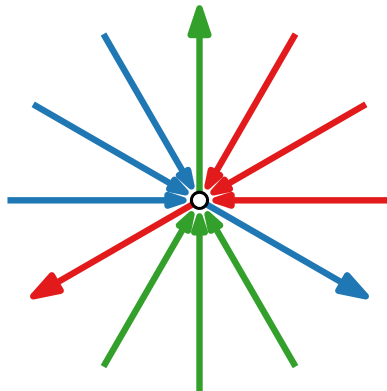
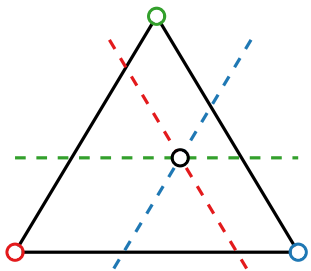


Visualization of Graphs

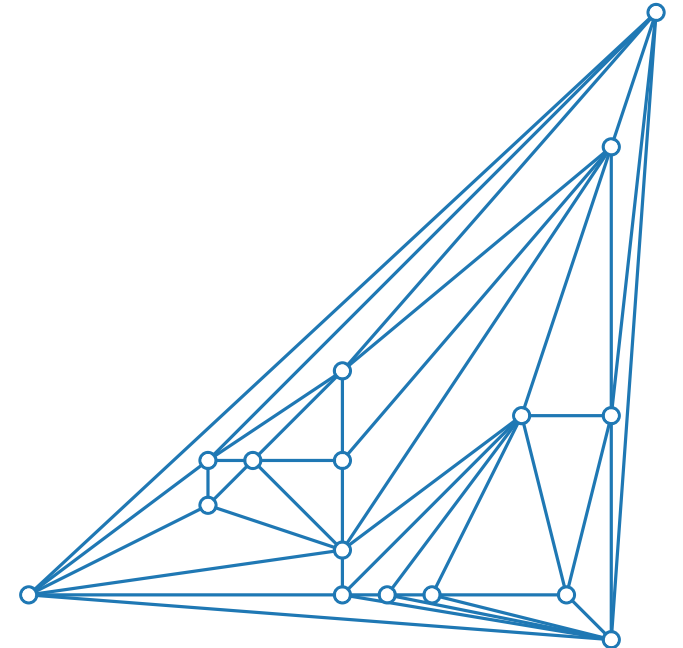
Lecture 4:

Straight-Line Drawings of Planar Graphs II: Schnyder Woods



Alexander Wolff

Summer term 2025



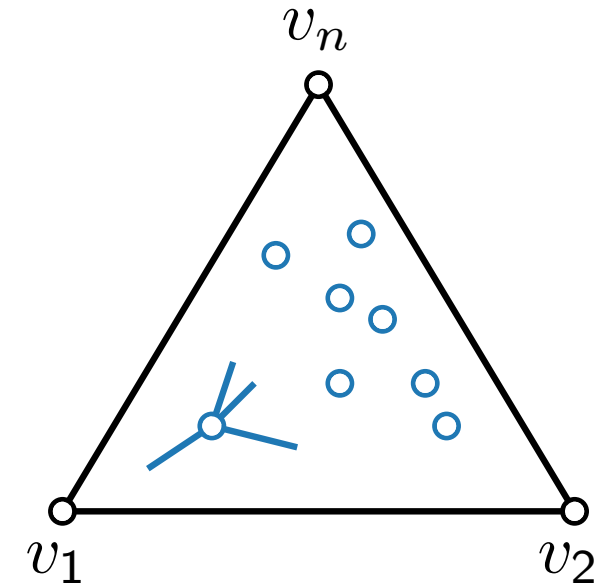
Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$.

Theorem. [Schneider '90]
Every n -vertex planar graph has a planar straight-line drawing of size ~~$(n - 2) \times (n - 2)$~~ $(2n - 5) \times (2n - 5)$.

Idea. (easier to show)

- Fix outer triangle.
- Compute coordinates of inner vertices
 - based on outer triangle and
 - how much space there should be for other vertices
 - using weighted barycentric coordinates.

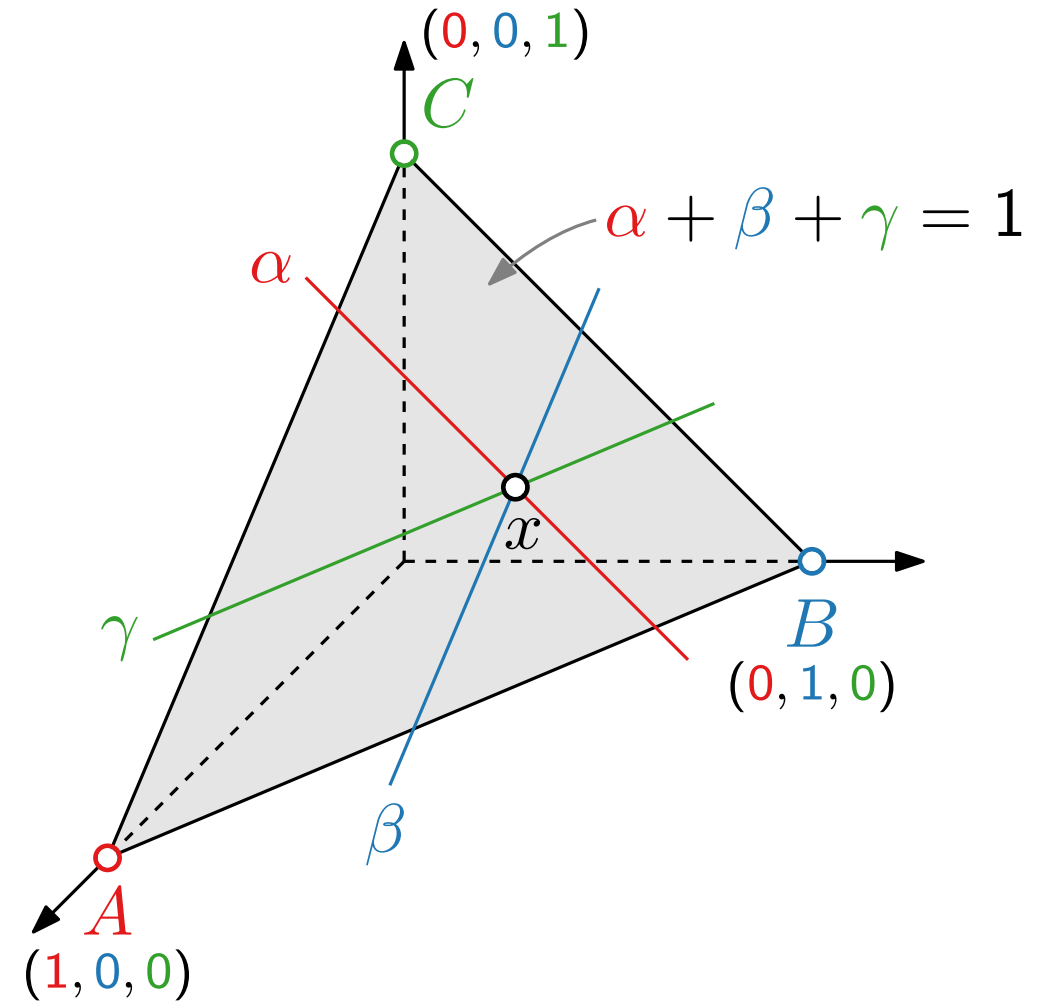
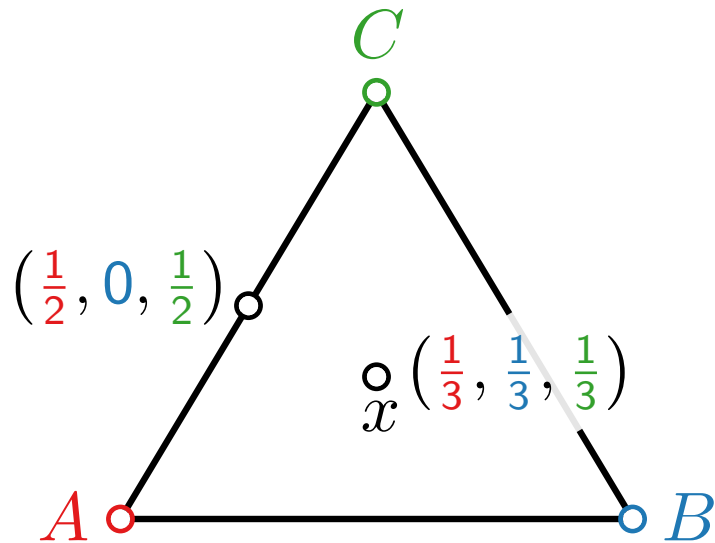


Barycentric Coordinates

Recall: $\text{barycenter}(x_1, \dots, x_k) = \sum_{i=1}^k x_i / k$

Let A, B, C form a triangle, and let x lie in $\triangle ABC$. The **barycentric coordinates** of x with respect to $\triangle ABC$ are a triple $(\alpha, \beta, \gamma) \in \mathbb{R}_{\geq 0}^3$ such that

- $\alpha + \beta + \gamma = 1$ and
- $x = \alpha A + \beta B + \gamma C$.



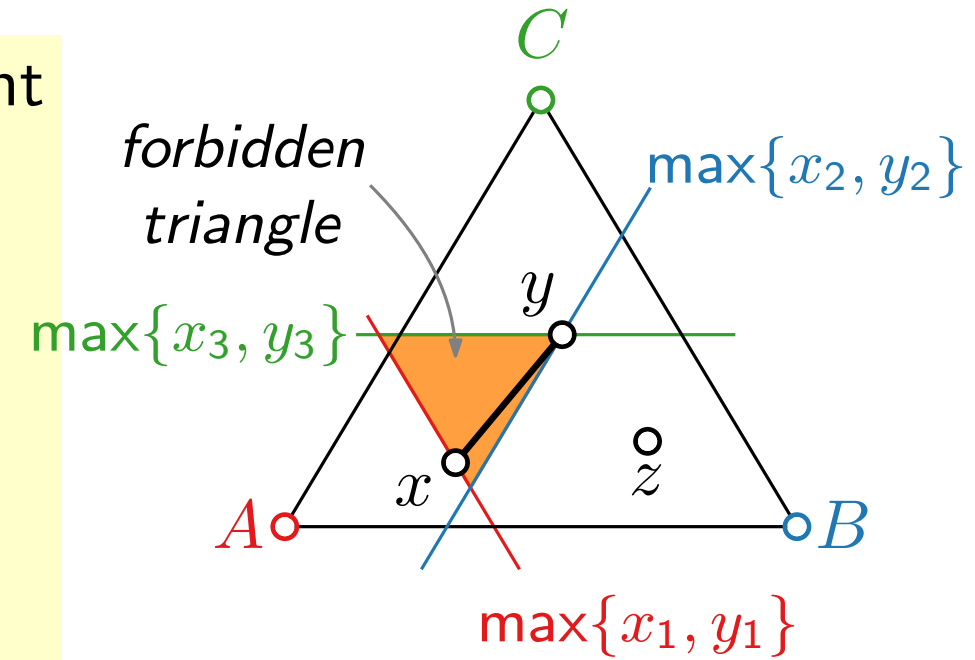
Barycentric Representation

A **barycentric representation** of a graph G is an assignment of barycentric coordinates to the vertices of G :

$$f: V(G) \rightarrow \mathbb{R}_{\geq 0}^3, \quad v \mapsto (\textcolor{red}{v}_1, \textcolor{blue}{v}_2, \textcolor{green}{v}_3)$$

with the following properties:

- (B1) $\textcolor{red}{v}_1 + \textcolor{blue}{v}_2 + \textcolor{green}{v}_3 = 1$ for all $v \in V(G)$,
- (B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$, there exists a $k \in \{\textcolor{red}{1}, \textcolor{blue}{2}, \textcolor{green}{3}\}$ with $x_k < z_k$ and $y_k < z_k$.



Barycentric Representations of Planar Graphs

Lemma.

Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G , and let $A, B, C \in \mathbb{R}^2$ be in general position. Then the mapping

$$\phi: v \in V \mapsto v_1 A + v_2 B + v_3 C$$

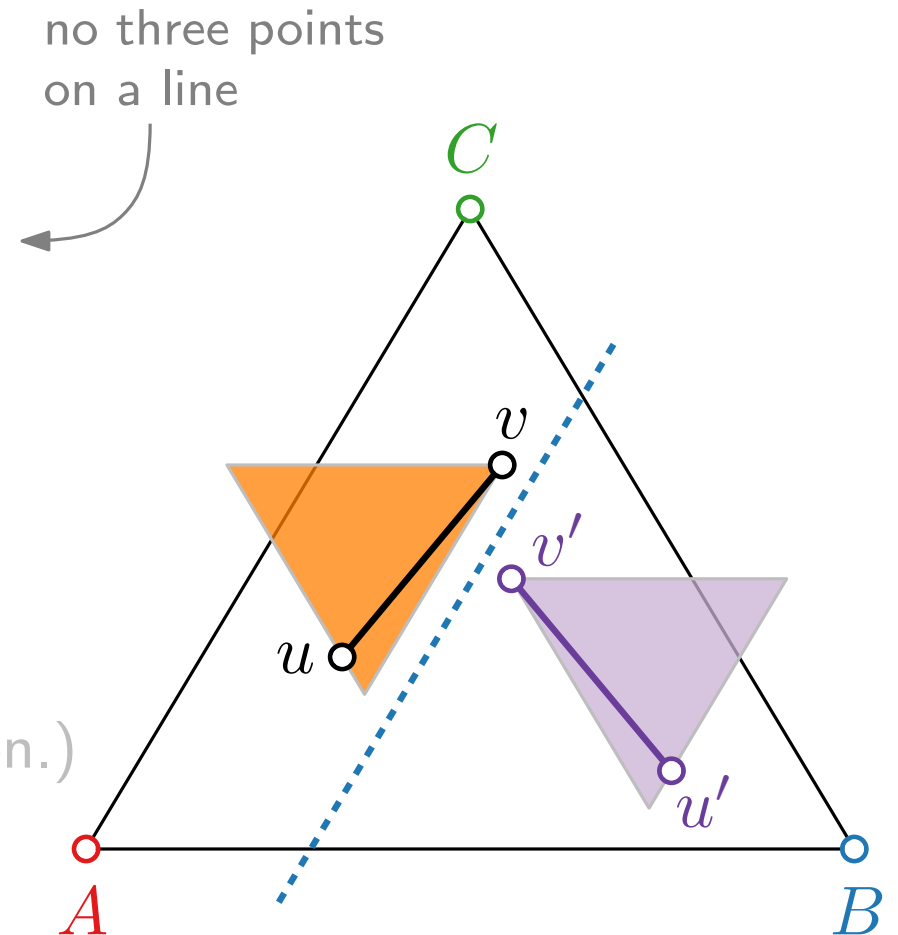
yields a **planar straight-line** drawing of G inside $\triangle ABC$.

