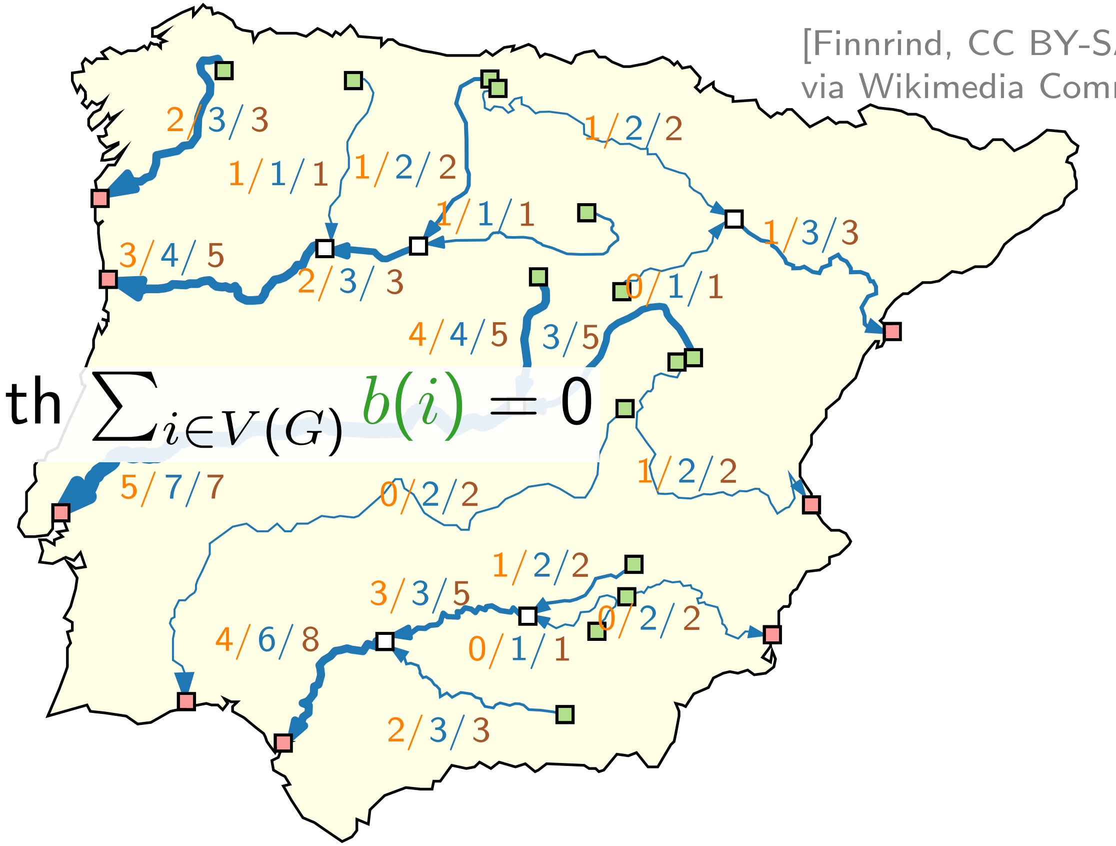
~~A **maximum** S–T flow is an S–T flow where $\sum_{(i,j)\in E(G), i \in S} X(i,j) - \sum_{(j,i)\in E(G), i \in S} X(j,i)$ is maximized.~~

General Flow Network



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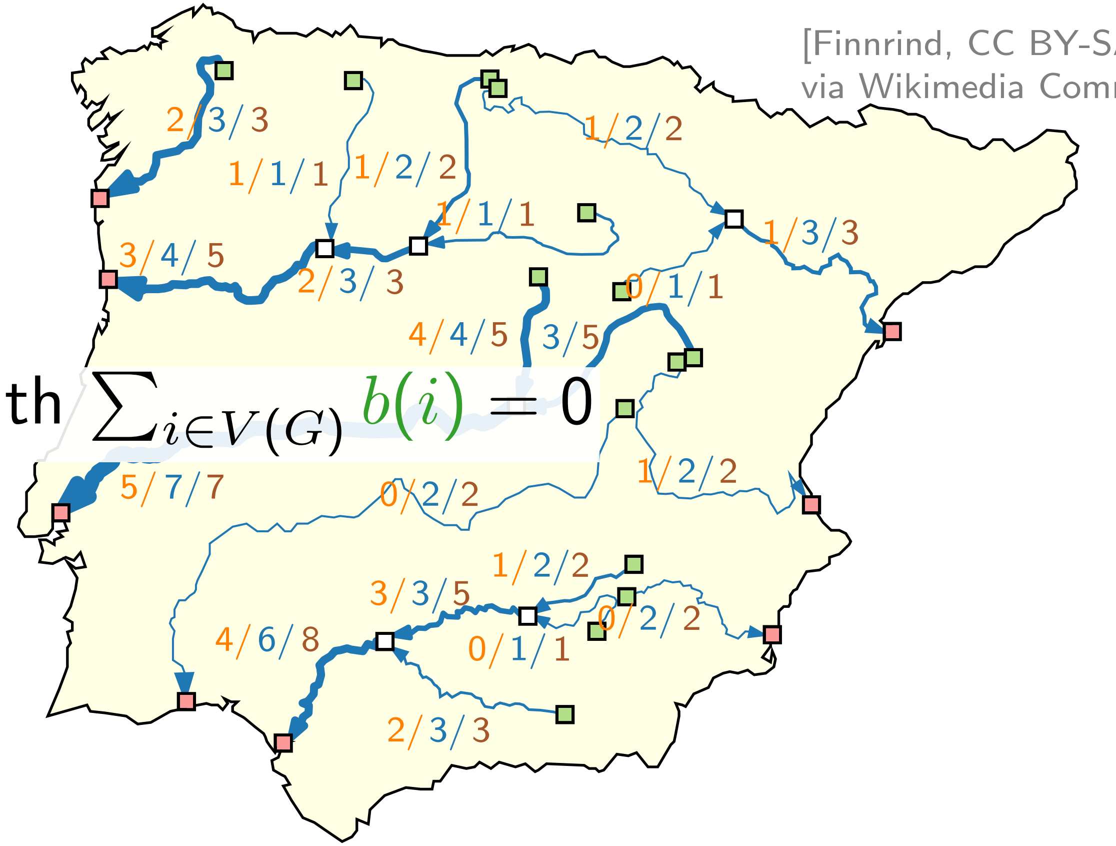
General Flow Network



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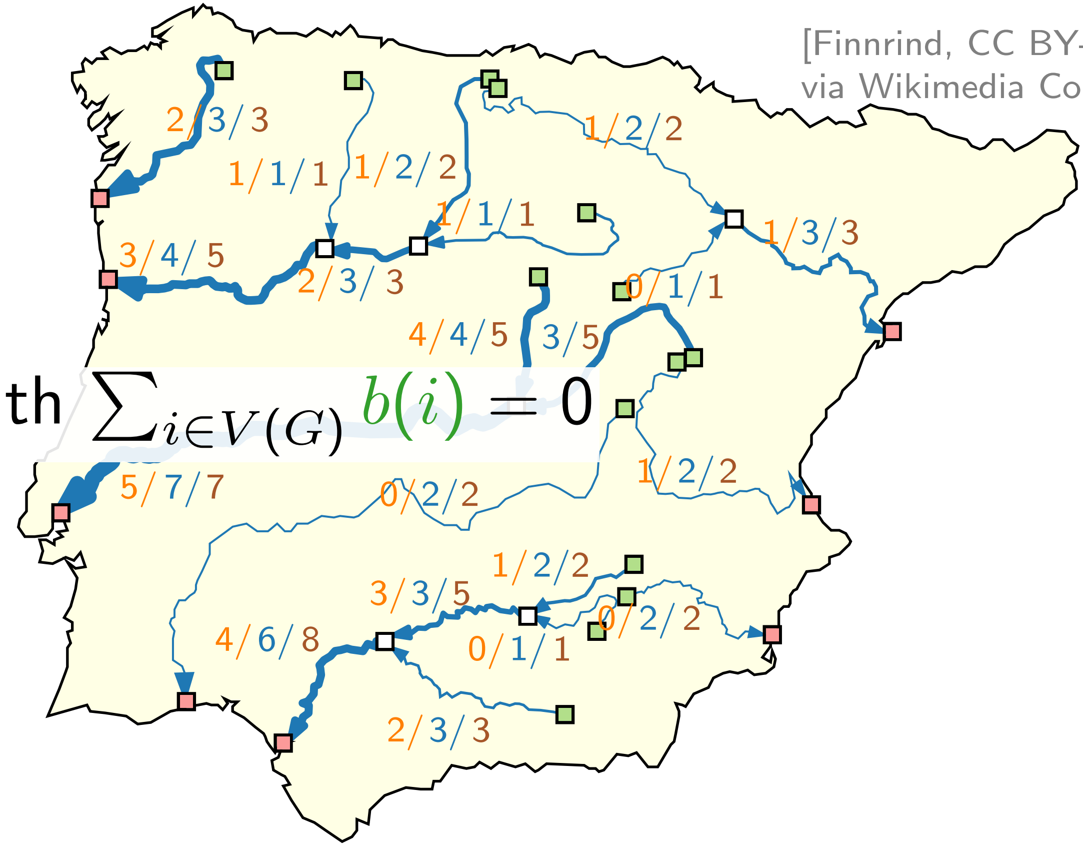
General Flow Network



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X is a **minimum-cost flow** if X is a valid flow that minimizes $\text{cost}(X)$.

General Flow Network – Algorithms

n: #vertices
m: #edges

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	O((n + m') log U S(n, m, nC))
2	Rock	1980	O((n + m') log U S(n, m, nC))
3	Rock	1980	O(n log C M(n, m, U))
4	Bland and Jensen	1985	O(m log C M(n, m, U))
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
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7	Ahuja, Goldberg, Orlin and Tarjan	1988	O(nm log log U log (nC))

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
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4	Galil and Tardos	1986	$O(n^2 \log n\, S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	O((n + m') log n S(n, m))

S(n, m)	= O(m + n log n)	Fredman and Tarjan [1984]
S(n, m, C)	$= O(\text{Min}\,(m + n\sqrt{\log C}),$ $(m \log\log C))$	Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977]
M(n, m)	$= O(\min\,(nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.	King, Rao, and Tarjan [1991]
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Theorem. [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
The minimum-cost flow problem with integral vertex demands, edge capacities, and edge costs can be solved in $O(m^{1+o(1)} \log U \log C)$ time, where U is the maximum capacity and C are the maximum costs.

Topology – Shape – Metrics

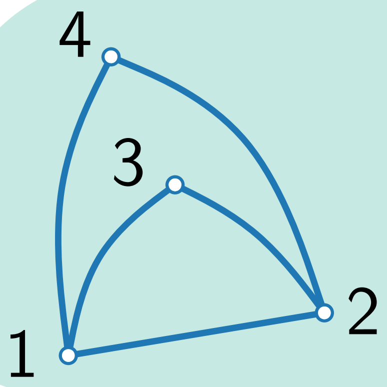
Three-step approach:

[Tamassia 1987]

$$V(G) = \{v_1, v_2, v_3, v_4\}$$
$$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$

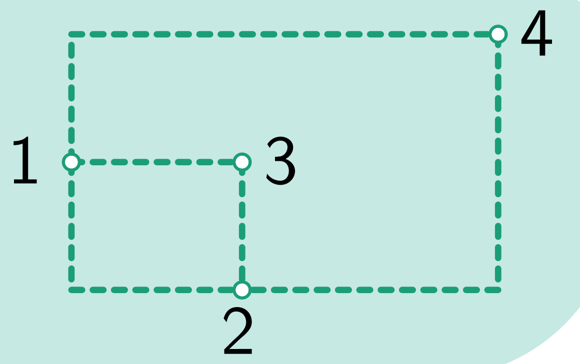
combinatorial embedding/ planarization

reduce crossings

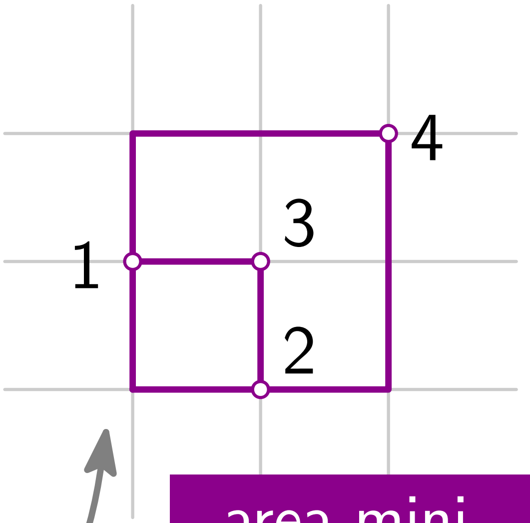


bend minimization

orthogonal representation



planar orthogonal drawing



area minimization

TOPOLOGY — SHAPE — METRICS

Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given:

Find:

Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given: ■ Plane graph G with maximum degree 4

Find:

Bend Minimization with Given Embedding

Geometric orthogonal bend minimization.

Given:
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Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

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Compare with the following variant:

Combinatorial orthogonal bend minimization.

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Given:
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Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

Bend Minimization with Given Embedding

How to solve the combinatorial orthogonal bend minimization problem?

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Idea.
Formulate as a network-flow problem:

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Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, the sequence δ_1 is reversed and inverted copy of δ_2.

(H3) For each **face** f, it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v, the sum of incident angles is 2π.

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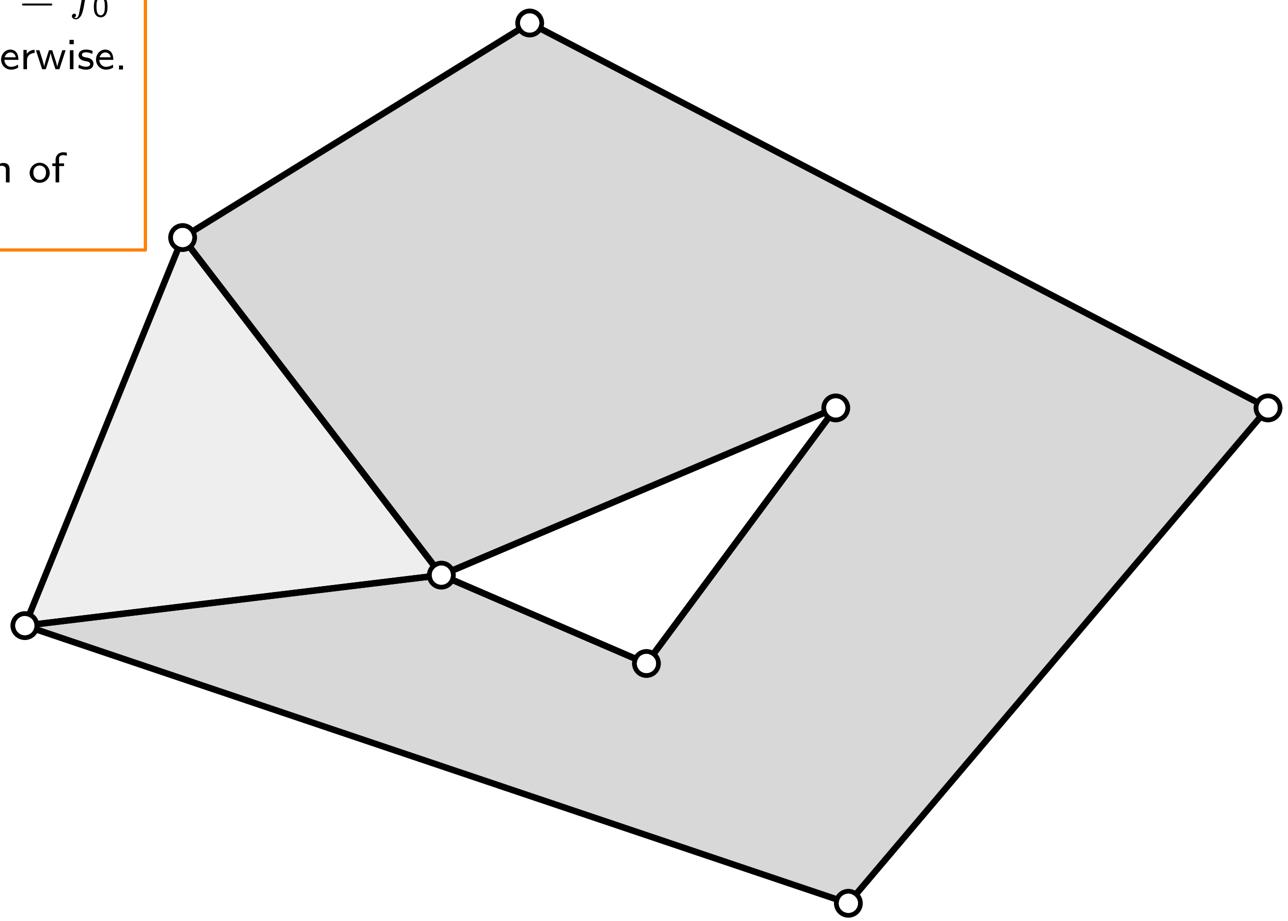

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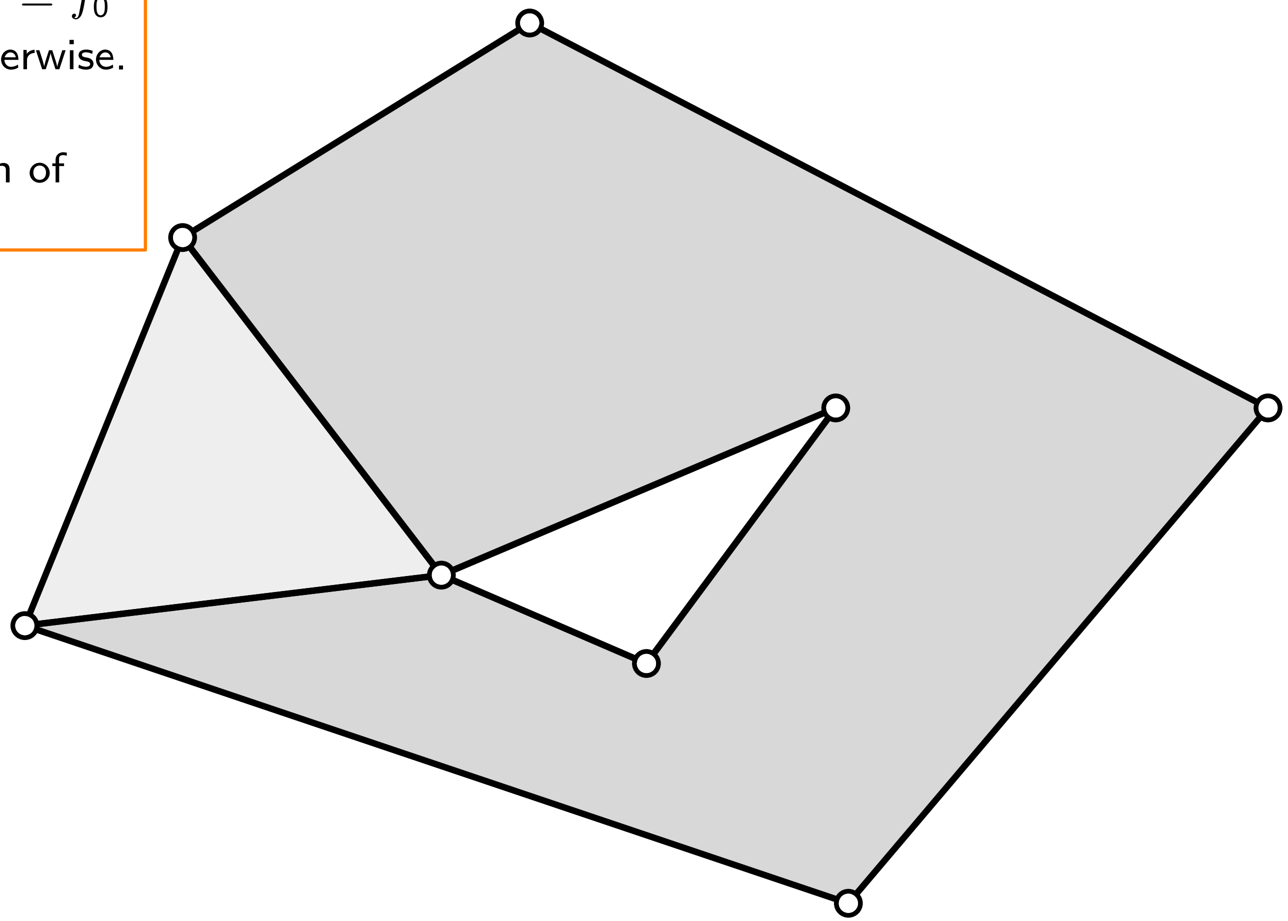

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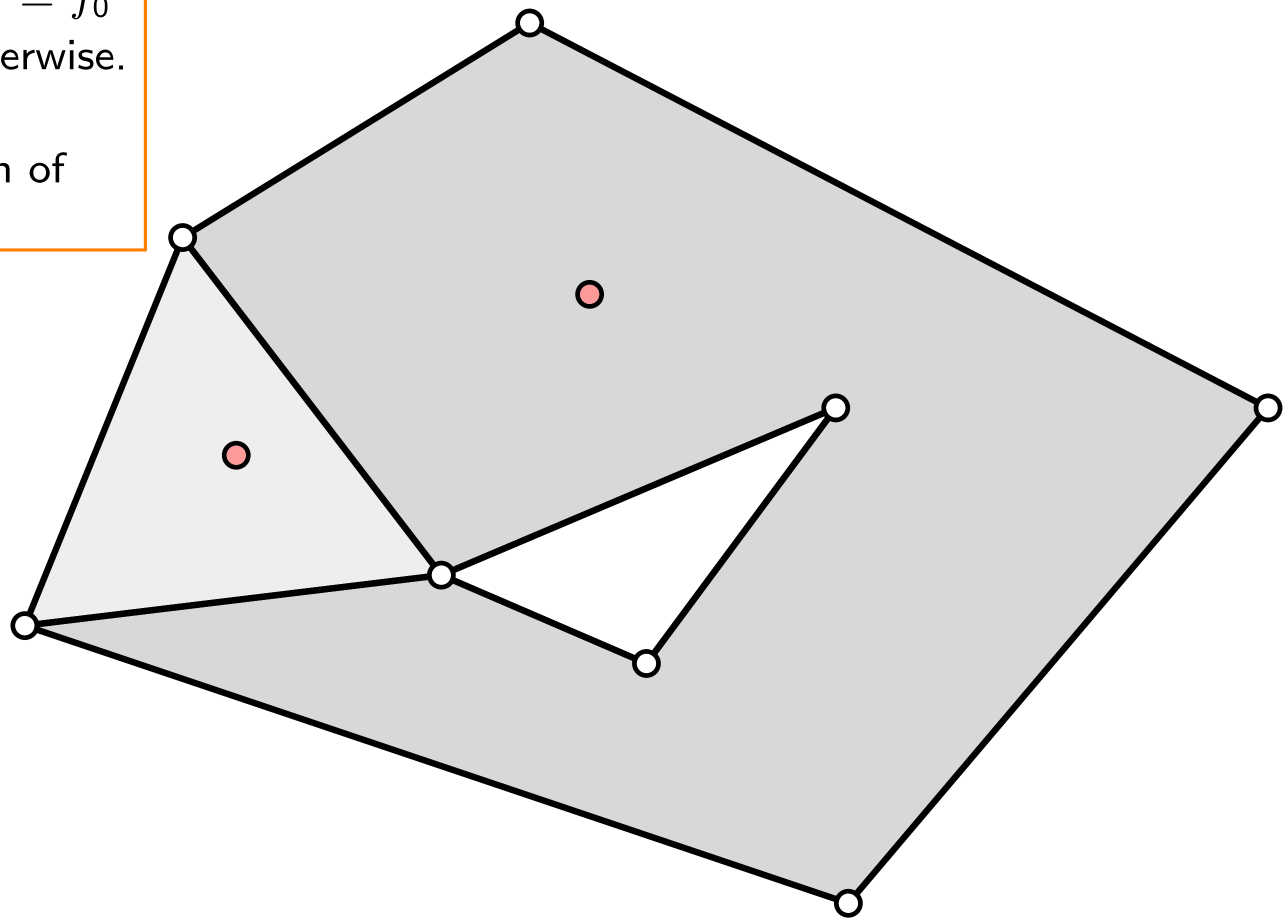

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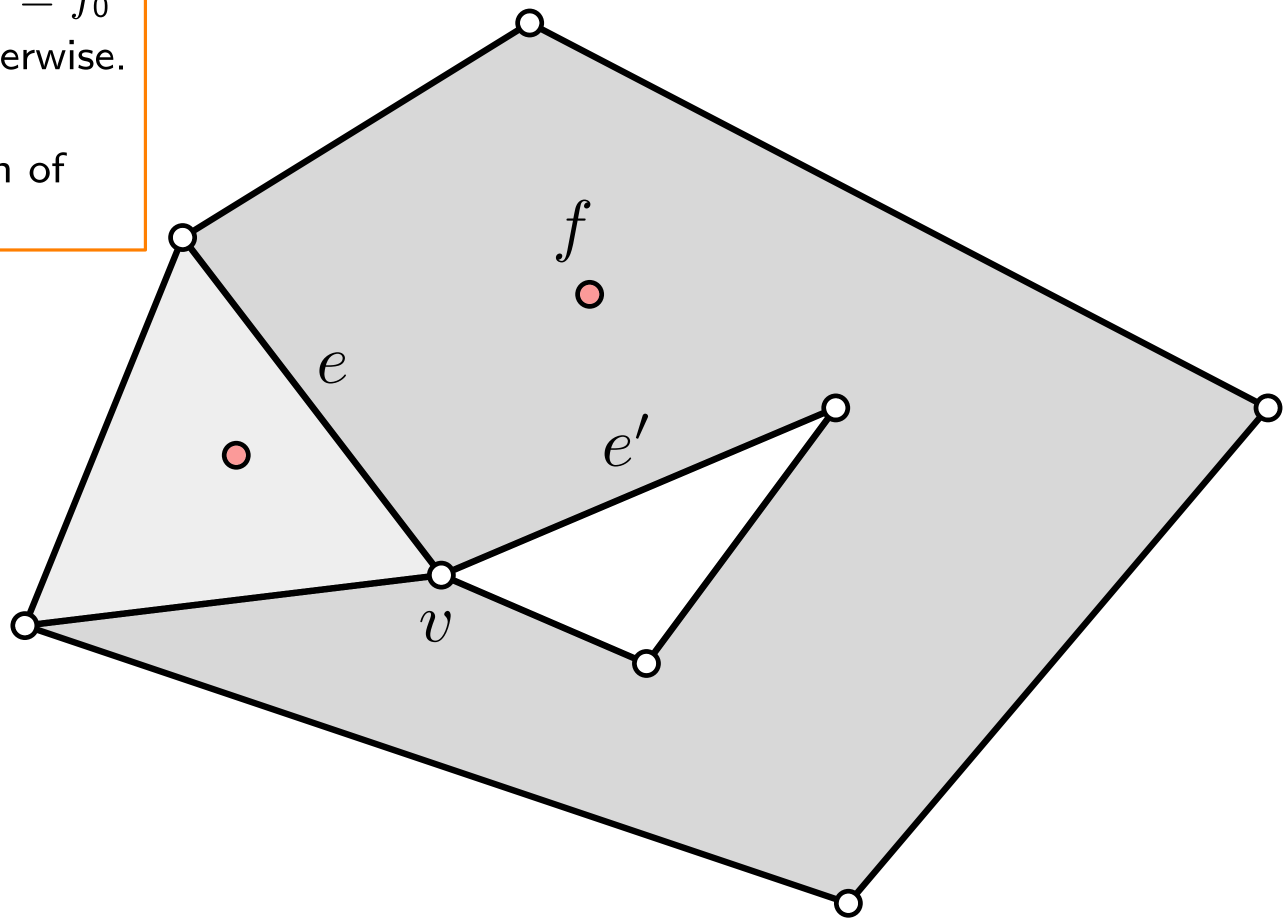

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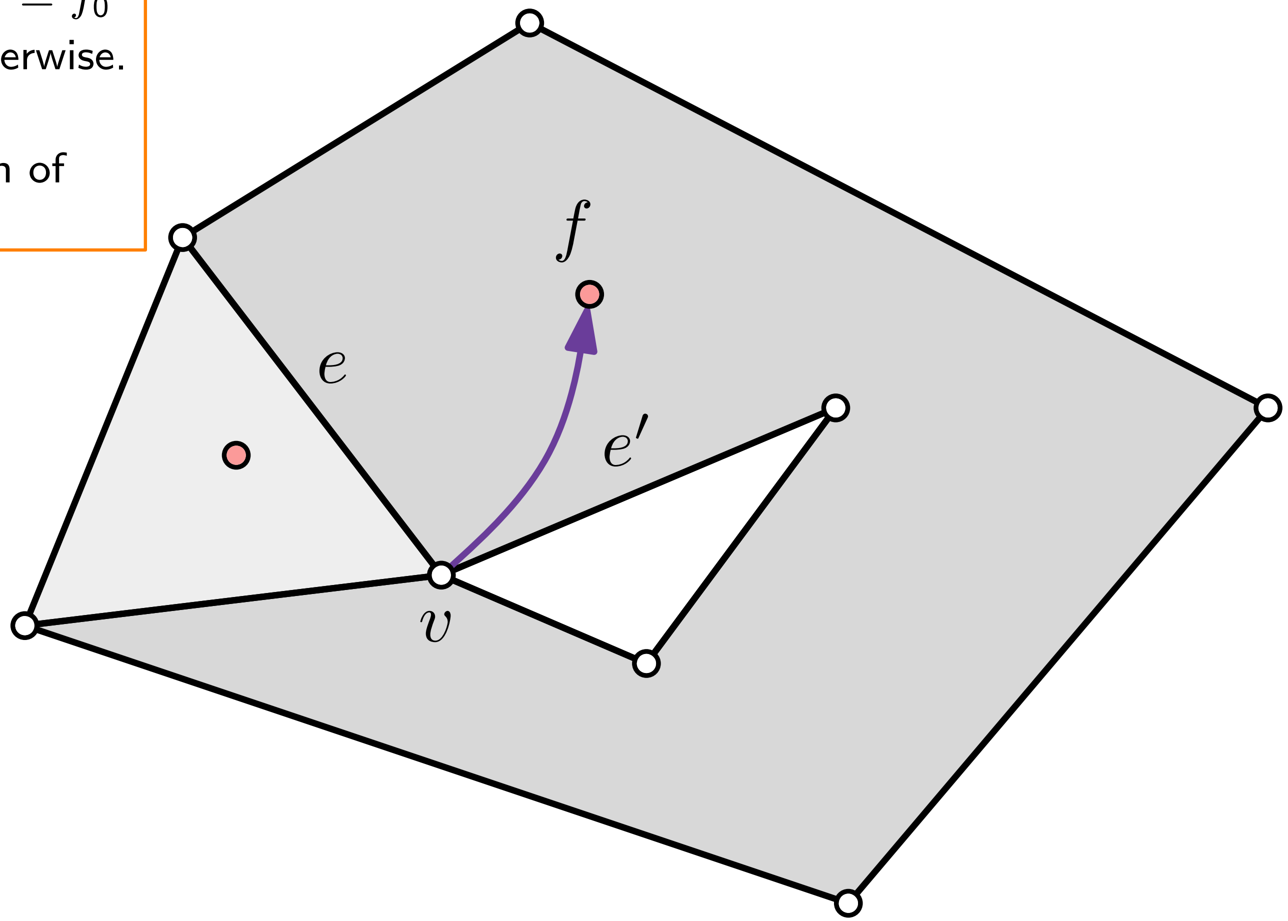

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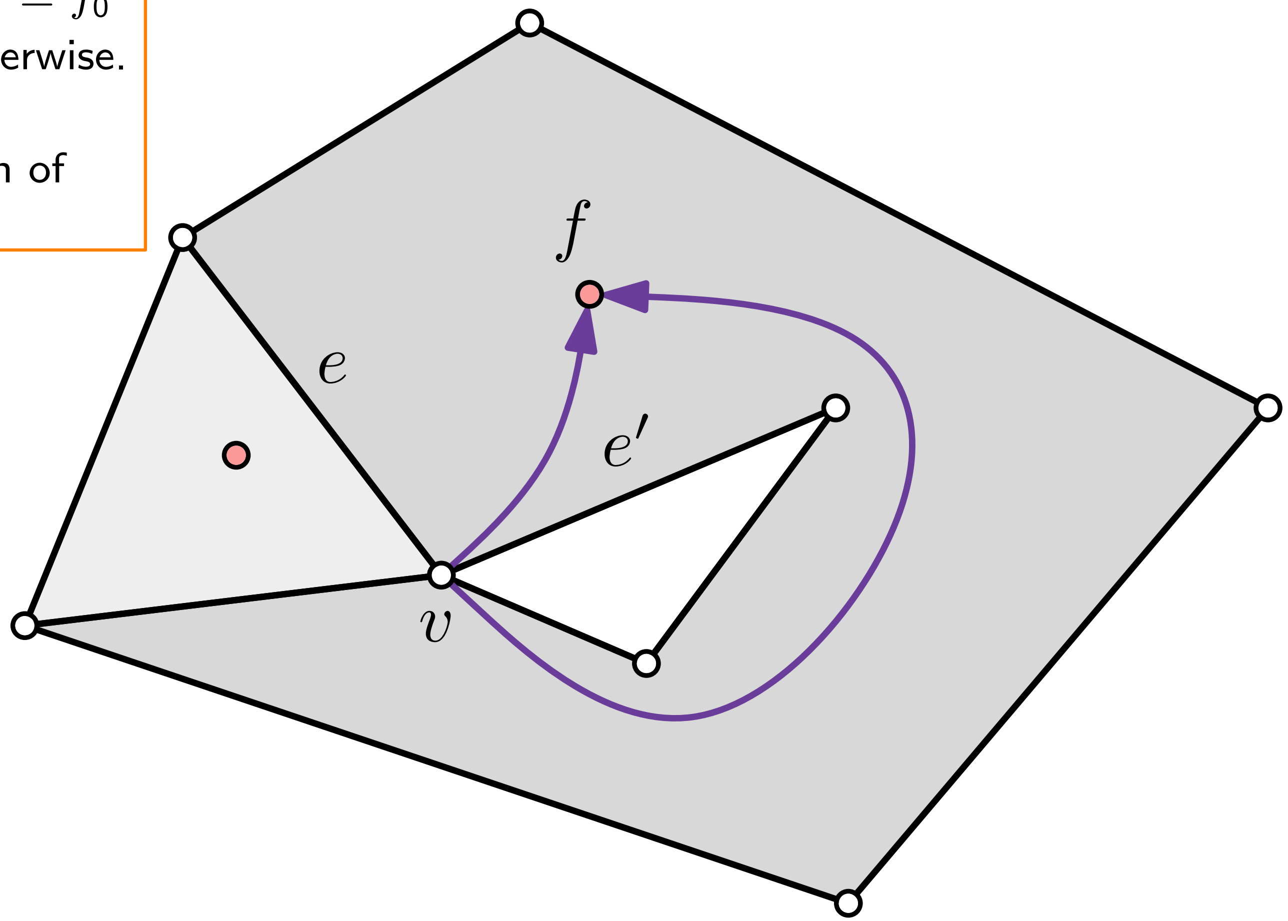

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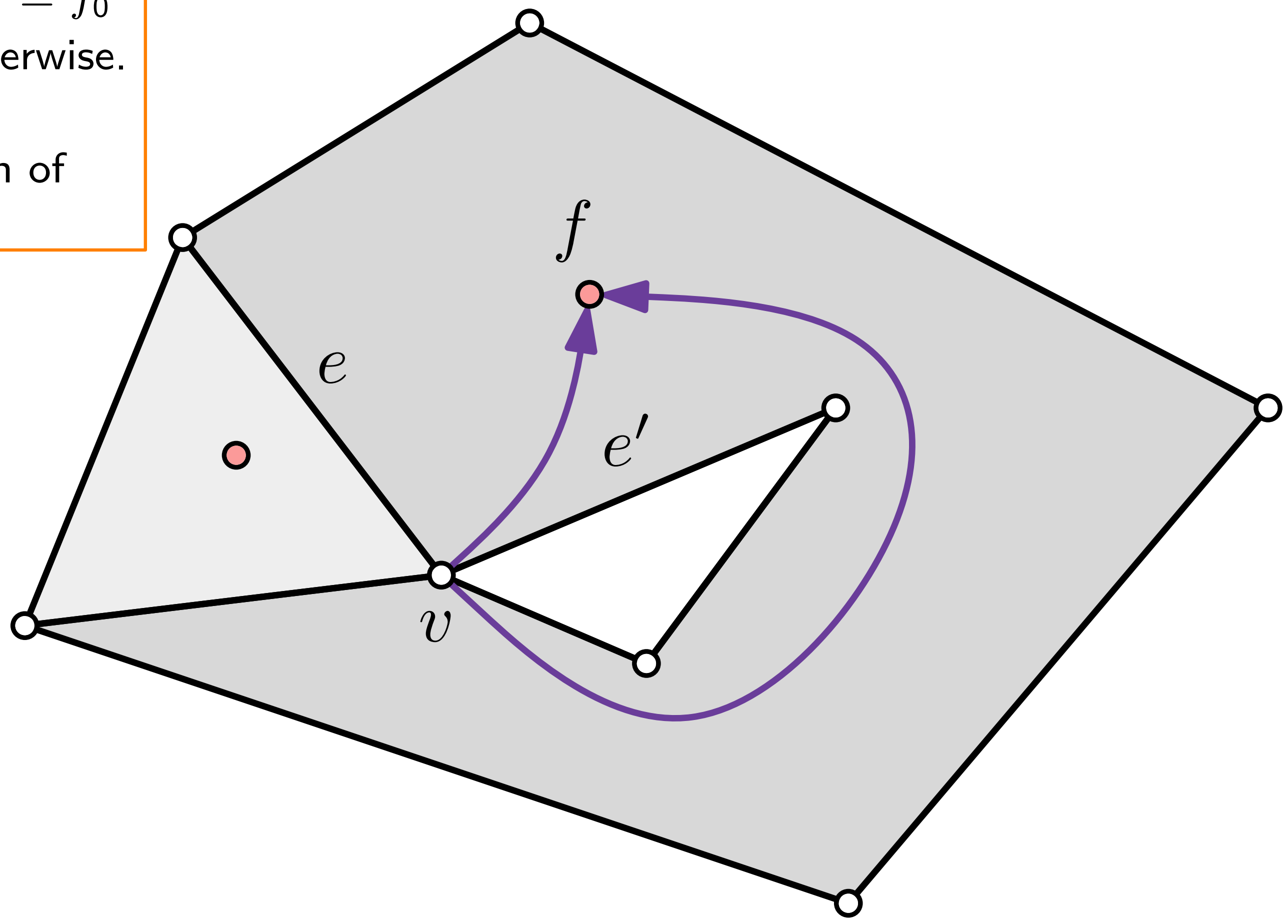

Directed multigraph!

Flow Network for Bend Minimization

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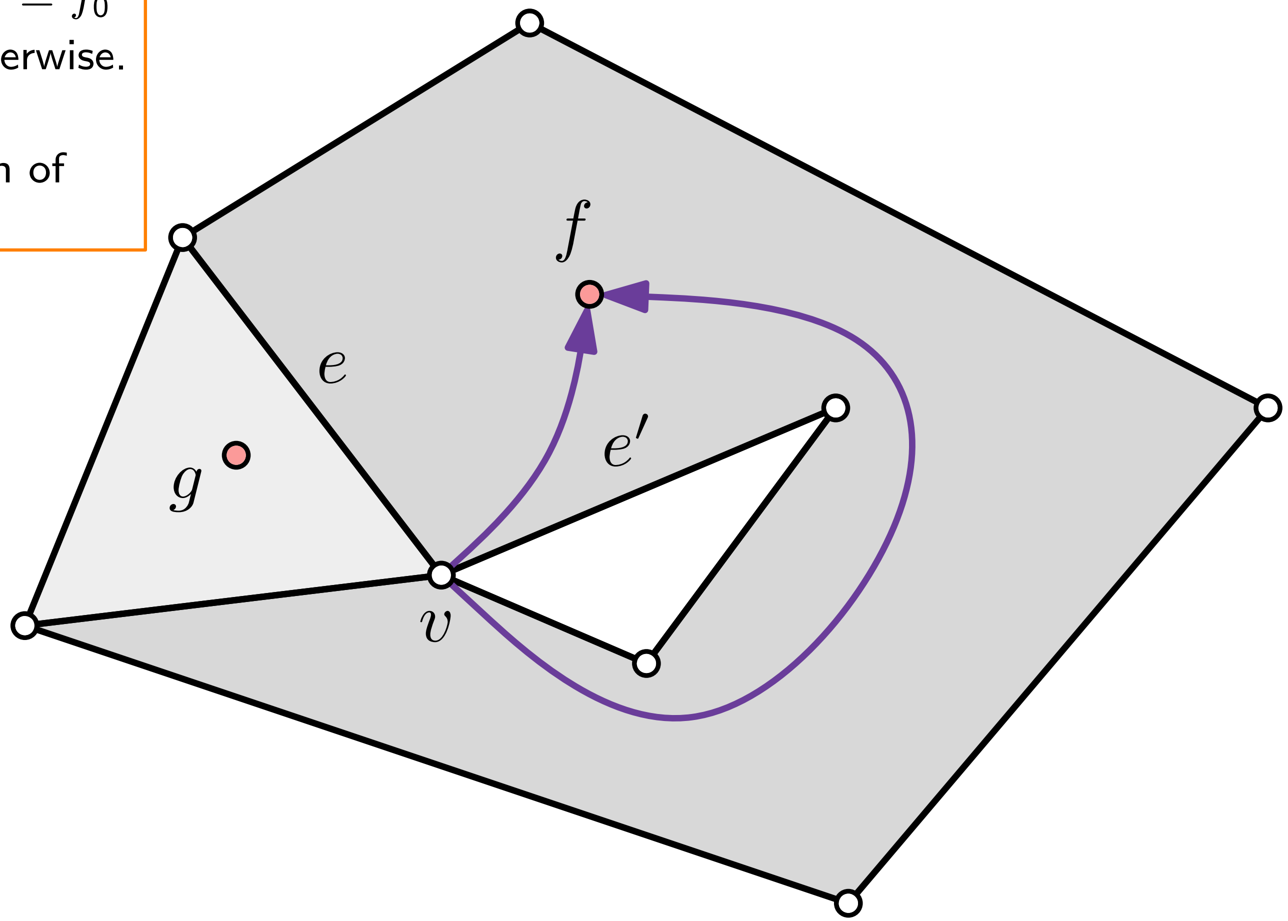

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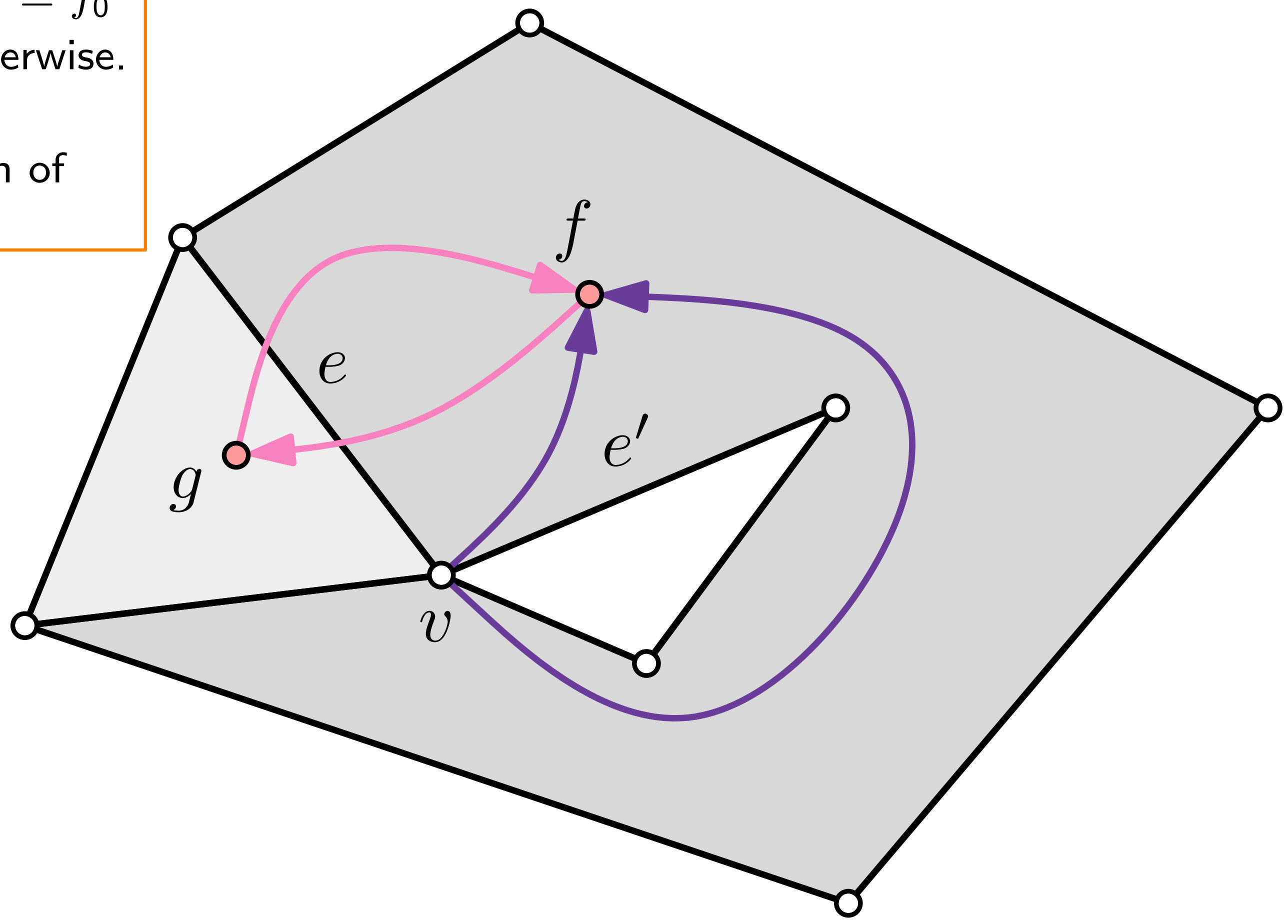

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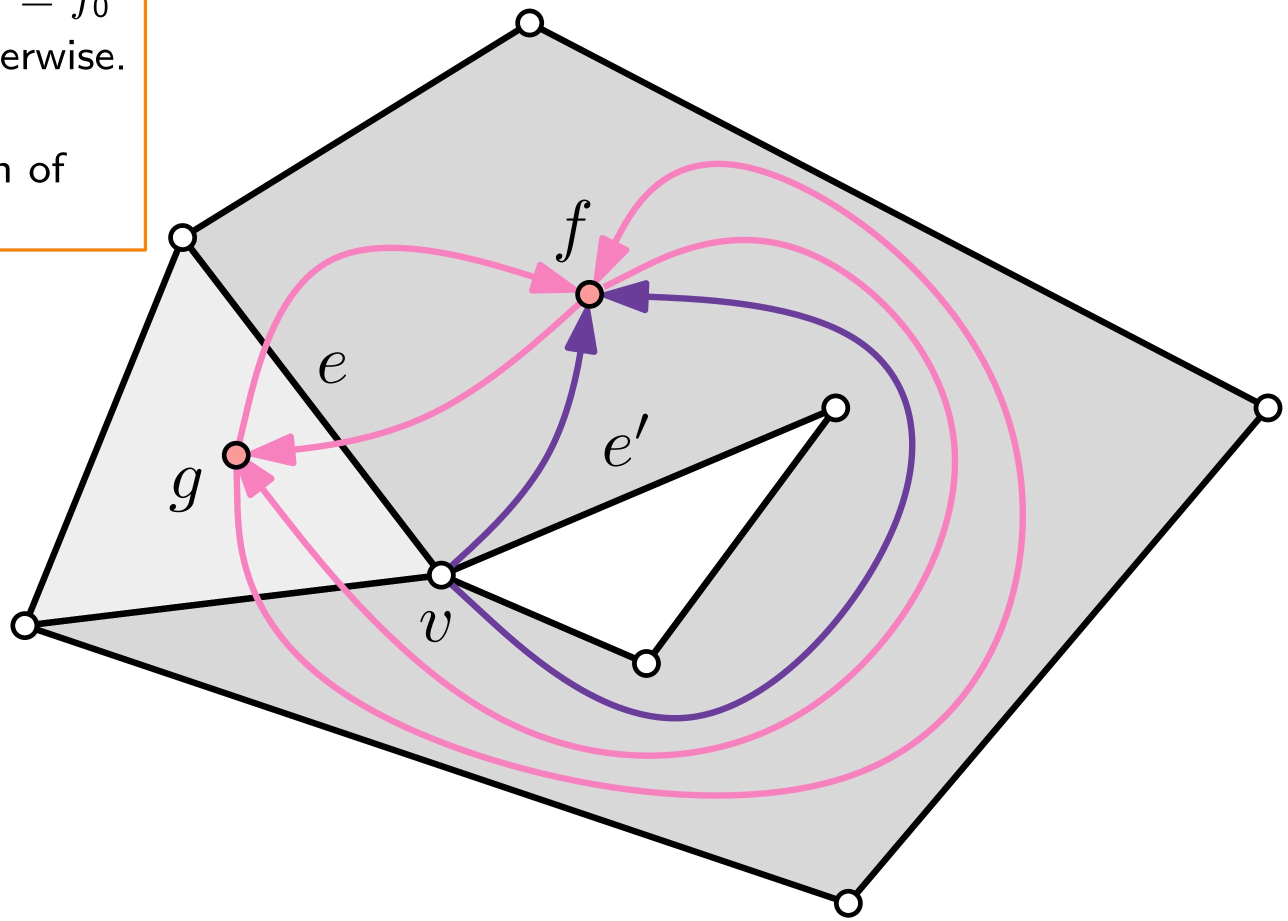

