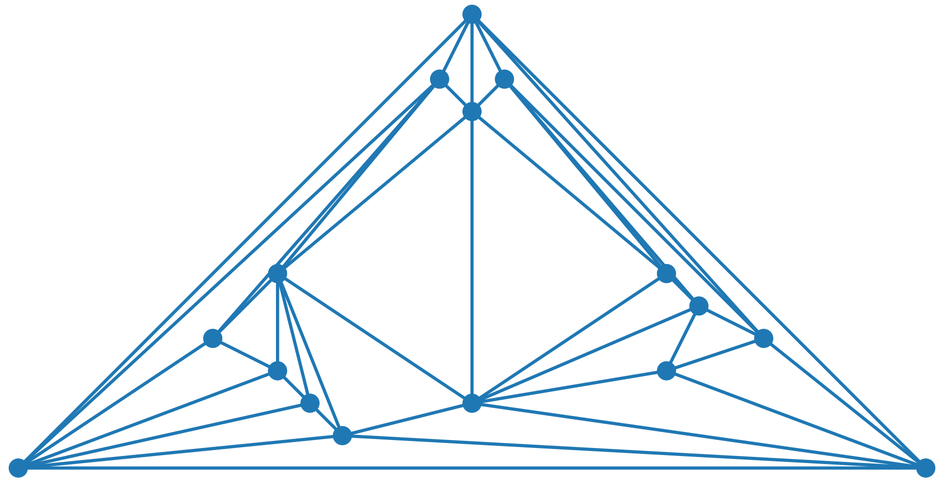
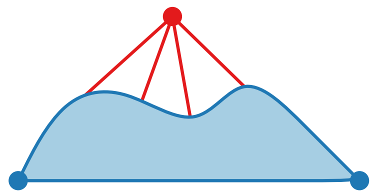
Visualization of Graphs

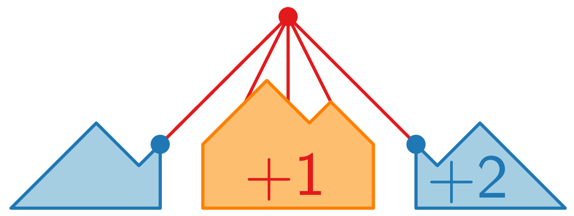
Lecture 3:
Straight-Line Drawings of Planar Graphs I: Canonical Orderings and the Shift Method



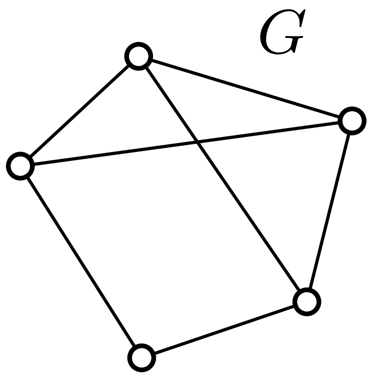
Samuel Wolf

Summer term 2025





Planar Graphs



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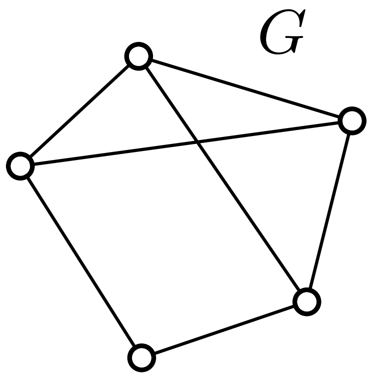

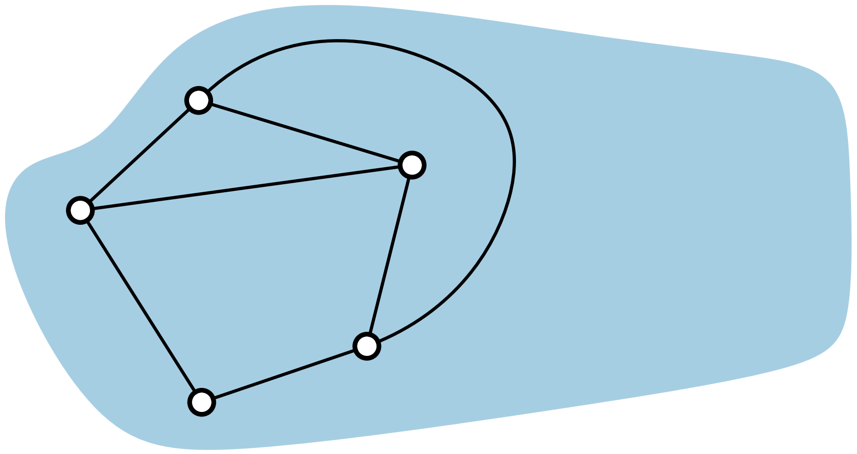

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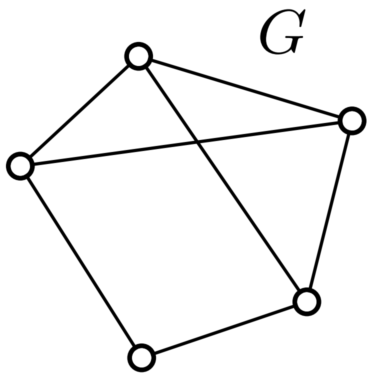

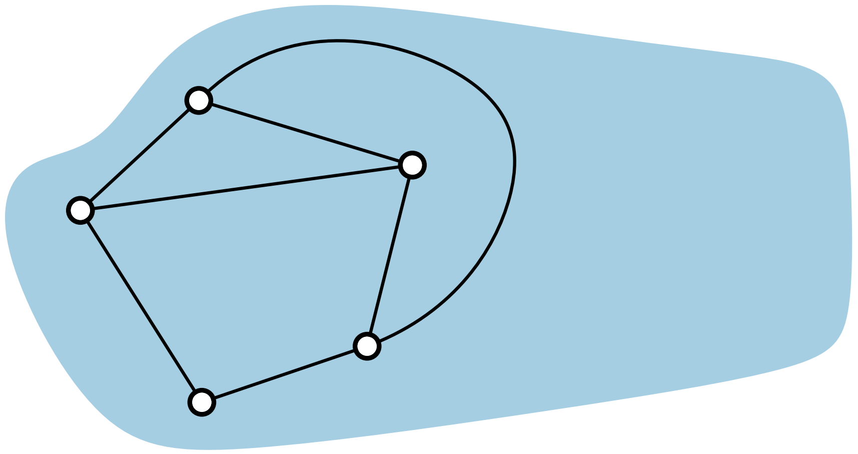

G is **planar**:
it can be drawn in such a way that
no two edges intersect each other.

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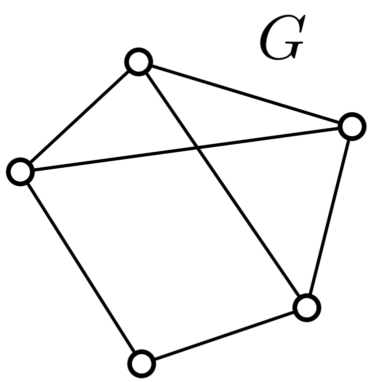

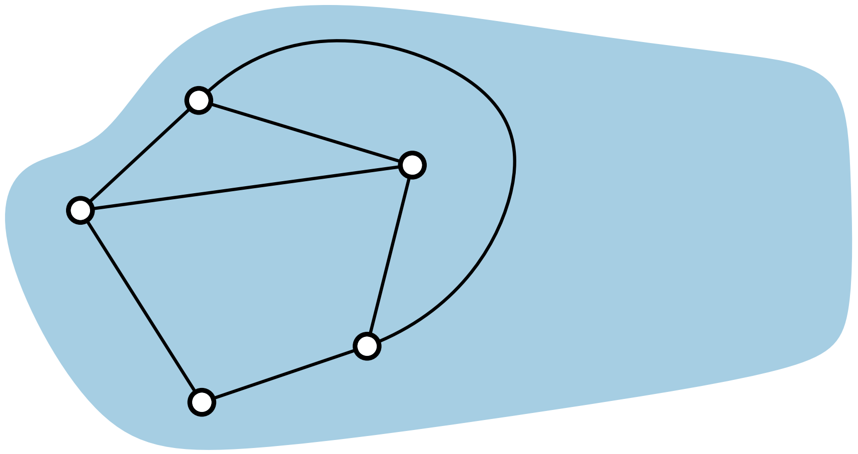

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clockwise orientation of adjacent vertices around each vertex

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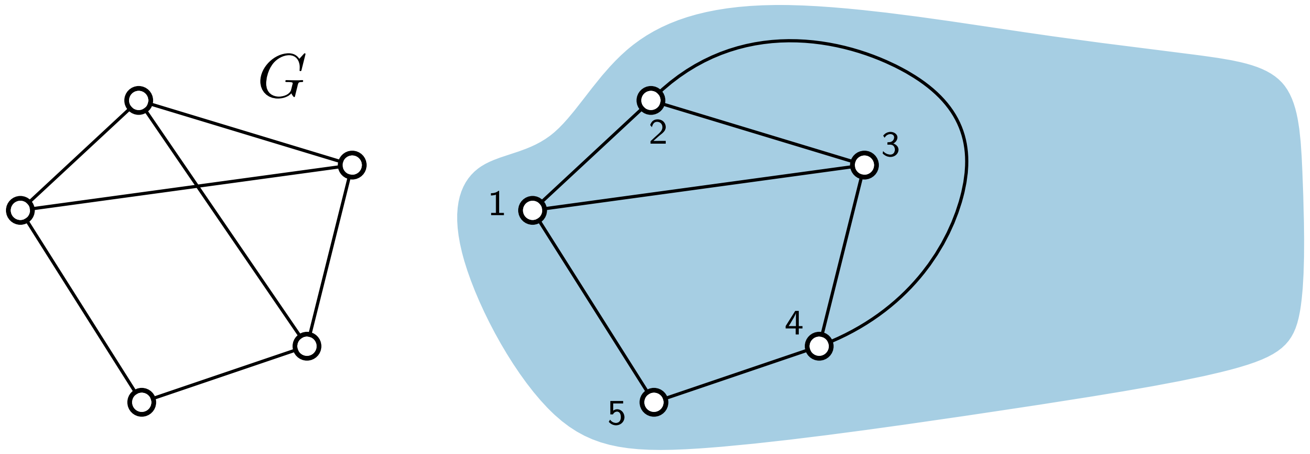

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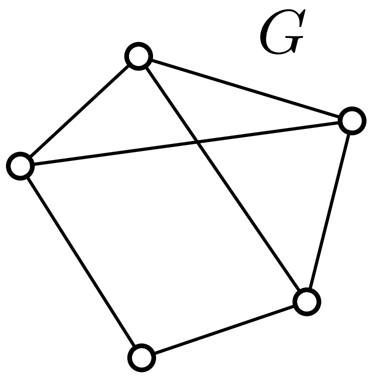

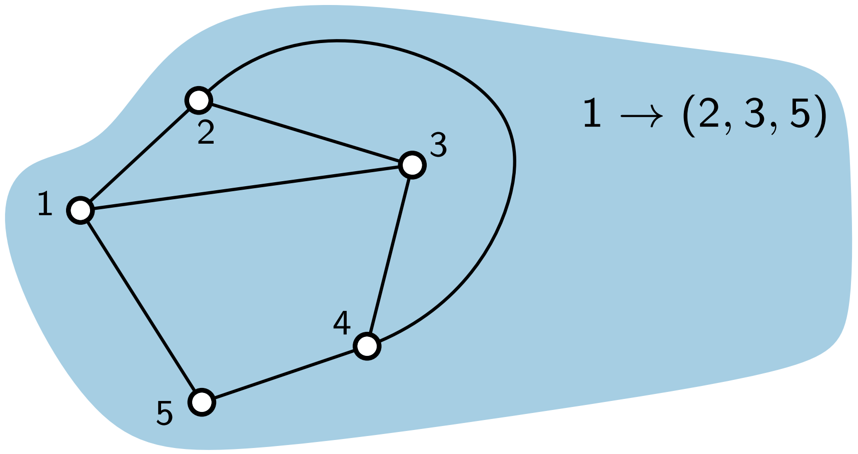

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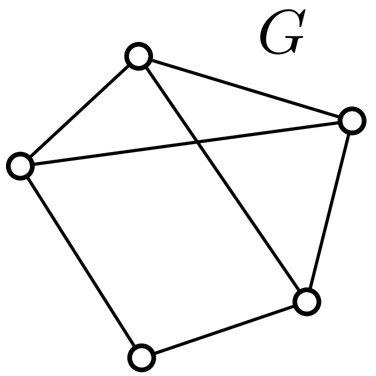

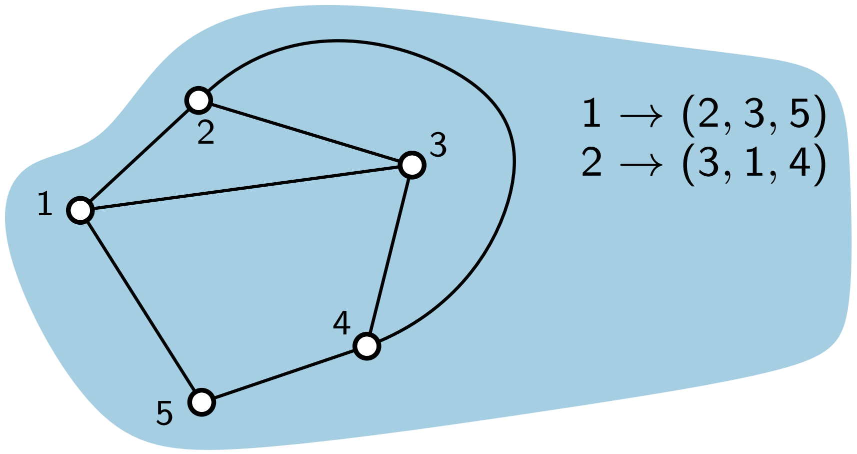

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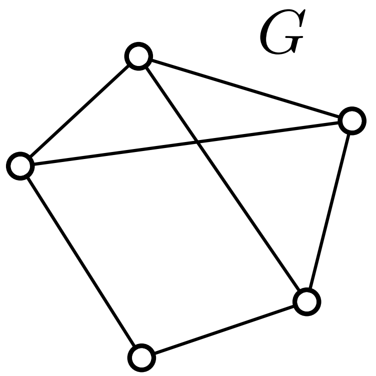

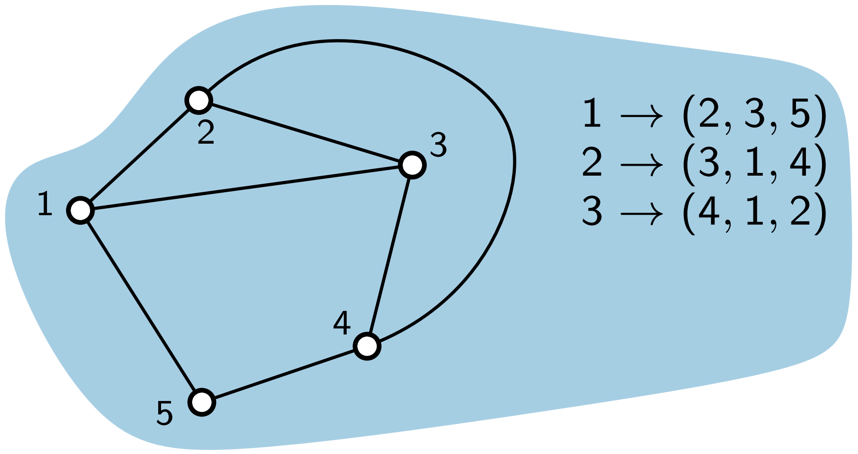

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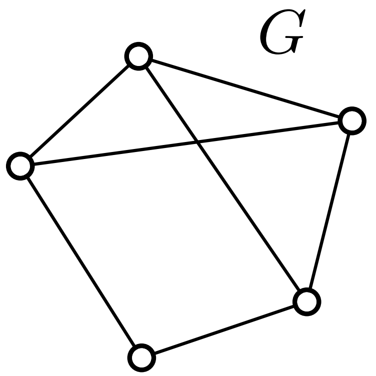

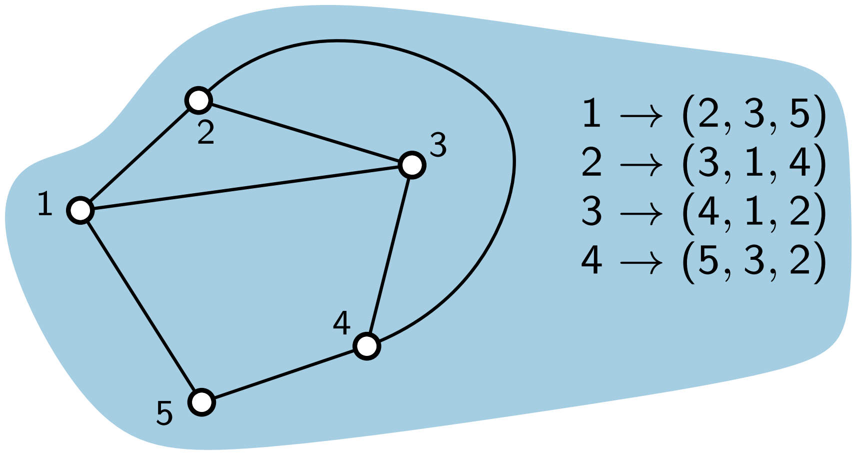

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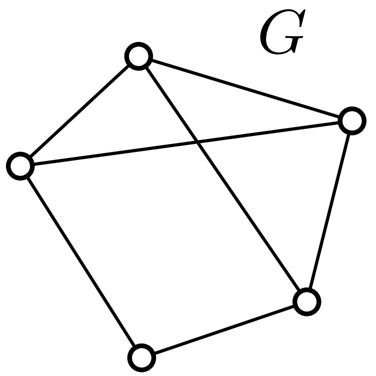

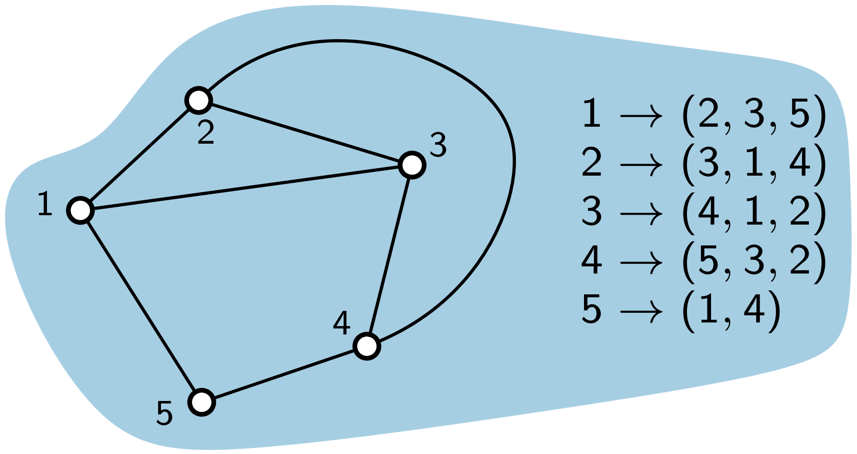

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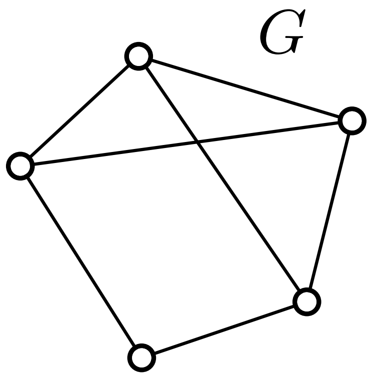
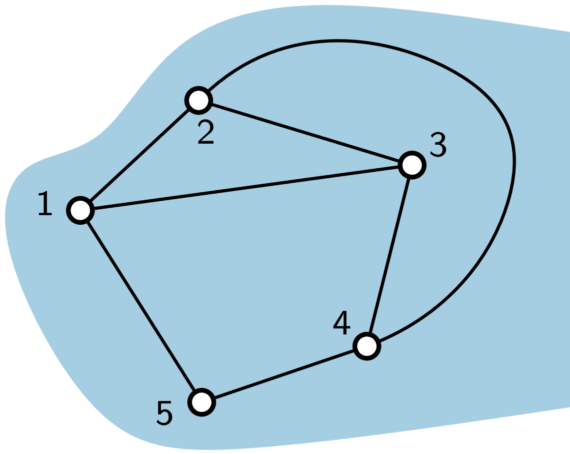
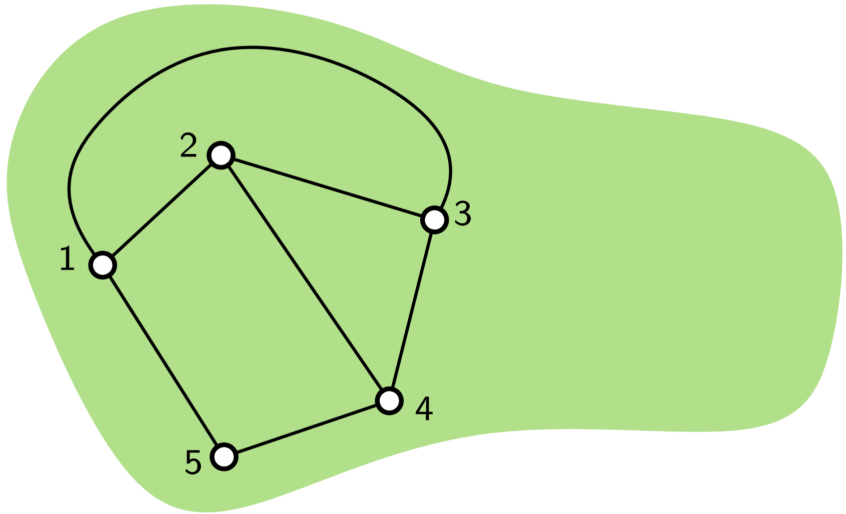




$1 \to (2, 3, 5)$
$2 \to (3, 1, 4)$
$3 \to (4, 1, 2)$
$4 \to (5, 3, 2)$
$5 \to (1, 4)$



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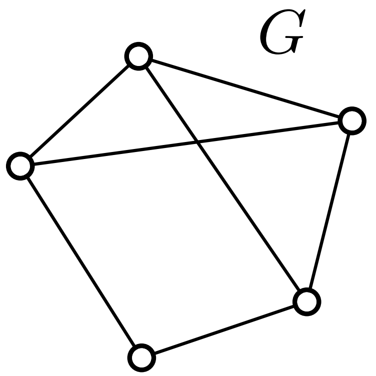

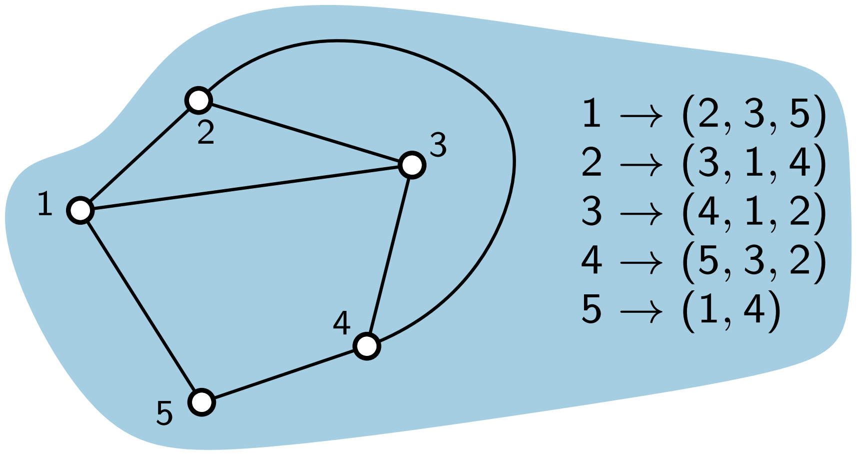

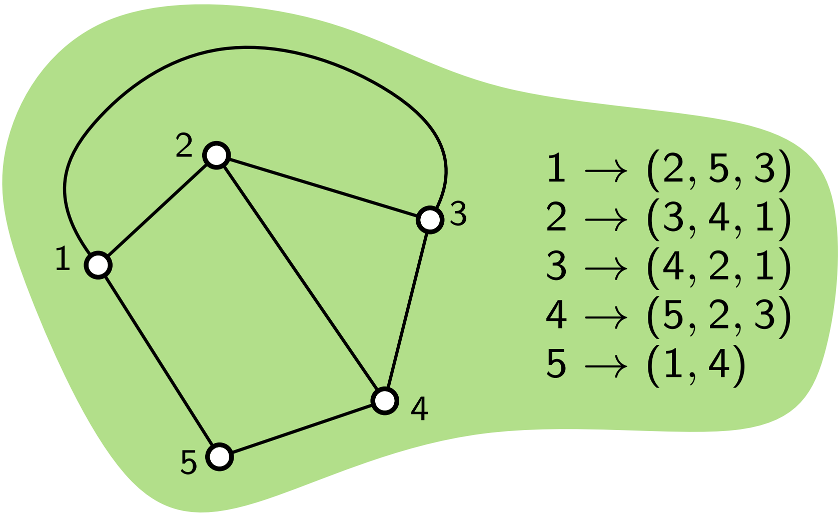

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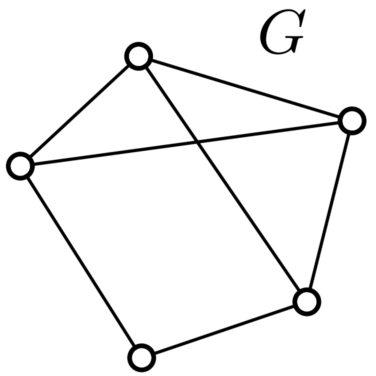

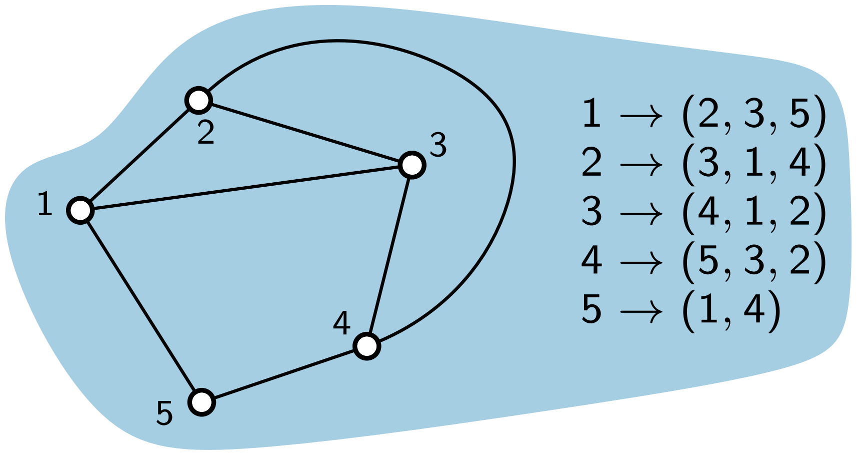

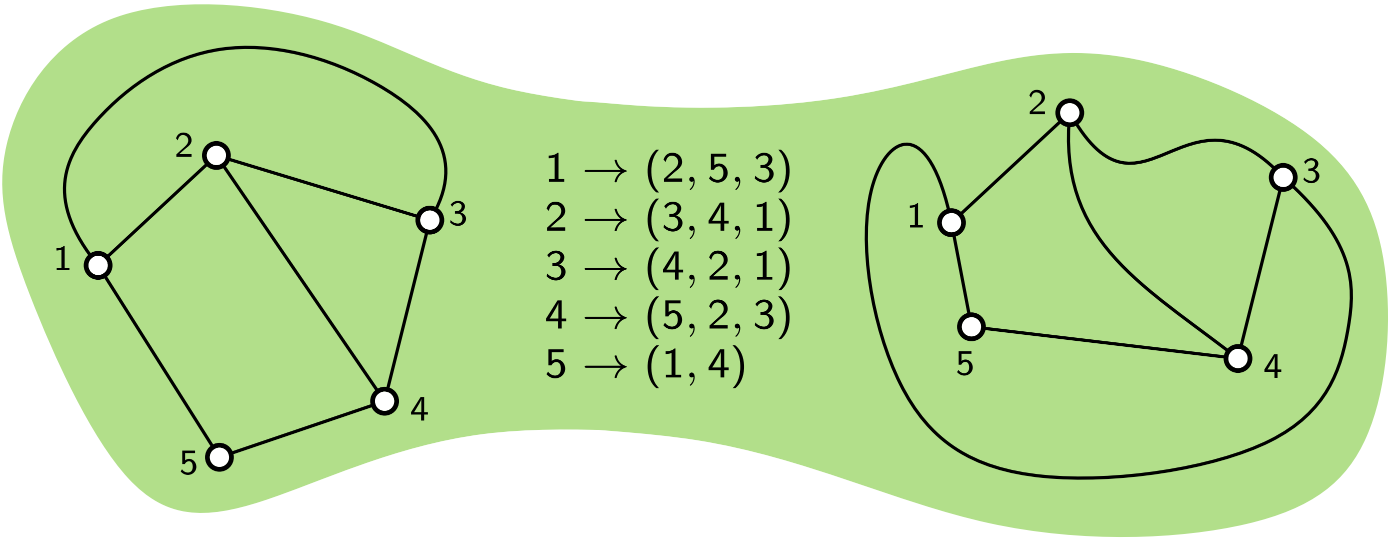

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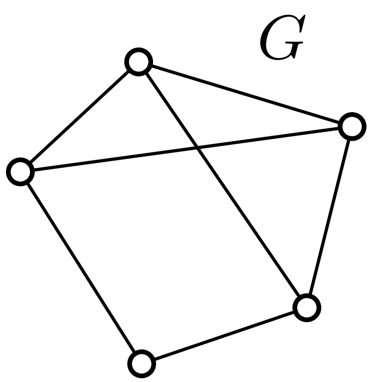

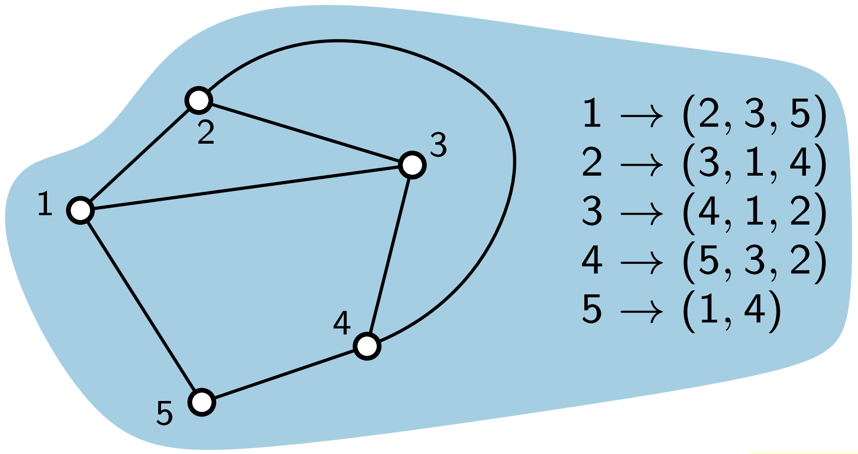

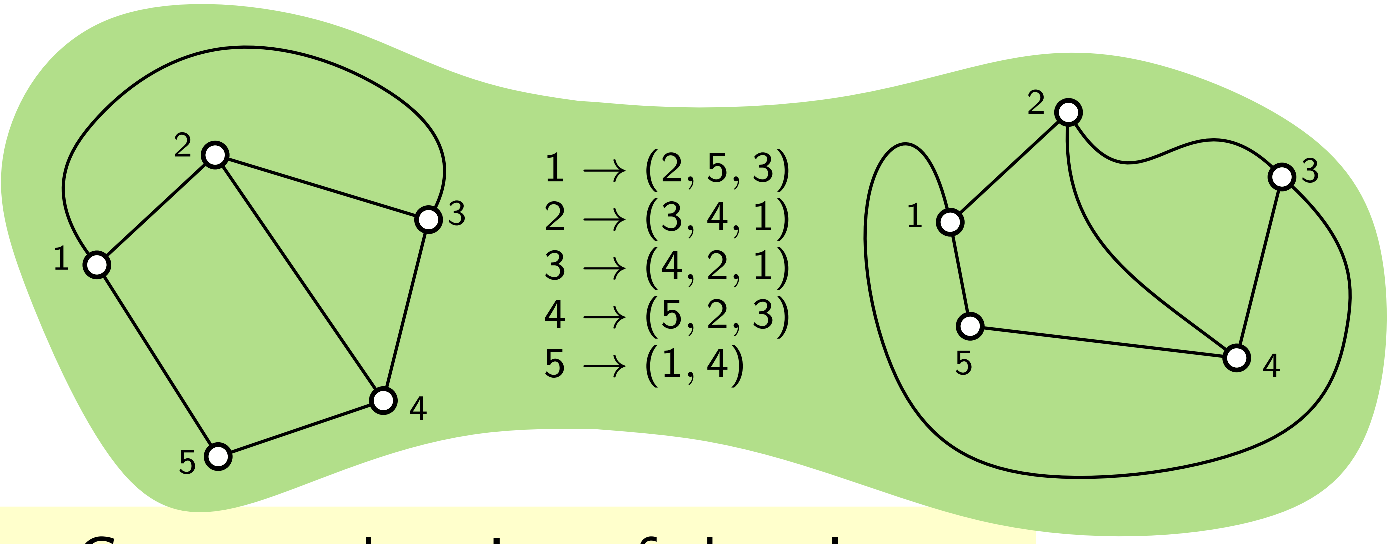

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faces: Connected region of the plane bounded by edges

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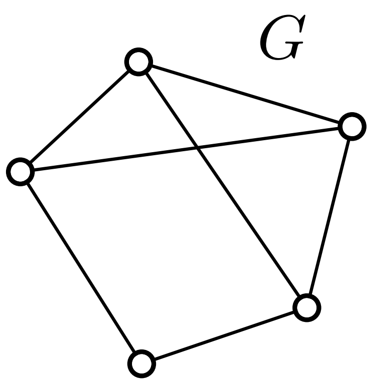

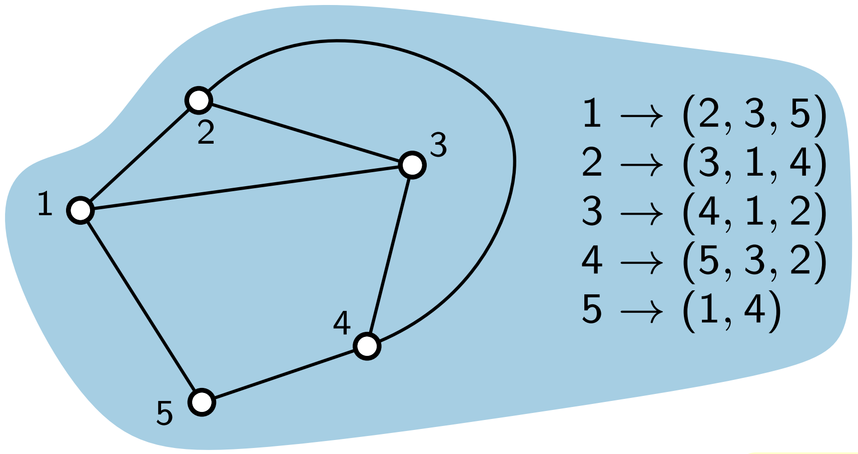

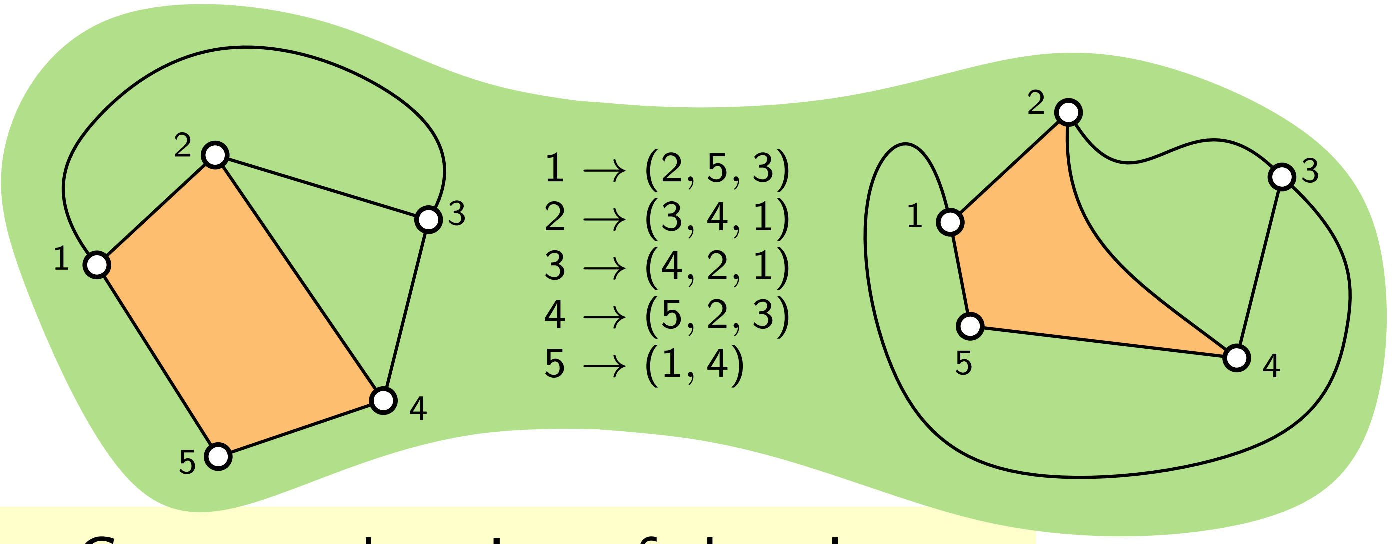

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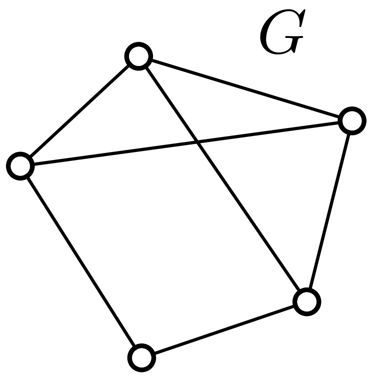

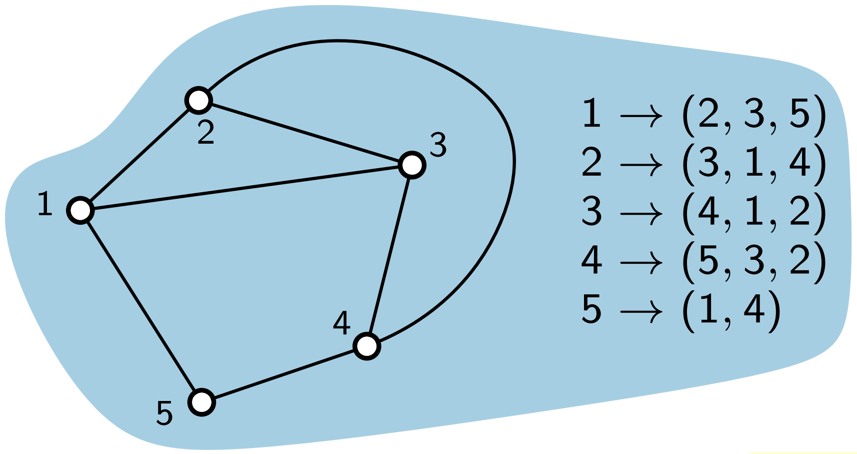

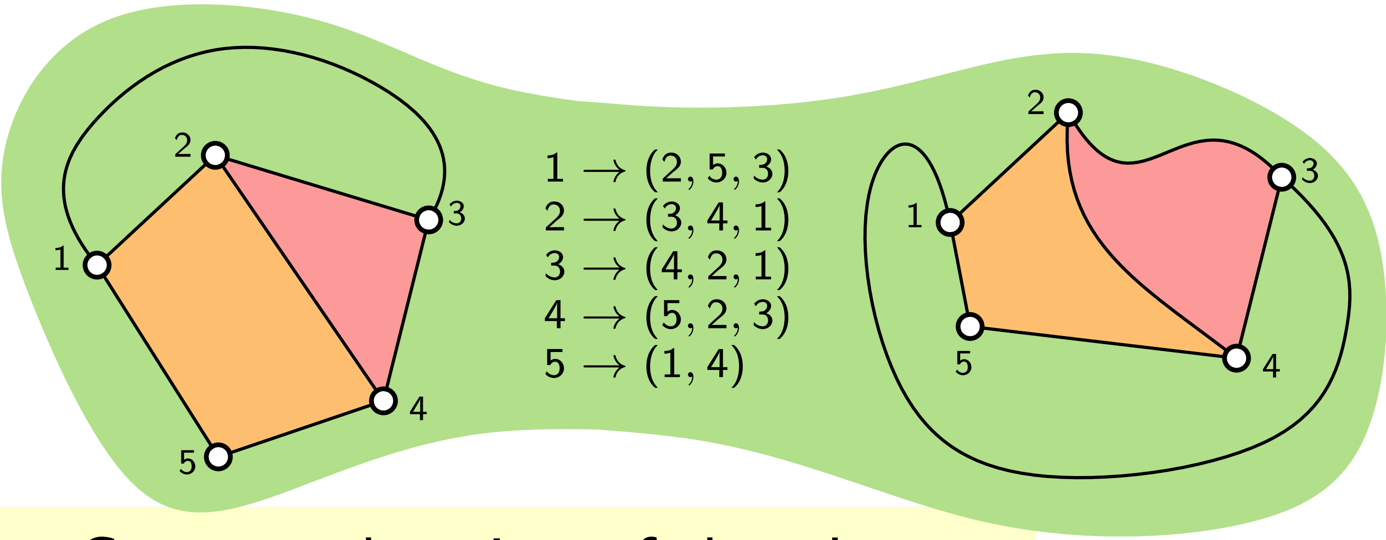

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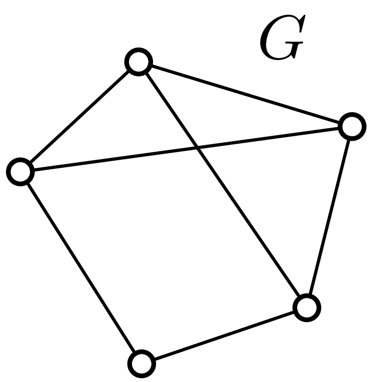

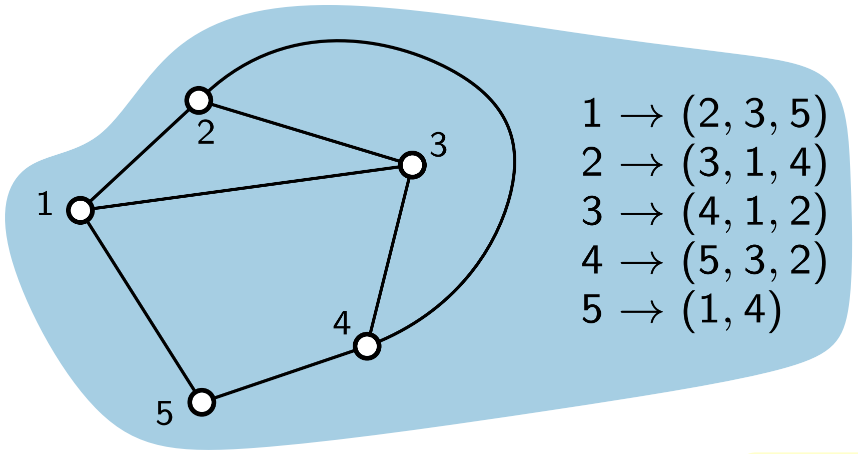

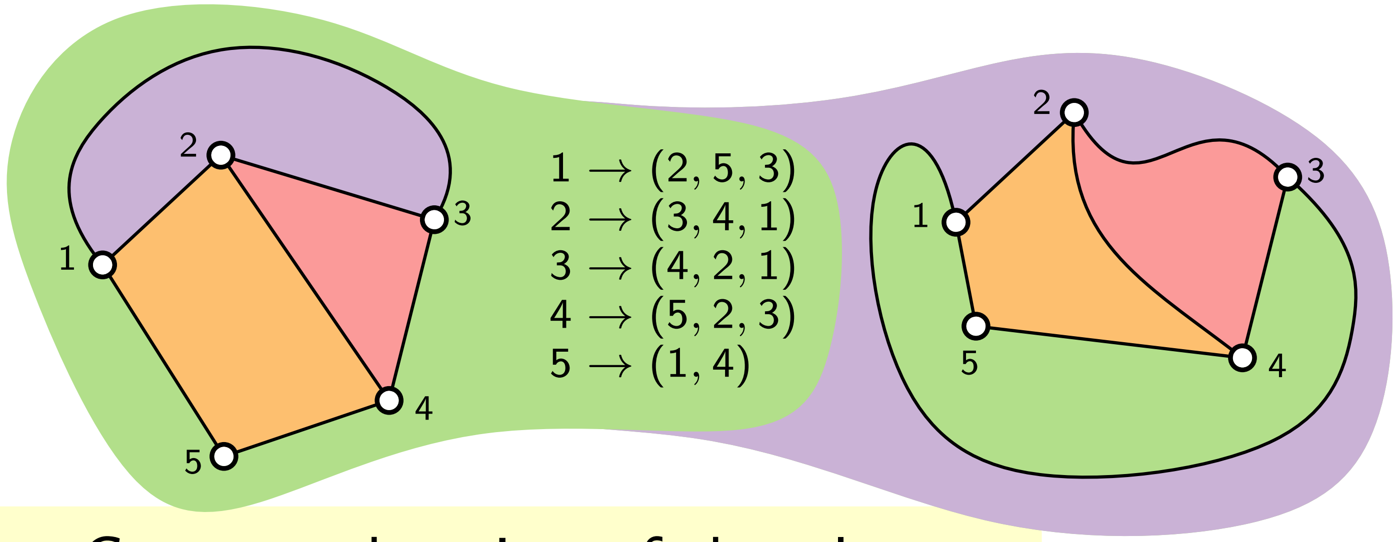

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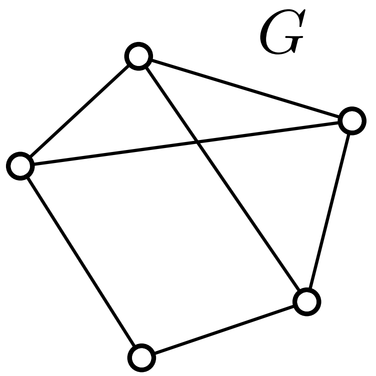

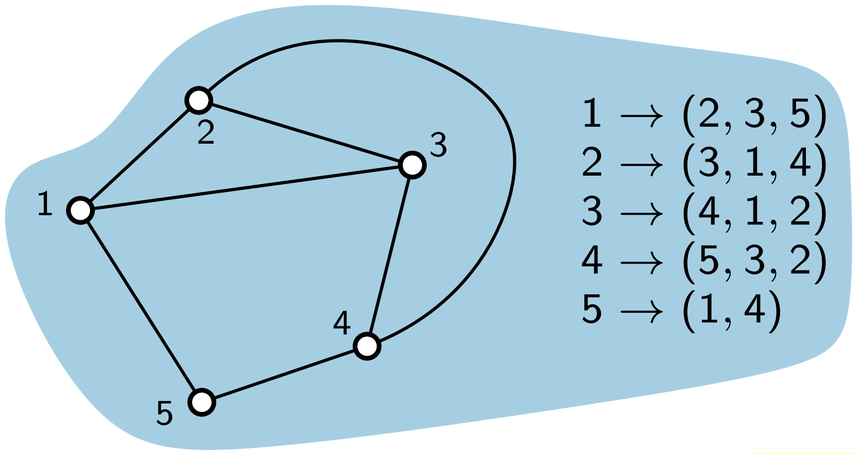

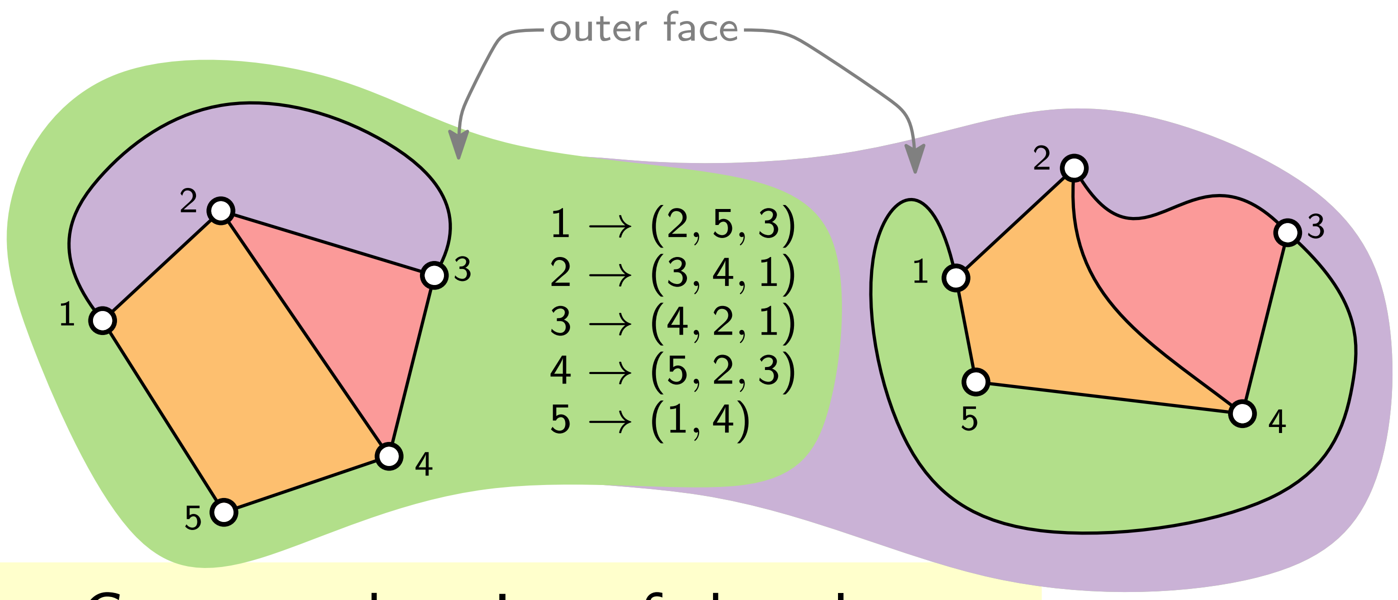

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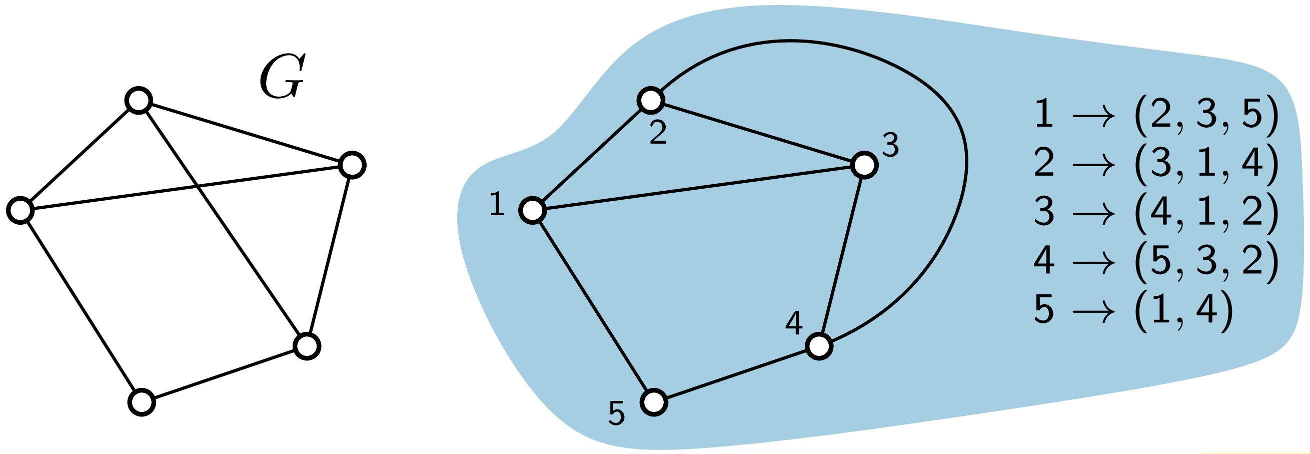

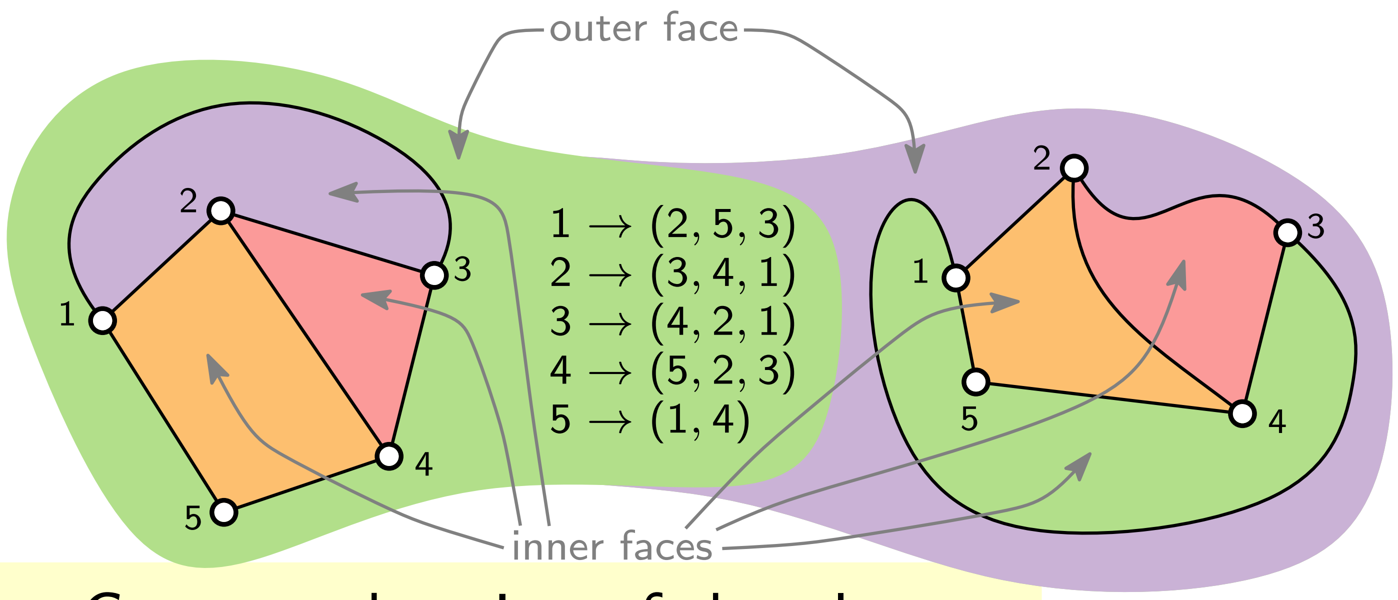

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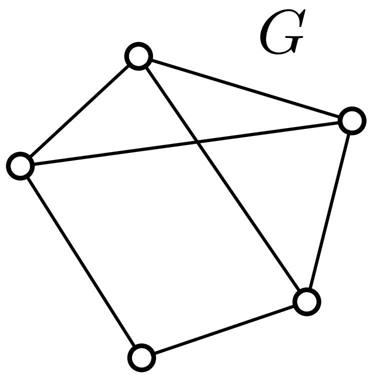

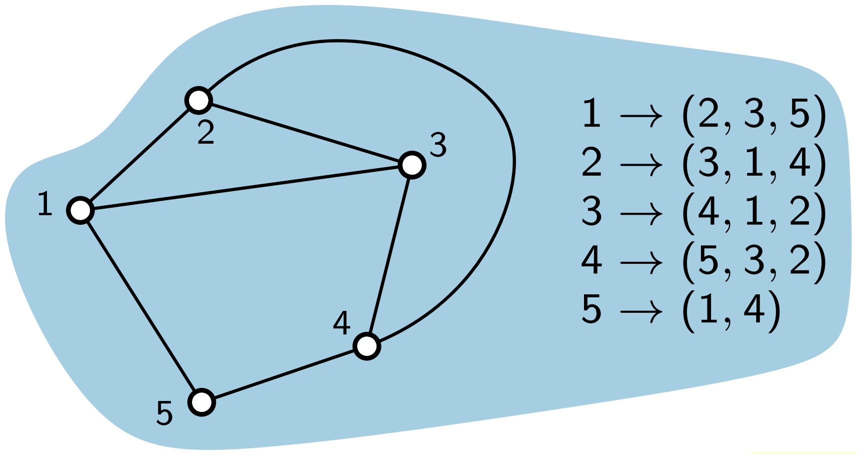

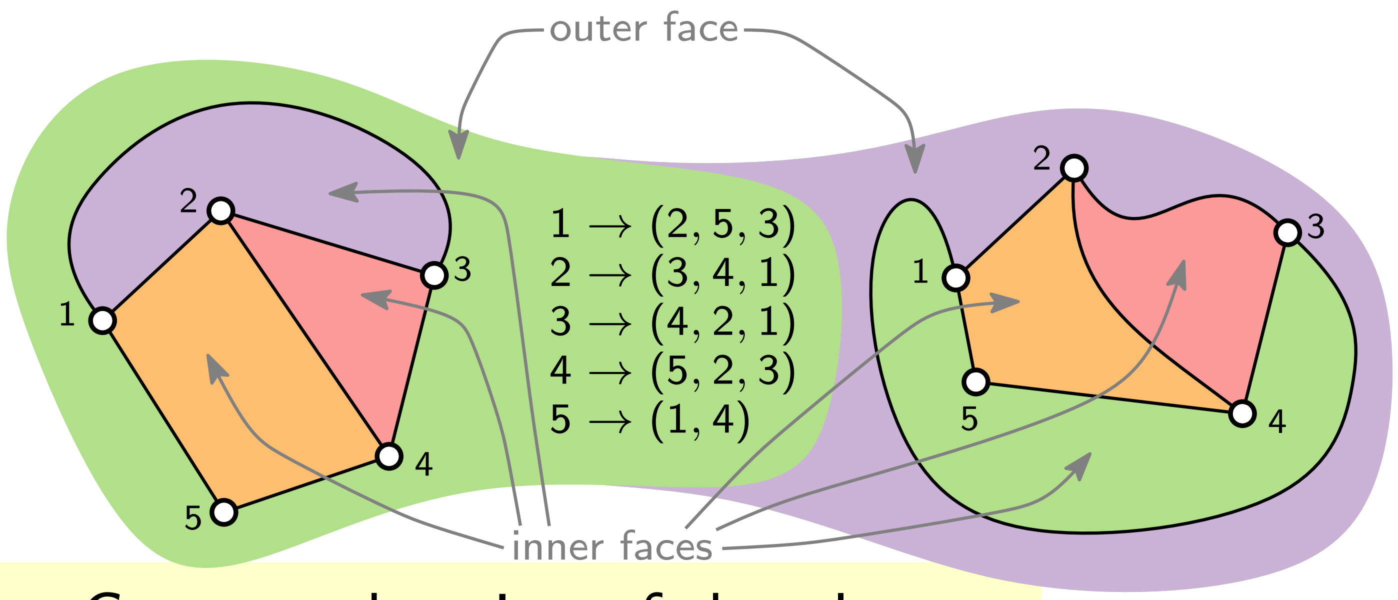

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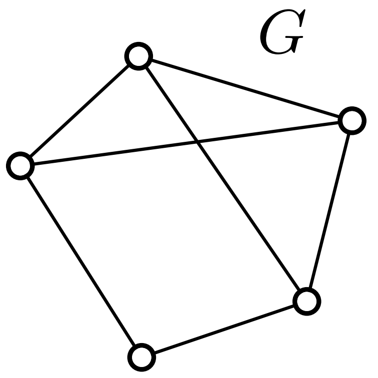
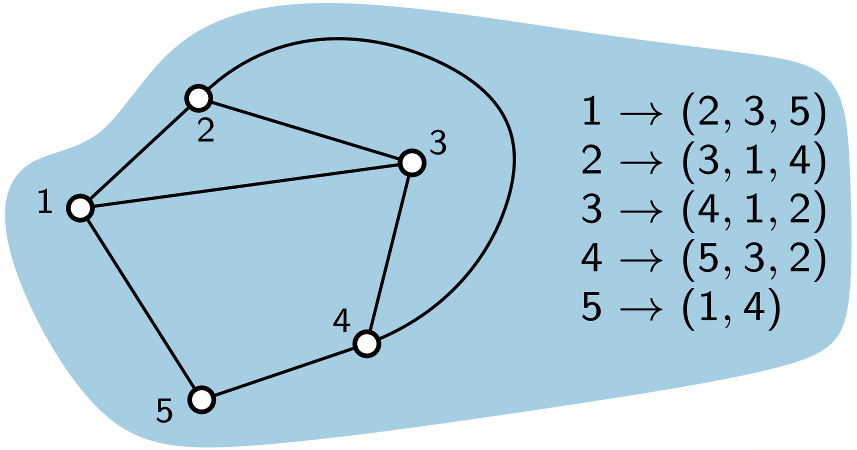
faces: Connected region of the plane bounded by edges

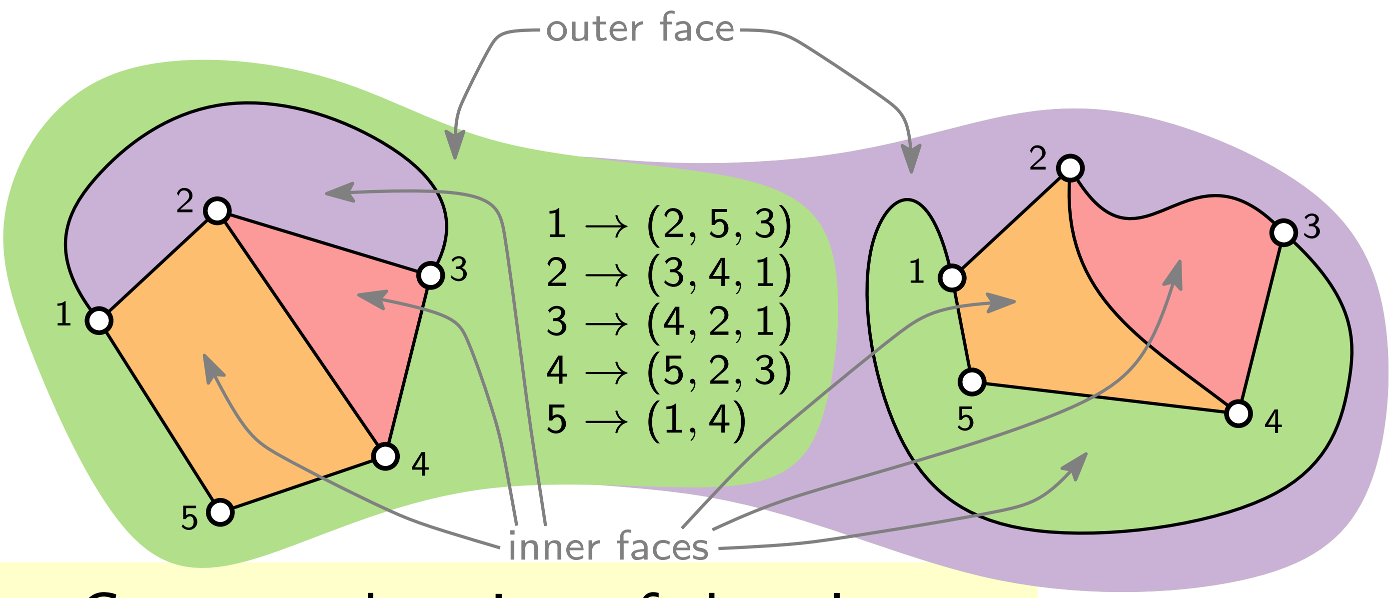
Euler's polyhedra formula.