- No vertex x can lie on an edge $\{u, v\}$. (Obvious by definition.)
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ crosses:

$$u'_i > u_i, v_i, \quad v'_j > u_j, v_j, \quad u_k > u'_k, v'_k, \quad v_l > u'_l, v'_l$$

$$\Rightarrow \{i, j\} \cap \{k, l\} = \emptyset$$

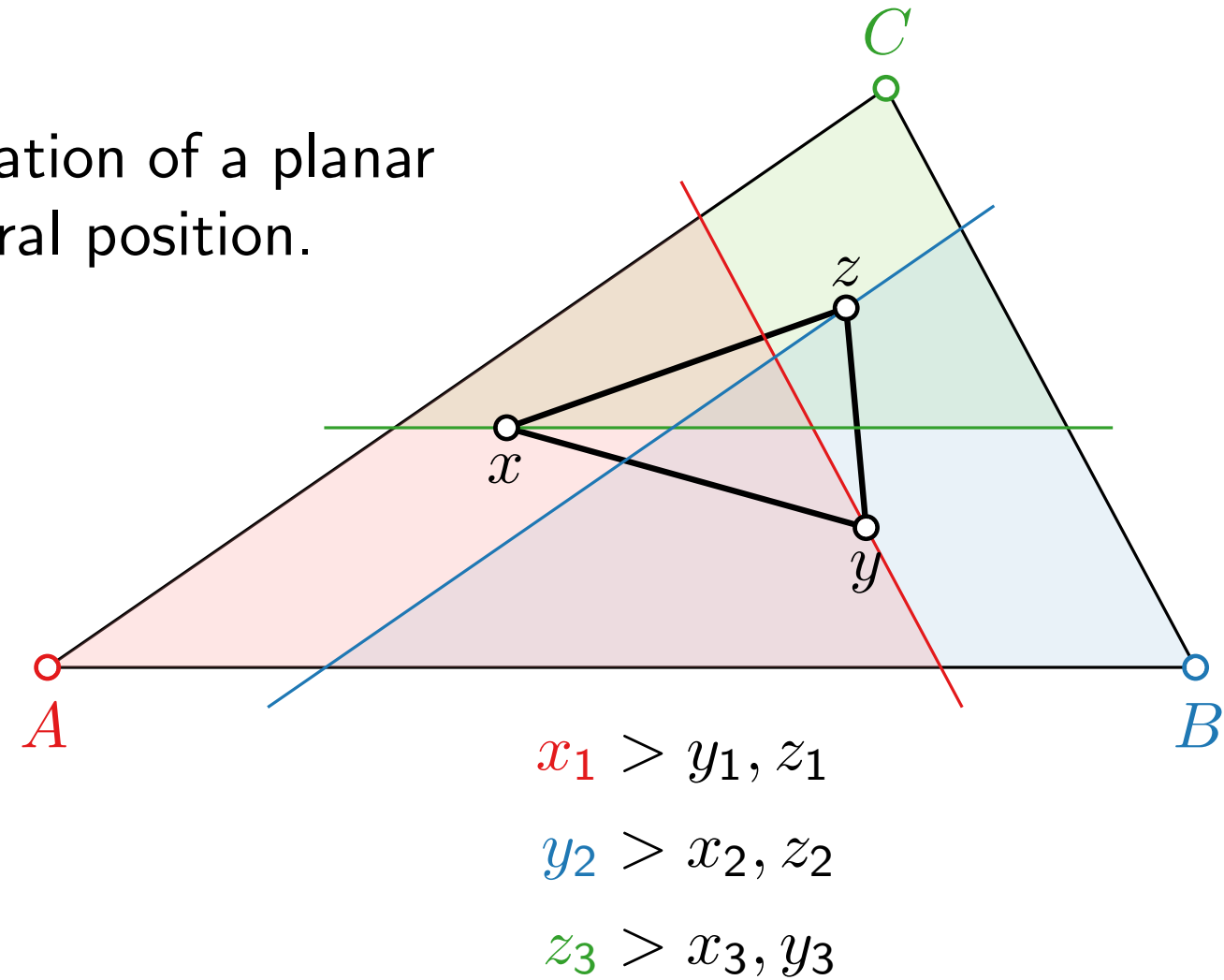
$$\text{w.l.o.g. } i = j = 2 \Rightarrow u'_2, v'_2 > u_2, v_2 \Rightarrow \text{separated by a straight line}$$



How to find a barycentric representation?

Schnyder Labeling

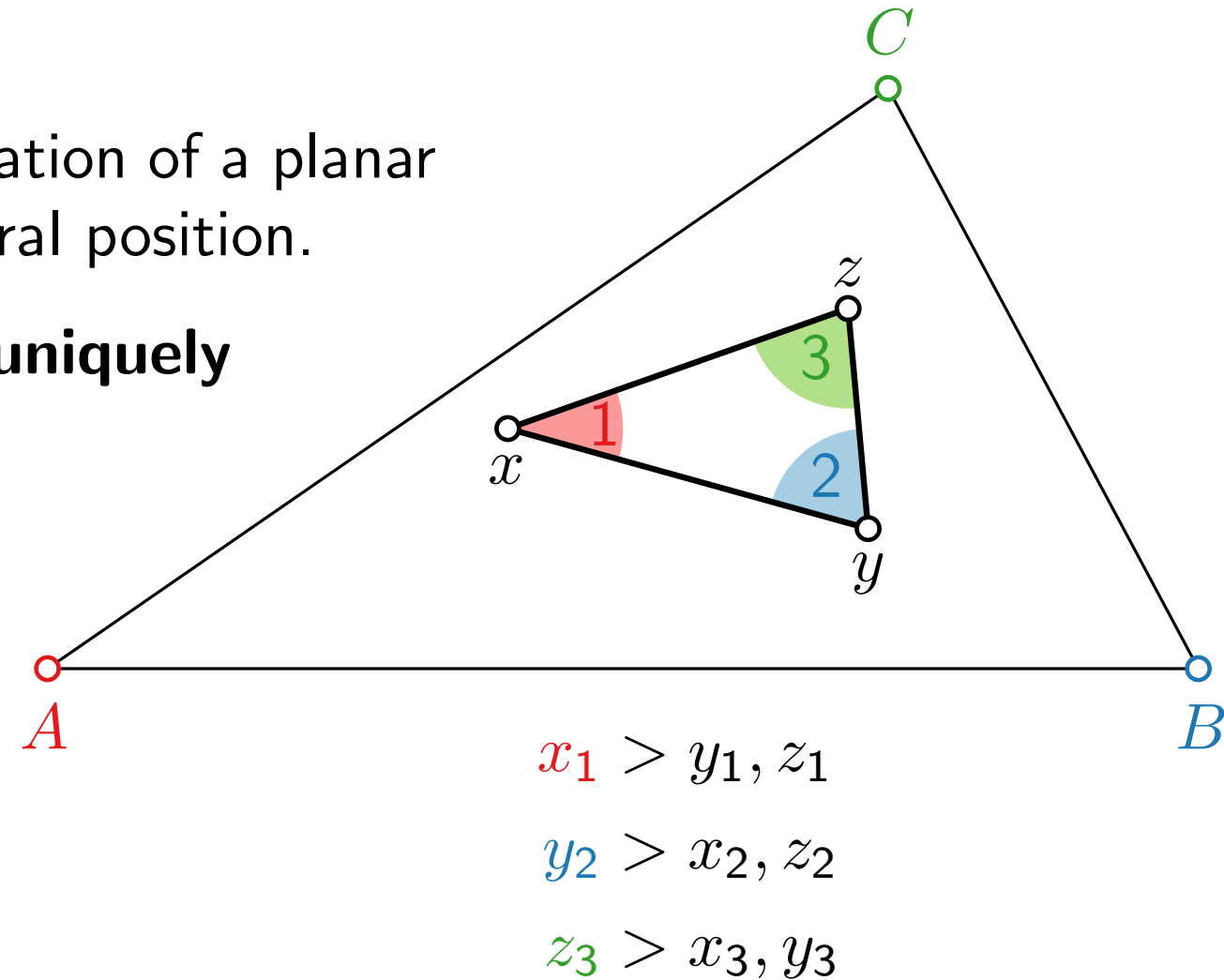
Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar triangulation G , and let $A, B, C \in \mathbb{R}^2$ be in general position.



Schnyder Labeling

Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar triangulation G , and let $A, B, C \in \mathbb{R}^2$ be in general position.

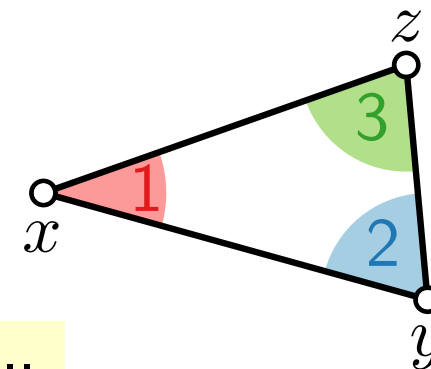
We can label each angle in each triangle $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.



Schnyder Labeling

Let $f: v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar triangulation G , and let $A, B, C \in \mathbb{R}^2$ be in general position.

We can label each angle in each triangle $\triangle xyz$ **uniquely** with $k \in \{1, 2, 3\}$.

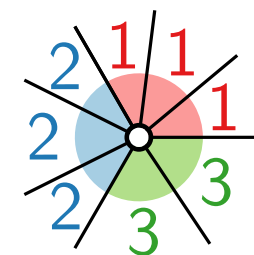


A **Schnyder labeling** of a plane triangulation G is a labeling of all internal angles with labels 1, 2, and 3 such that:

Faces: The three angles of an internal face are labeled 1, 2, and 3 in counterclockwise (ccw) order.

Vertices: The ccw order of labels around each vertex consists of

- a non-empty interval of 1s,
- followed by a non-empty interval of 2s,
- followed by a non-empty interval of 3s.

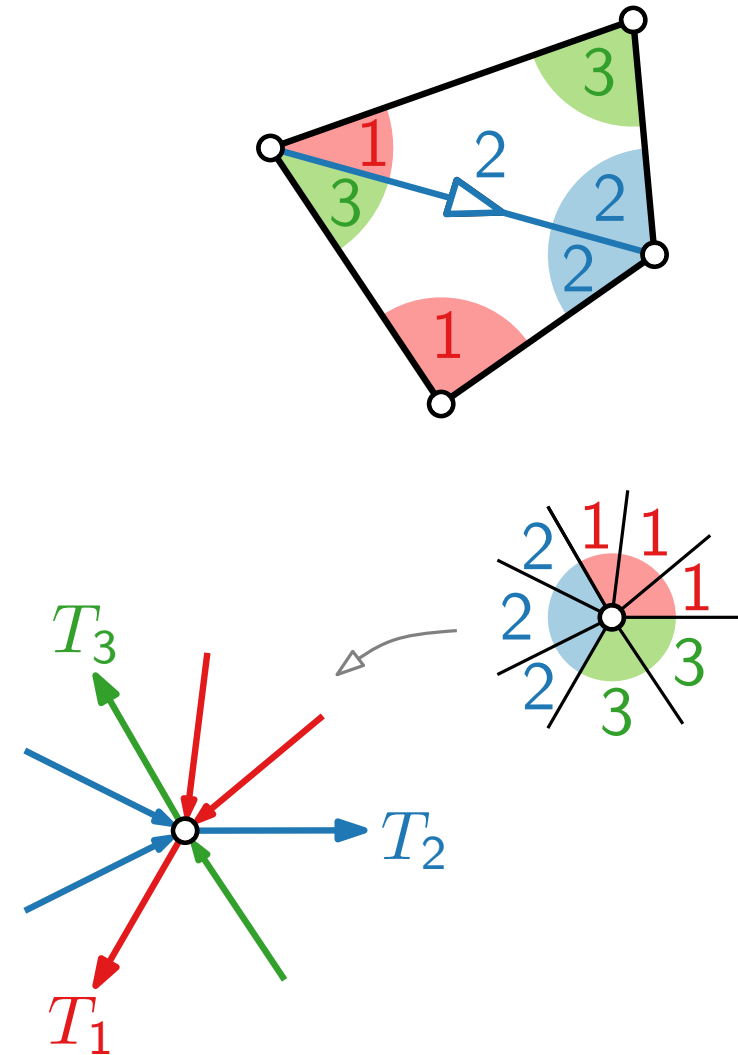


Schnyder Wood

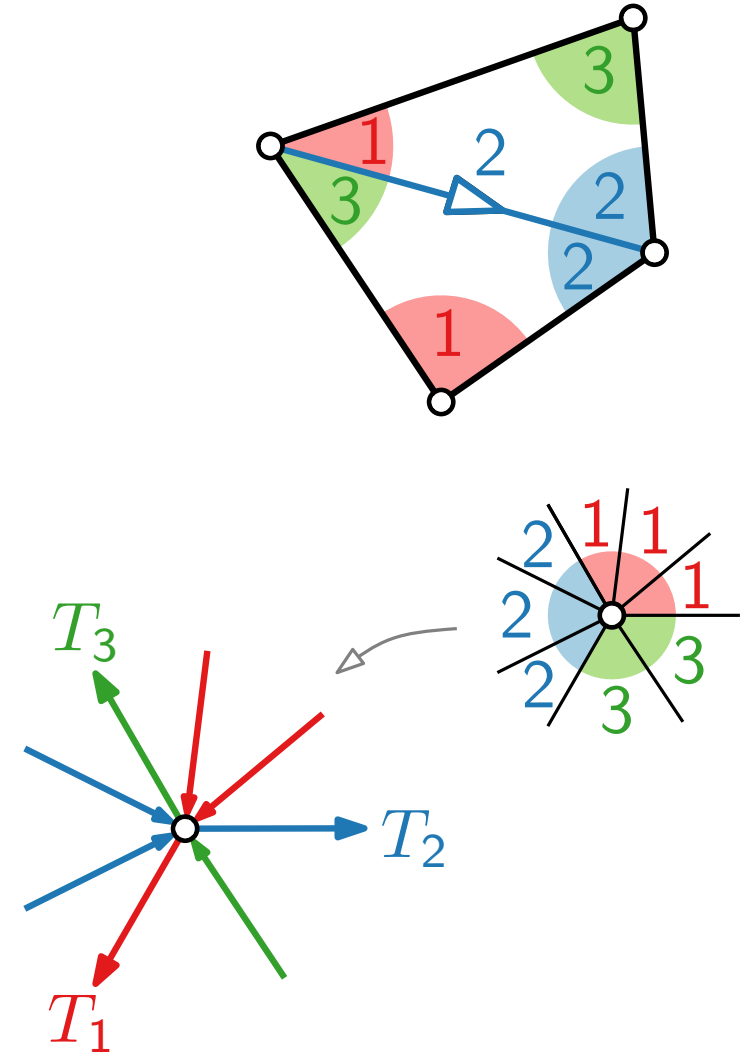
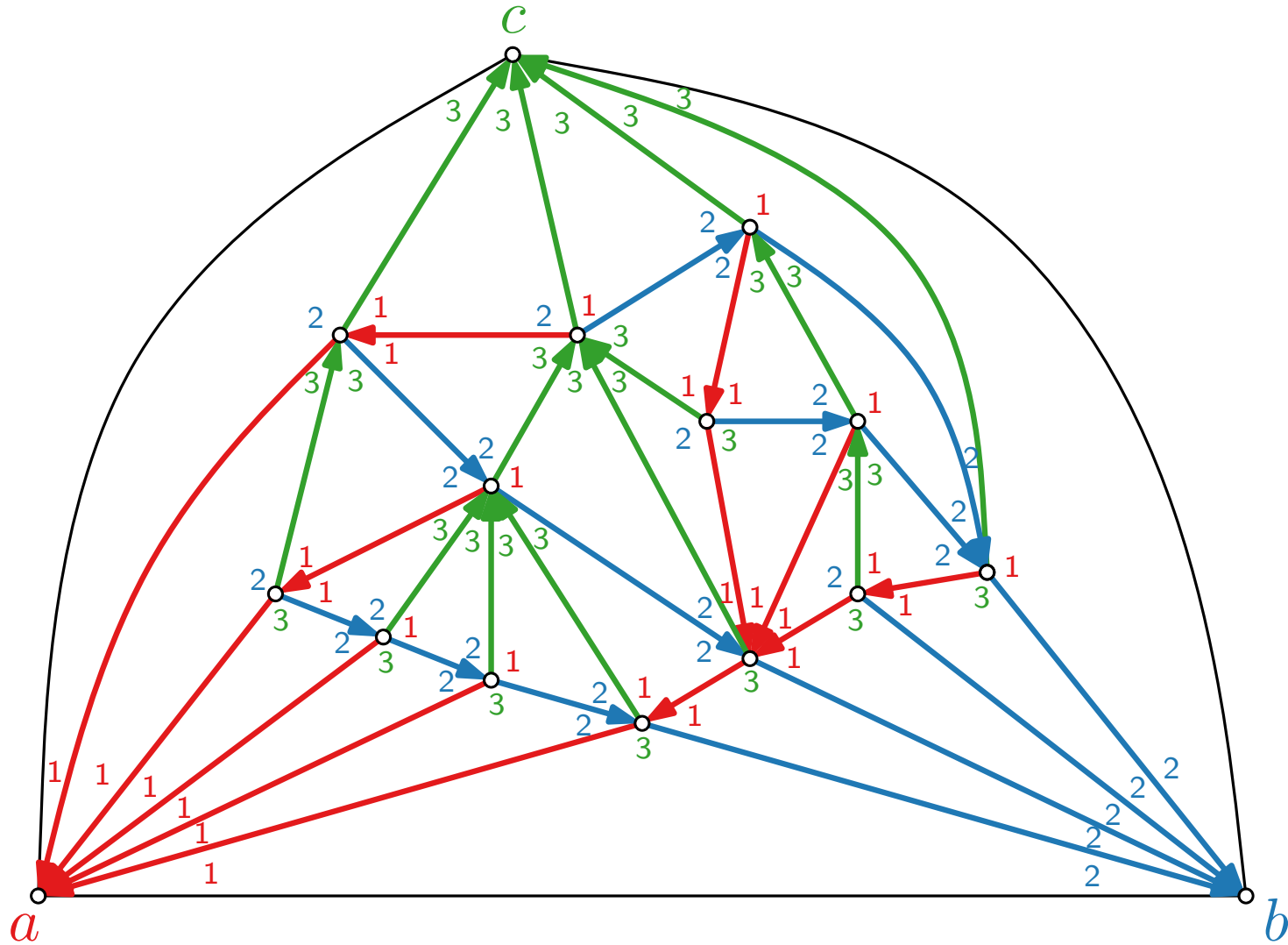
A Schnyder labeling induces an edge labeling.

