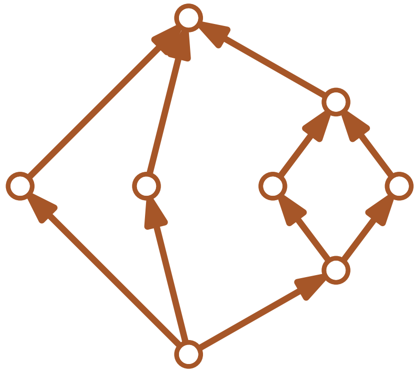
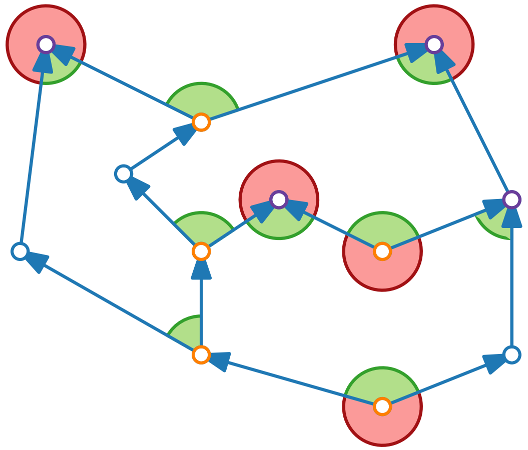
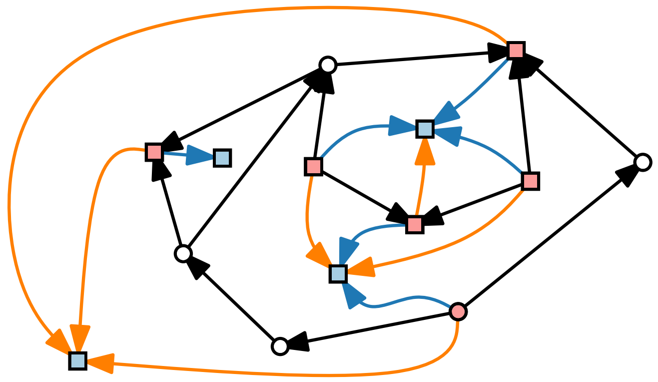
Visualization of Graphs



Lecture 5:
Upward Planar Drawings



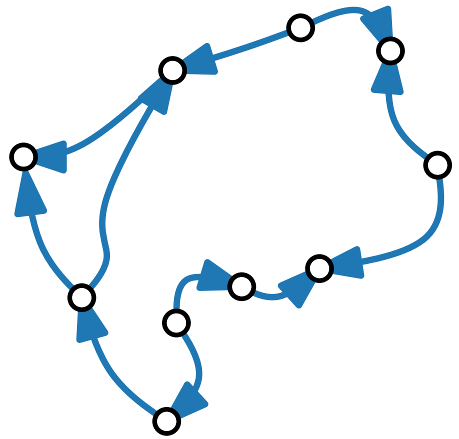
Part I:
Recognition



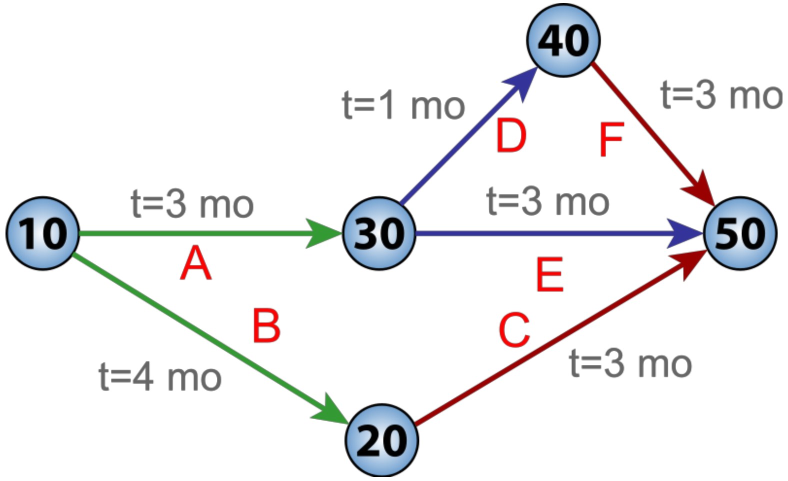
Alexander Wolff

Summer term 2025

Upward Planar Drawings – Motivation

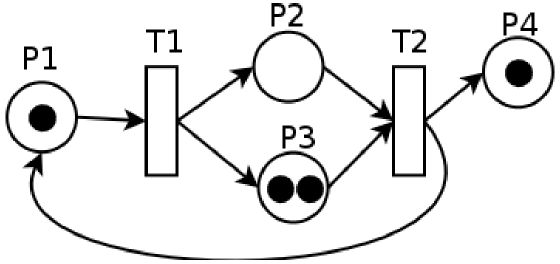


- What may the direction of edges in a directed graph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .
- We aim for drawings where the general direction is preserved.



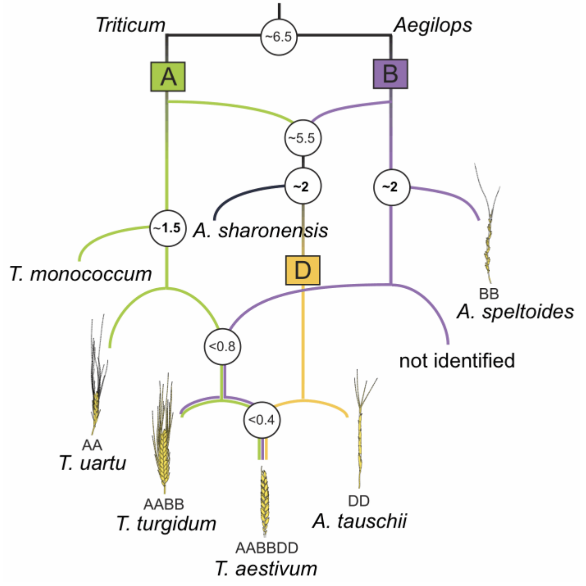
PERT diagram

Program Evaluation and Review Technique
(Project management)



Petri net

Place/Transition net
(Modeling languages for distributed systems)



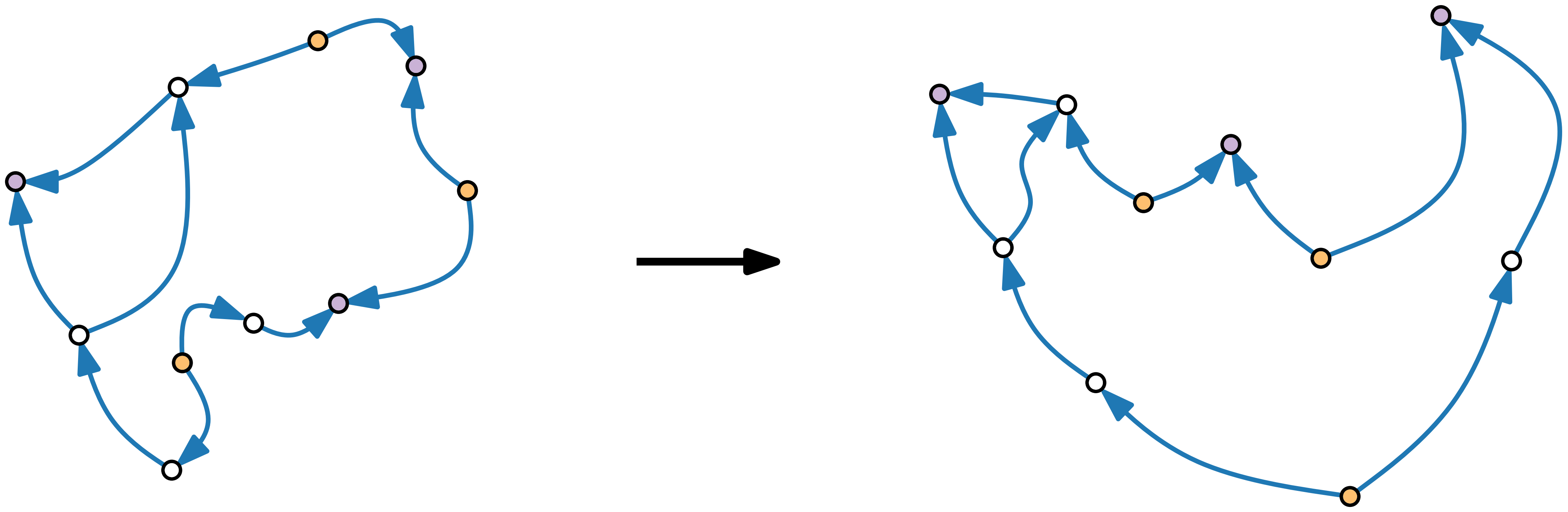
Phylogenetic network

Ancestral trees / networks
(Biology)

Upward Planar Drawings – Definition

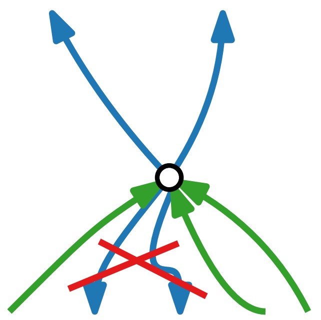
A directed graph (*digraph*) is **upward planar** when it admits a drawing

- that is planar and
- where each edge is drawn as an upward y-monotone curve.

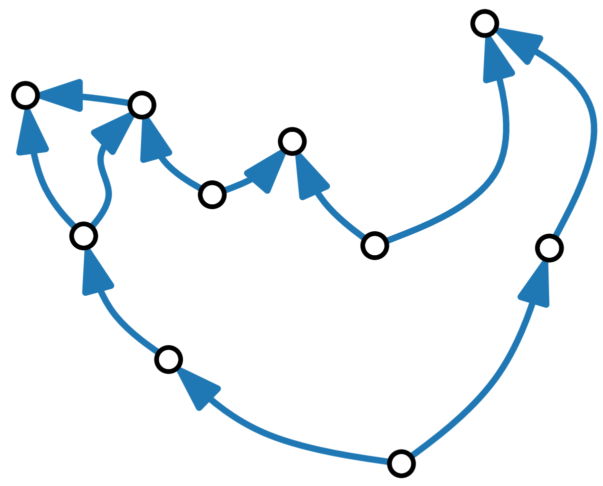


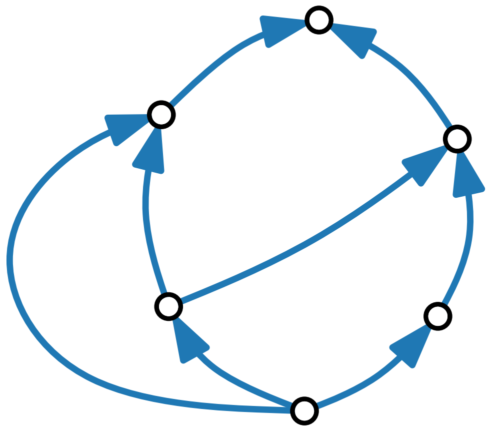
Upward Planarity – Necessary Conditions

- For an (embedded) digraph to be upward planar, it needs to . . .
  - be planar
  - be acyclic
  - have a bimodal embedding

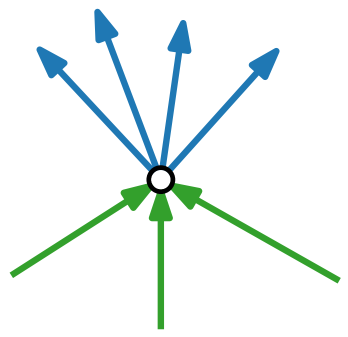


- . . . but these conditions are *not sufficient*. $\rightarrow$ **Exercise**

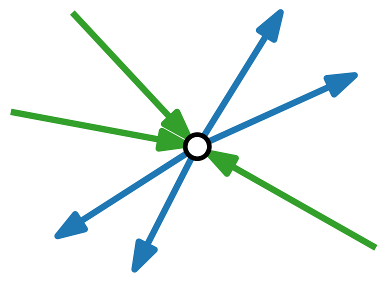




bimodal vertex



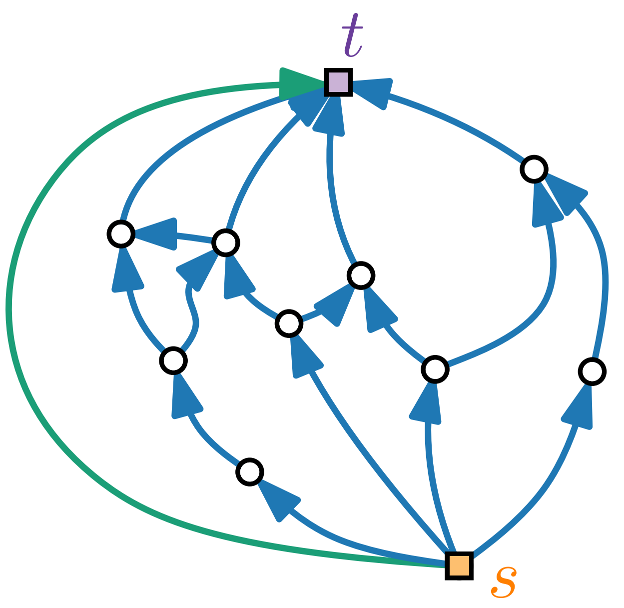
not bimodal



Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G, the following statements are equivalent:

(1) G is upward planar.
(2) G admits an upward planar straight-line drawing.
(3) G is a spanning subgraph of a planar st-digraph.



