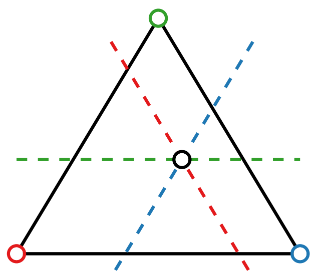
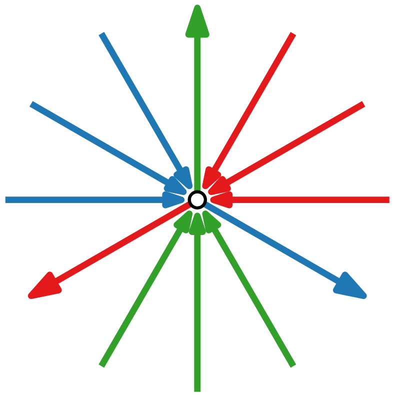
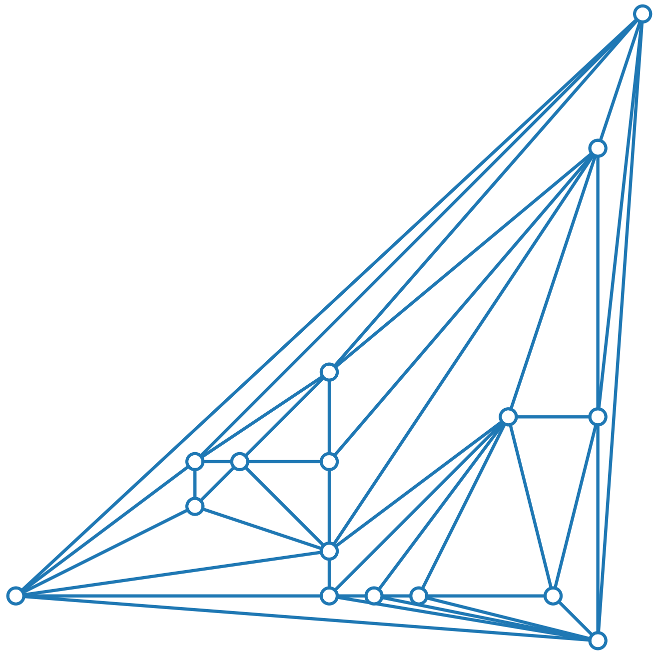
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 4:
Straight-Line Drawings of Planar Graphs II: Schnyder Woods







Alexander Wolff

Summer term 2025

Planar Straight-Line Drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

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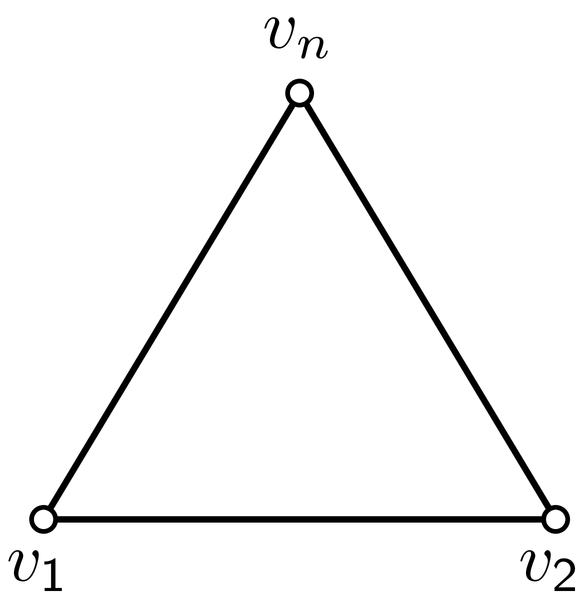
- Fix outer triangle.



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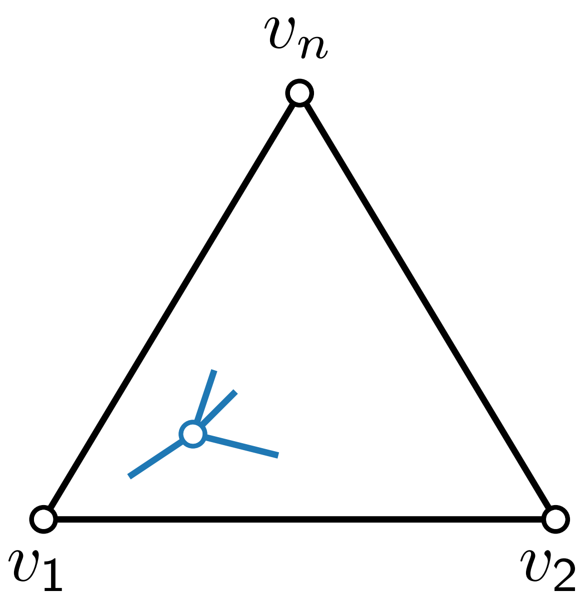

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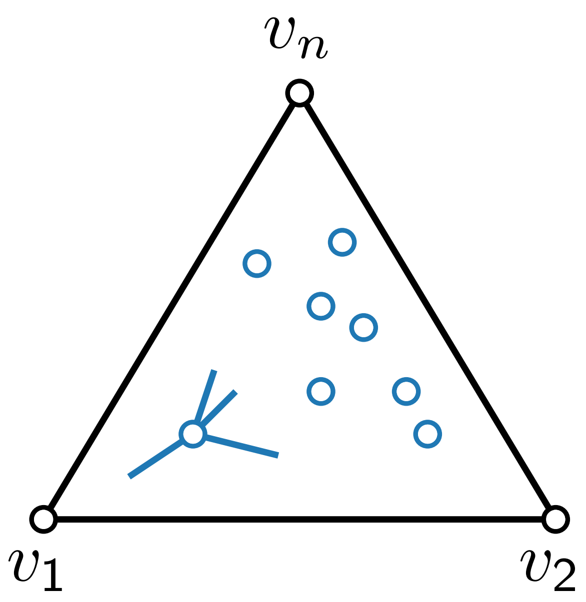

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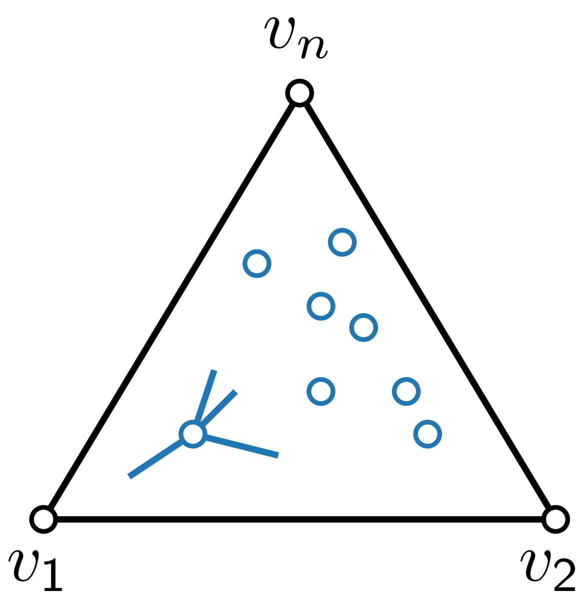

Idea.

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  - using weighted barycentric coordinates.

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(easier to show)

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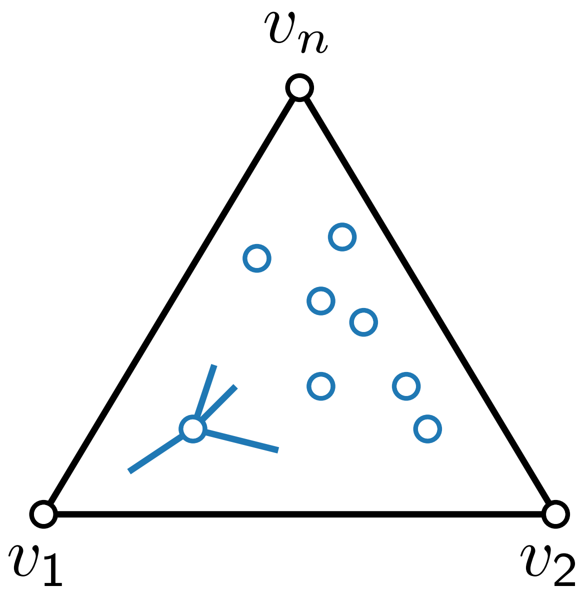
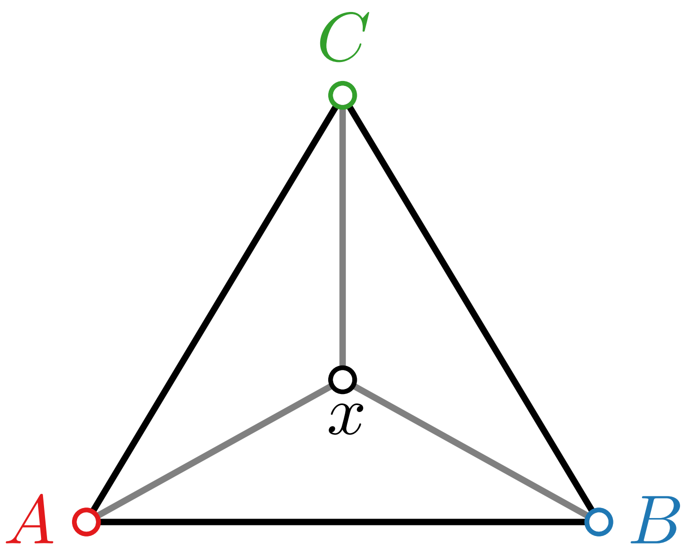


Barycentric Coordinates

Recall: barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$



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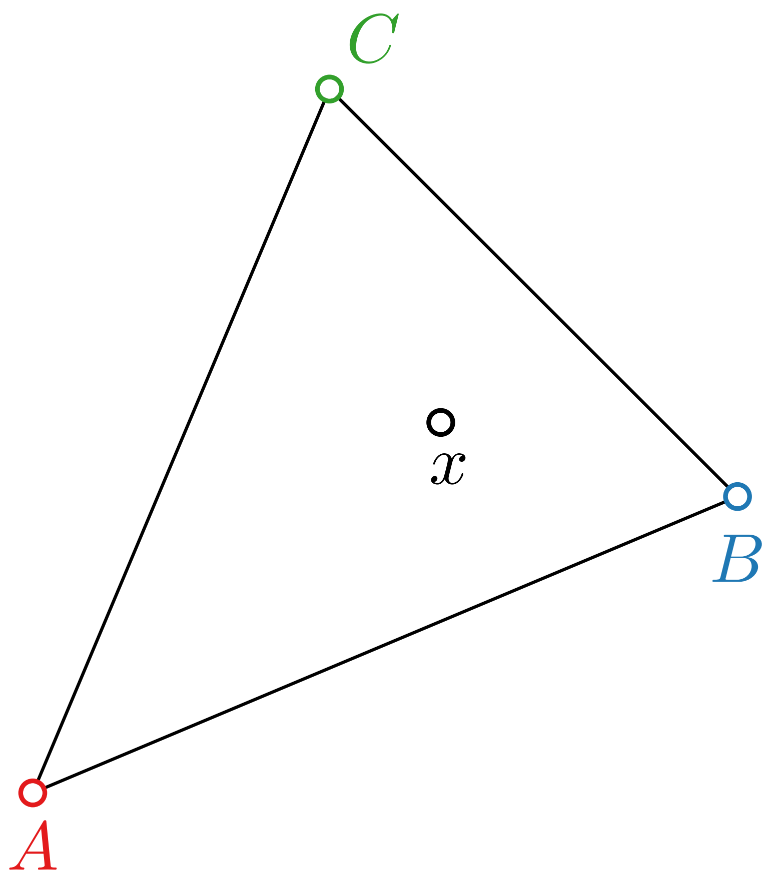

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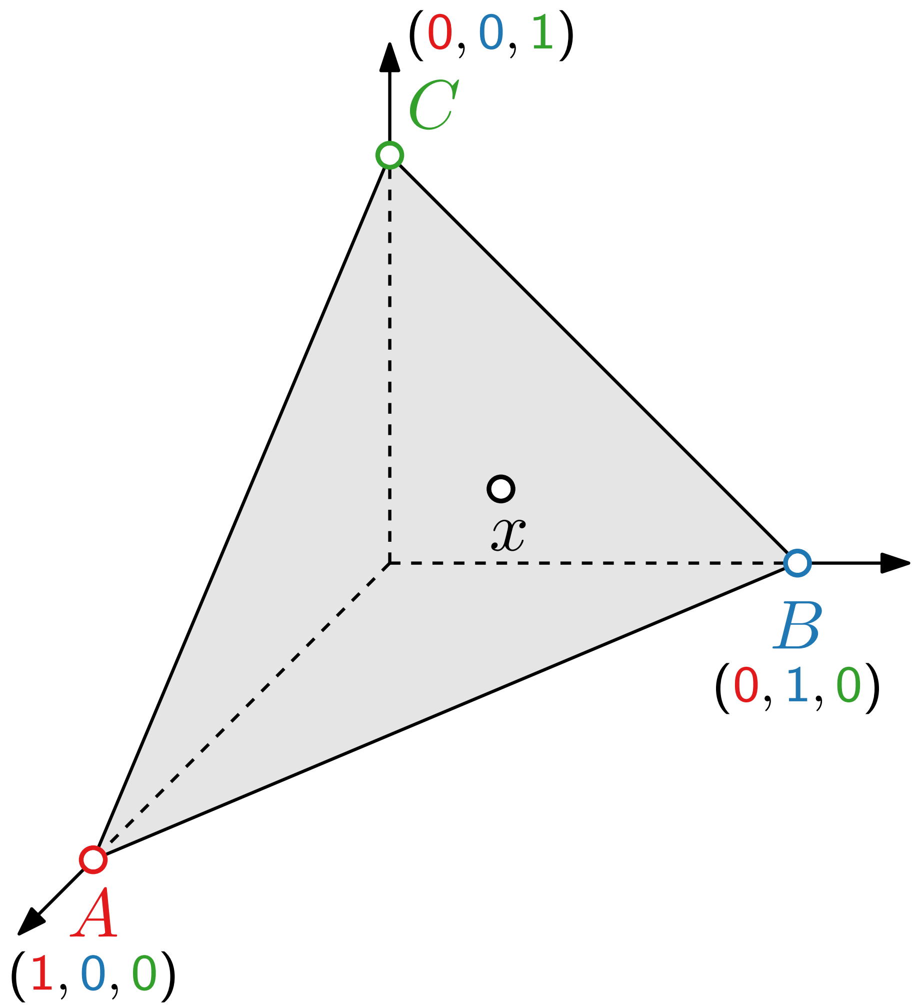

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- $\alpha + \beta + \gamma = 1$ and
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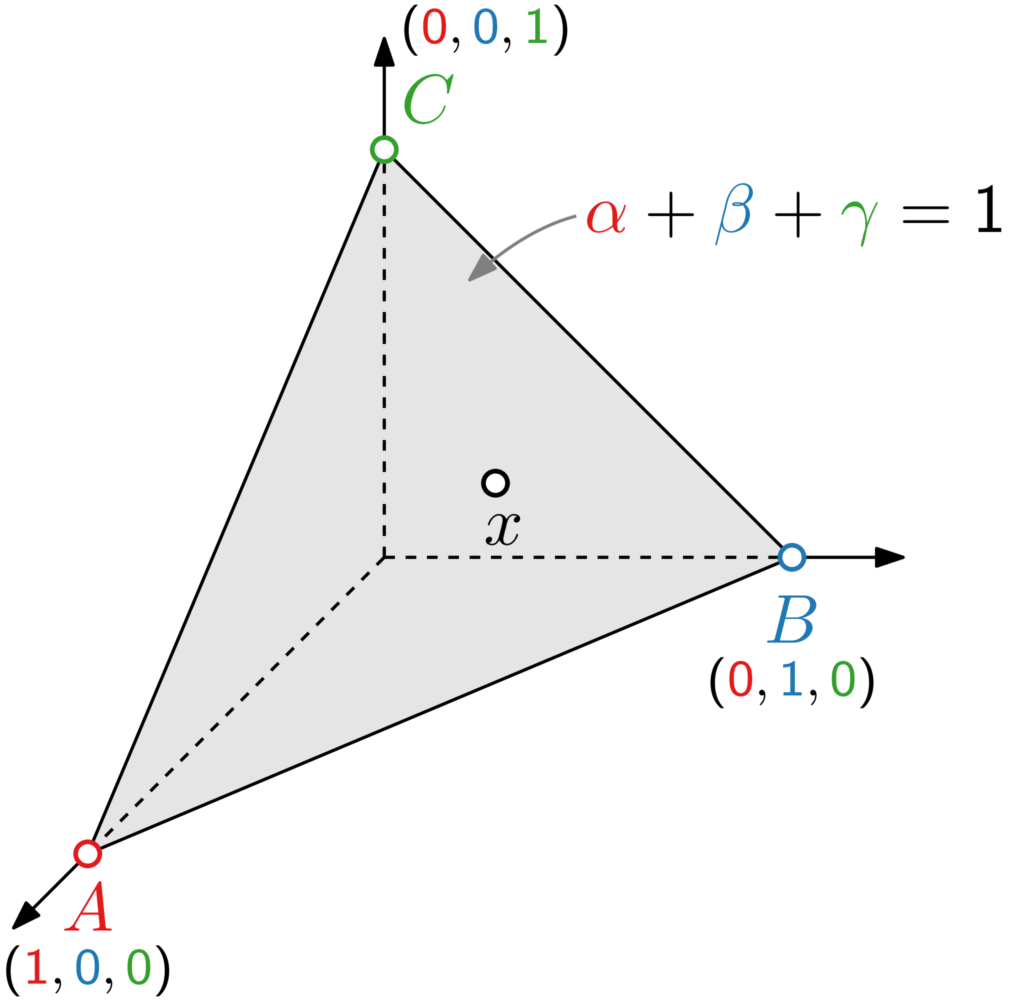

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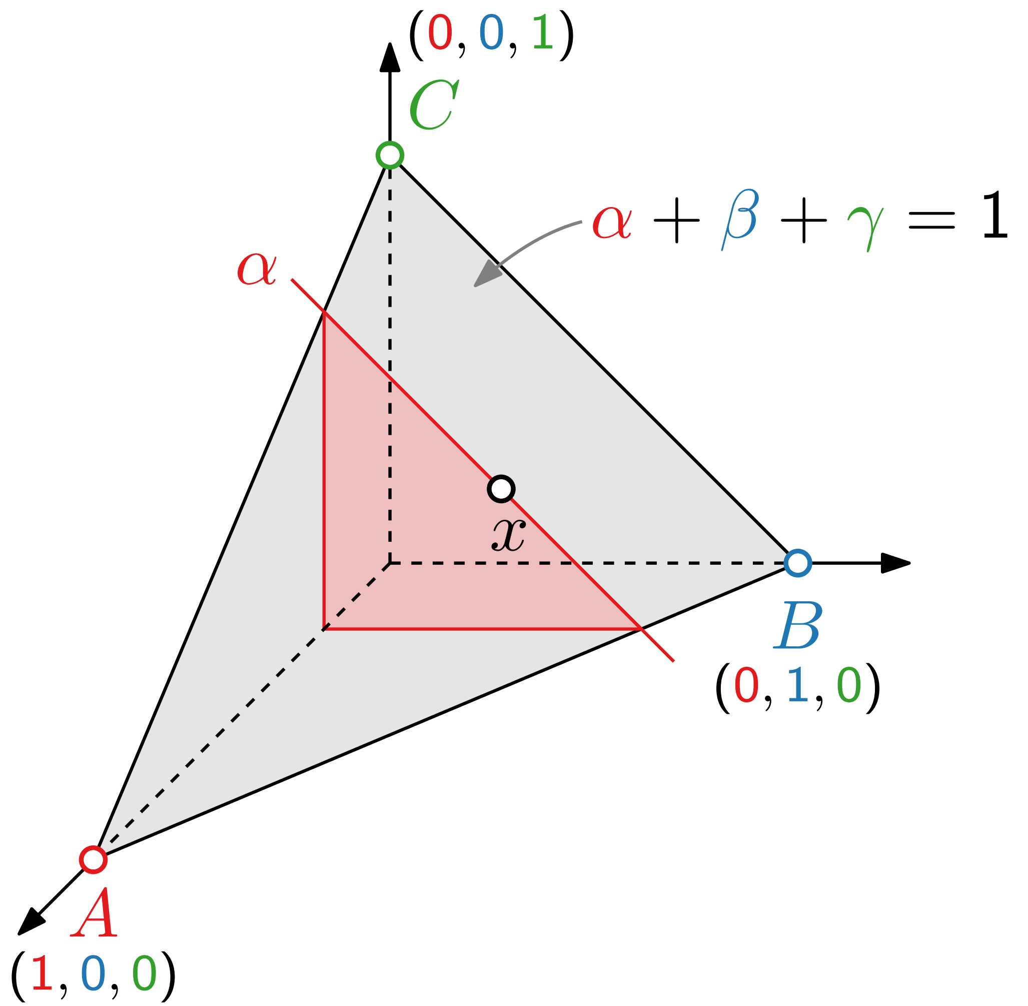

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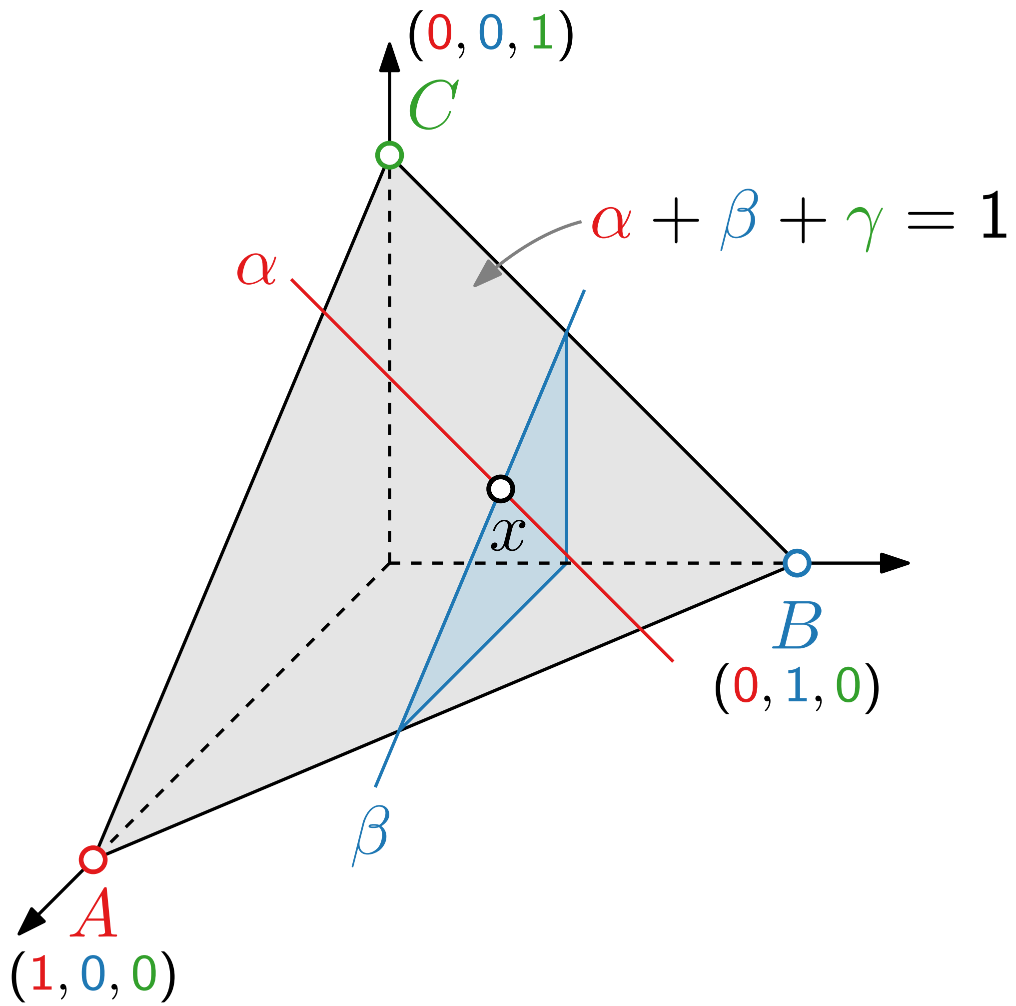

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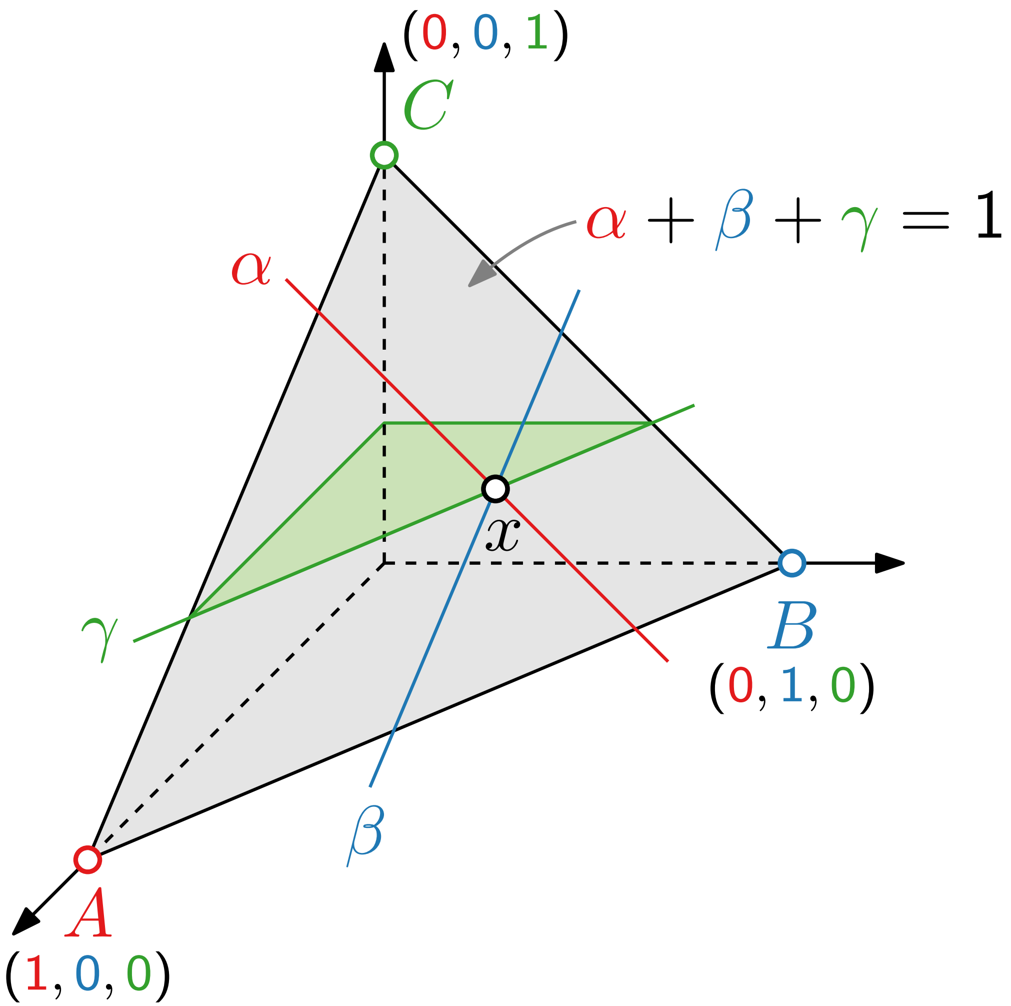

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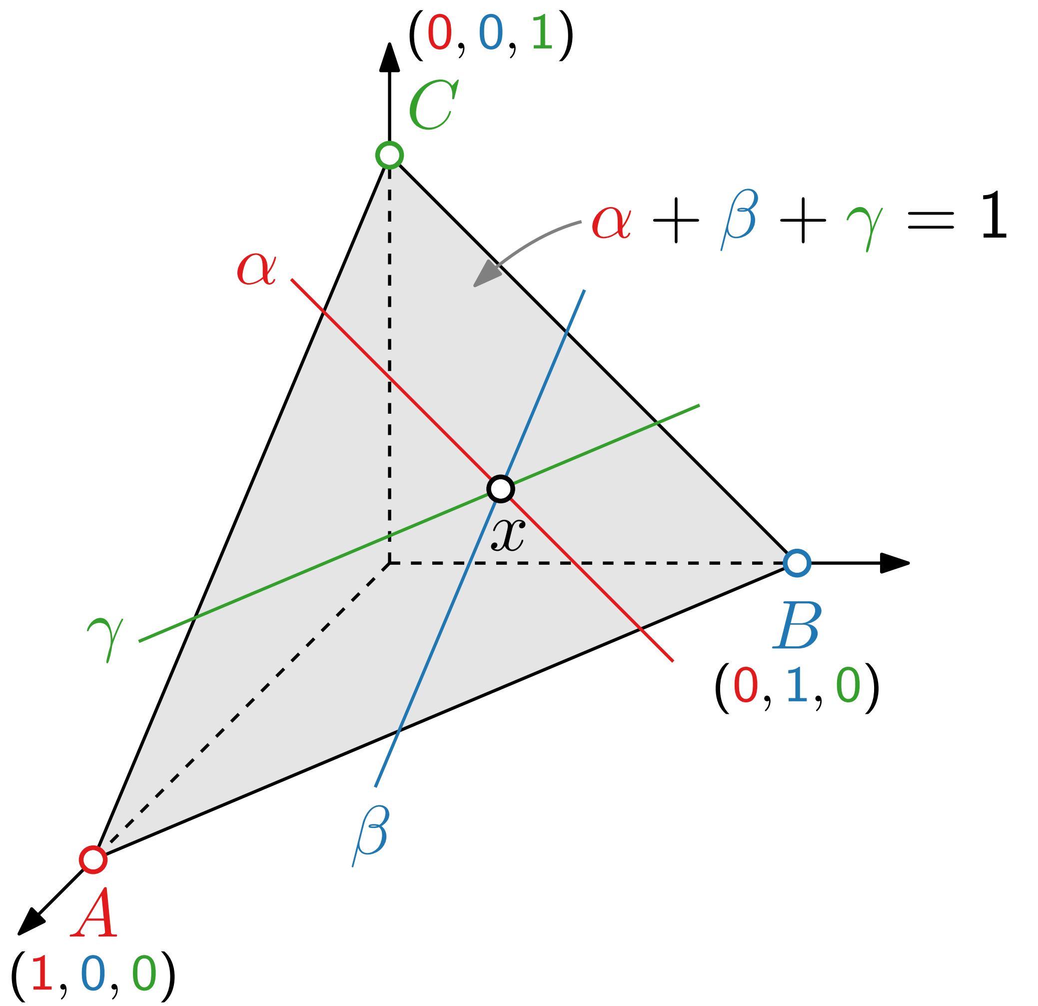

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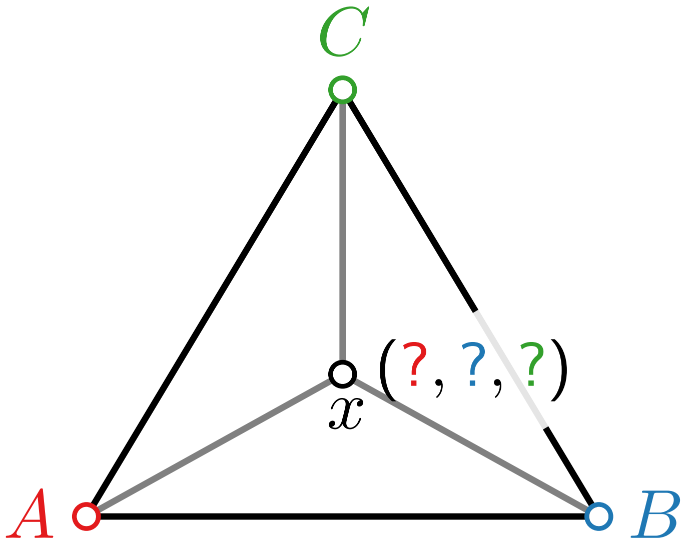

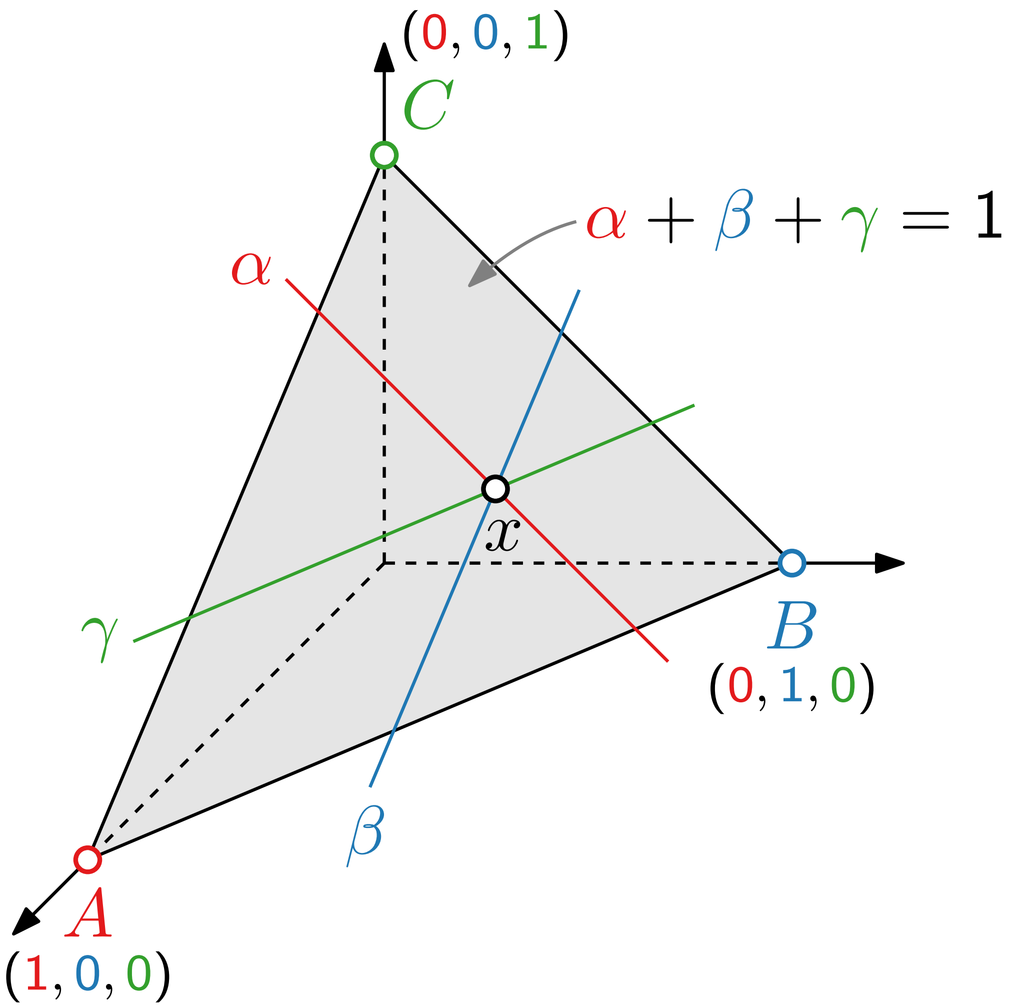

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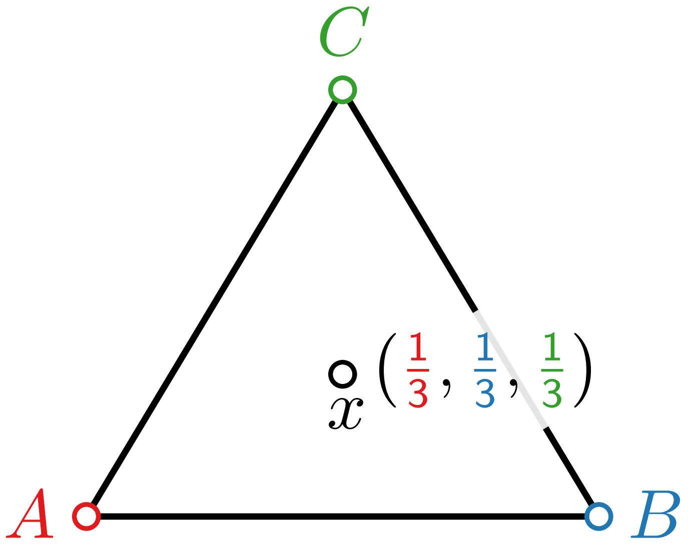

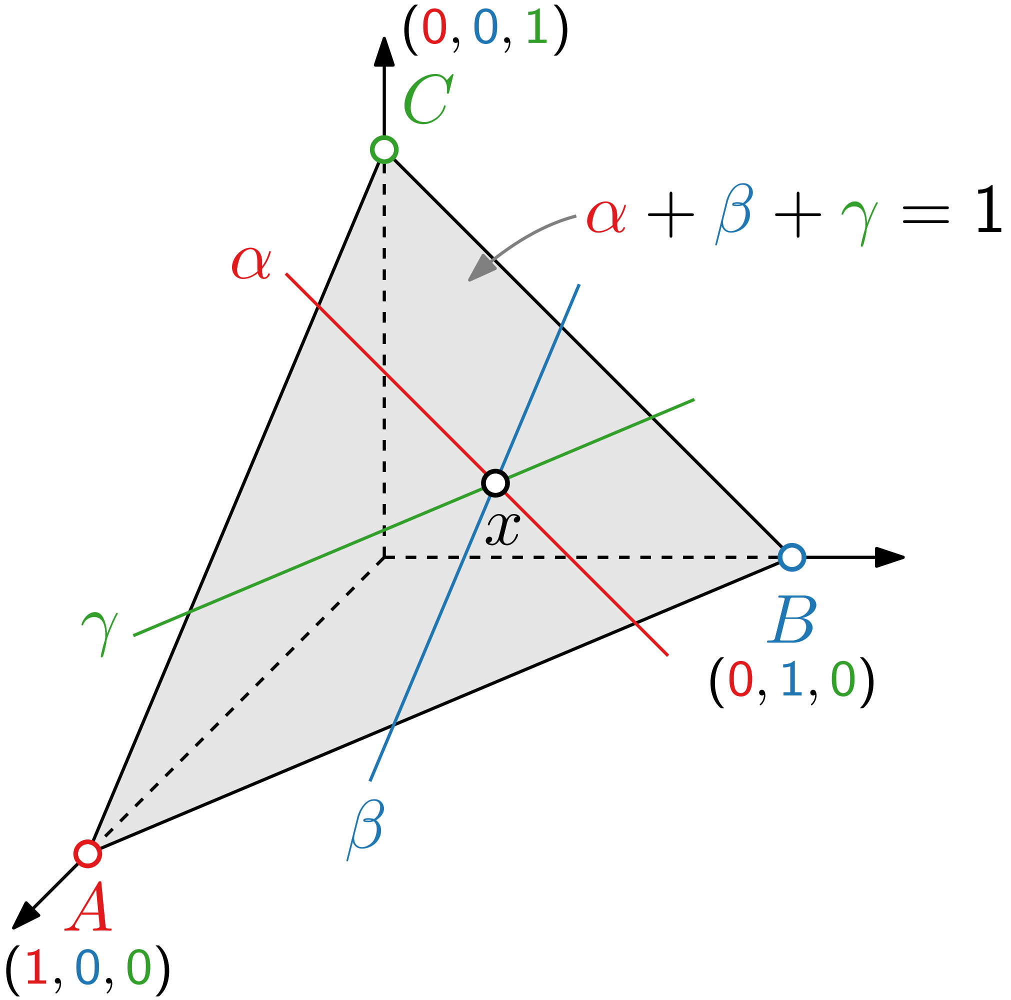

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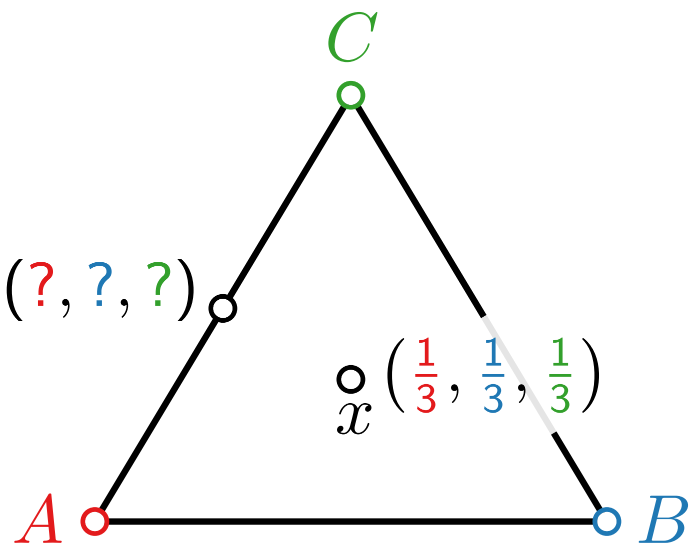

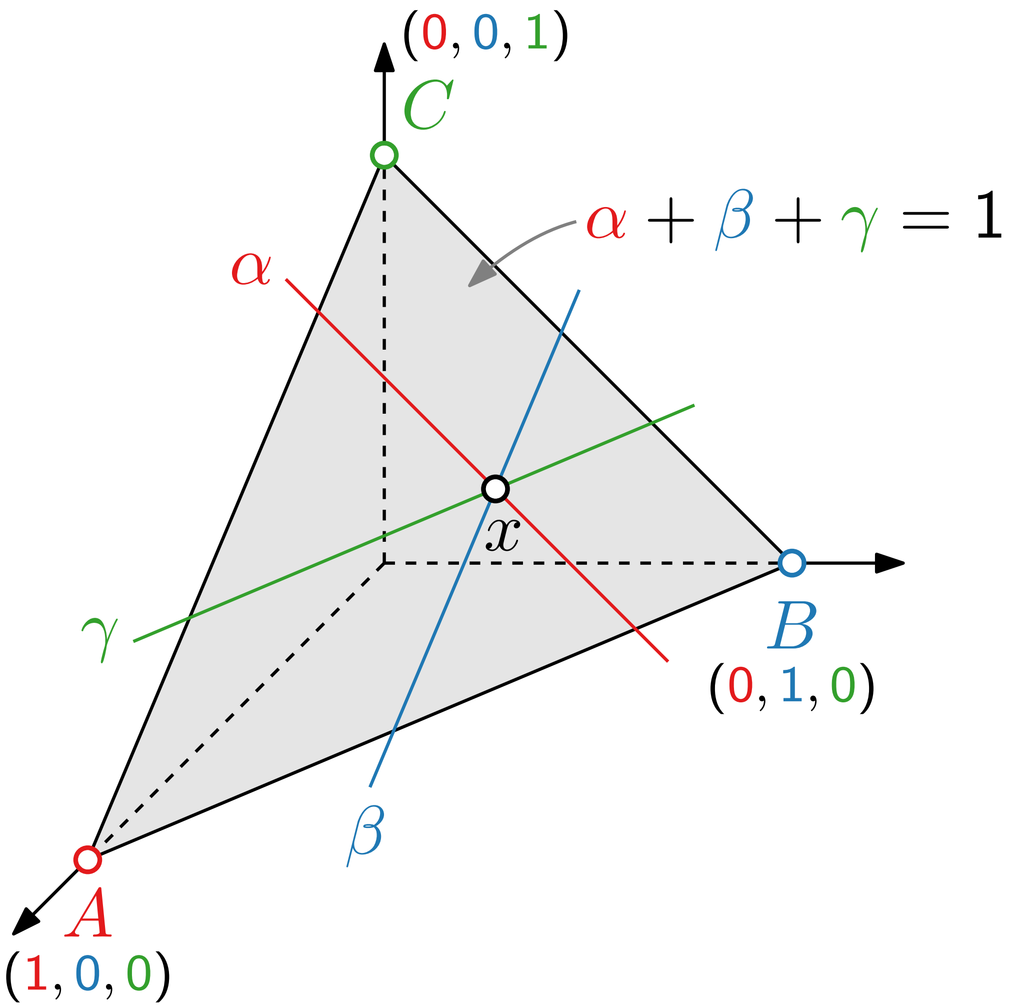

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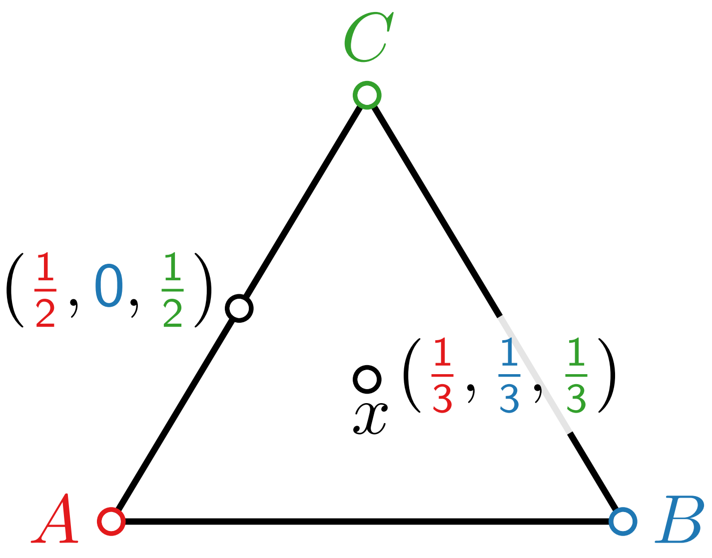

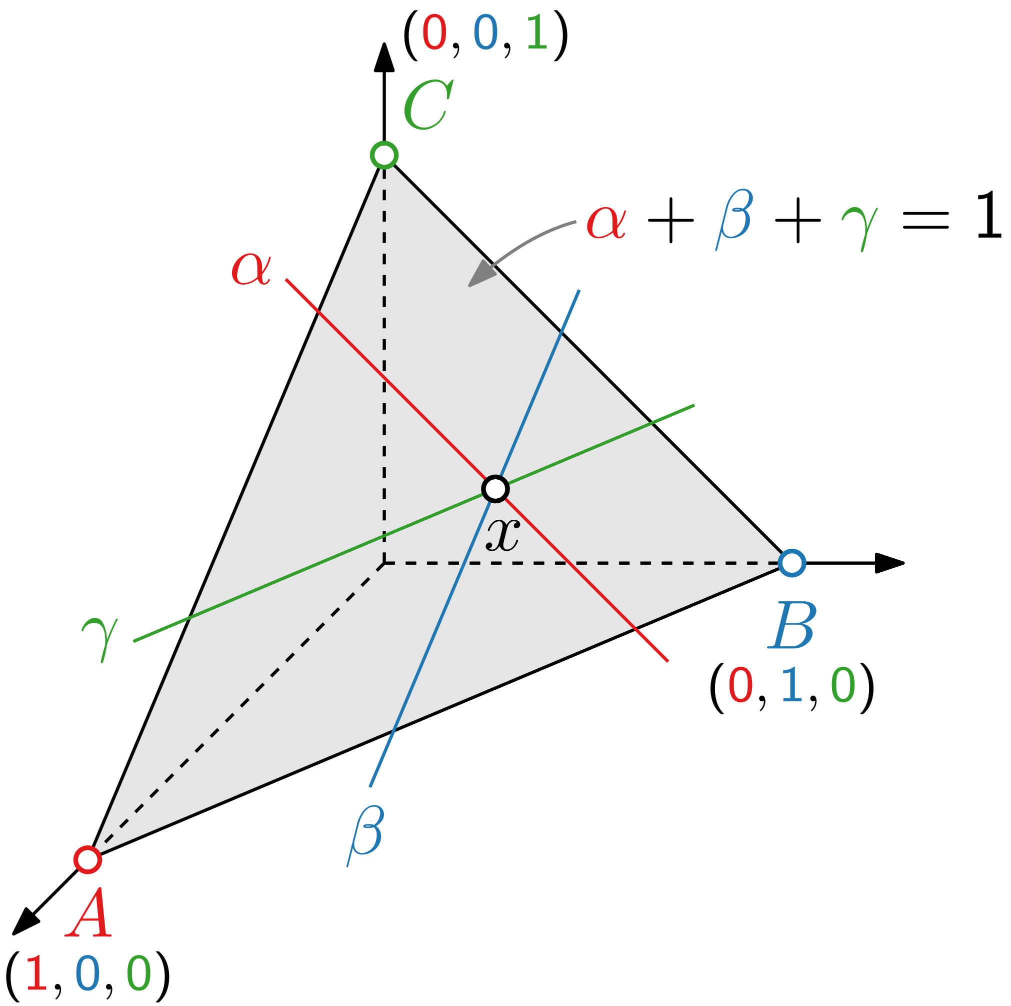

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A **barycentric representation** of a graph G is an assignment of barycentric coordinates to the vertices of G:

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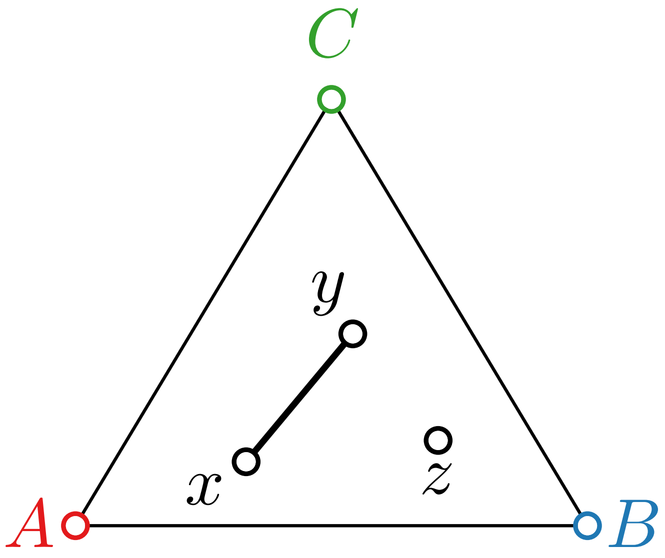

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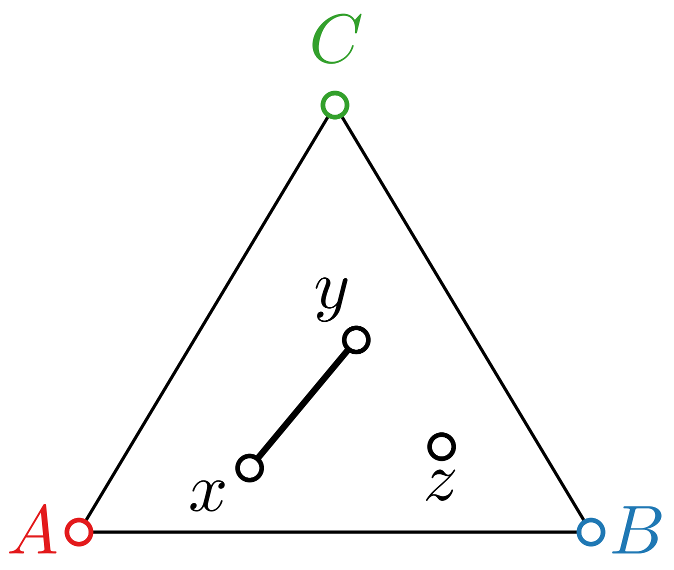

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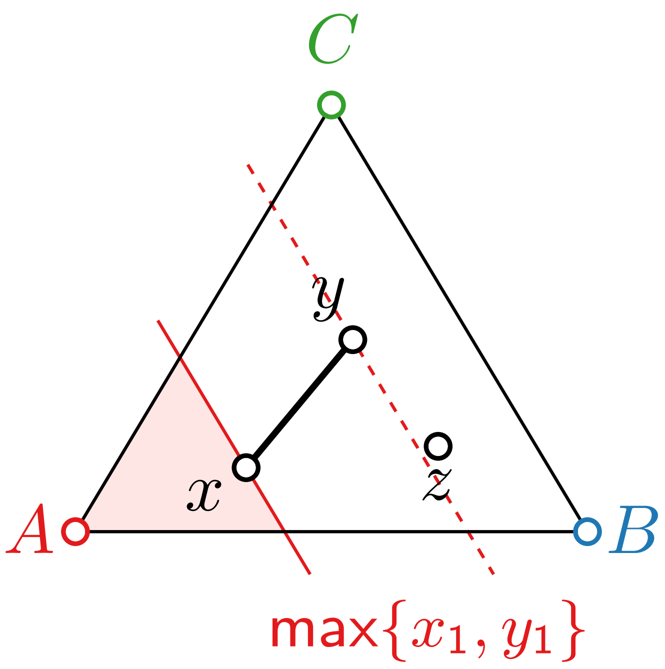

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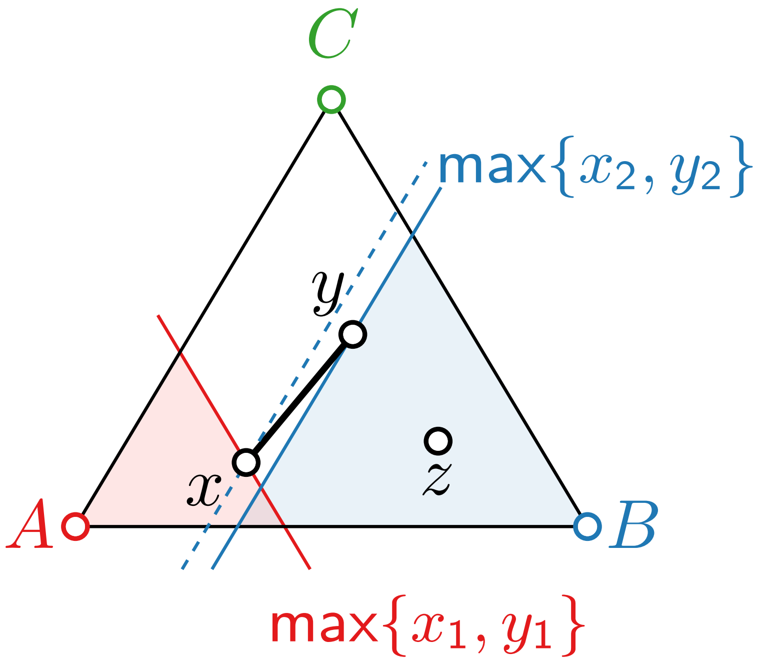

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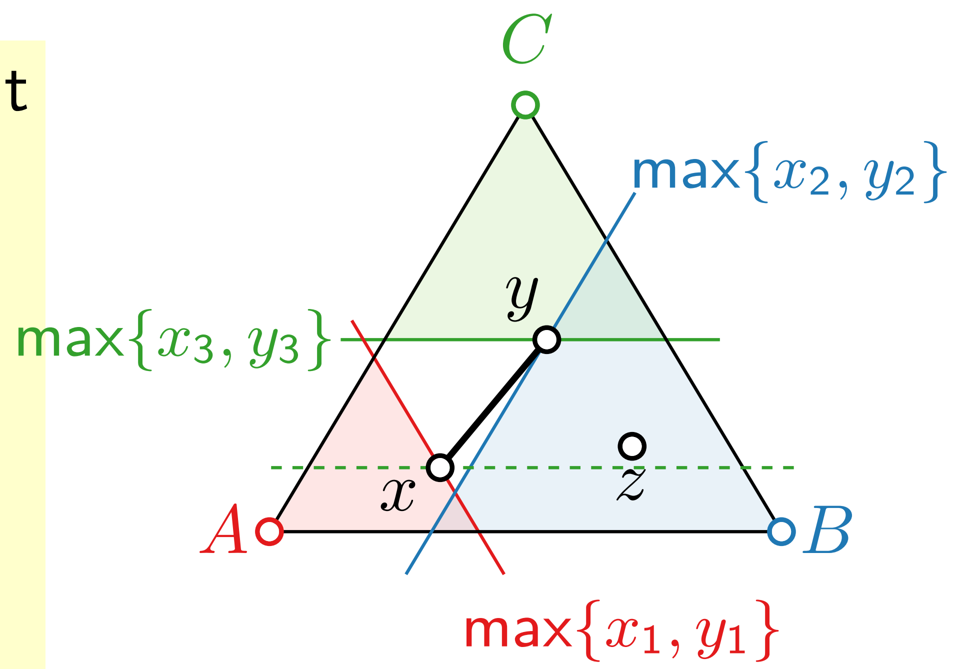

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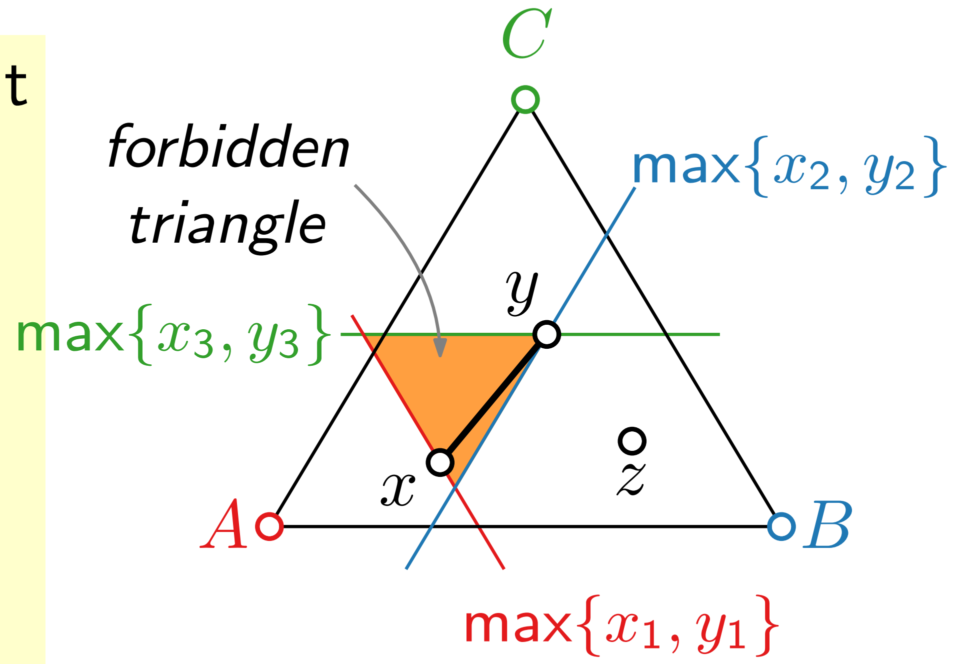
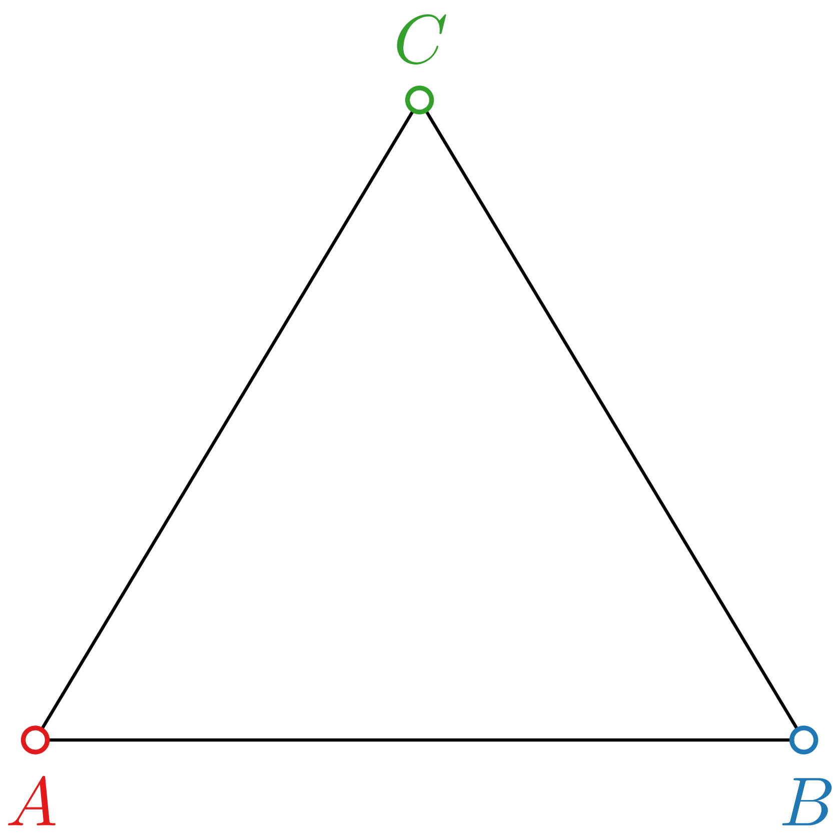


Barycentric Representations of Planar Graphs

Lemma.
Let $f : v \mapsto (v_1, v_2, v_3)$ be a barycentric representation of a planar graph G, and let $A, B, C \in \mathbb{R}^2$ be in general position.



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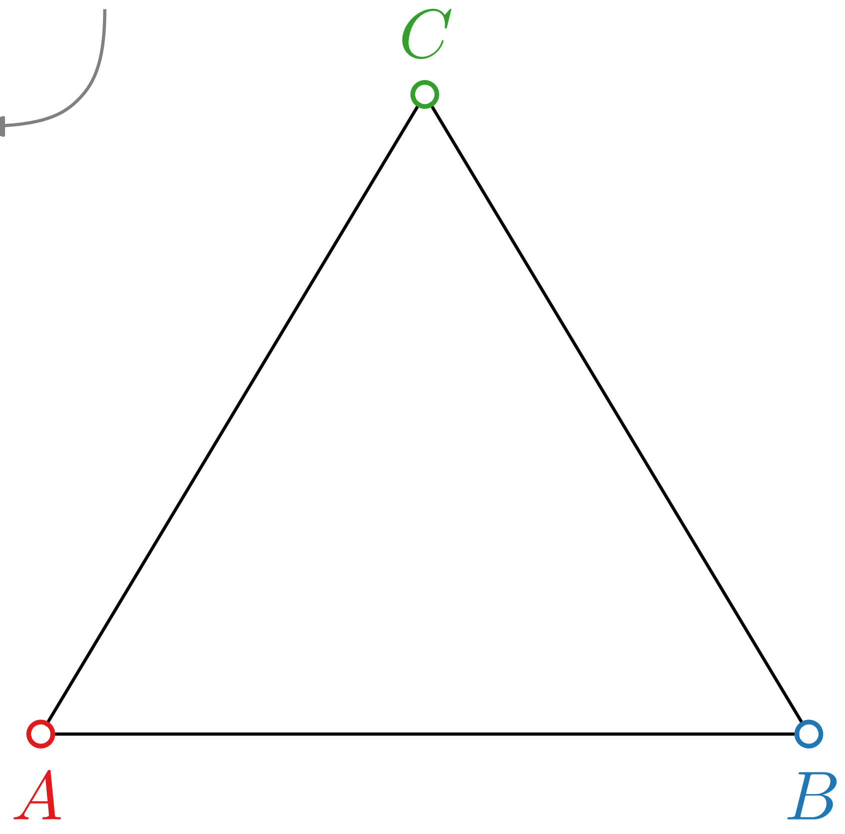
no three points on a line



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$$\phi\colon v \in V \mapsto v_1 A + v_2 B + v_3 C$$

yields a **planar straight-line** drawing of G inside $\triangle ABC$.

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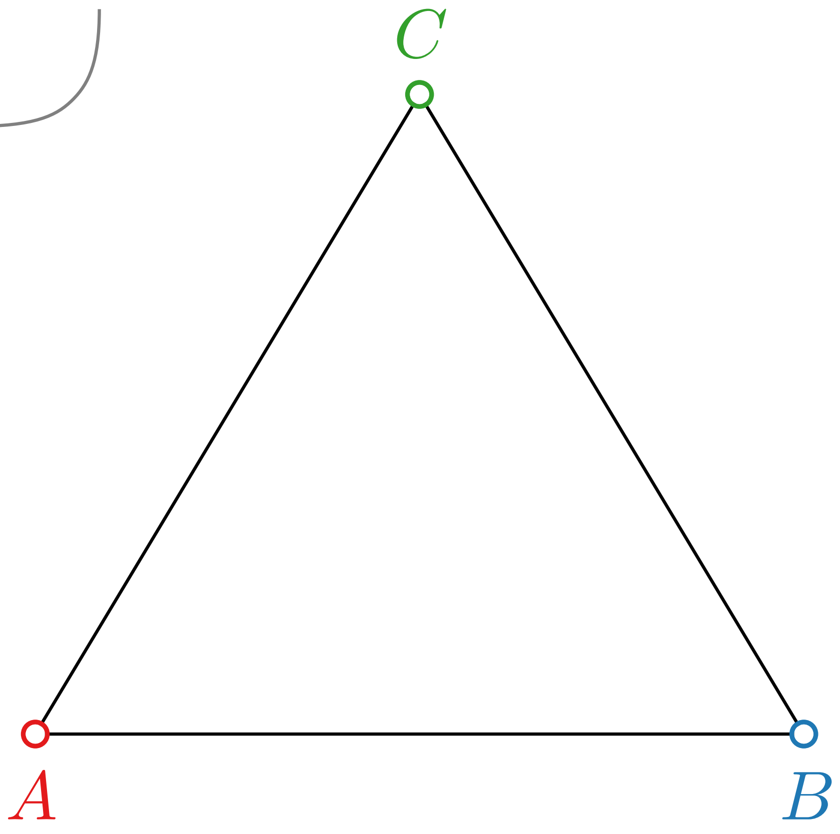

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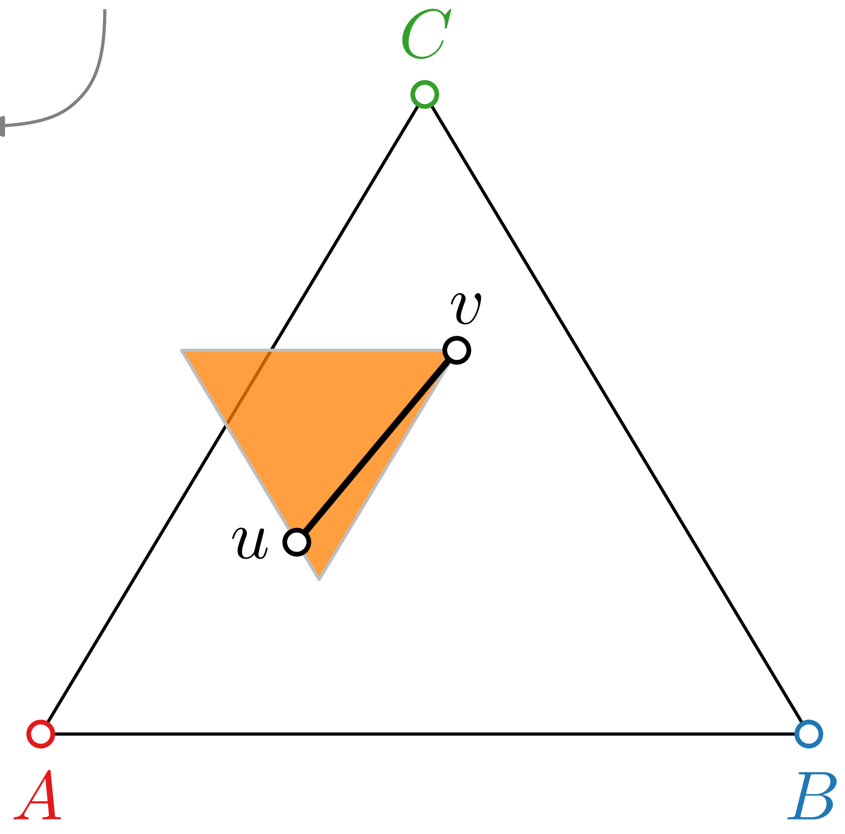

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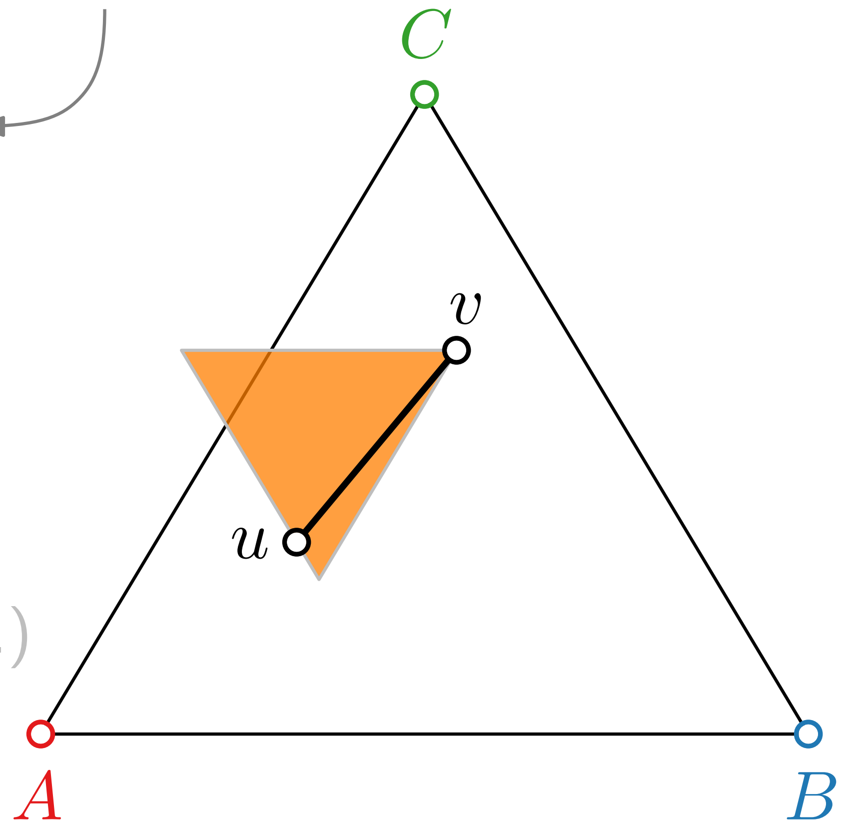

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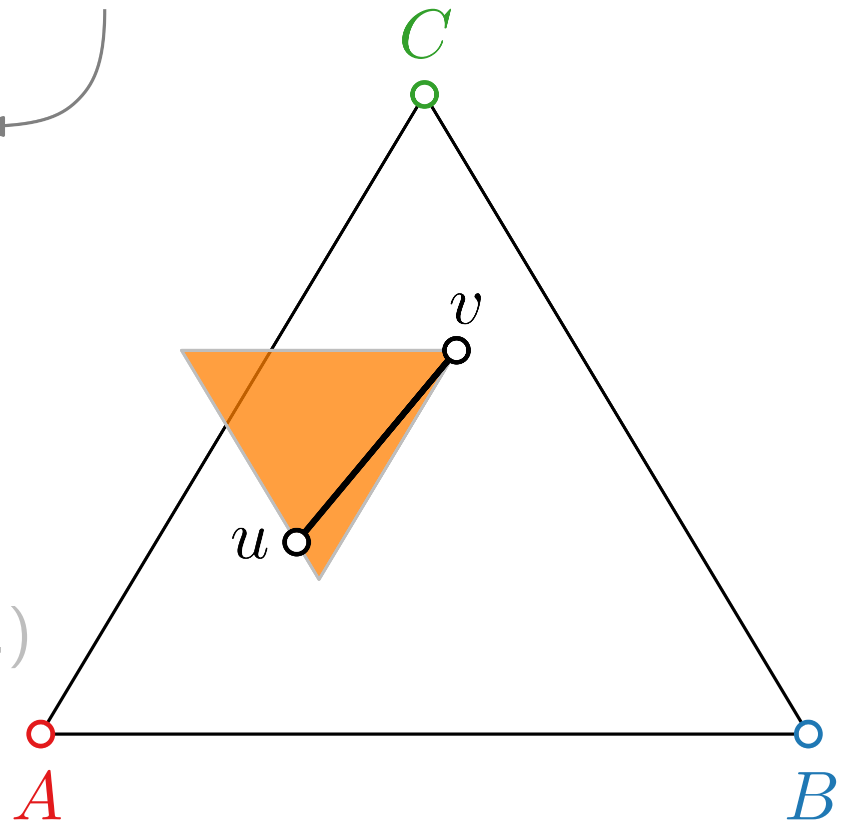

- No vertex x can lie on an edge $\{u, v\}$. (Obvious by definition.)
- No pair of edges $\{u, v\}$ and $\{u', v'\}$ crosses:

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yields a **planar straight-line** drawing of G inside $\triangle ABC$.

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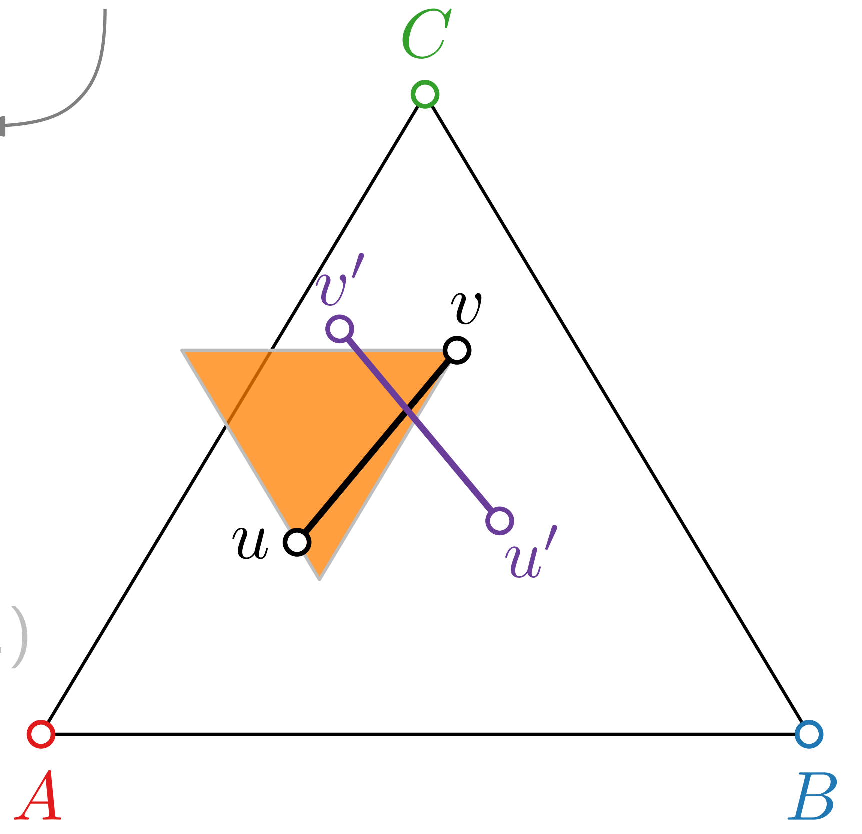

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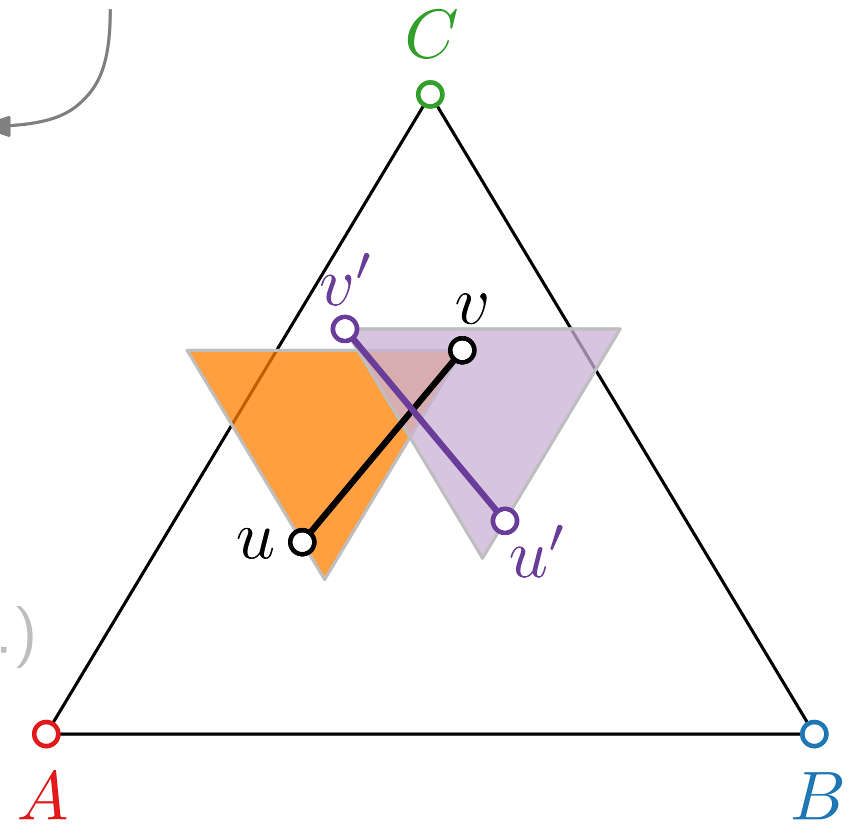

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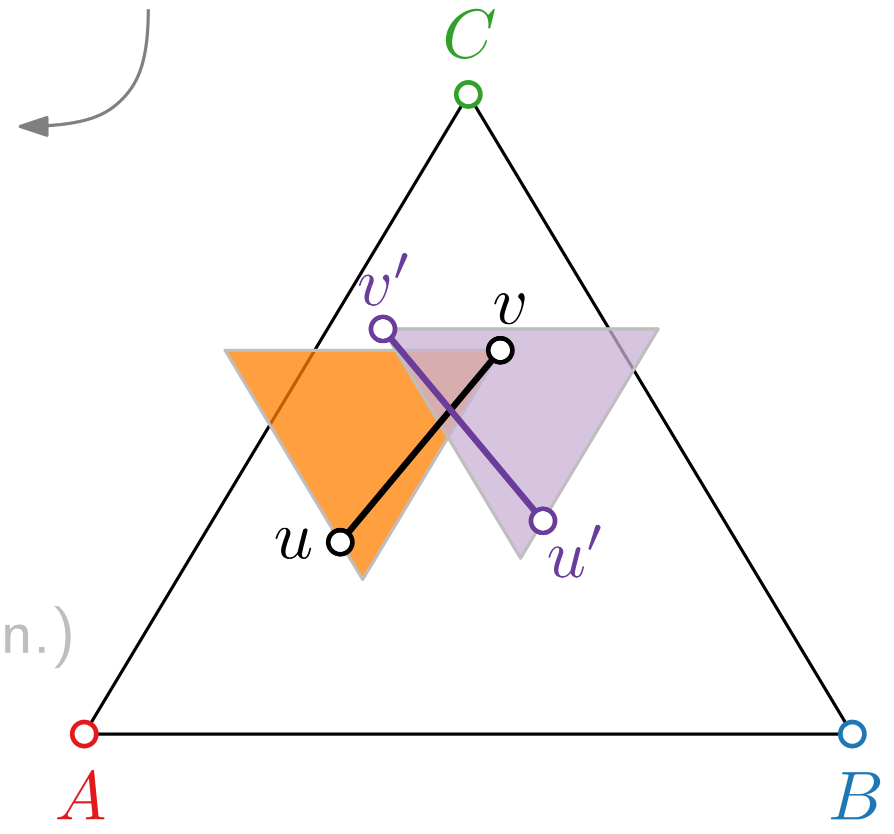

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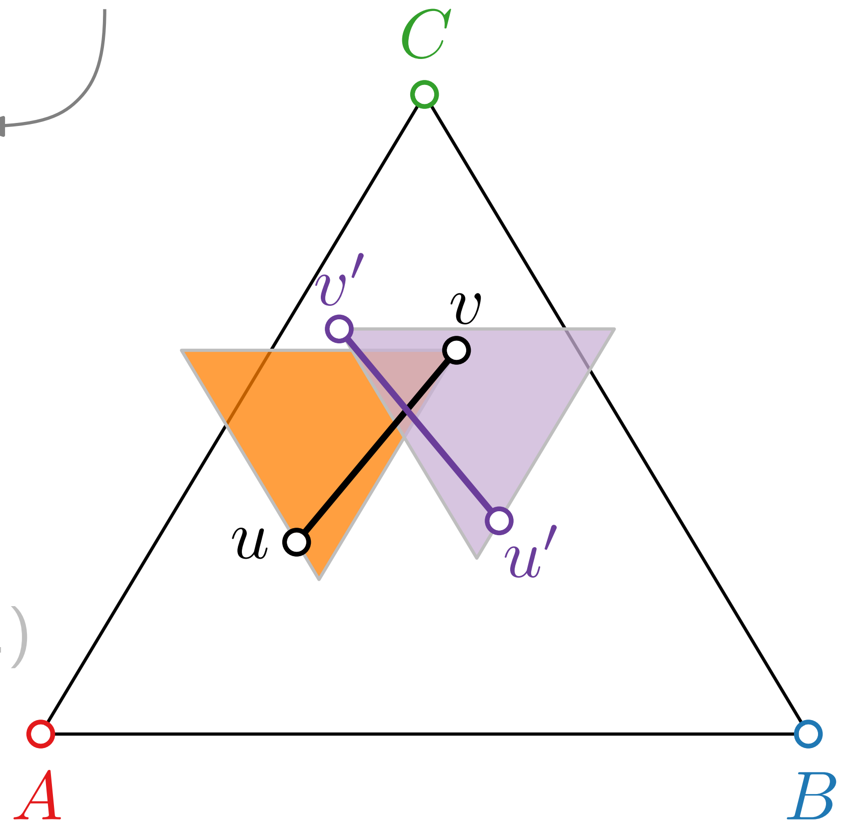

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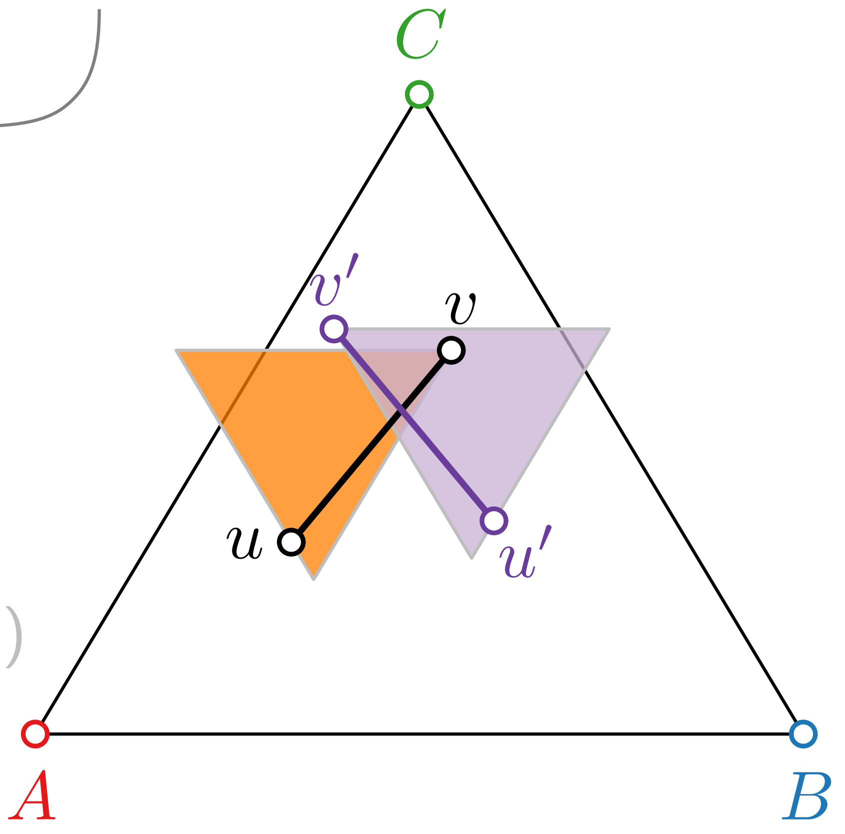

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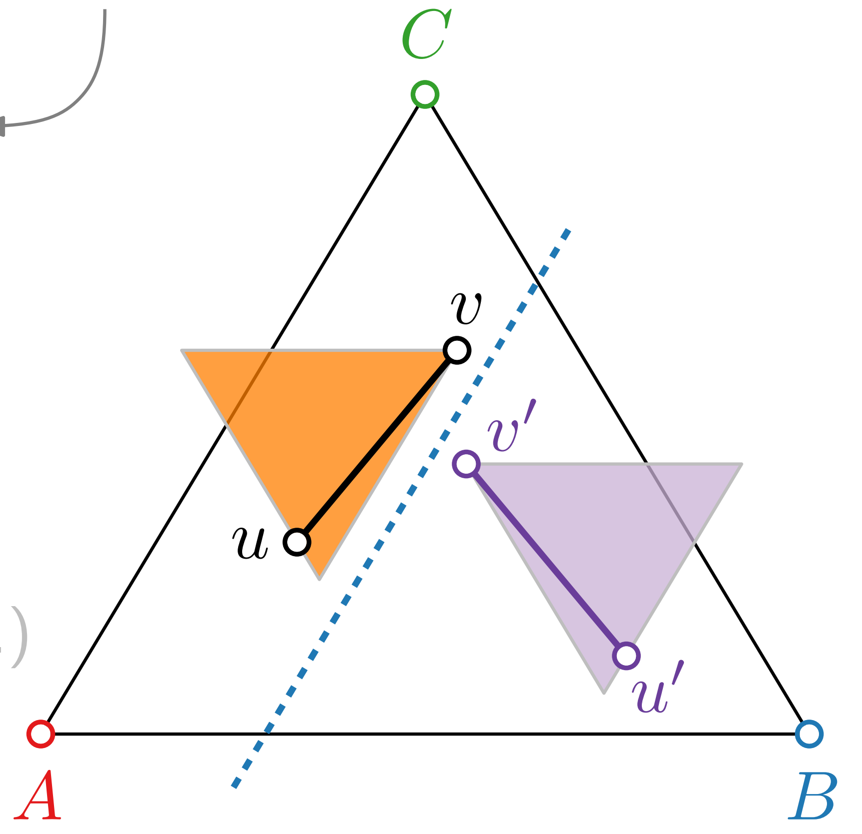

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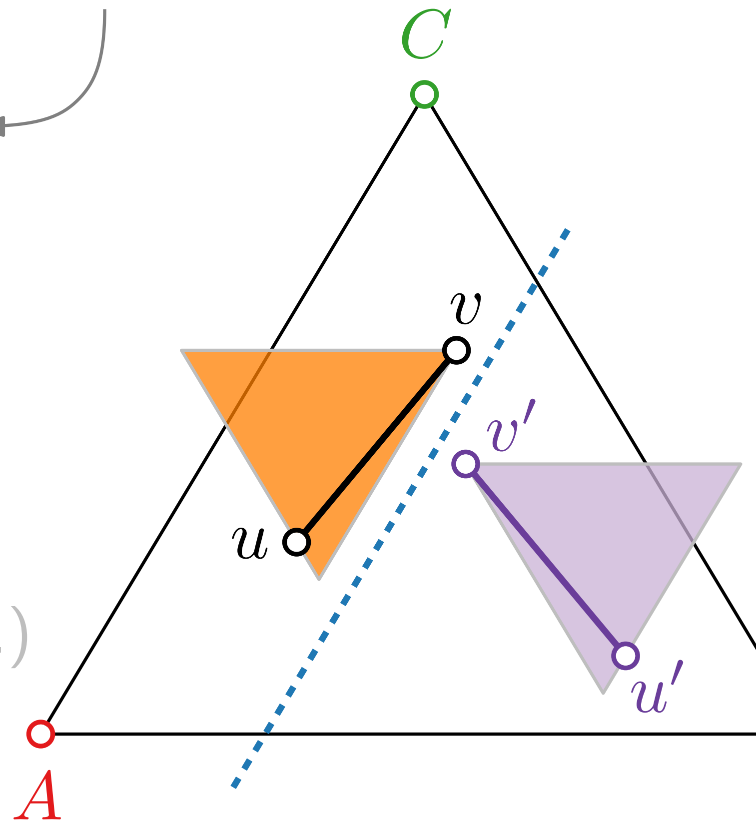

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How to find a barycentric representation?