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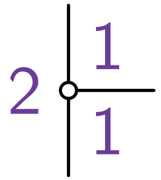

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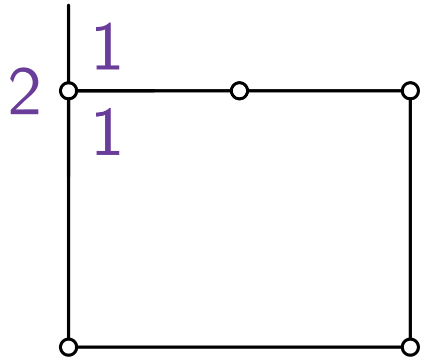

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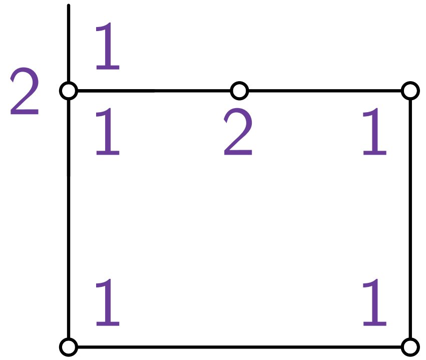

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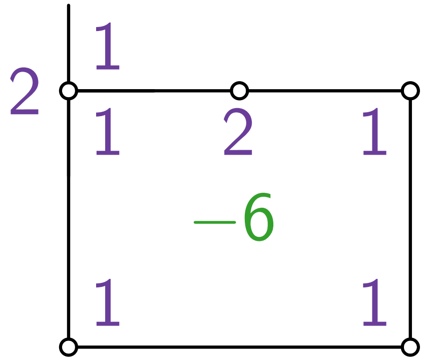

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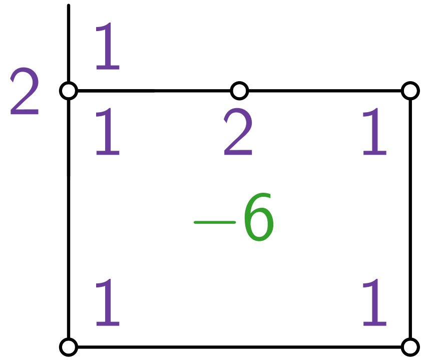

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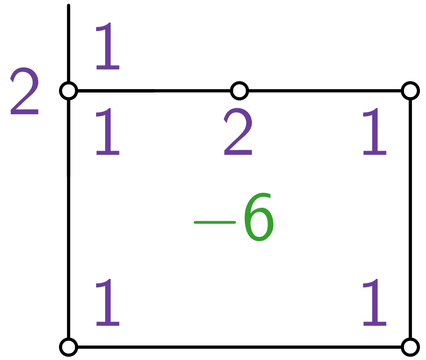

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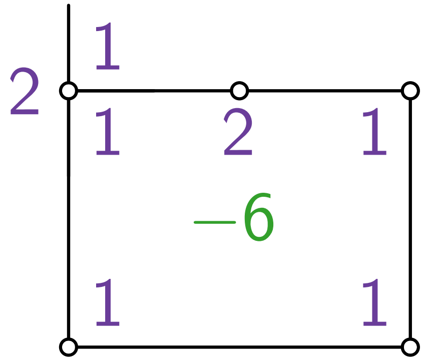

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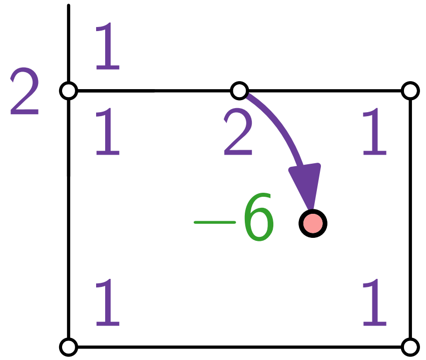

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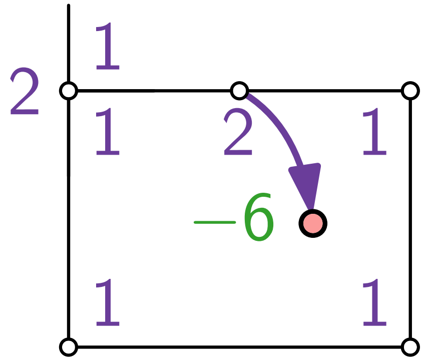

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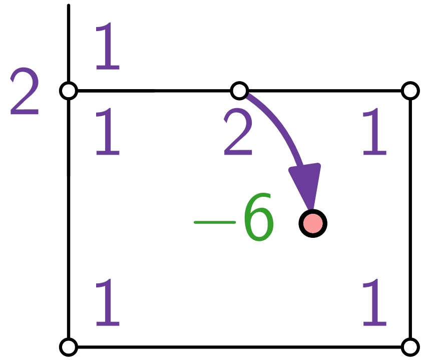

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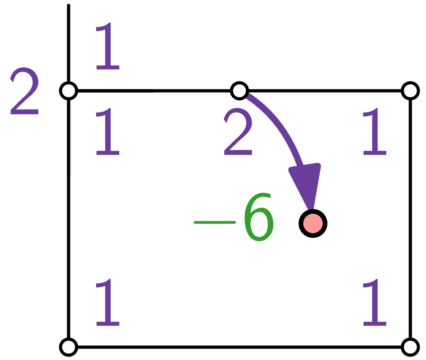

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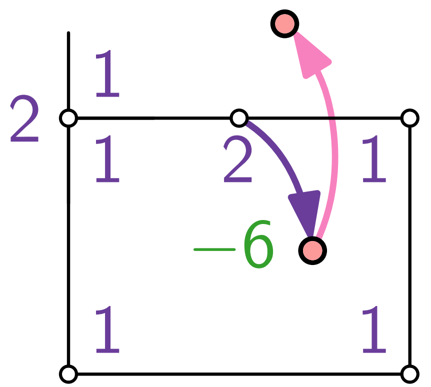

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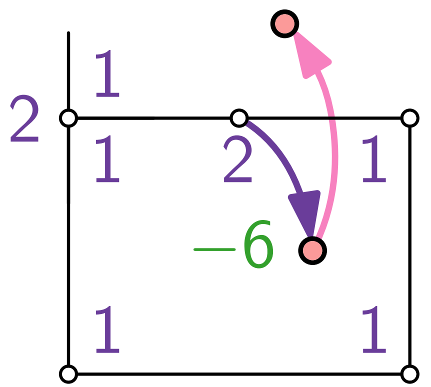

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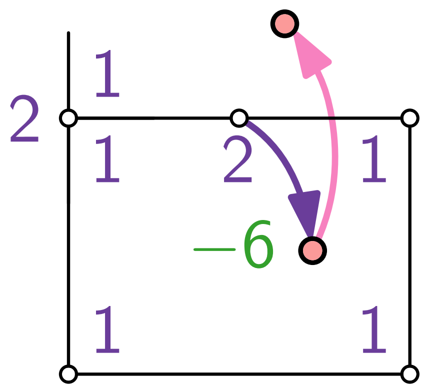
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

$\Rightarrow \sum_{w \in V(G) \cup F} b(w) = 0$ (Euler)



$\forall (v, f) \in E', v \in V(G), f \in F \qquad \ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$

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Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

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Define flow network $N(G) = ((V(G) \cup F, E'); b; \ell; u; \text{cost})$:

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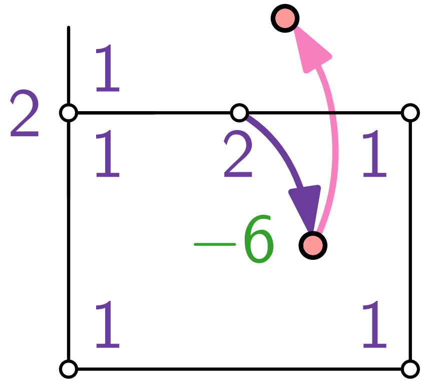

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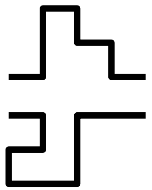

We model only the *number* of bends. Why is it enough?

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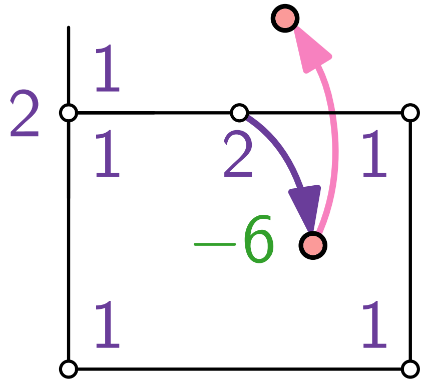

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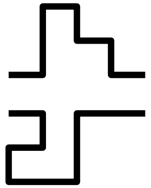

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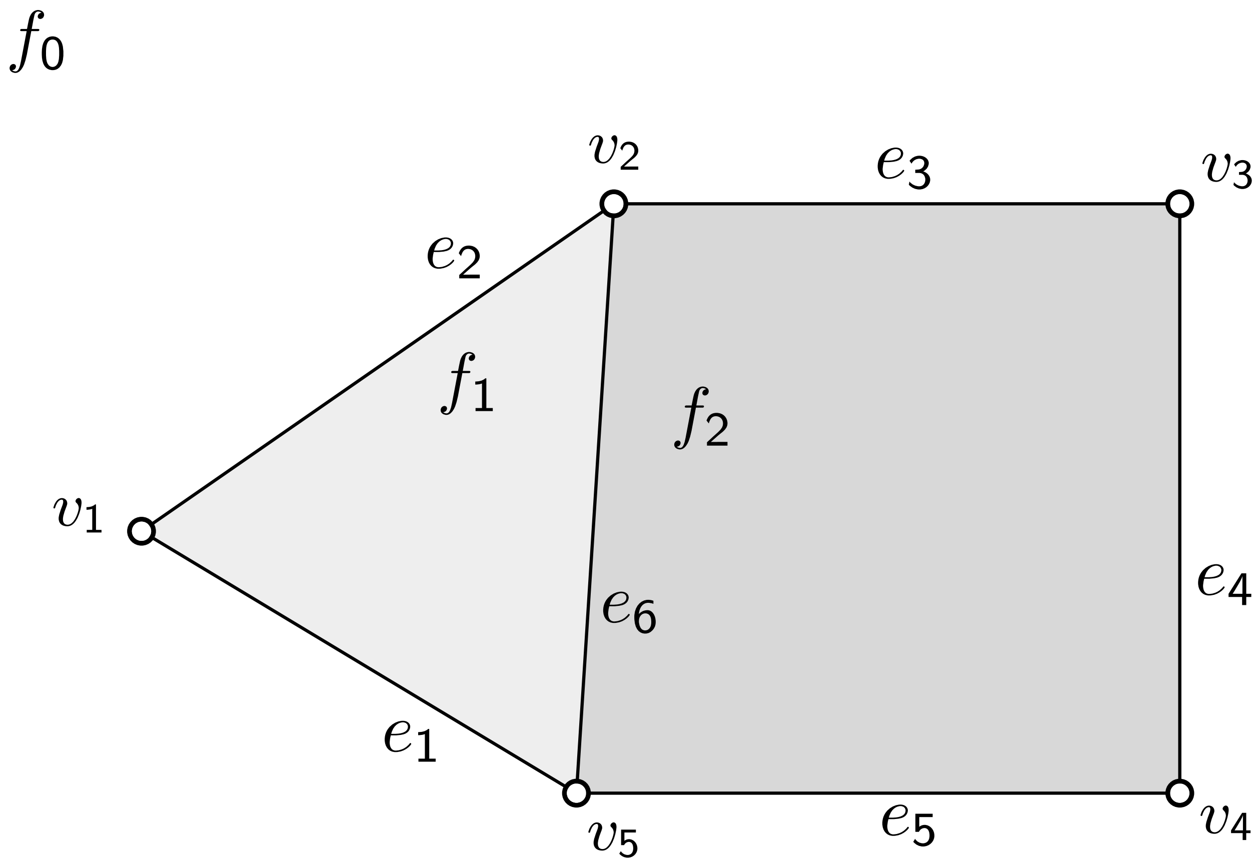
→ *Exercise!*

Flow Network Example



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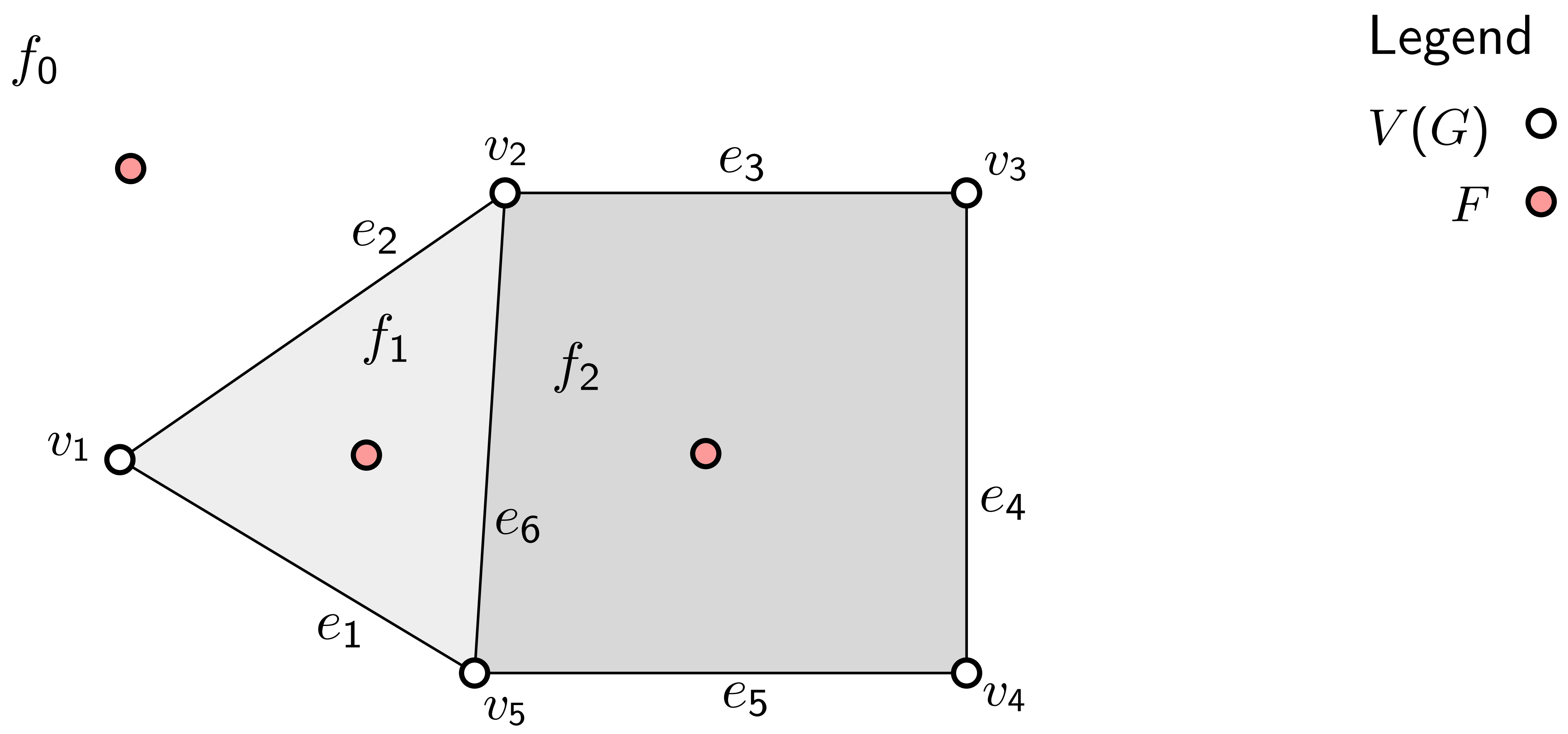

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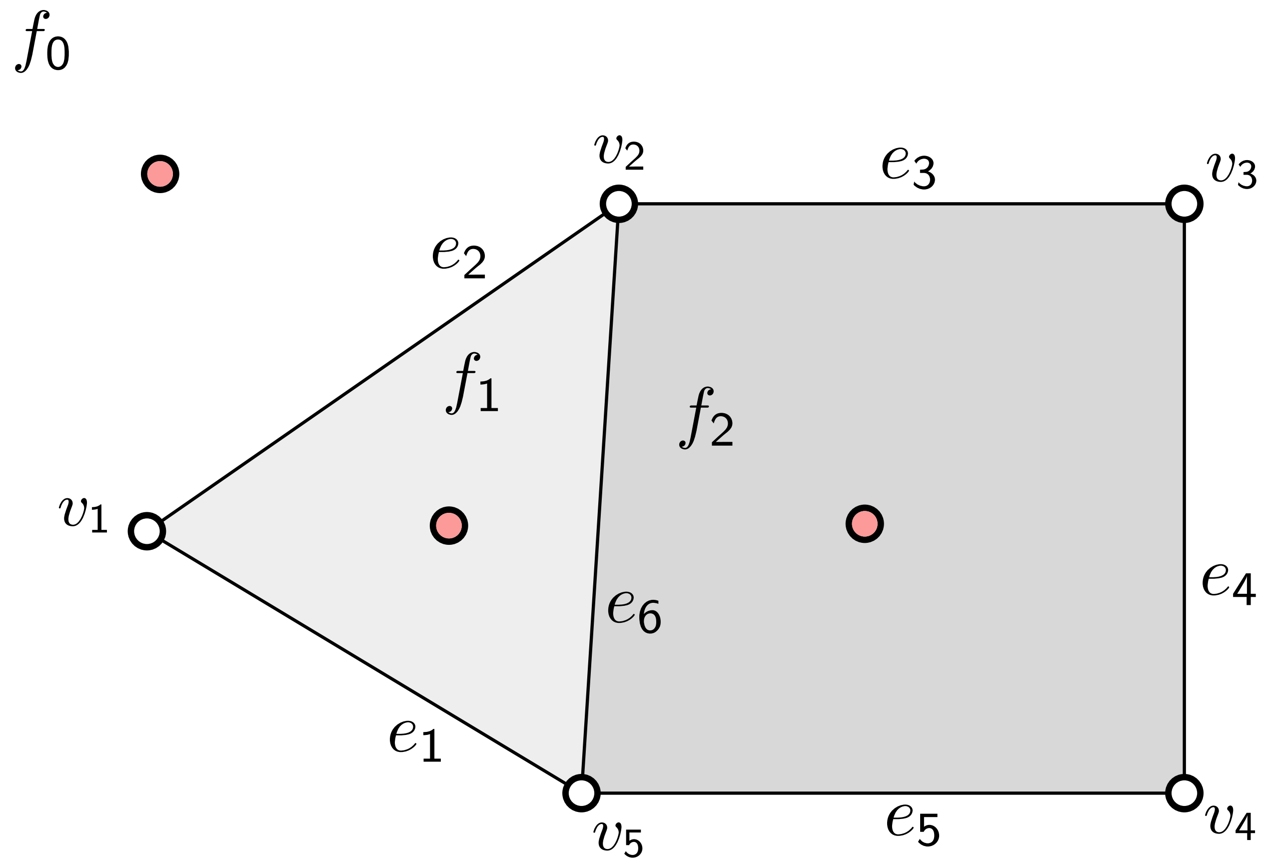

Legend

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F ●

$\ell/u/$cost

$V(G) \times F \supseteq$ 1/4/0 →

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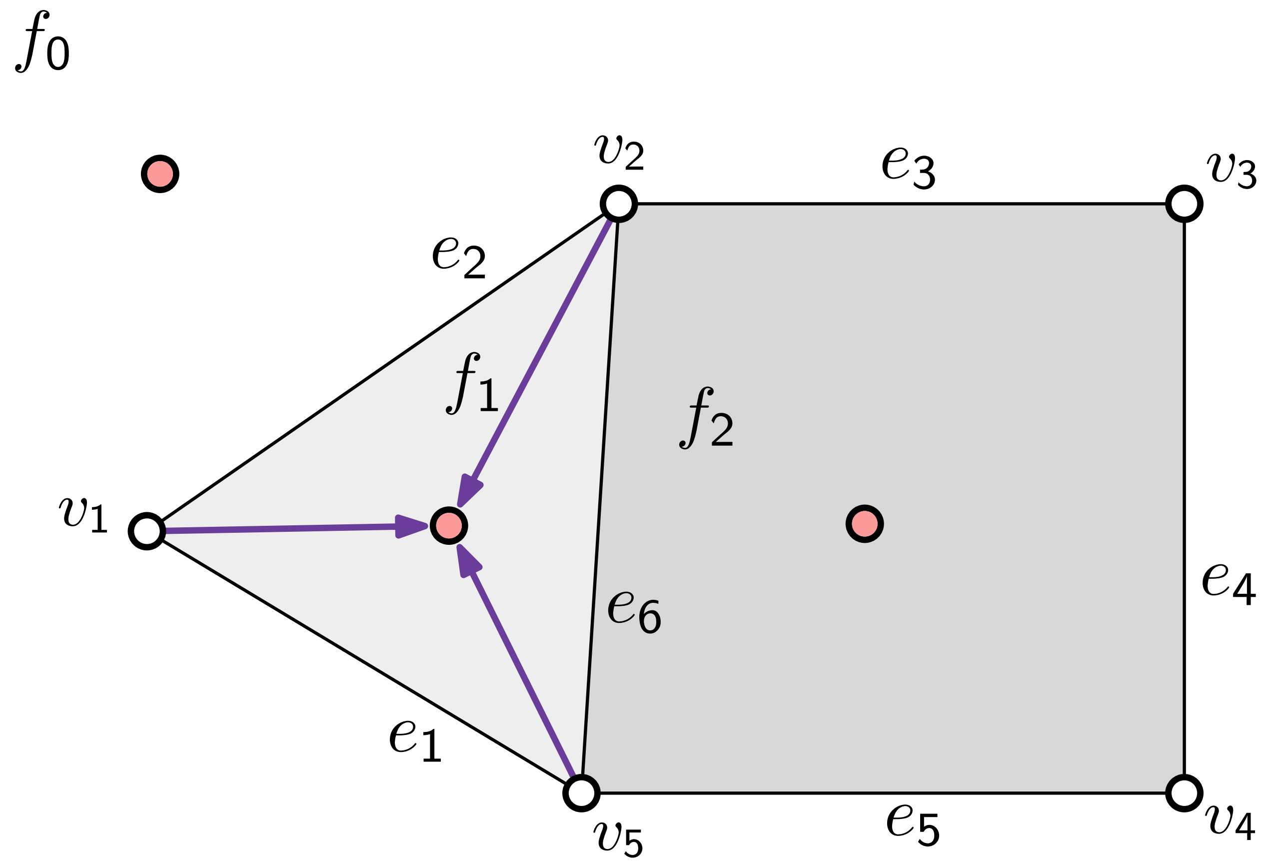

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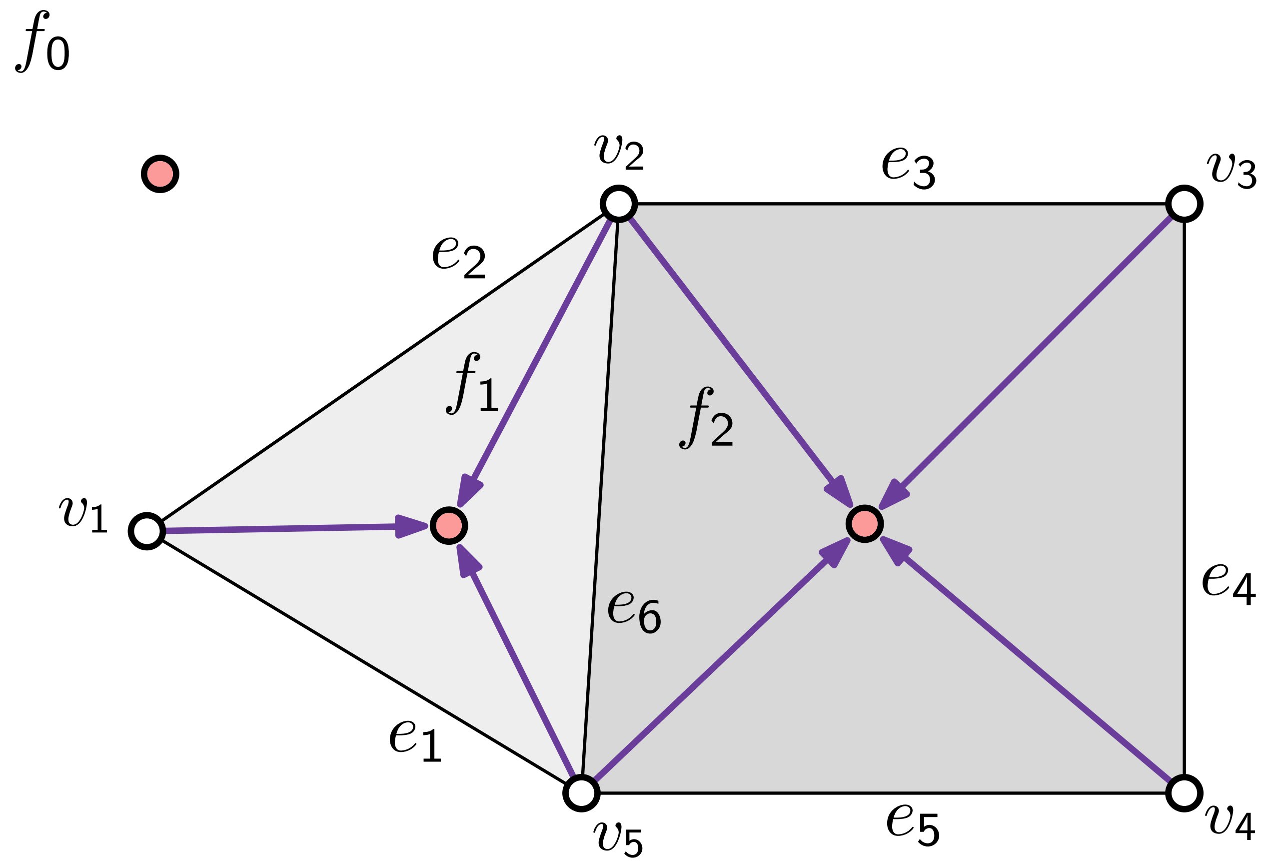

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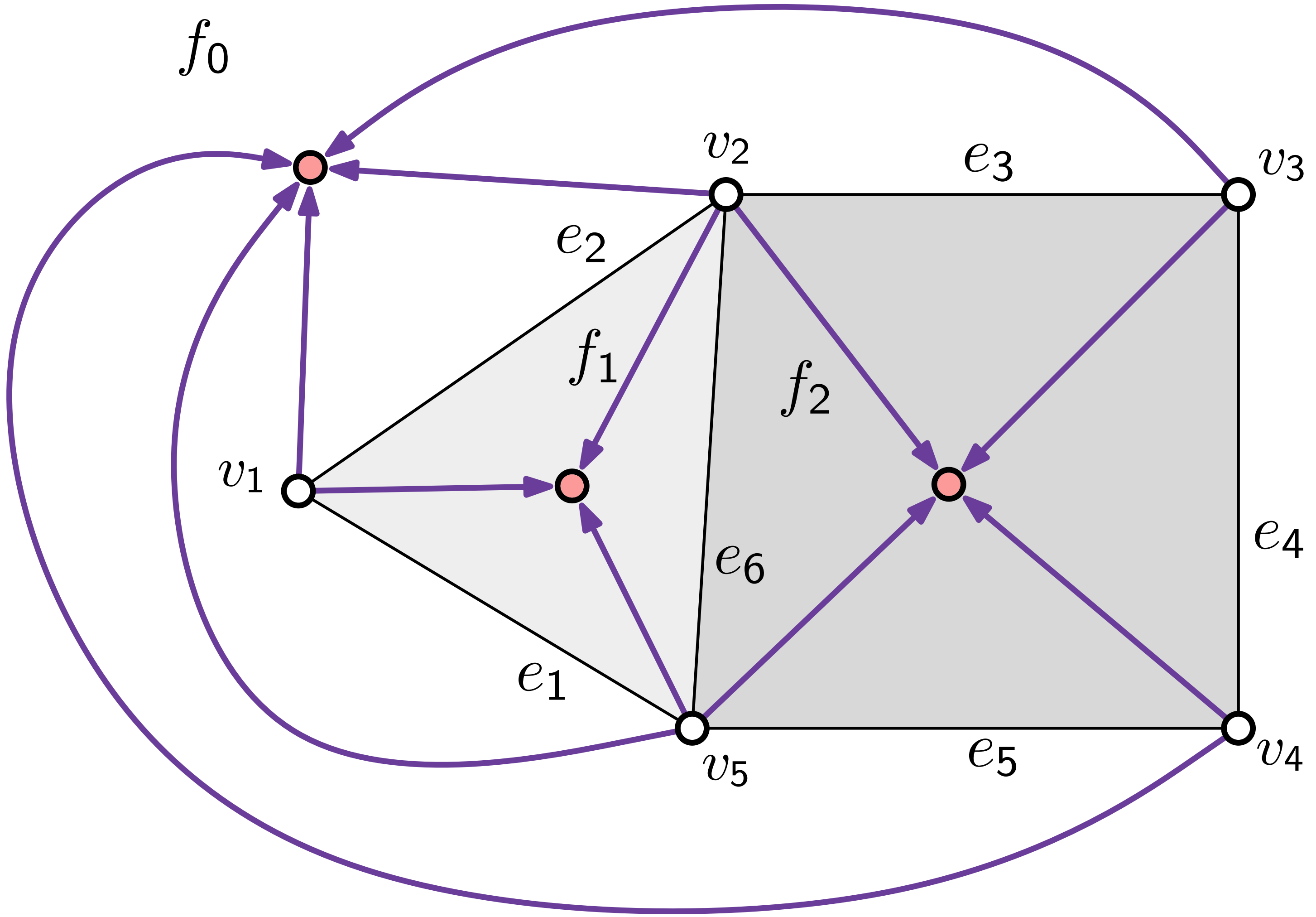

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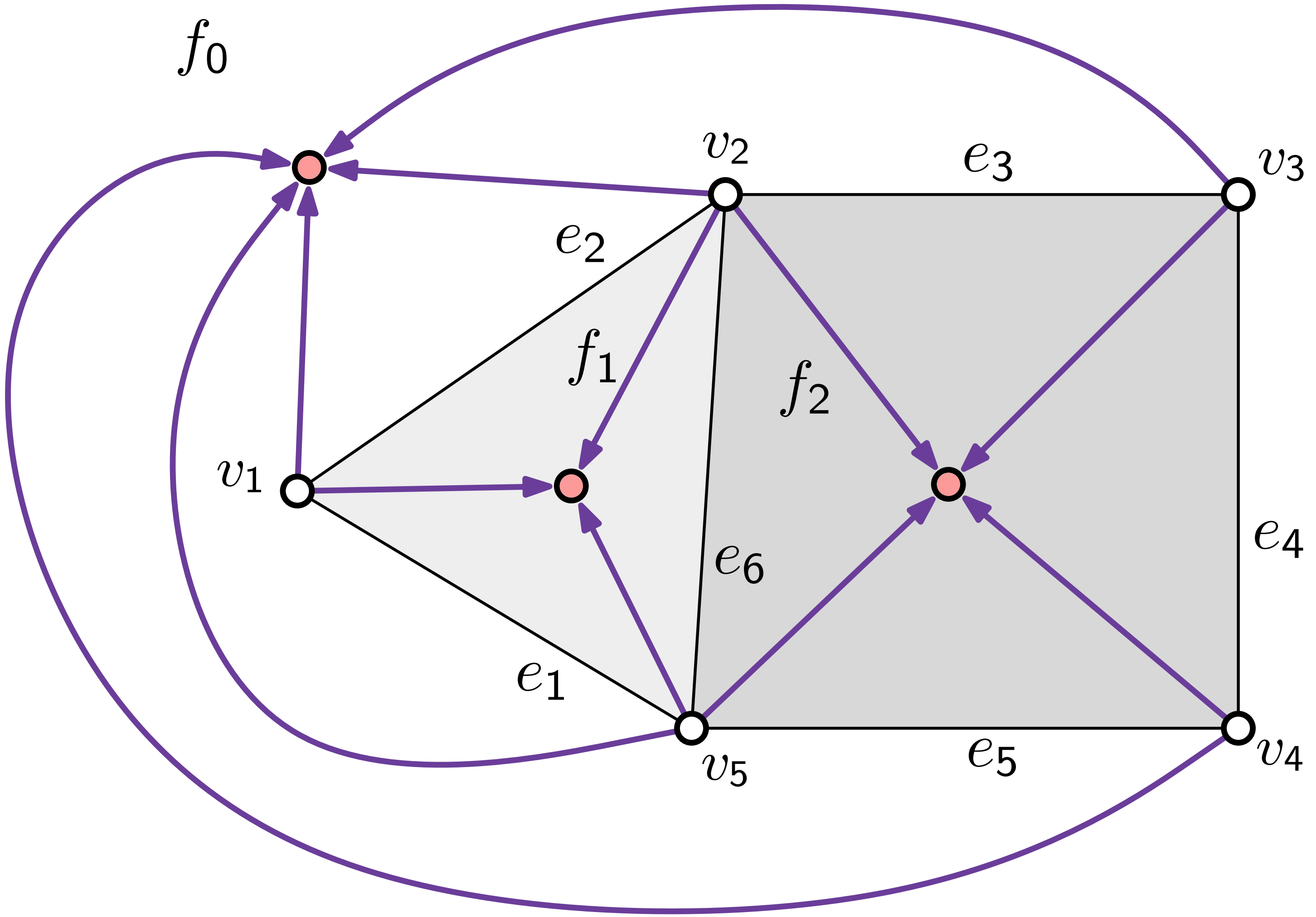

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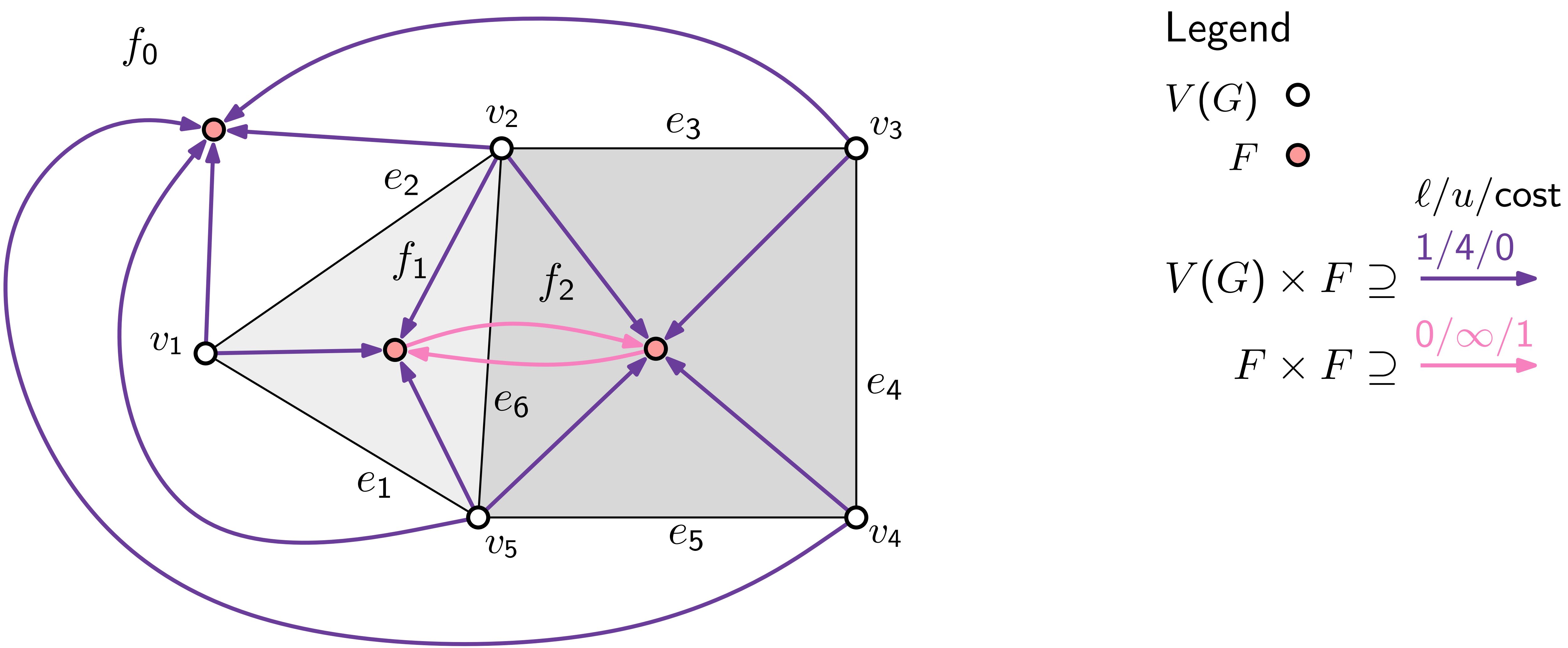

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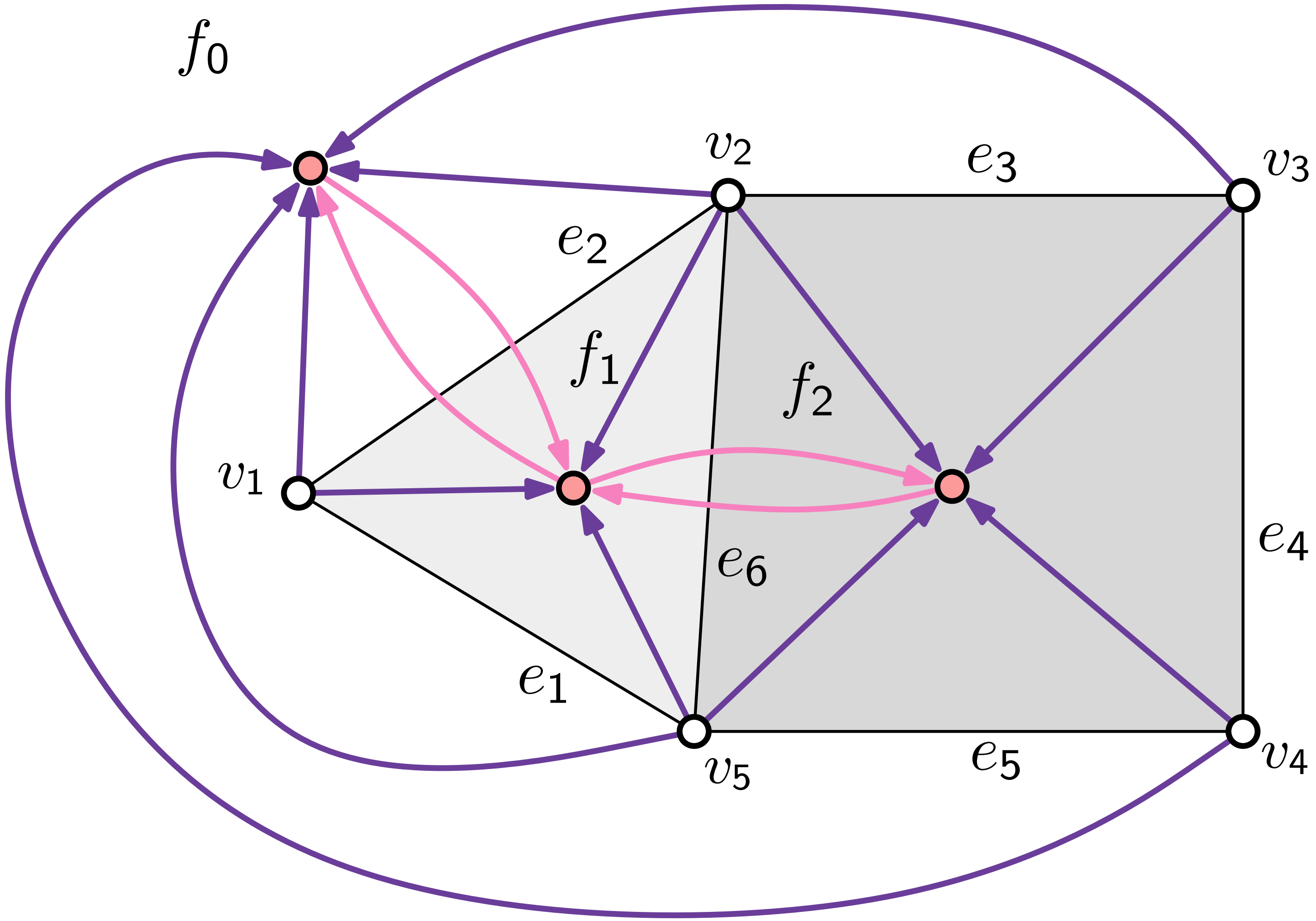

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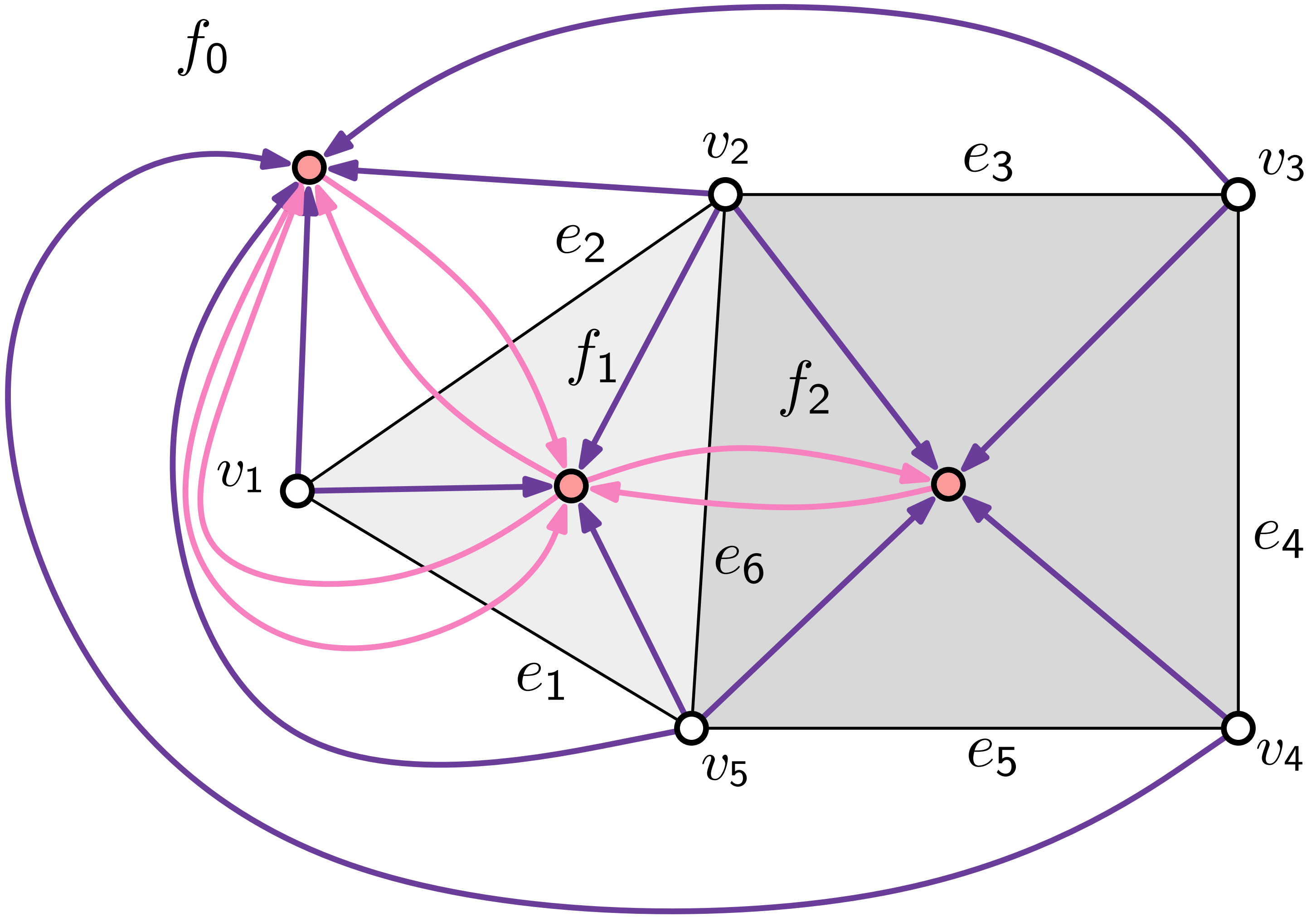

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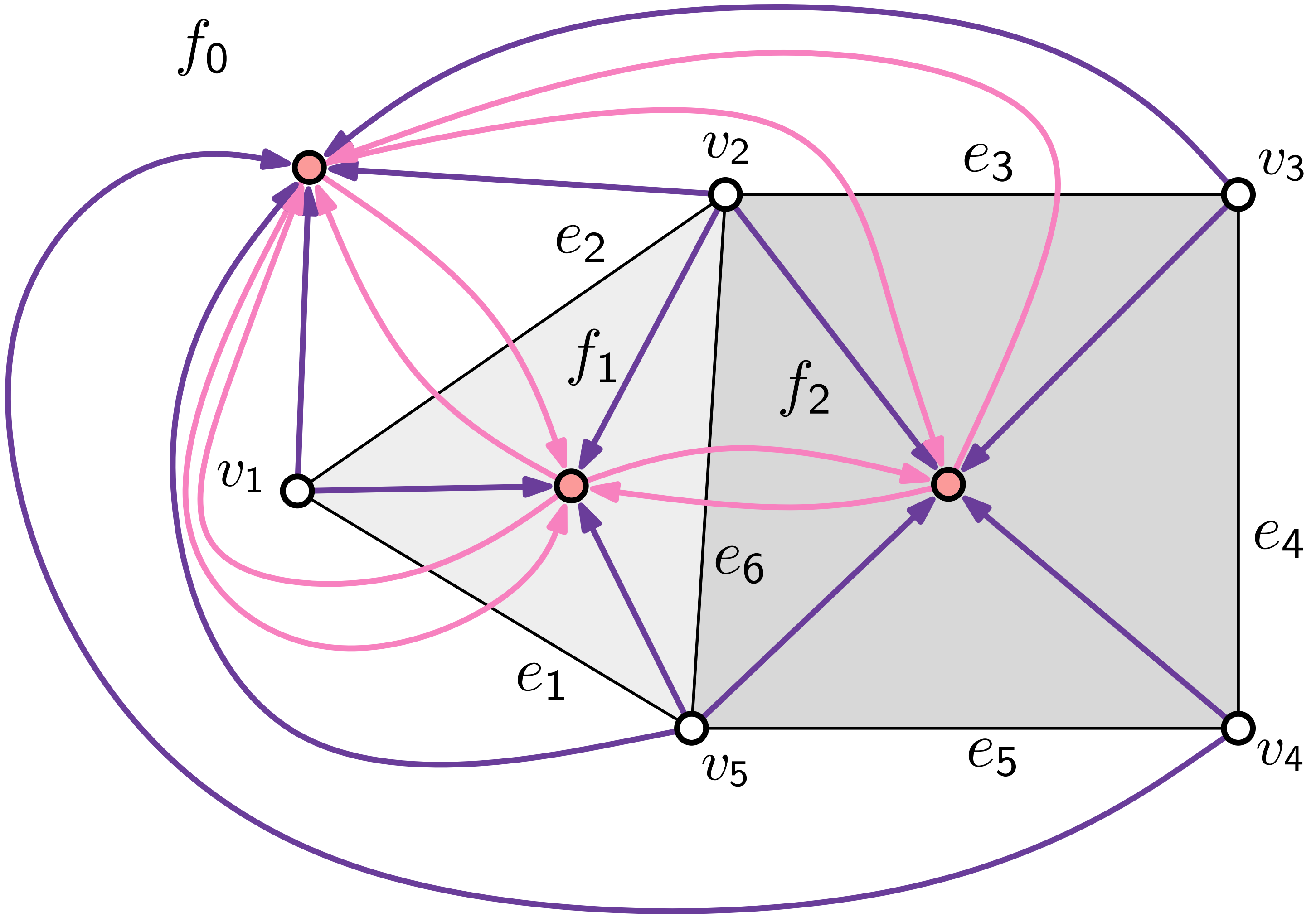