Additionally:
Embedded such that s and t are on the outer face f_0.

or:
Edge (s, t) exists.

no crossings

acyclic digraph with
a single source s and a single sink t

Upward Planarity – Characterization

Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
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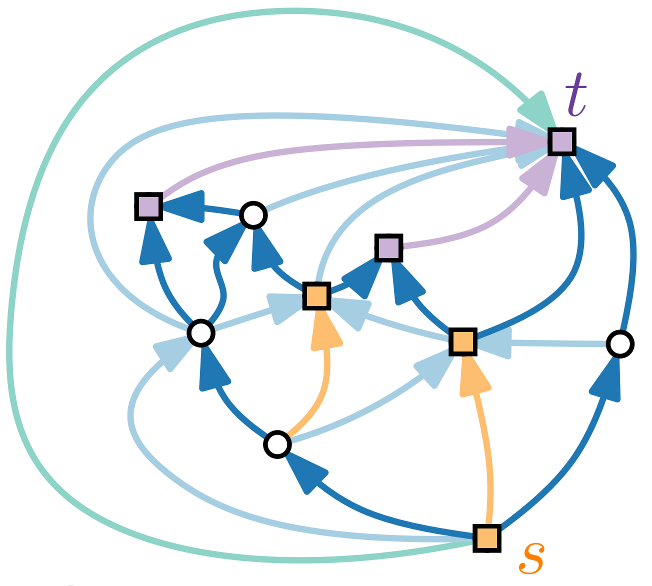
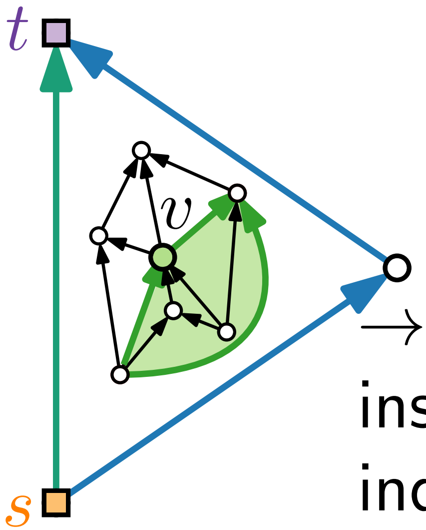


Proof.
(2) $\Rightarrow$ (1) By definition. (1) $\Rightarrow$ (3) For the proof idea, see the example above.
(3) $\Rightarrow$ (2) Triangulate & construct drawing:

Claim.
Can be drawn in pre-specified triangle.

Induction on the number of vertices n.

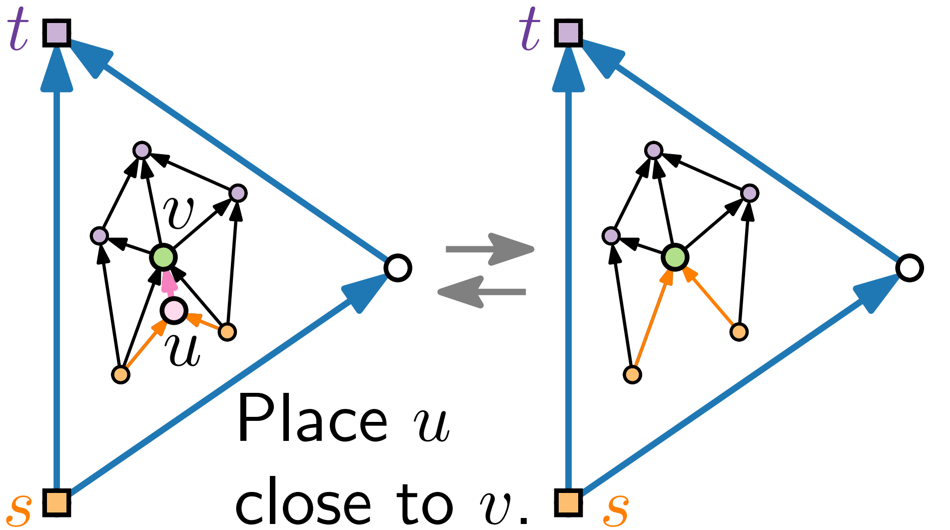
Case 1: chord



$\rightarrow$ two smaller instances; solve inductively

Case 2: no chord

Idea: Contract uv!



Place u close to v.

Upward Planarity – Complexity

Theorem. [Garg & Tamassia, 1995]
Given a *planar acyclic* digraph G,
it is NP-hard to decide whether G is upward planar.

Theorem 2. [Bertolazzi, Di Battista, Mannino, Tamassia, 1994]
Given an *embedded* planar digraph G,
it can be tested in quadratic time whether G is upward planar.

Corollary.
Given a *triconnected* planar digraph G,
it can be tested in quadratic time whether G is upward planar.

Theorem. [Hutton & Lubiw, 1996]
Given an acyclic *single-source* digraph G,
it can be tested in linear time whether G is upward planar.

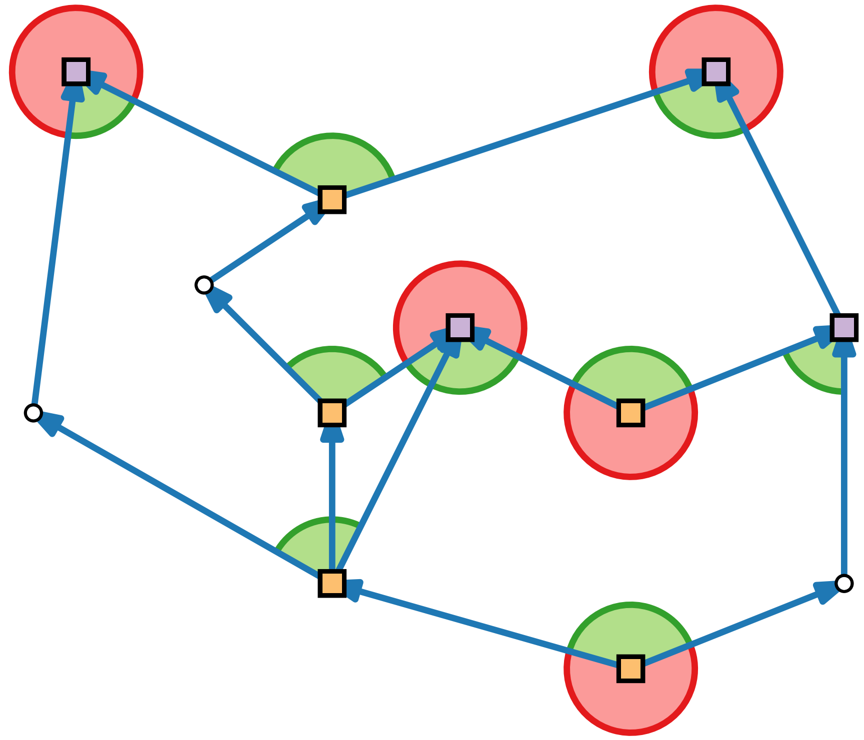
The Problem

Fixed Embedding Upward Planarity Testing.
Let G be a plane digraph, let F be the set of faces of G, and let f_0 be the outer face of G.
Test whether G is upward planar (w.r.t. to F and f_0).

Plan.

- Find a property that any upward planar drawing of G satisfies.
- Formalize this property.
- Specify an algorithm to test this property.

Angles, Local Sources & Sinks



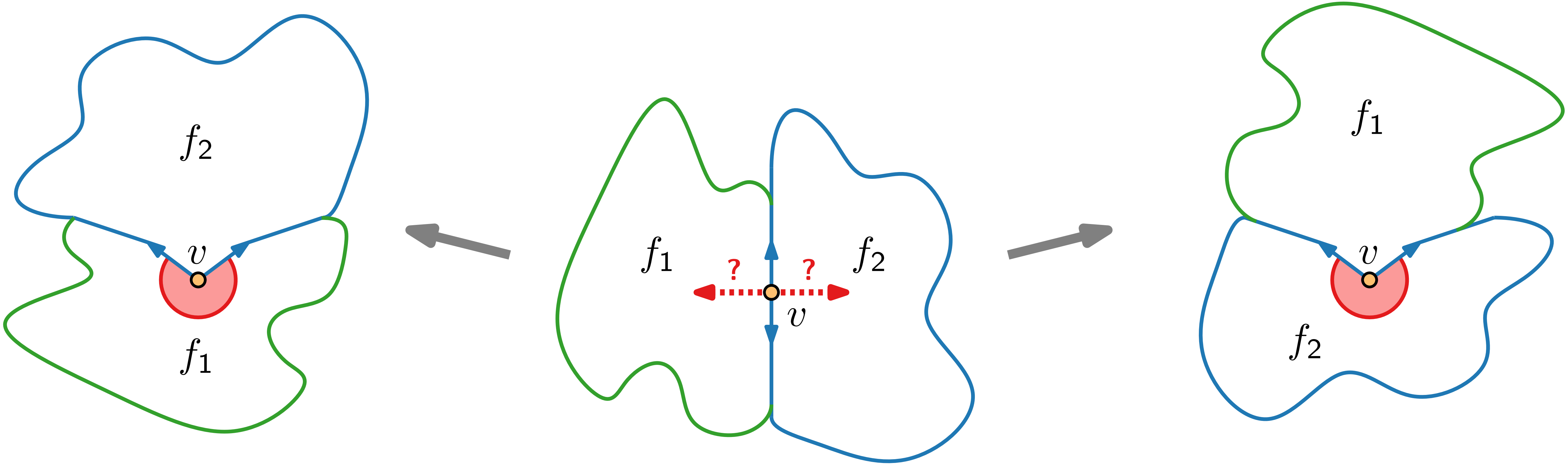
Definitions.

- A vertex v is a **local source** w.r.t. to a face f if v has two outgoing edges on ∂f. ← boundary of f
- A vertex v is a **local sink** w.r.t. to a face f if v has two incoming edges on ∂f.
- An angle α at a local source/sink is **large** if $\alpha > \pi$ and **small** otherwise.
- $L(v) = \#$ large angles at v
- $L(f) = \#$ large angles in f
- $S(v) = \#$ small angles at v
- $S(f) = \#$ small angles at f
- $A(f) = \#$ local sources w.r.t. to f
  $= \#$ local sinks w.r.t. to f

Lemma 1.
$L(f) + S(f) = 2A(f)$

Assignment Problem

- Observe that the **global sources** and **global sinks** have precisely one large angle.
- All other vertices have only small angles.
- Let v be a global source and let it be incident to faces f_1 and f_2.
- Does v have a large angle in f_1 or f_2?



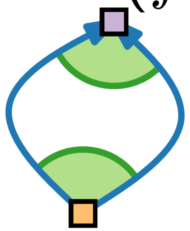
Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$$

Proof by induction on $L(f)$.