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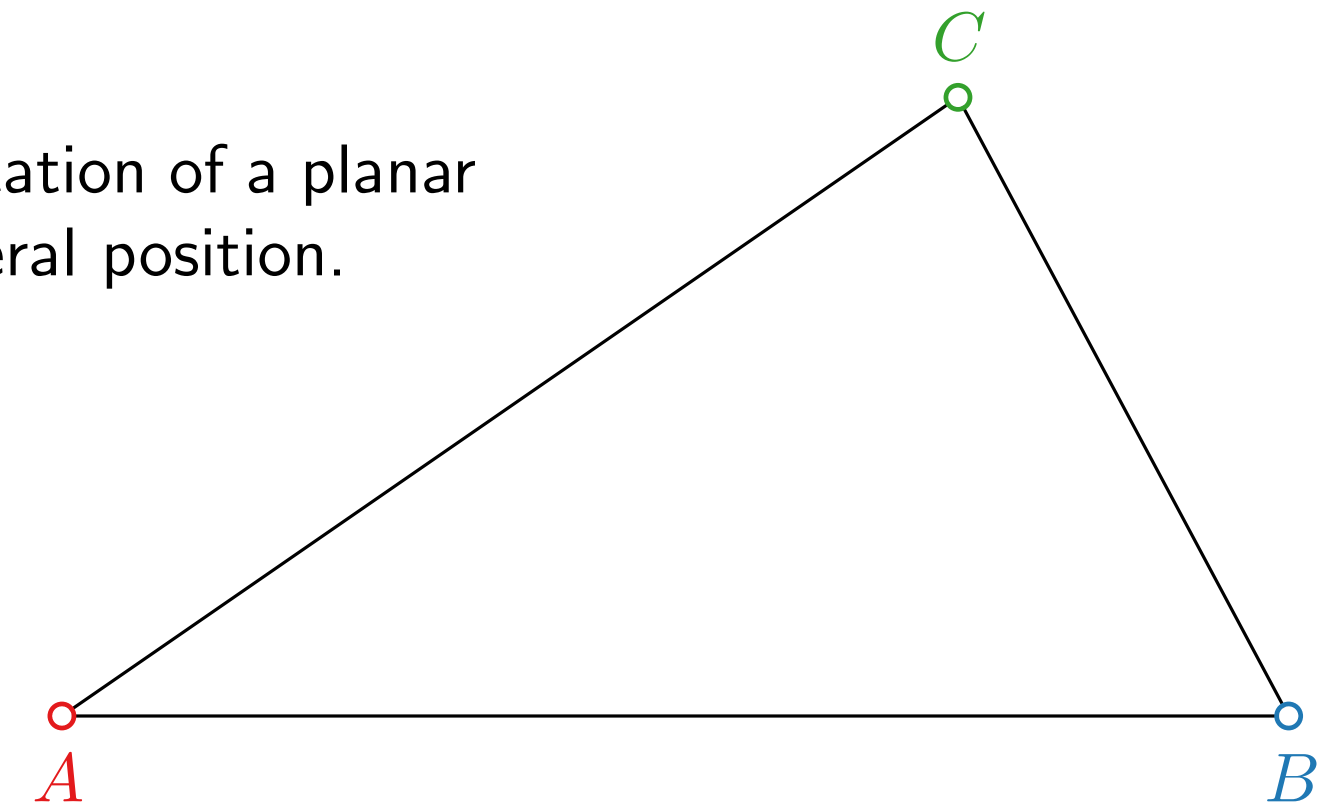

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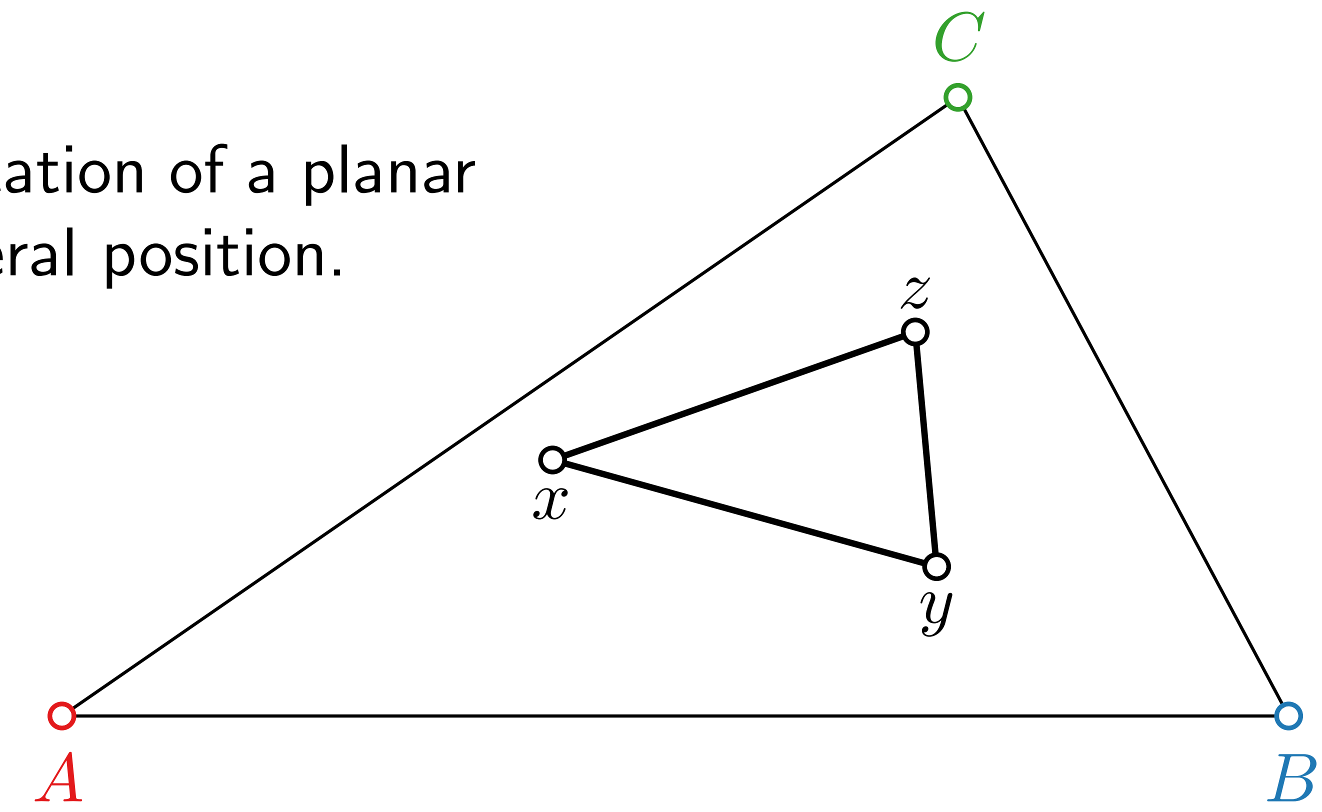

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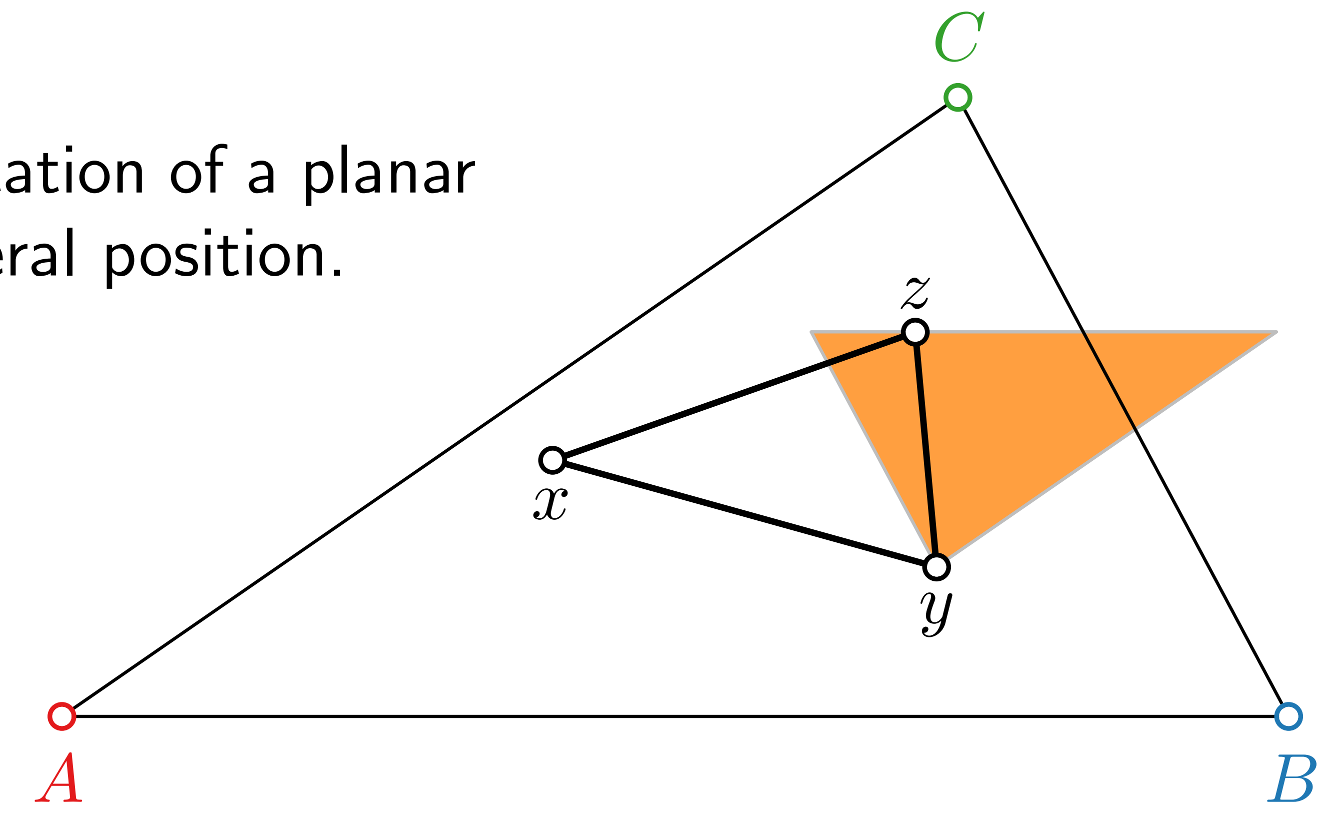

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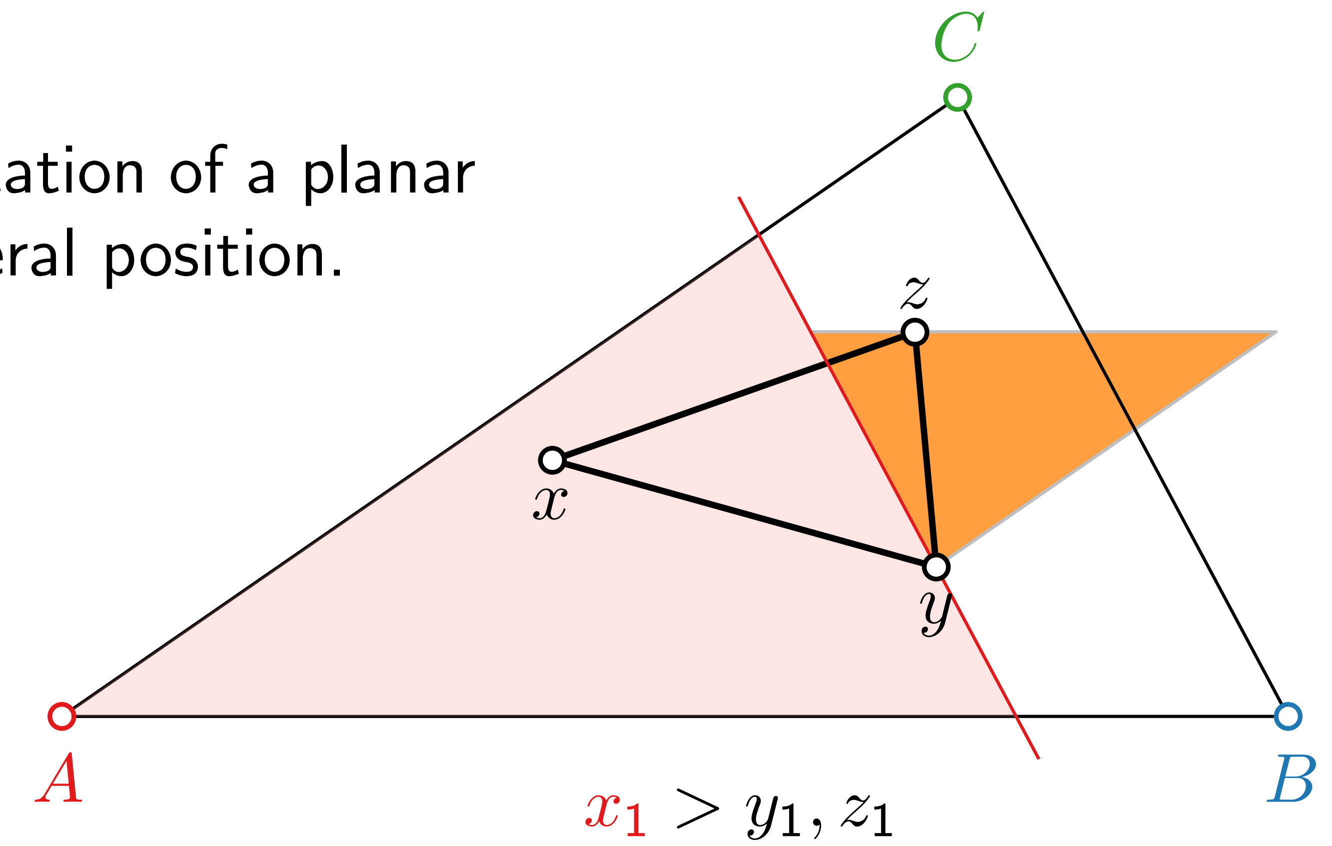

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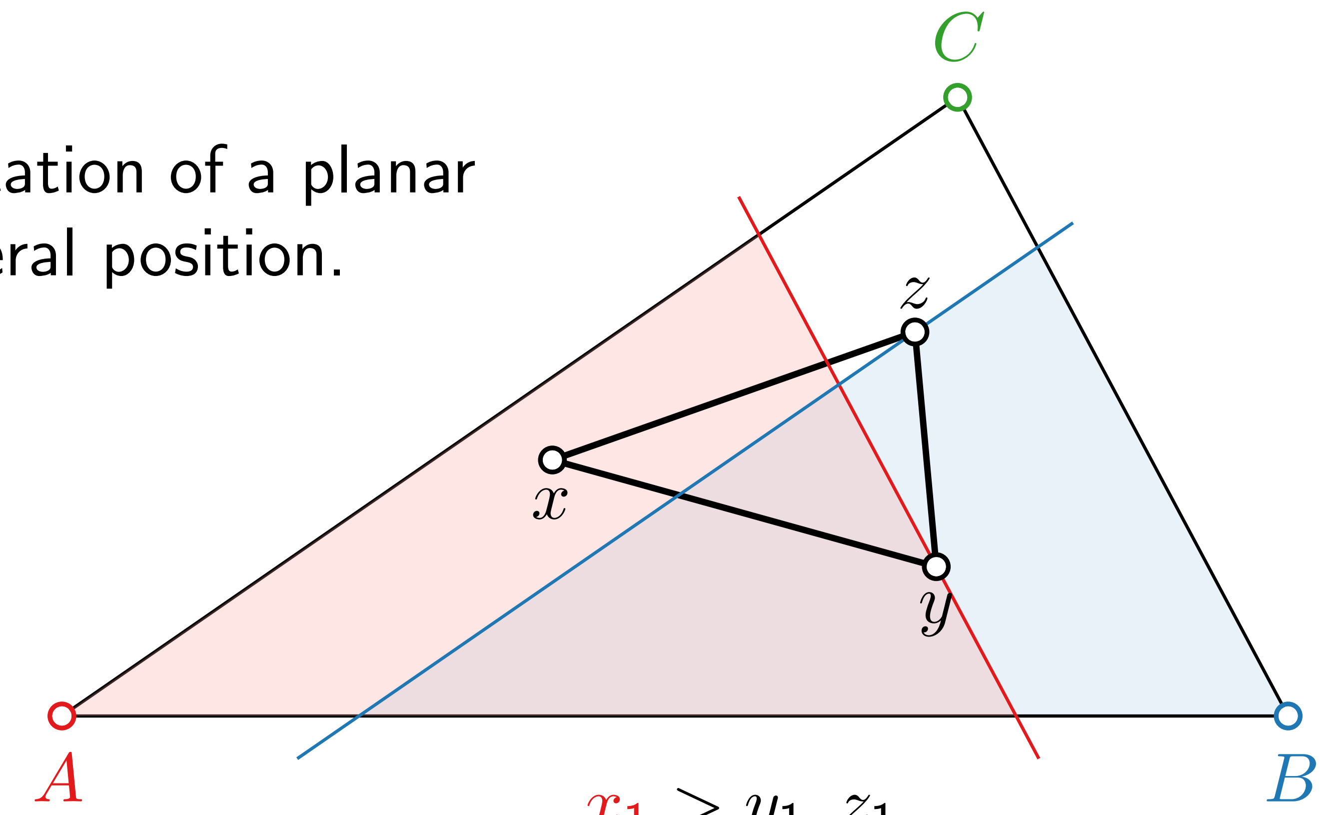

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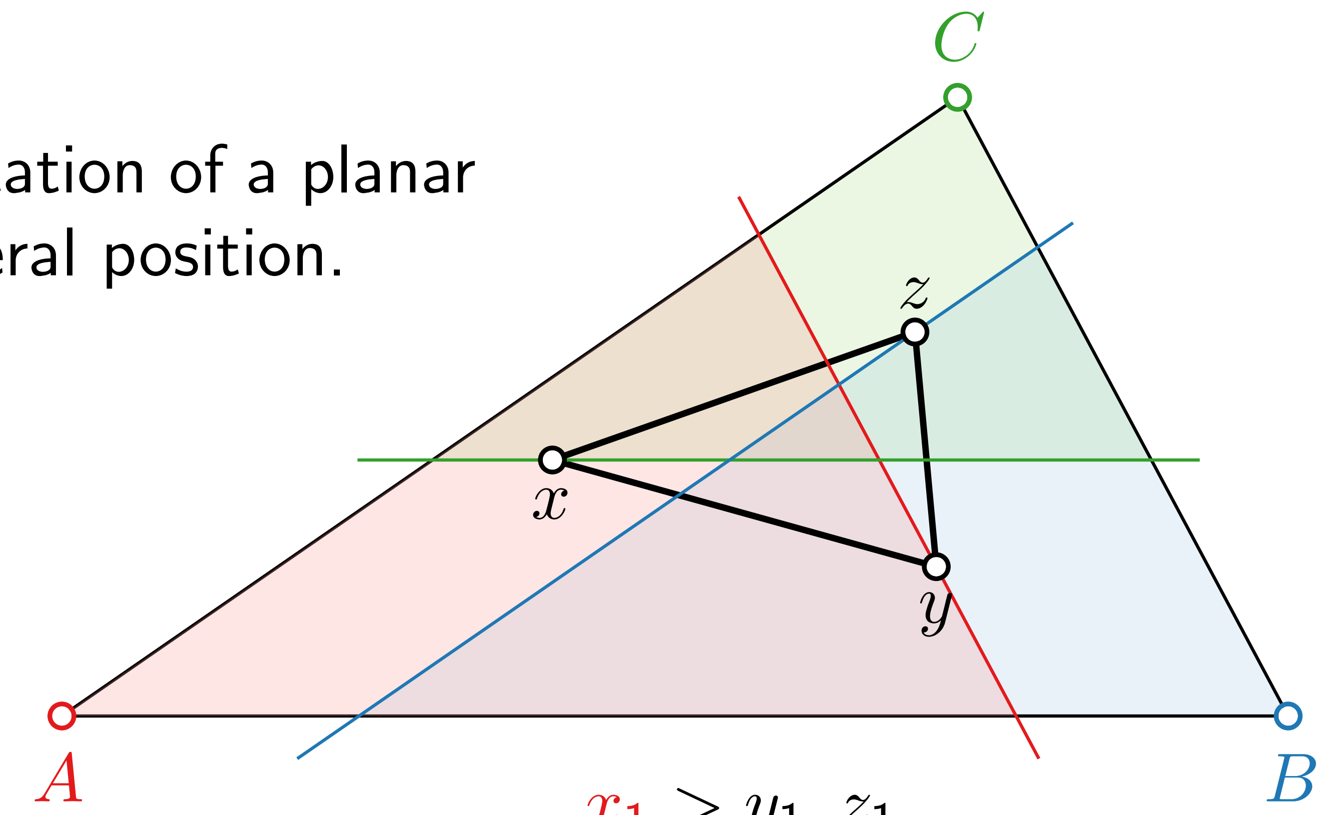

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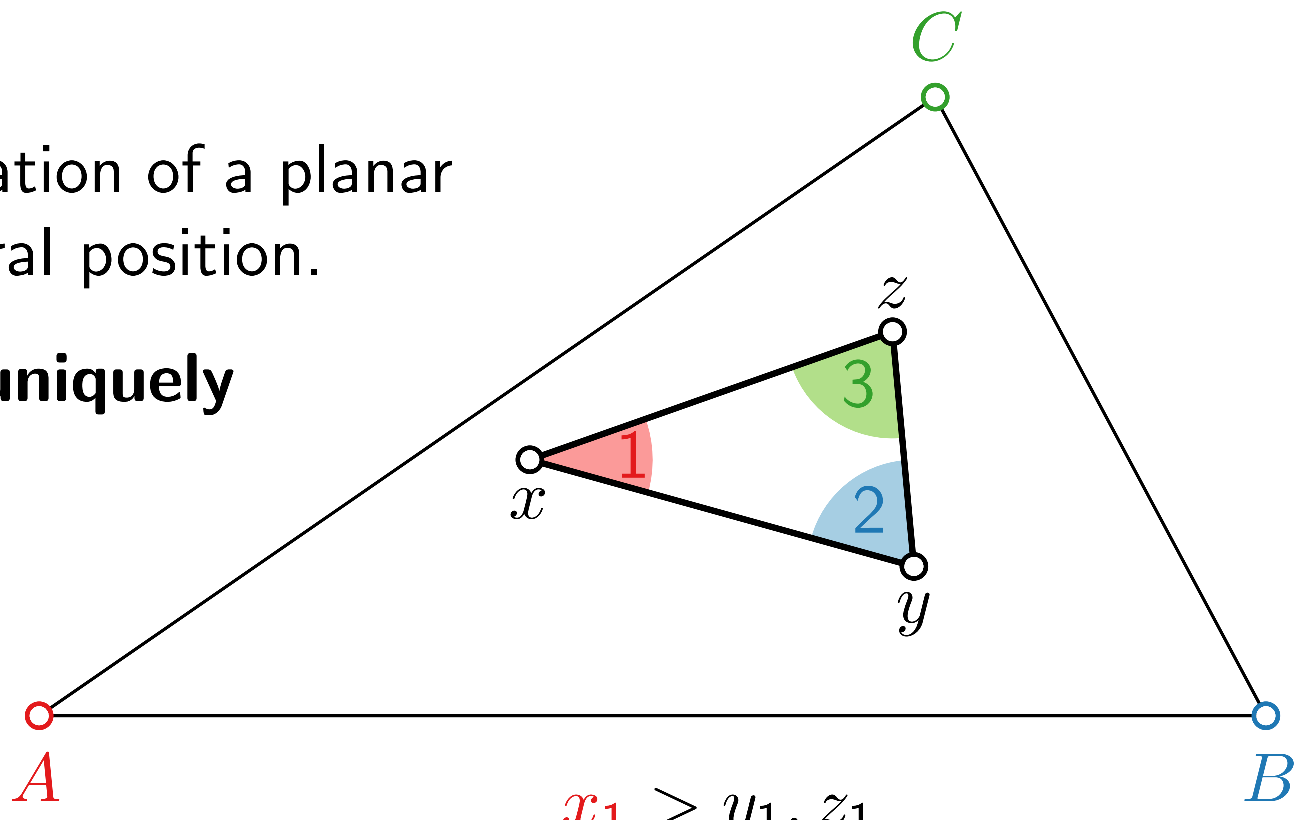

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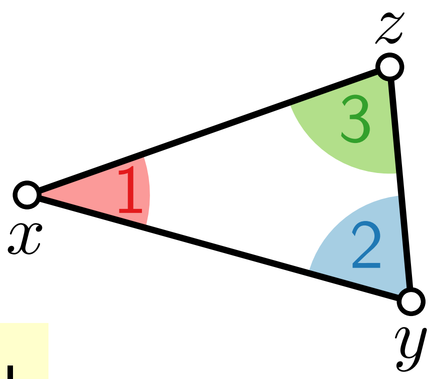

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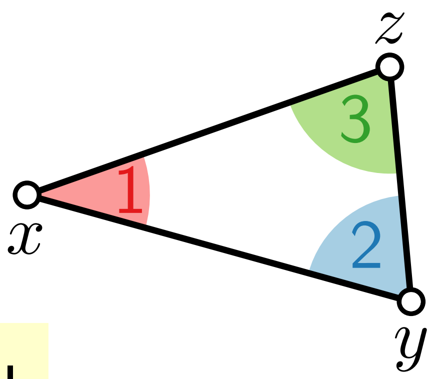

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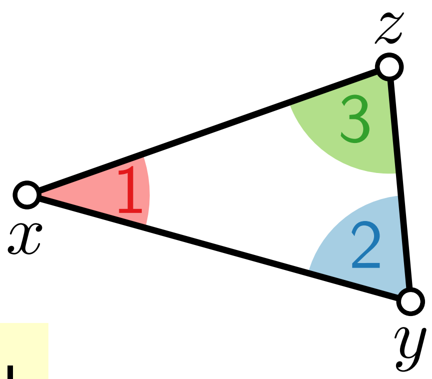

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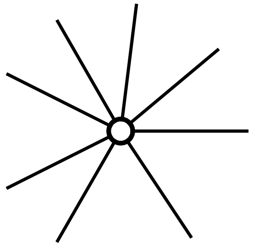

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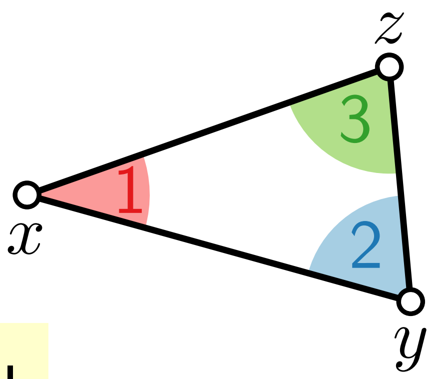

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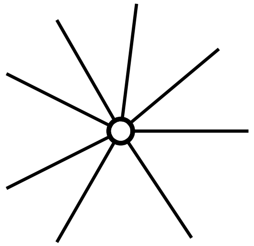

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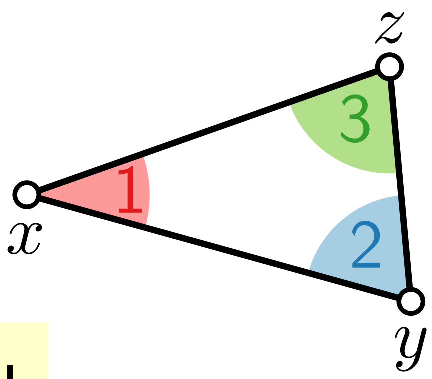

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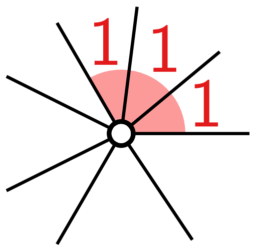

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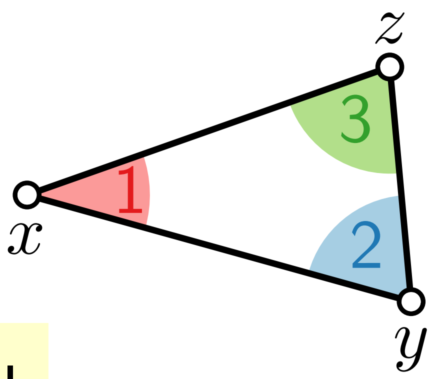

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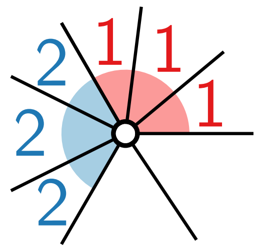

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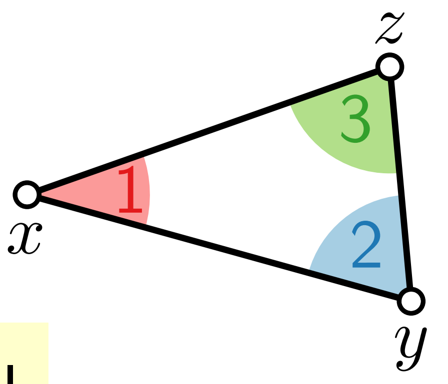

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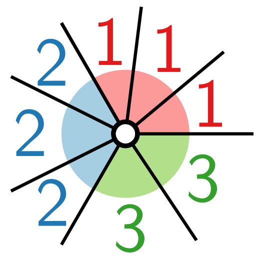

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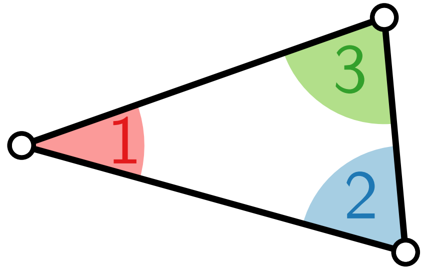
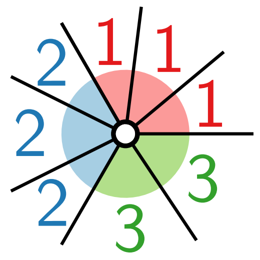
A Schnyder labeling induces an edge labeling.





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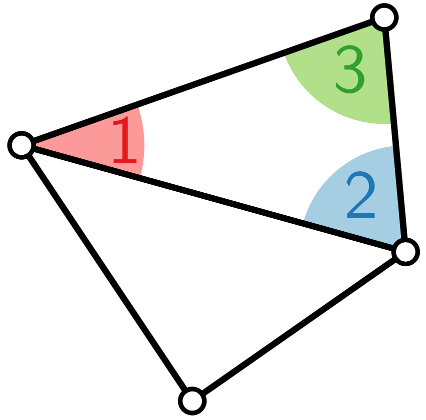

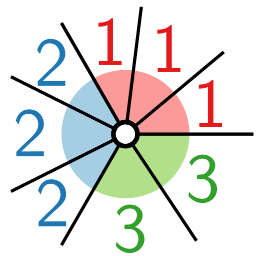

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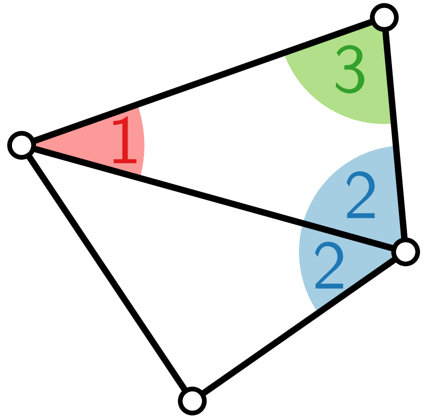

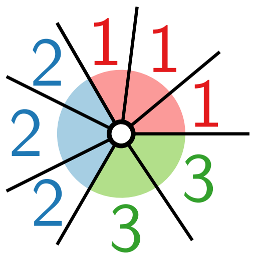

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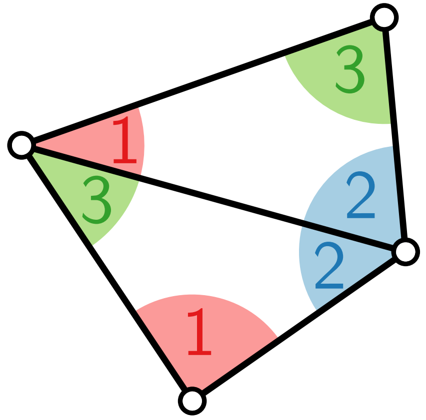

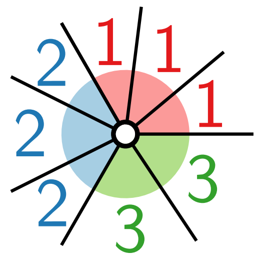

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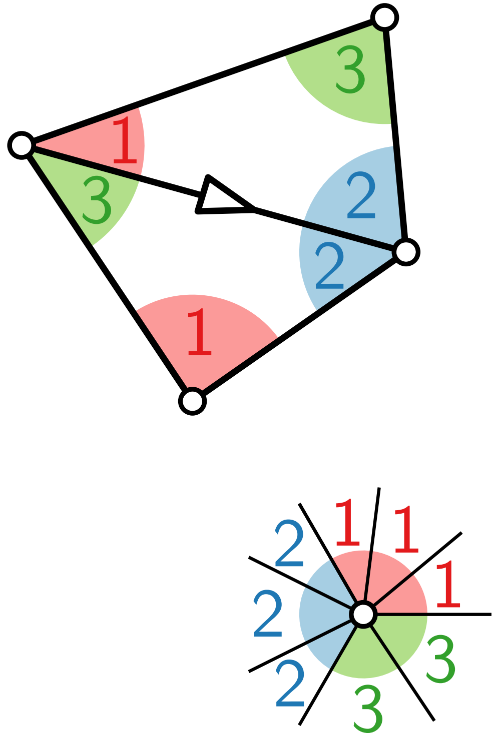

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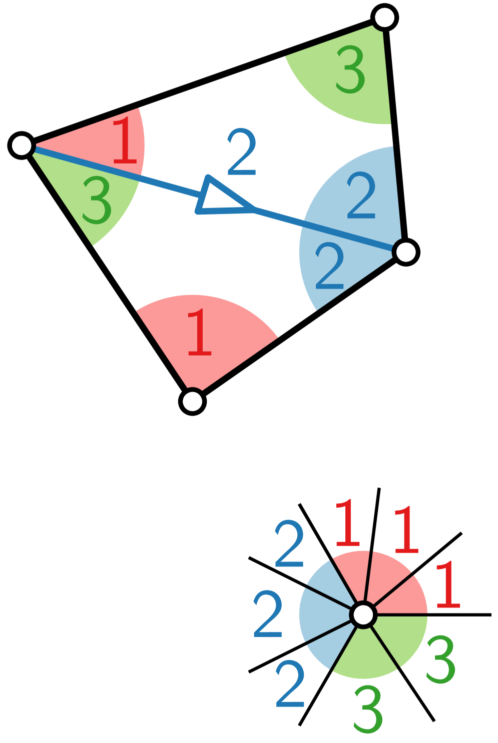

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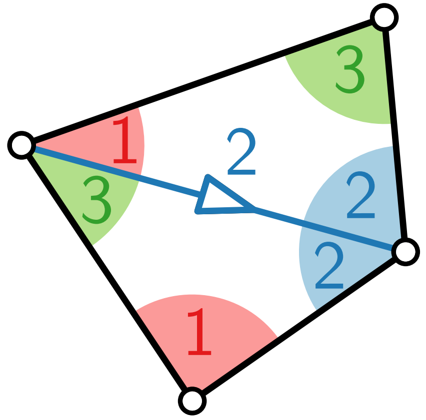

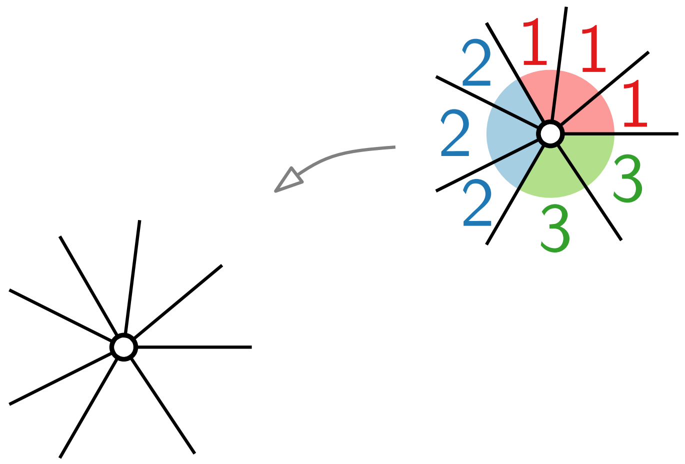

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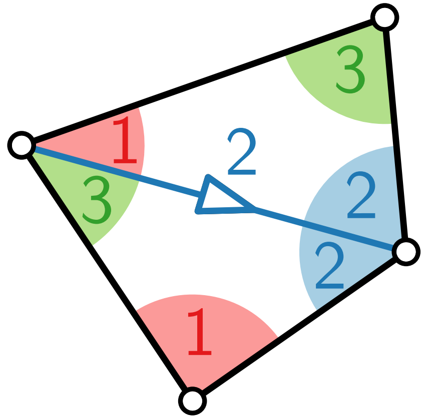

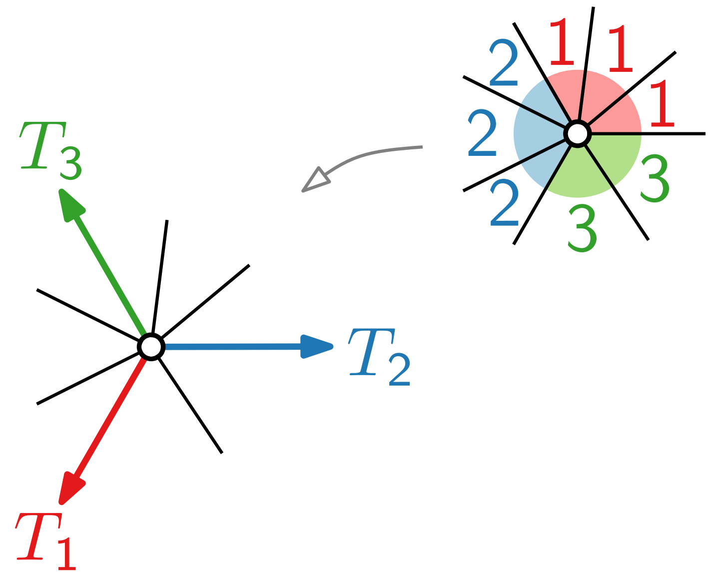

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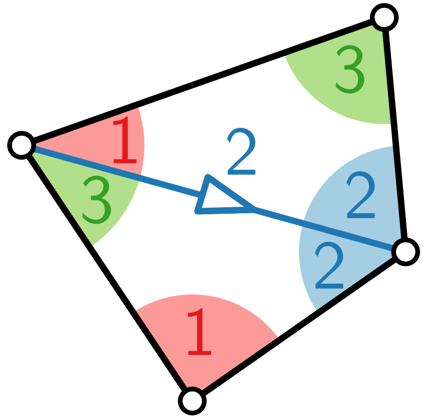

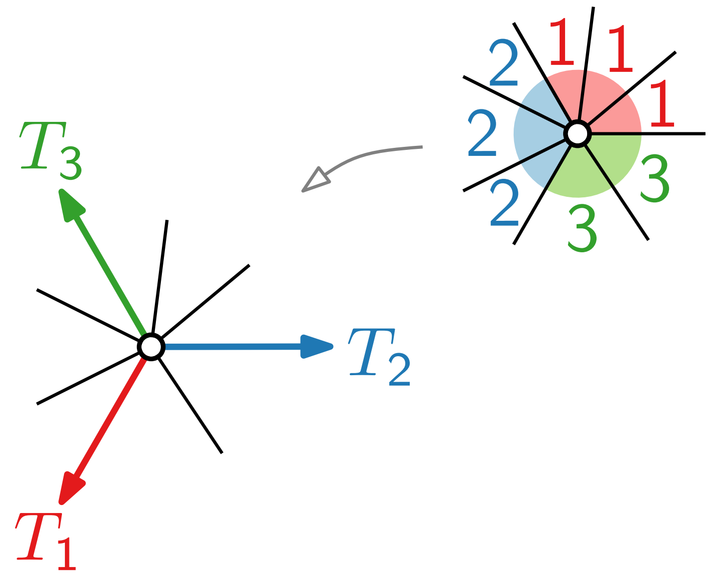

Schnyder Wood

A Schnyder labeling induces an edge labeling.

A **Schnyder wood** (or **Schnyder realizer**) of a plane triangulation G is a partition of the inner edges of G into three sets of oriented edges T_1, T_2, T_3 such that, for each inner vertex v of G, it holds that

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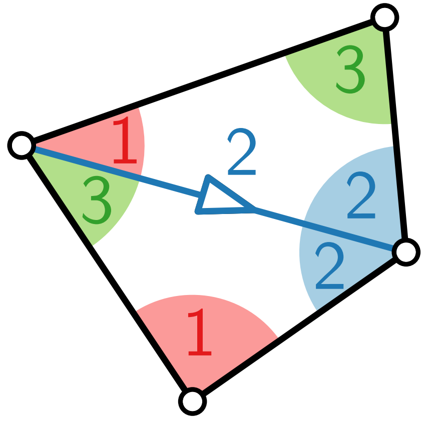

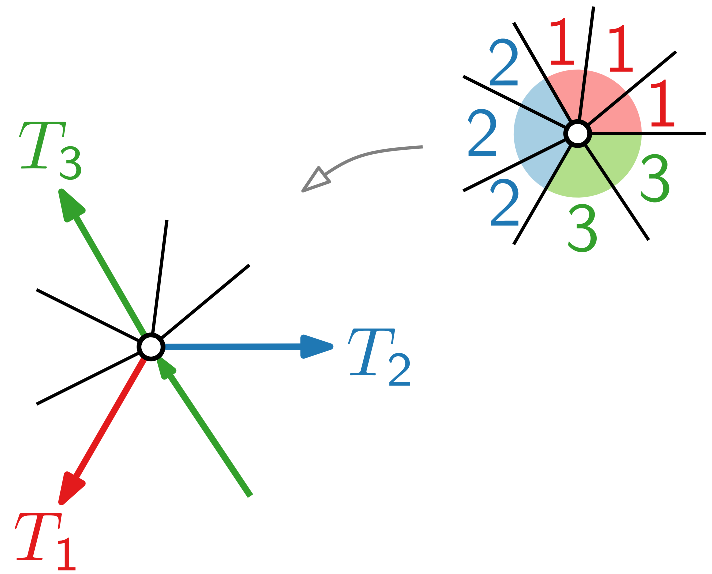

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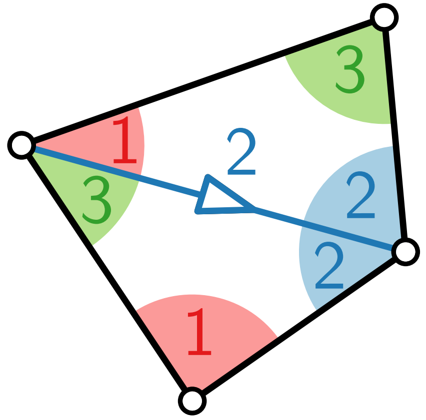

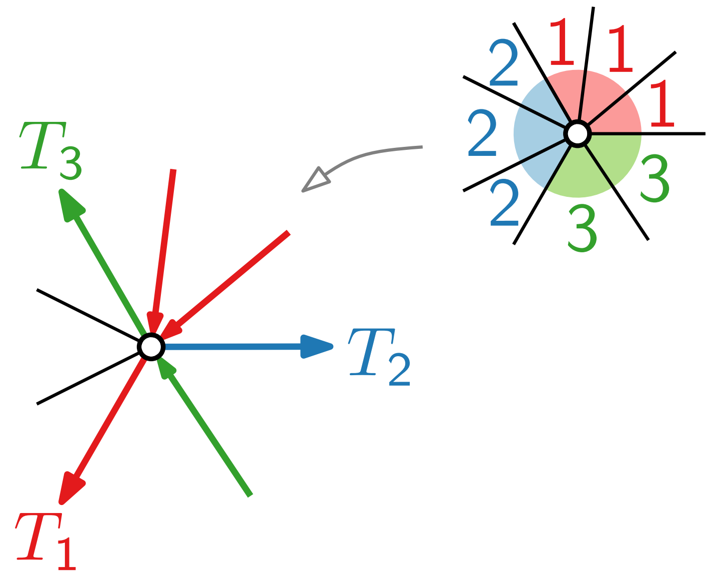

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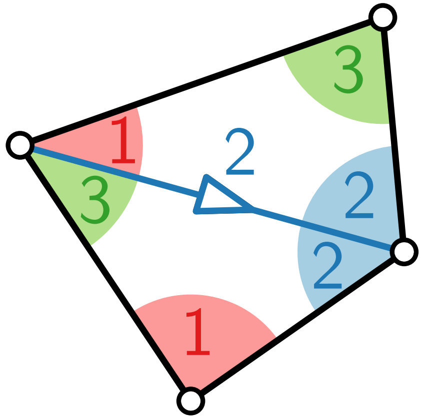
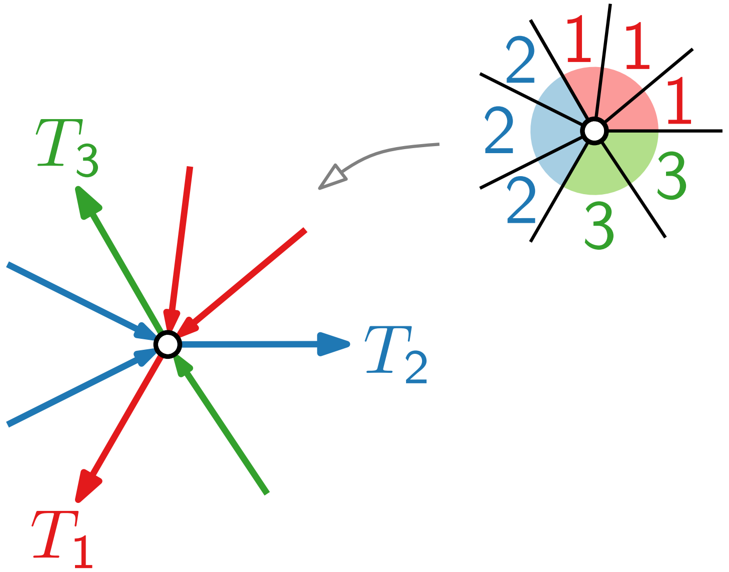
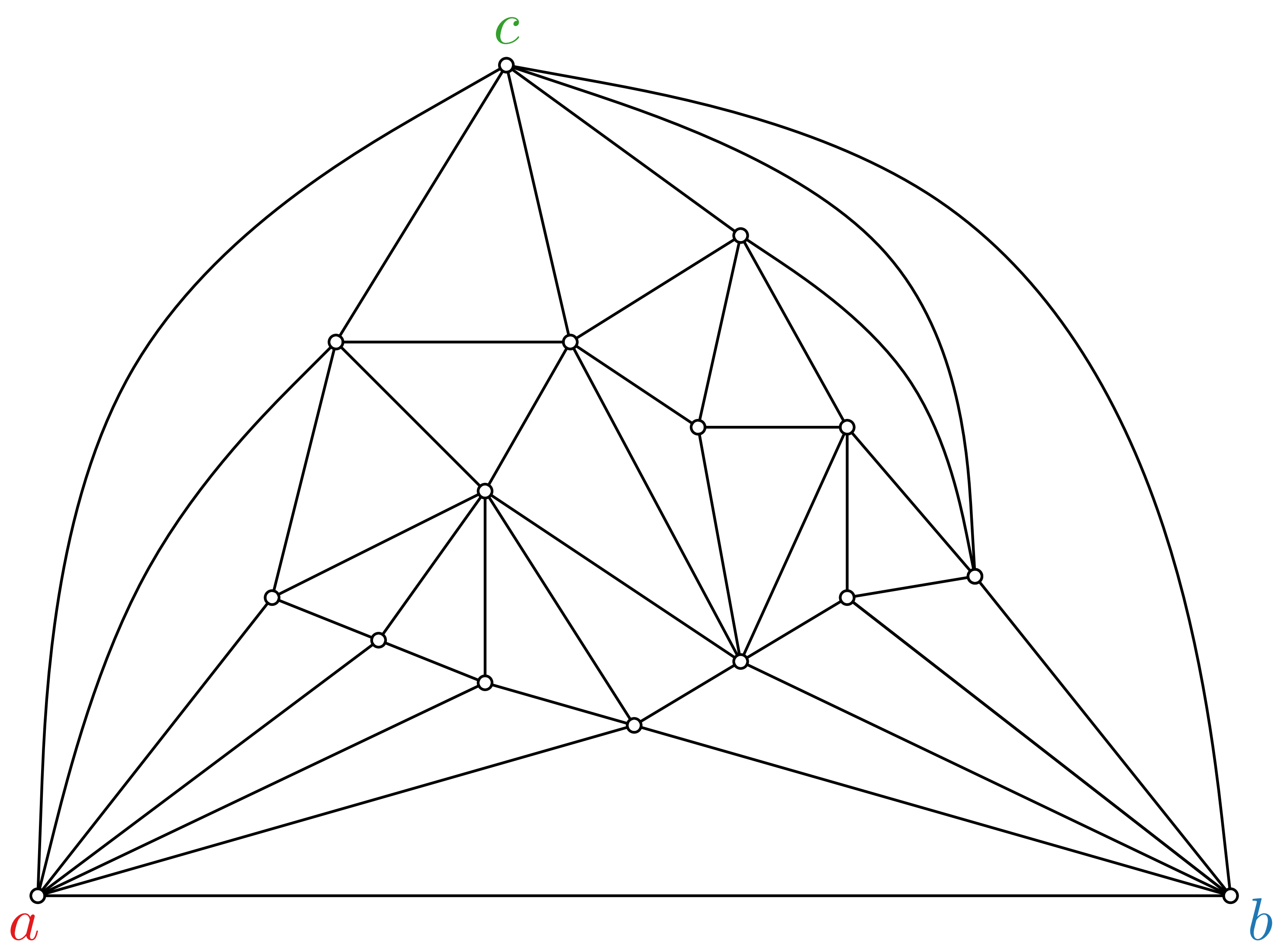
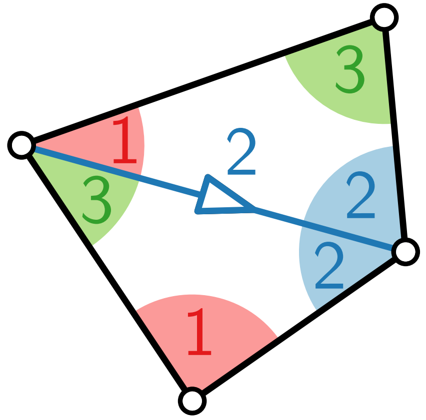
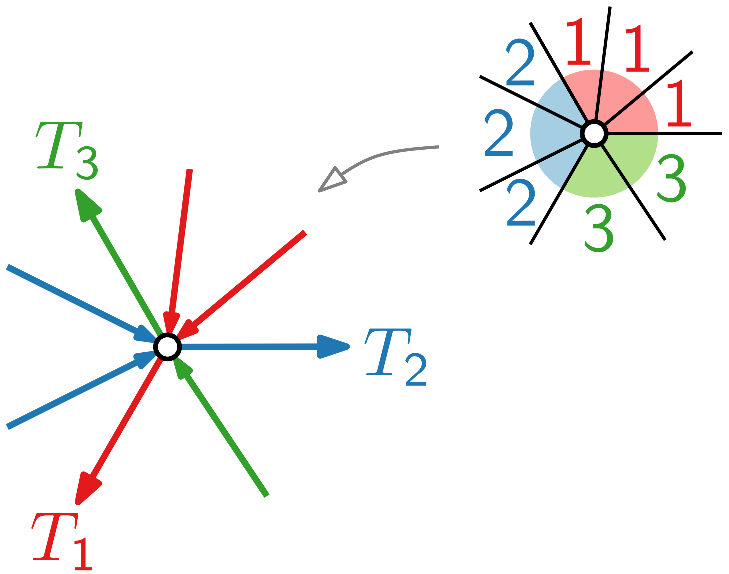




Schnyder Wood – Example and Properties







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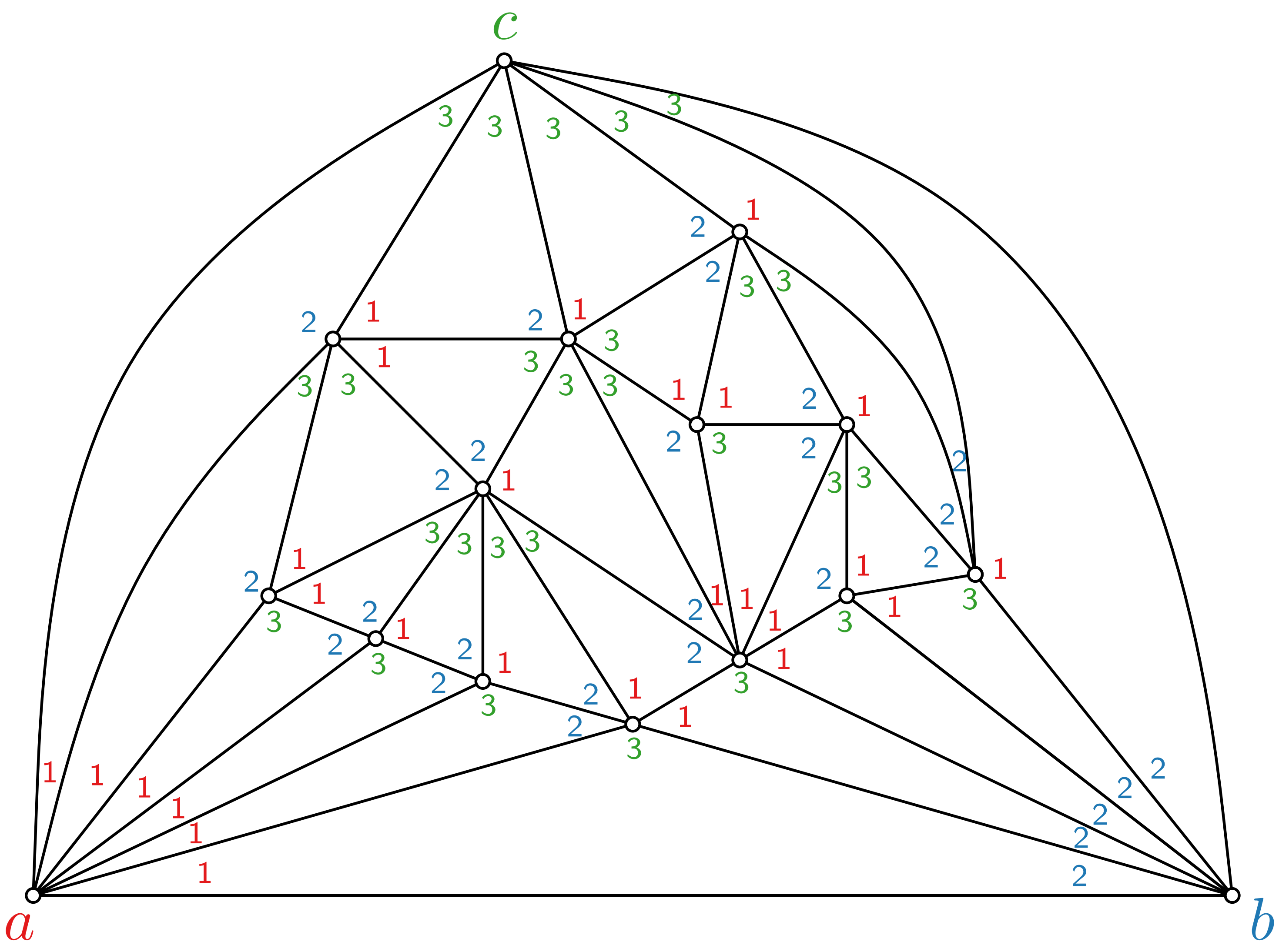

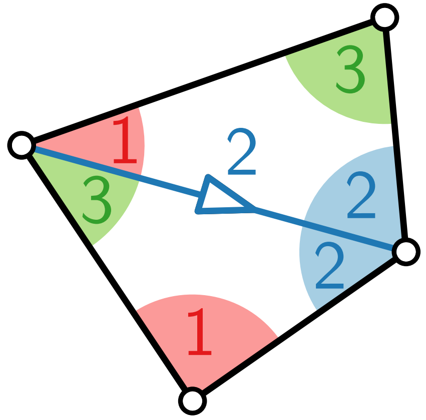

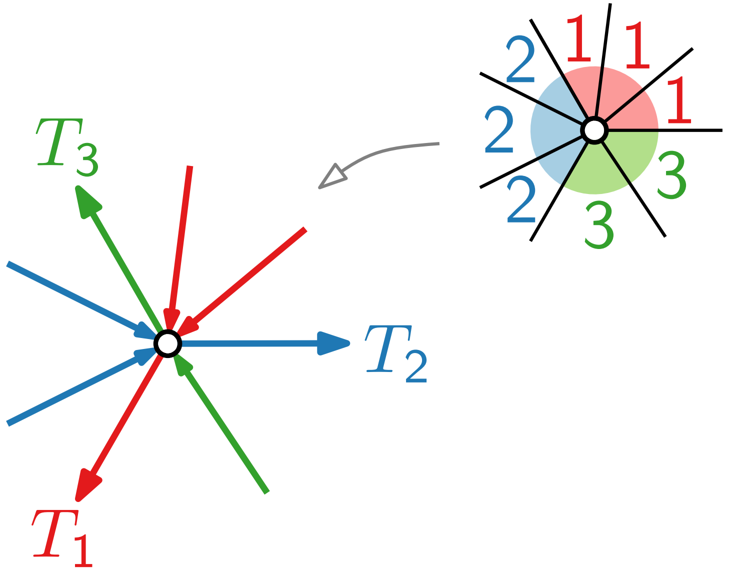

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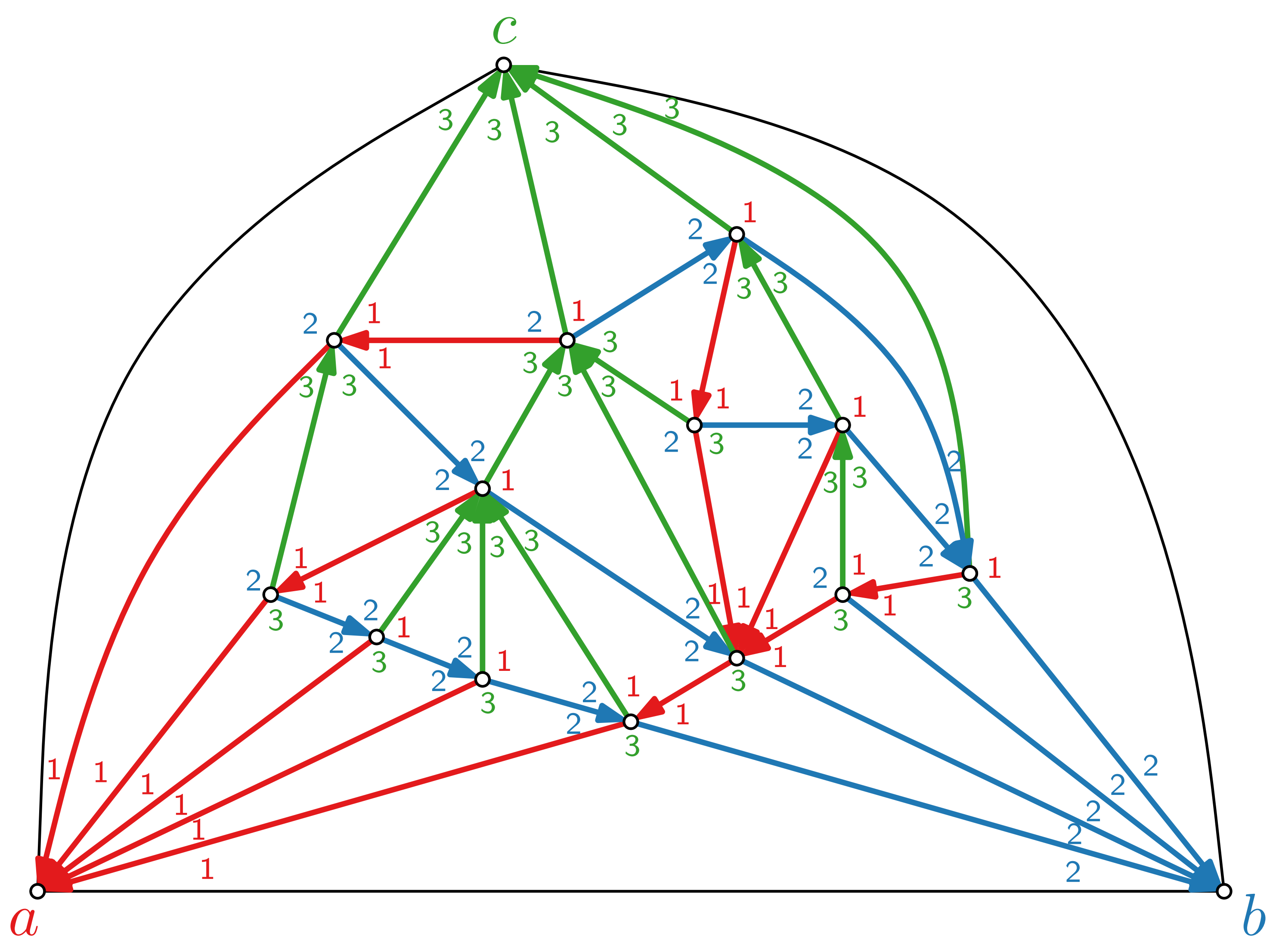

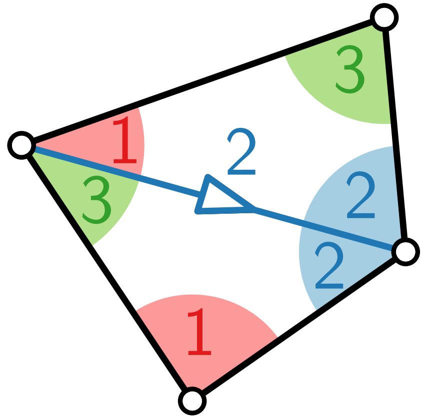

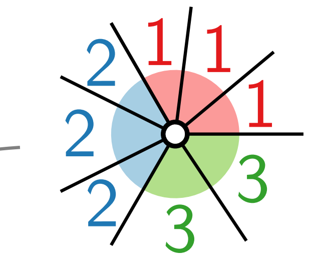

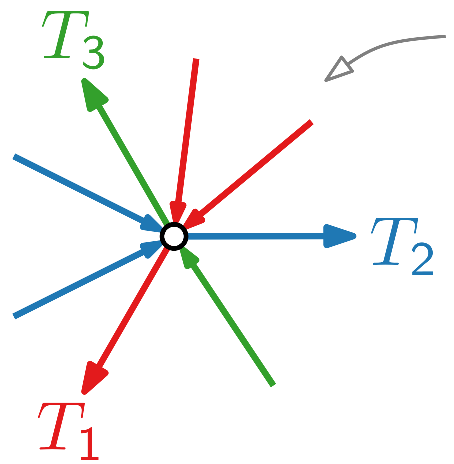

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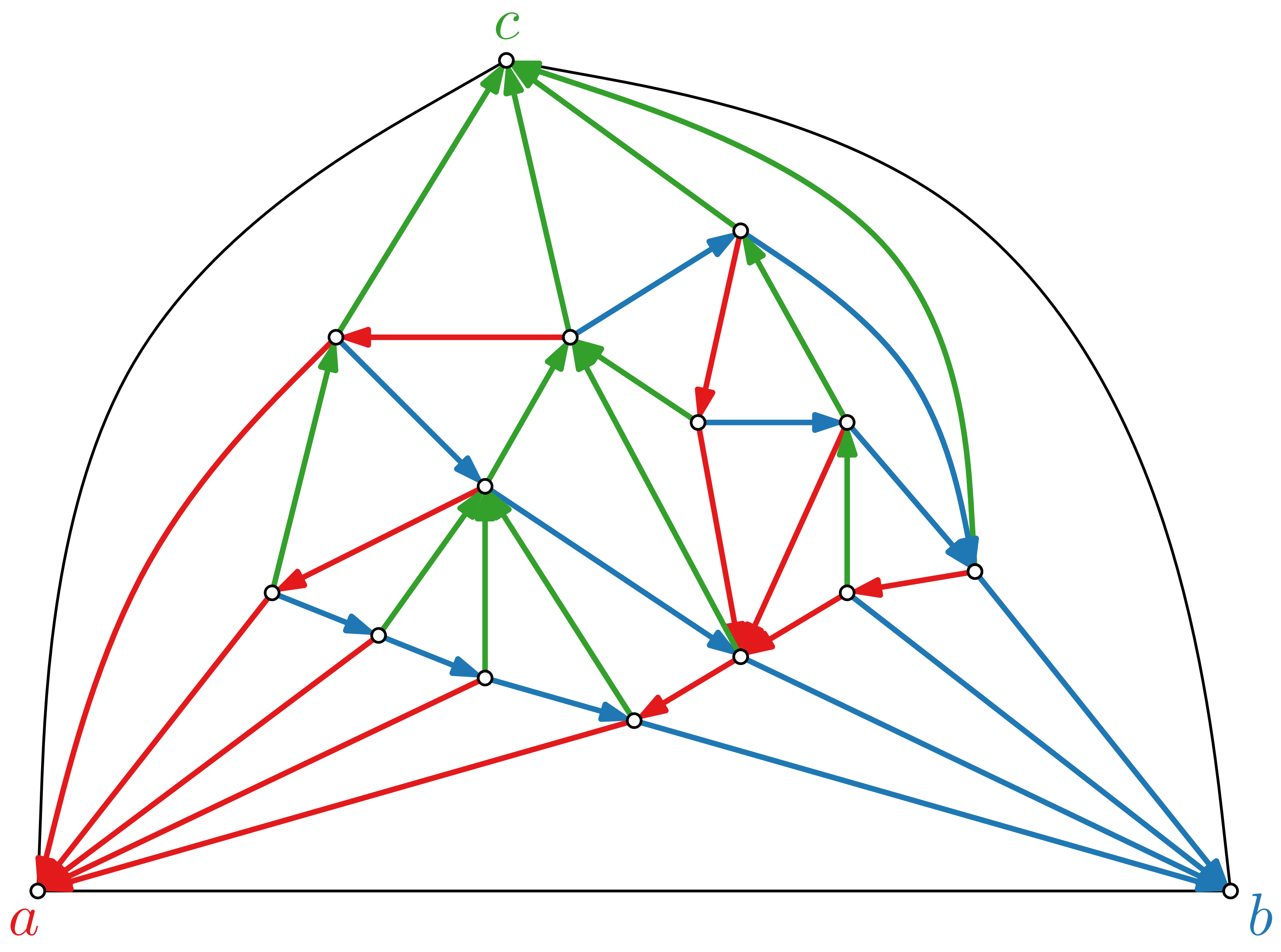

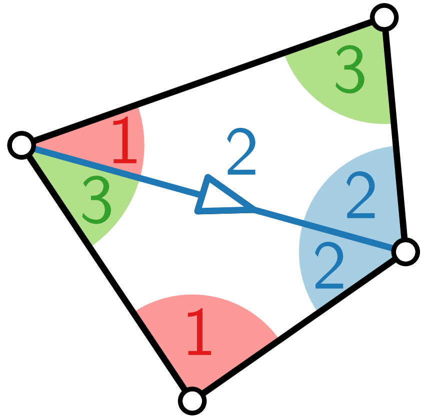

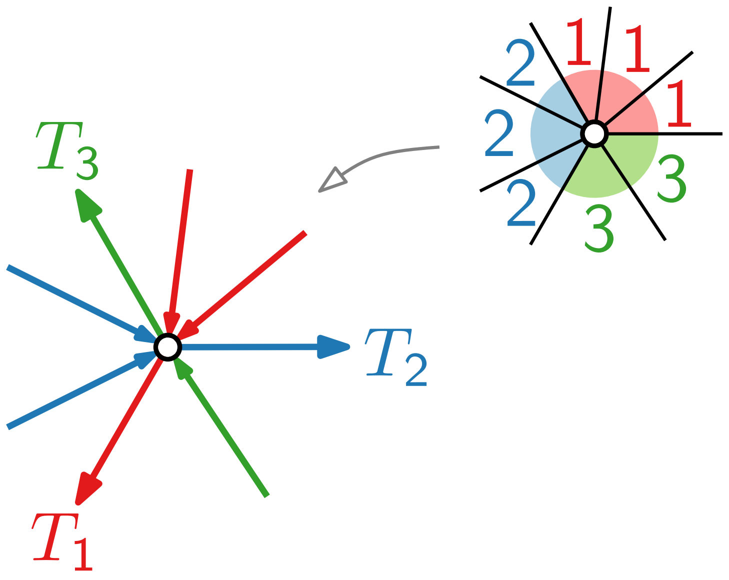

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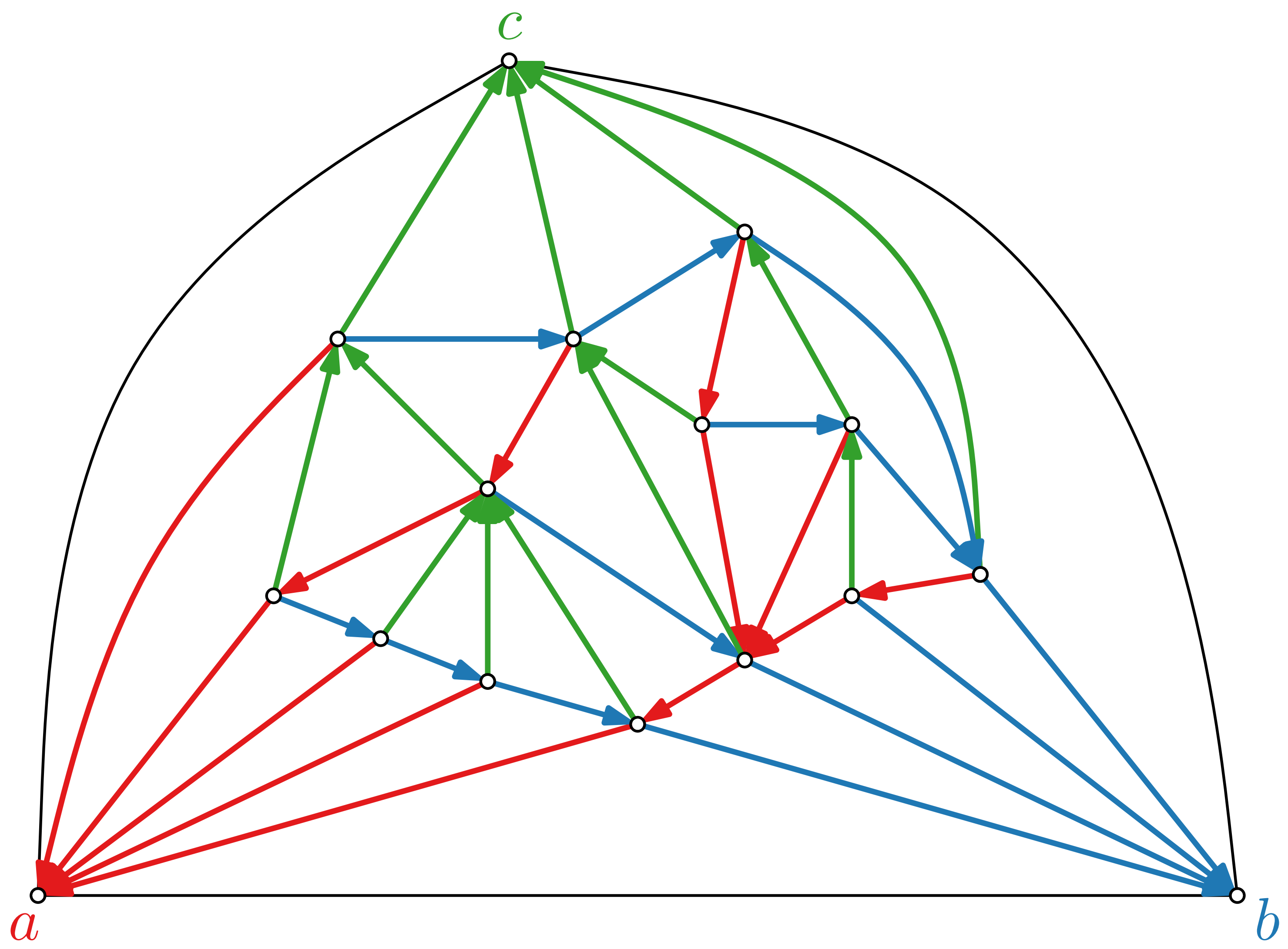
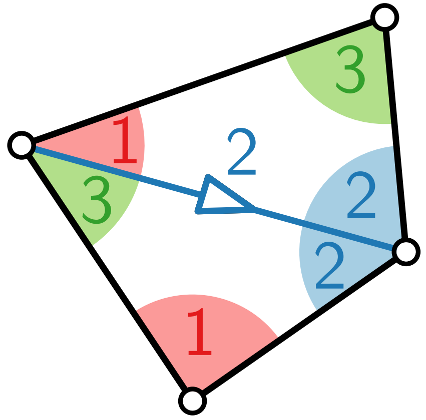
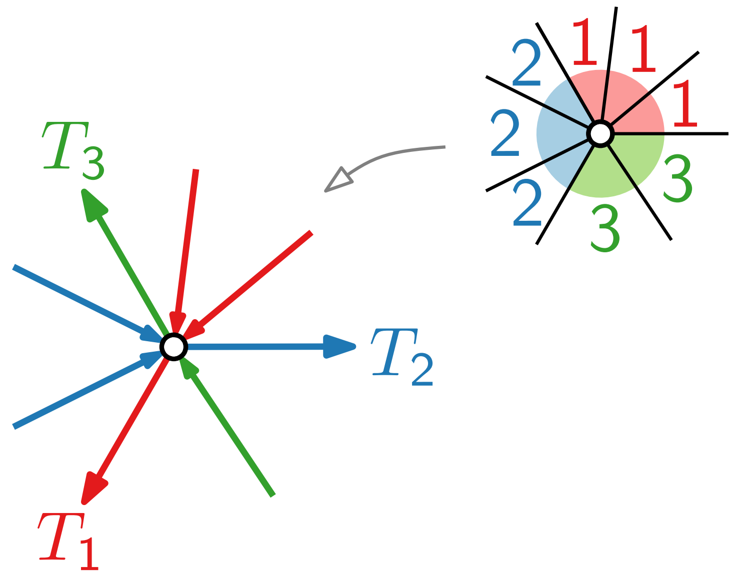


(a Schnyder labeling is not unique)





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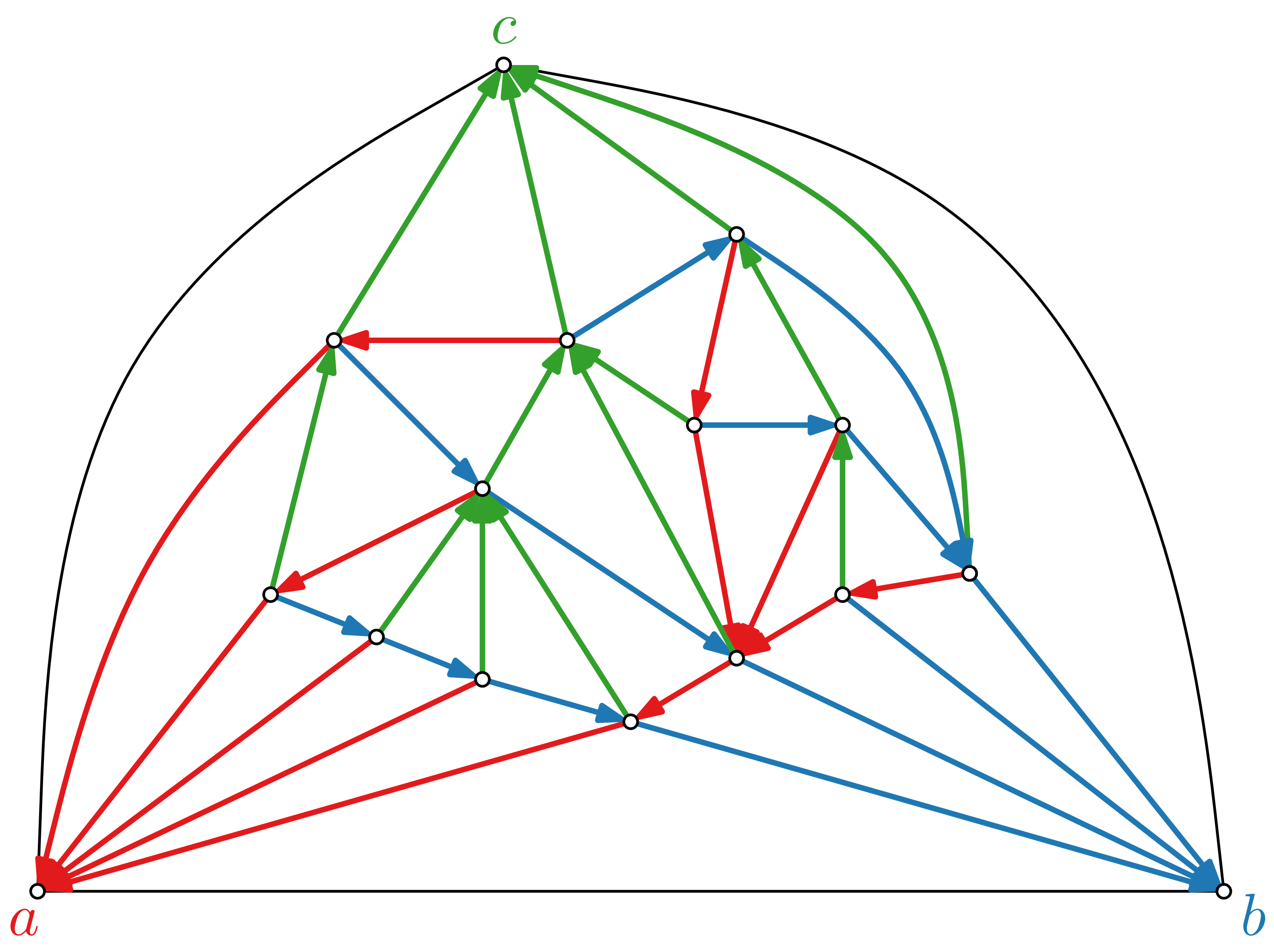

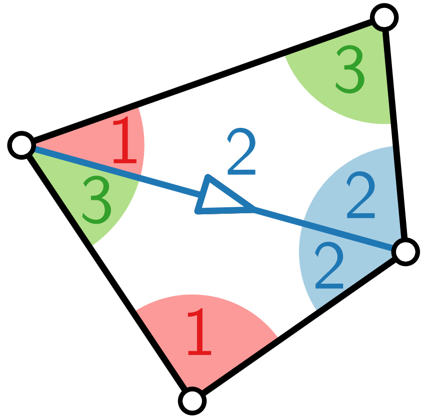

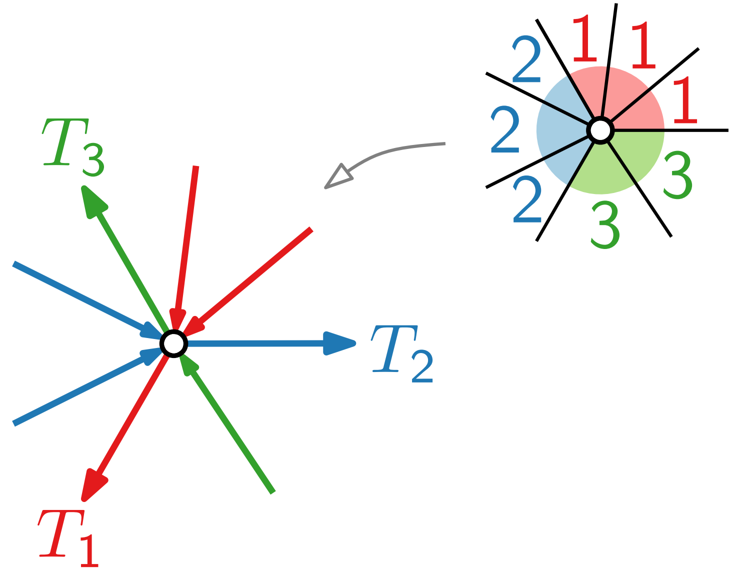

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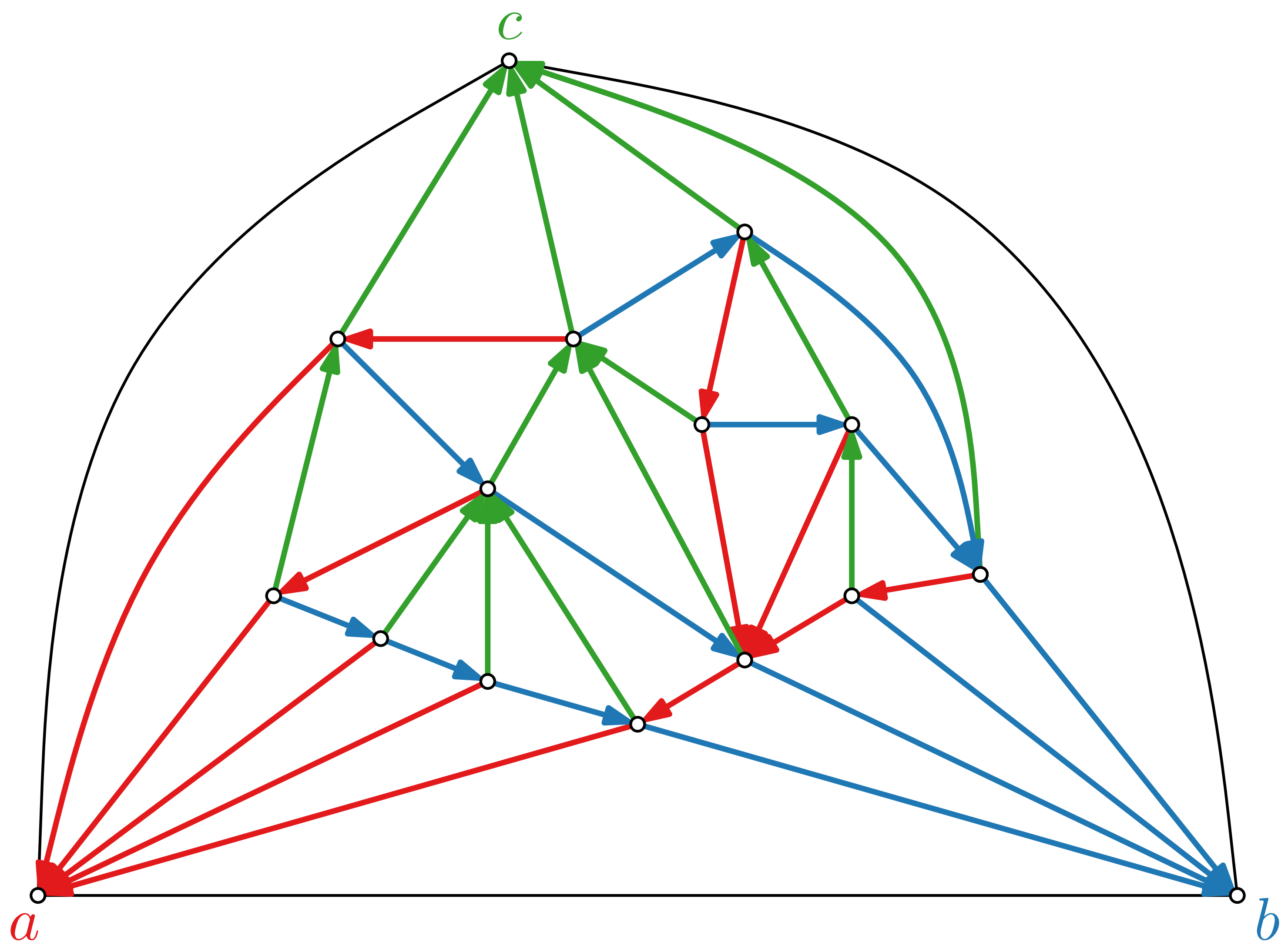
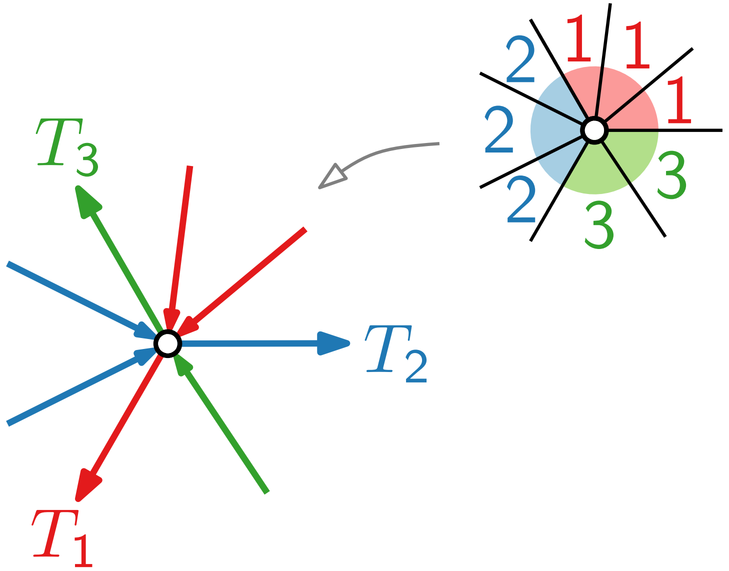


- All inner edges incident to a, b, and c are incoming in the same set (color).