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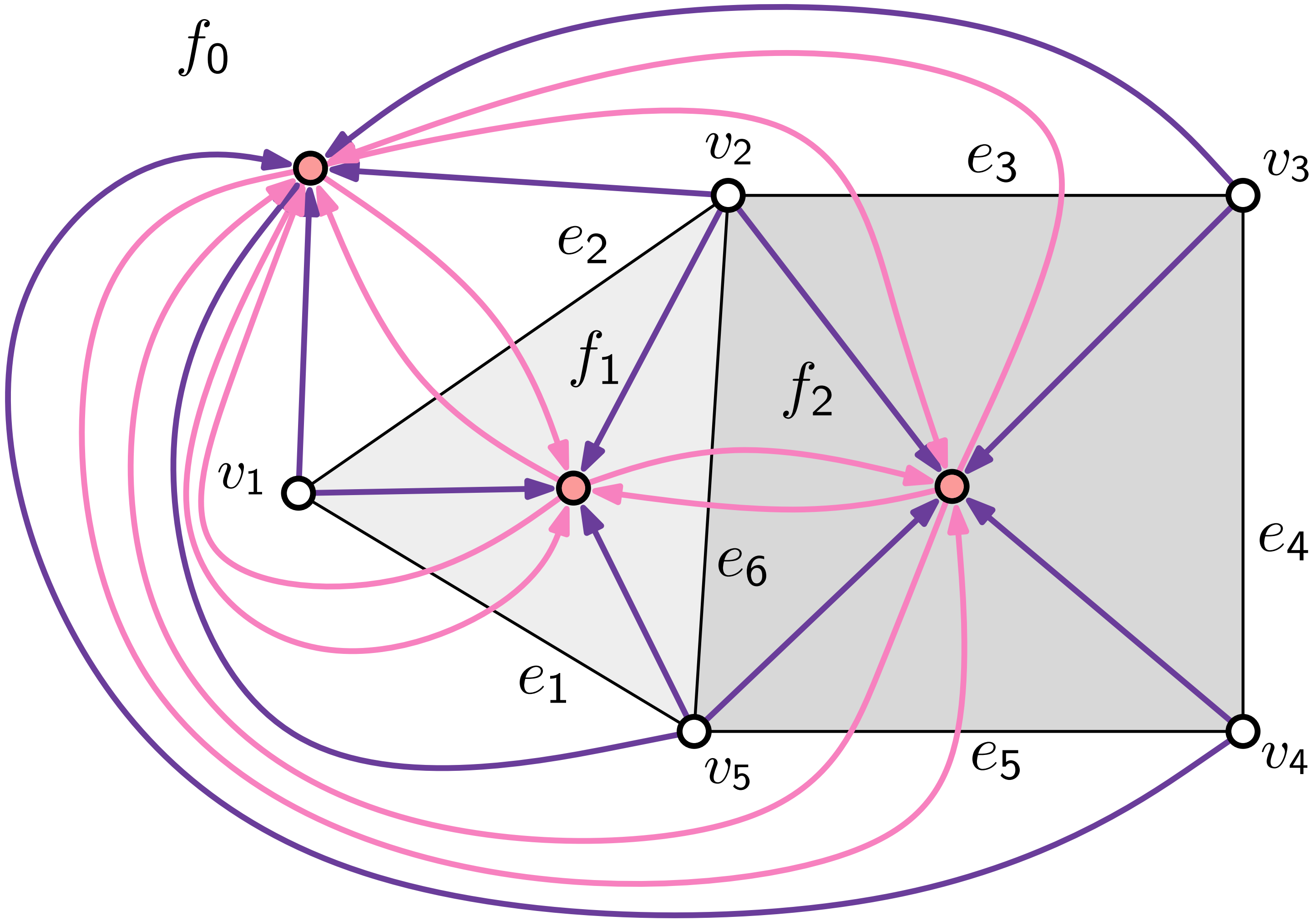

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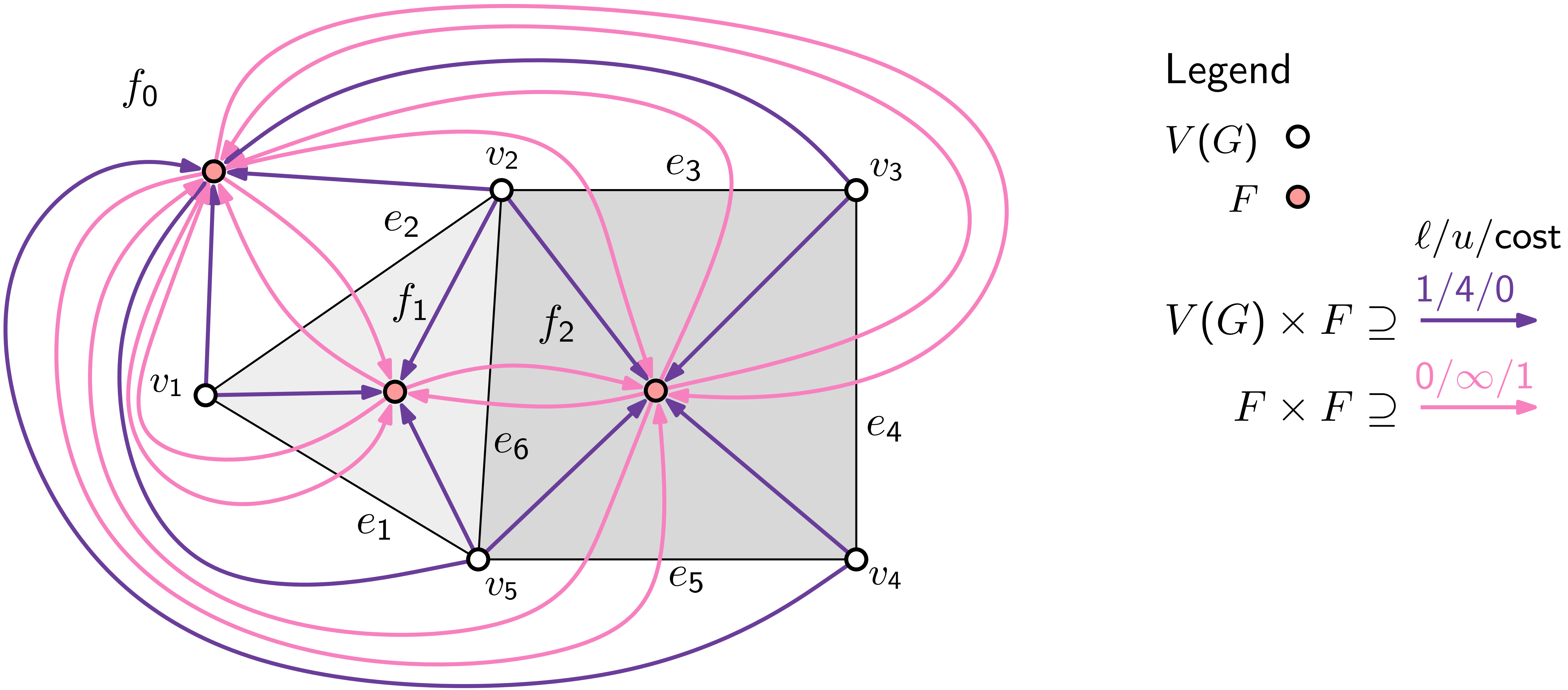

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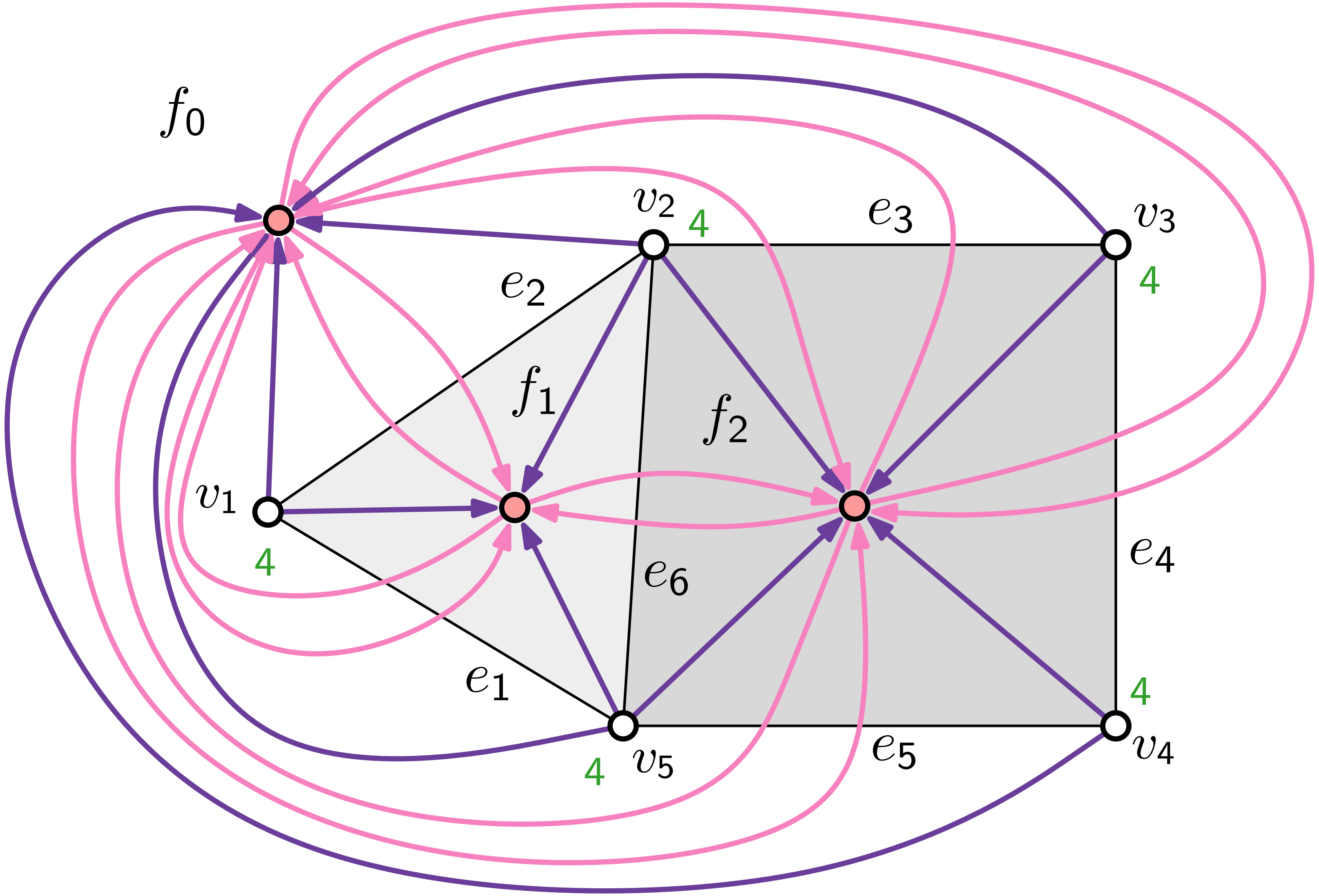

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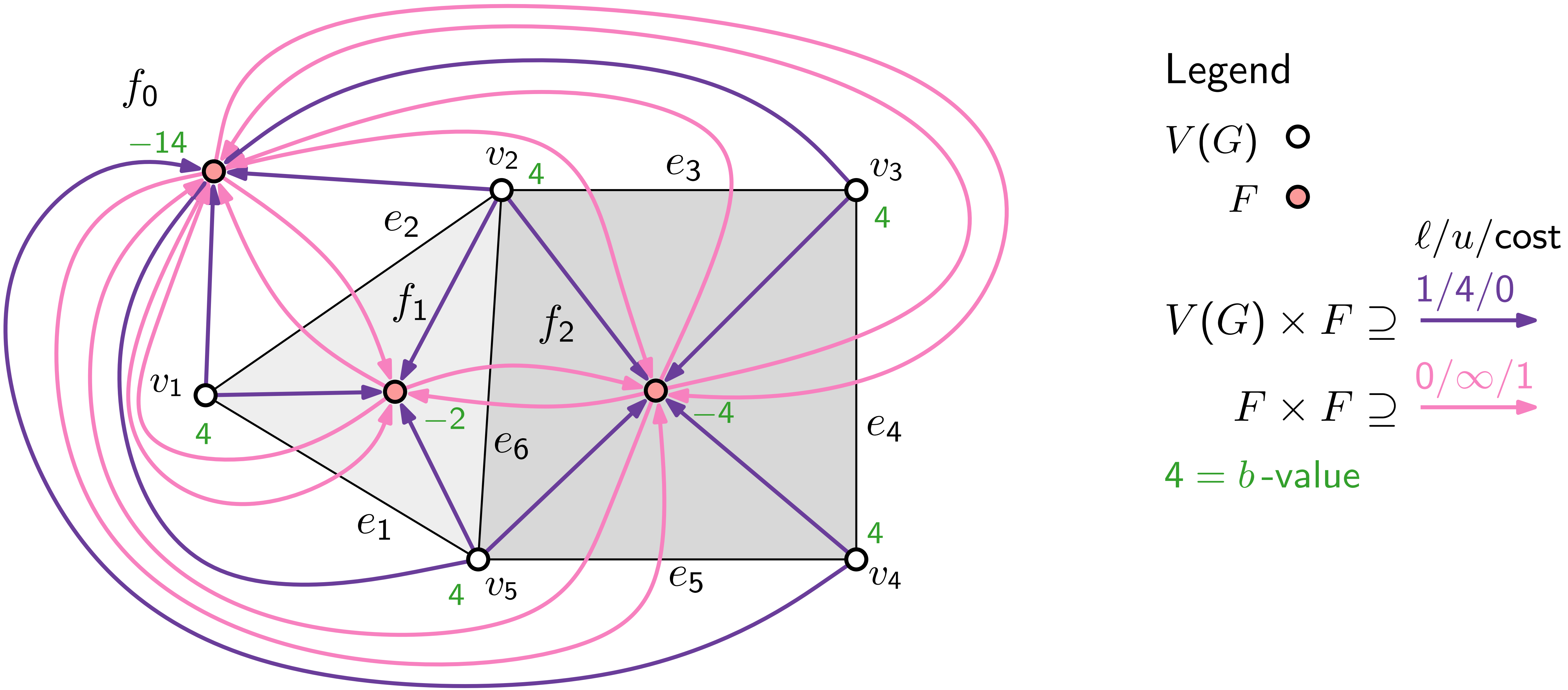
Flow Network Example



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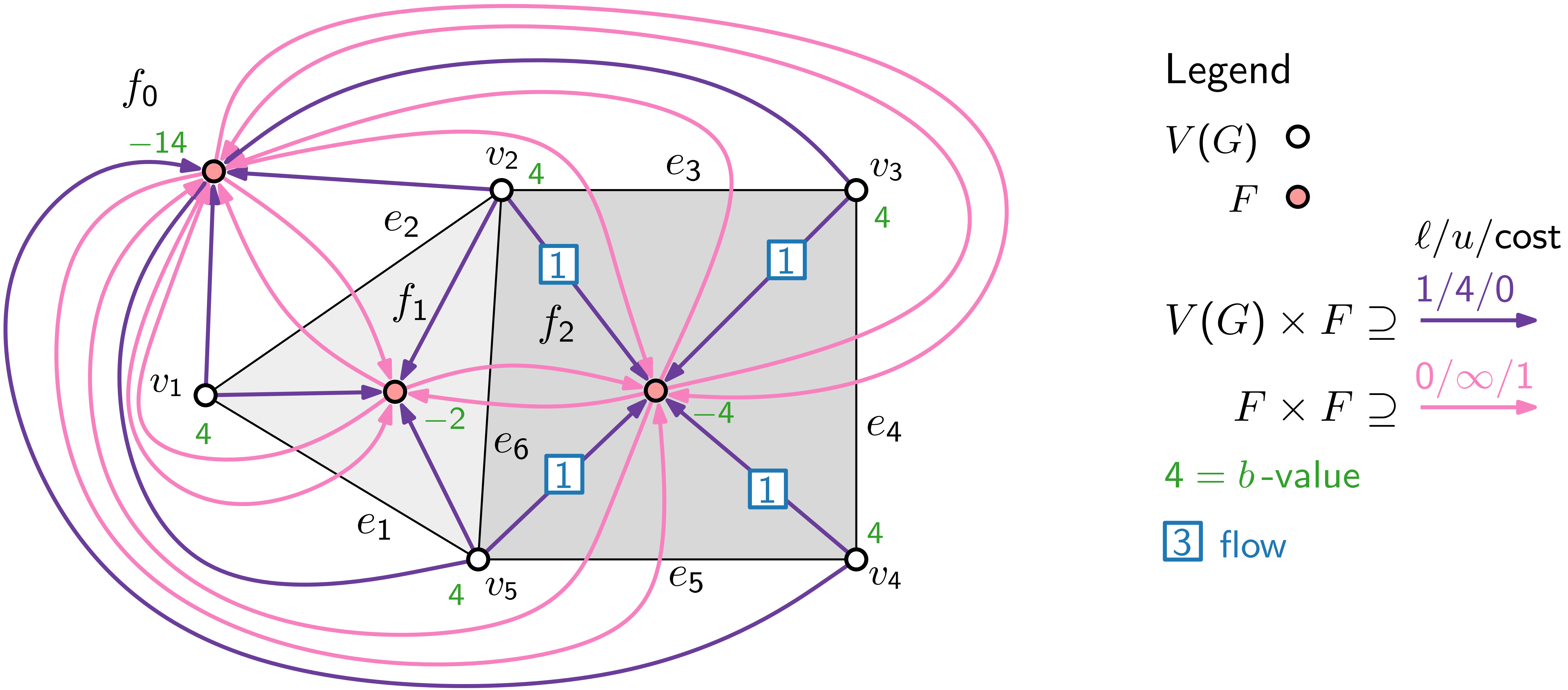

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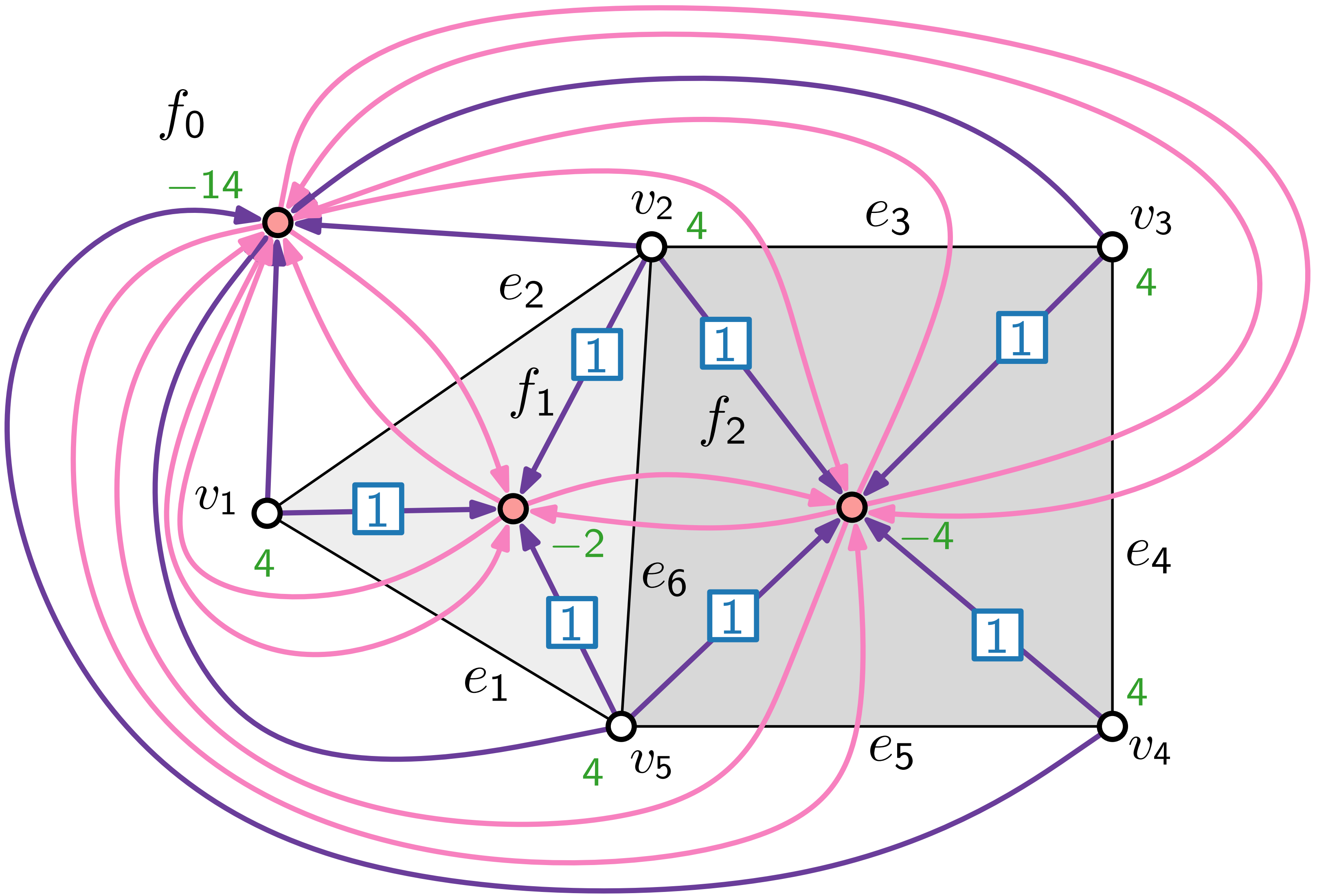

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3 flow

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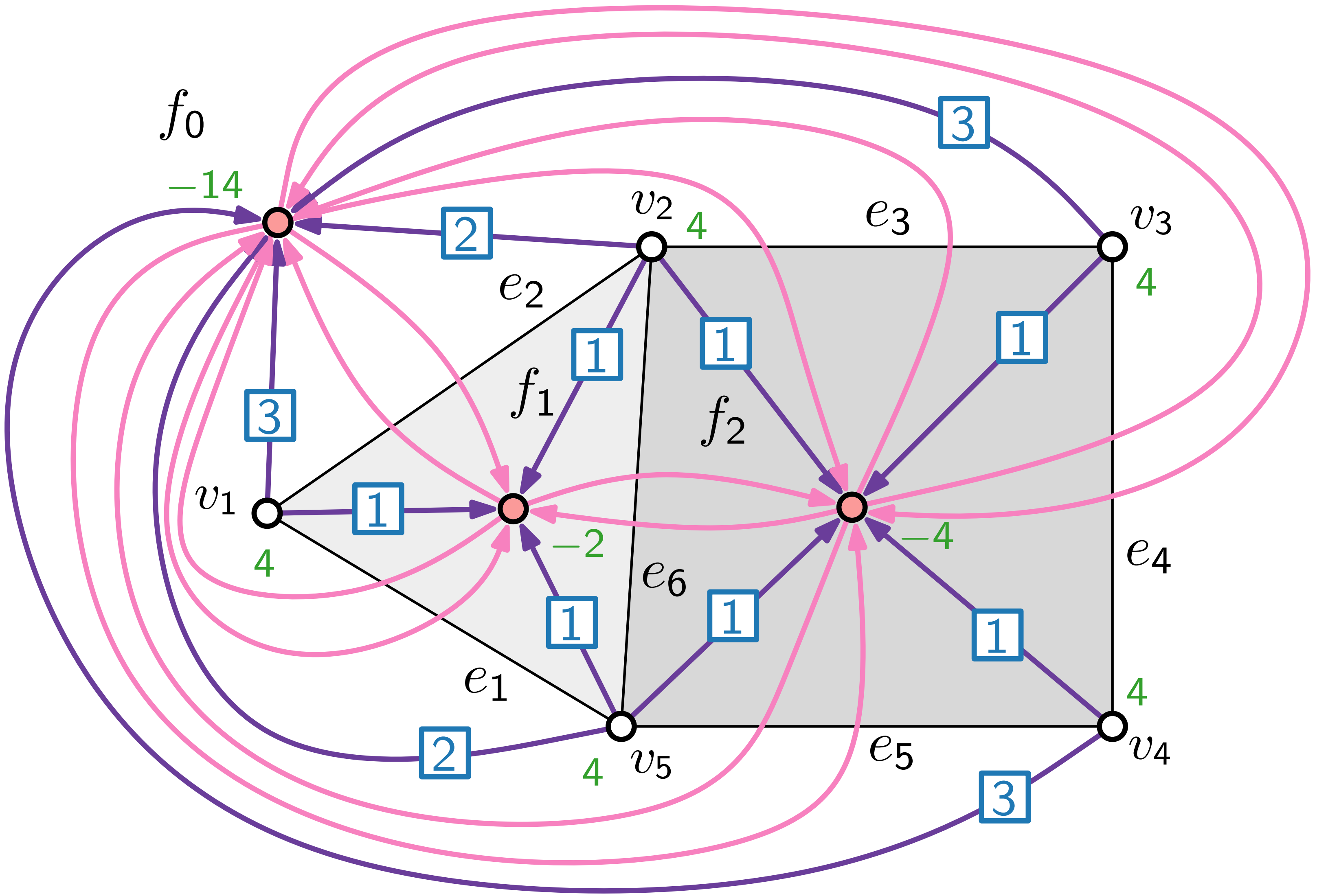

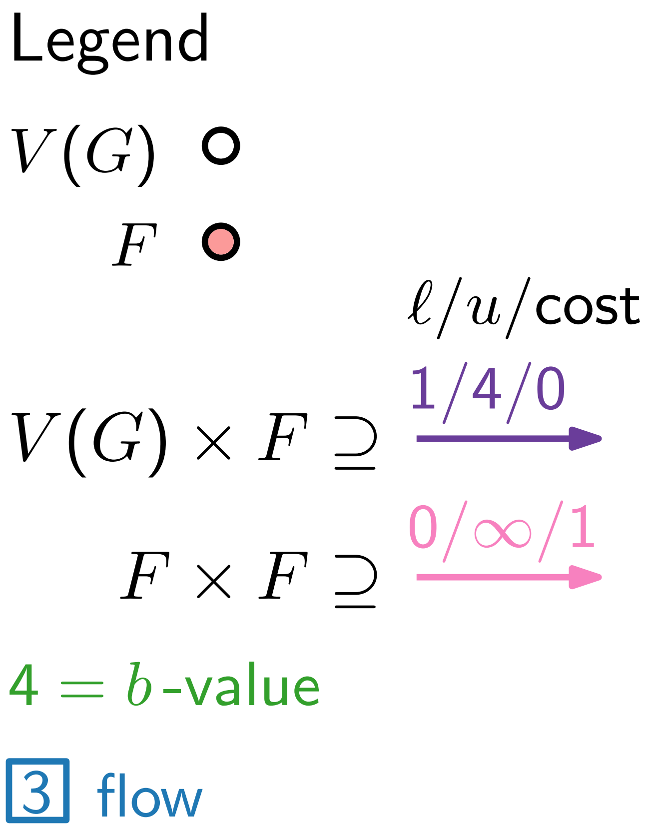

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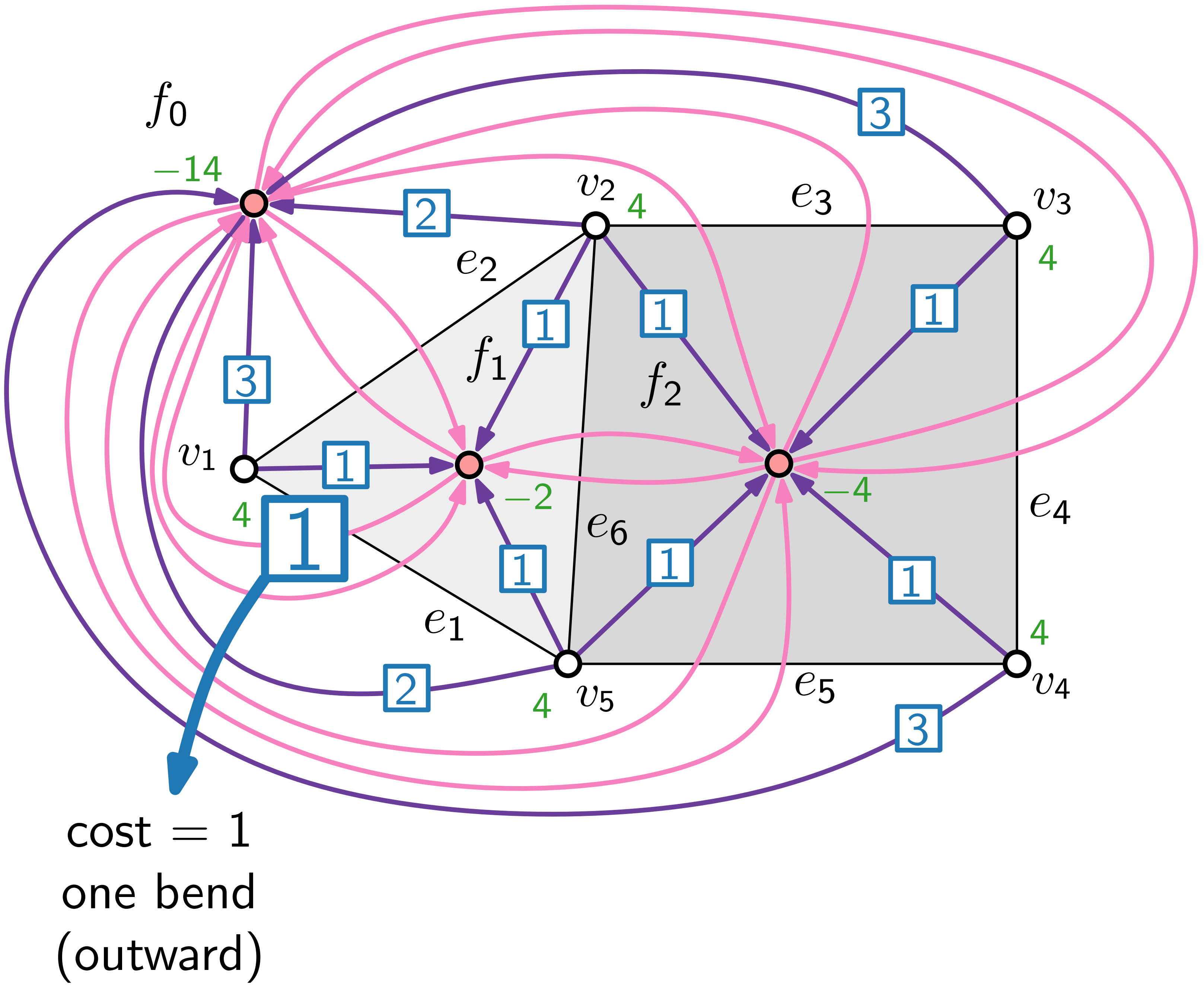

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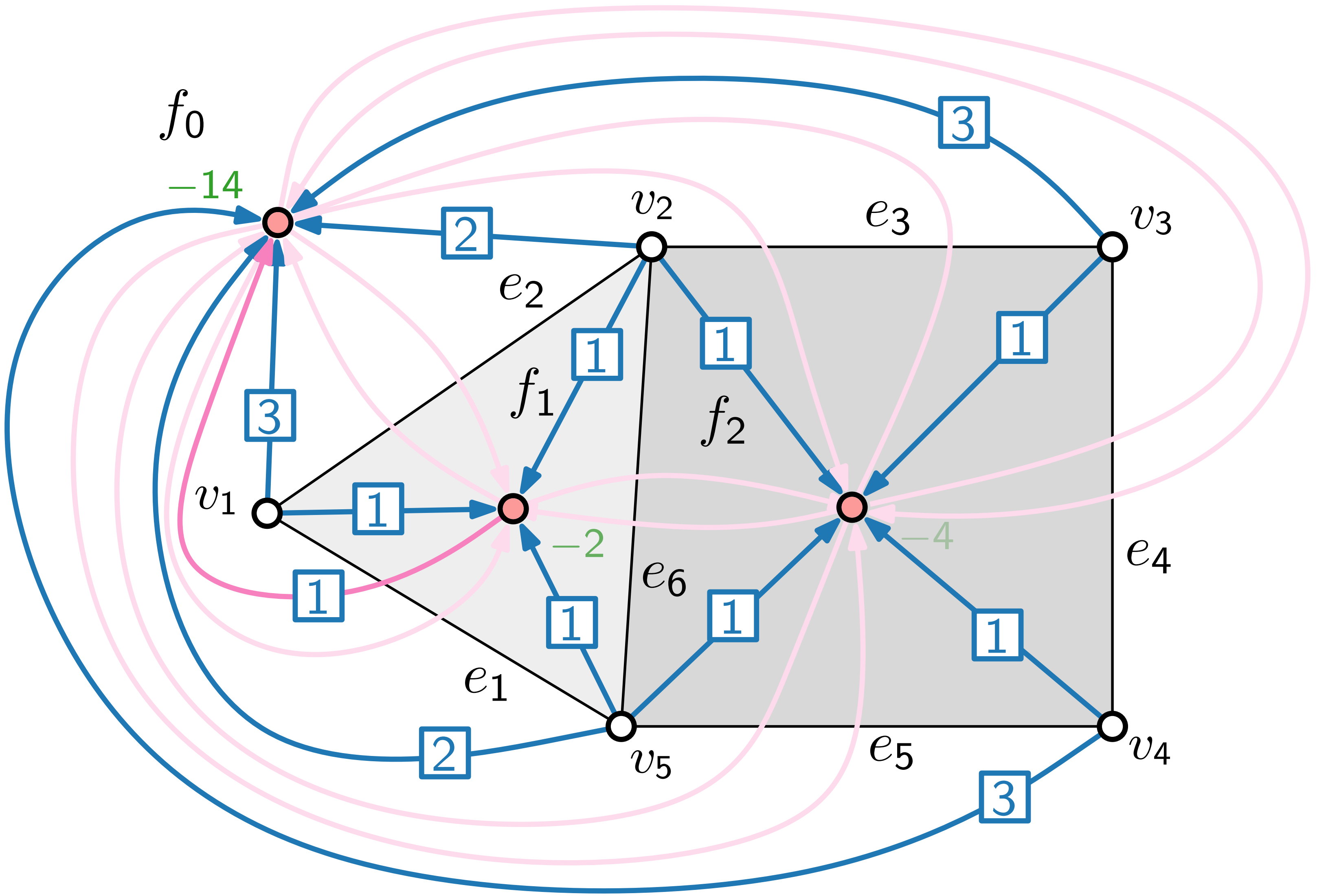

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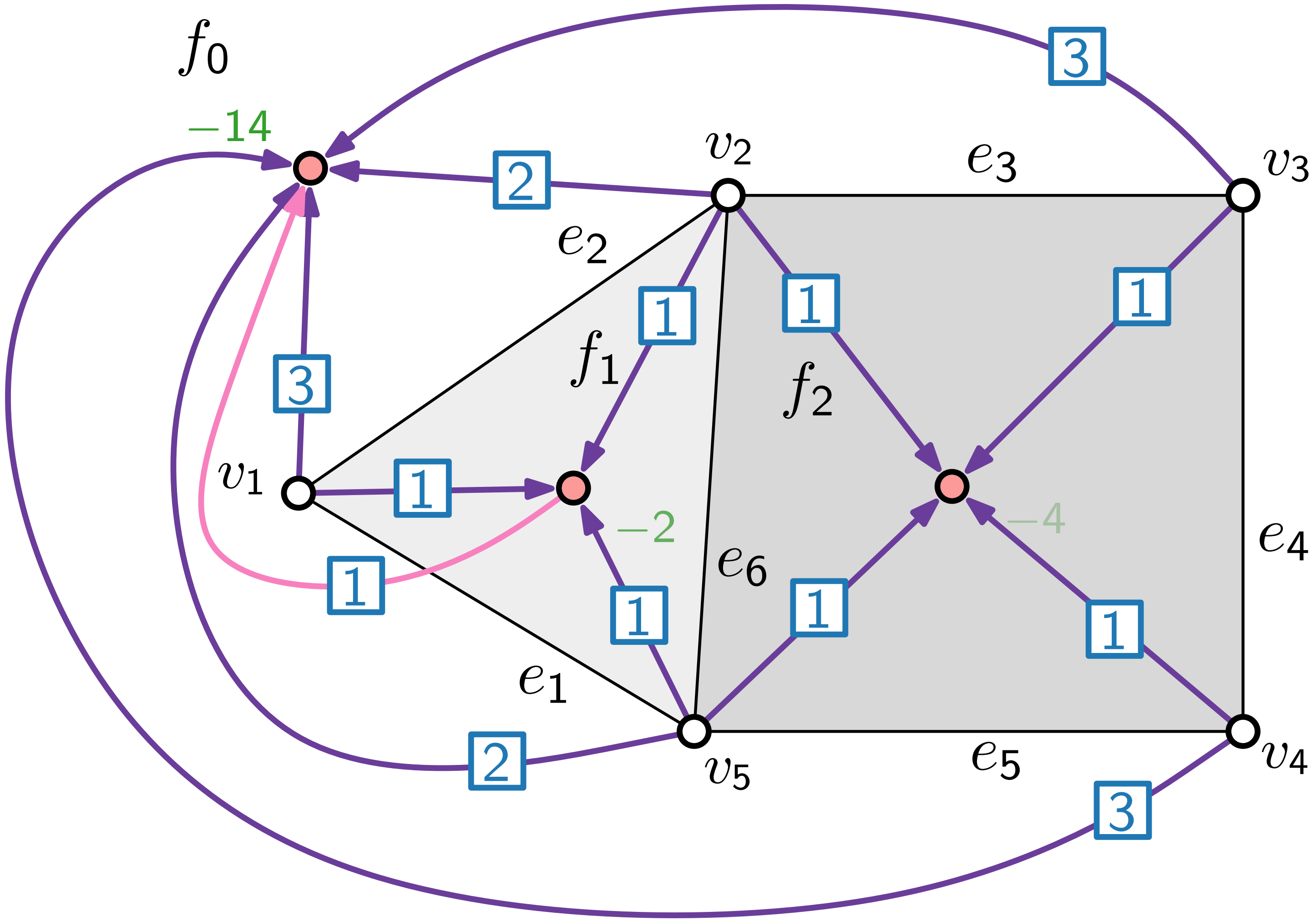

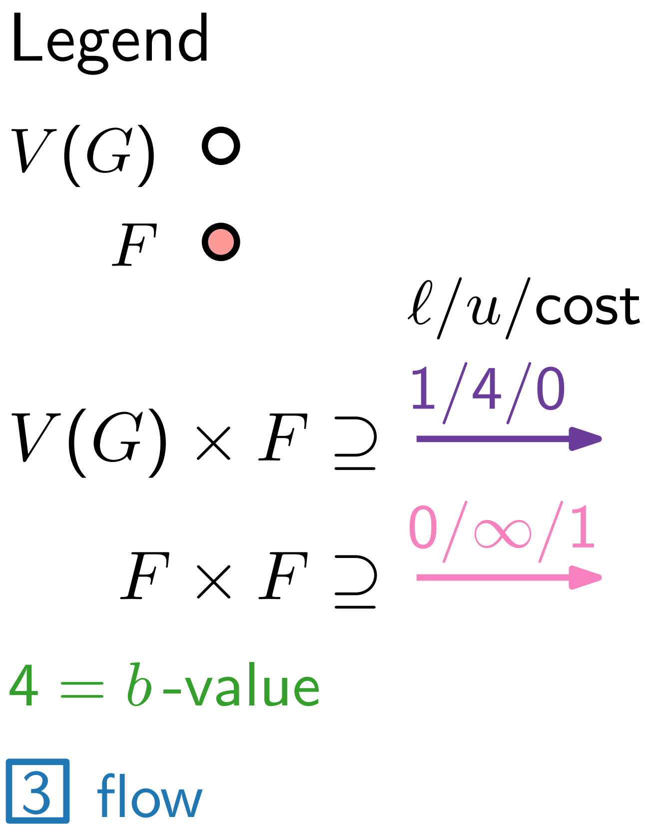

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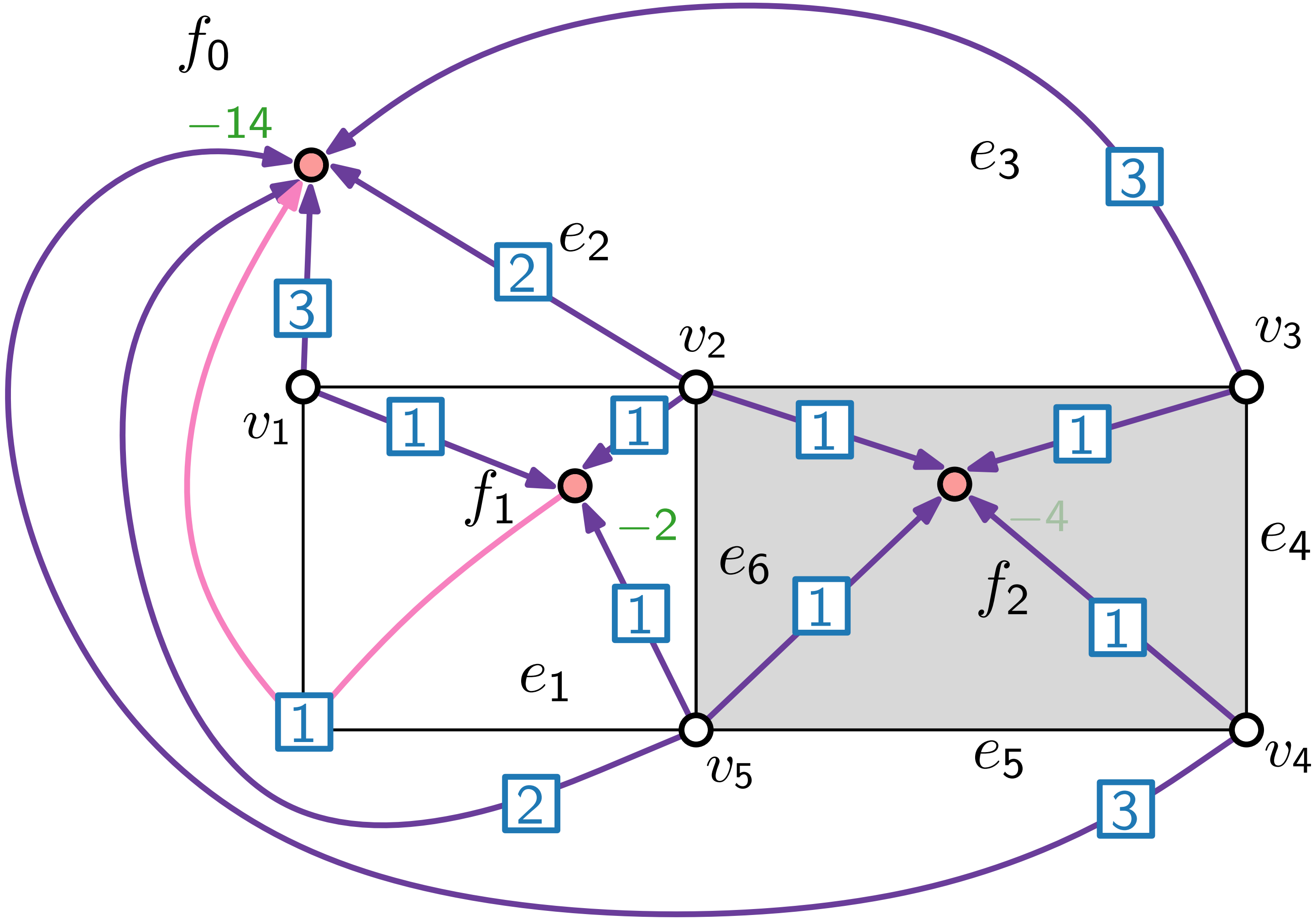

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Bend Minimization – Result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends. $\Leftrightarrow$
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(H3) Angle sum of $f = \pm 4$ ✓ $\rightarrow$ *Exercise.*

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"$\Rightarrow$": Given an orthogonal representation $H(G)$ with k bends, construct a valid flow X in $N(G)$ of cost k.

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- Show that X is a valid flow and has cost k.

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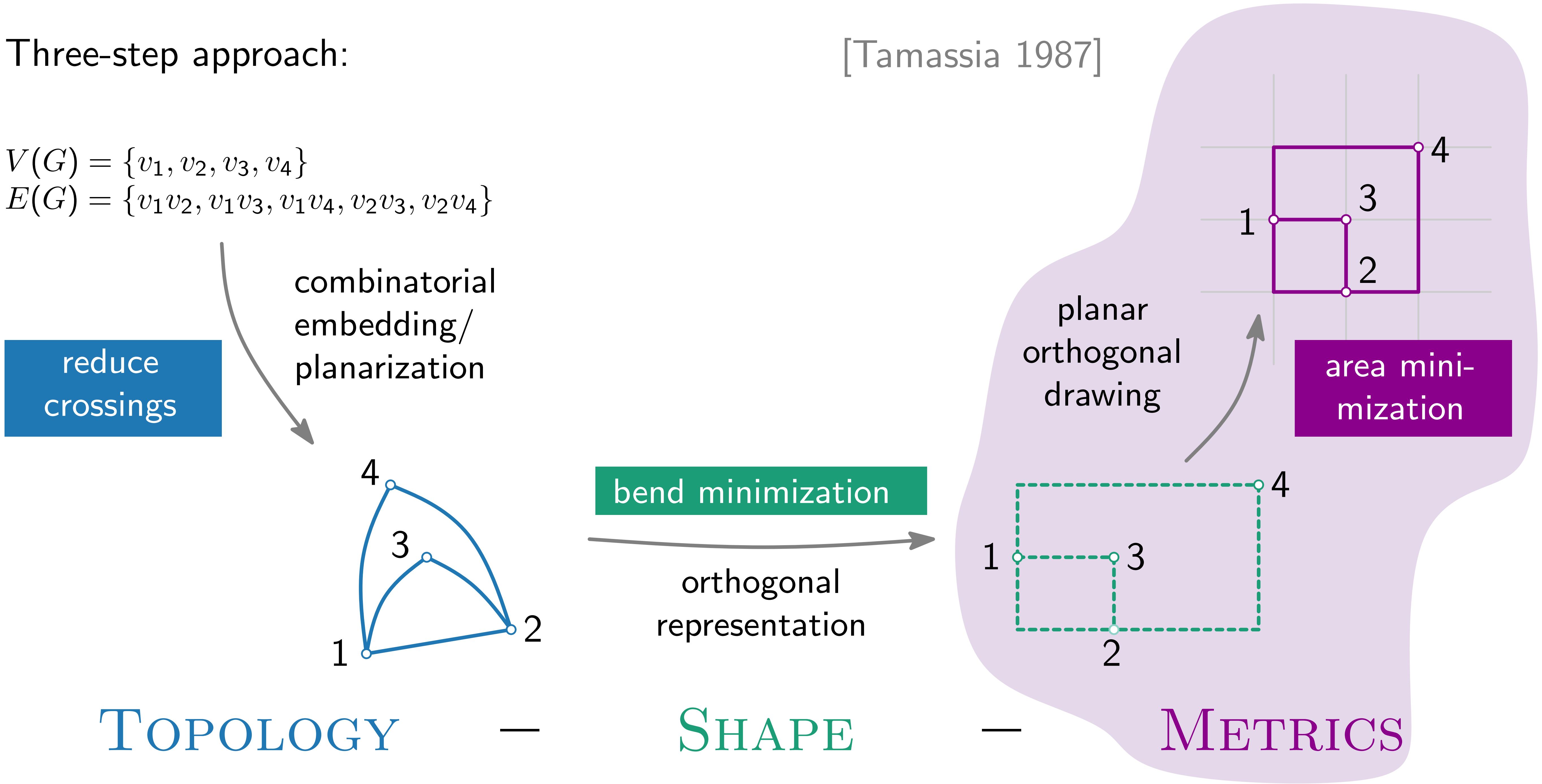
Theorem. [Garg & Tamassia 2001]
Bend minimization without given combinatorial embedding is NP-hard.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V(G) = \{v_1, v_2, v_3, v_4\}$
$E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



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Compaction problem.

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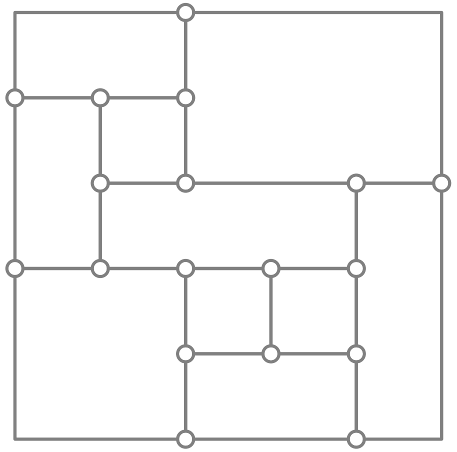
Idea.

- Formulate flow network for horizontal/vertical compaction

Flow Network for Edge-Length Assignment



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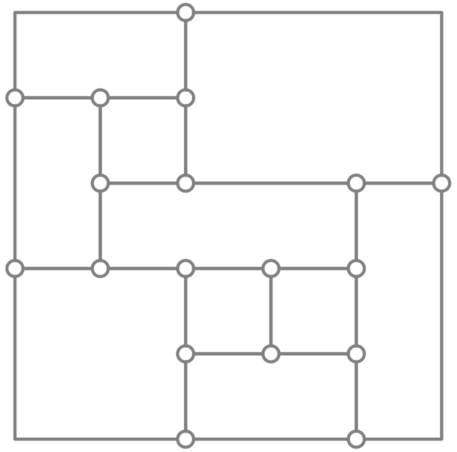
Definition.