- $L(f) = 0$

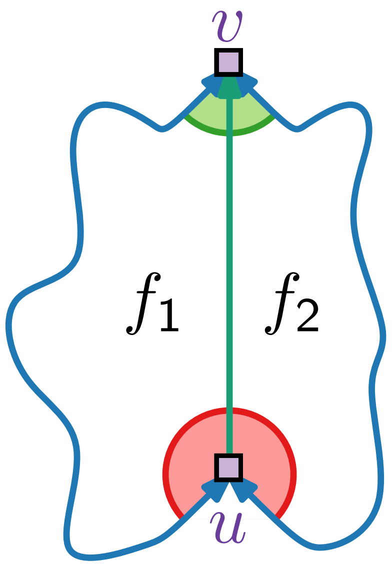


$\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to...

- sink v with small angle:



$$L(f) - S(f) = \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 1 - (S(f_1) + S(f_2) - 1)$$

$$= -2 - 2 + 2 = -2$$

Angle Relations

Lemma 2.

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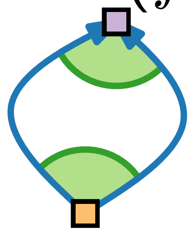
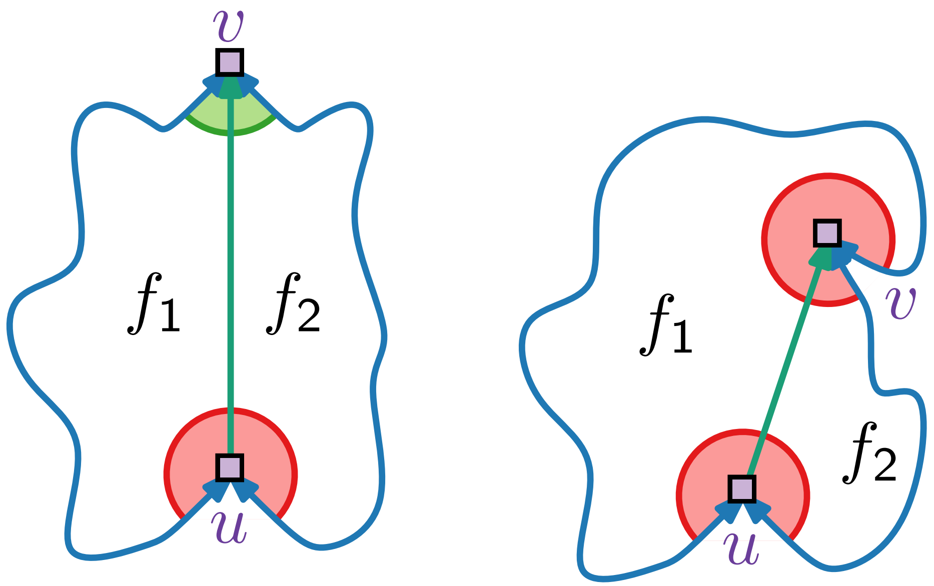
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to. . .

- sink v with small/large angle:



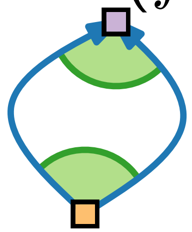
$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 - 2 + 2 = -2 \end{aligned}$$

Angle Relations

Lemma 2. $L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$

Proof by induction on $L(f)$.

- $L(f) = 0$

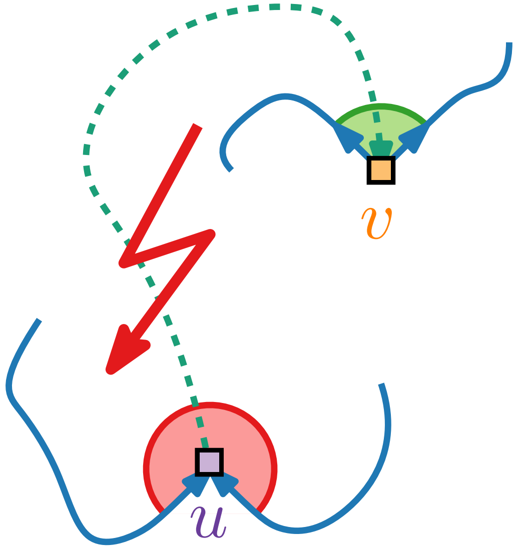


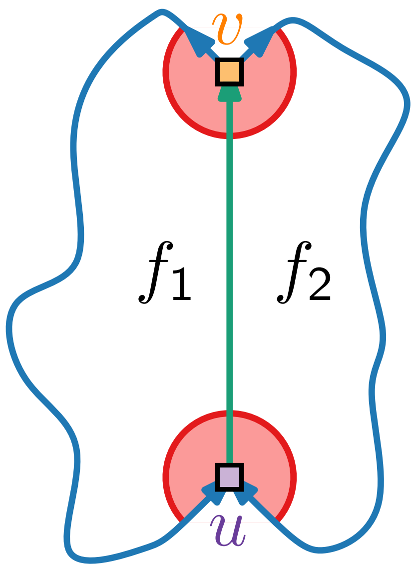
$\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to...

- source v with ~~small~~/large angle:





$$\begin{aligned} L(f) - S(f) &= \overset{-2}{L(f_1)} + \overset{-2}{L(f_2)} + 2 \\ &\quad - (S(f_1) + S(f_2)) \\ &= -2 - 2 + 2 = -2 \end{aligned}$$

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2 & \text{if } f \neq f_0, \\ +2 & \text{if } f = f_0. \end{cases}$$

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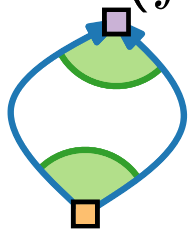
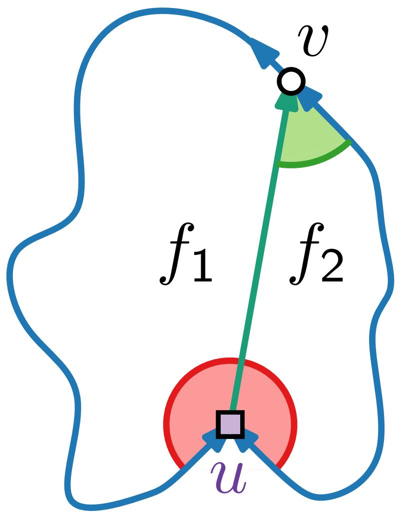
- $L(f) = 0$  $\Rightarrow S(f) = 2$ ✓

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to...

- vertex v that is neither source nor sink:



$$L(f) - S(f) = \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 - (S(f_1) + S(f_2) - 1)$$
$$= -2 - 2 + 2 = -2$$

- Otherwise "high" source u exists. $\rightarrow$ symmetric
- Similar argument for the outer face f_0.

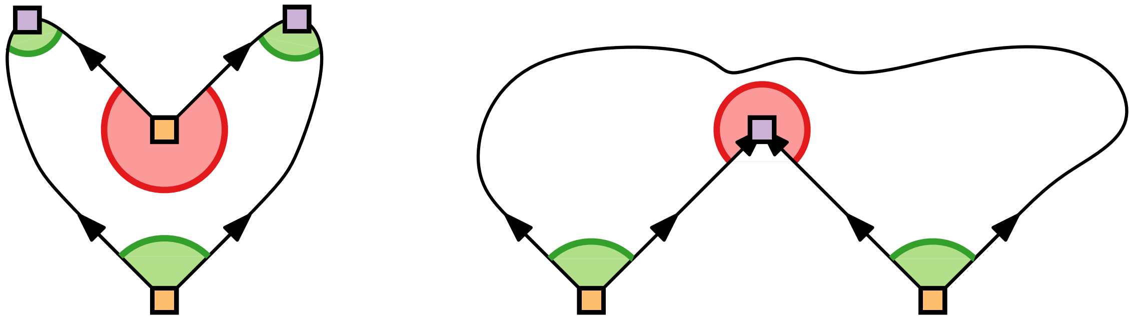
Number of Large Angles

Lemma 3.
In every upward planar drawing of G, it holds that

- for each vertex v: $L(v) = \begin{cases} 0 & \text{if } v \text{ is an inner vertex,} \\ 1 & \text{if } v \text{ is a gobal source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$

Proof.

$$\begin{array}{lrl} \text{Lemma 1:} & L(f) + S(f) & = 2A(f) \\ \text{Lemma 2:} & L(f) - S(f) & = \pm 2. \\ \hline & \Rightarrow 2L(f) & = 2A(f) \pm 2. \end{array}$$



Assignment of Large Angles to Faces

Let S be the set of (global) sources, and let T be the set of (global) sinks.

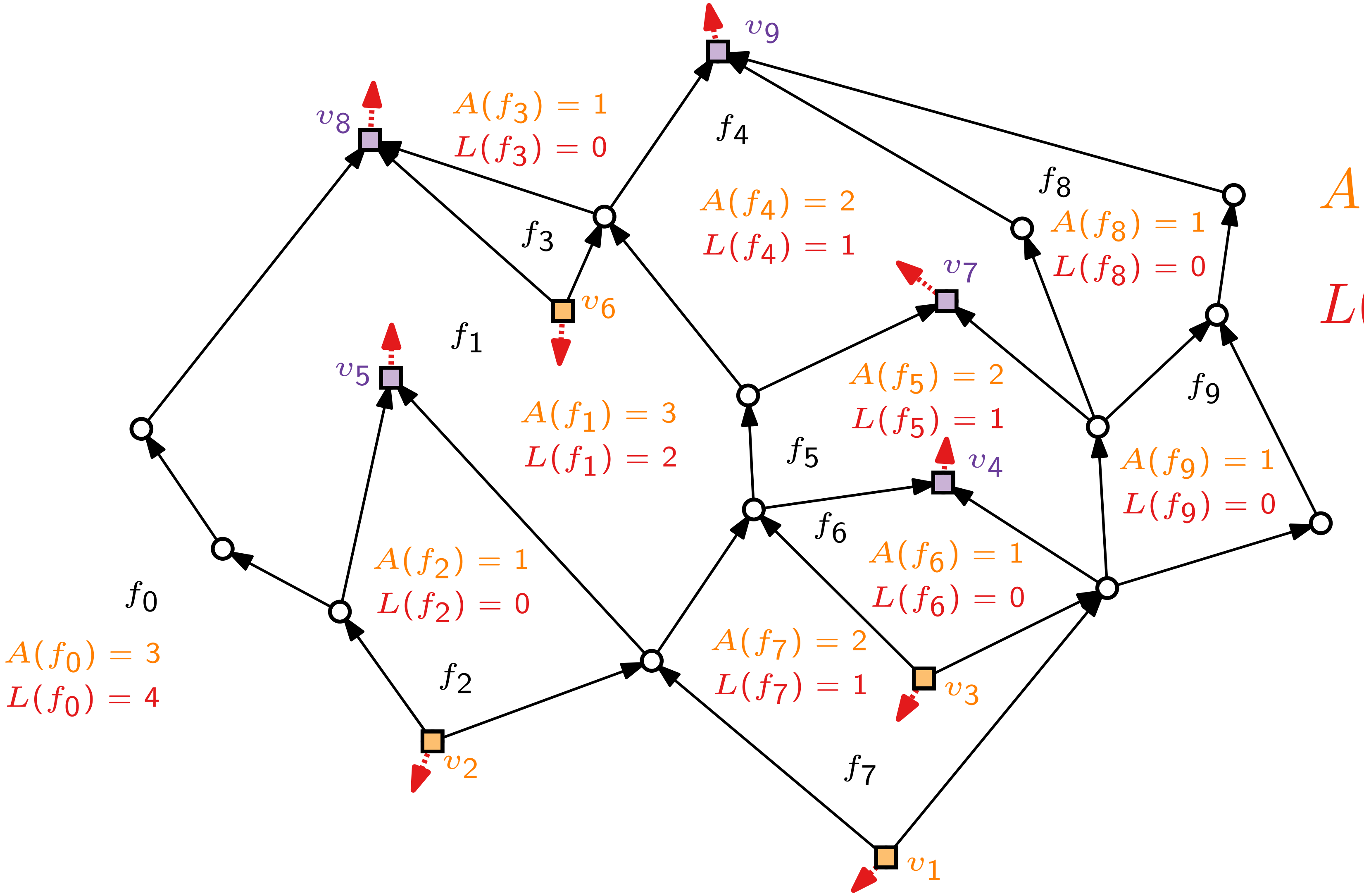
Definition.
A **consistent assignment** $\Phi\colon S \cup T \to F$ is a mapping with

$$\Phi\colon v \mapsto \text{ incident face, where } v \text{ forms a large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

Example of Angle-to-Face Assignment



global sources & sinks

$A(f) =$ # local sources/sinks of f

$L(f) =$ # large angles of f

assignment
$\Phi\colon S \cup T \to F$

Result Characterization

Theorem 3.
Let G be an acyclic plane digraph with embedding given by F and f_0.