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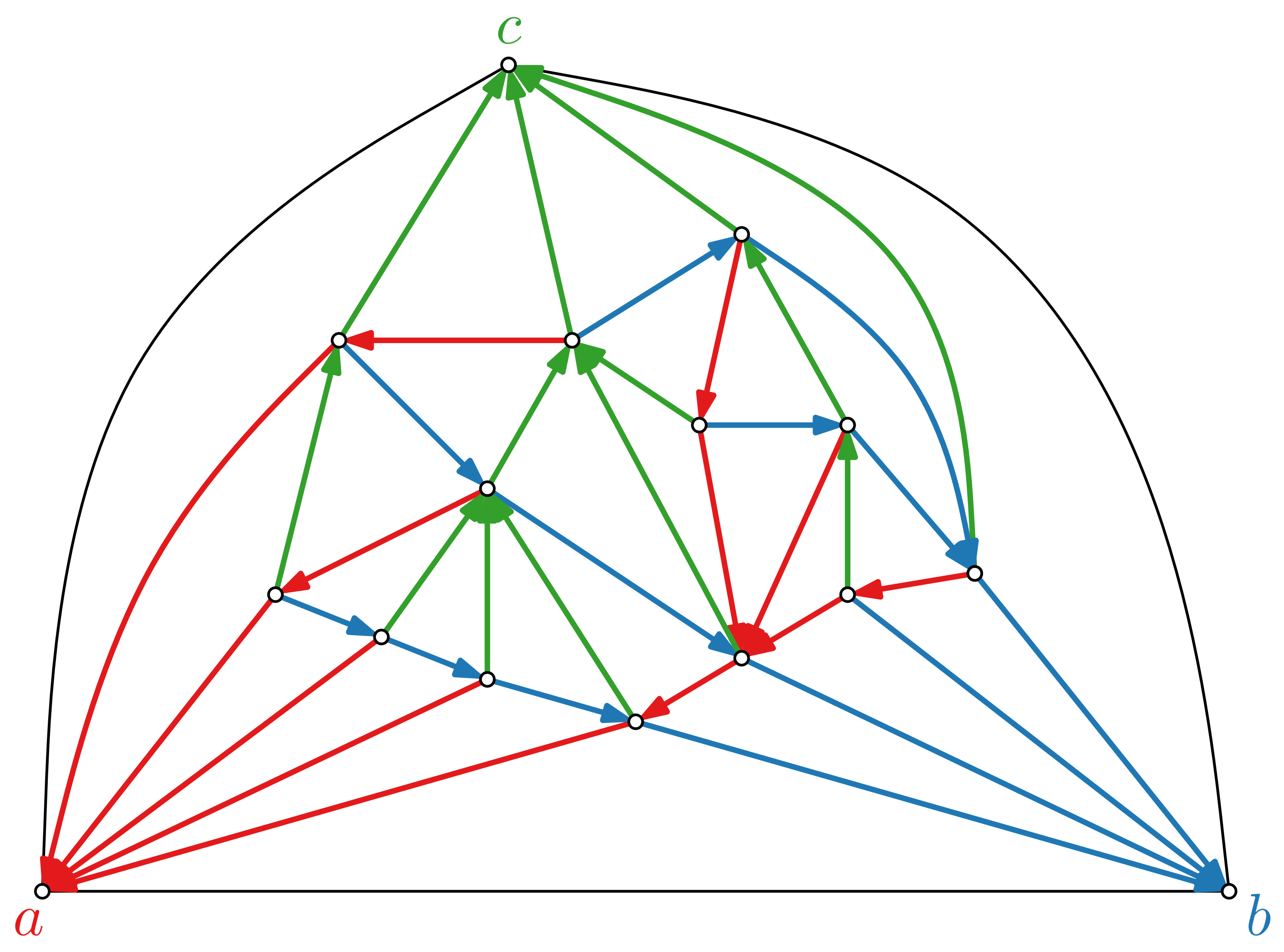

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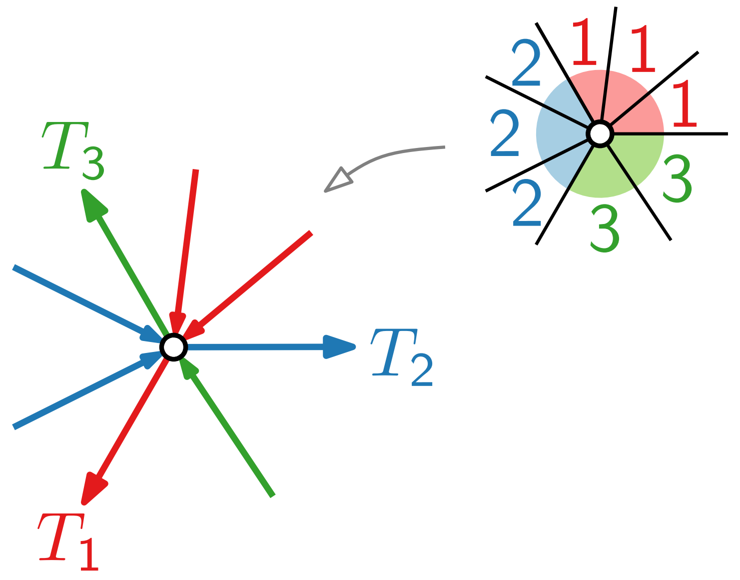
- T_1, T_2, and T_3 are trees.
  Each spans all inner vertices and one outer vertex (its root).



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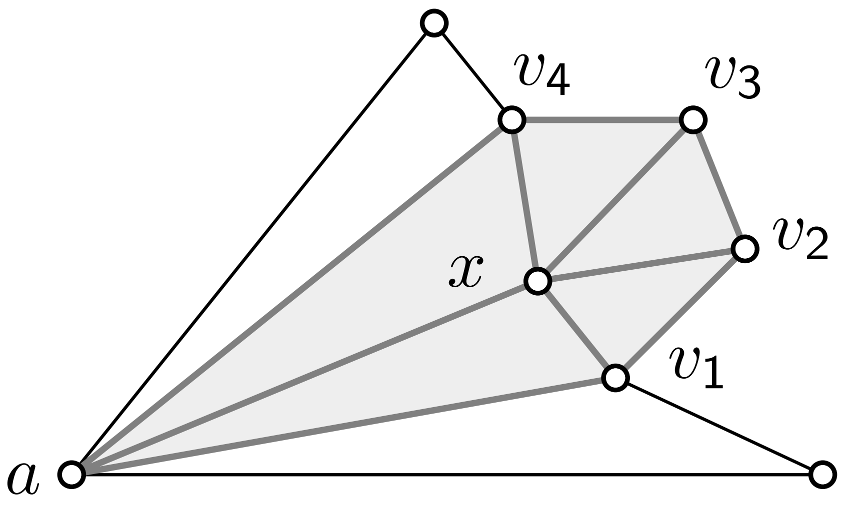

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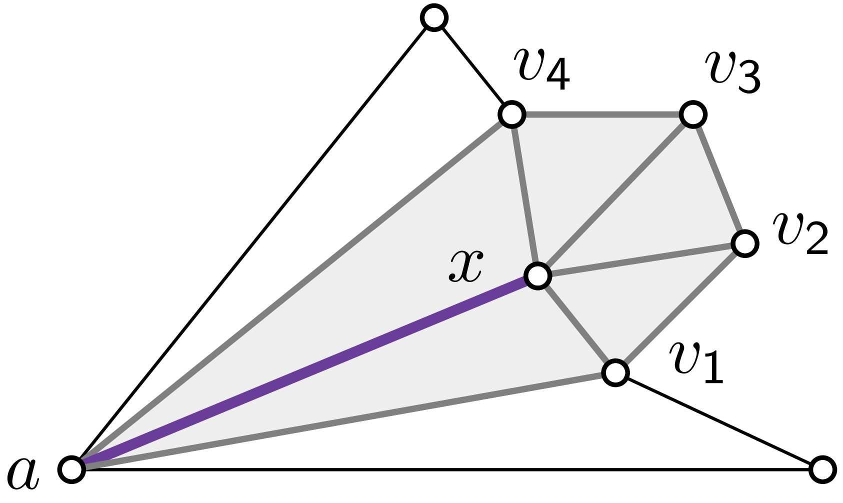

$\{a, x\}$

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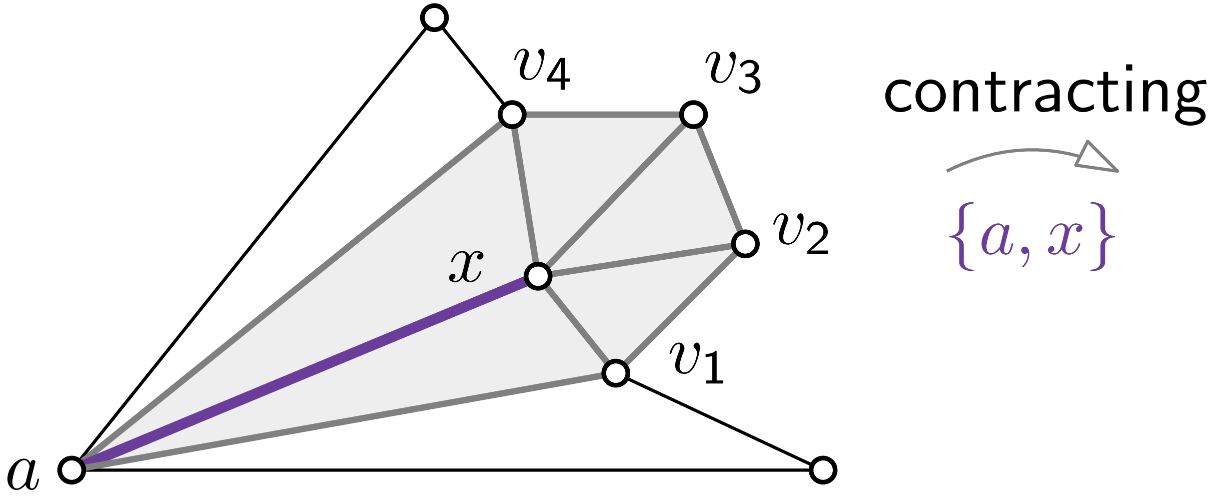

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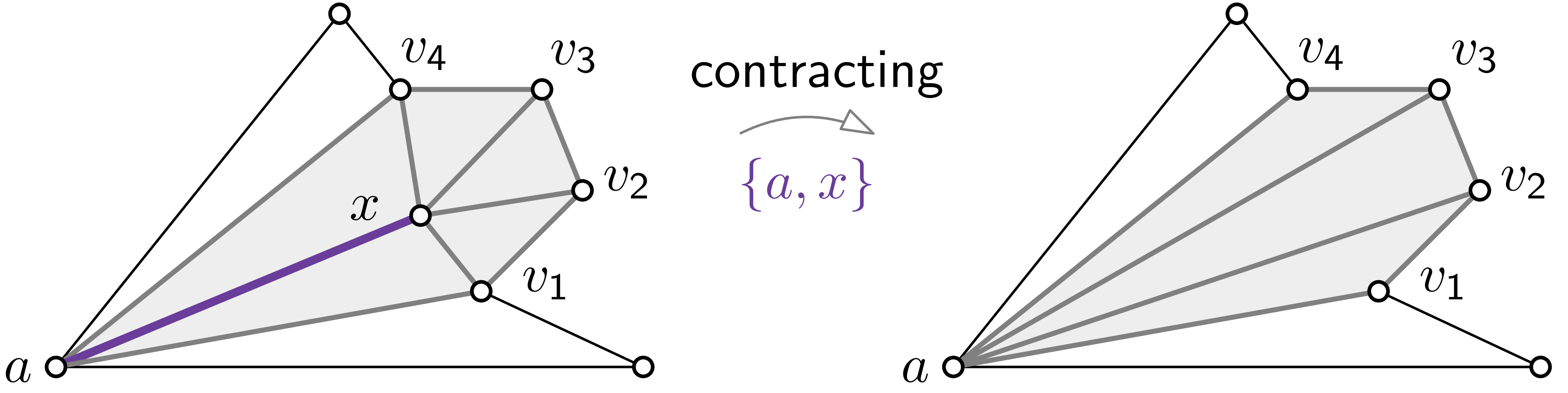

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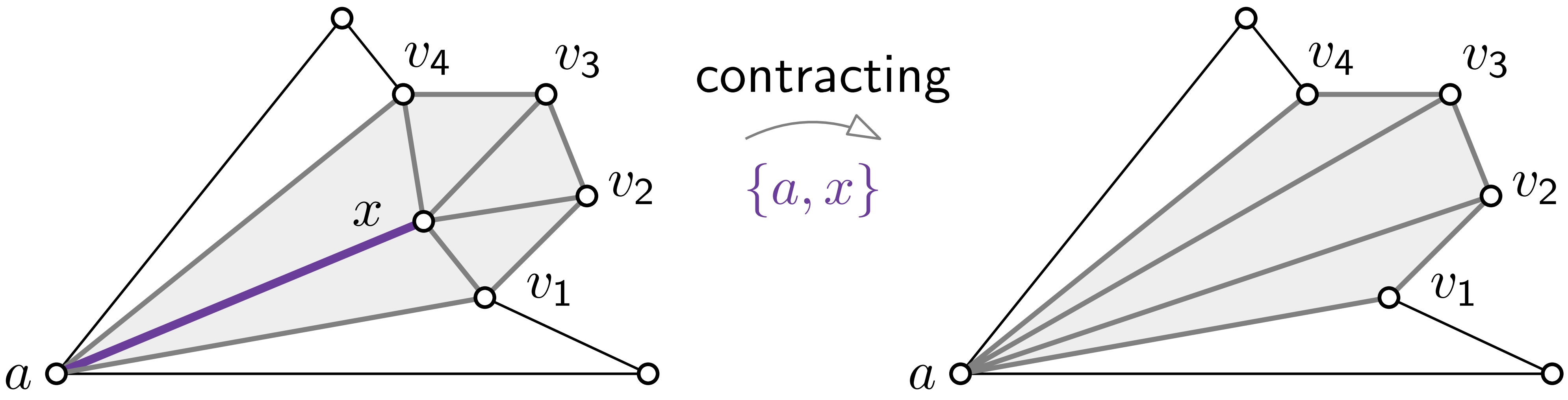

. . . requires that a and x have exactly two common neighbors.

Schnyder Wood – Existence

Lemma. [Kampen 1976]
Let G be a plane triangulation with vertices a, b, c on the outer face.
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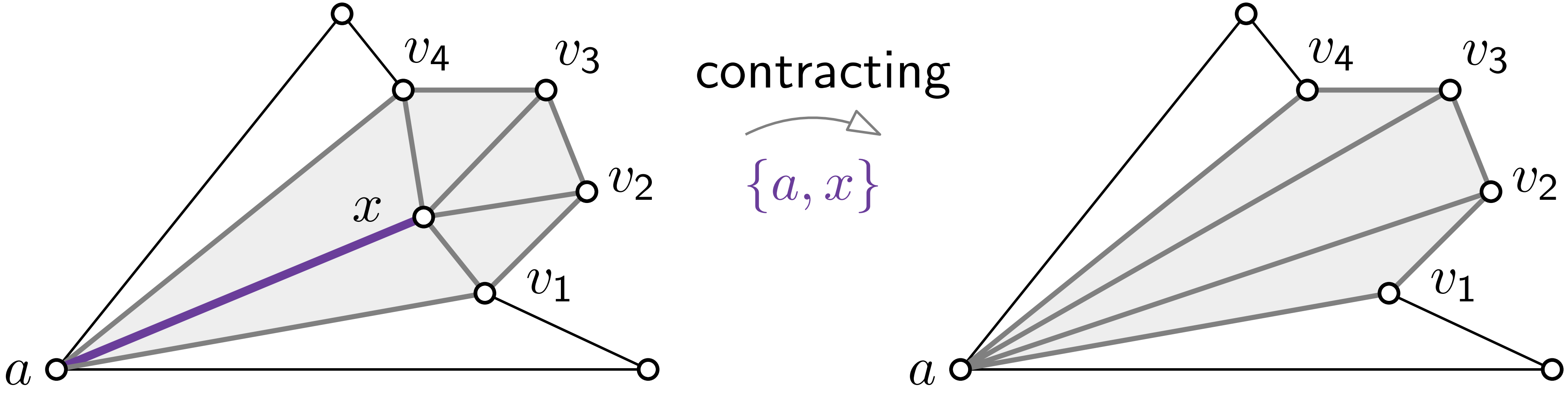

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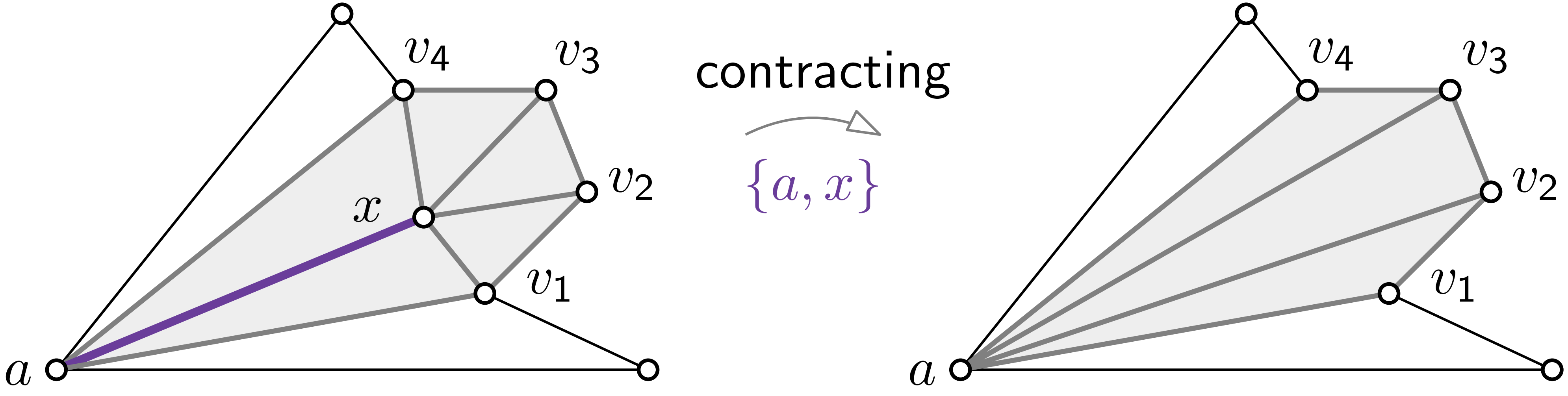

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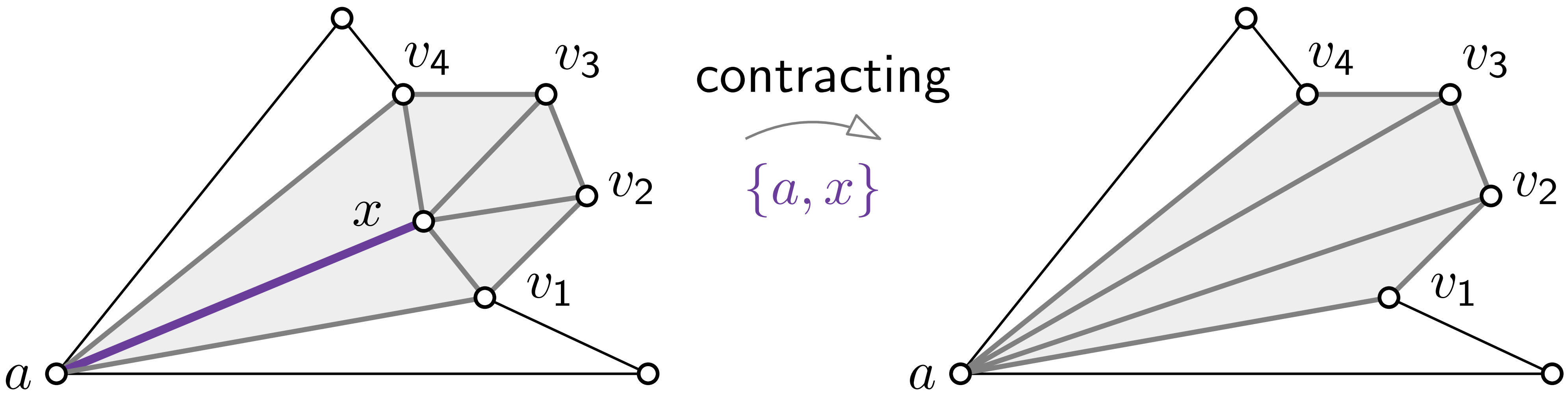

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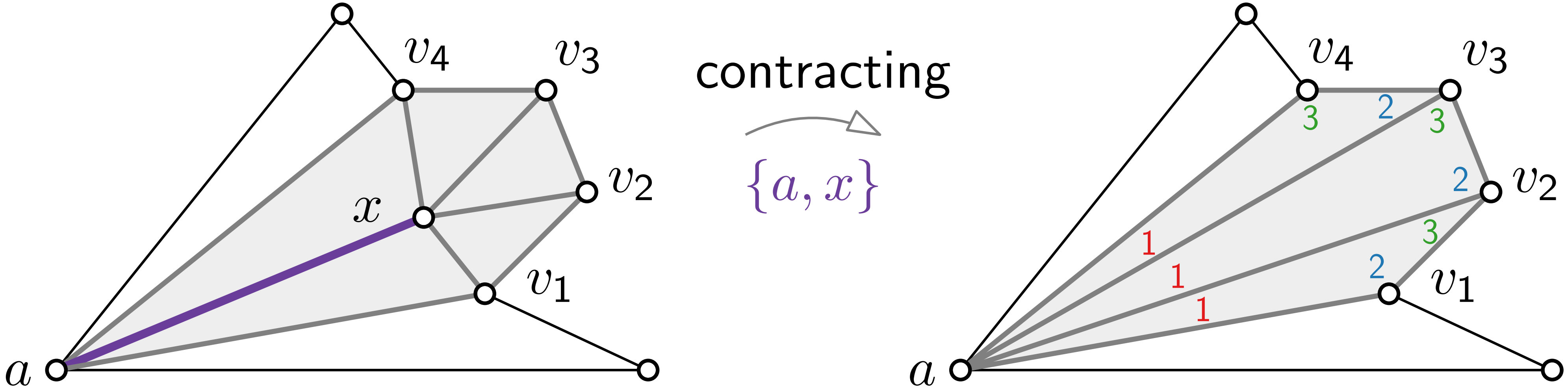

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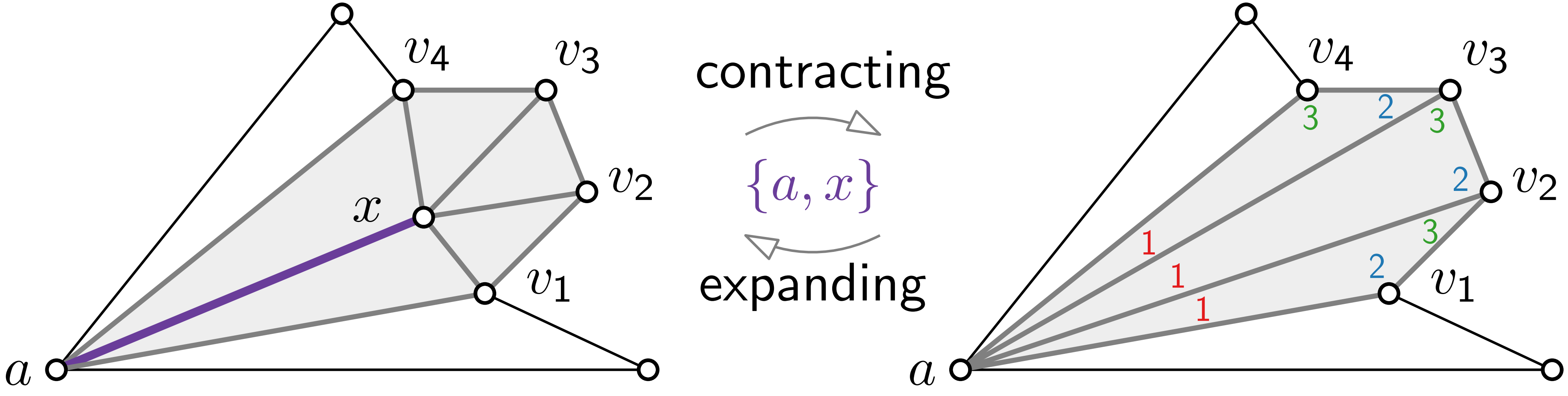

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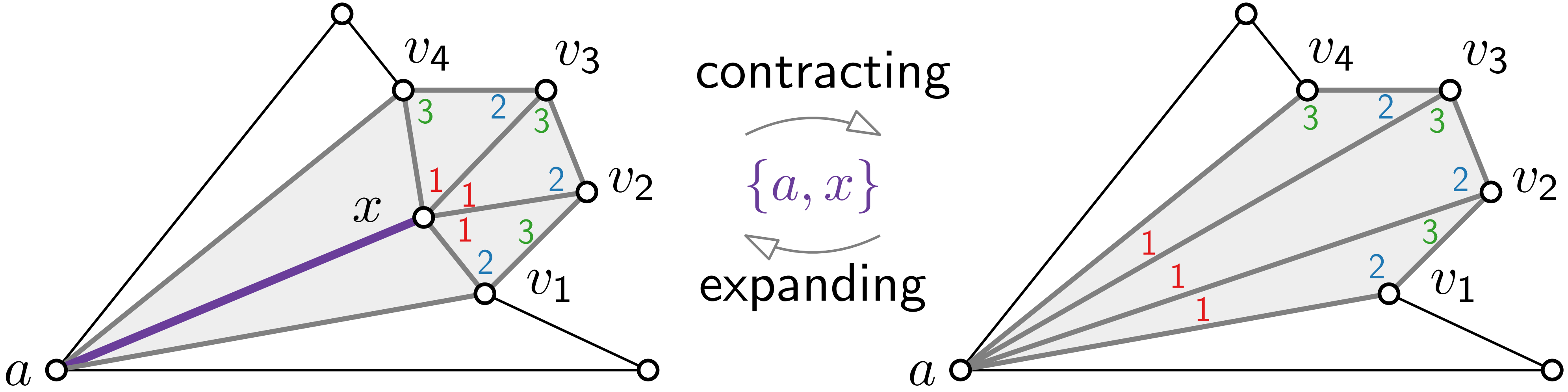

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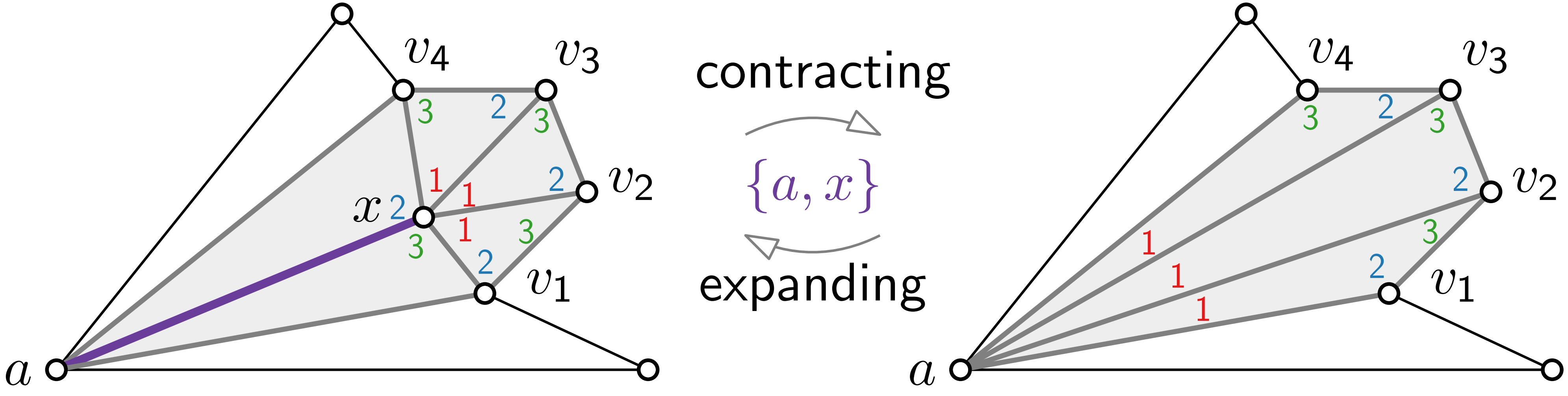

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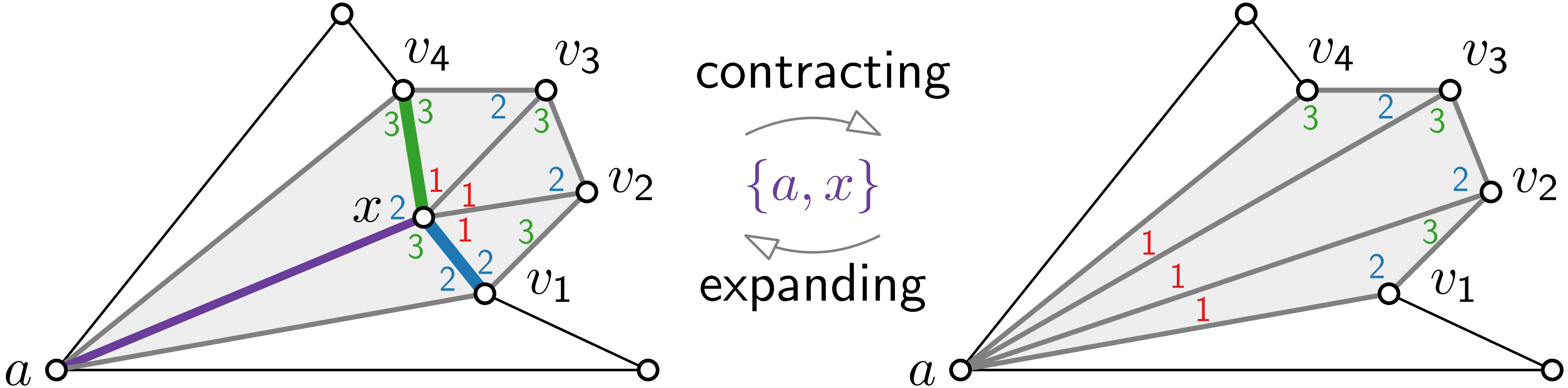

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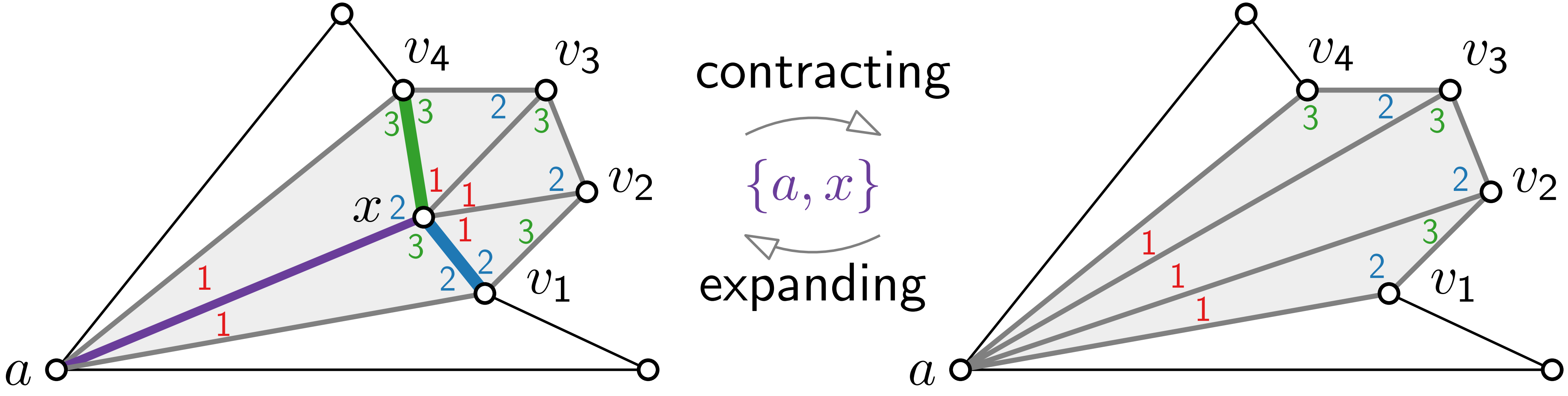

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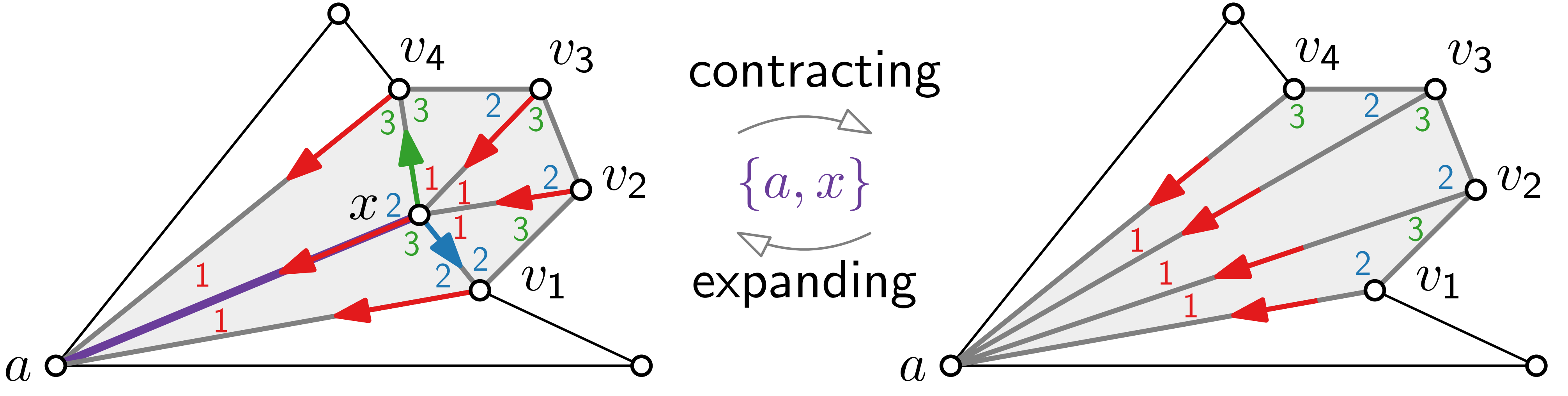

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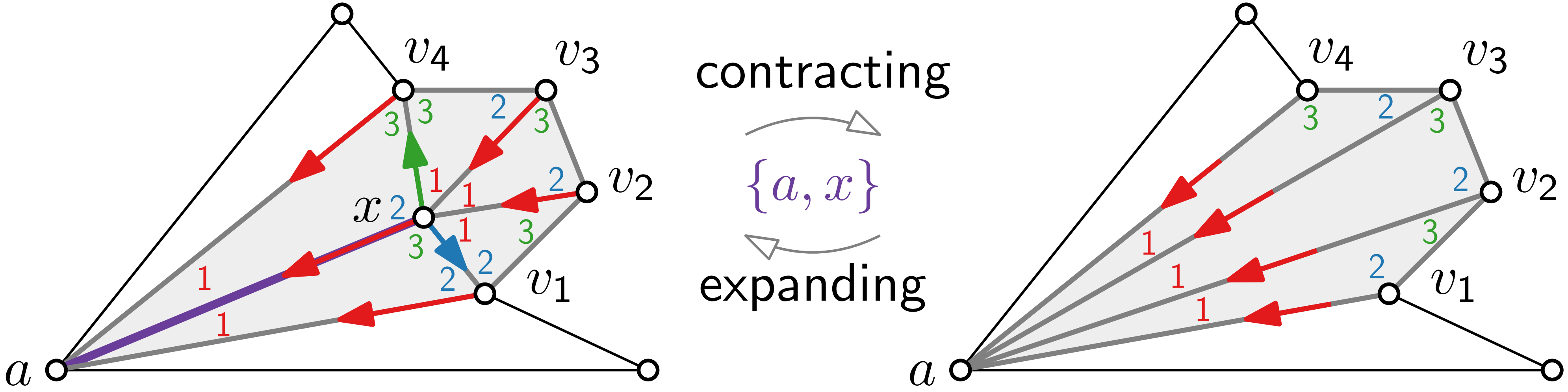

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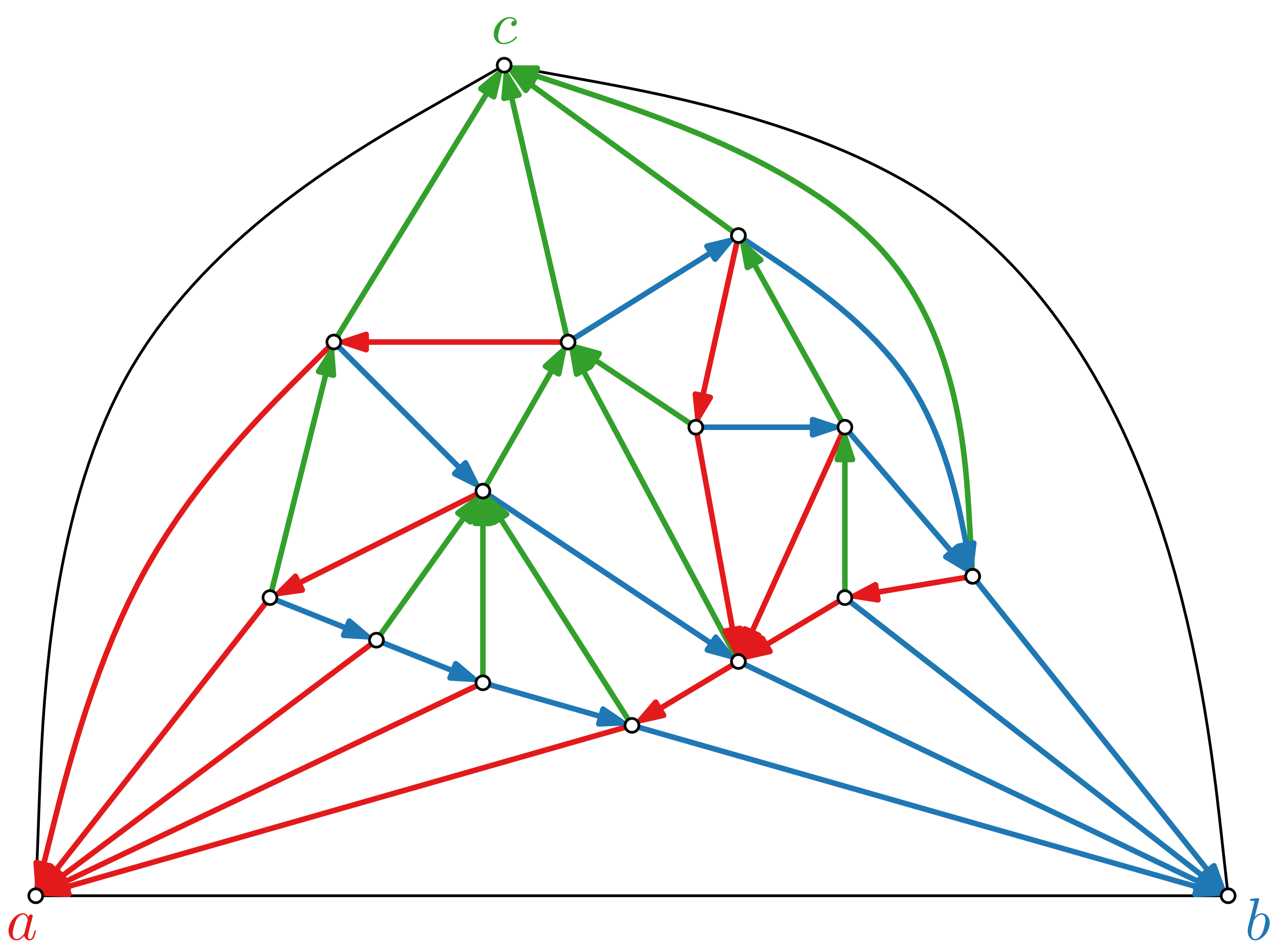
This constructive proof yields an algorithm for computing a Schnyder labeling. It can be implemented to run in $\mathcal{O}(n)$ time.
→ *Exercise* ☺

Schnyder Wood – More Properties



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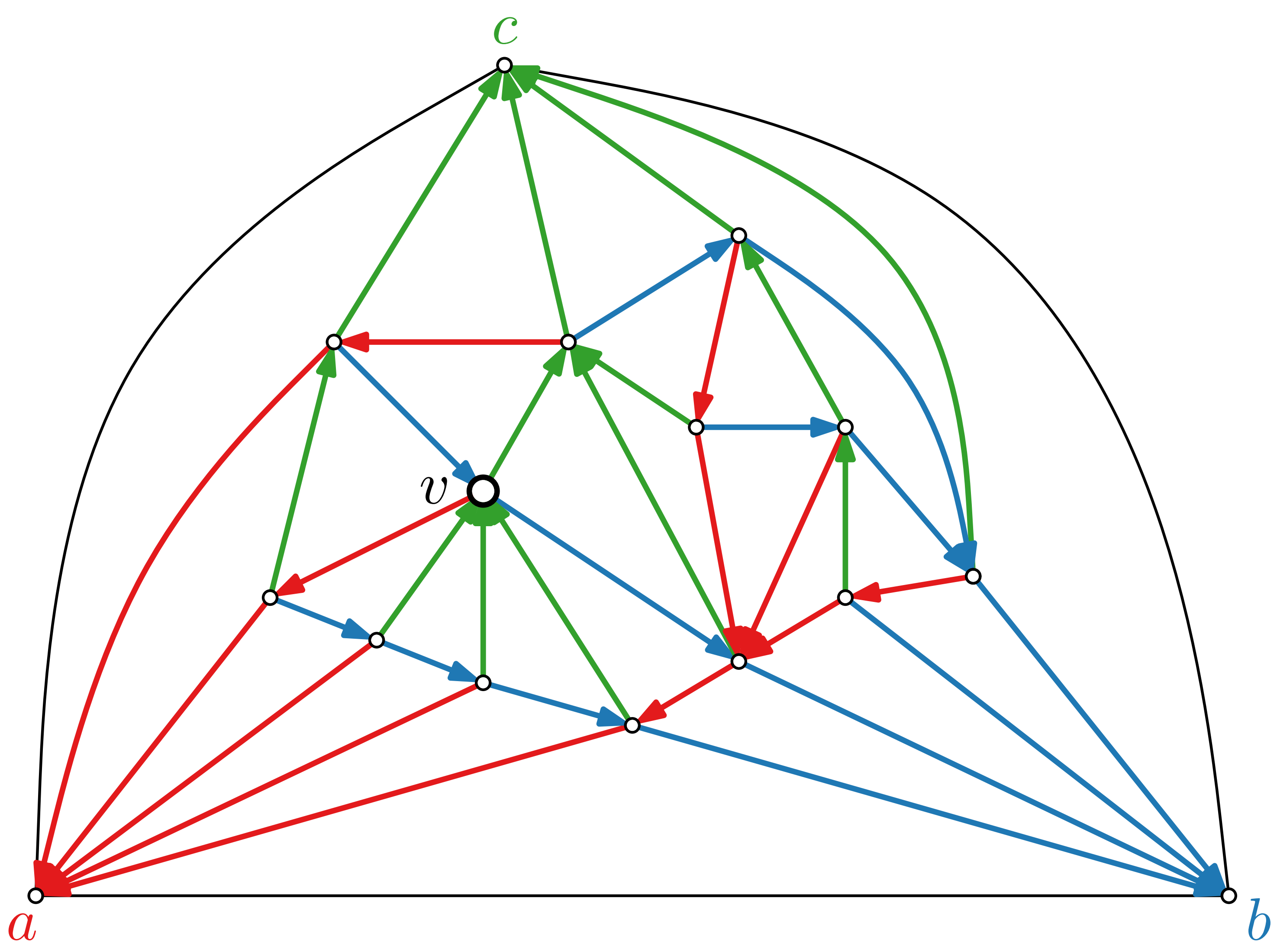

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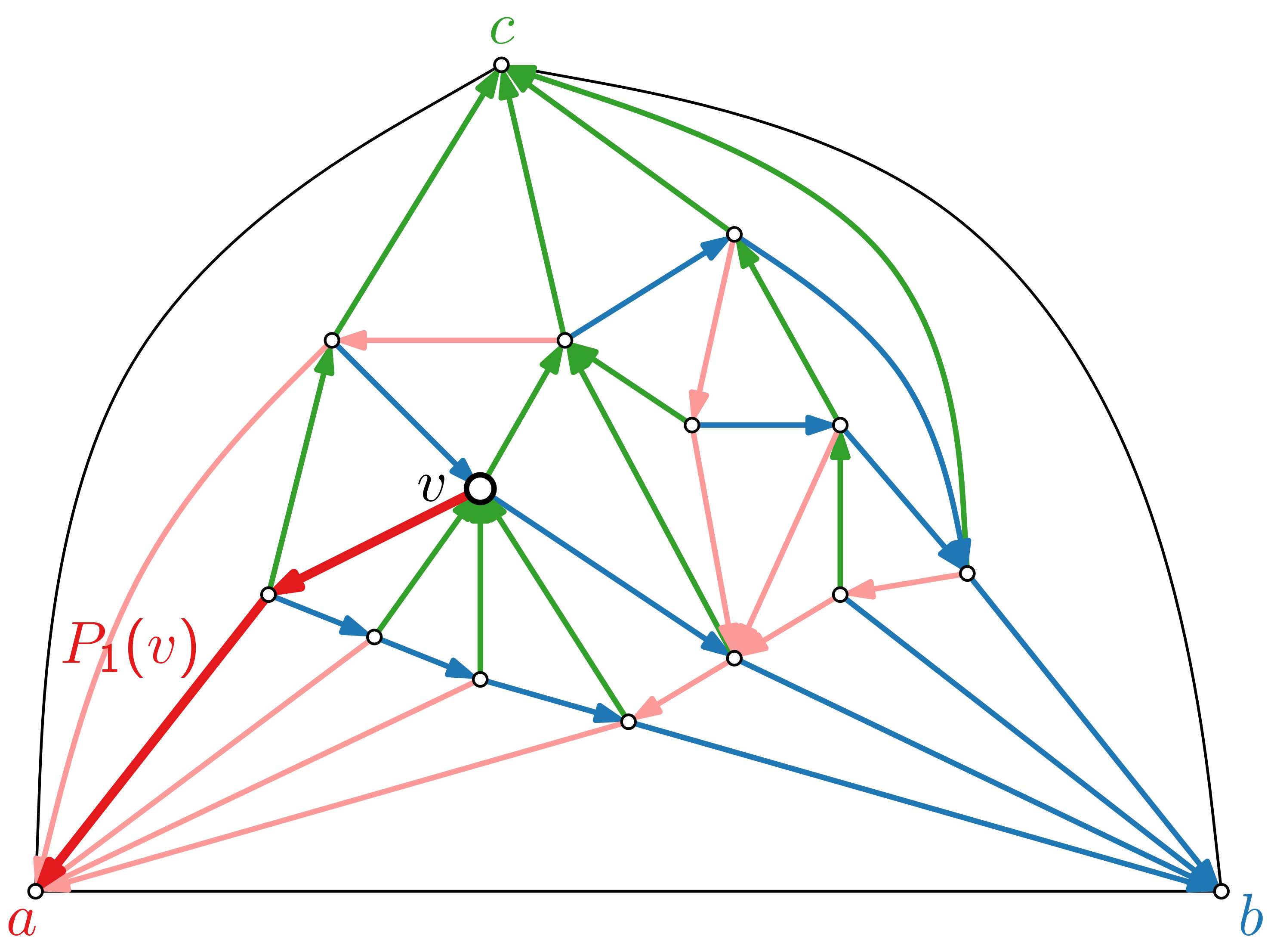

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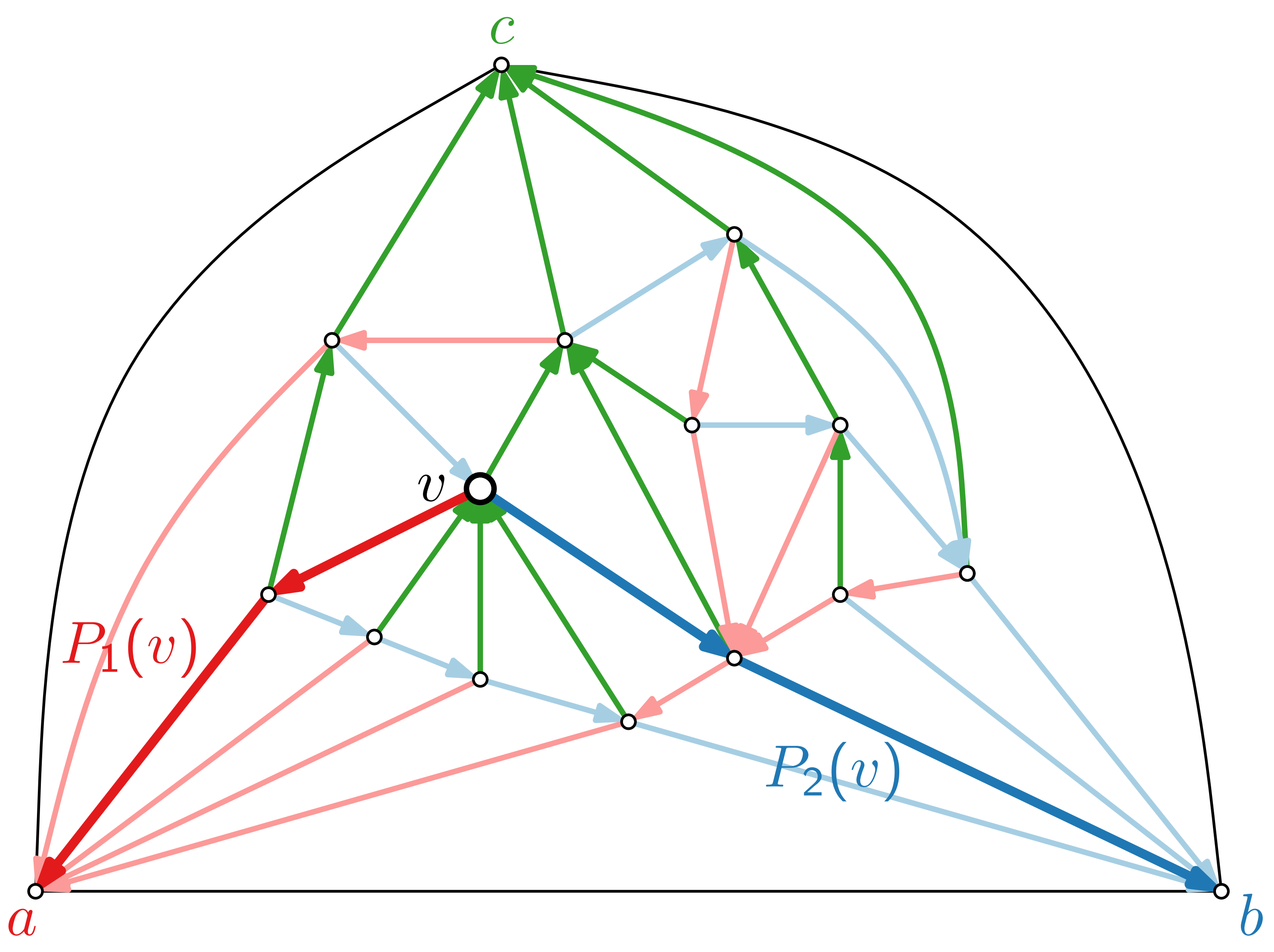

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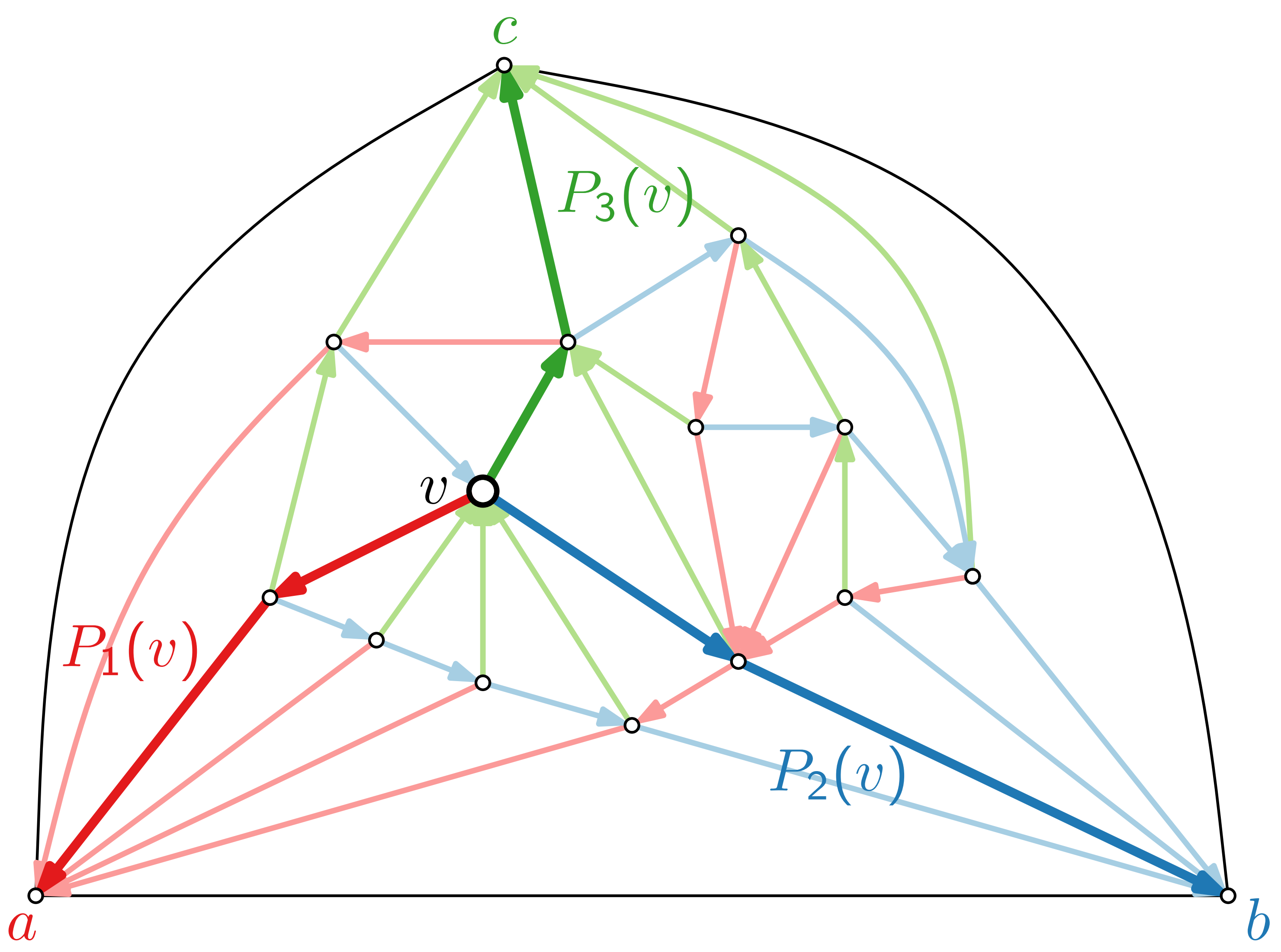

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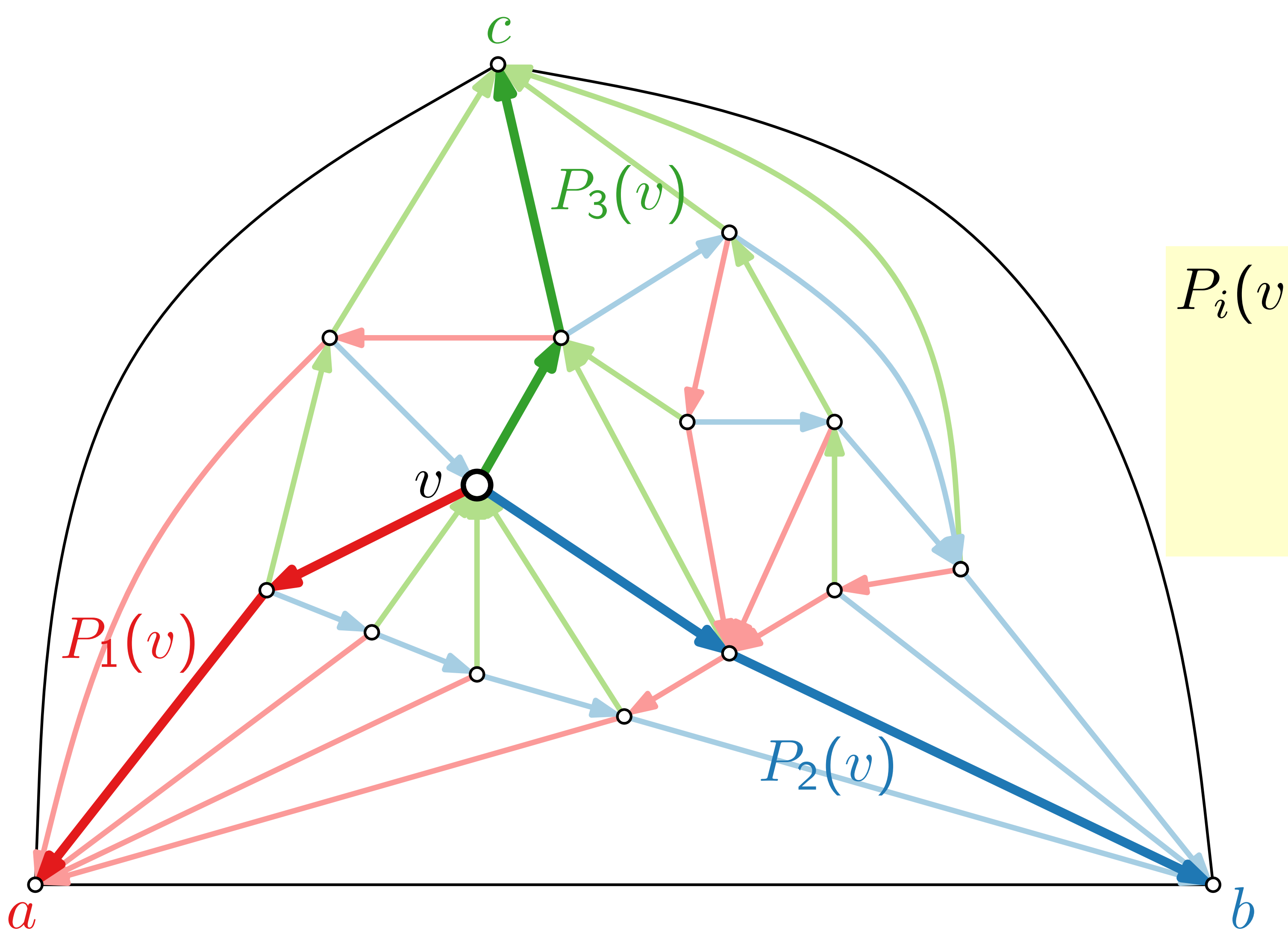

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$P_i(v)$: unique path from v to root of T_i

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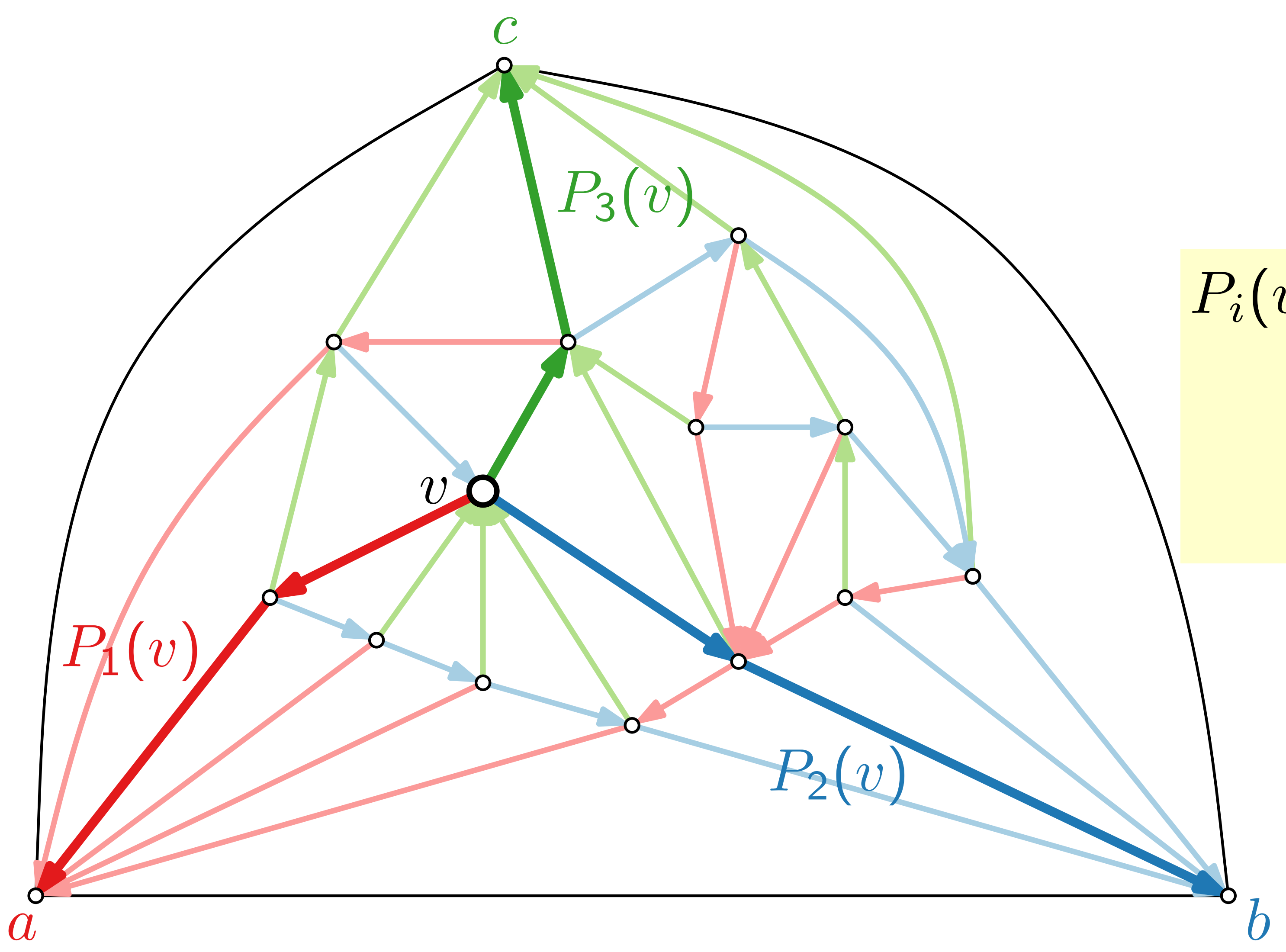

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Lemma.

- $P_1(v)$, $P_2(v)$, $P_3(v)$ cross only at v.

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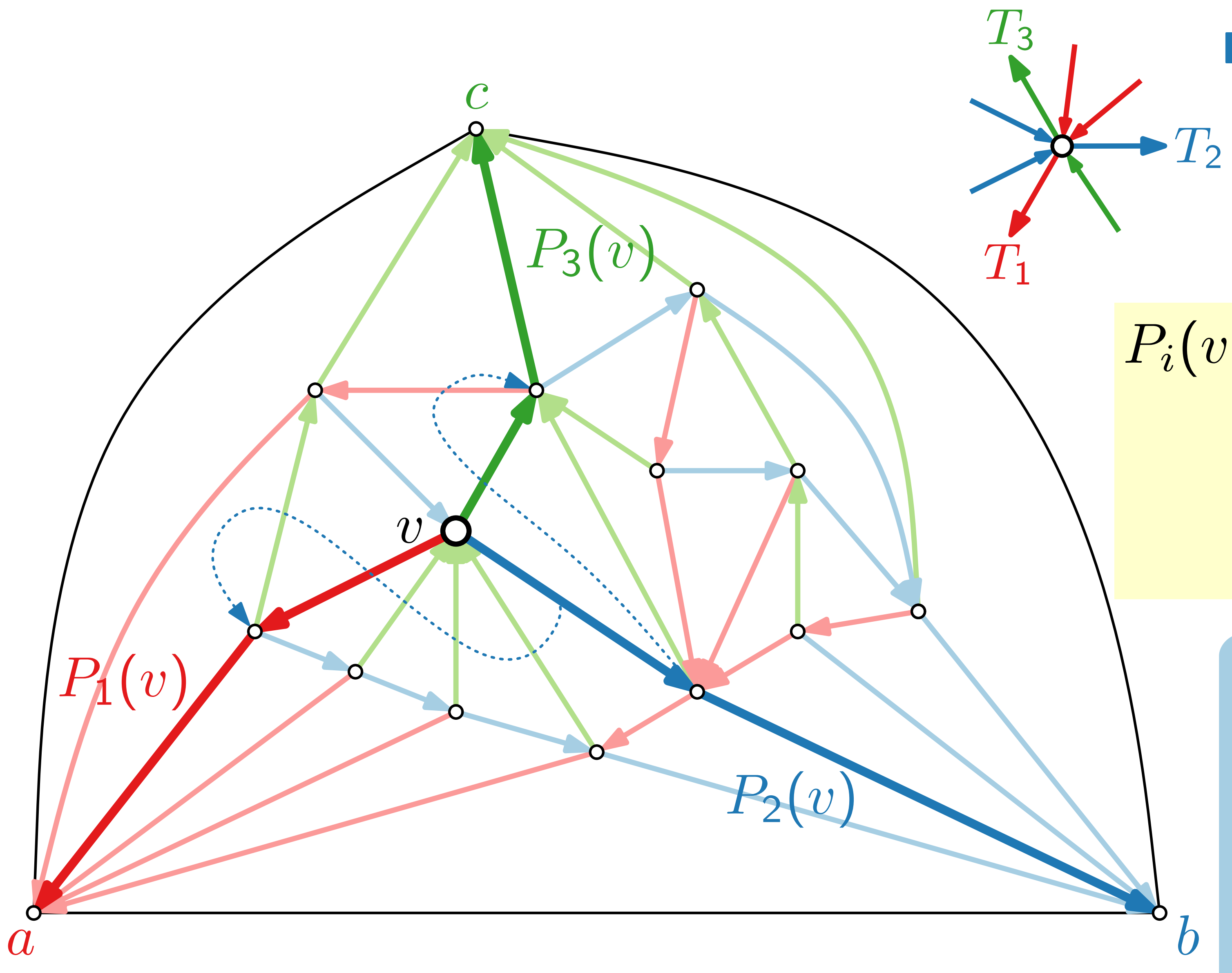

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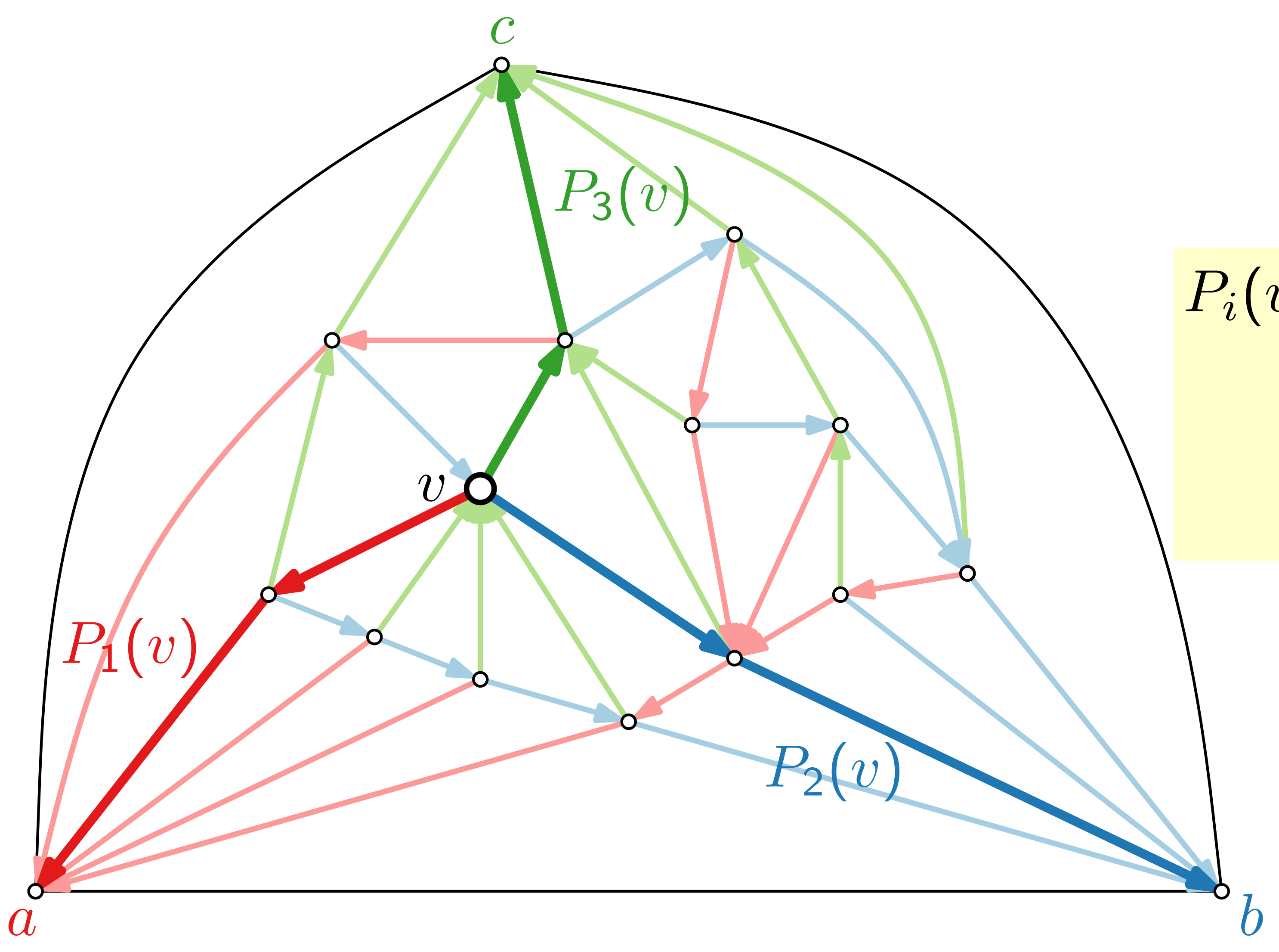

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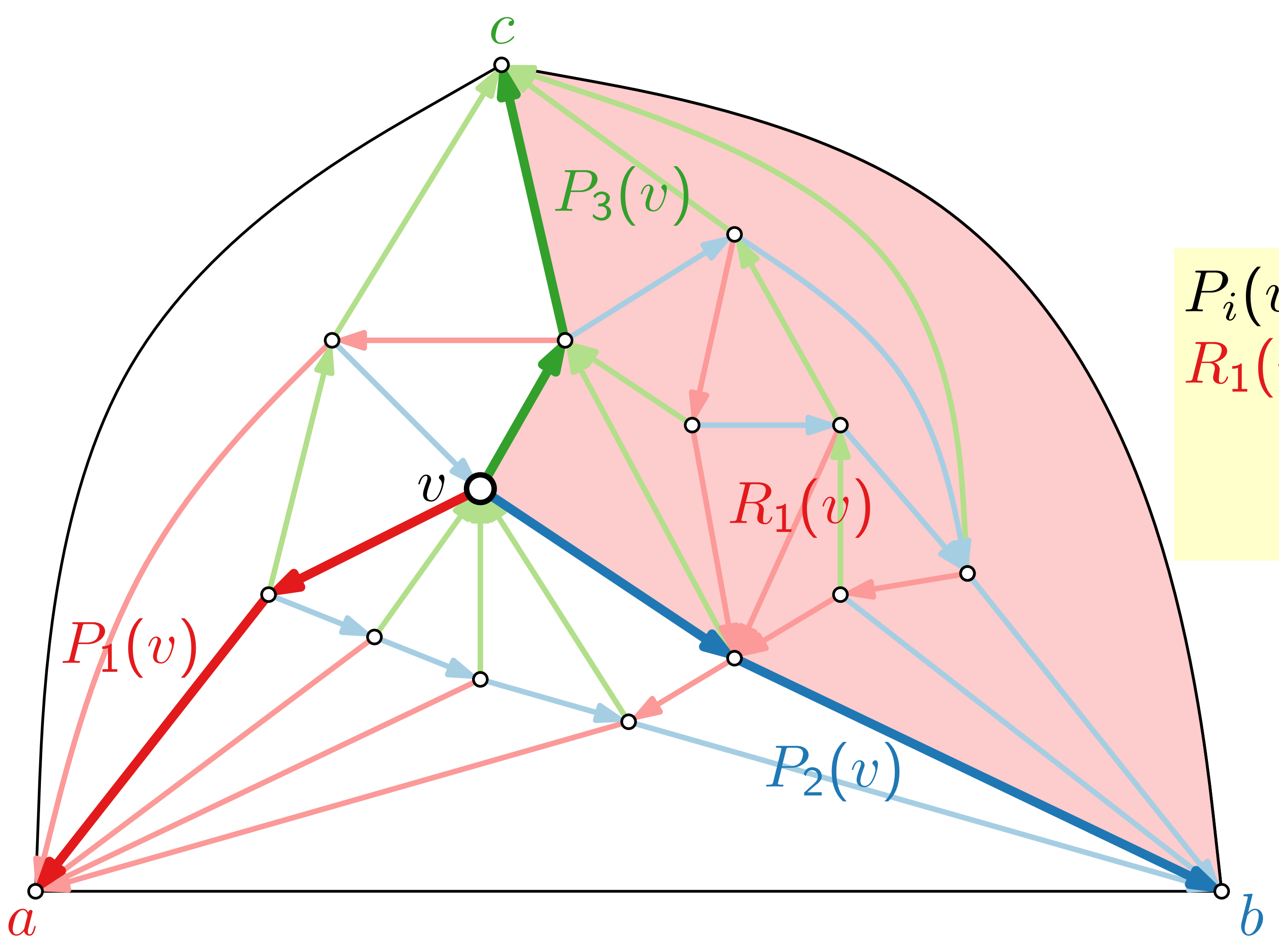

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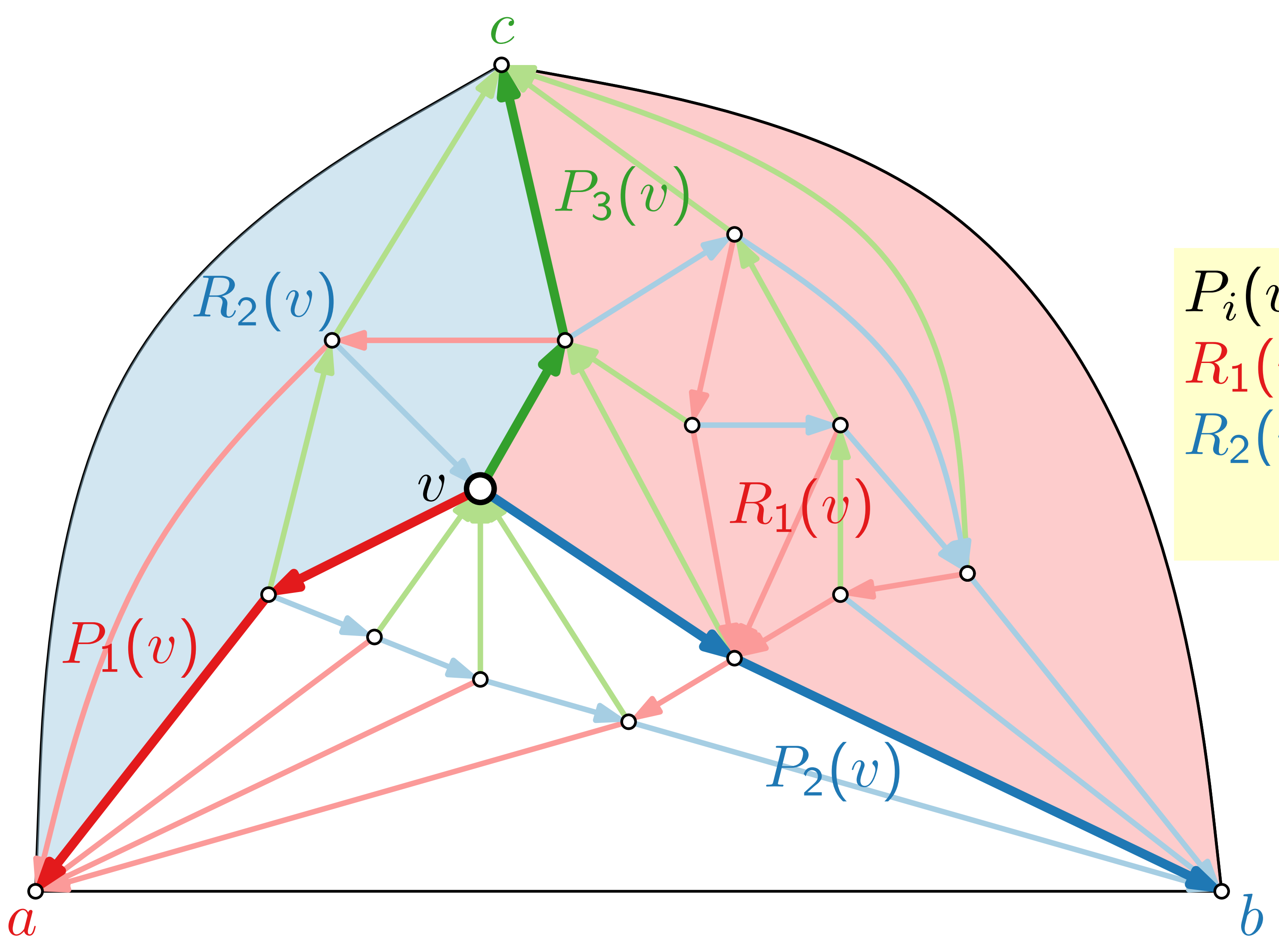

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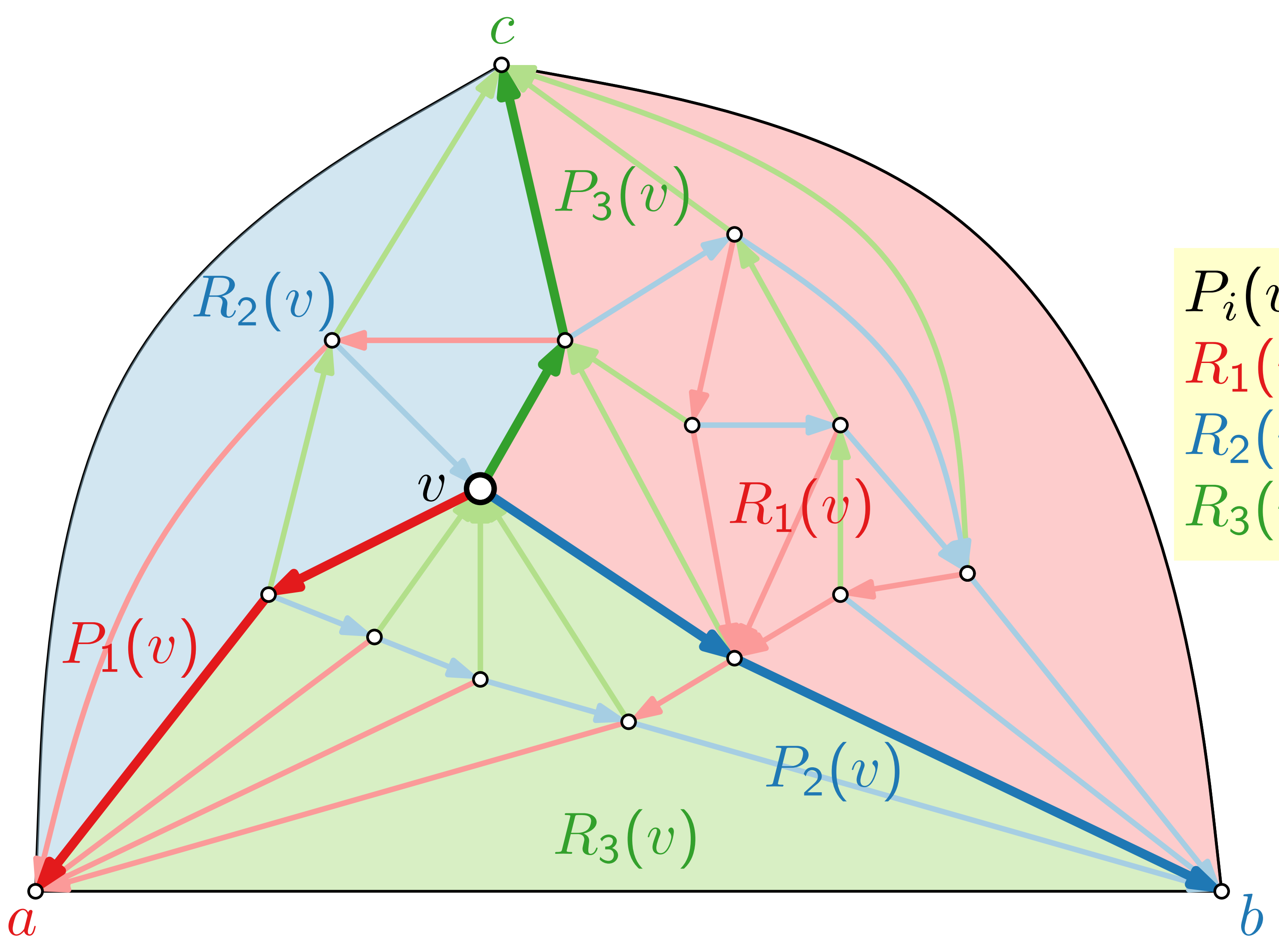