Flow Network $N_{\mathrm{hor}} = ((W_{\mathrm{hor}}, E_{\mathrm{hor}}); b; \ell; u; \mathrm{cost})$



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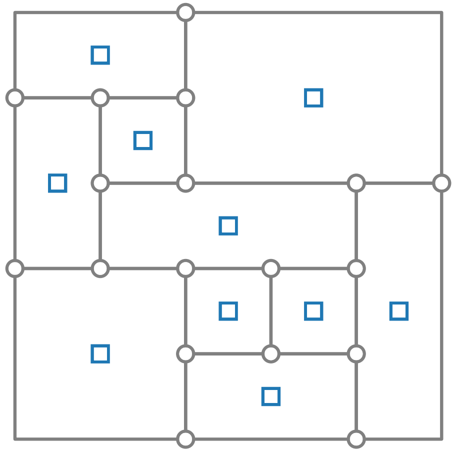

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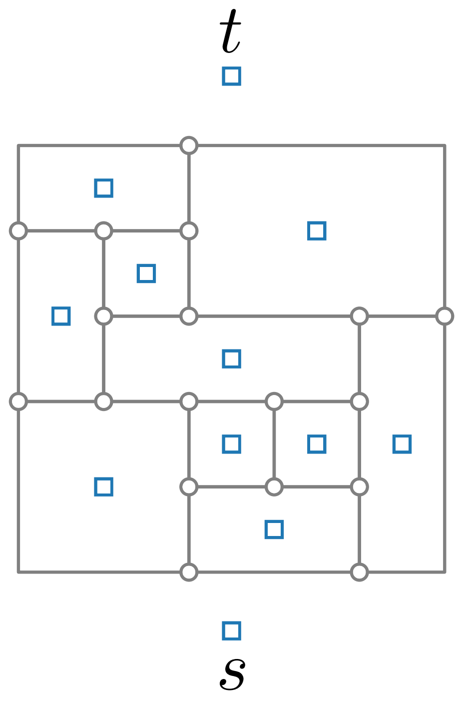

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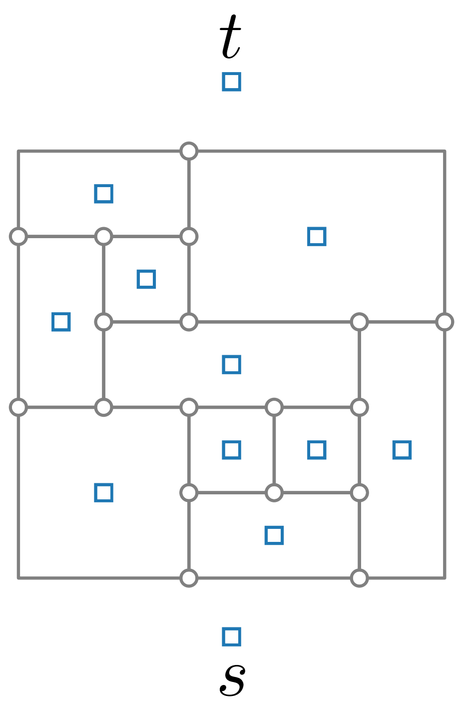

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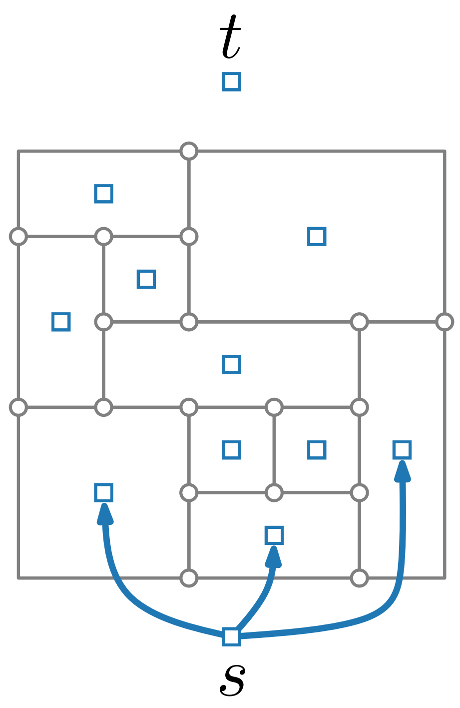

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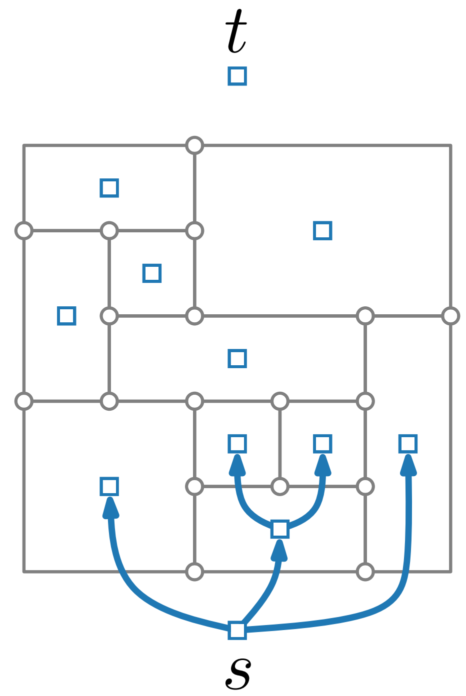

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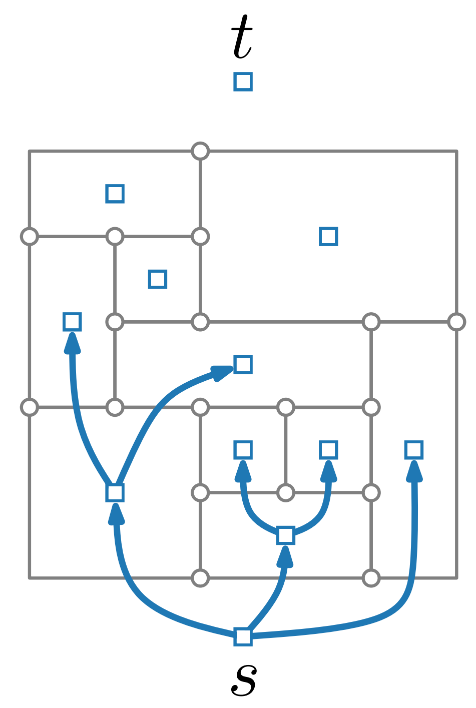

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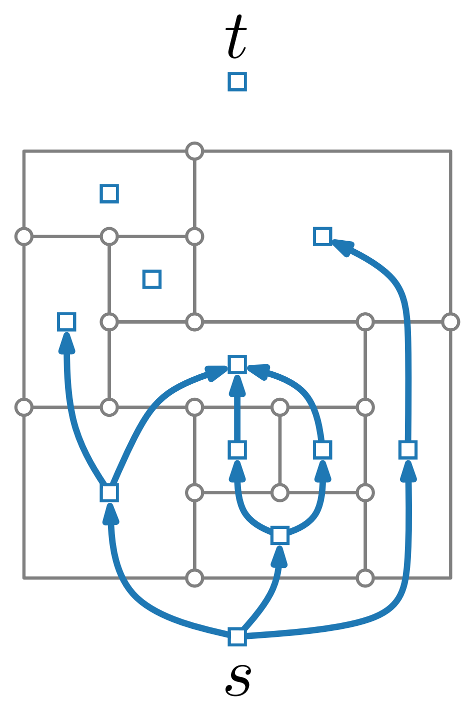

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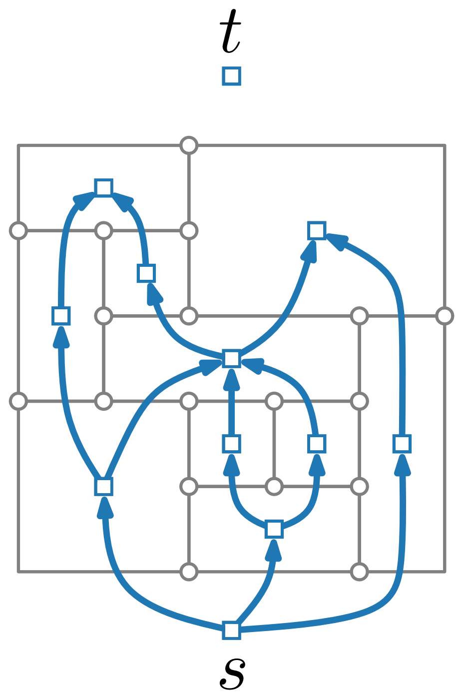

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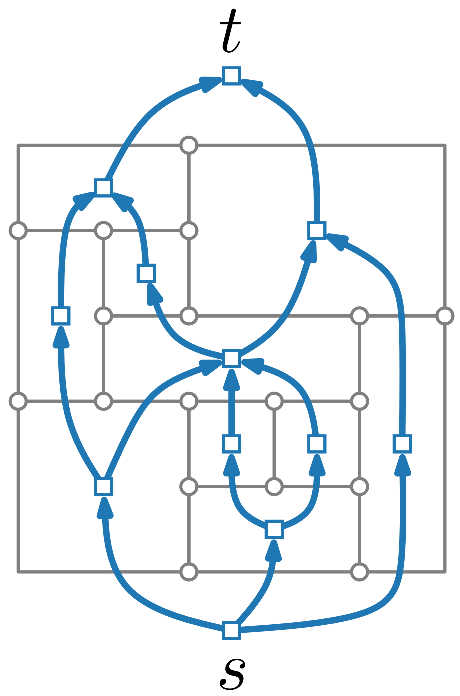

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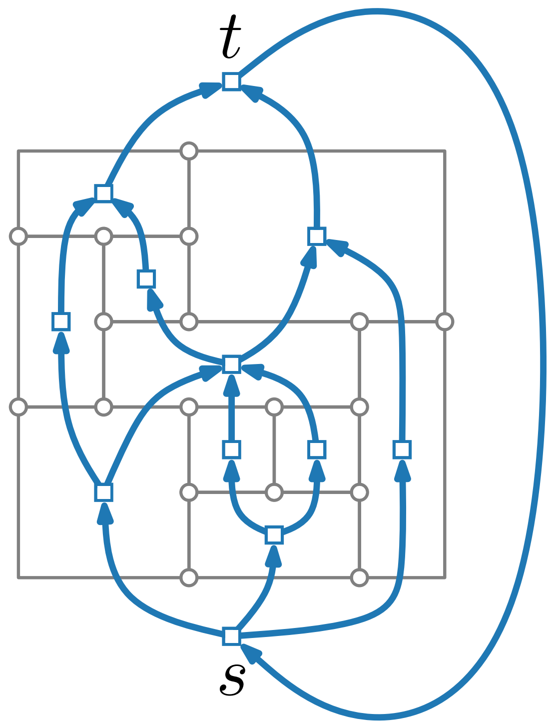

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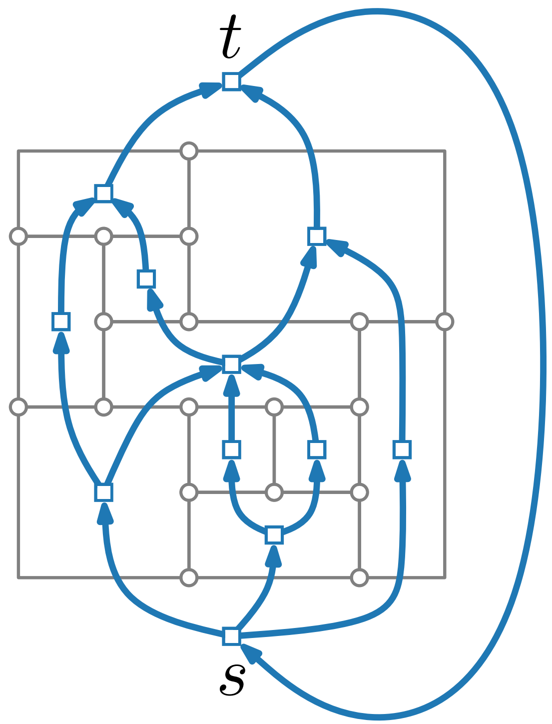

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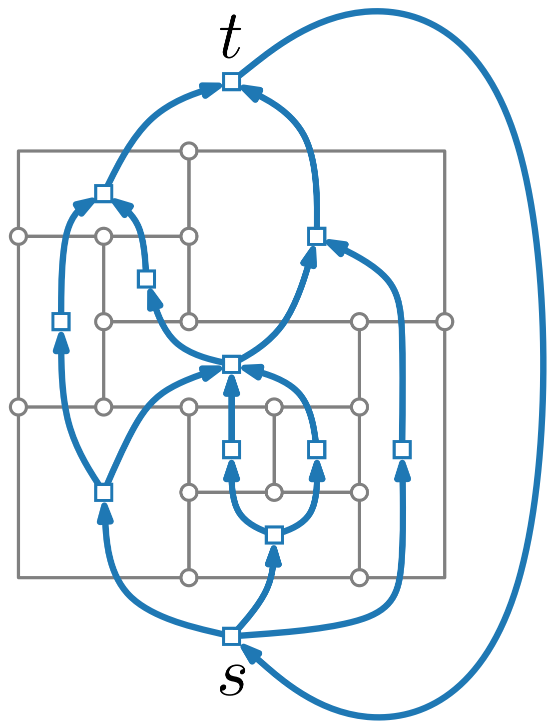

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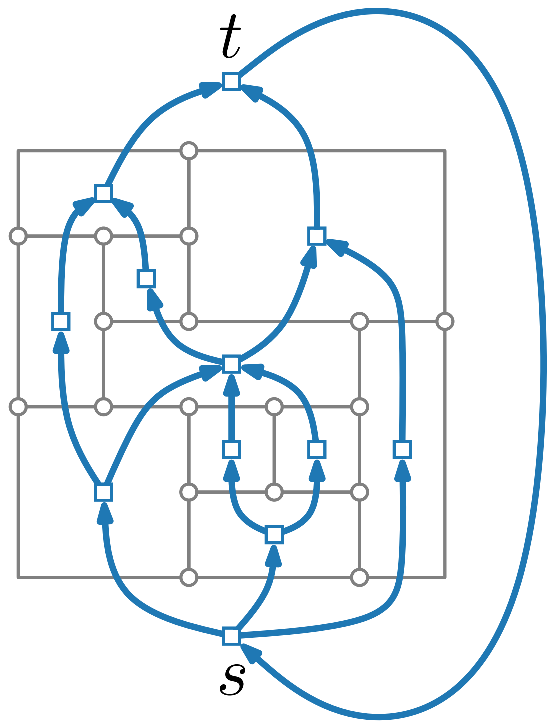

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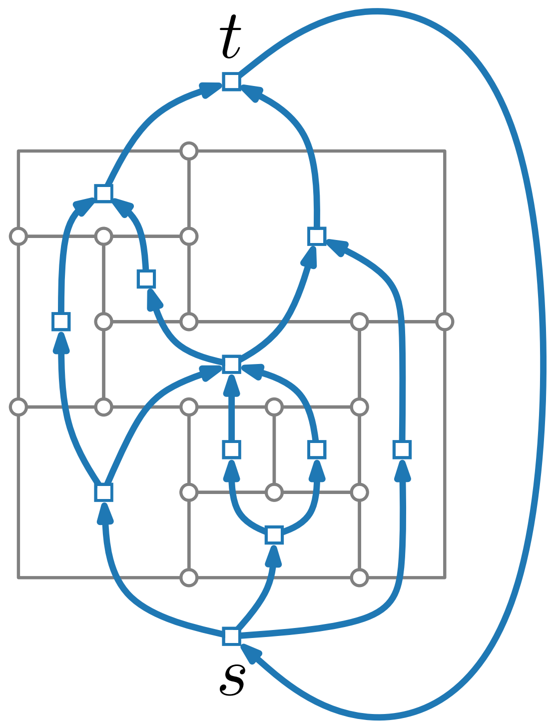

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- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and f lies to the *left* of $g\} \cup \{(t, s)\}$
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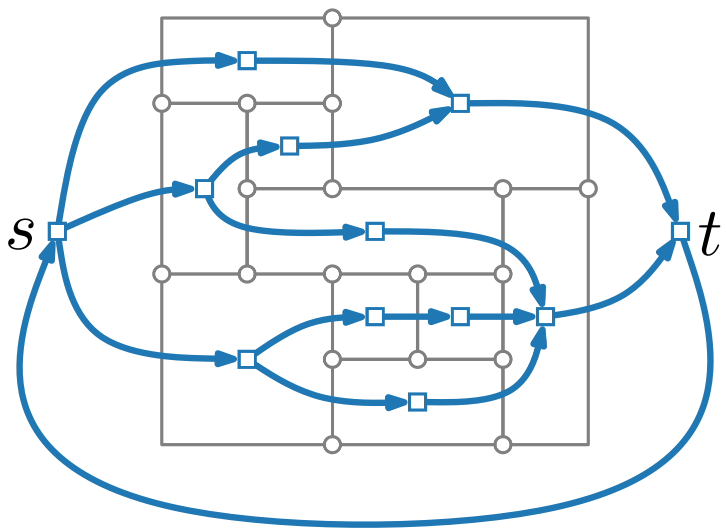
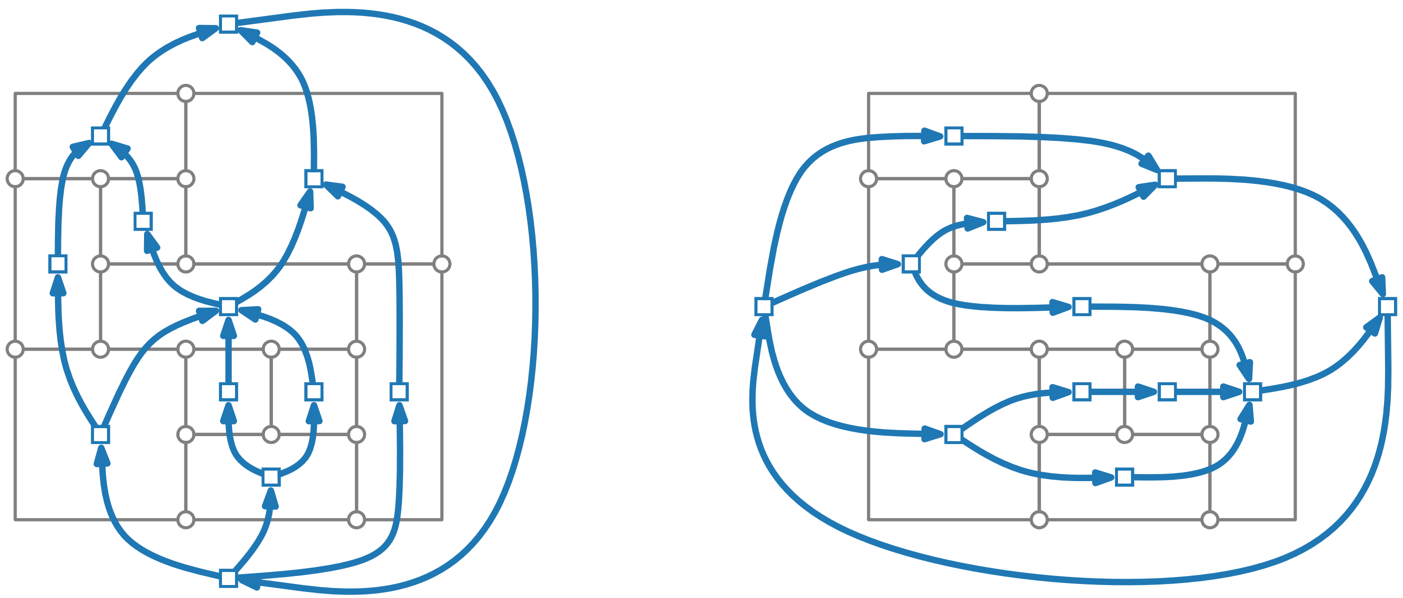
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$

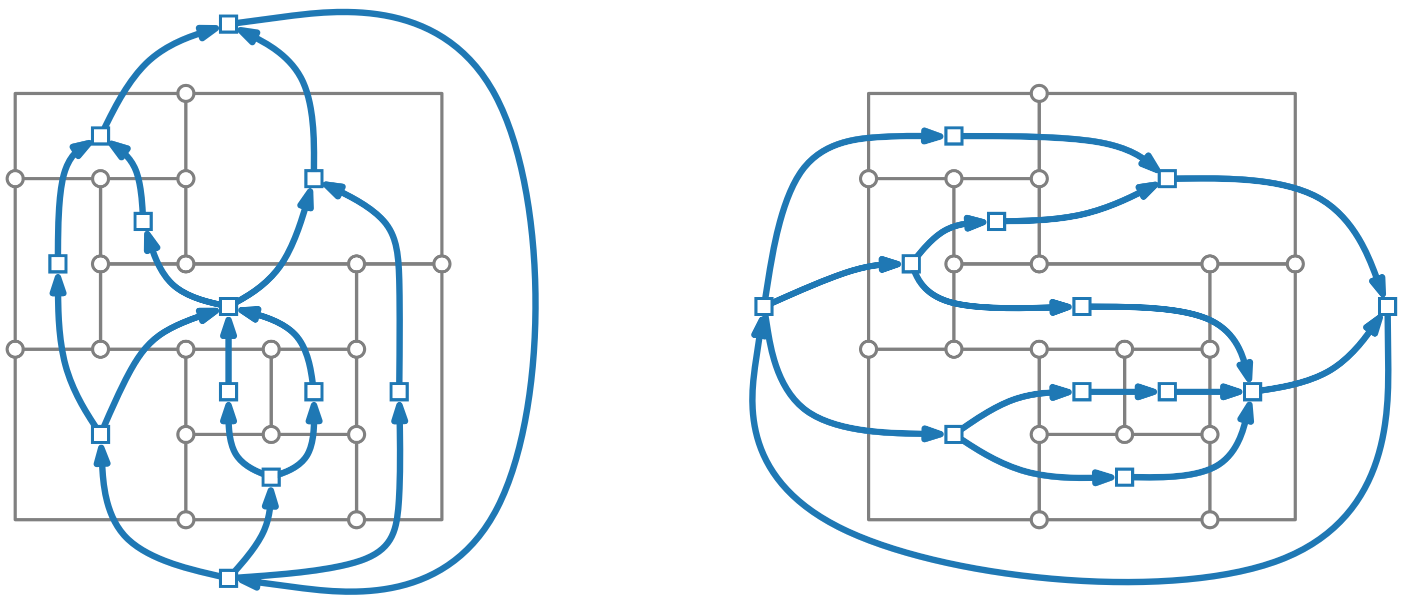


Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

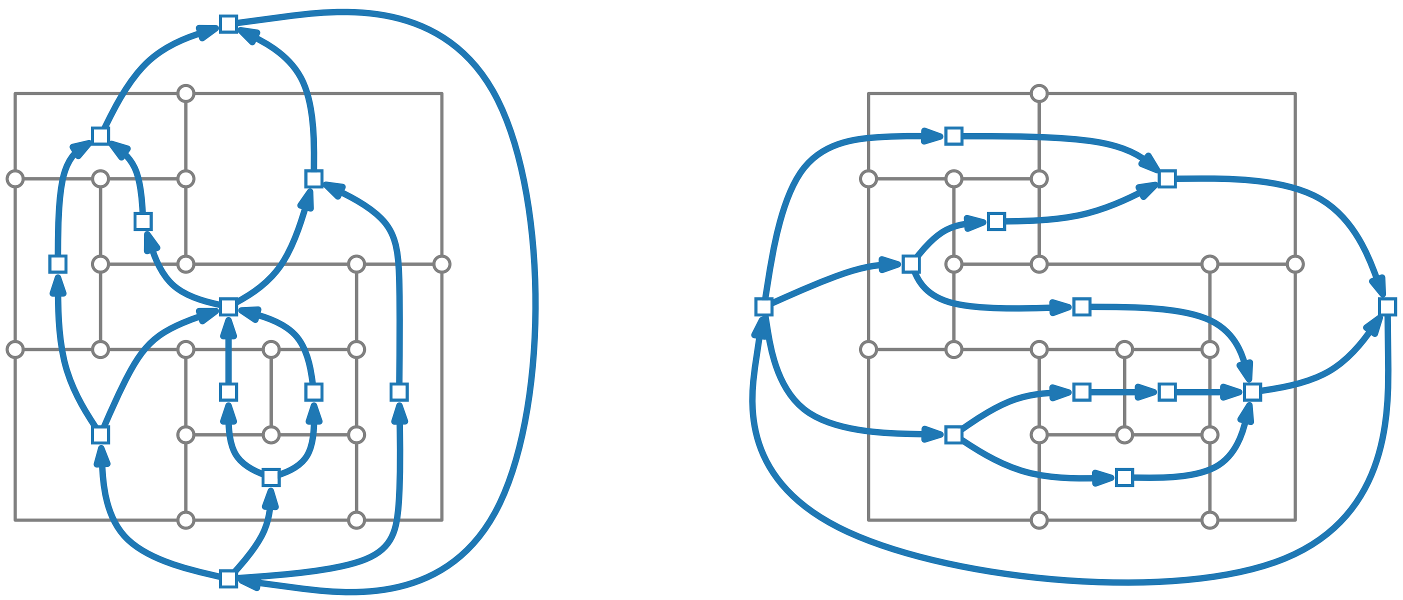
Compaction – Result



Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

Compaction – Result

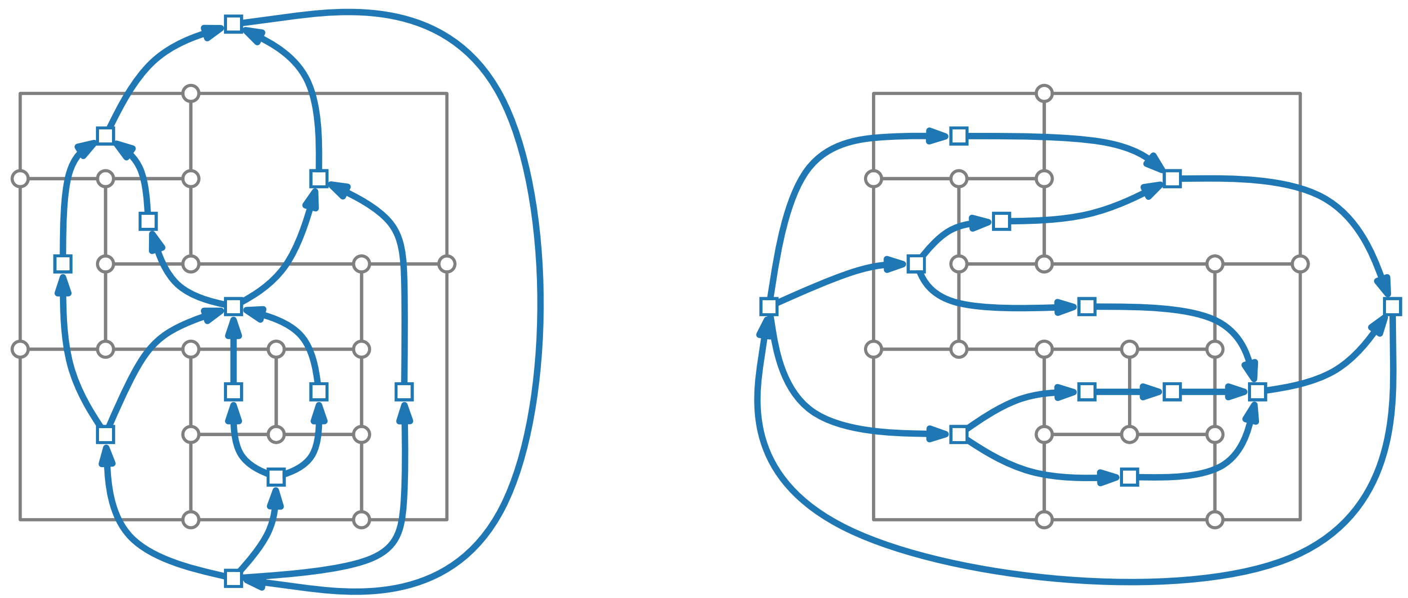


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$?

Compaction – Result

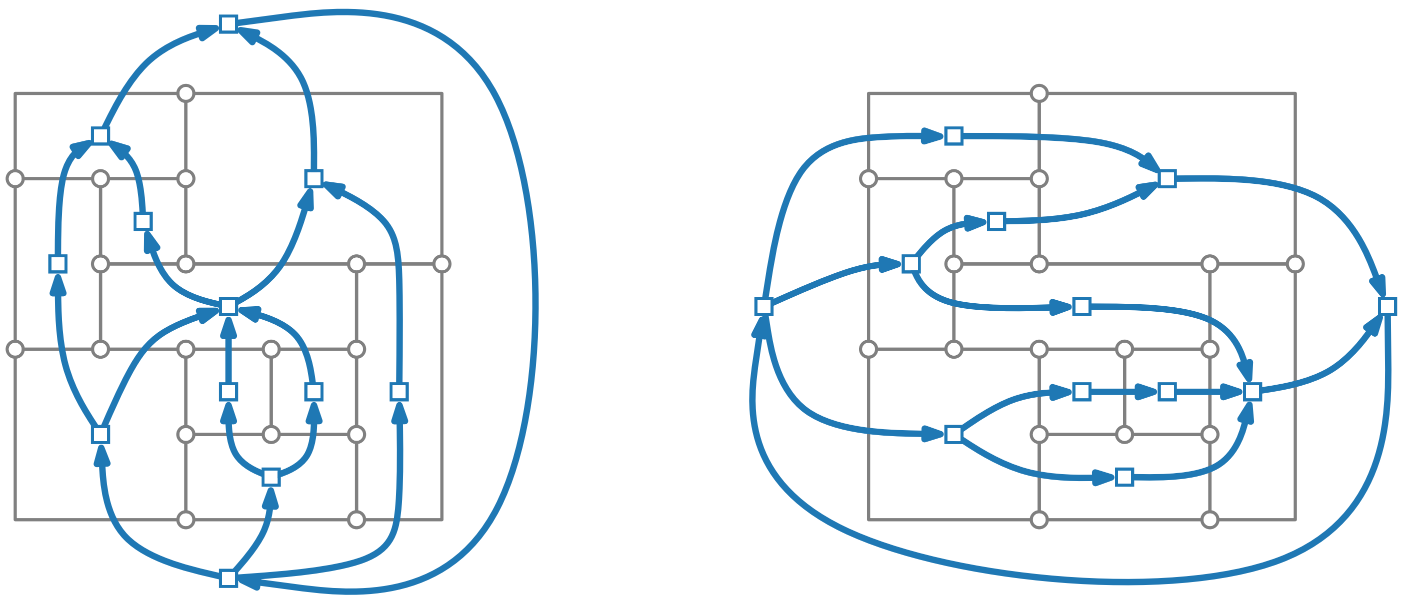


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of the drawing

Compaction – Result

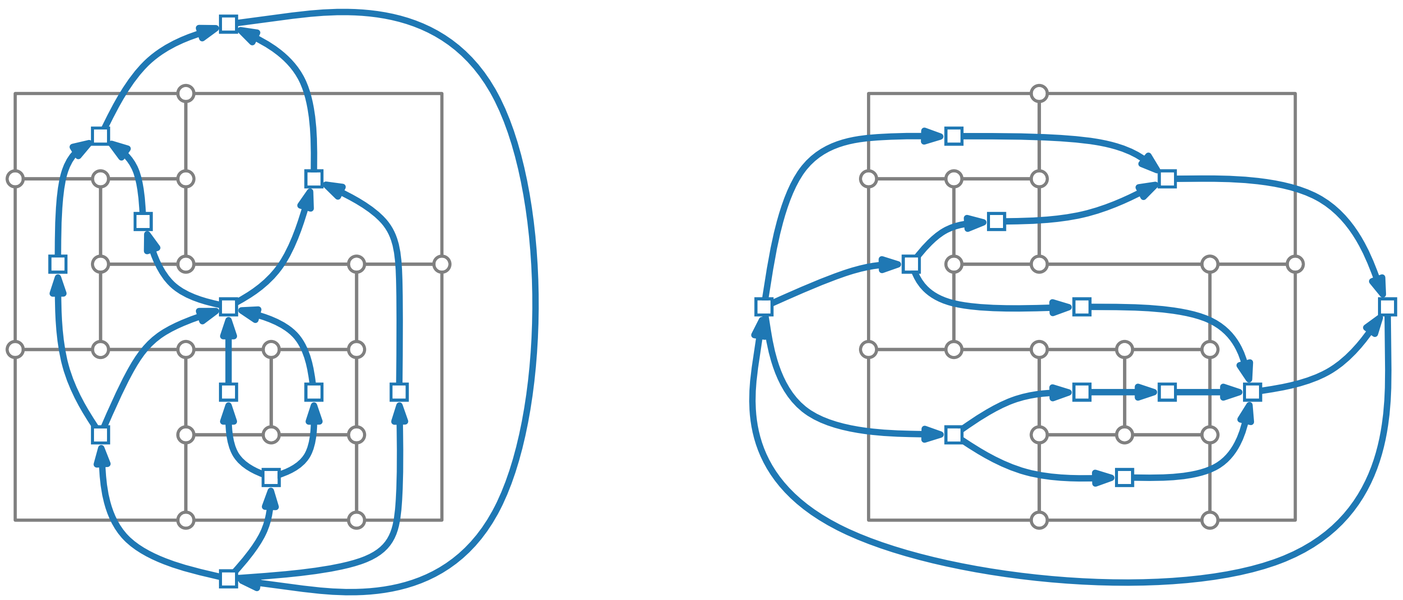


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$

Compaction – Result

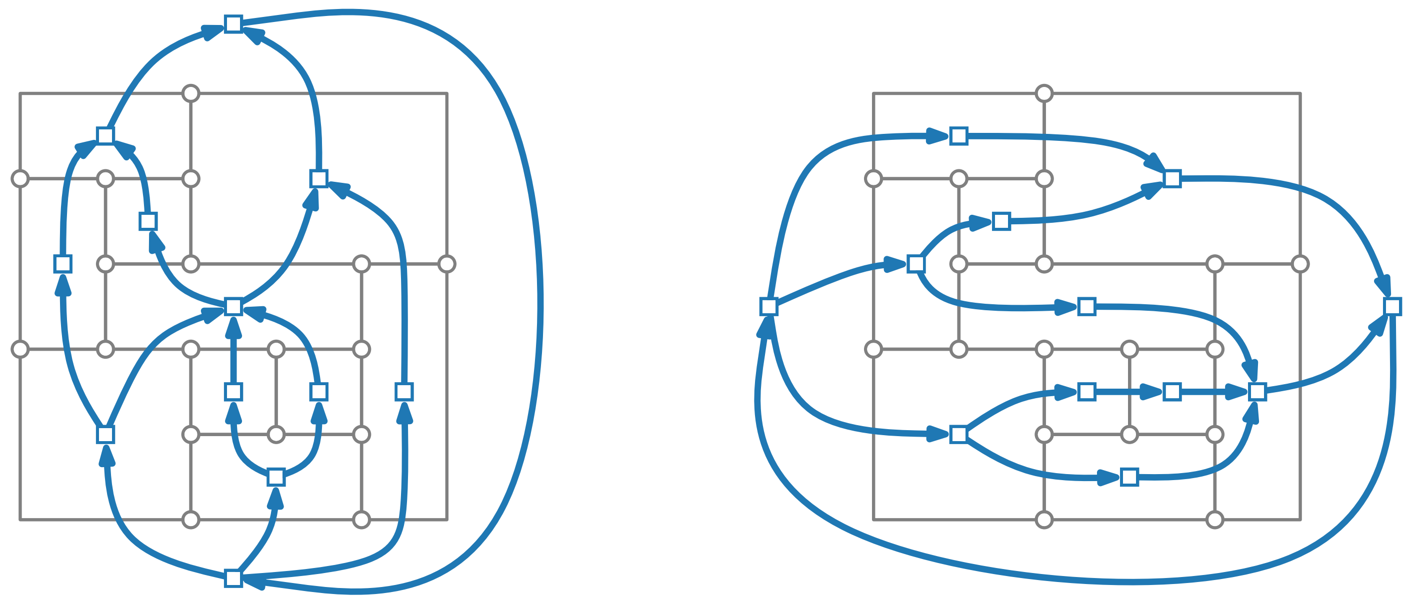


Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Compaction – Result



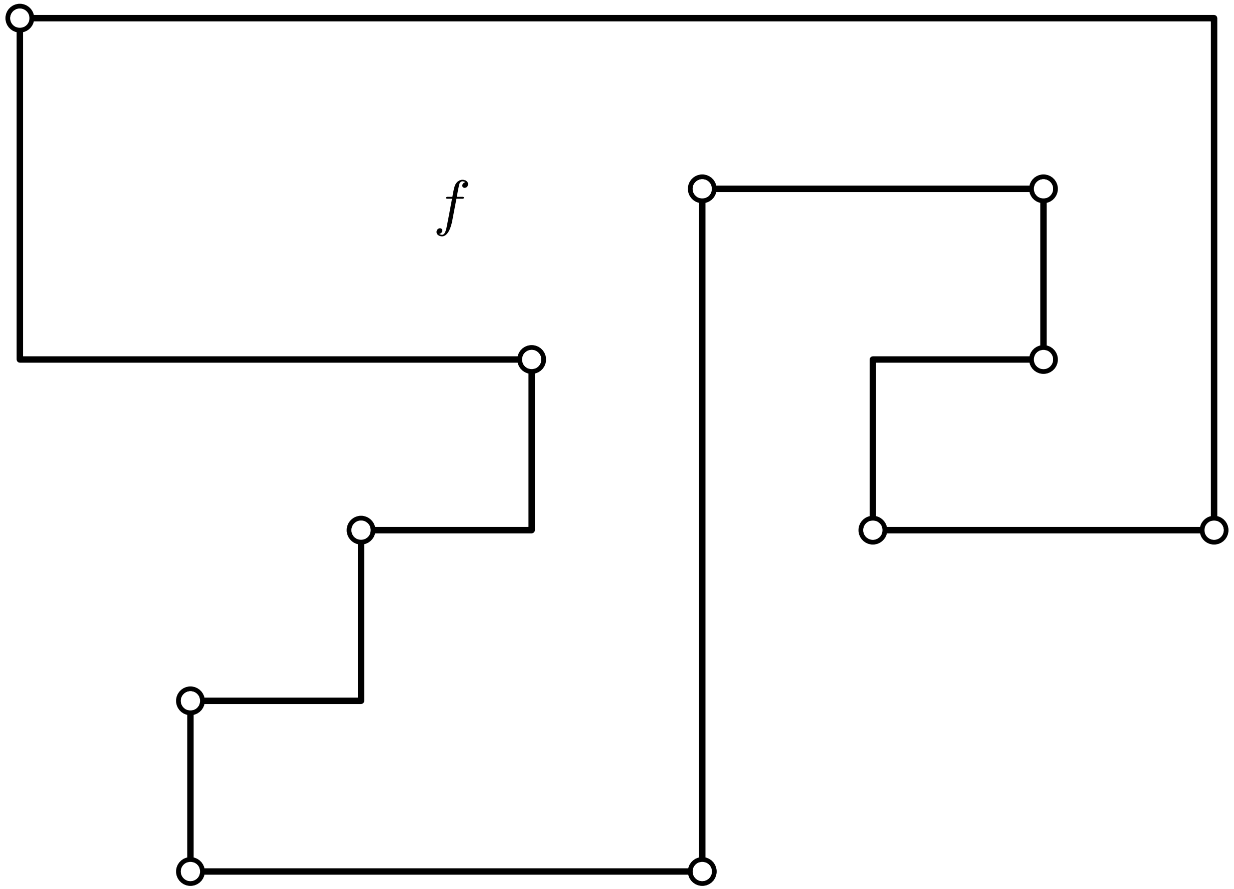
What if not all faces are rectangular?

Theorem.
A valid flow for N_{hor} and N_{ver} exists $\Leftrightarrow$
corresponding edge lengths induce an orthogonal drawing.

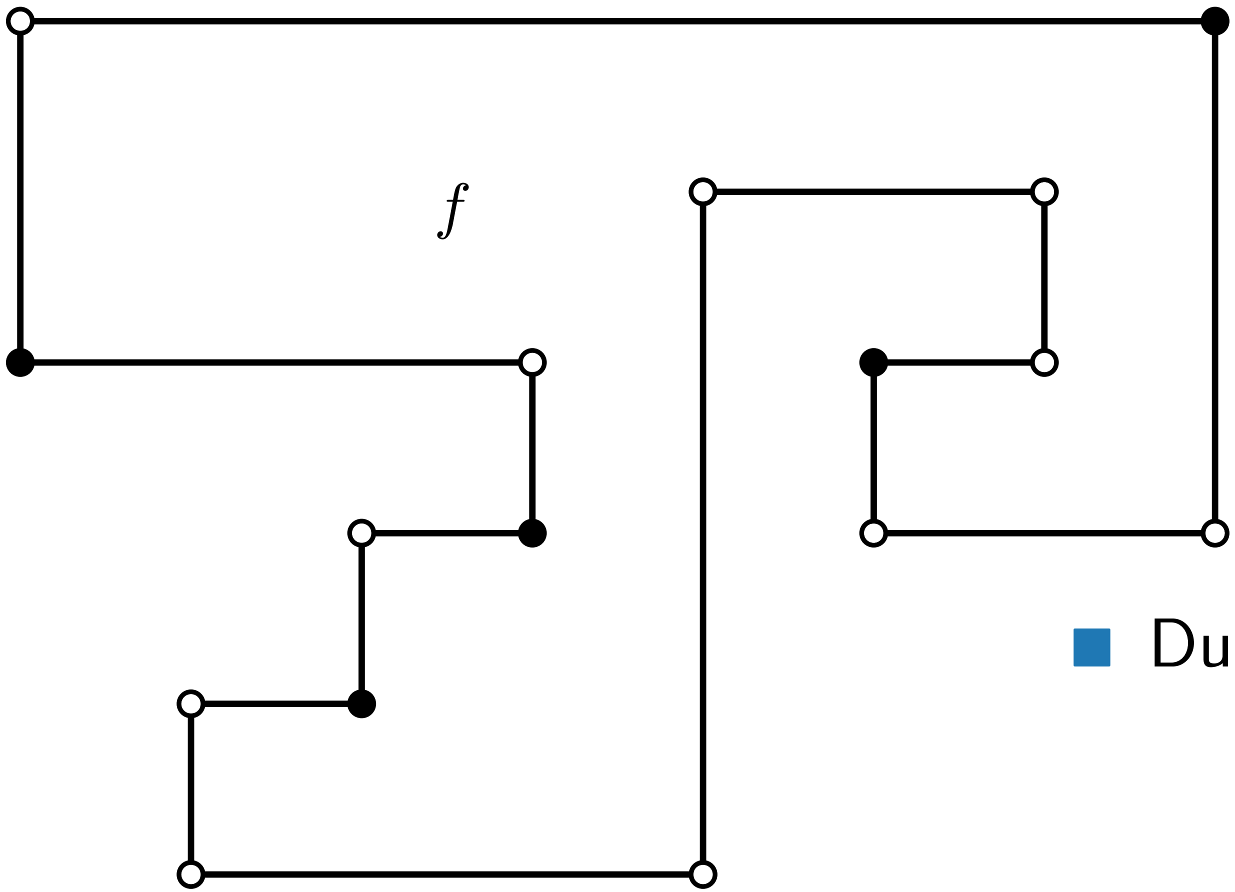
What values of the drawing do the following quantities represent?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of the drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Refinement of G and $H(G)$ – Inner Face

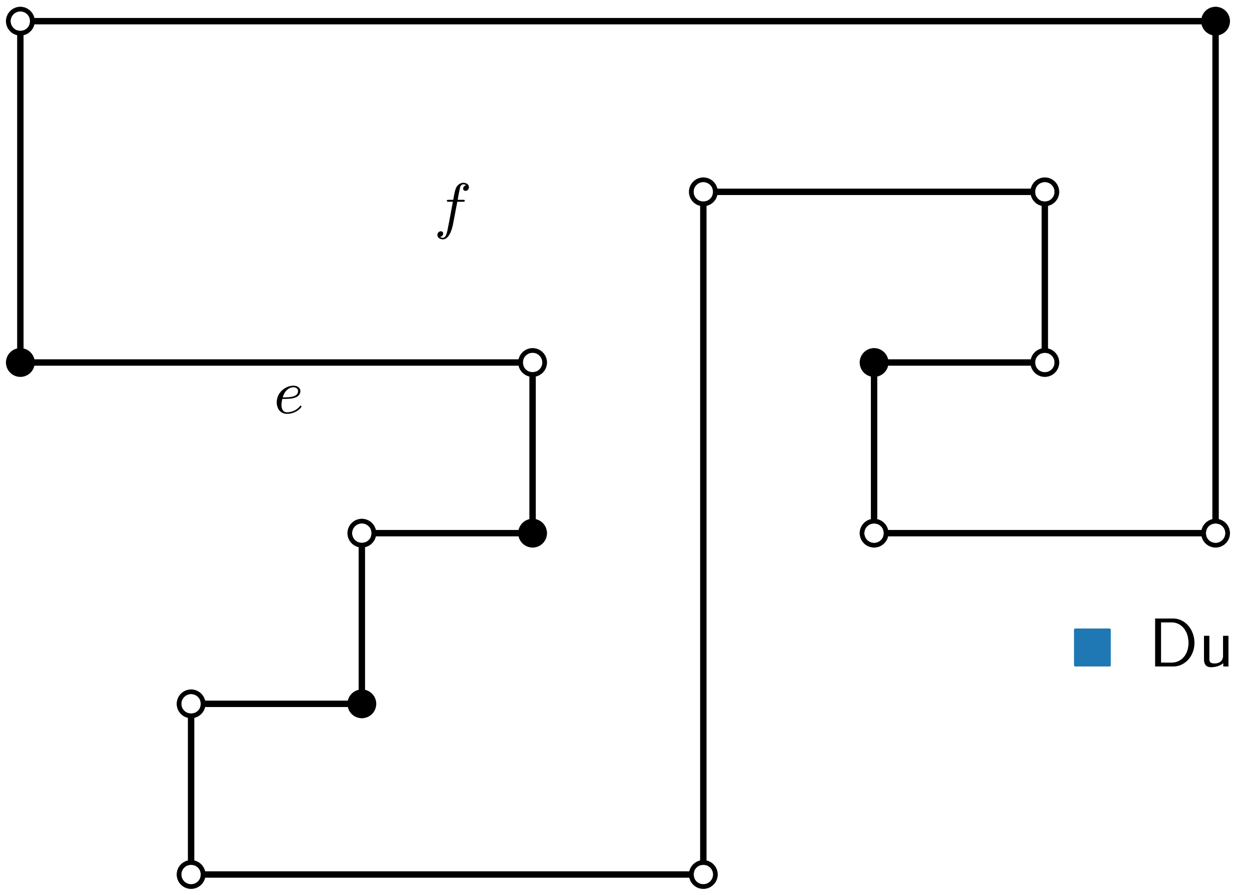


Refinement of G and $H(G)$ – Inner Face



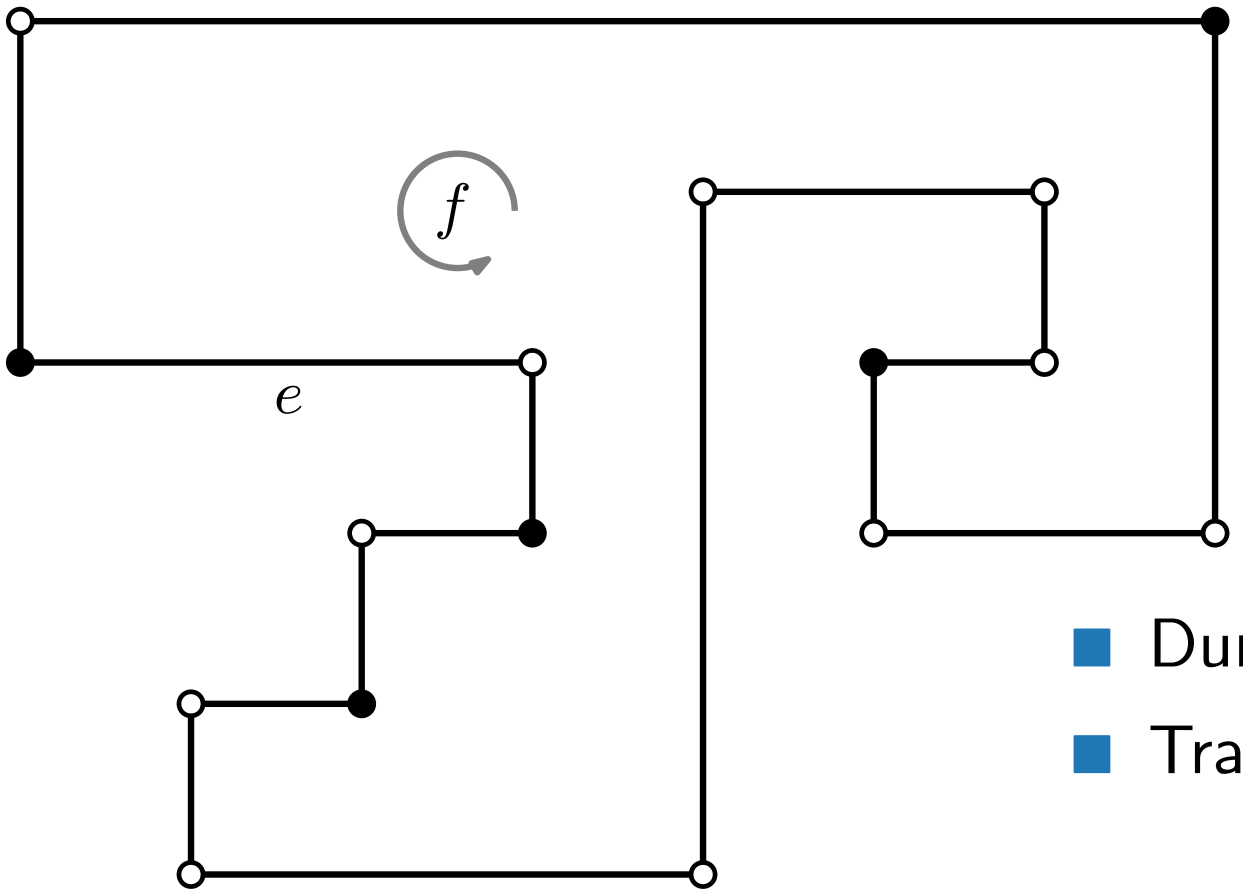
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