A **Schnyder wood** (or **Schnyder realizer**) of a plane triangulation G is a partition of the inner edges of G into three sets of oriented edges T_1 , T_2 , T_3 such that, for each inner vertex v of G , it holds that

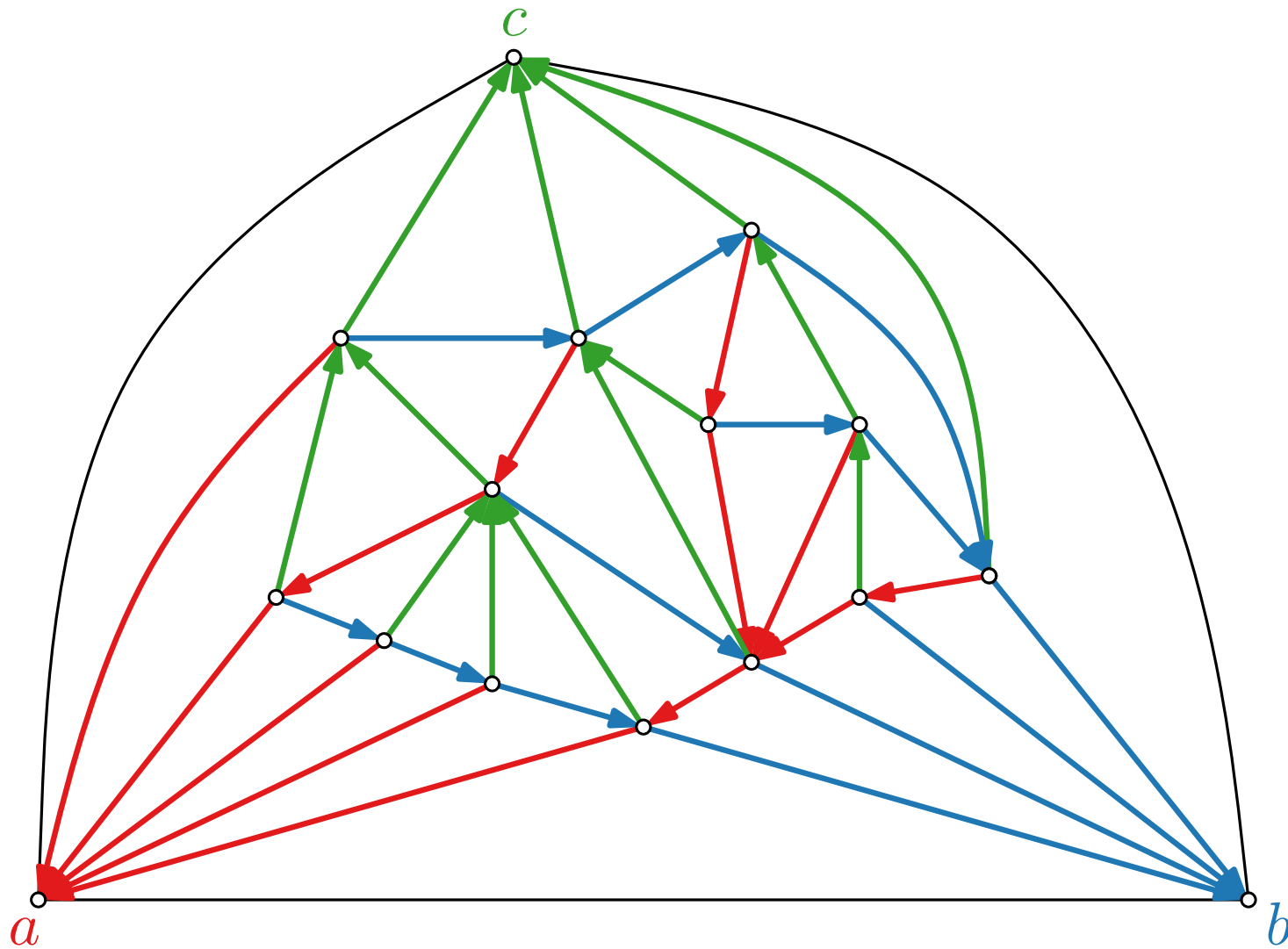
- v has one outgoing edge in each of T_1 , T_2 , and T_3 .
- The ccw order of edges around v is:
 leaving in T_1 , entering in T_3 , leaving in T_2 ,
 entering in T_1 , leaving in T_3 , entering in T_2 .



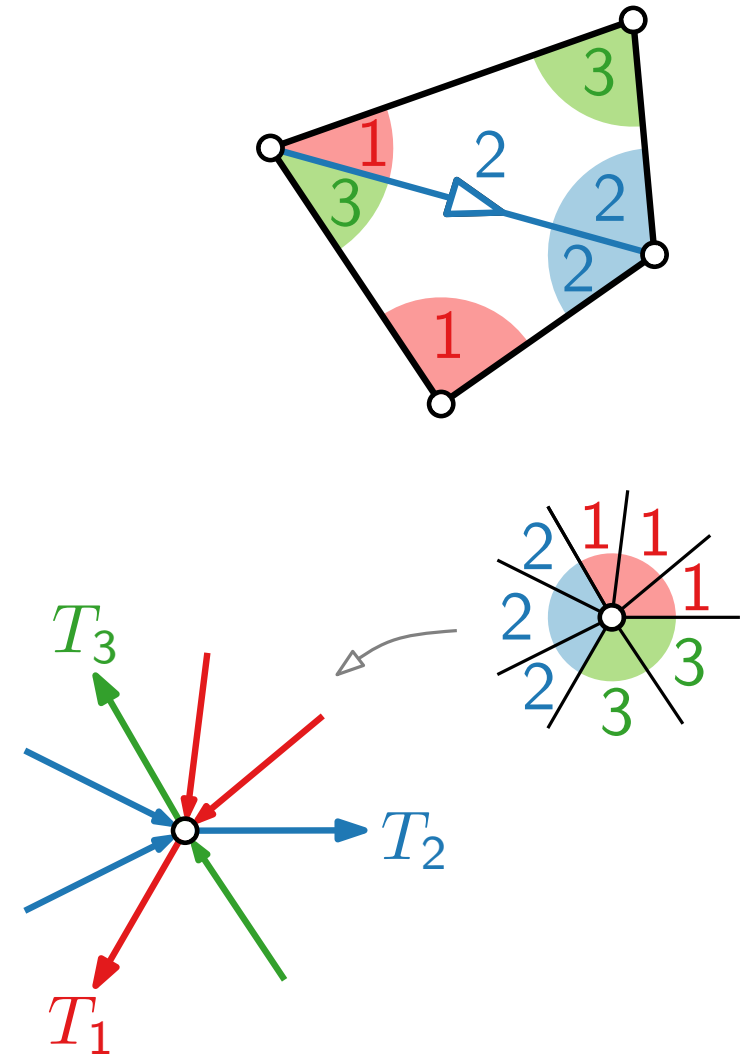
Schnyder Wood – Example and Properties



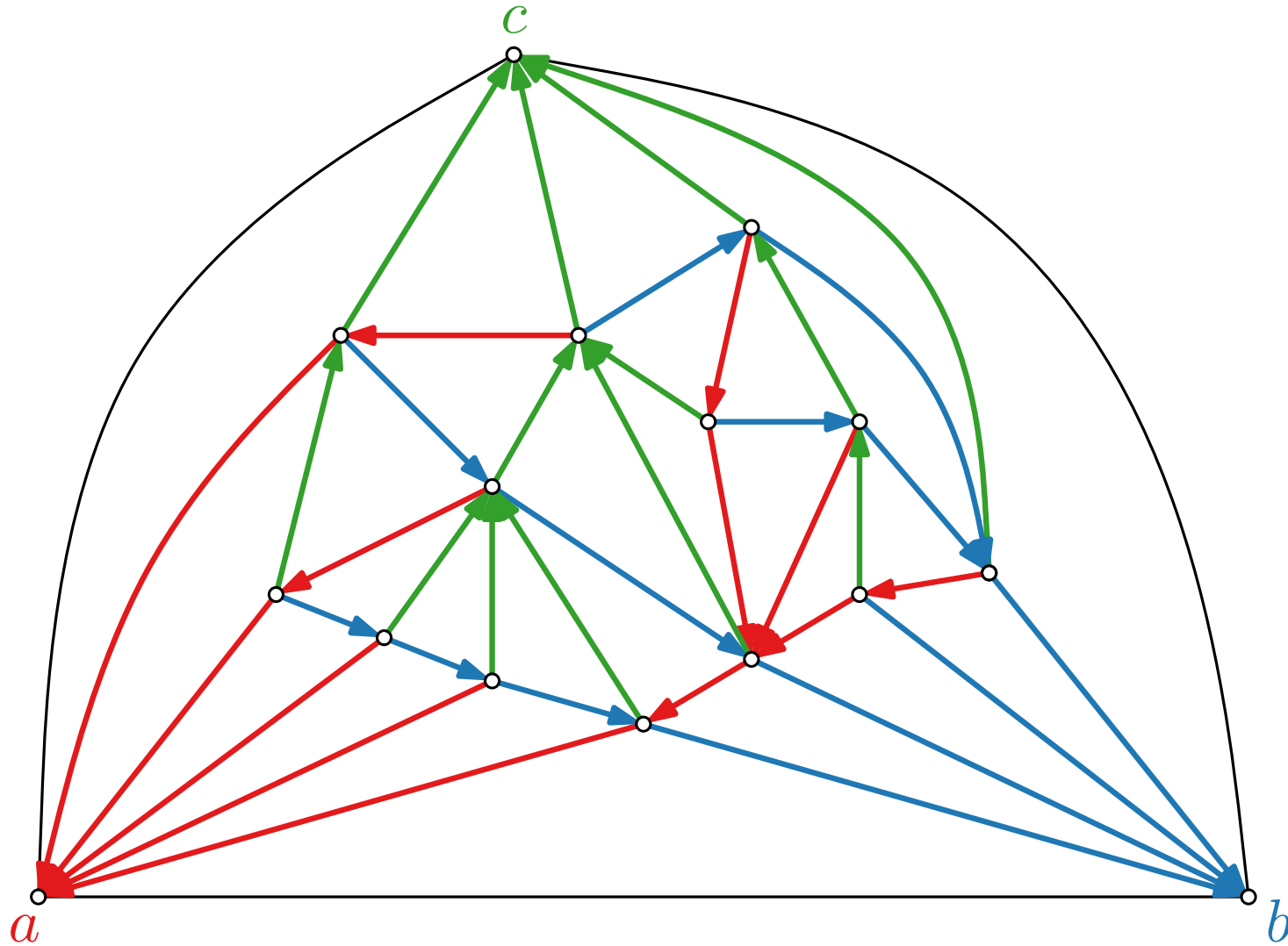
Schnyder Wood – Example and Properties



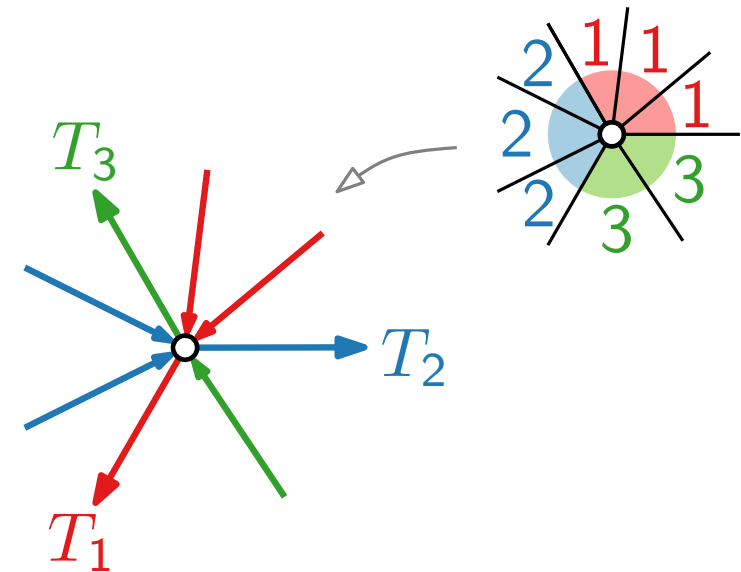
(a Schnyder labeling is not unique)



Schnyder Wood – Example and Properties



- All inner edges incident to a , b , and c are incoming in the same set (color).
- T_1 , T_2 , and T_3 are trees. Each spans all inner vertices and one outer vertex (its root).