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Planar Graphs







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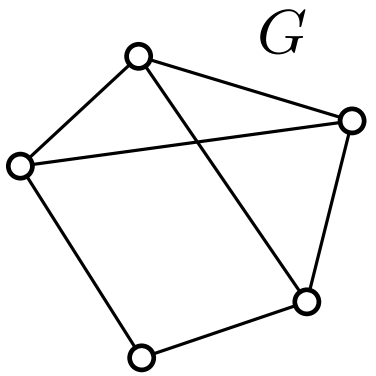
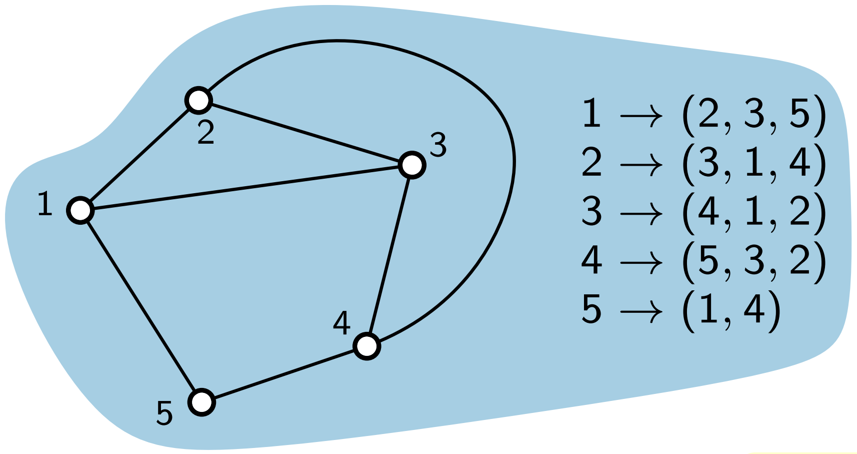
Euler's polyhedra formula.

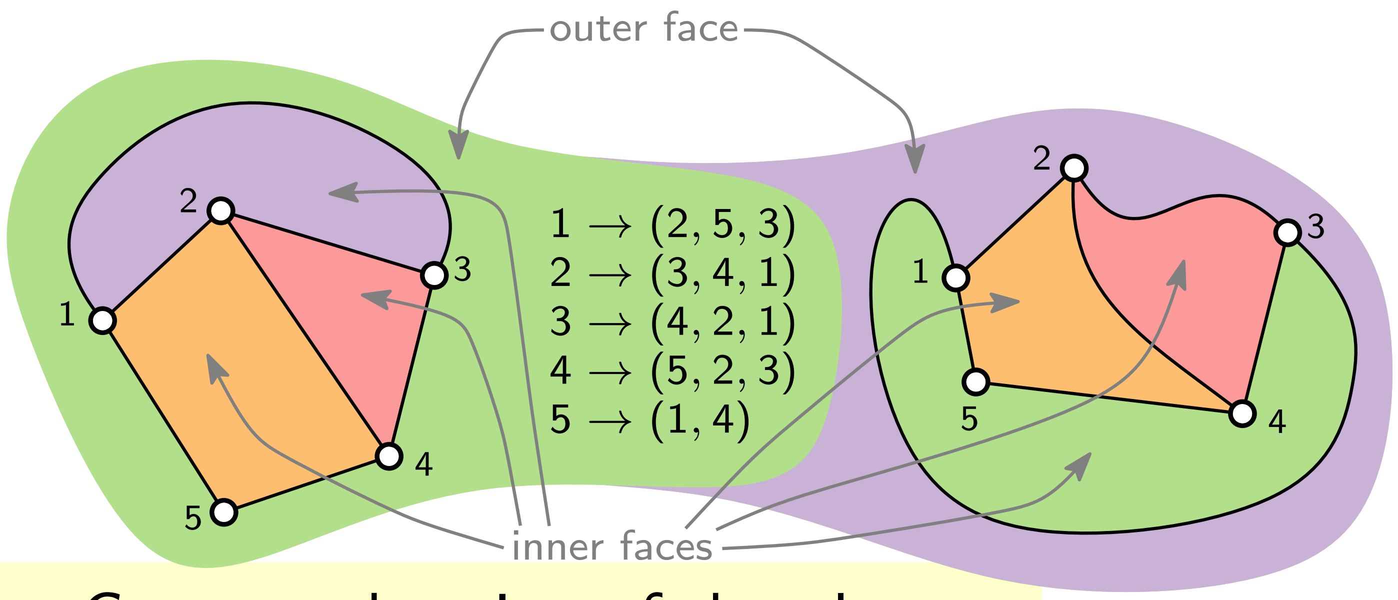
$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof.

Planar Graphs







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Euler's polyhedra formula.

$$\#\text{faces} - \#\text{edges} + \#\text{vertices} = \#\text{conn.comp.} + 1$$
$$f - m + n = c + 1$$

Proof. By induction on m:

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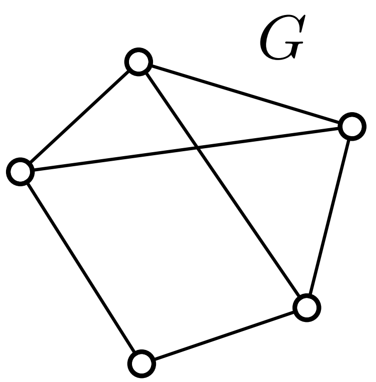

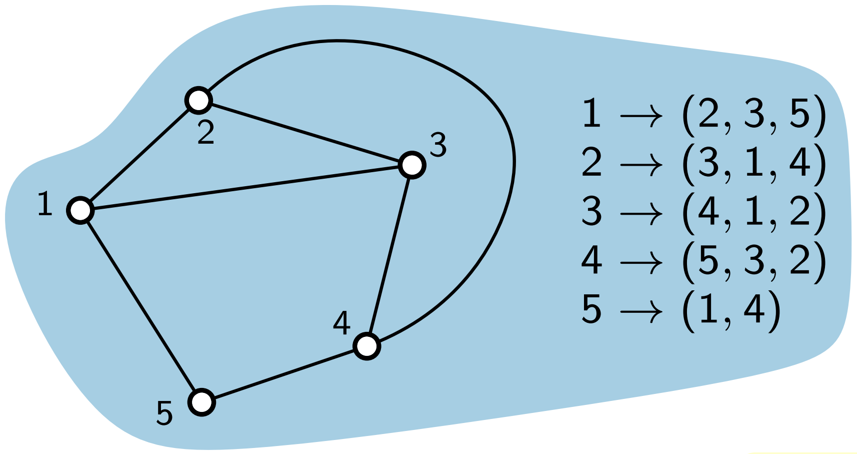

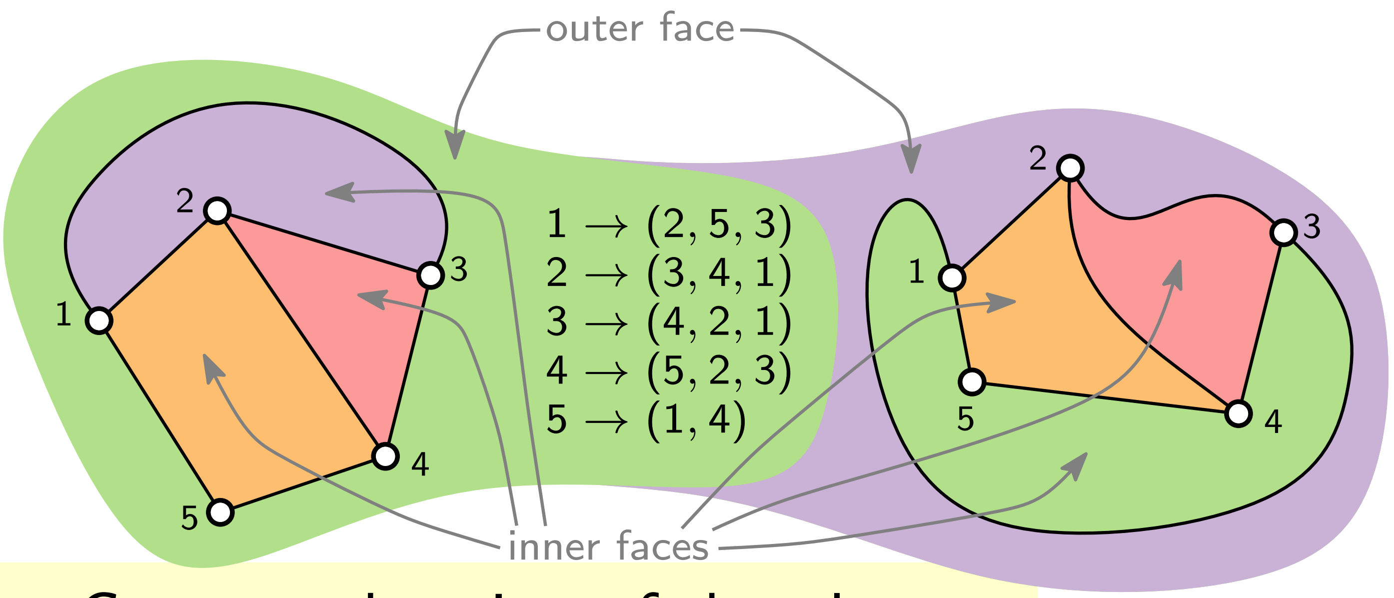

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Euler's polyhedra formula.

#faces − #edges + #vertices = #conn.comp. + 1

$$f \quad - \quad m \quad + \quad n \quad = \quad c \quad + 1$$

Proof. By induction on m:

$m = 0 \Rightarrow$

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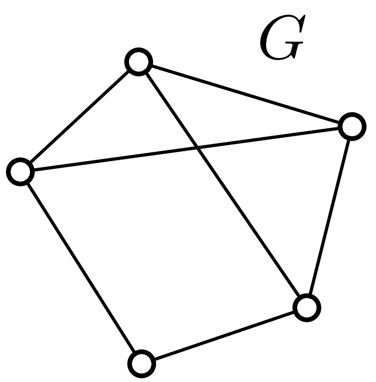

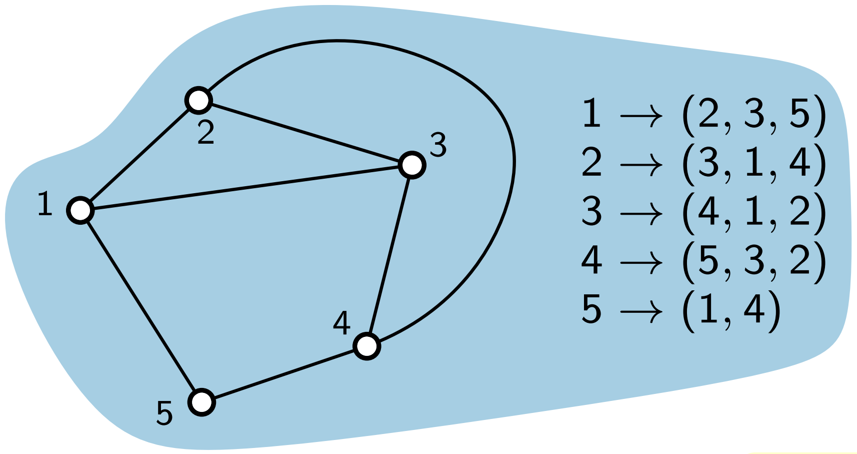

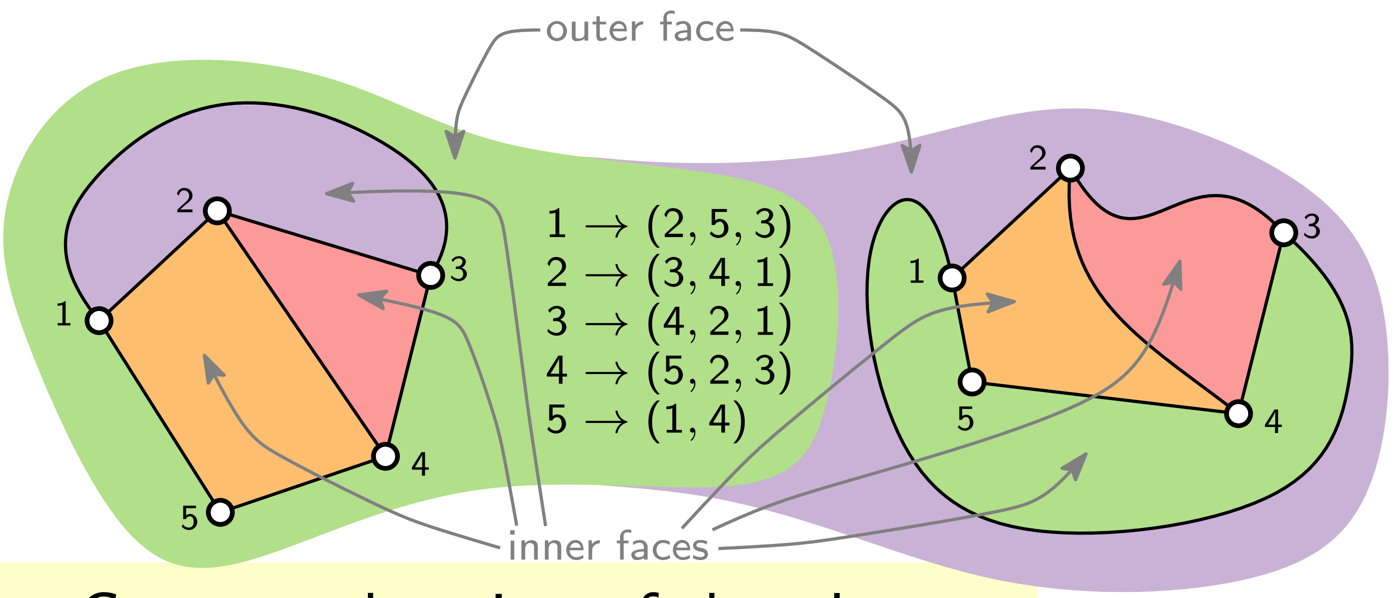

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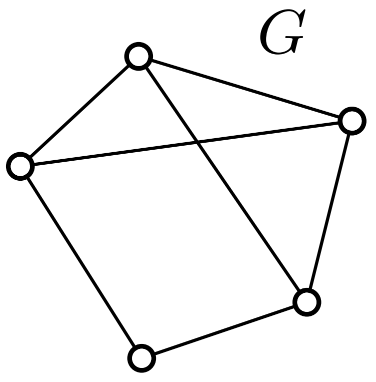
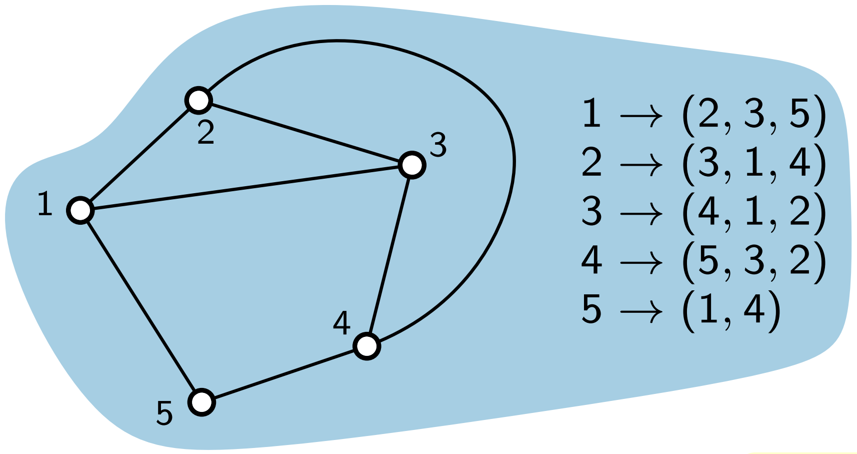
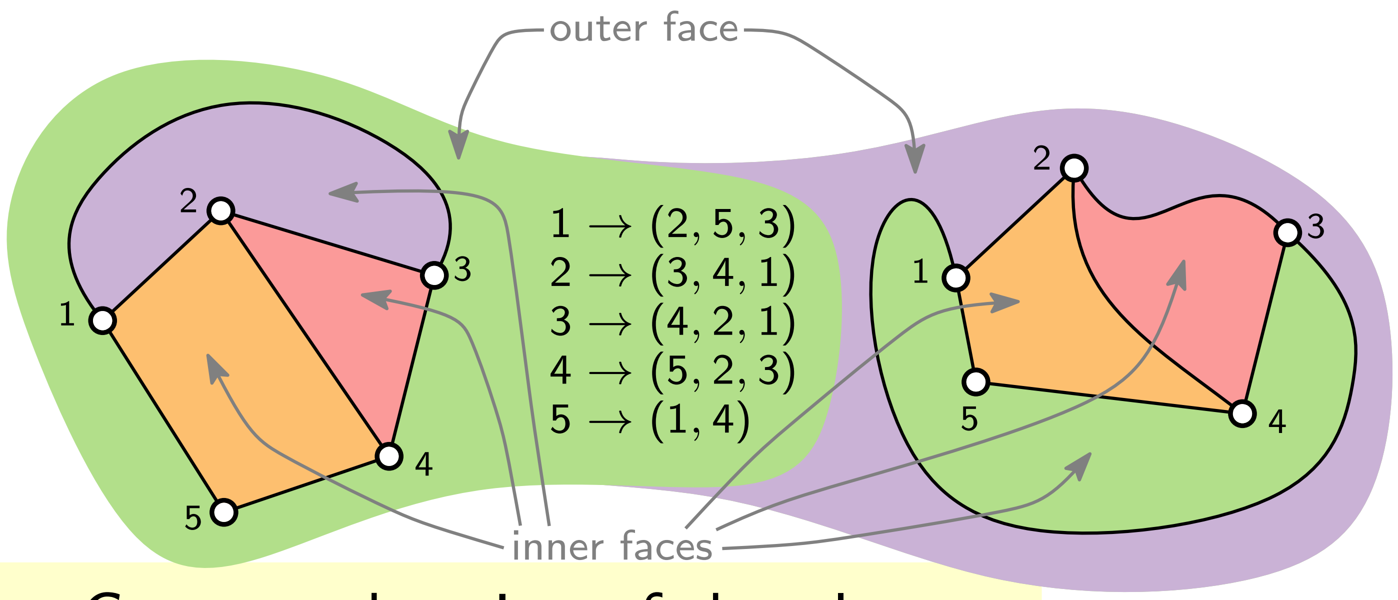
Planar Graphs







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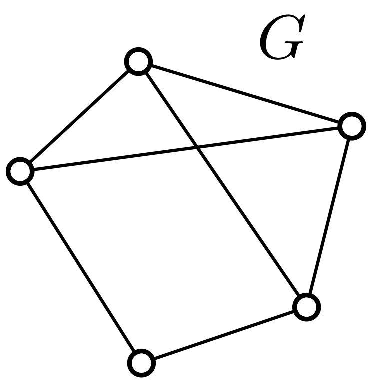

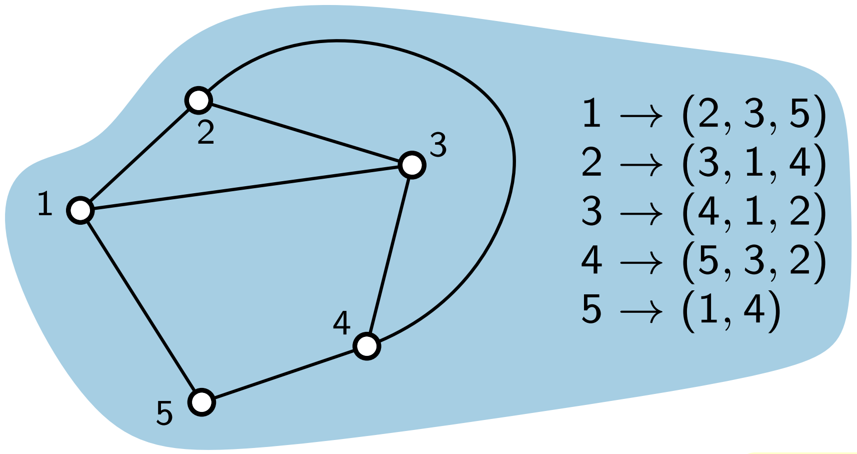

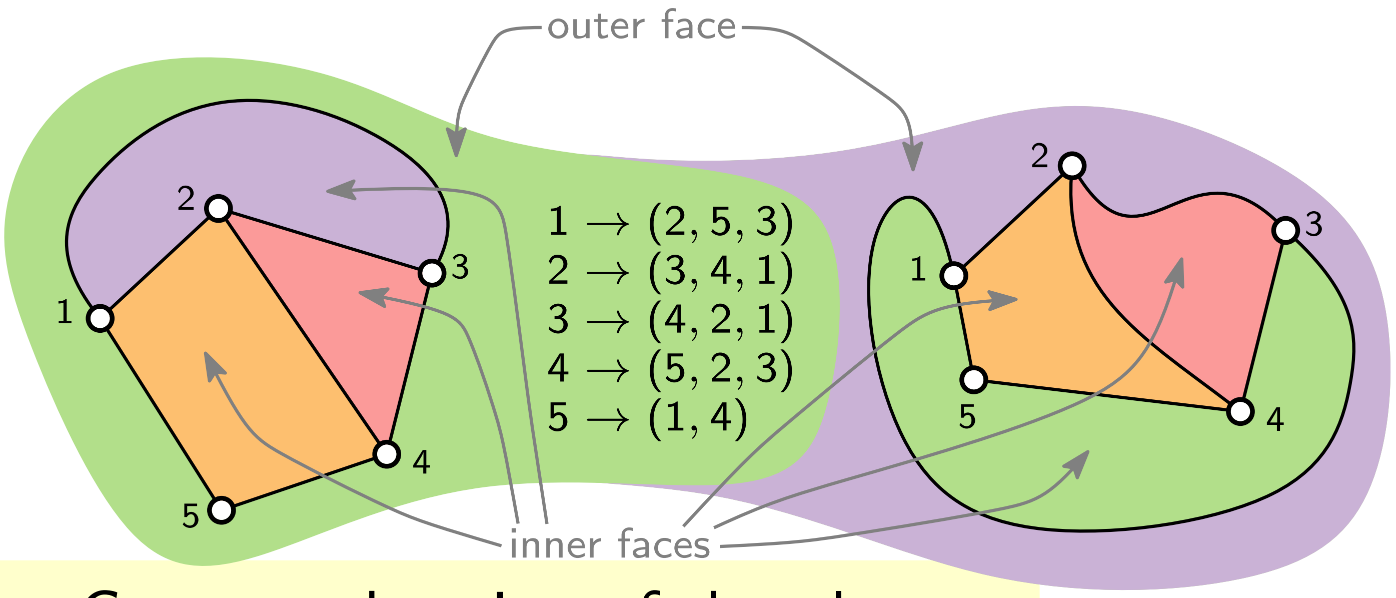

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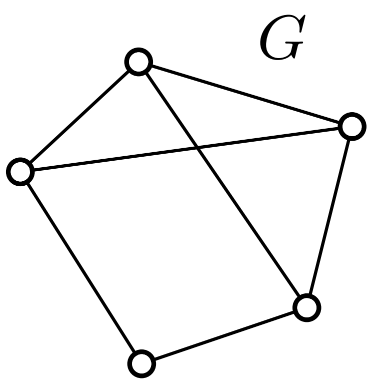

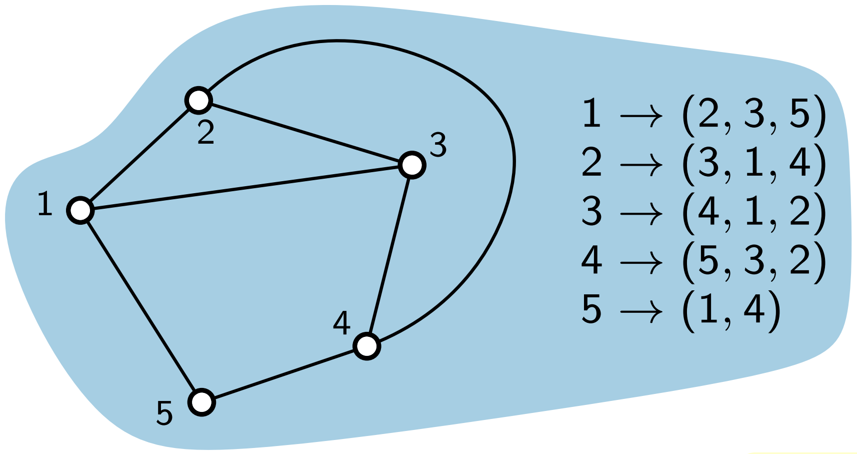

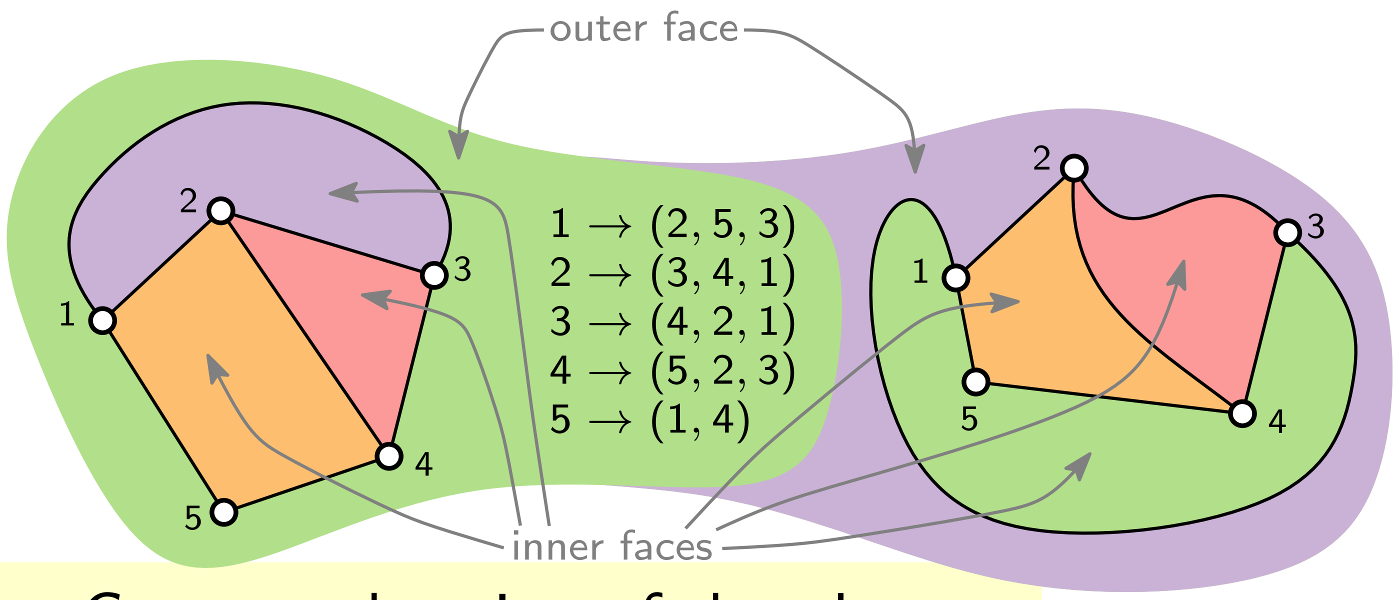

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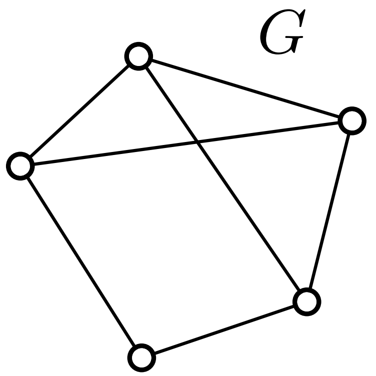

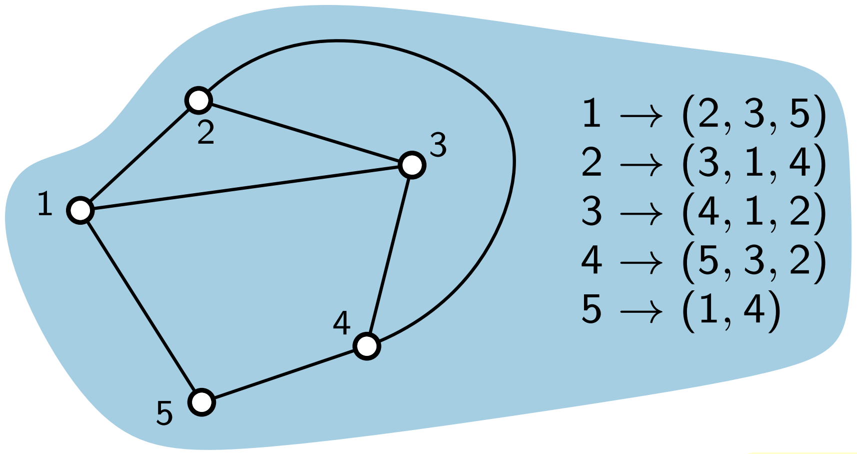

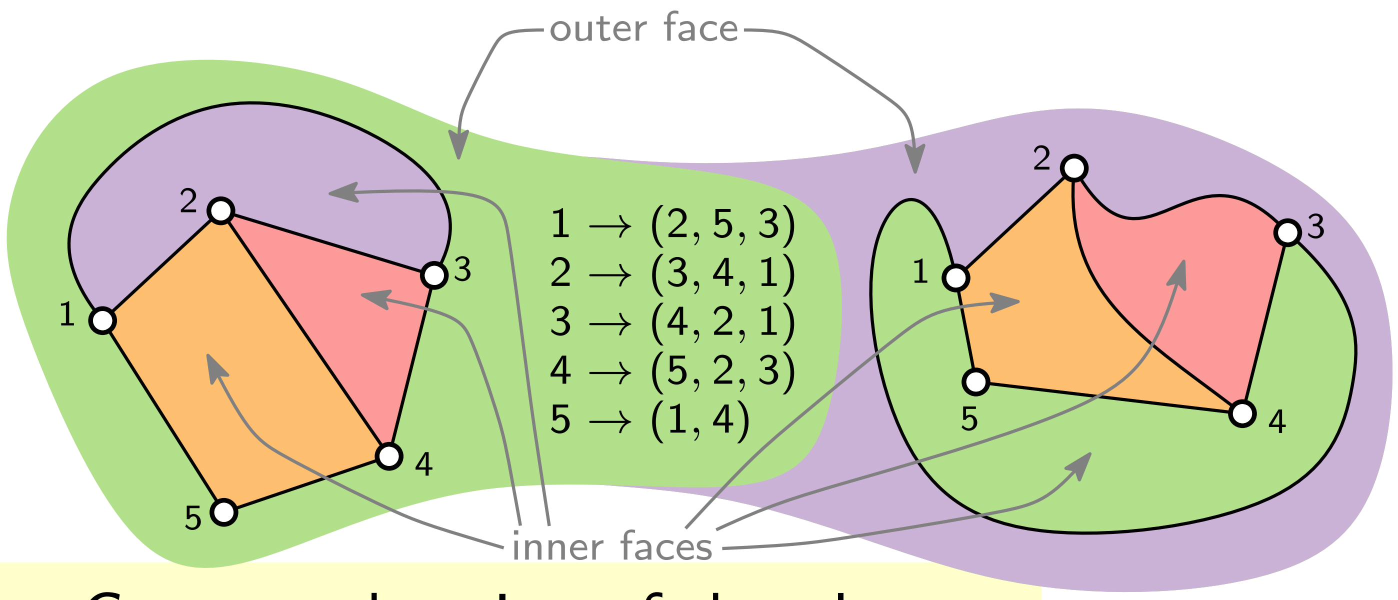

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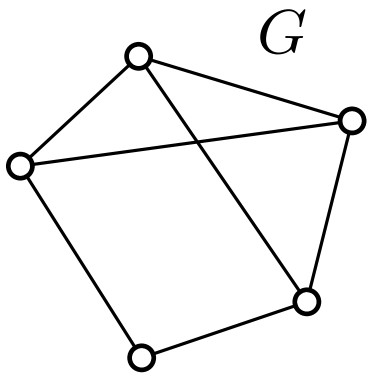

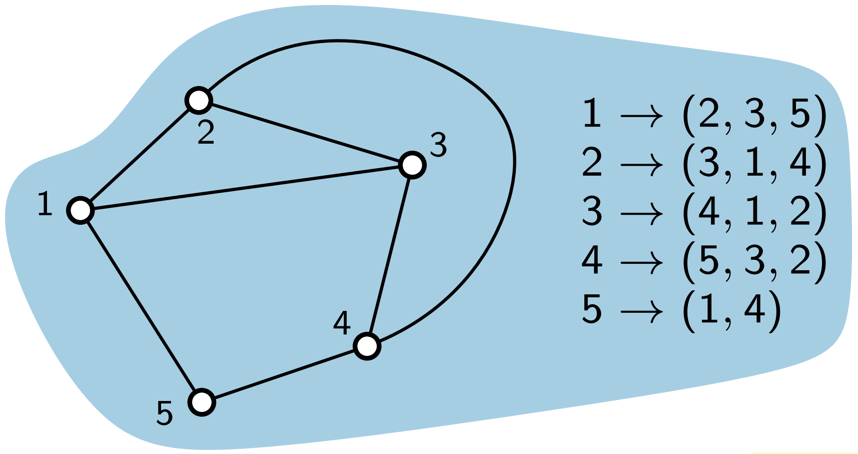

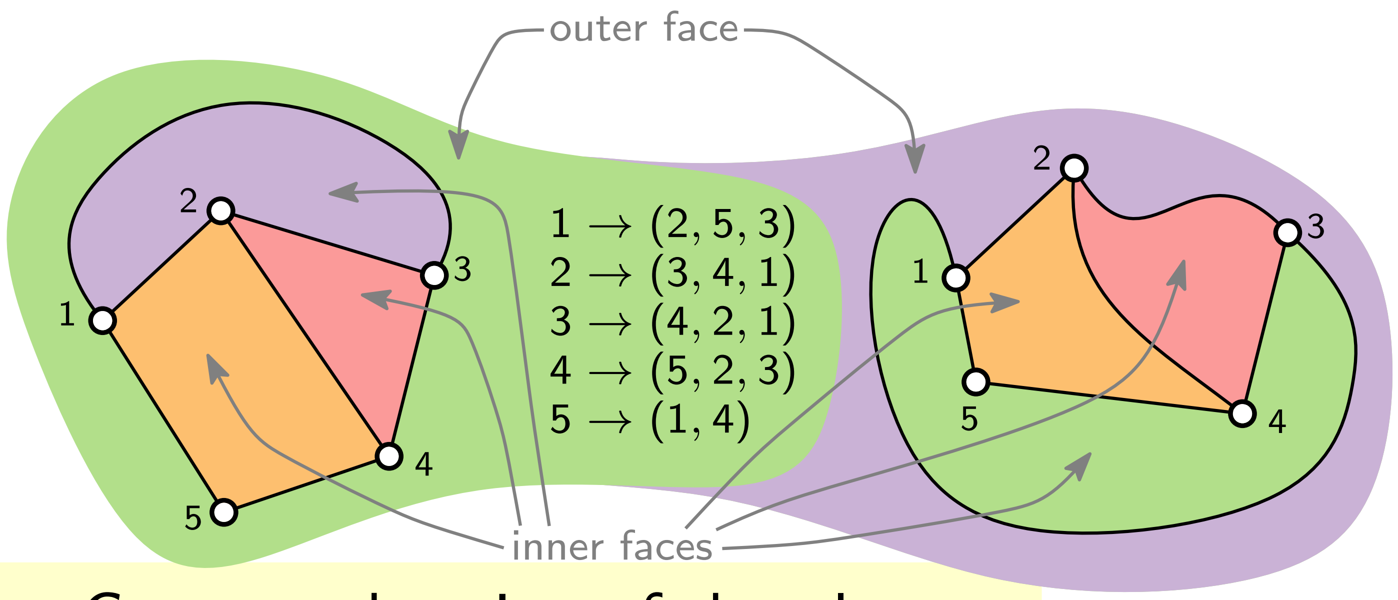

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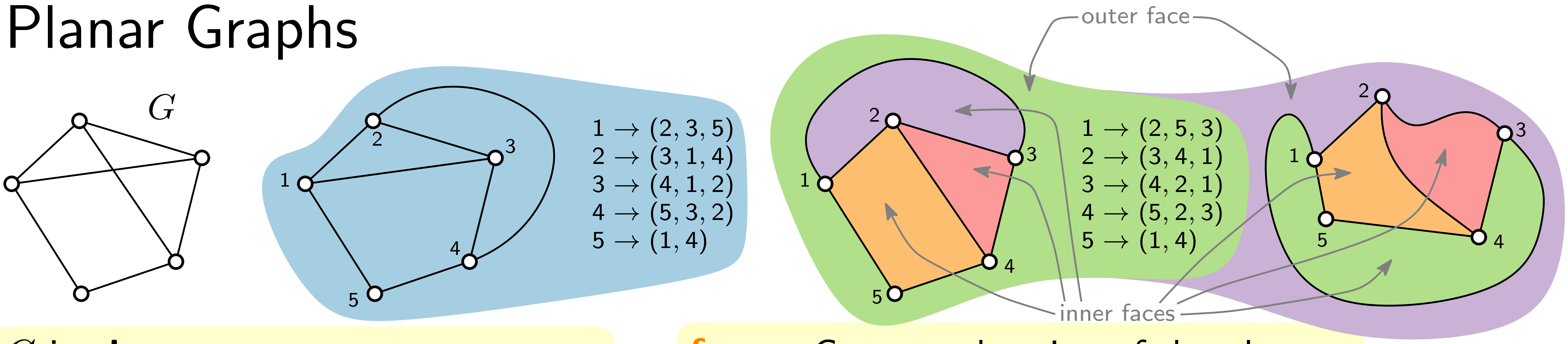

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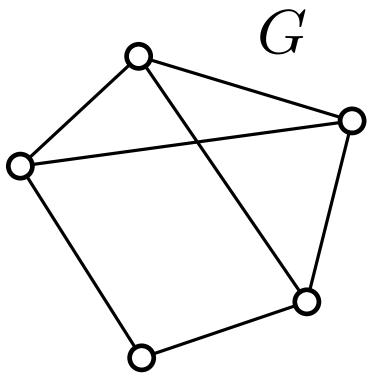

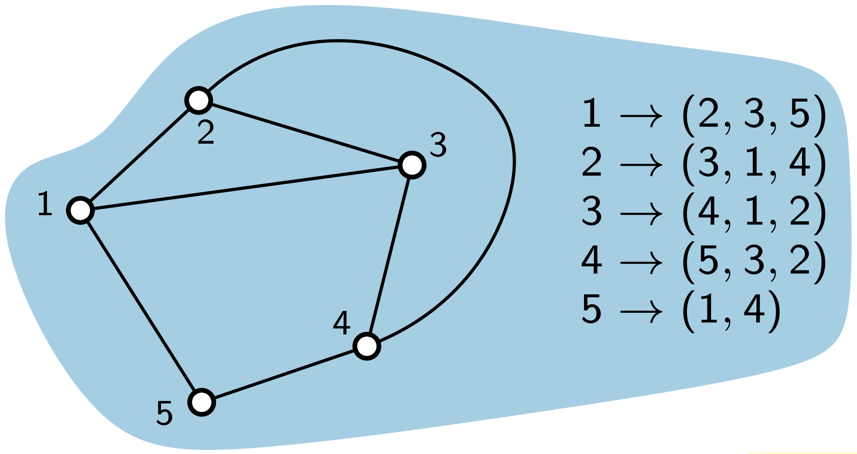

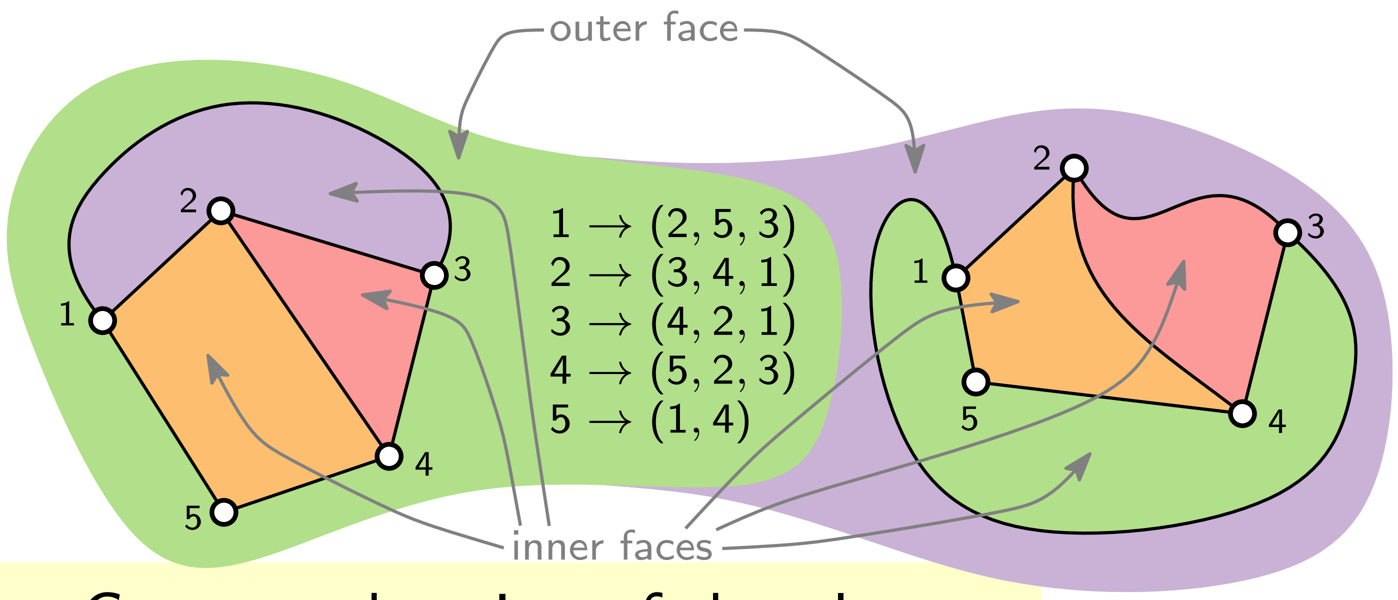

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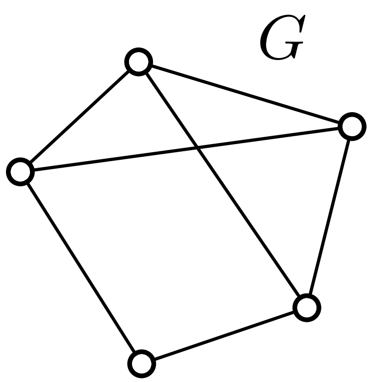

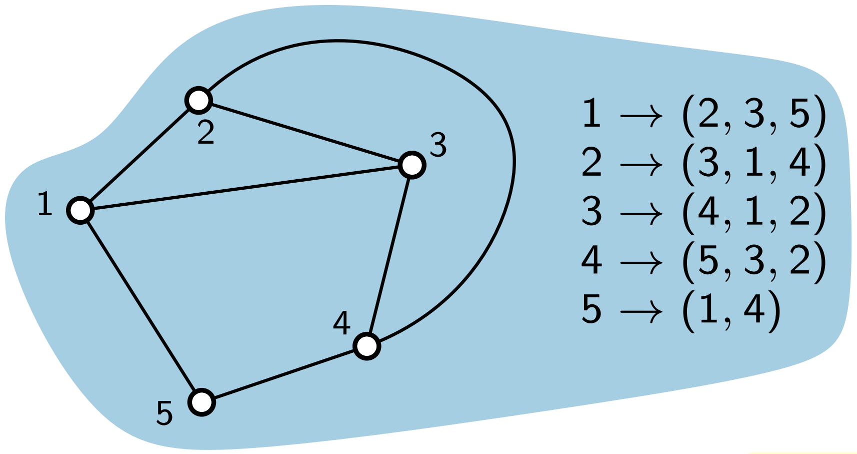

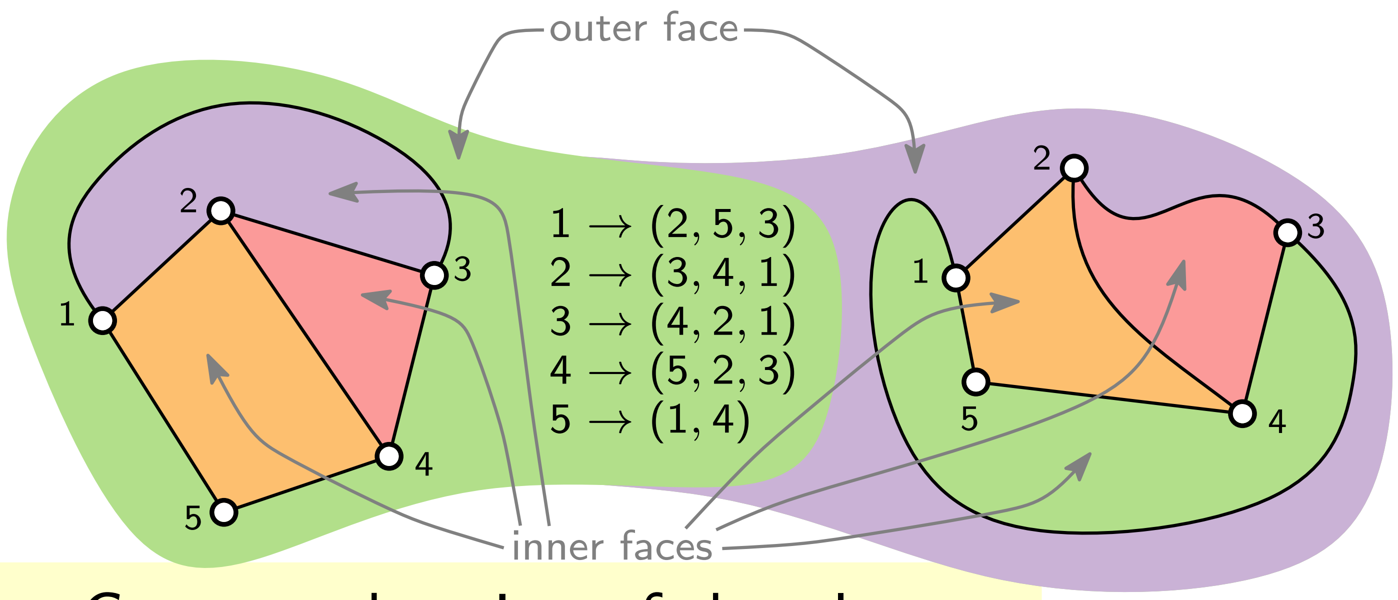

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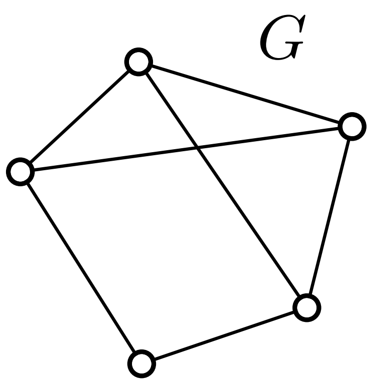

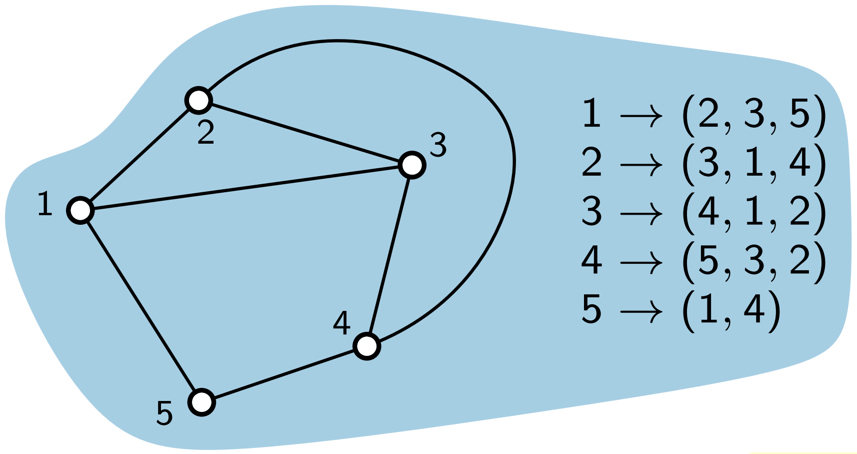

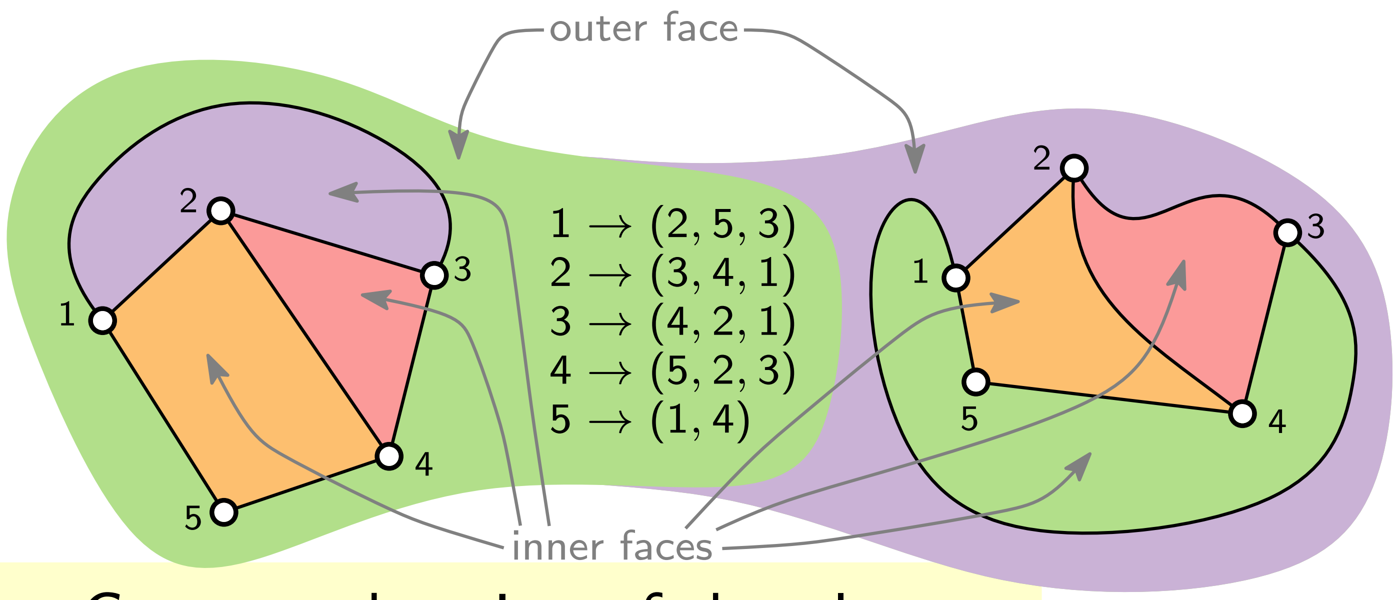

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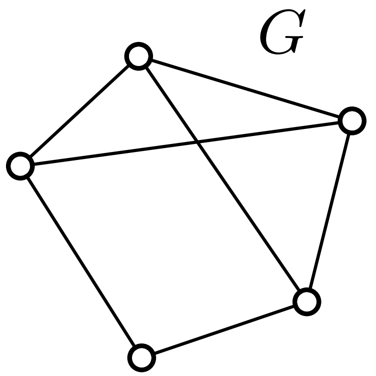

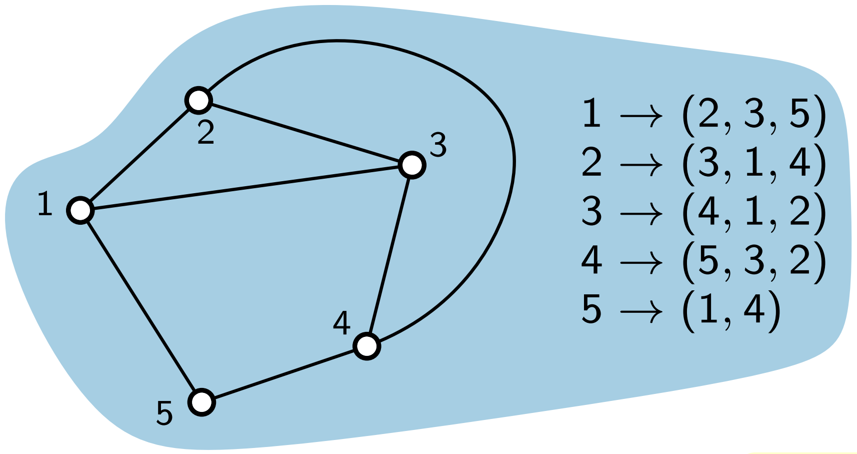

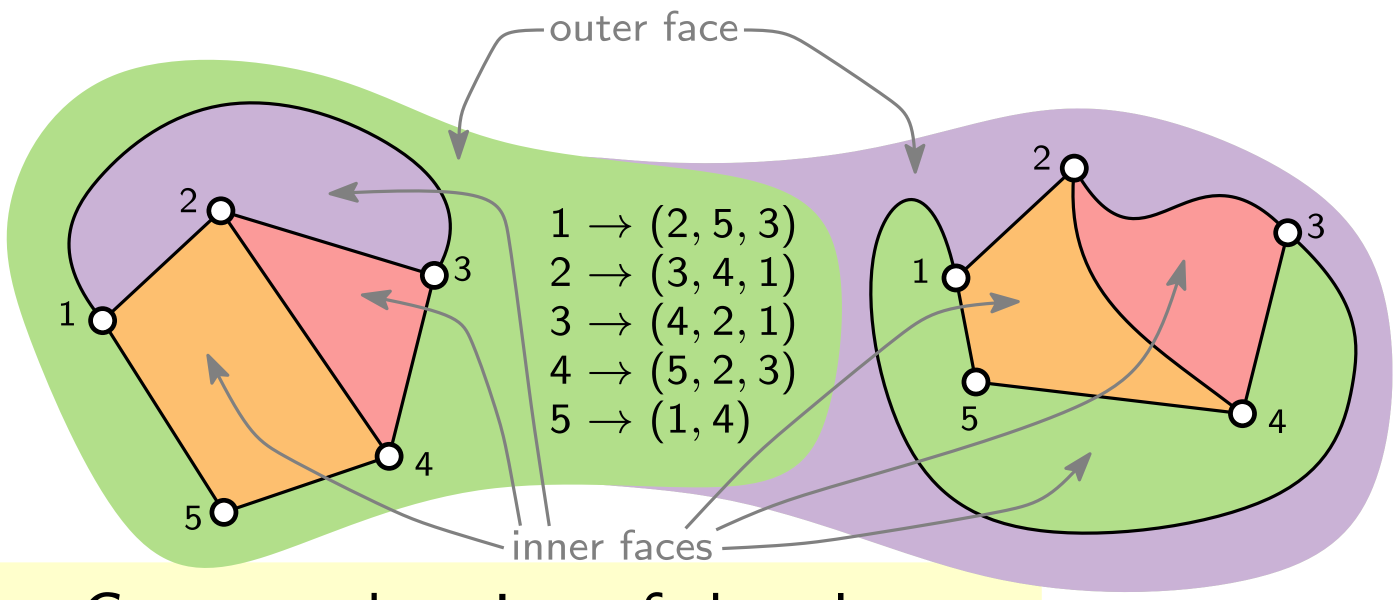

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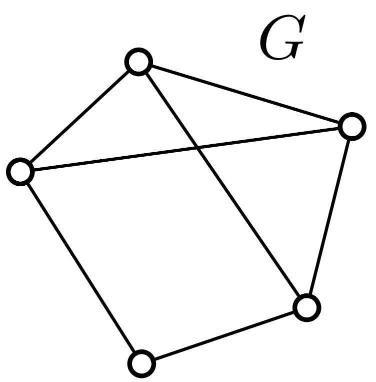

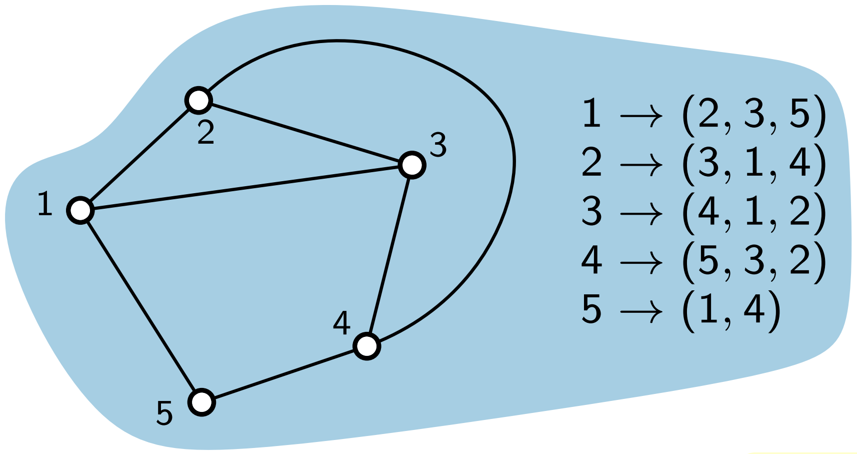

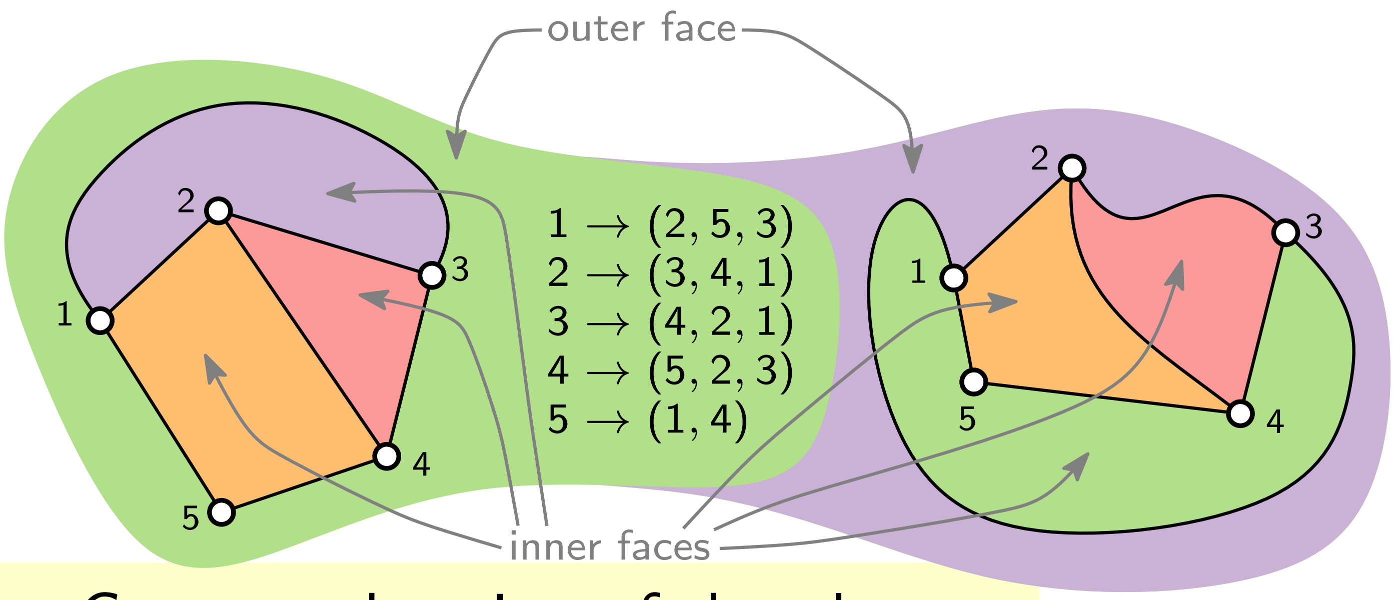

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Properties of Planar Graphs

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Theorem. G simple planar graph with $n \geq 3$ vtc.