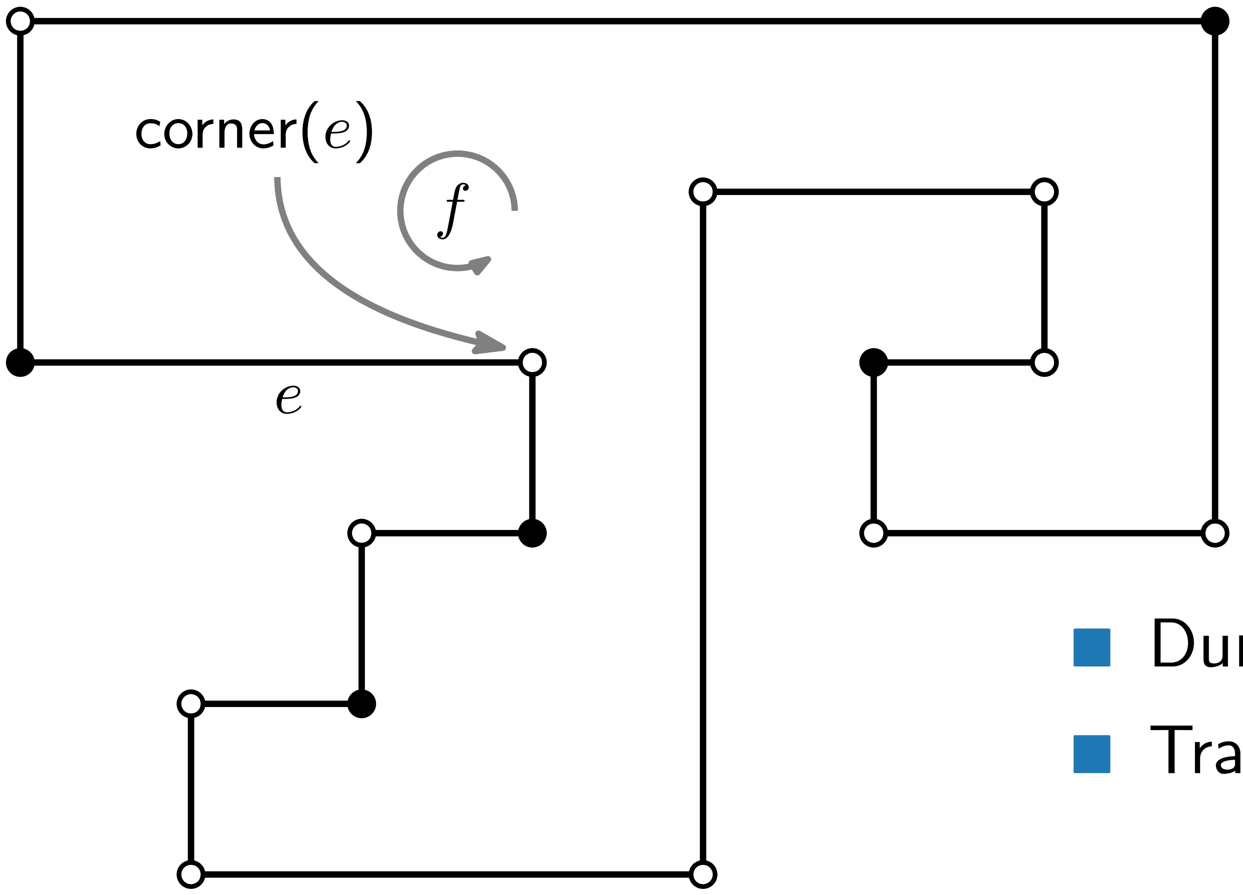
- Dummy vertices for bends

Refinement of G and $H(G)$ – Inner Face



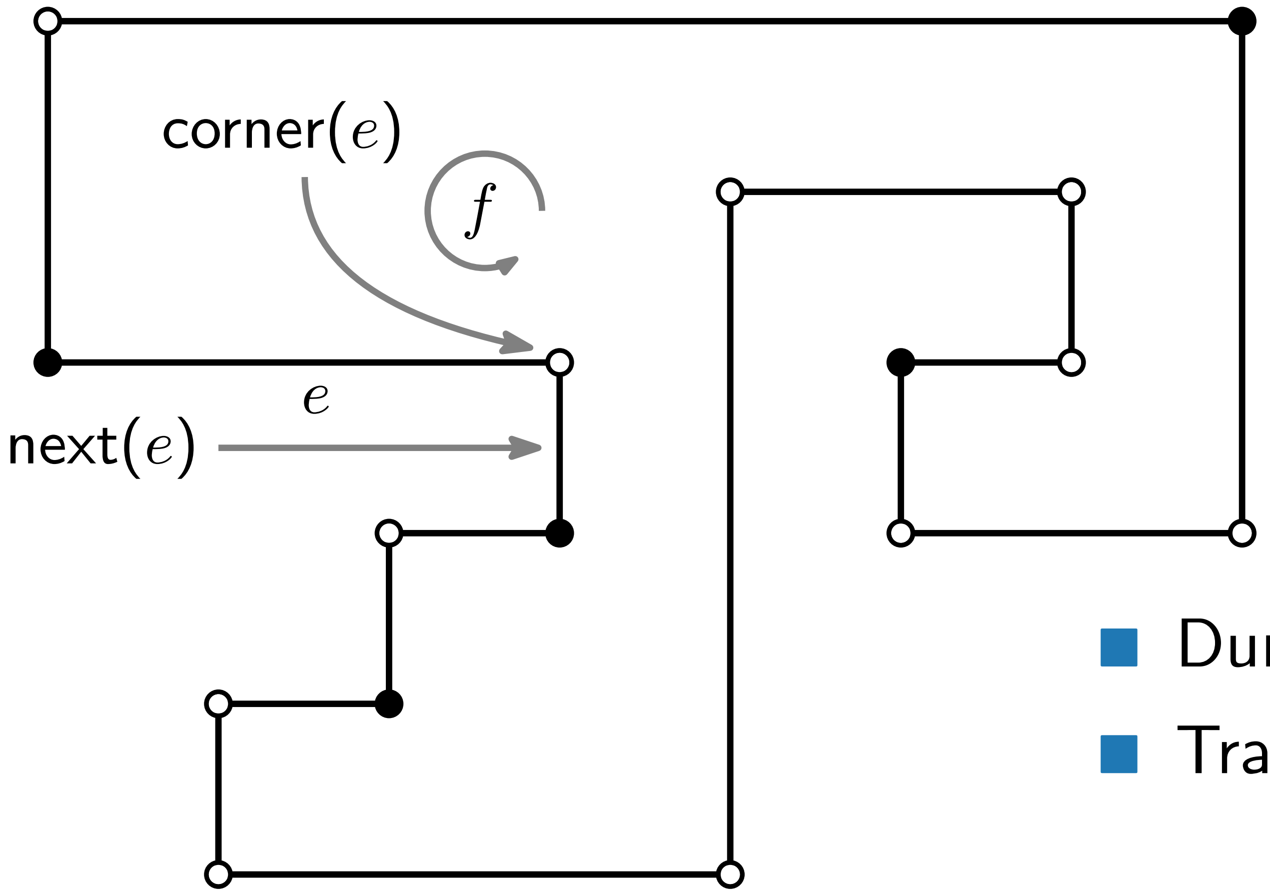
- Dummy vertices for bends
- Traverse counter-clockwise

Refinement of G and $H(G)$ – Inner Face



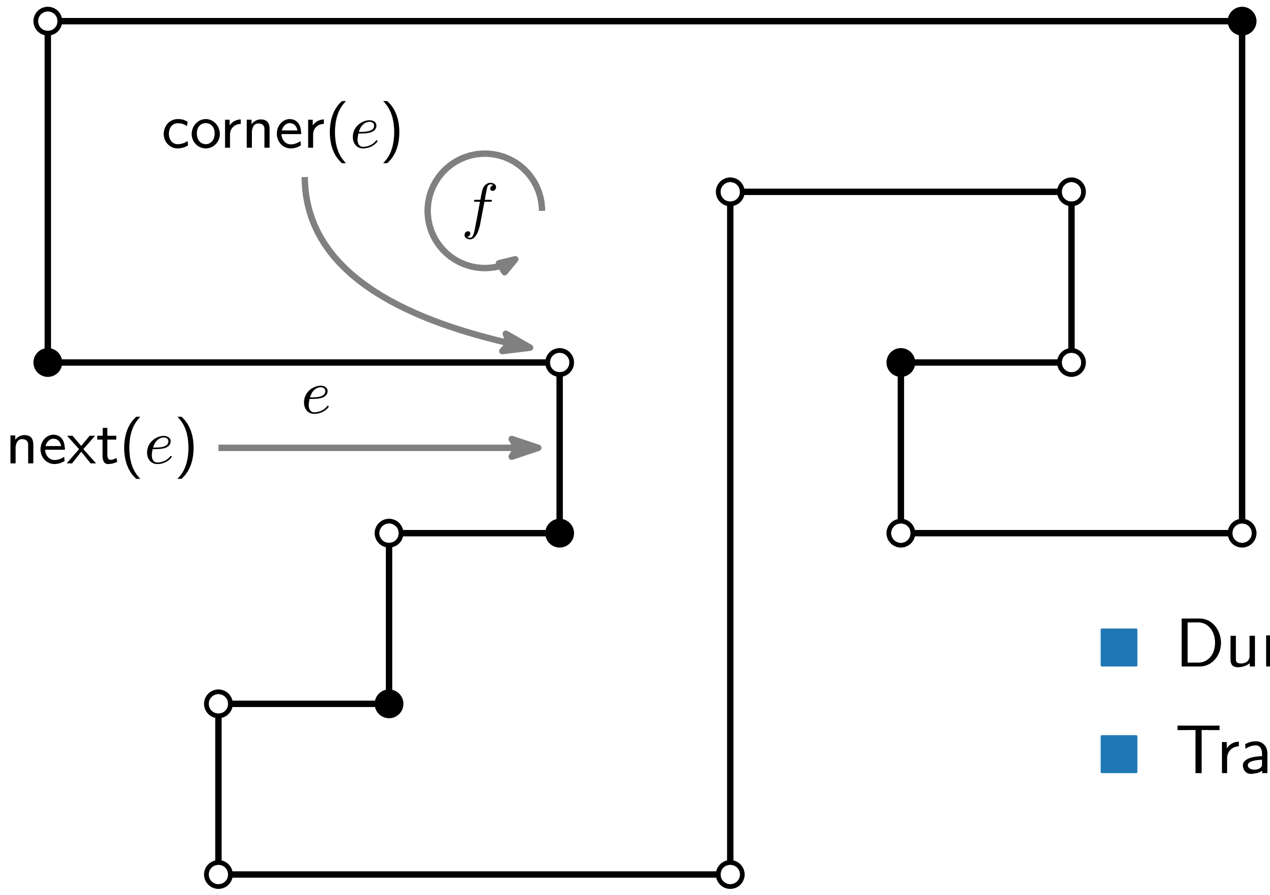
- Dummy vertices for bends
- Traverse counter-clockwise

Refinement of G and $H(G)$ – Inner Face



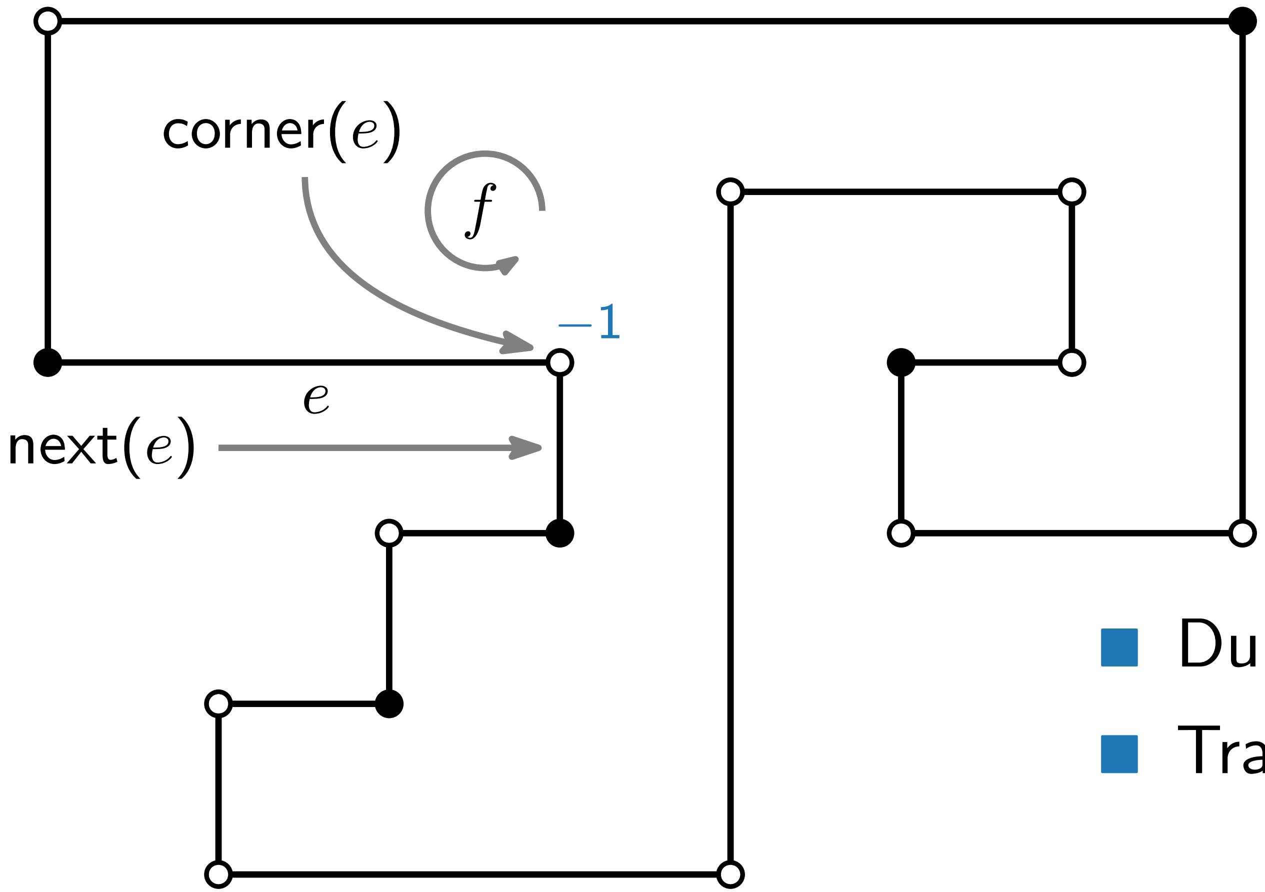
- Dummy vertices for bends
- Traverse counter-clockwise

Refinement of G and $H(G)$ – Inner Face



- Dummy vertices for bends
- Traverse counter-clockwise
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

Refinement of G and $H(G)$ – Inner Face



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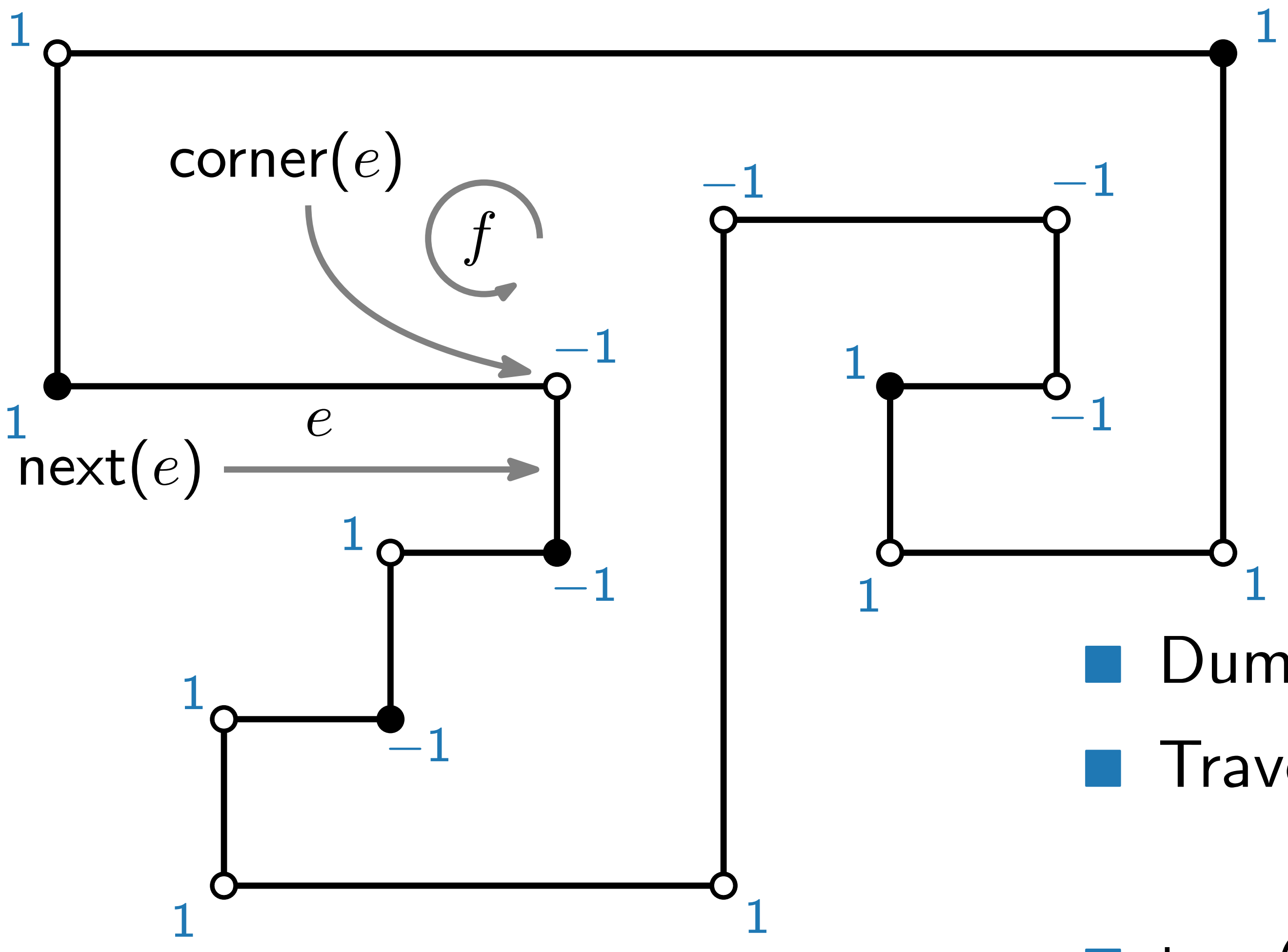

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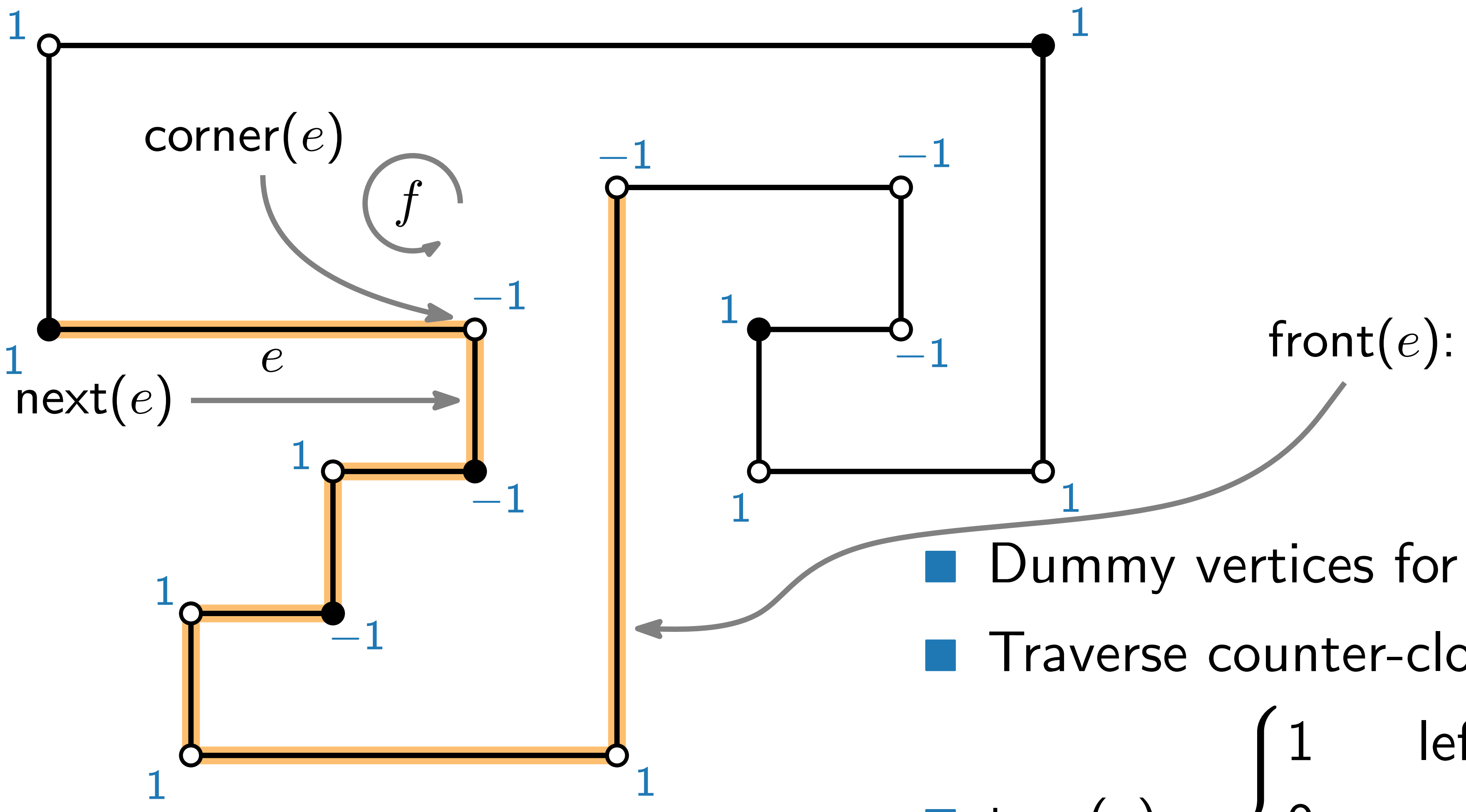
Refinement of G and $H(G)$ – Inner Face



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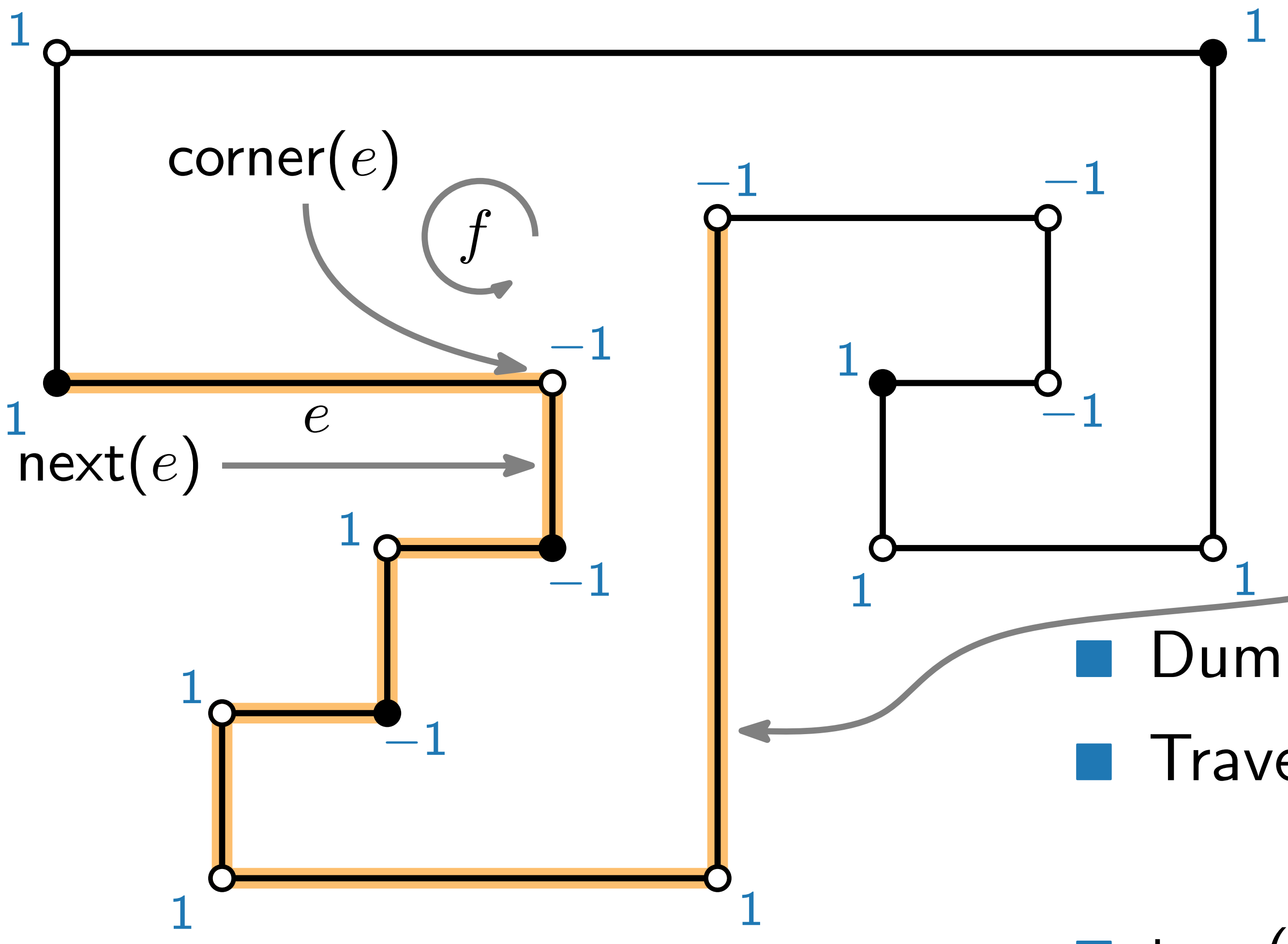
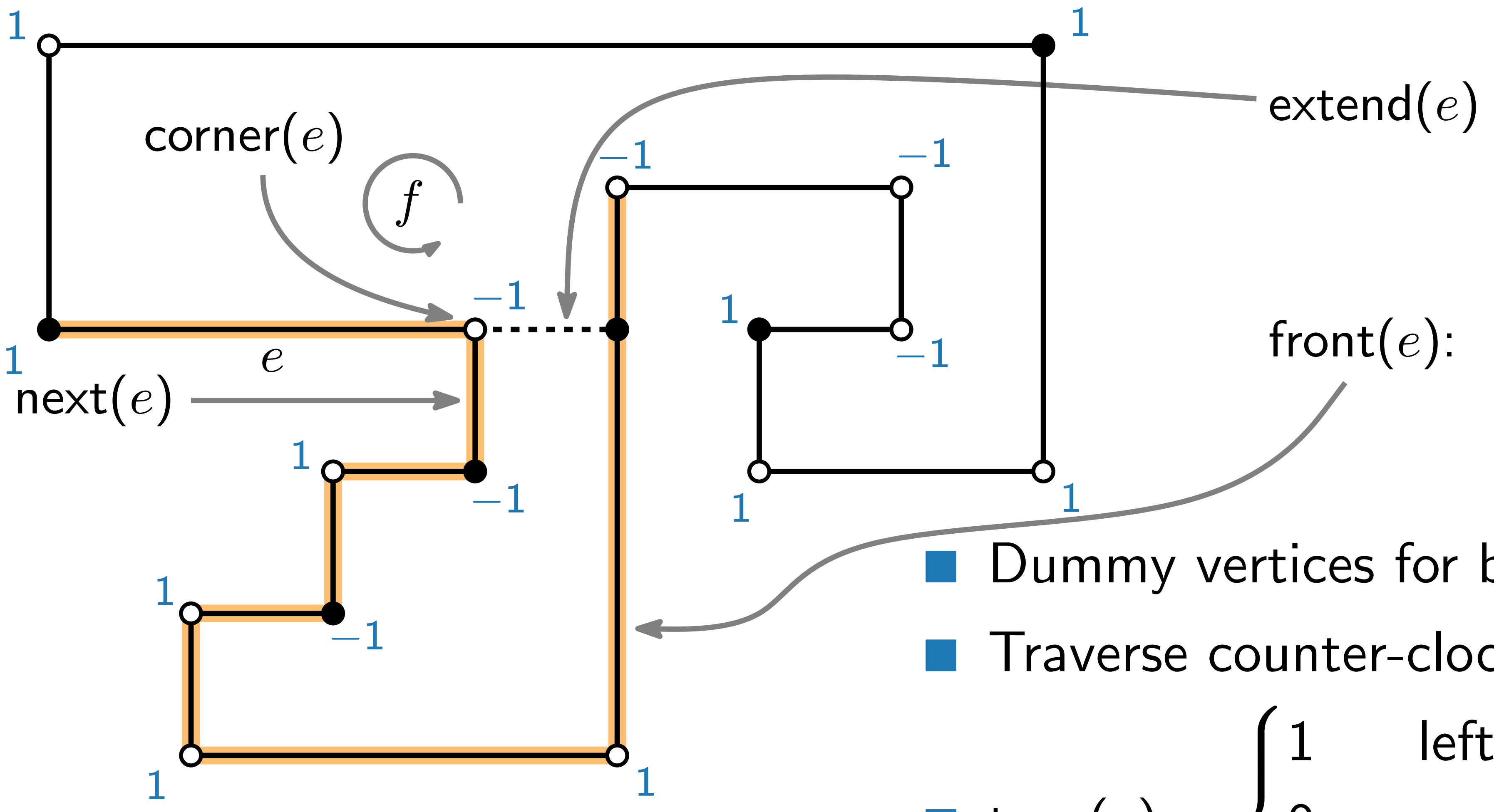
Refinement of G and $H(G)$ – Inner Face



front(e): First edge e after e such that the angle sum between e and e' is positive.

- Dummy vertices for bends
- Traverse counter-clockwise
- $$\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$$

Refinement of G and $H(G)$ – Inner Face



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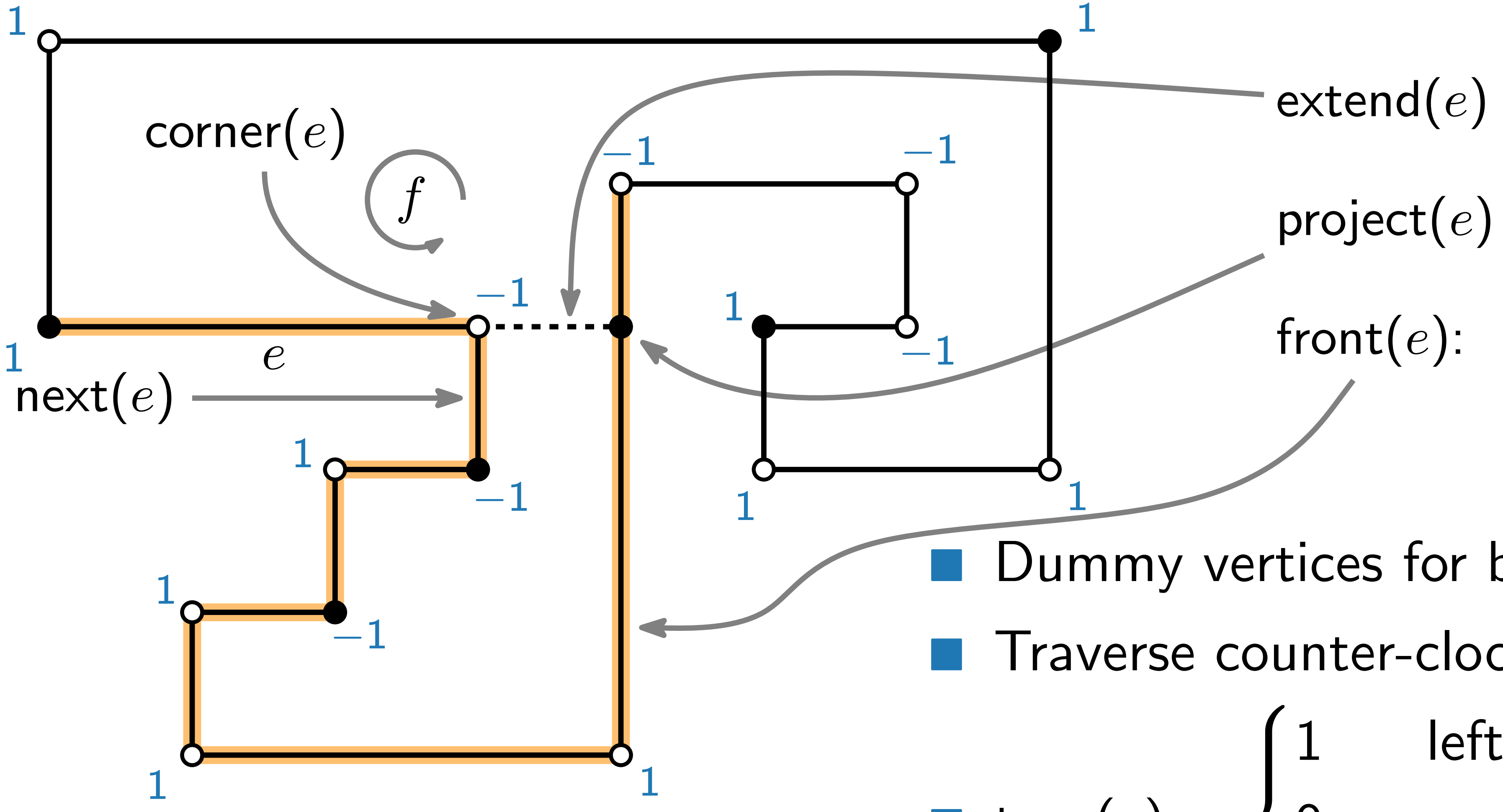

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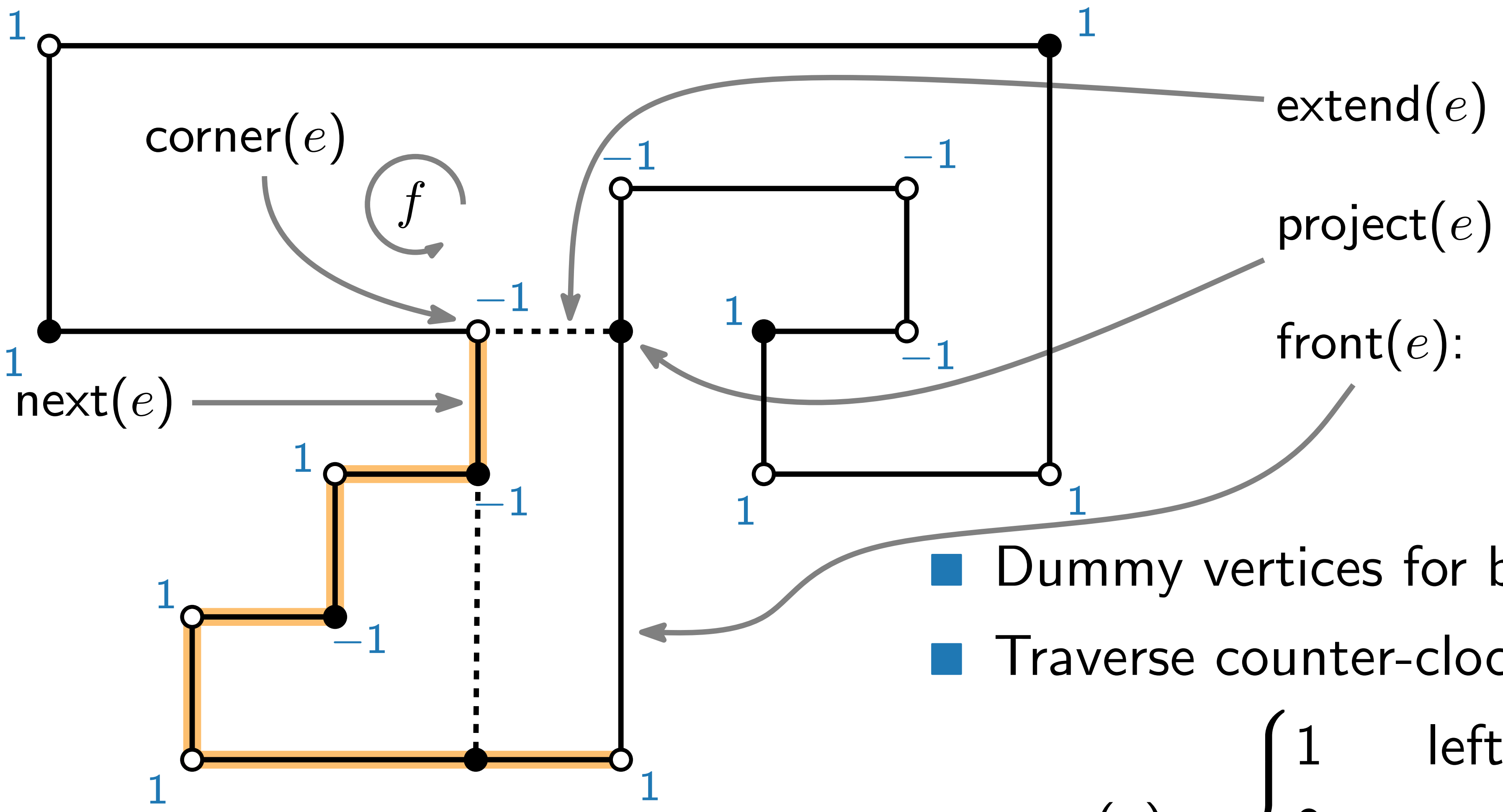

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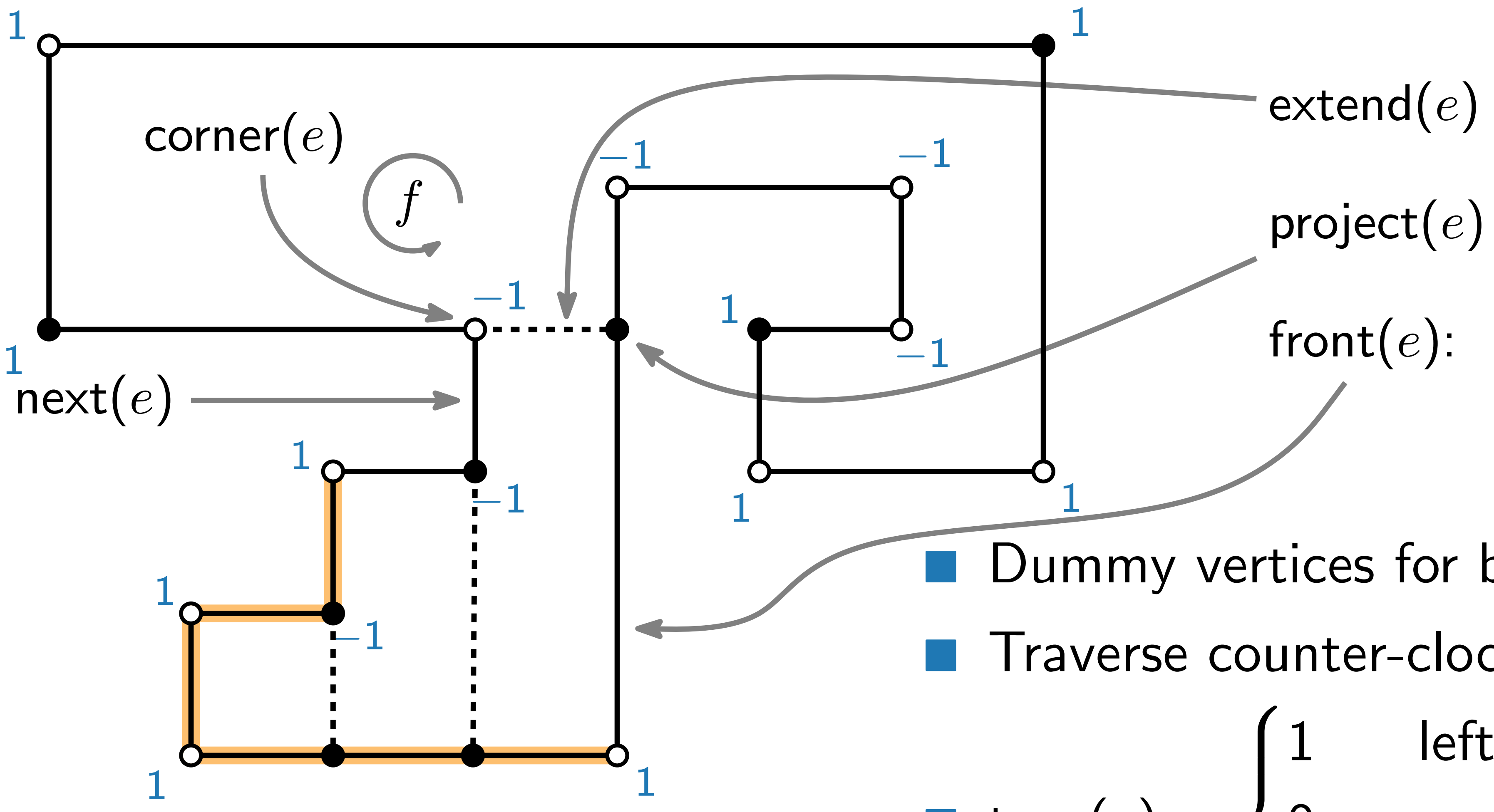

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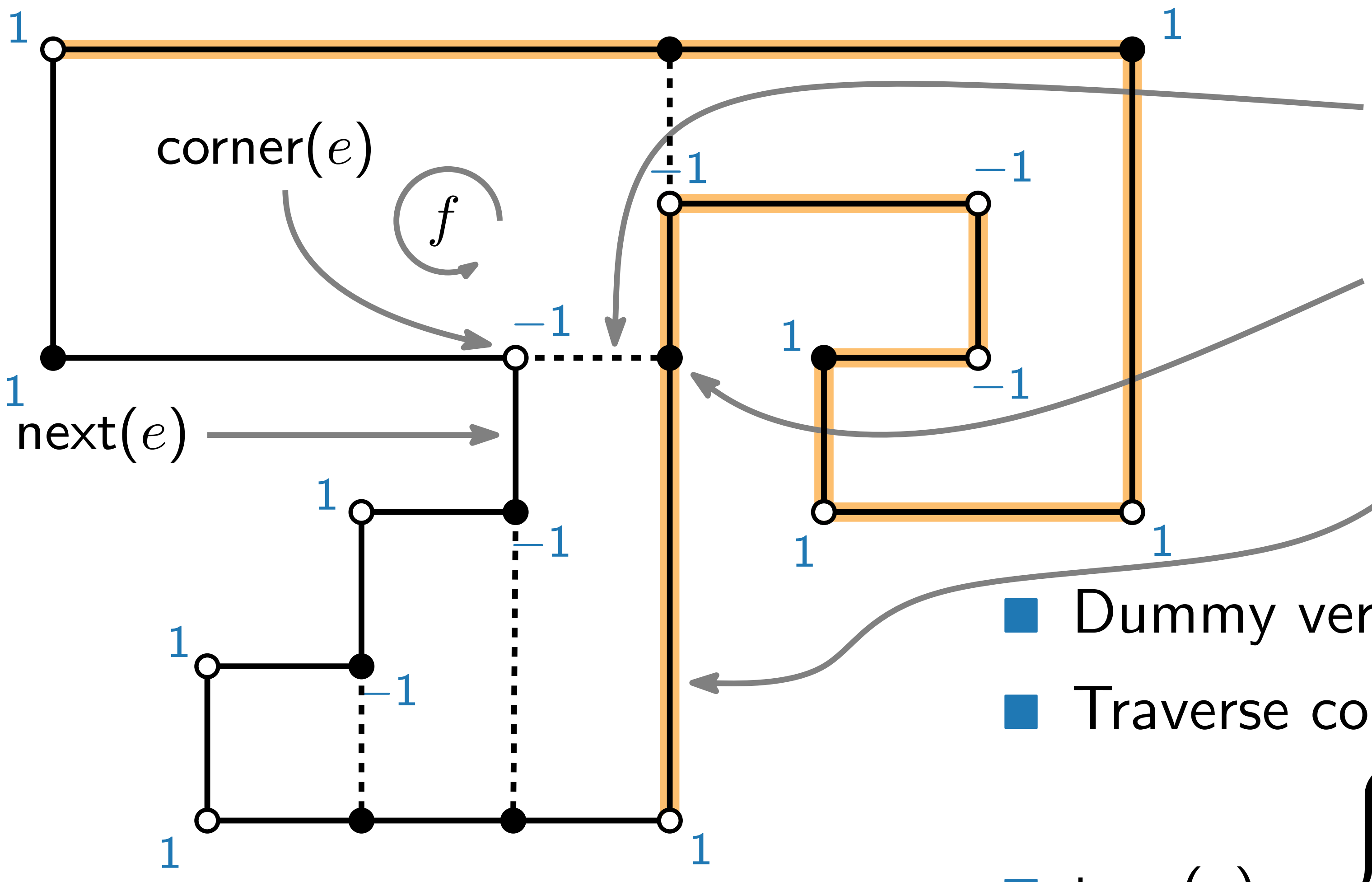

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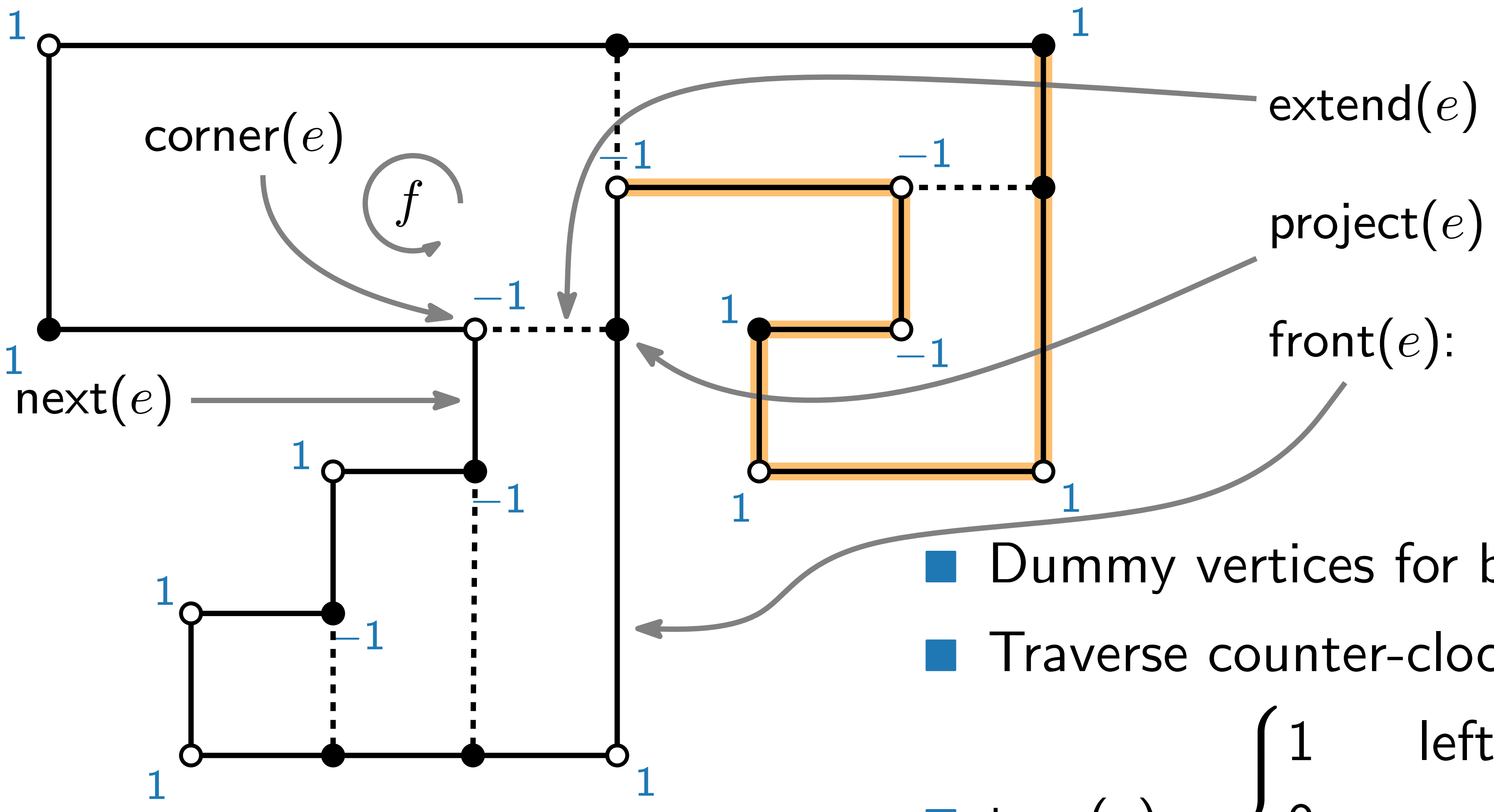