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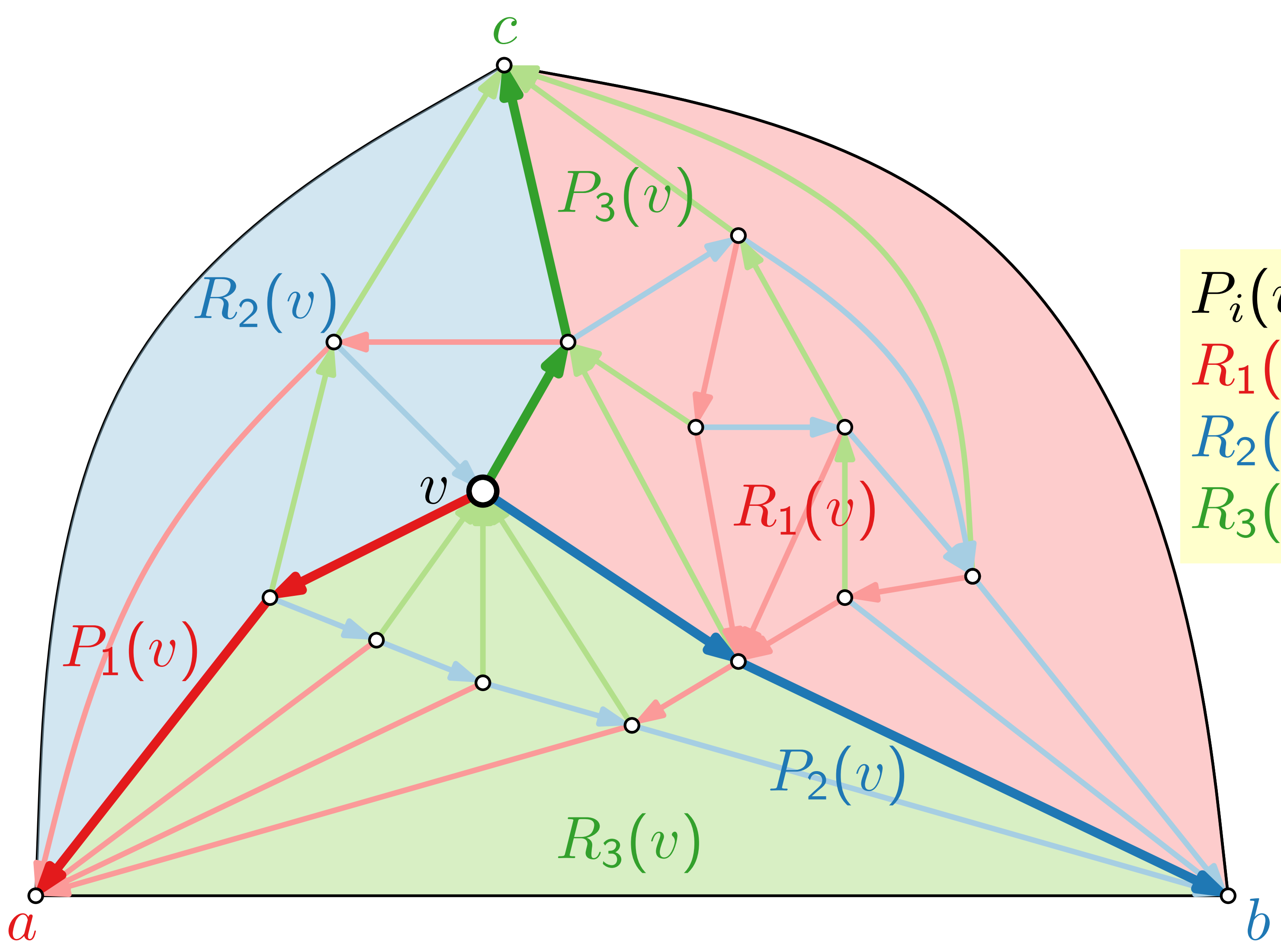

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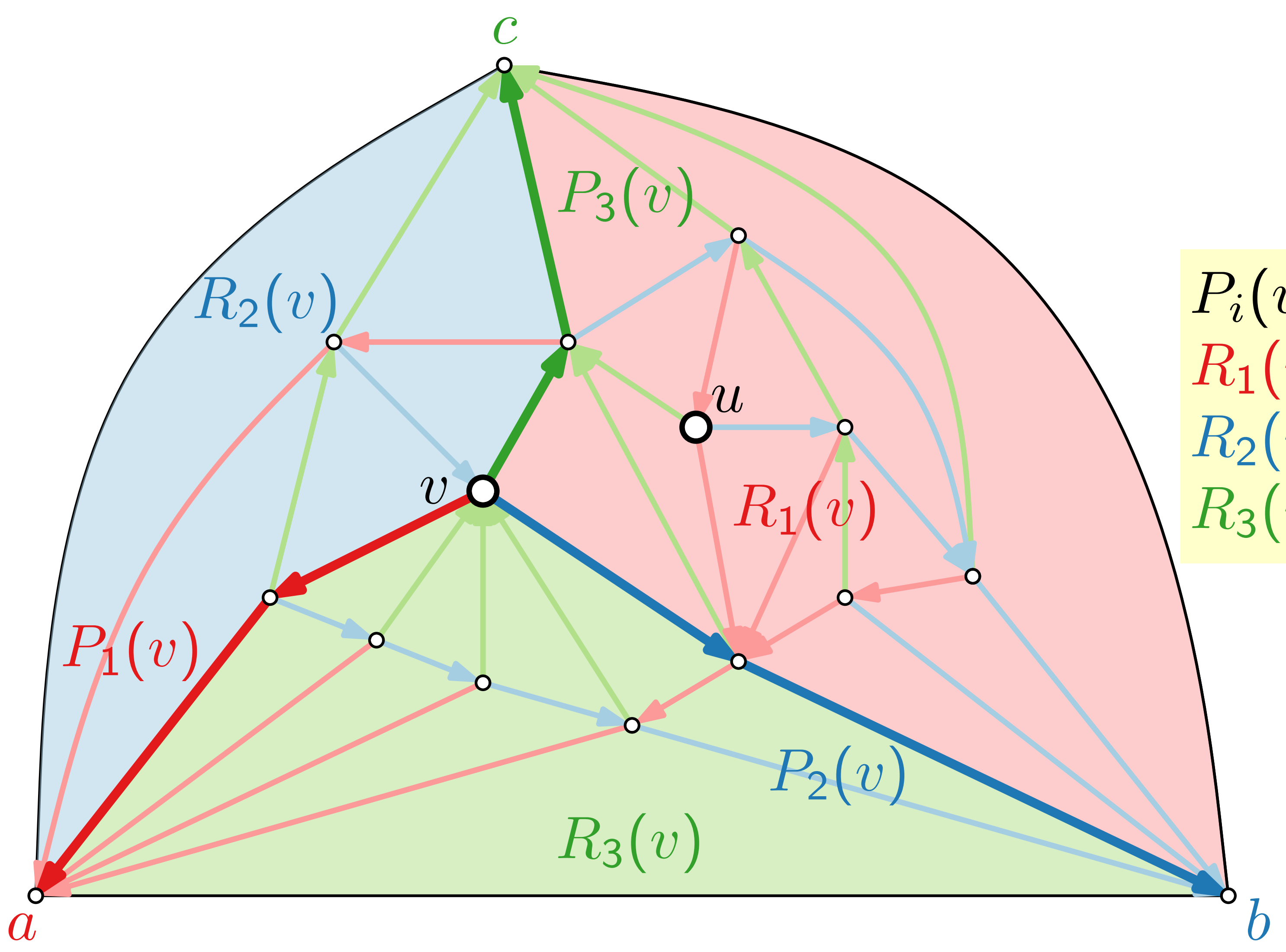

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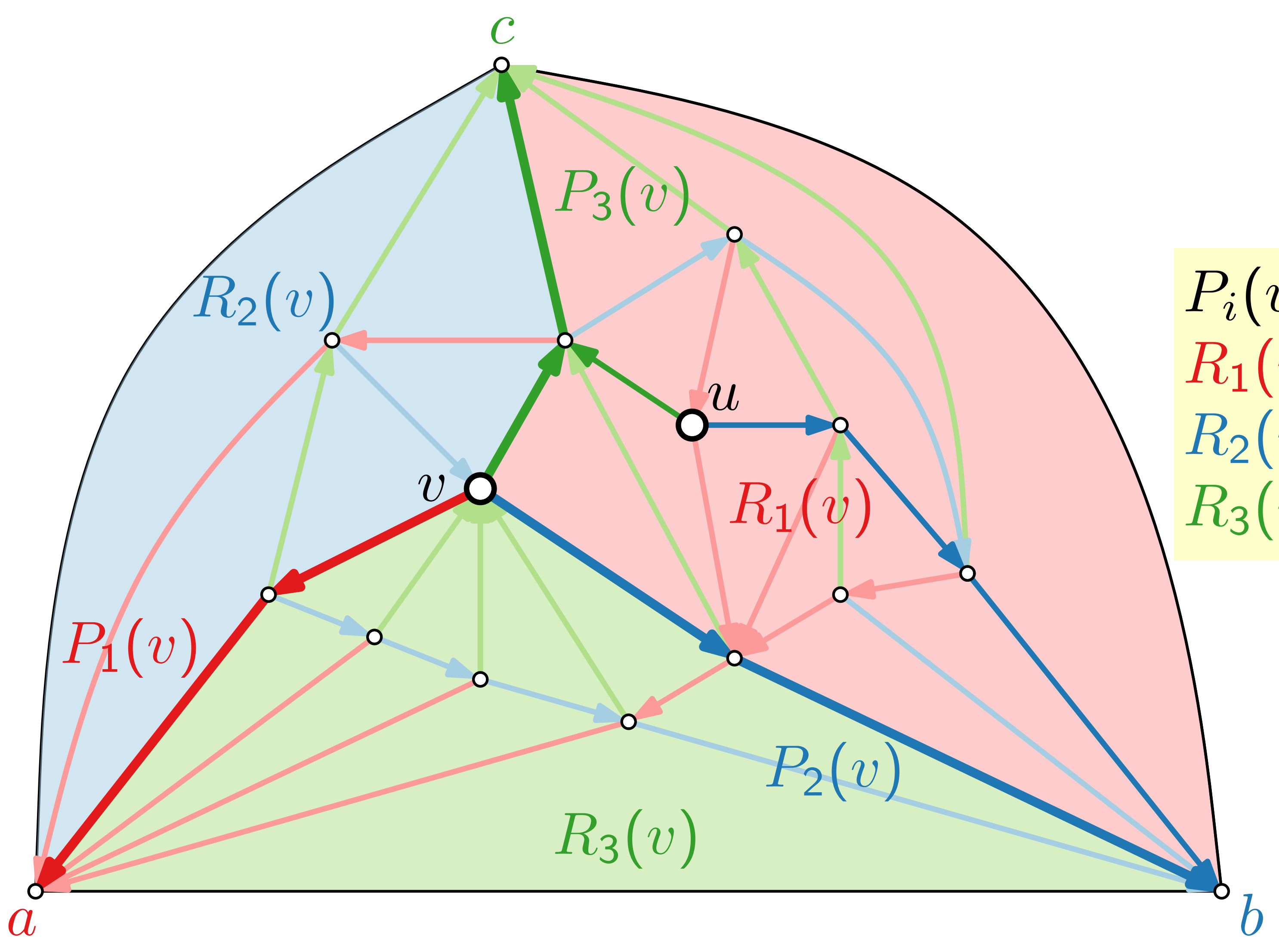

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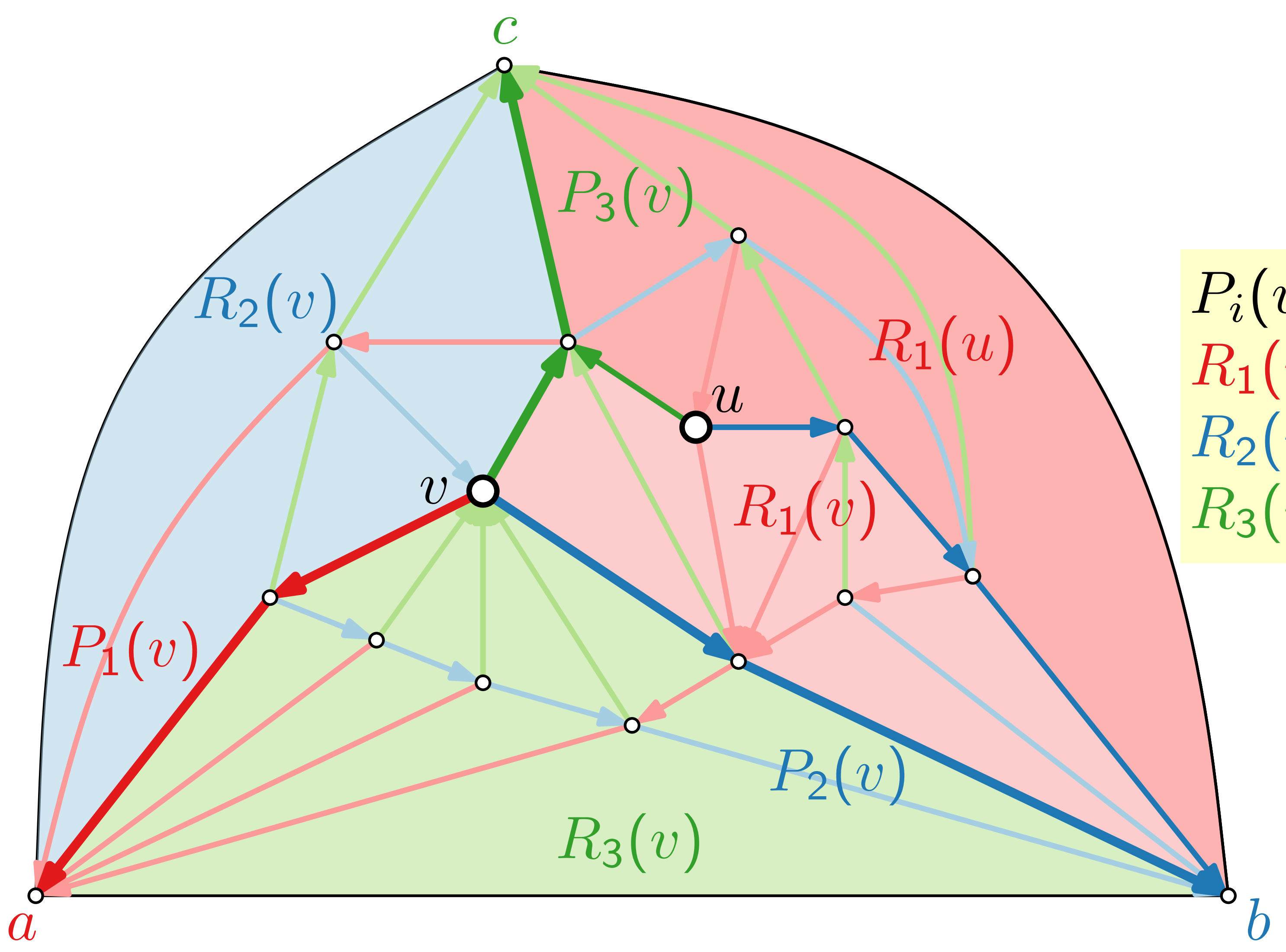

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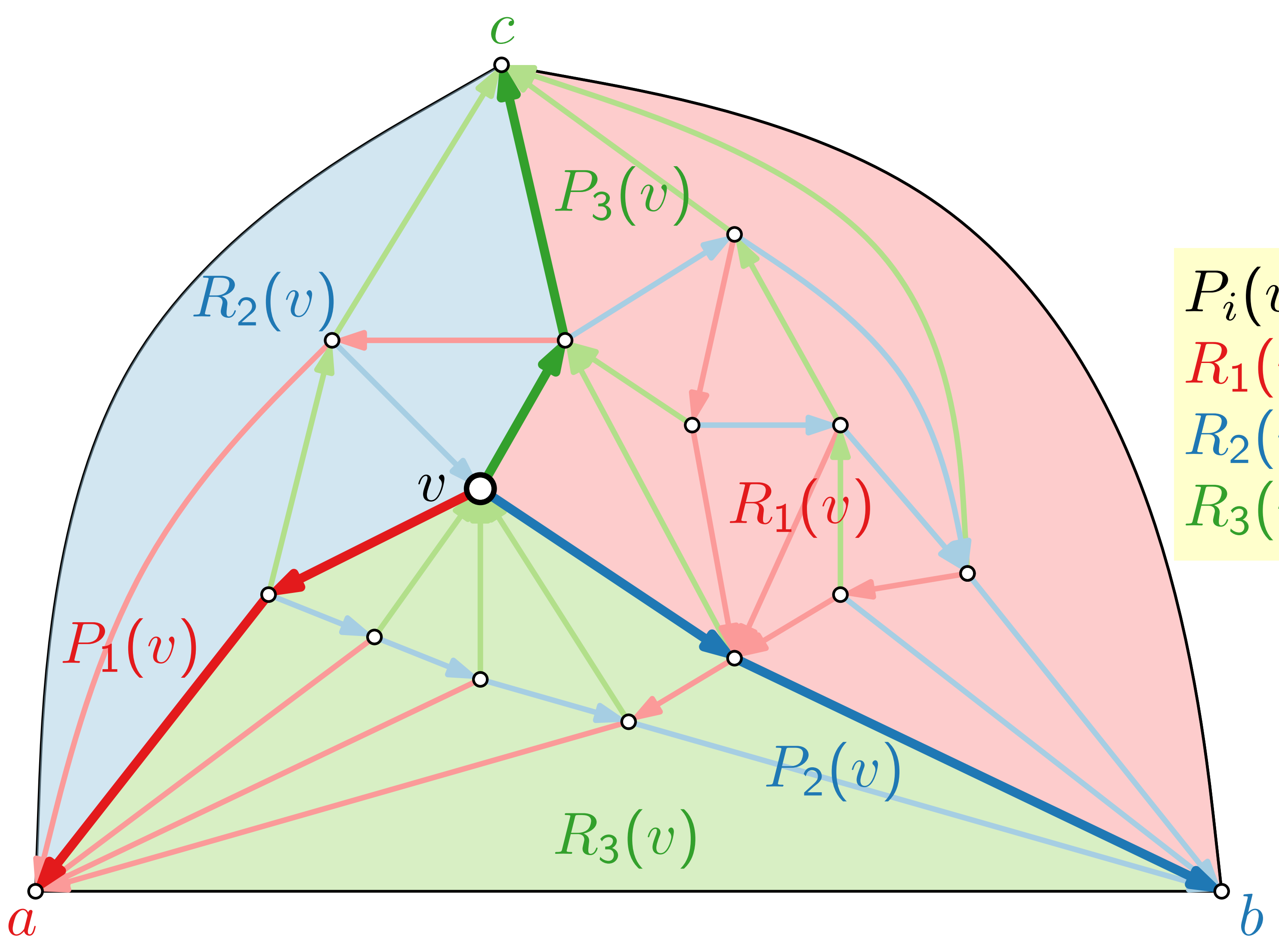

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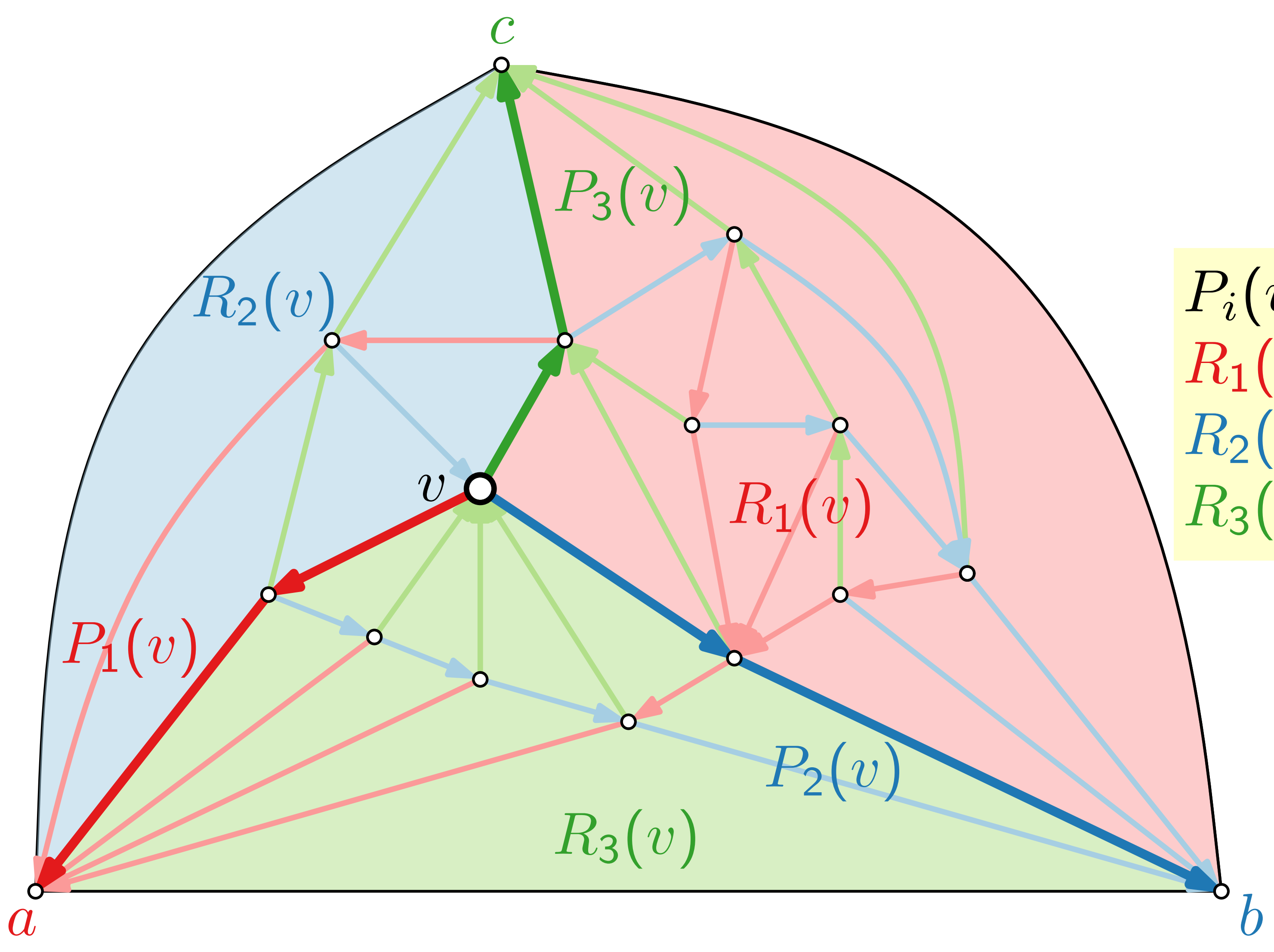

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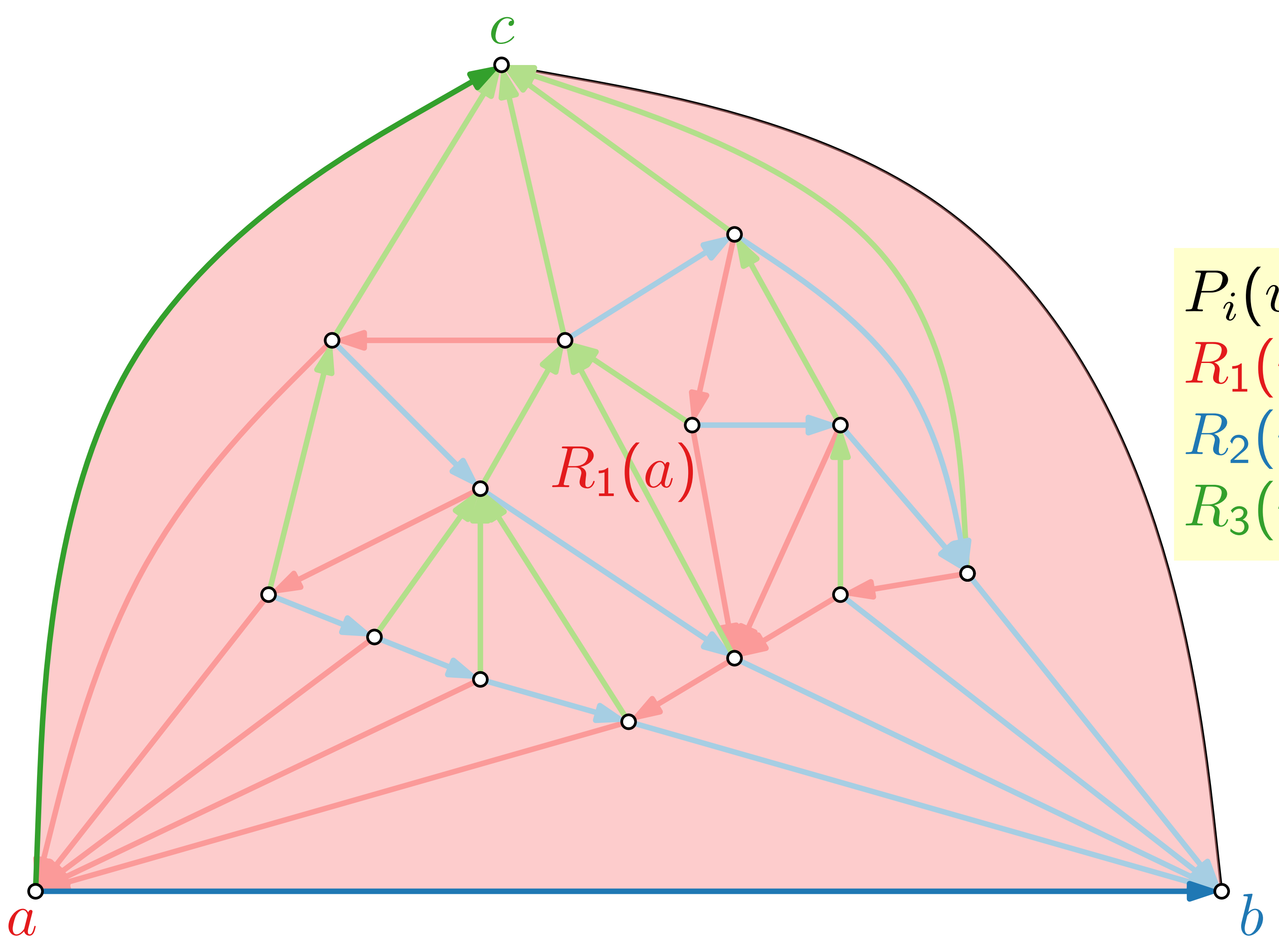

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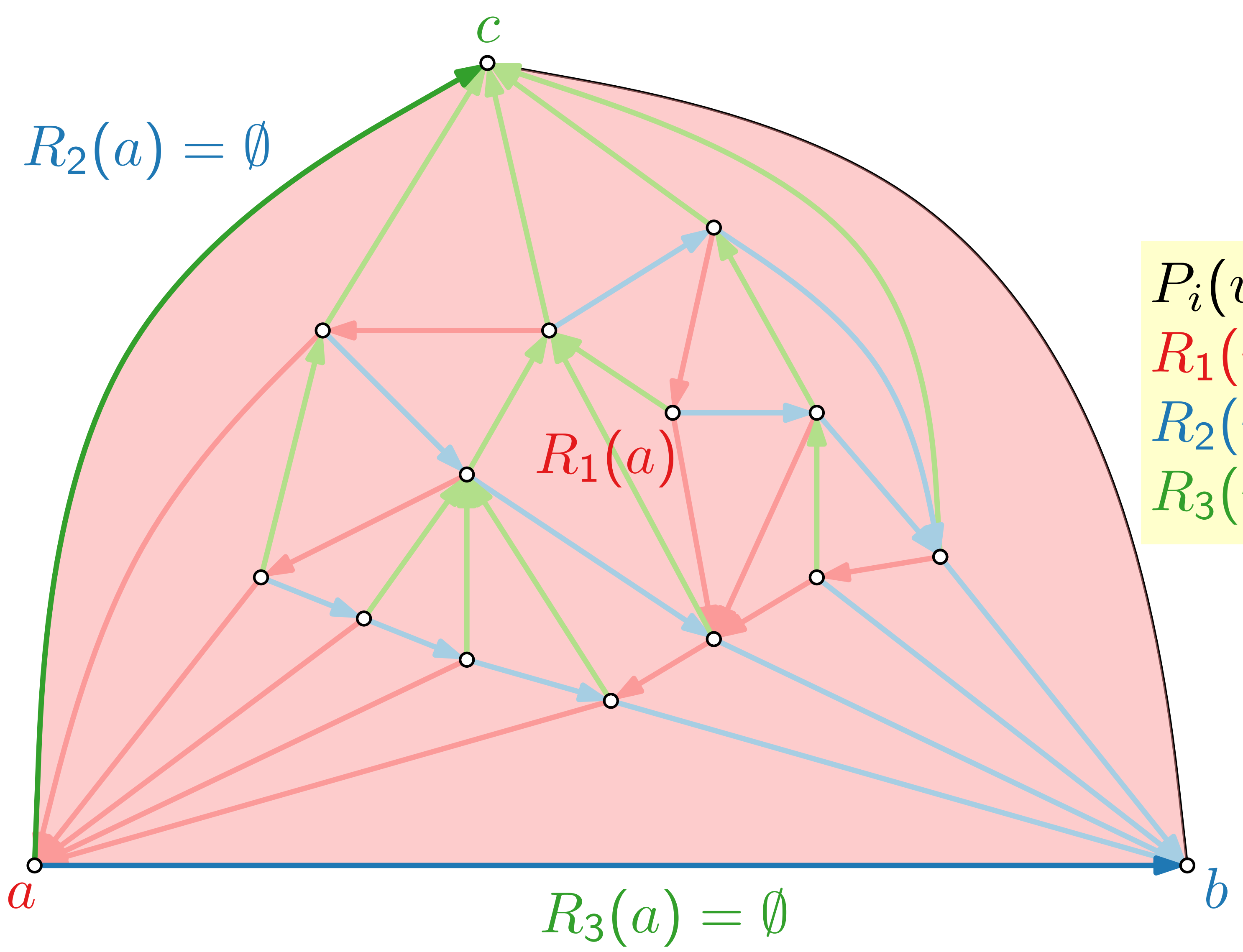

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Theorem. [Schnyder '90]
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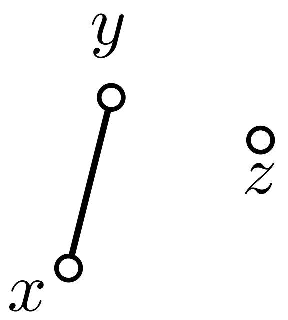

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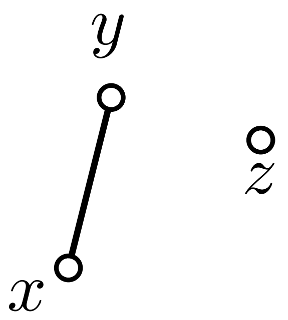

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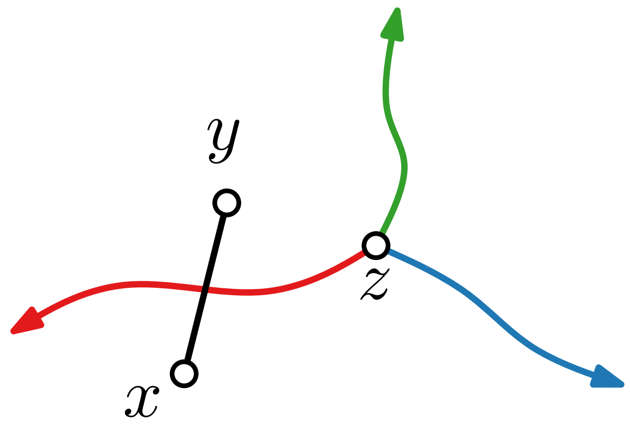

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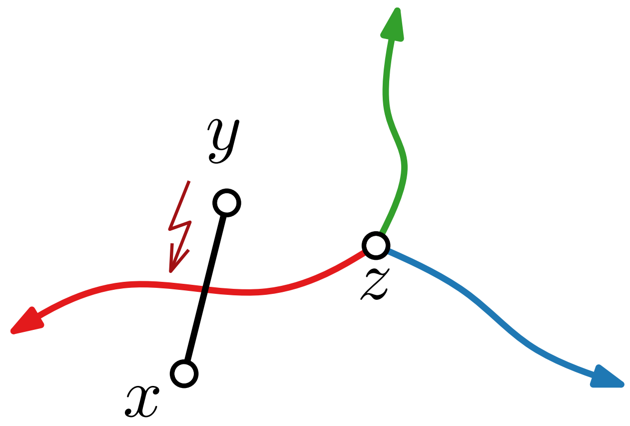

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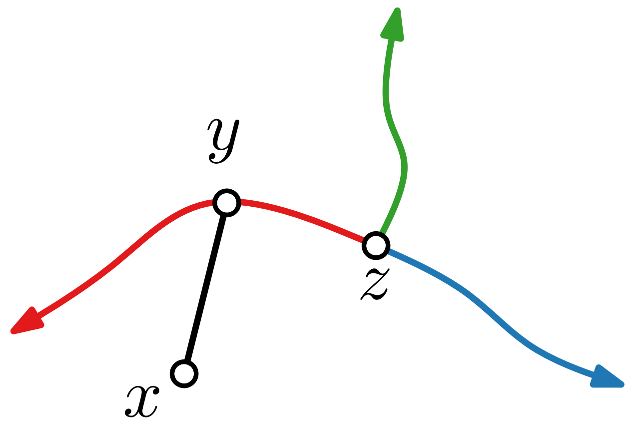

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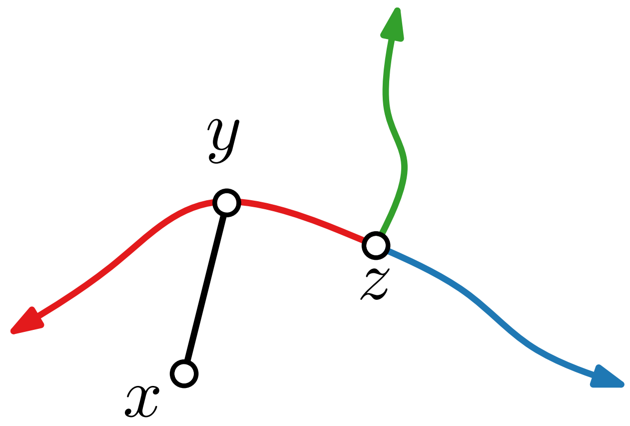

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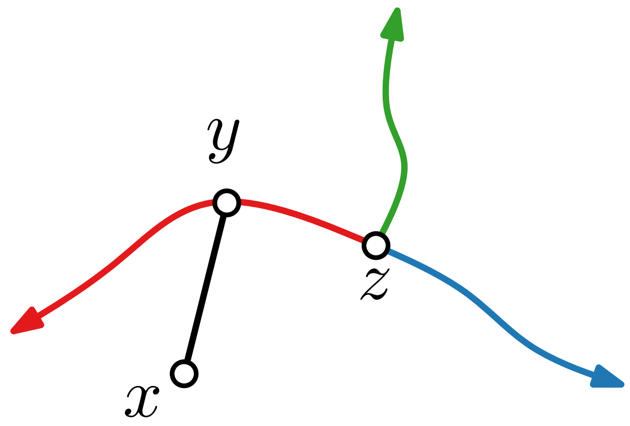

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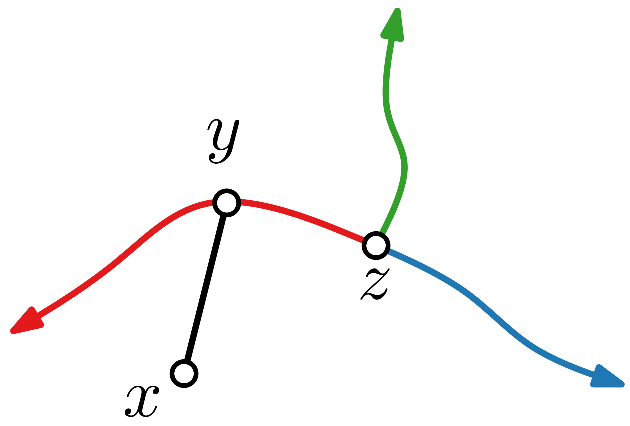

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Set $A = (0,0)$, $B = (2n-5, 0)$, and $C = (0, 2n-5)$.

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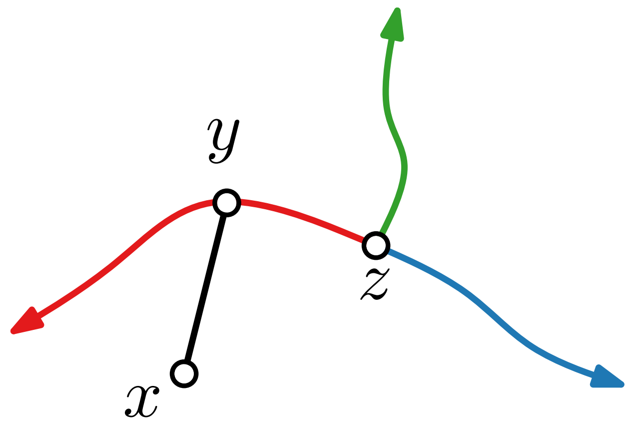

Schnyder Drawing

Set $A = (0, 0)$, $B = (2n - 5, 0)$, and $C = (0, 2n - 5)$.

Theorem. [Schnyder '90]

For a plane triangulation G, the mapping

$$f \colon v \mapsto (v_1, v_2, v_3) = \frac{1}{2n-5}(|R_1(v)|, |R_2(v)|, |R_3(v)|)$$

is a barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(2n-5) \times (2n-5)$ grid.

(B1) $v_1 + v_2 + v_3 = 1$ for all $v \in V(G)$ ✓

(B2) for each $\{x, y\} \in E(G)$ and each $z \in V(G) \setminus \{x, y\}$ there exists $k \in \{1, 2, 3\}$ with $x_k < z_k$ and $y_k < z_k$ ✓

- $\{x, y\}$ must lie in $R_i(z)$ for some $i \in \{1, 2, 3\}$
- $x, y \in R_i(z) \Rightarrow R_i(x), R_i(y) \subsetneq R_i(z)$

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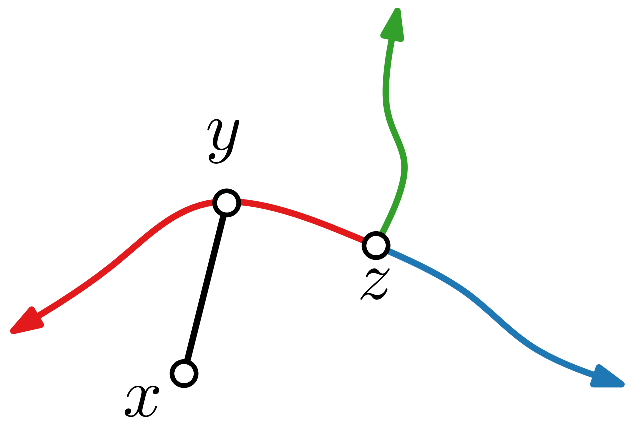

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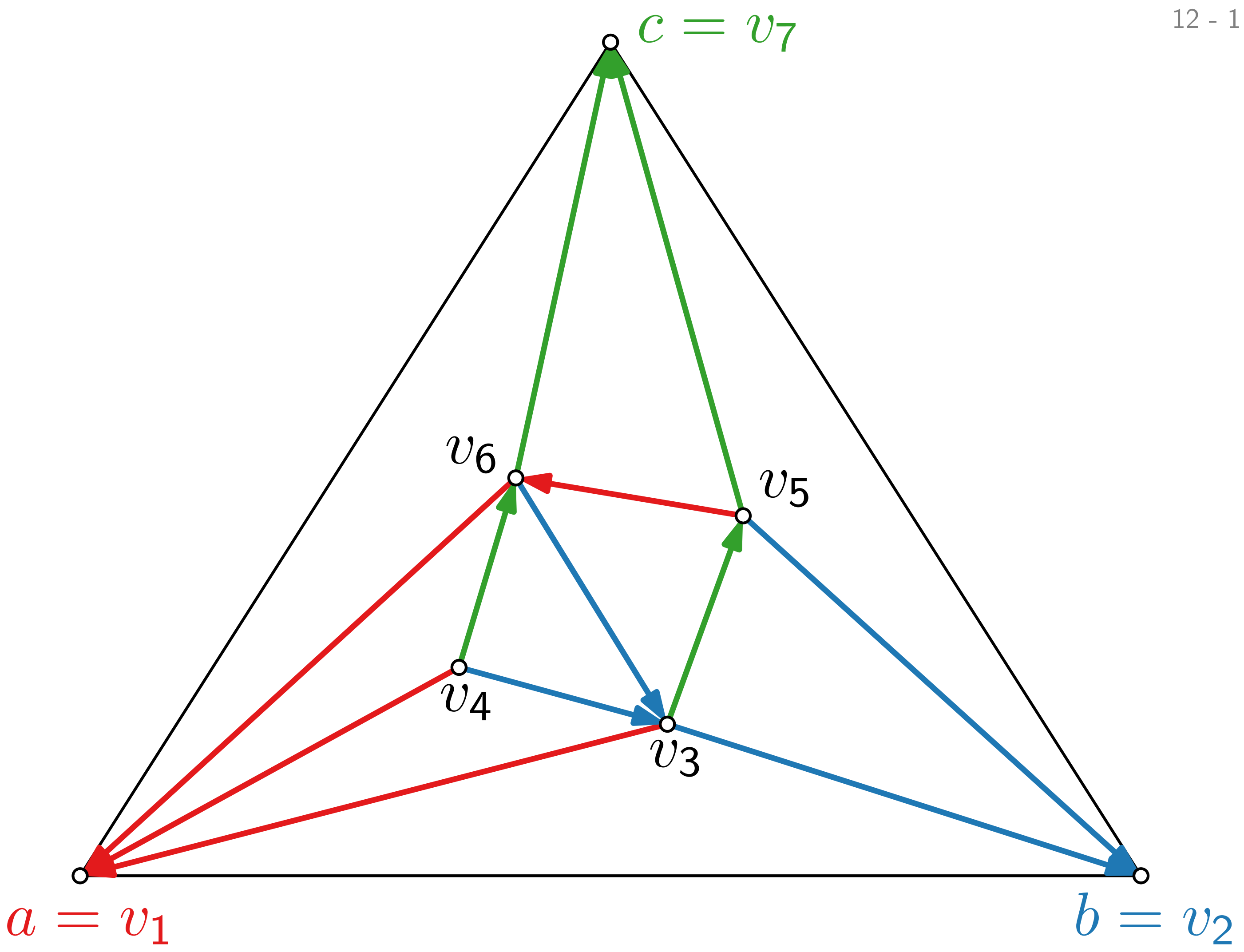

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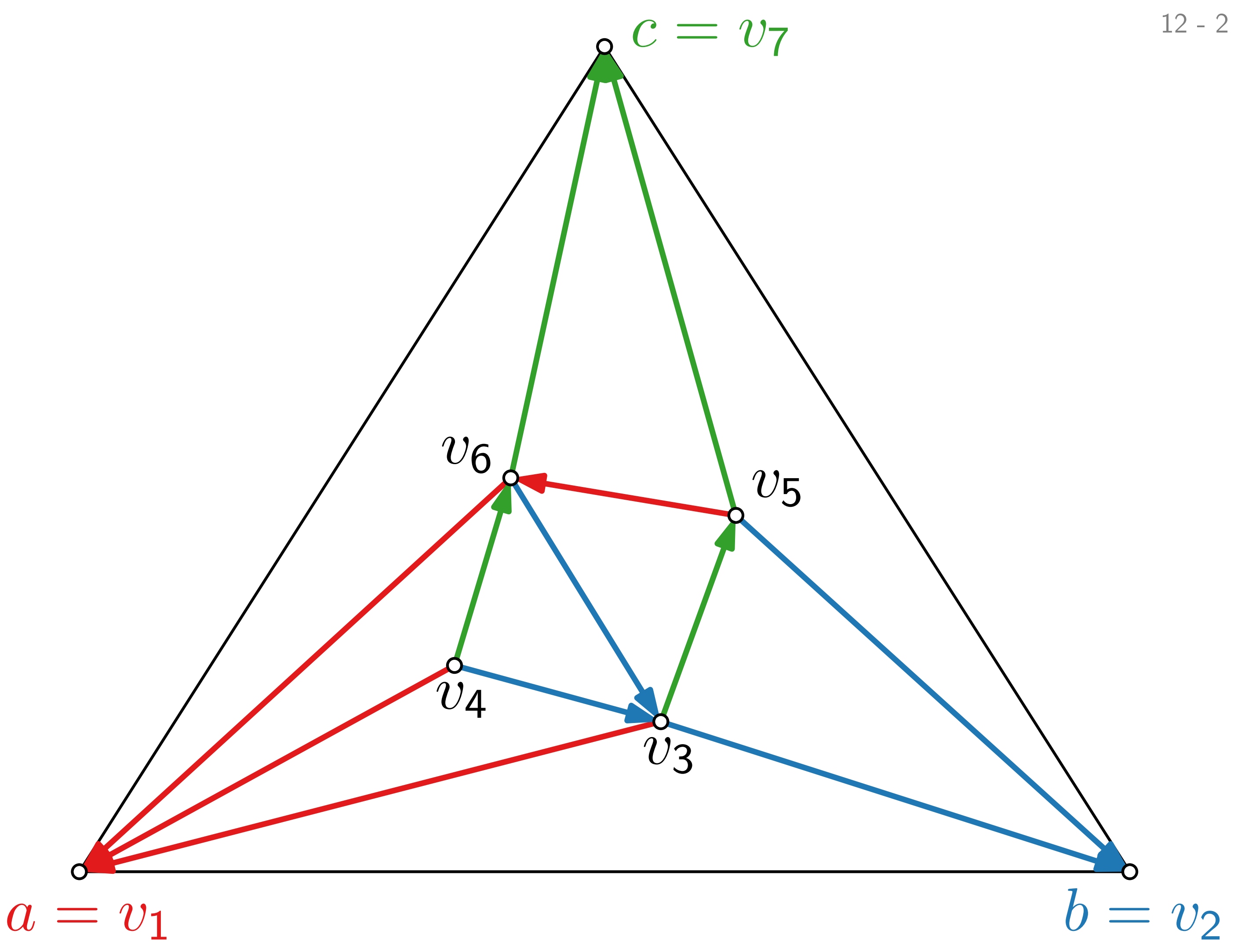

$n = 7; \quad 2n - 5 = 9$

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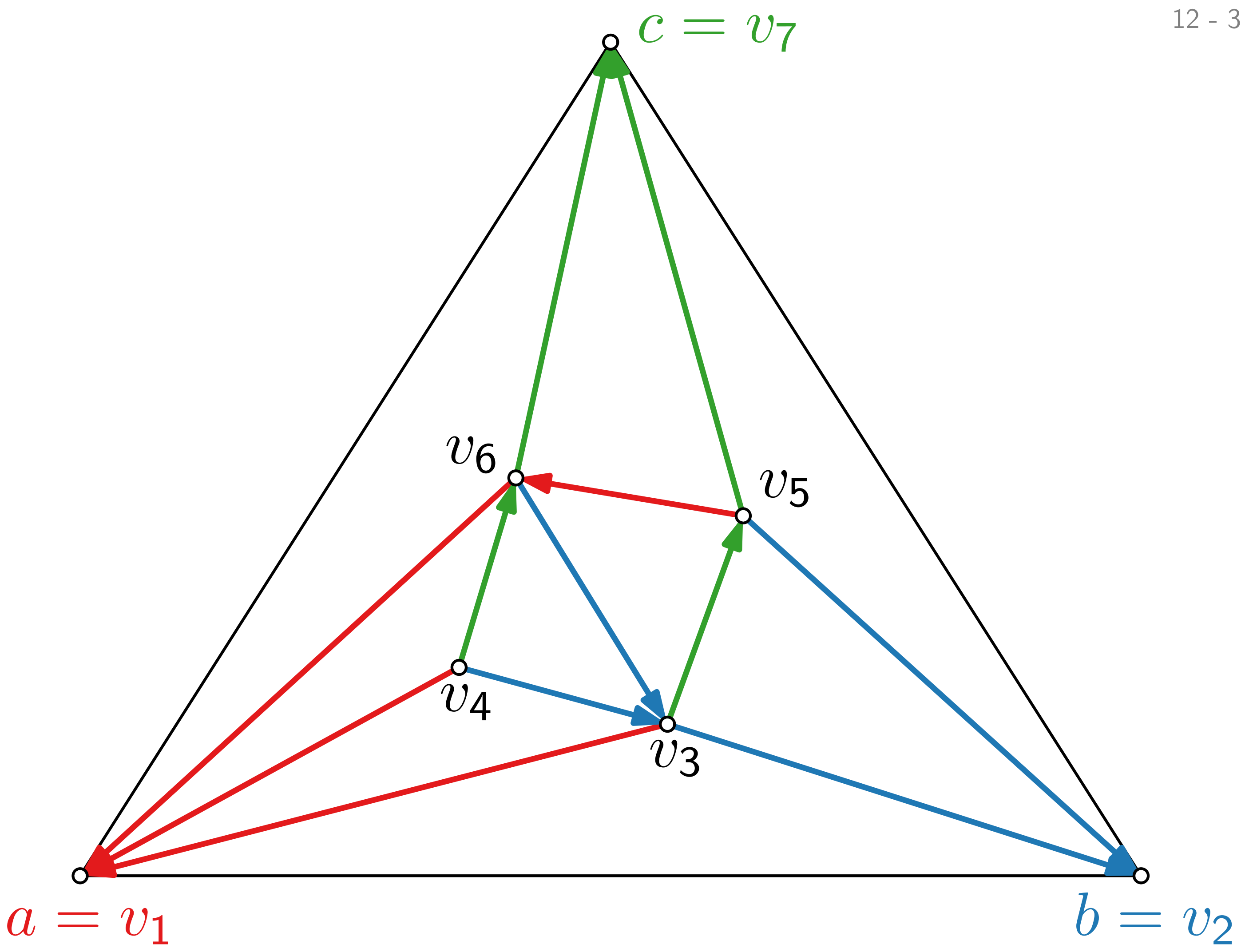

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$f(v_1) = (9, 0, 0)$

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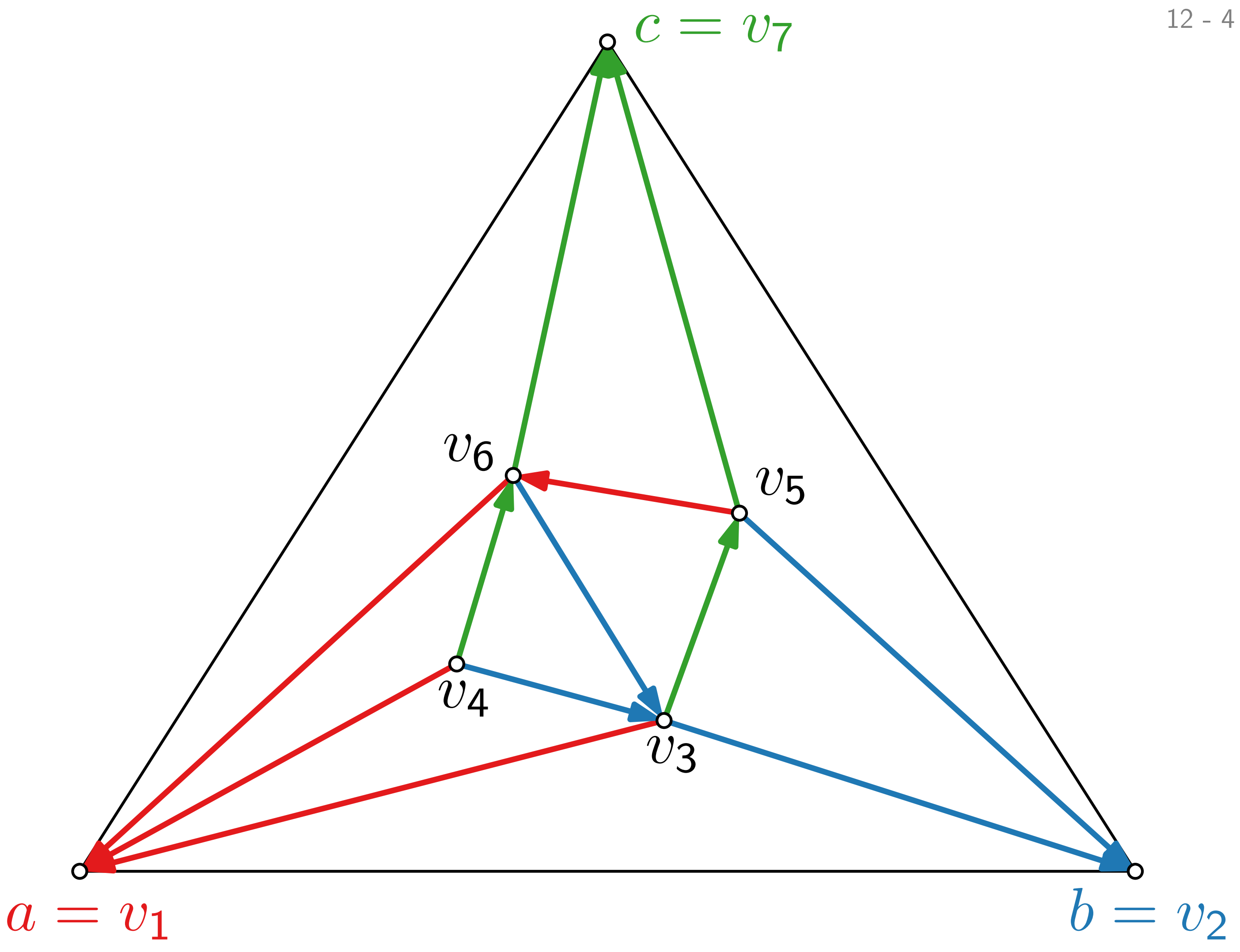

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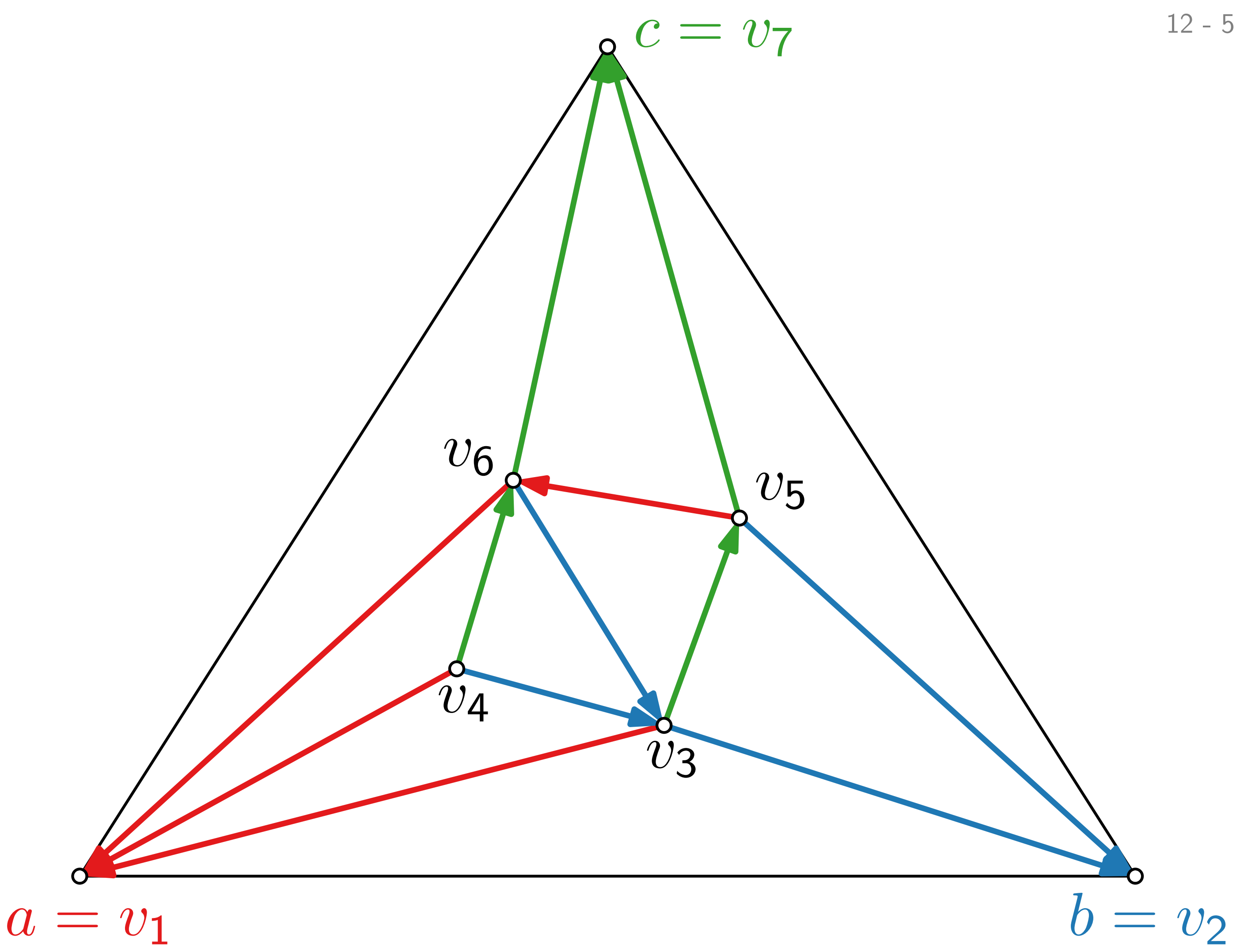

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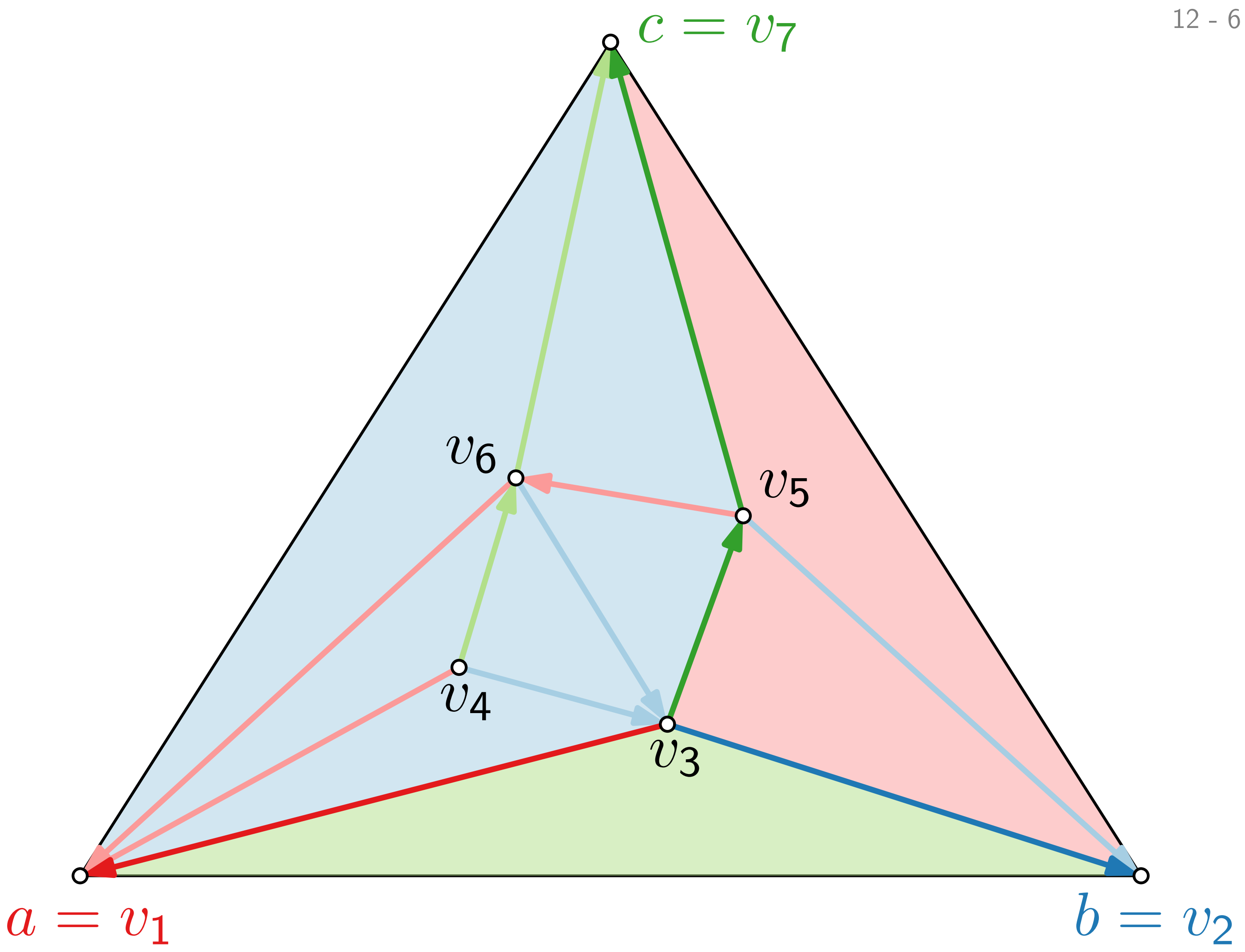

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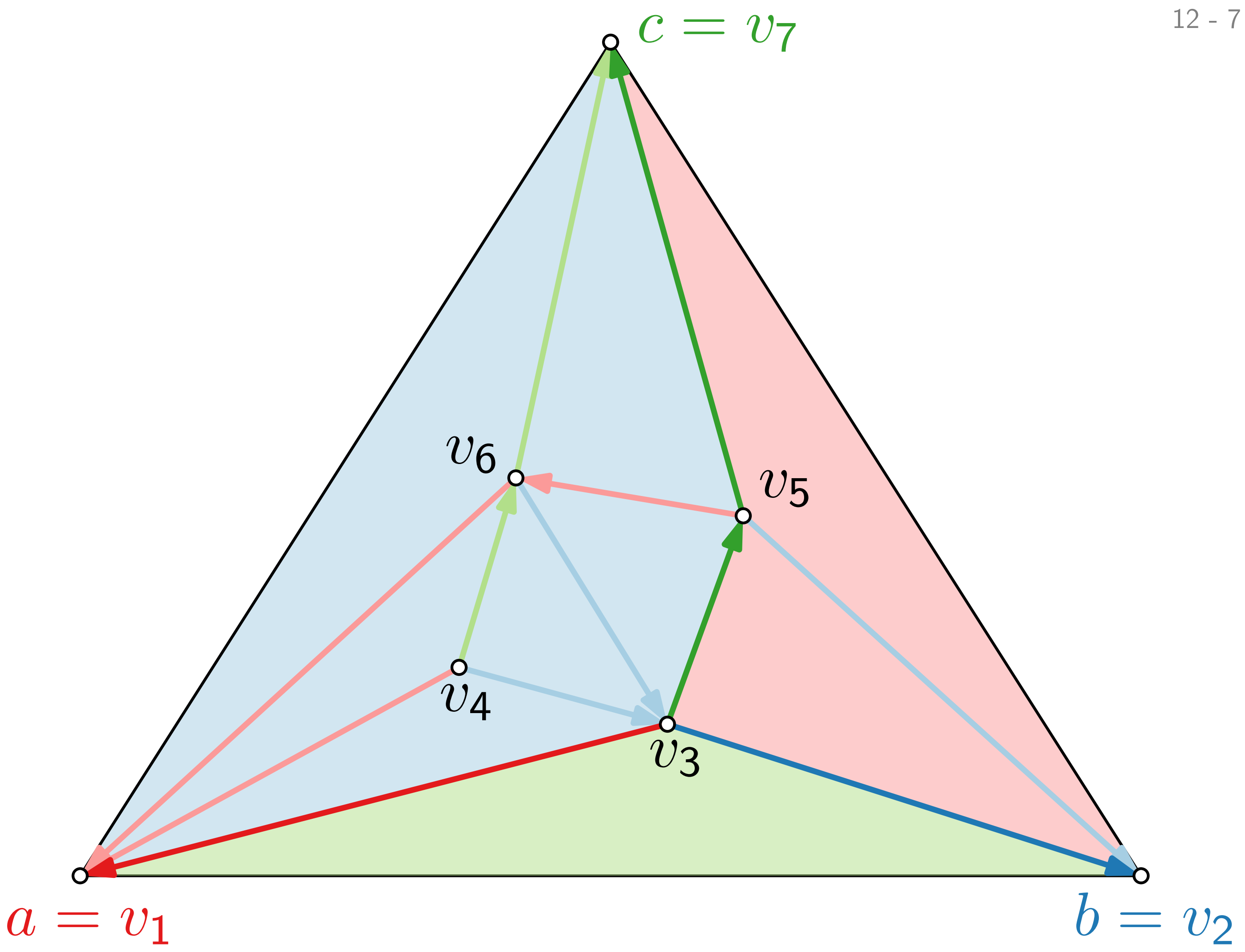

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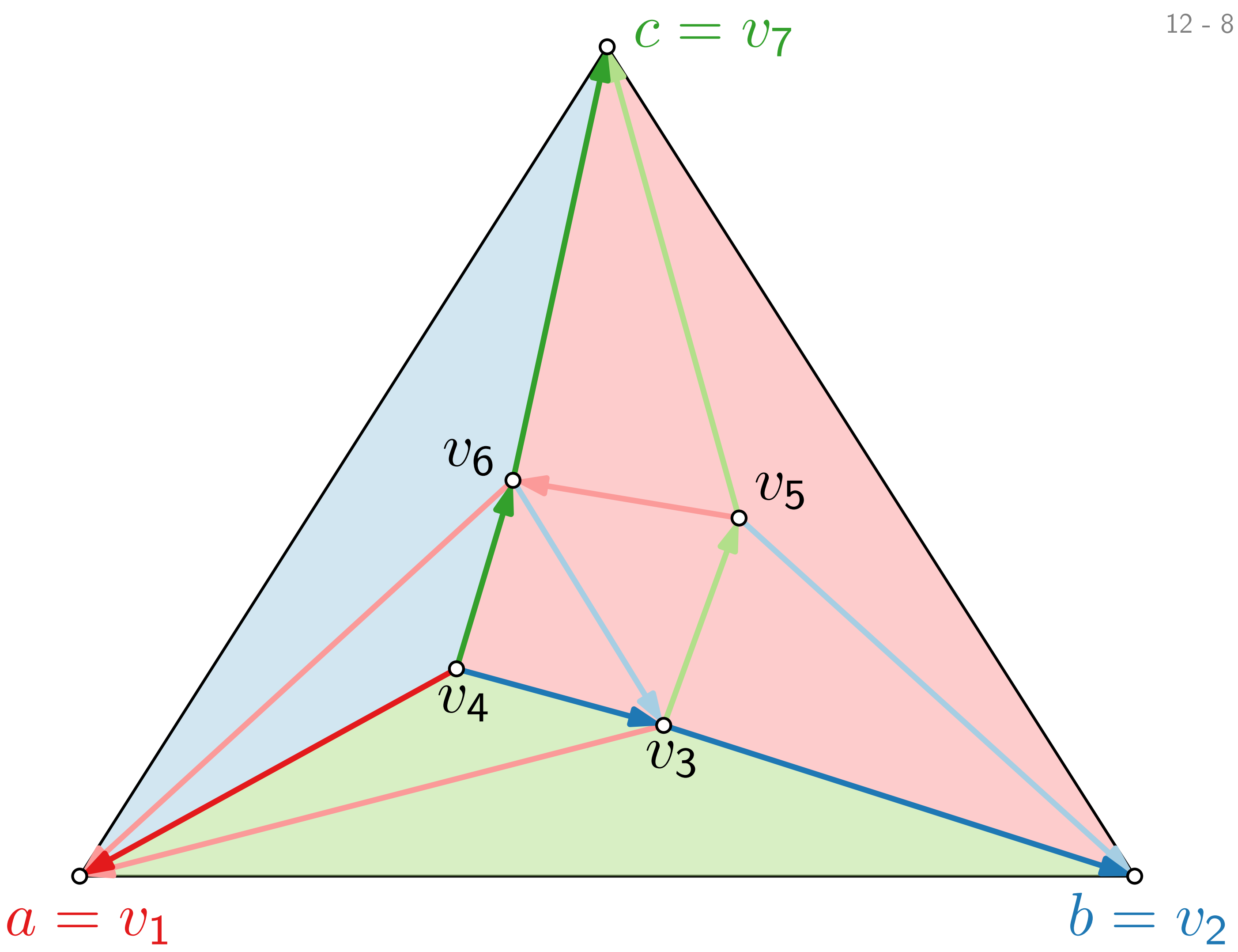

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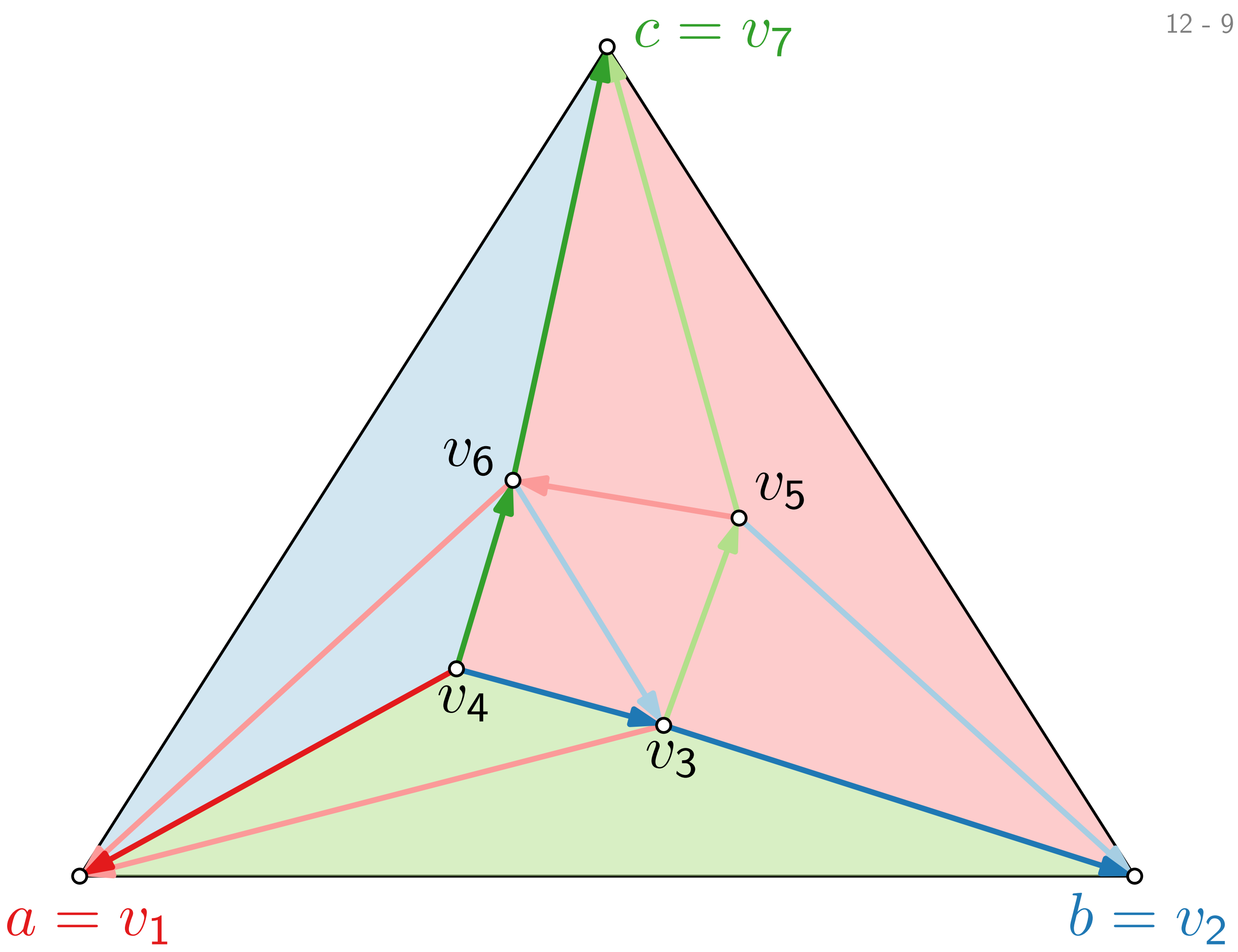

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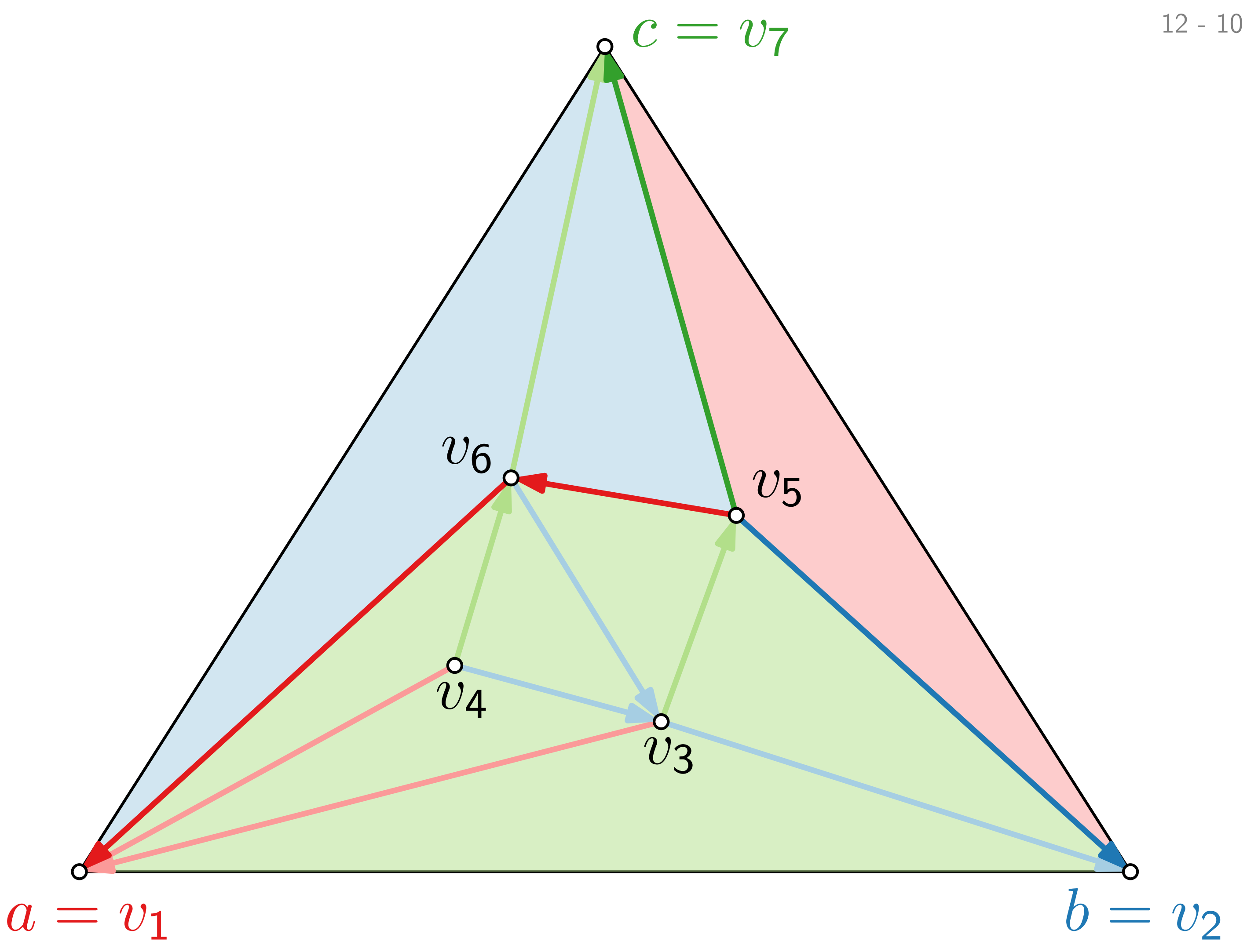

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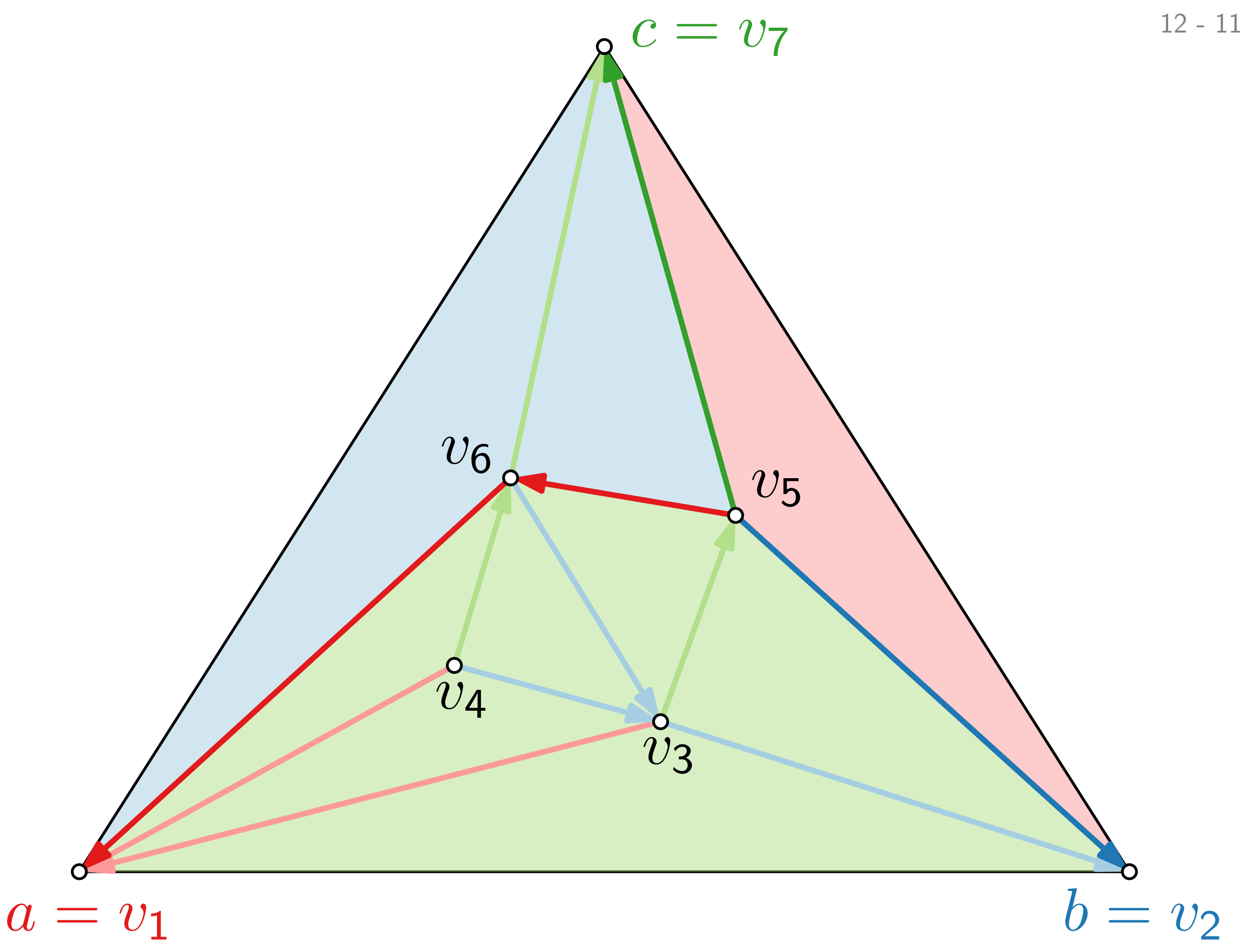

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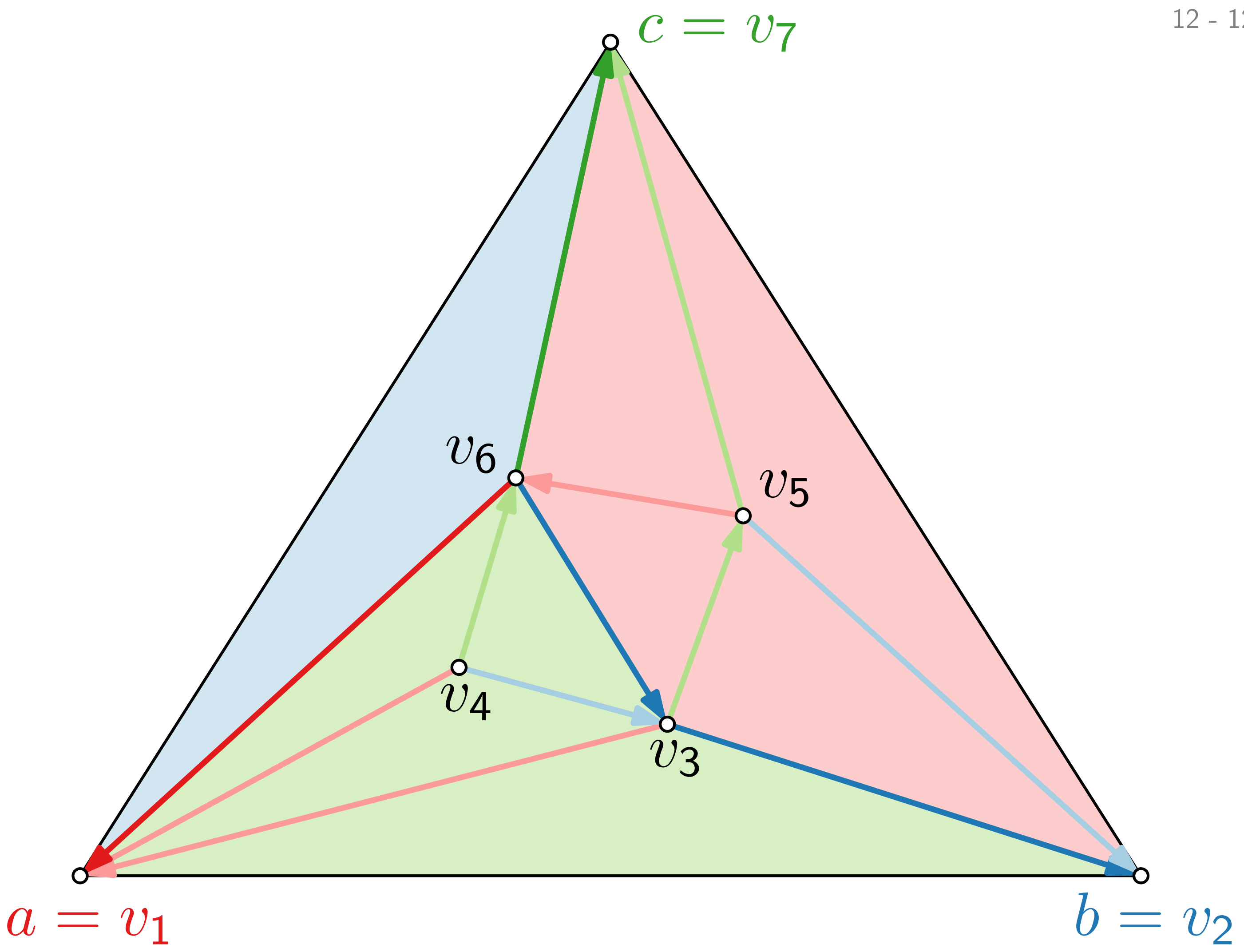