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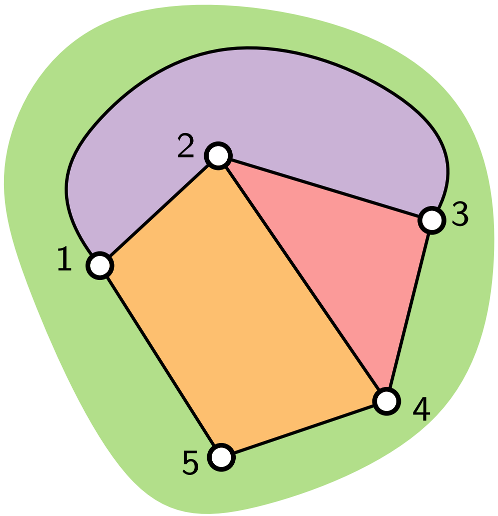

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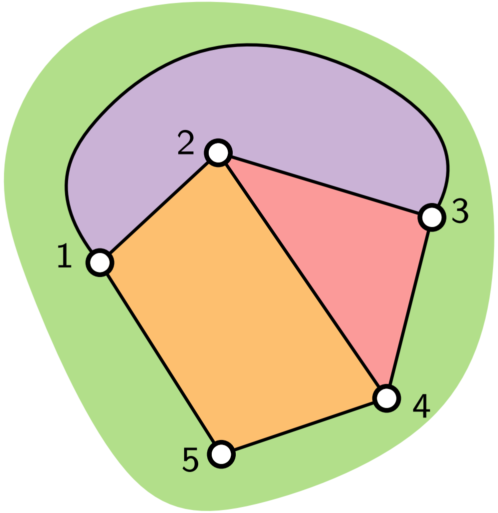
idea: count edge–face incidences



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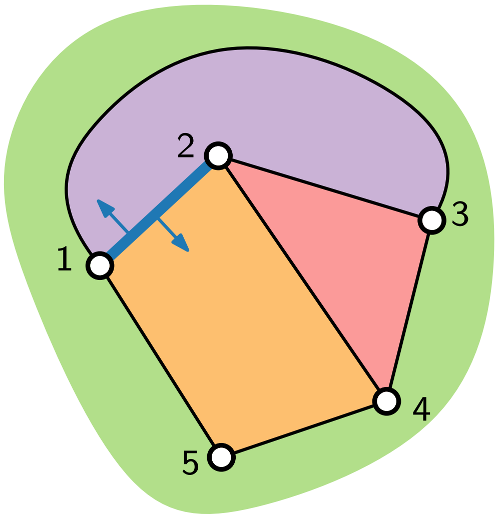

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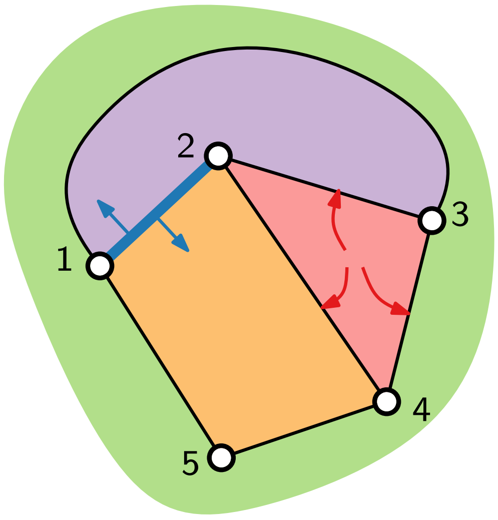

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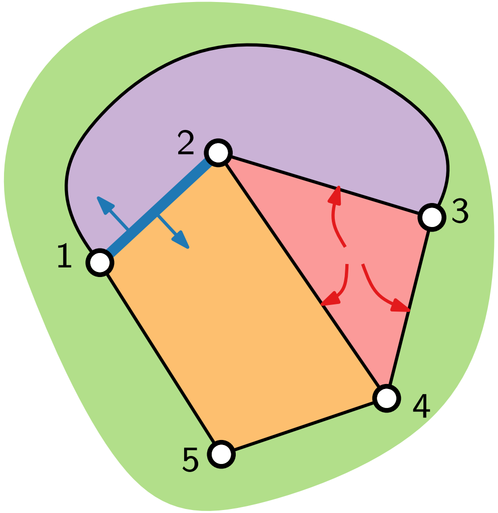

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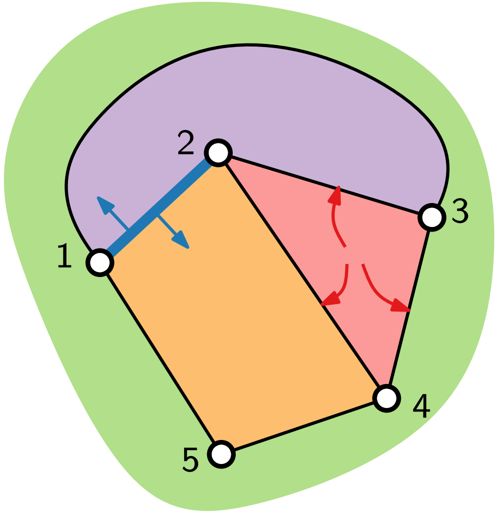

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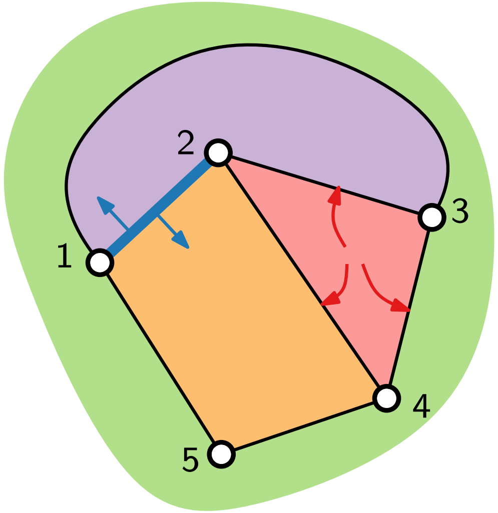

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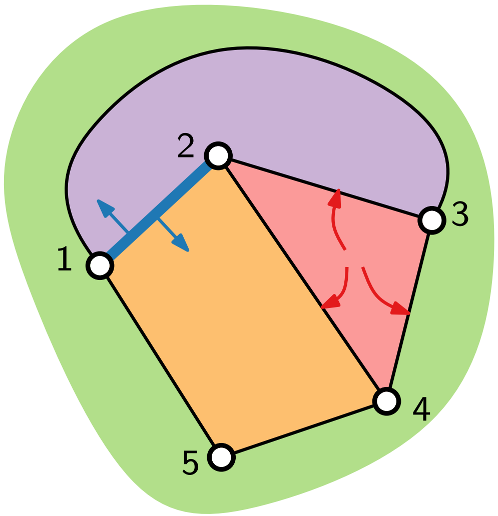

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1. $m \leq 3n - 6$

Proof. 1. Every edge incident to ≤ 2 faces

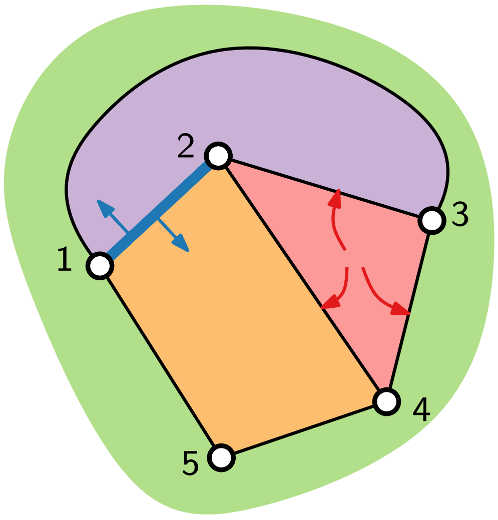
Every face incident to ≥ 3 edges

$\Rightarrow 3f \leq \#\,\text{incidences} \leq 2m$

$\Rightarrow \quad 3c + 3 = 3f - 3m + 3n$

$c \geq 1$

idea: count edge–face incidences



Properties of Planar Graphs

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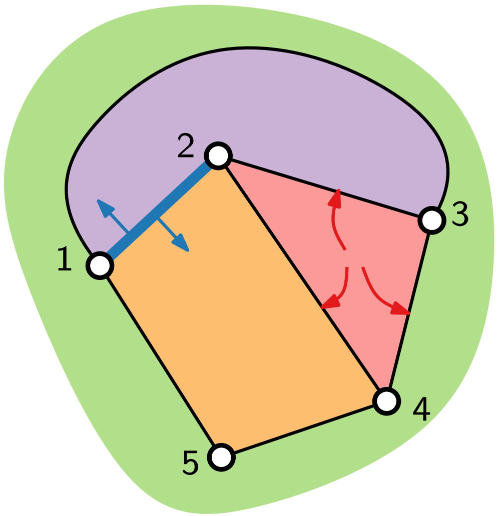

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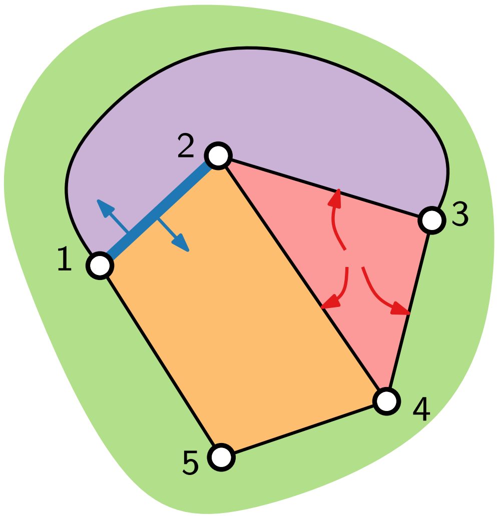

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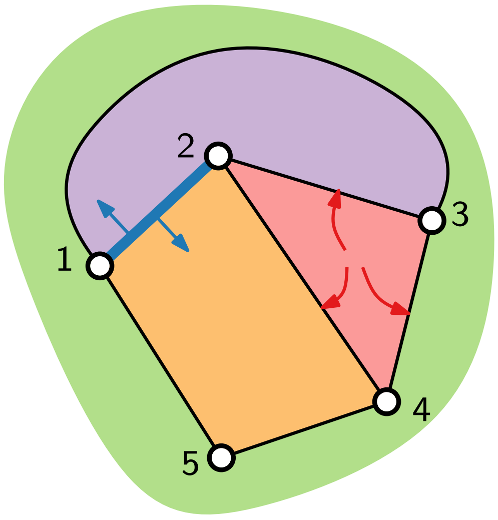

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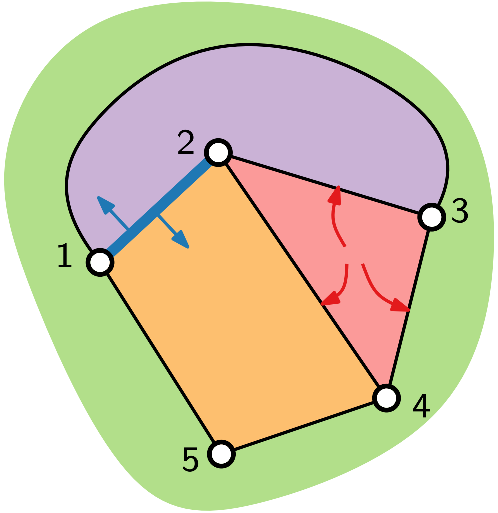

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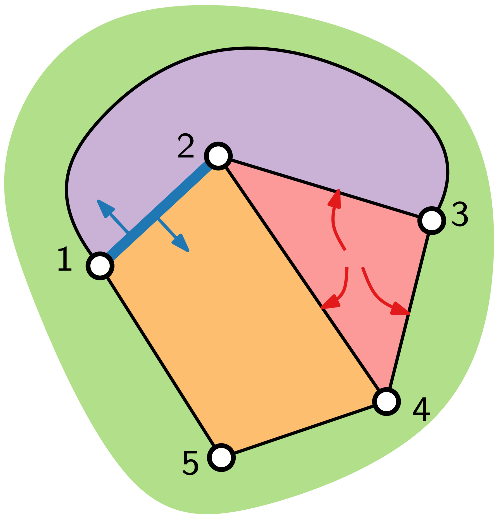

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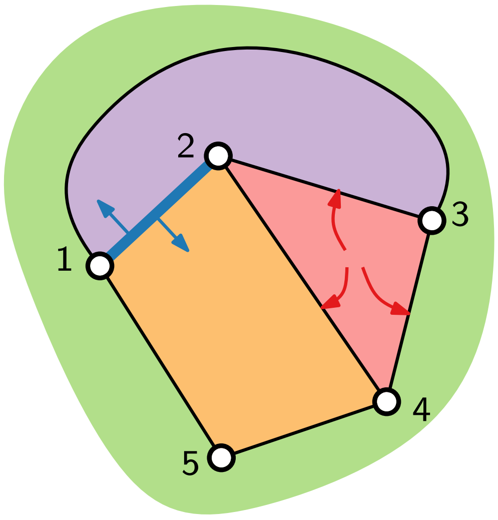

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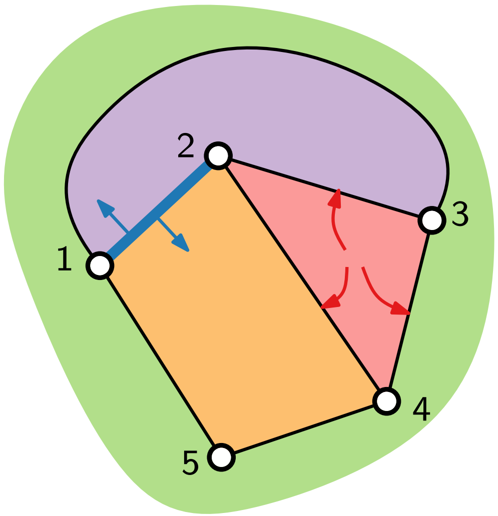

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idea: count edge–face incidences

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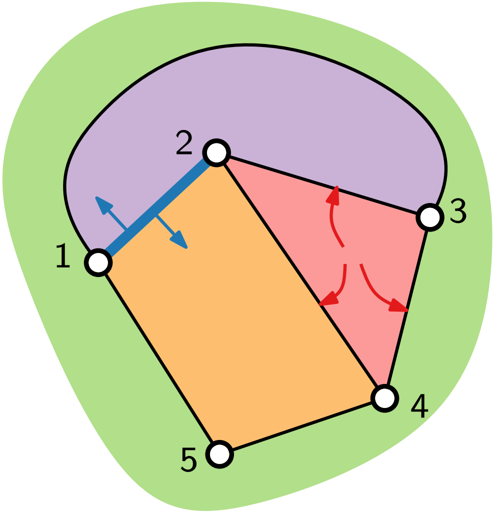

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2. $3f \leq 2m \leq 6n - 12$

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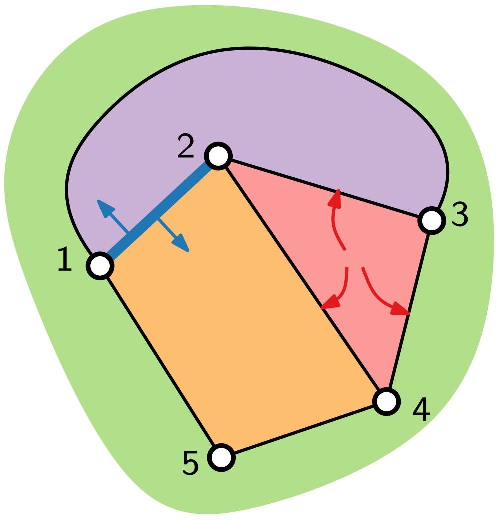

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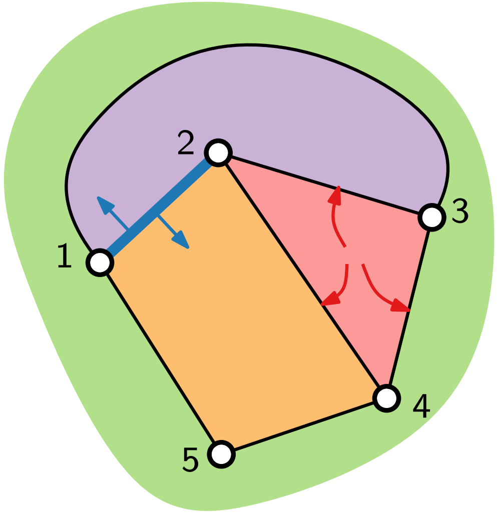

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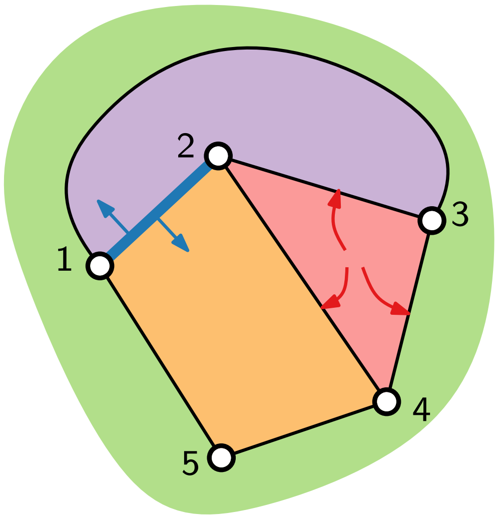

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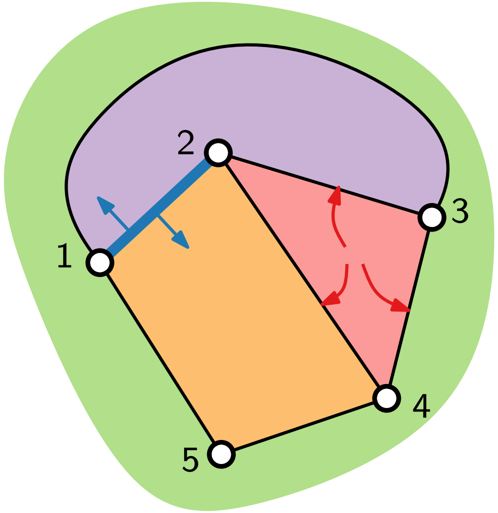

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$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

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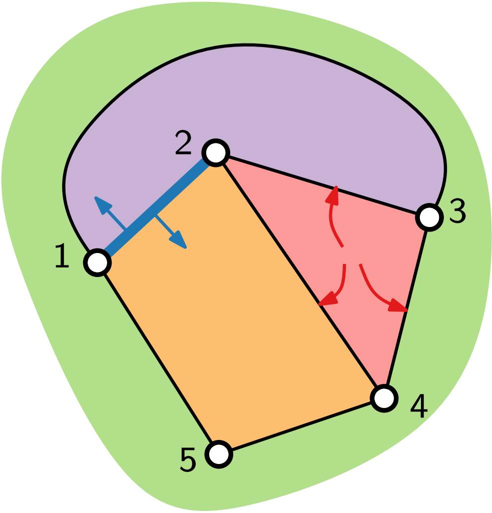

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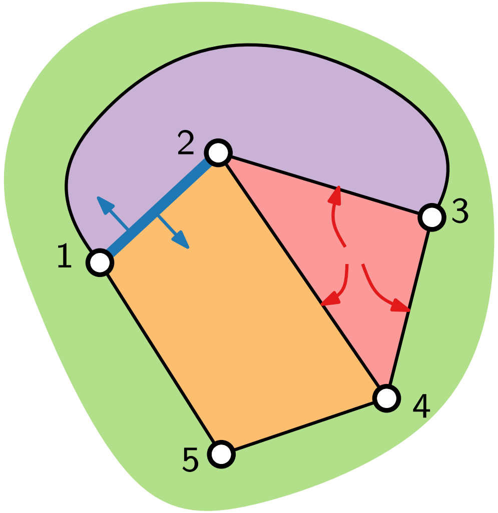

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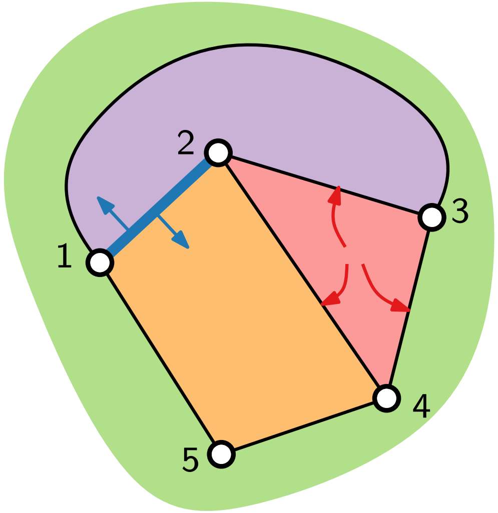

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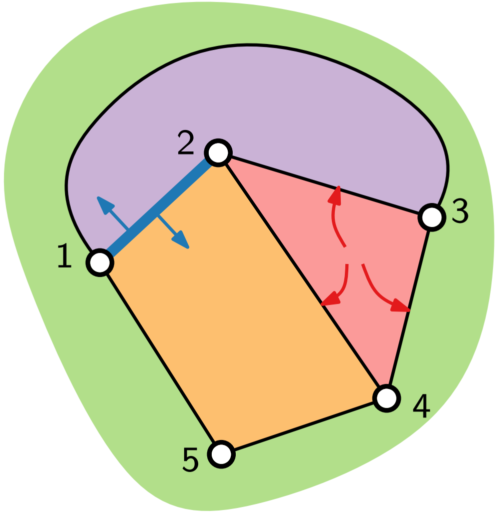

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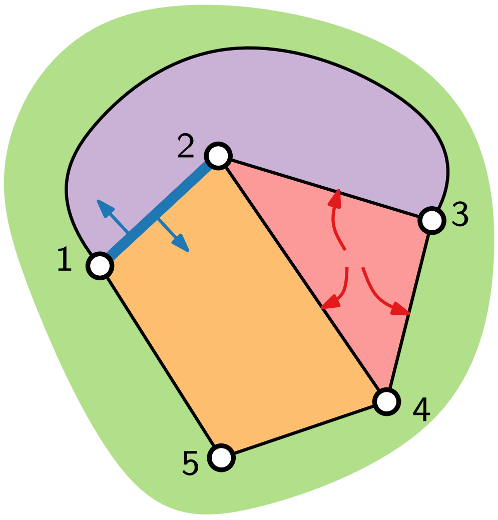

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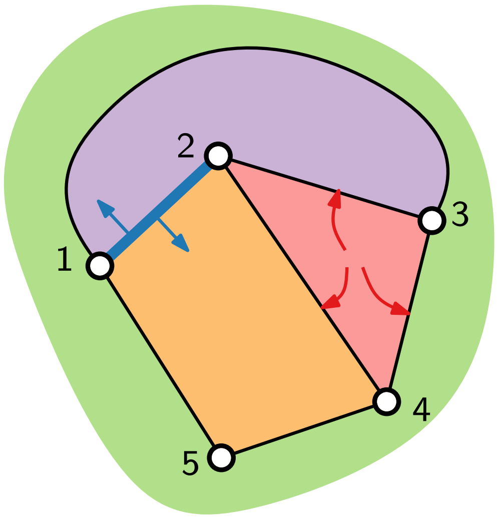

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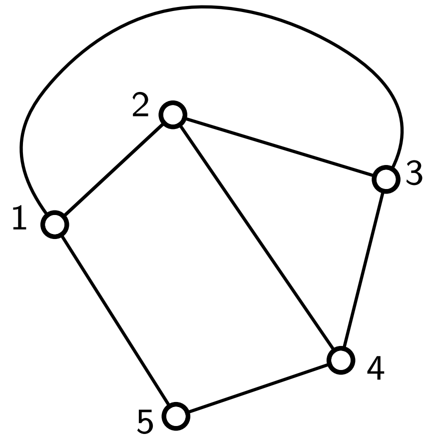
Triangulations

A **plane triangulation** is a plane graph where every face is a triangle.



Triangulations

planar graph given with a planar embedding

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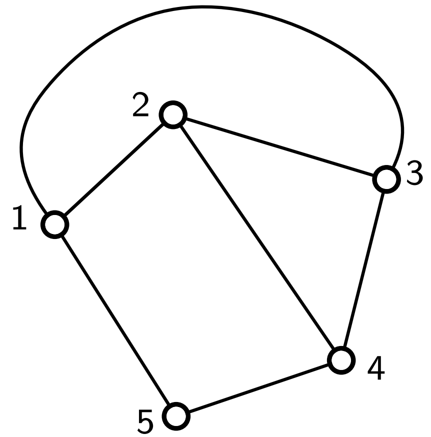

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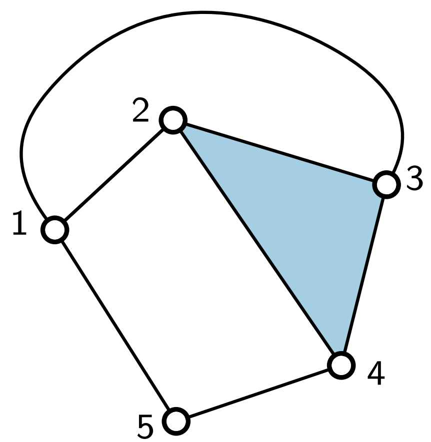

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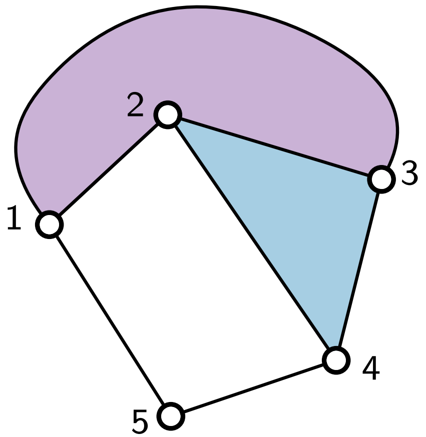

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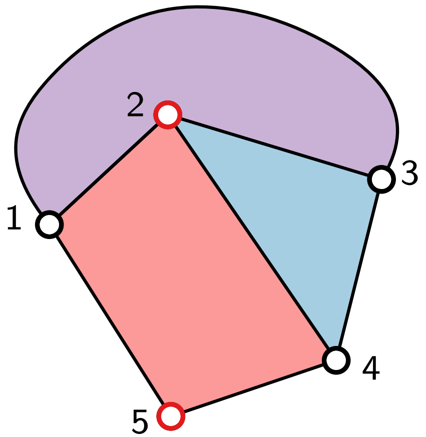

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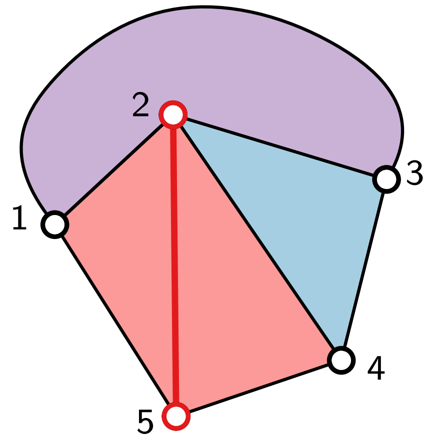

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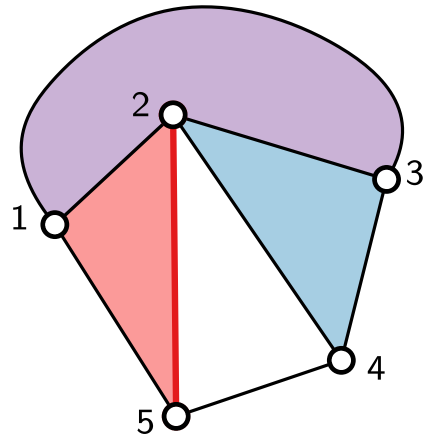

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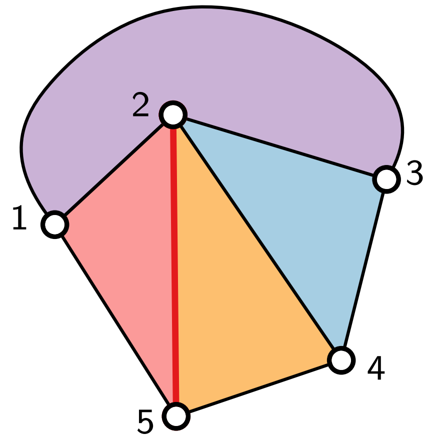

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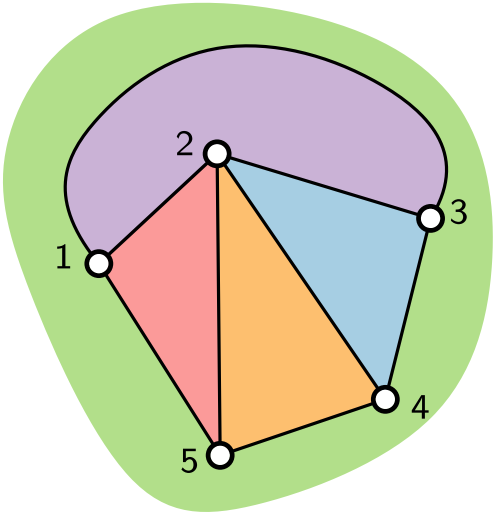

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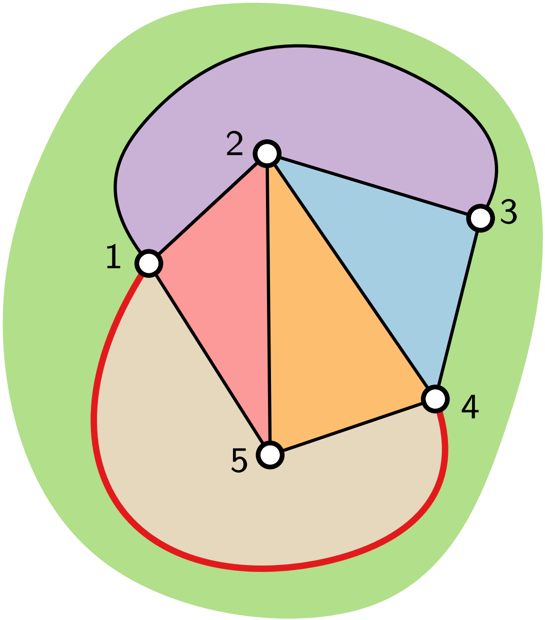

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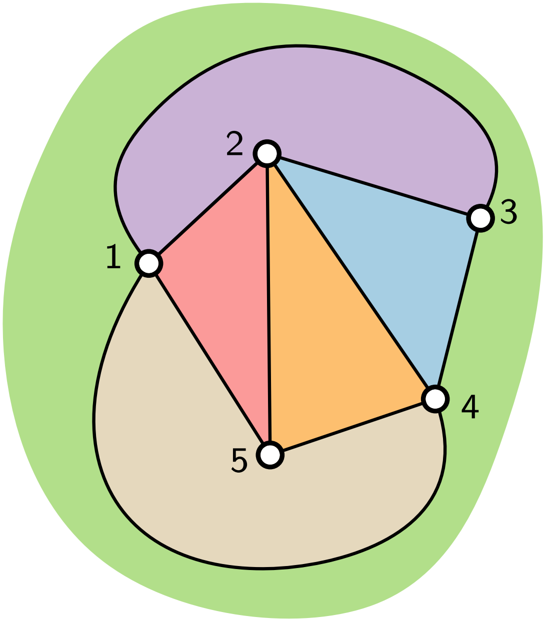
A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.



Triangulations

planar graph given with a planar embedding

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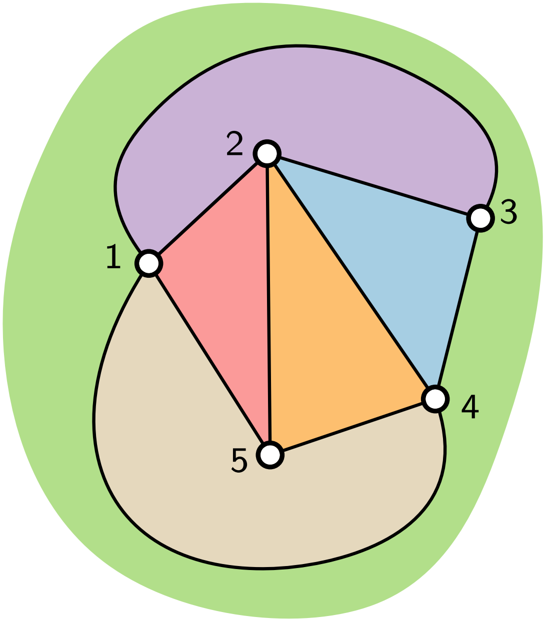

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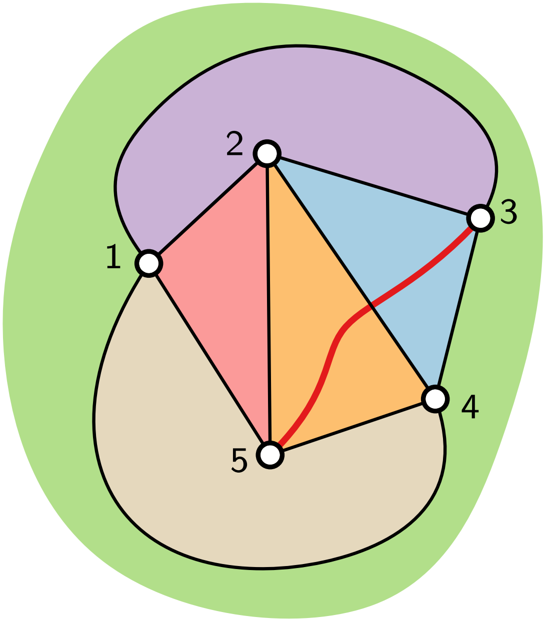

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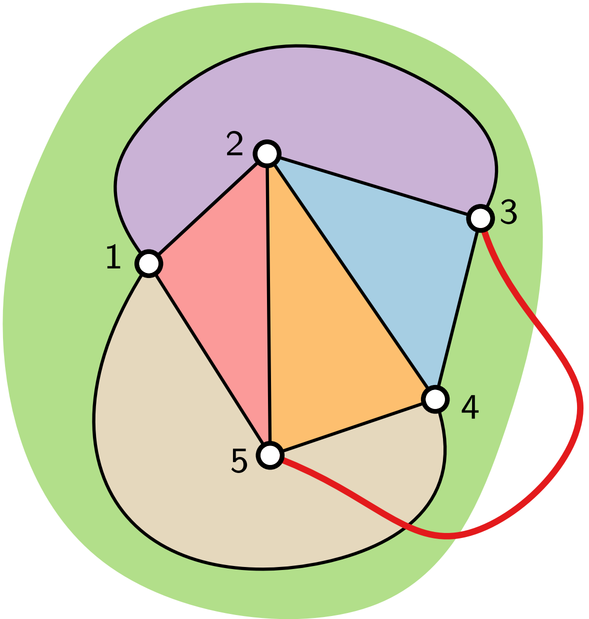

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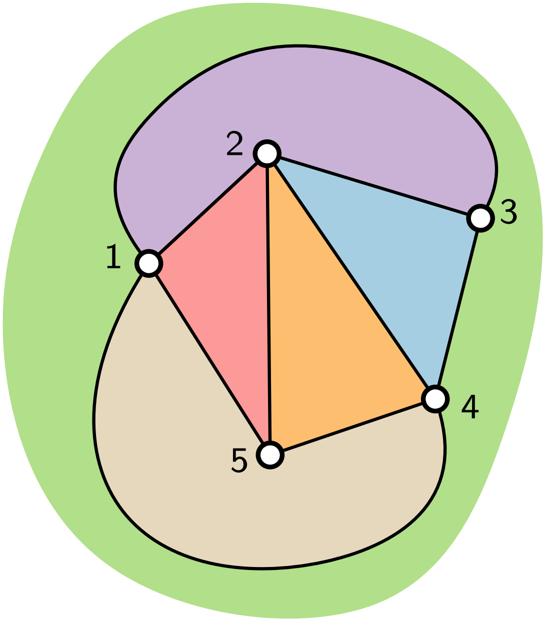

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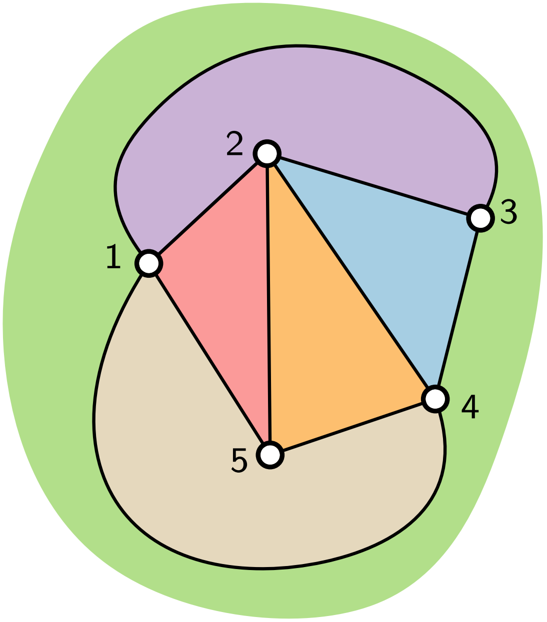

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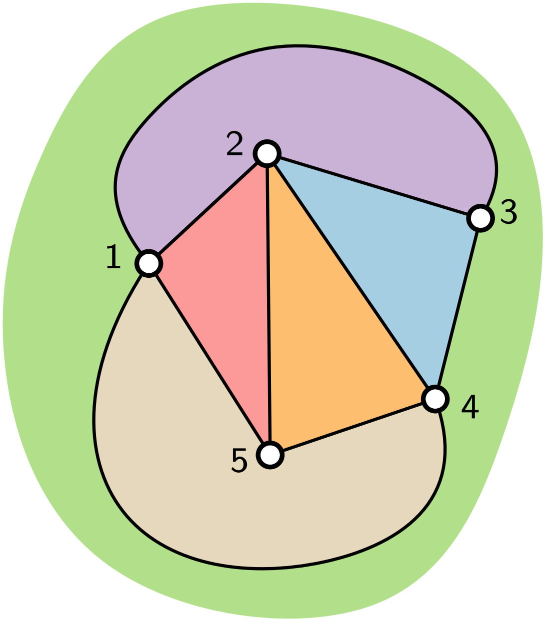

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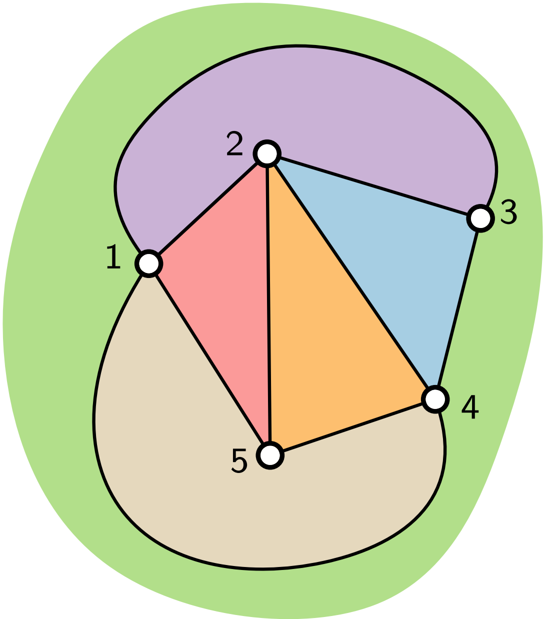

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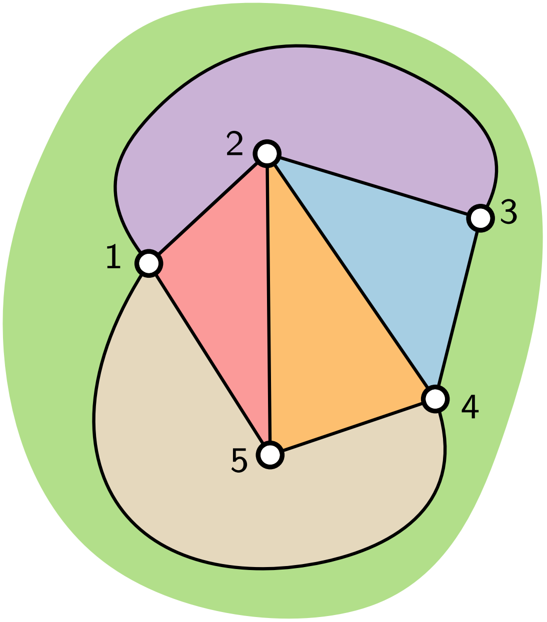

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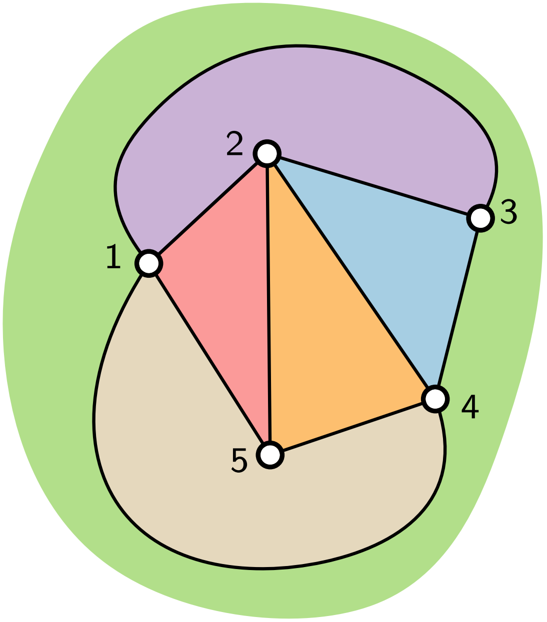

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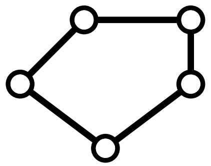

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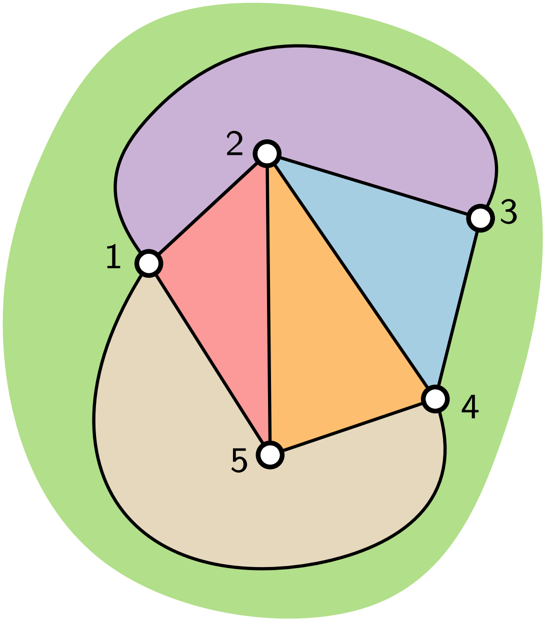

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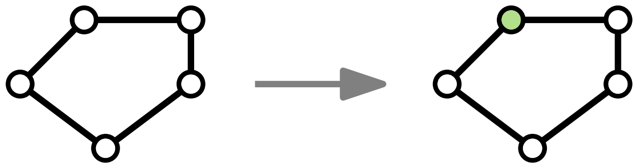

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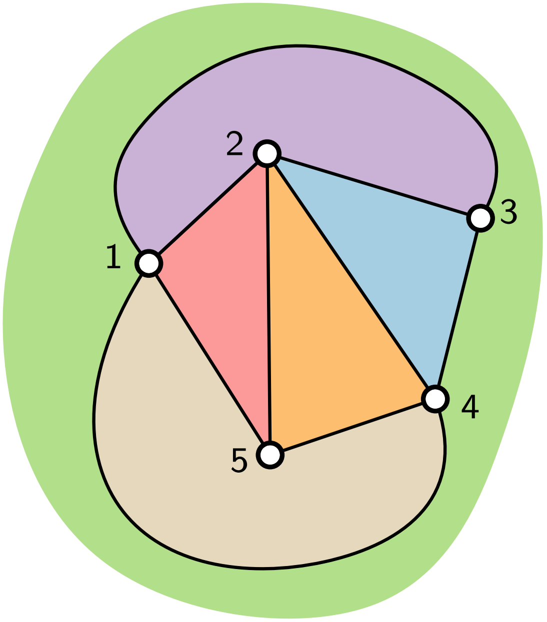

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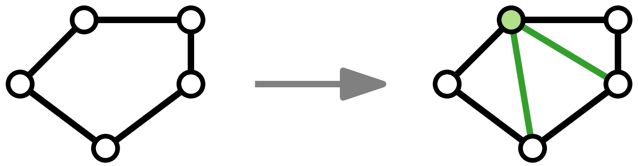

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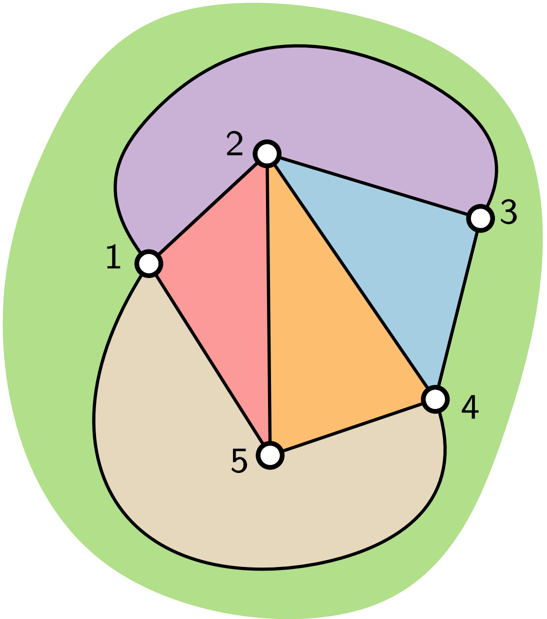

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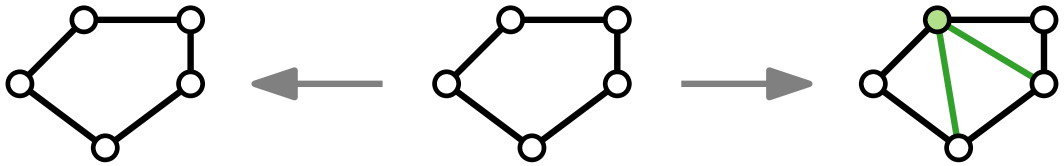

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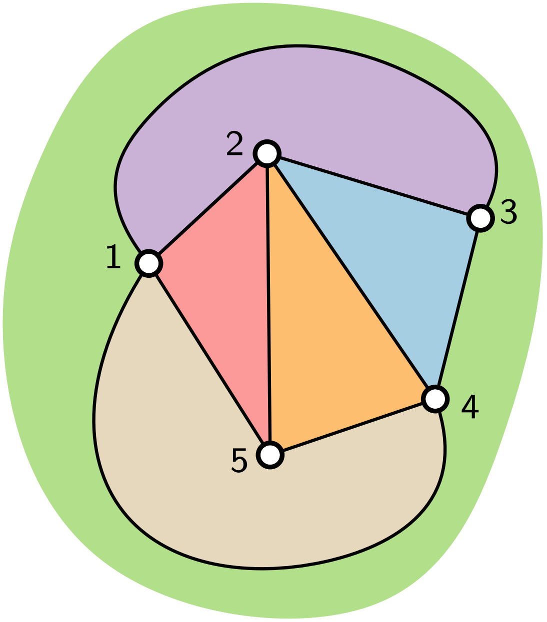

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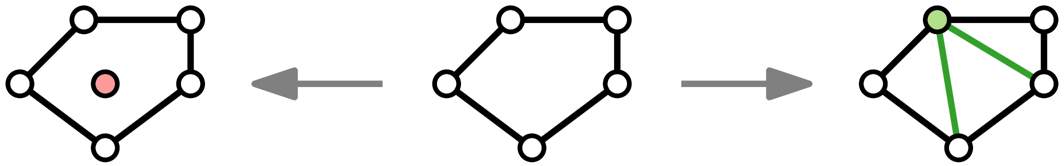

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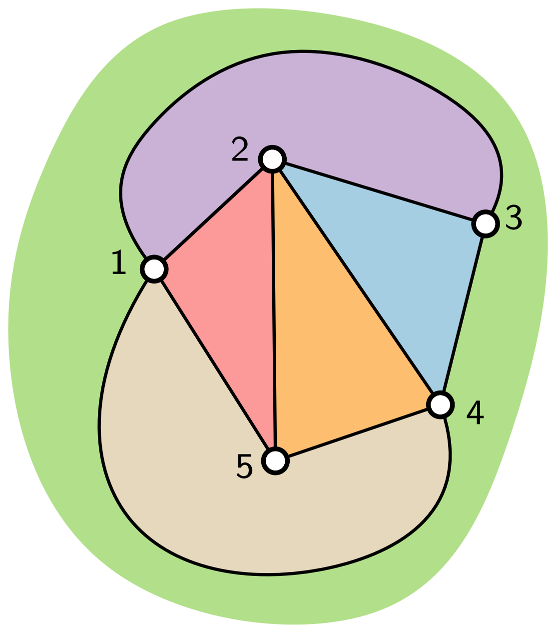

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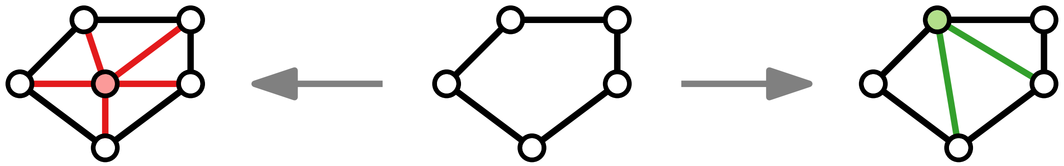

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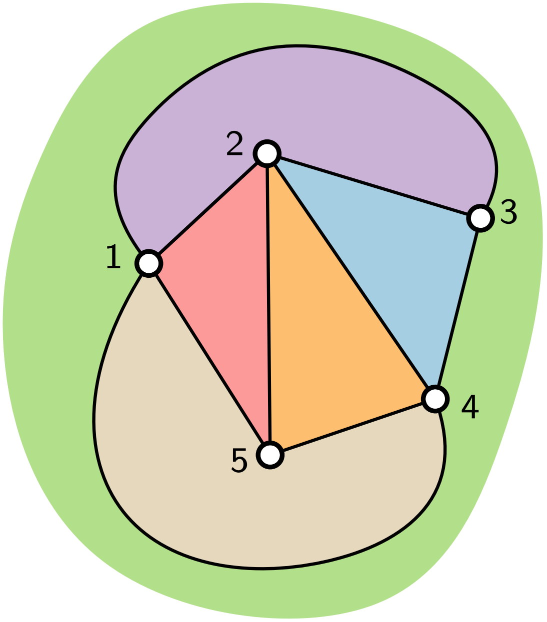

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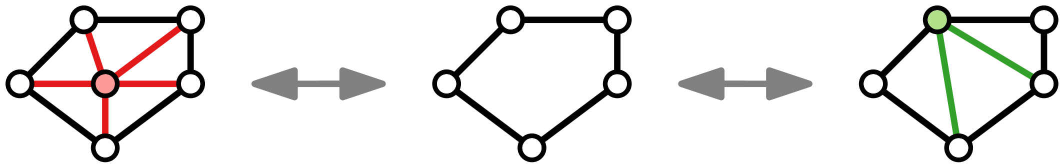

Motivation

- Why planar and straight-line?