Then G is upward planar (respecting F and f_0)
$\Leftrightarrow$ G is bimodal and there exists a consistent assignment Φ.

Proof.
$\Rightarrow$: As constructed before.

$\Leftarrow$: Idea:

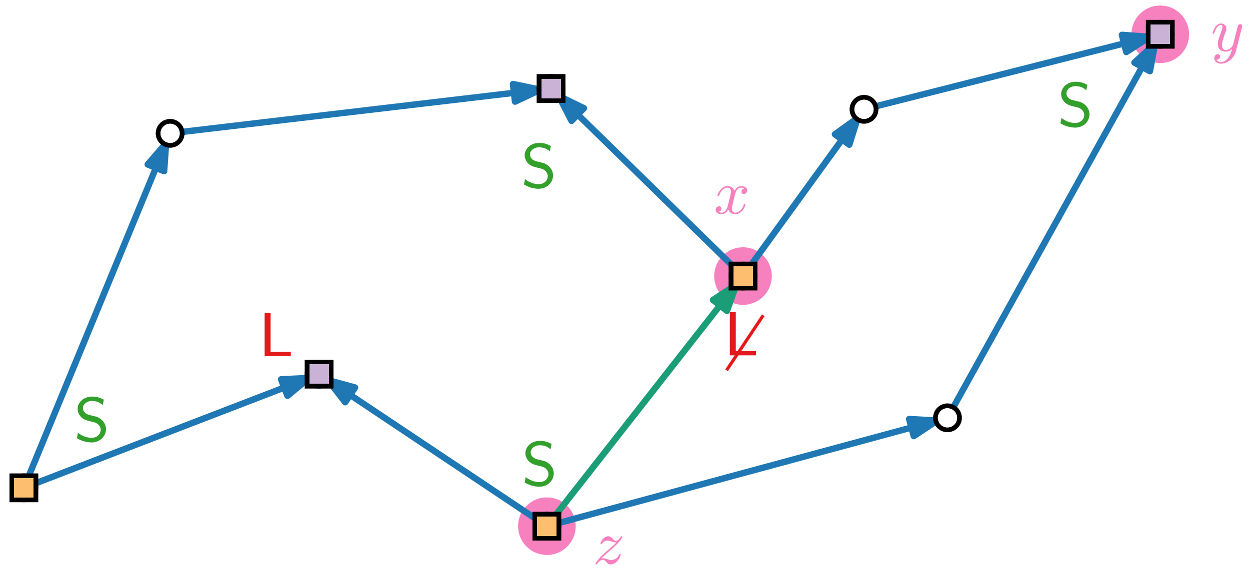
- Construct planar st-digraph that is a supergraph of G.
- Apply equivalence from Theorem 1.

G is upward planar $\Leftrightarrow$ G is a spanning subgraph of a planar st-digraph.
$\Leftrightarrow$ G admits a straight-line upward planar drawing.
(Note: Proof was constructive!)

Refinement Algorithm: $\Phi, F, f_0 \to$ st-digraph

Let f be a face.
Consider the clockwise angle sequence σ_f of L / S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



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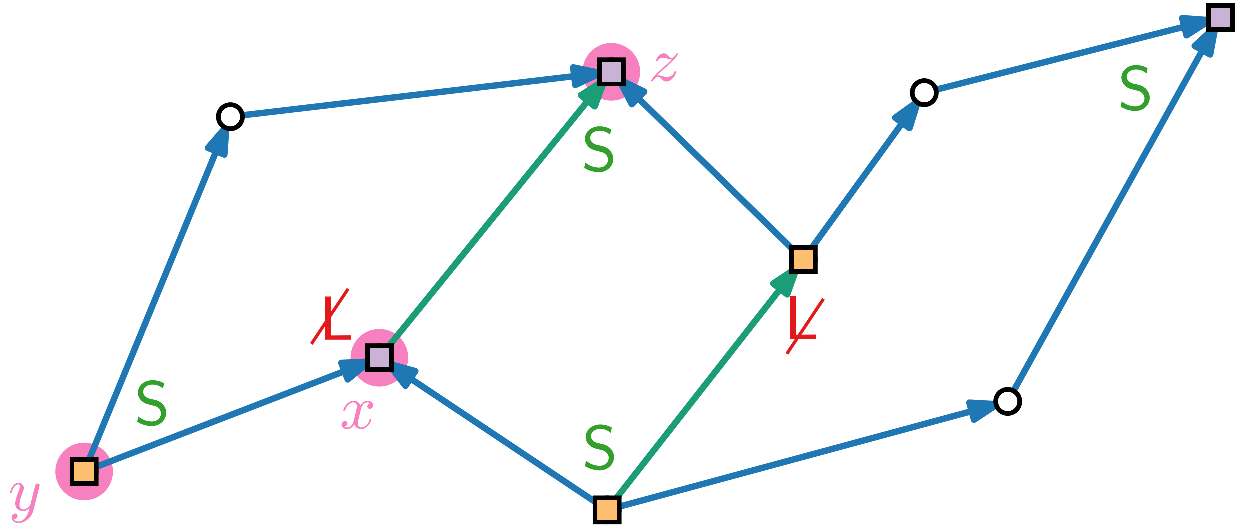
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).



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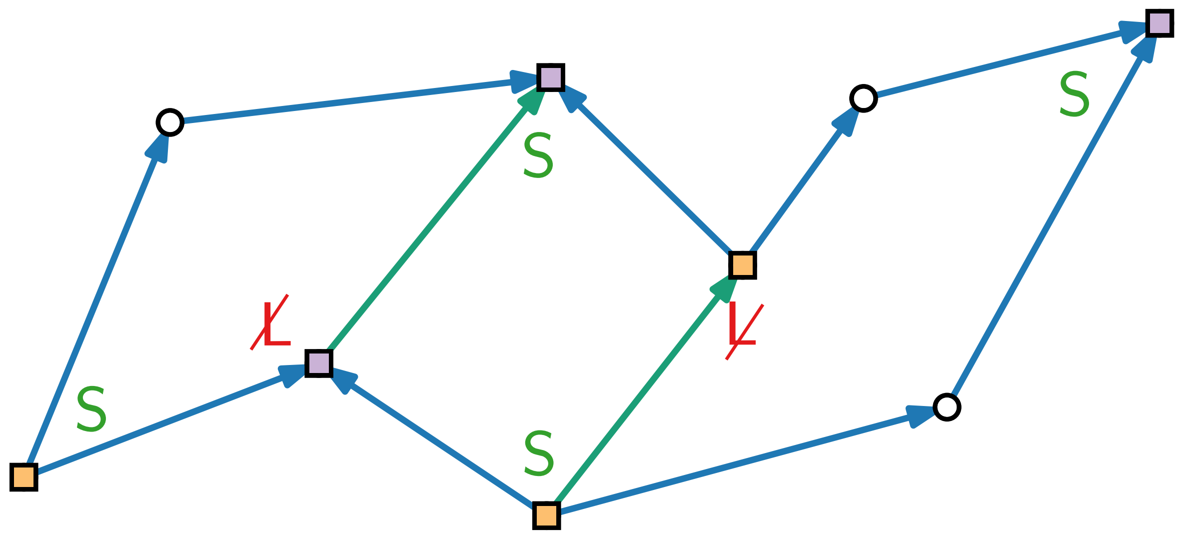
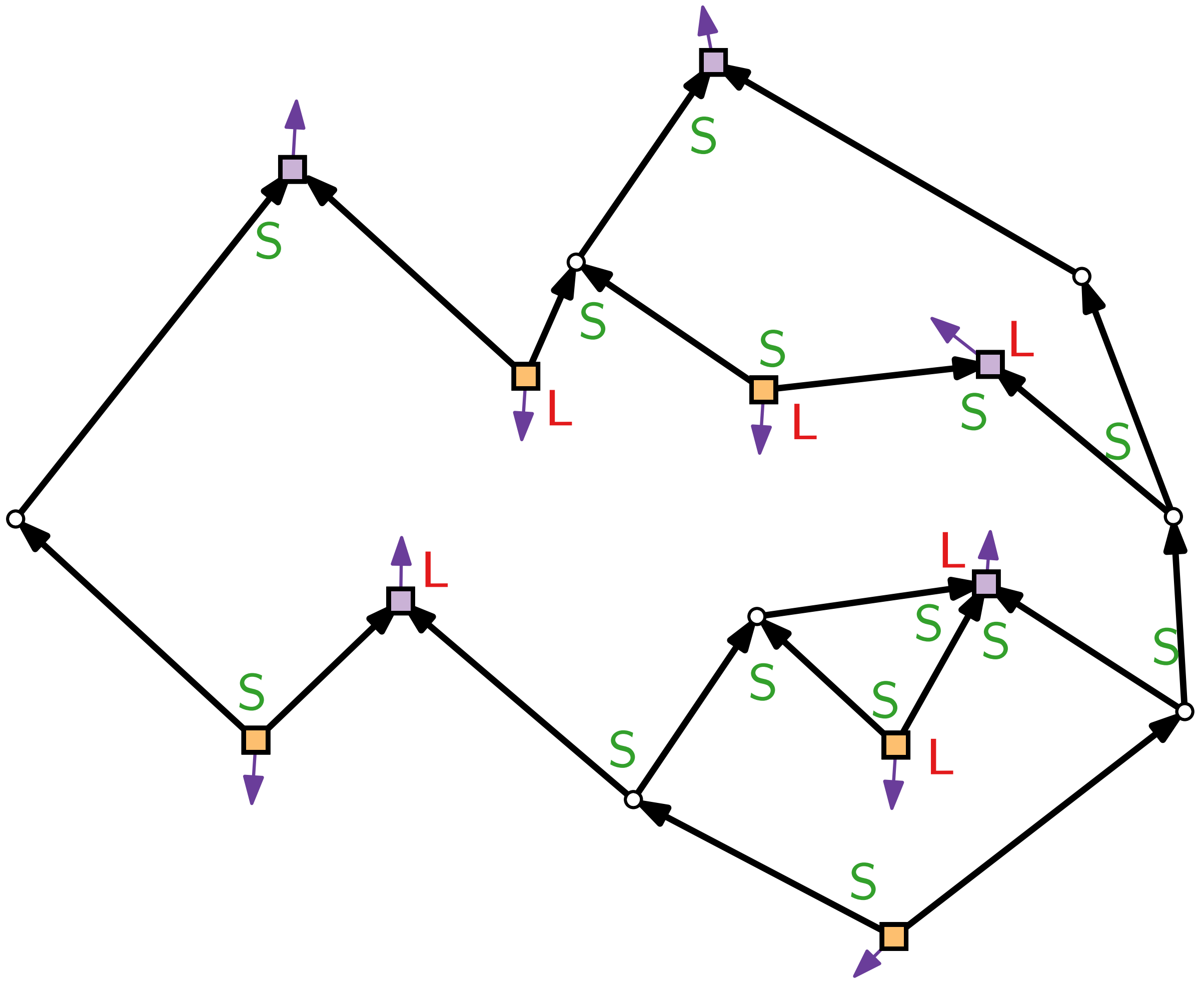
- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0 similarly.