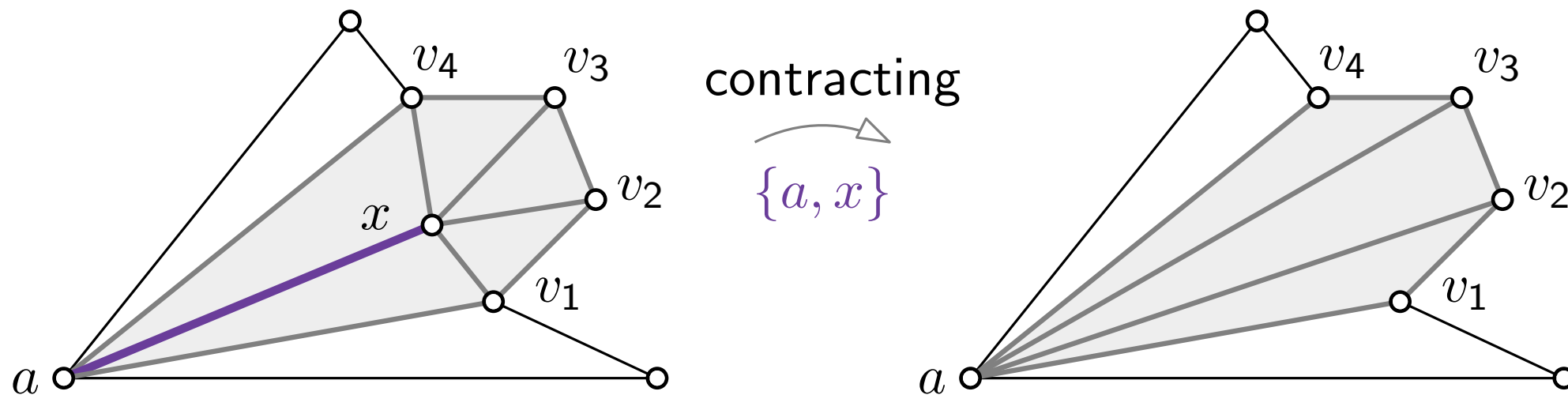


Schnyder Wood – Existence

Lemma.

[Kampen 1976]

Let G be a plane triangulation with vertices a, b, c on the outer face. Then there exists a **contractible edge** $\{a, x\}$ in G with $x \notin \{b, c\}$.



...requires that a and x have exactly two common neighbors.

Schnyder Wood – Existence

Lemma.

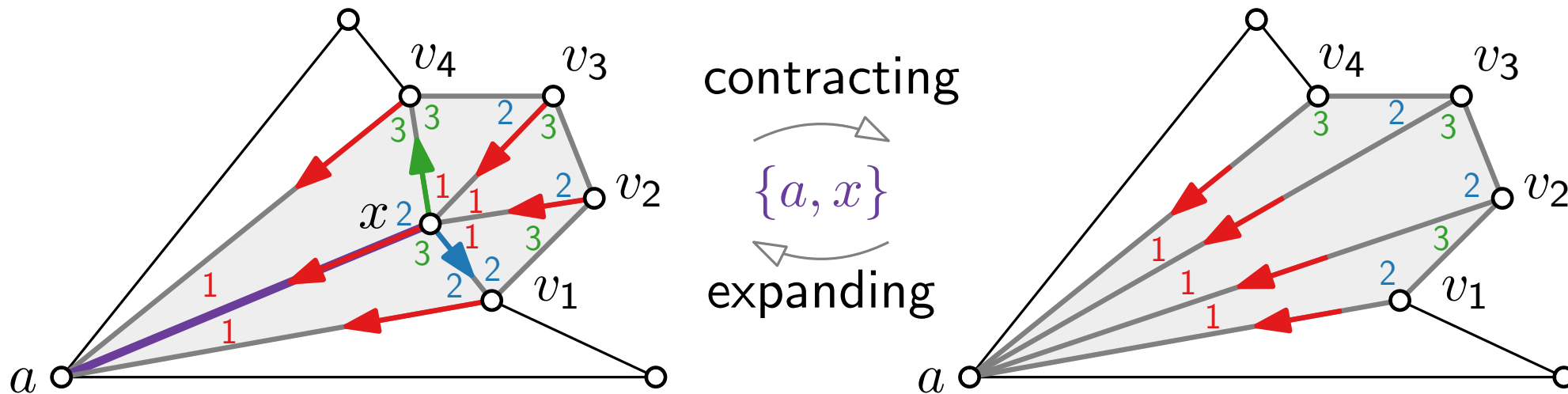
[Kampen 1976]

Let G be a plane triangulation with vertices a, b, c on the outer face. Then there exists a **contractible edge** $\{a, x\}$ in G with $x \notin \{b, c\}$.

Theorem.

Every plane triangulation has a Schnyder labeling and a Schnyder wood.

Proof by induction on $\#$ vertices via edge contractions.

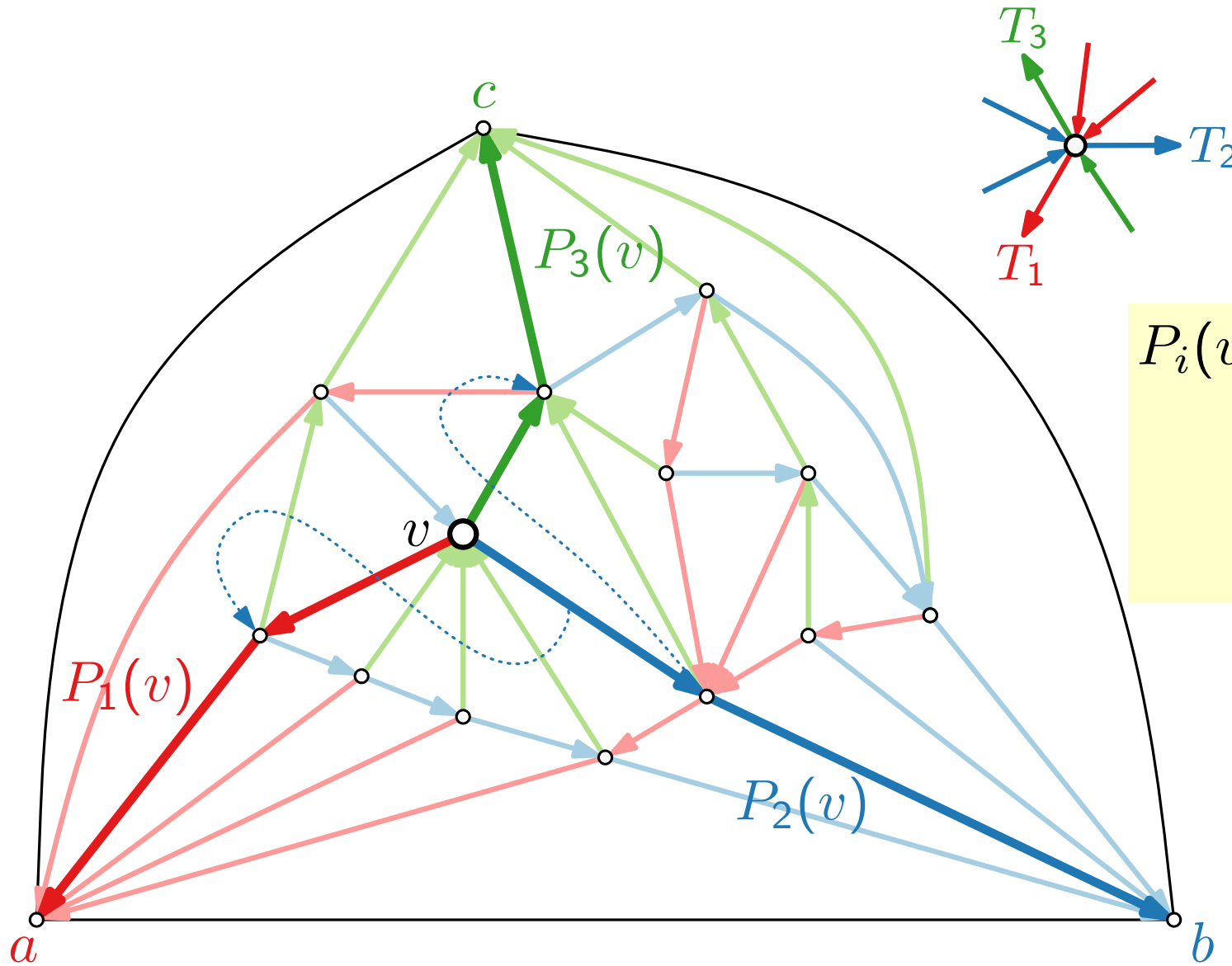


This constructive proof yields an algorithm for computing a Schnyder labeling. It can be implemented to run in $\mathcal{O}(n)$ time.

...requires that a and x have exactly two common neighbors.

→ Exercise 😊

Schnyder Wood – More Properties



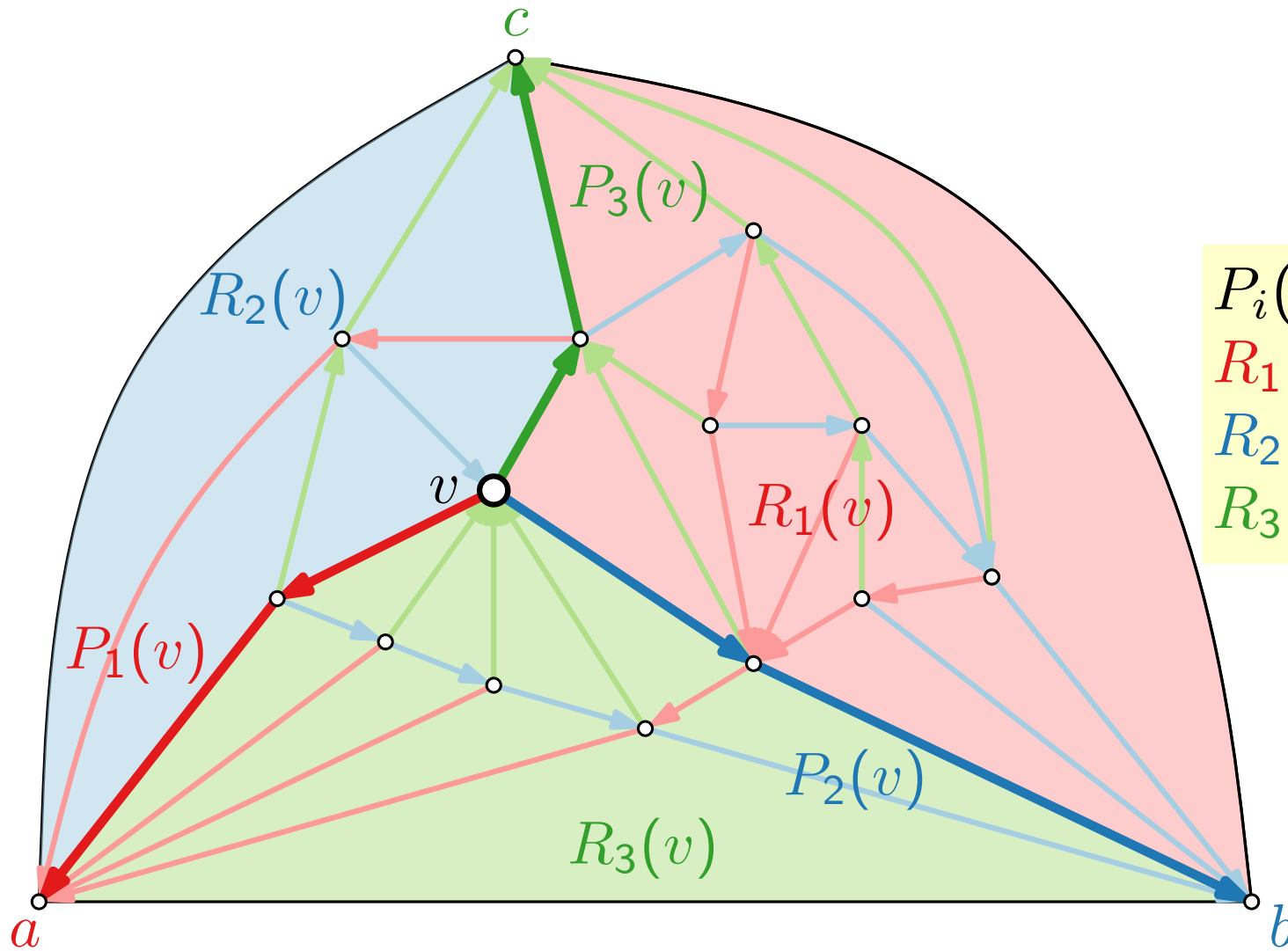
- From each vertex v there exists a unique
 - directed **red** path $P_1(v)$ to a ,
 - directed **blue** path $P_2(v)$ to b , and
 - directed **green** path $P_3(v)$ to c .

$P_i(v)$: unique path from v to root of T_i

Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v .

Schnyder Wood – More Properties



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 - directed **red** path $P_1(v)$ to a ,
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$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: set of faces bounded by $\langle P_2(v), bc, P_3(v) \rangle$

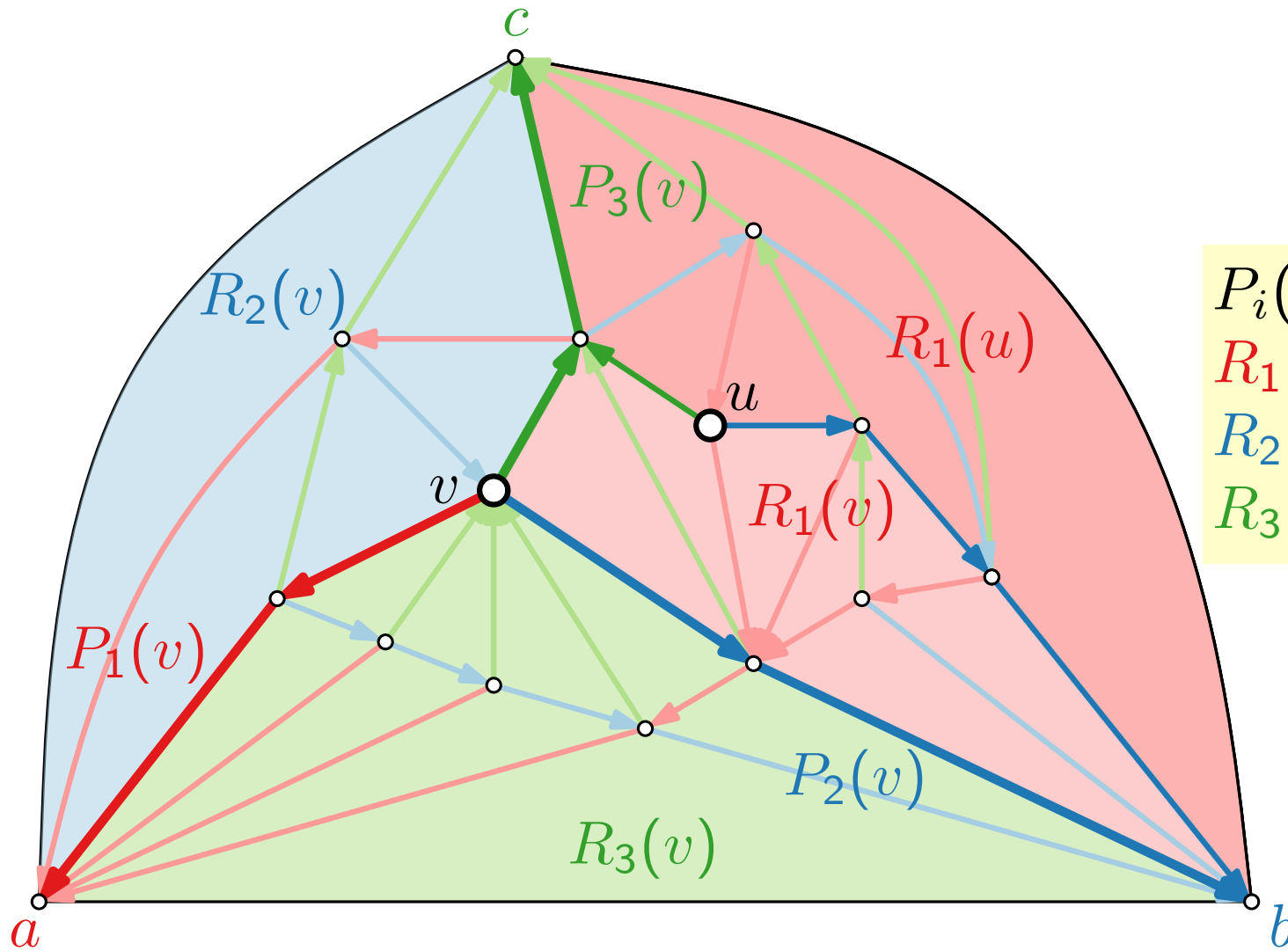
$R_2(v)$: set of faces bounded by $\langle P_3(v), ca, P_1(v) \rangle$