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[Bennett, Ryall, Spaltzeholz and Gooch '07]

The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

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- No crossings $\Rightarrow$ planar
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Drawing aesthetics to optimize

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Towards Straight-Line Drawings

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Characterization

Towards Straight-Line Drawings

Characterization

Recognition

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Towards Straight-Line Drawings

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
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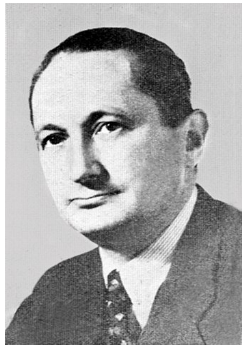
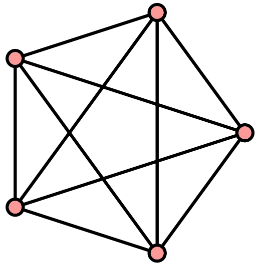
Kazimierz Kuratowski (1896–1980)





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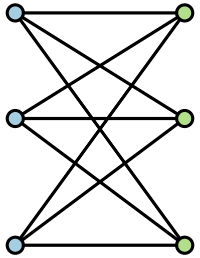

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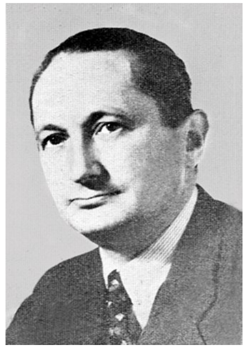
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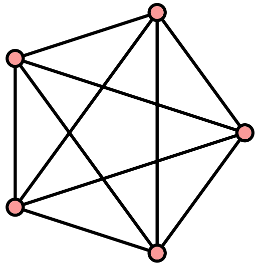
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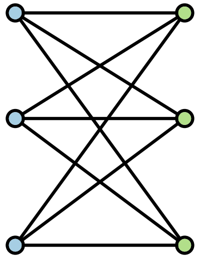
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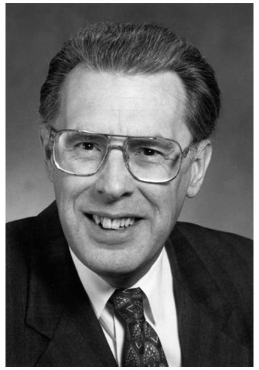
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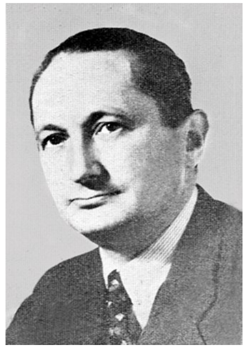

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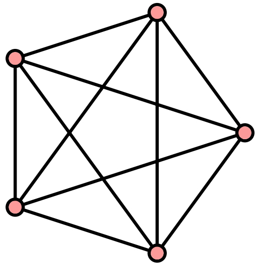

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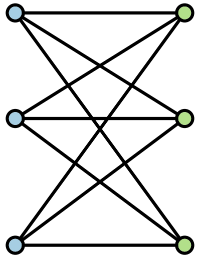

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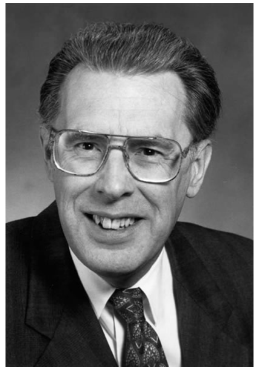

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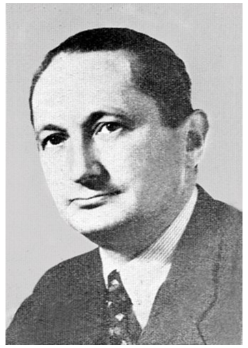
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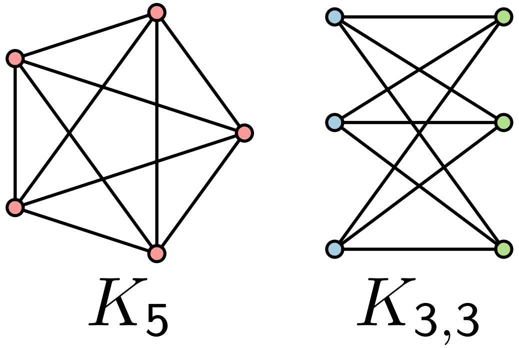

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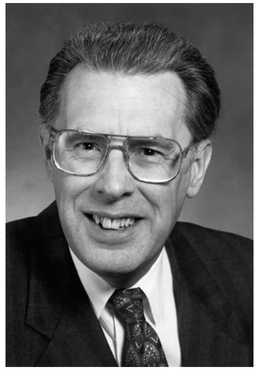
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Theorem. [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has a planar drawing where the edges are straight-line segments.


Klaus Wagner (1910–2000)
Autor: Konrad Jacobs, wikipedia

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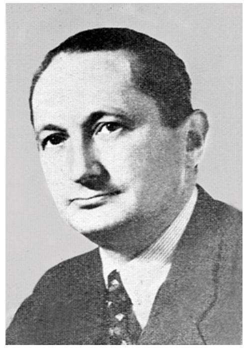

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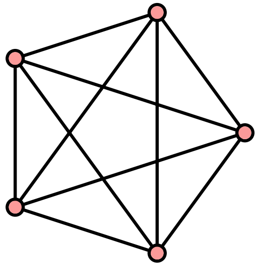

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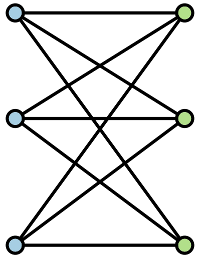

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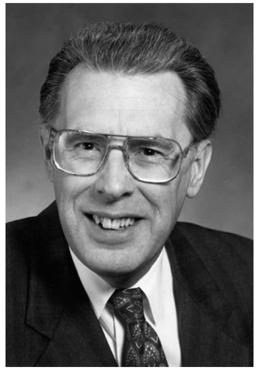

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Every planar graph has a planar drawing where the edges are straight-line segments.

The algorithms implied by these theorems produce drawings whose area is **not** bounded by any polynomial in n.

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Drawing

Planar Straight-Line Drawings

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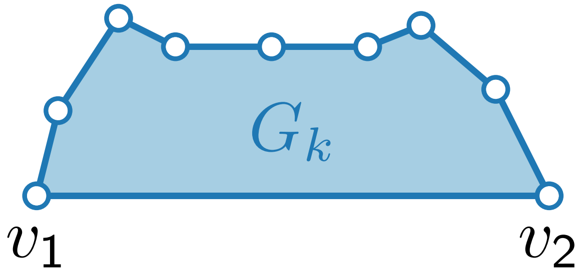

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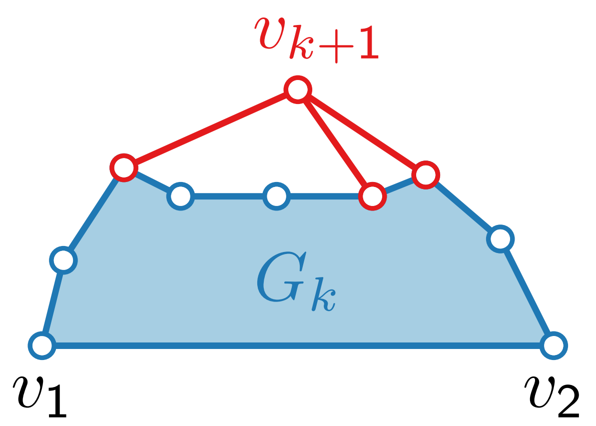

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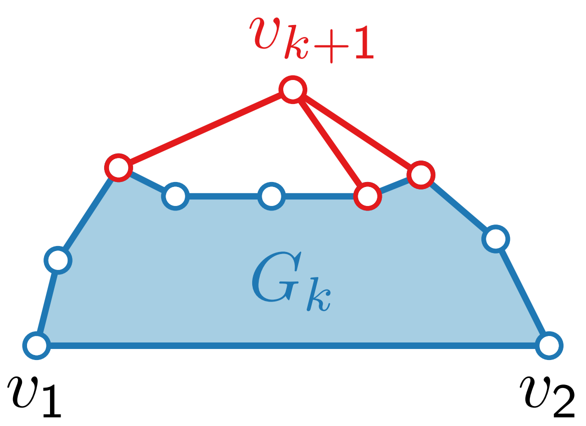

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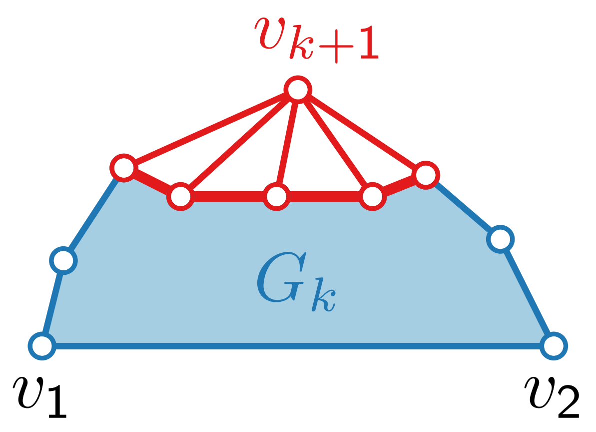

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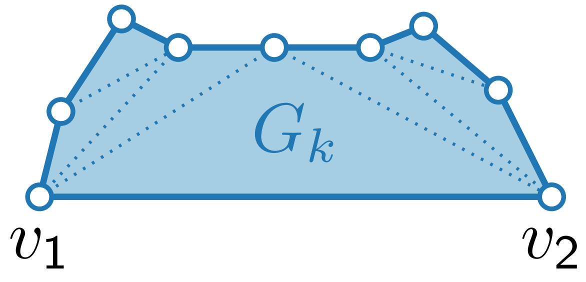

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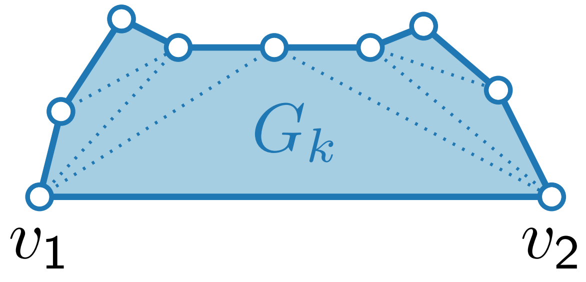

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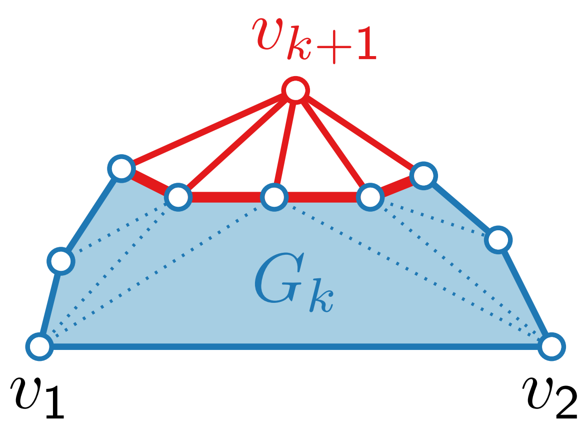

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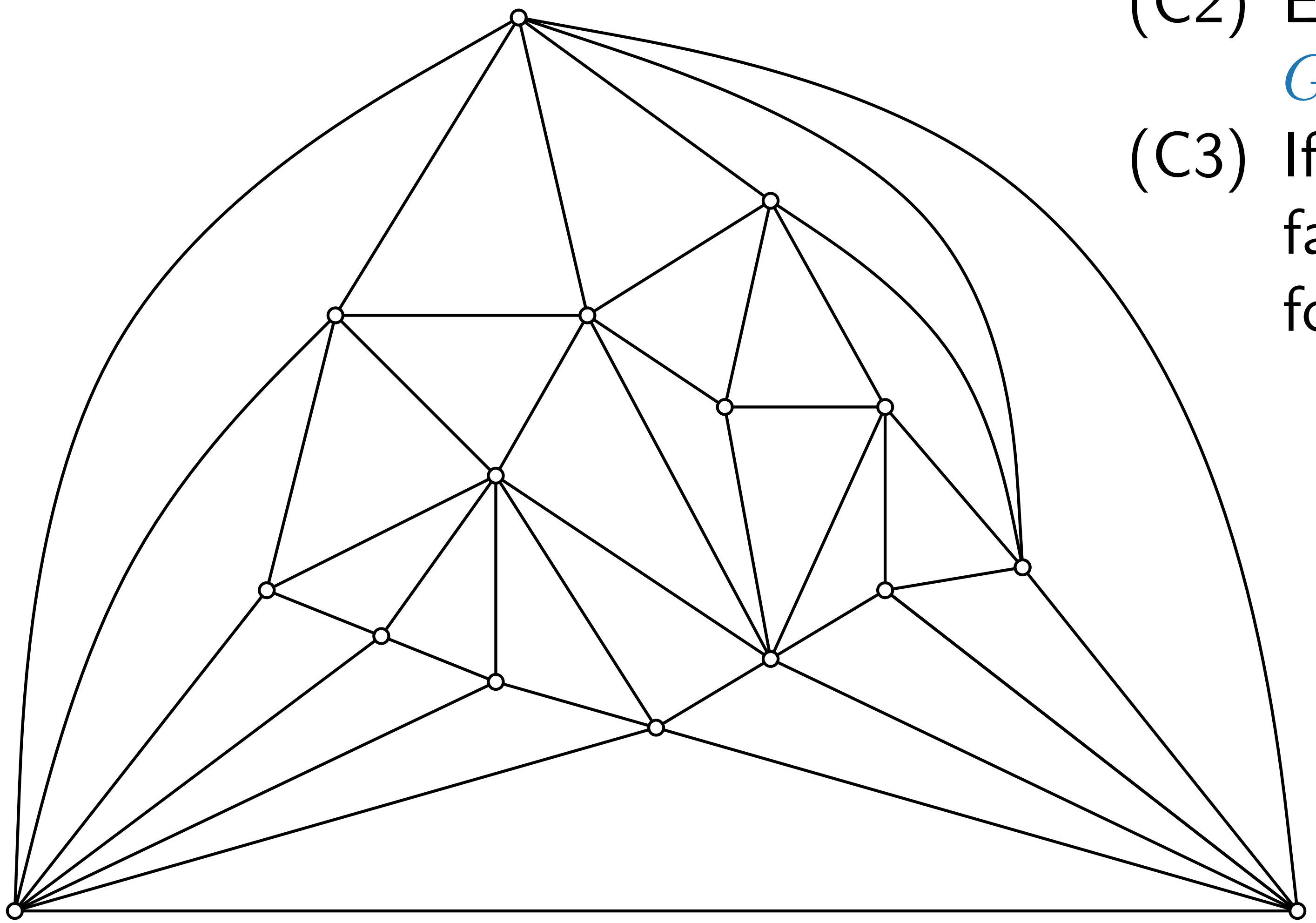

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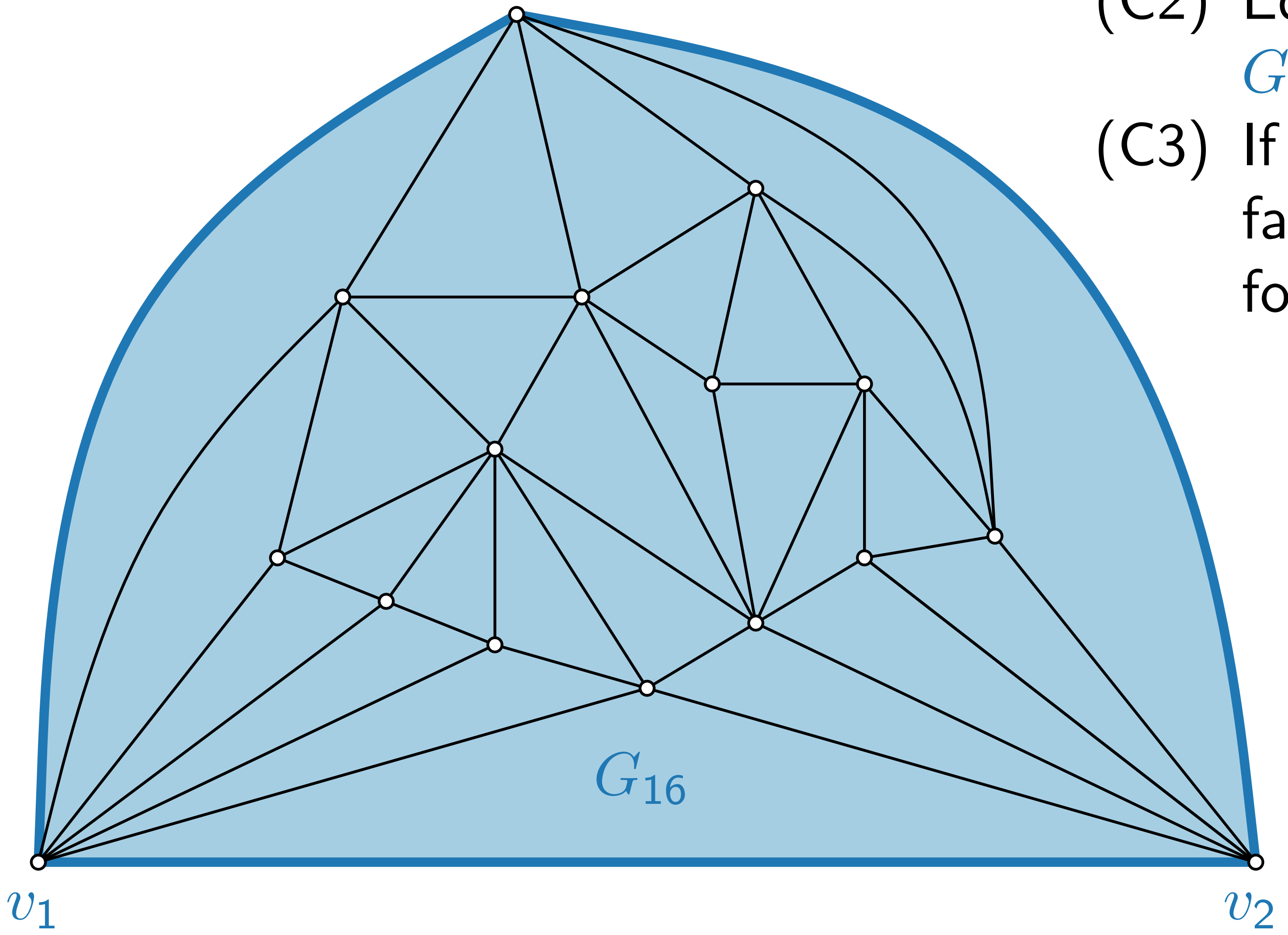

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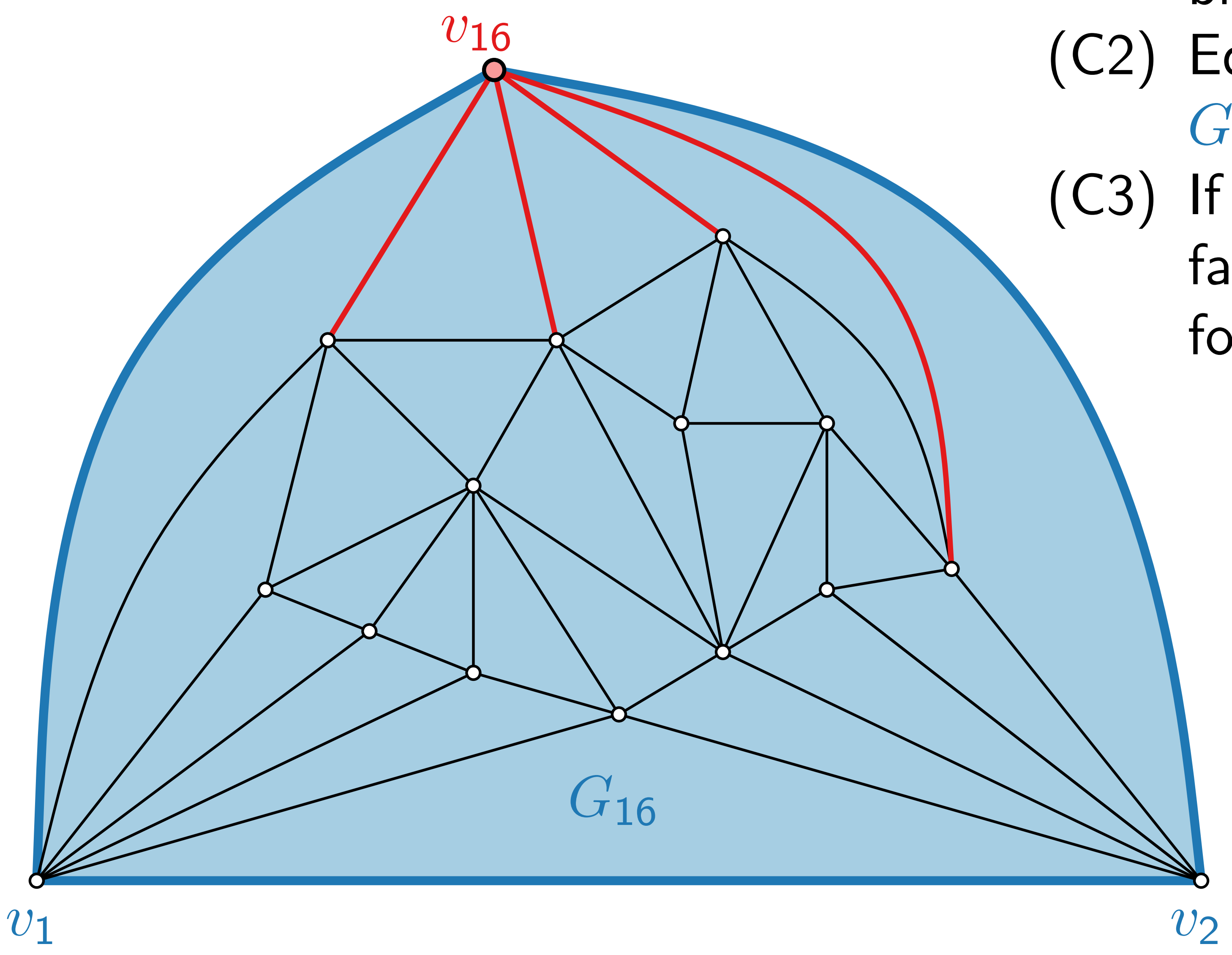

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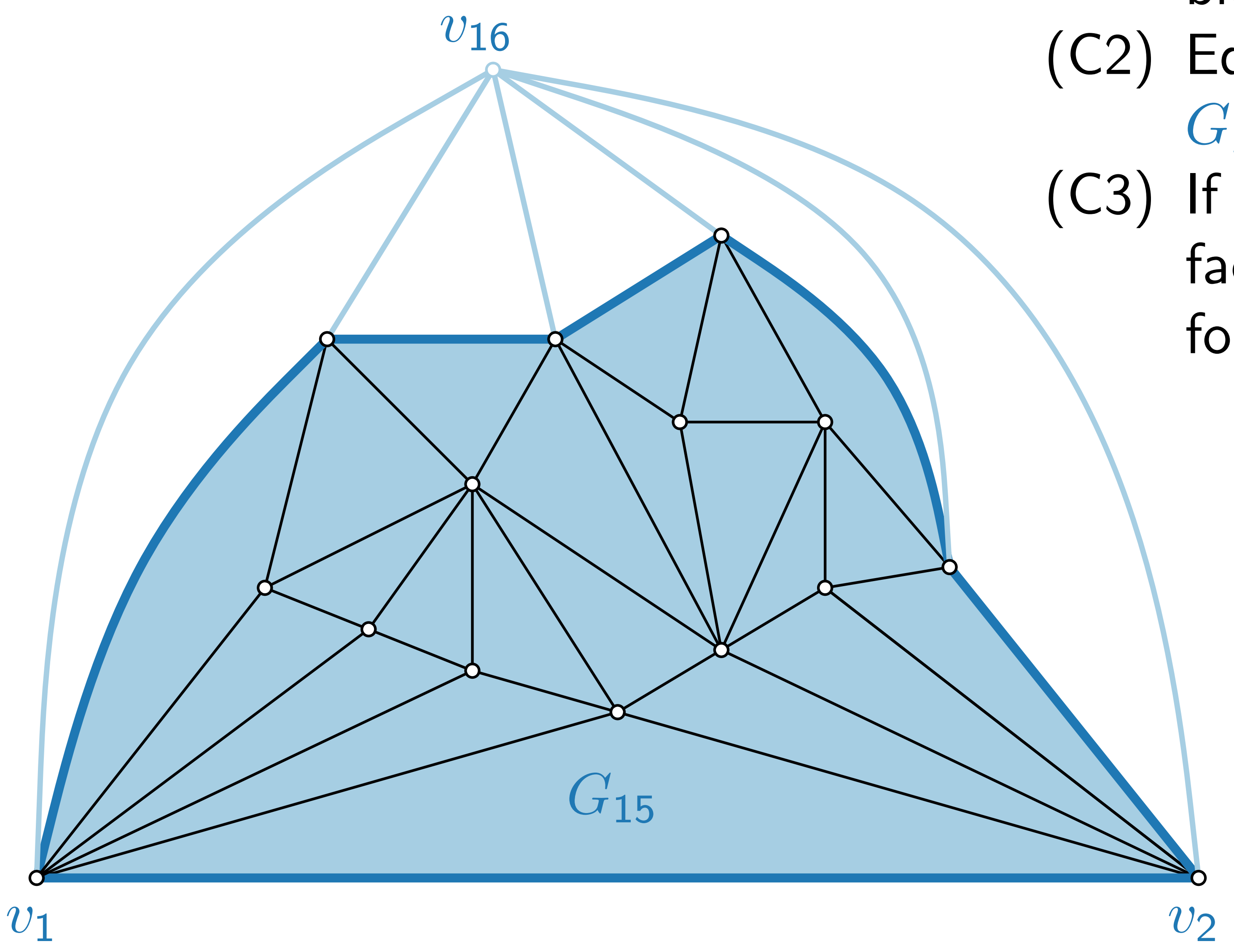

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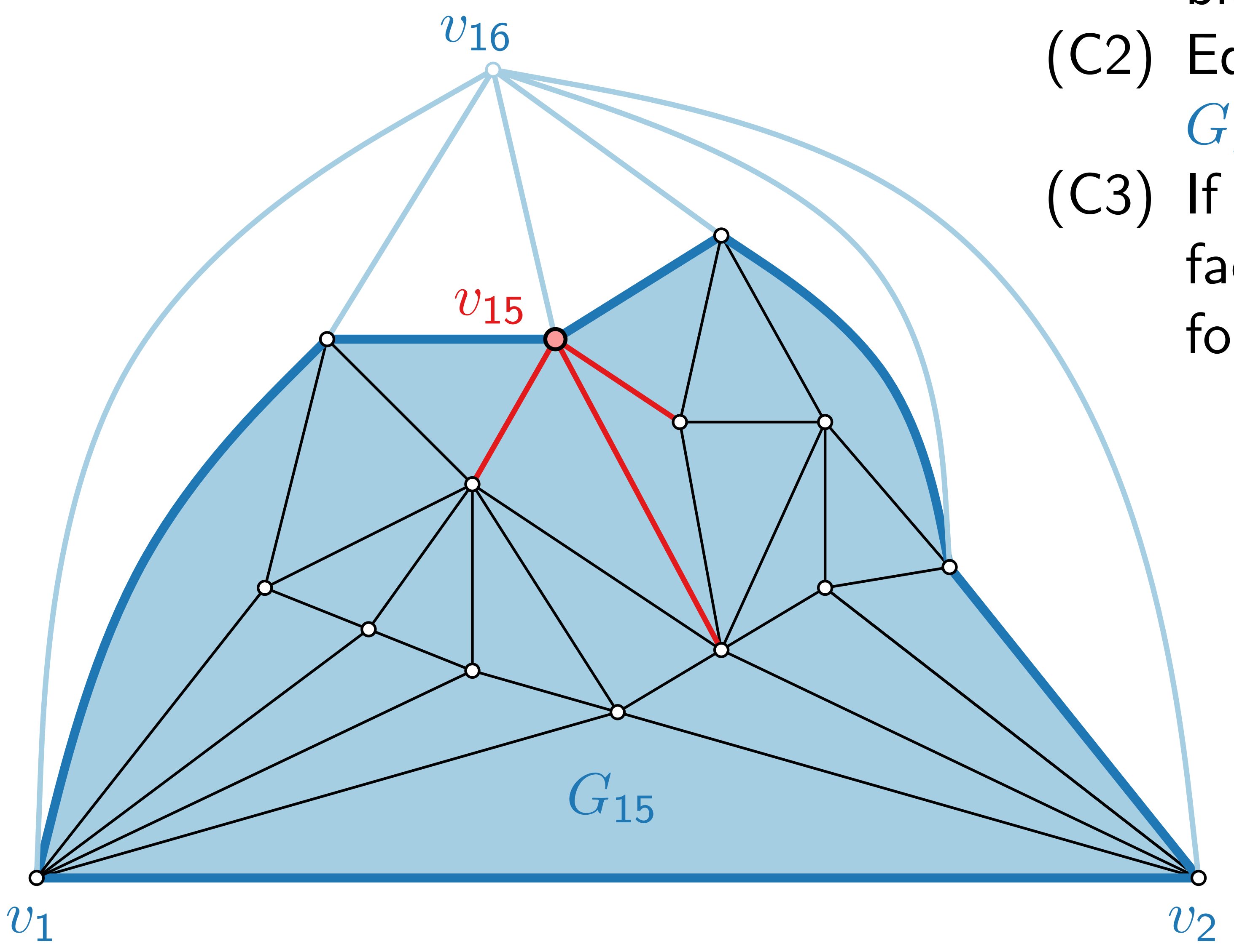

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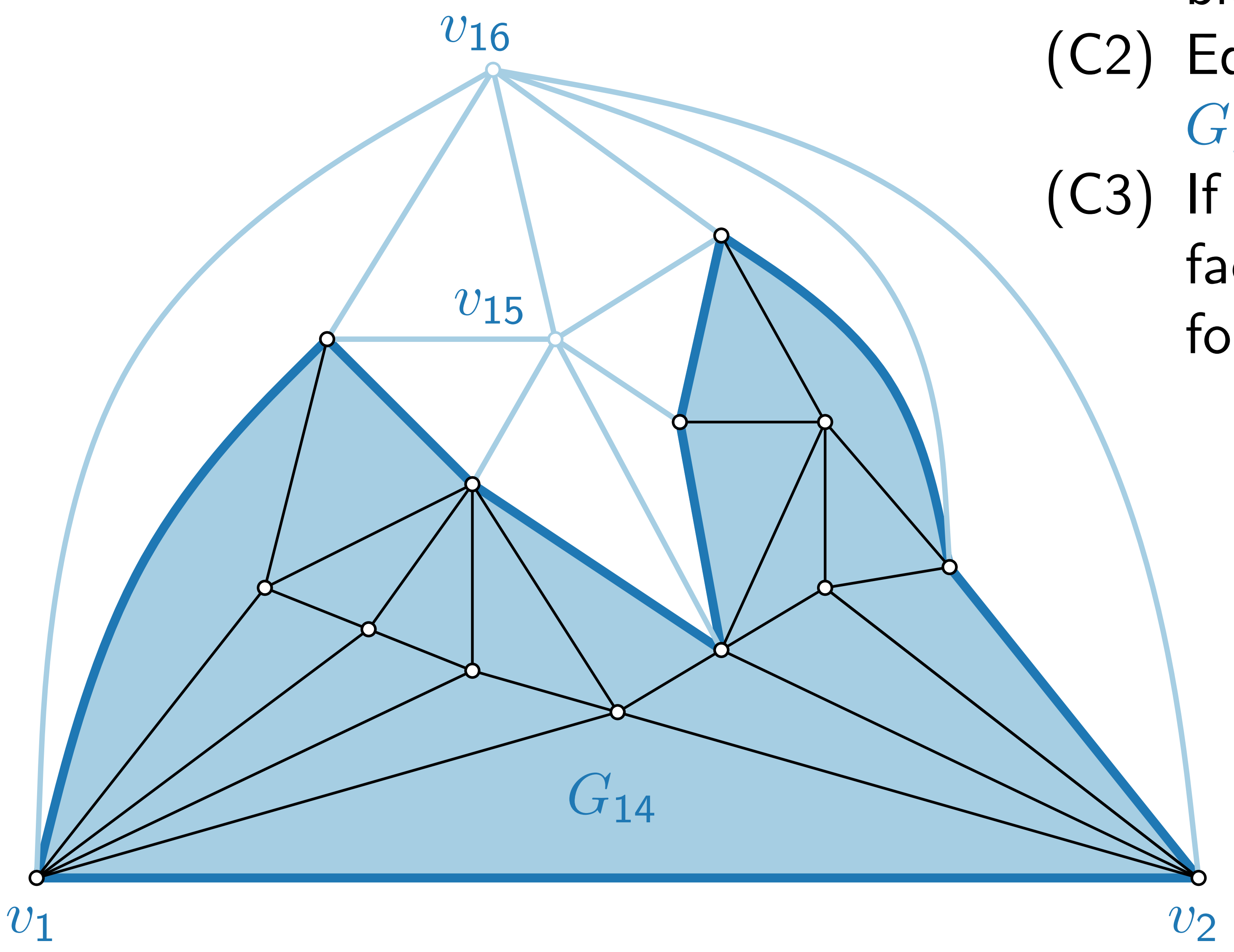

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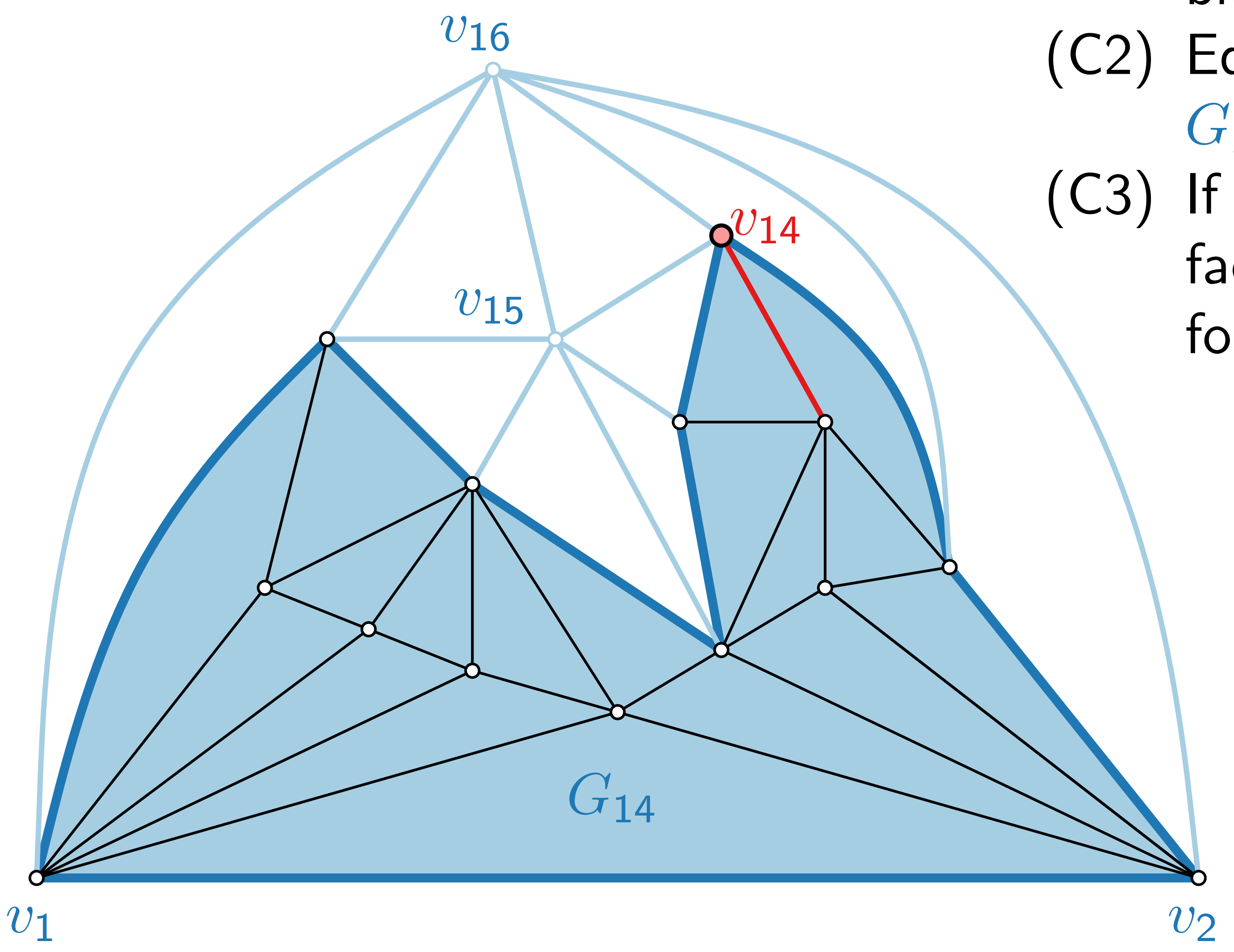

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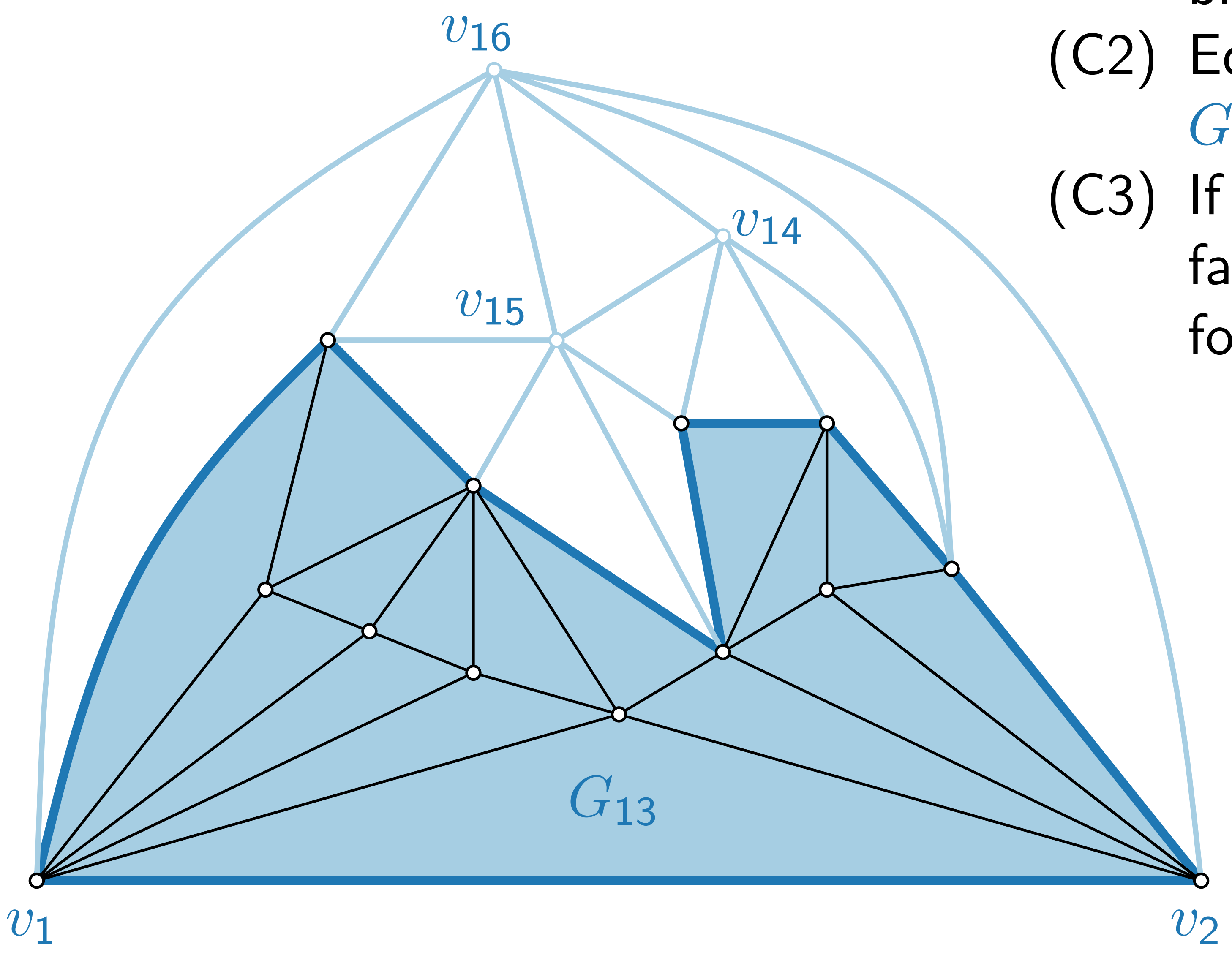

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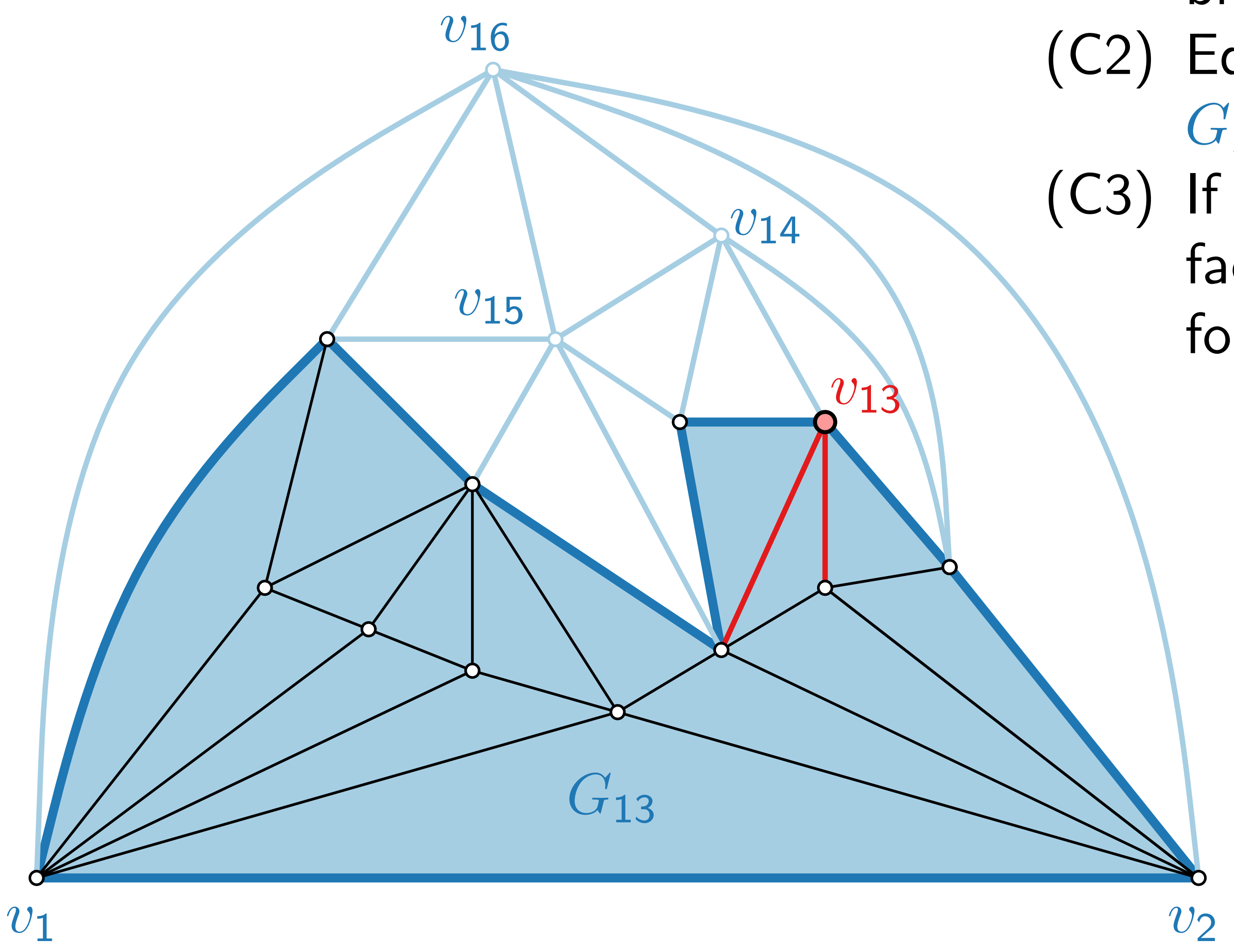

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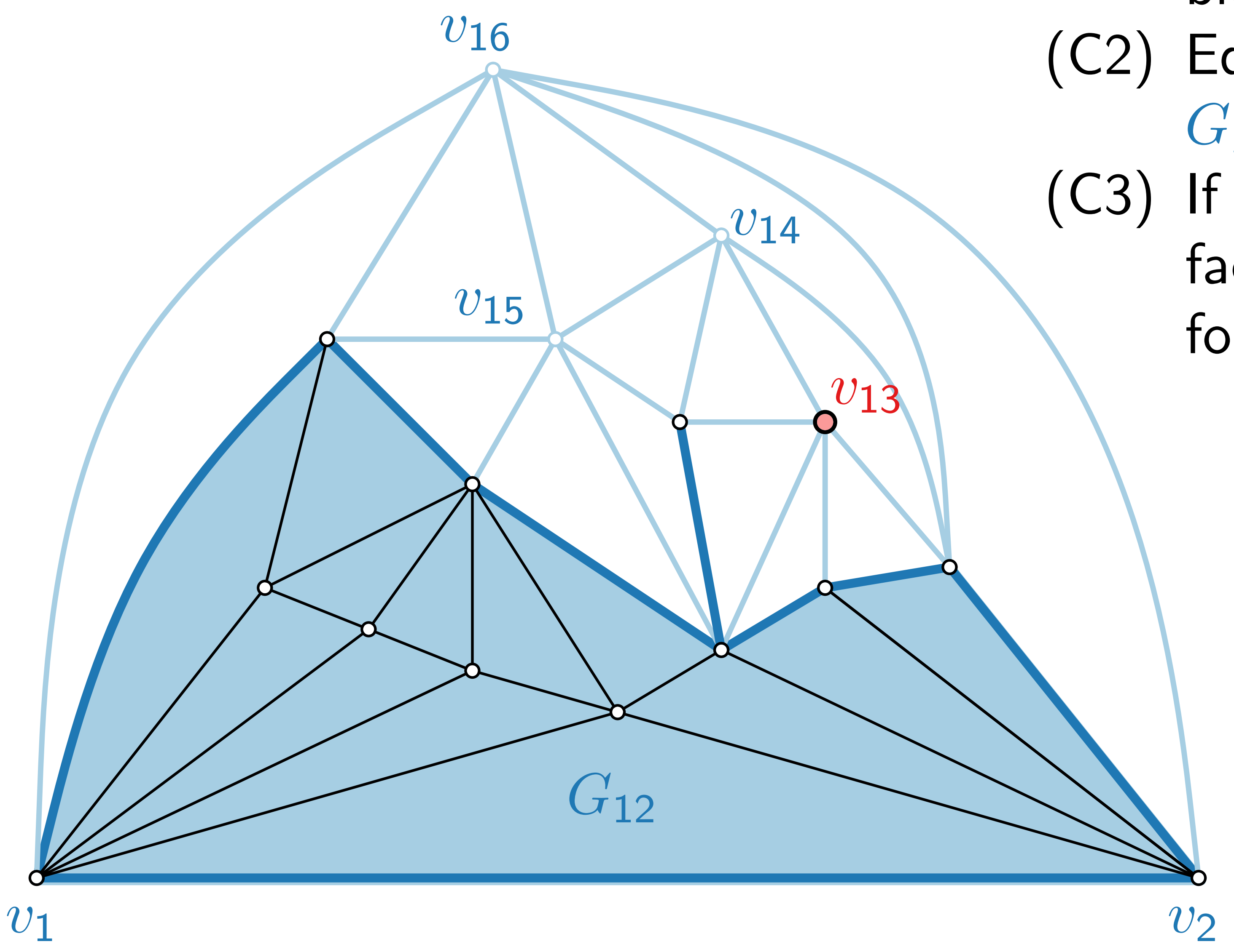

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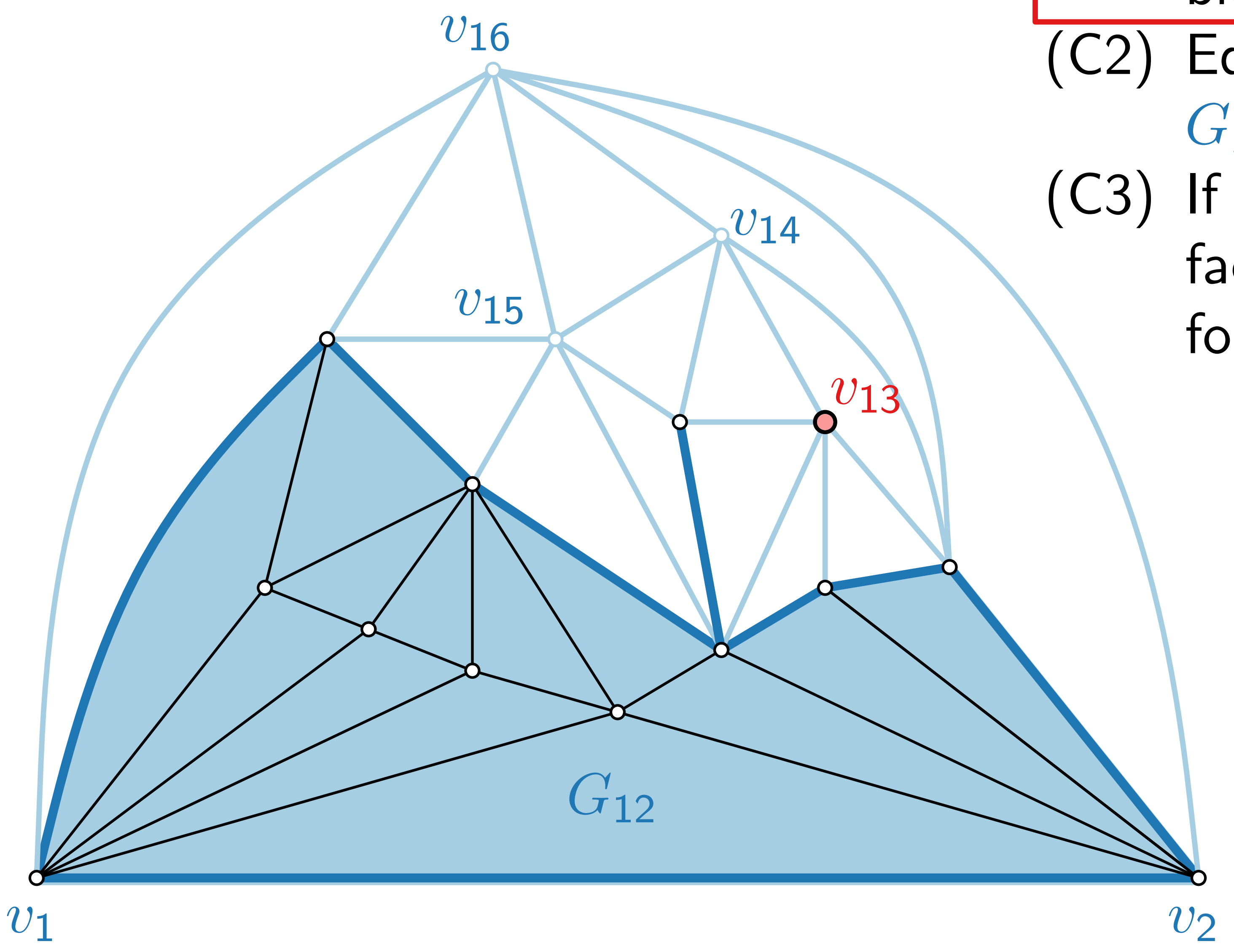

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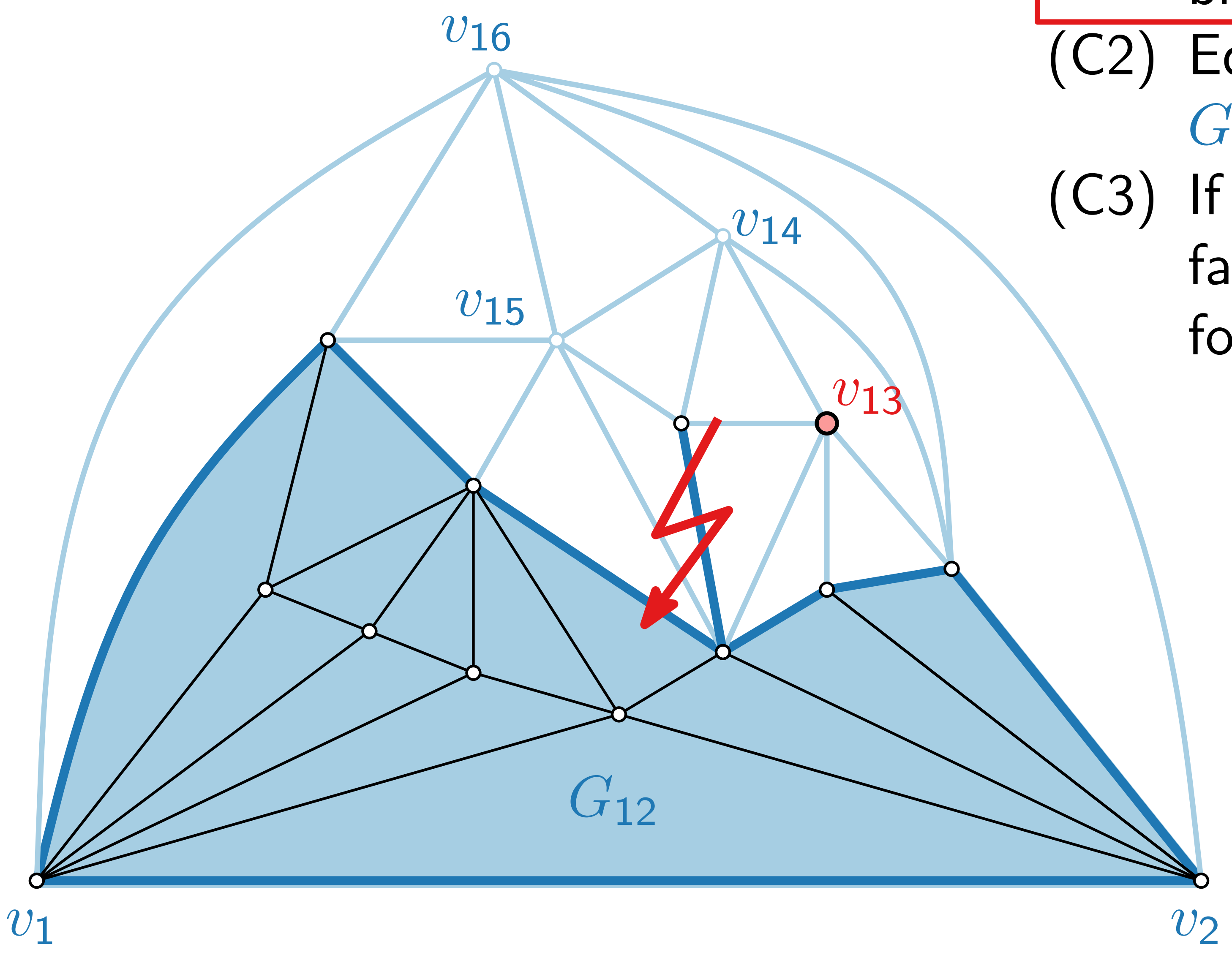

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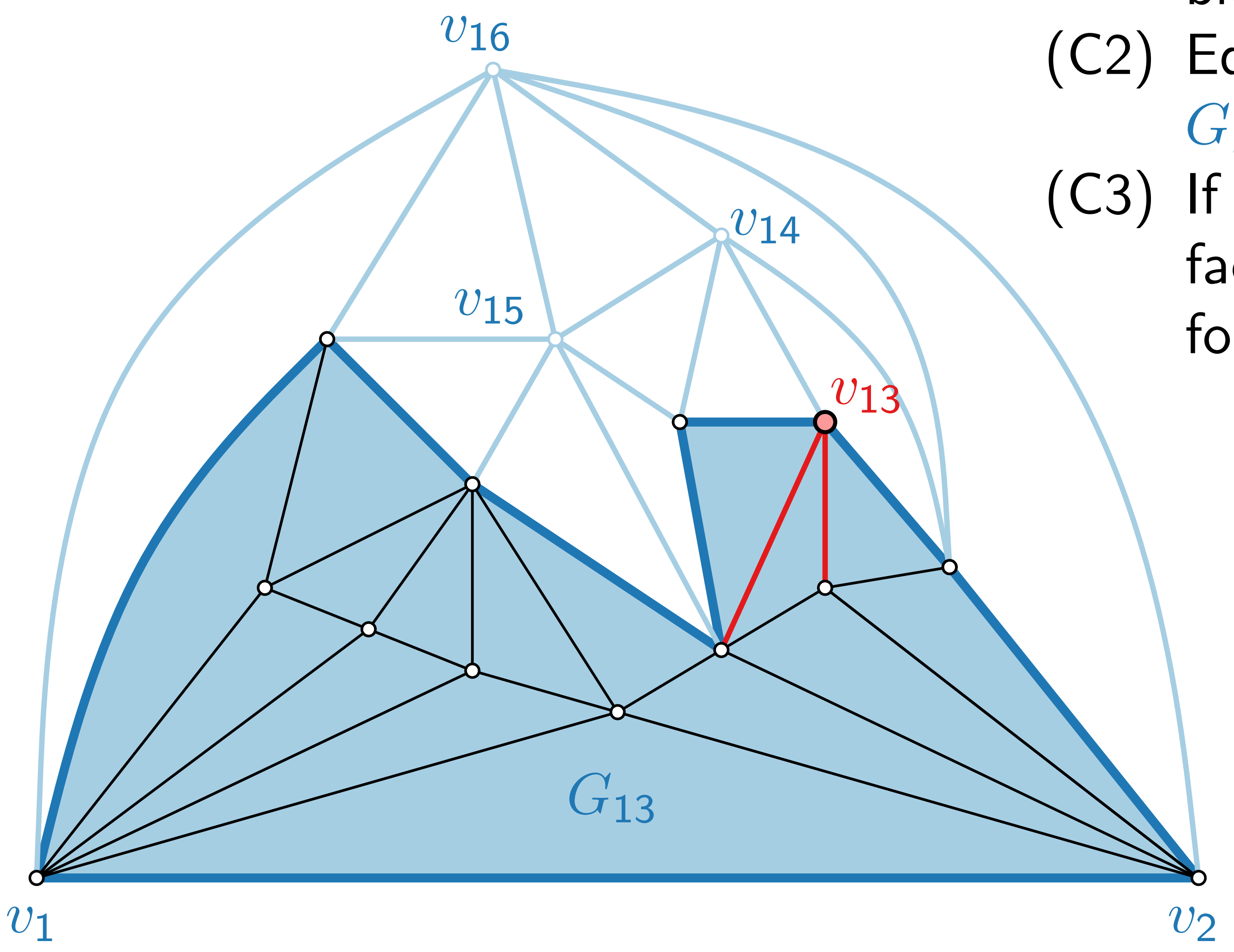