  $\rightarrow$ **Exercise**

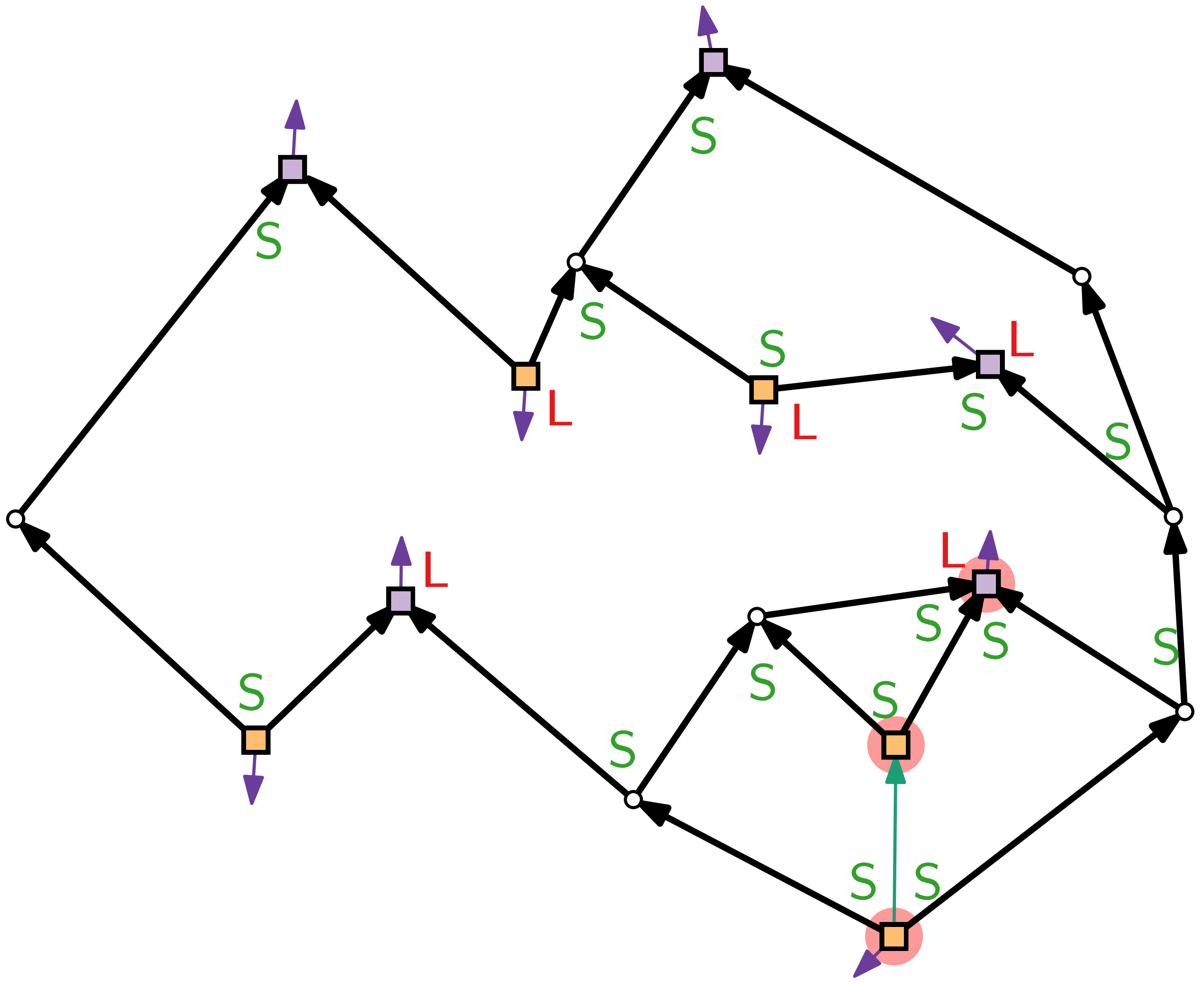


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

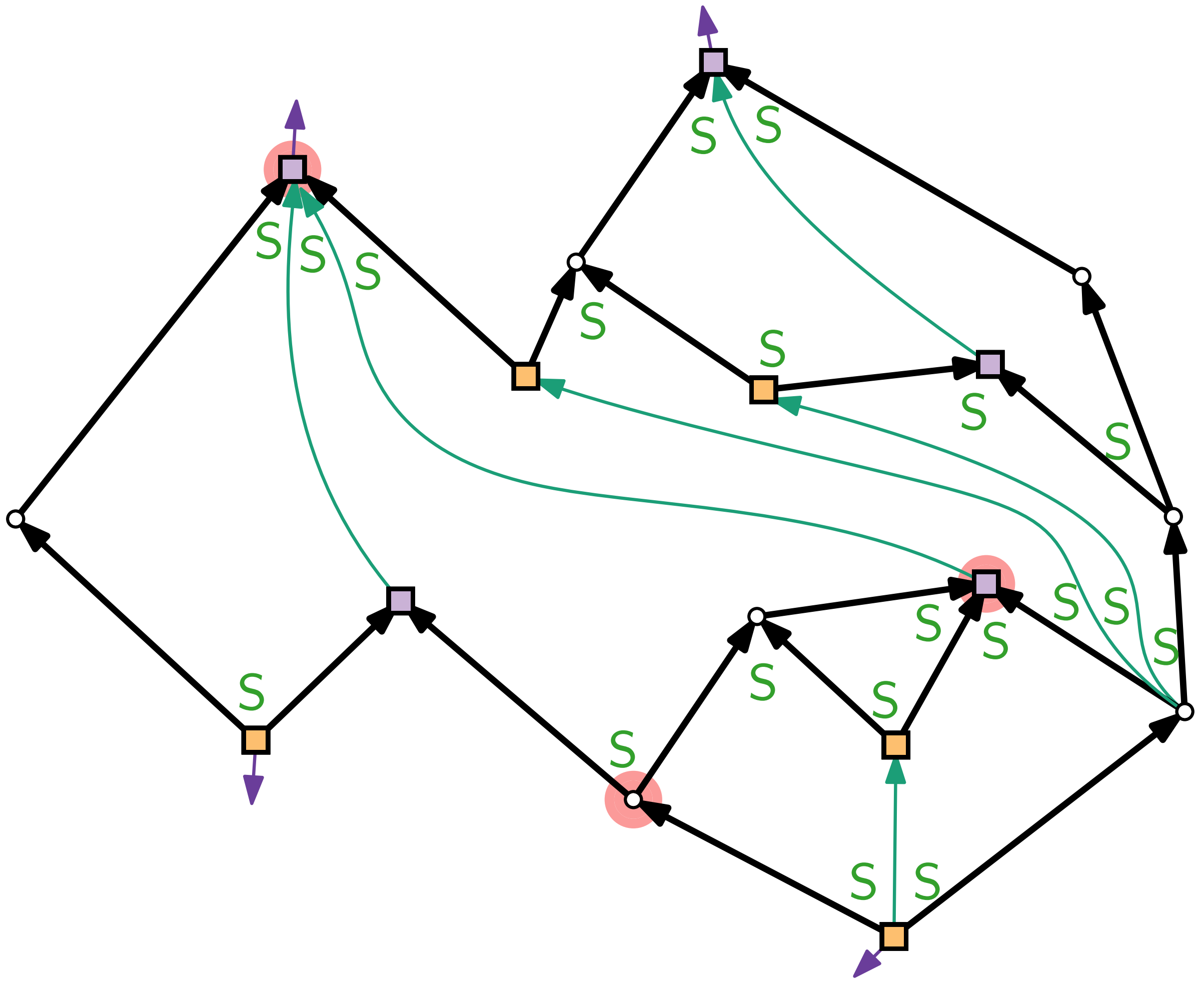
Refinement Example



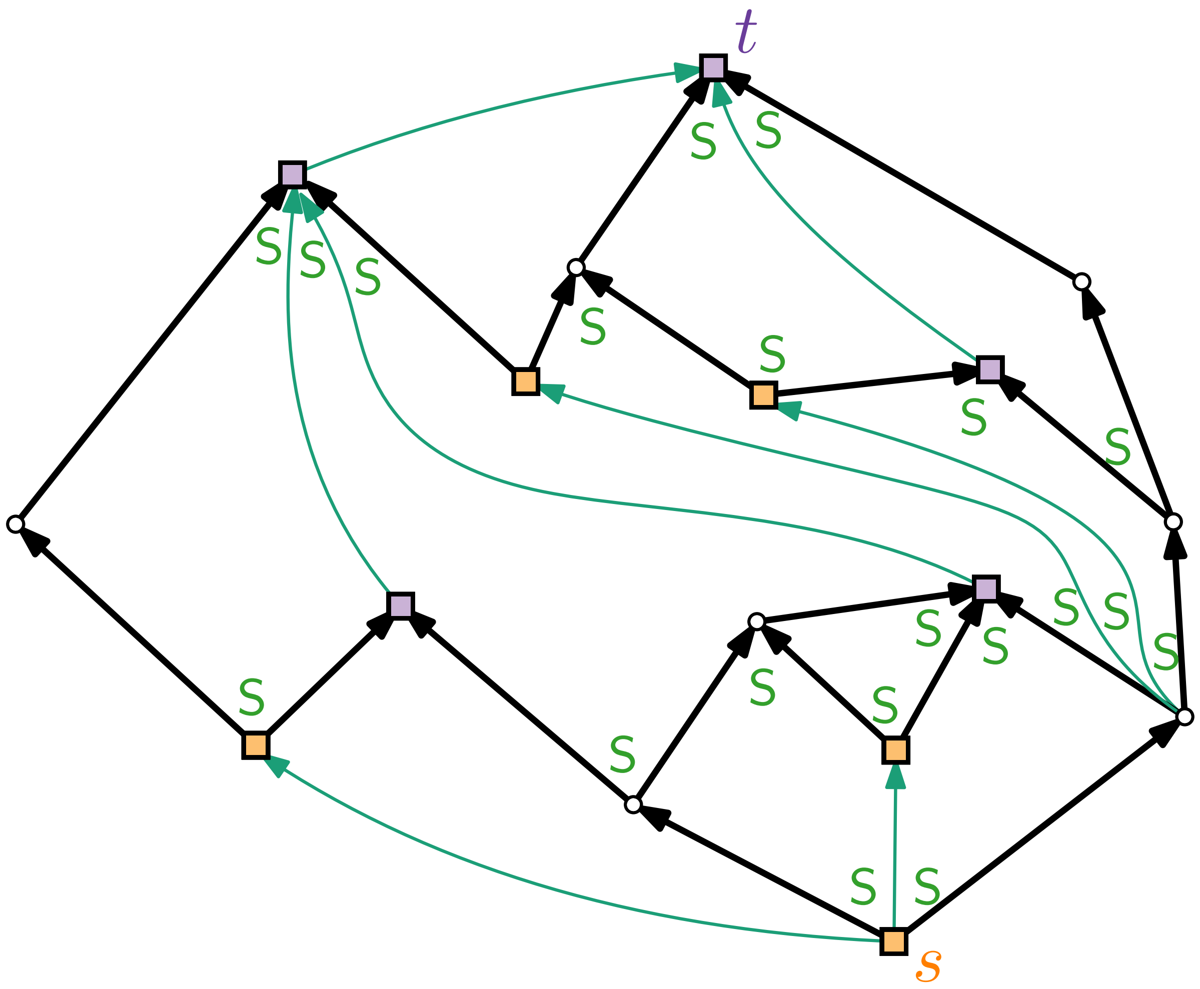
Refinement Example



Refinement Example



Refinement Example



Result Upward Planarity Test

Theorem 2. [Bertolazzi, Di Battista, Mannino, Tamassia '94]
Given an *embedded* planar digraph G,
we can test in quadratic time whether G is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.
- Deleted edges added in refinement step.

Finding a Consistent Assignment

Idea. Flow $(v, f) = 1$
from global source / sink v to the incident face f its large angle gets assigned to.