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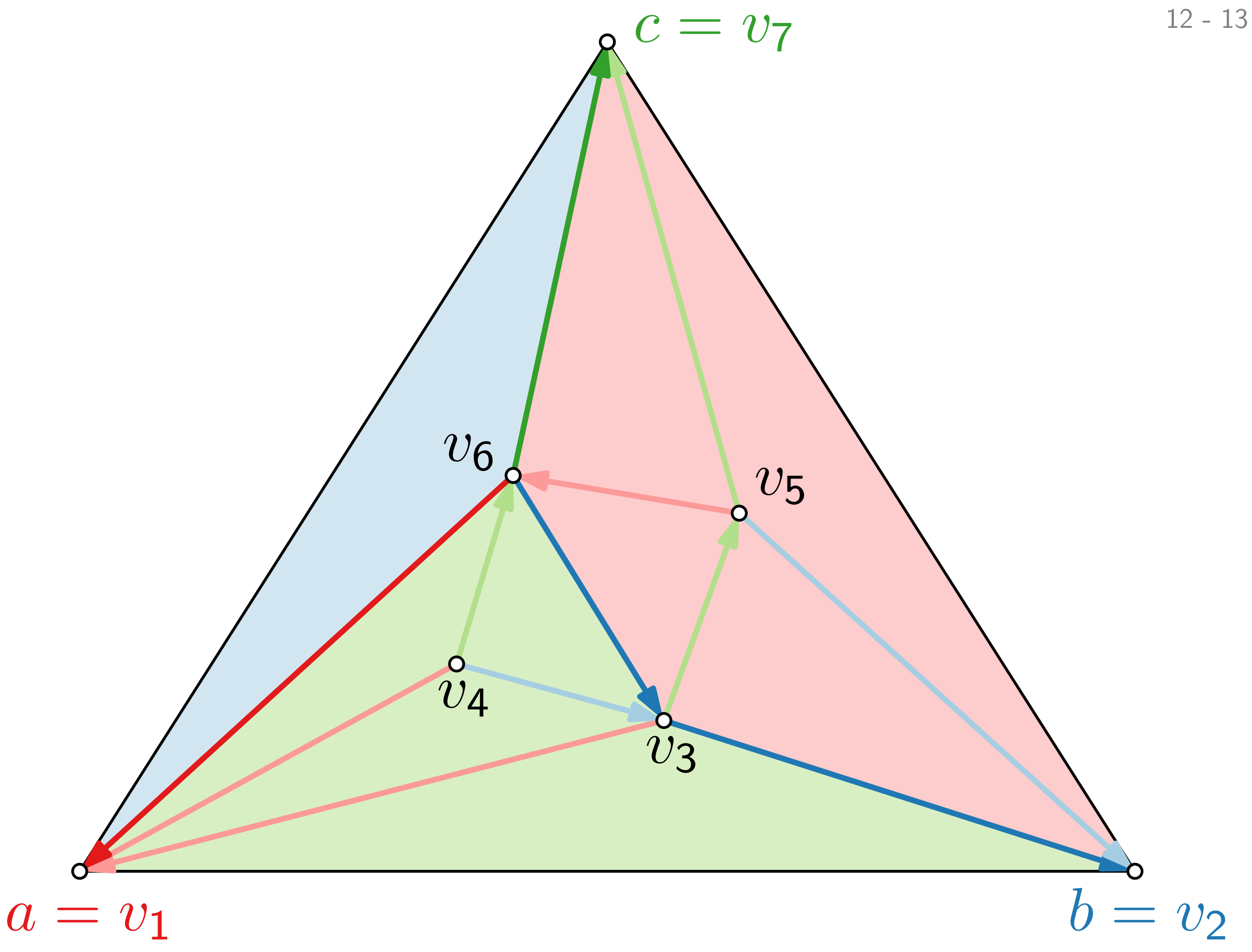

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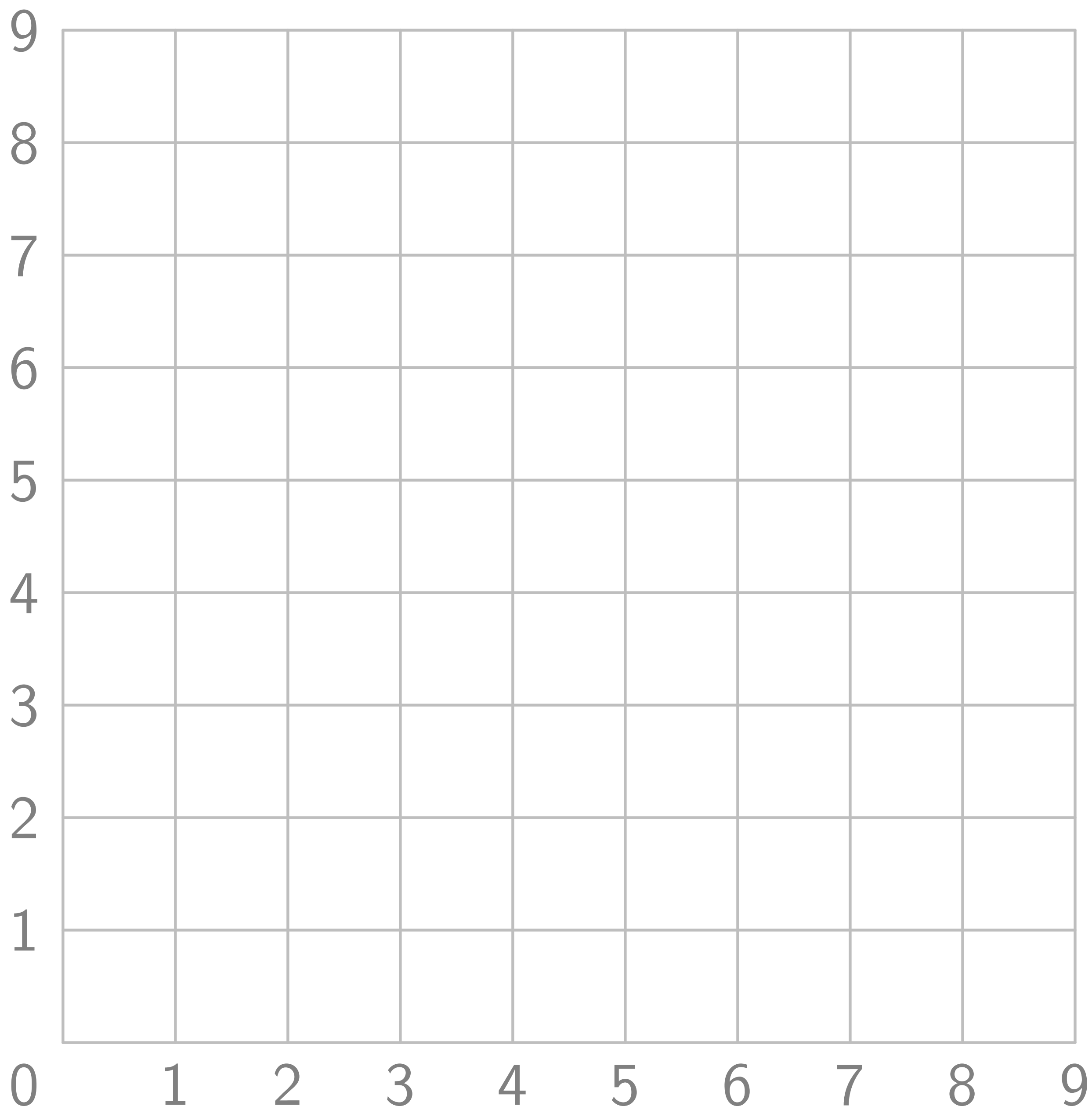

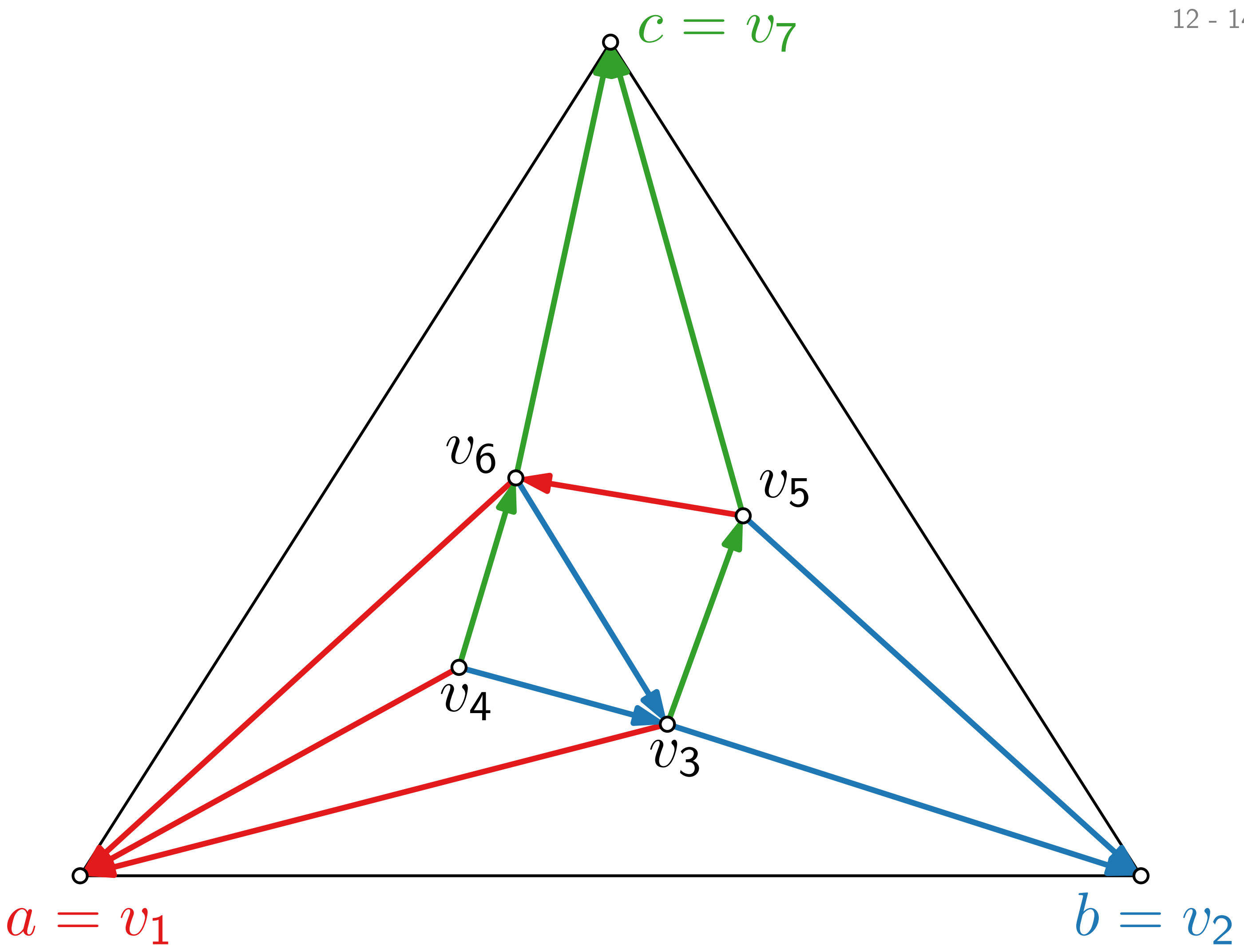

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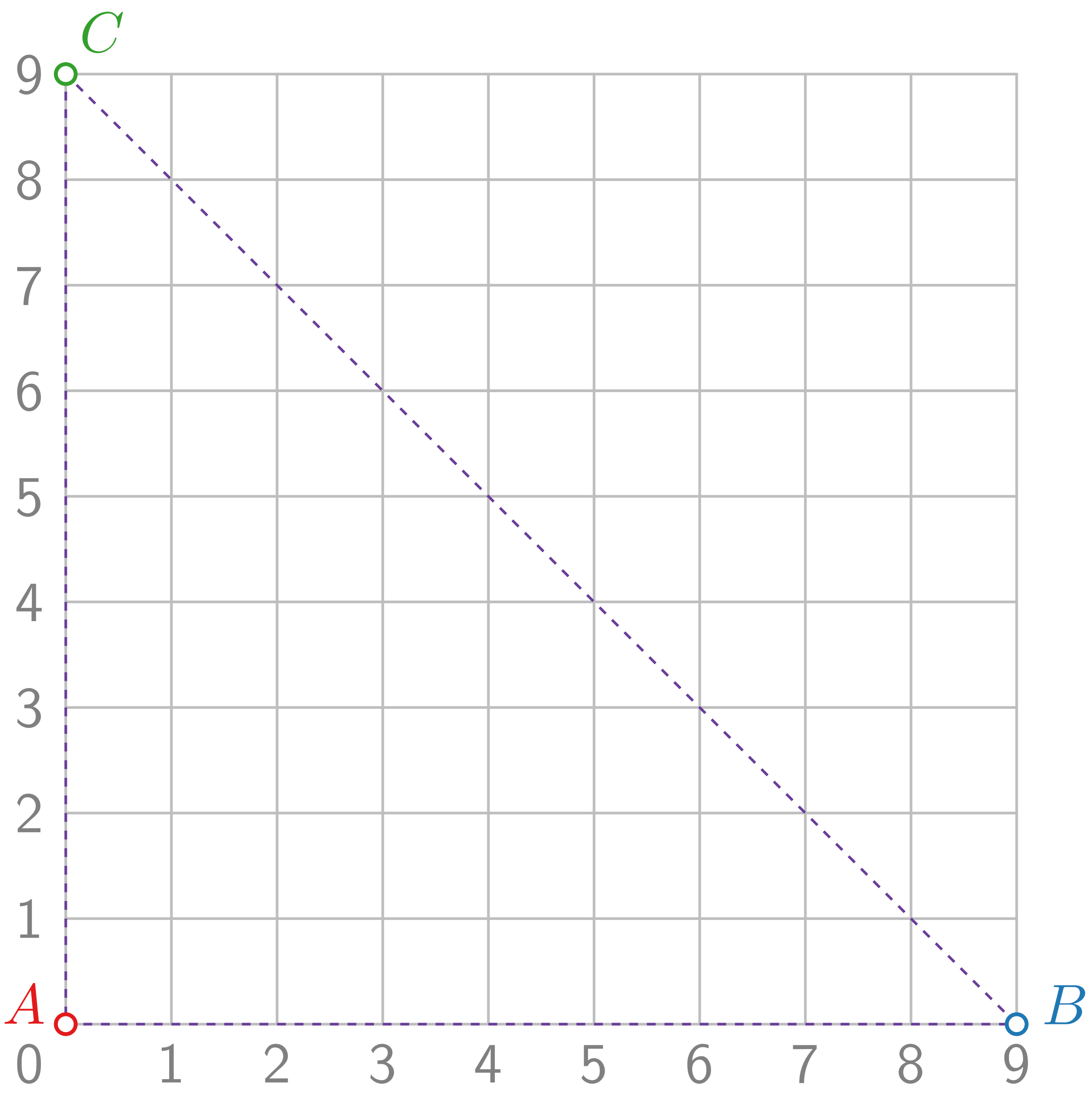

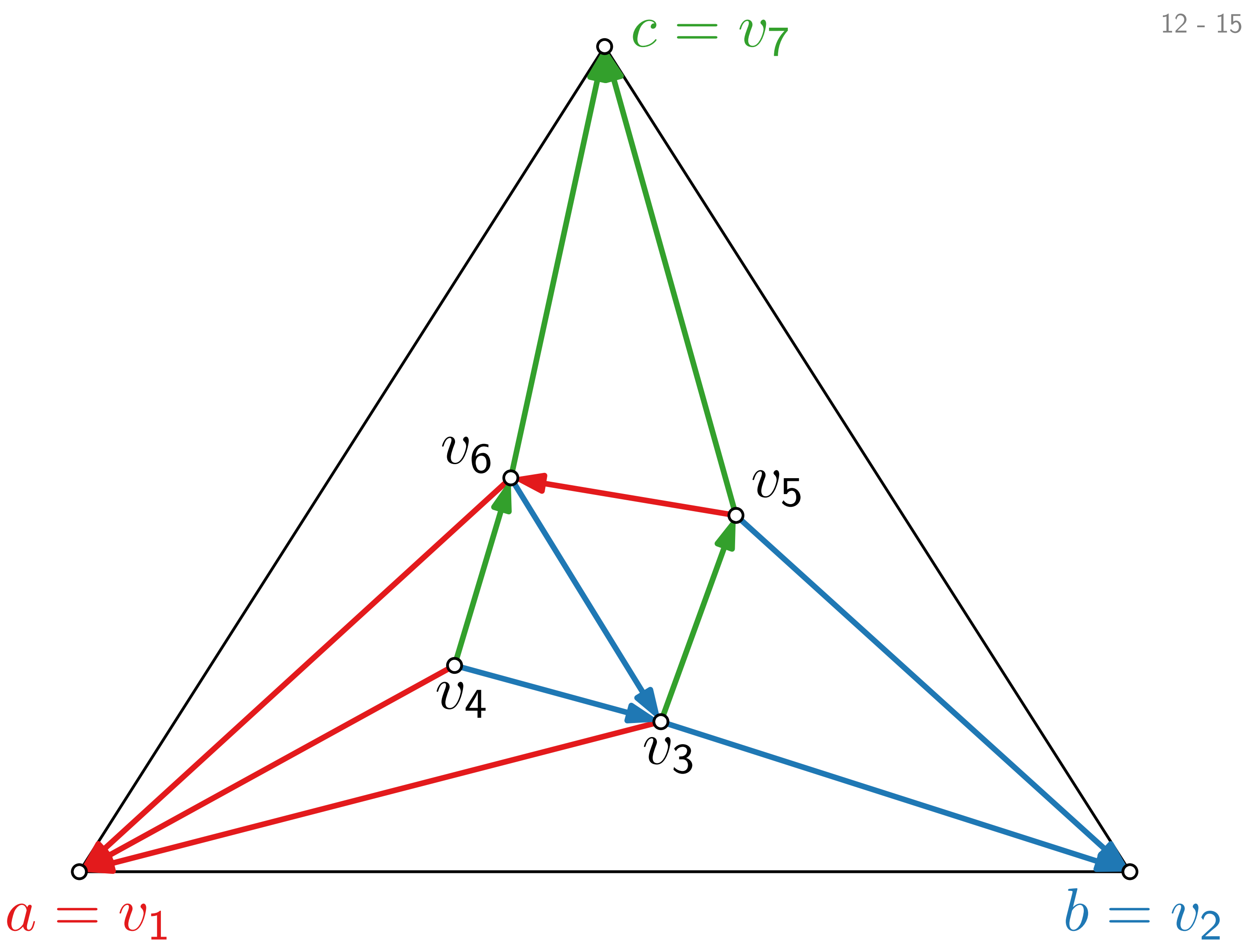

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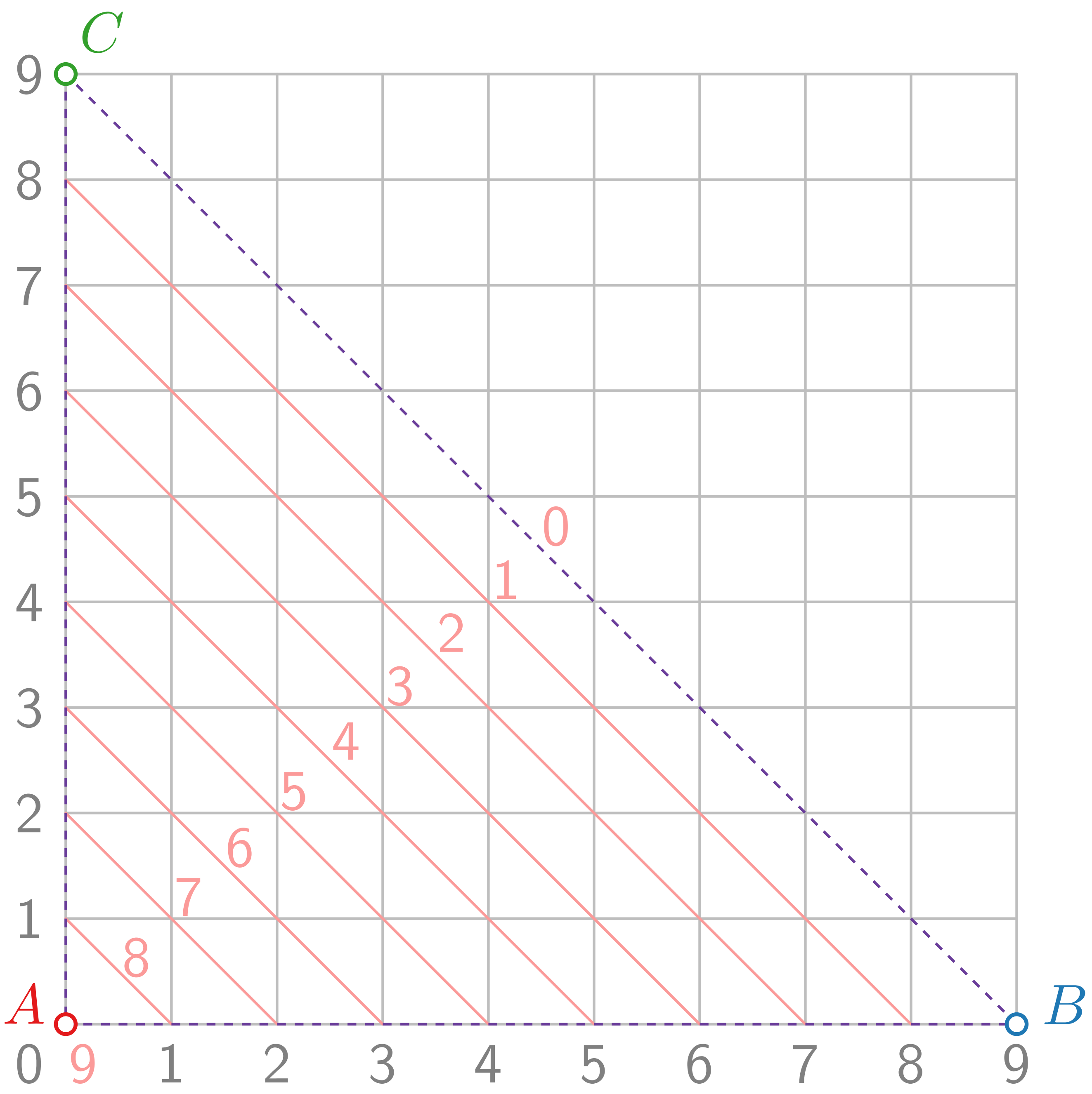

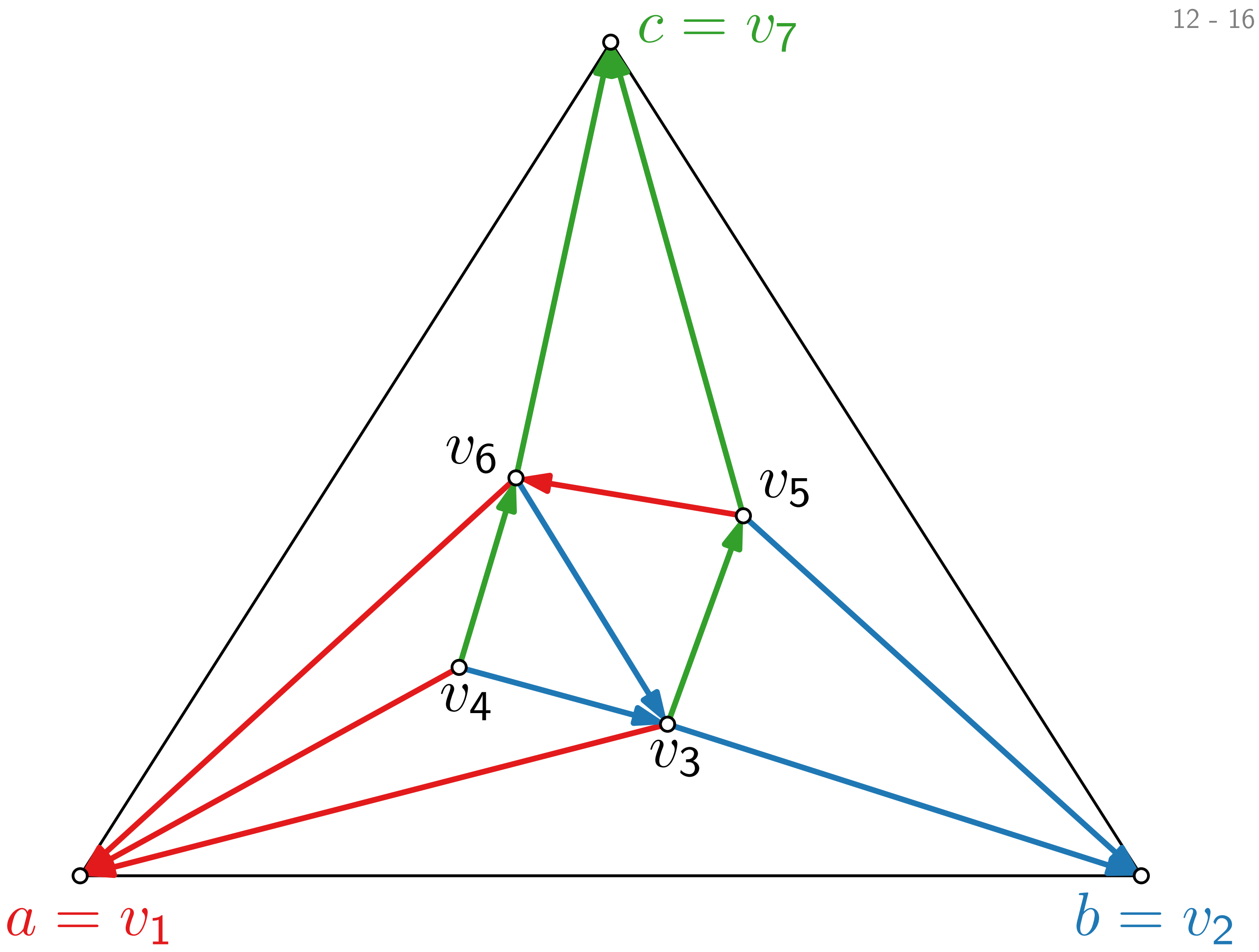

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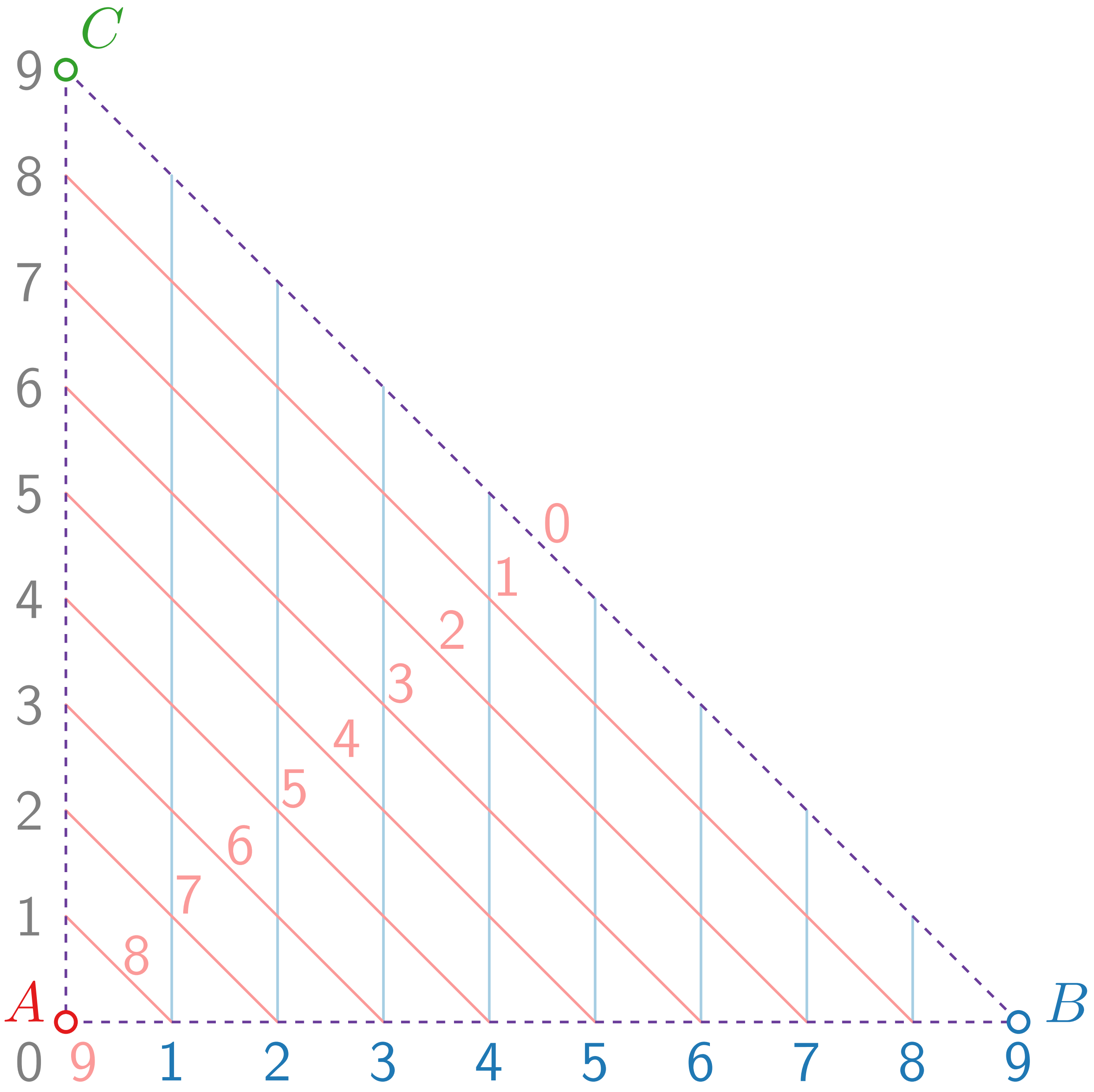

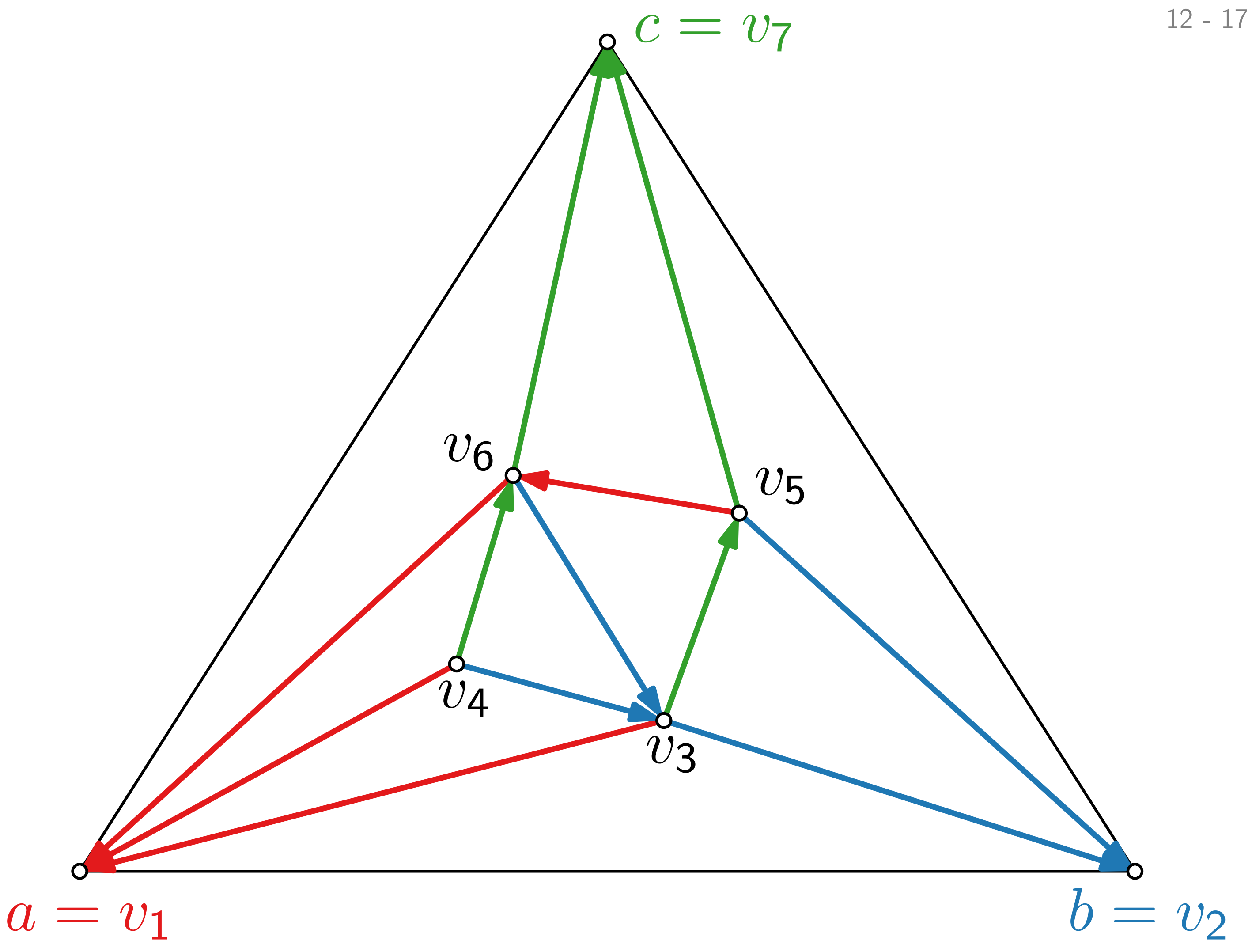

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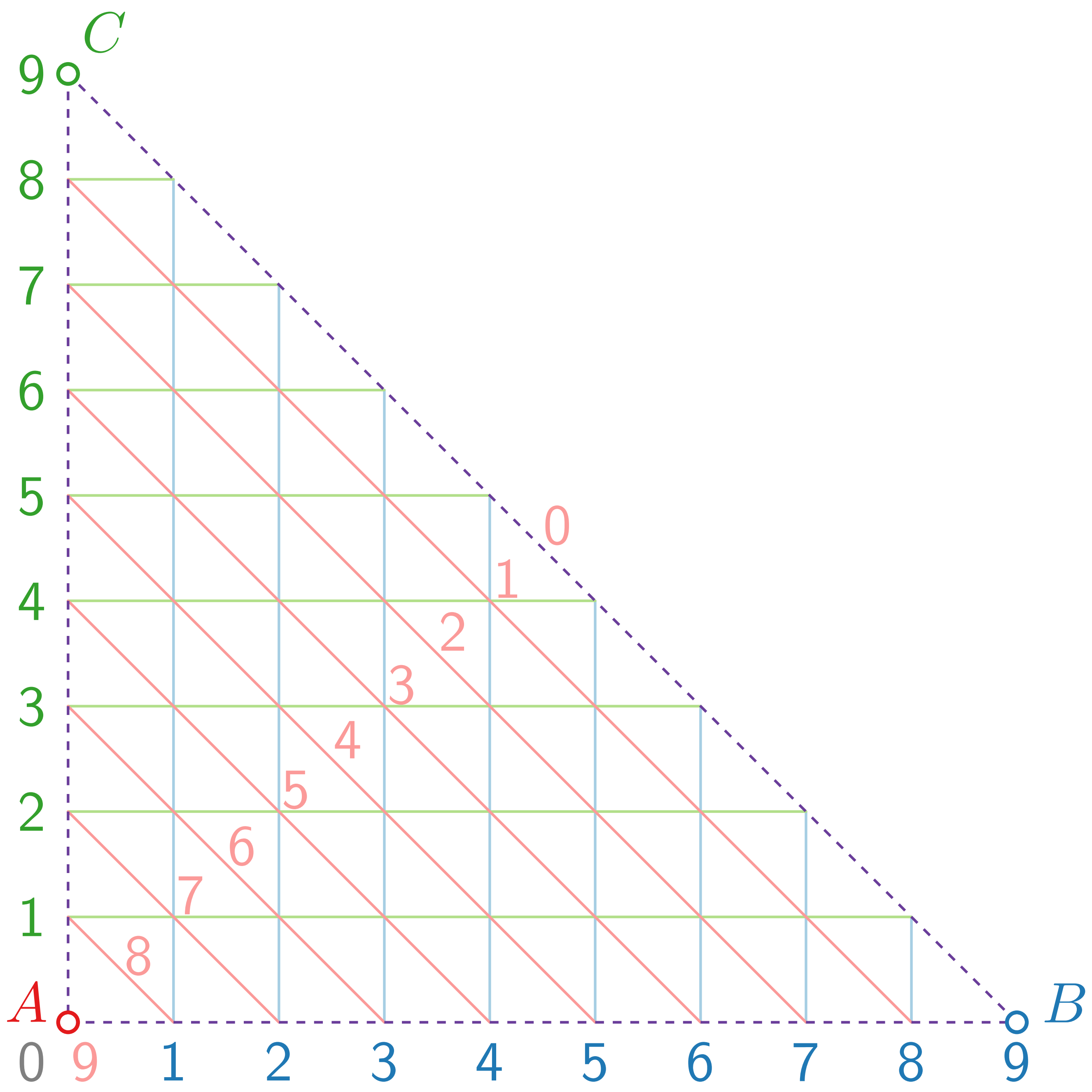

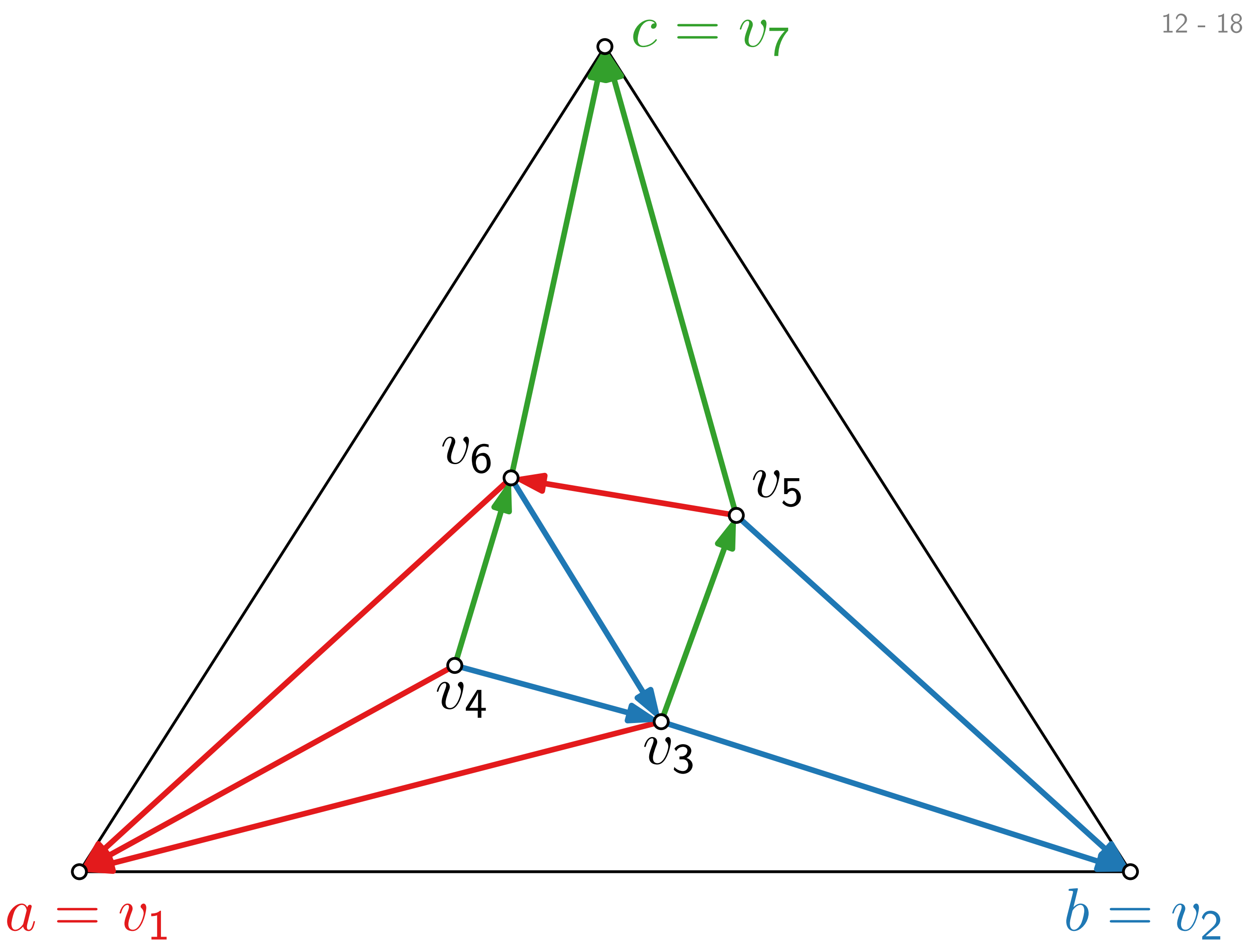

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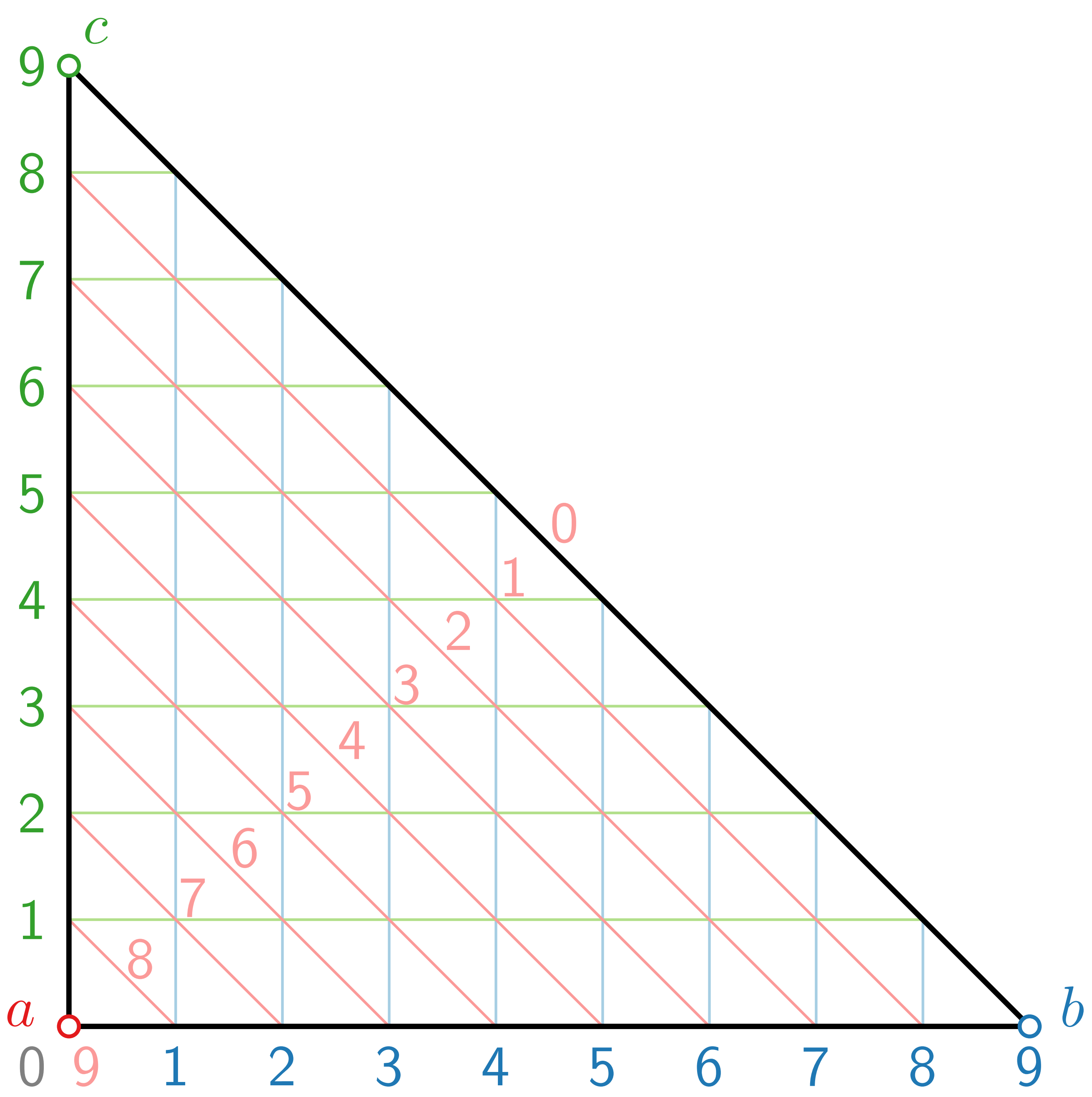

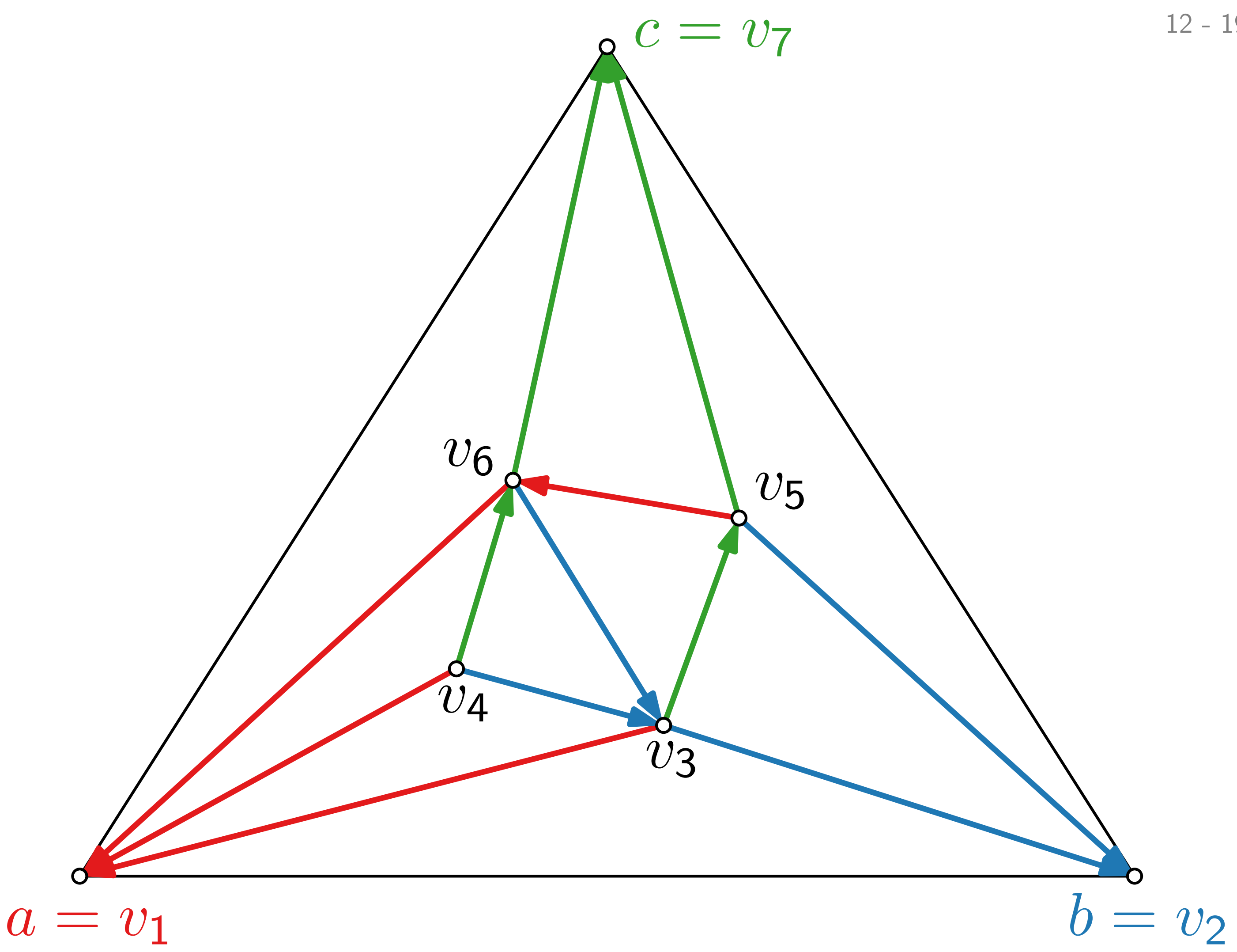

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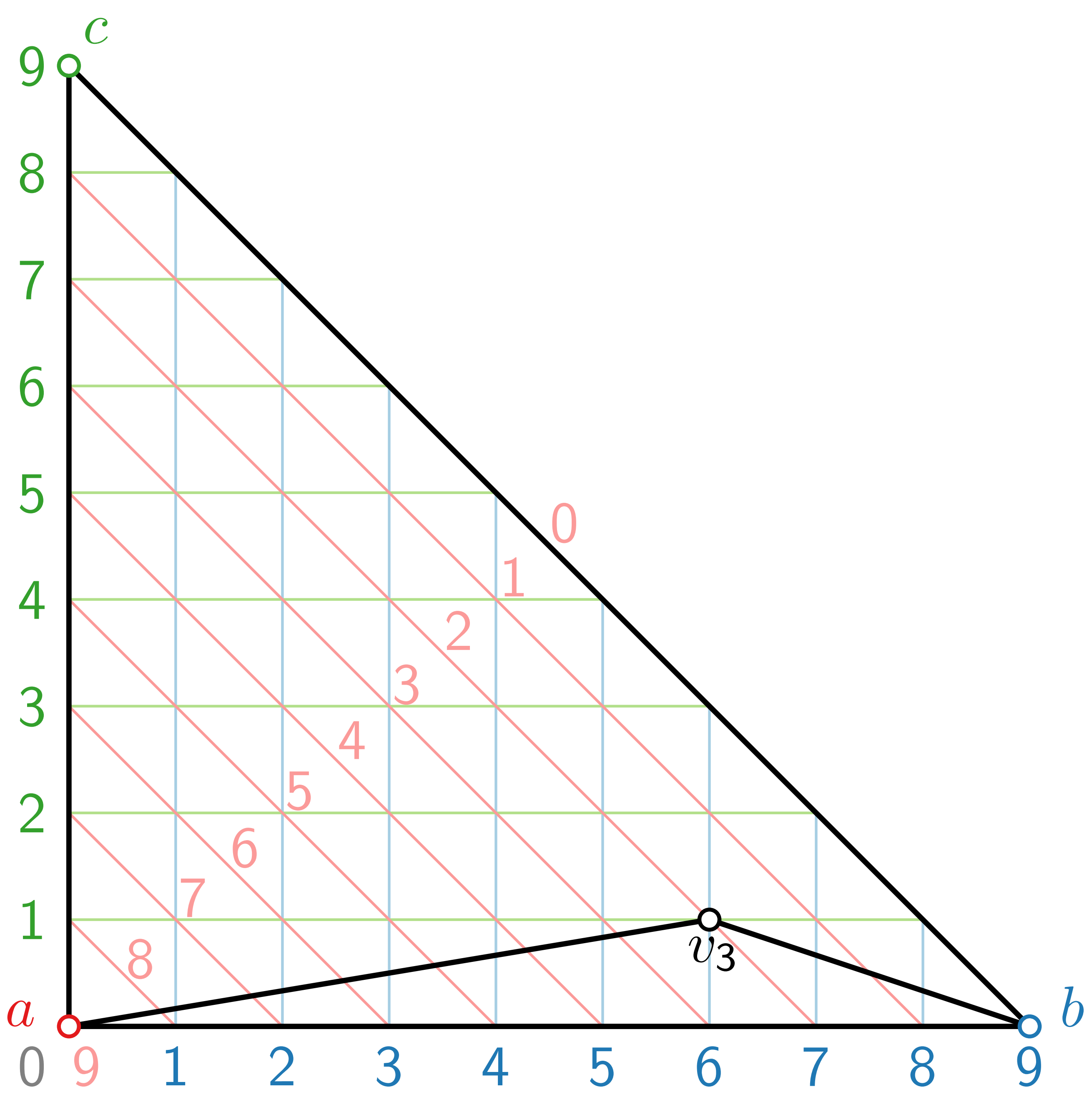

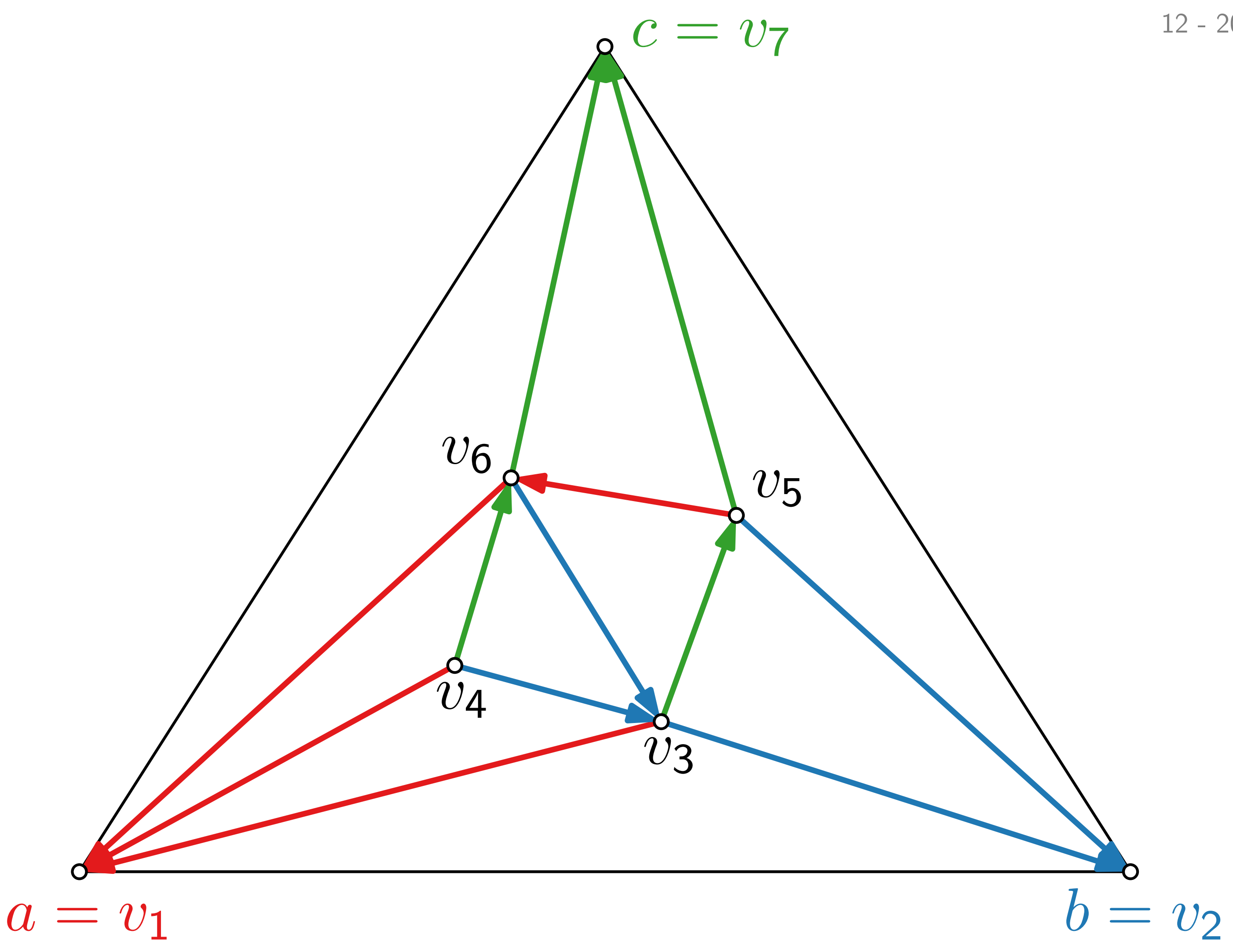

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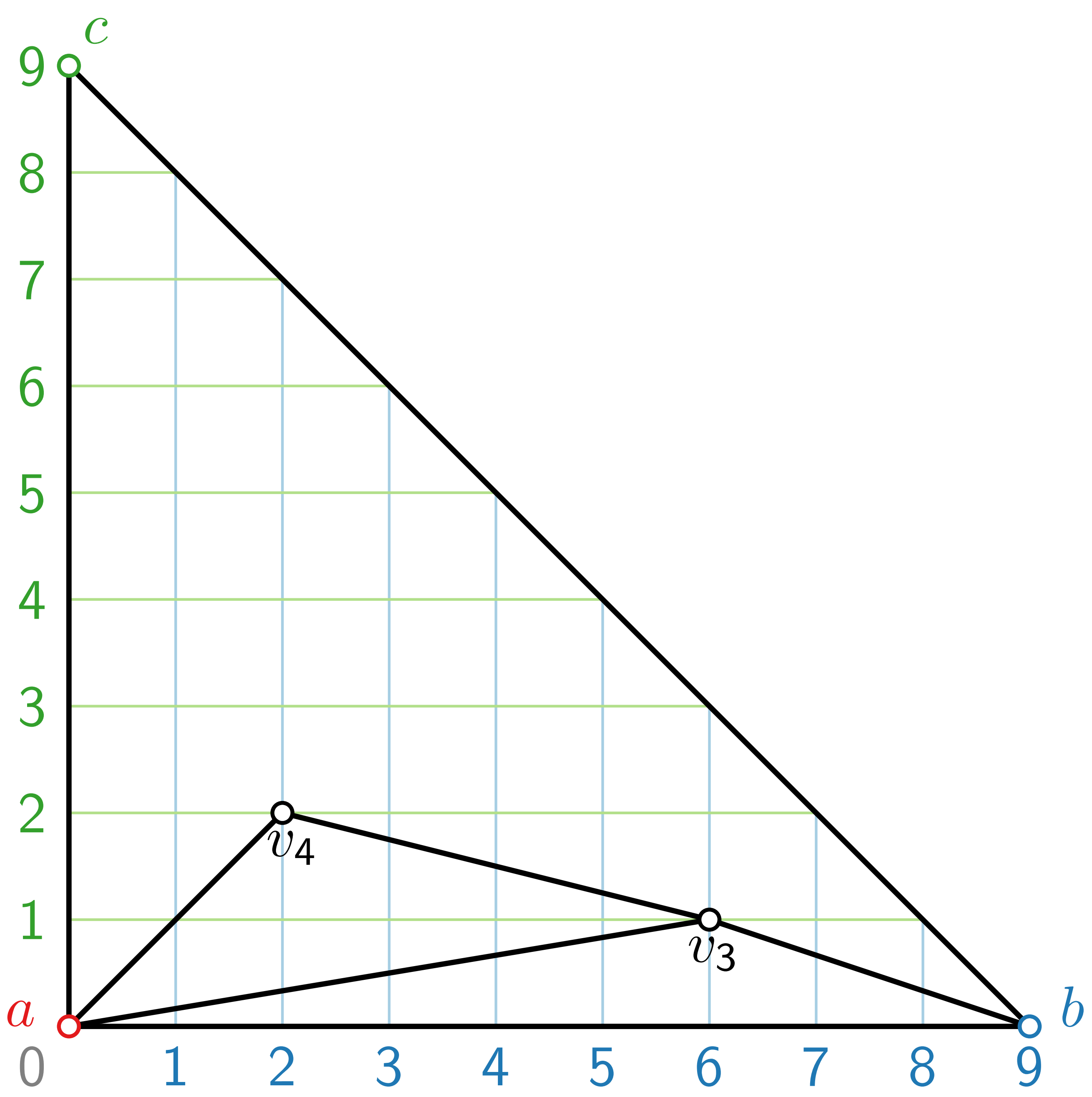

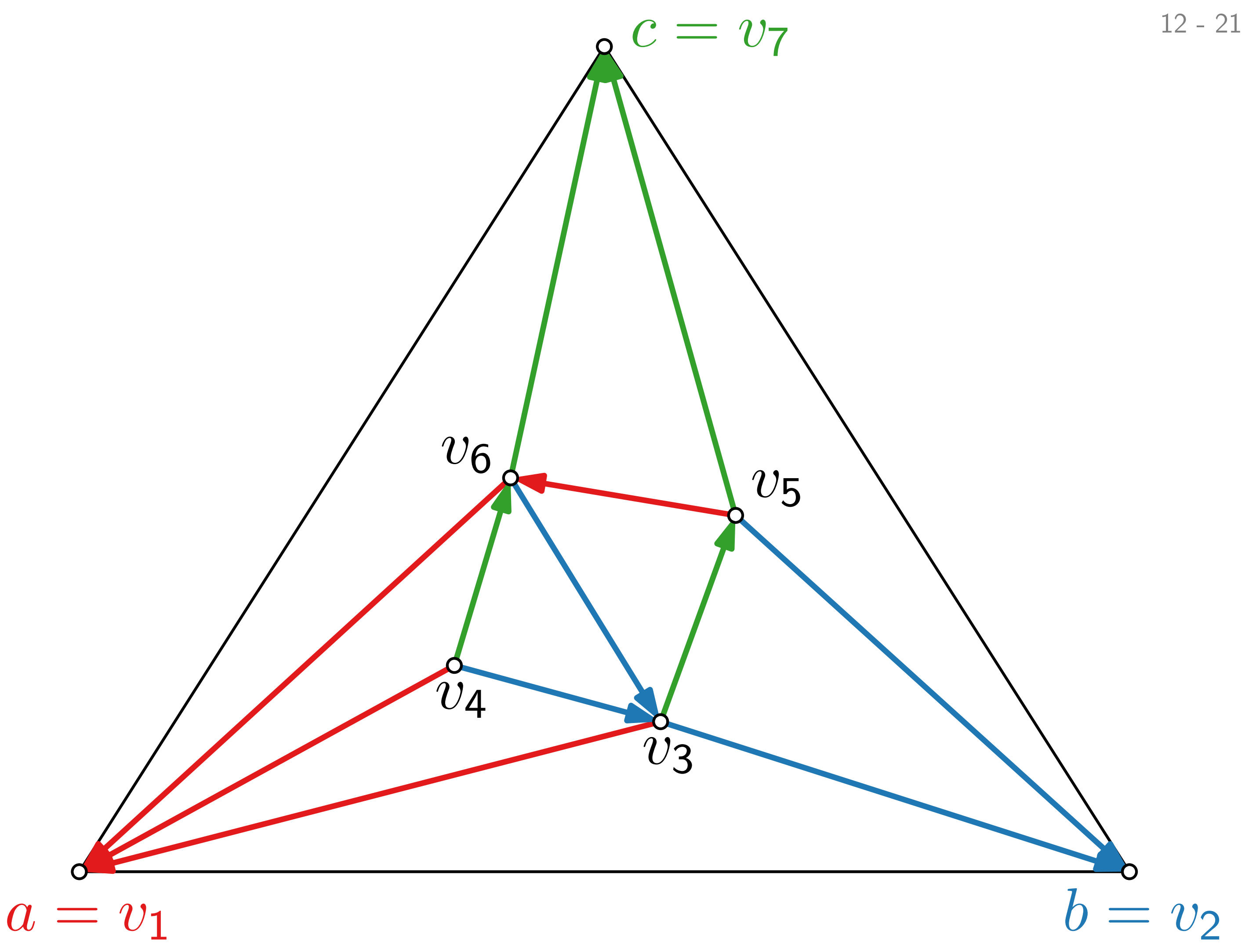

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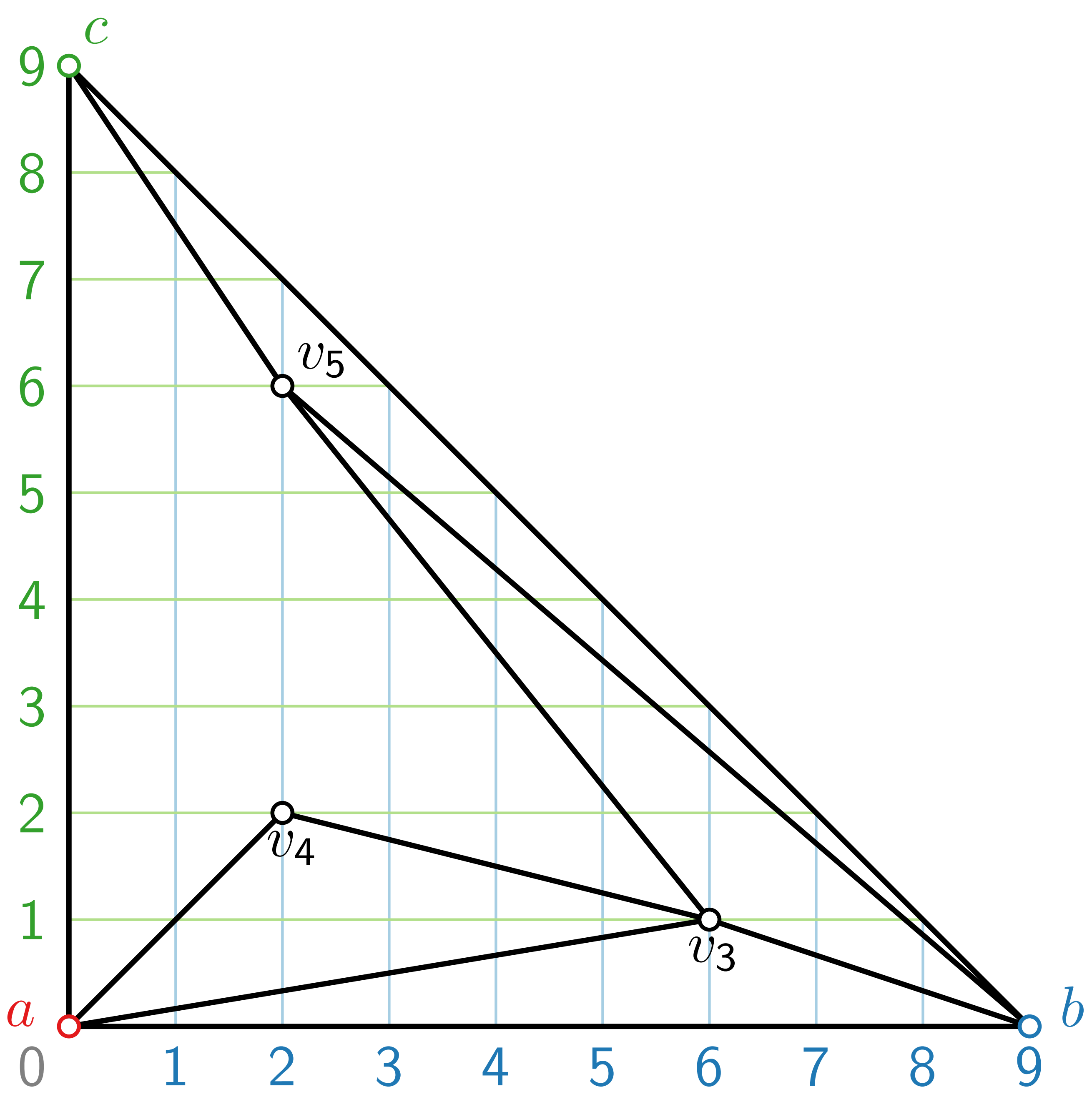

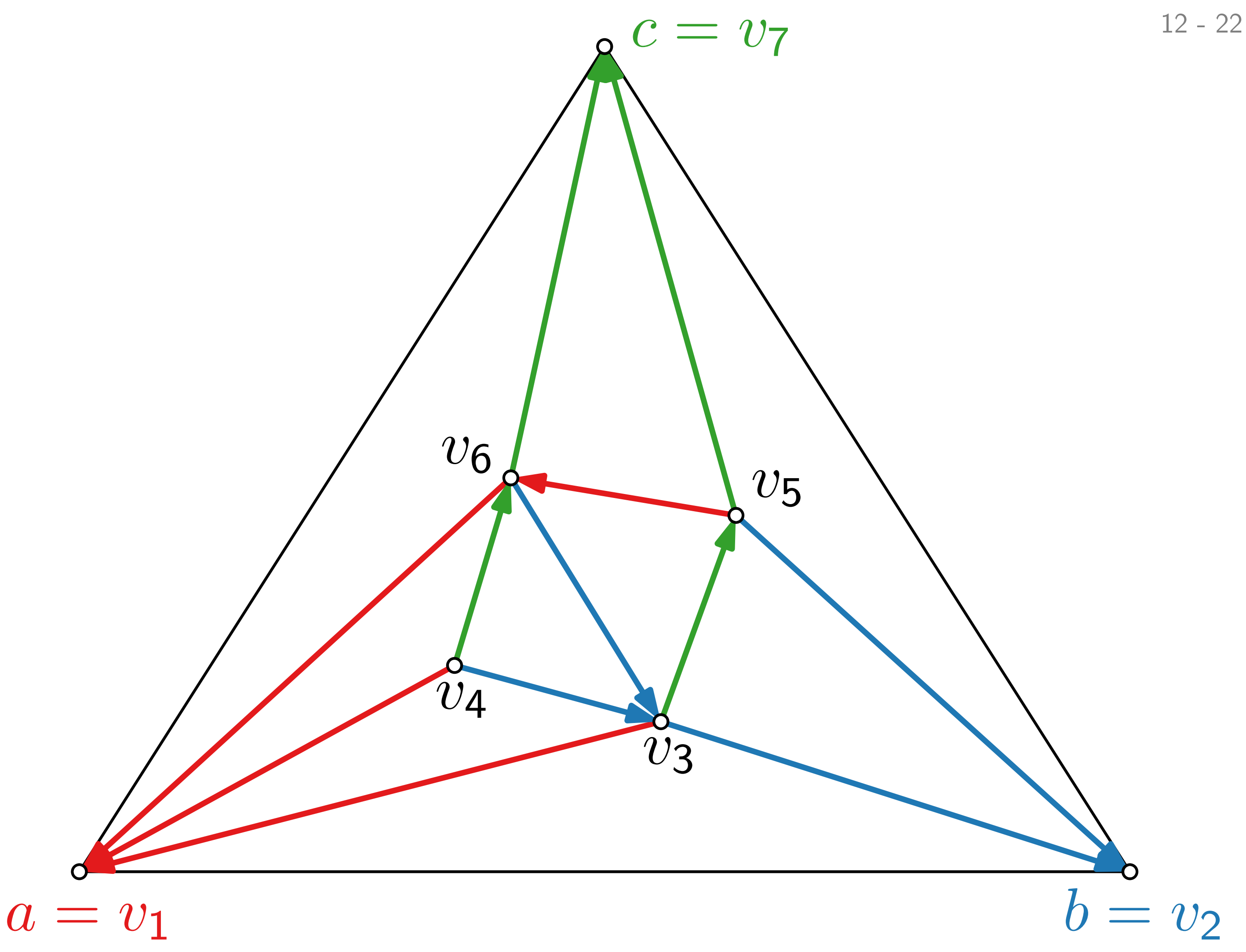

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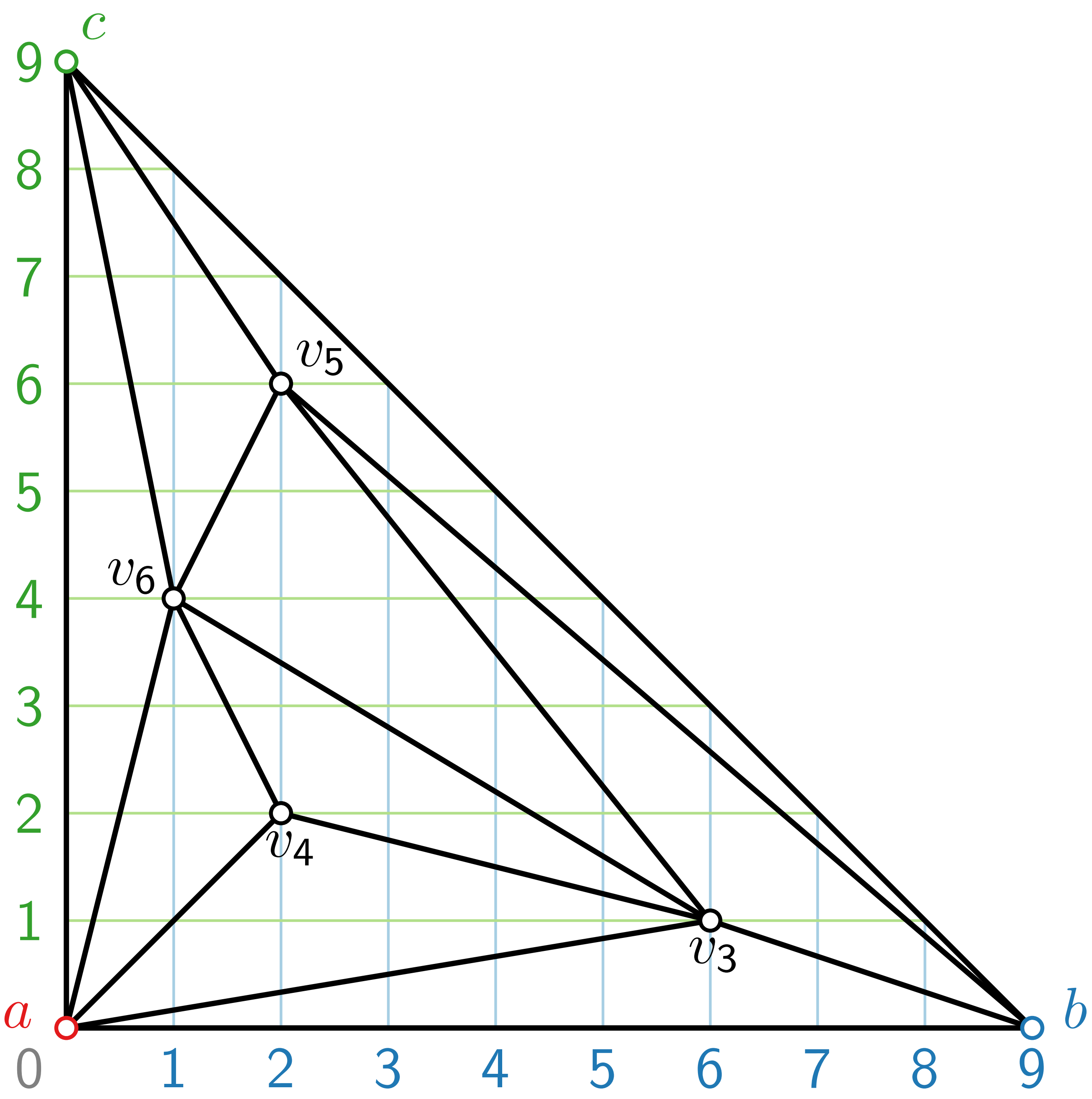

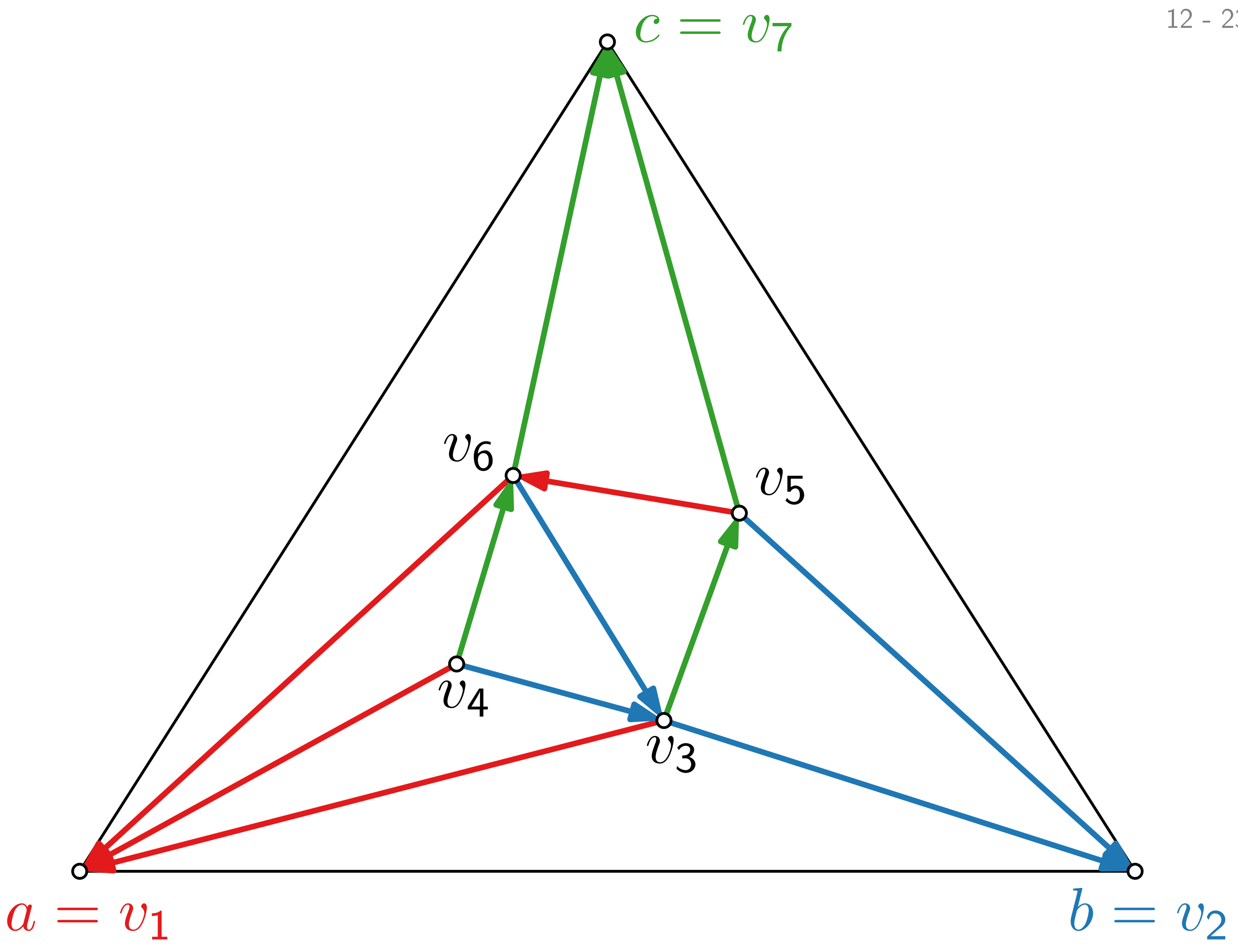

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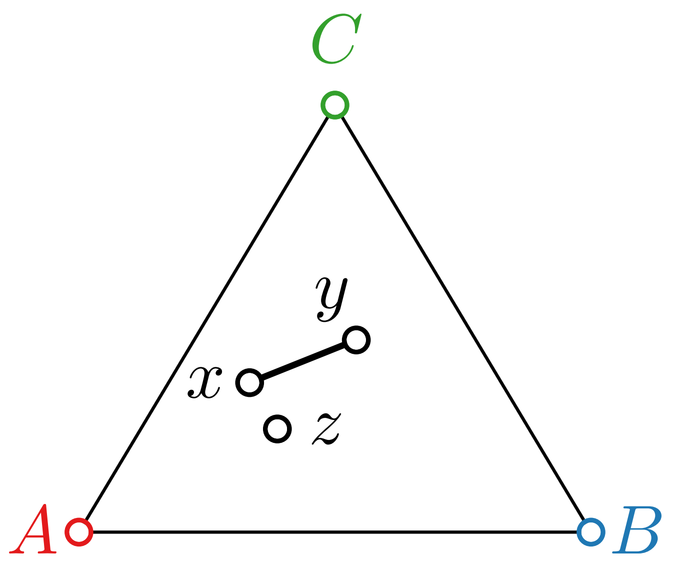

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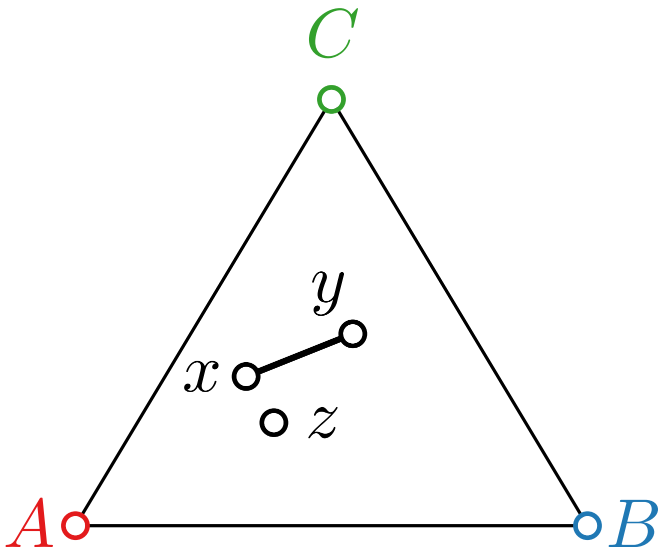

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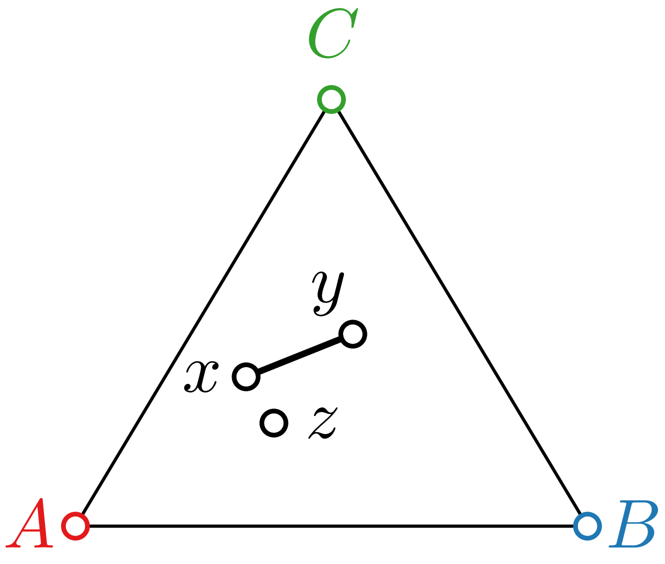
i.e., either $y_k < z_k$ or $(y_k = z_k$ and $y_{k+1} < z_{k+1})$



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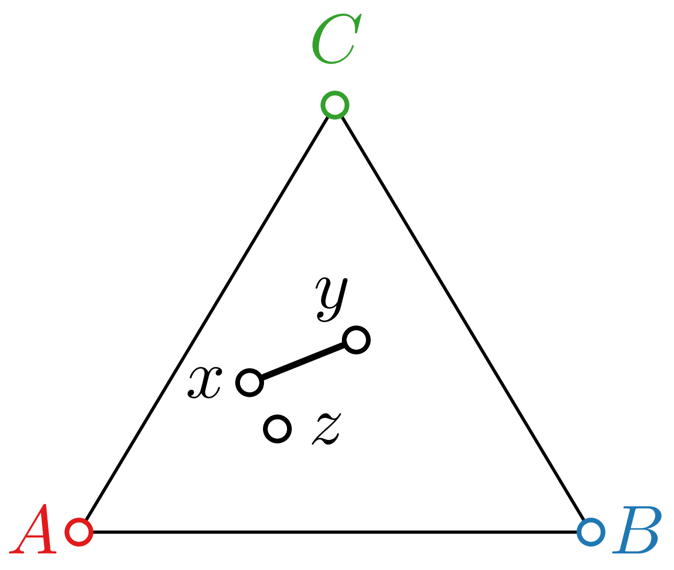

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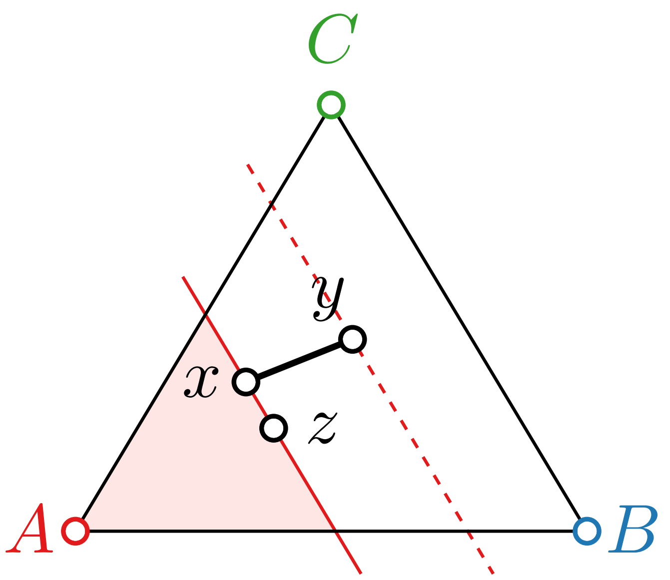

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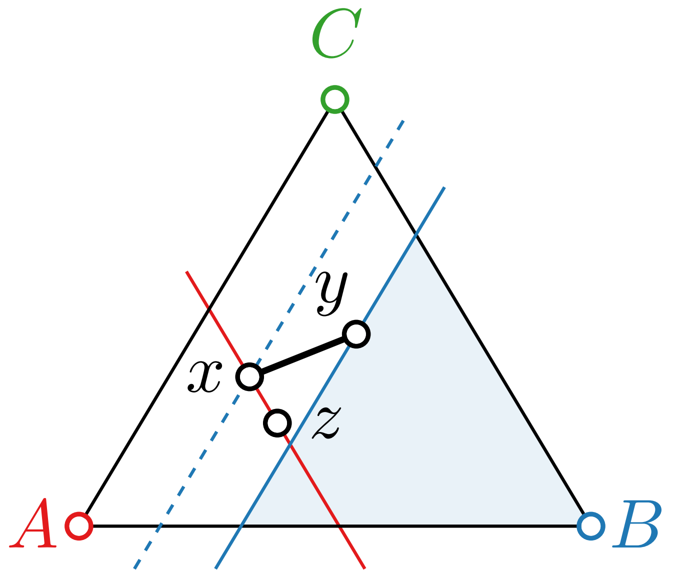