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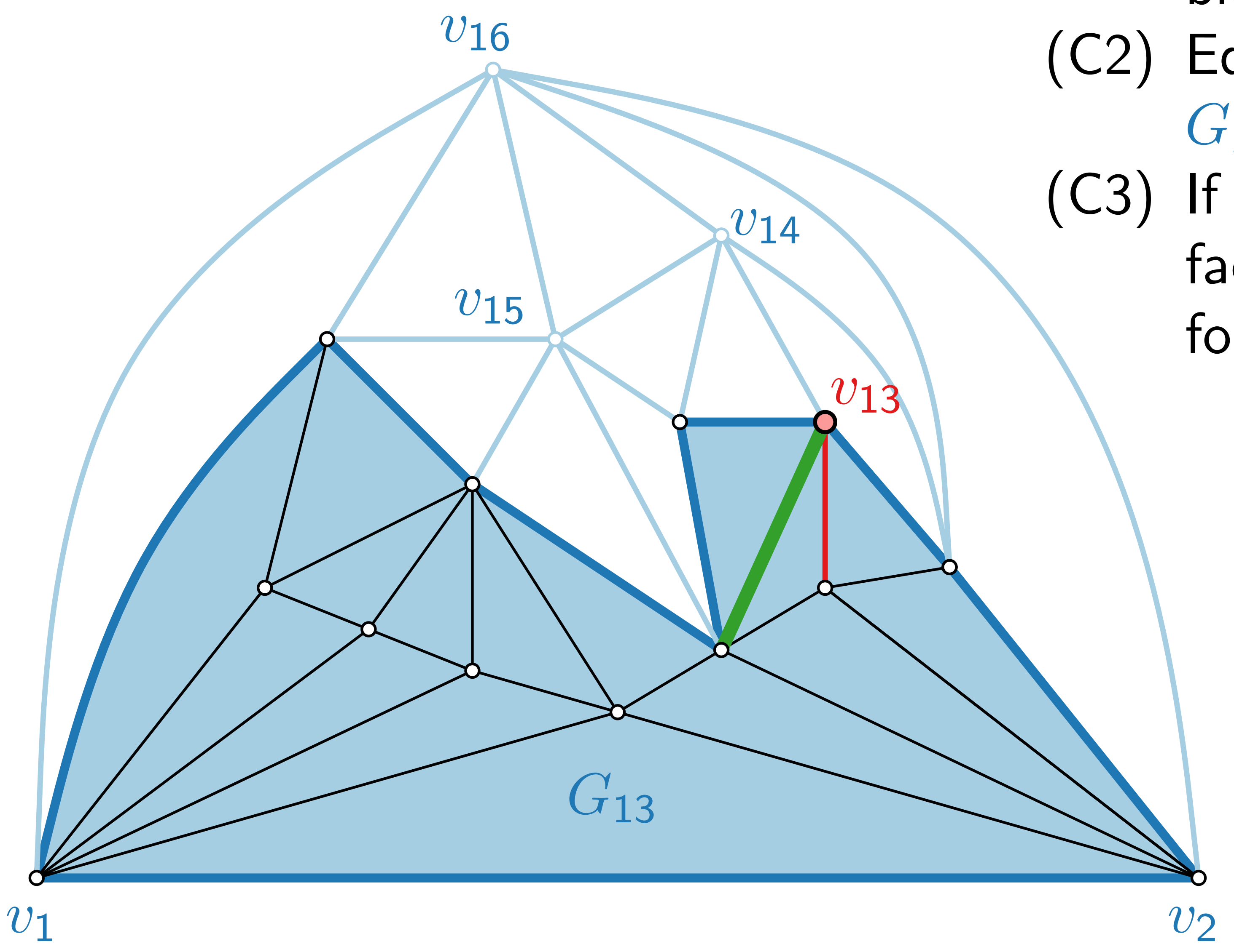

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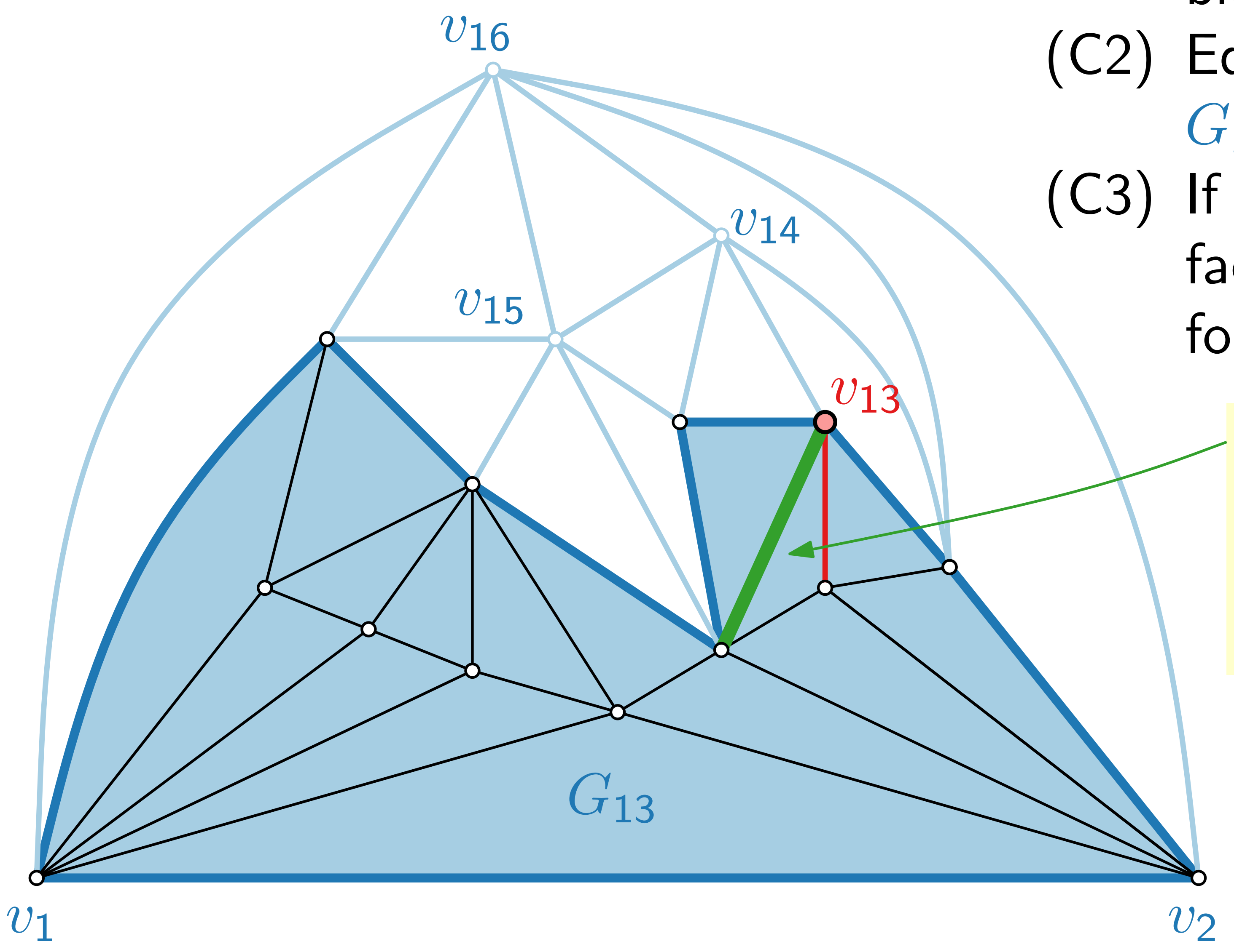

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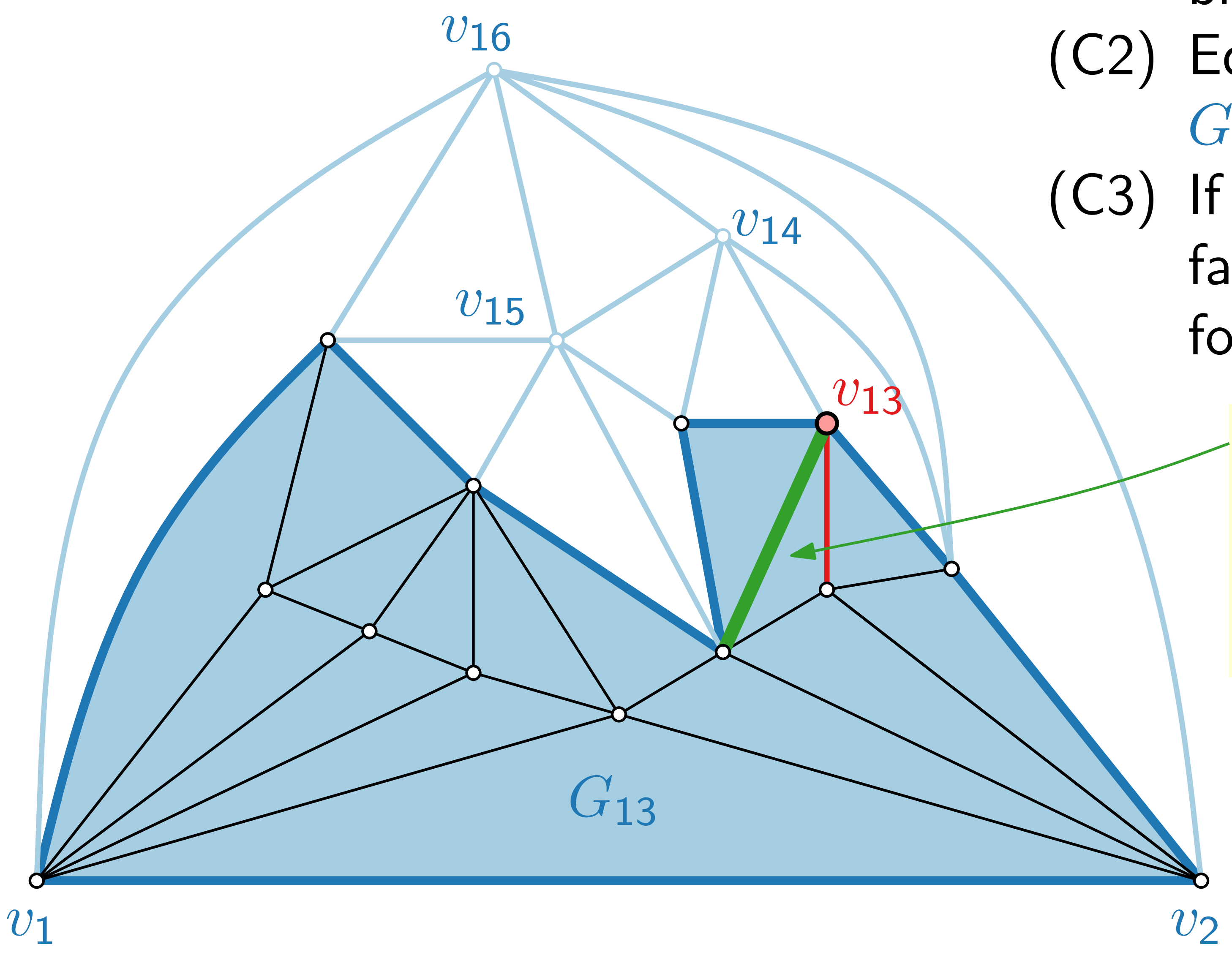

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chord:
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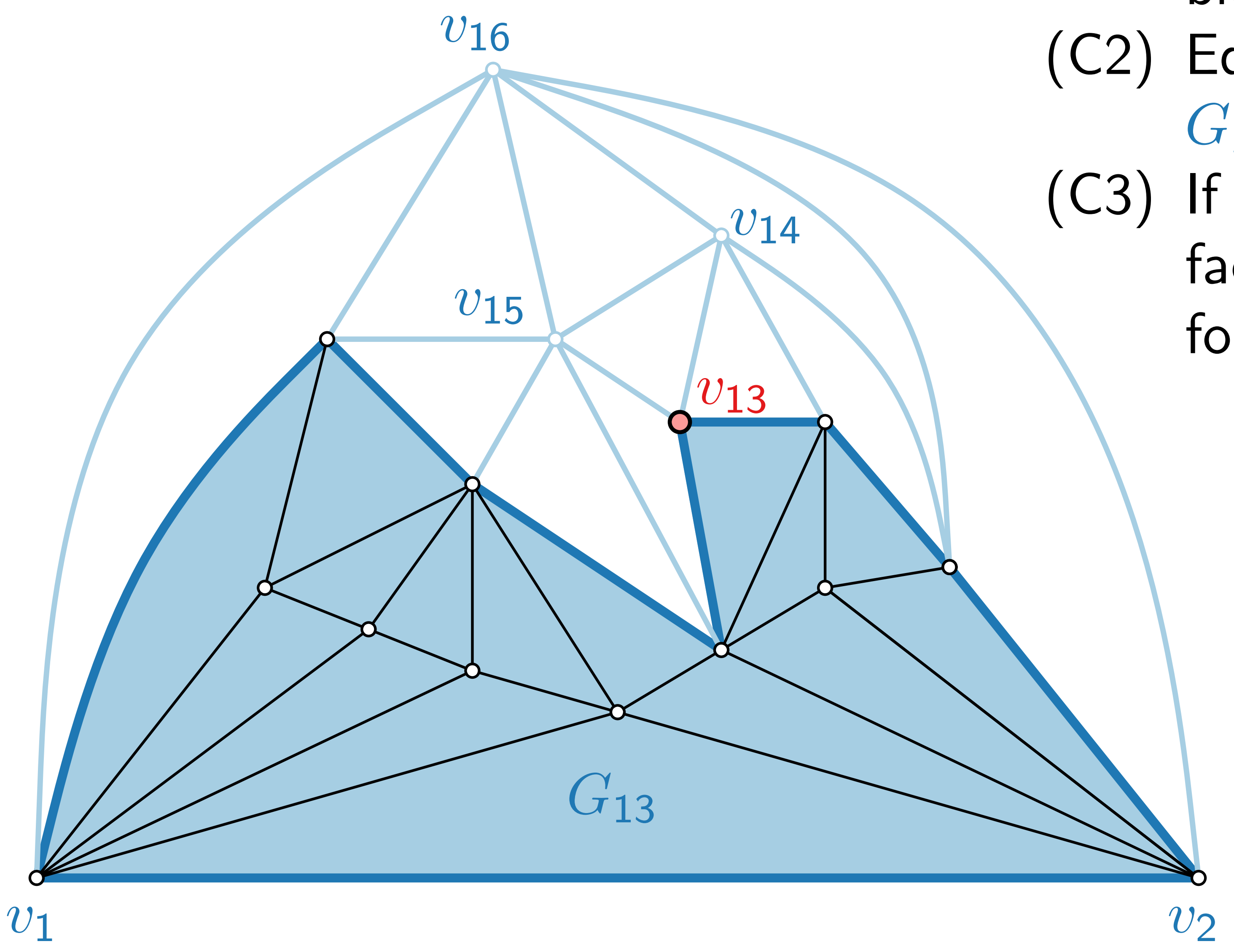

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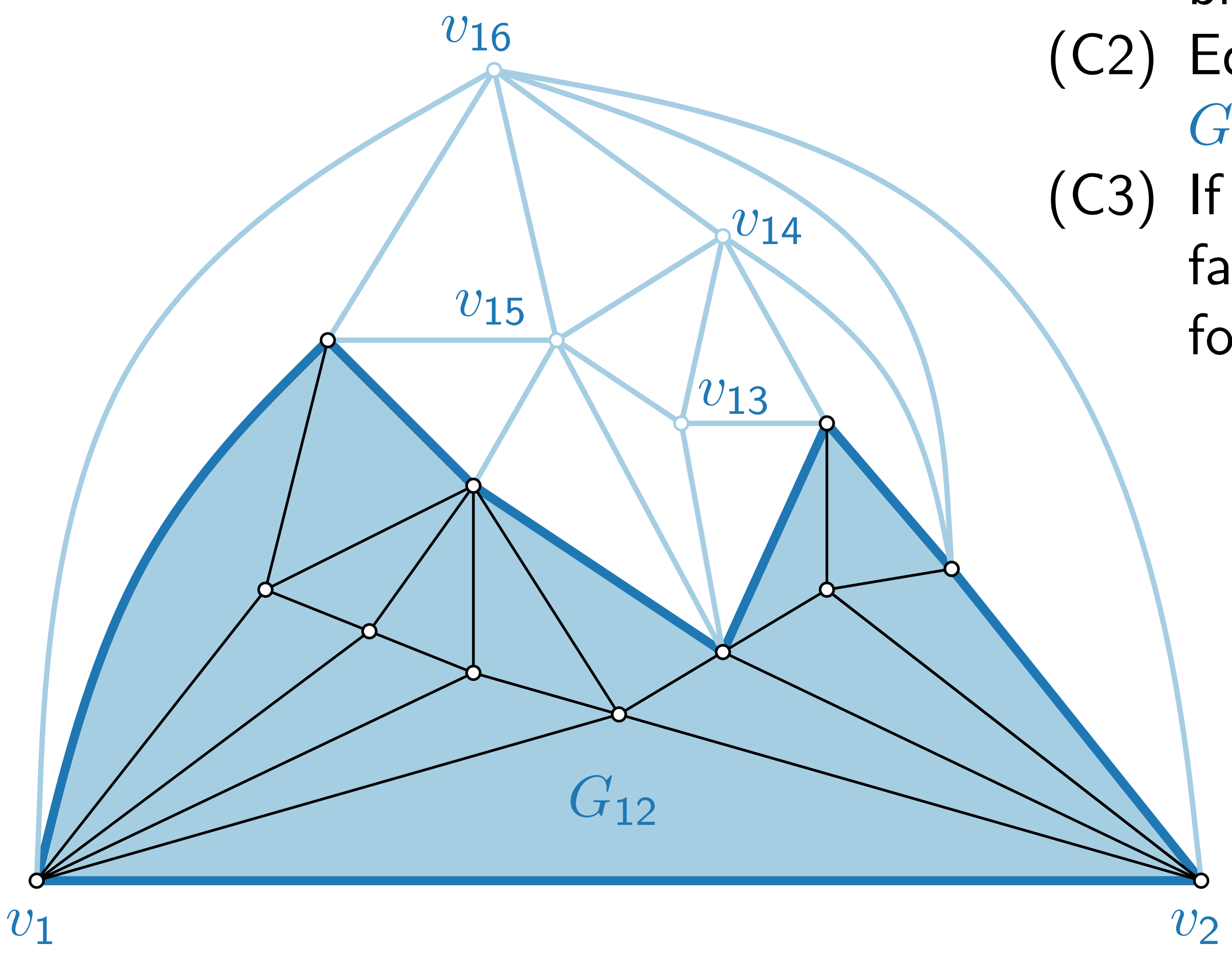

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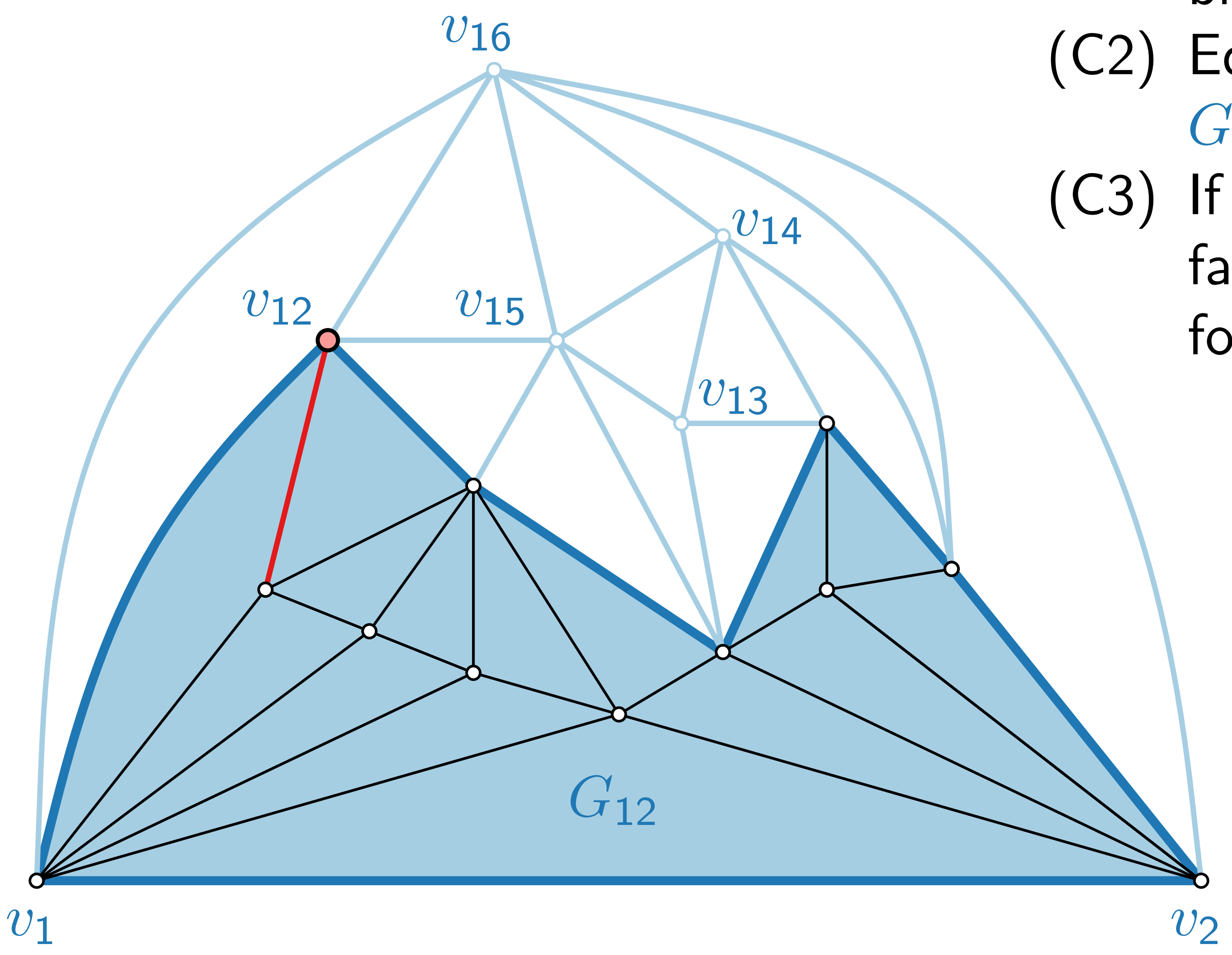

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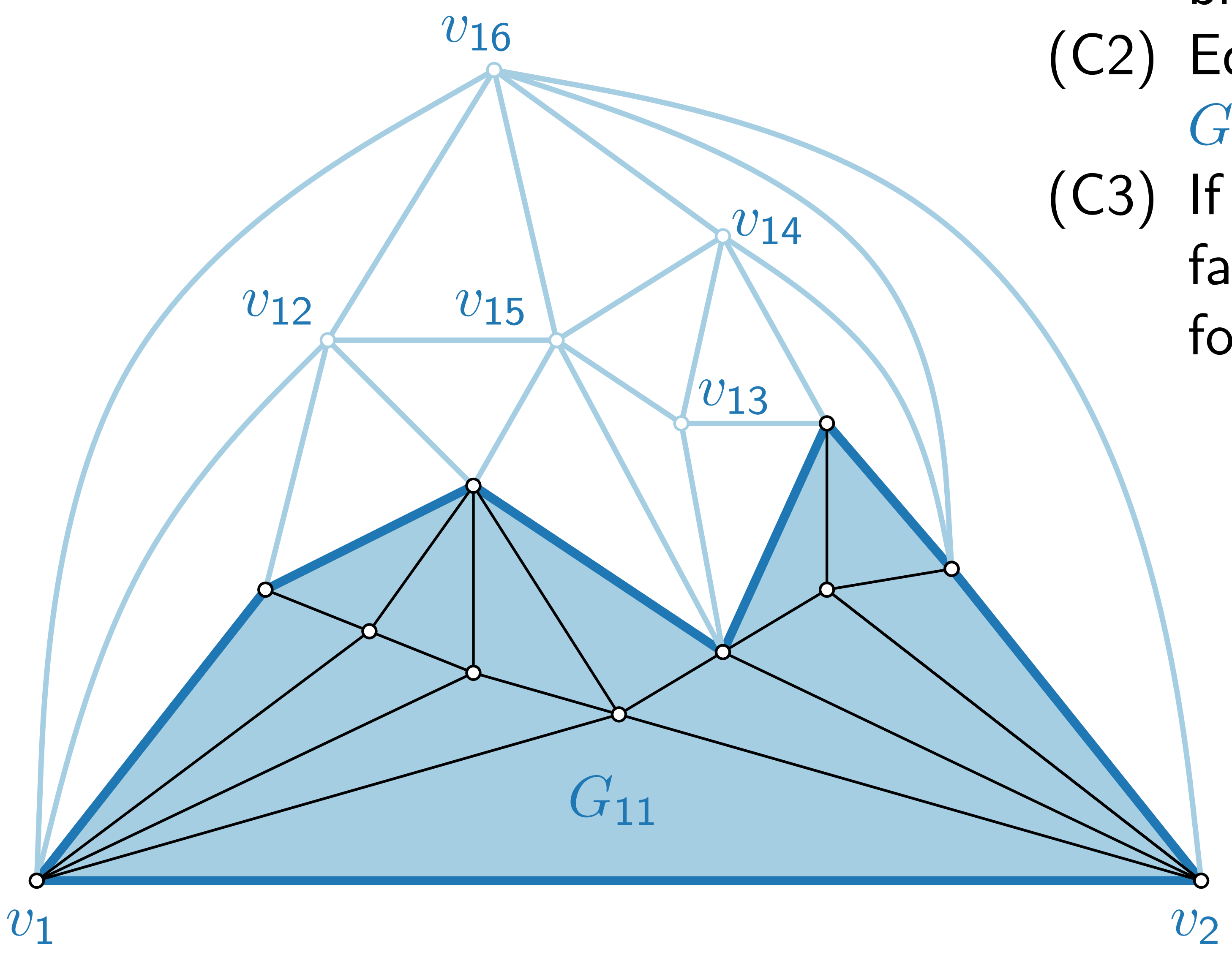

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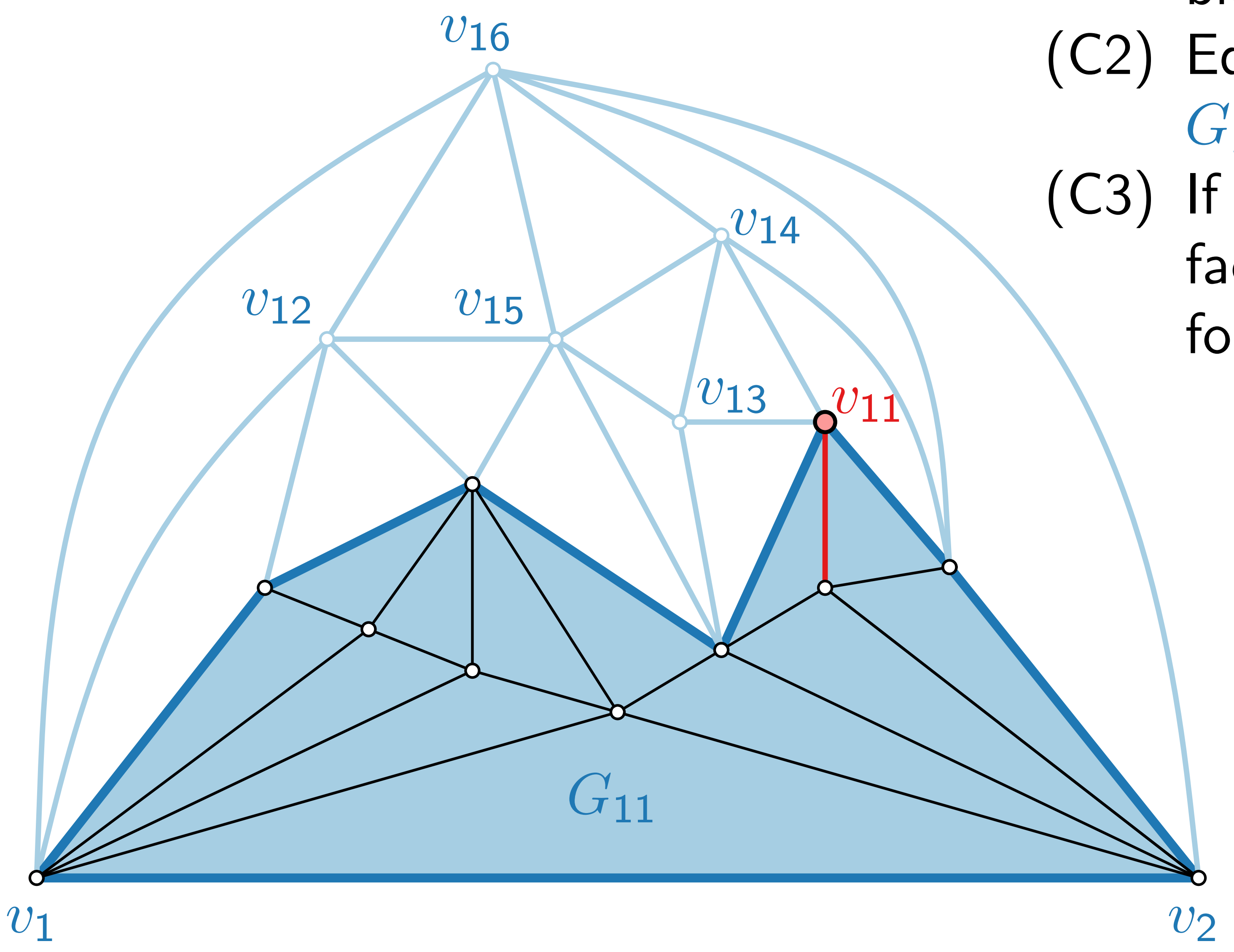

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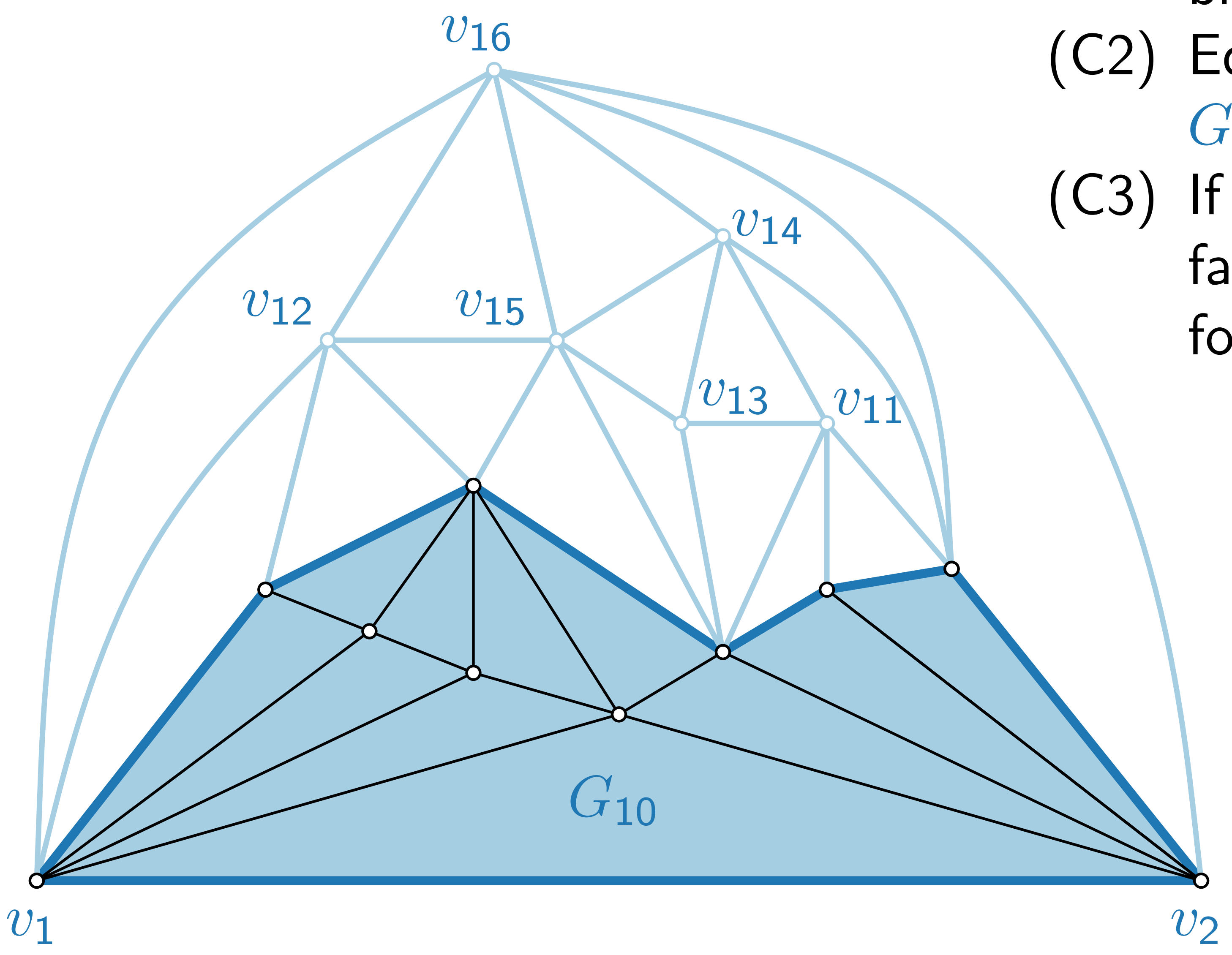

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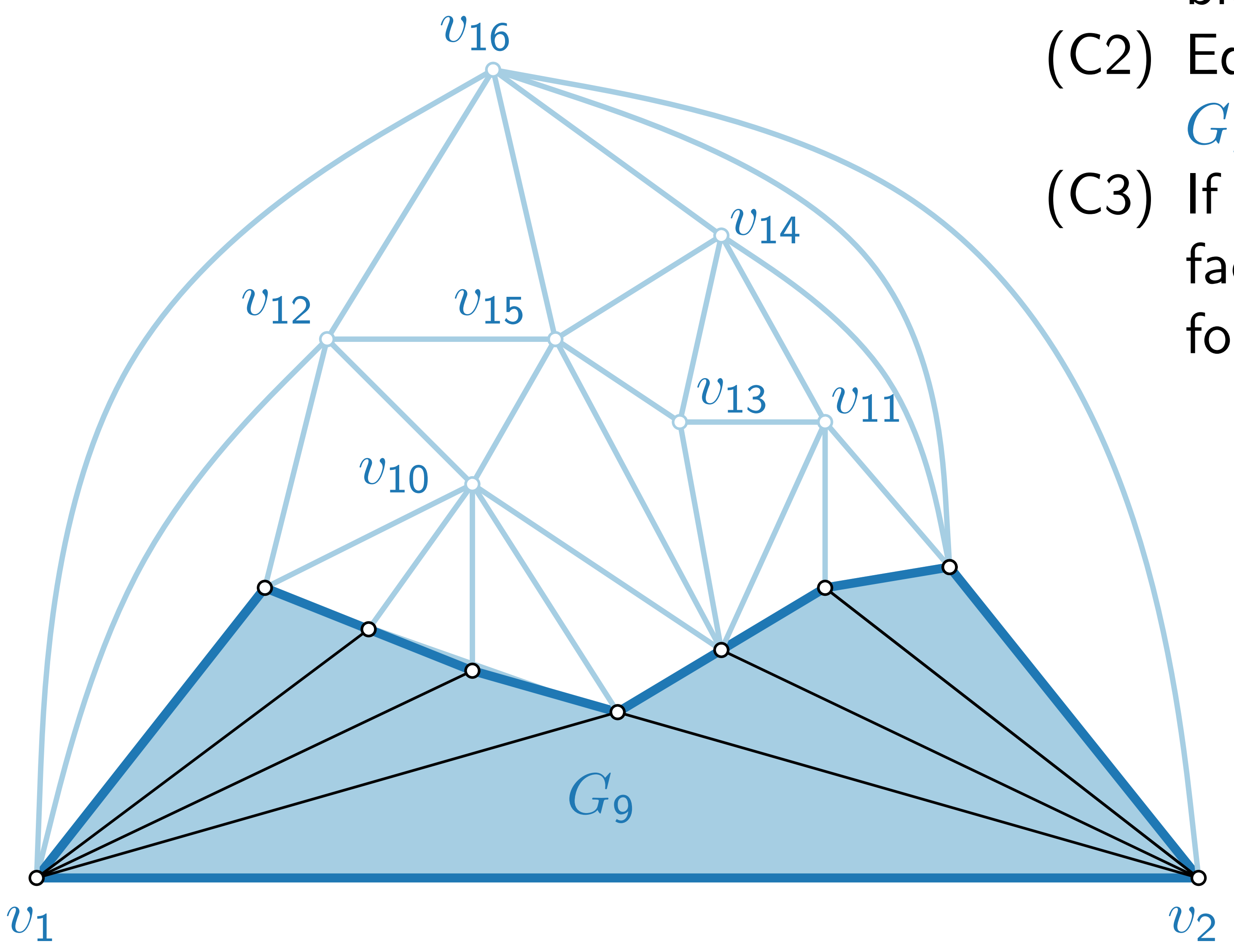

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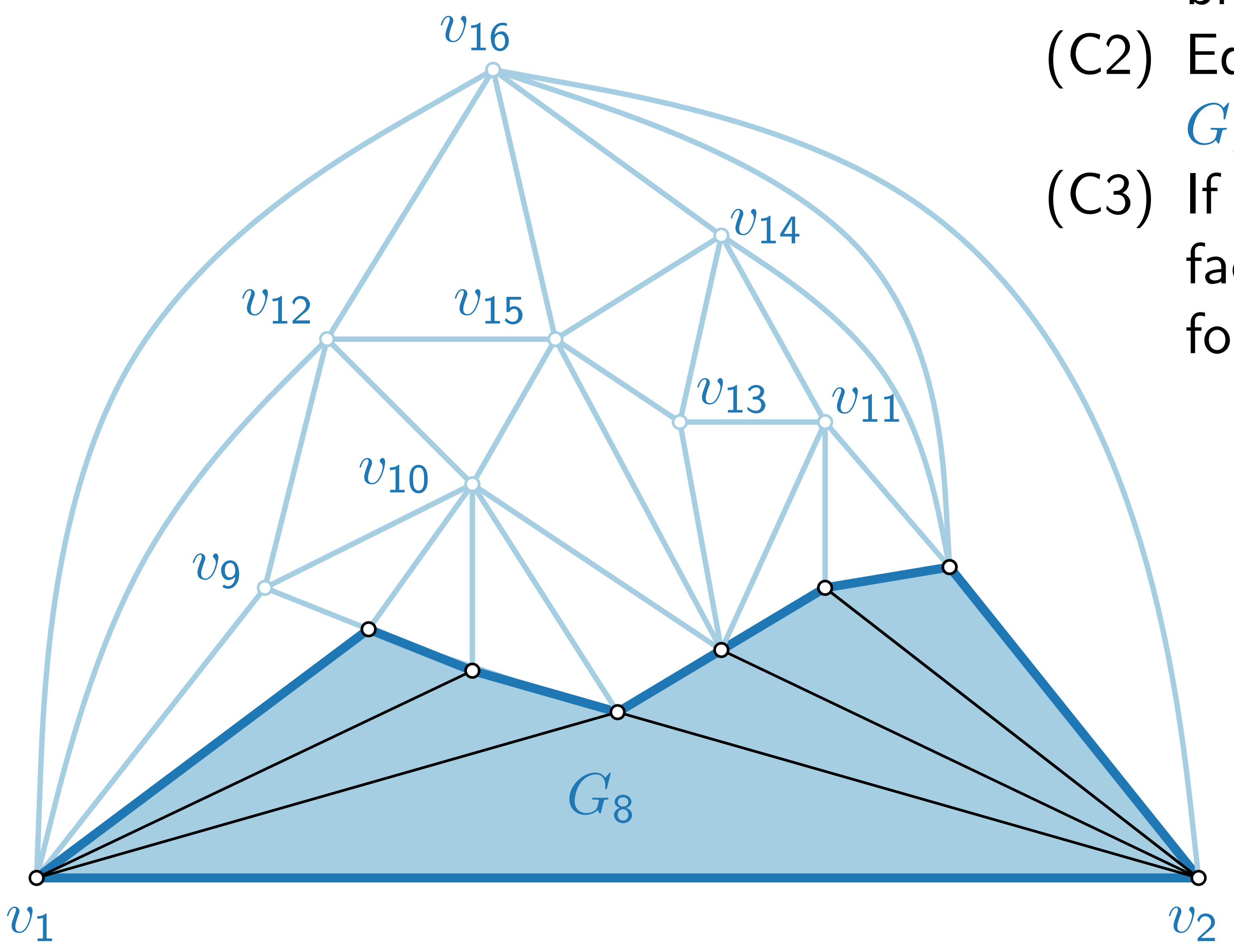

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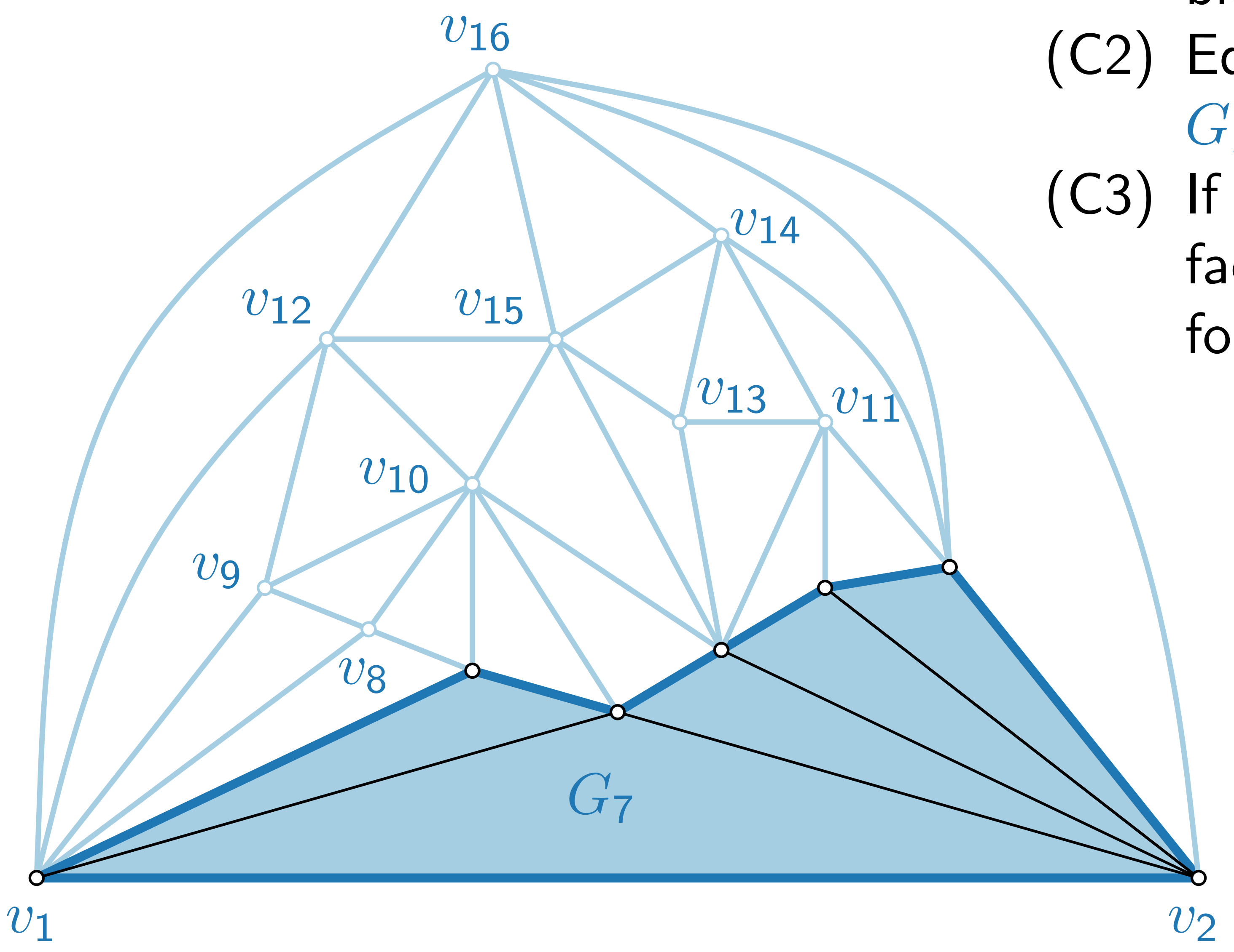

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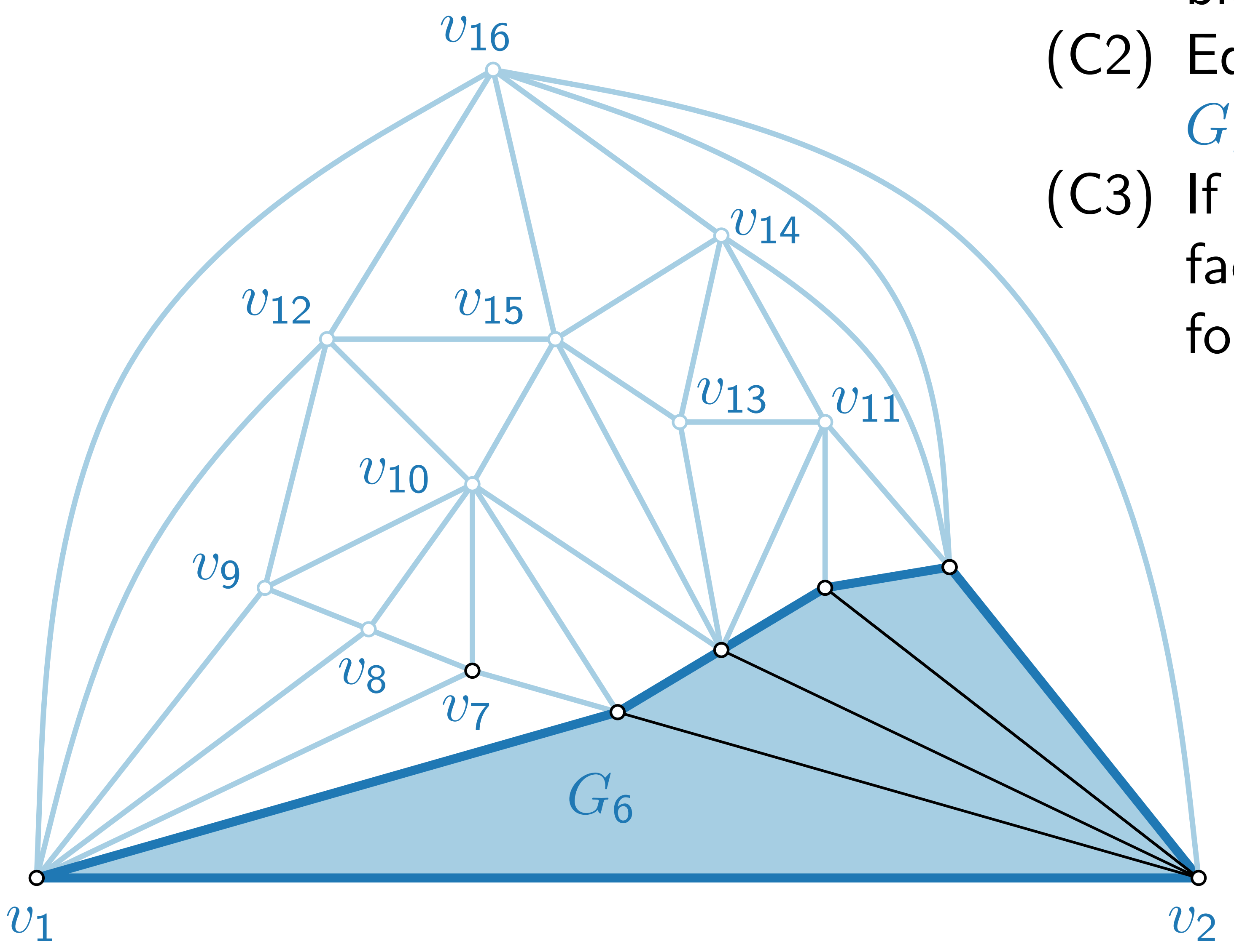

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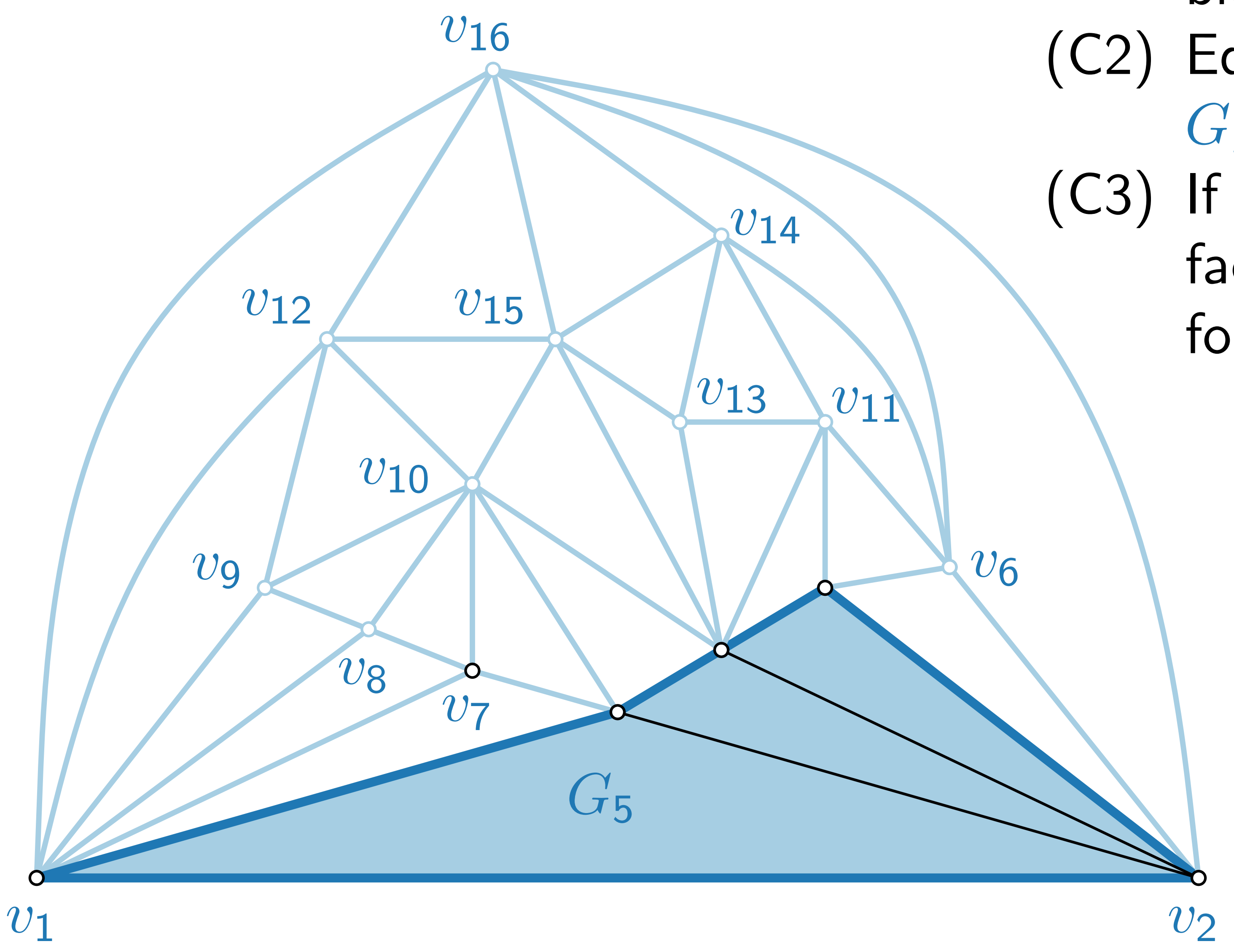

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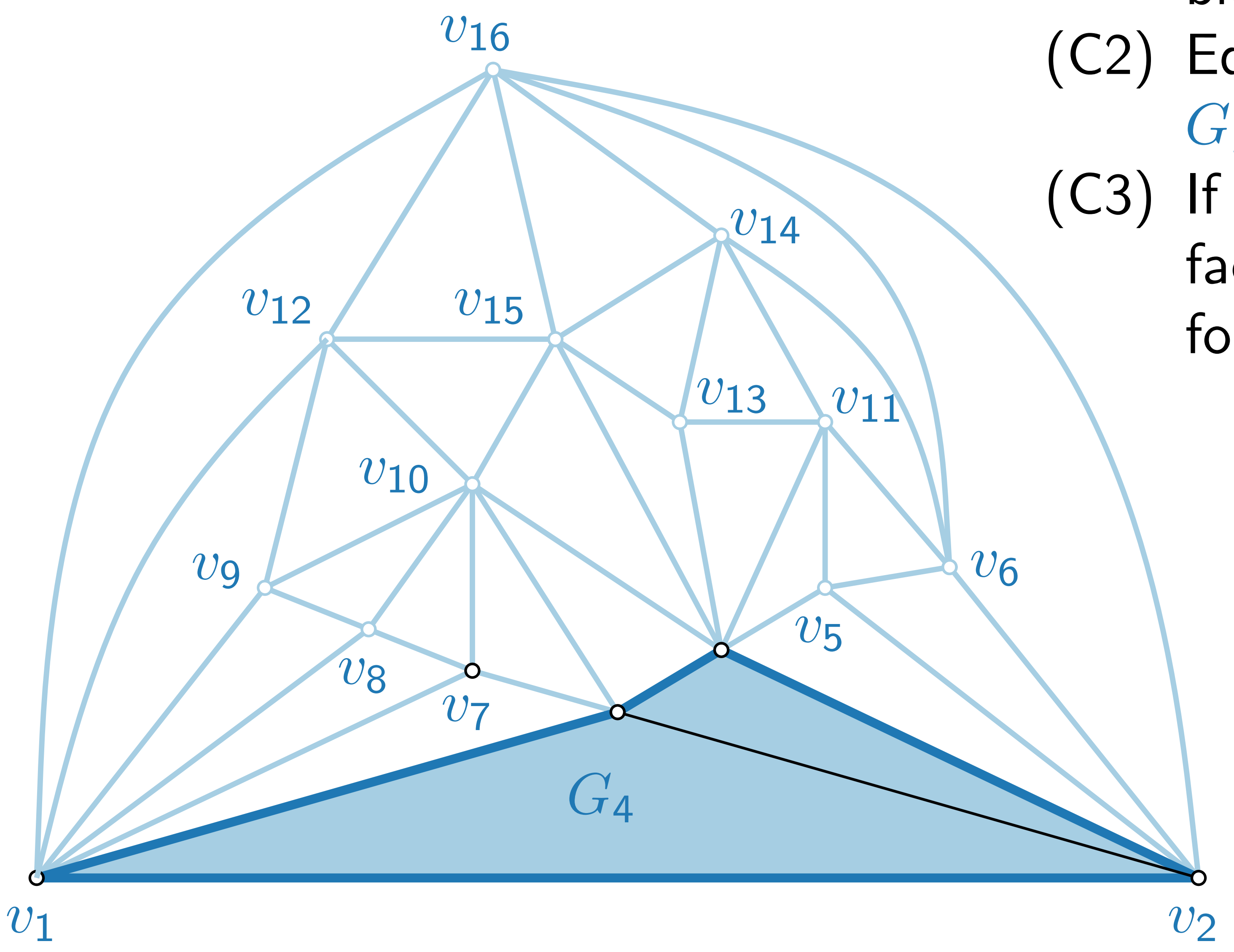

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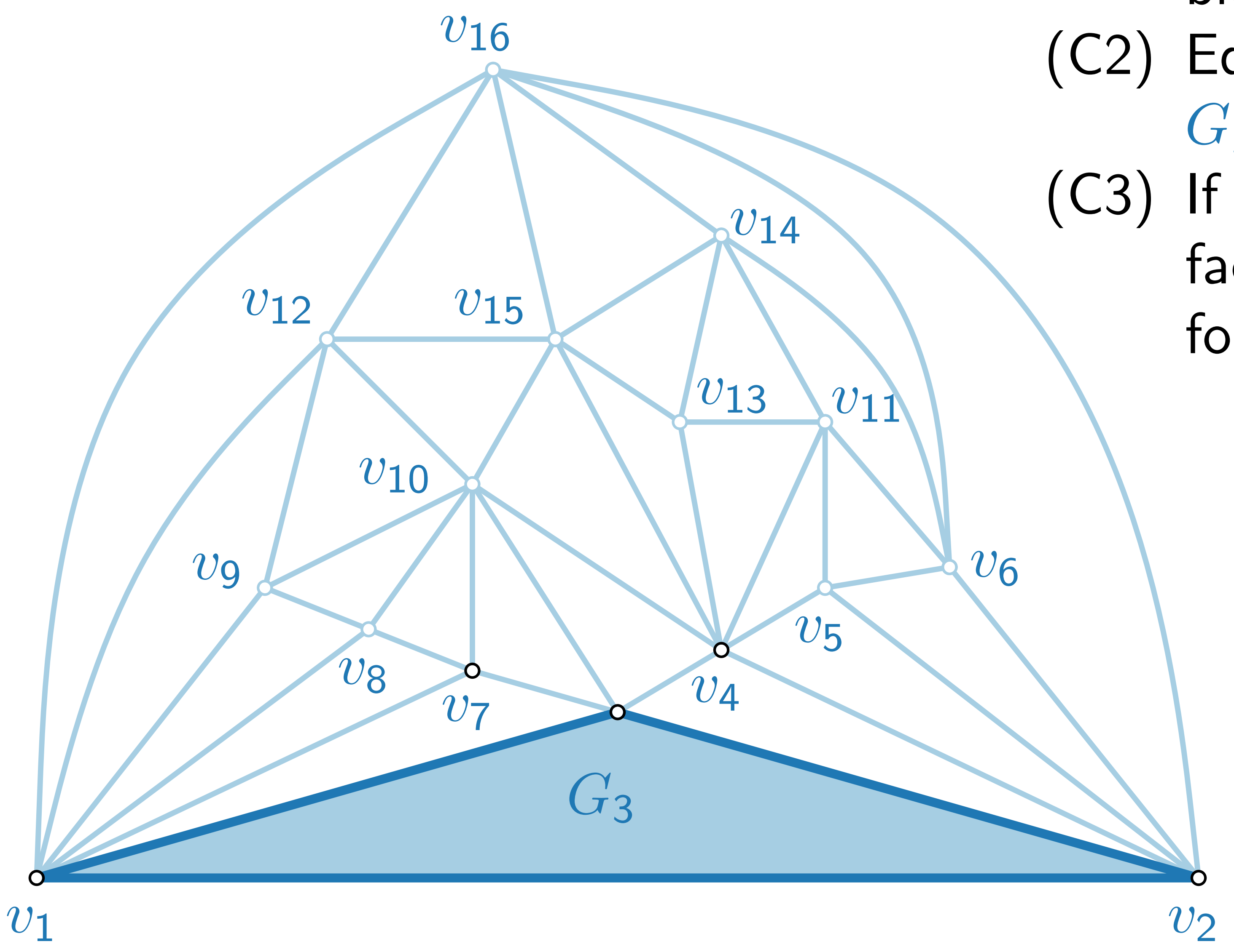