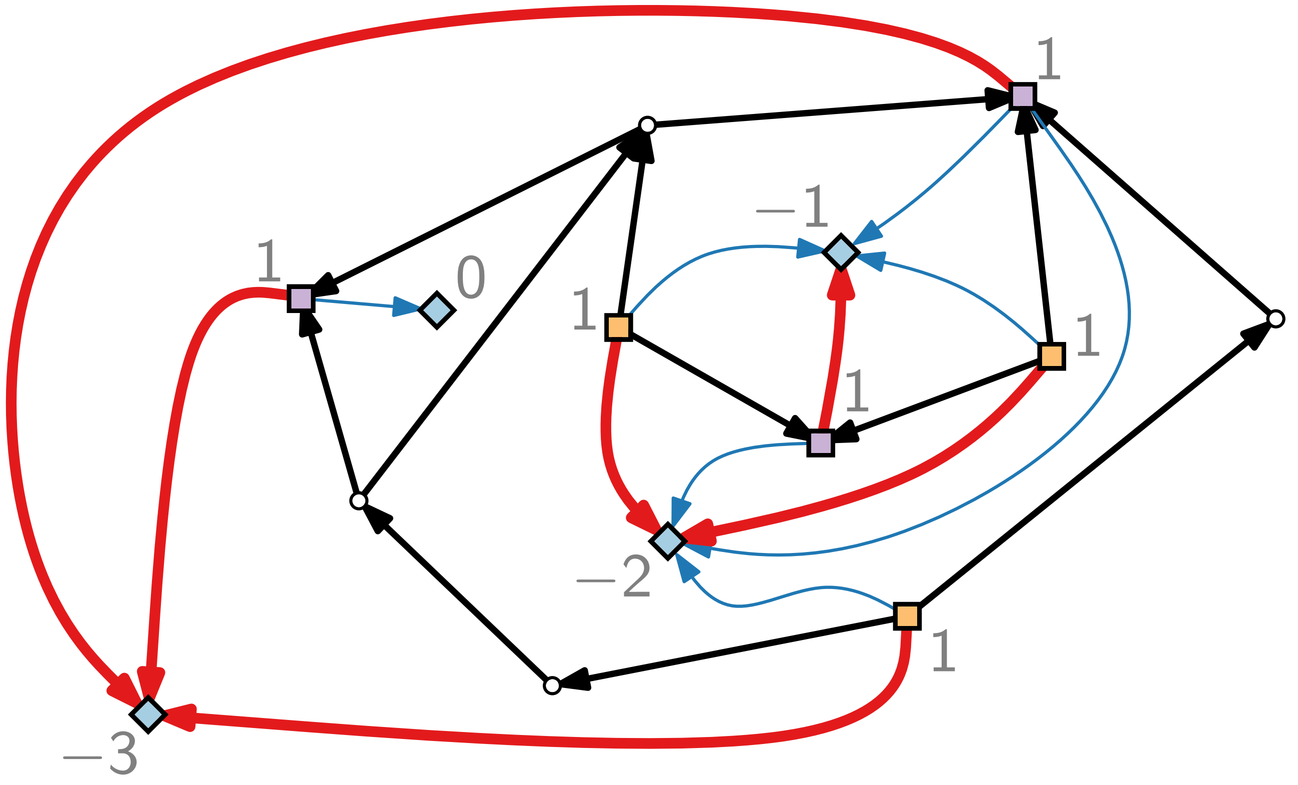
nodes of flow network edges of flow network lower/upper bounds on edge capcities
supplies/demands of nodes

Flow network.

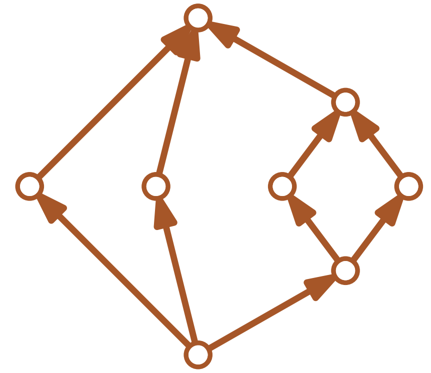
$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V(G) \mid v$ source or sink$\} \cup F(G)$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0\ \forall e \in E'$
- $u(e) = 1\ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V(G) \\ -(A(w) - 1) & \forall w \in F(G) \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

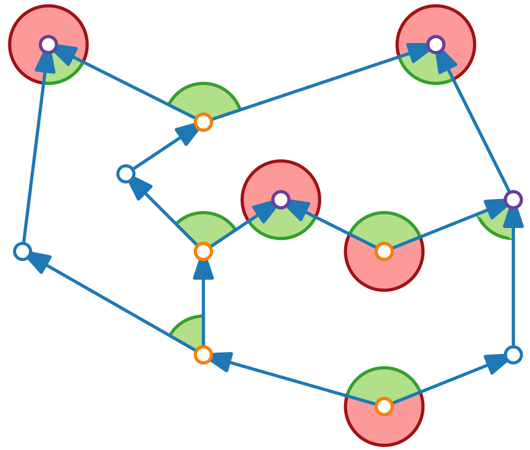
Example.



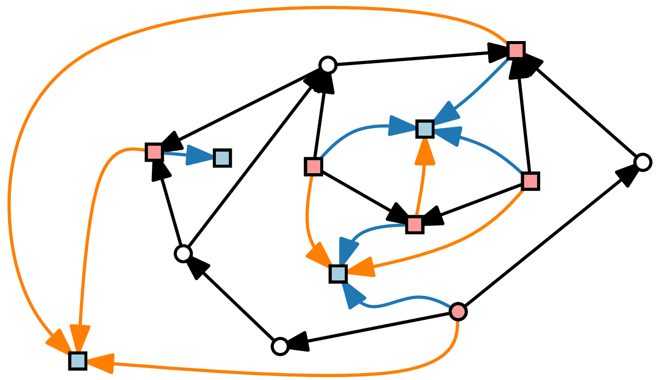
Visualization of Graphs



Lecture 5:
Upward Planar Drawings



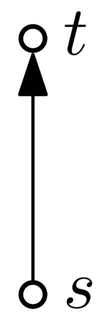
Part II:
Series-Parallel Graphs



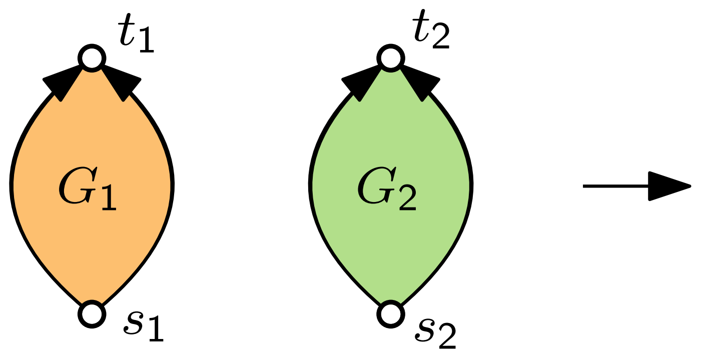
Series-Parallel Graphs

A graph G is **series-parallel** if

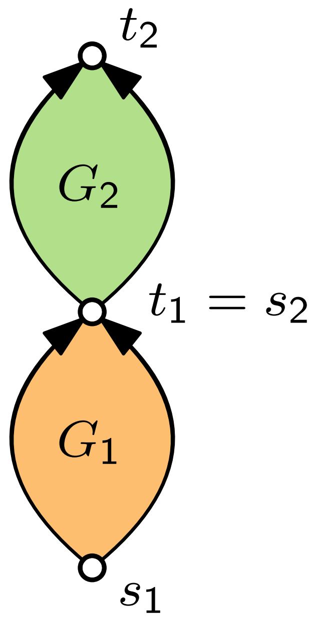
- it contains a single (directed) edge (s, t), or
- it consists of two series-parallel graphs G_1, G_2 with sources s_1, s_2 and sinks t_1, t_2 that are combined using one of the following rules:



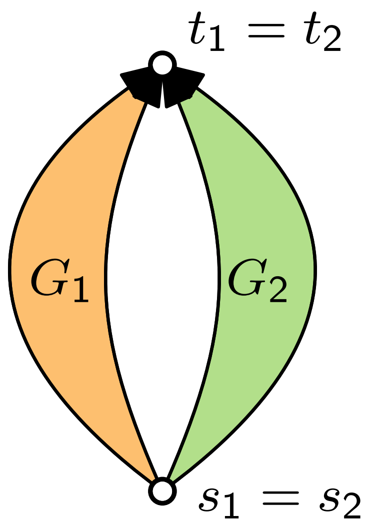
Convince yourself that series-parallel graphs are (upward) planar!



Series composition



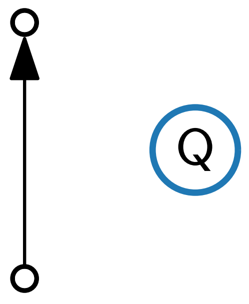
Parallel composition

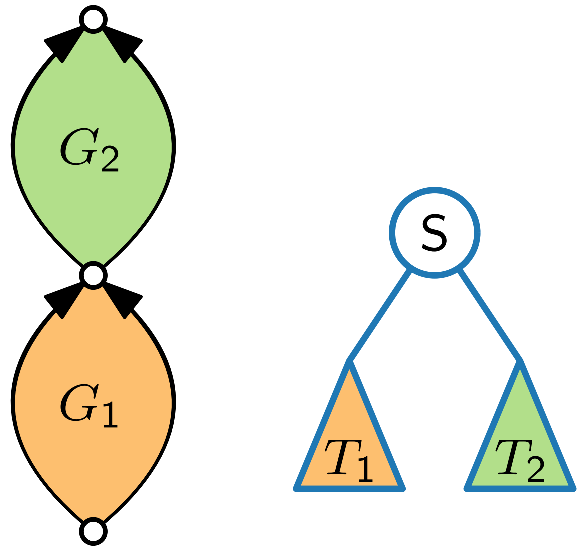


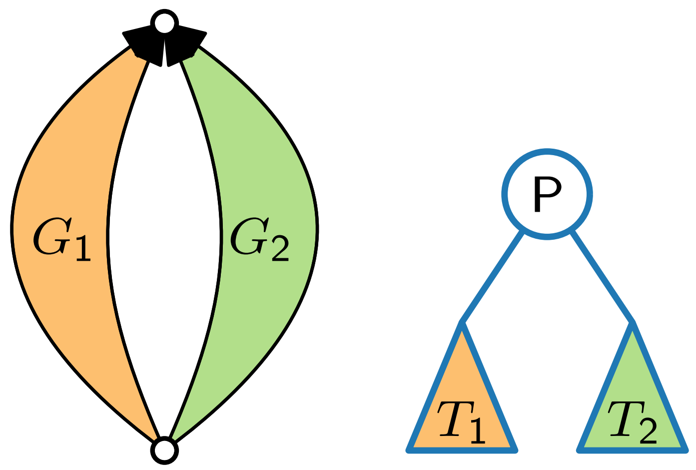
Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**.