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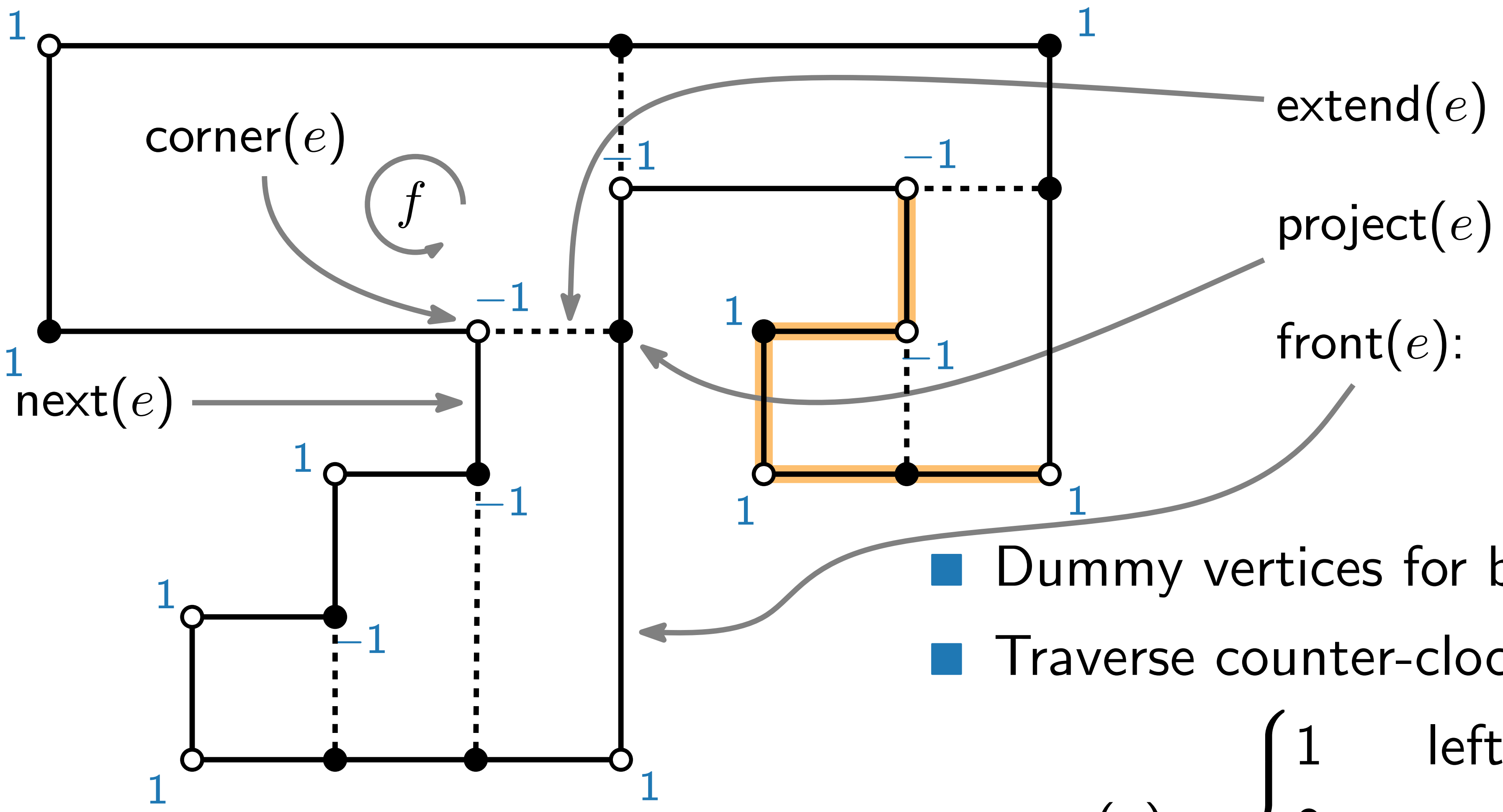

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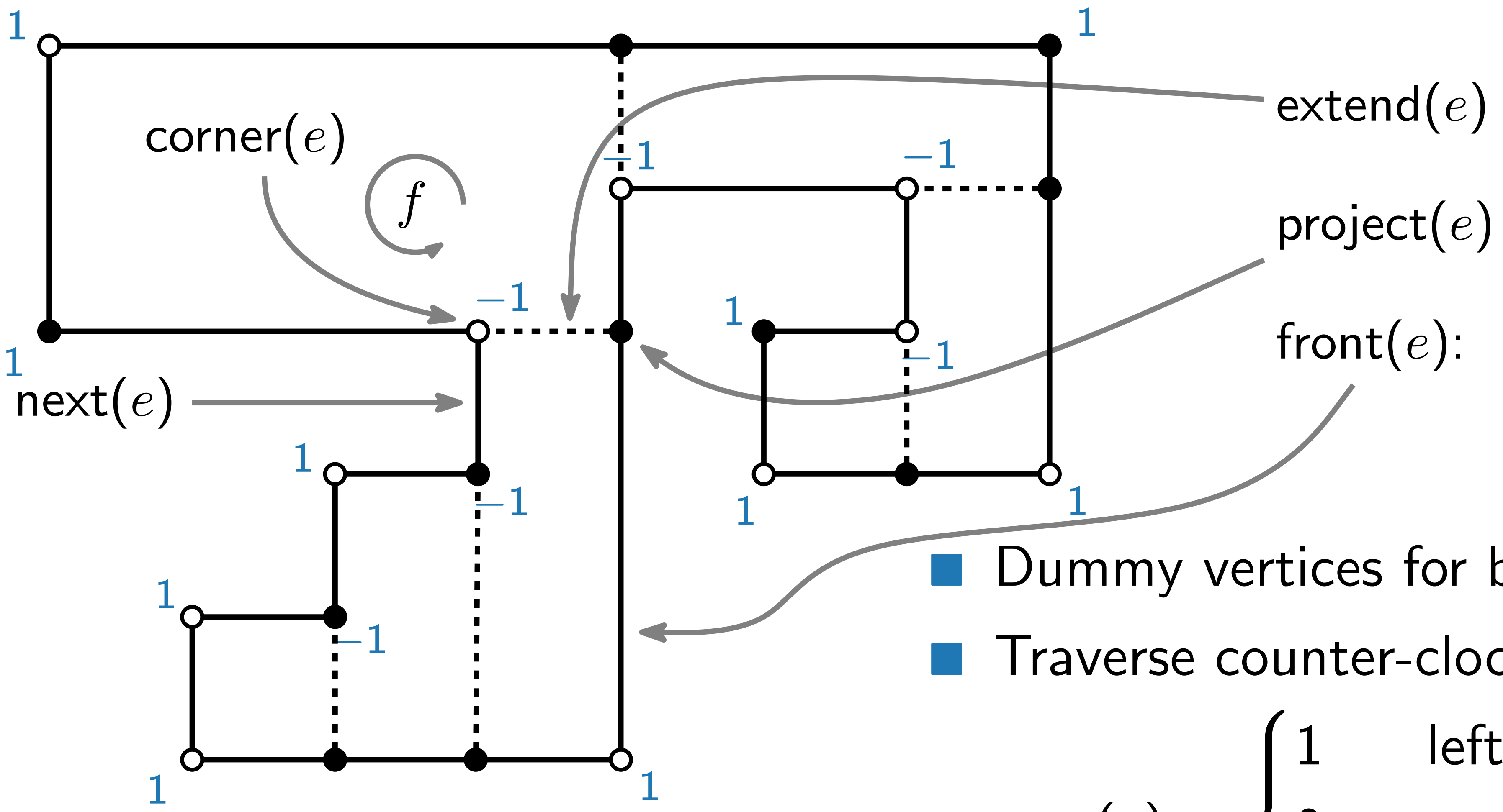

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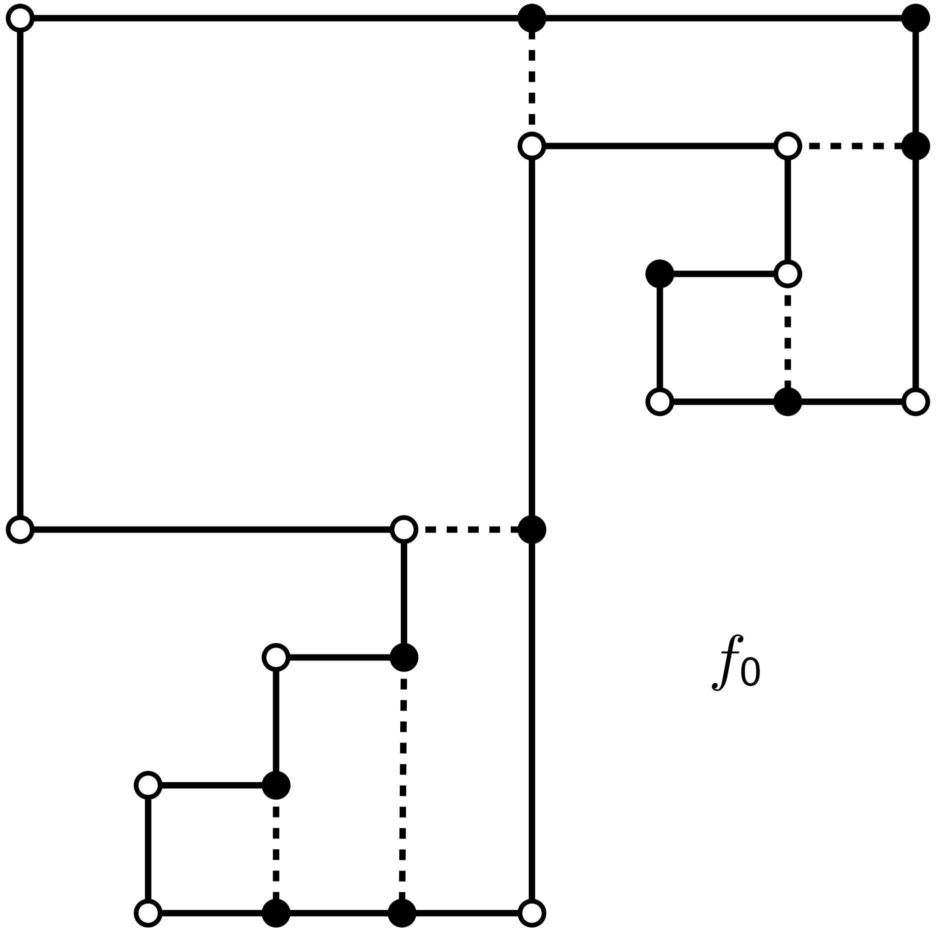
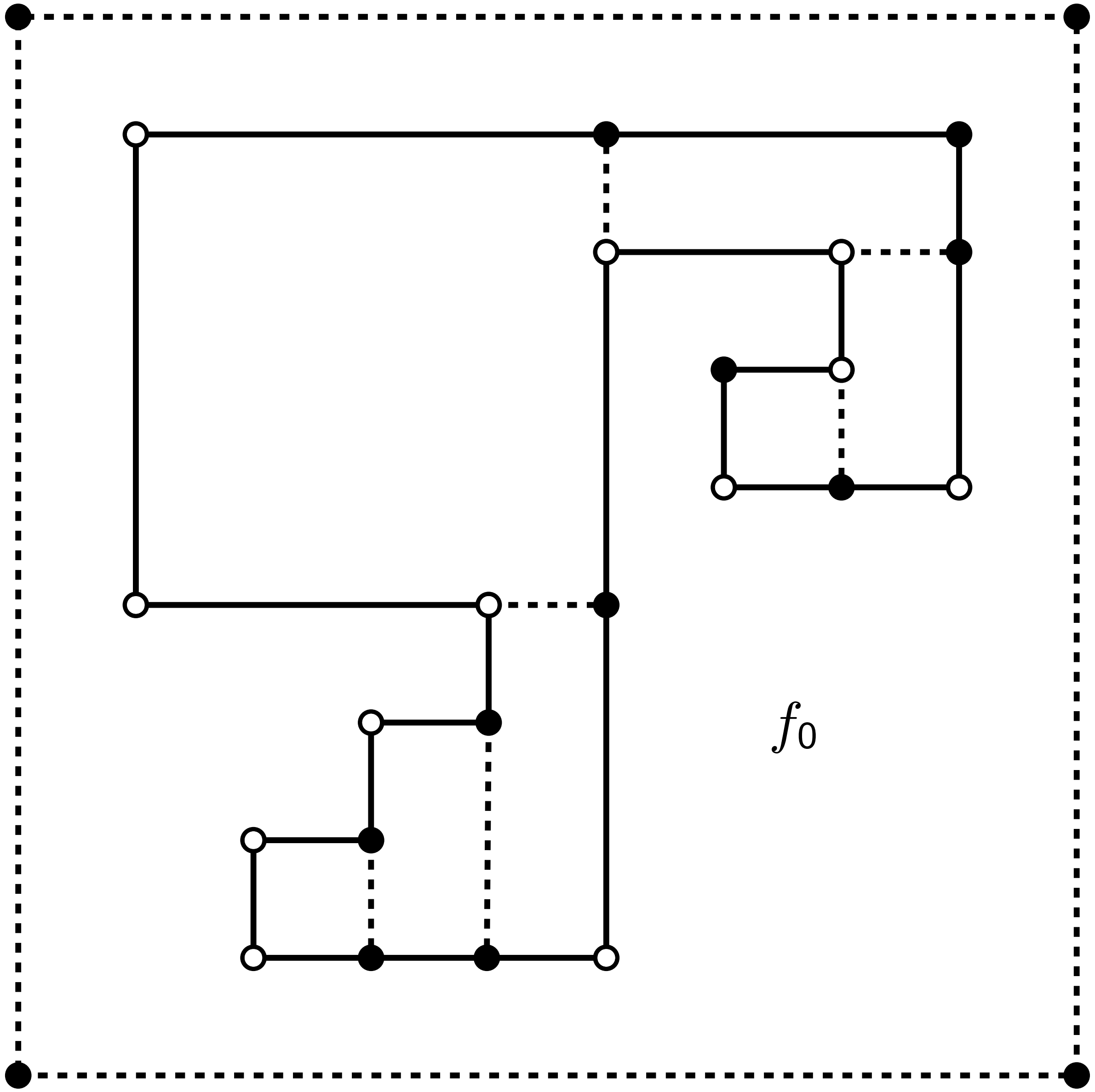
Refinement of G and $H(G)$ – Outer Face

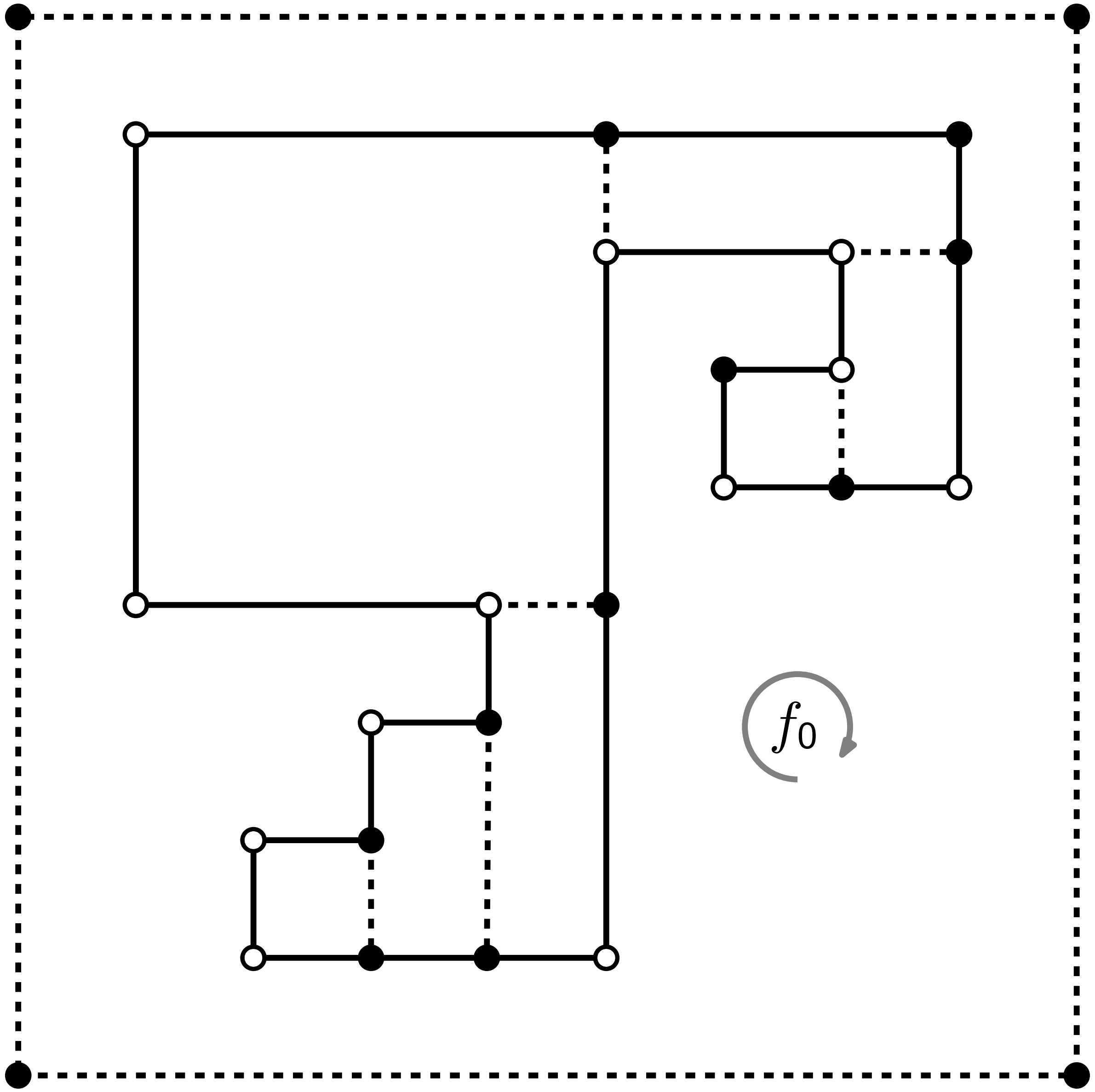


Refinement of G and $H(G)$ – Outer Face



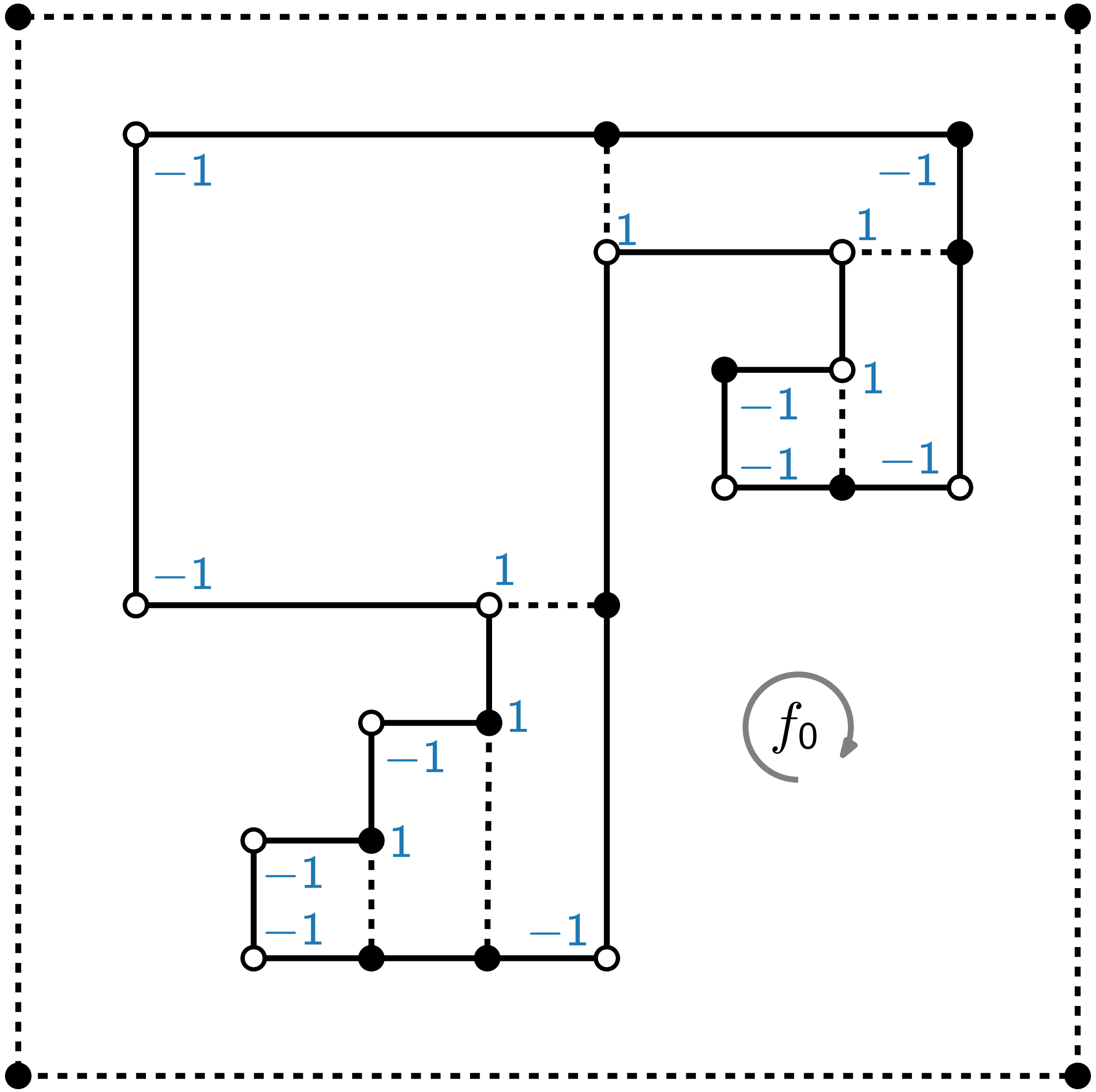
- Add an outer rectangle

Refinement of G and $H(G)$ – Outer Face



- Add an outer rectangle
- Traverse clockwise

Refinement of G and $H(G)$ – Outer Face



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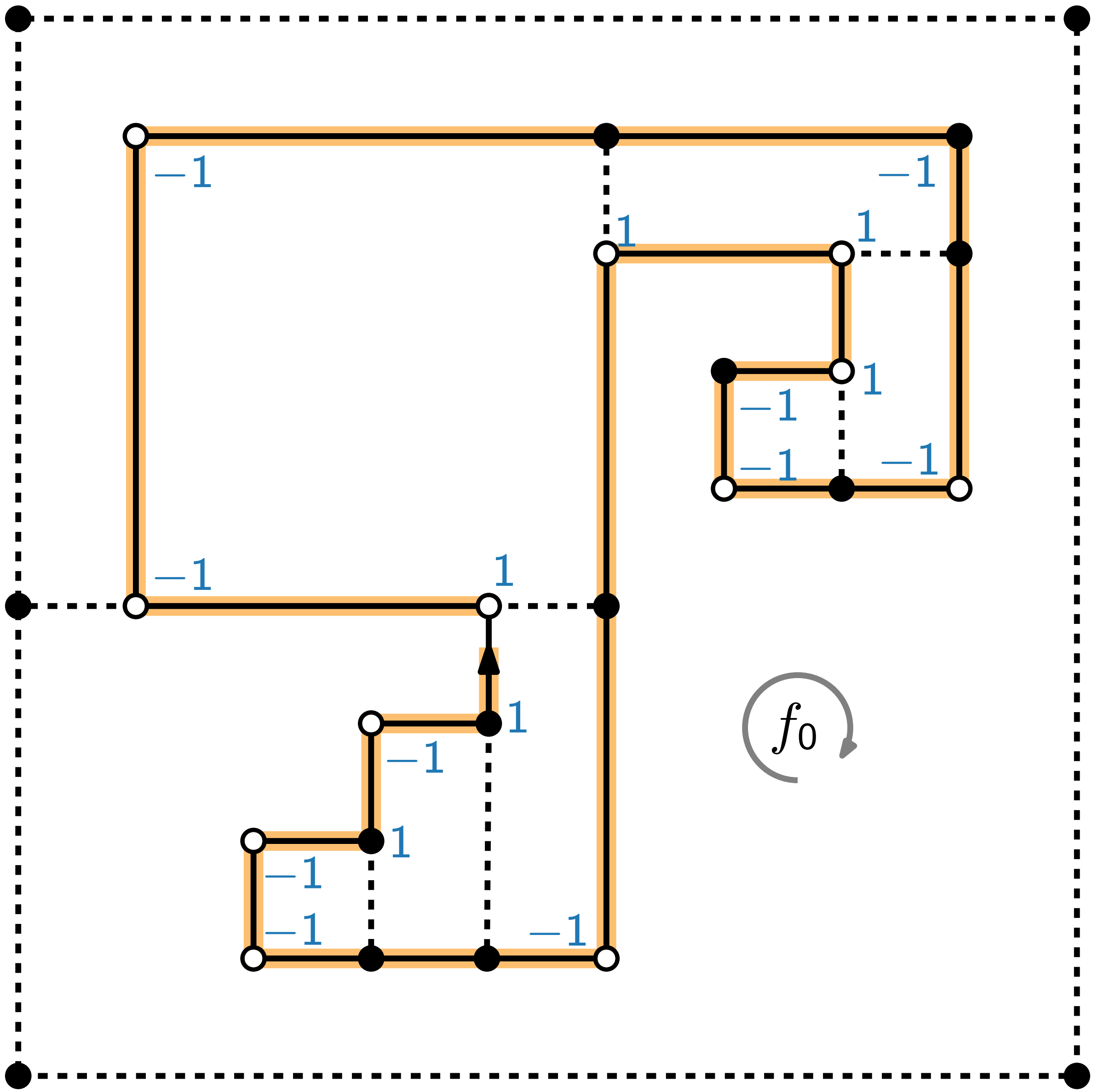
Refinement of G and $H(G)$ – Outer Face



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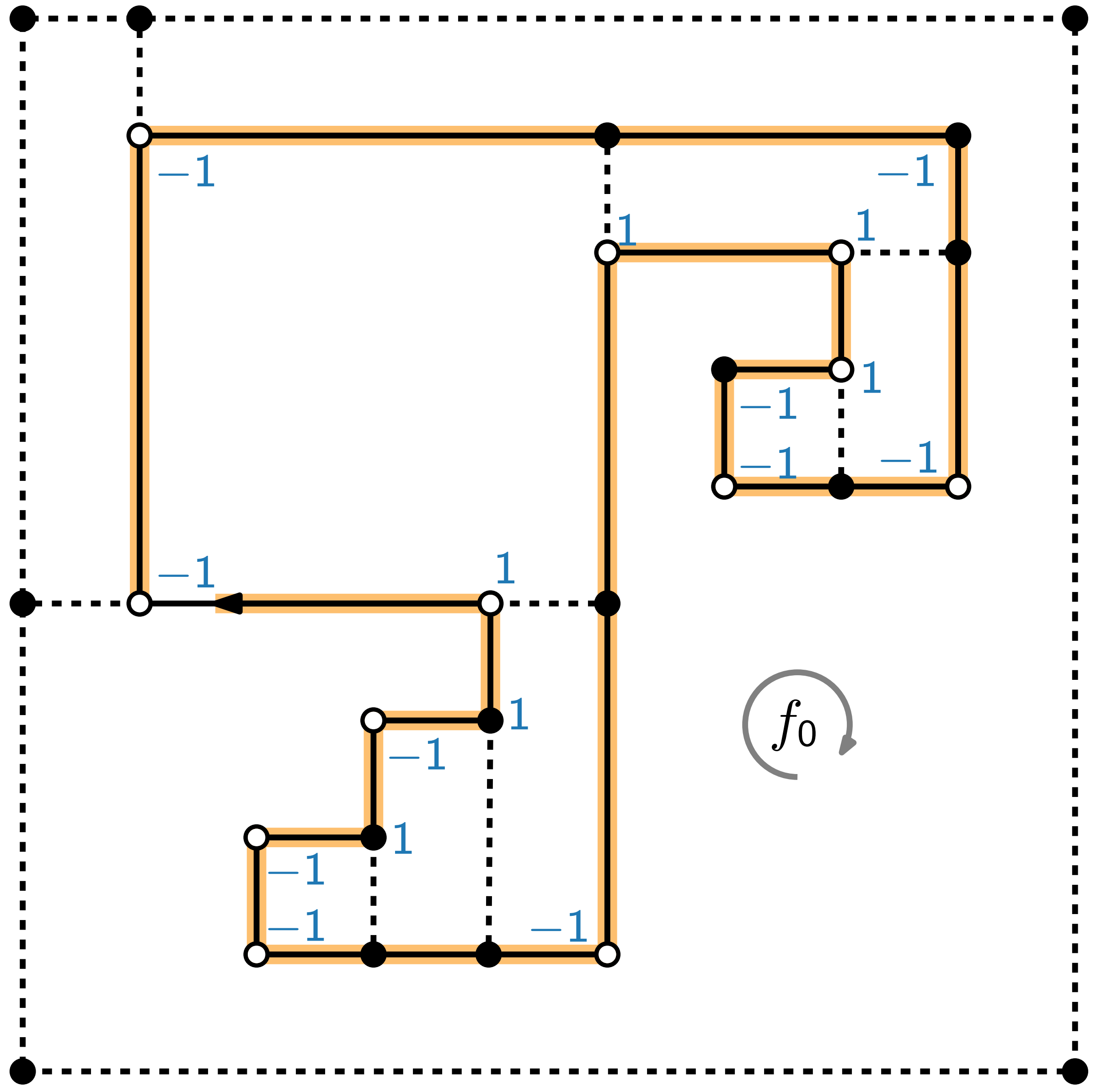
Refinement of G and $H(G)$ – Outer Face



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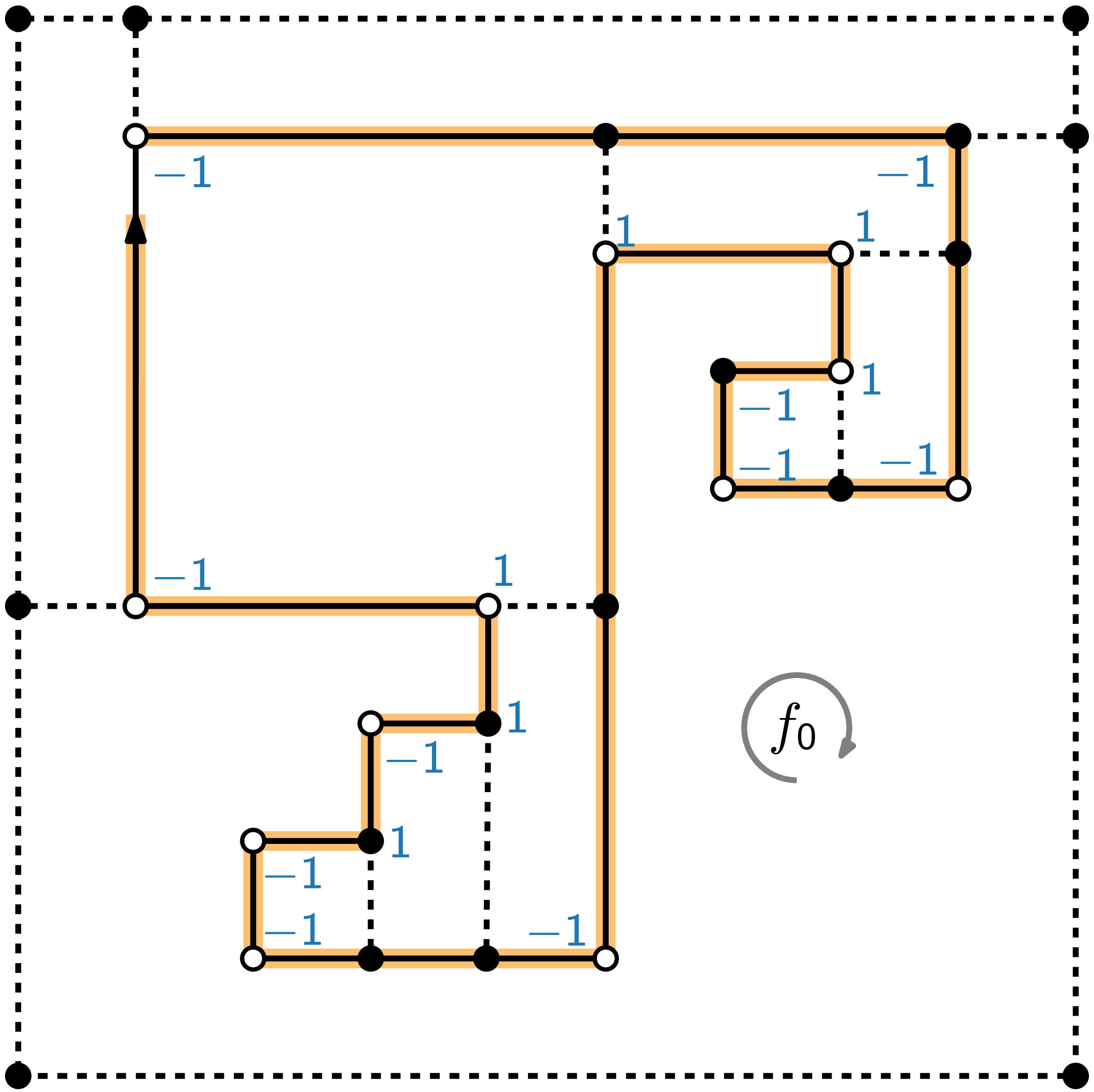
Refinement of G and $H(G)$ – Outer Face



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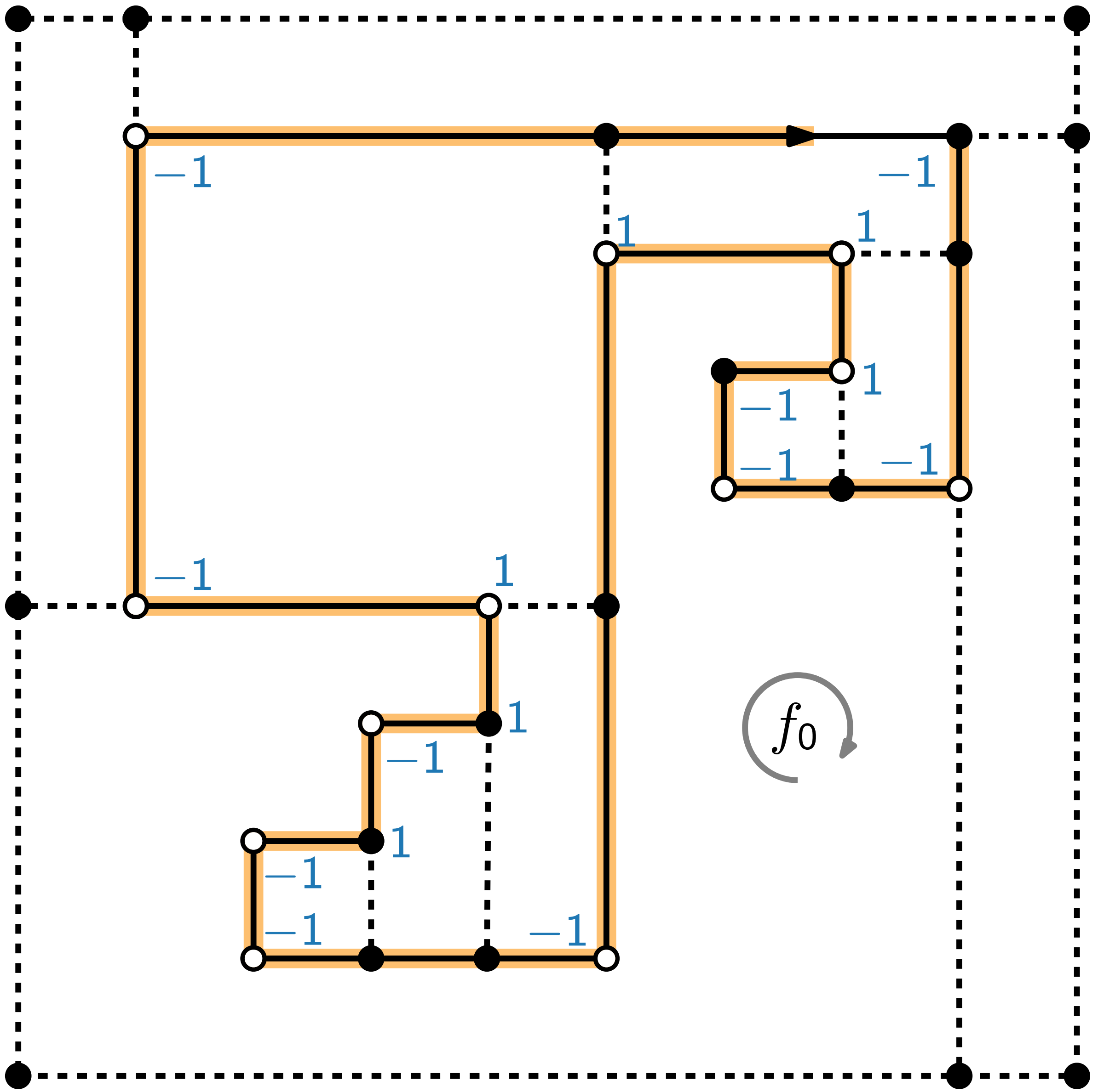
Refinement of G and $H(G)$ – Outer Face



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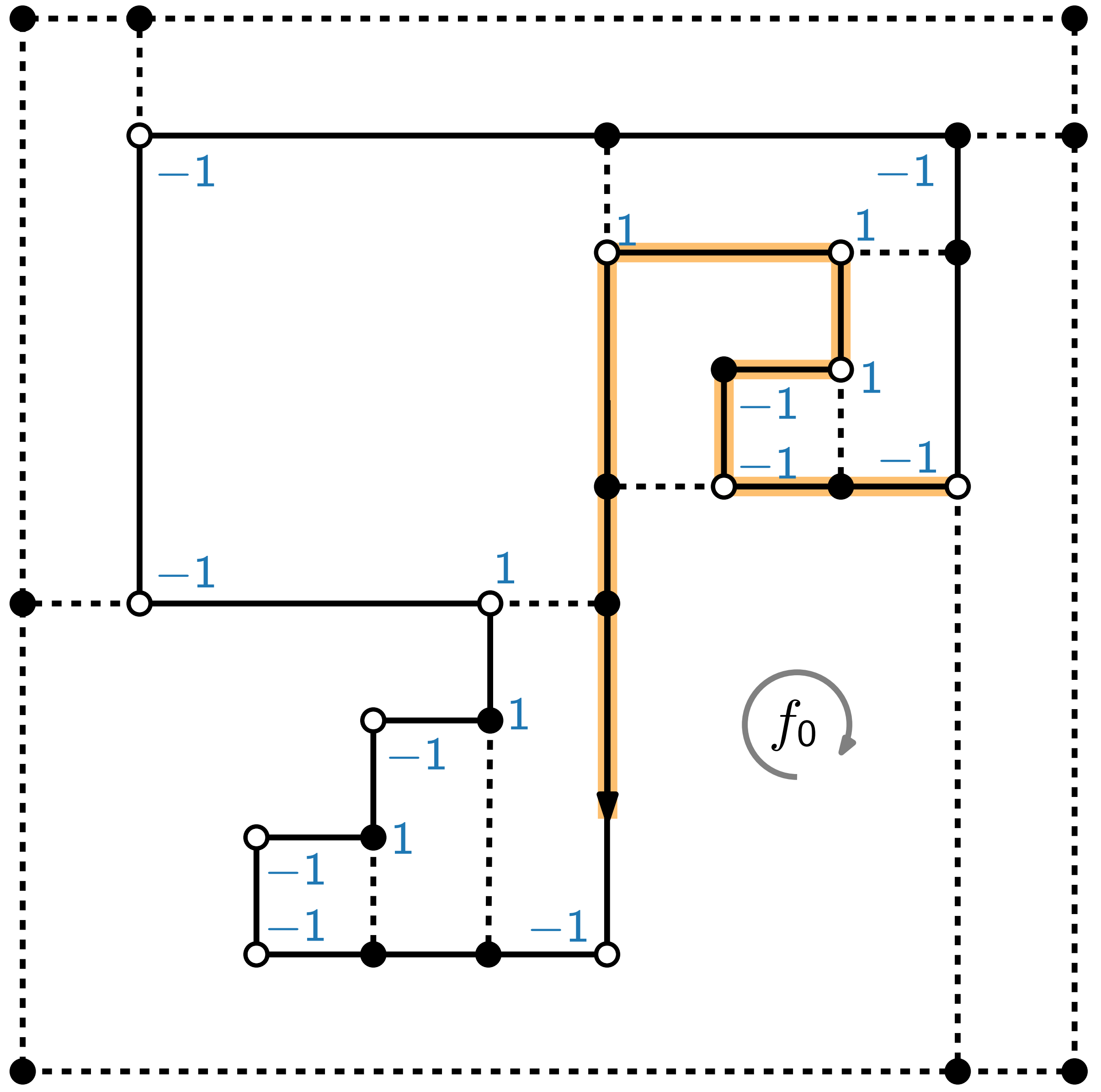

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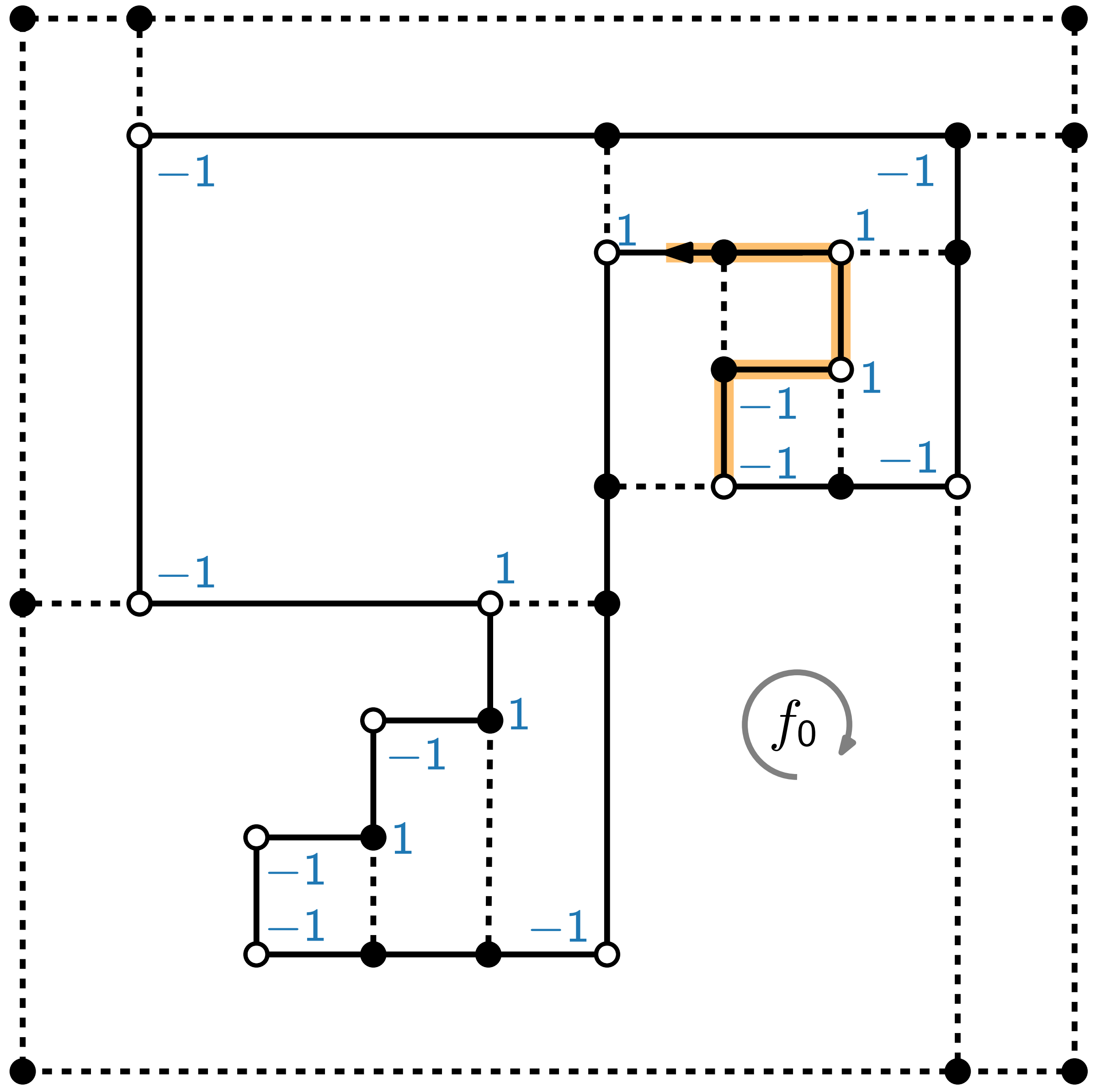
Refinement of G and $H(G)$ – Outer Face



- Add an outer rectangle
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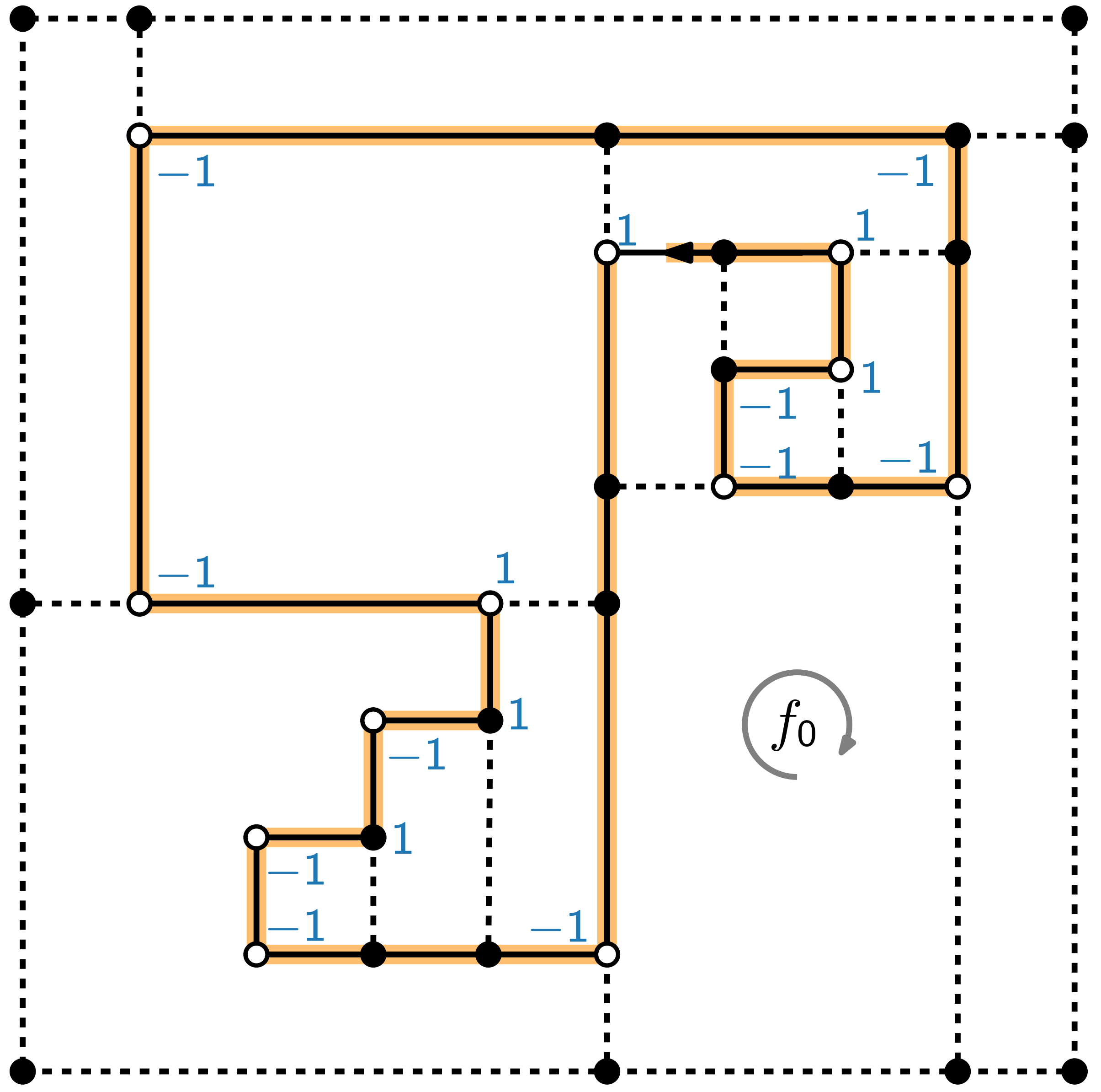
Refinement of G and $H(G)$ – Outer Face