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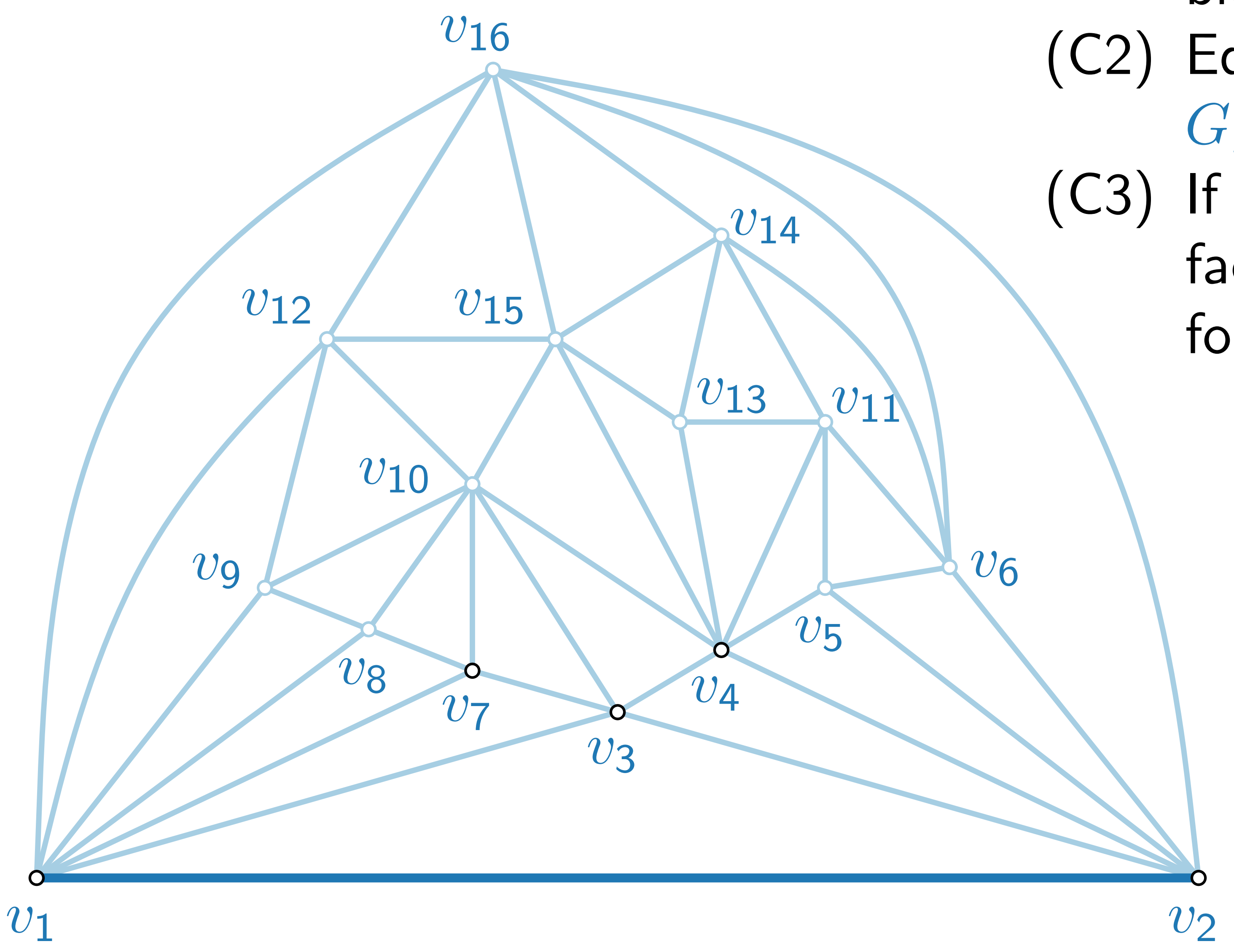

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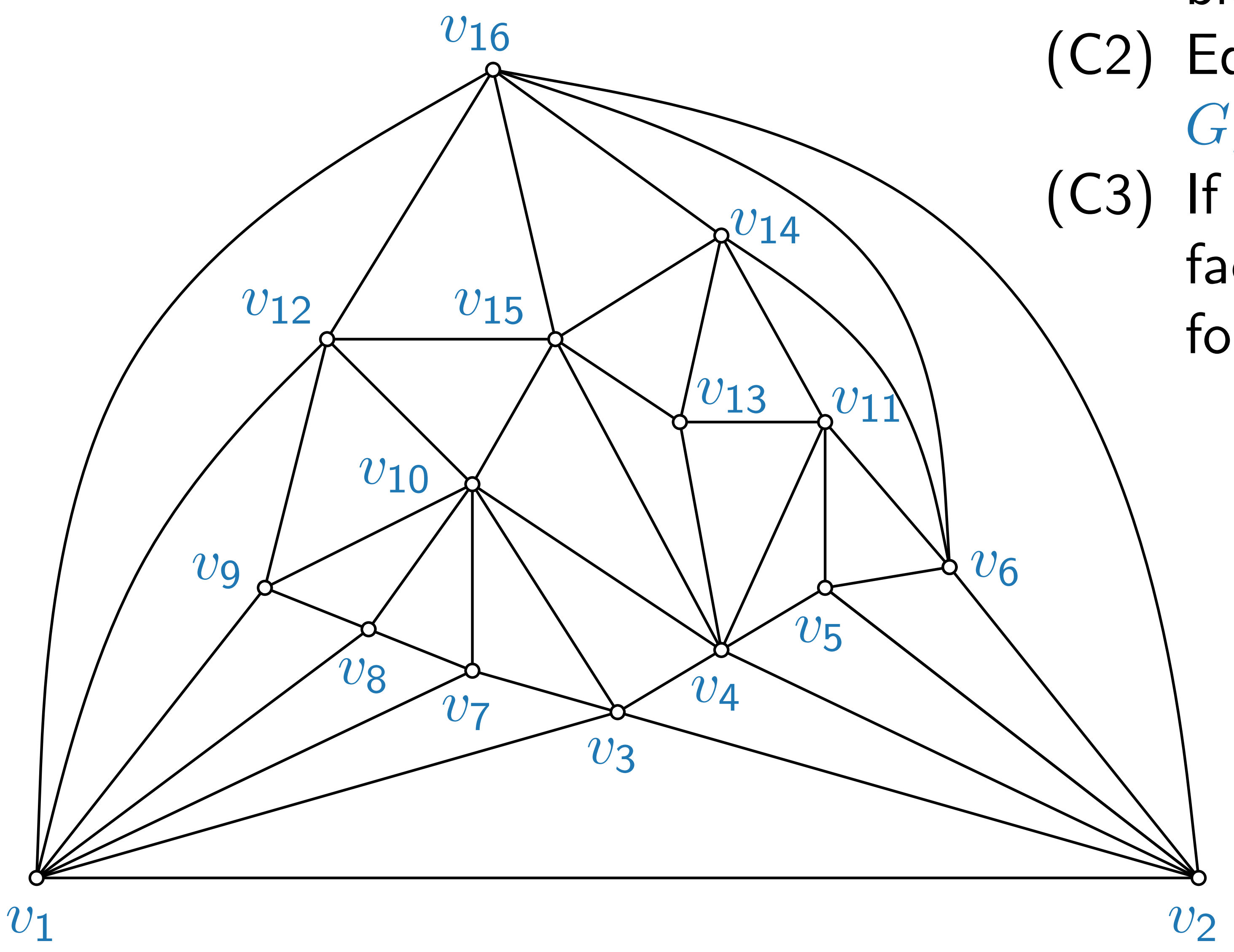

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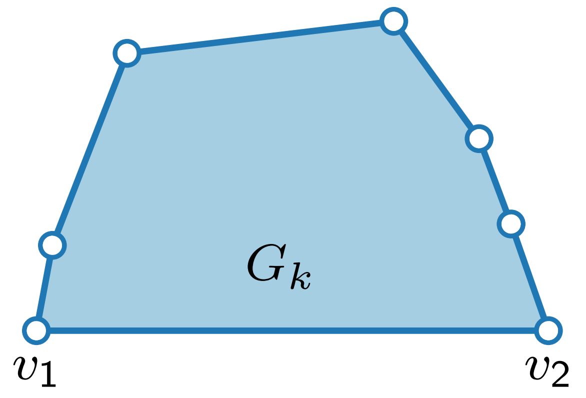

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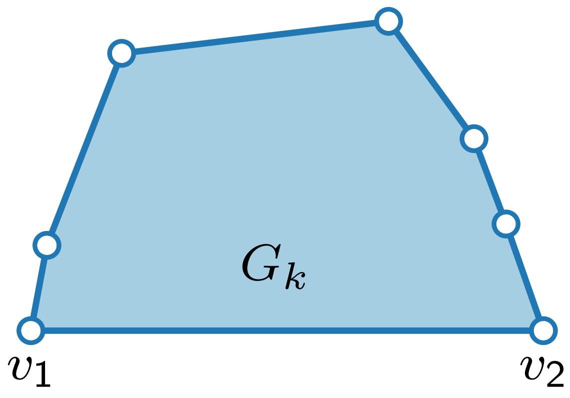

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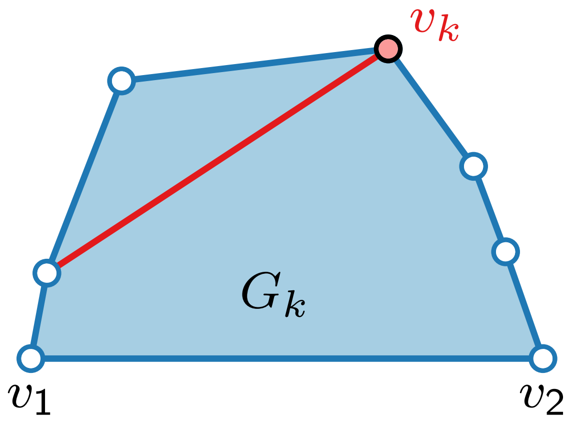

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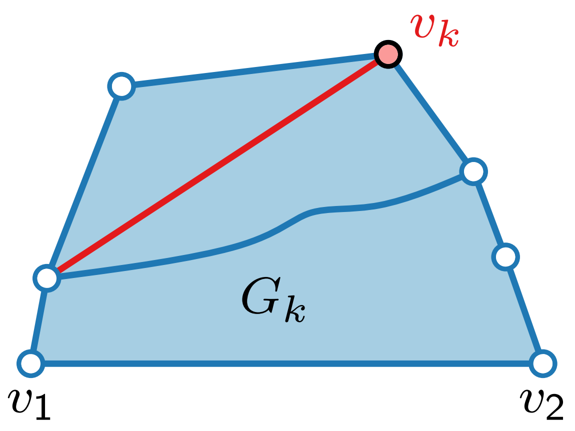

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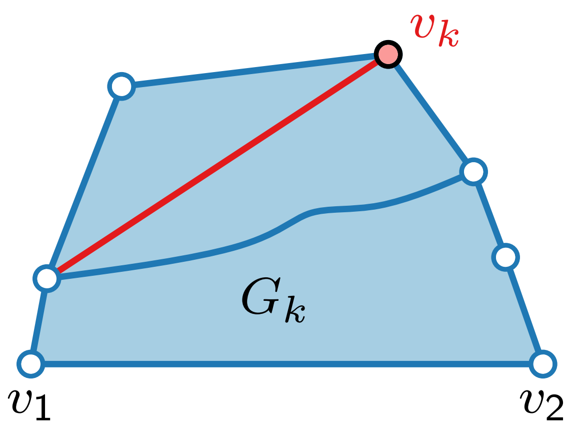

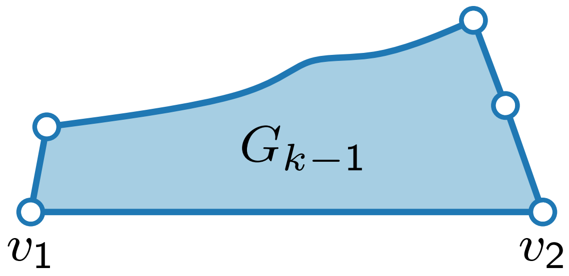

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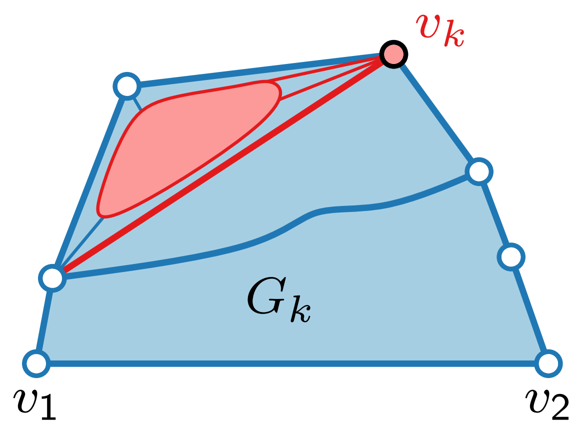

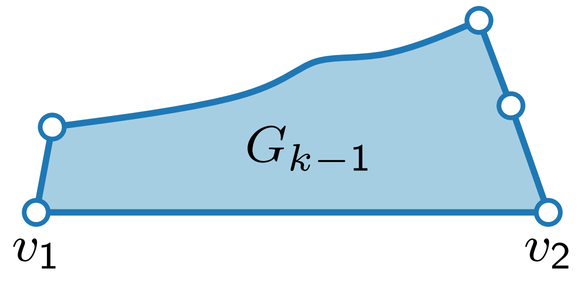

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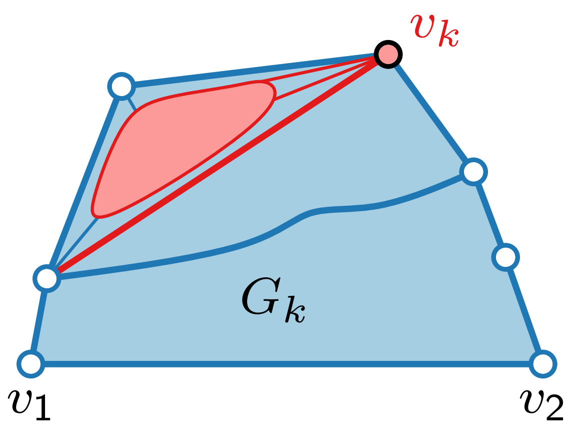

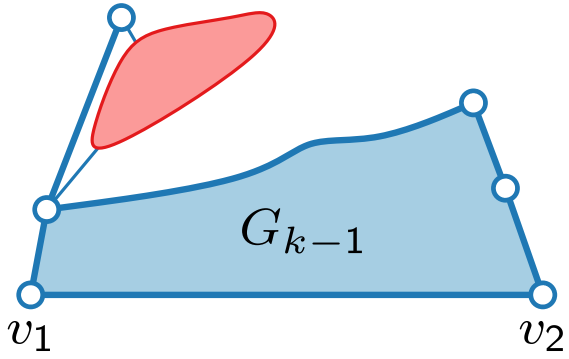

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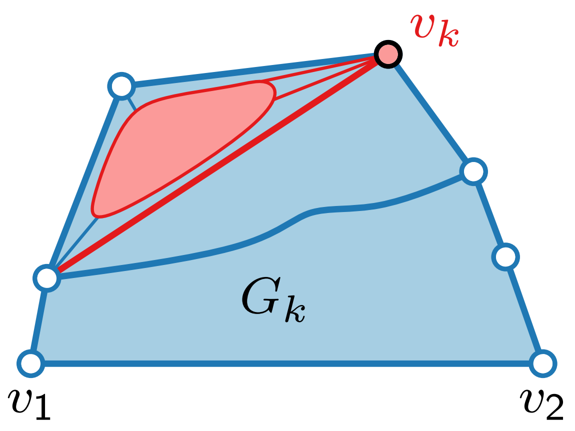

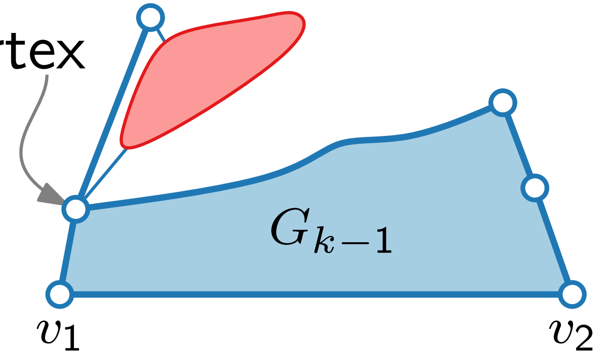

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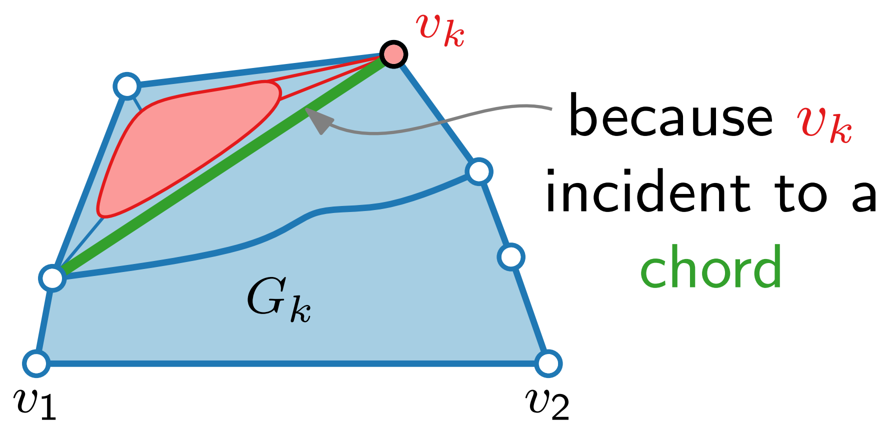

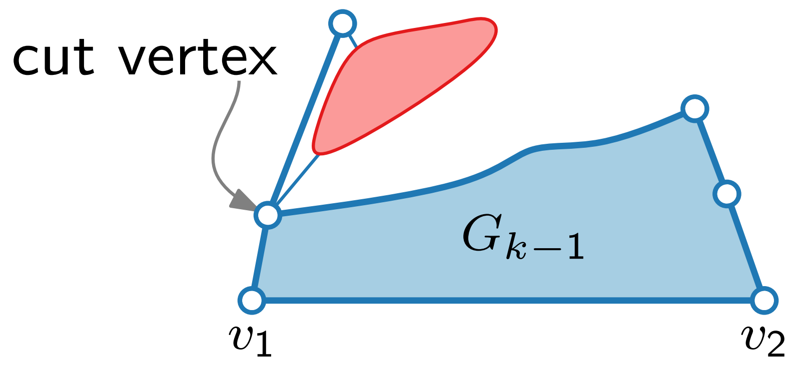

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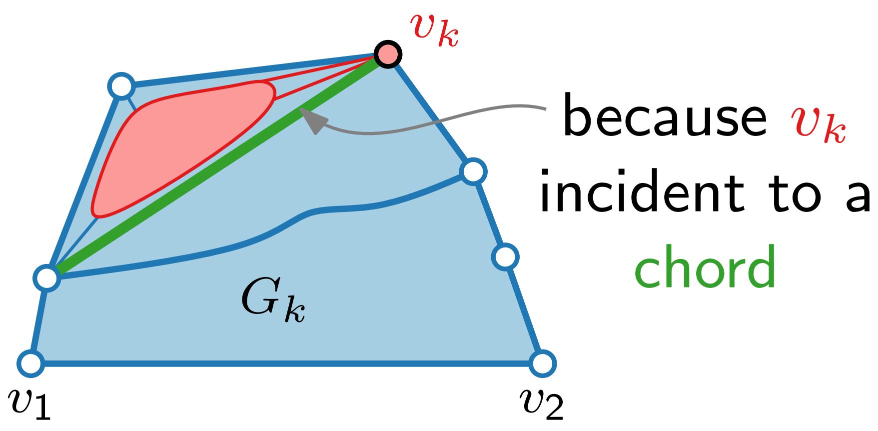

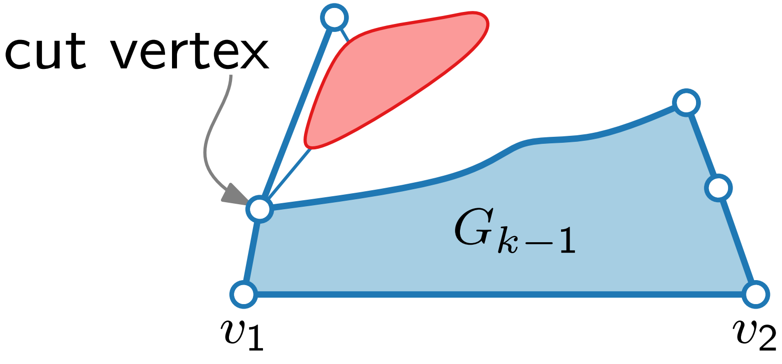

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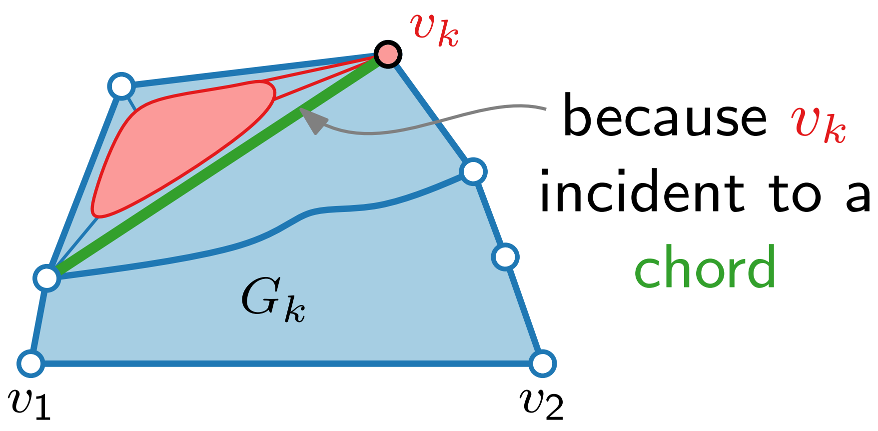

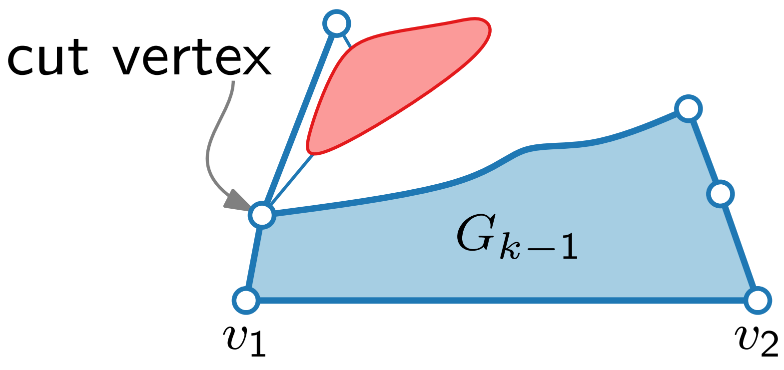

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1. v_k not incident to chord is sufficient.

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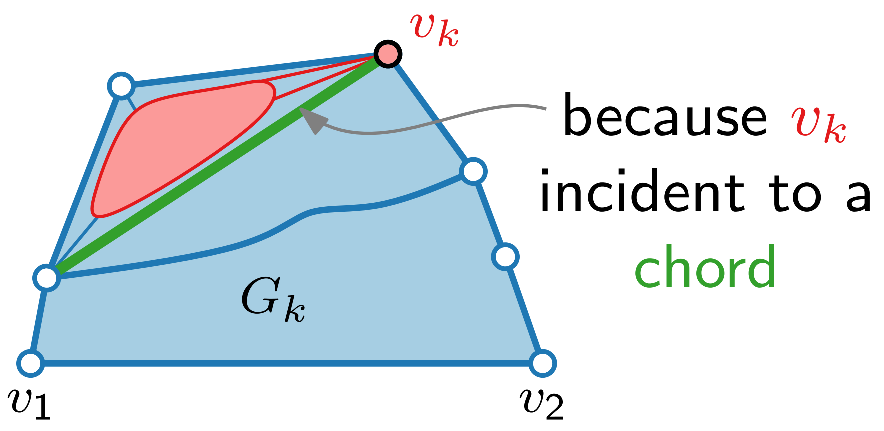
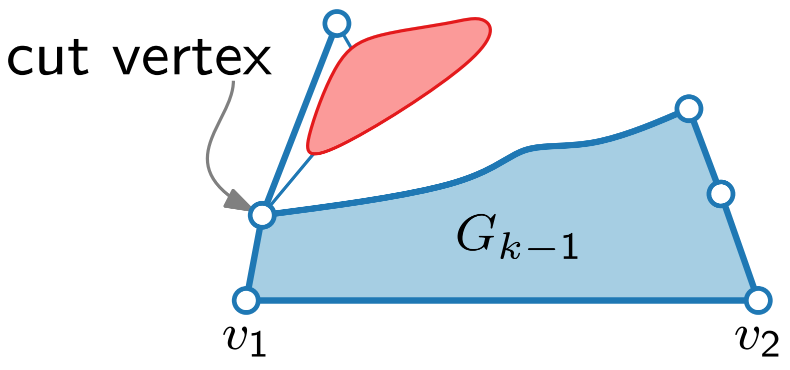
Induction base $(k = n)$: Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions (C1)–(C3) hold.

Induction hypothesis: Vertices $v_{n-1}, \dots, v_{k+1}$ have been chosen such that conditions (C1)–(C3) hold for every $i \in \{k+1, \dots, n\}$.

Induction step: Consider G_k. We search for v_k.





We need to show:

1. v_k not incident to chord is sufficient.
2. Such v_k exists.

Canonical Order – Existence

Claim 1.
If v_k is not incident to a chord,
then G_{k-1} is biconnected.

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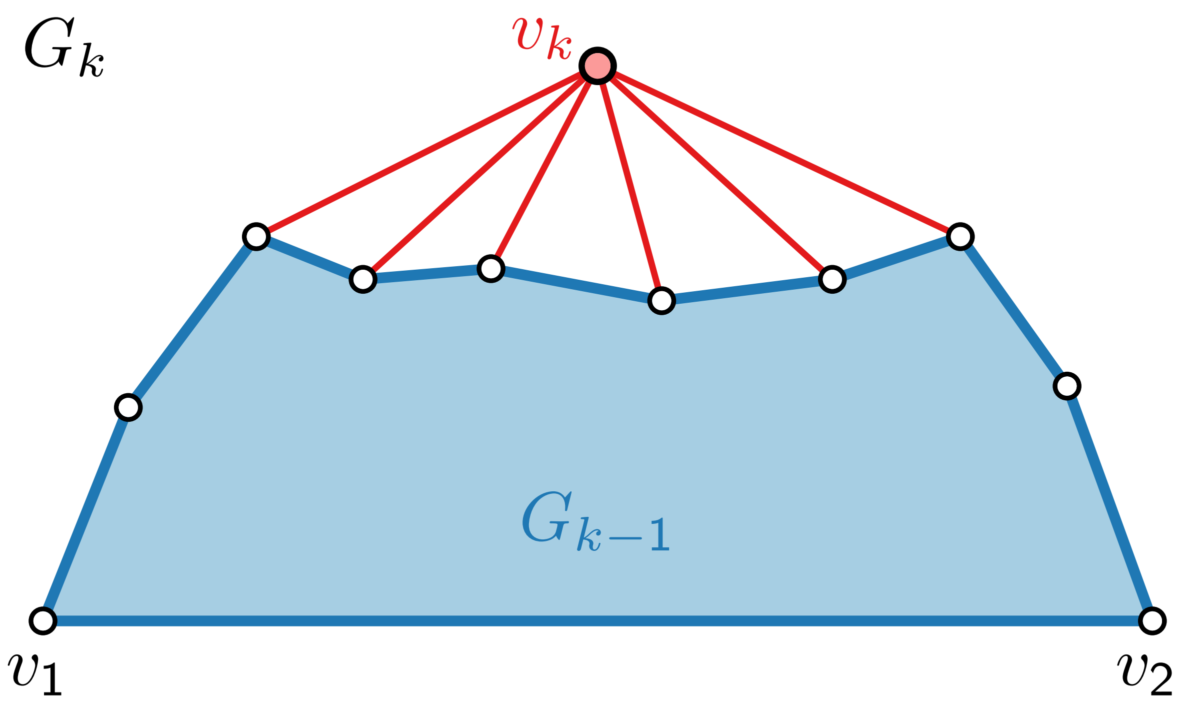

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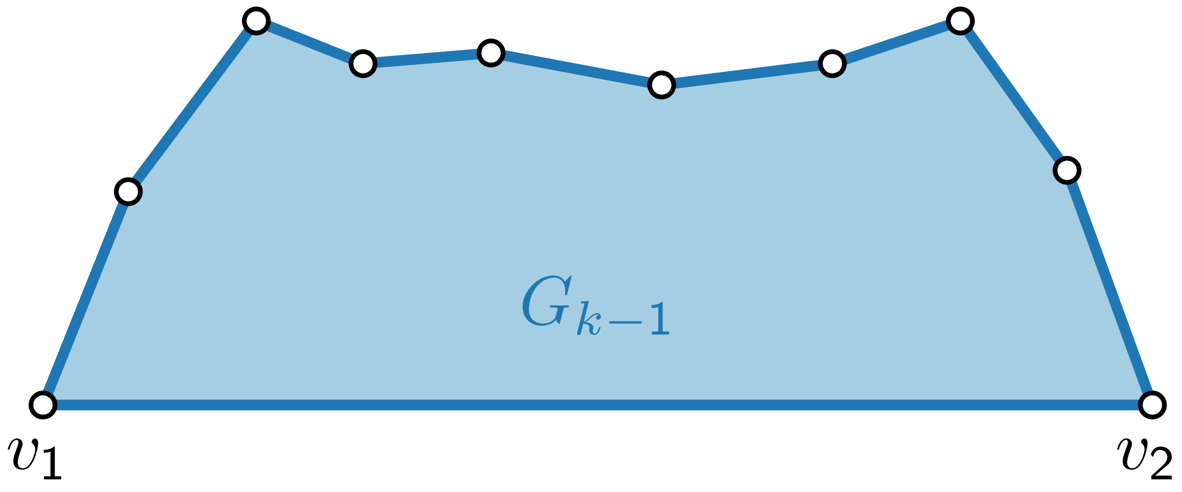

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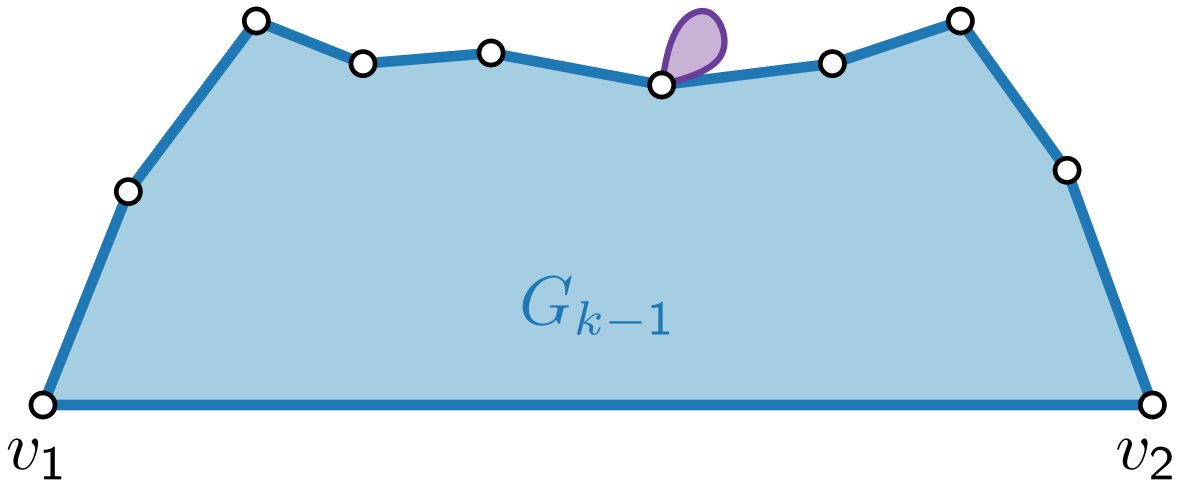

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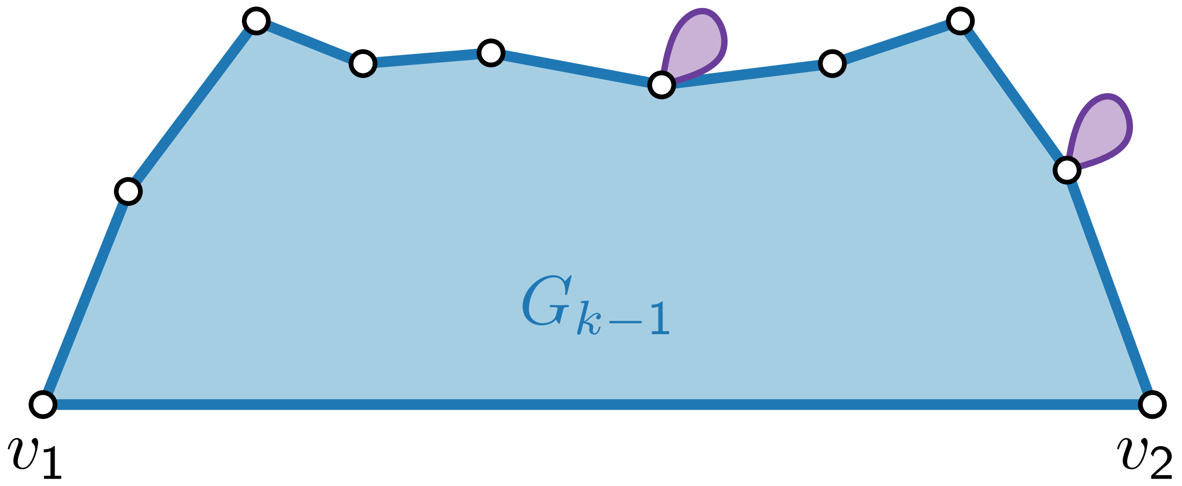

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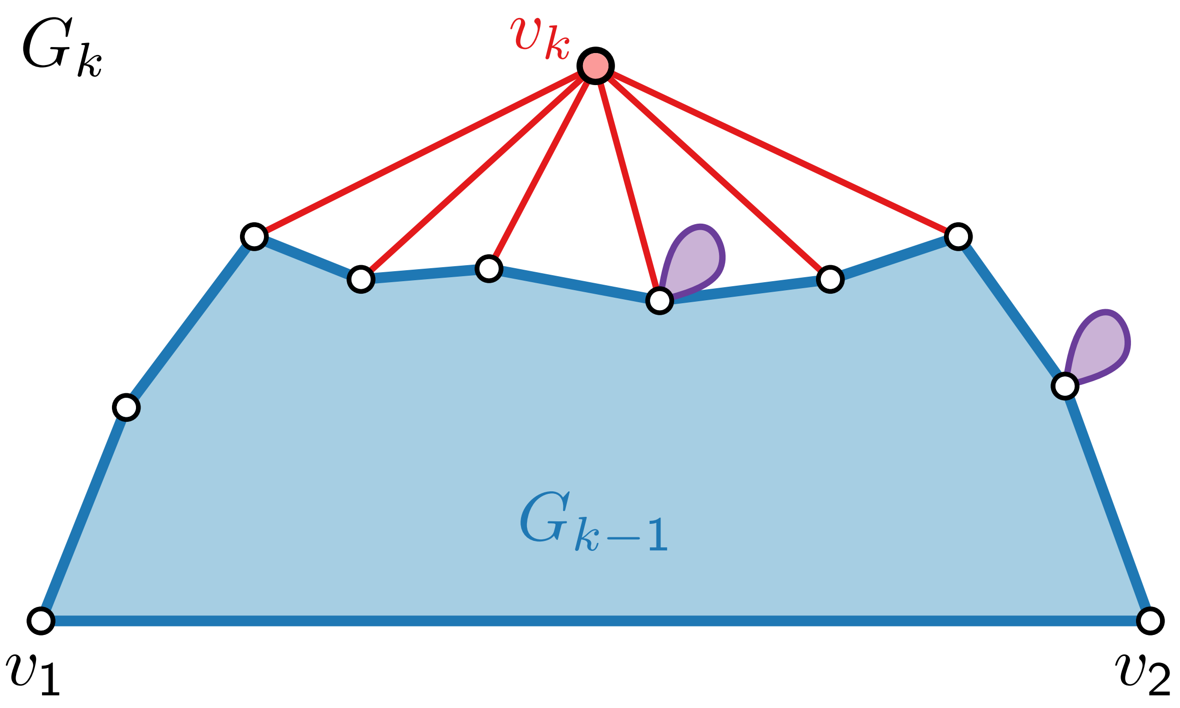

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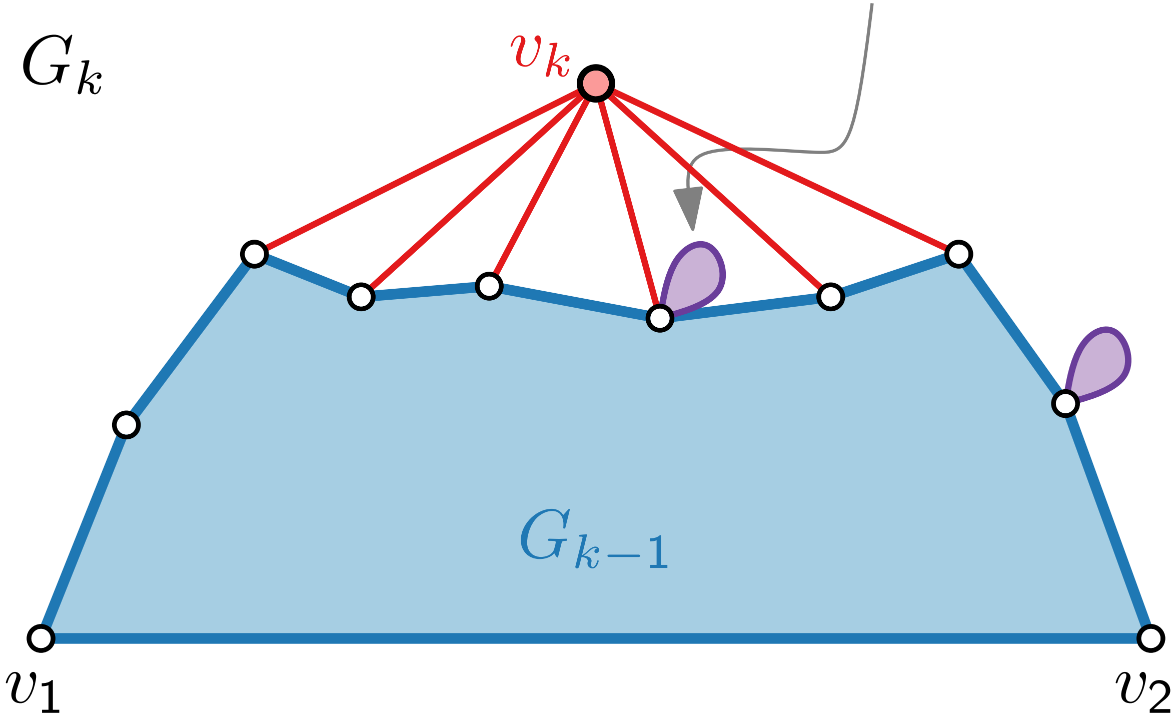
Contradiction to neighbors of v_k forming a path on ∂G_{k-1}!



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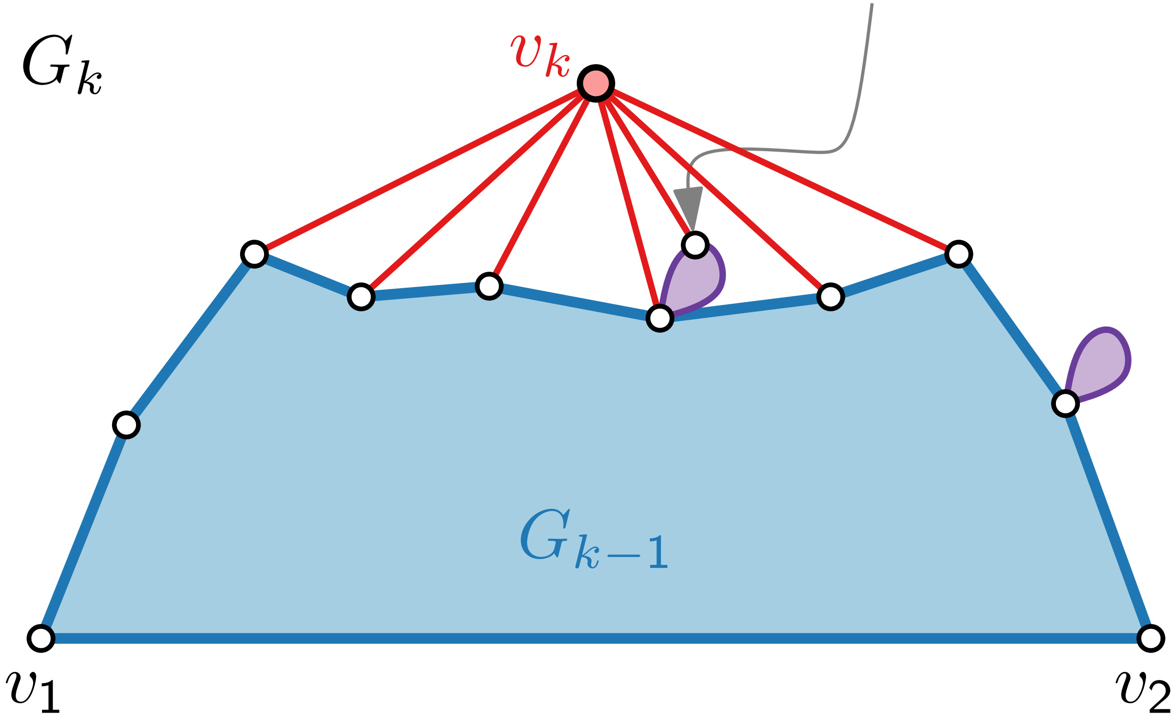

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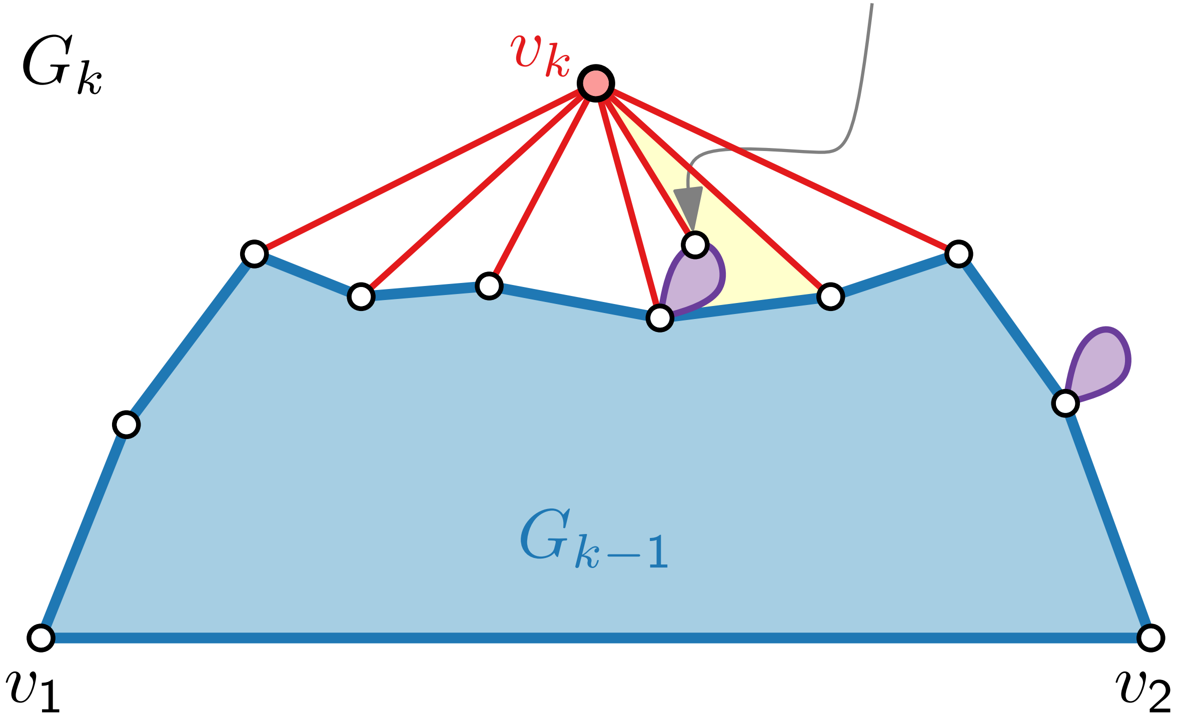

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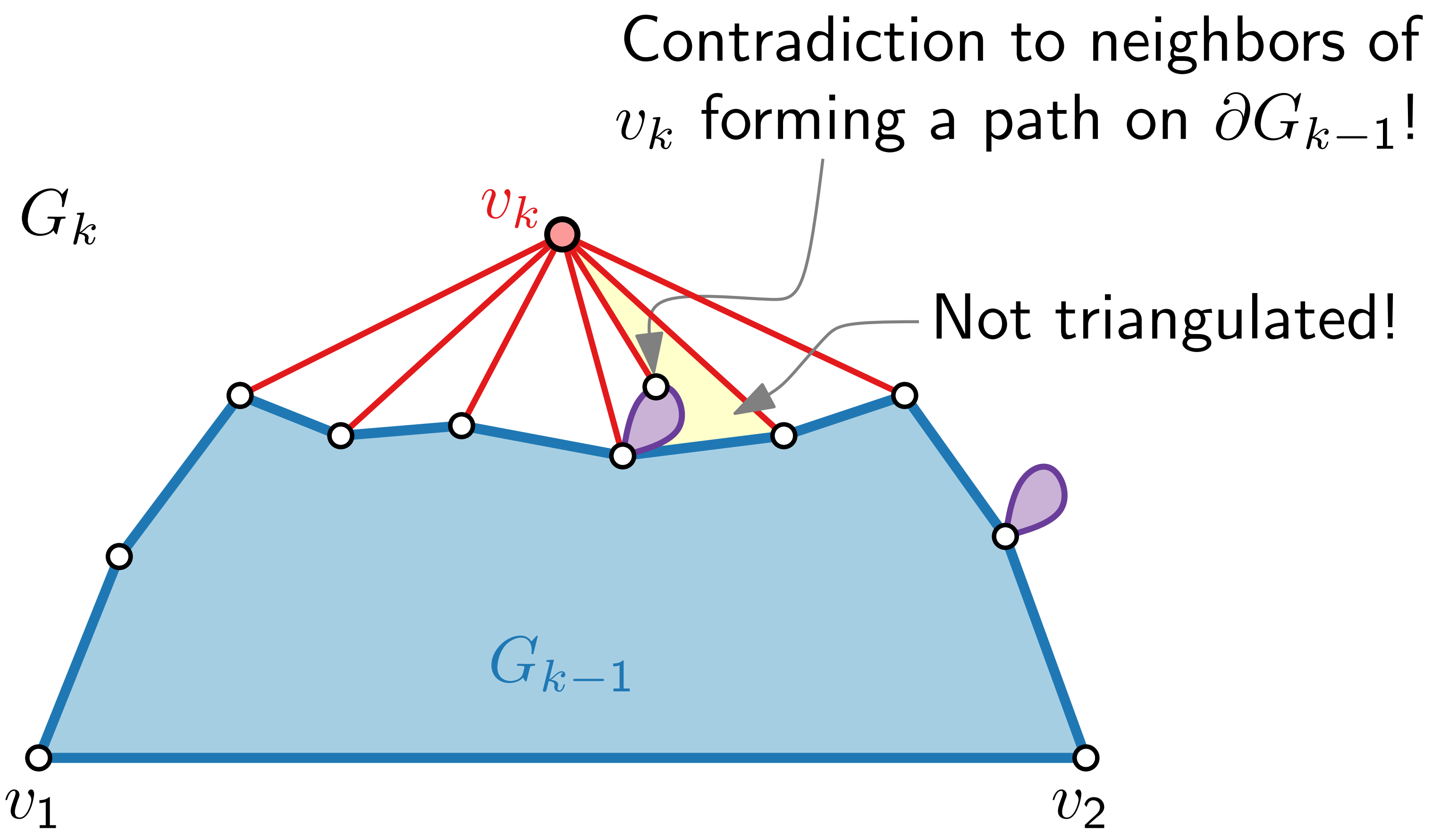

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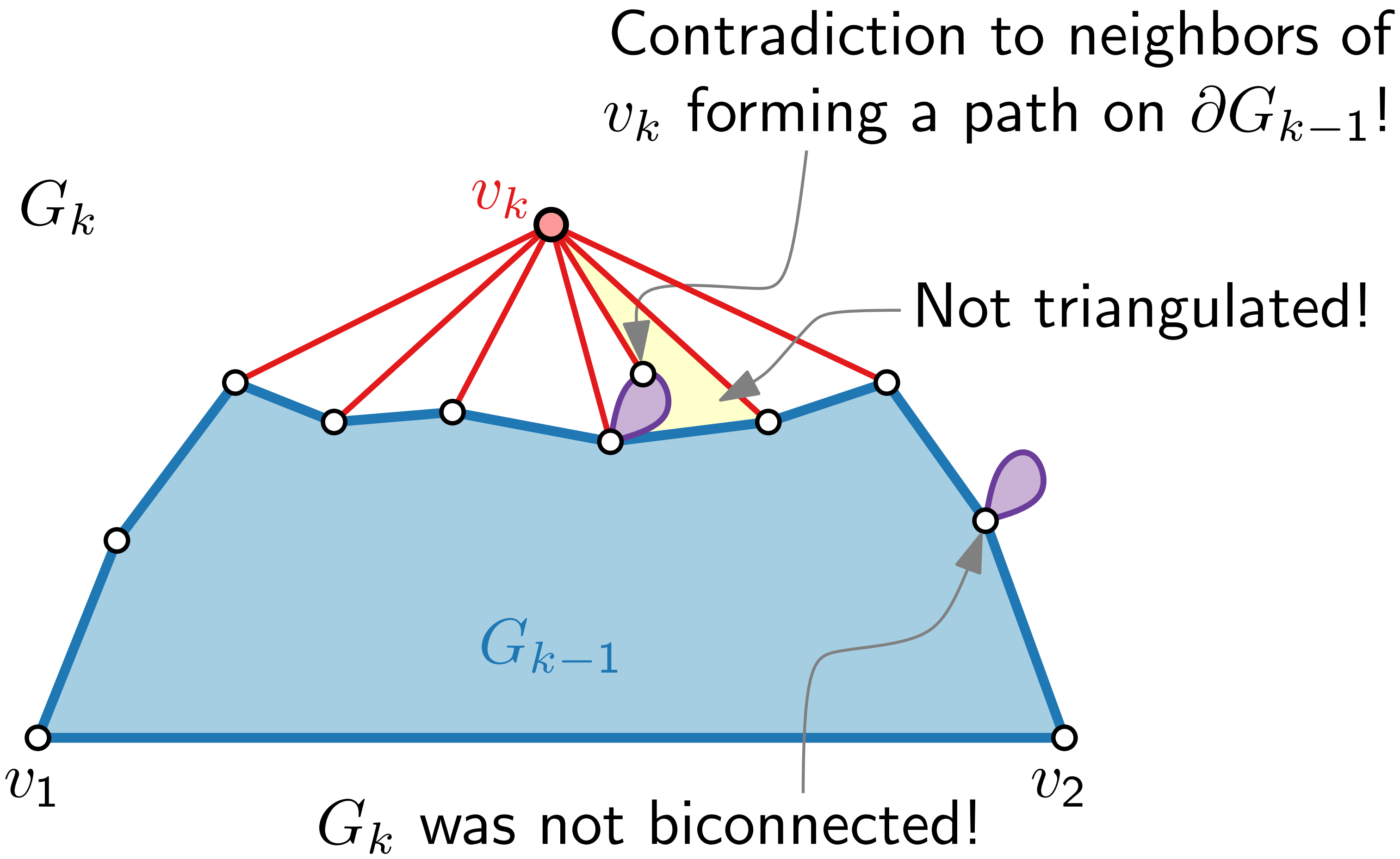

Canonical Order – Existence

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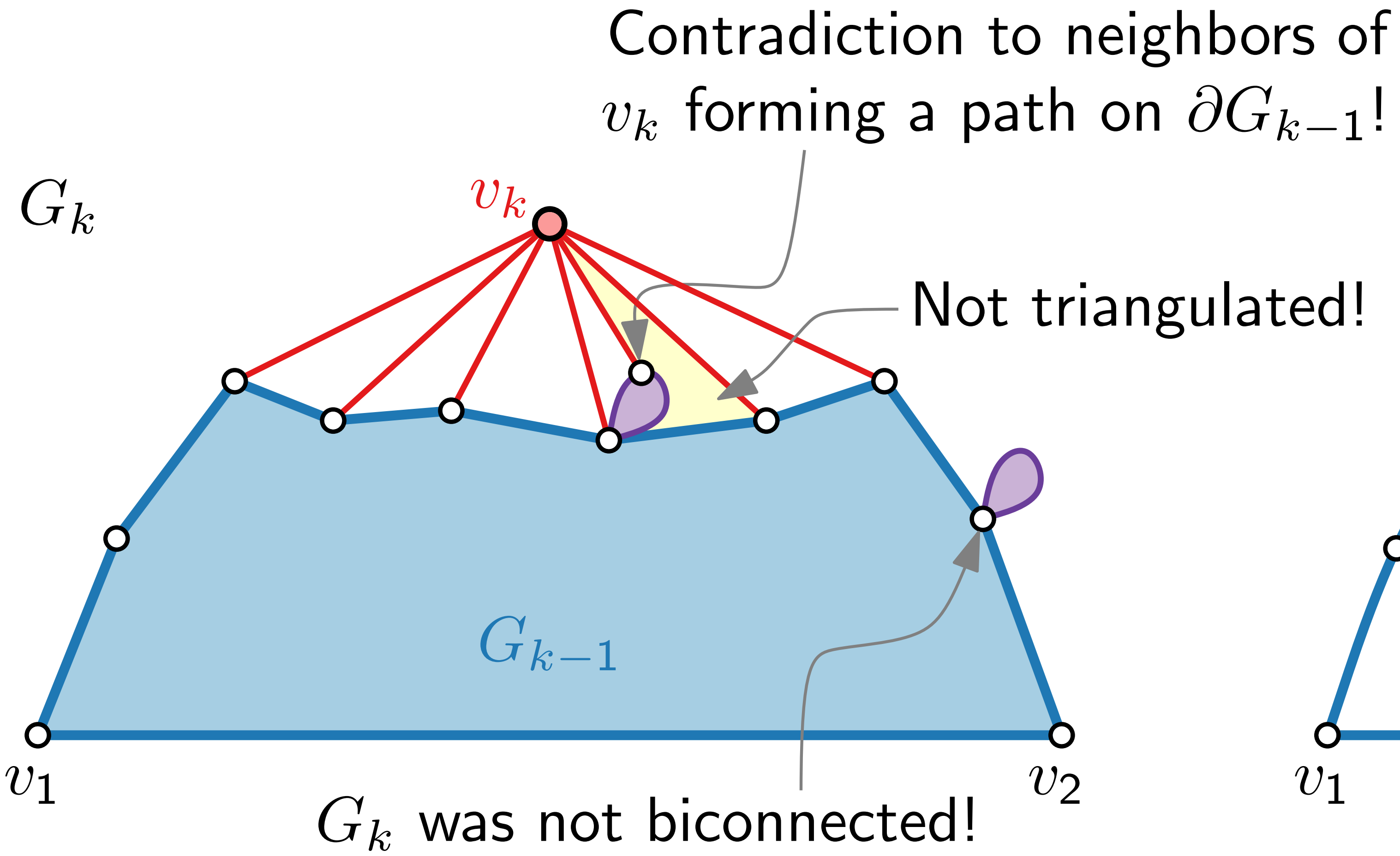