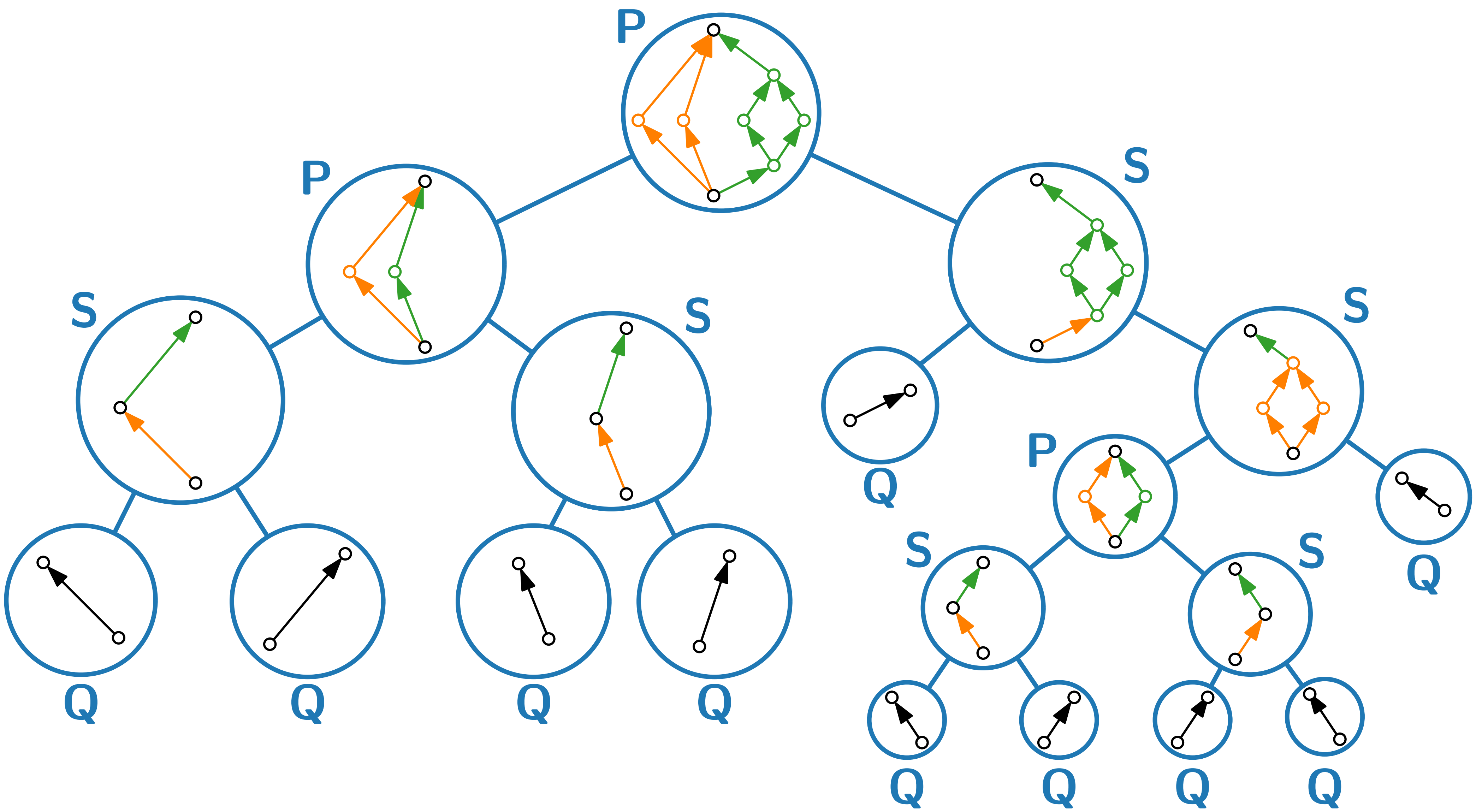
- A **Q**-node represents a single edge.
- An **S**-node represents a series composition;
  its children T_1 and T_2 represent G_1 and G_2.
- A **P**-node represents a parallel composition;
  its children T_1 and T_2 represent G_1 and G_2



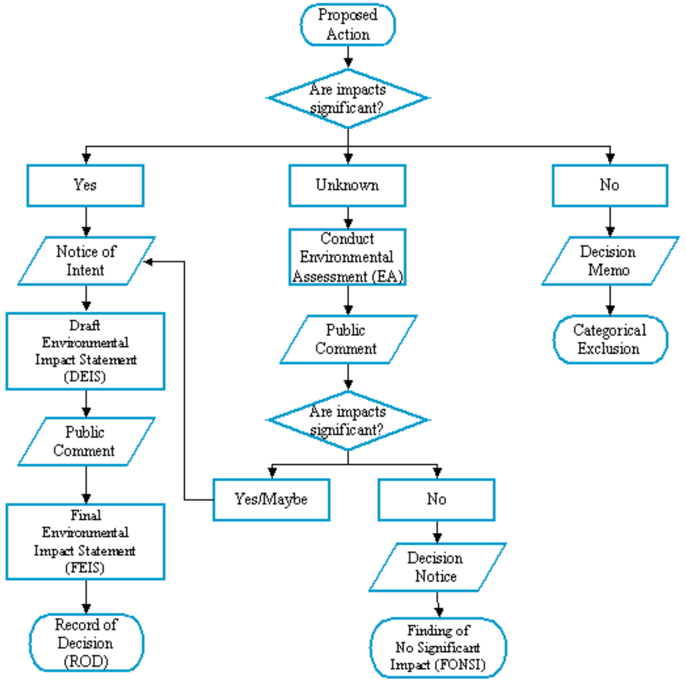




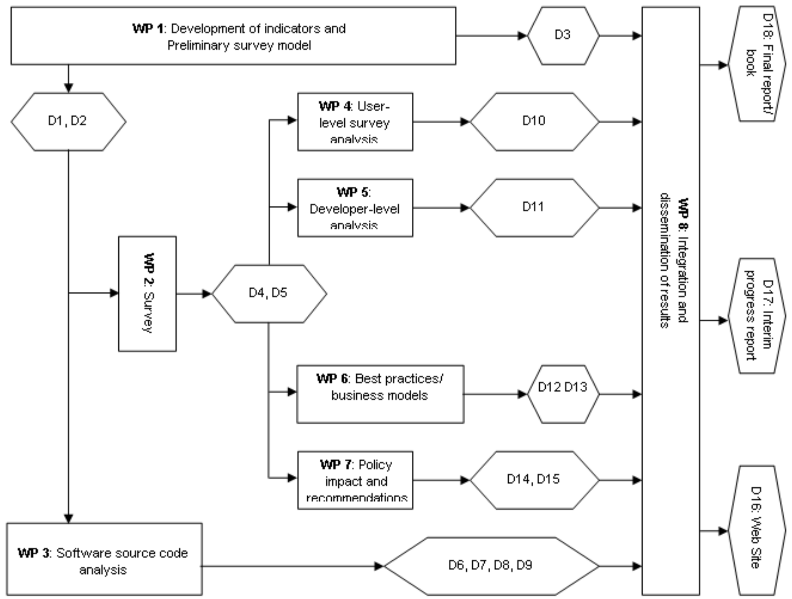
Series-Parallel Graphs – Decomposition Example



Series-Parallel Graphs – Applications



Flowcharts



PERT-Diagrams

(Program Evaluation and Review Technique)

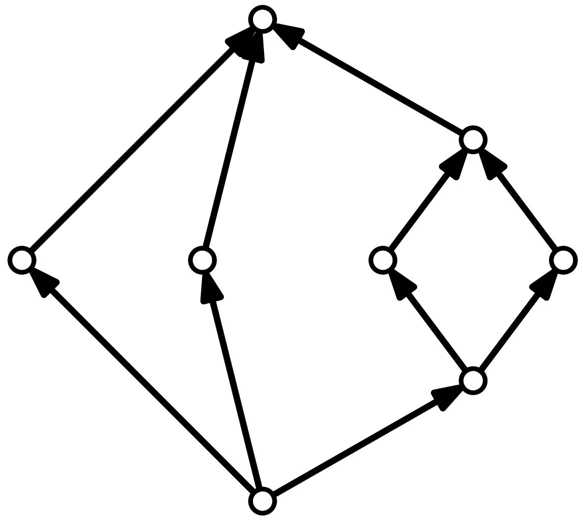
Computational complexity:

Series-parallel graphs often admit linear-time algorithms for problems that are NP-hard in general, e.g., minimum maximal matching, maximum independent set, Hamiltonian completion.

Series-Parallel Graphs – Drawing Style

Drawing conventions

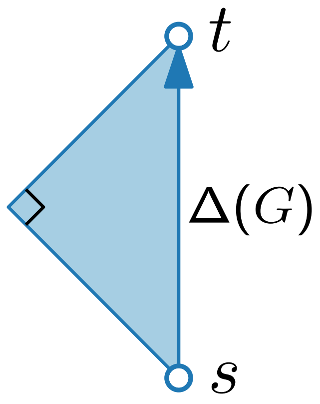
- Planarity
- Straight-line edges
- Upward



Drawing aesthetics to optimize

- Area
- Symmetry

Series-Parallel Graphs – Straight-Line Drawings

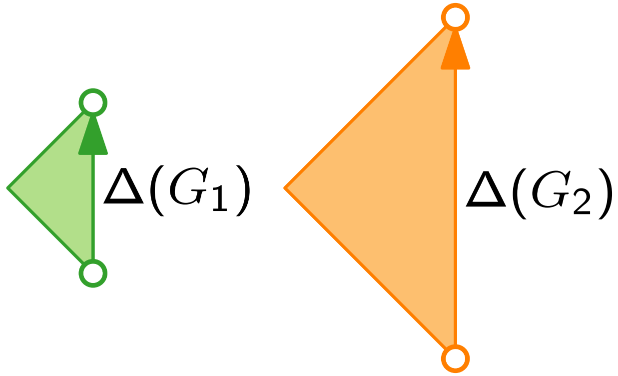


Divide-and-conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top

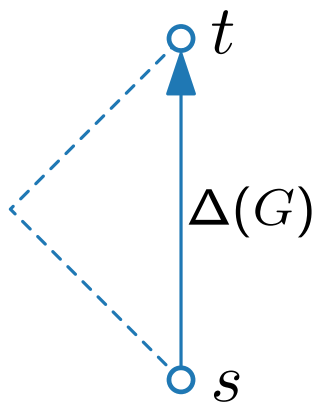
Base case: Q-nodes

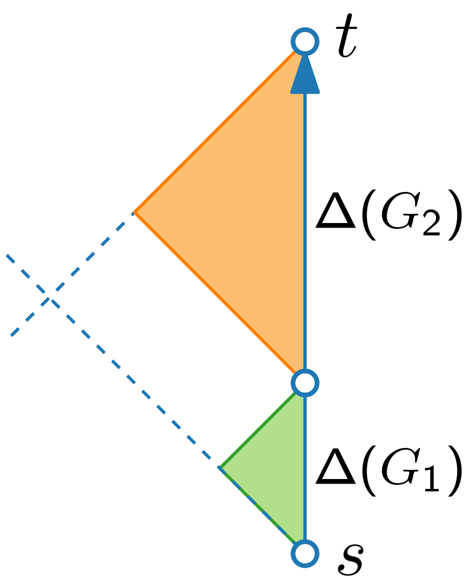
Divide: Draw G_1 and G_2 first

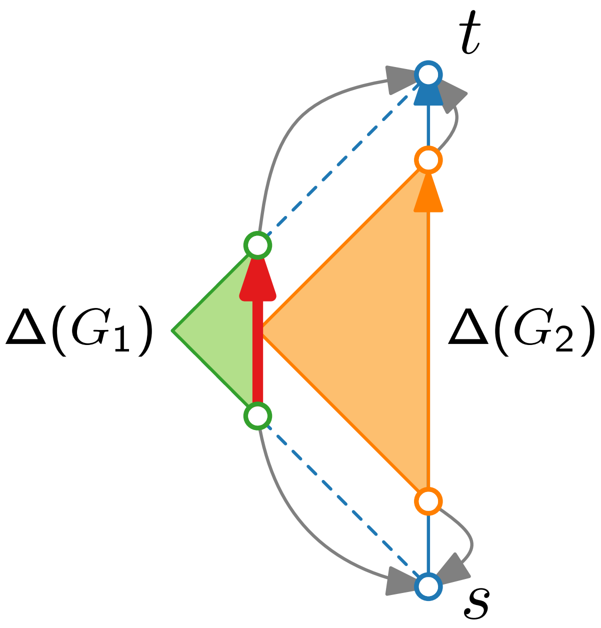


Conquer:

- S-nodes: series compositions
- P-nodes: parallel compositions







Do you see any problem?

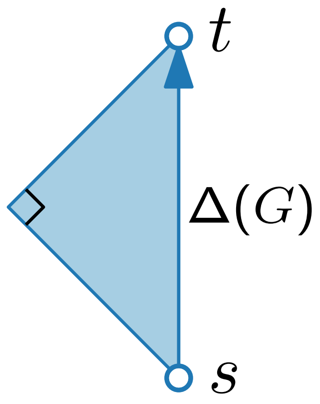
single edge

change embedding!