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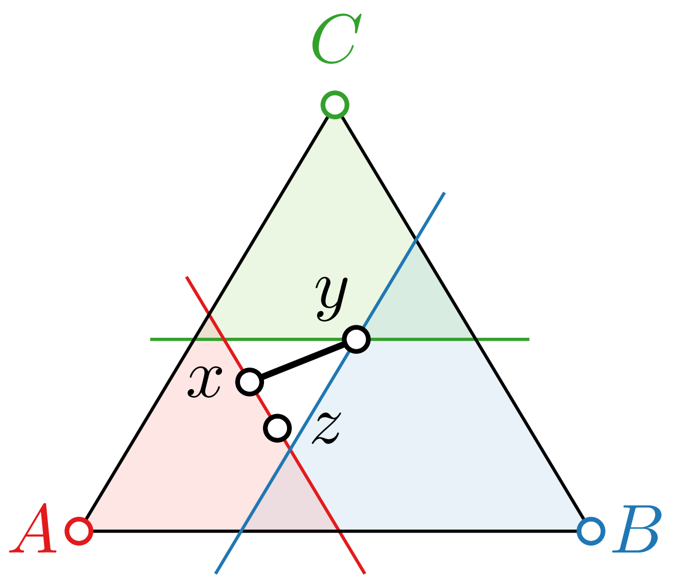

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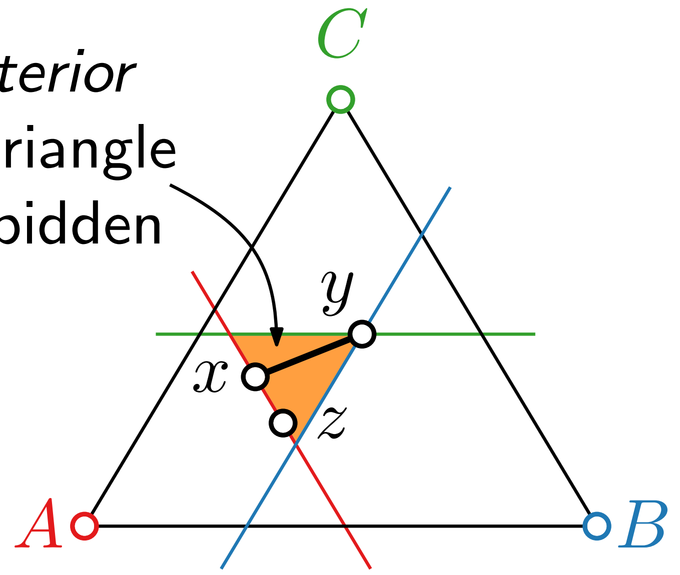

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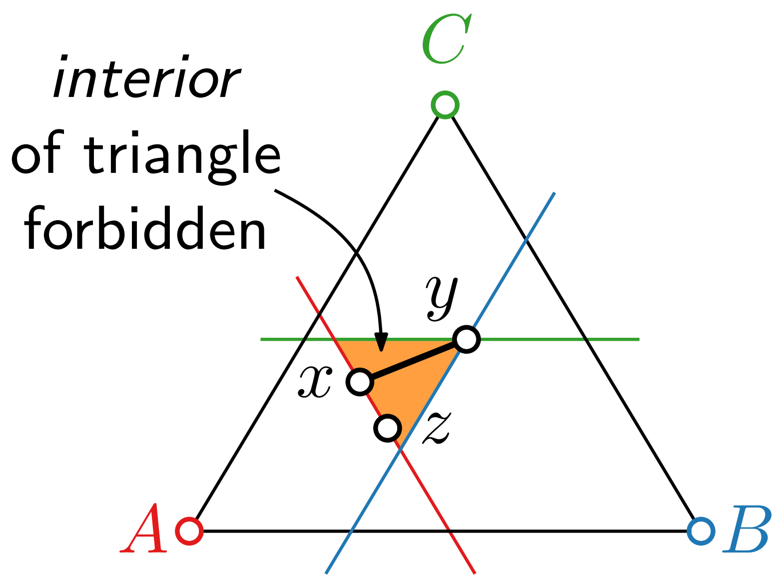

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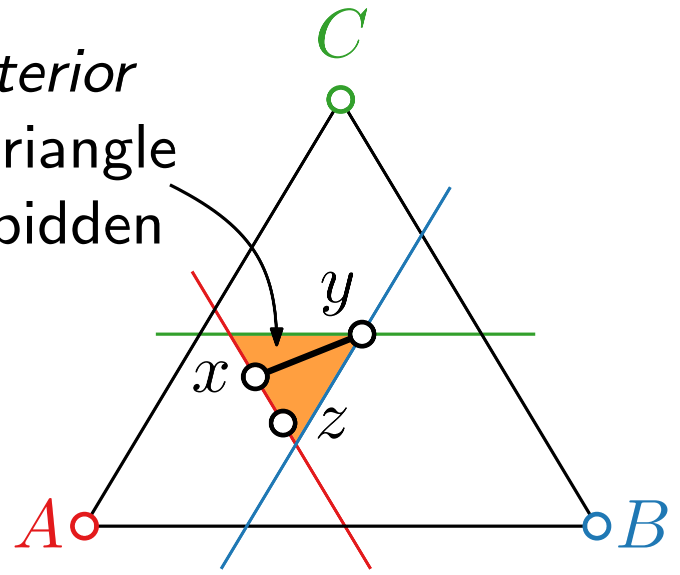

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Proof. $\to$ *Exercise!*

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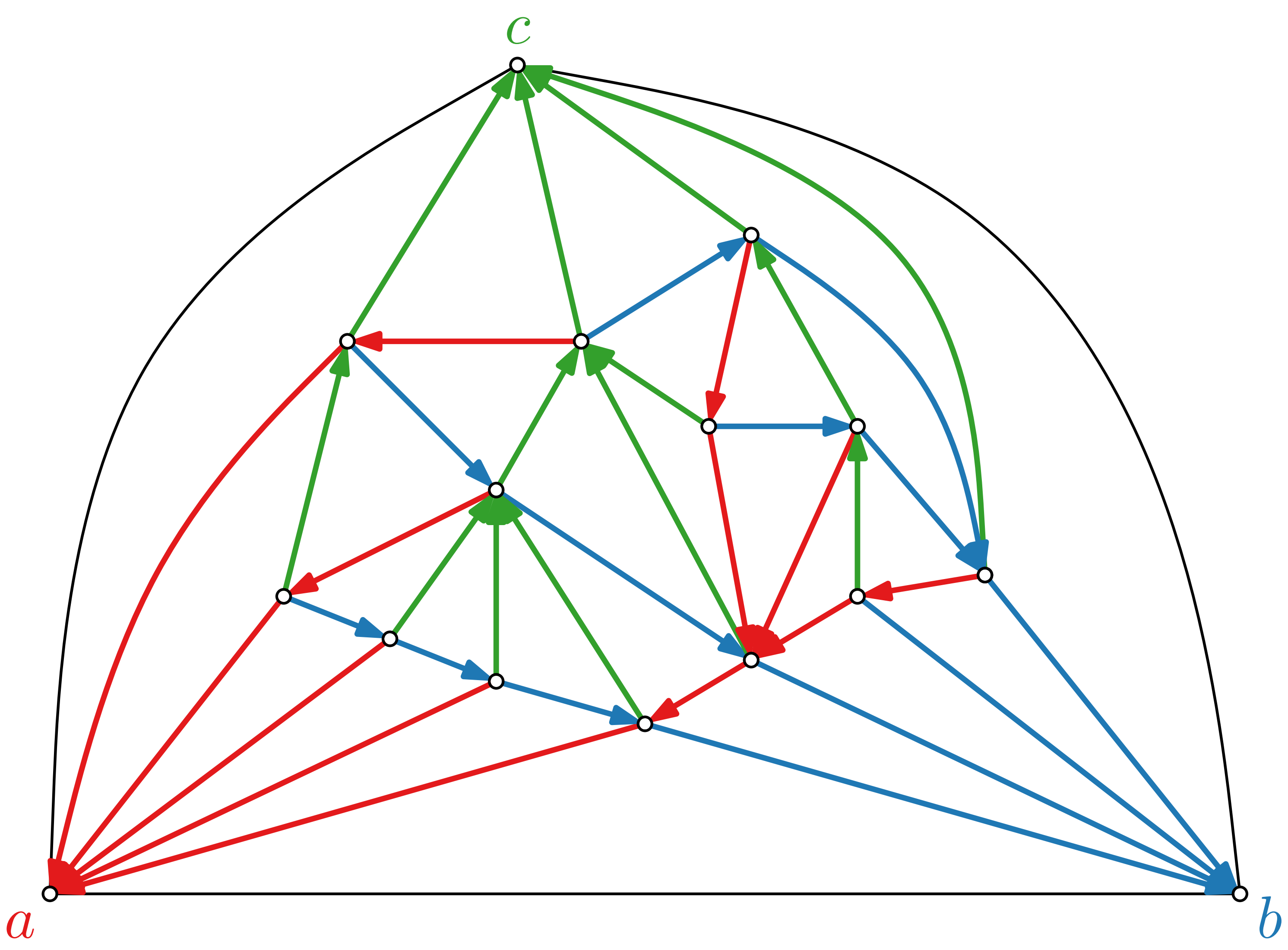

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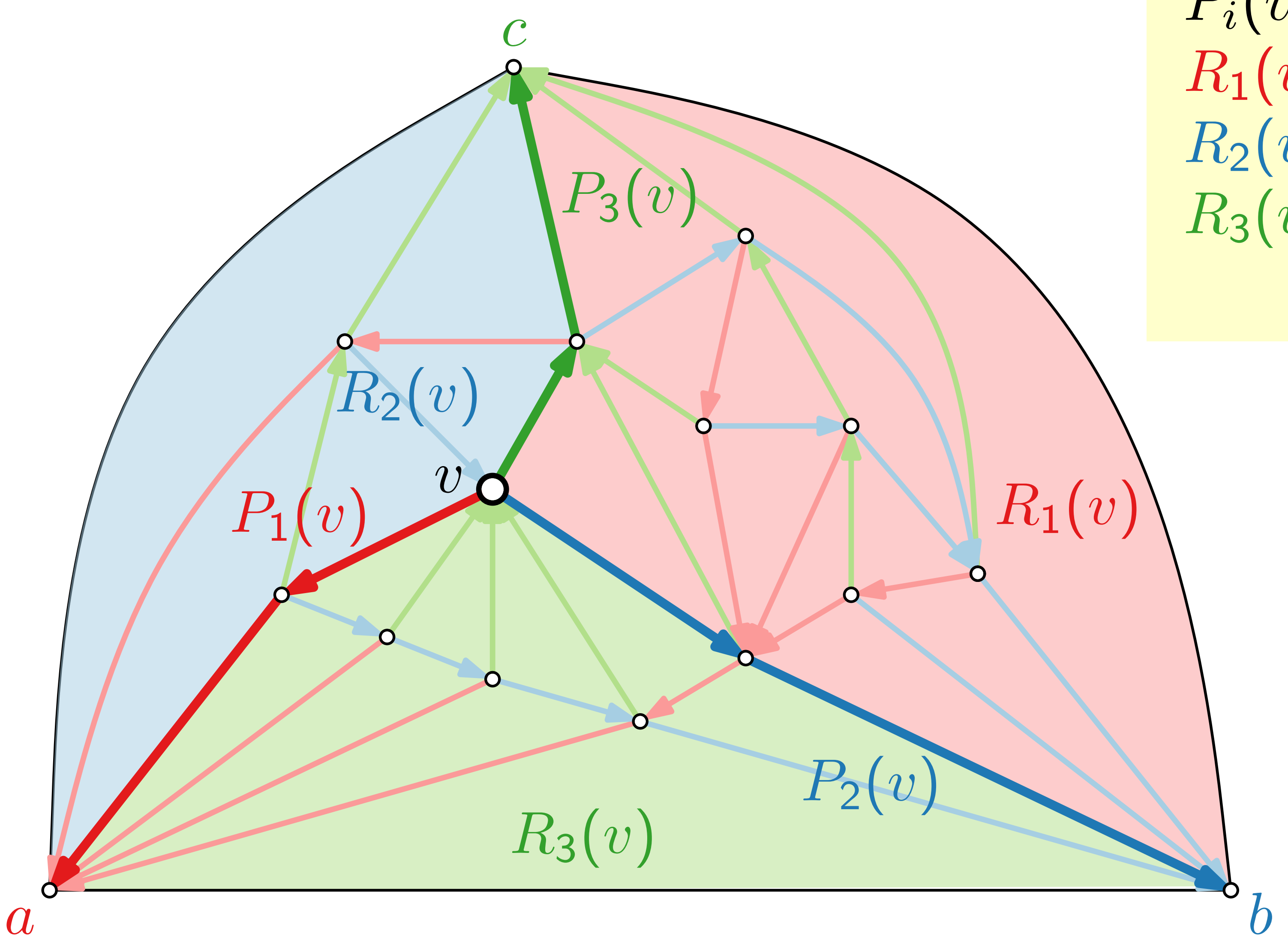

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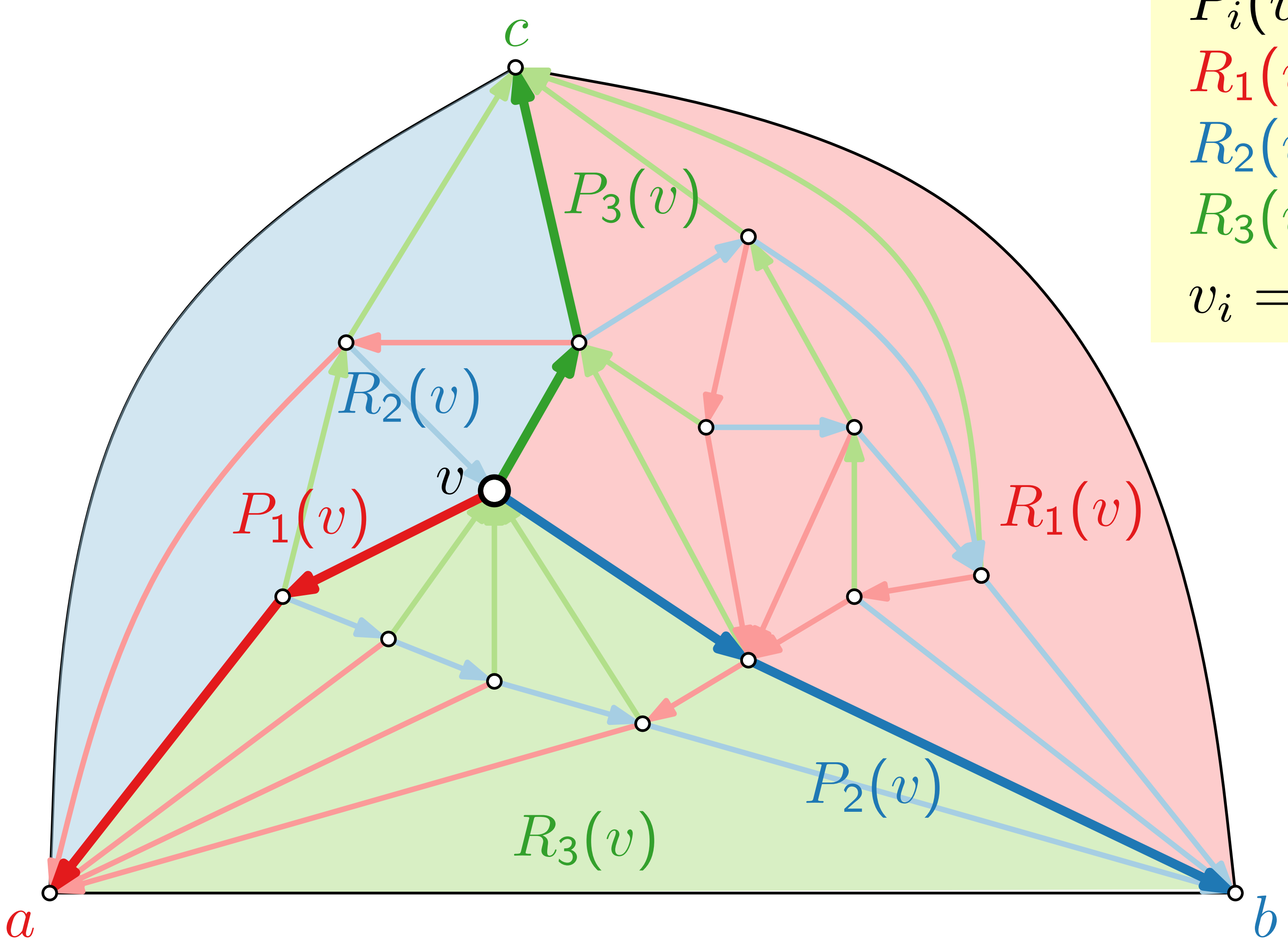

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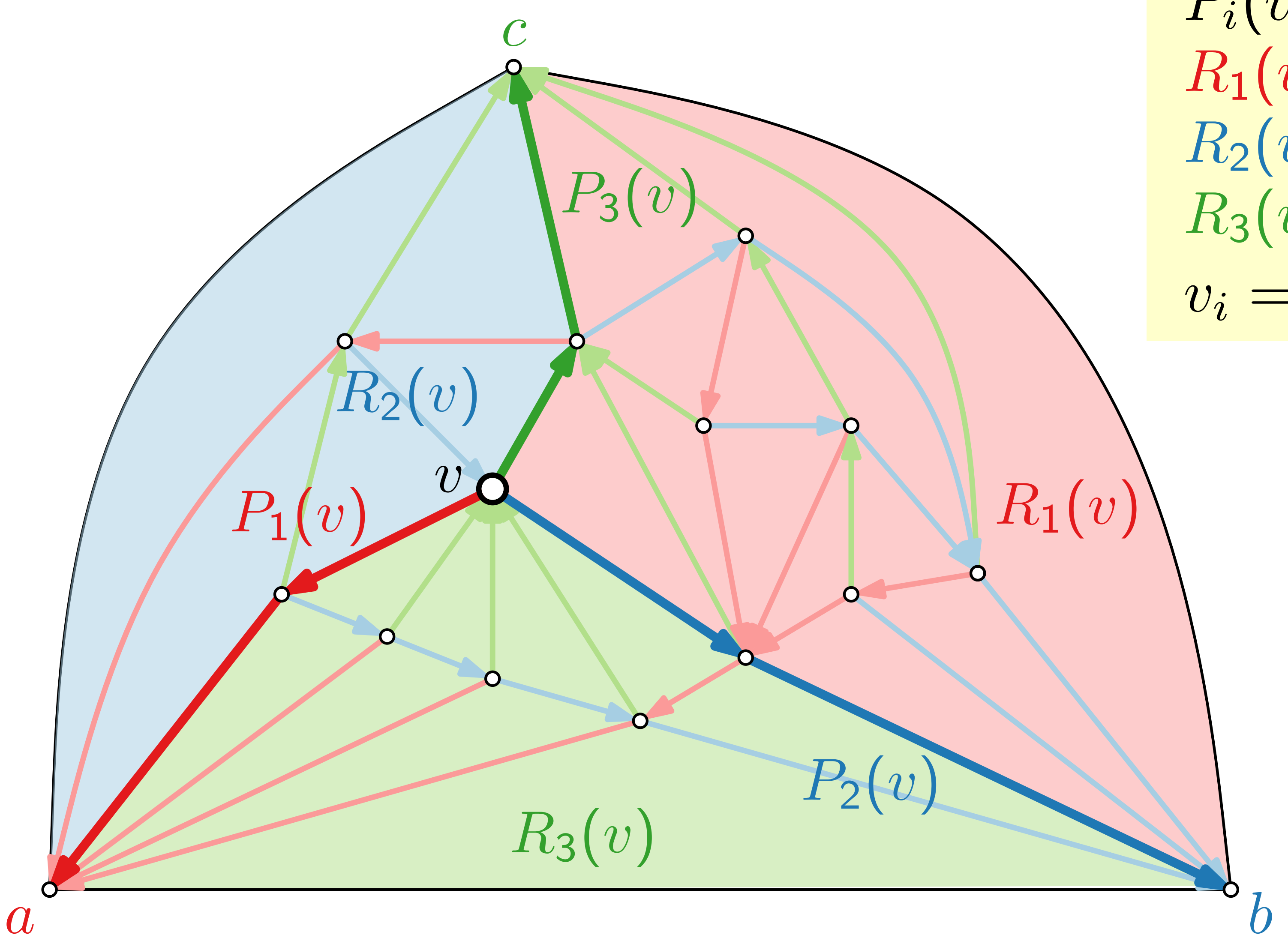

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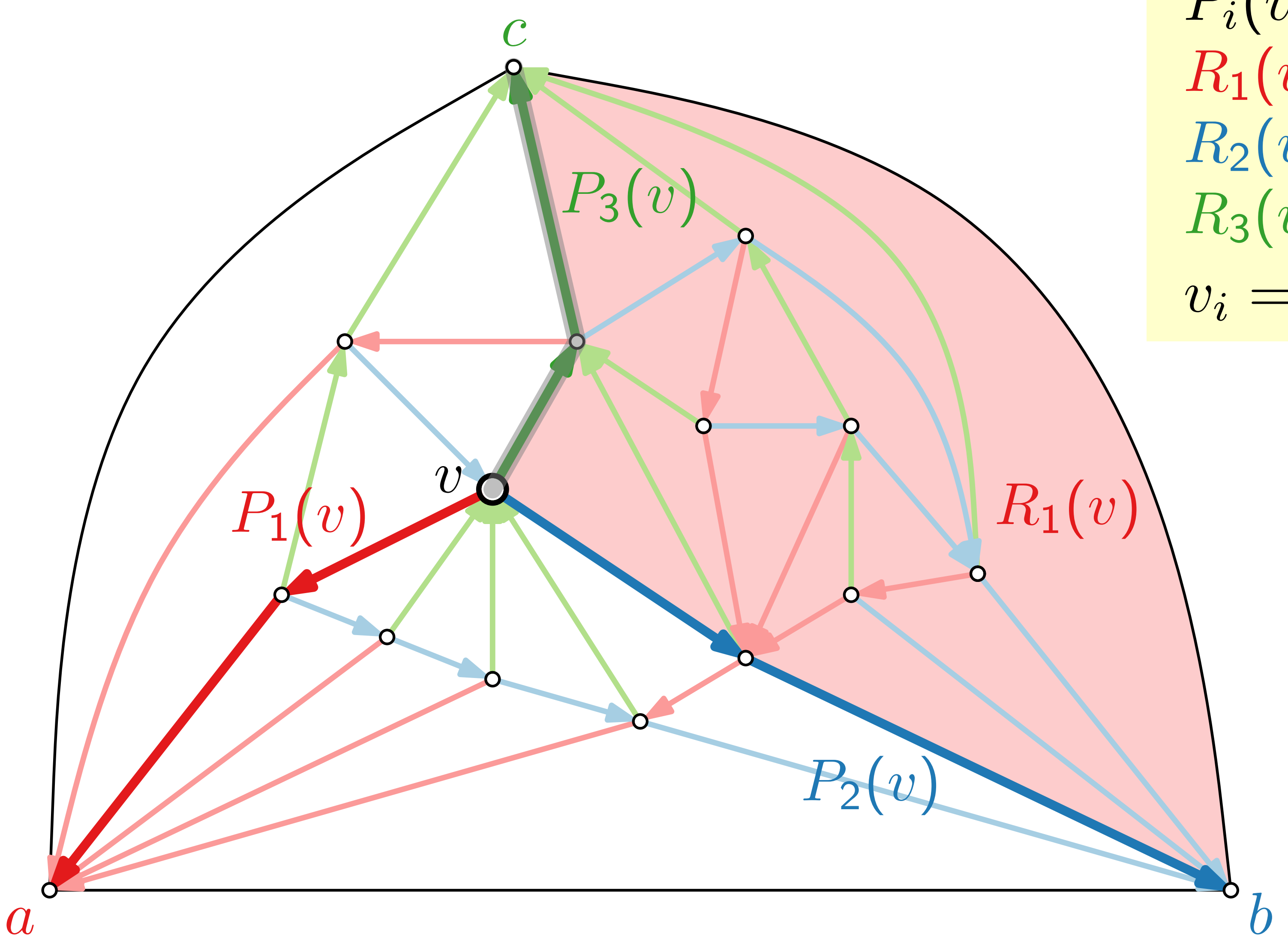

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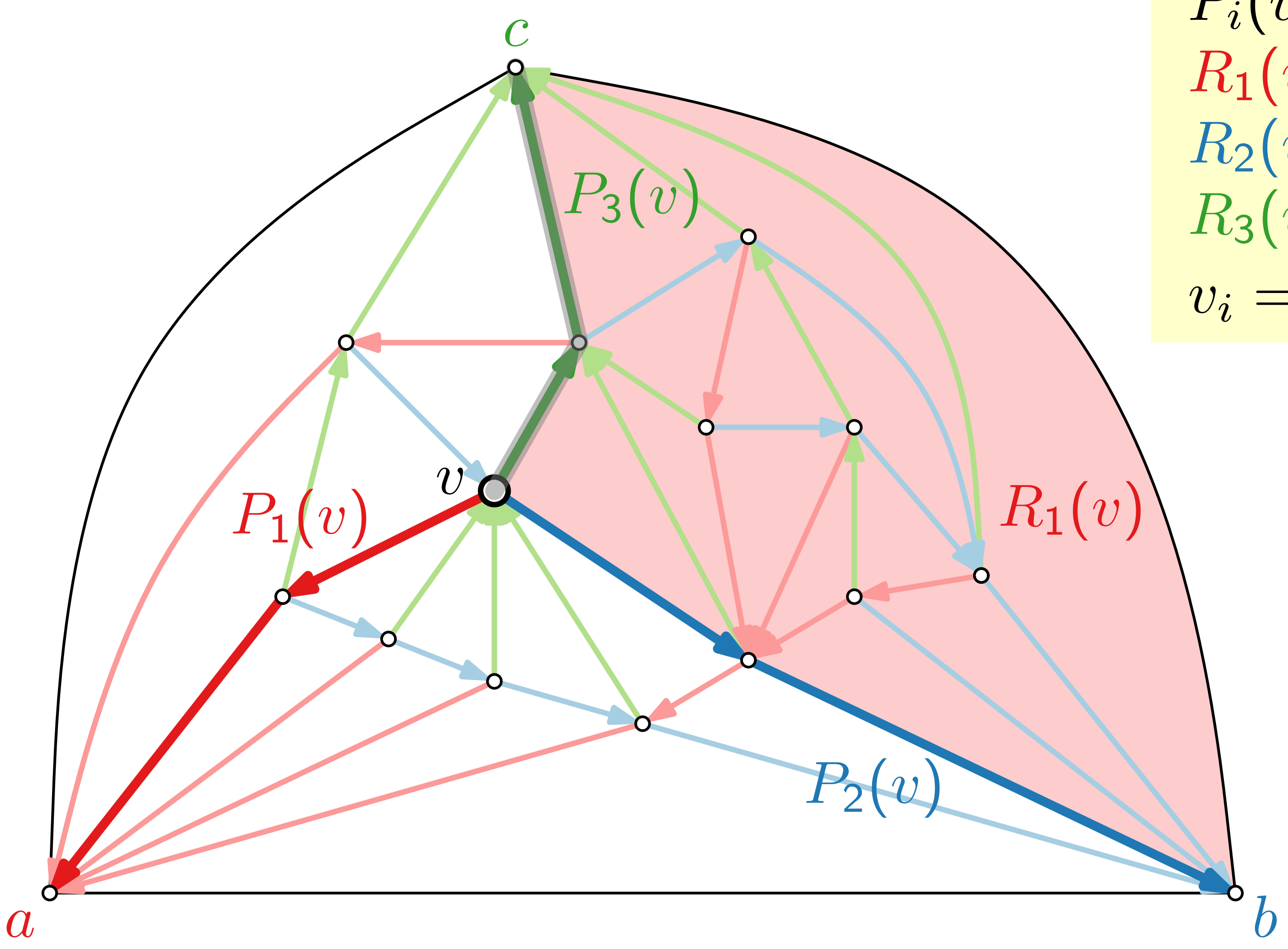

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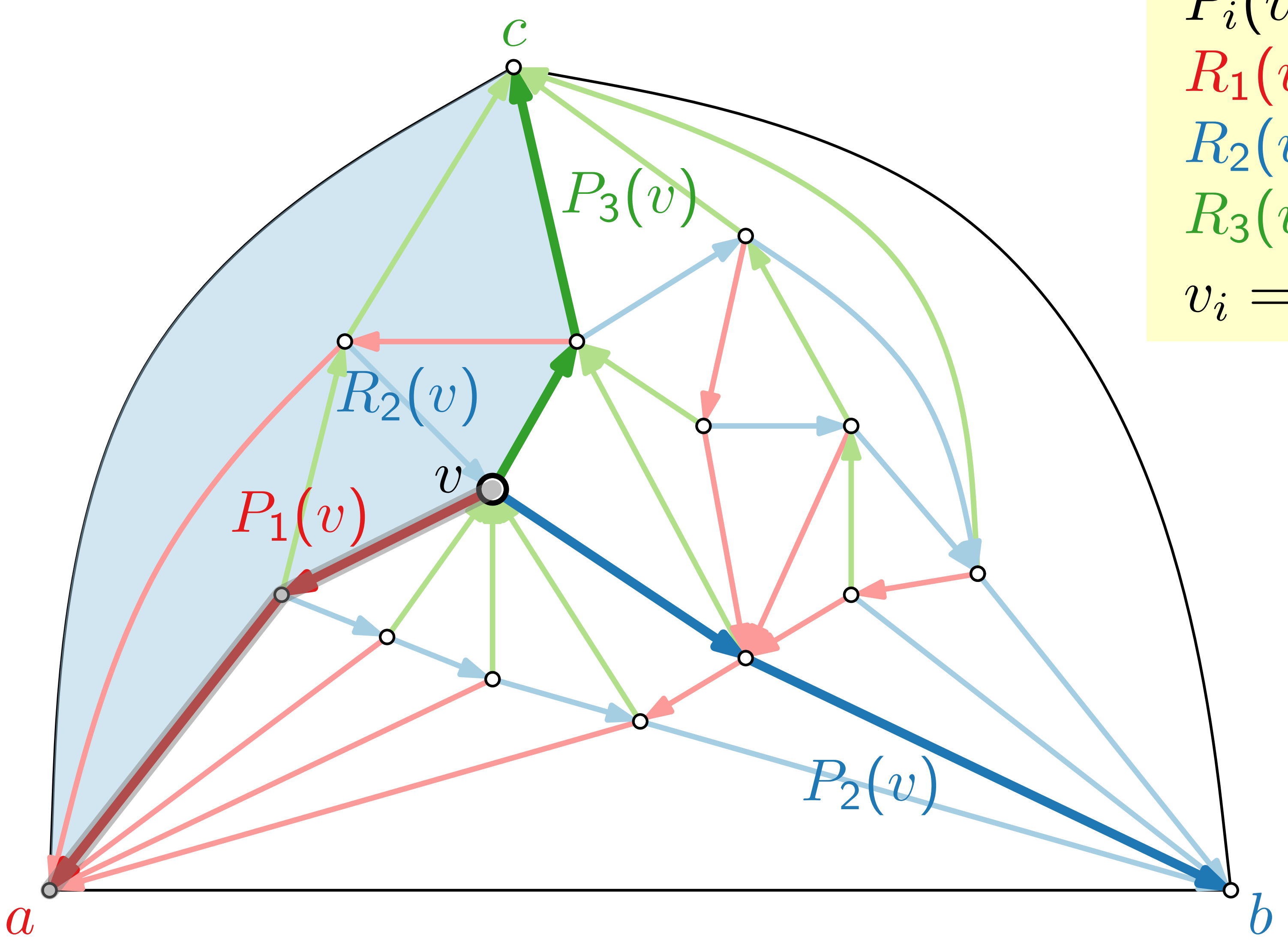

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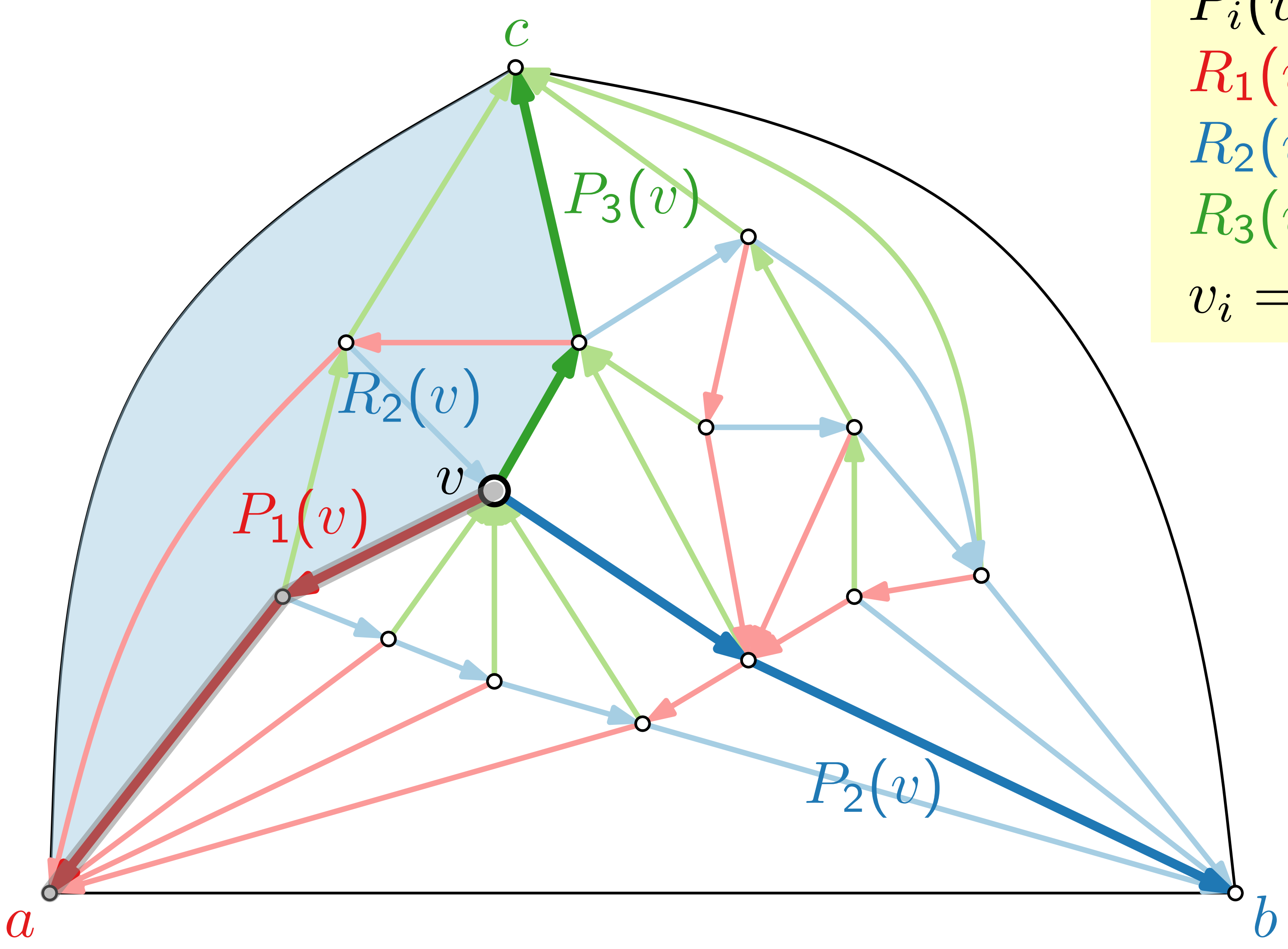

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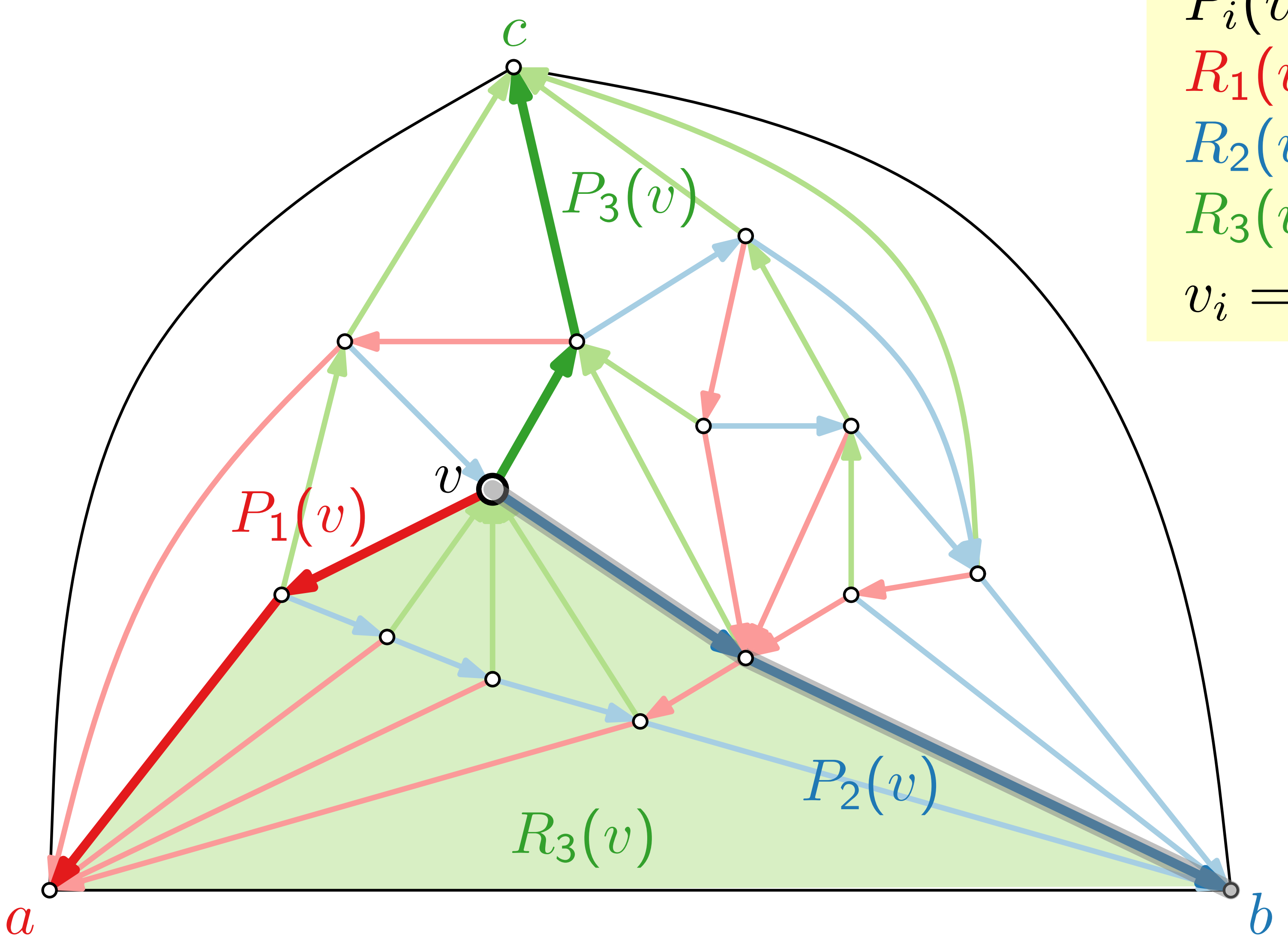

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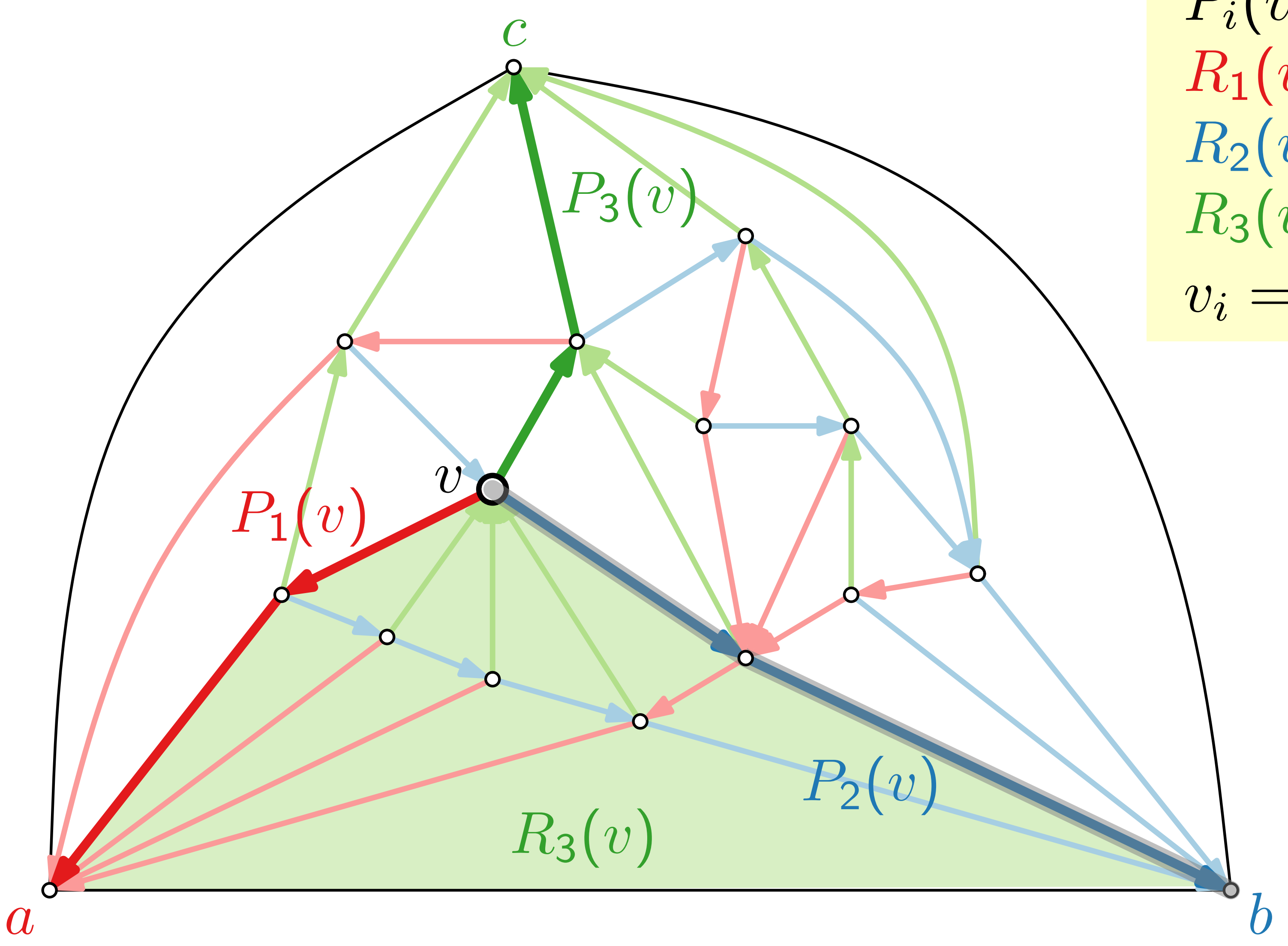

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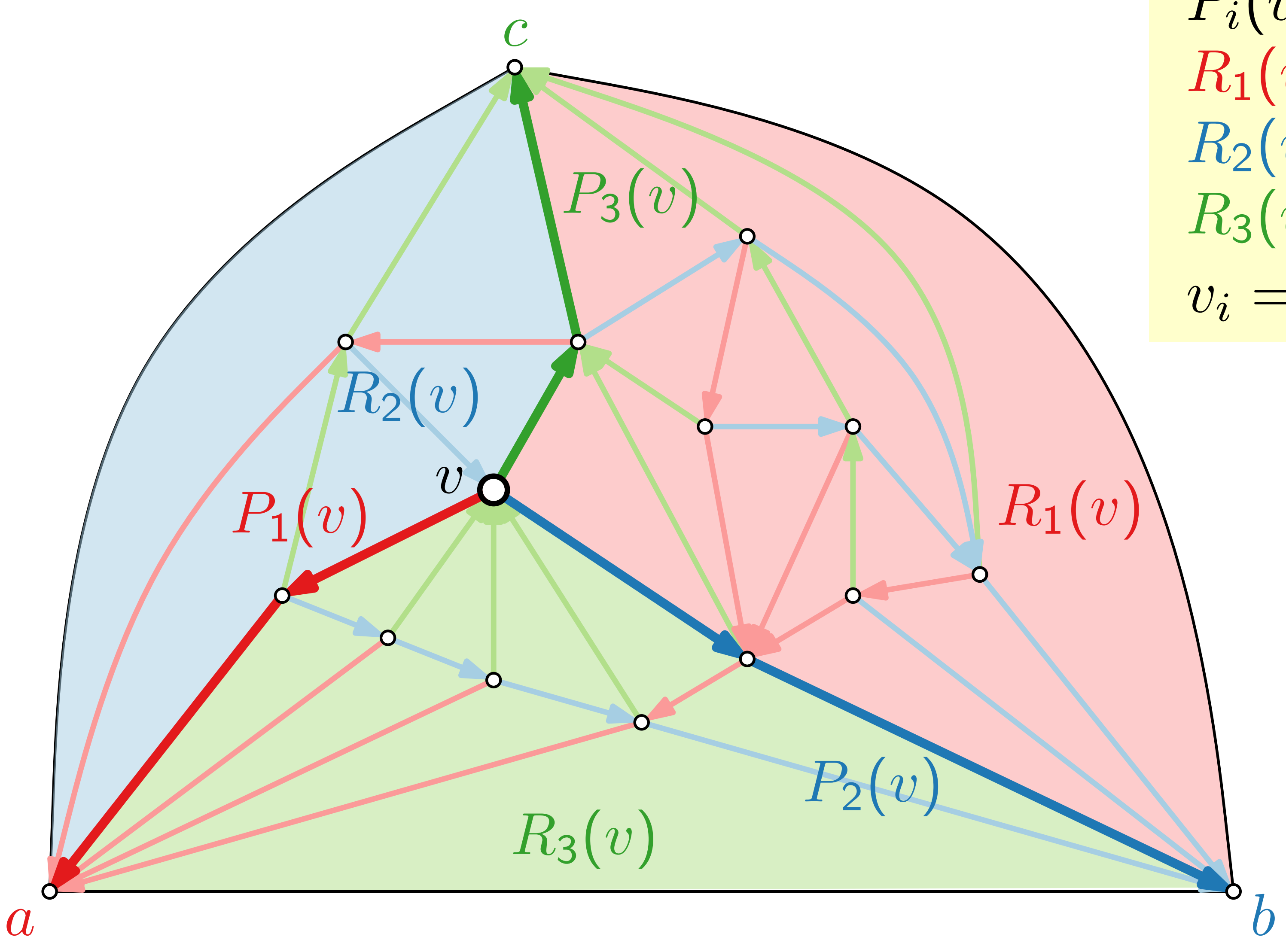

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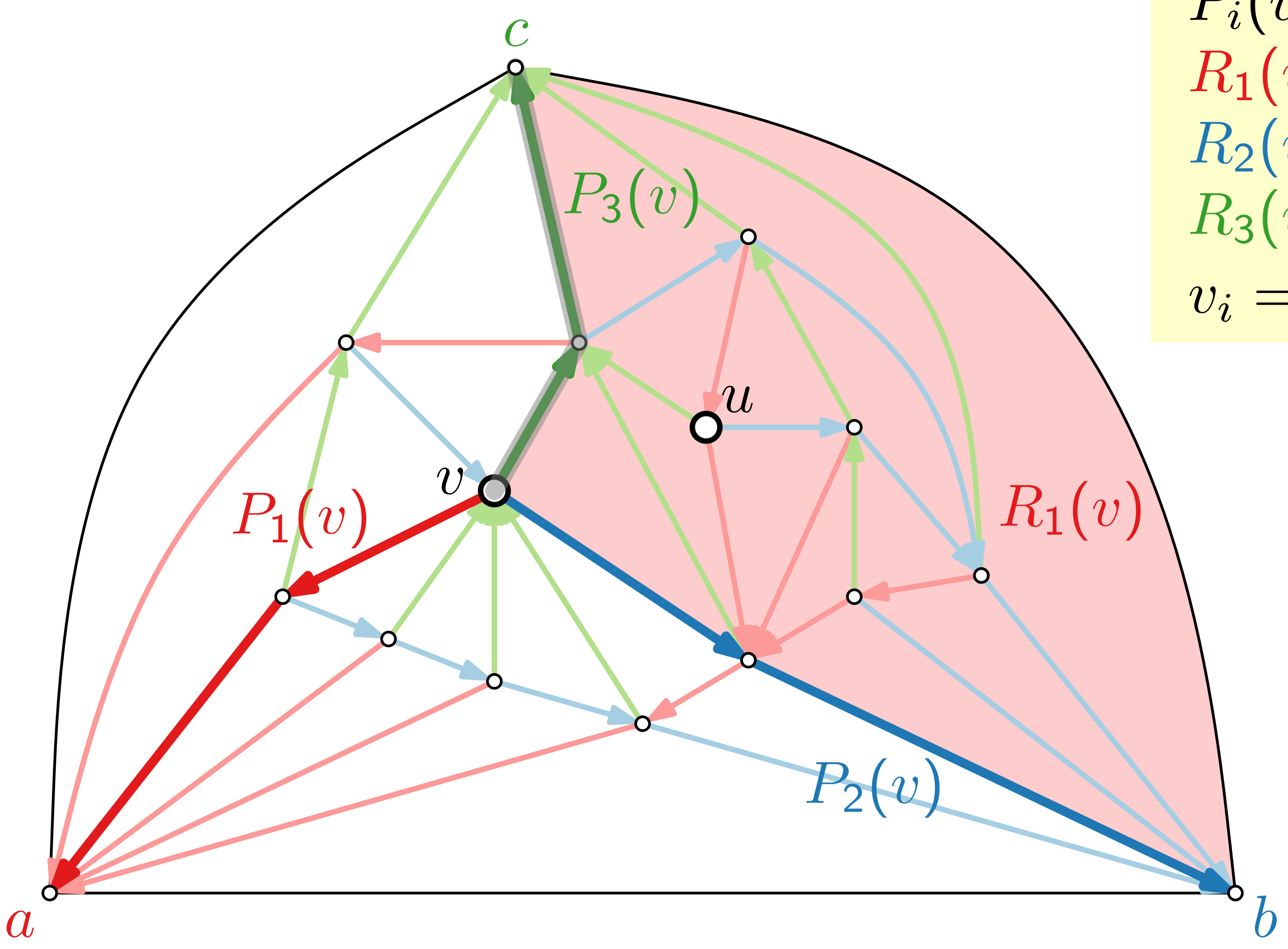

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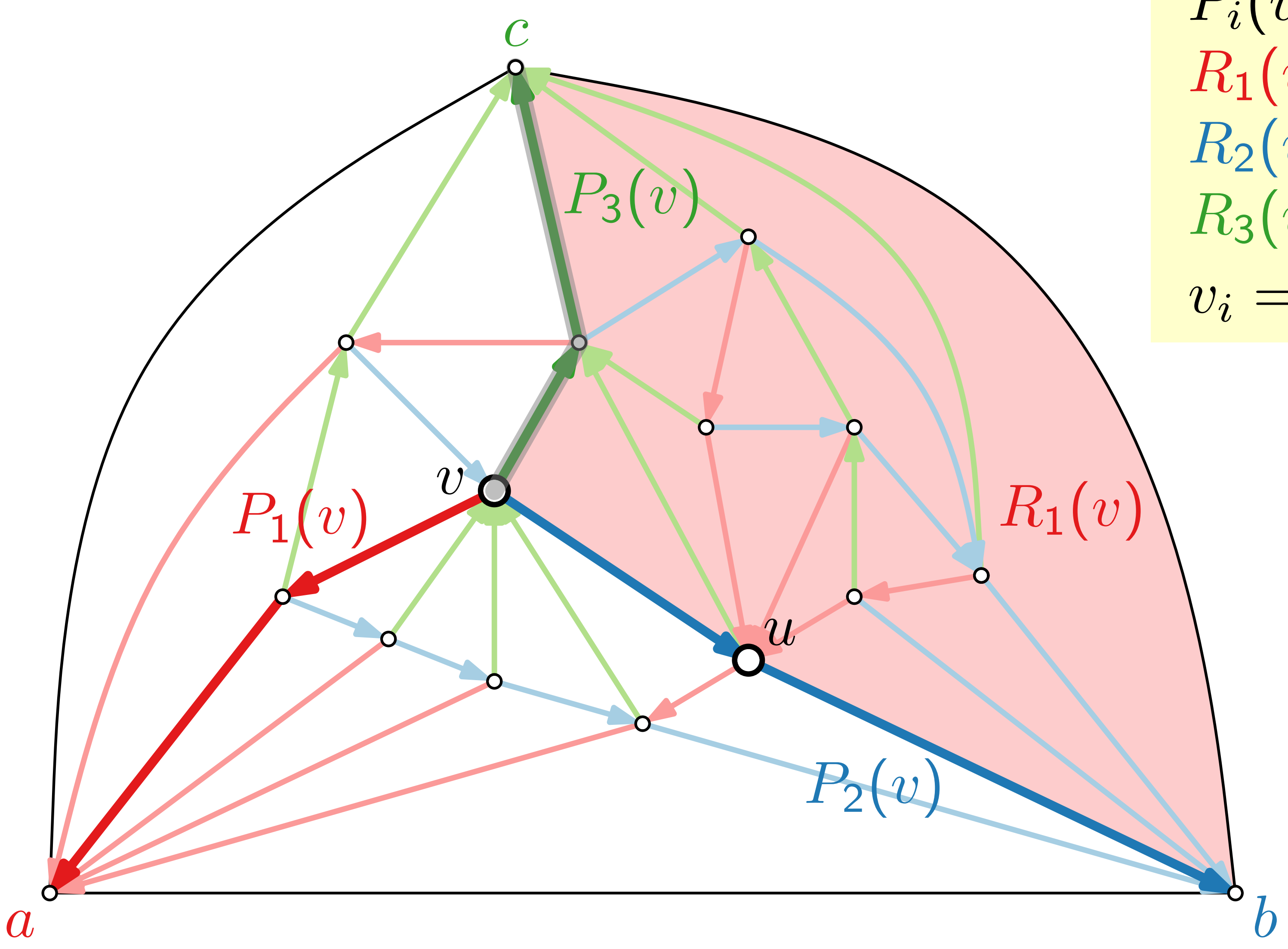

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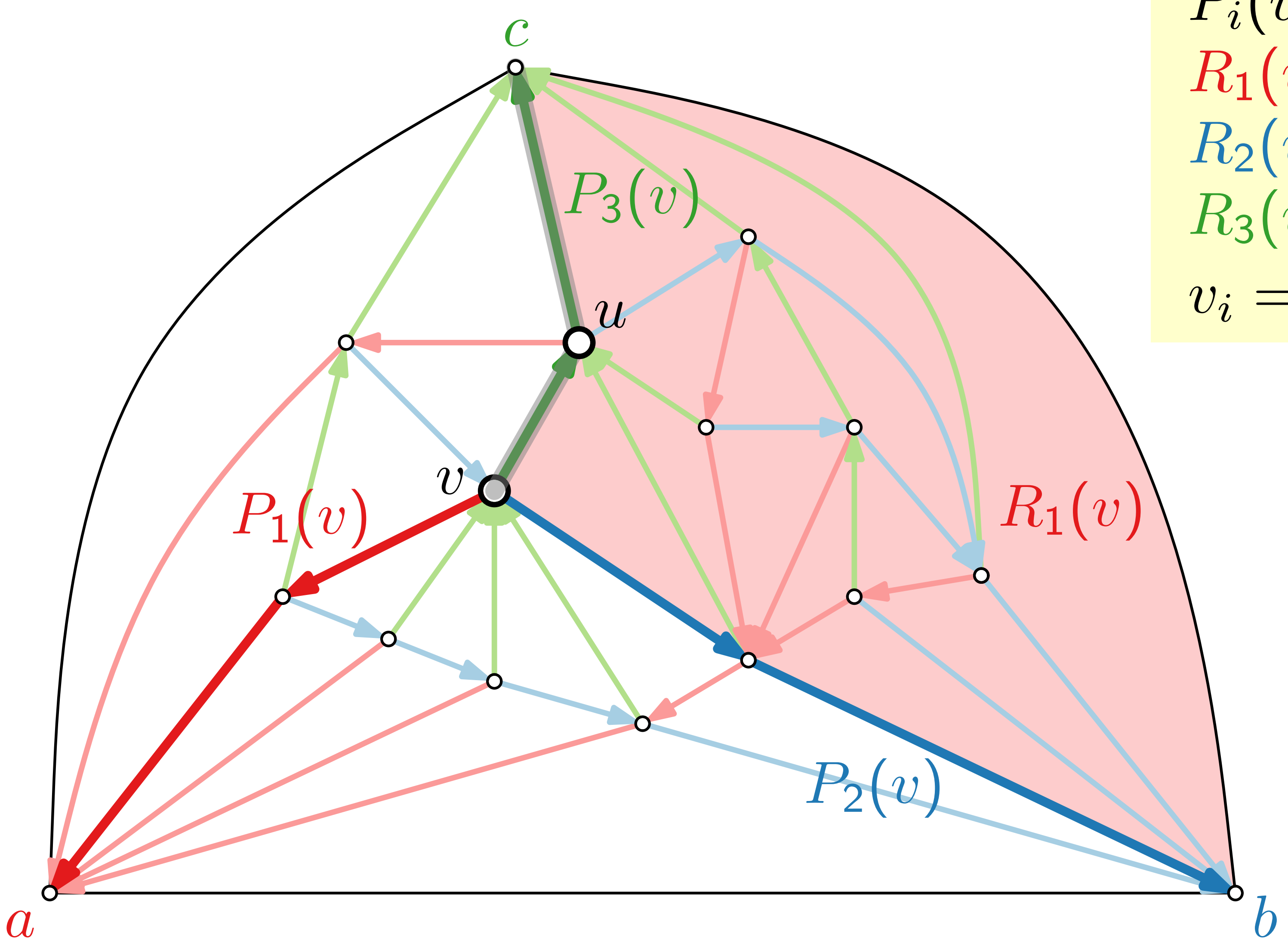

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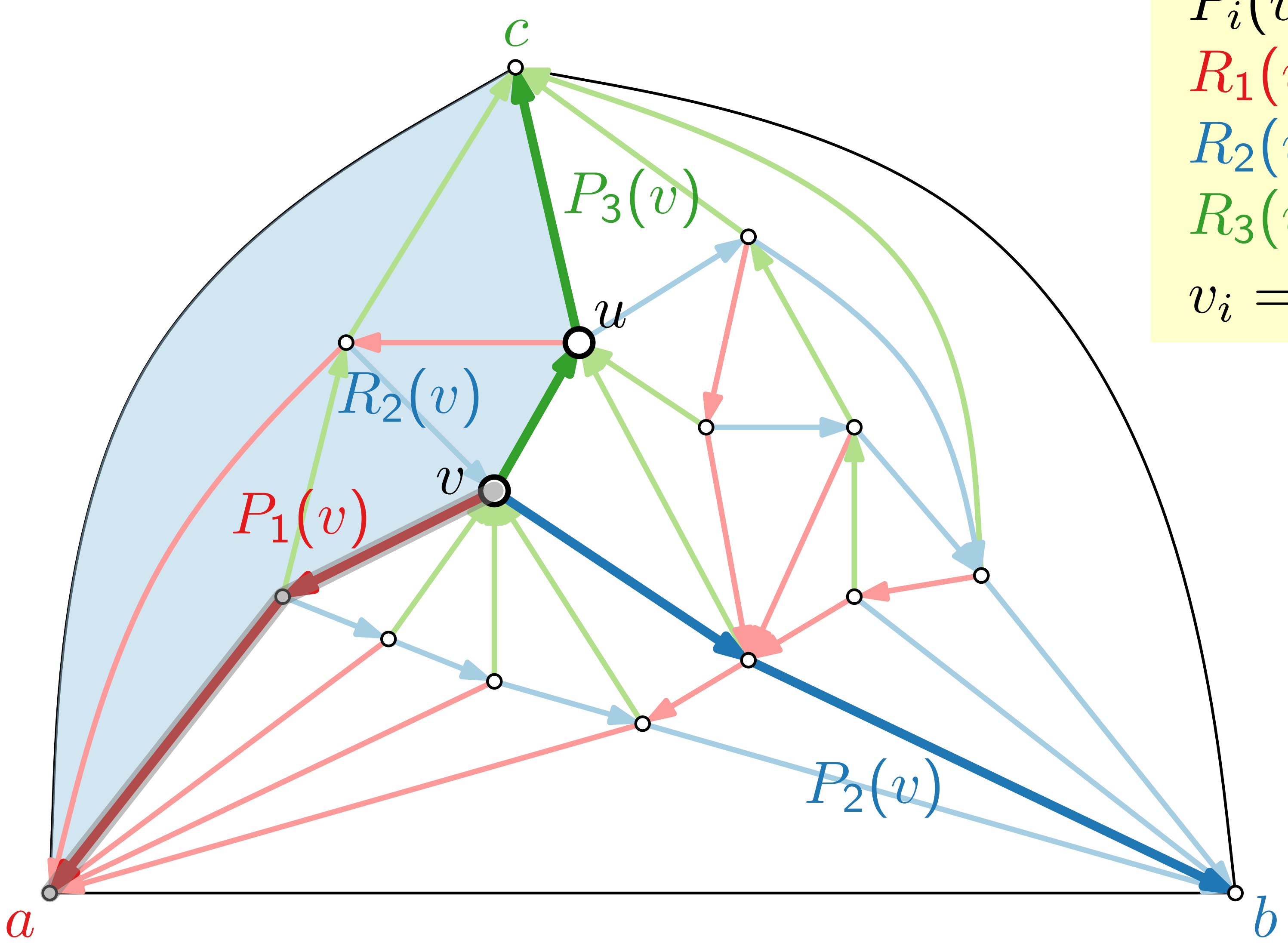

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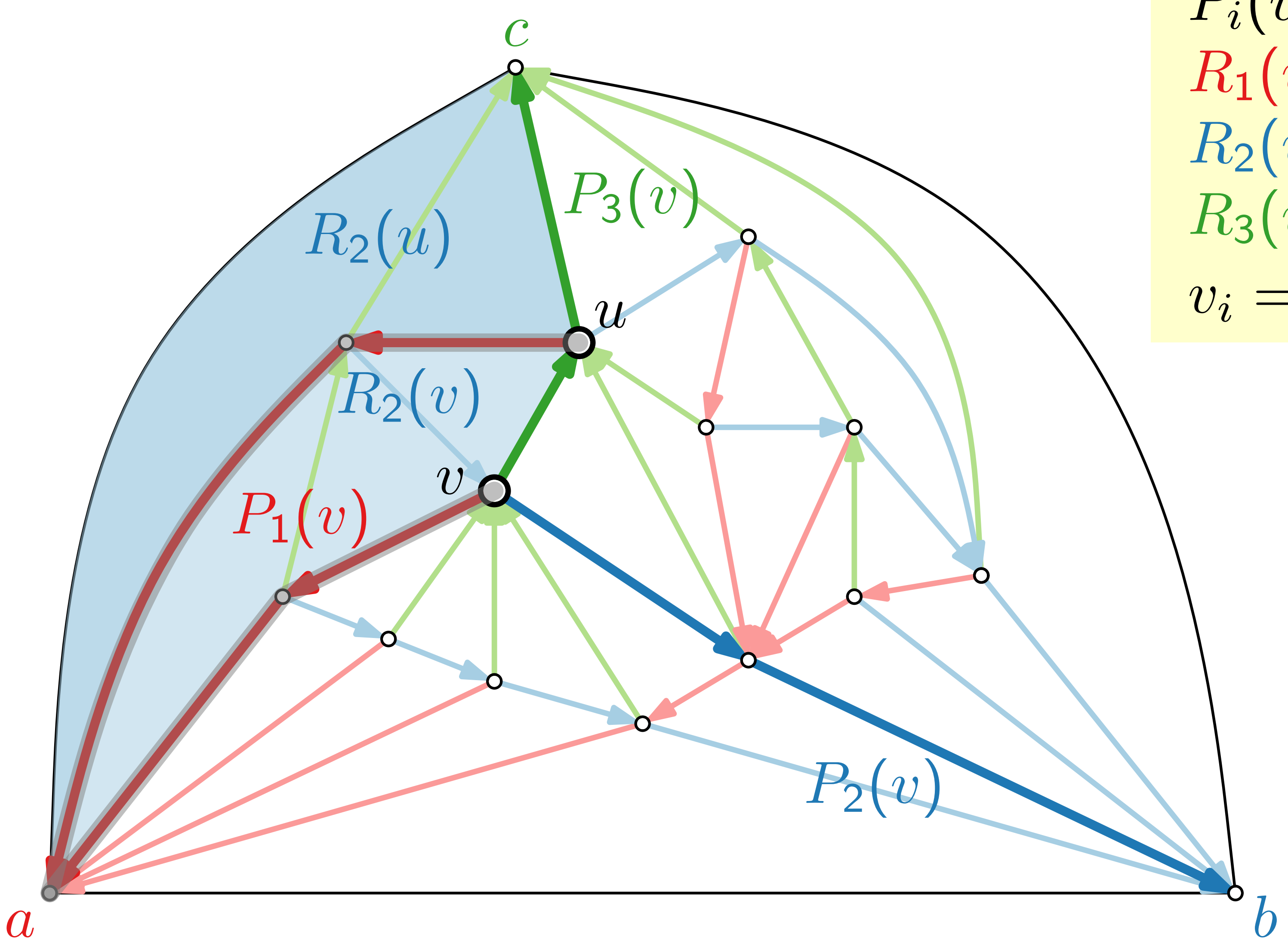

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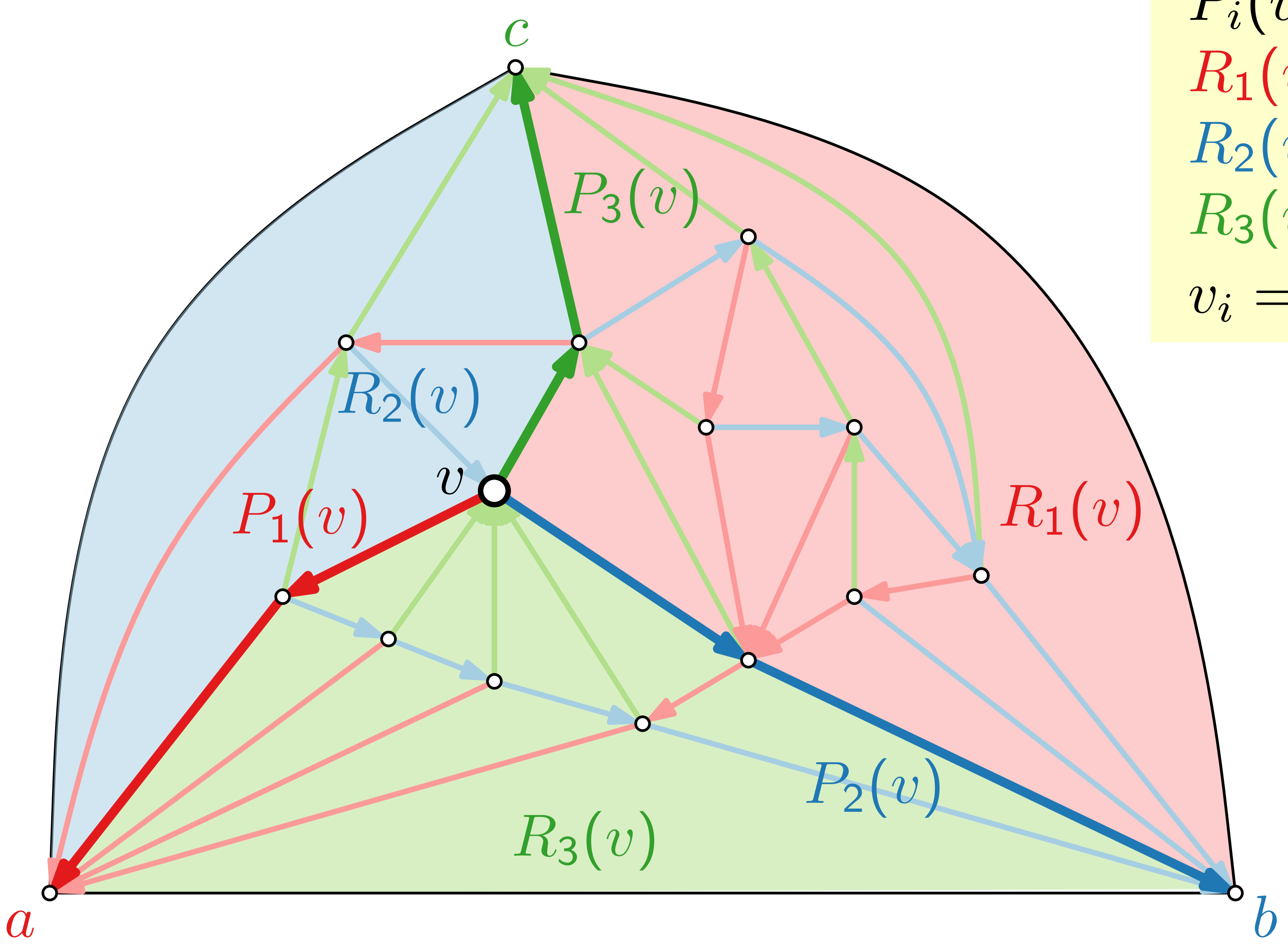

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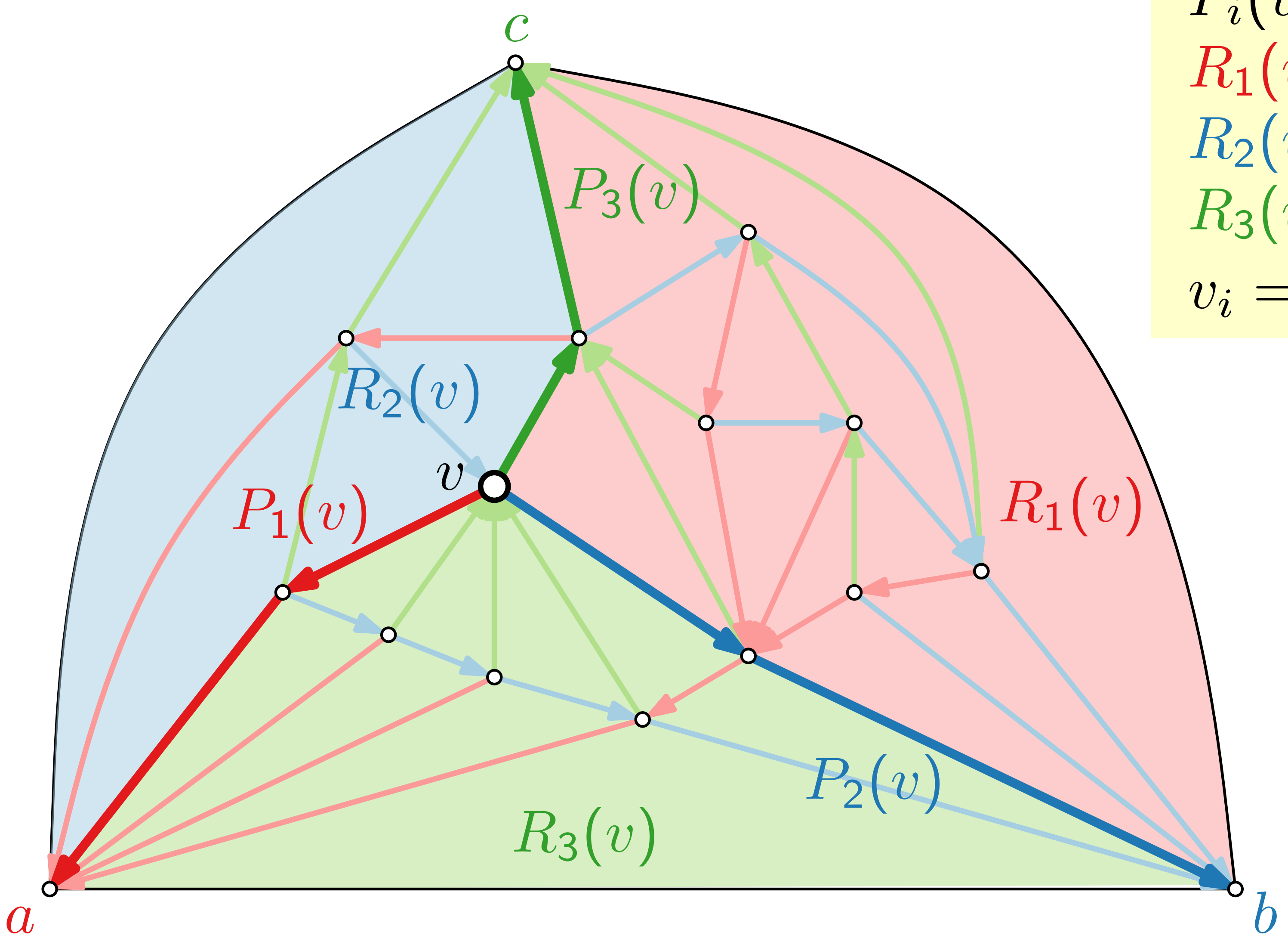

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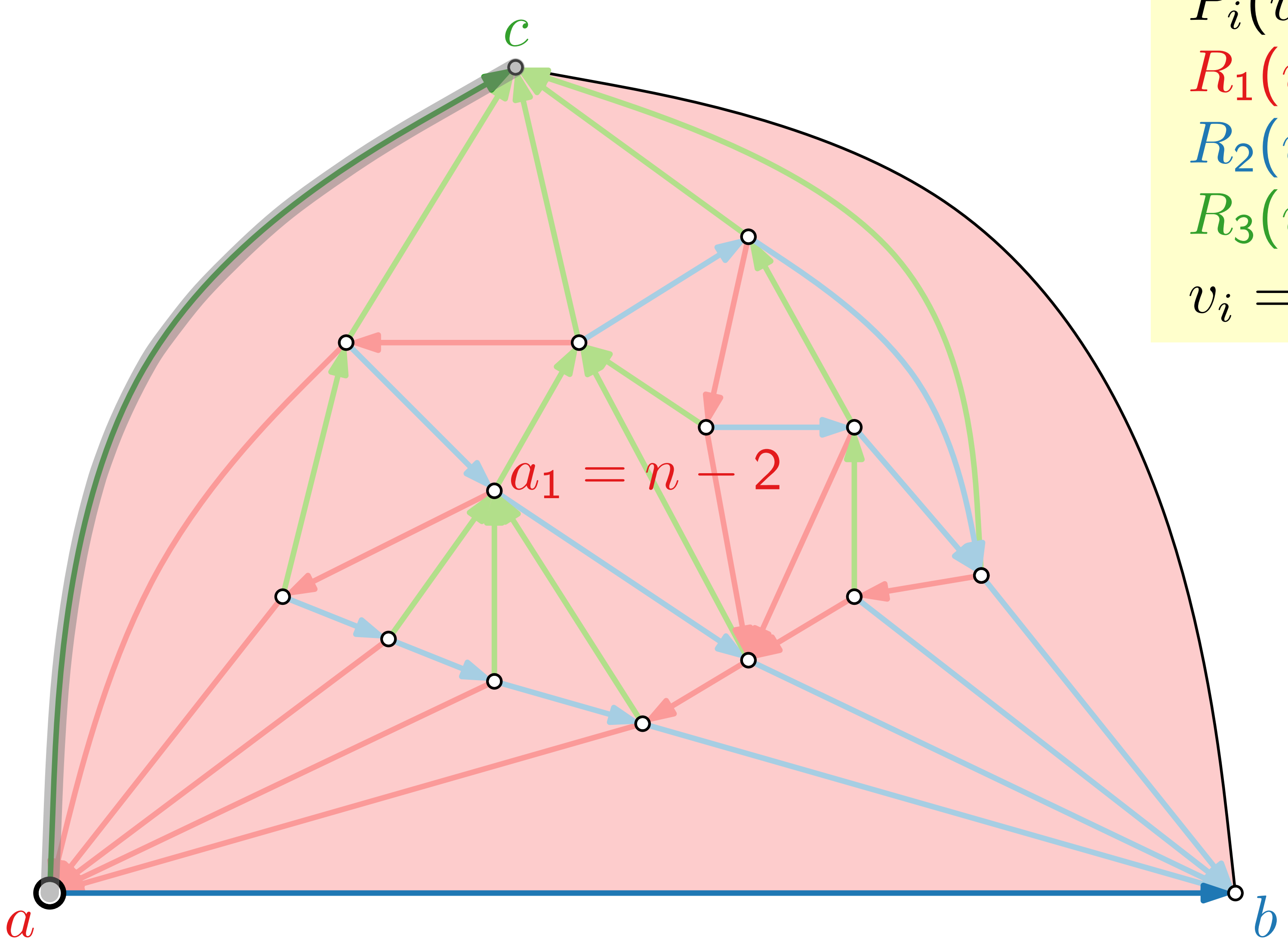
- For inner vertices $u \neq v$, it holds that $u \in R_i(v) \Rightarrow (u_i, u_{i+1}) <_{\text{lex}} (v_i, v_{i+1})$.
- $v_1 + v_2 + v_3 = n - 1$

Counting Vertices



$P_i(v)$: unique path from v to root of T_i

$R_1(v)$: subgraph bounded by $\langle P_2(v), bc, P_3(v)\rangle$

$R_2(v)$: subgraph bounded by $\langle P_3(v), ca, P_1(v)\rangle$

$R_3(v)$: subgraph bounded by $\langle P_1(v), ab, P_2(v)\rangle$

$v_i = |V(R_i(v))| - |V(P_{i-1}(v))|$ (indices modulo 3)

$v_1 = 10 - 3 = 7$

$v_2 = 6 - 3 = 3$

$v_3 = 8 - 3 = 5$

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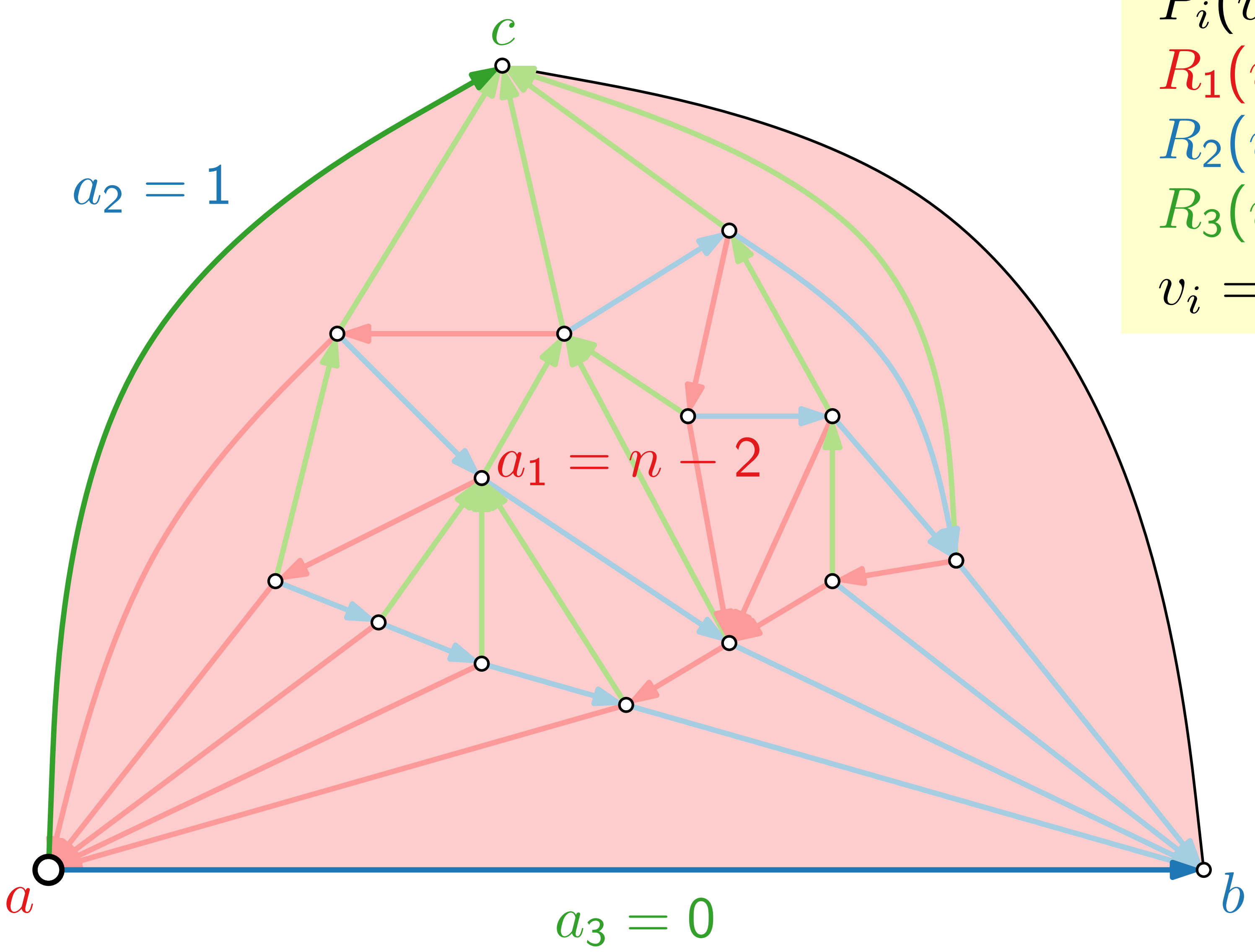

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Schnyder Drawing*

Set $A = (0, 0)$, $B = (n-1, 0)$, and $C = (0, n-1)$.

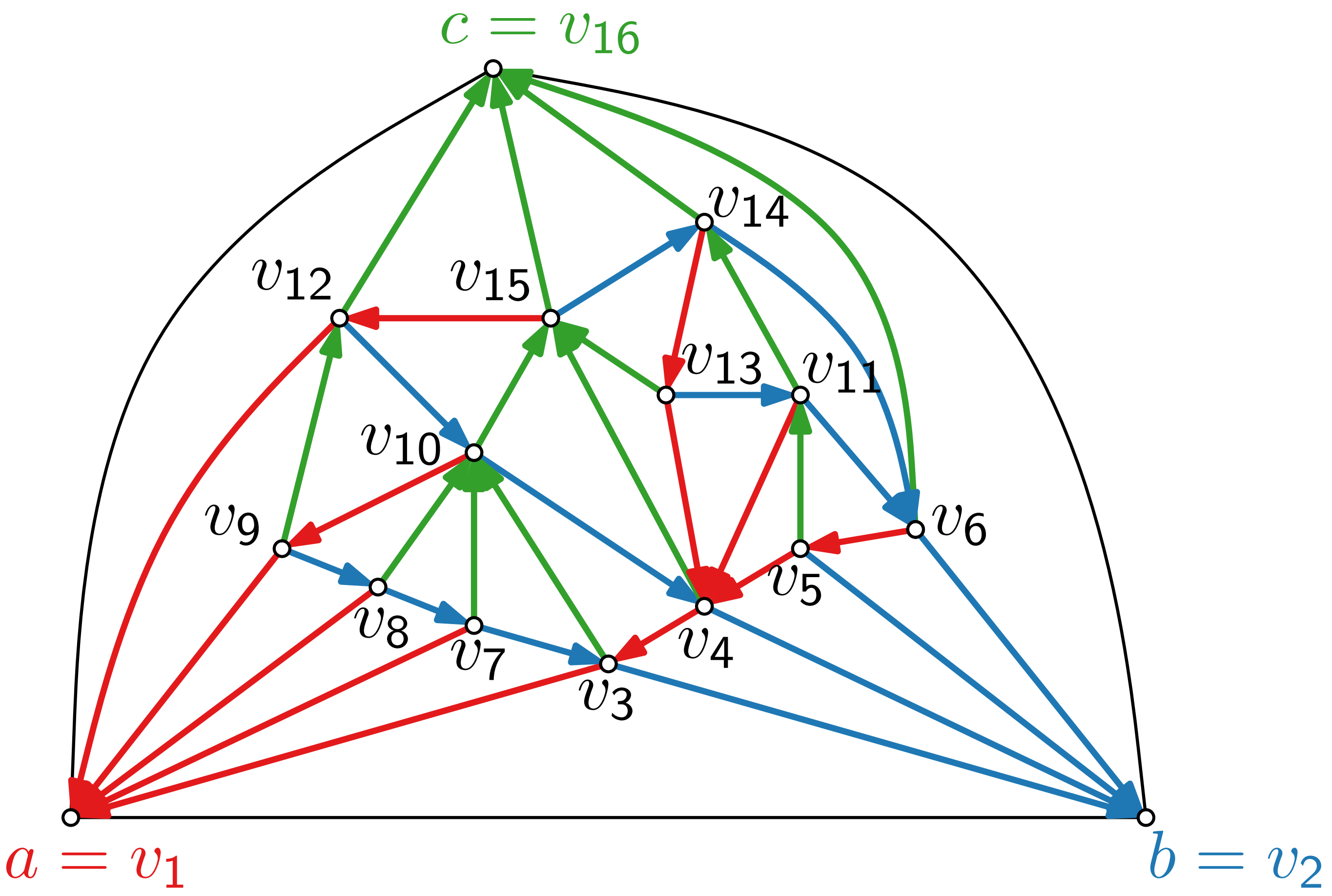
Theorem. [Schnyder '90]

For a plane triangulation G, the mapping

$$f: v \mapsto \frac{1}{n-1}(v_1, v_2, v_3)$$

is a weak barycentric representation of G and, thus, yields a planar straight-line drawing of G on the $(n-2) \times (n-2)$ grid.