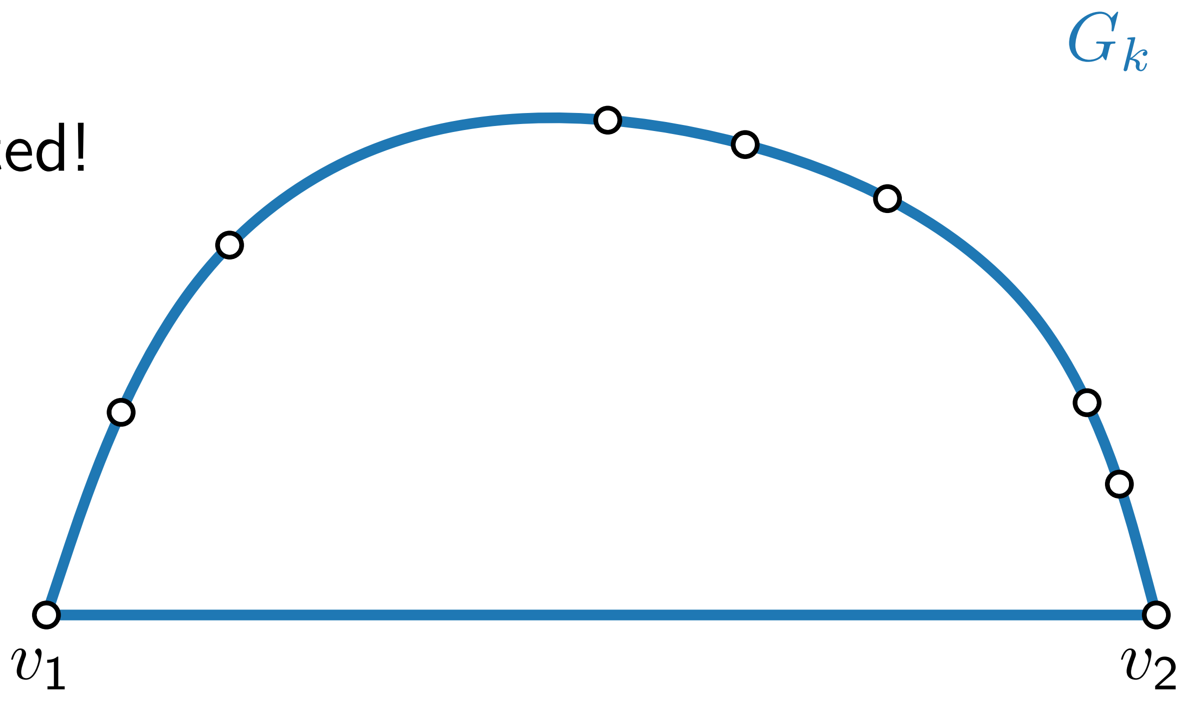

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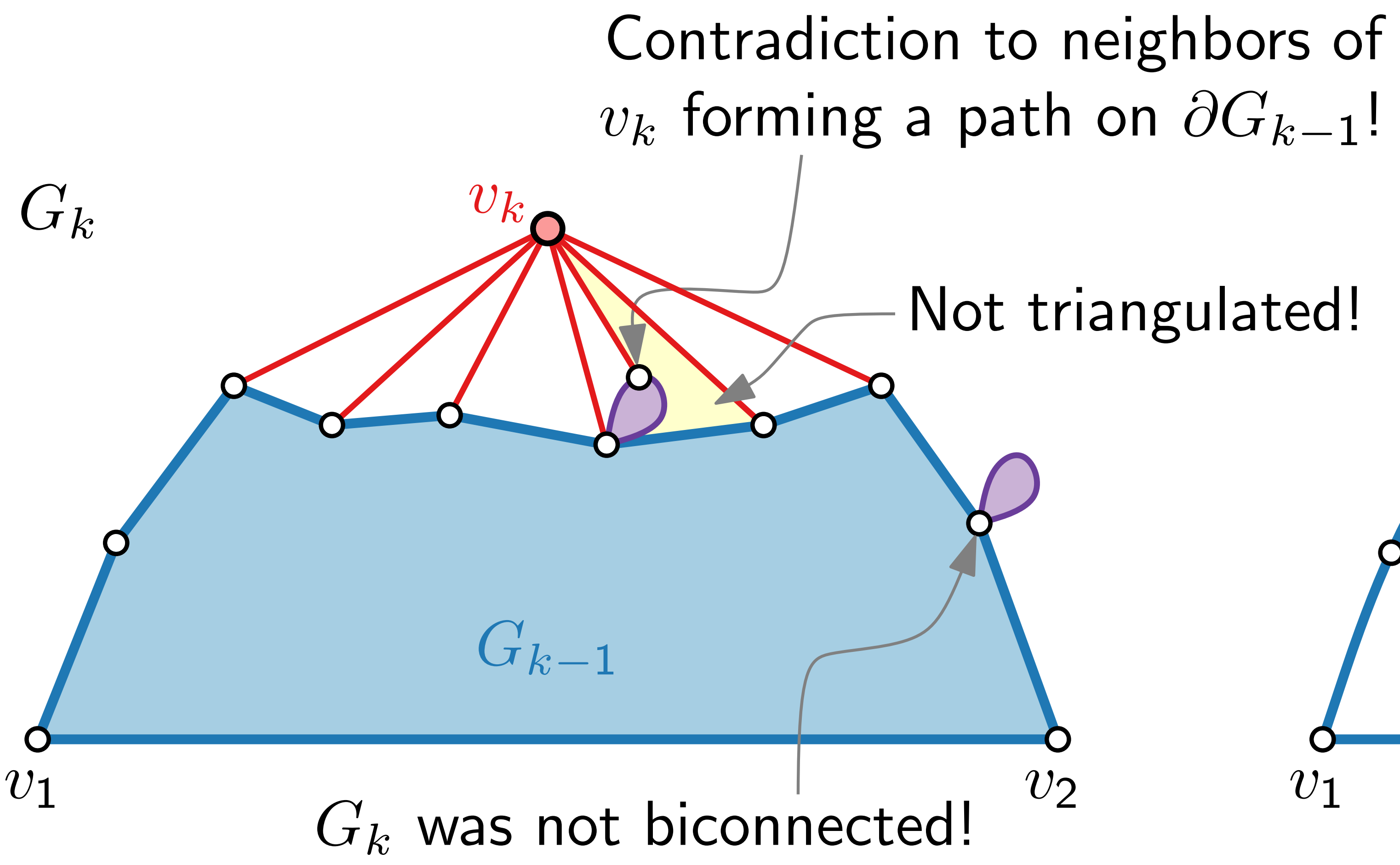

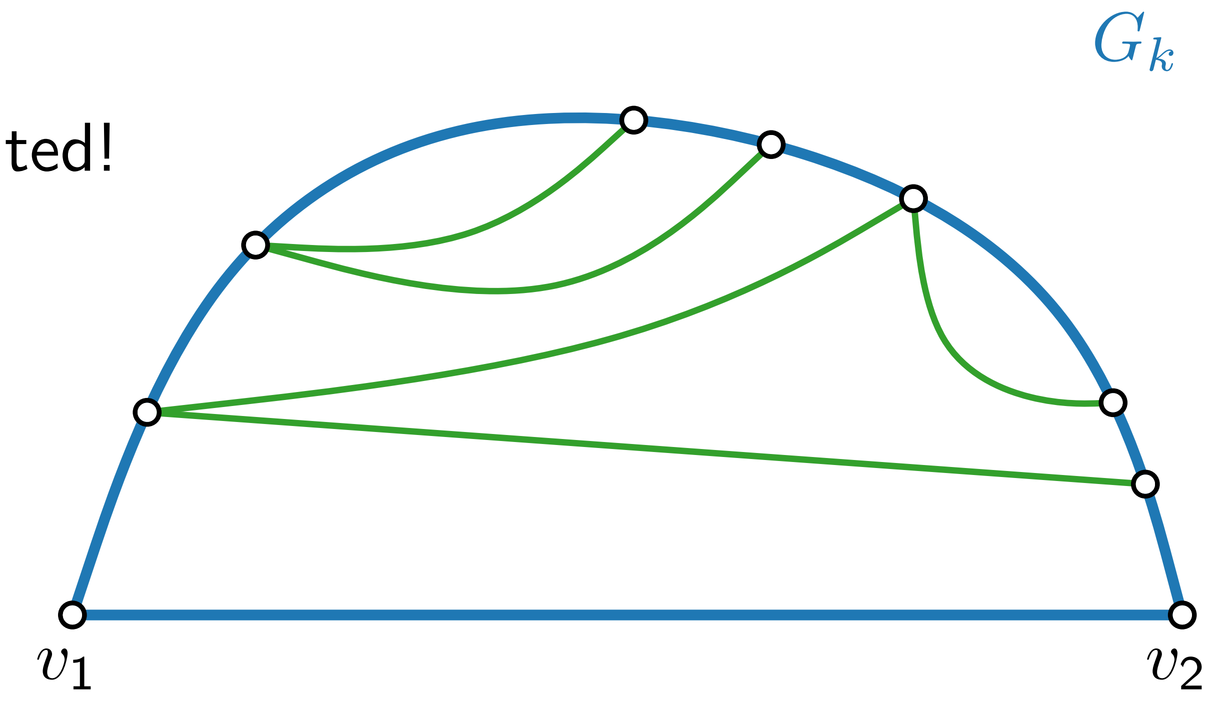

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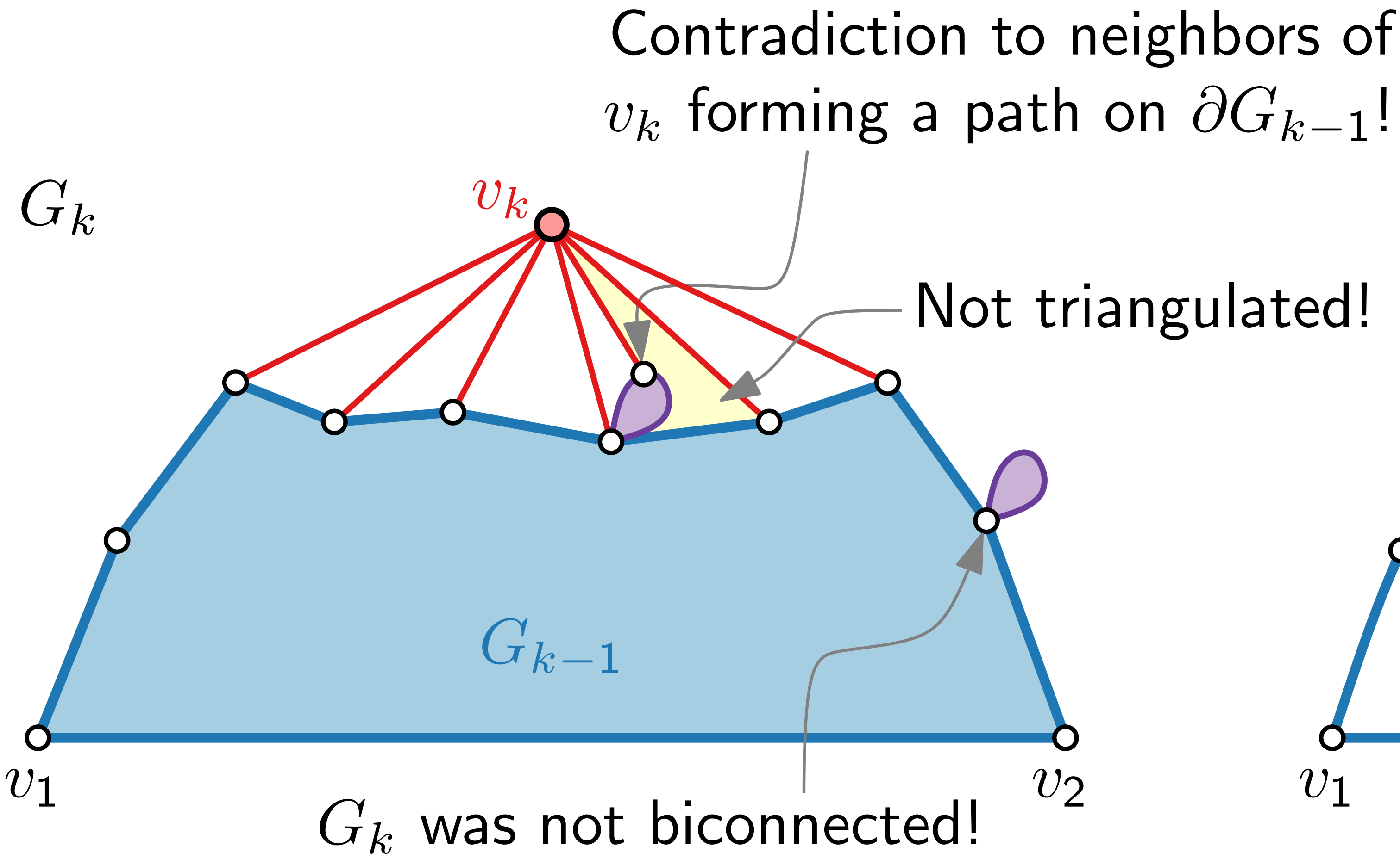

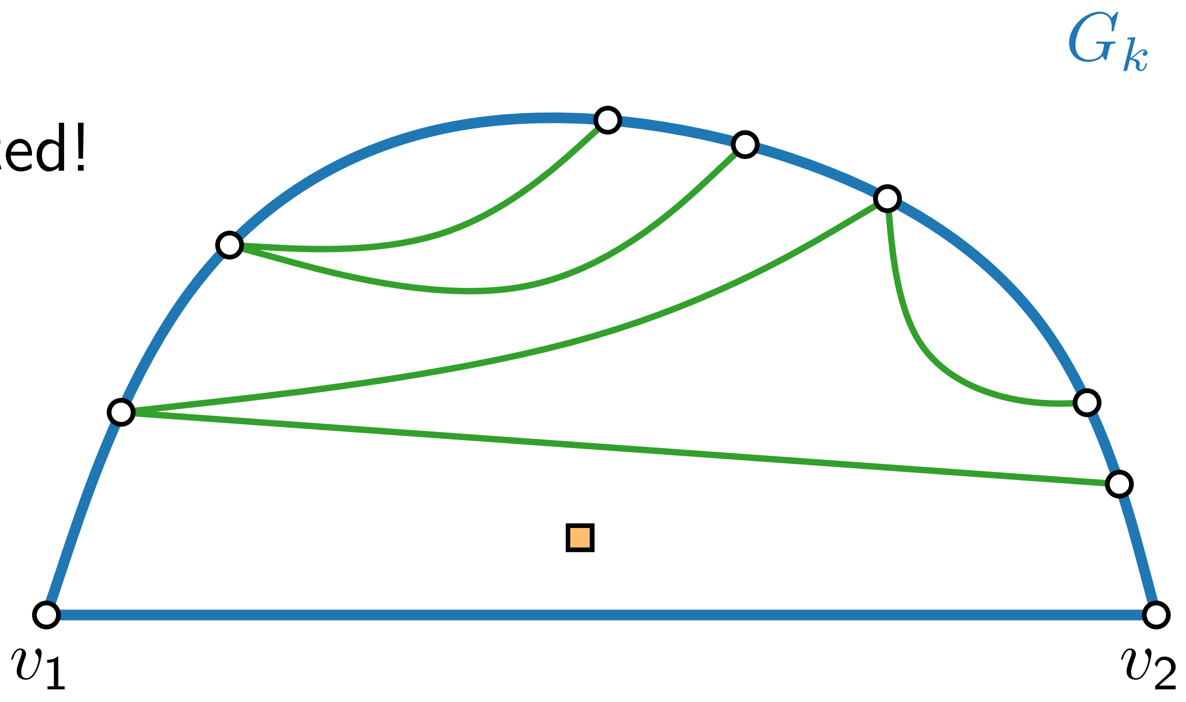

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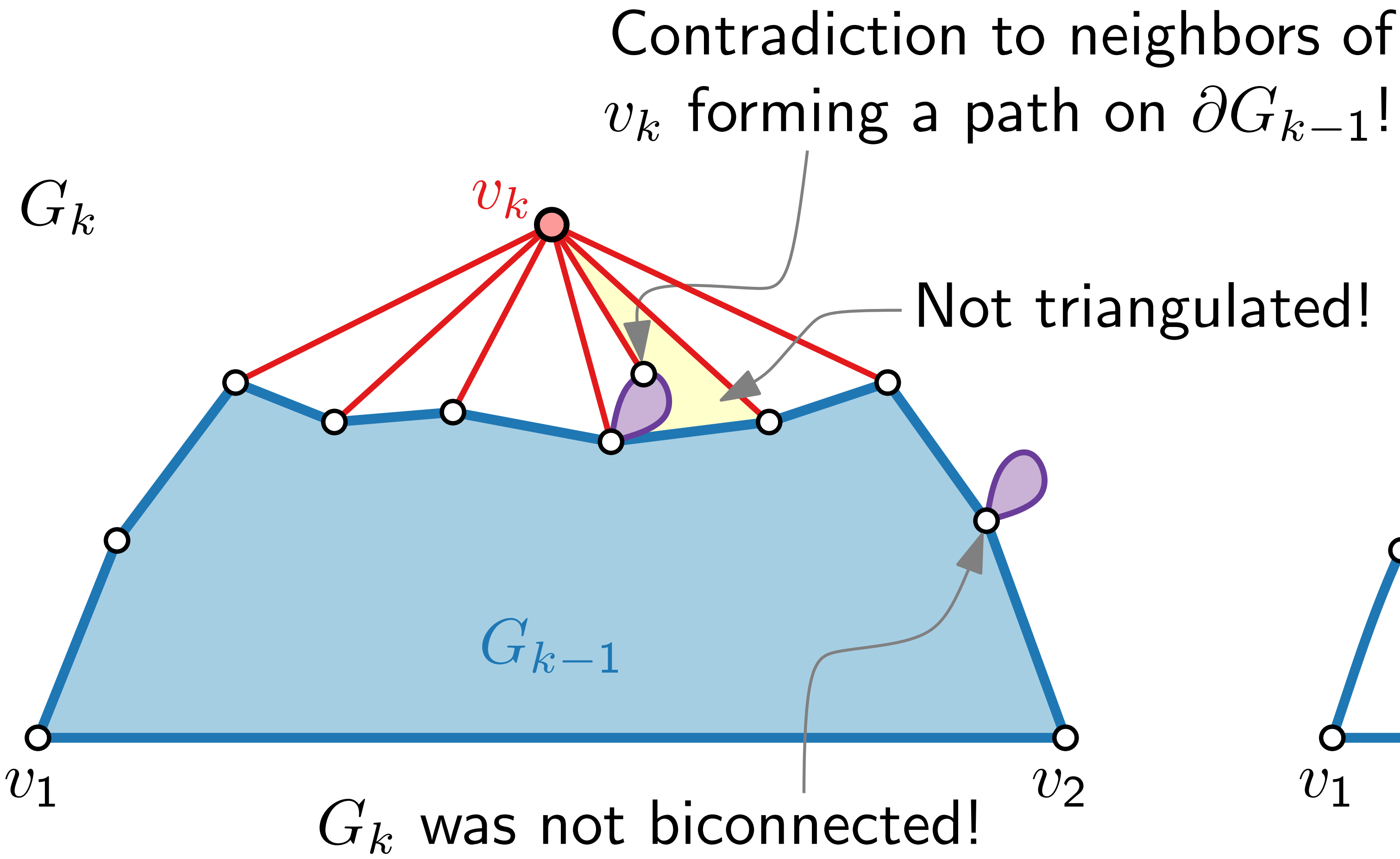

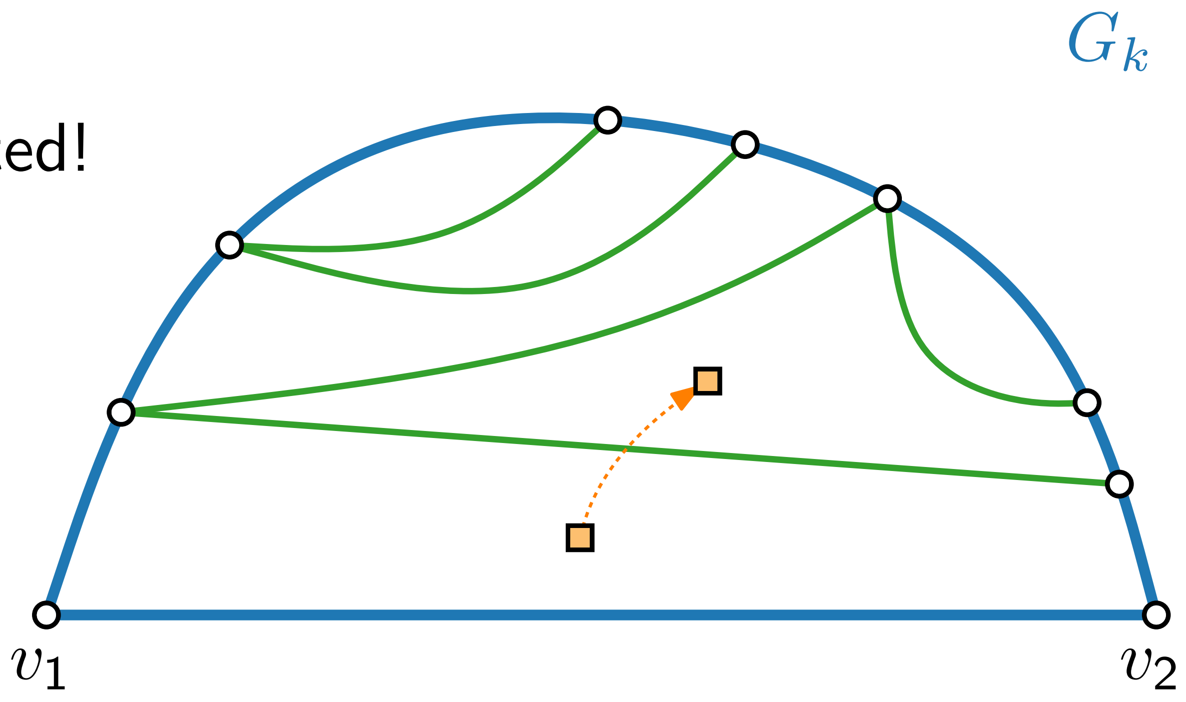

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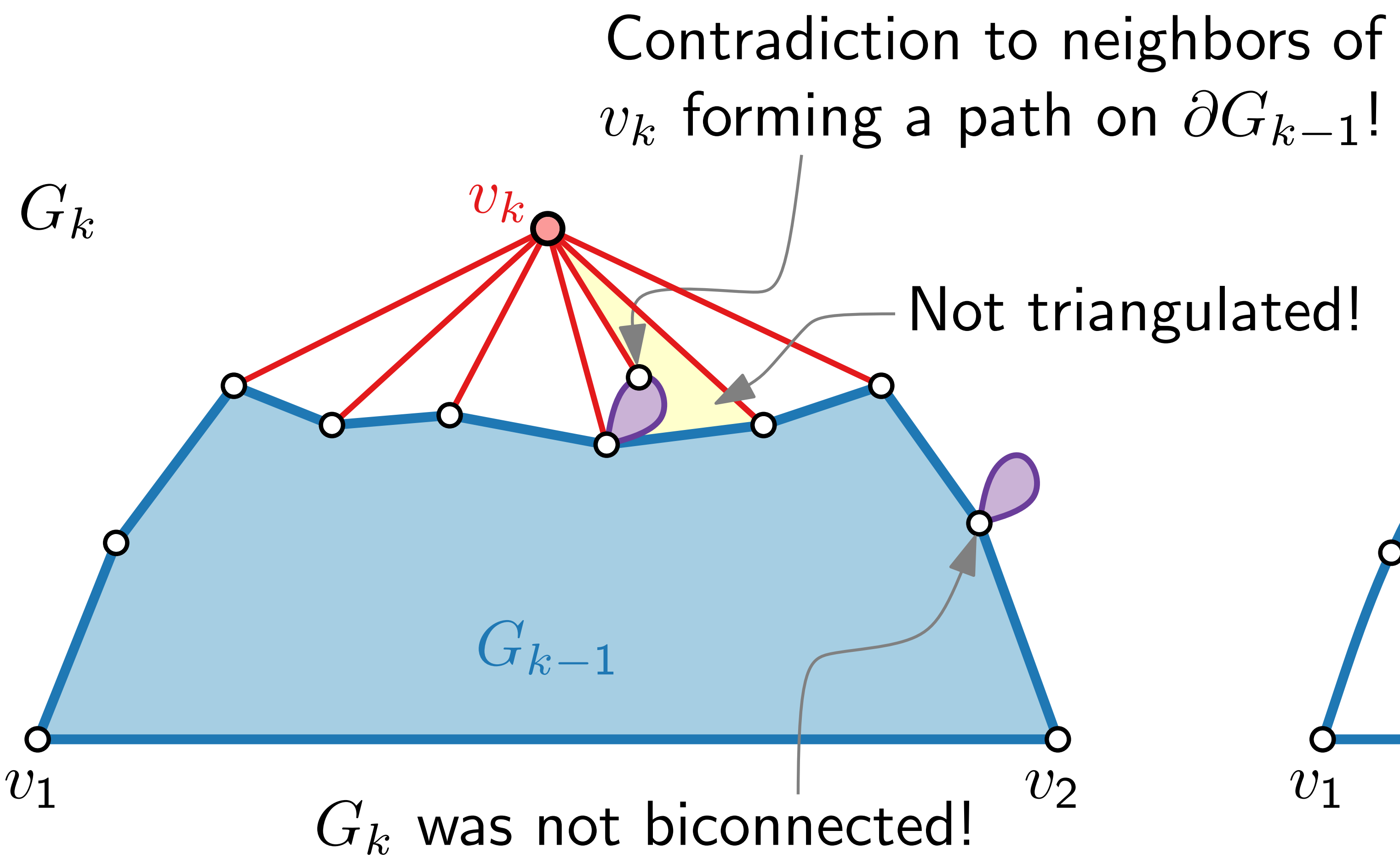

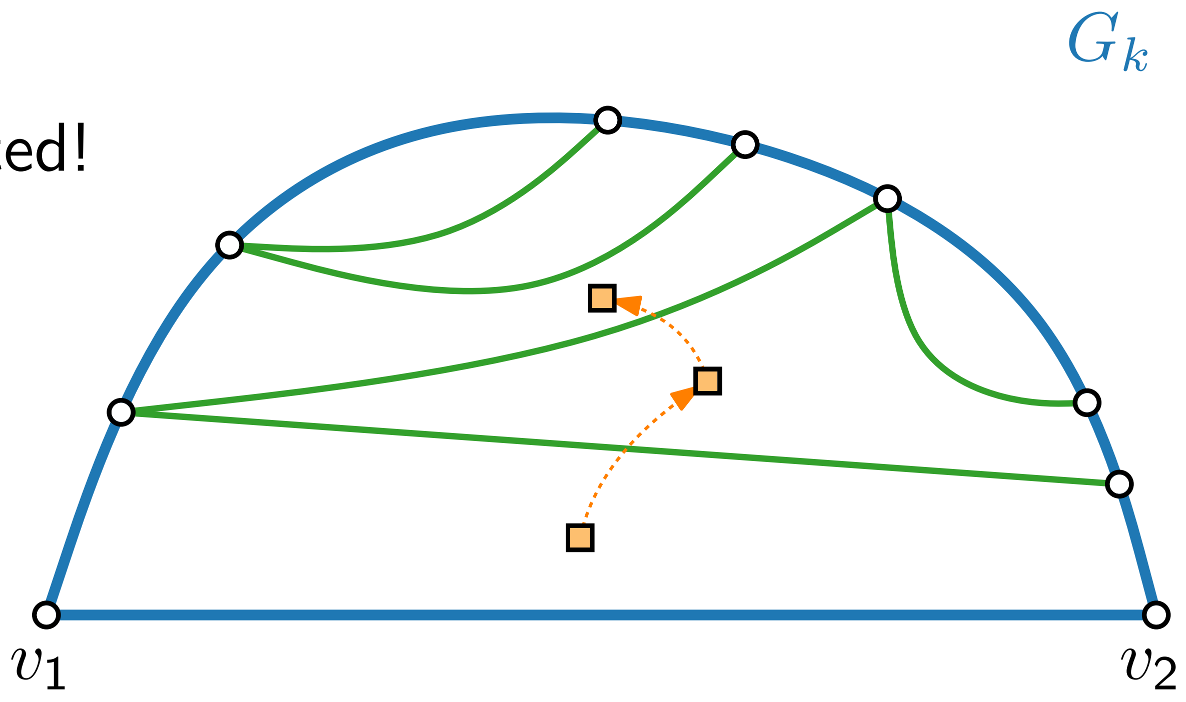

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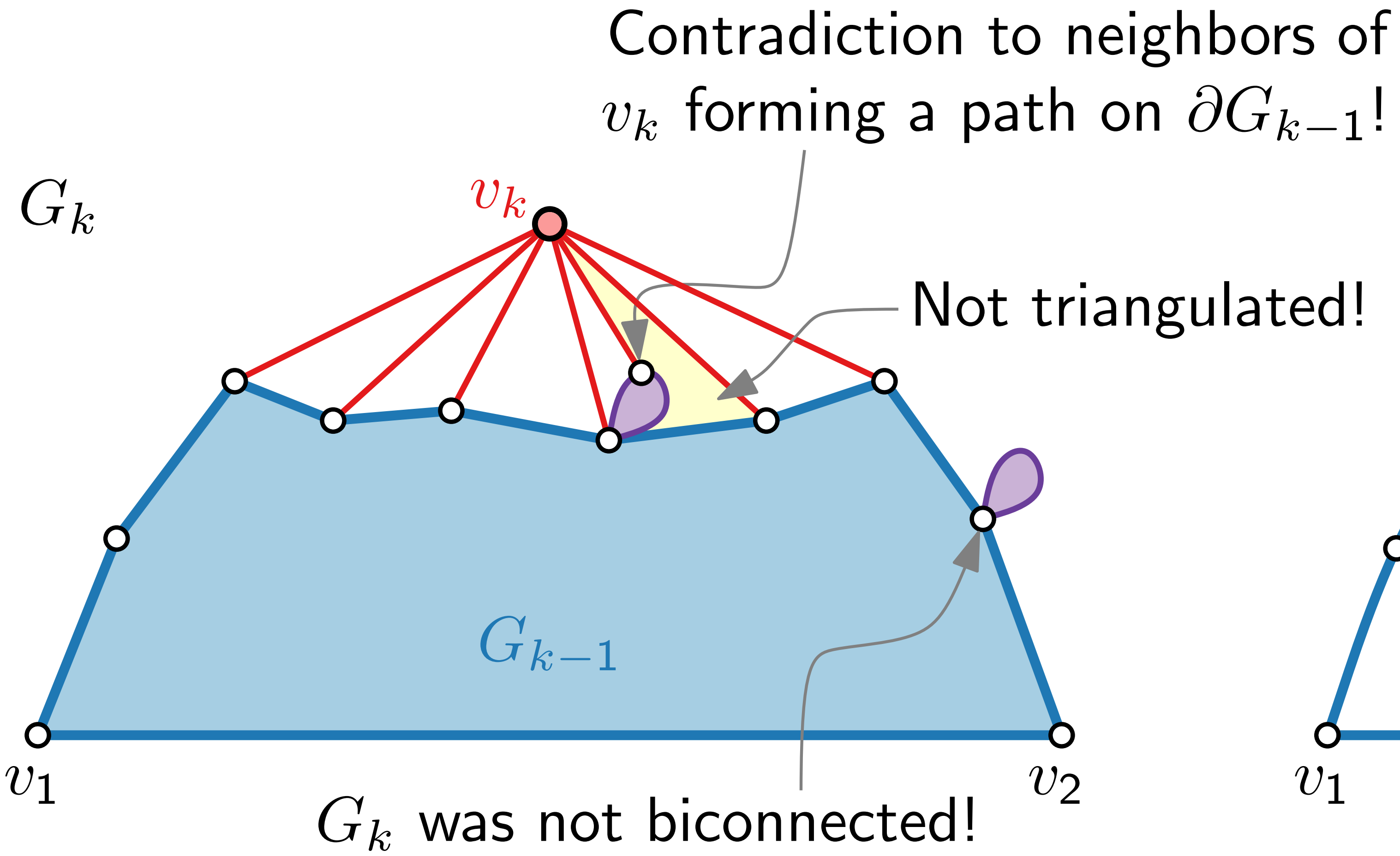

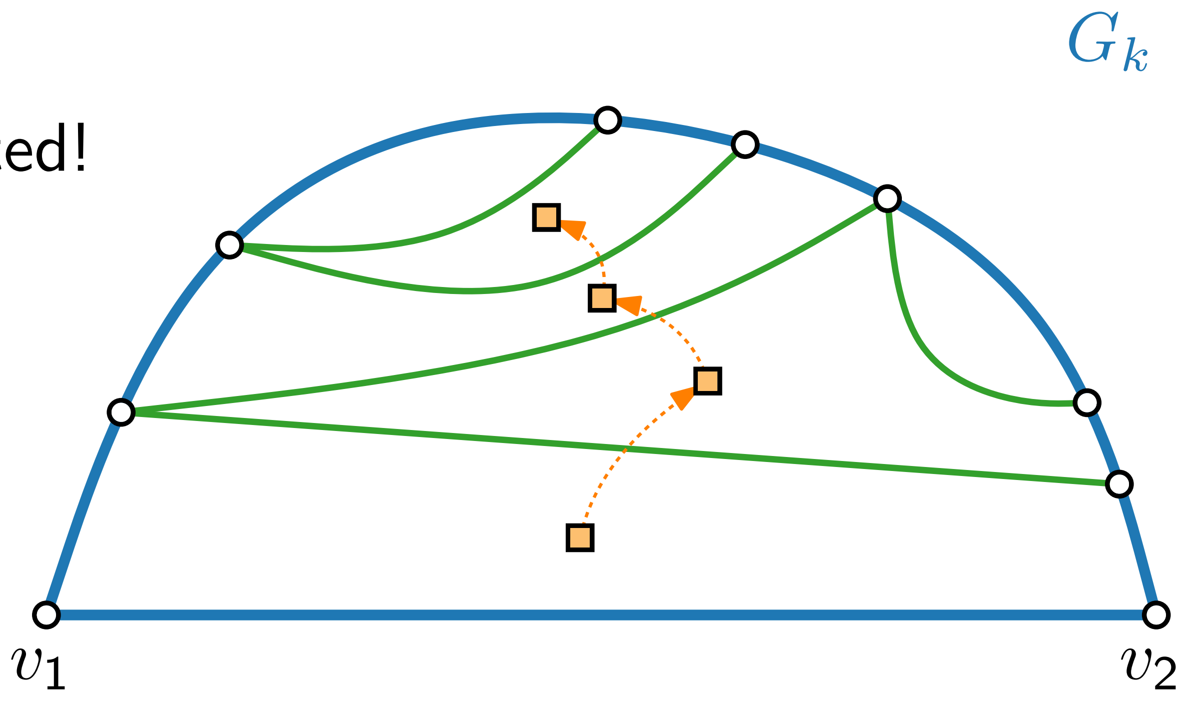

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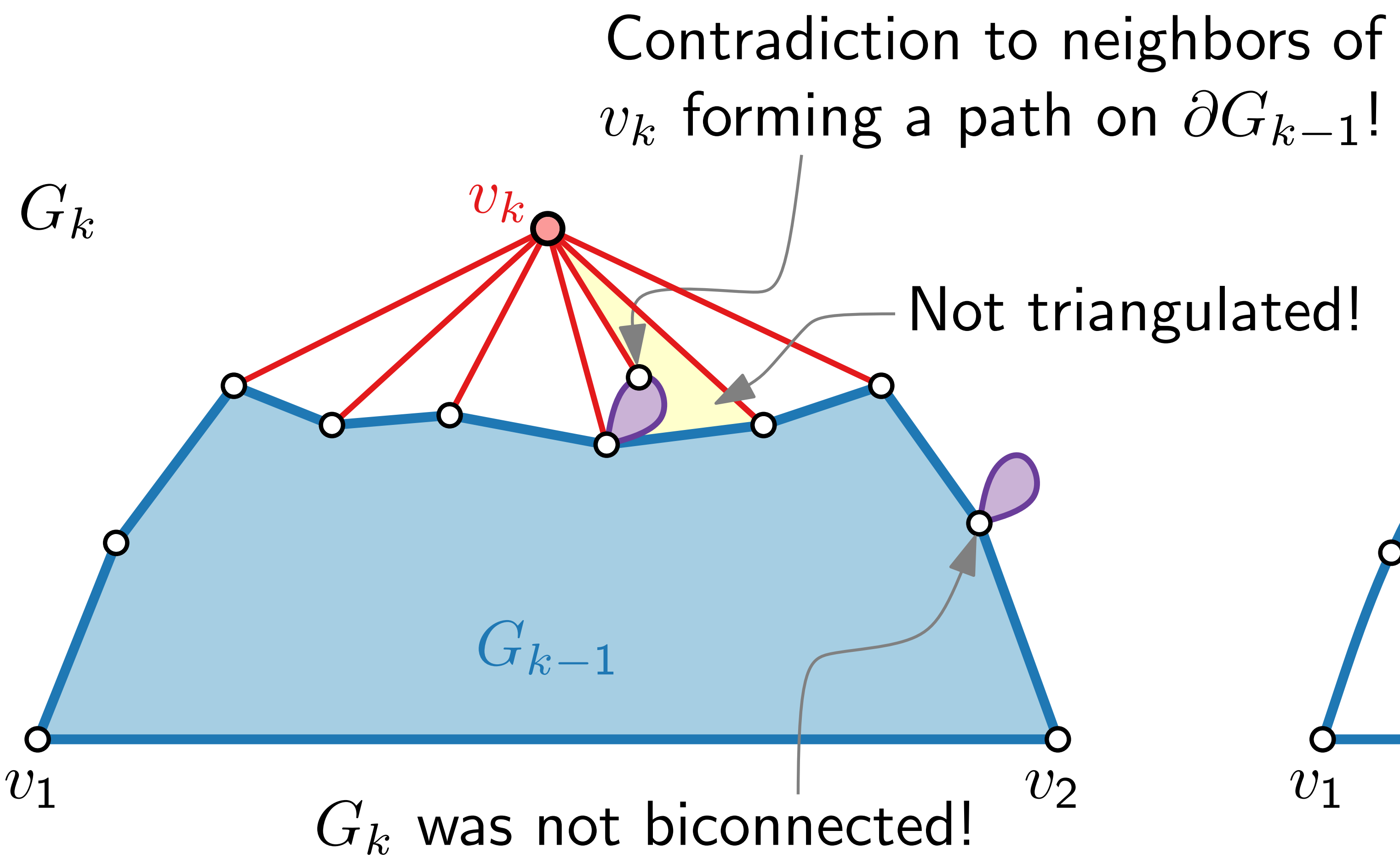

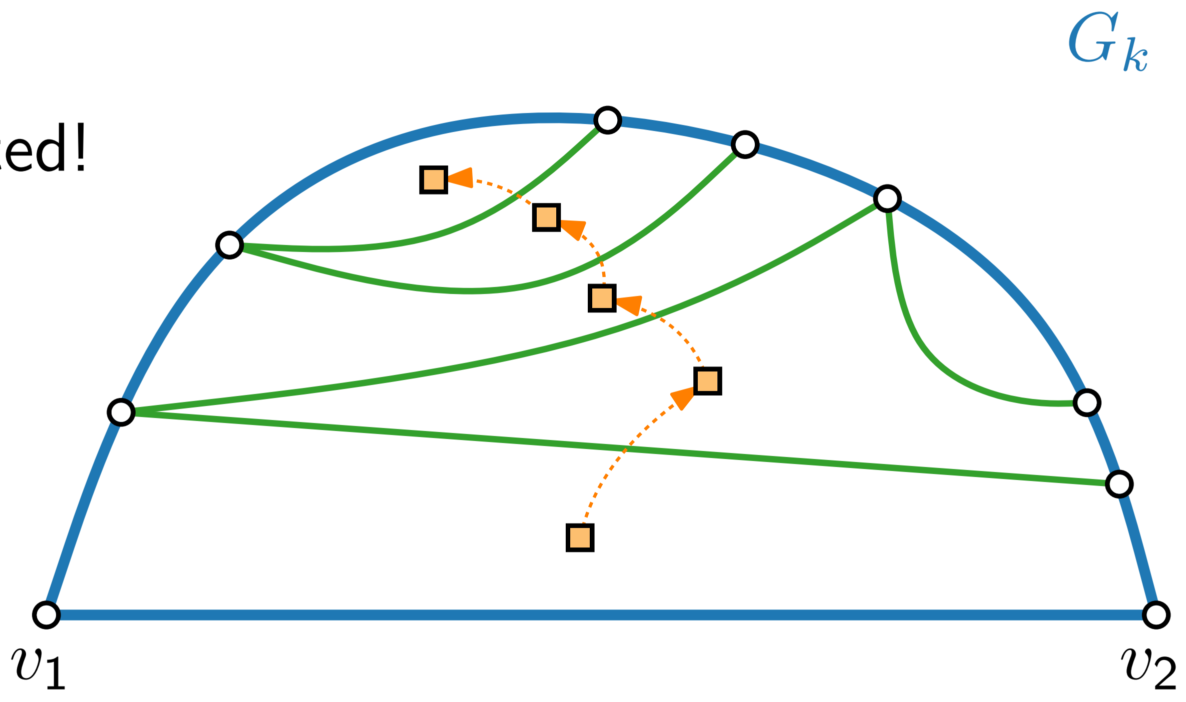

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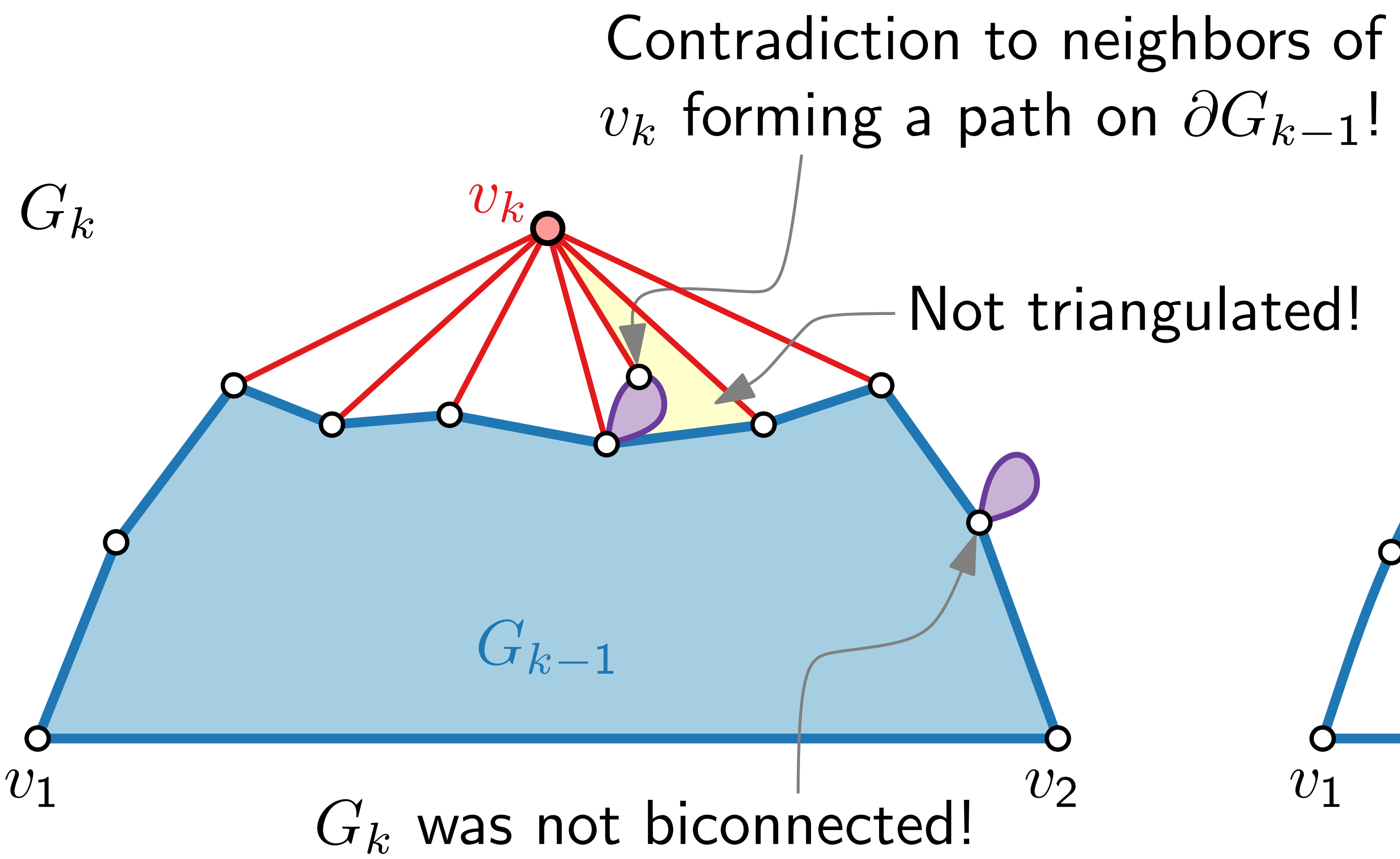

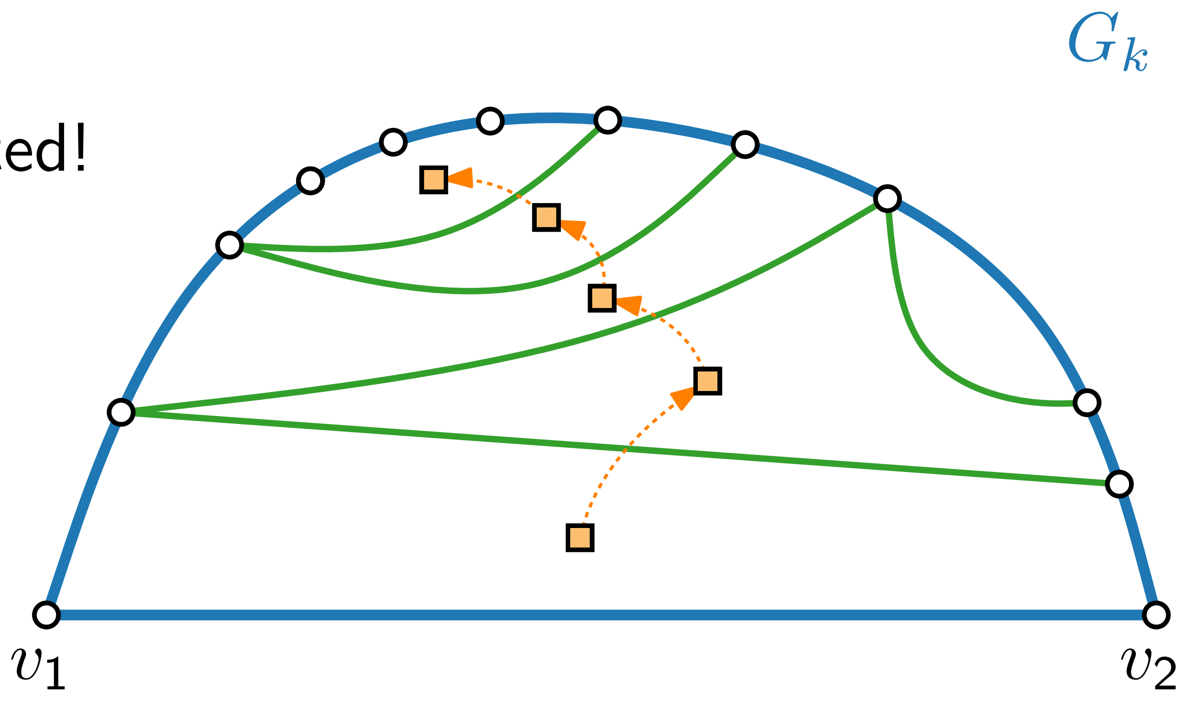

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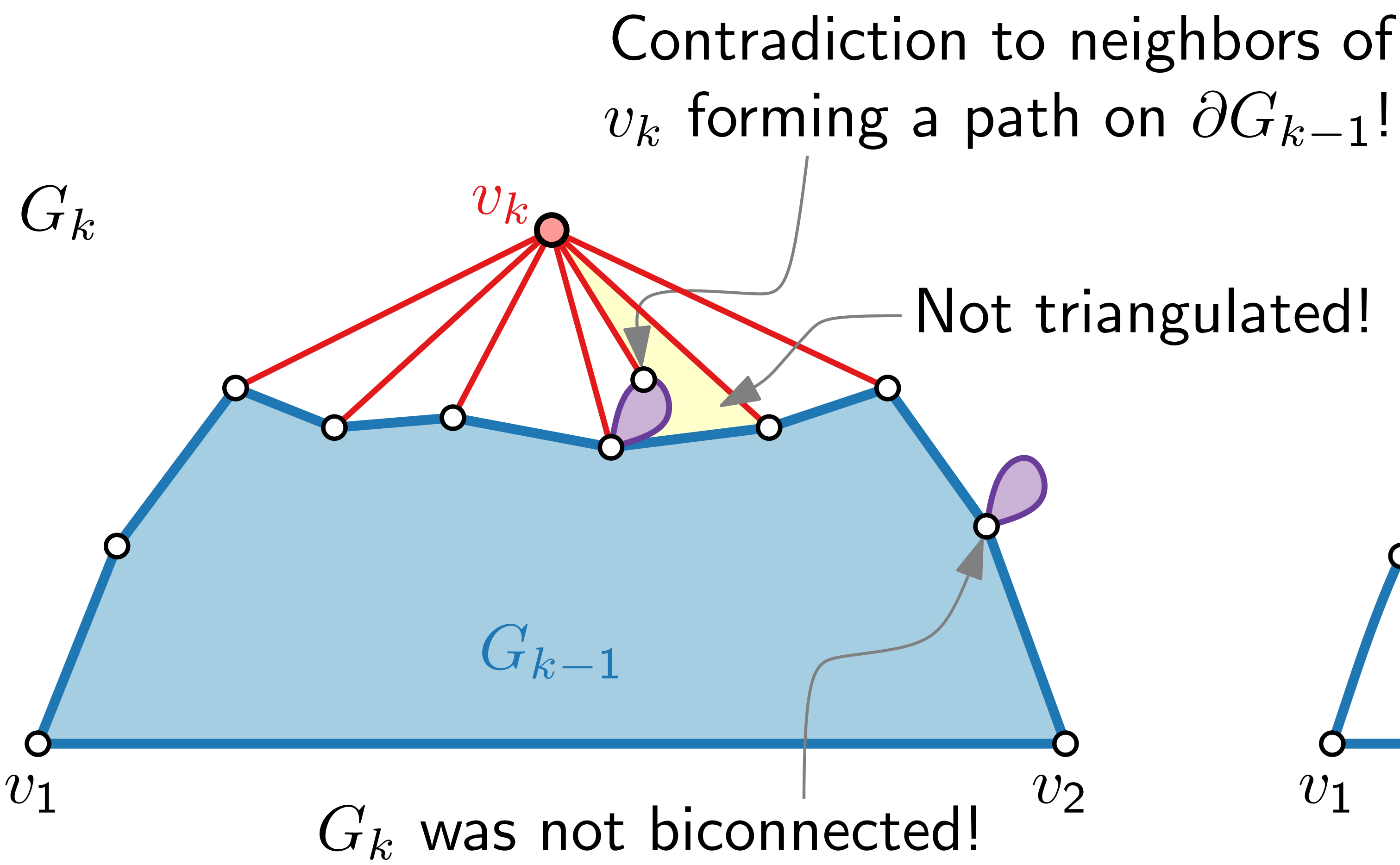

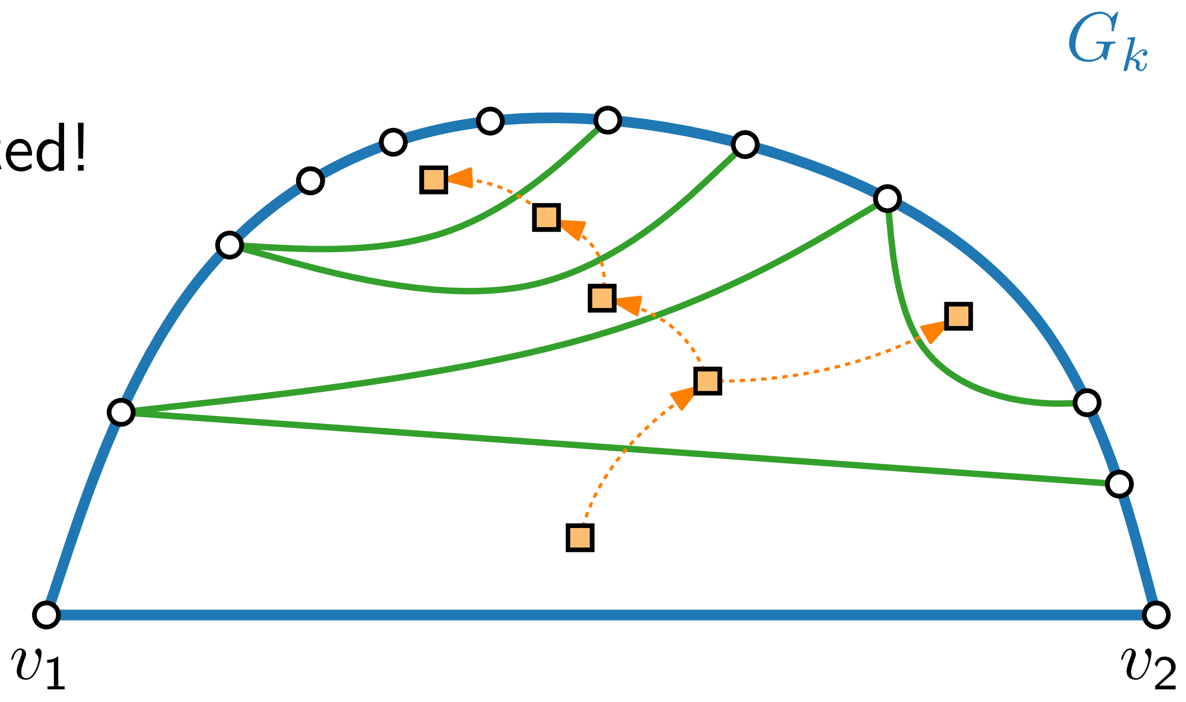

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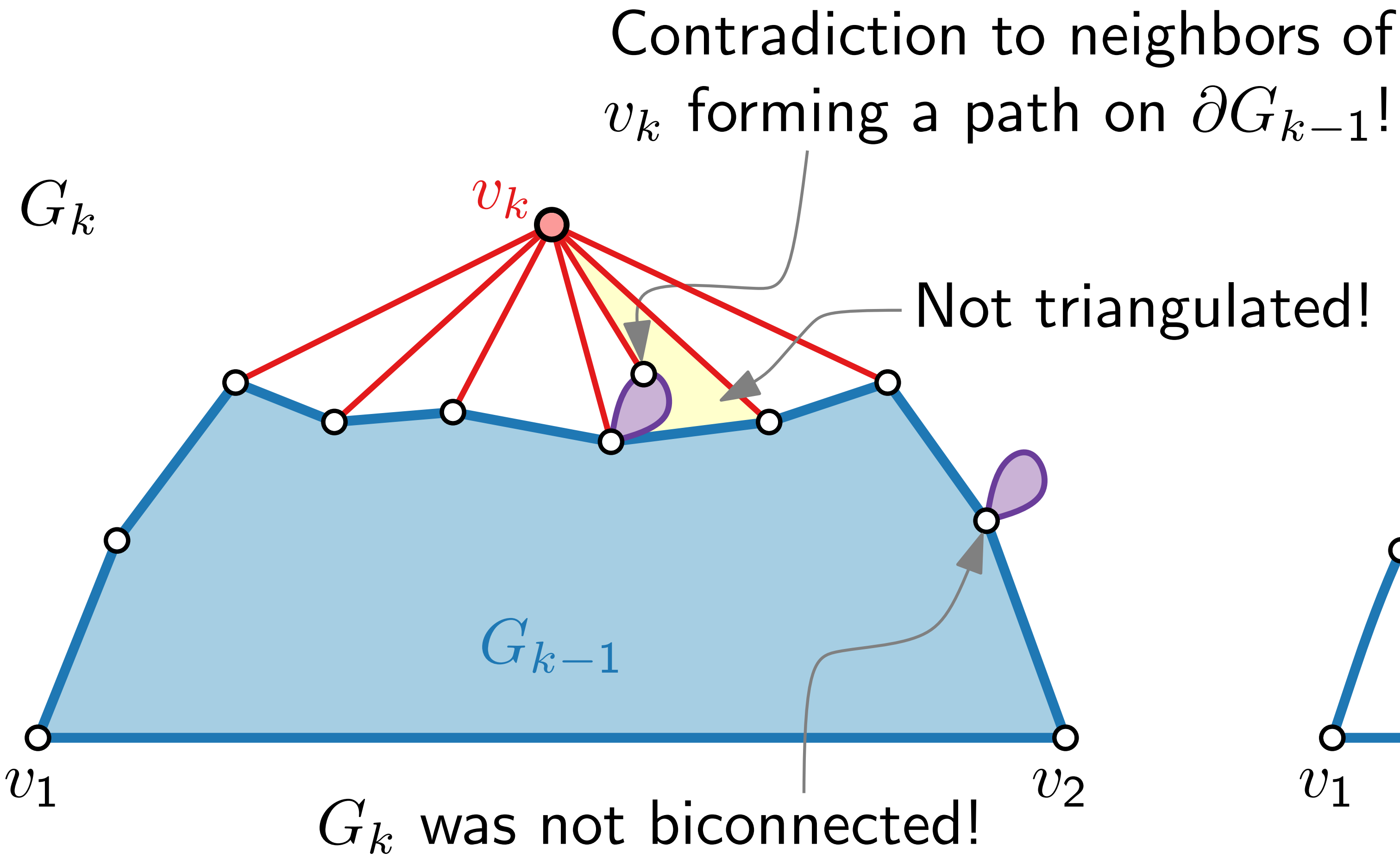