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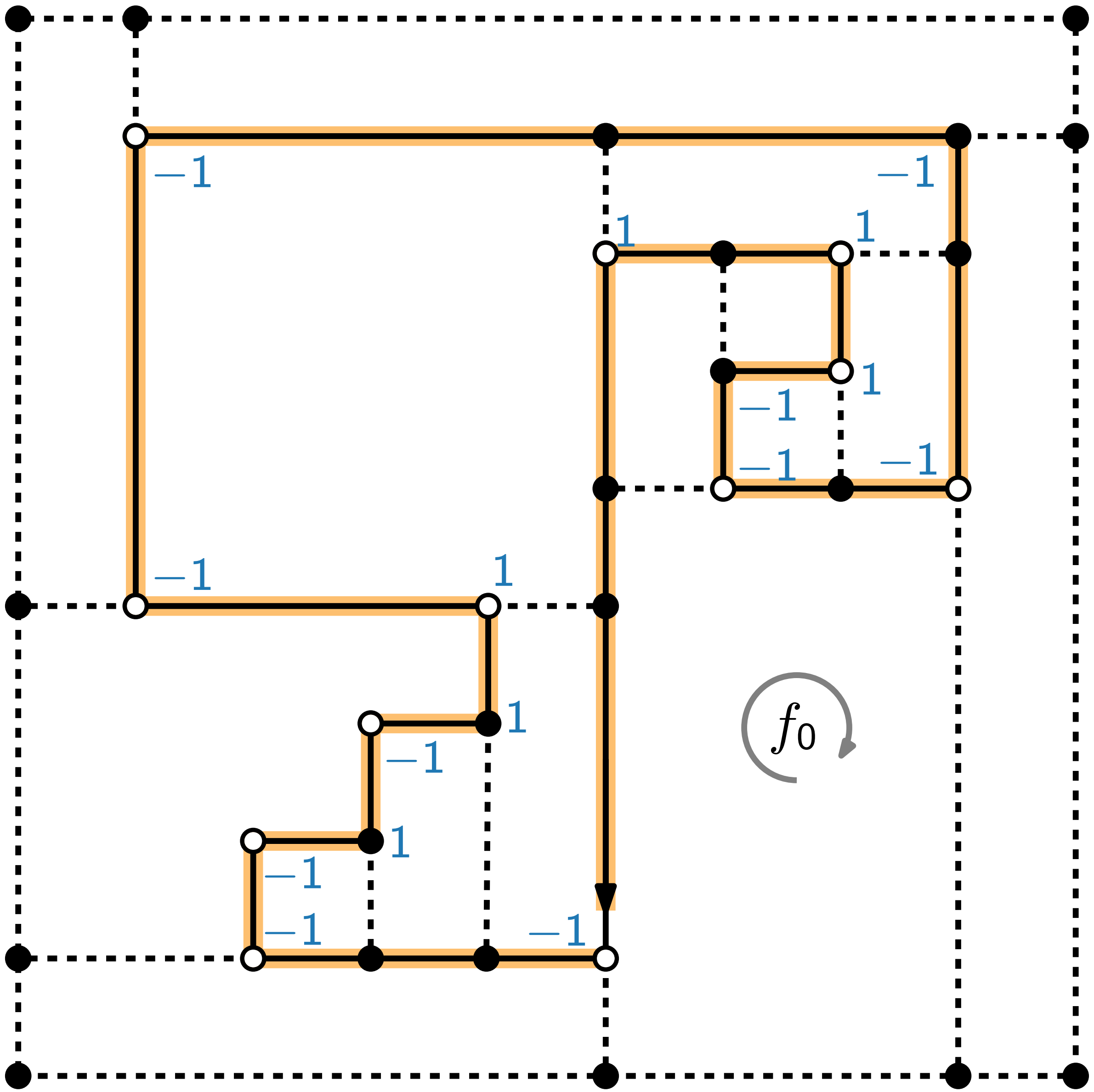
Refinement of G and $H(G)$ – Outer Face



- Add an outer rectangle
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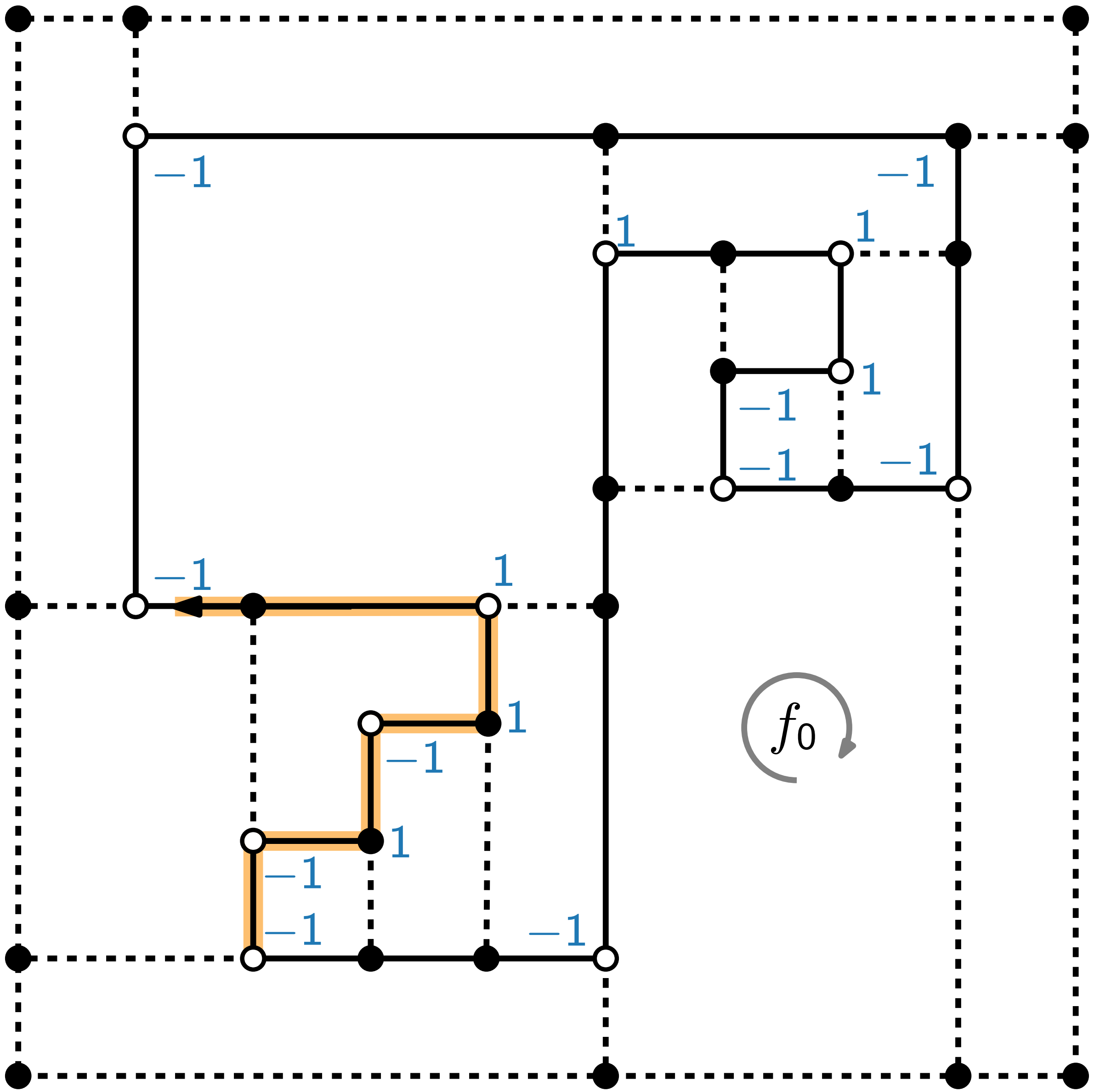
Refinement of G and $H(G)$ – Outer Face



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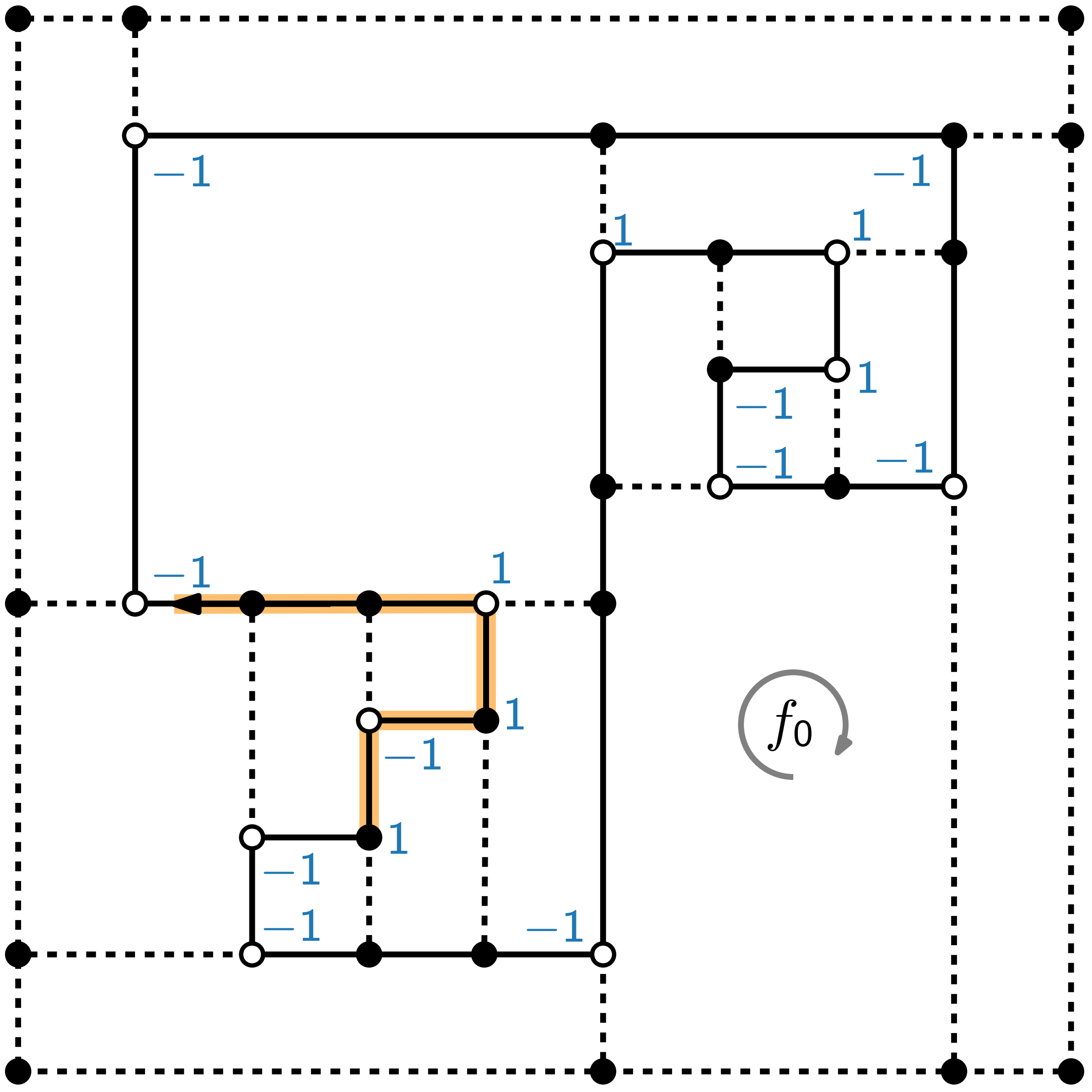
Refinement of G and $H(G)$ – Outer Face



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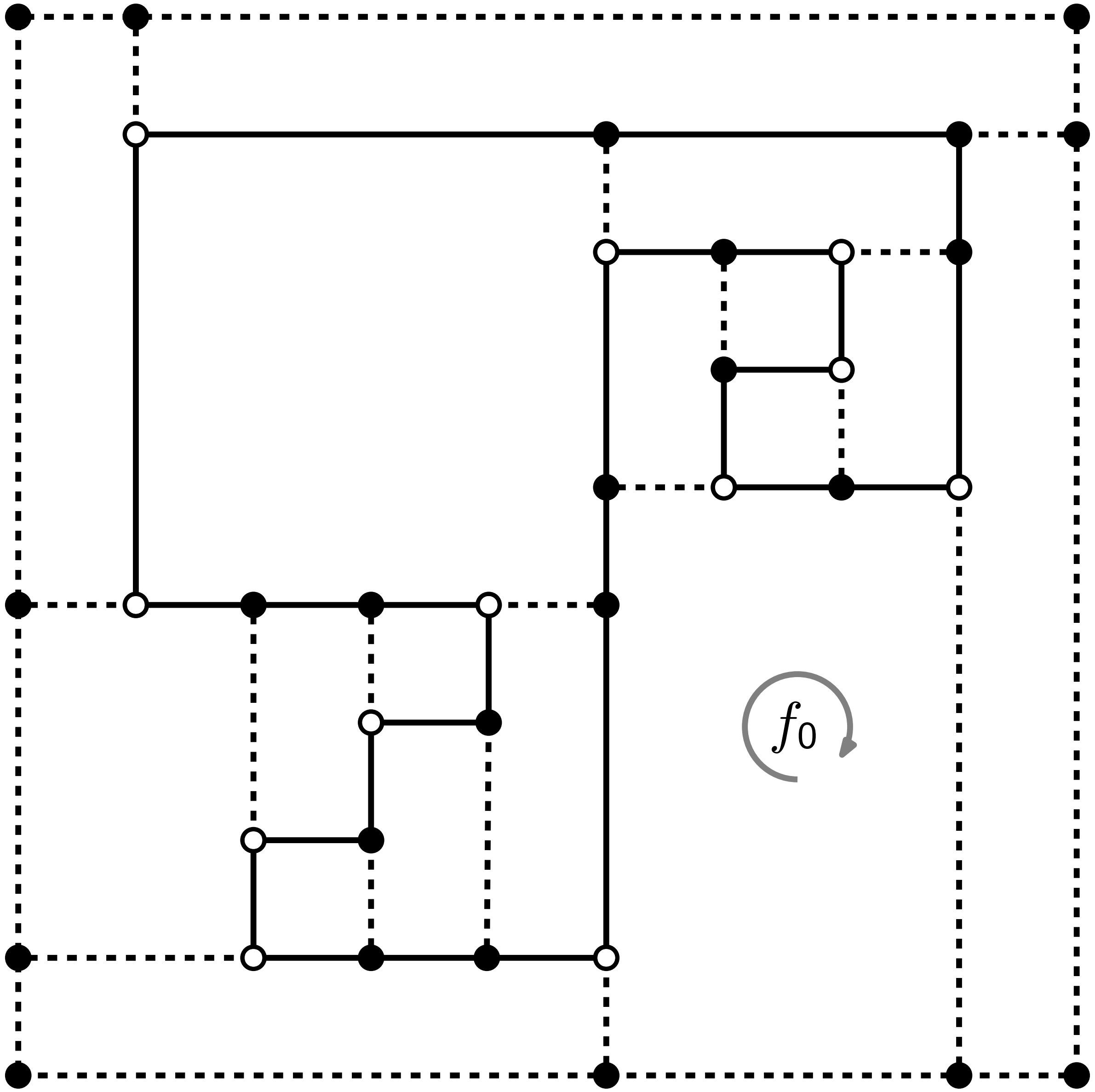
Refinement of G and $H(G)$ – Outer Face



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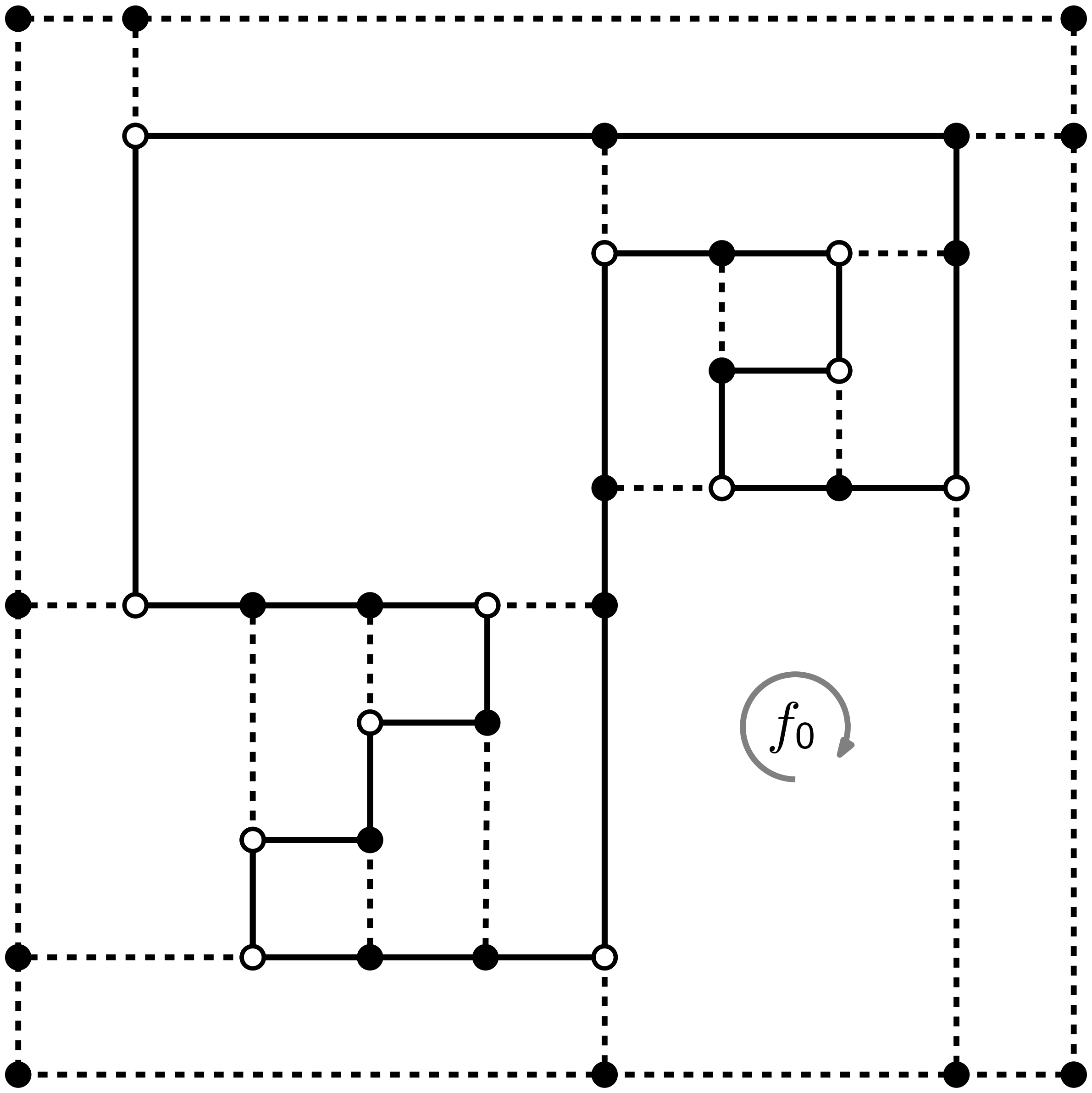
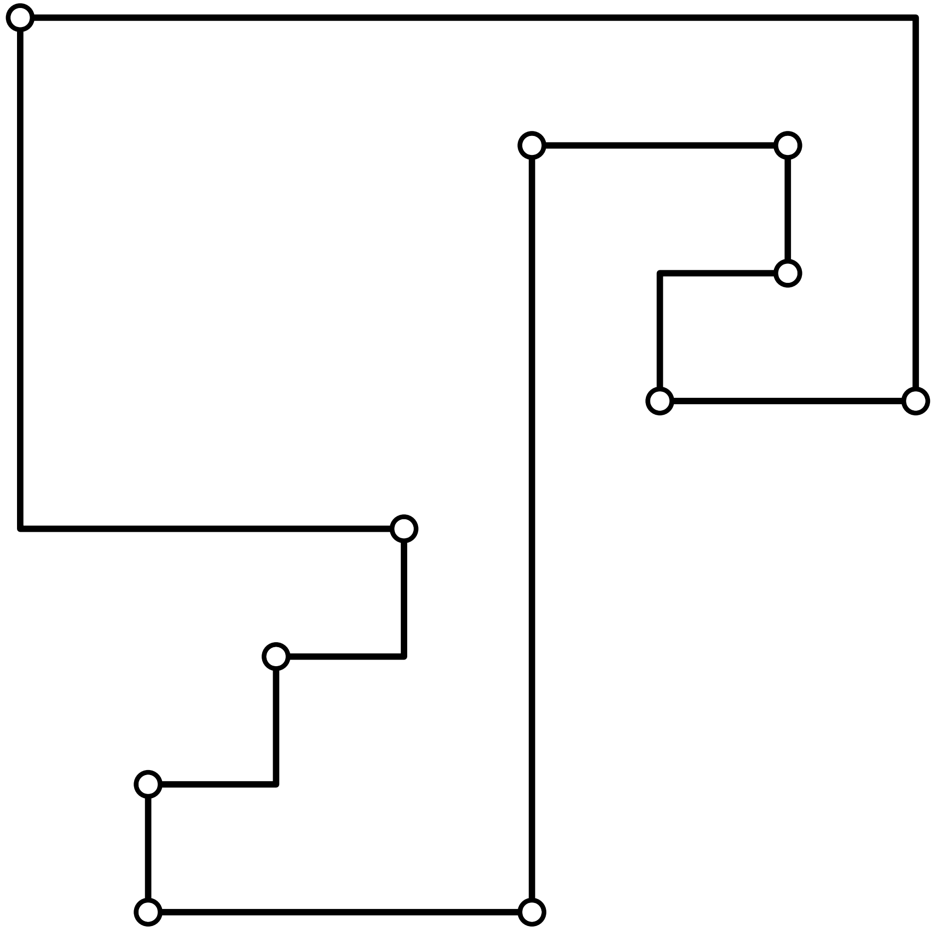
Refinement of G and $H(G)$ – Outer Face



- Add an outer rectangle
- Traverse clockwise

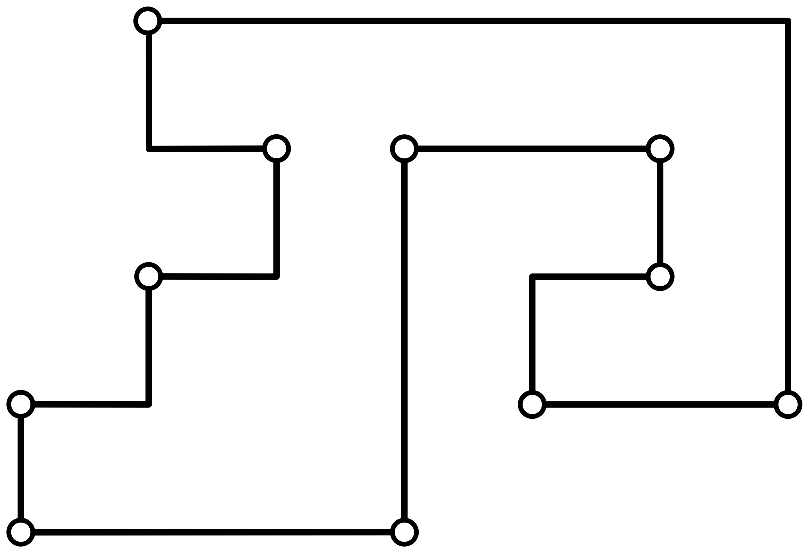
Area minimized?

Refinement of G and $H(G)$ – Outer Face



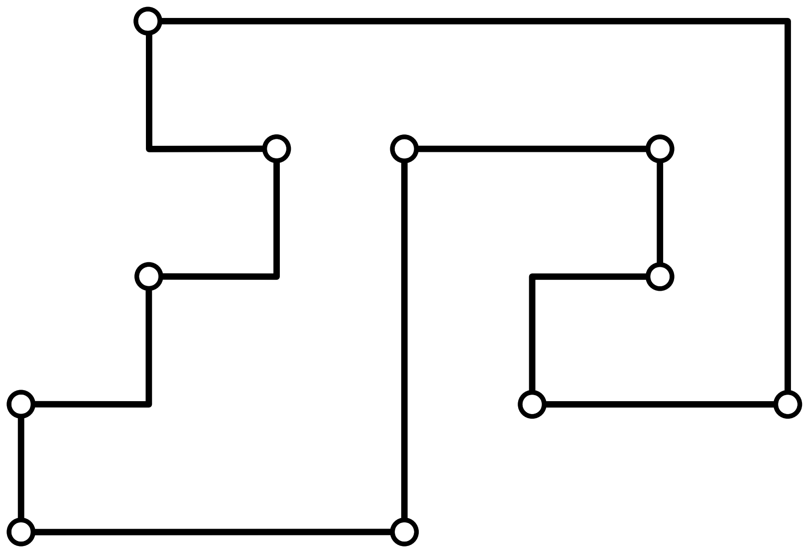
Area minimized?

Refinement of G and $H(G)$ – Outer Face



Area minimized? No!

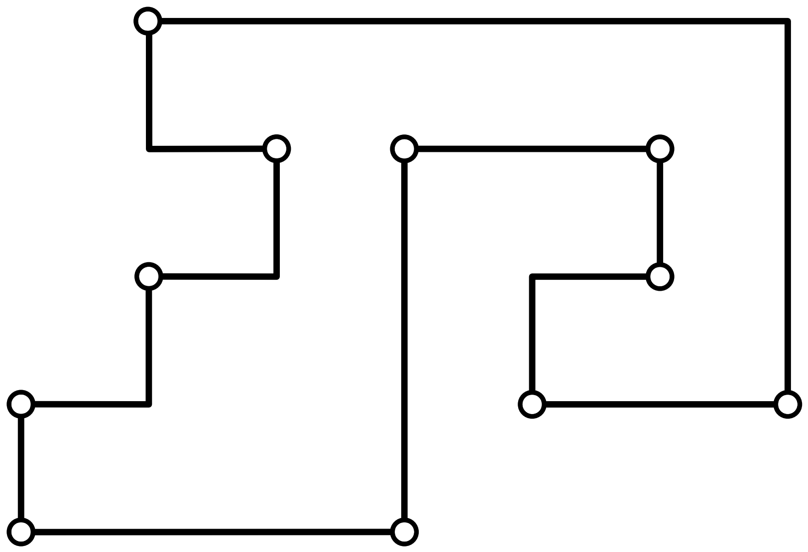
Refinement of G and $H(G)$ – Outer Face



Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

Refinement of G and $H(G)$ – Outer Face



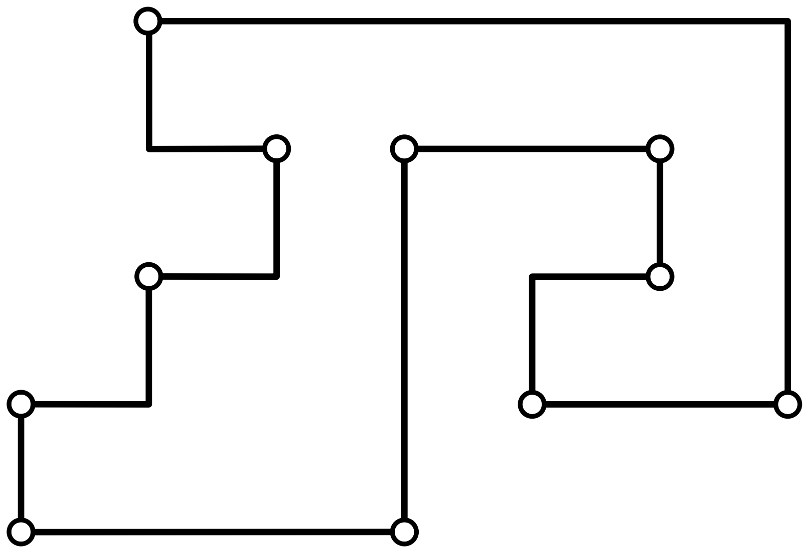
Area minimized? No!

vertices

bends

But we get bound $O((n + b)^2)$ on the area.

Refinement of G and $H(G)$ – Outer Face



Area minimized? No!

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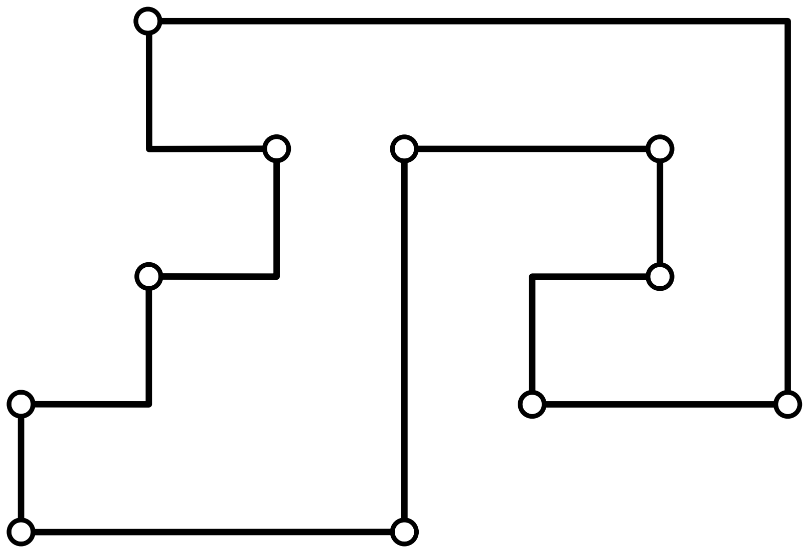
But we get bound $O((n+b)^2)$ on the area.

Theorem. [Patrignani 2001]
Compaction for a given orthogonal representation is NP-hard in general.

Refinement of G and $H(G)$ – Outer Face



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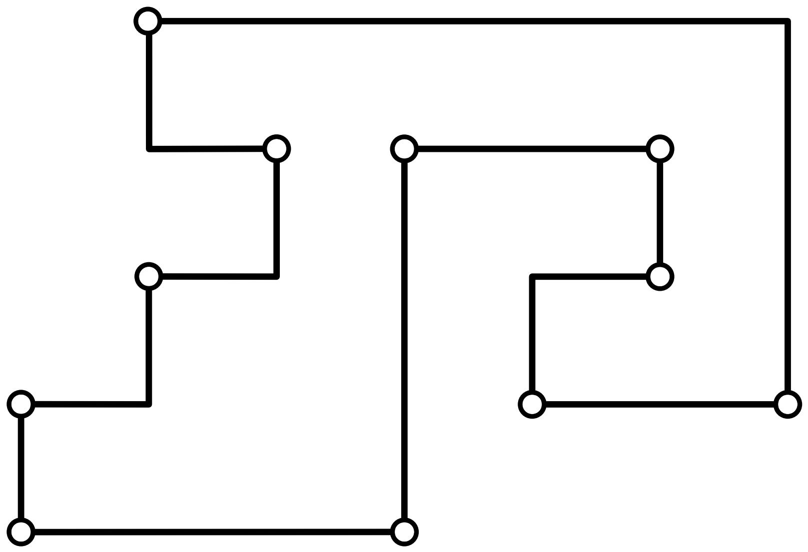
But we get bound $O((n + b)^2)$ on the area.

Theorem. [Patrignani 2001]
Compaction for a given orthogonal representation is NP-hard in general.

Theorem. [EFKSSW 2022]
Compaction is NP-hard even for orthogonal representations of *cycles*.

Refinement of G and $H(G)$ – Outer Face



Area minimized? No!

vertices

bends

But we get bound $O((n + b)^2)$ on the area.

Theorem. [Patrignani 2001]
Compaction for a given orthogonal representation is NP-hard in general.

Theorem. [EFKSSW 2022]
Compaction is NP-hard even for orthogonal representations of *cycles*.

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Polynomial-time reduction from the NP-complete satisfiability problem (SAT).

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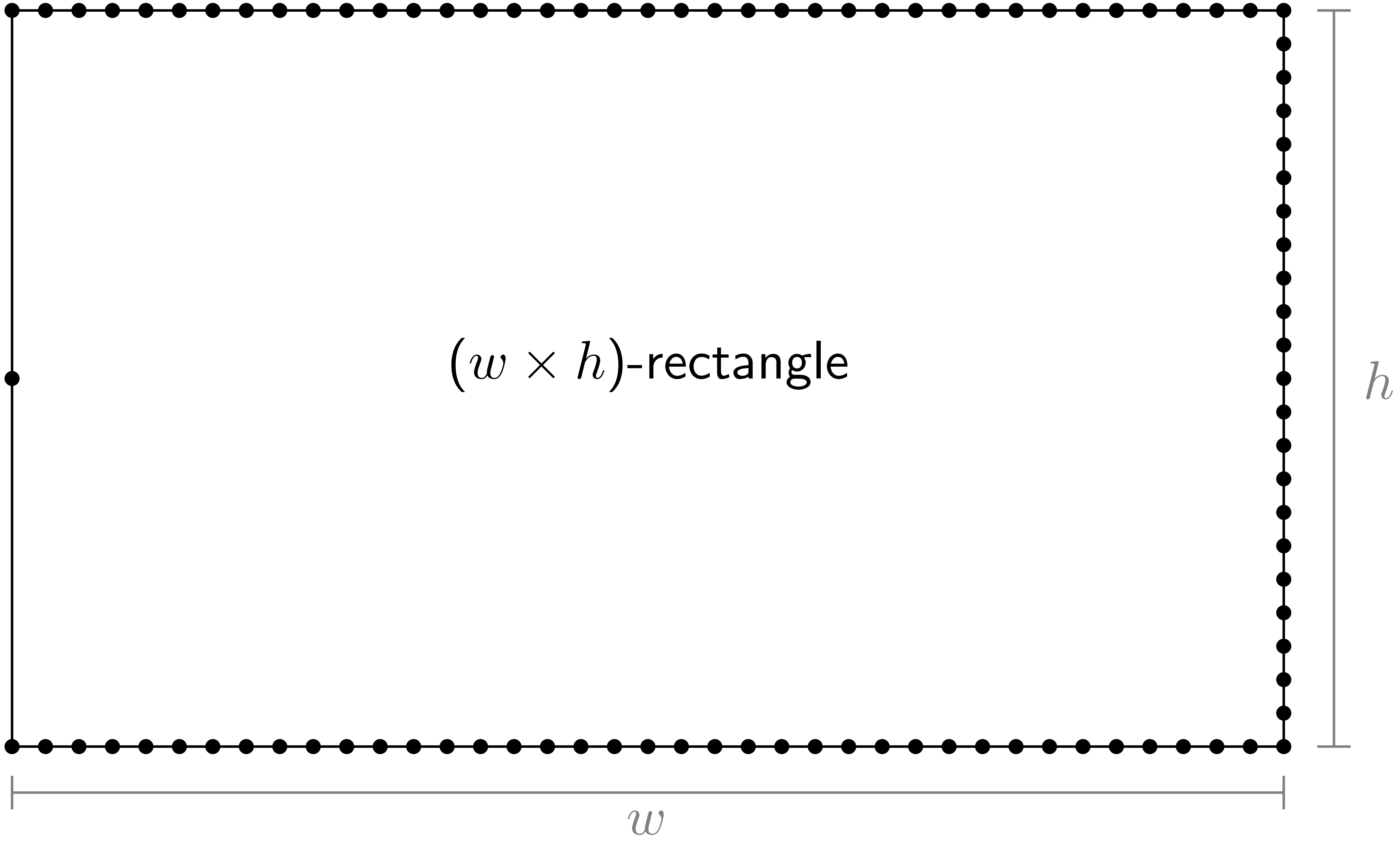
- Given SAT instance $\Phi \Rightarrow$ construct a plane graph G and a orthogonal description $H(G)$
- Φ is satisfiable $\Leftrightarrow$ G can be drawn w.r.t. $H(G)$ in area K for some specific number K

Boundary, Belt, and “Piston” Gadget



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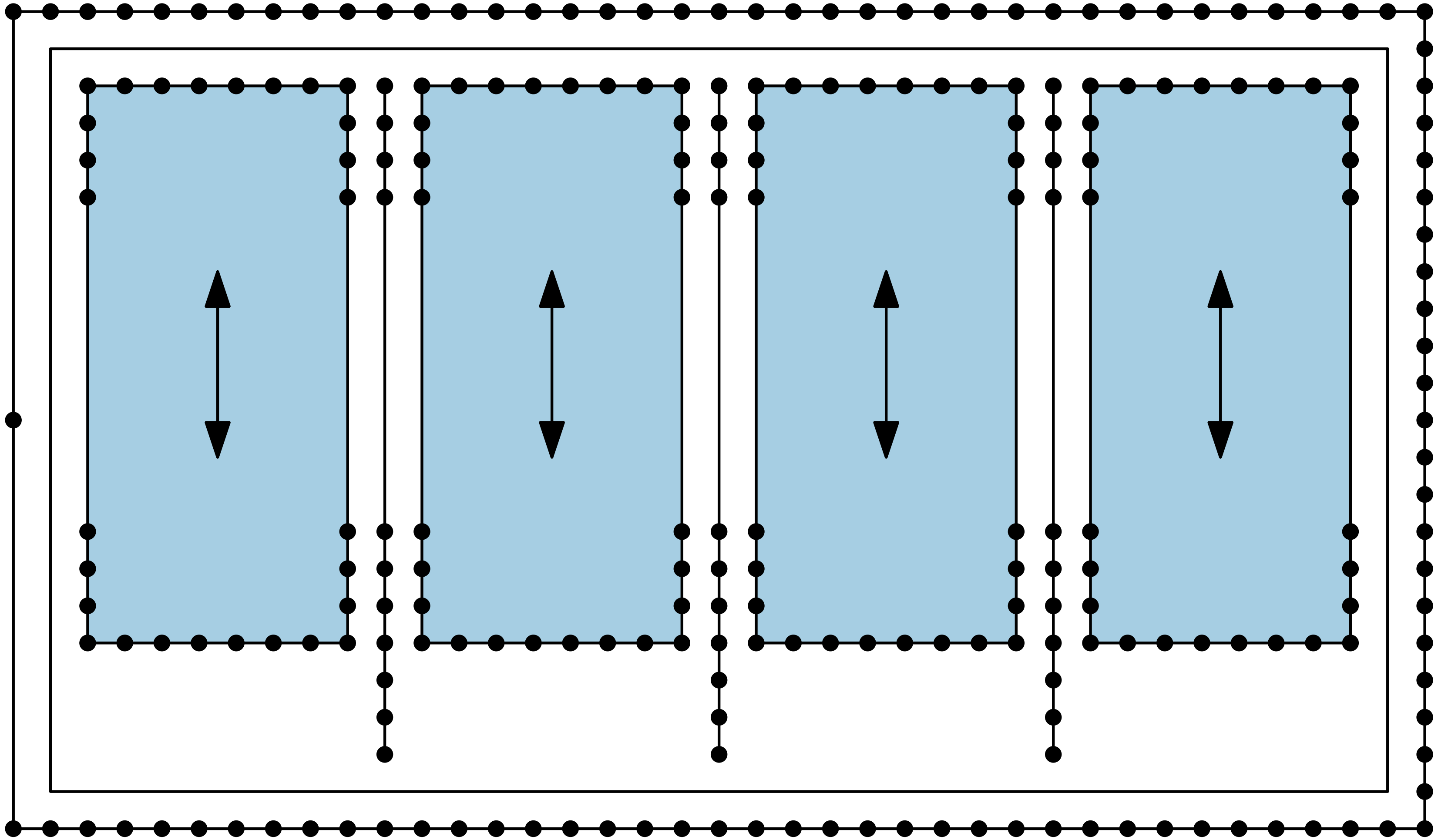

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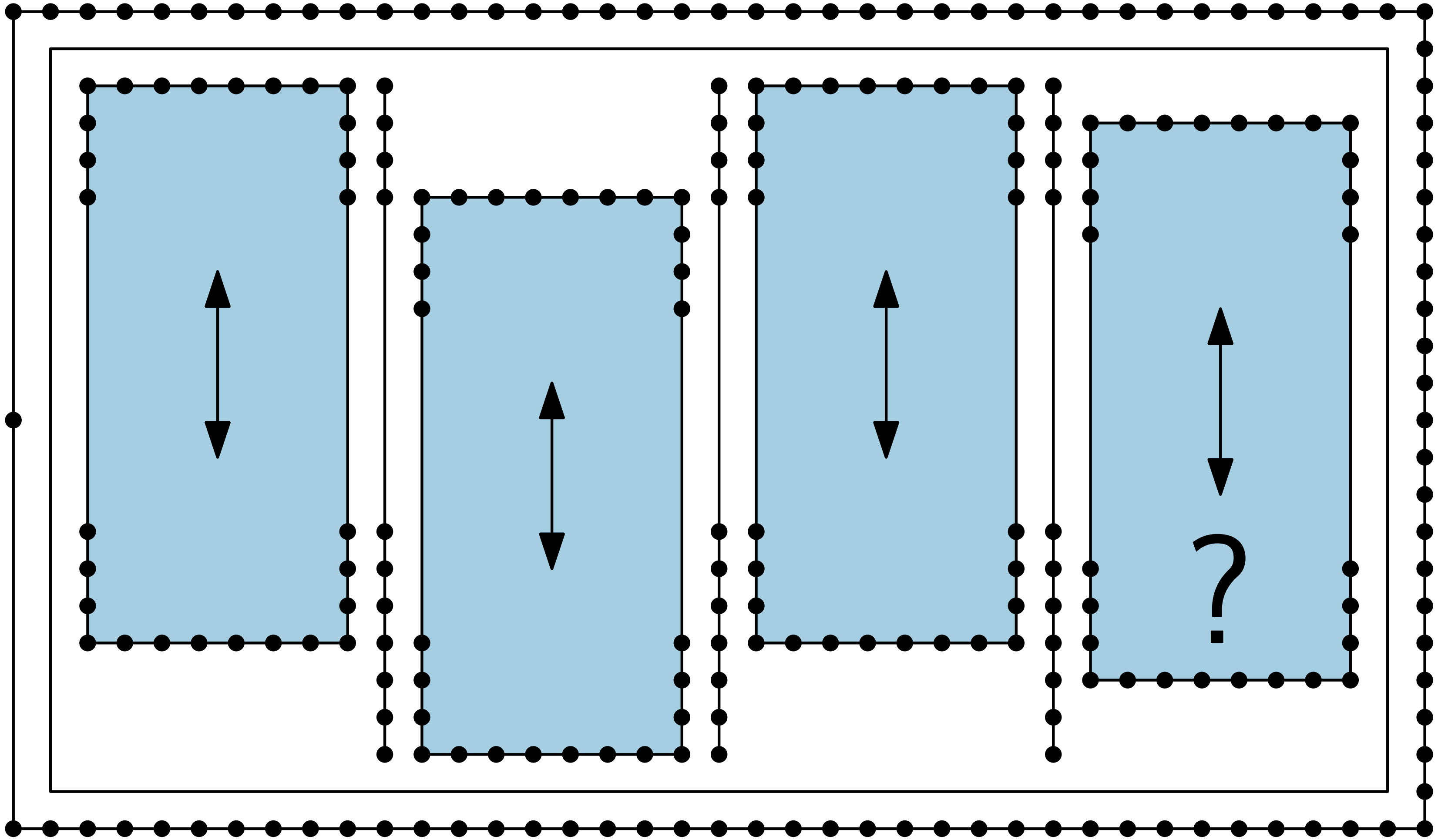

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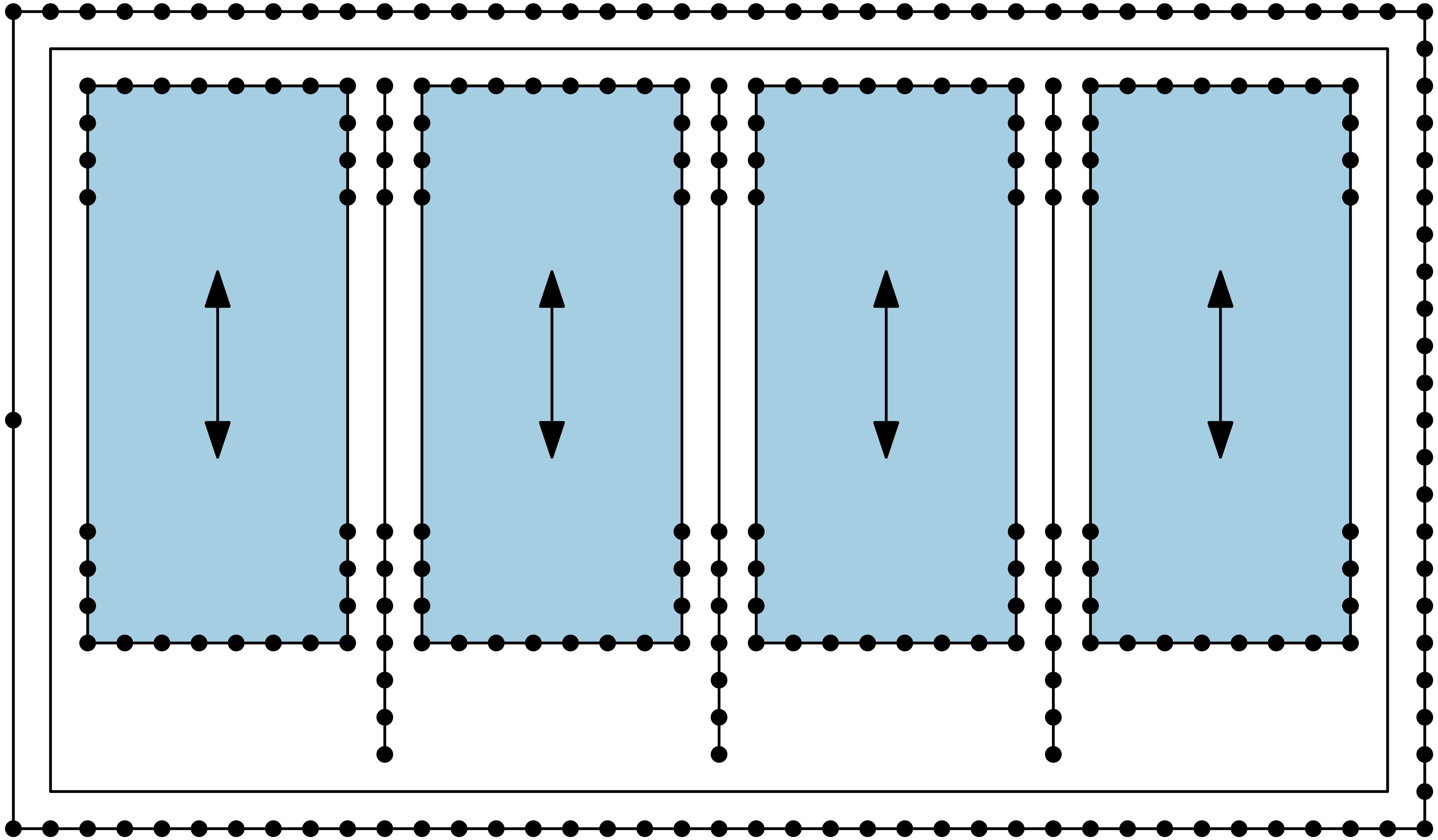

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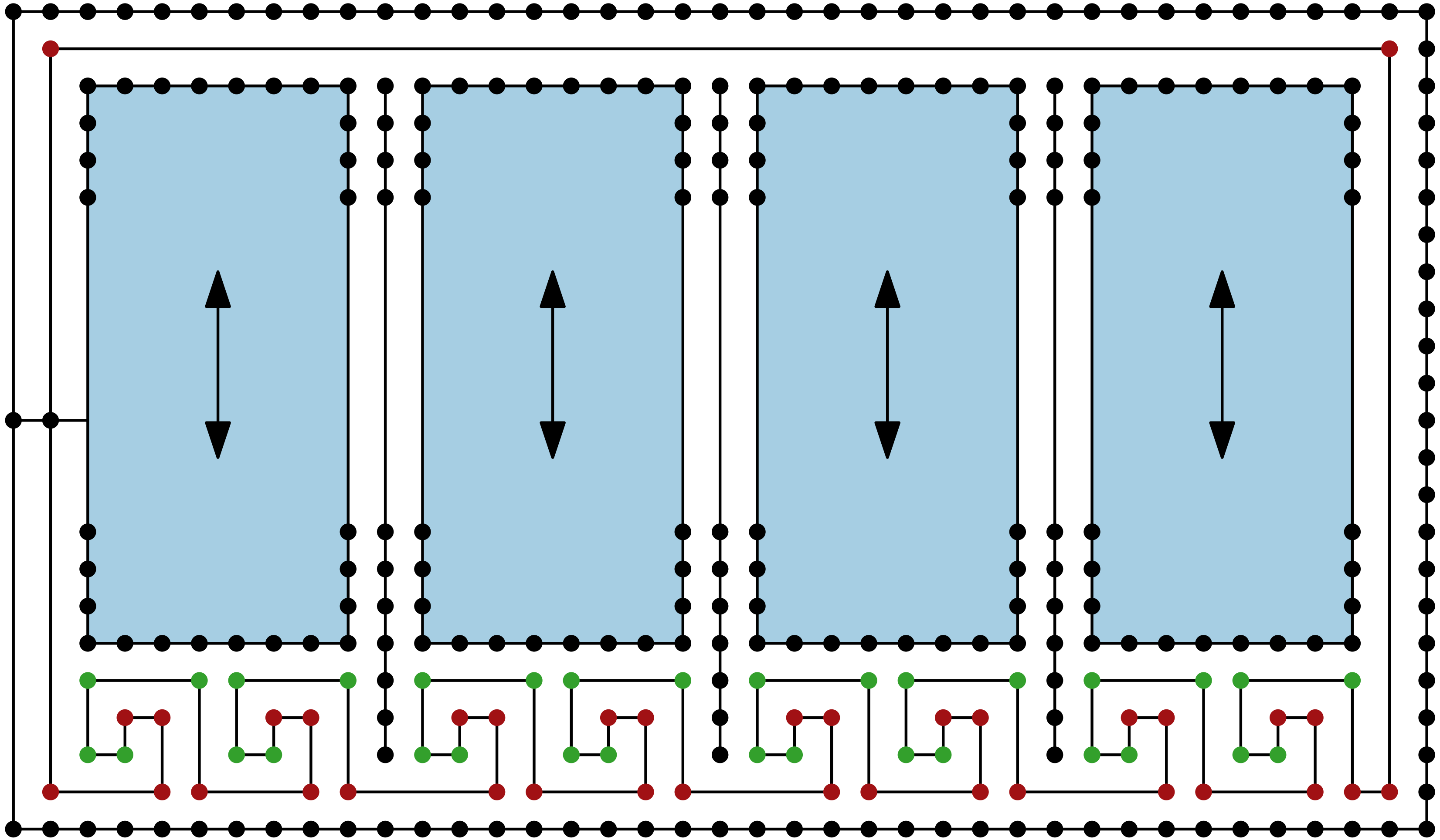