$R_3(v)$: set of faces bounded by $\langle P_1(v), ab, P_2(v) \rangle$

Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v .
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

Schnyder Wood – More Properties



- From each vertex v there exists a unique
 - directed **red** path $P_1(v)$ to a ,
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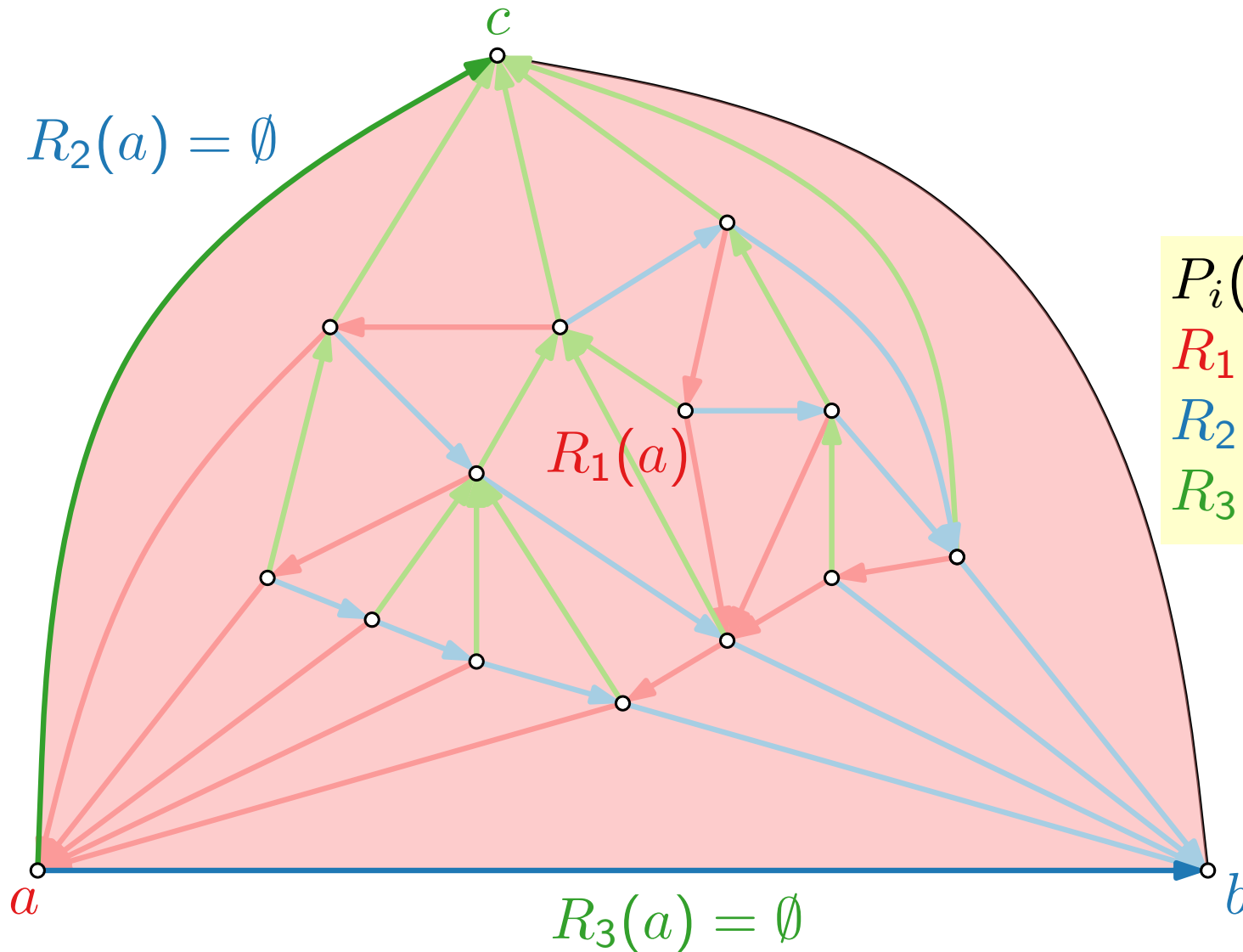
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Schnyder Wood – More Properties



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Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v .
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.
- $|R_1(v)| + |R_2(v)| + |R_3(v)| = 2n - 5$

Schnyder Drawing

Theorem.

[Schnyder '90]

For a plane triangulation G , the mapping

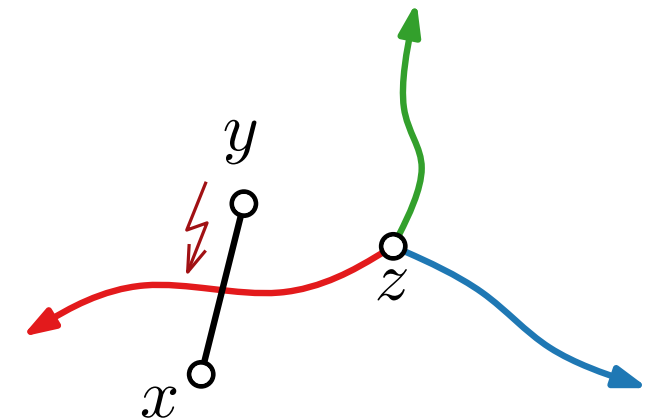
$$f: v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5} (|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G and, thus, yields a planar straight-line drawing of G

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$ ✓

(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$

■ $\{x, y\}$ must lie in $R_i(z)$ for some $i \in \{1, 2, 3\}$



Schnyder Drawing

Set $A = (0, 0)$, $B = (2n - 5, 0)$, and $C = (0, 2n - 5)$.

Theorem.

[Schnyder '90]

For a plane triangulation G , the mapping

$$f: v \mapsto (v_1, v_2, v_3) = \frac{1}{2n - 5} (|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

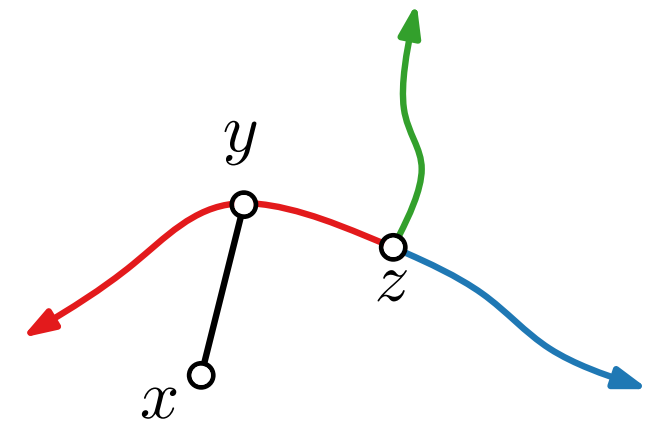
is a barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(2n - 5) \times (2n - 5)$ grid.

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$ ✓

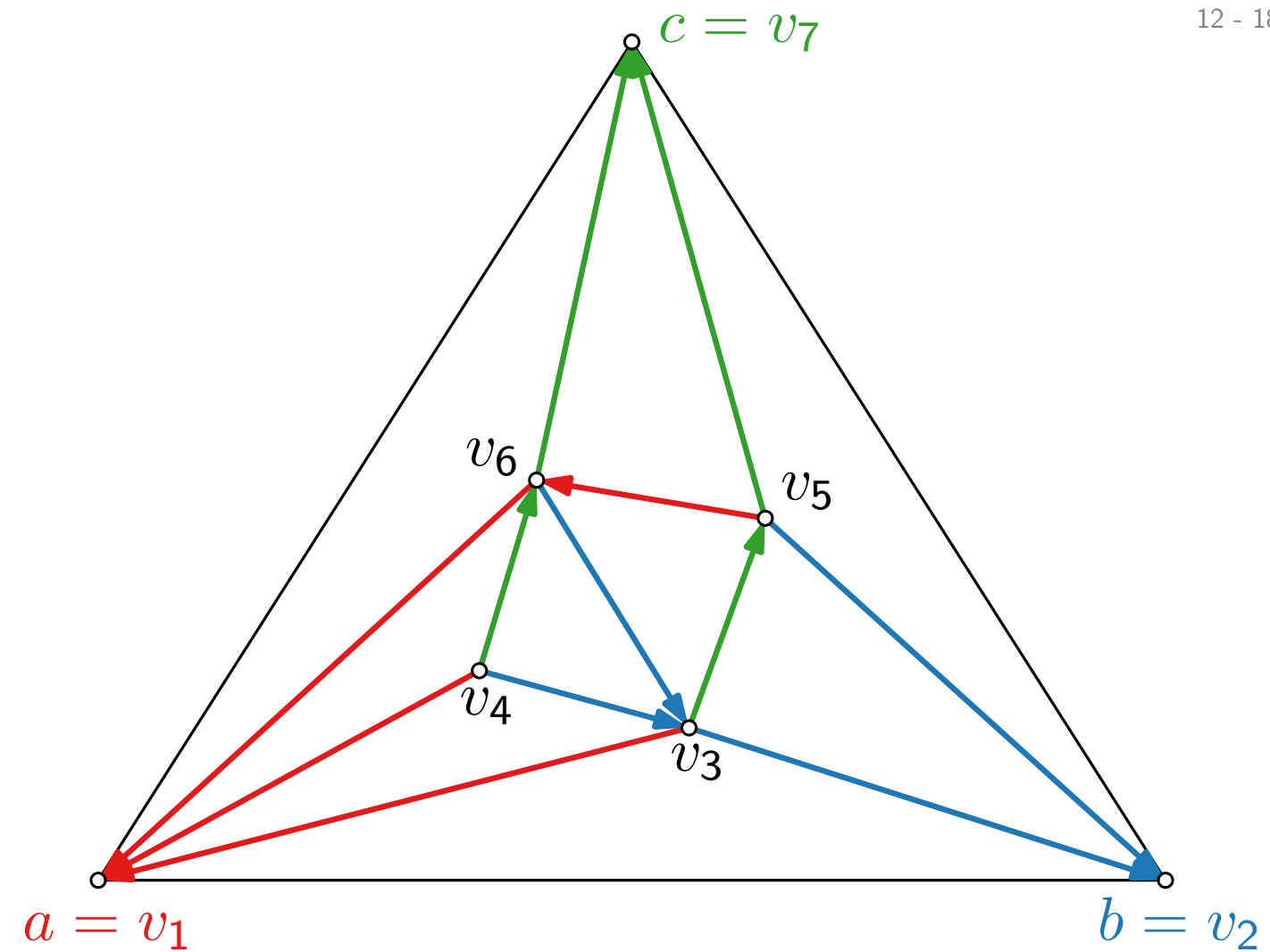
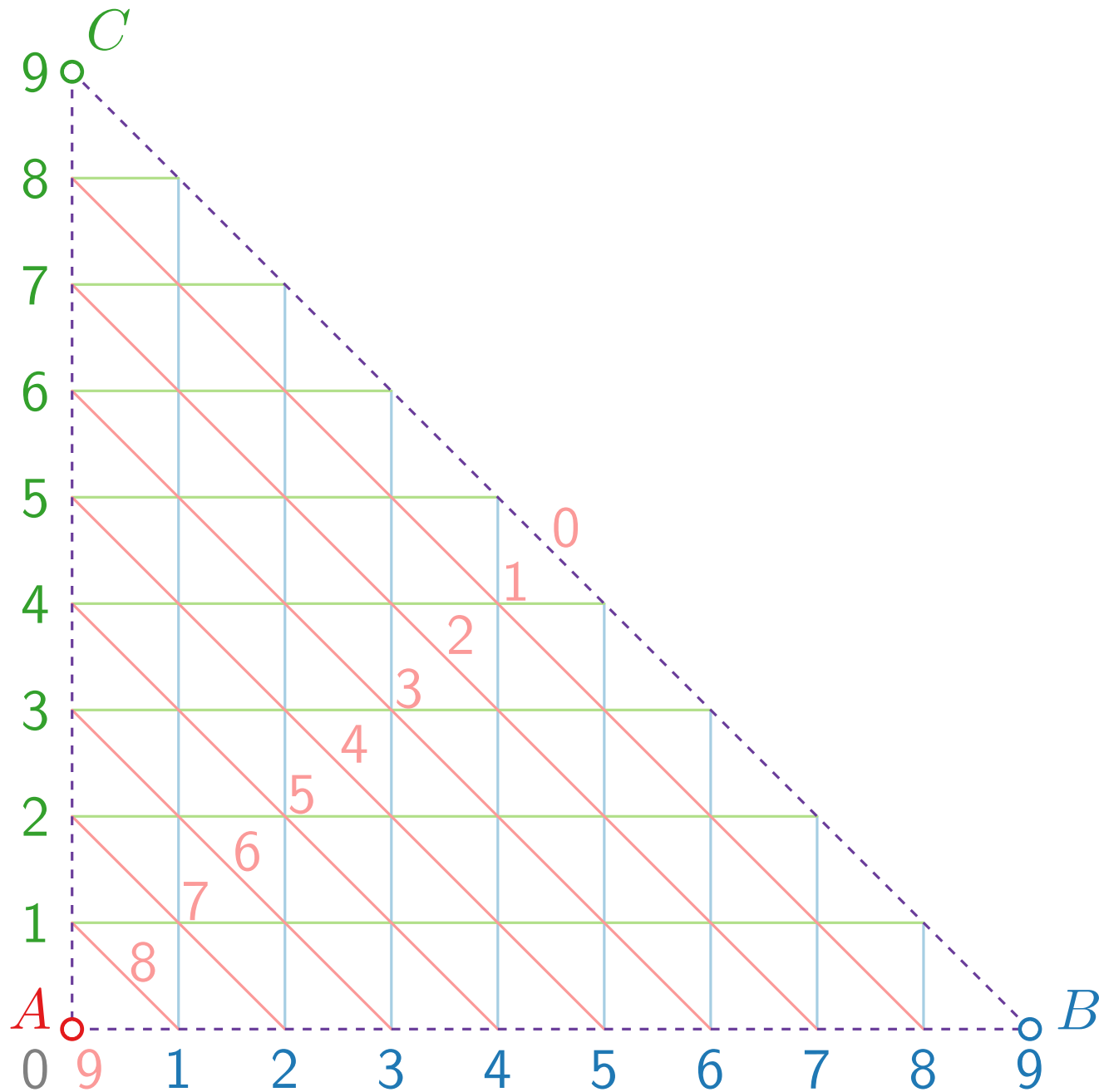
(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$ ✓

■ $\{x, y\}$ must lie in $R_i(z)$ for some $i \in \{1, 2, 3\}$

■ $x, y \in R_i(z) \Rightarrow R_i(x), R_i(y) \subsetneq R_i(z)$
 $\Rightarrow |R_i(x)|, |R_i(y)| < |R_i(z)|$