Schnyder Drawing* – Example



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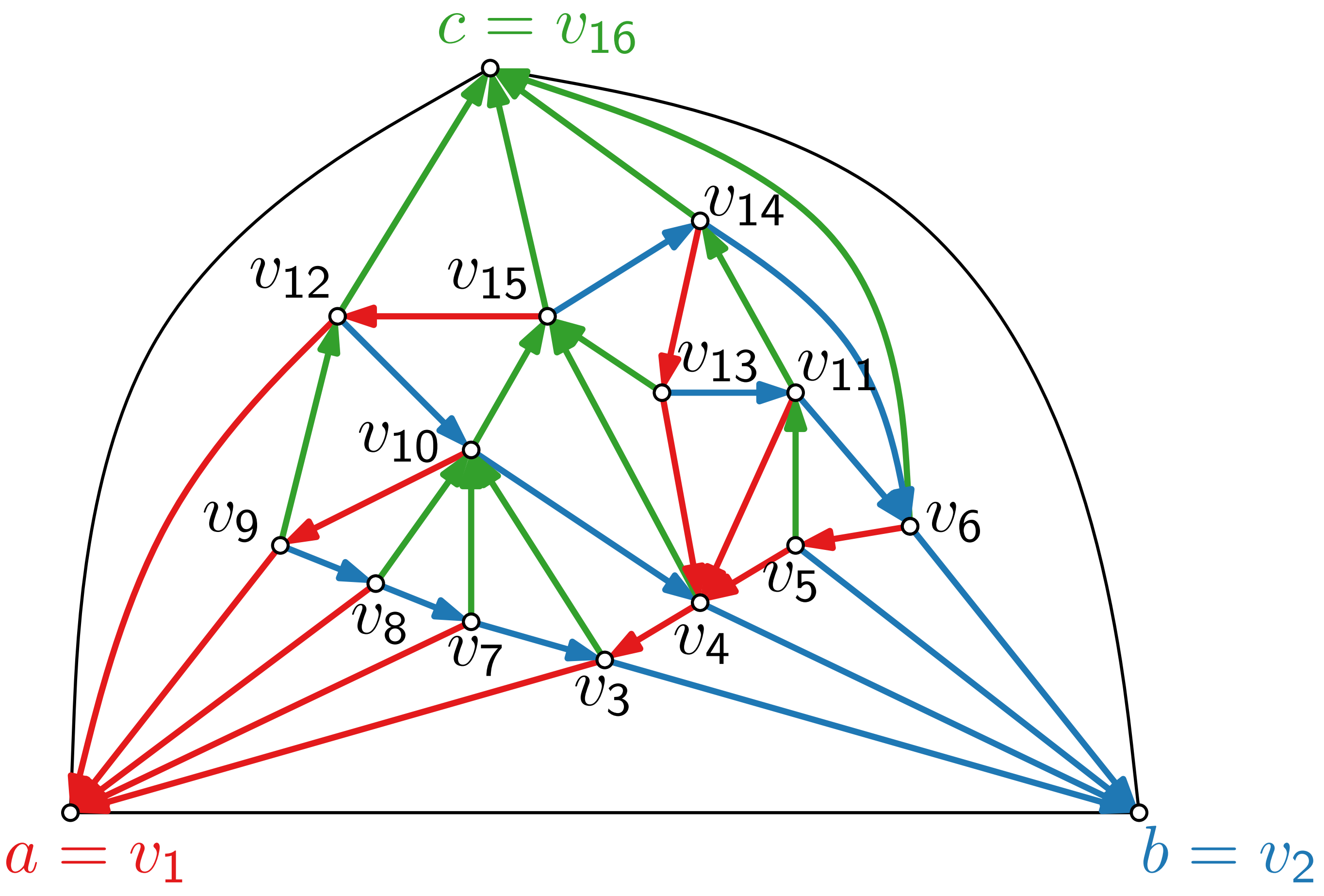
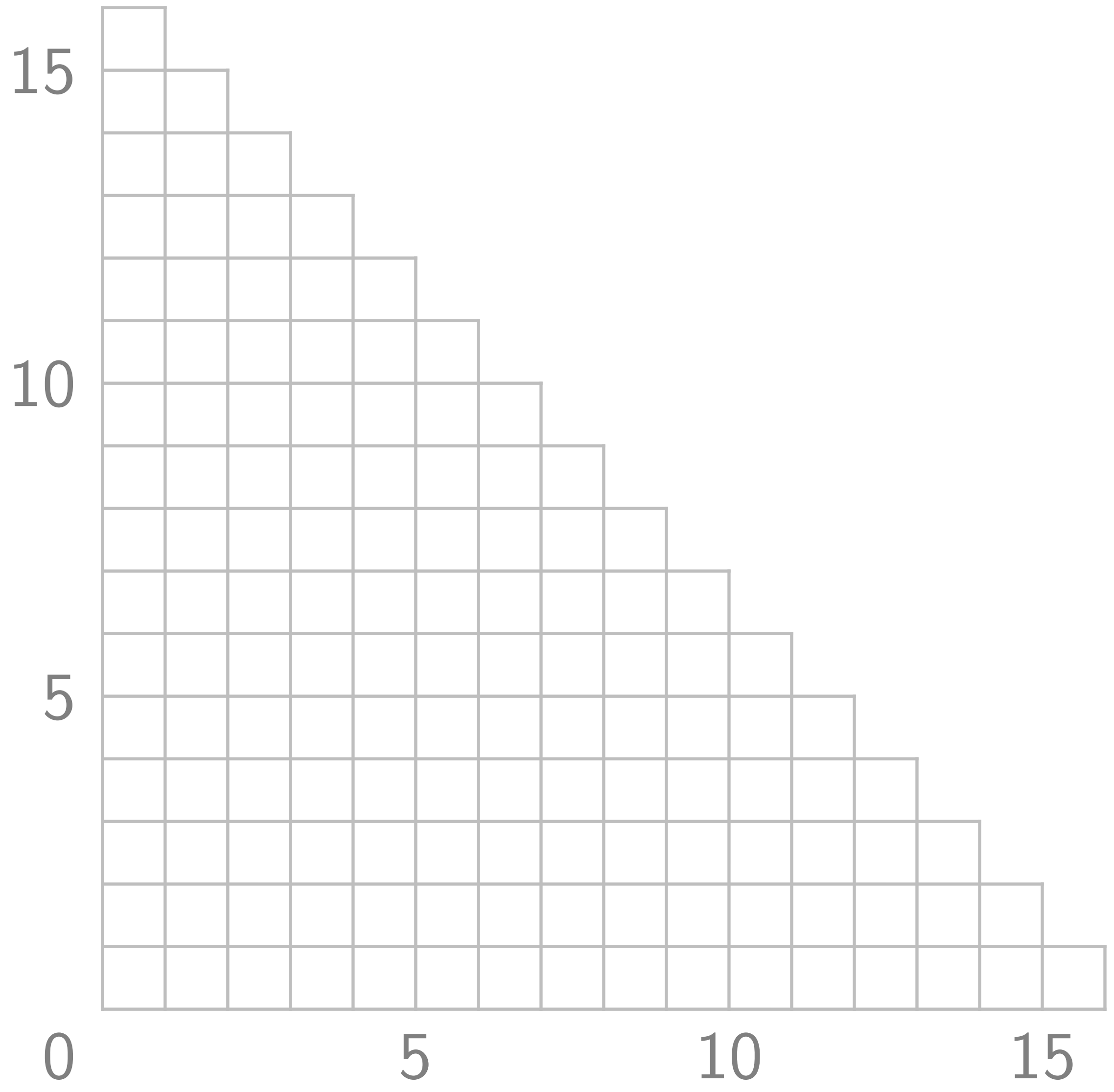
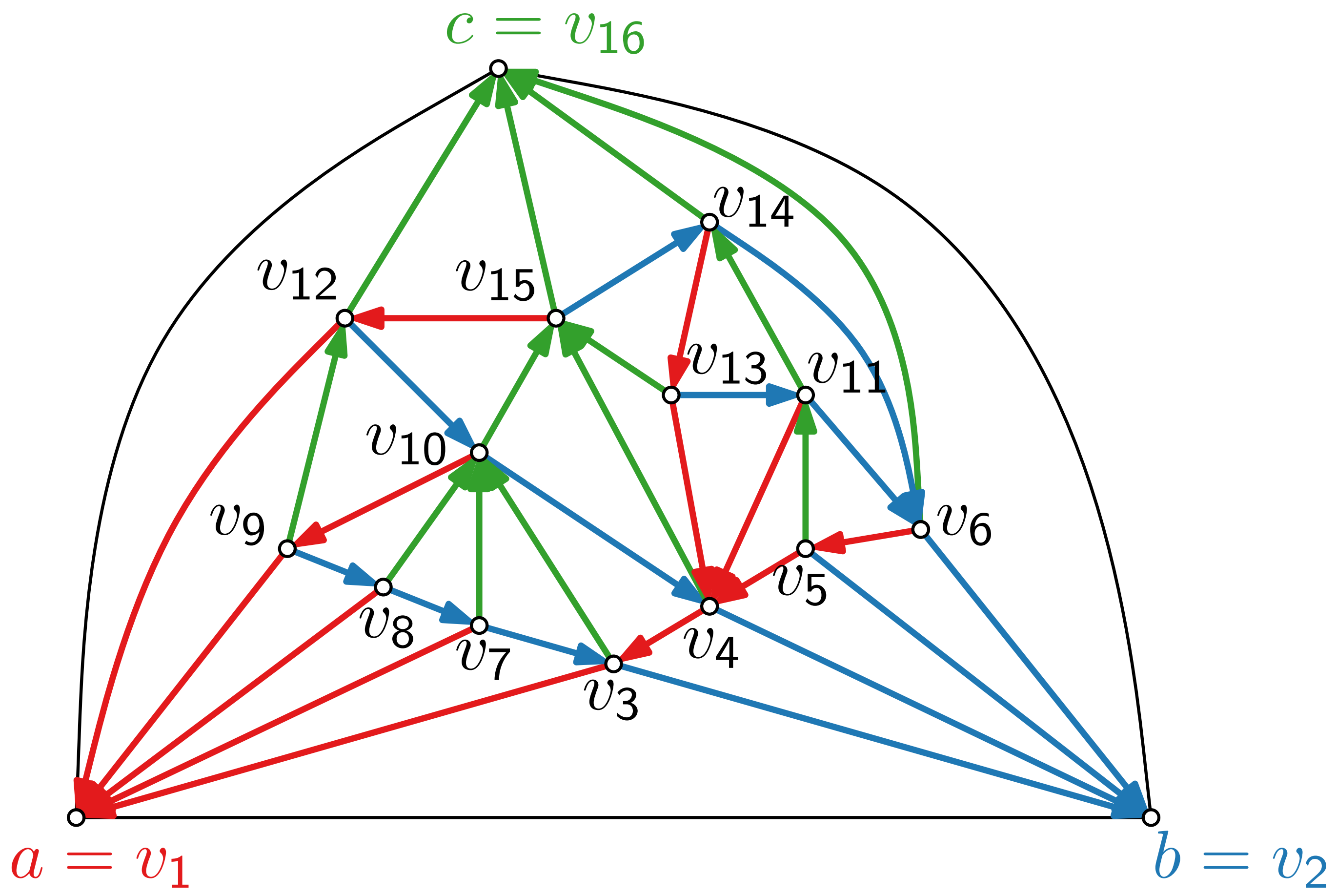


$n = 16,\ n-1 = 15,\ n-2 = 14$

Schnyder Drawing* – Example





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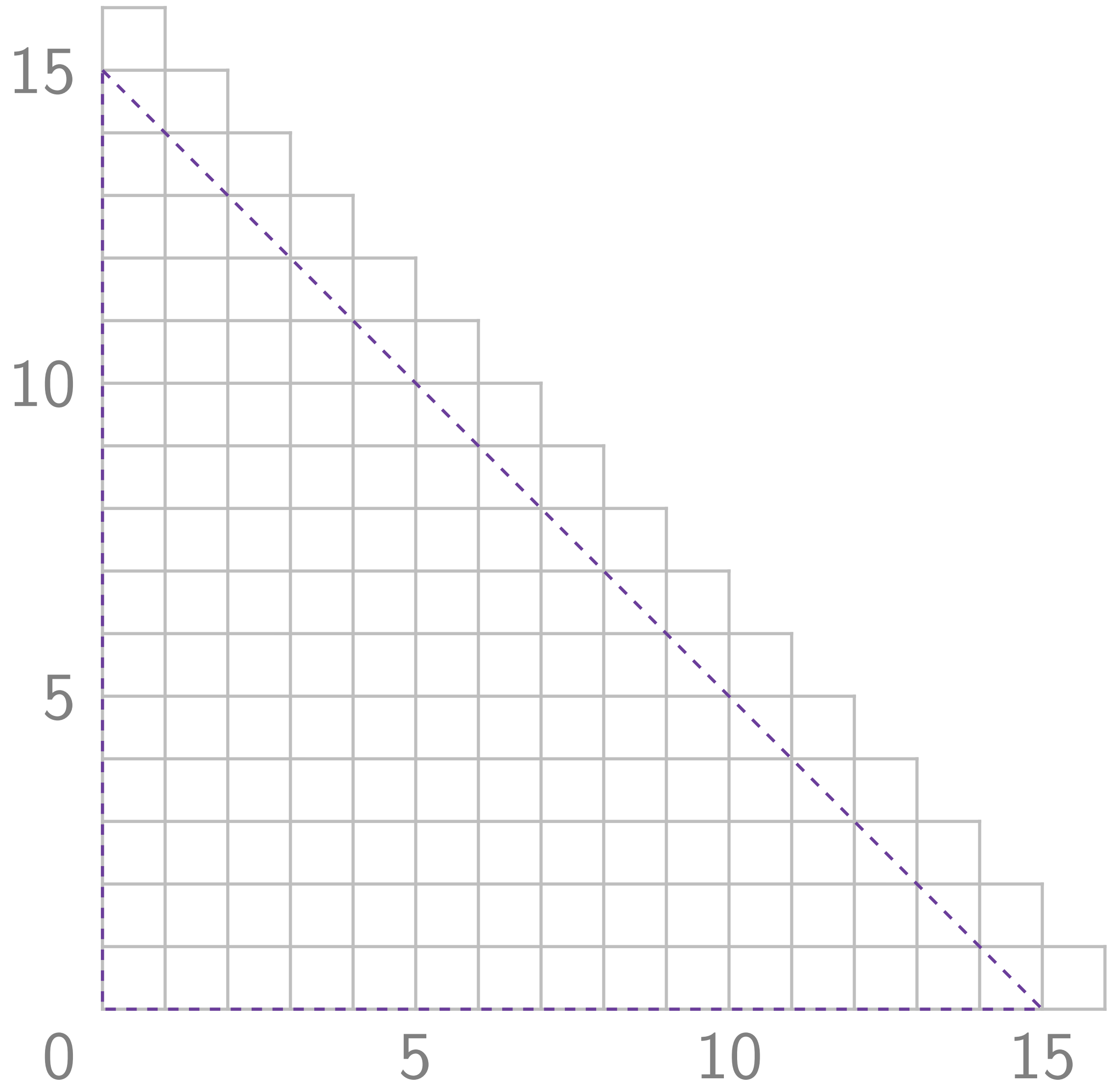

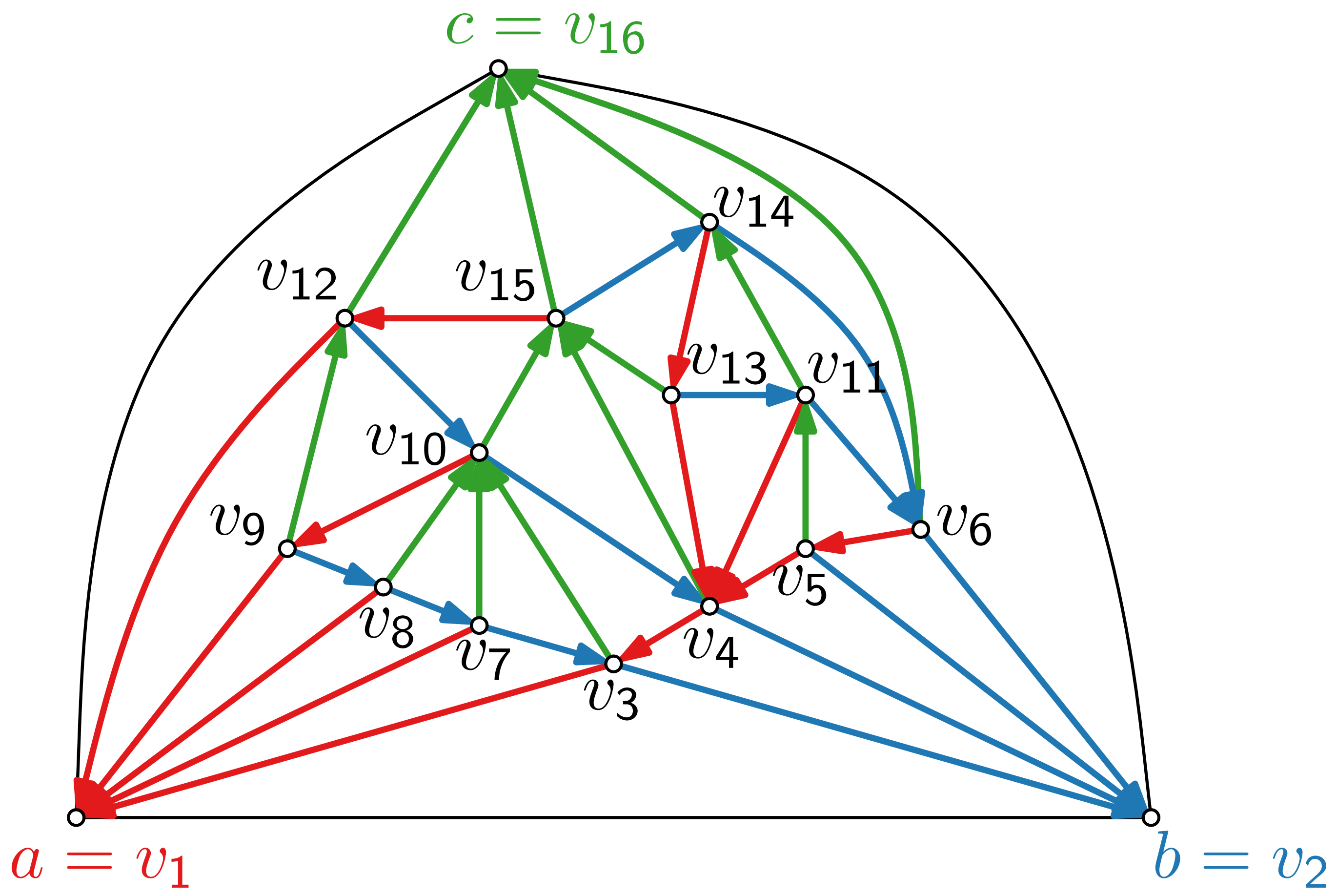

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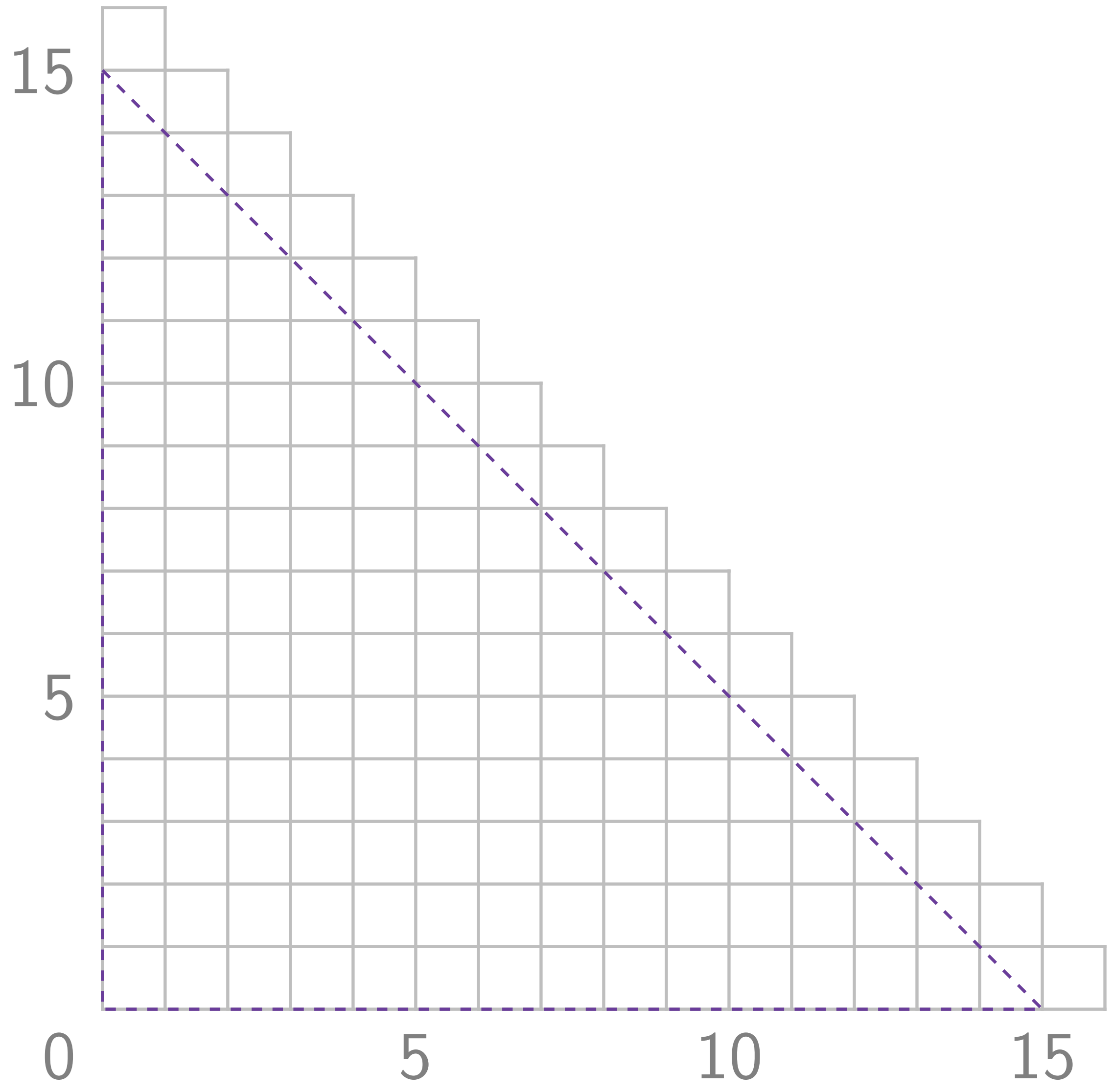
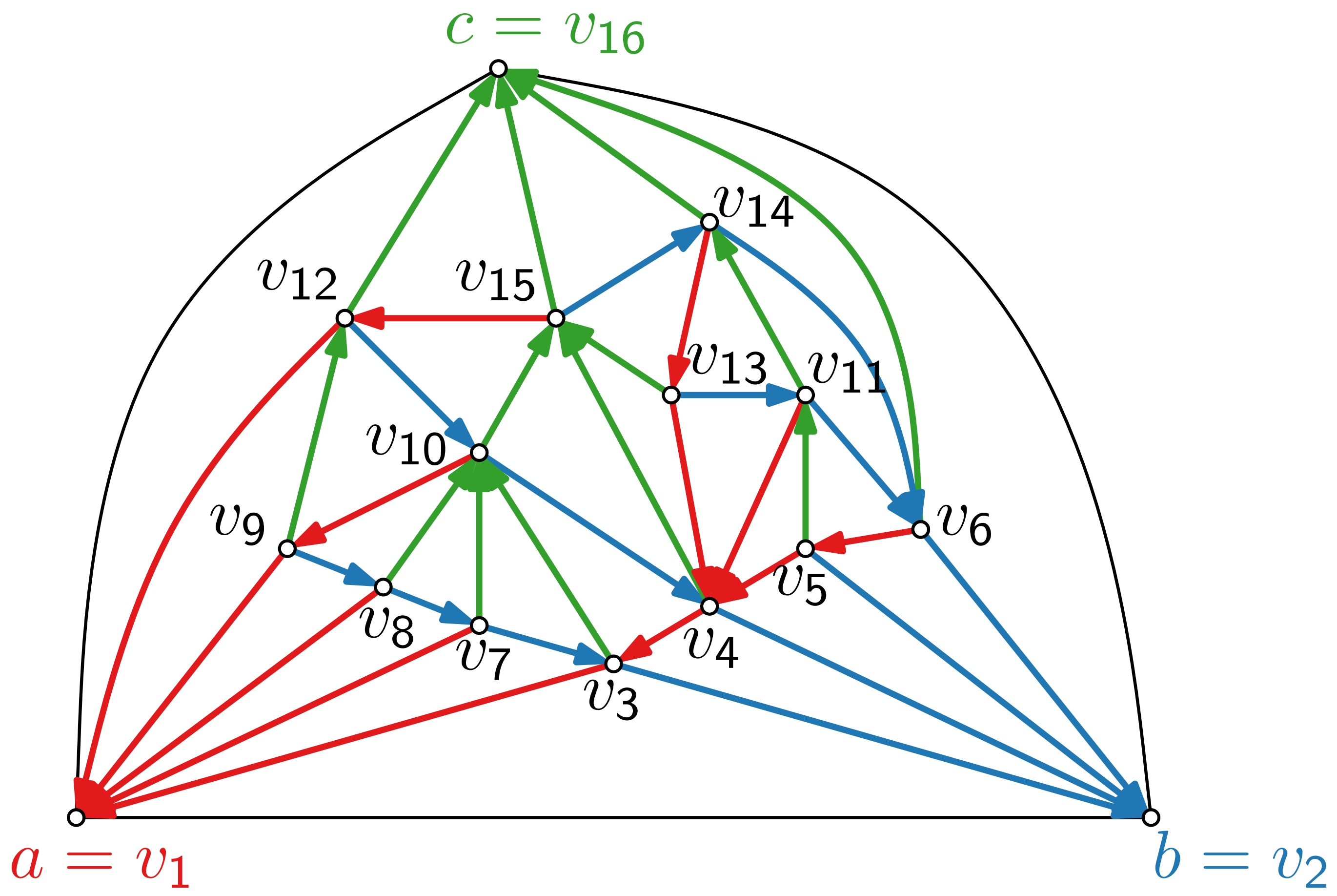
Schnyder Drawing⋆ – Example





$n = 16$, $n - 1 = 15$, $n - 2 = 14$

$f(a) =$

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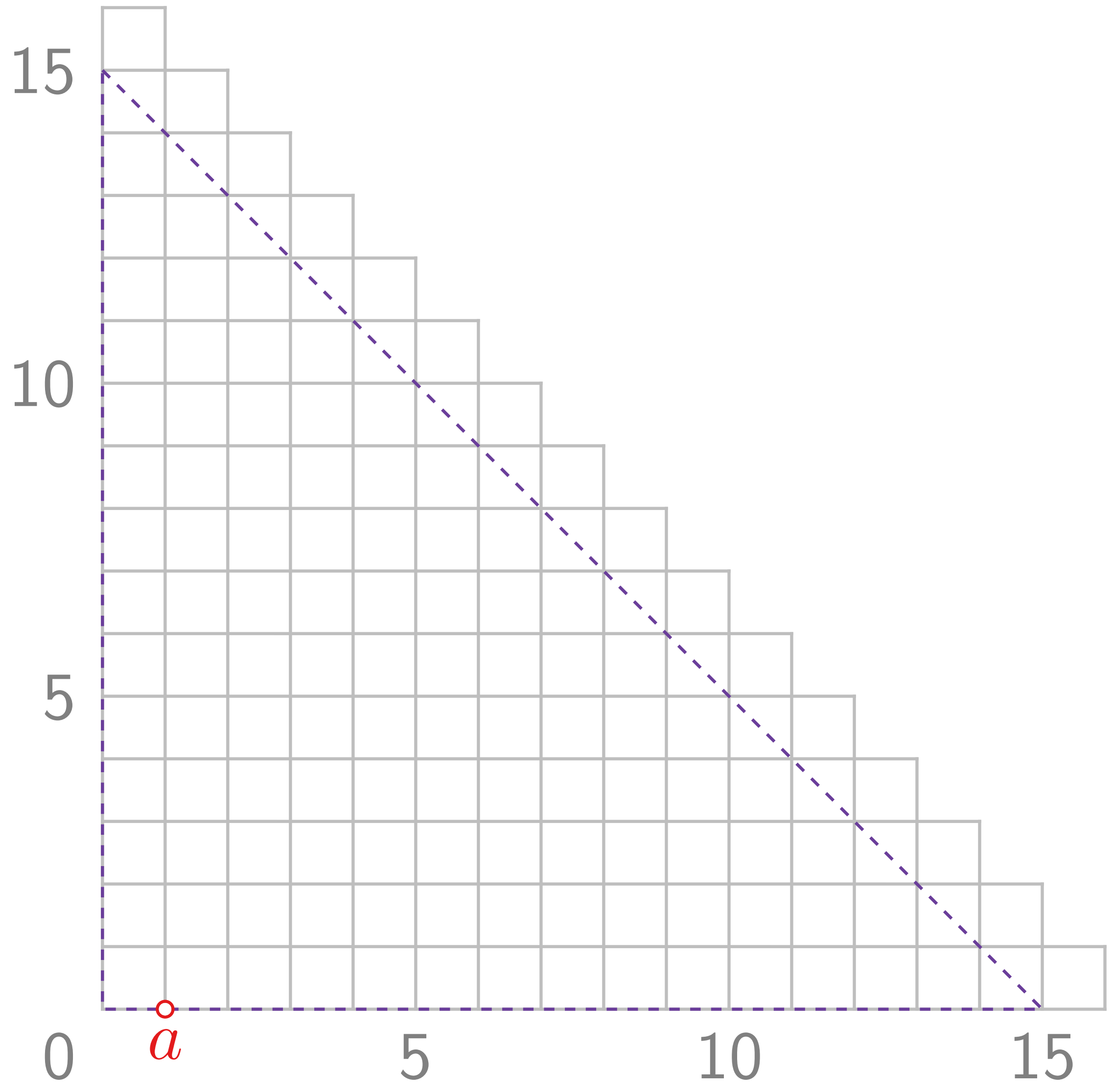

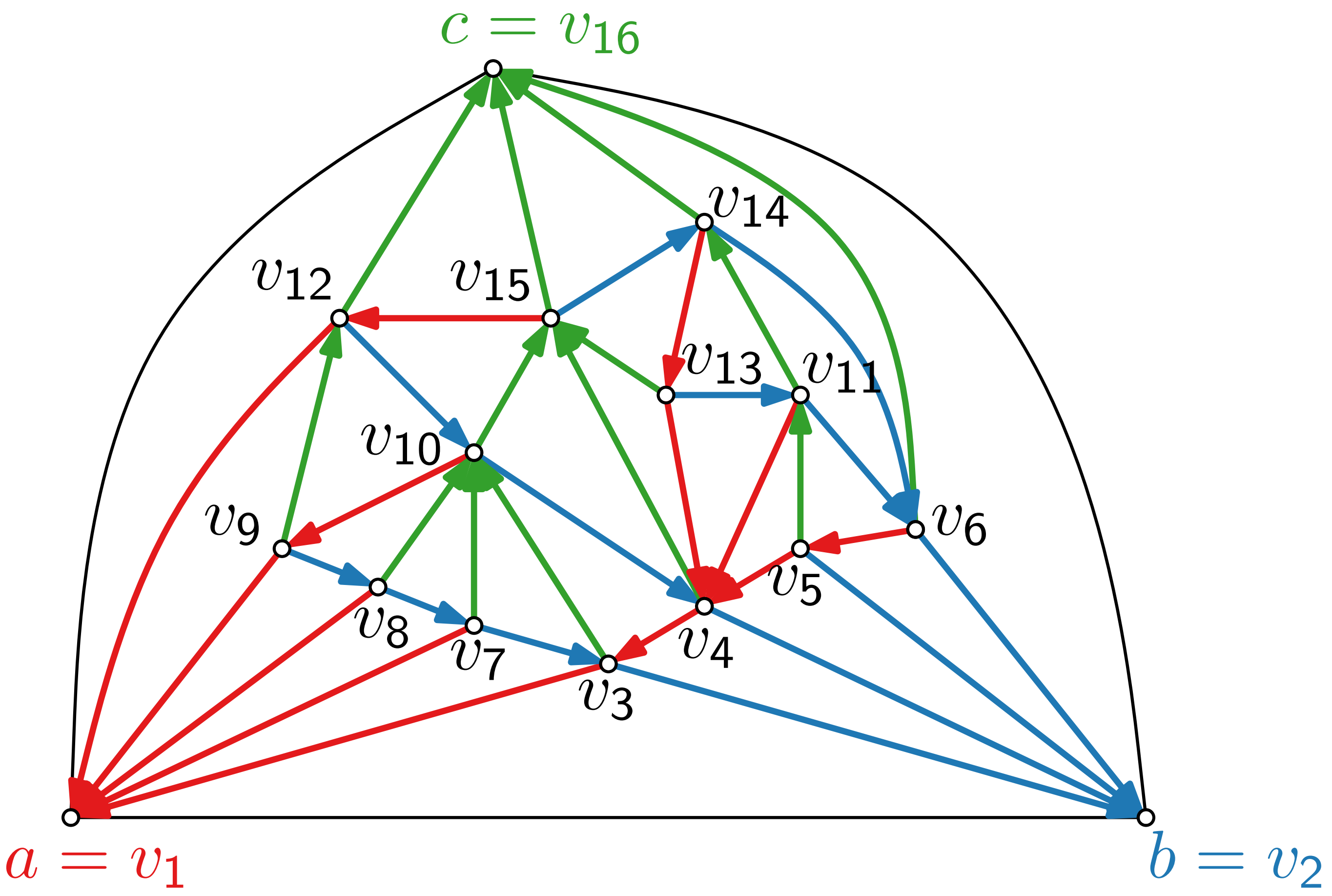

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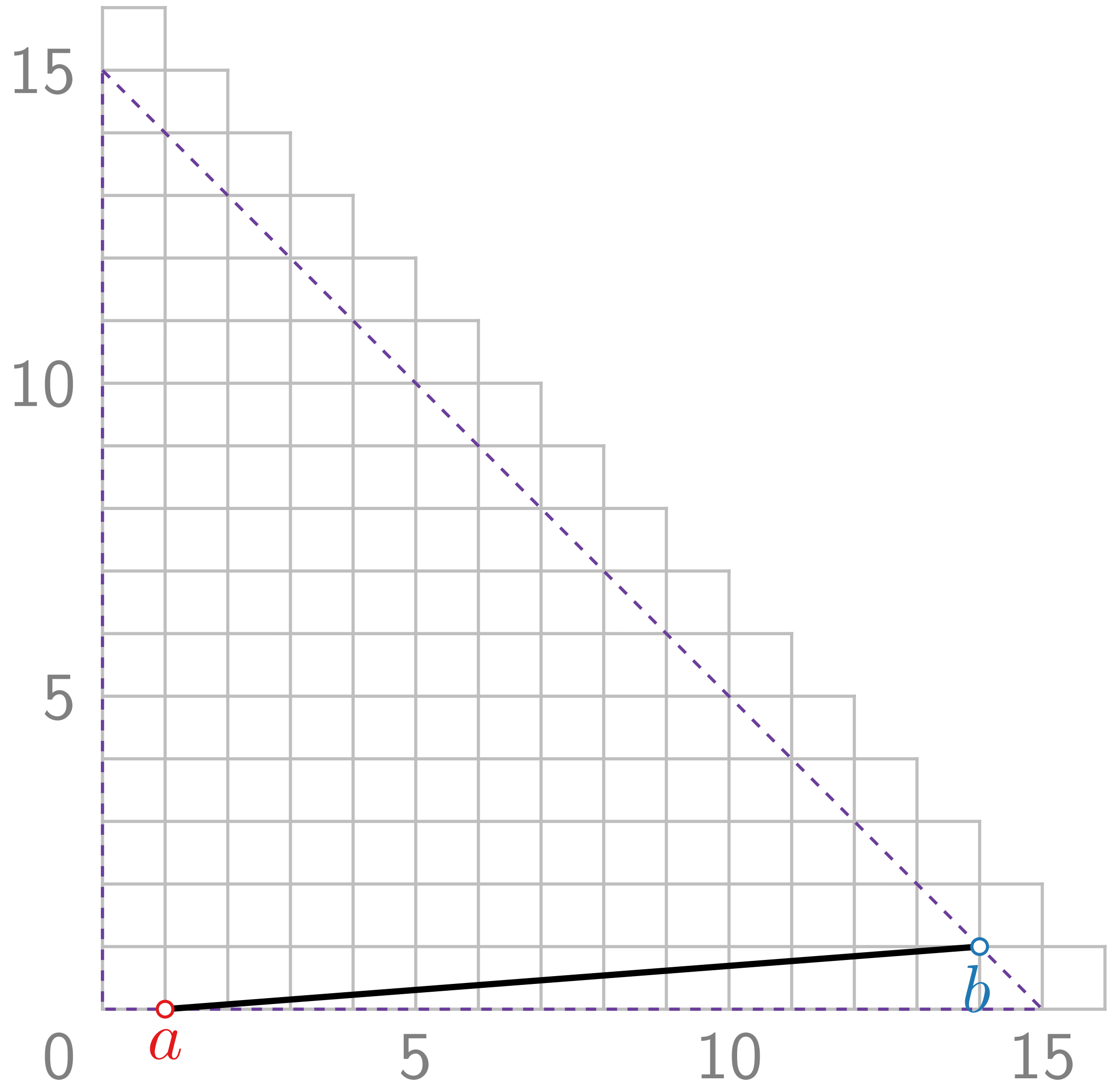
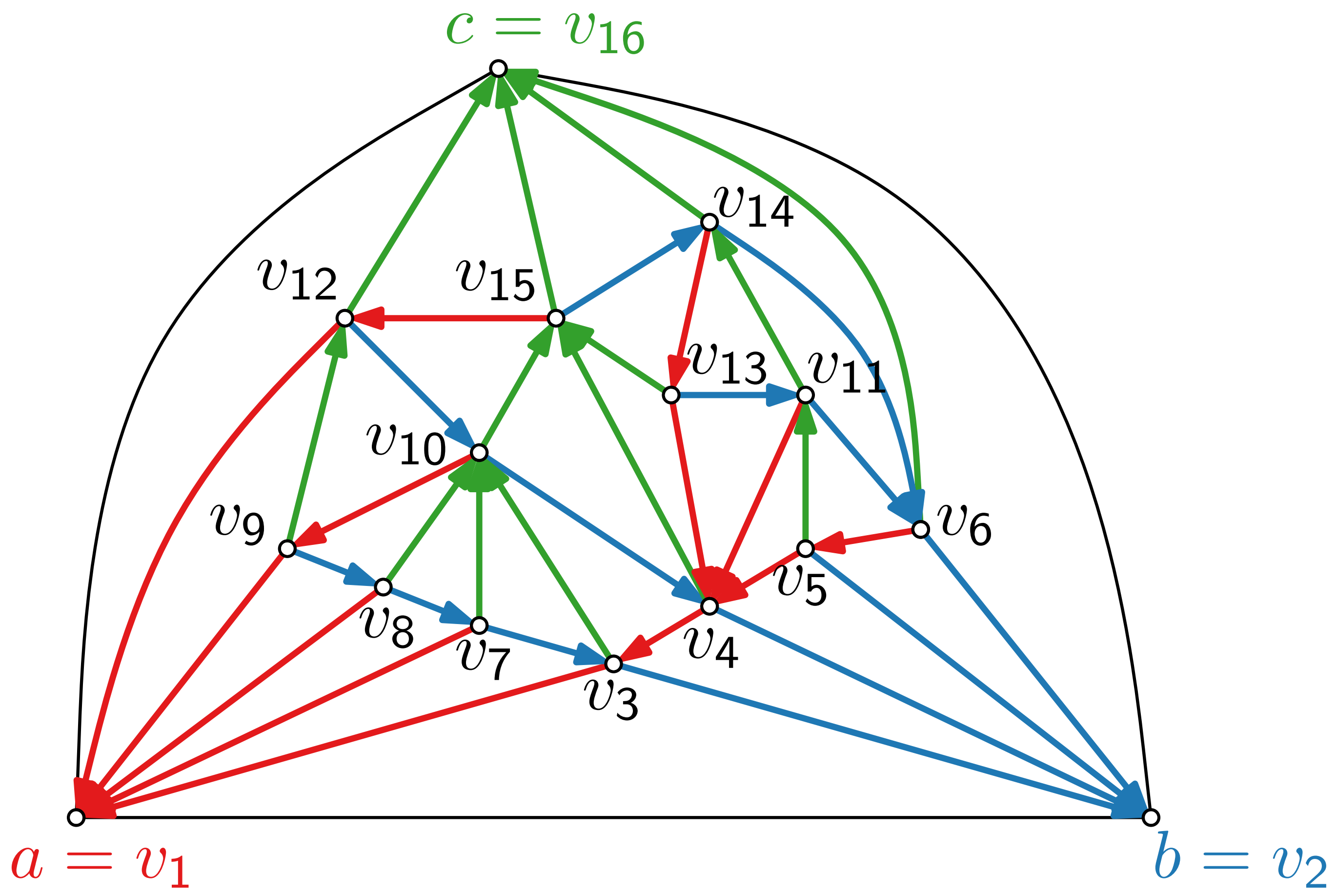
$f(a) = (14, 1, 0)$

Schnyder Drawing* – Example



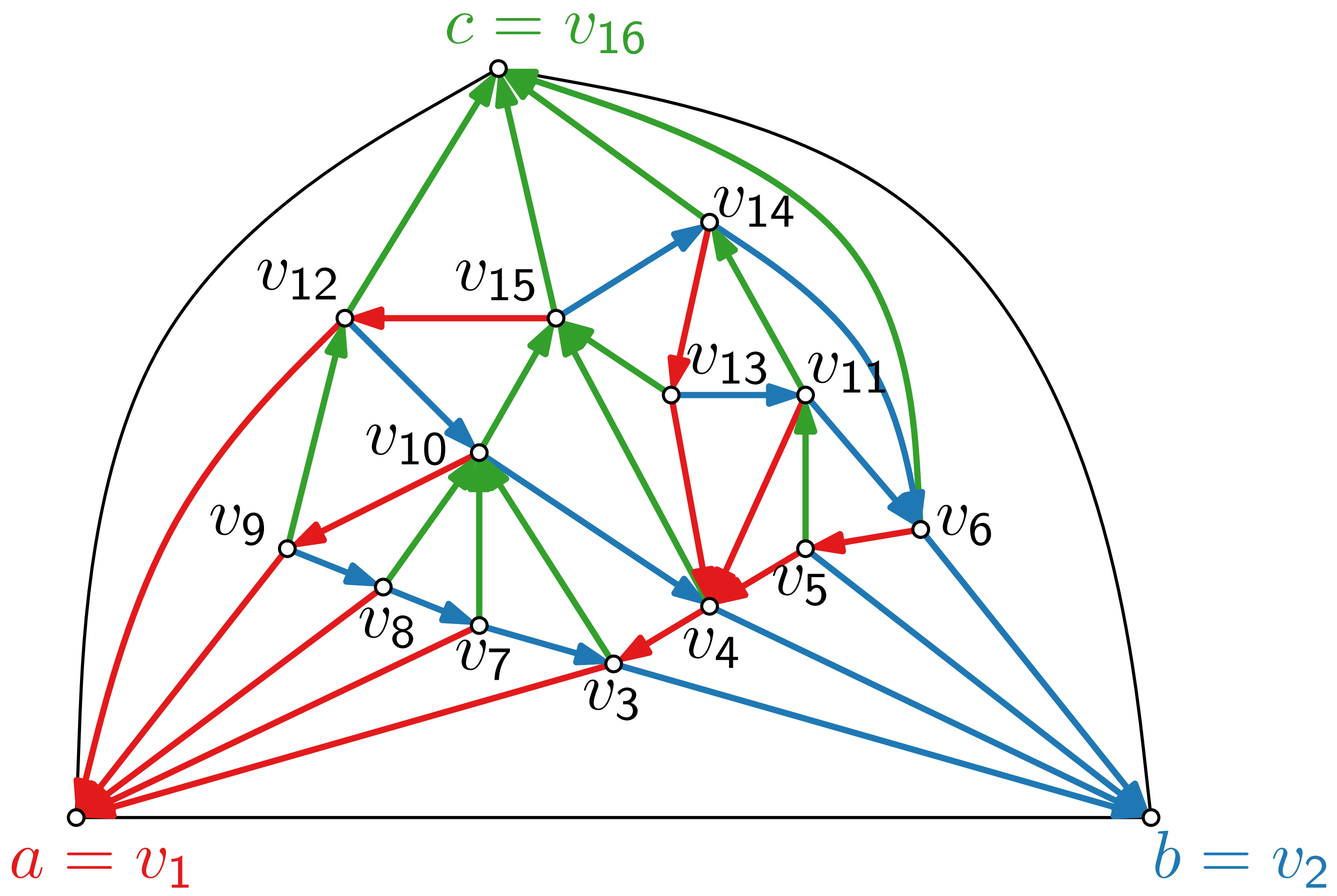


$n = 16,\ n-1 = 15,\ n-2 = 14$

$f(a) = (14, 1, 0)$

$f(b) = (0, 14, 1)$

Schnyder Drawing* – Example



$n = 16$, $n - 1 = 15$, $n - 2 = 14$

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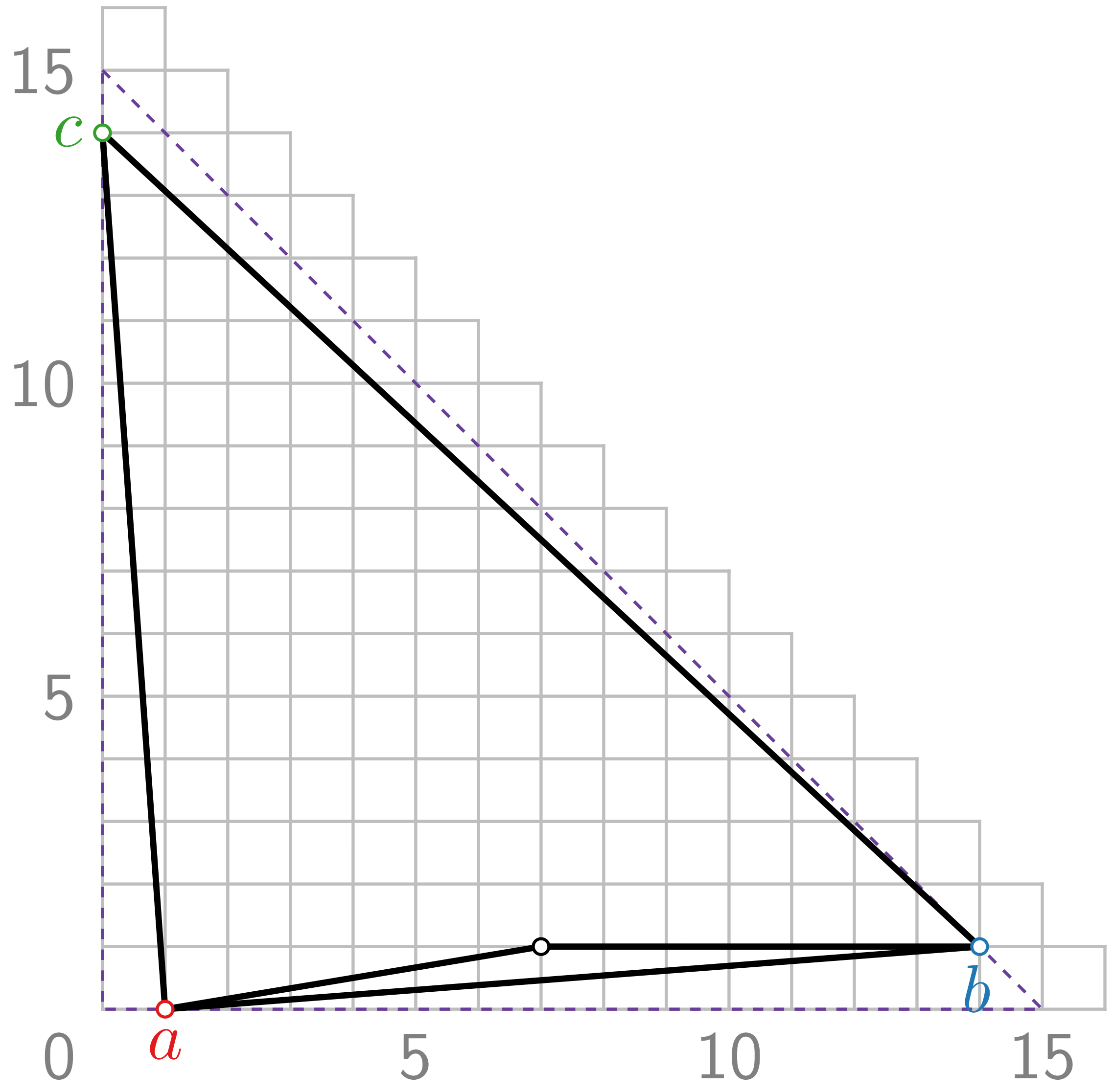

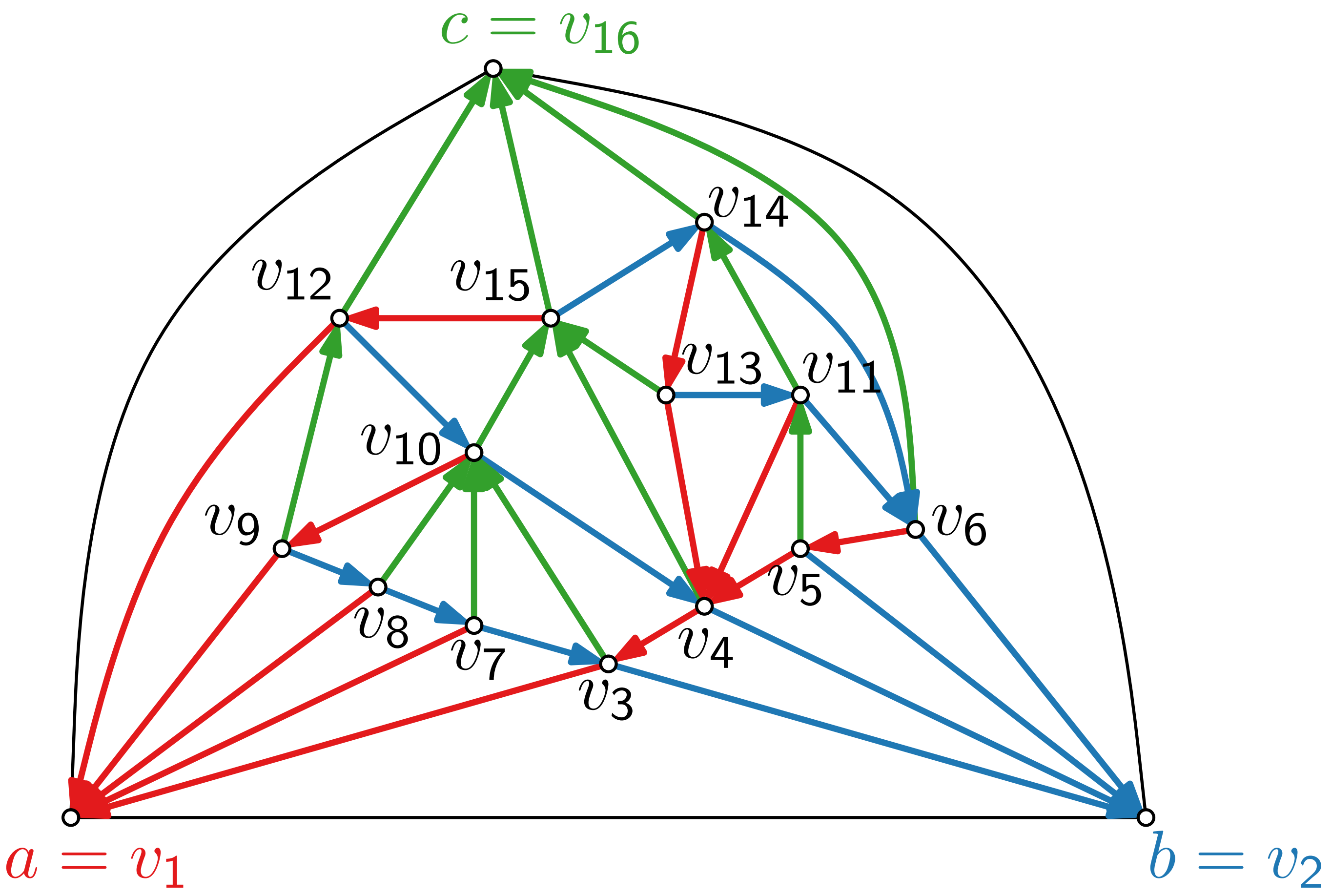

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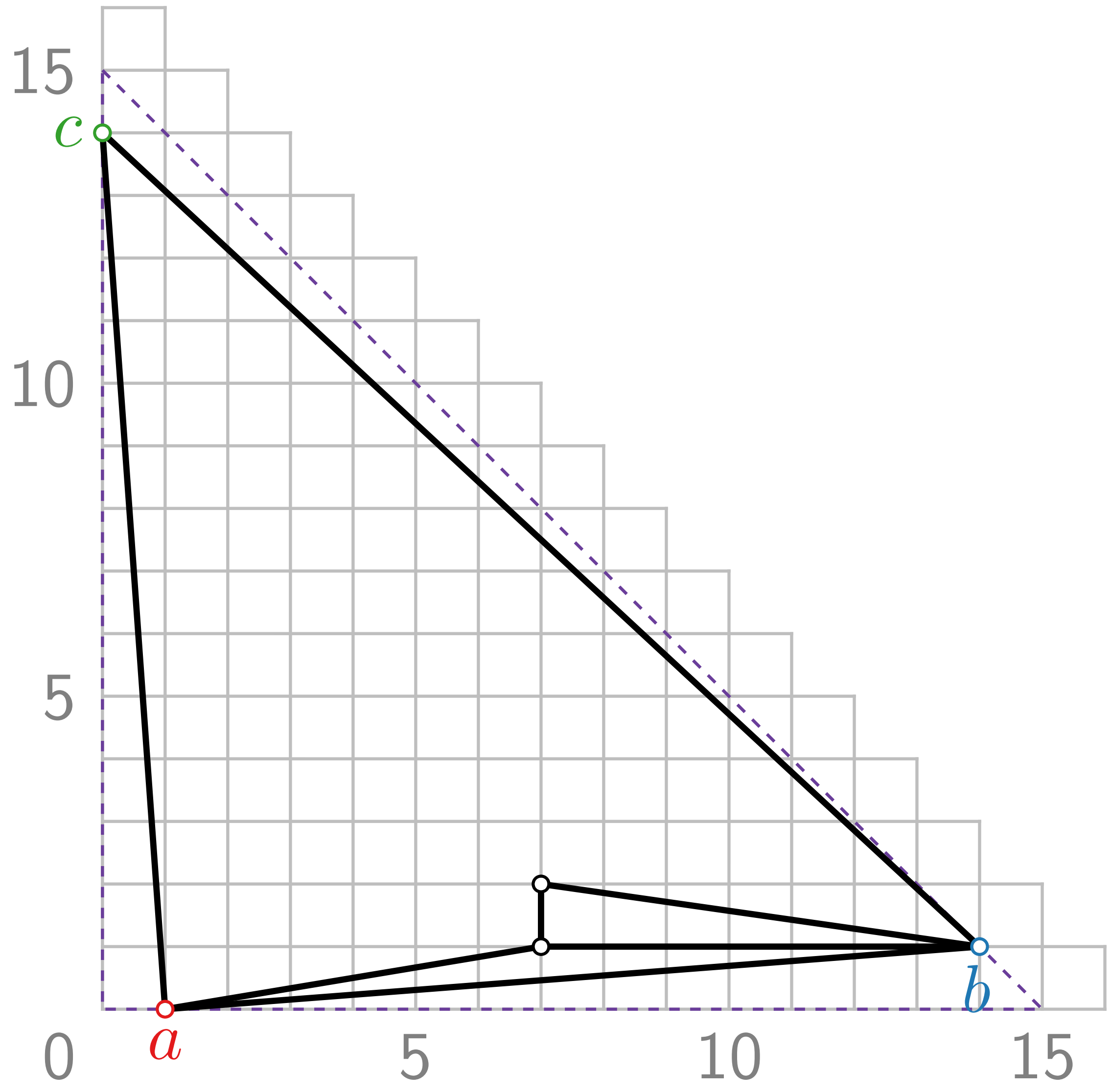

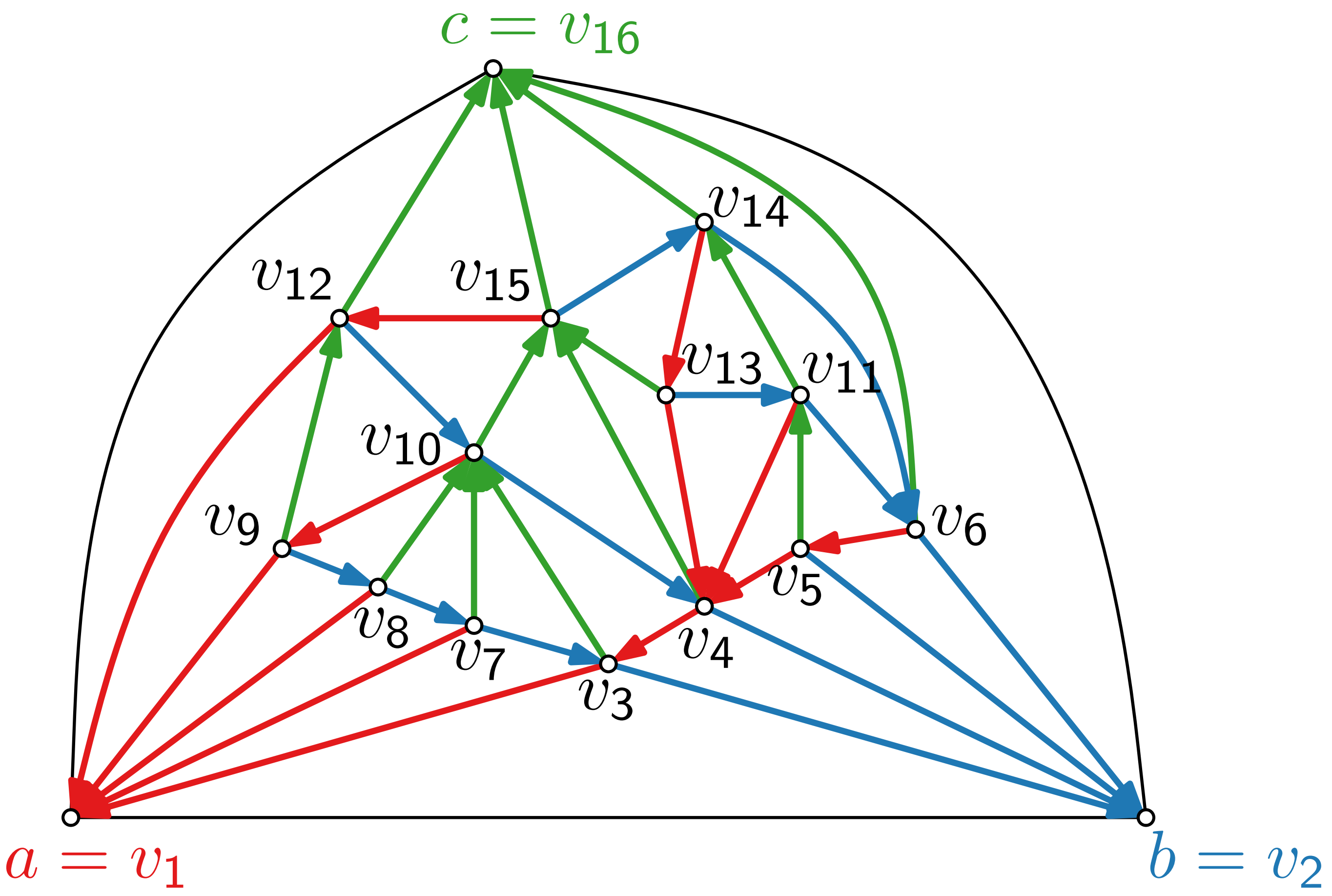

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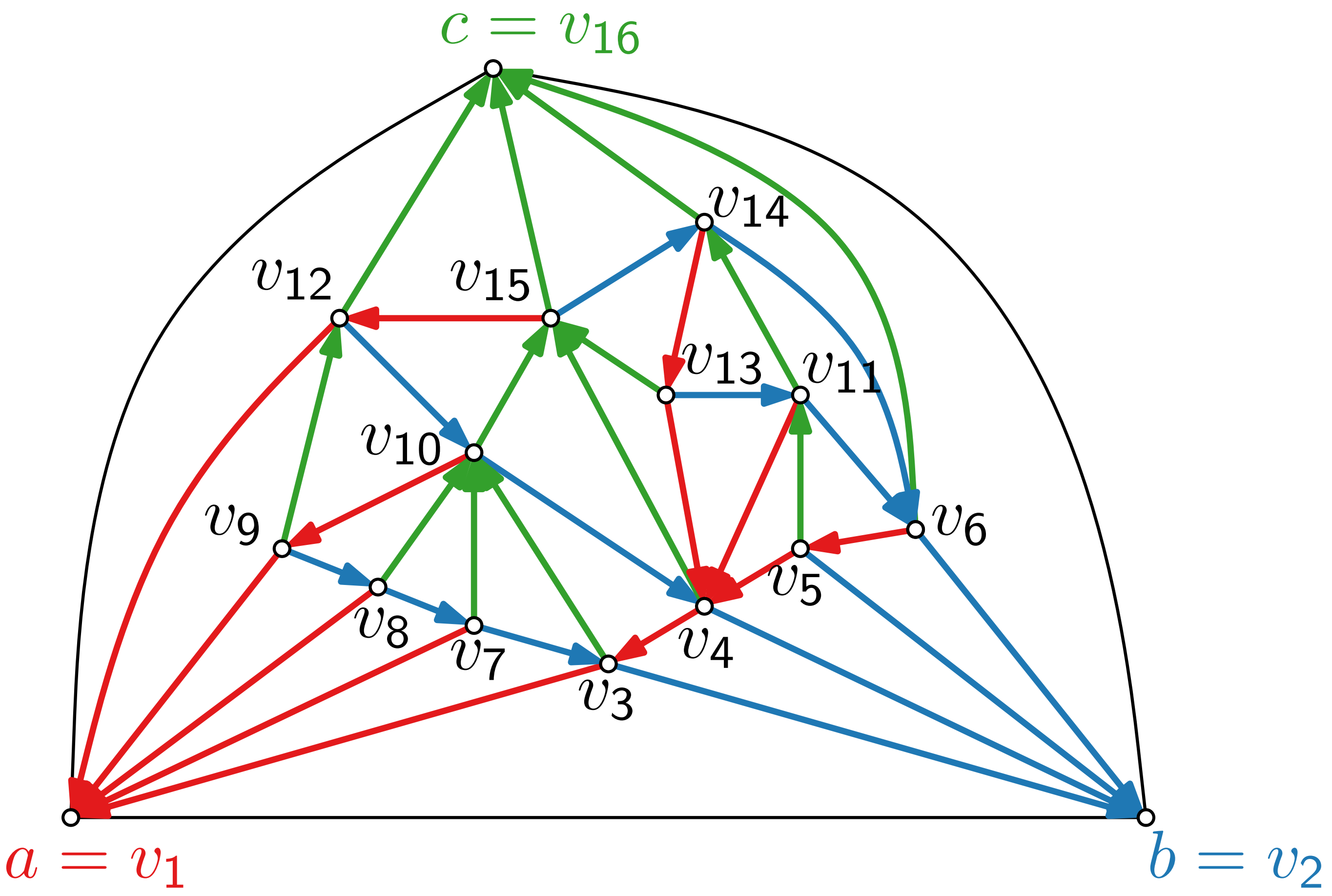

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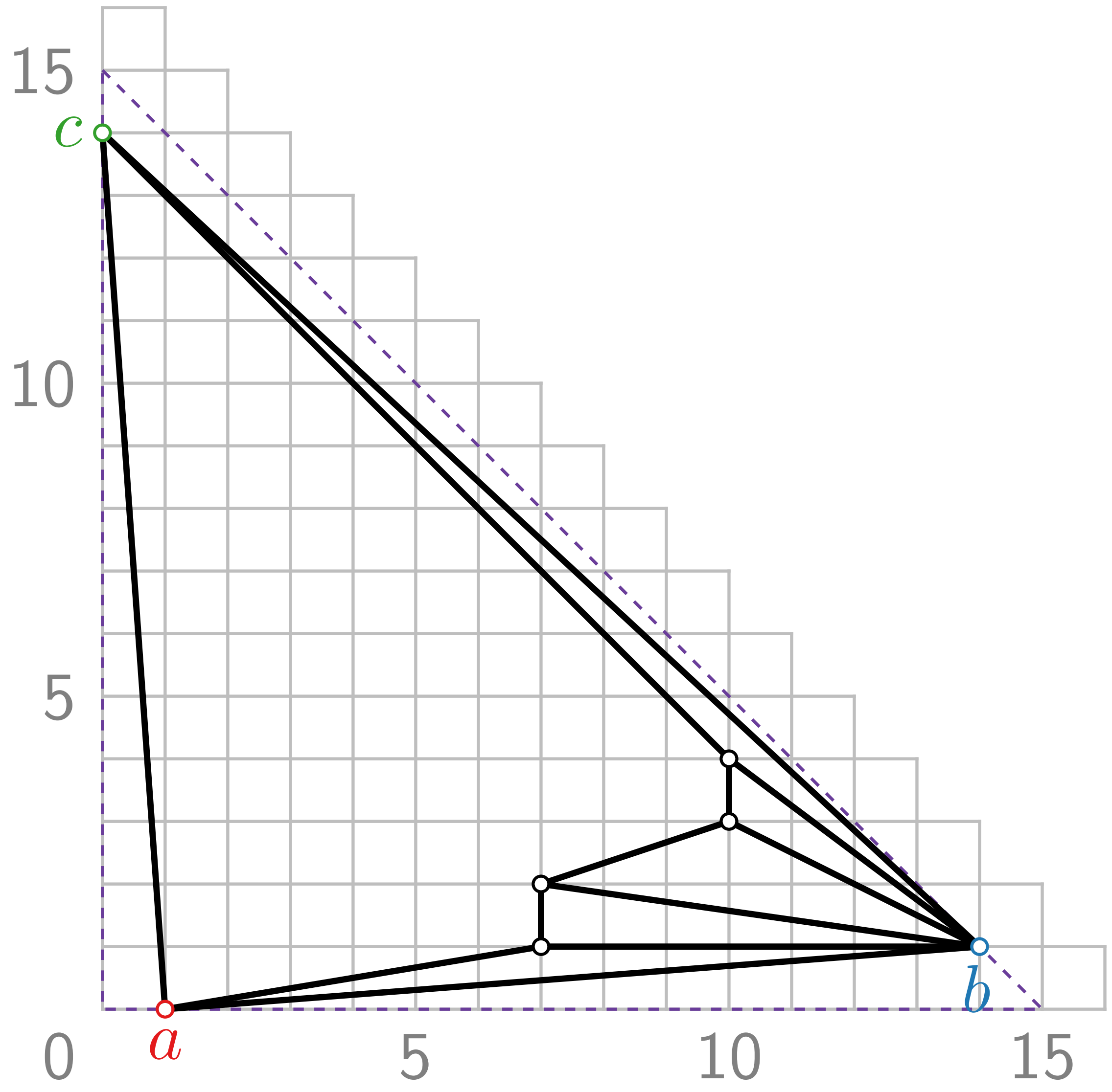

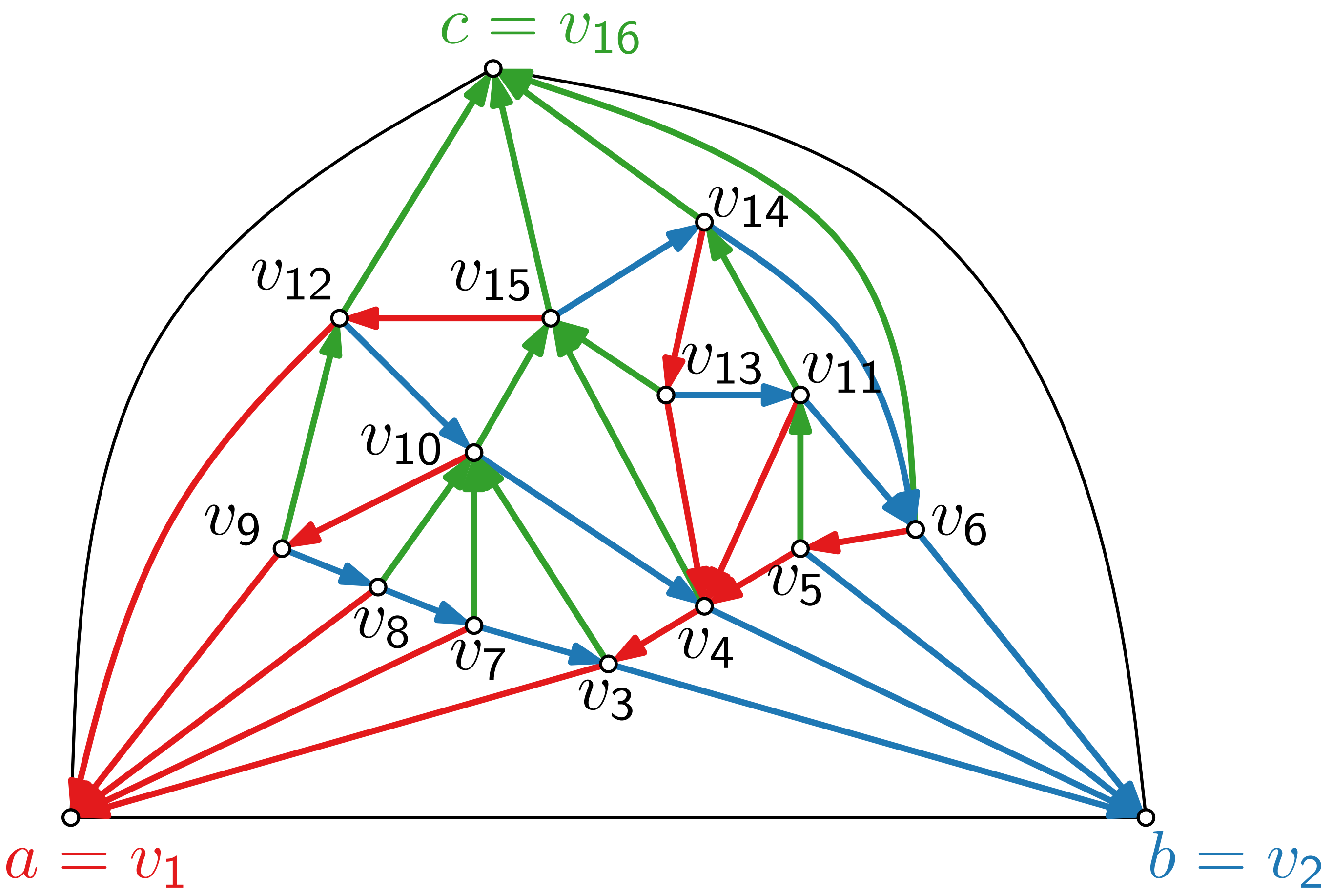

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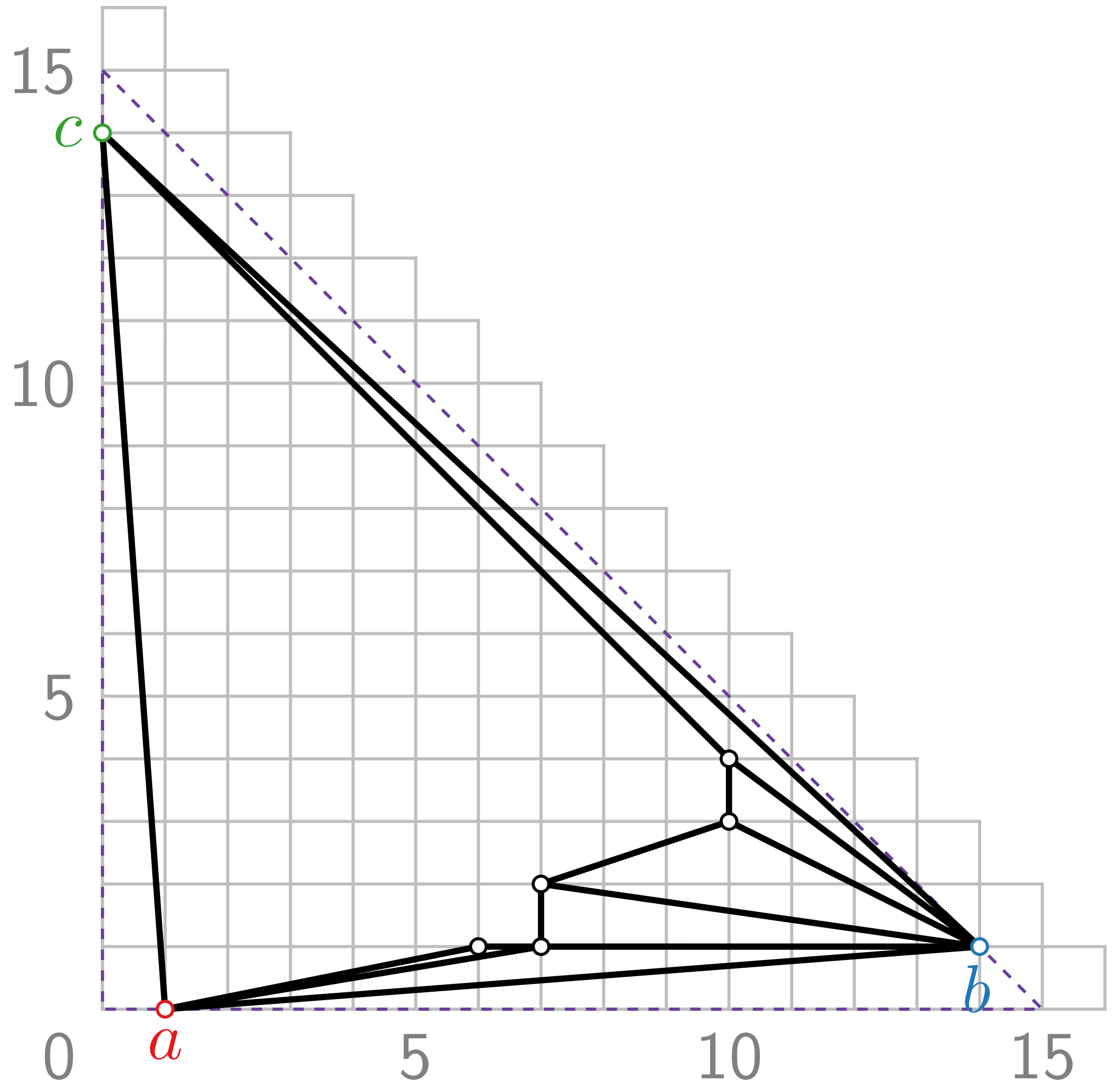

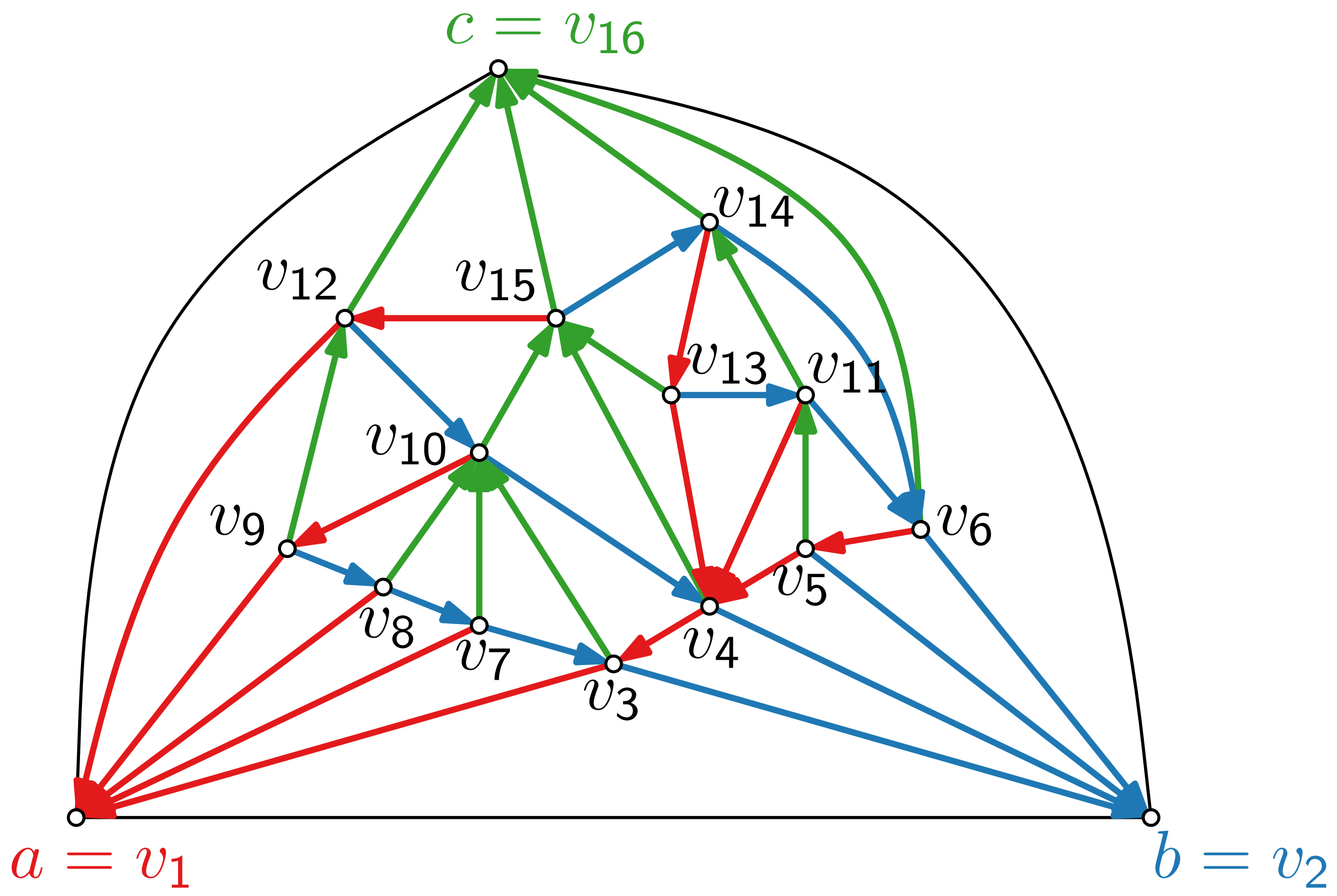

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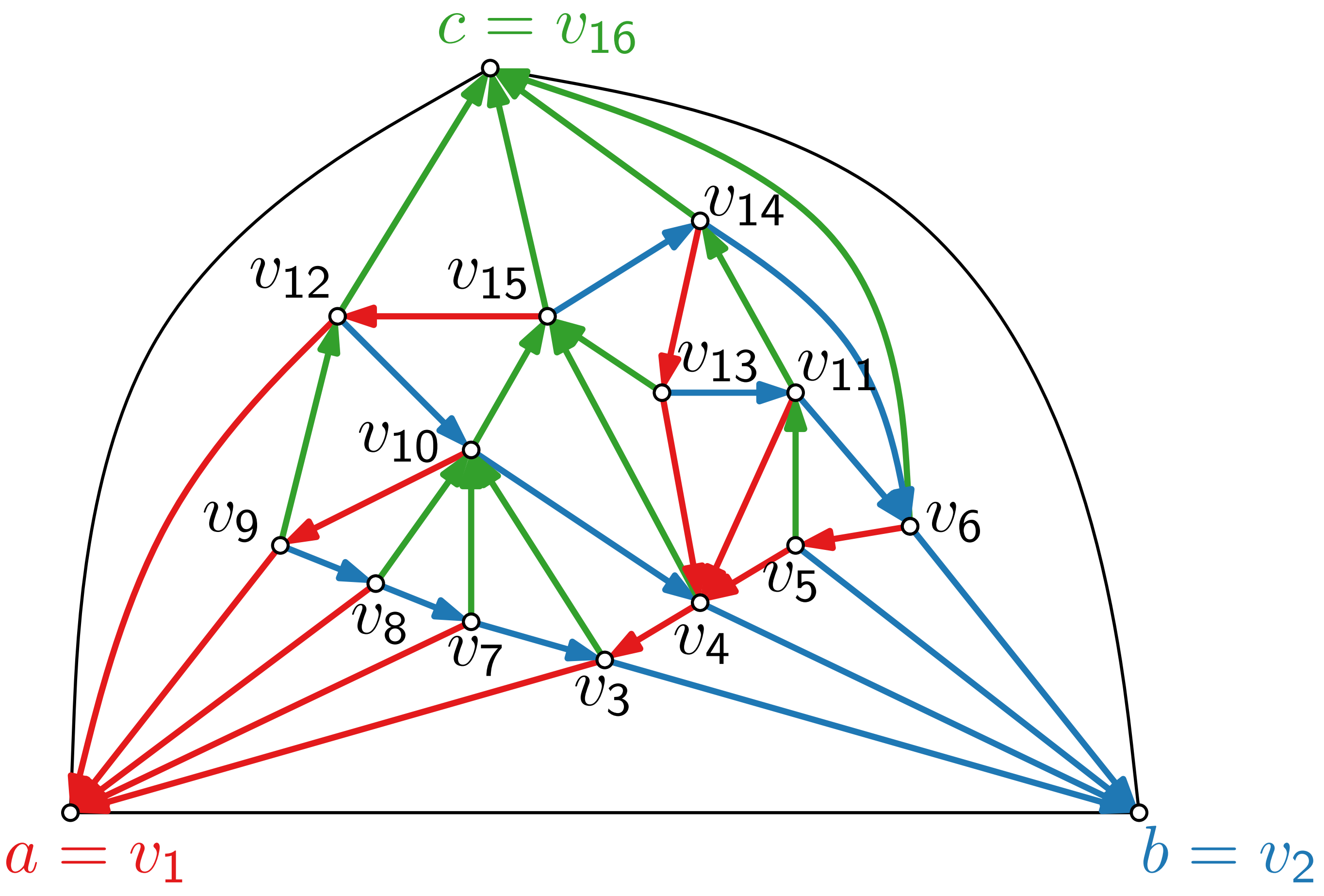

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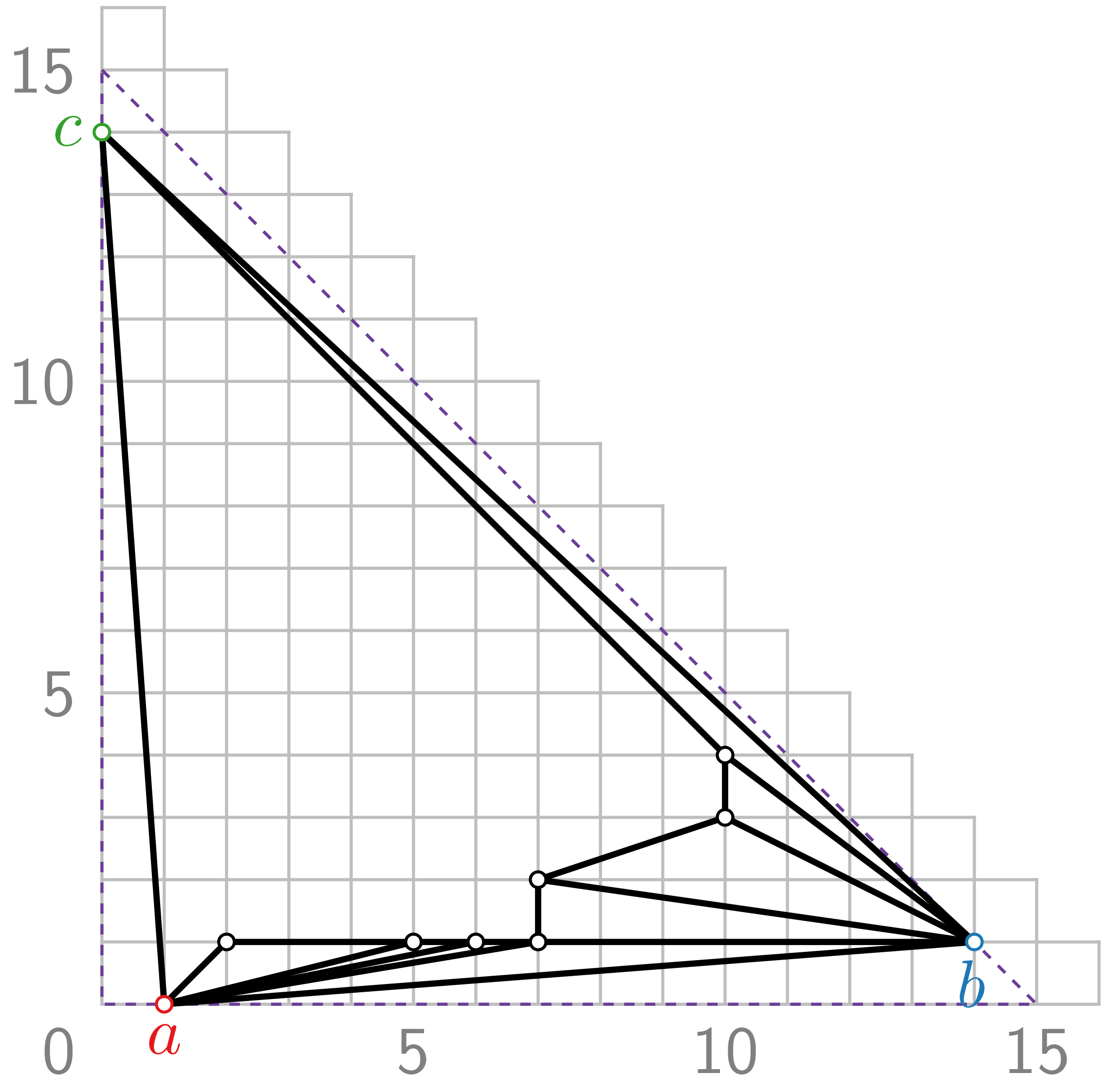