Series-Parallel Graphs – Straight-Line Drawings

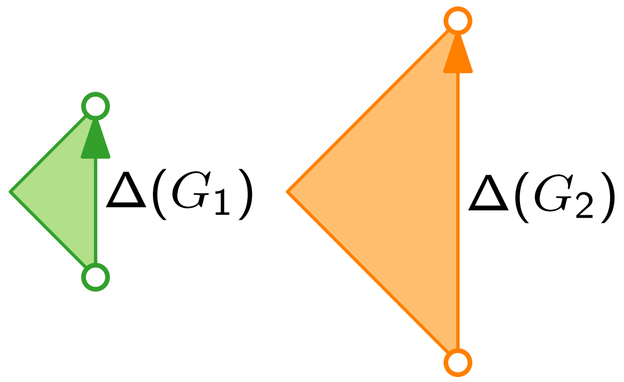
Divide-and-conquer algorithm using the decomposition tree

- Invariant: draw G inside a right-angled isosceles bounding triangle $\Delta(G)$ with s at the bottom and t at the top



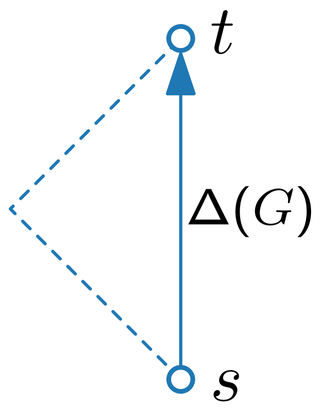
Base case: Q-nodes

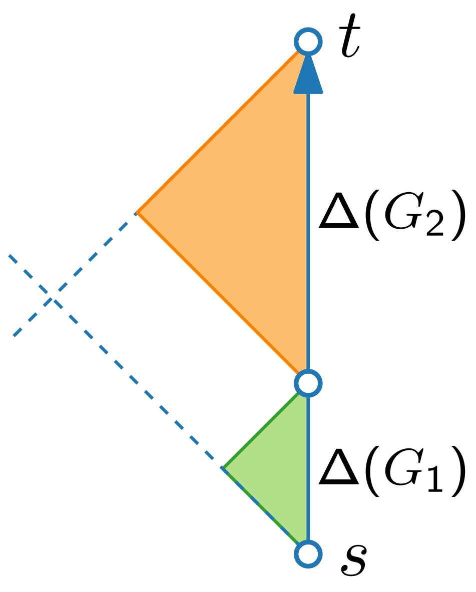
Divide: Draw G_1 and G_2 first

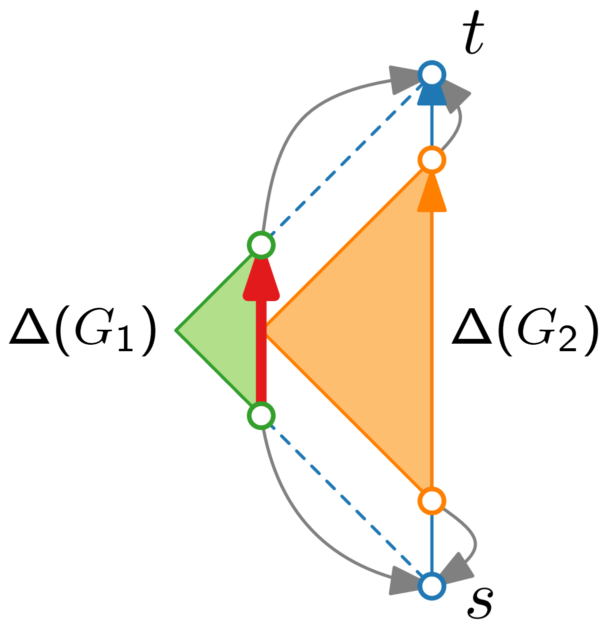


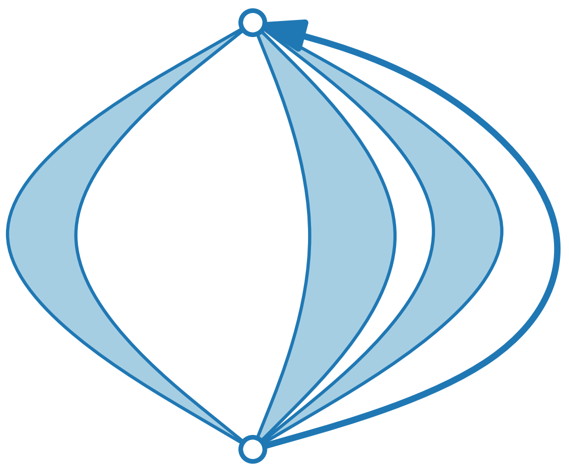
Conquer:

- S-nodes: series compositions
- P-nodes: parallel compositions



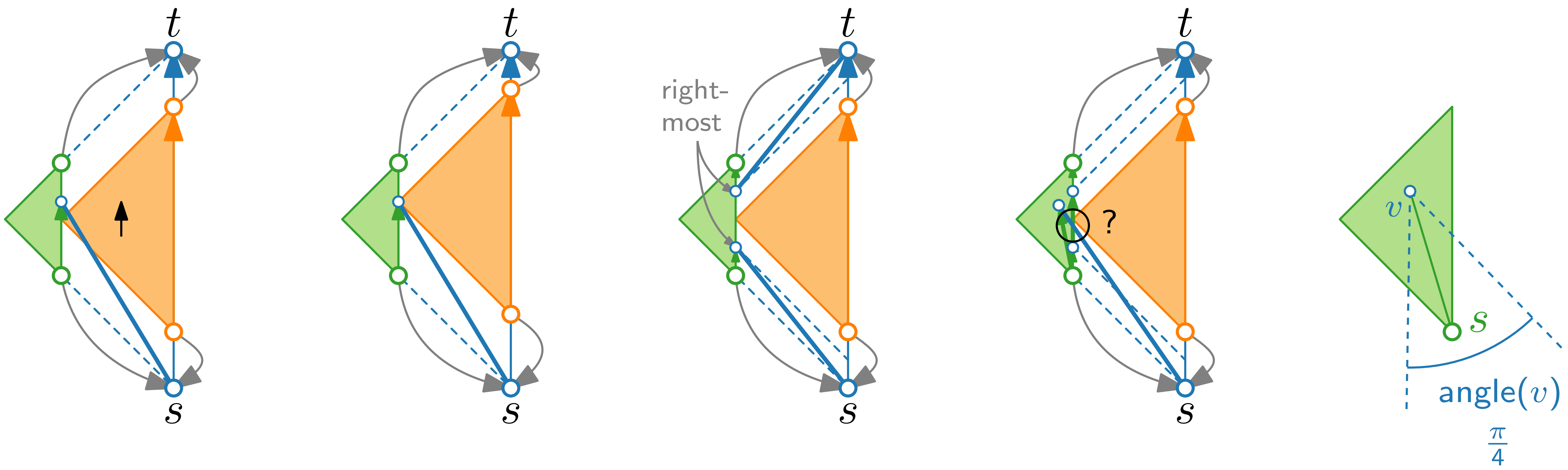






Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?



- This condition **is** preserved during the induction step.

Assume the following holds:
the only vertex in angle(v) is s

Lemma.
The drawing produced by the algorithm is planar.

Series-Parallel Graphs – Result

Theorem.
Let G be a series-parallel graph. Then G (with **variable embedding**) admits a drawing Γ that

- is upward planar,
- is straight-line, and
- uses quadratic area.
- Isomorphic components of G have congruent drawings up to translation.

Γ can be computed in linear time.

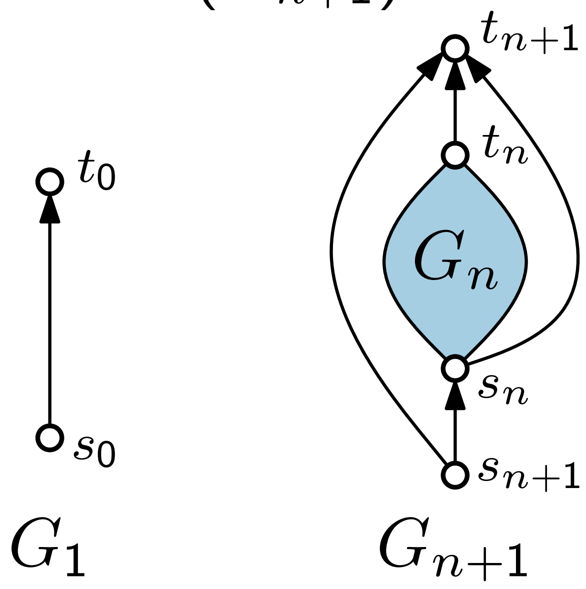
Series-Parallel Graphs – Fixed Embedding

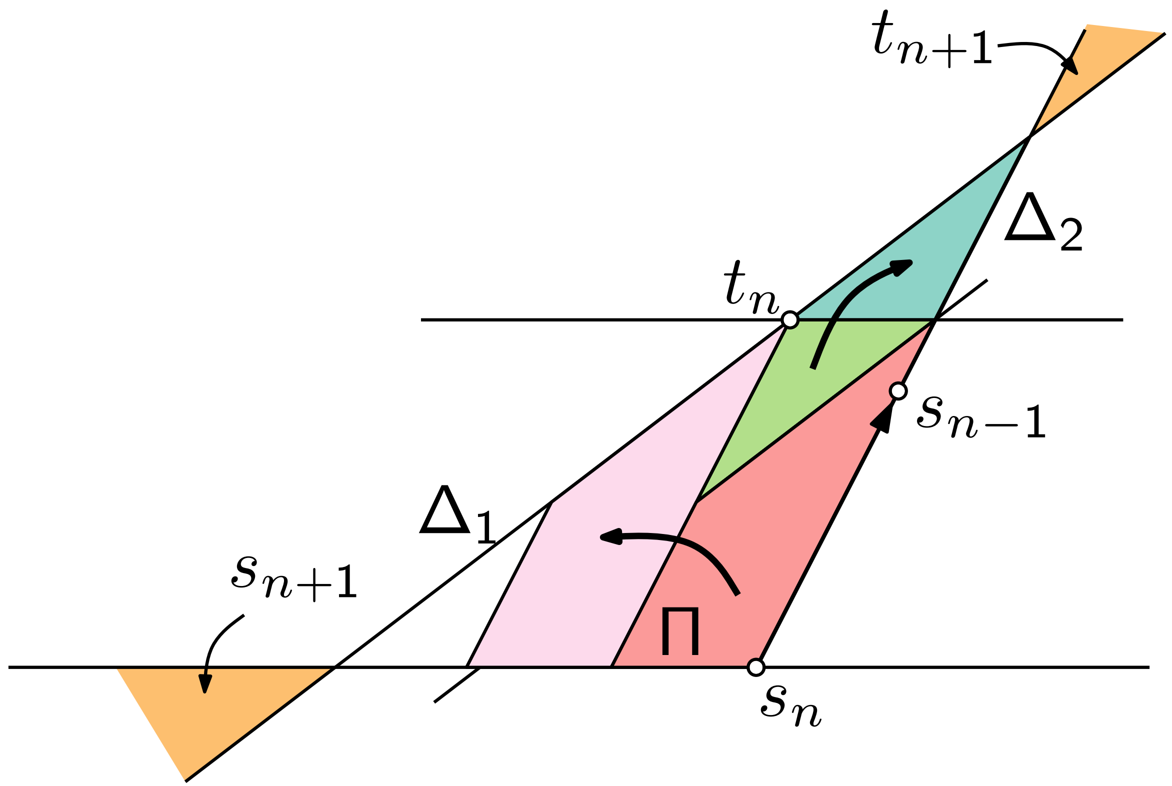
Theorem. [Bertolazzi, Di Battista, Mannino, Tamassia '94]

For any $n \geq 1$, there exists a $2n$-vertex series-parallel graph G_n in an embedding such that any upward planar straight-line drawing of G_n that respects the given embedding requires $\Omega(4^n)$ area.

- $2 \cdot \text{Area}(G_n) < \text{Area}(\Pi)$
- $2 \cdot \text{Area}(\Pi) \leq \text{Area}(G_{n+1})$

$\Rightarrow 4 \cdot \text{Area}(G_n) < \text{Area}(G_{n+1})$





Discussion

- There exist fixed-parameter (FPT) algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy & Lynch 2005, Didimo et al. 2009]

- Finding a consistent assignment (Theorem 2) can be sped up to $\mathcal{O}(n + r^{1.5})$, where $r = \#$ sources.
  [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied:
  upward drawings of mixed graphs, upward drawings with layers for the vertices, upward planarity on cylinder/torus, upward k-planarity, ...

Literature

- [GD Ch. 6] Detailed explanation on upward planarity.
- [GD Ch. 3] Divide-and-conquer methods for series-parallel graphs.

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista &Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg &Tamassia '95]
  On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton & Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94]
  Upward Drawings of Triconnected Digraphs
- [Healy & Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giordano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10]
  Improving the running time of embedded upward planarity testing