Schnyder Drawing – Example



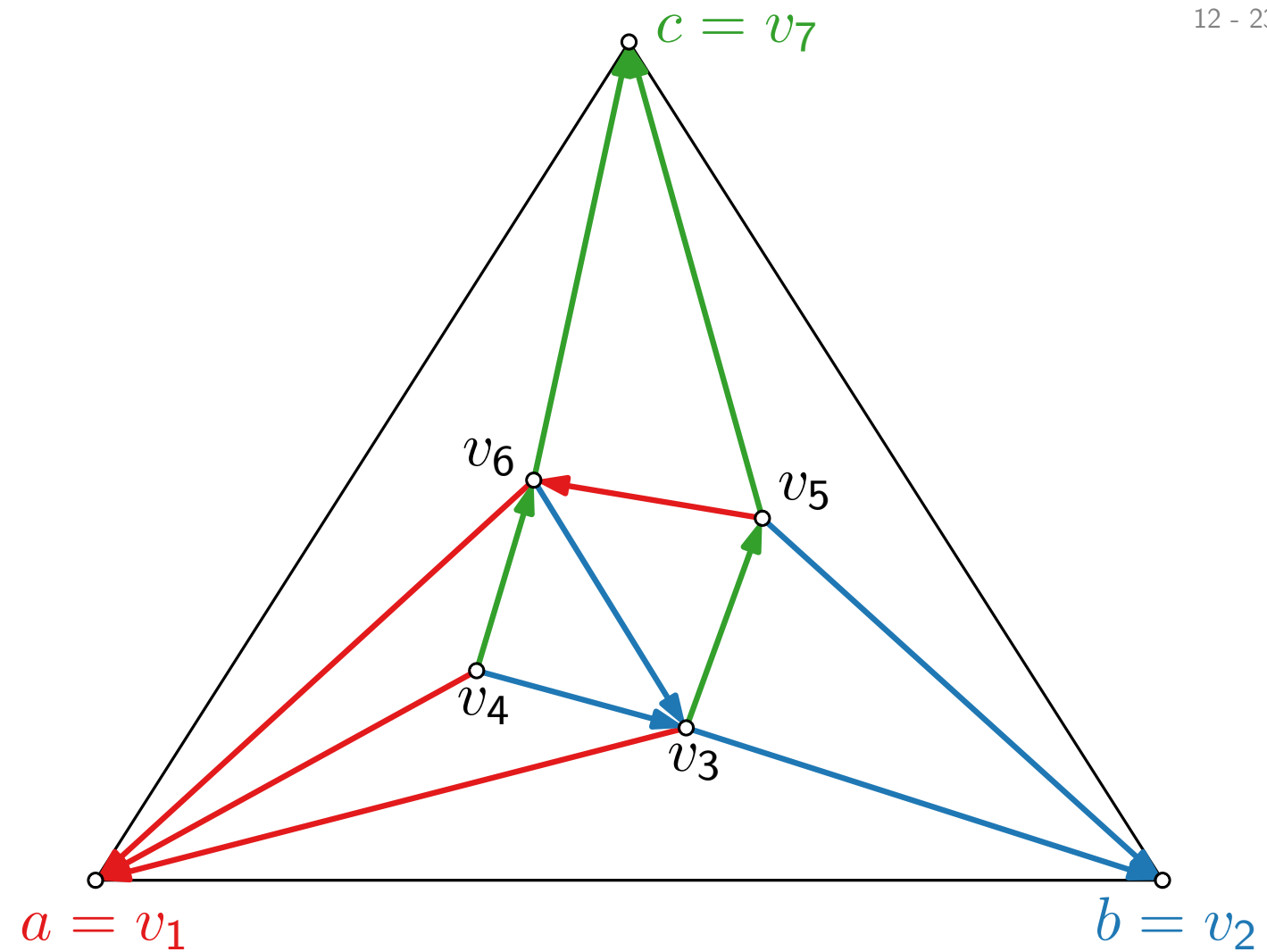
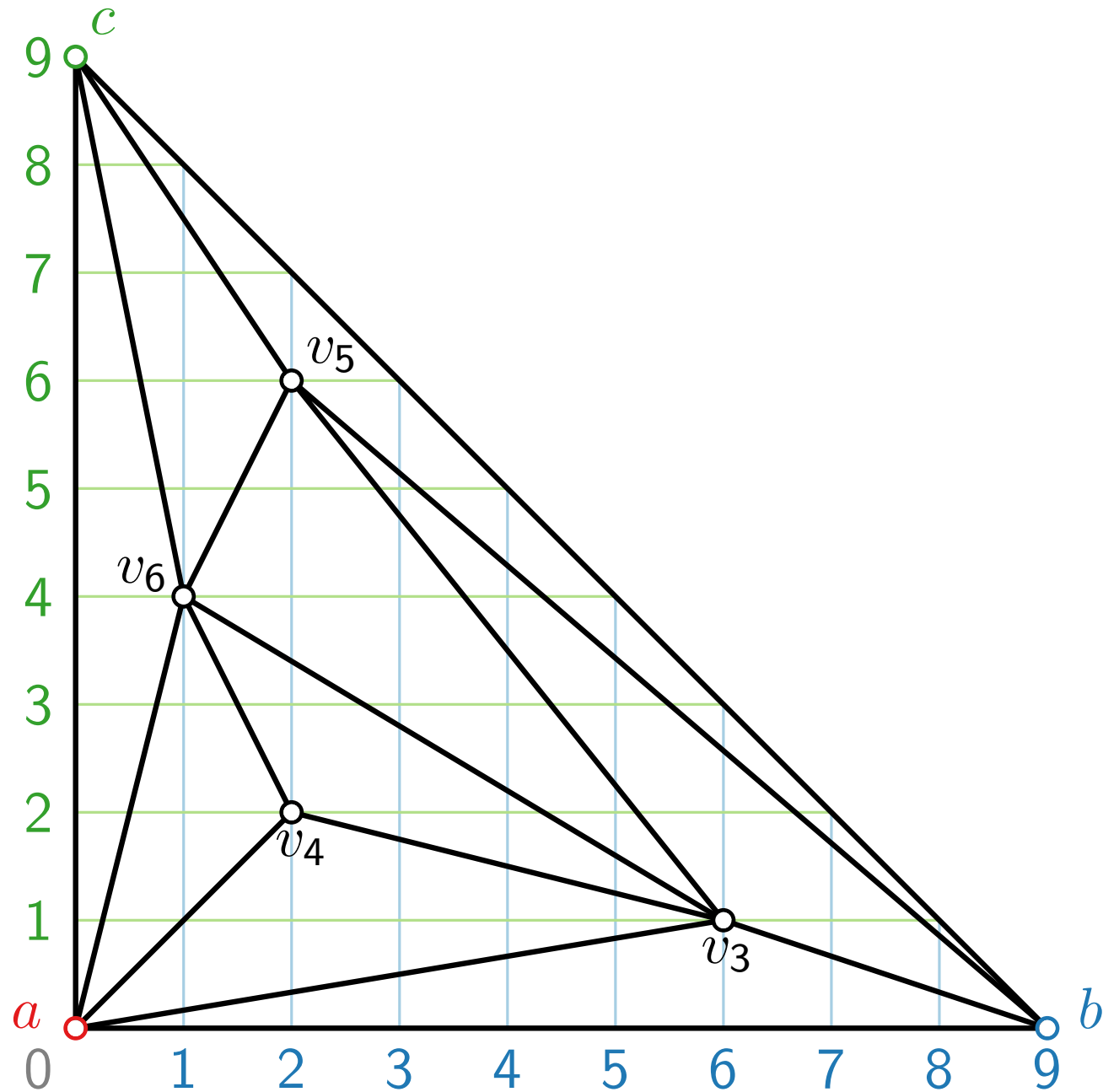
$$n = 7; \quad 2n - 5 = 9 \quad f(v_4) = (5, 2, 2)$$

$$f(v_1) = (9, 0, 0) \quad f(v_5) = (1, 2, 6)$$

$$f(v_2) = (0, 9, 0) \quad f(v_6) = (4, 1, 4)$$

$$f(v_3) = (2, 6, 1) \quad f(v_7) = (0, 0, 9)$$

Schnyder Drawing – Example



$$n = 7; \quad 2n - 5 = 9 \quad f(v_4) = (5, 2, 2)$$

$$f(v_1) = (9, 0, 0) \quad f(v_5) = (1, 2, 6)$$

$$f(v_2) = (0, 9, 0) \quad f(v_6) = (4, 1, 4)$$

$$f(v_3) = (2, 6, 1) \quad f(v_7) = (0, 0, 9)$$

Weak Barycentric Representation

A **weak barycentric representation** of a graph G is an assignment of barycentric coordinates to $V(G)$:

$$f: V(G) \rightarrow \mathbb{R}_{\geq 0}^3, \quad v \mapsto (v_1, v_2, v_3)$$

with the following properties:

(W1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$,

(W2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$, there exists a $k \in \{1, 2, 3\}$ with

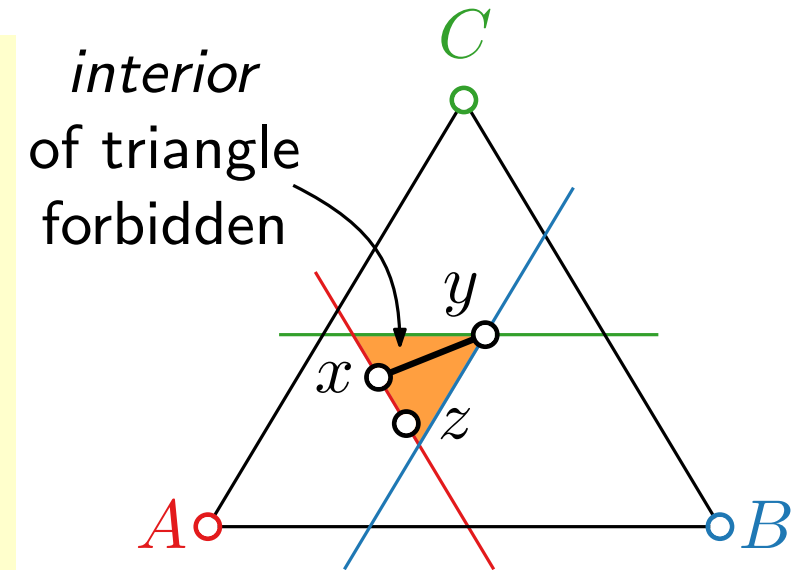
$$(x_k, x_{k+1}) <_{\text{lex}} (z_k, z_{k+1}) \text{ and } (y_k, y_{k+1}) <_{\text{lex}} (z_k, z_{k+1}).$$

Lemma.

For a weak barycentric representation $f: v \mapsto (v_1, v_2, v_3)$ and a triangle $\triangle ABC$, the mapping $\phi: V(G) \rightarrow \mathbb{R}^3$ with

$$v \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar** drawing of G inside $\triangle ABC$.

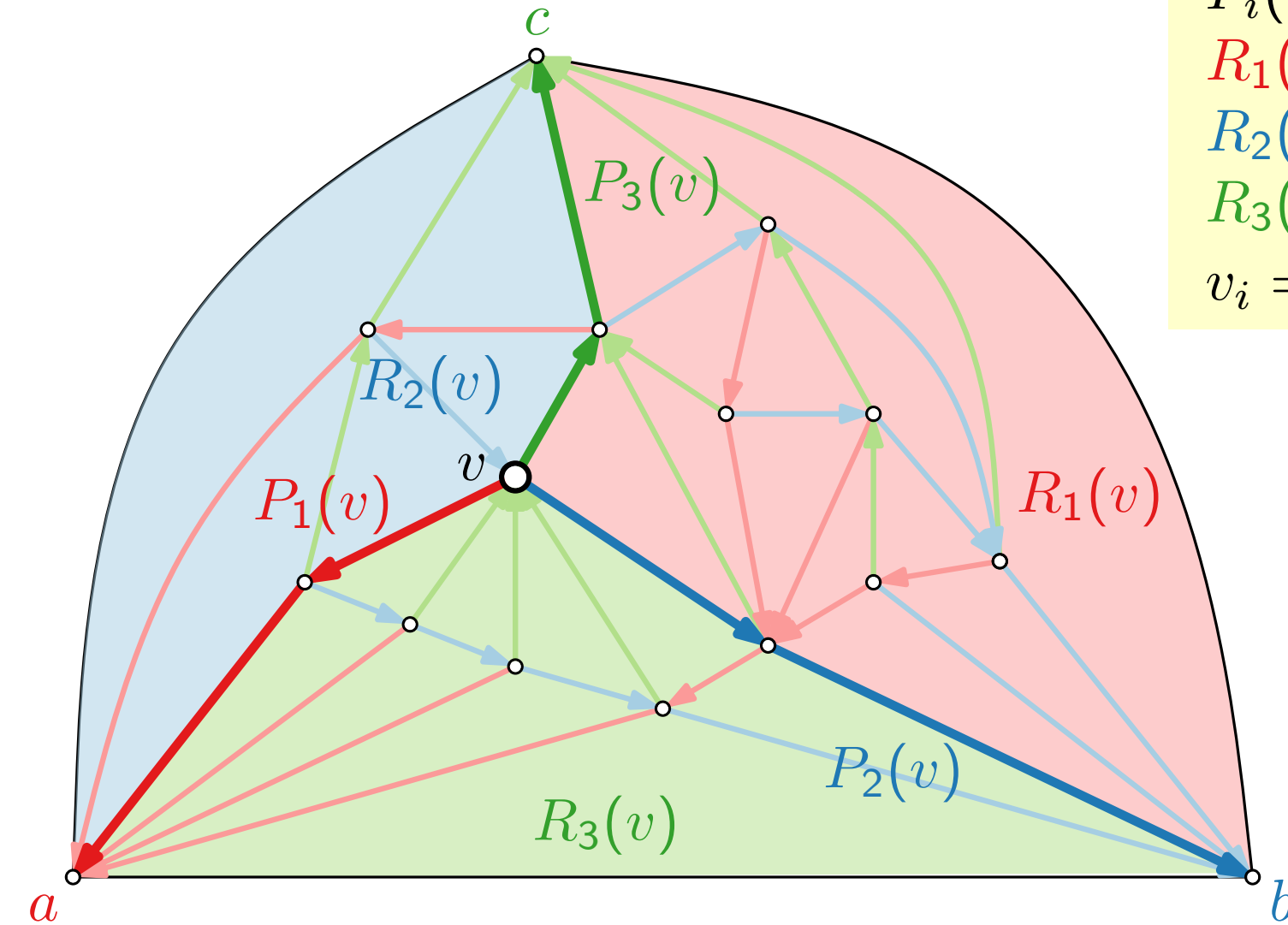


i.e., either $y_k < z_k$ or
 $(y_k = z_k \text{ and } y_{k+1} < z_{k+1})$

indices modulo 3

Proof. \rightarrow Exercise!