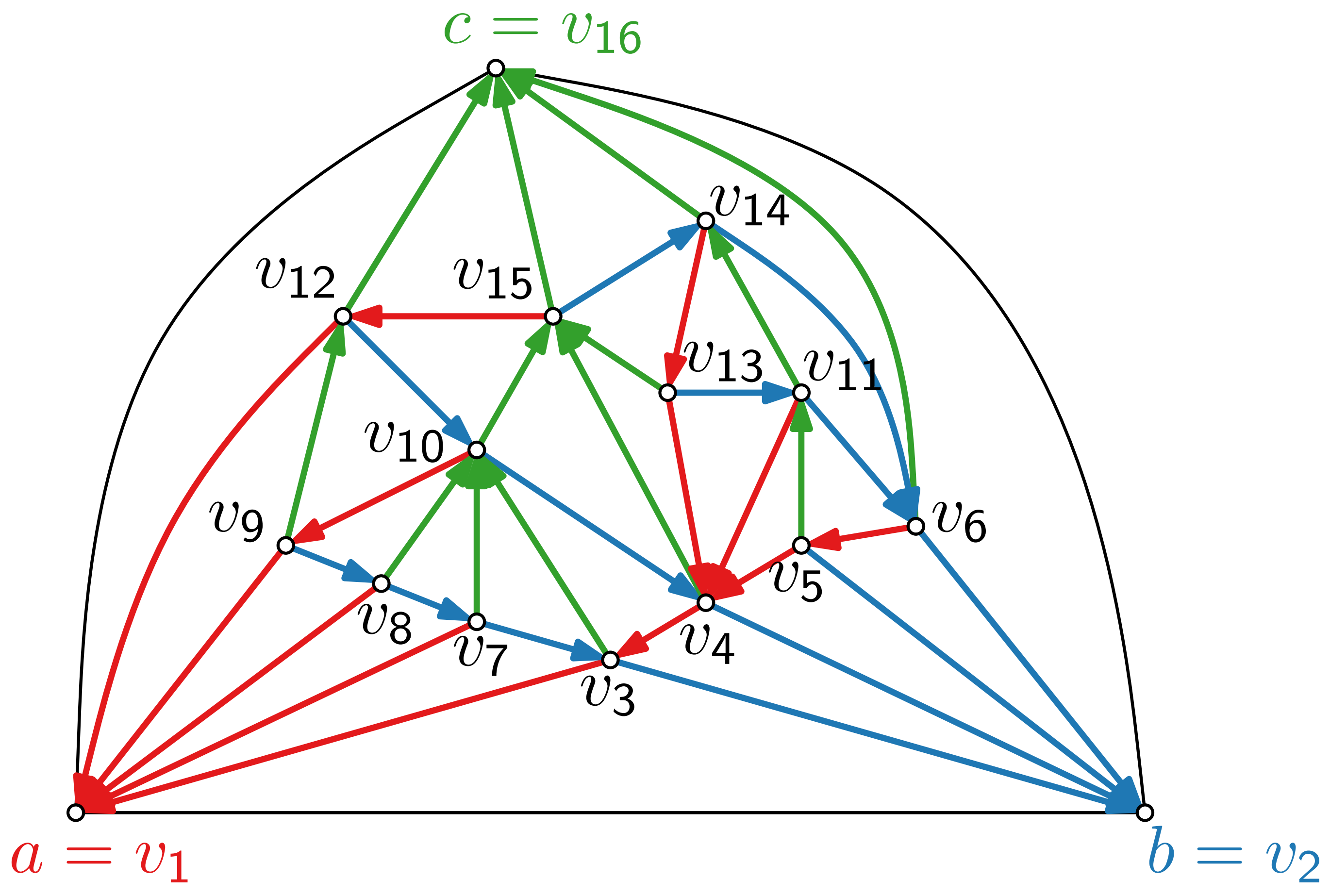

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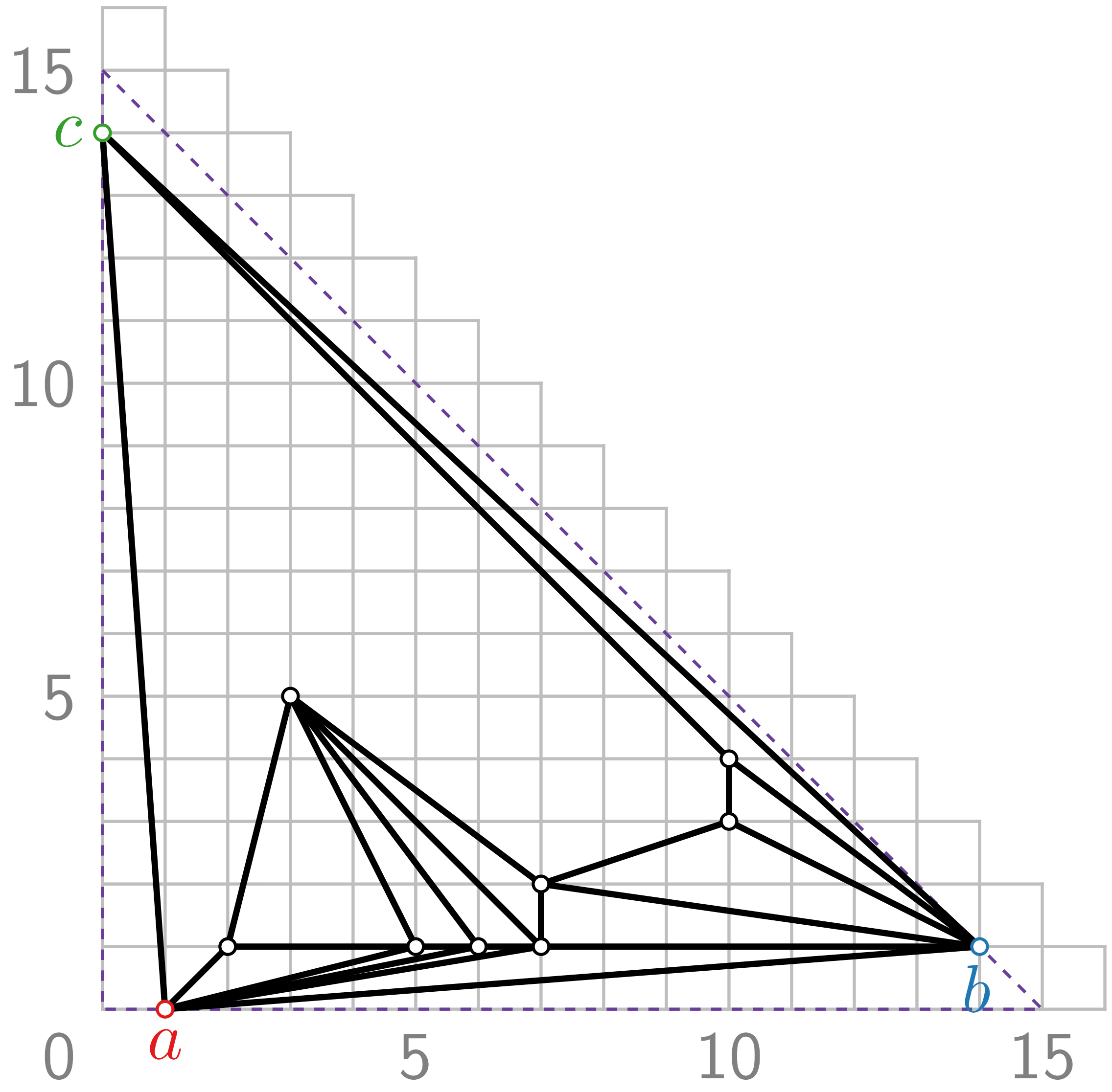

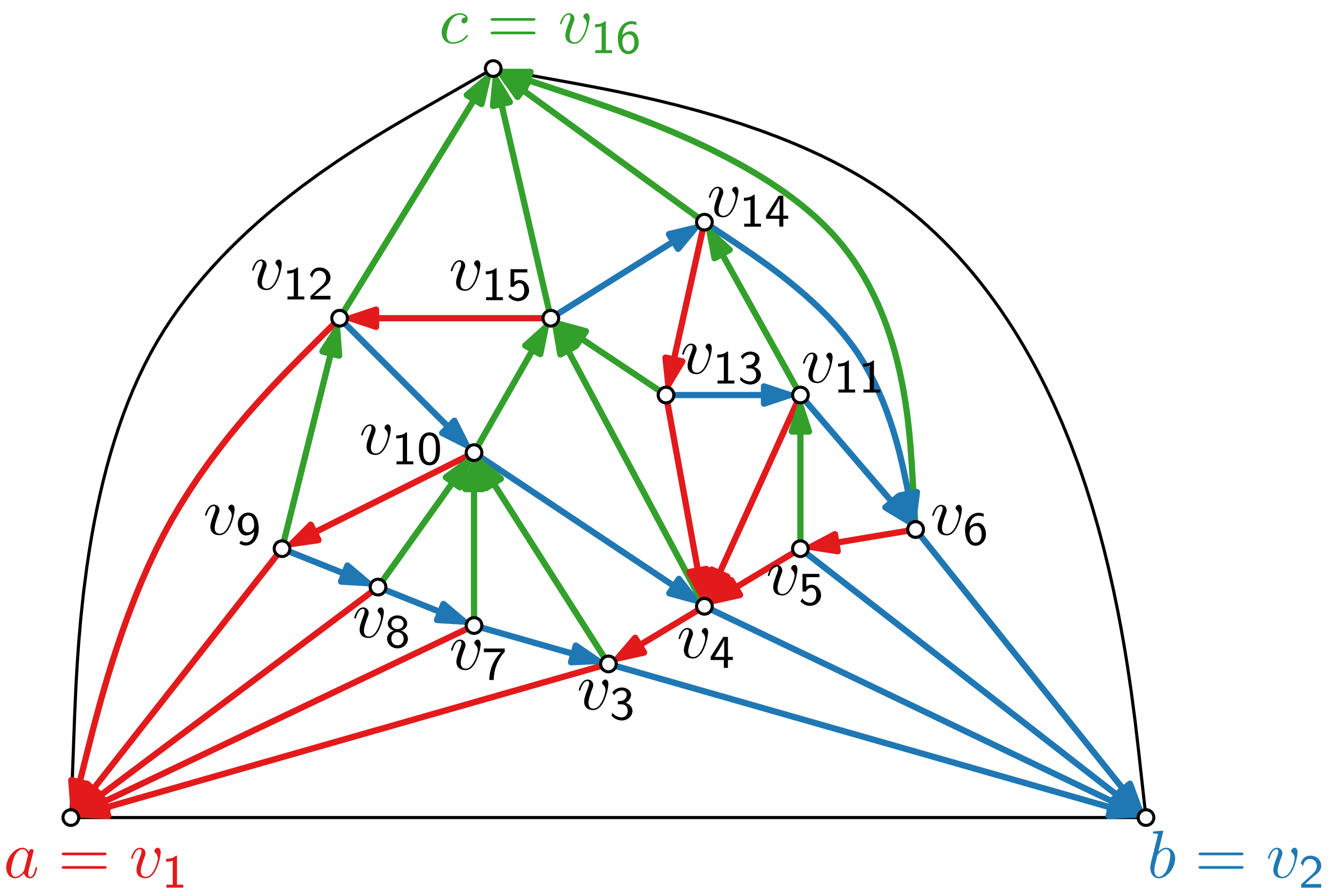

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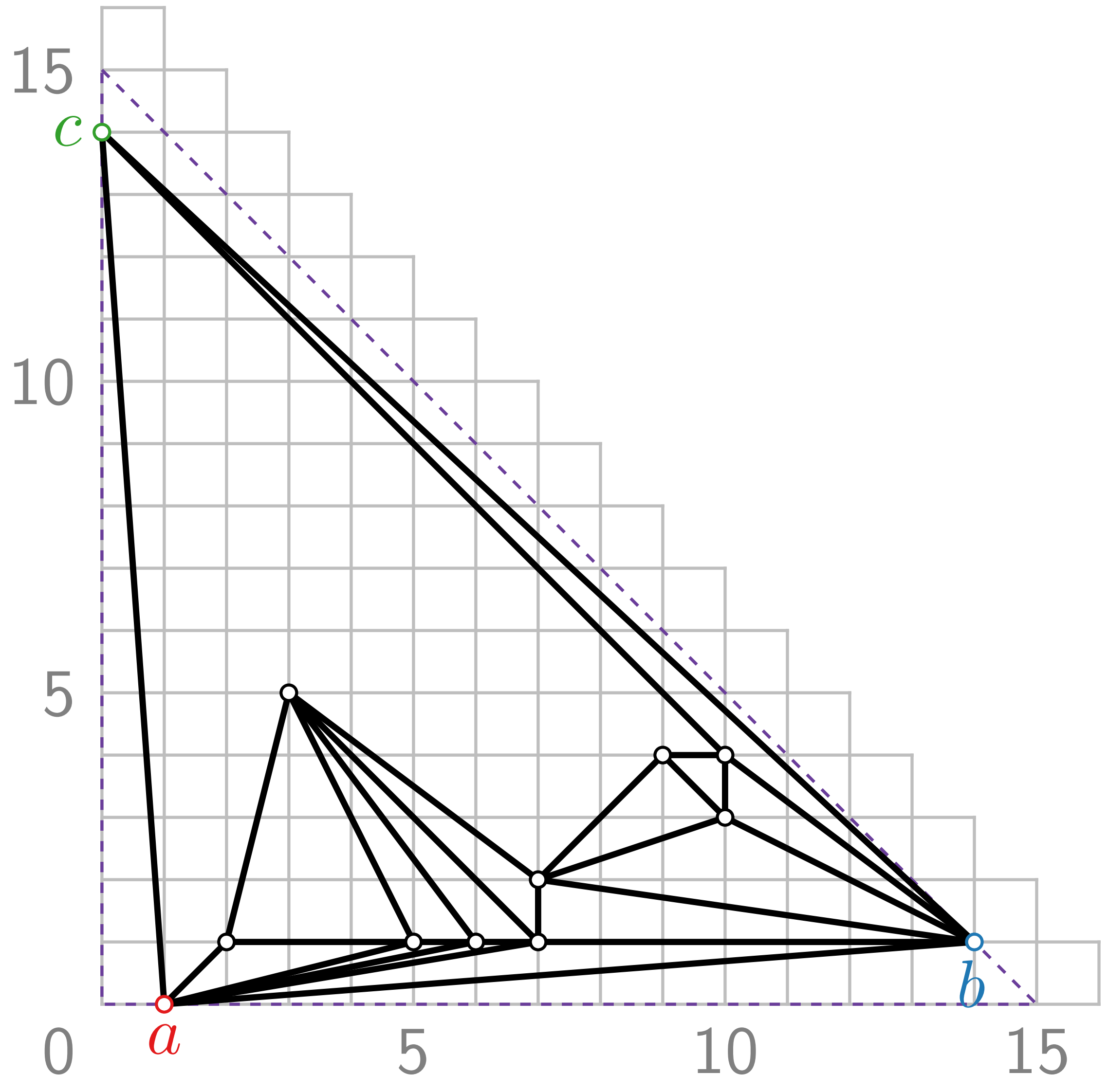

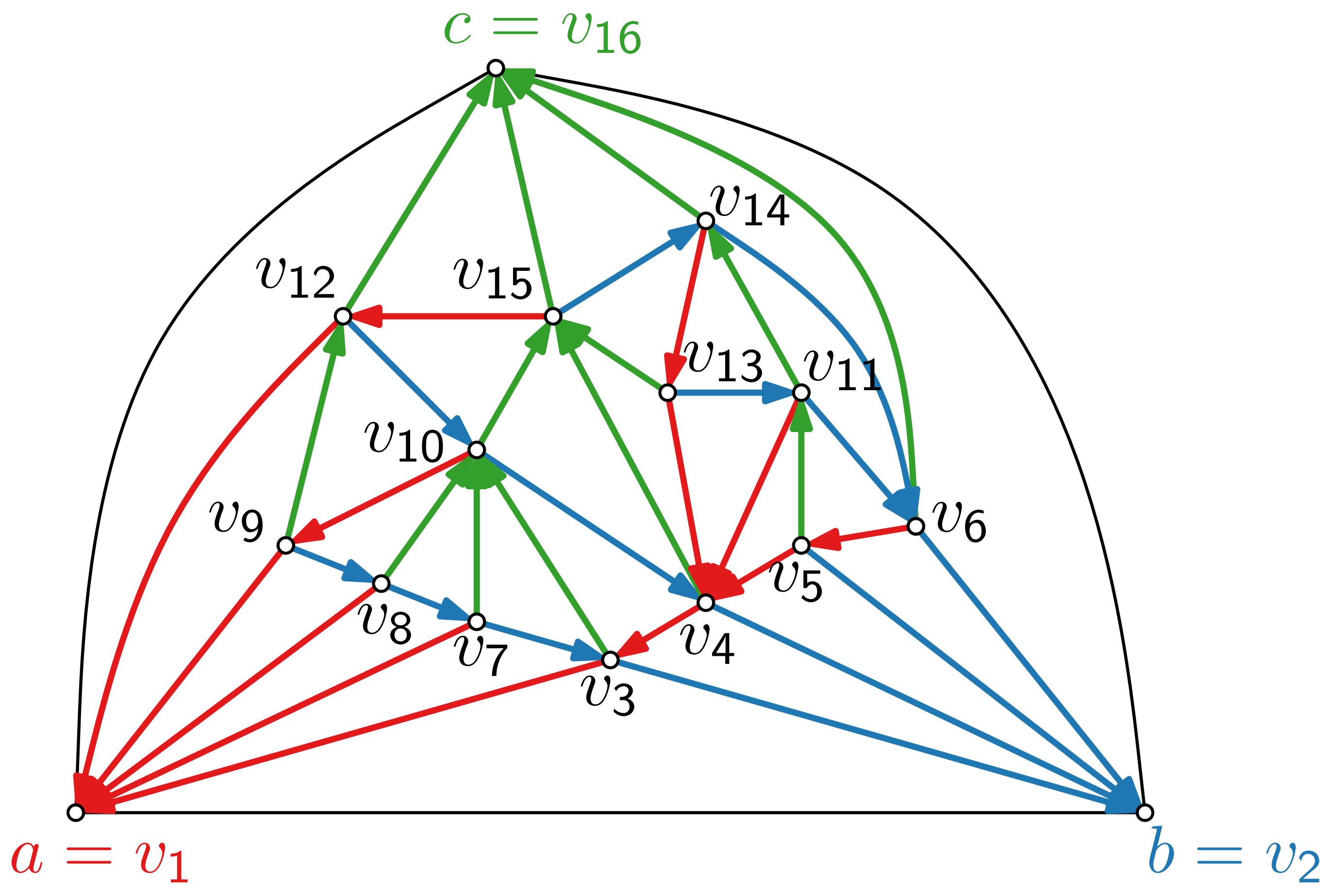

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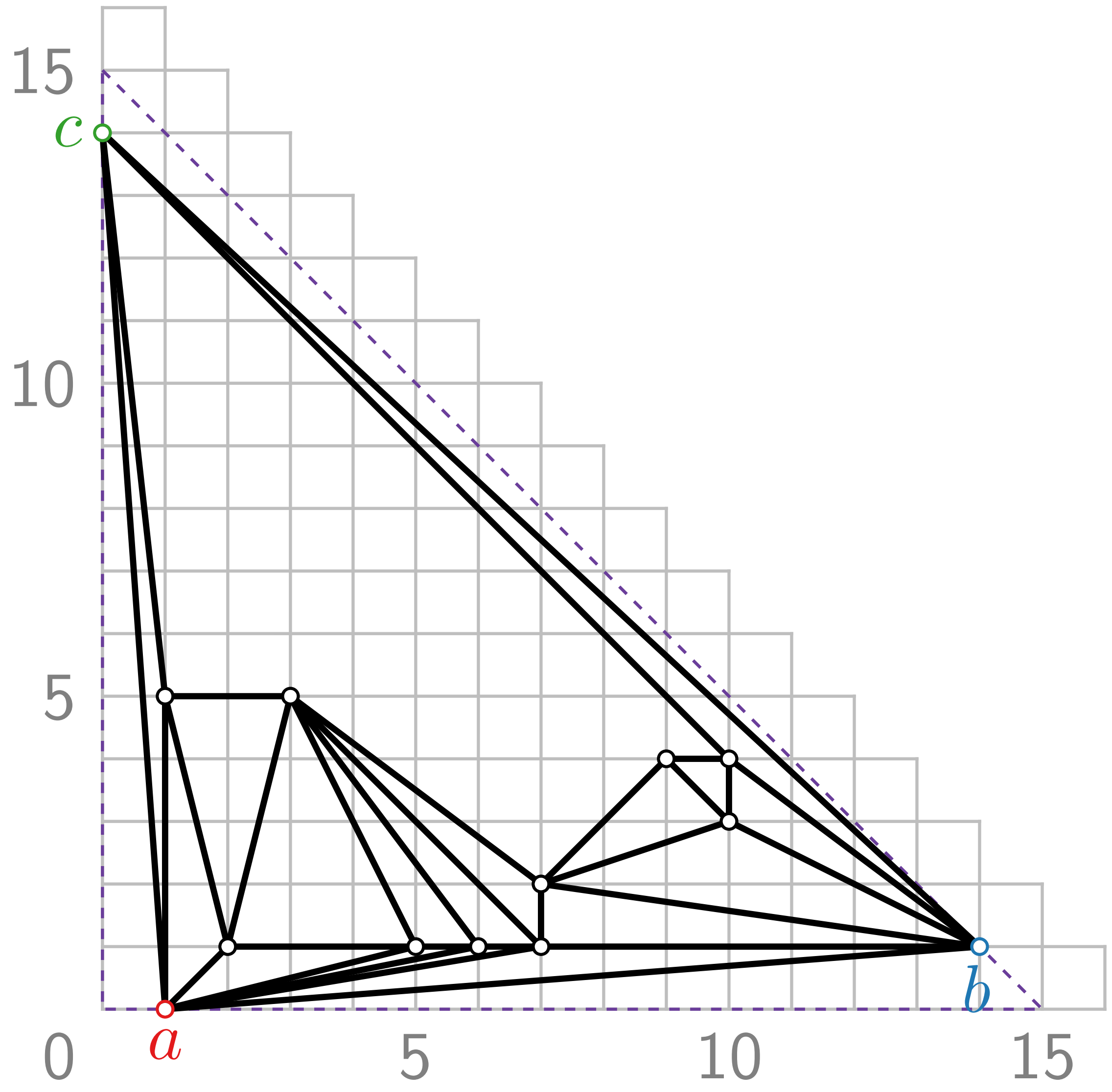

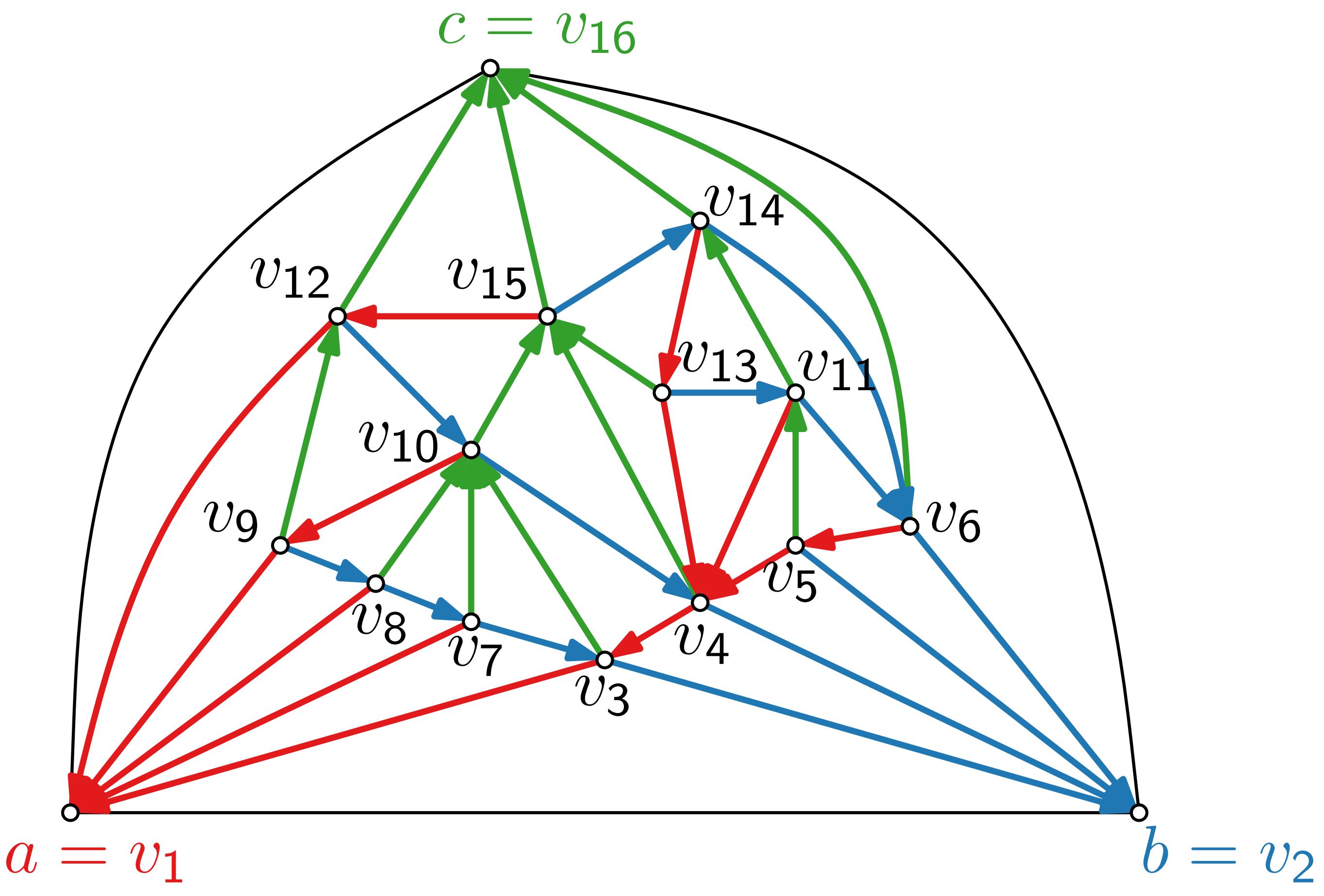

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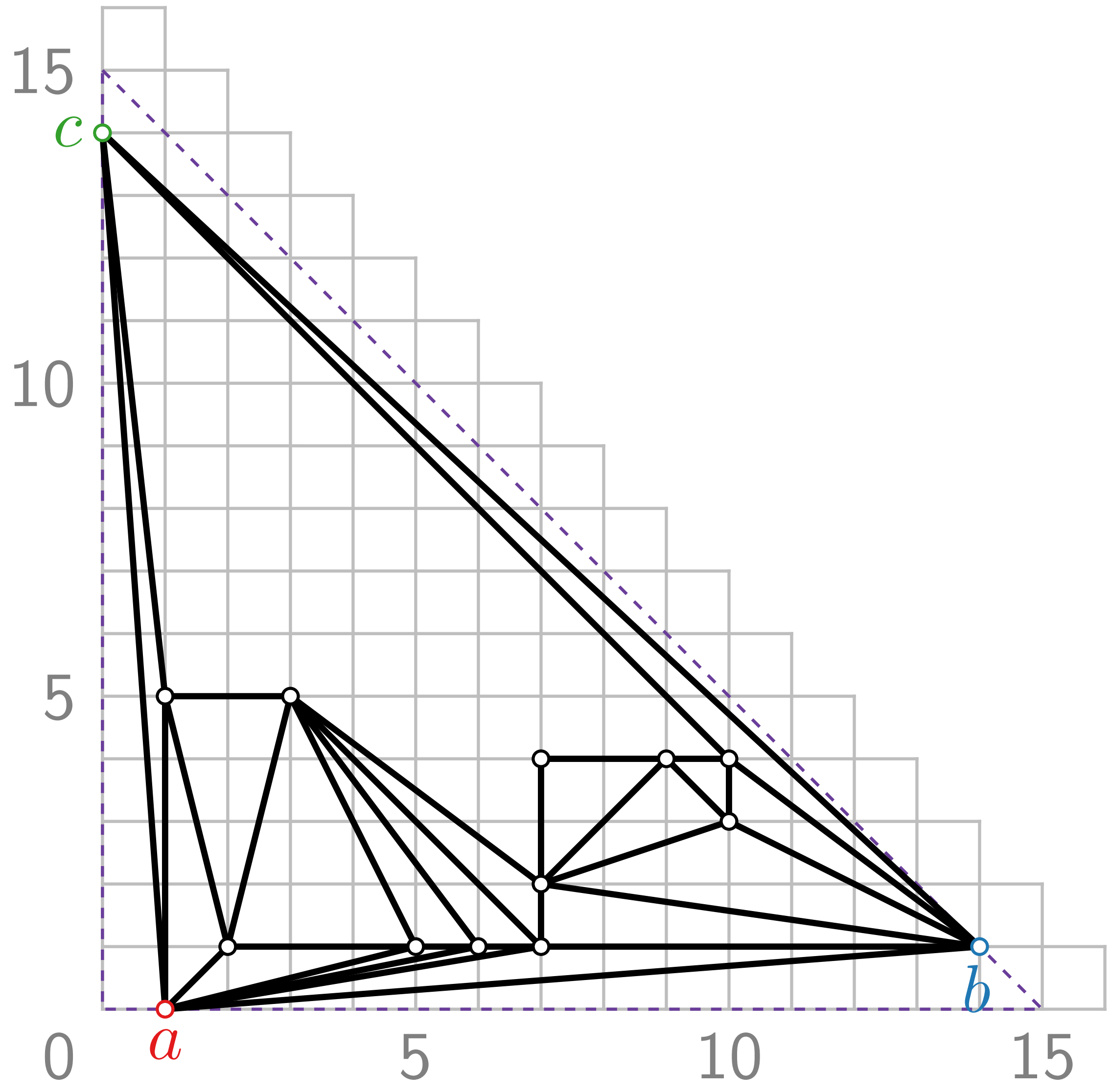

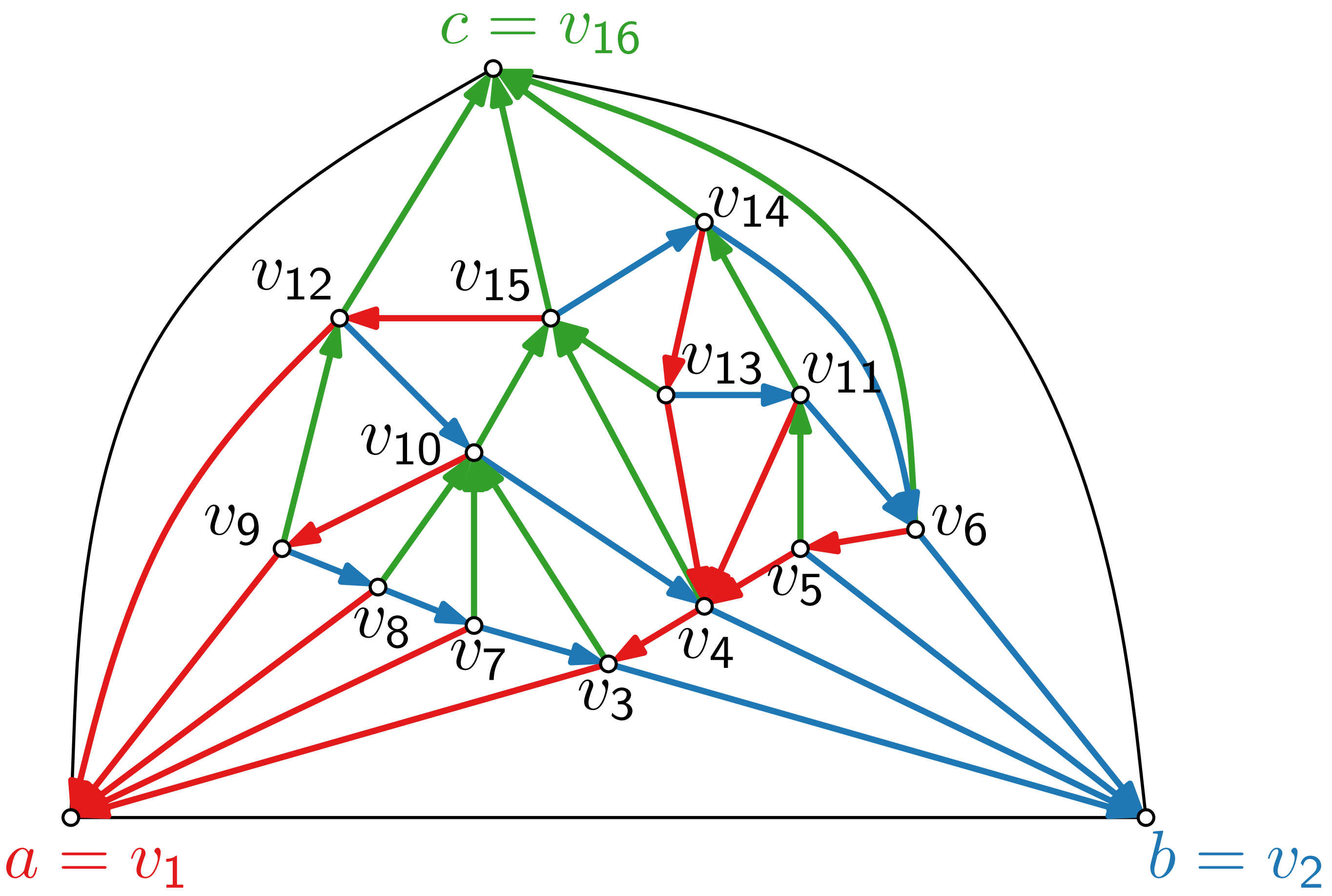

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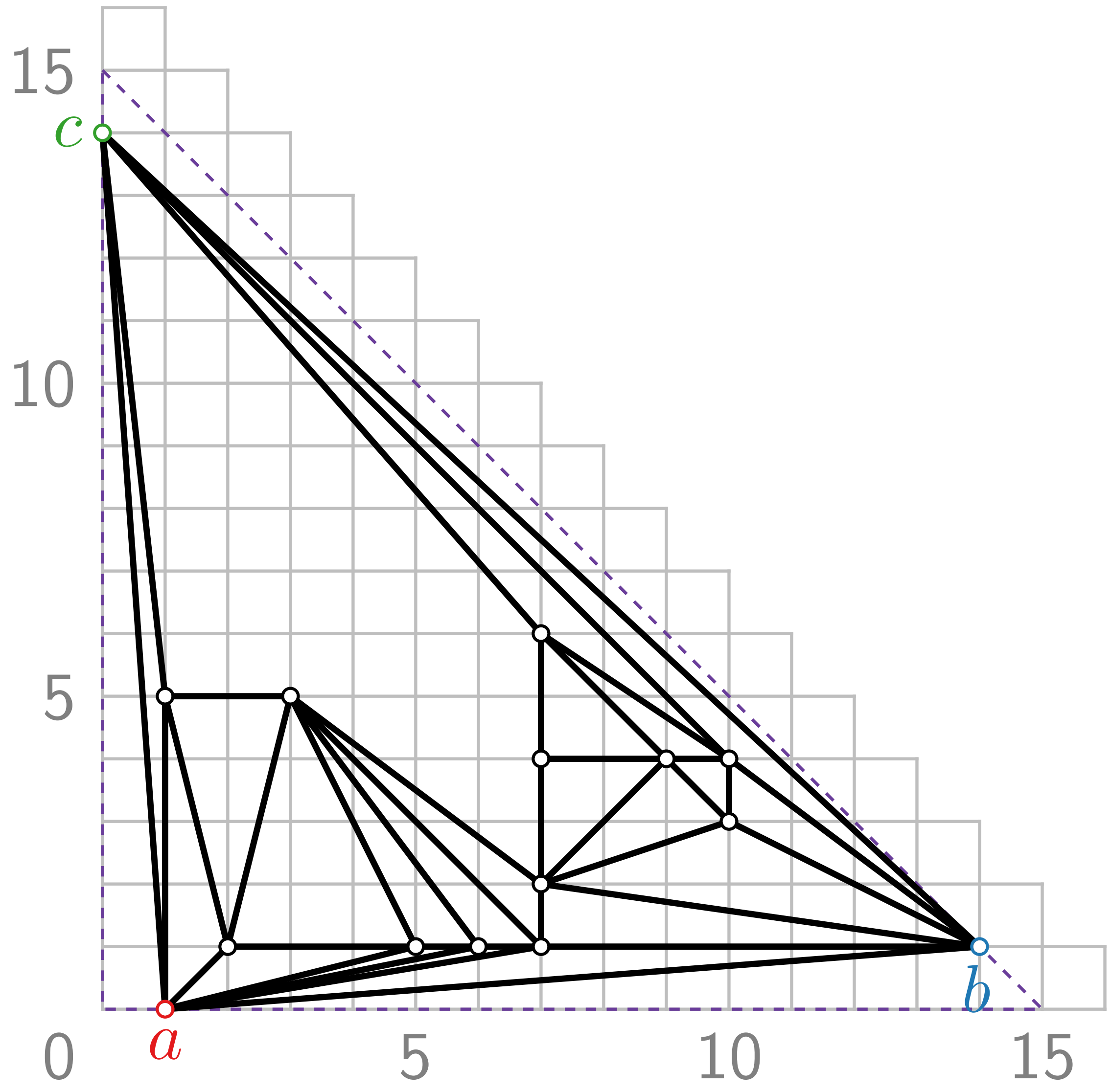

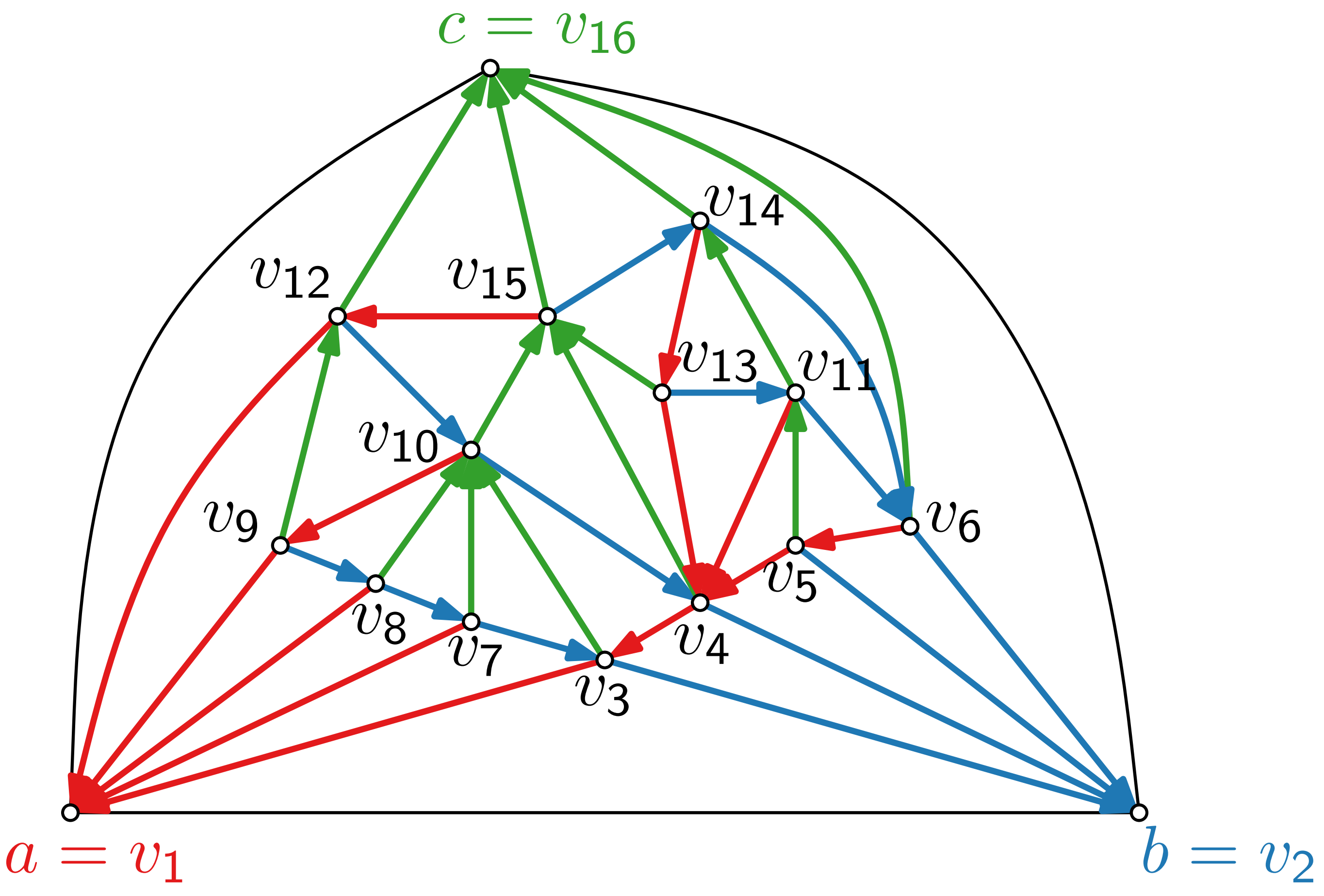

$n = 16,\ n - 1 = 15,\ n - 2 = 14$

$f(v_3) = ({\color{red}7}, {\color{blue}7}, {\color{green}1})$

$f(a) = ({\color{red}14}, {\color{blue}1}, {\color{green}0})$ $\qquad f(v_4) = ({\color{red}6}, {\color{blue}7}, {\color{green}2})$

$f(b) = ({\color{red}0}, {\color{blue}14}, {\color{green}1})$ $\qquad f(v_5) = ({\color{red}2}, {\color{blue}10}, {\color{green}3})$

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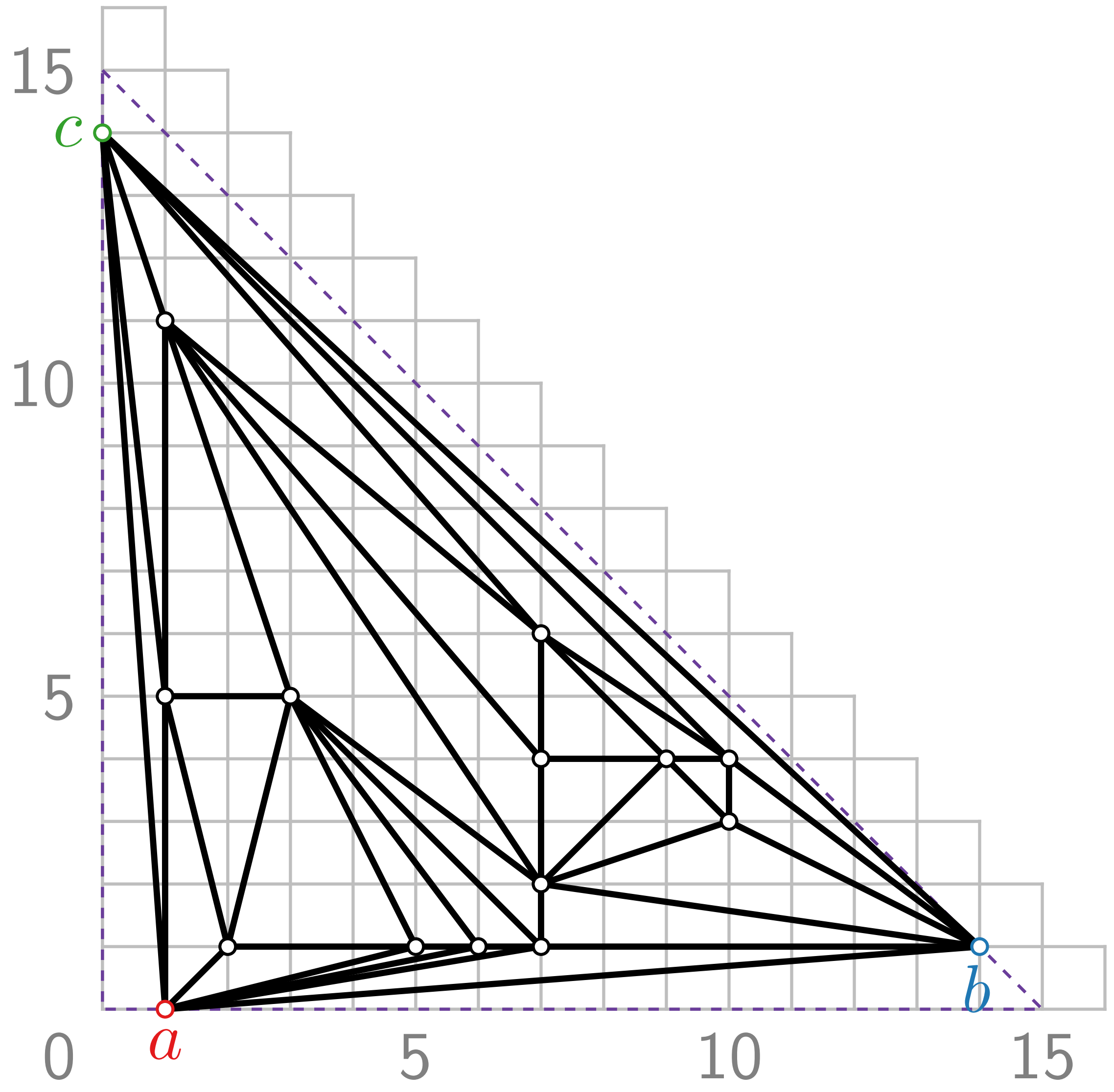

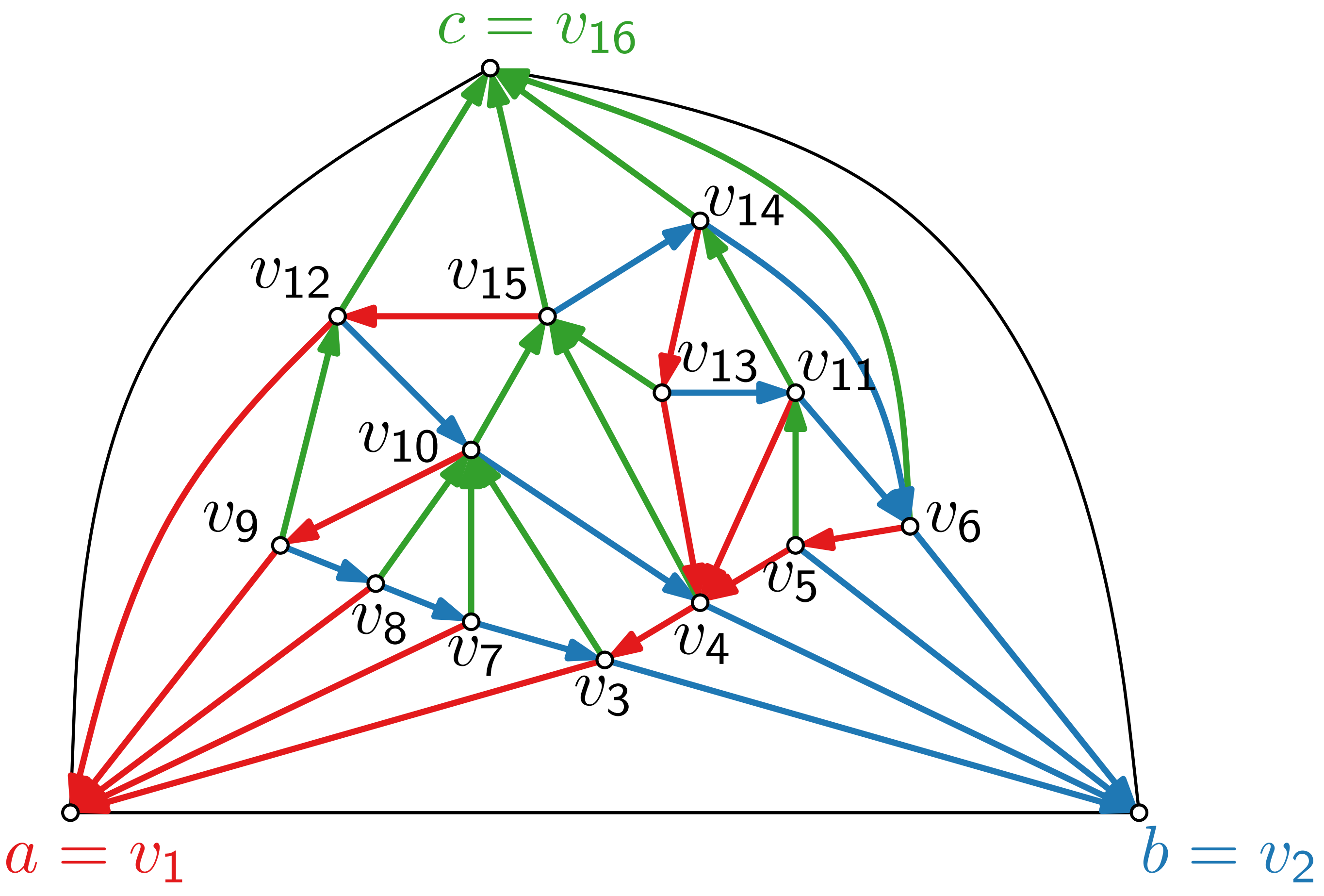

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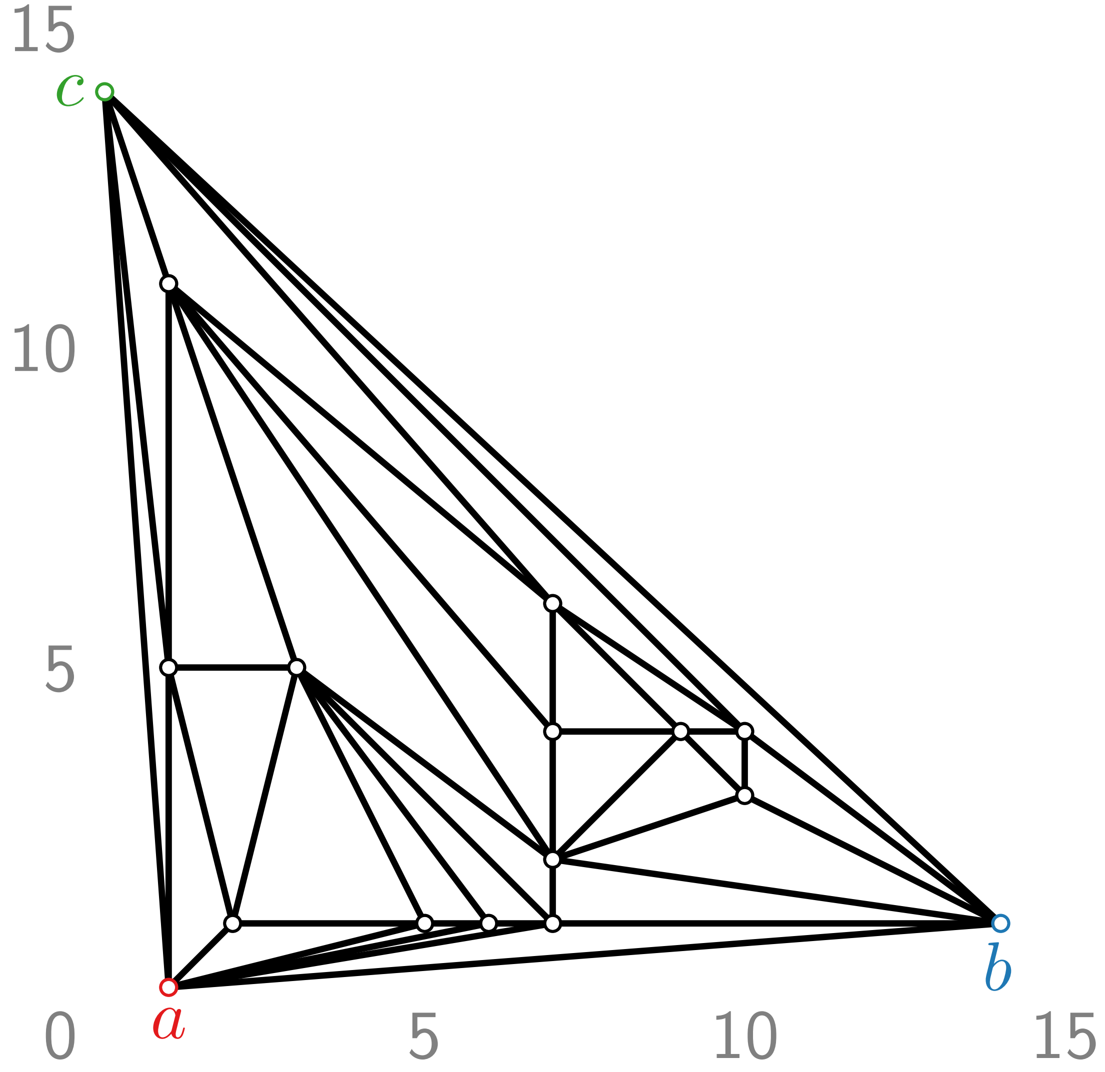

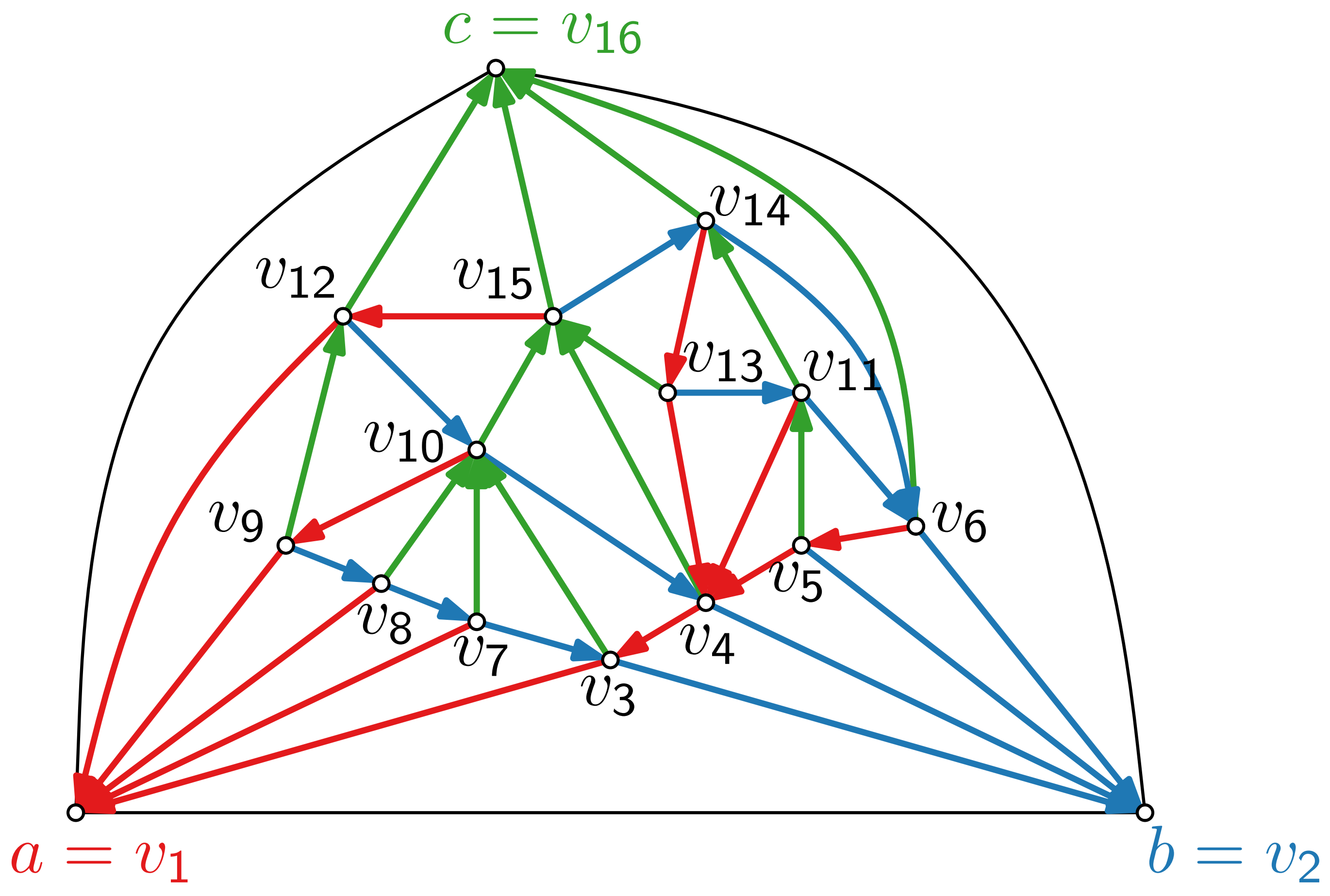

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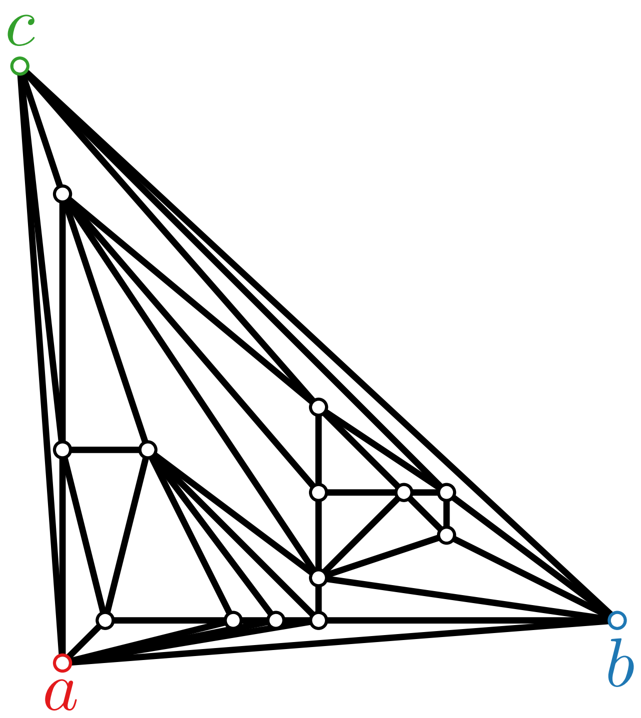
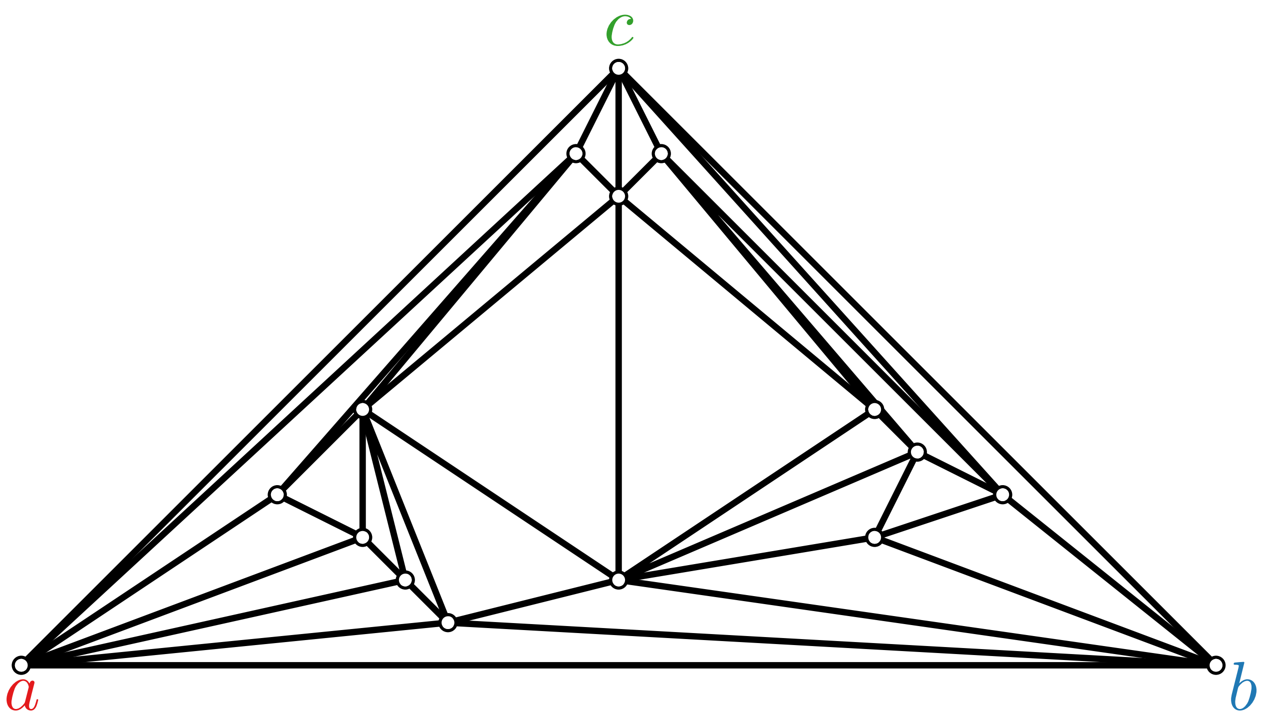
Results & Variations

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Schnyder '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$. Such a drawing can be computed in $O(n)$ time.





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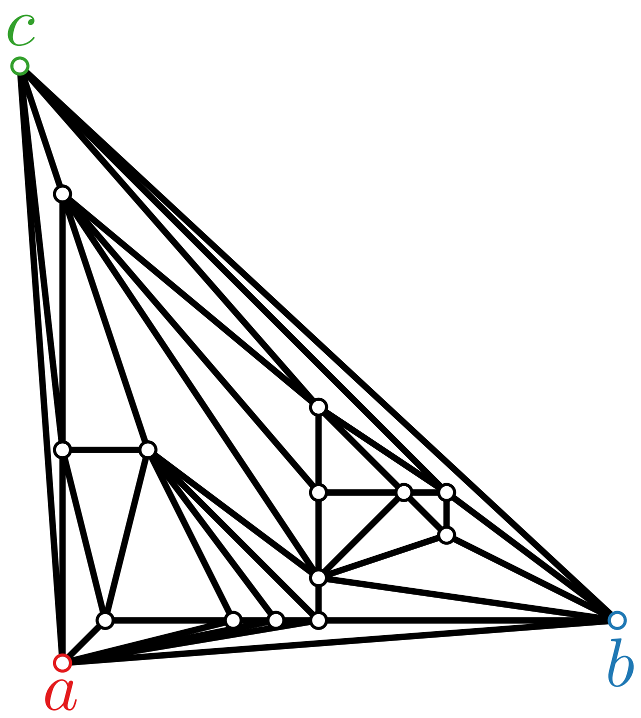

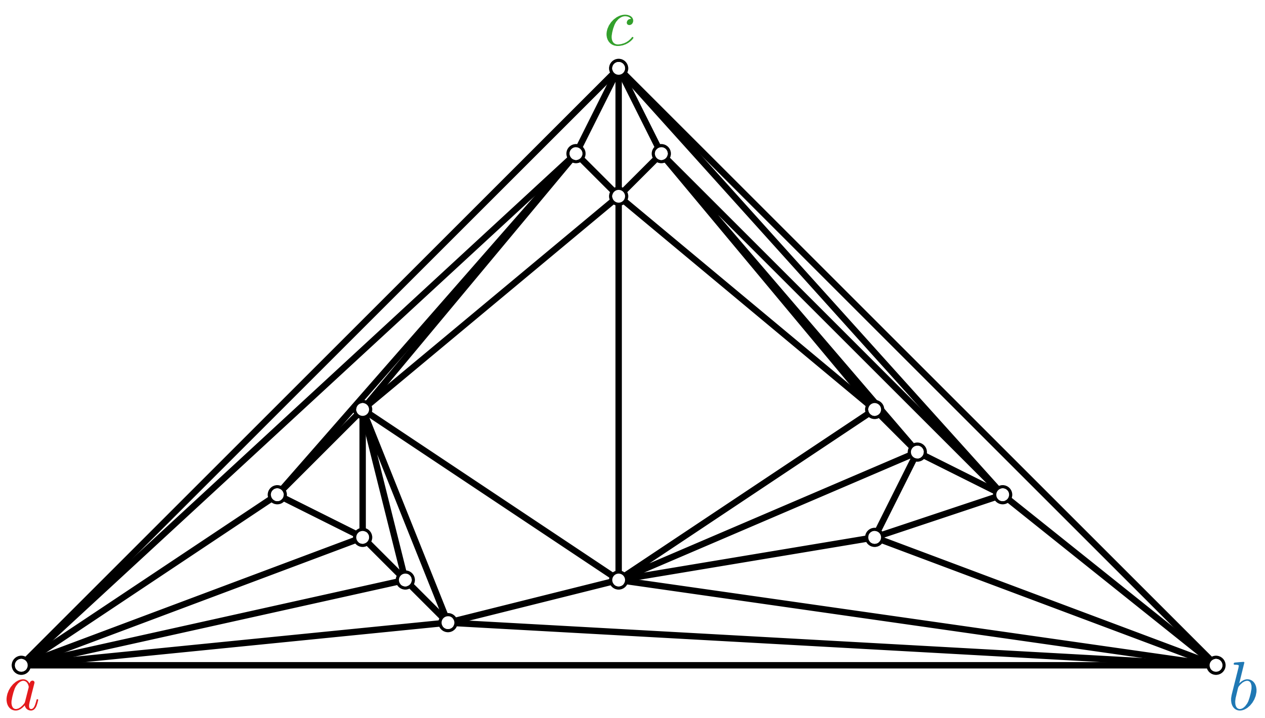

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Theorem. [Brandenburg '08]
Every n-vertex planar graph has a planar straight-line drawing of size $4n/3 \times 2n/3$. Such a drawing can be computed in $O(n)$ time.

Results & Variations

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Theorem. [Brandenburg '08]
Every n-vertex planar graph has a planar straight-line drawing of size $4n/3 \times 2n/3$. Such a drawing can be computed in $O(n)$ time.

Theorem. [Dolev, Leighton, Trickey '84]
There exist n-vertex plane graphs such that any planar straight-line drawing of them has an area of at least $(2n/3-1) \times (2n/3-1)$.

Results & Variations

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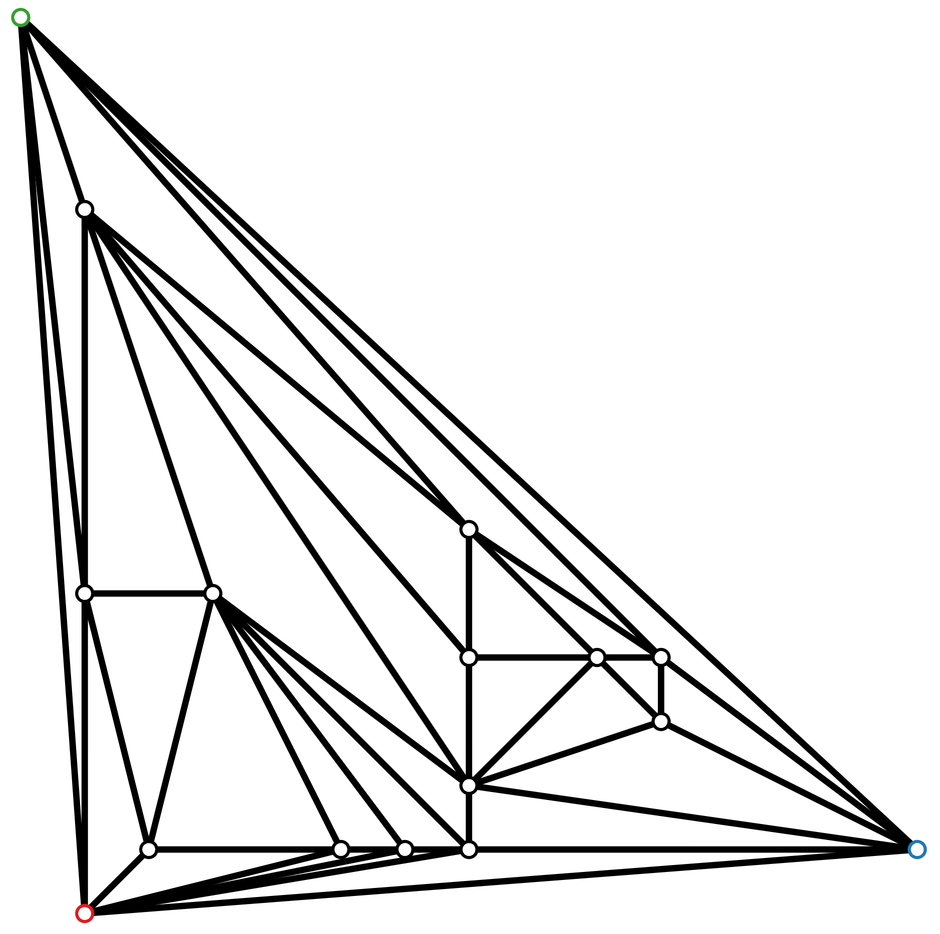
[Frati, Patrignani '07] Area at least $n^2/9 + \Omega(n)$ in the variable-embedding setting.

Results & Variations



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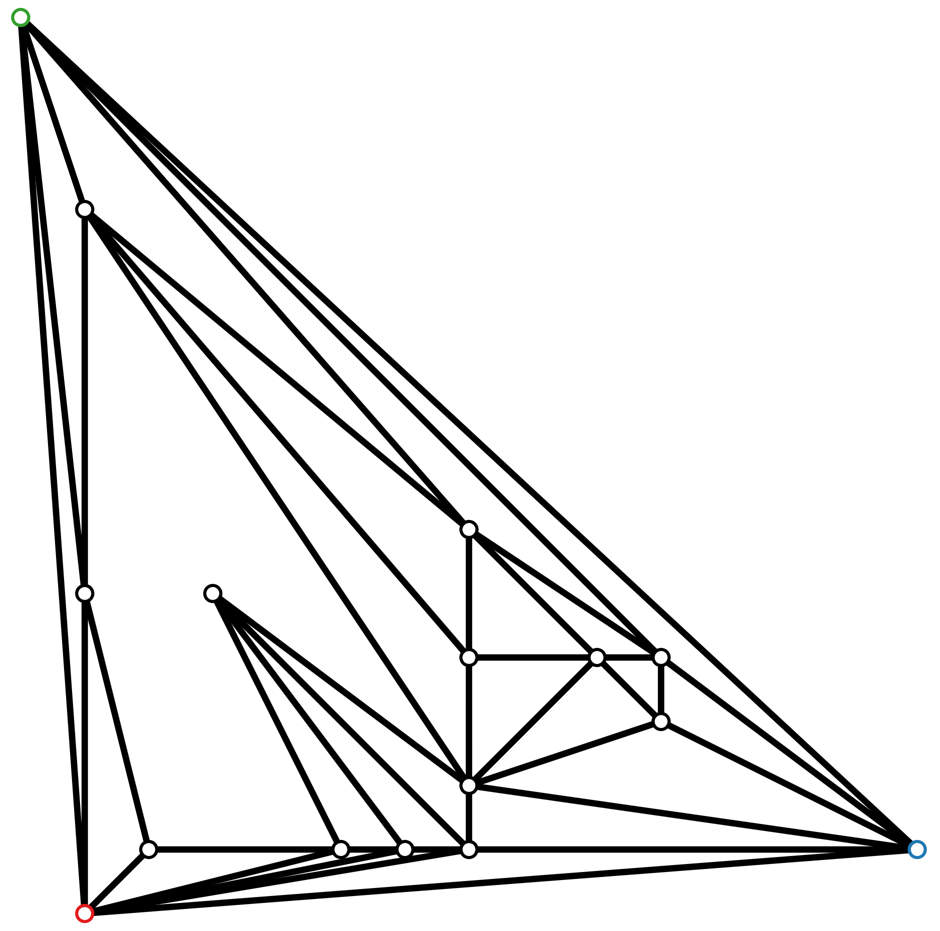

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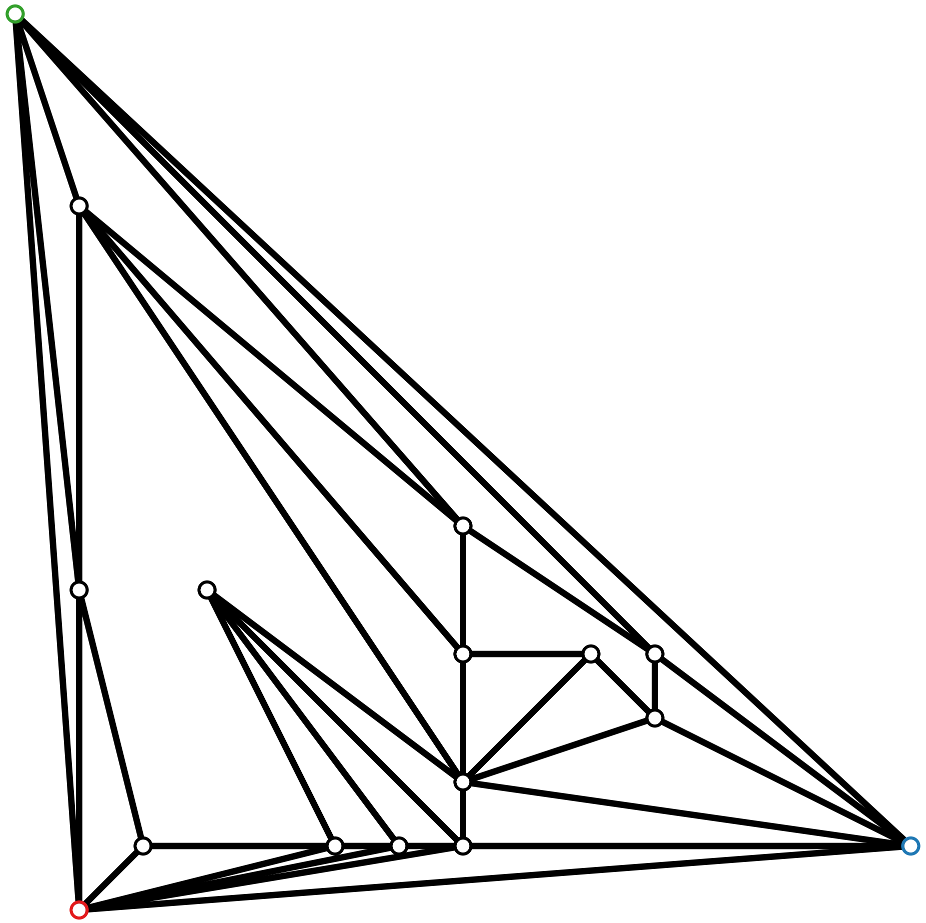

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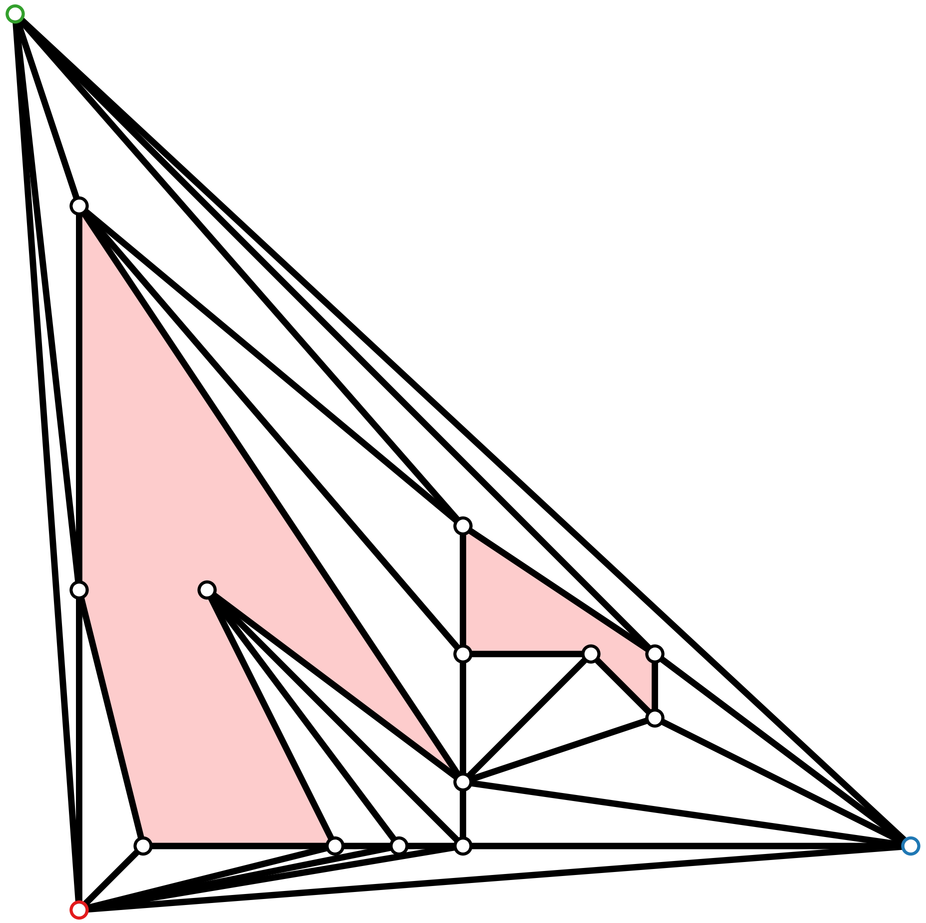

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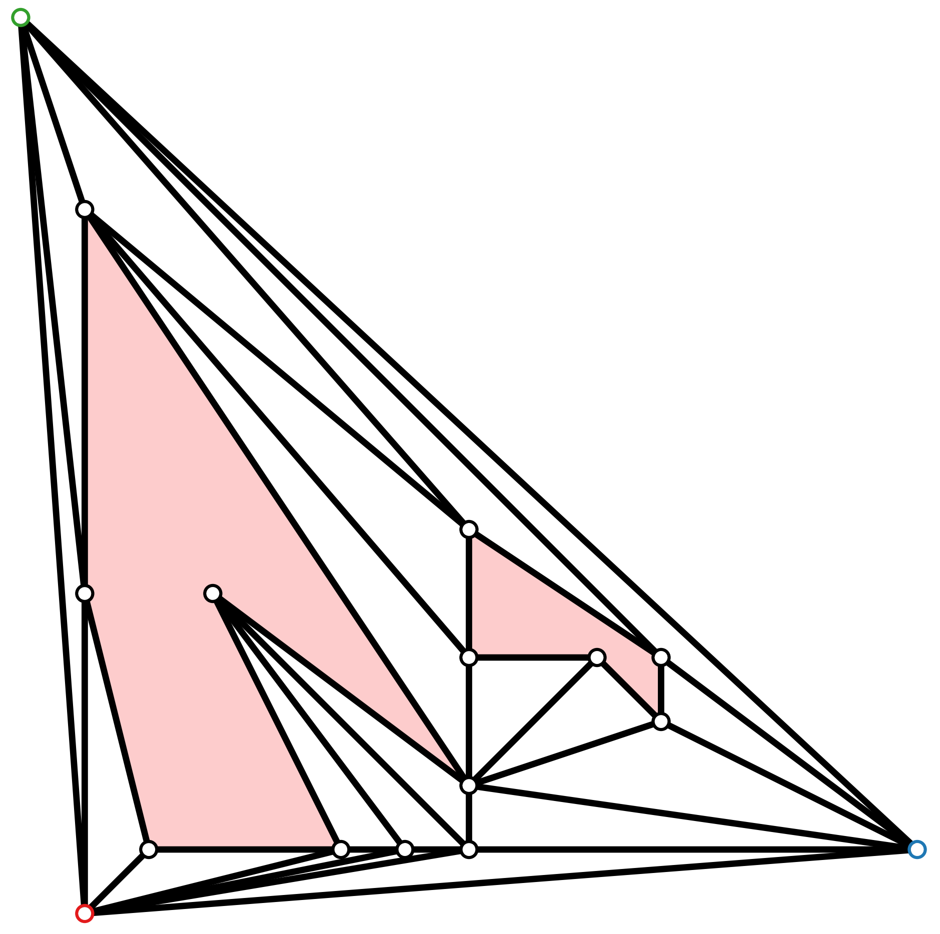

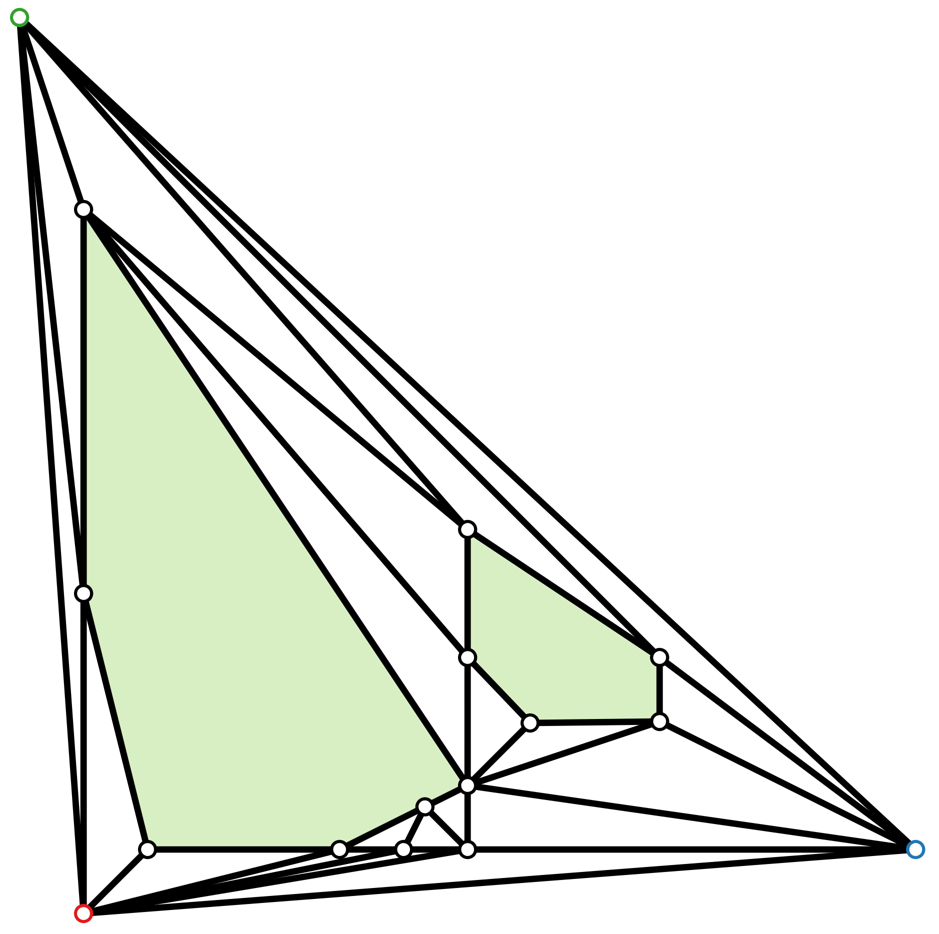

Results & Variations

Theorem. [Kant '96]
Every n-vertex 3-connected planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$ where all faces are drawn convex.
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Theorem. [Felsner '01]
Every 3-connected planar graph with f faces has a planar straight-line drawing of size $(f-1) \times (f-1)$ where all faces are drawn convex.
Such a drawing can be computed in $O(n)$ time.

Literature

- [PGD Ch. 4.3] for detailed explanation of Schnyder woods etc.
- [Sch90] "Embedding planar graphs on the grid", Walter Schnyder, SoCG 1990 – original paper on Schnyder realizer method.