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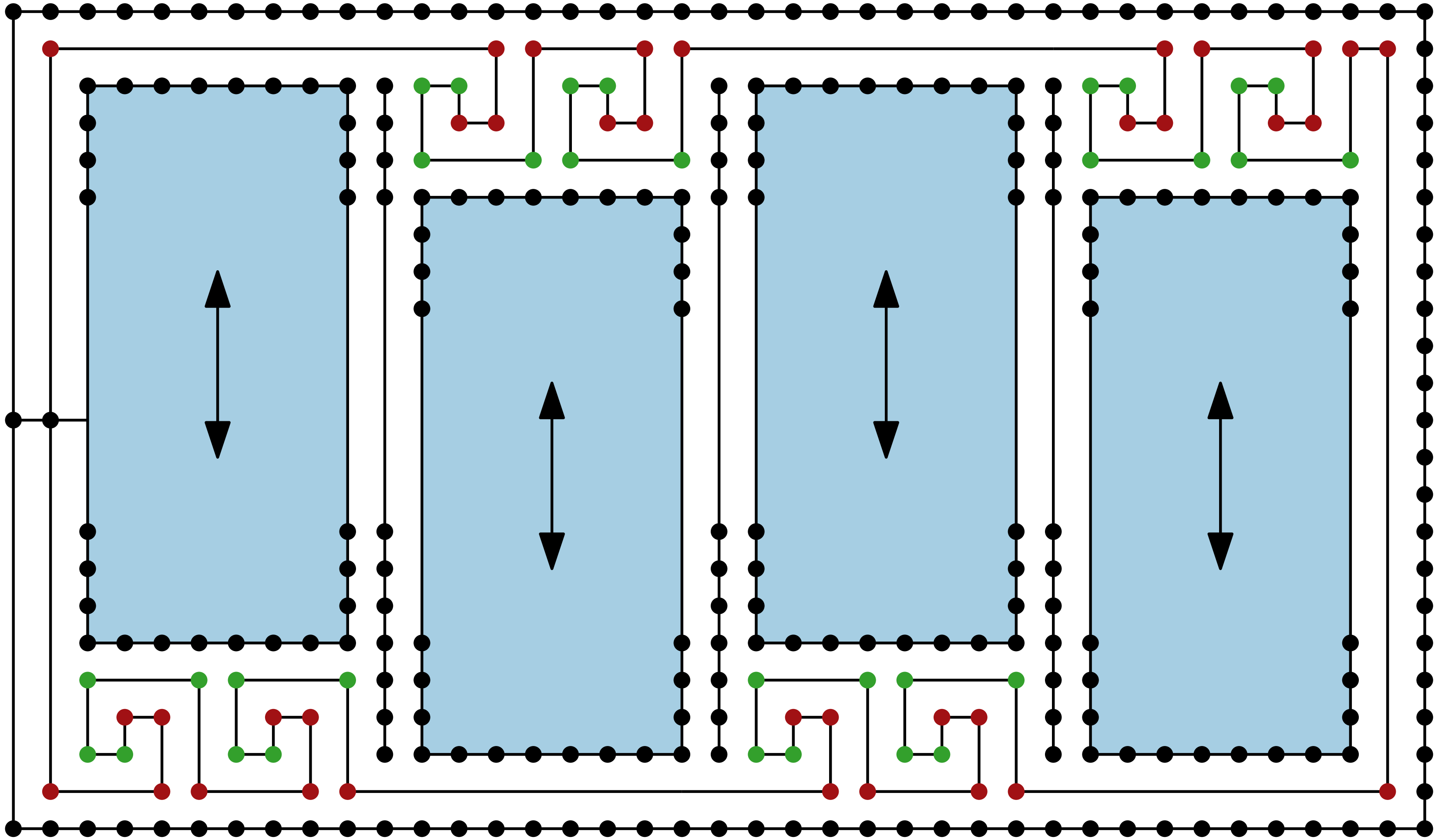

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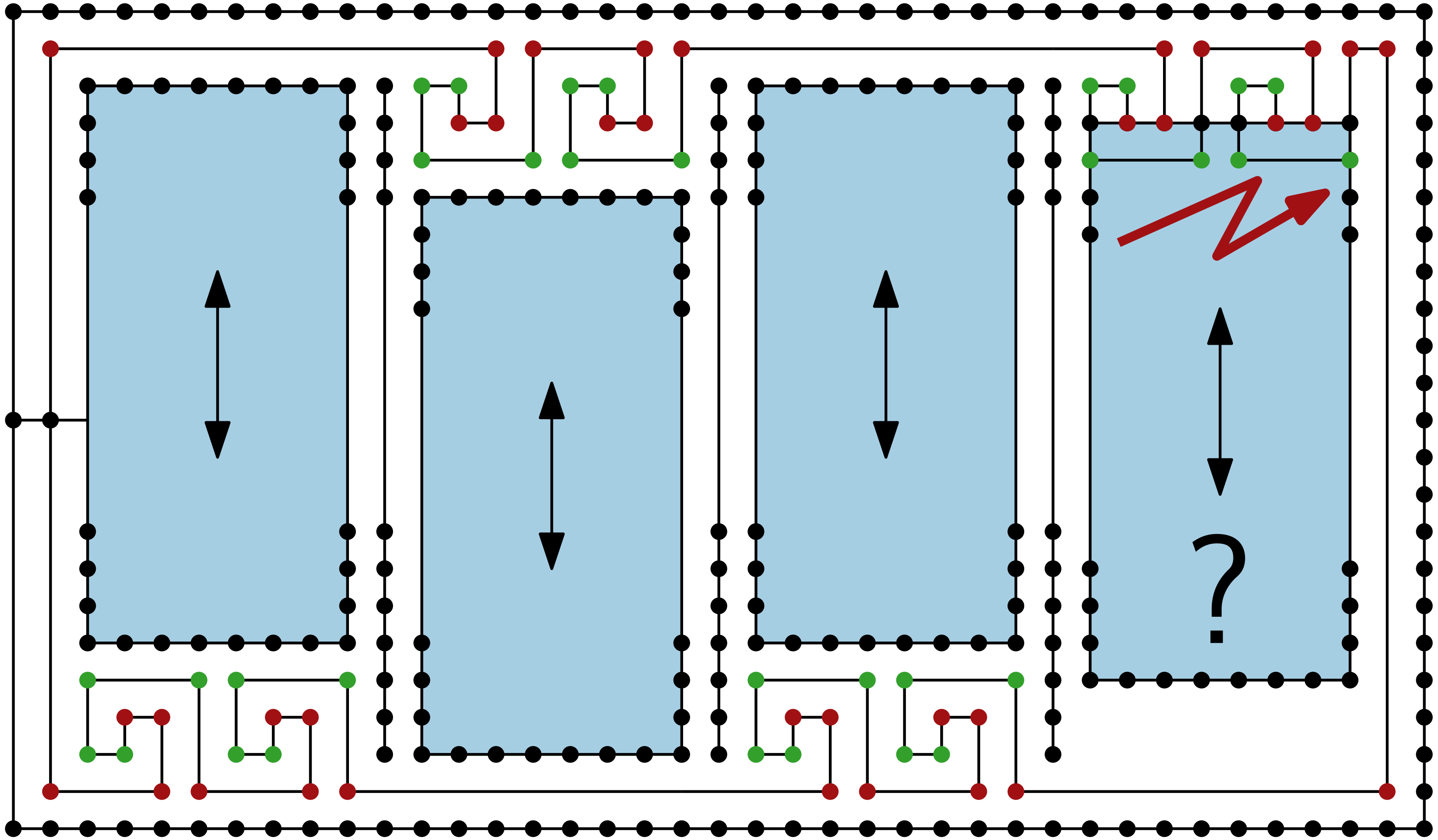

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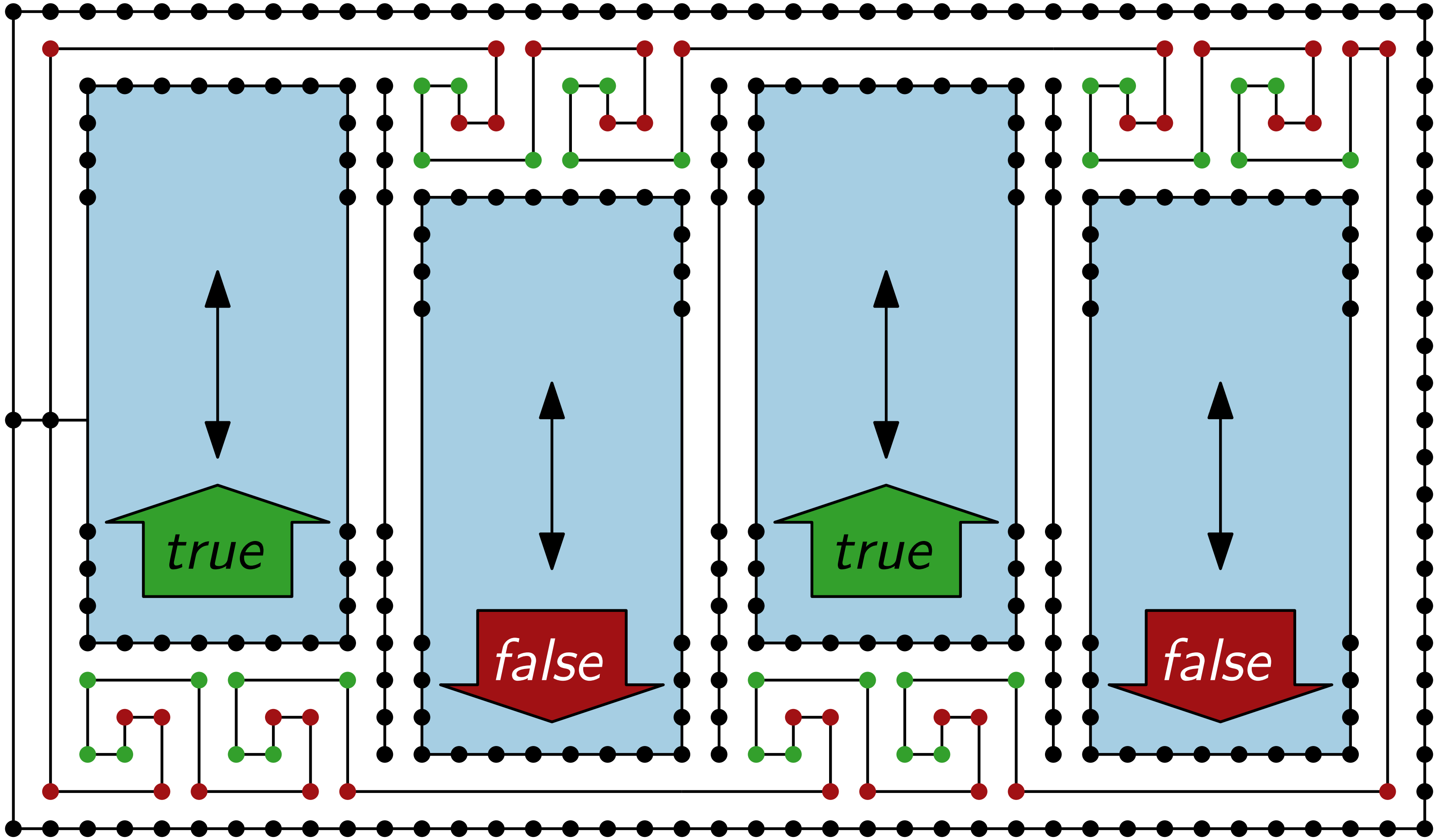
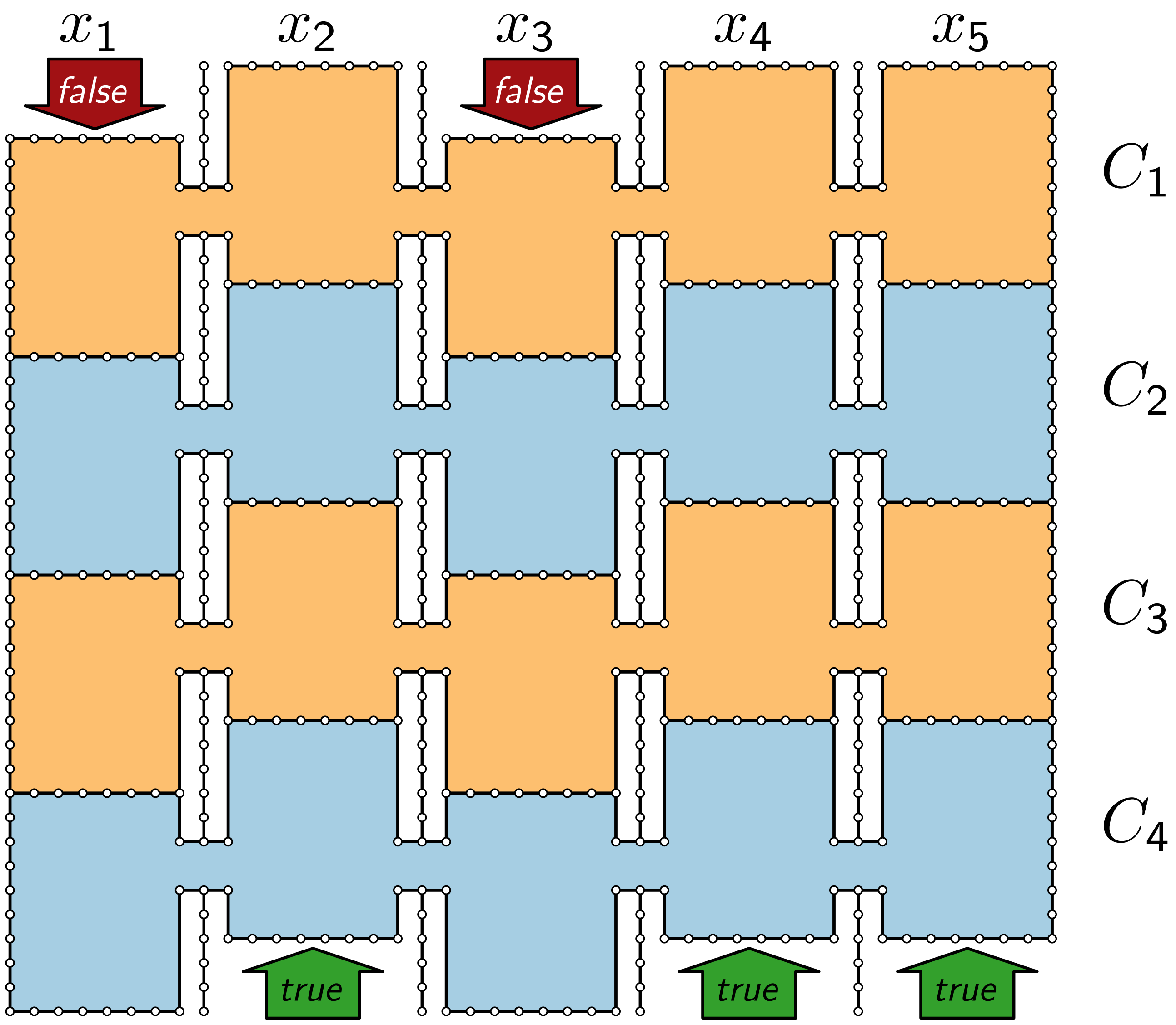


Clause Gadgets



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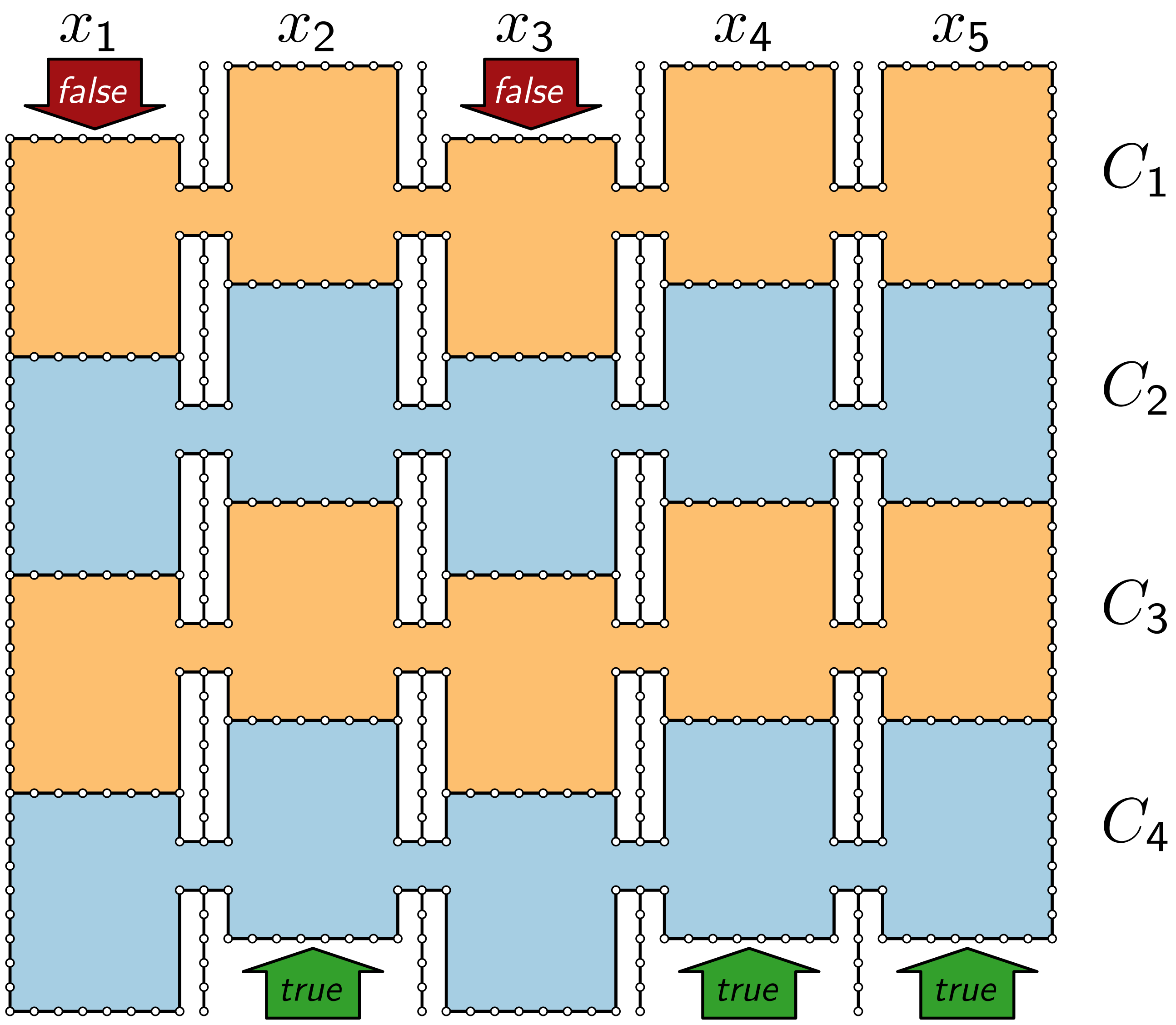

Example:

$C_1 = x_2 \lor \neg x_4$

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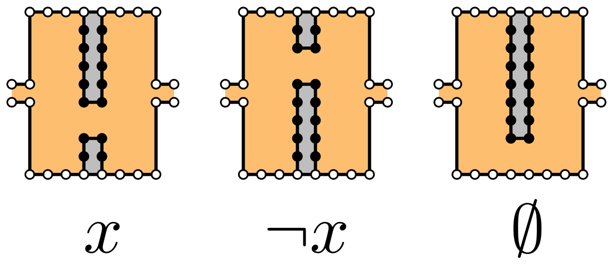

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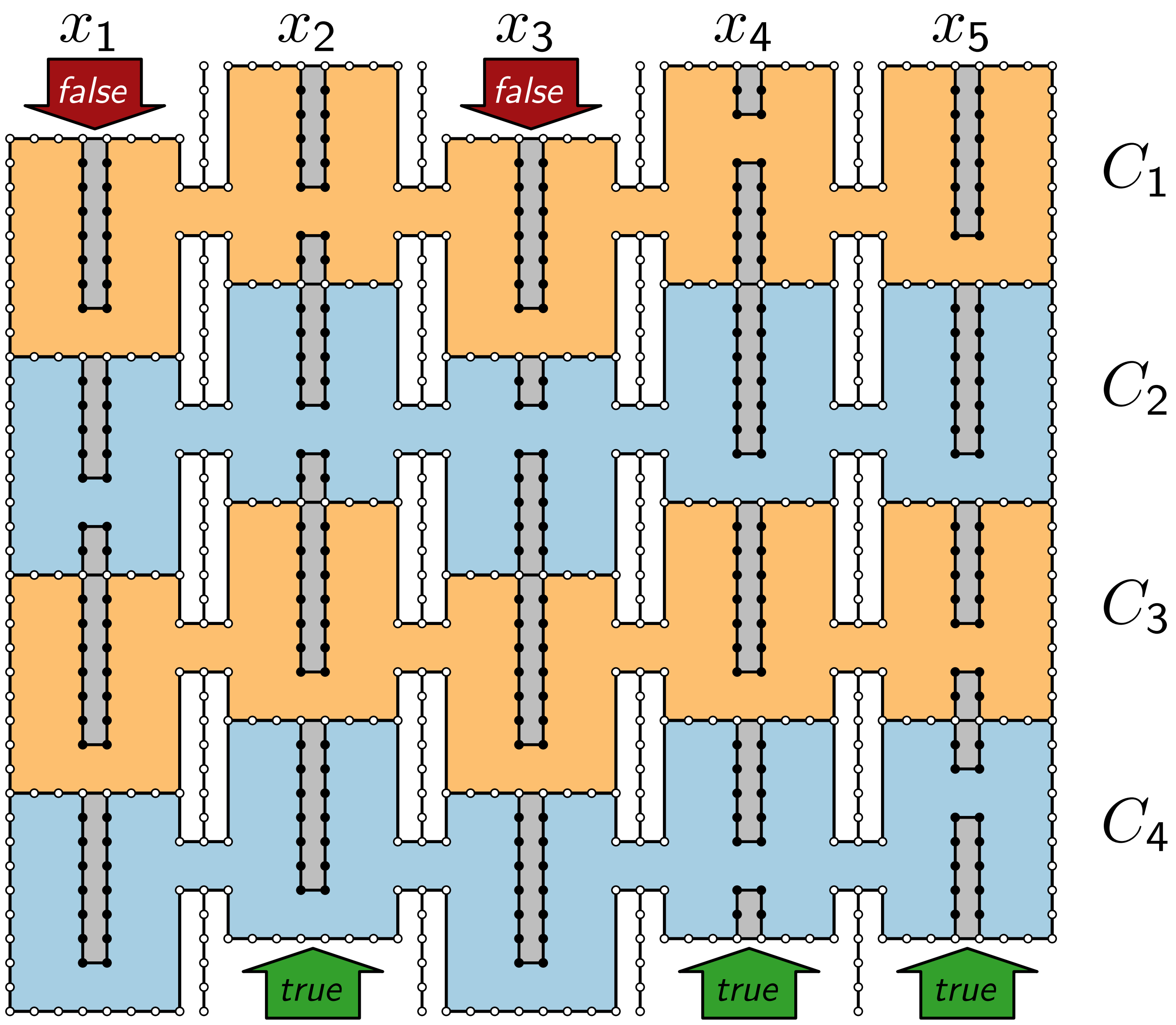

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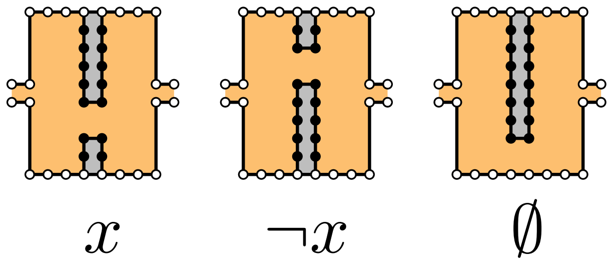

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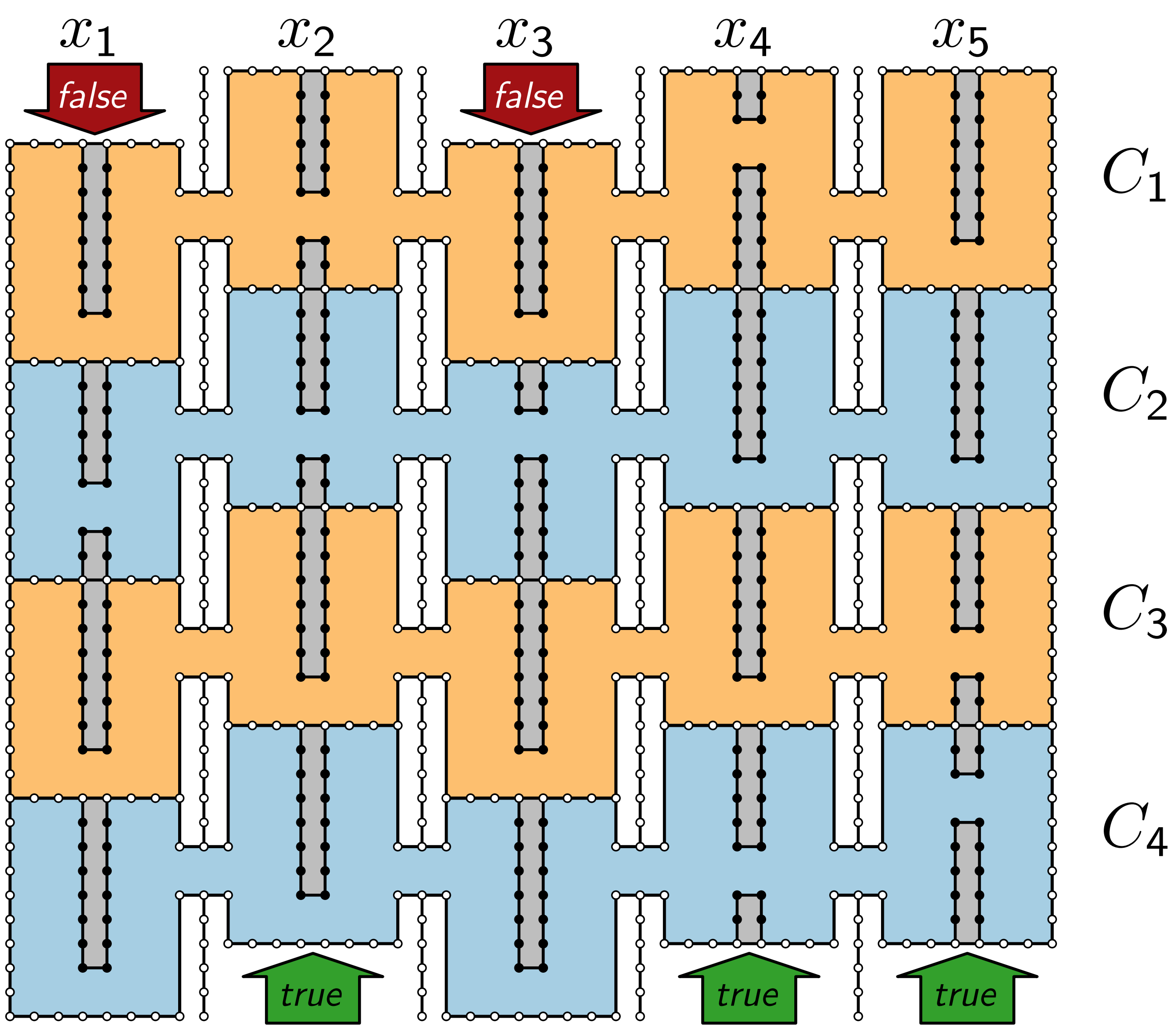

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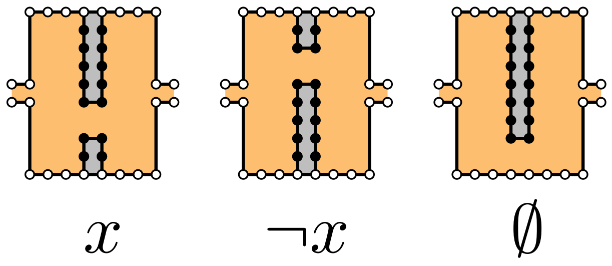
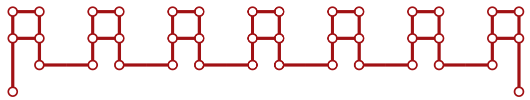
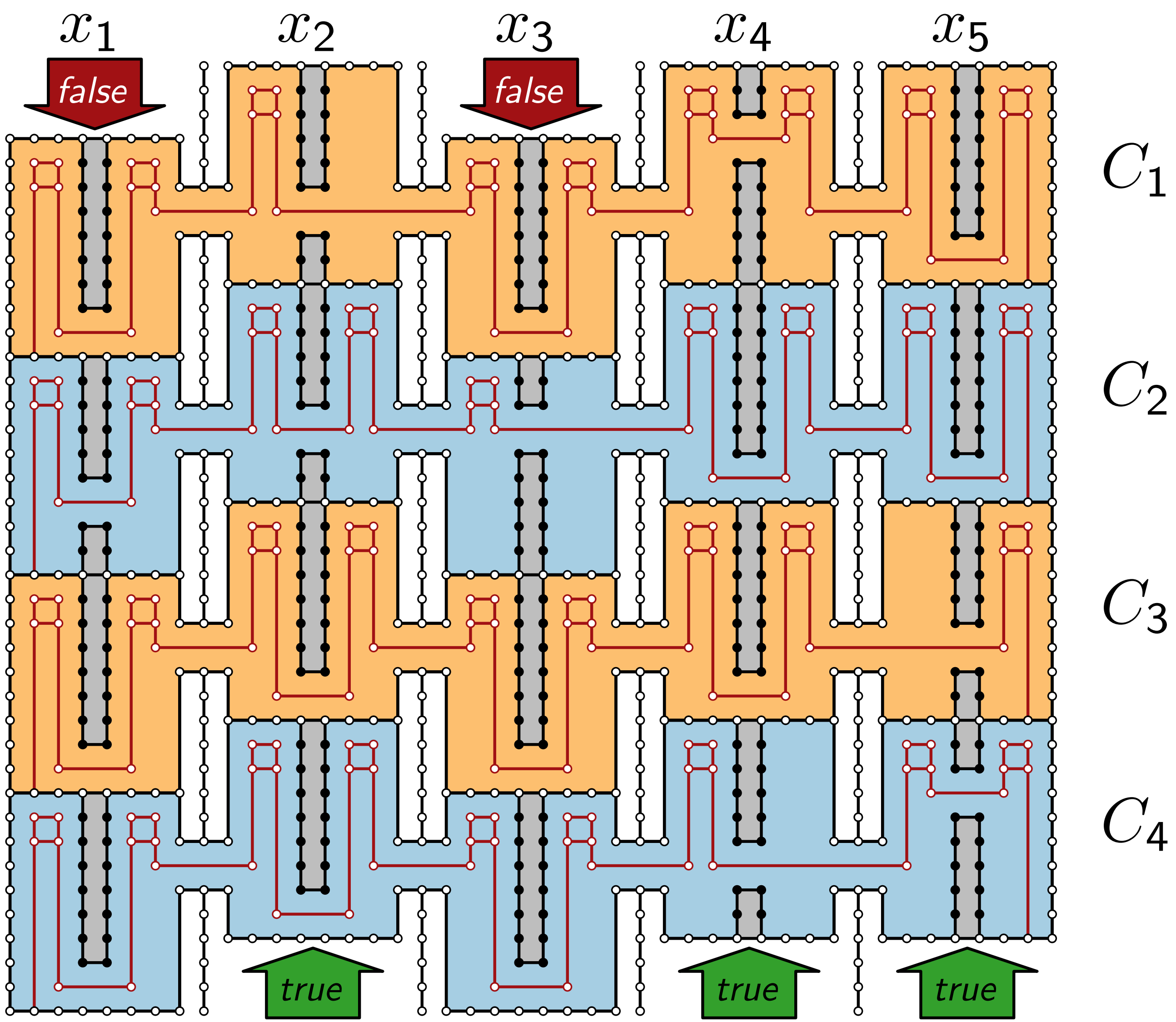




insert $(2n-1)$-chain through each clause

Clause Gadgets



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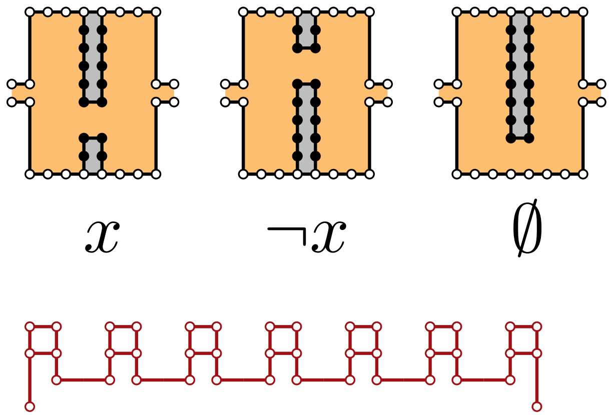

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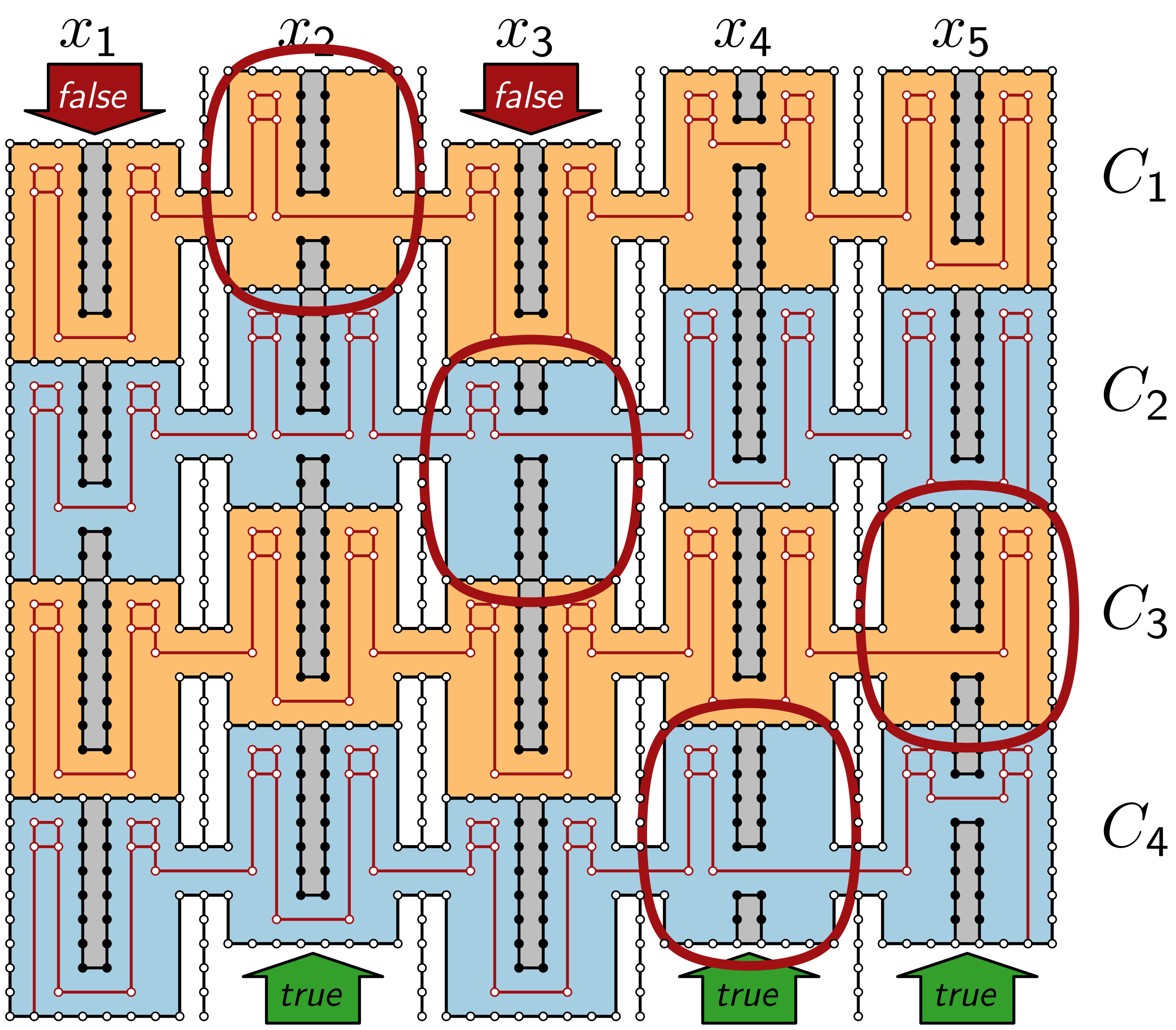

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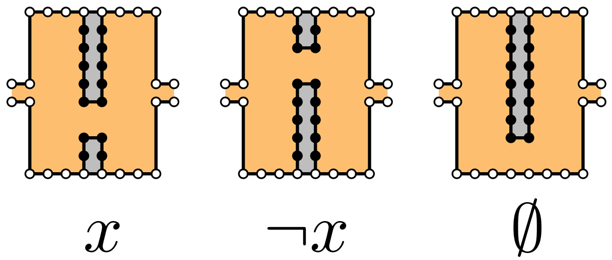

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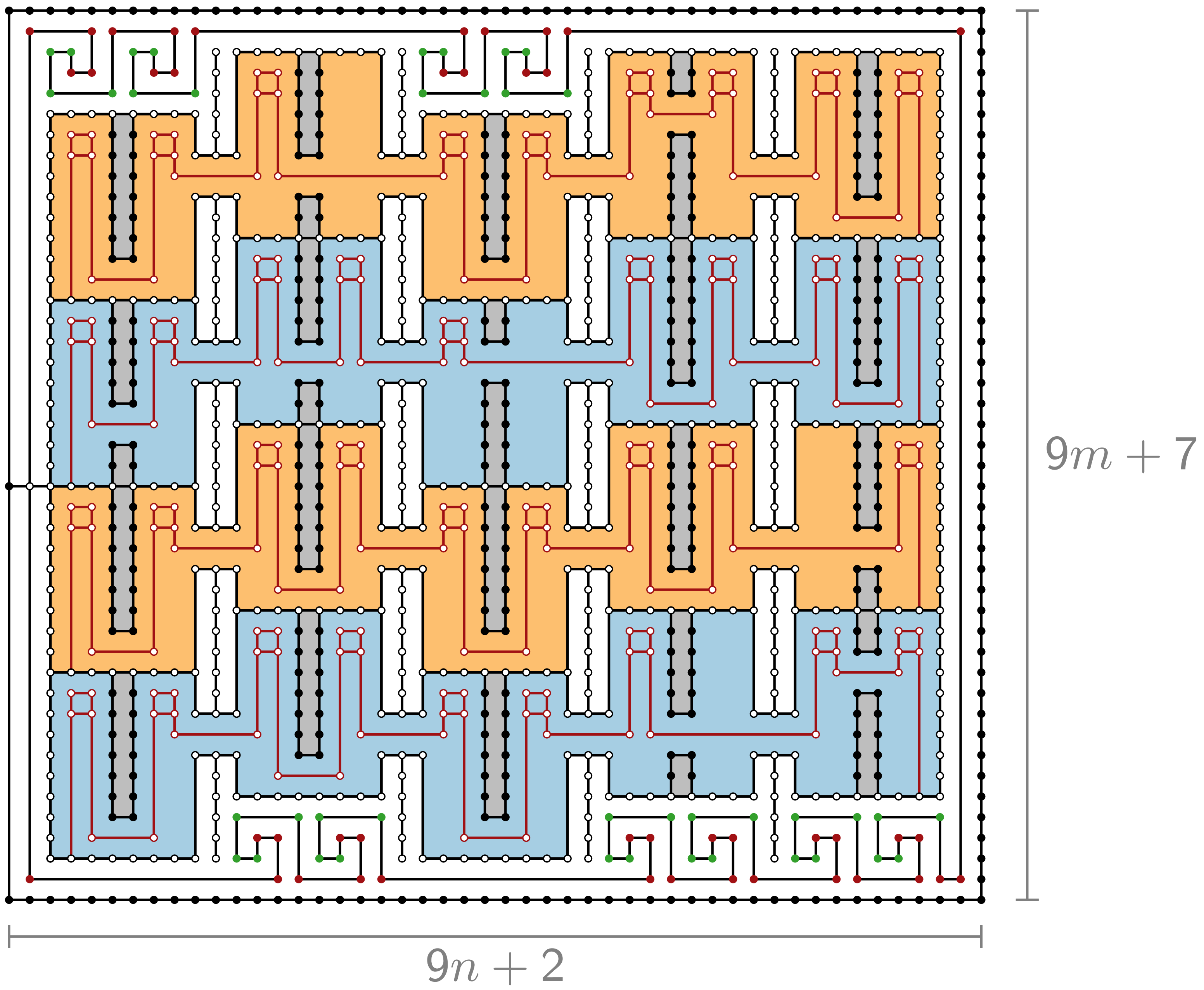
→ for every clause, there needs to be ≥ 1 "gap of a literal" to be on the same height as the "tunnel" to the next literal

Complete Reduction



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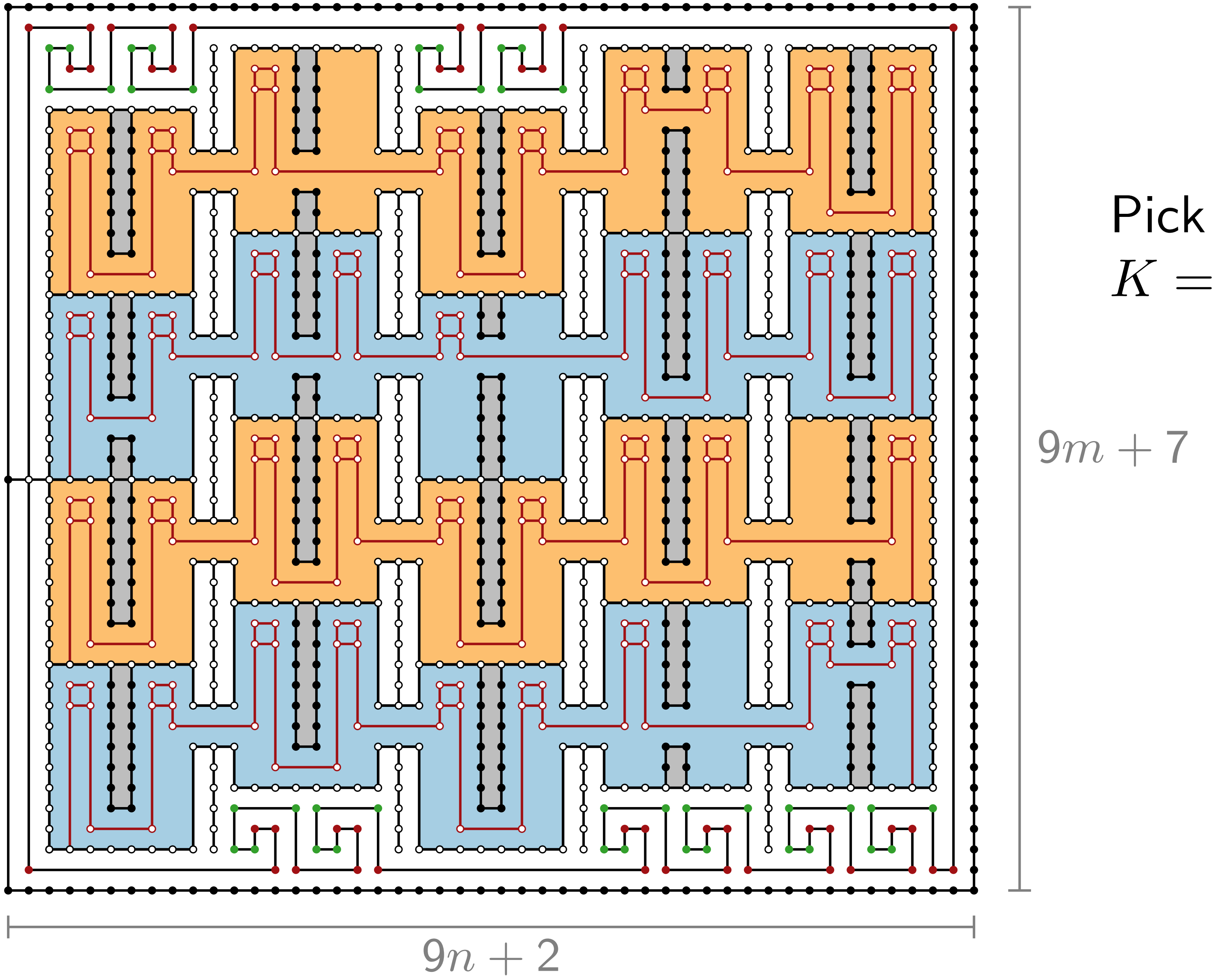

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$K = (9n+2) \times (9m+7)$

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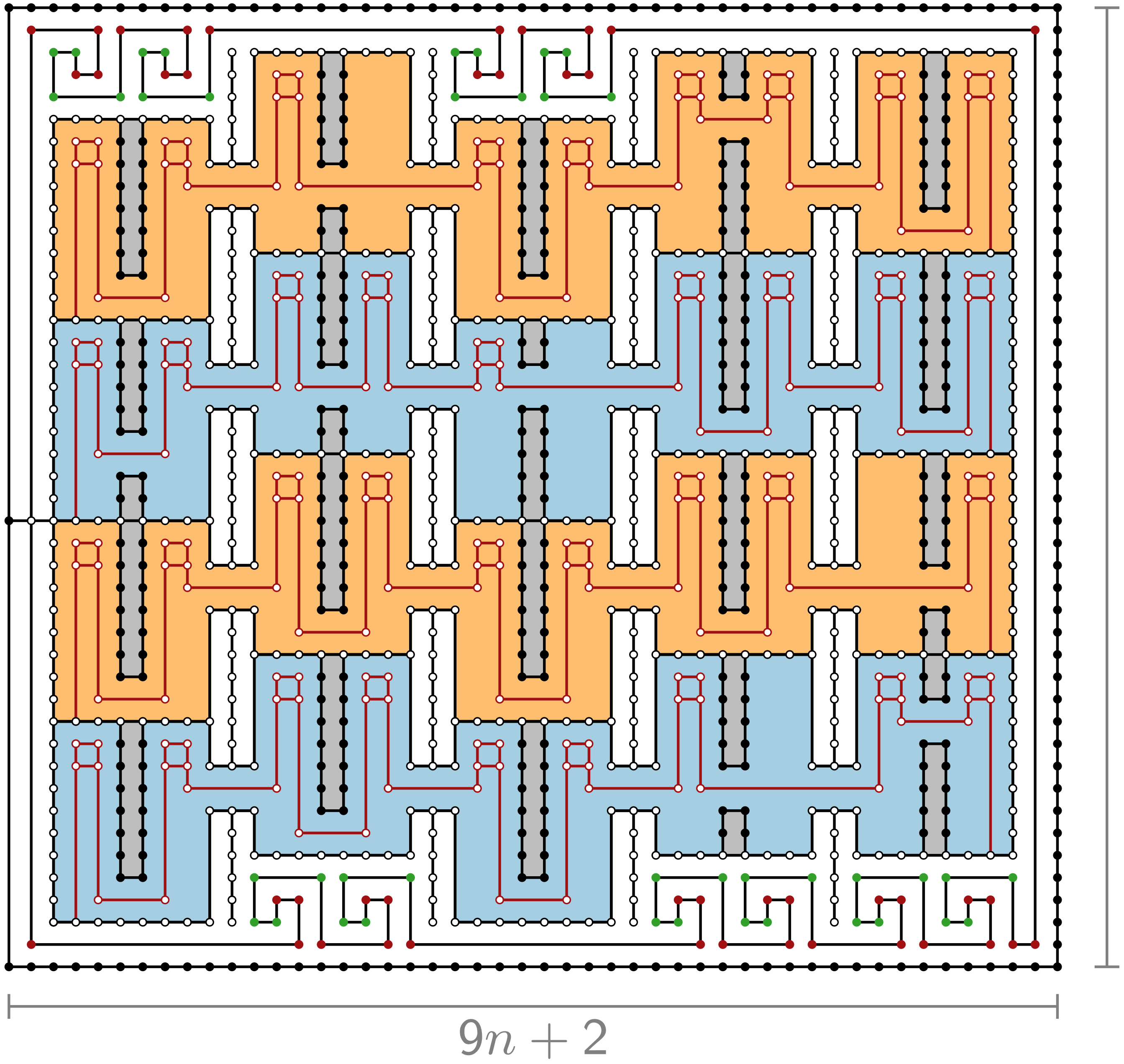

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Then:

G under $H(G)$ has an orthogonal drawing in area K

$\Leftrightarrow$

Φ satisfiable

□

Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] “On embedding a graph in the grid with the minmum number of bends”
  Original paper on flow for bend minimization.
- [van den Brand, Chen, Kyng, Liu, Peng, Probst, Sachdeva, Sidford 2023]
  “A Deterministic Almost-Linear Time Algorithm for Minimum-Cost Flow”
  State-of-the-art algorithm for solving the minimum-cost flow problem
  (published recently in the proceedings of the FOCS 2023 conference).
- [Patrignani 2001] “On the complexity of orthogonal compaction”
  NP-hardness proof for orthogonal representation of planar max-degree-4 graphs.
- [Evans, Fleszar, Kindermann, Saeedi, Shin, Wolff 2022]
  “Minimum rectilinear polygons for given angle sequences”: Compacting cycles is NP-hard.
- [Antić, Liotta, Masařík Ortali, Pfretzschner, Stumpf, Wolff, Zink 2025]
  “Unbent Collections of Orthogonal Drawings”: It is NP-hard to find two drawings such that each edge is straight in one and the total number of bends is minimum.