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

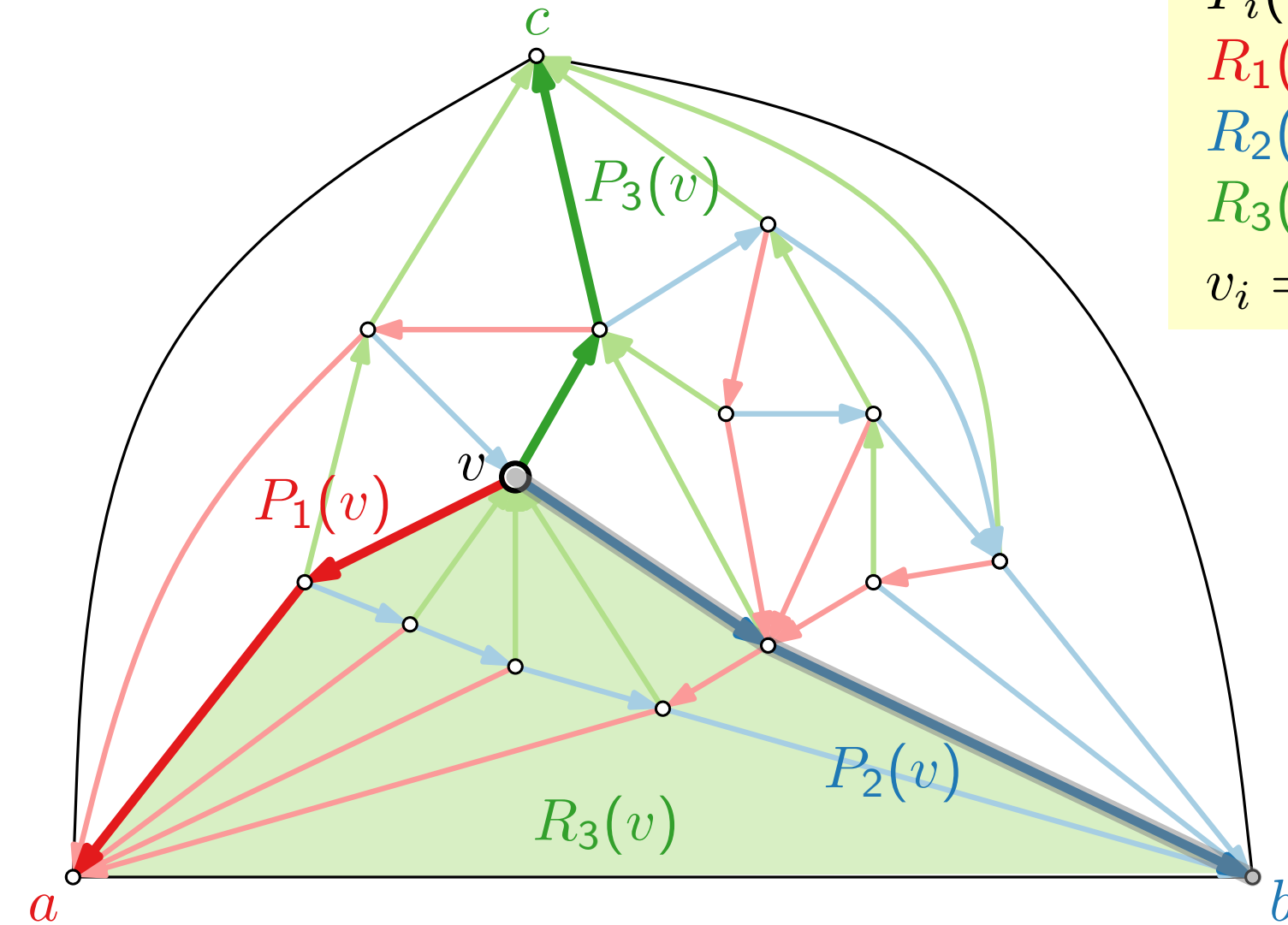
$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v) \rangle$

$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

Counting Vertices



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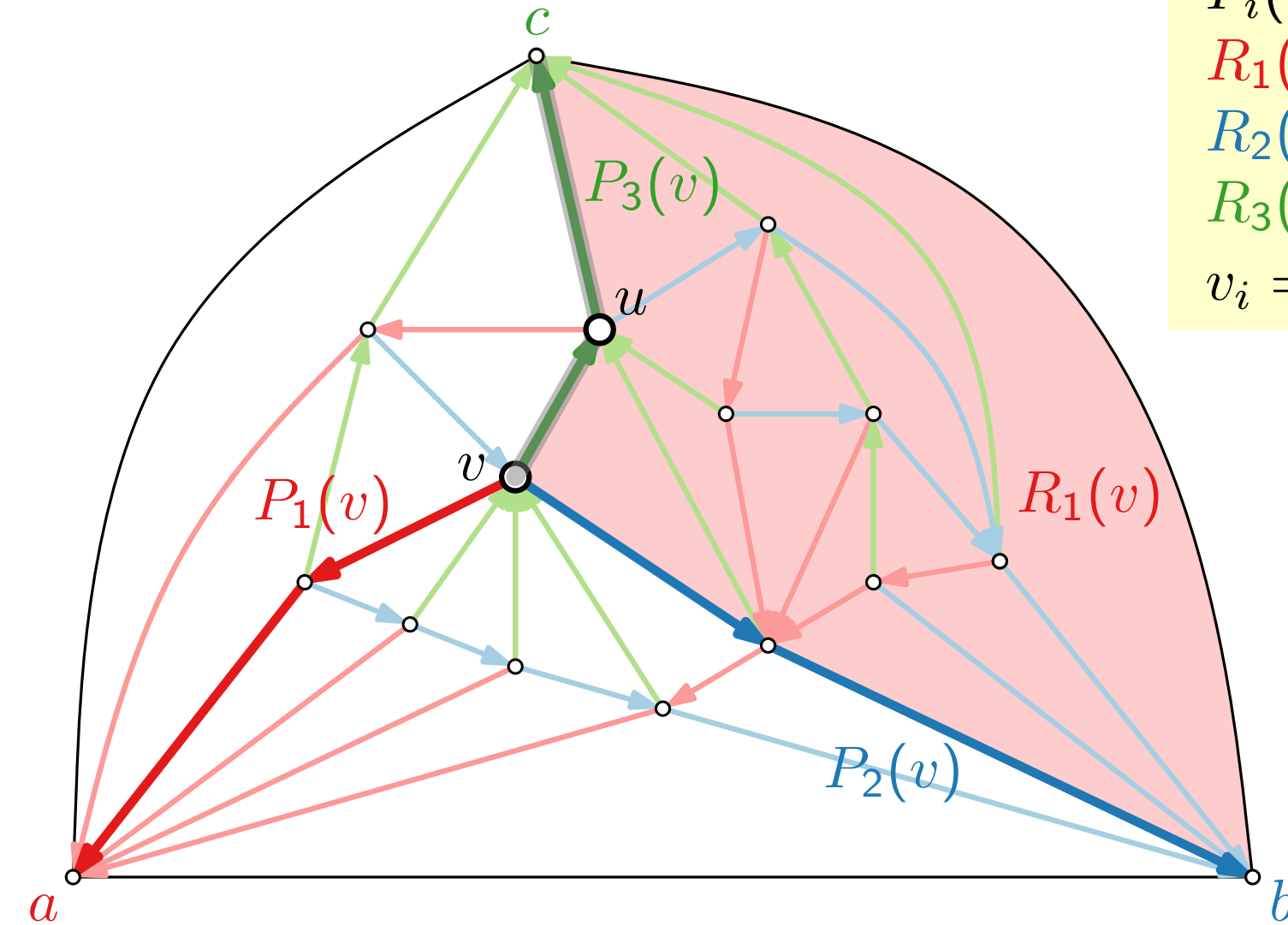
$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Counting Vertices



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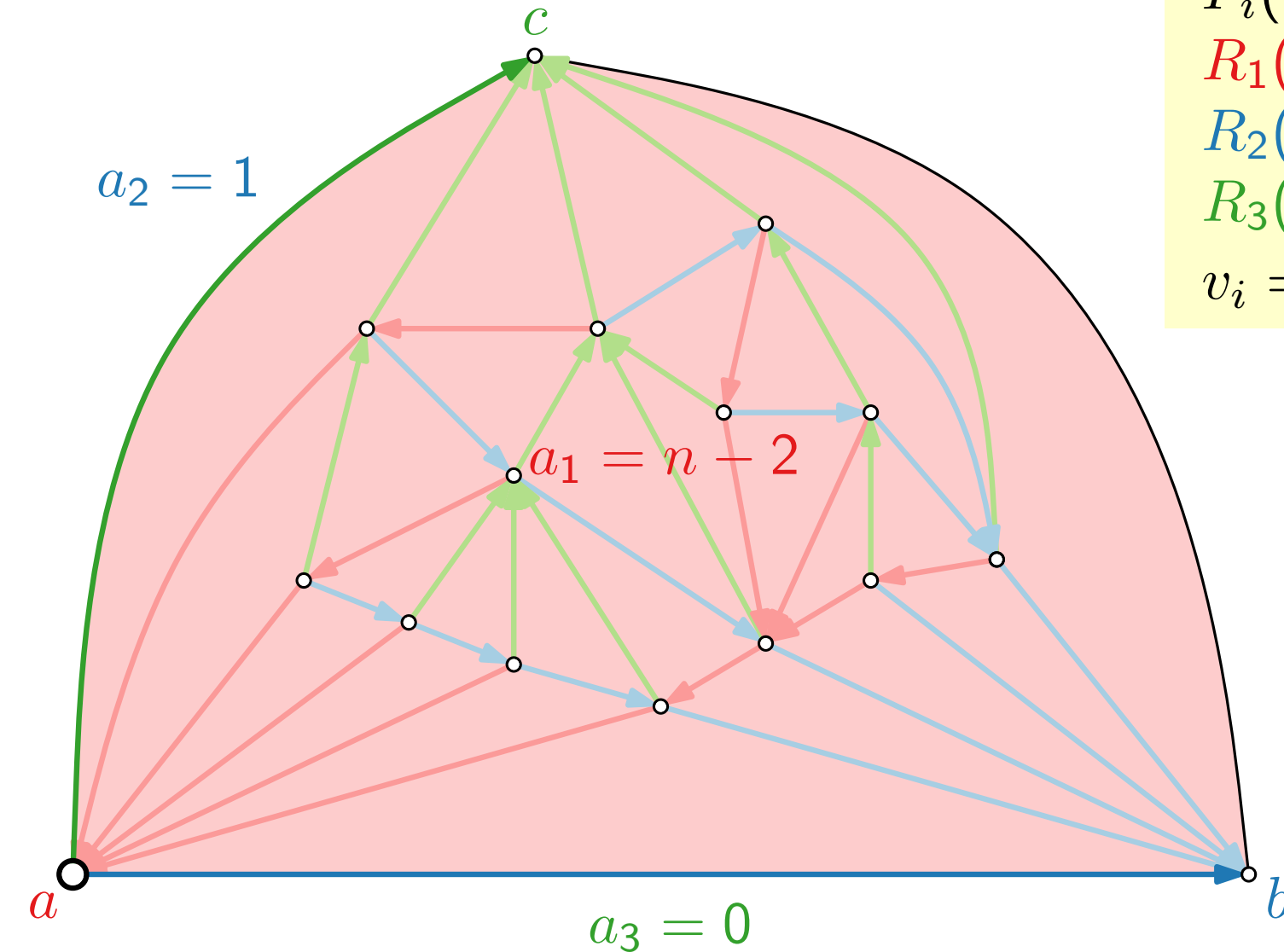
$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v) \rangle$

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$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v) \rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$$v_1 = 10 - 3 = 7$$

$$v_2 = 6 - 3 = 3$$

$$v_3 = 8 - 3 = 5$$

Lemma.

■ For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.

■ $v_1 + v_2 + v_3 = n - 1$

Schnyder Drawing[★]

Set $A = (0, 0)$, $B = (n - 1, 0)$, and $C = (0, n - 1)$.

Theorem.

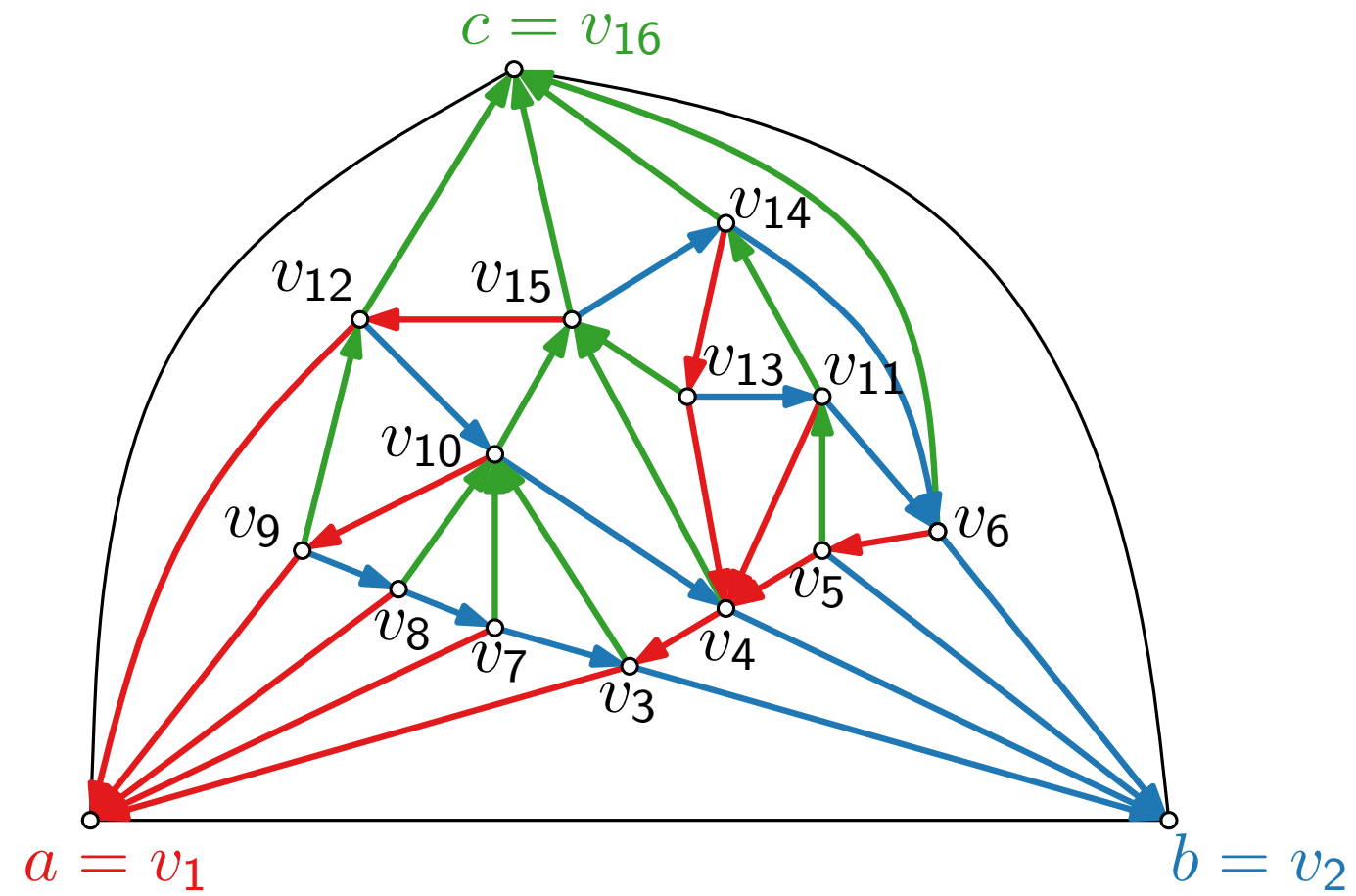
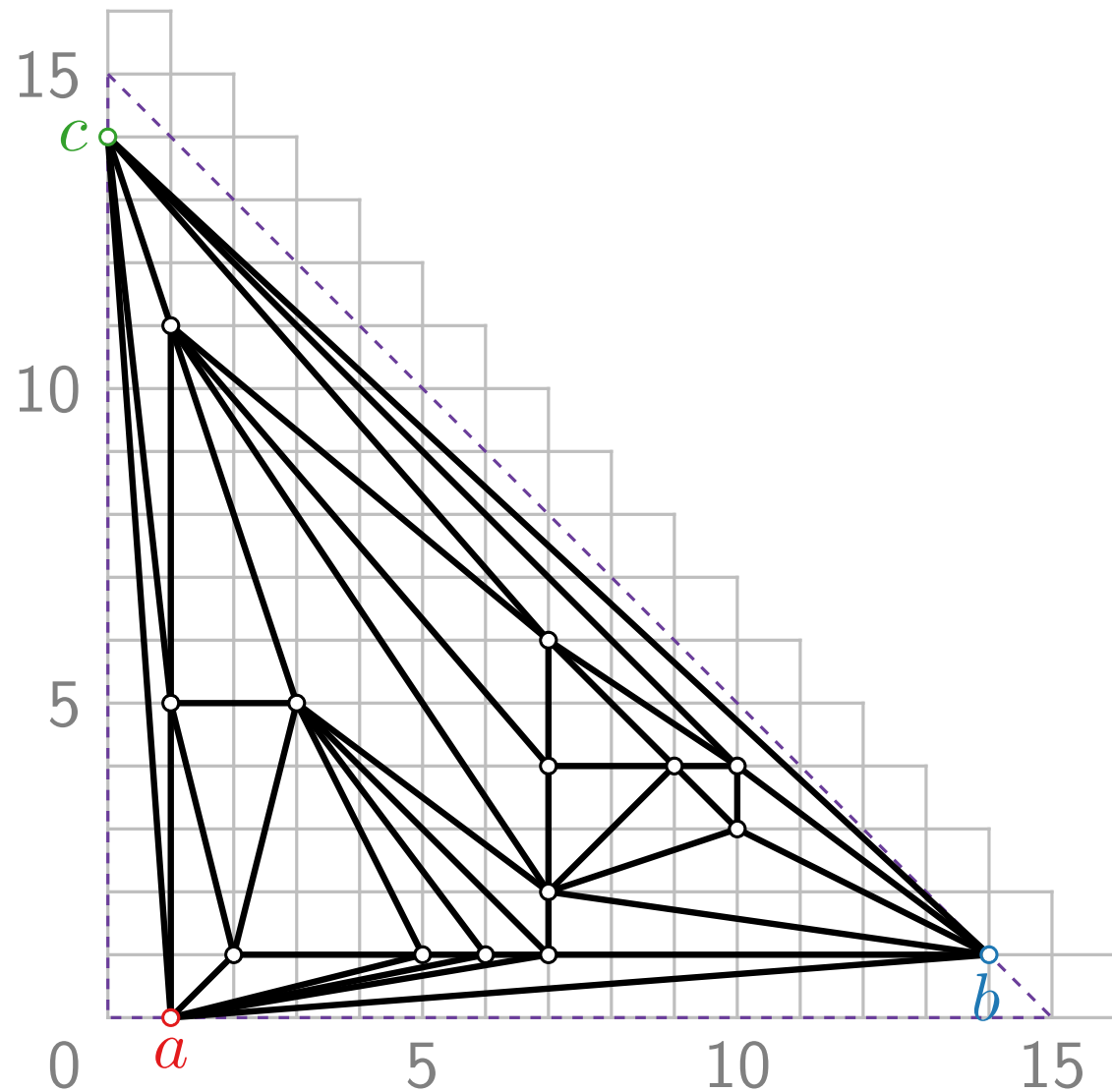
[Schnyder '90]

For a plane triangulation G , the mapping

$$f: v \mapsto \frac{1}{n-1}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

is a weak barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(n - 2) \times (n - 2)$ grid.

Schnyder Drawing^{*} – Example



$$\begin{aligned}
 n &= 16, n-1 = 15, nf(v_2) = (7, 7, 1) \\
 f(a) &= (14, 1, 0) & f(v_4) &= (6, 7, 2) \\
 f(b) &= (0, 14, 1) & f(v_5) &= (2, 10, 3) \\
 f(c) &= (1, 0, 14) & & \vdots
 \end{aligned}$$

Results & Variations

Theorem.

[De Fraysseix, Pach, Pollack '90]

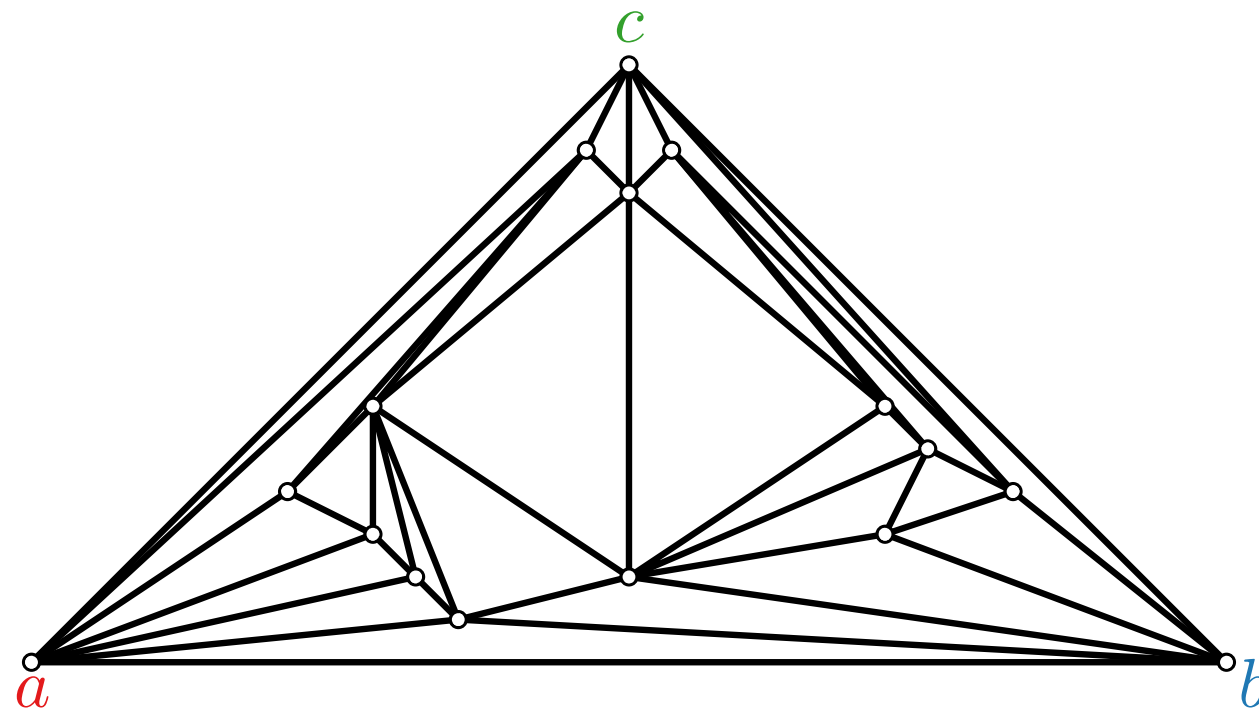
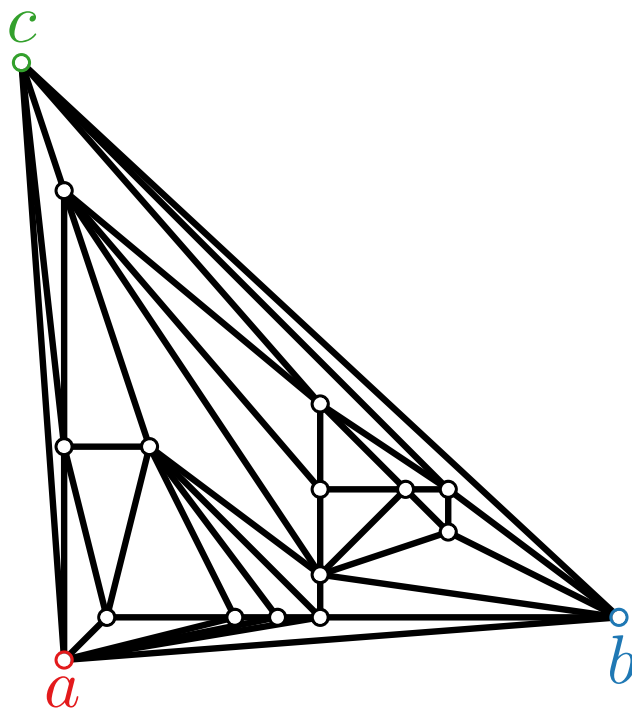
Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Theorem.

[Schnyder '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Exercise!



Results & Variations

Theorem.

[De Fraysseix, Pach, Pollack '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Theorem.

[Schnyder '90]

Every n -vertex planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$. Such a drawing can be computed in $O(n)$ time.

Exercise!

Theorem.

[Brandenburg '08]

Every n -vertex planar graph has a planar straight-line drawing of size $4n/3 \times 2n/3$. Such a drawing can be computed in $O(n)$ time.

Theorem.

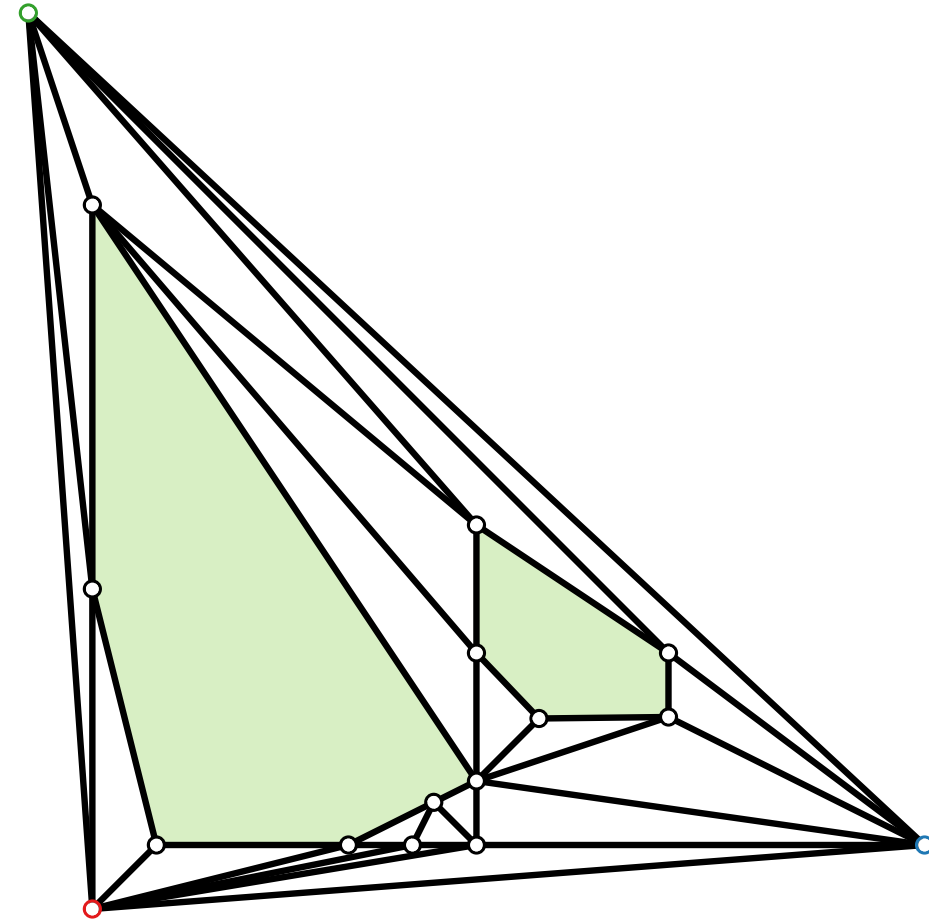
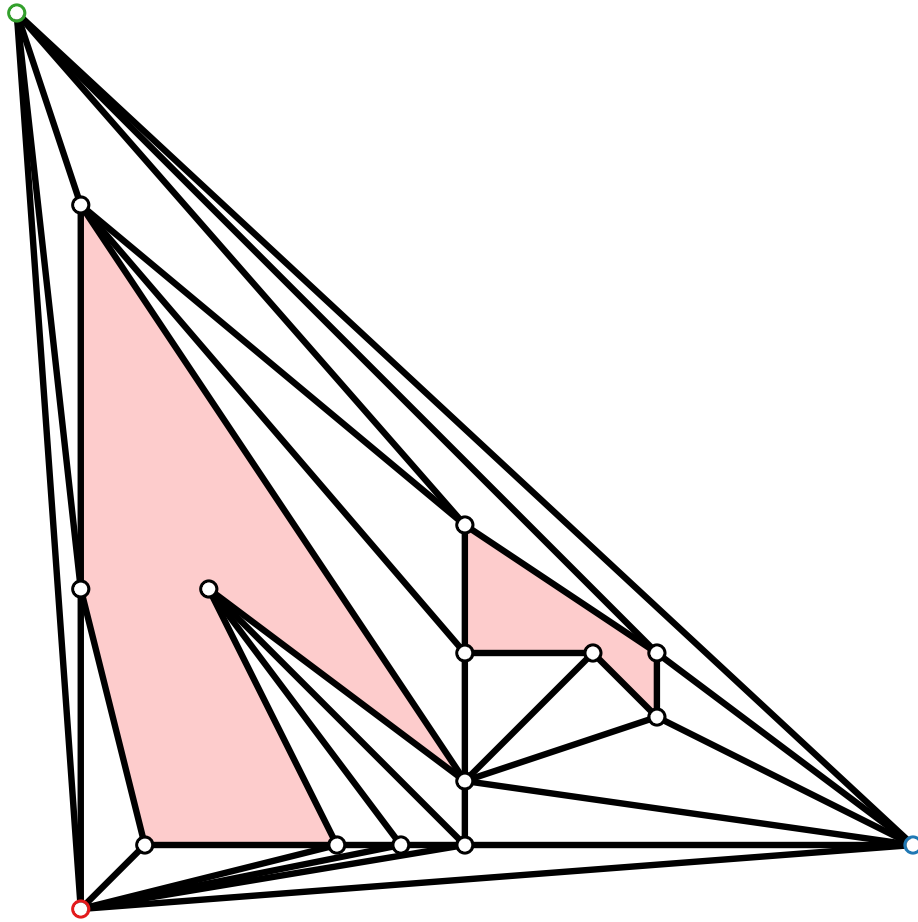
[Dolev, Leighton, Trickey '84]

There exist n -vertex plane graphs such that any planar straight-line drawing of them has an area of at least $(2n/3 - 1) \times (2n/3 - 1)$.

[Fрати, Patrignani '07]

Area at least $n^2/9 + \Omega(n)$ in the variable-embedding setting.

Results & Variations



Results & Variations

Theorem.

[Kant '96]

Every n -vertex 3-connected planar graph has a planar straight-line drawing of size $(2n - 4) \times (n - 2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Theorem.

[Chrobak & Kant '97]

Every n -vertex 3-connected planar graph has a planar straight-line drawing of size $(n - 2) \times (n - 2)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Theorem.

[Felsner '01]

Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f - 1) \times (f - 1)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Literature

- [PGD Ch. 4.3] for detailed explanation of Schnyder woods etc.
- [Sch90] “Embedding planar graphs on the grid”, Walter Schnyder, SoCG 1990 – original paper on Schnyder realizer method.