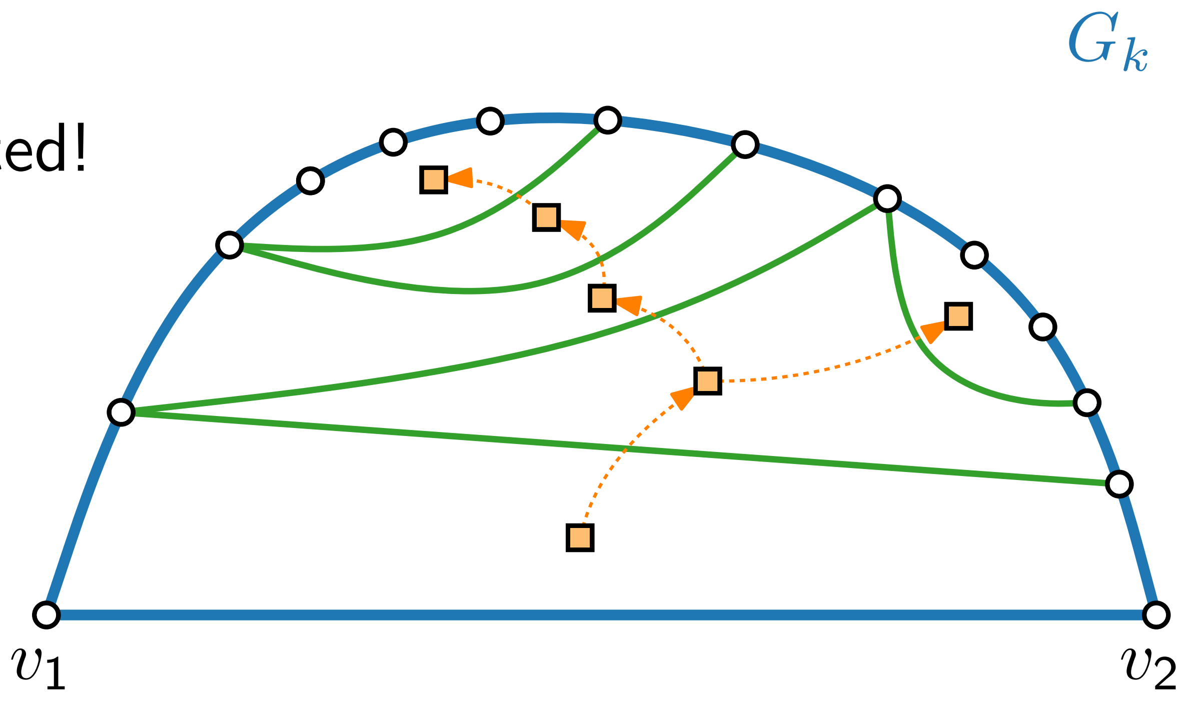

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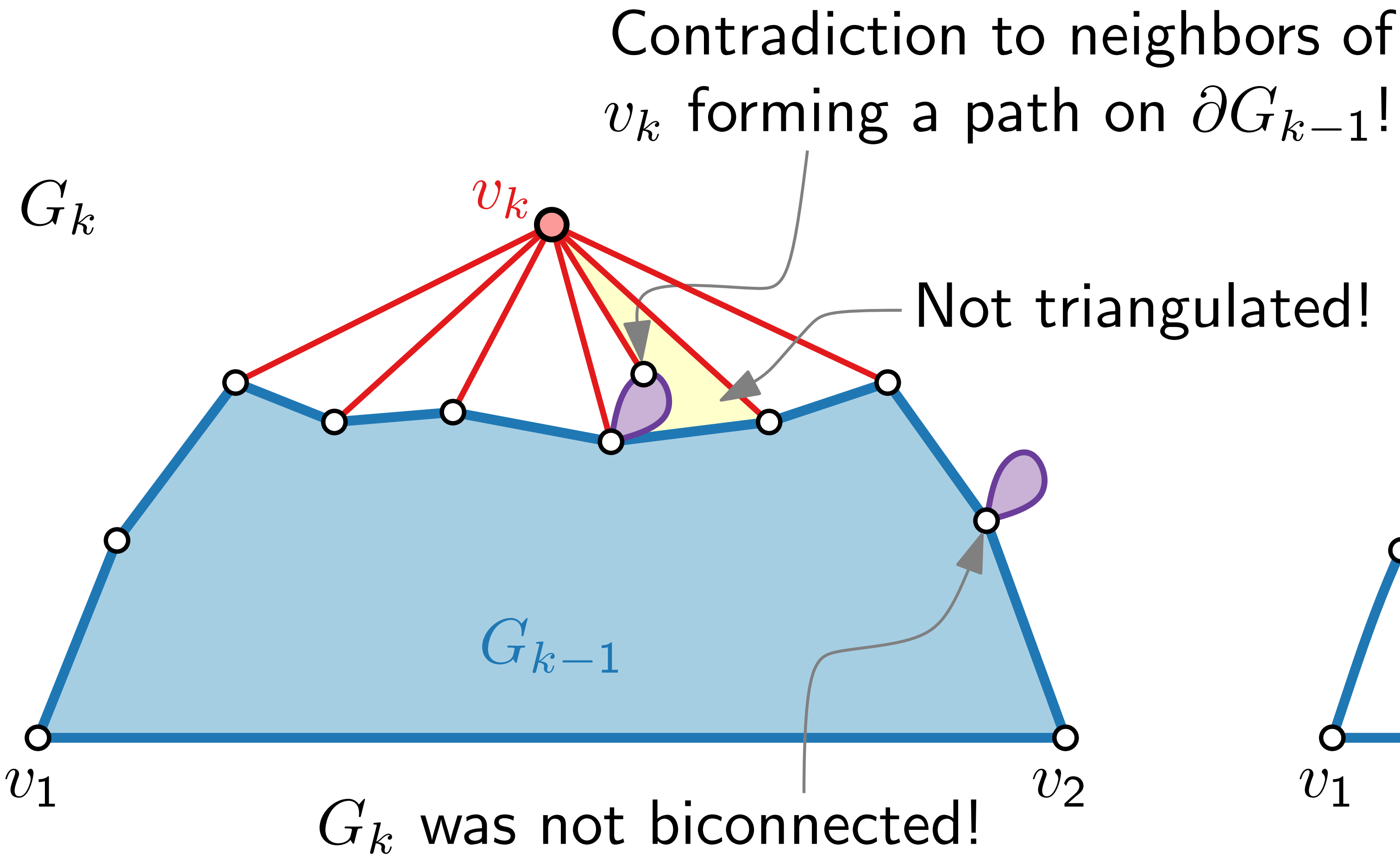

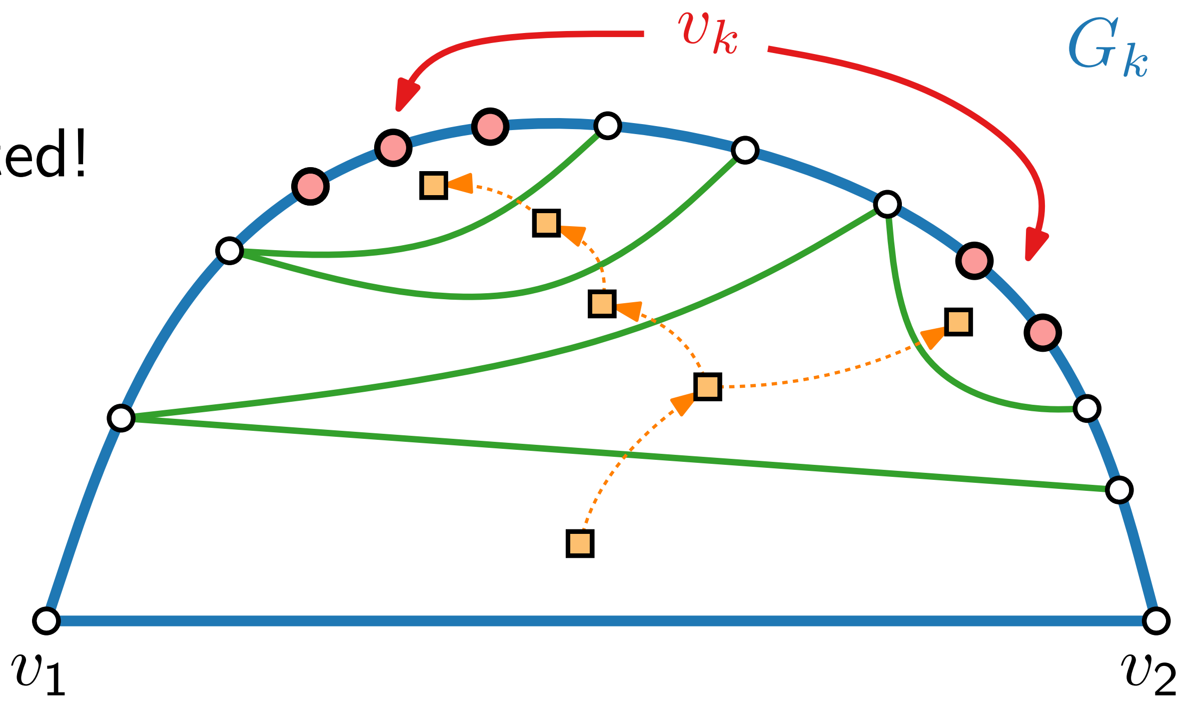

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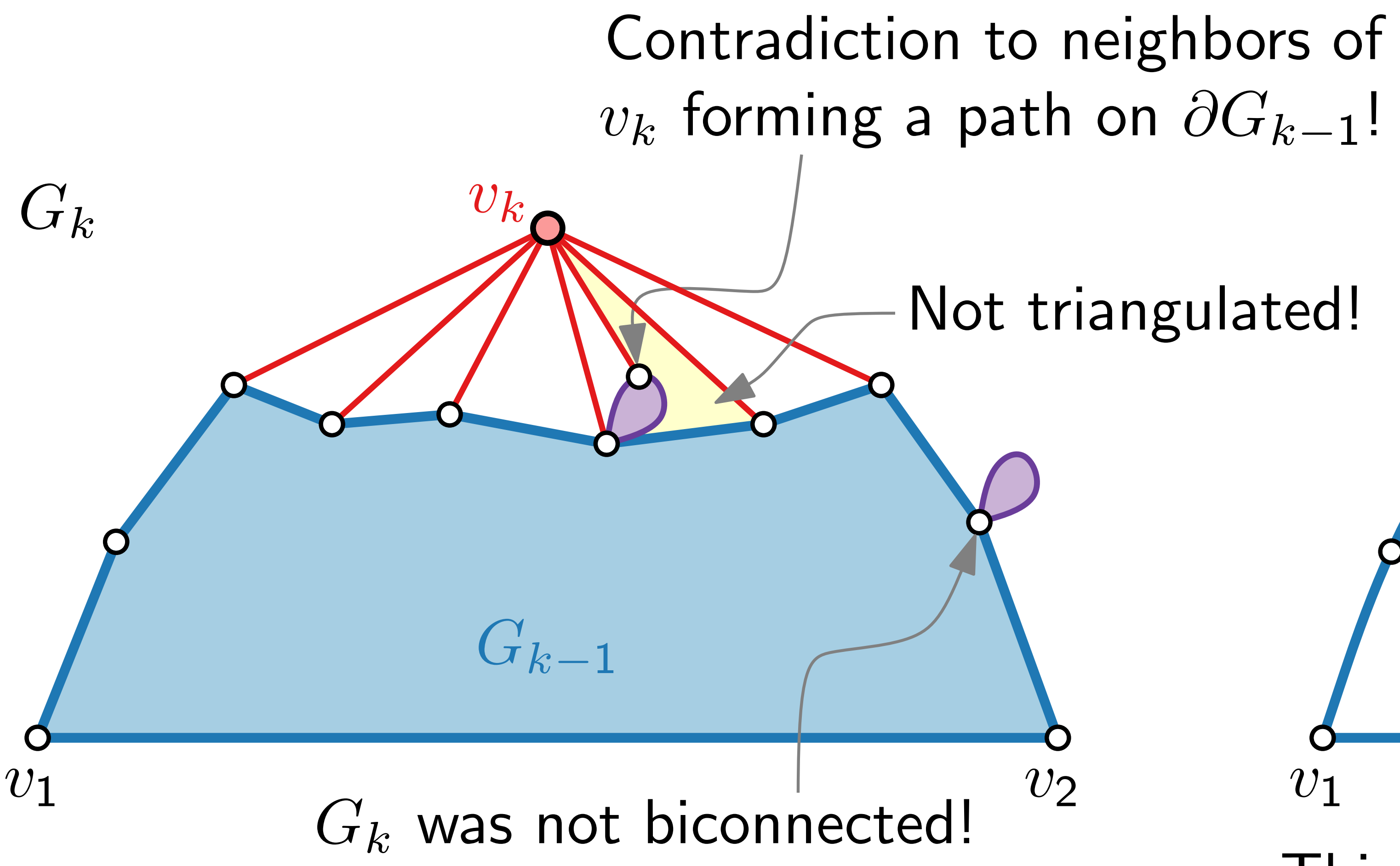

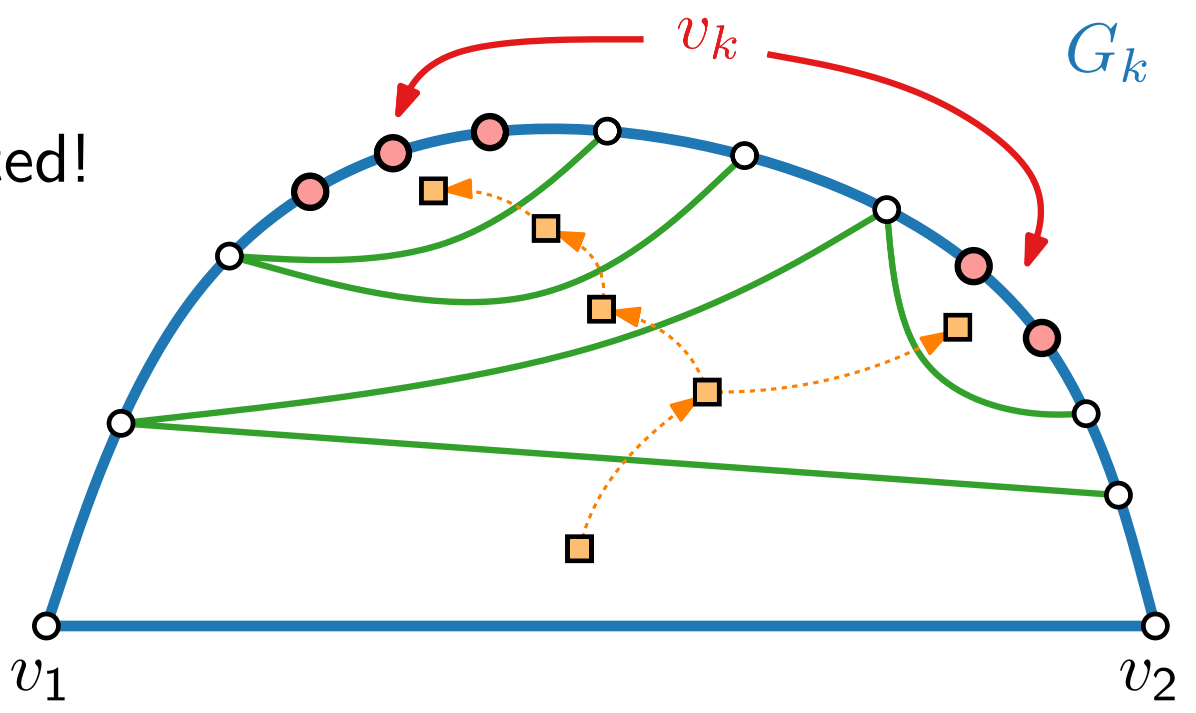

This completes the proof of the lemma. □

Canonical Order – Implementation

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

Canonical Order – Implementation

outer face

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foreach $v \in V(G)$ **do**

Canonical Order – Implementation

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CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**
  chords(v) $\leftarrow$ 0;

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foreach $v \in V(G)$ **do**
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- chord(v)= # chords incident to v

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foreach $v \in V(G)$ **do**
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foreach $v \in V(G)$ **do**

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- out(v) $=$ true iff v on boundary of current outer face

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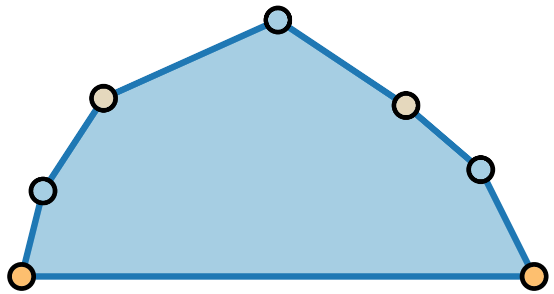

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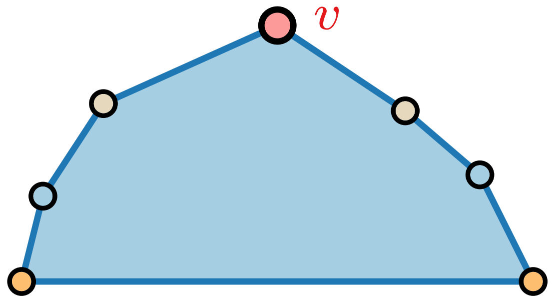

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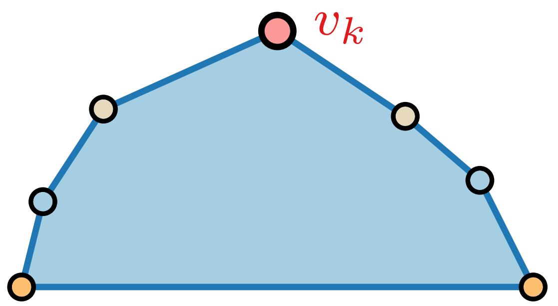

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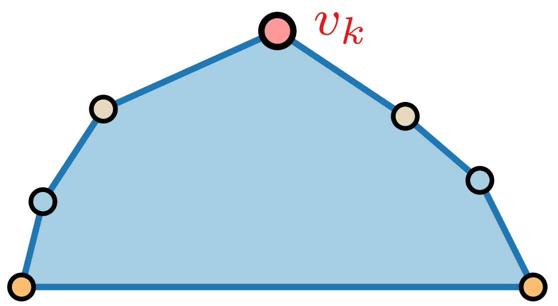

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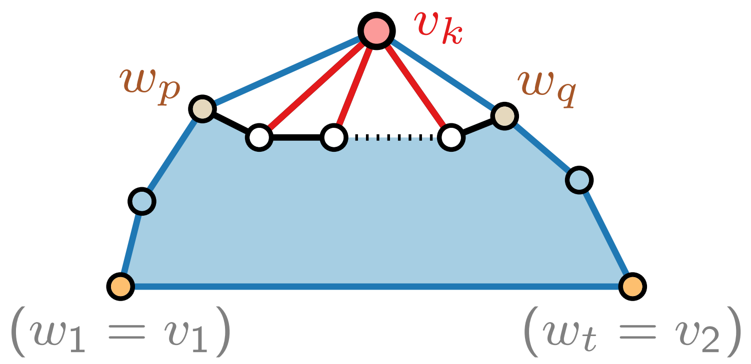

Canonical Order – Implementation

outer face

CanonicalOrder($G, \langle v_1, v_2, v_n \rangle$)

foreach $v \in V(G)$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false

out(v_1), out(v_2), out(v_n) $\leftarrow$ true

for $k = n$ **downto** 3 **do**
  choose $v \in V(G) \setminus \{v_1, v_2\}$ such that mark(v) $=$ false, out(v) $=$ true, chords(v) $= 0$
  $v_k \leftarrow v$; mark(v_k) $\leftarrow$ true; out(v_k) $\leftarrow$ false
  let $w_p, \ldots, w_q$ be the ordered unmarked neighbors of v_k
  for $i = p + 1$ **to** $q - 1$ **do**

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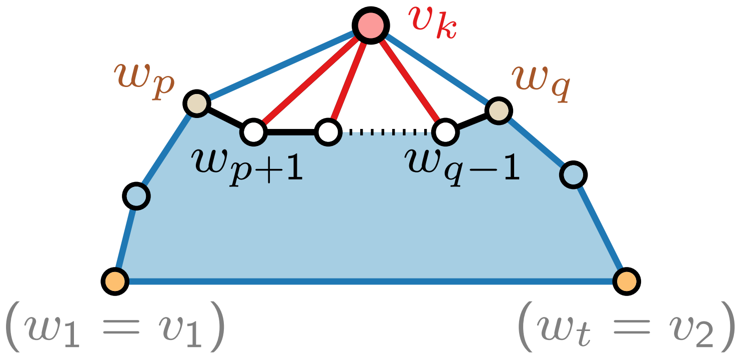

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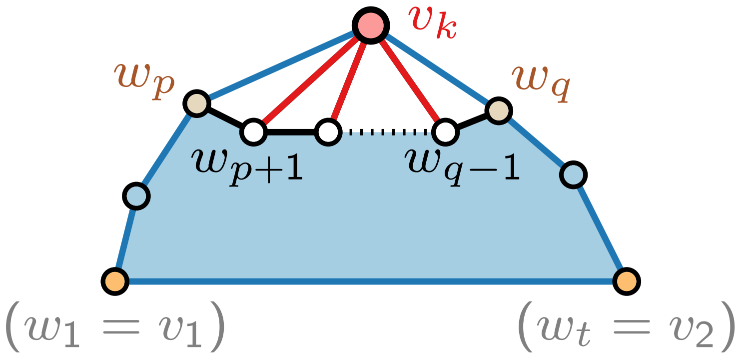

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out(v_1), out(v_2), out(v_n) ← true
for k = n downto 3 do
  | choose v ∈ V(G) \ {v_1, v_2} such that mark(v) = false,
  |   out(v) = true, chords(v) = 0
  | v_k ← v; mark(v_k) ← true; out(v_k) ← false
  | let w_p, ..., w_q be the ordered unmarked neighbors of v_k
  | for i = p + 1 to q − 1 do
  |   | out(w_i) ← true
  |   | foreach u ∈ Adj[w_i] \ {w_{i−1}, w_{i+1}} do
  |   |   └ if out(u) then chords(w_i)++, chords(u)++
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  └ if p + 1 = q then chords(w_p)--, chords(w_q)--
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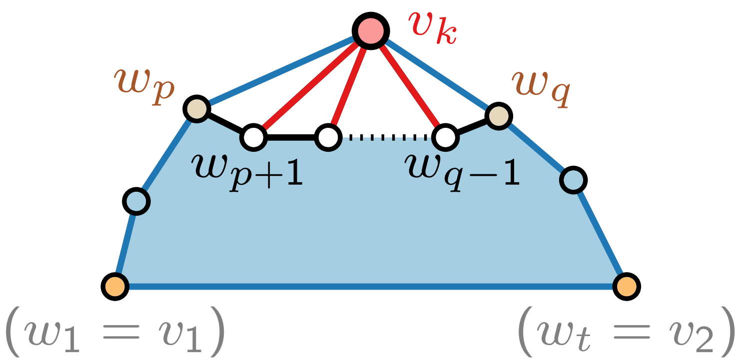

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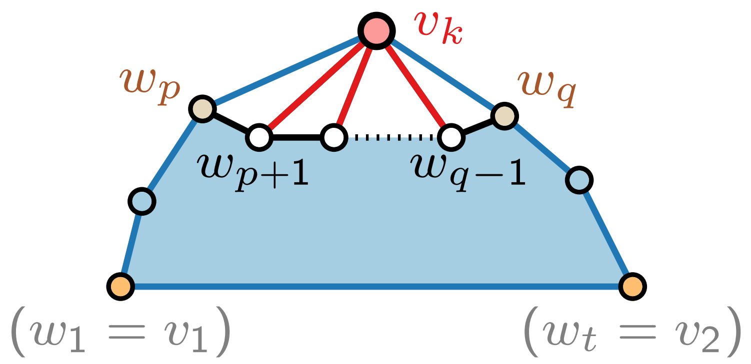

Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

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     out(v) = true, chords(v) = 0  // use list of candidates
    v_k ← v; mark(v_k) ← true; out(v_k) ← false
    let w_p, ..., w_q be the ordered unmarked neighbors of v_k
    for i = p + 1 to q − 1 do
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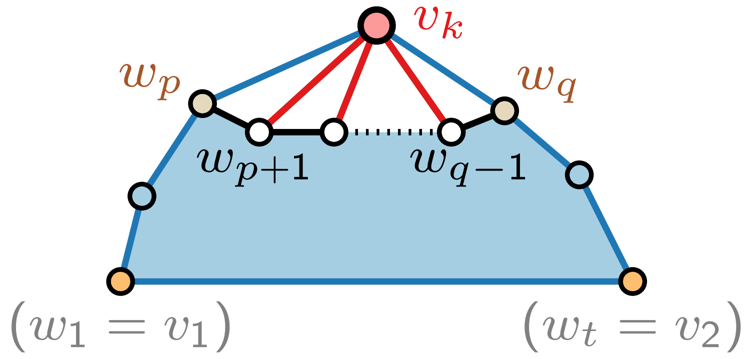

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outer face

```
CanonicalOrder(G, ⟨v1, v2, vn⟩)
foreach v ∈ V(G) do
  chords(v) ← 0; out(v) ← false; mark(v) ← false
out(v1), out(v2), out(vn) ← true
for k = n downto 3 do
    choose v ∈ V(G) \ {v1, v2} such that mark(v) = false,
     out(v) = true, chords(v) = 0 // use list of candidates
    vk ← v; mark(vk) ← true; out(vk) ← false
    let wp, ..., wq be the ordered unmarked neighbors of vk
    for i = p + 1 to q − 1 do   // O(n) time in total
        out(wi) ← true
        foreach u ∈ Adj[wi] \ {wi−1, wi+1} do
            if out(u) then chords(wi)++, chords(u)++
    if p + 1 = q then chords(wp)--, chords(wq)--
```

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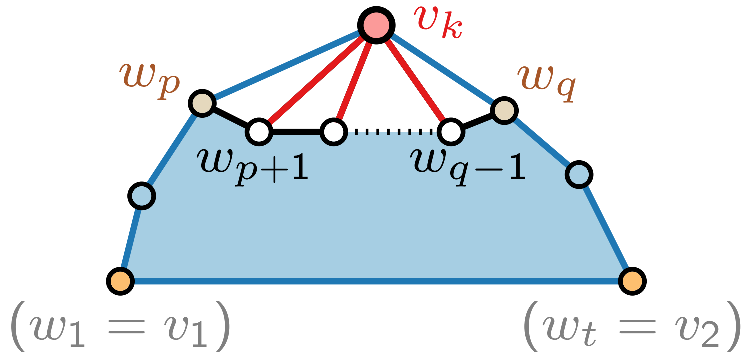

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Canonical Order – Implementation

outer face

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foreach $v \in V(G)$ **do**
  chords(v) $\leftarrow 0$; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false
out(v_1), out(v_2), out(v_n) $\leftarrow$ true
for $k = n$ **downto** 3 **do**
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   out(v) $=$ true, chords(v) $= 0$ // *use list of candidates*
  $v_k \leftarrow v$; mark(v_k) $\leftarrow$ true; out(v_k) $\leftarrow$ false
  let $w_p, \dots, w_q$ be the ordered unmarked neighbors of v_k
  for $i = p + 1$ **to** $q - 1$ **do** // *$O(n)$ time in total*
    out(w_i) $\leftarrow$ true // *$O(m) = O(n)$ in total*
    foreach $u \in$ Adj$[w_i] \setminus \{w_{i-1}, w_{i+1}\}$ **do**
      if out(u) **then** chords(w_i)++, chords(u)++
  if $p + 1 = q$ **then** chords(w_p)−−, chords(w_q)−−

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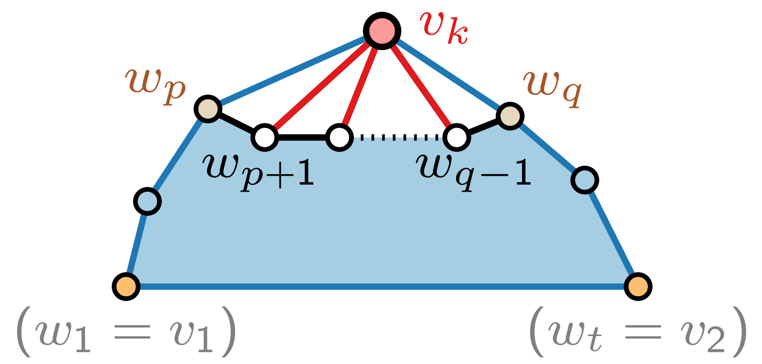

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Shift Method – Idea

Drawing invariants:

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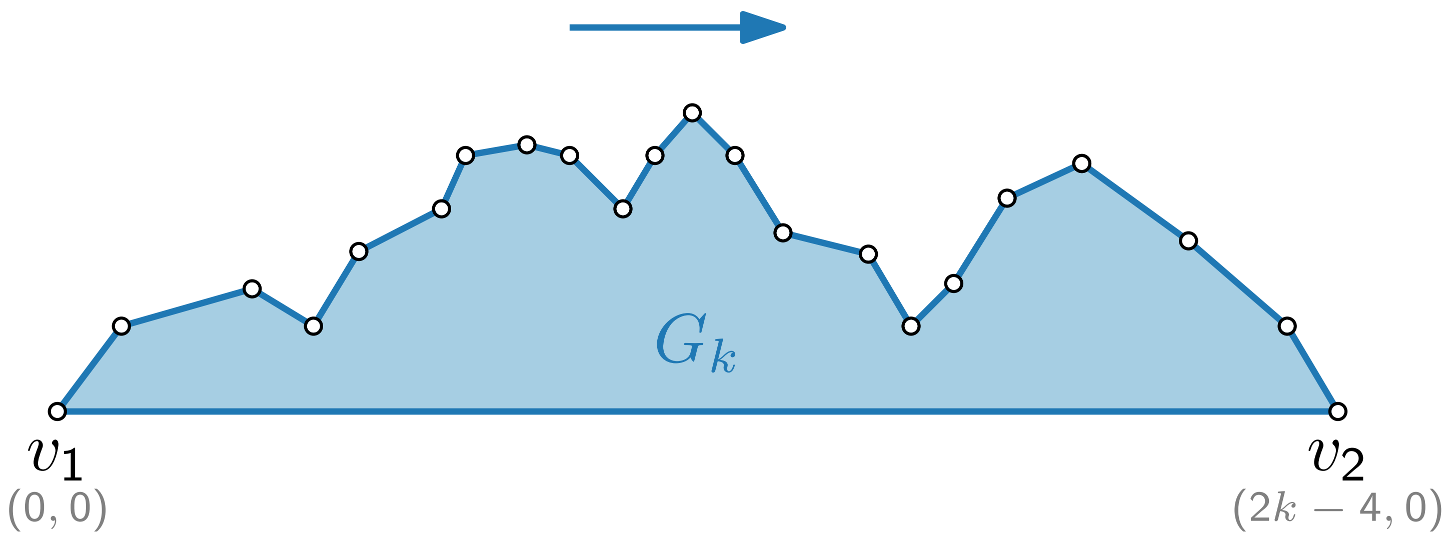

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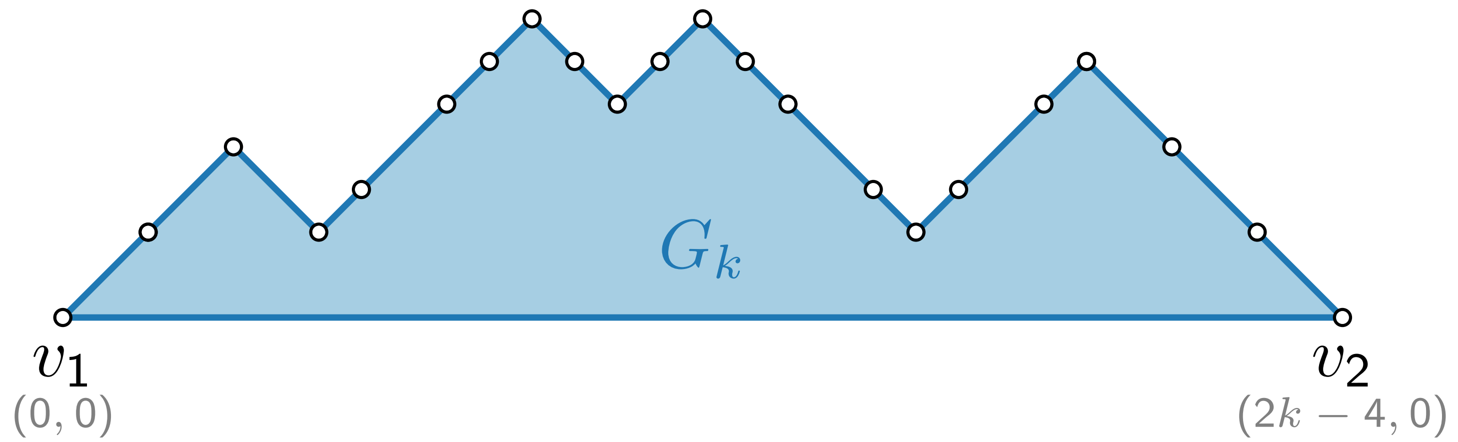

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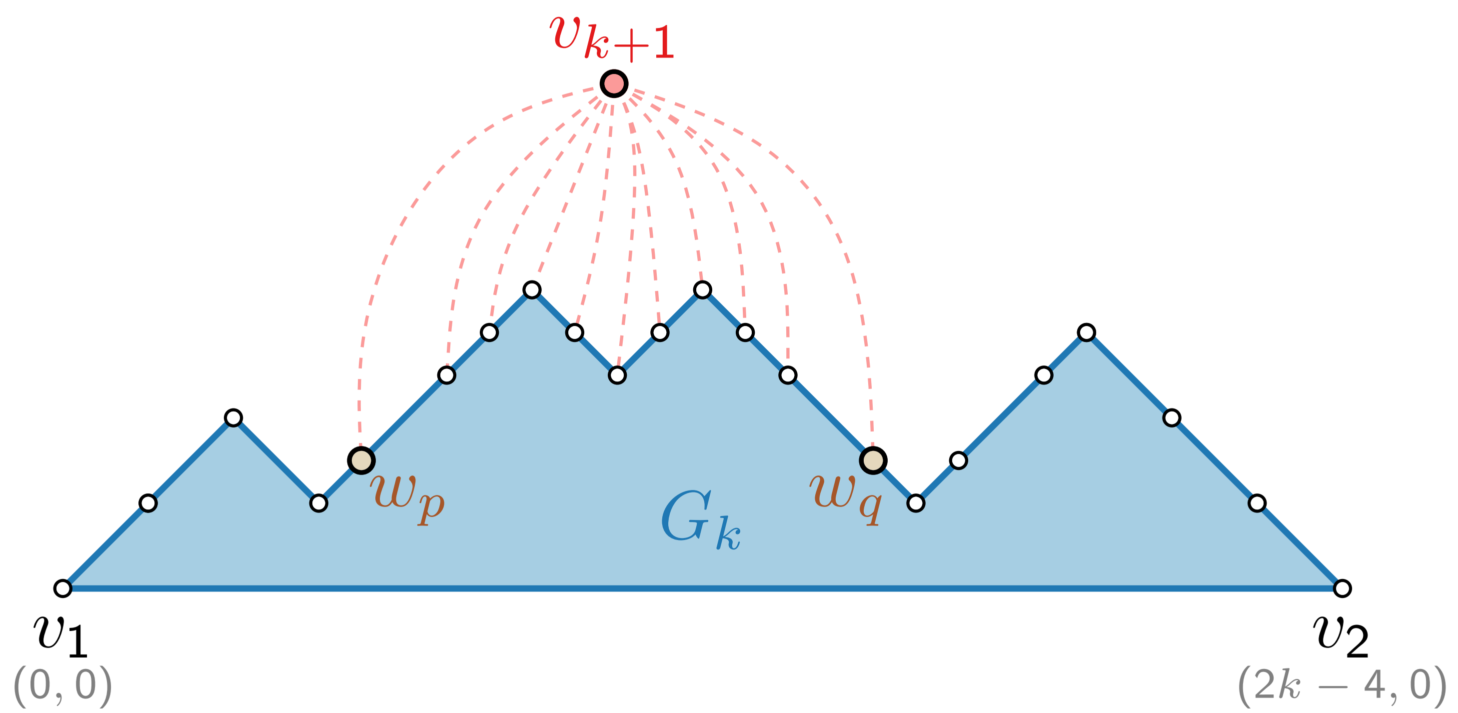

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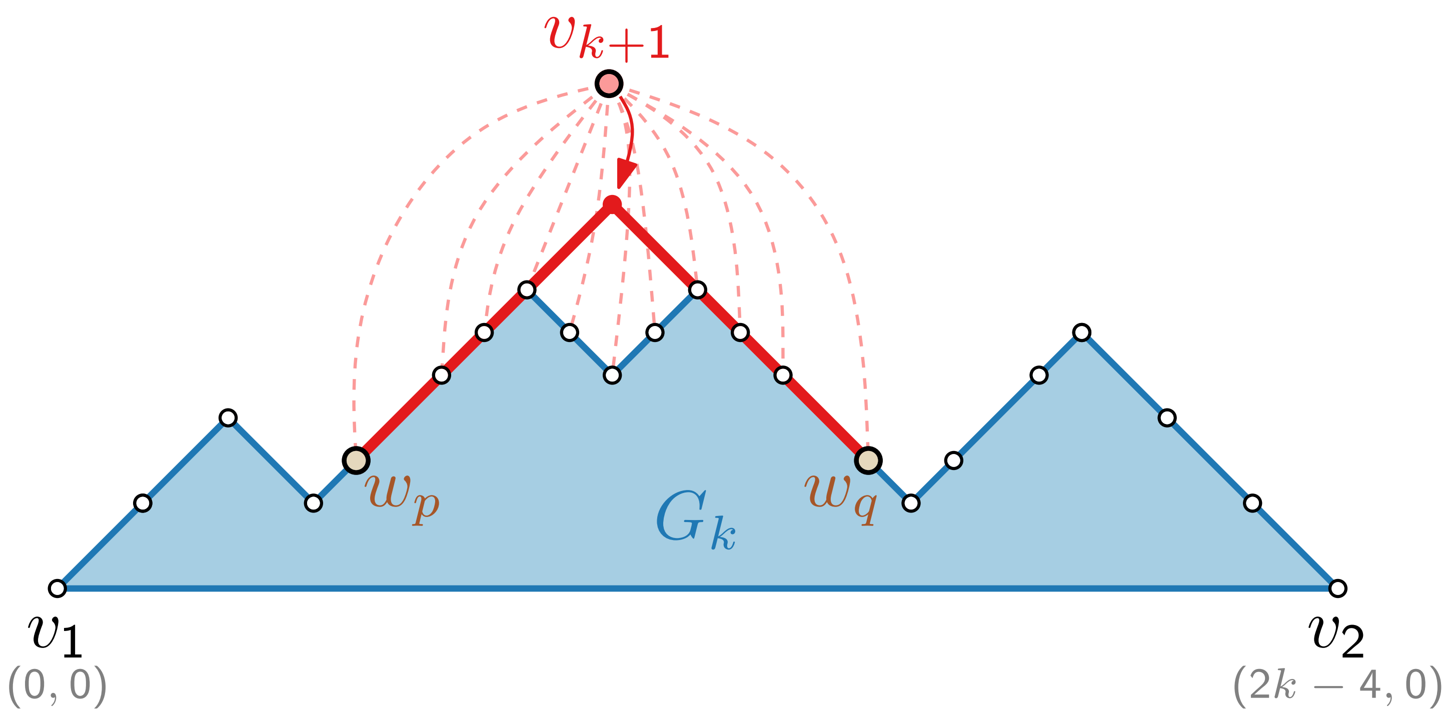

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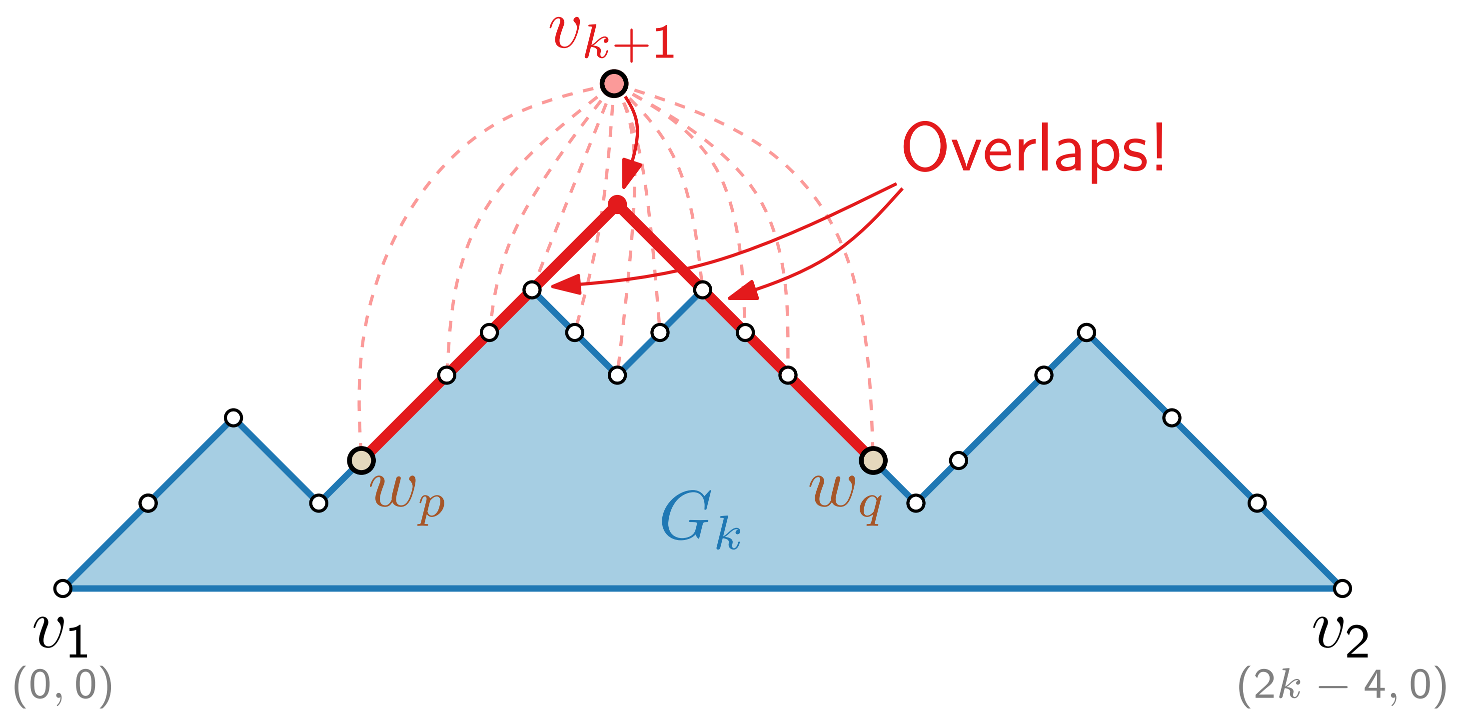

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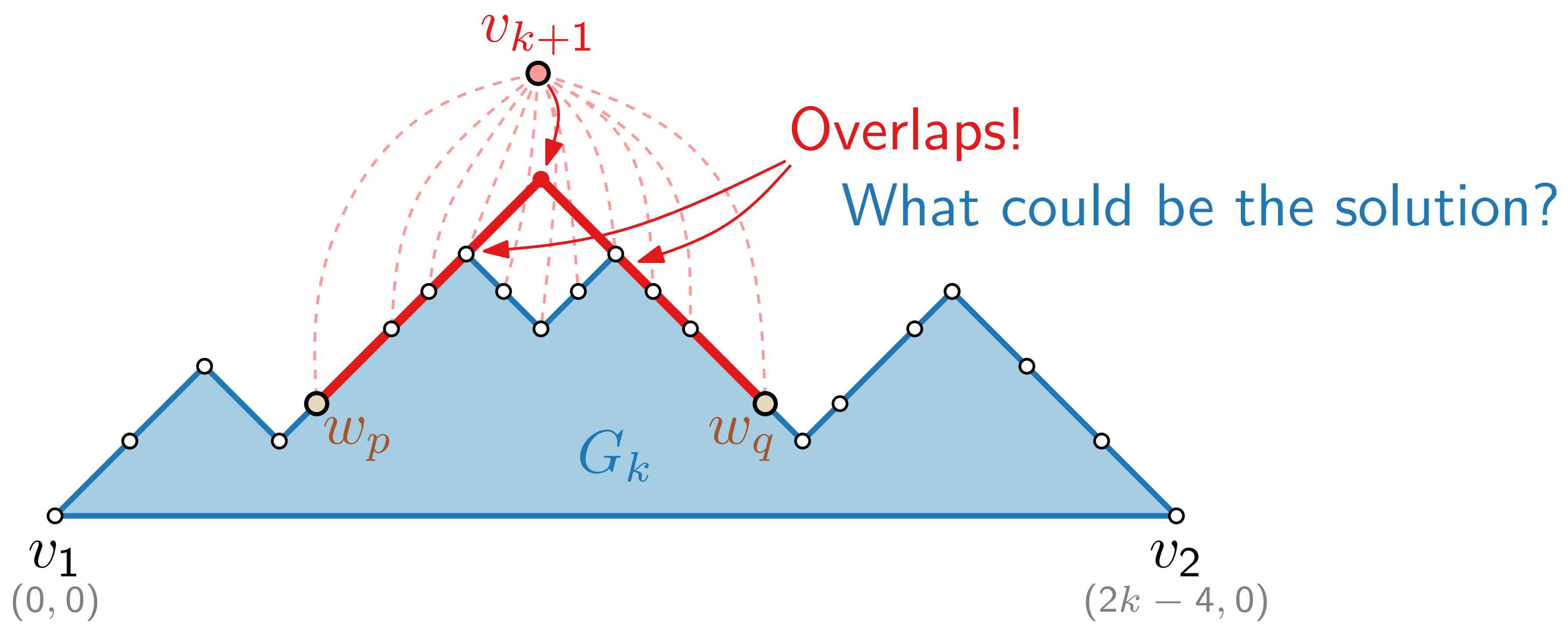

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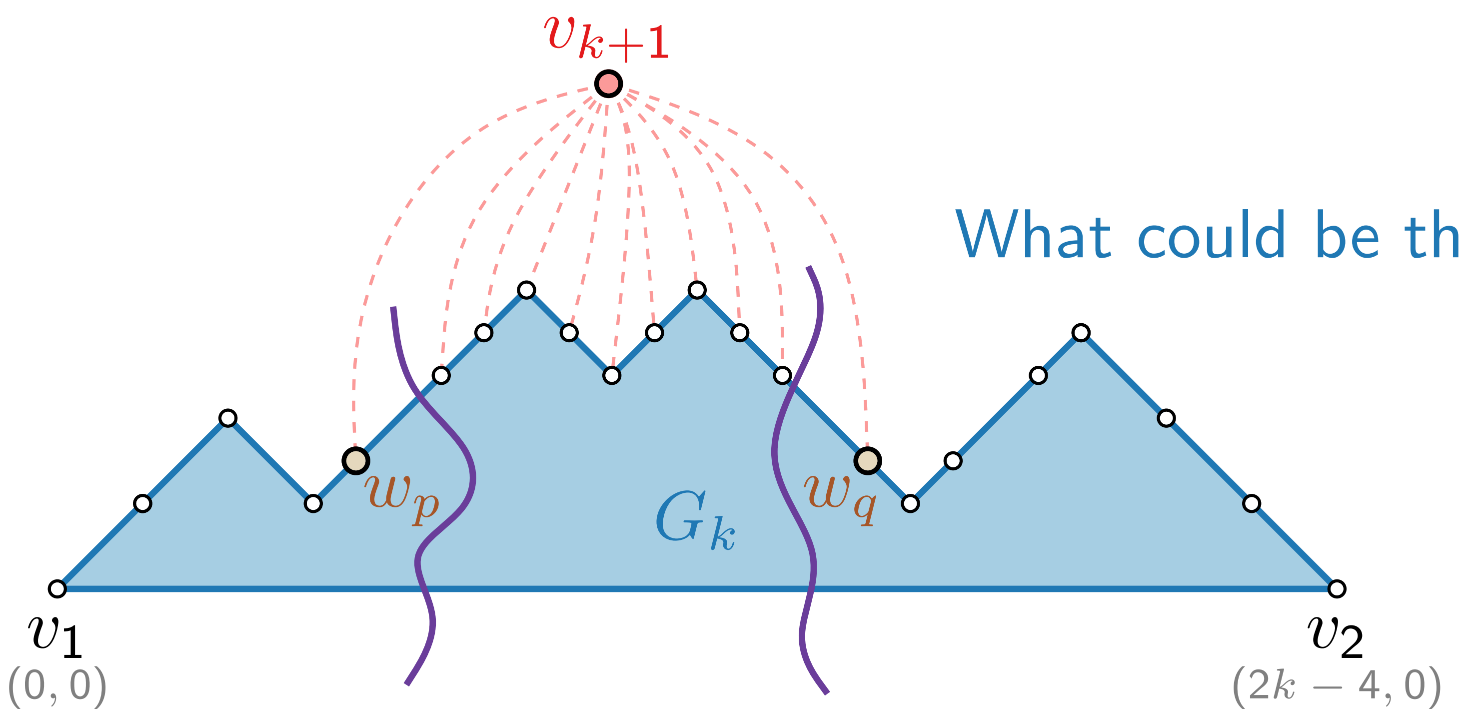

What could be the solution?

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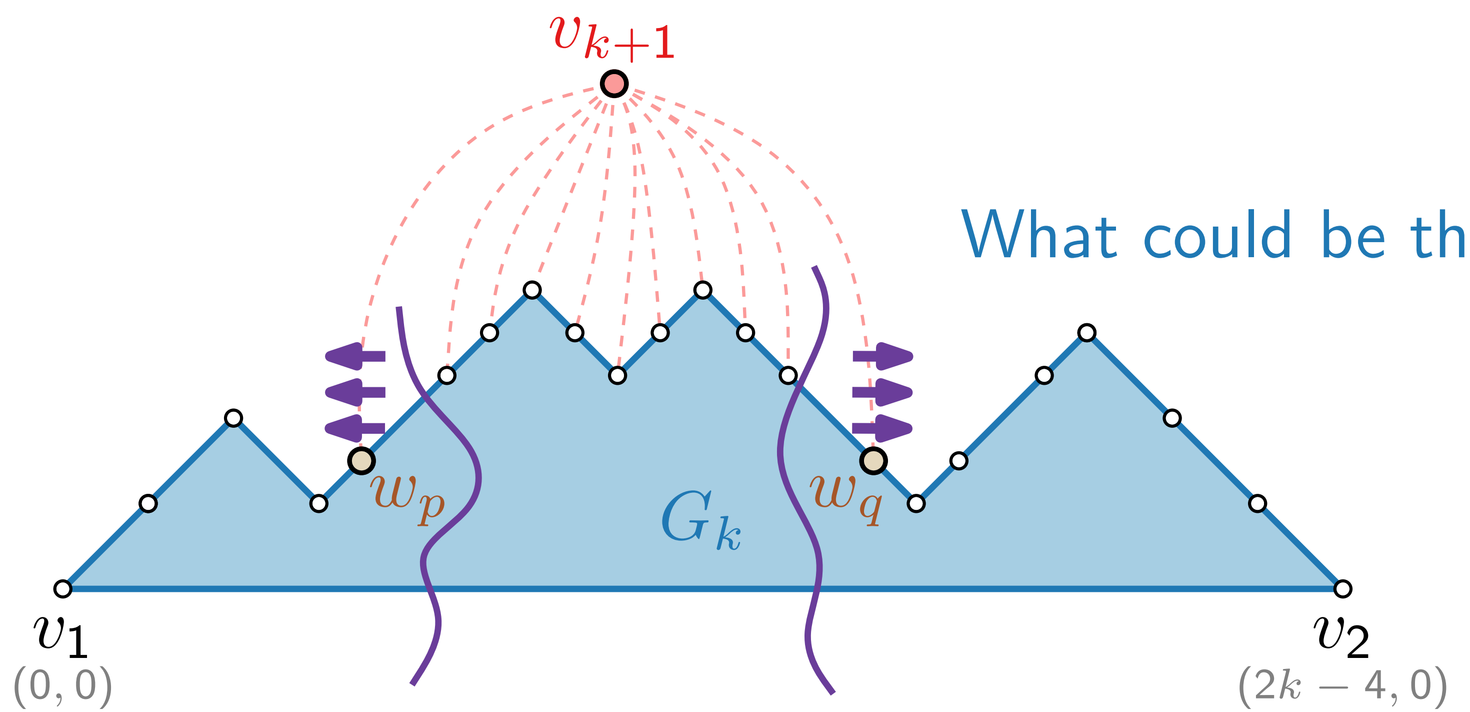

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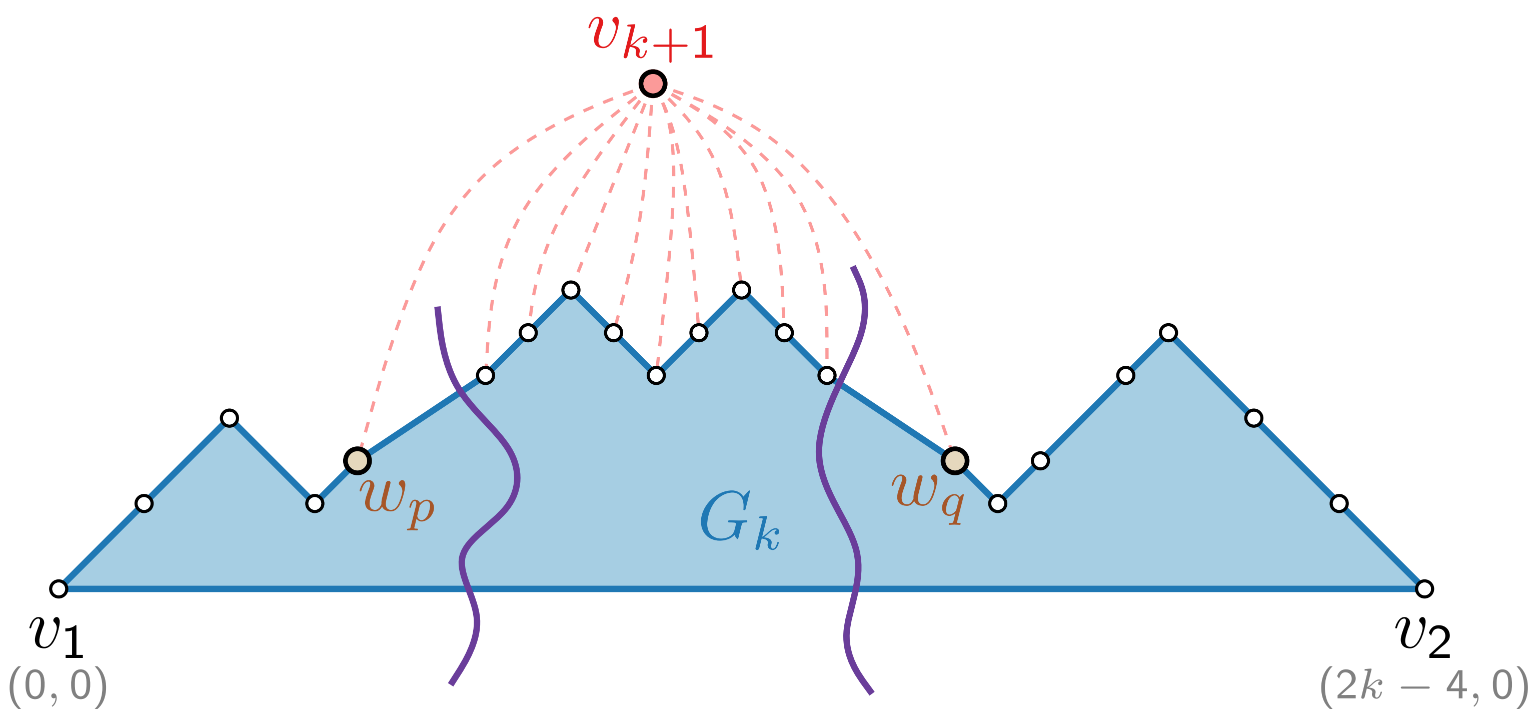

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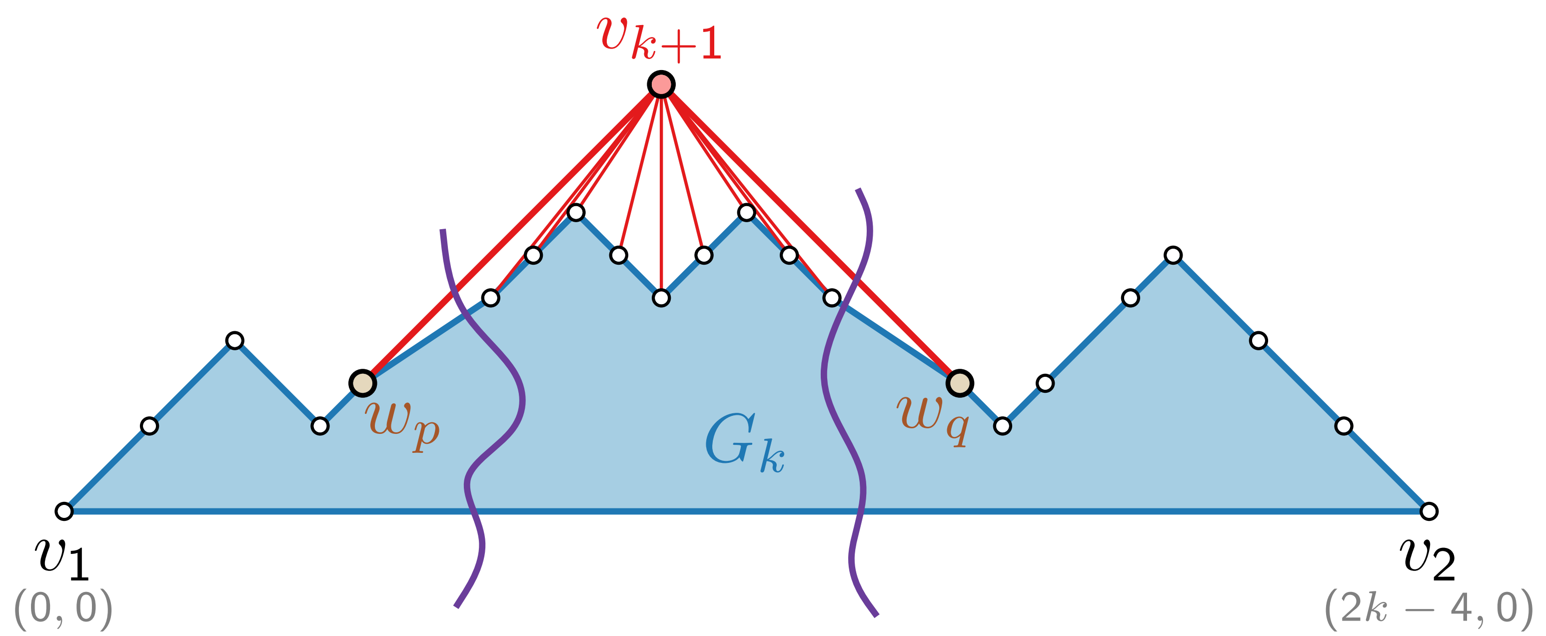

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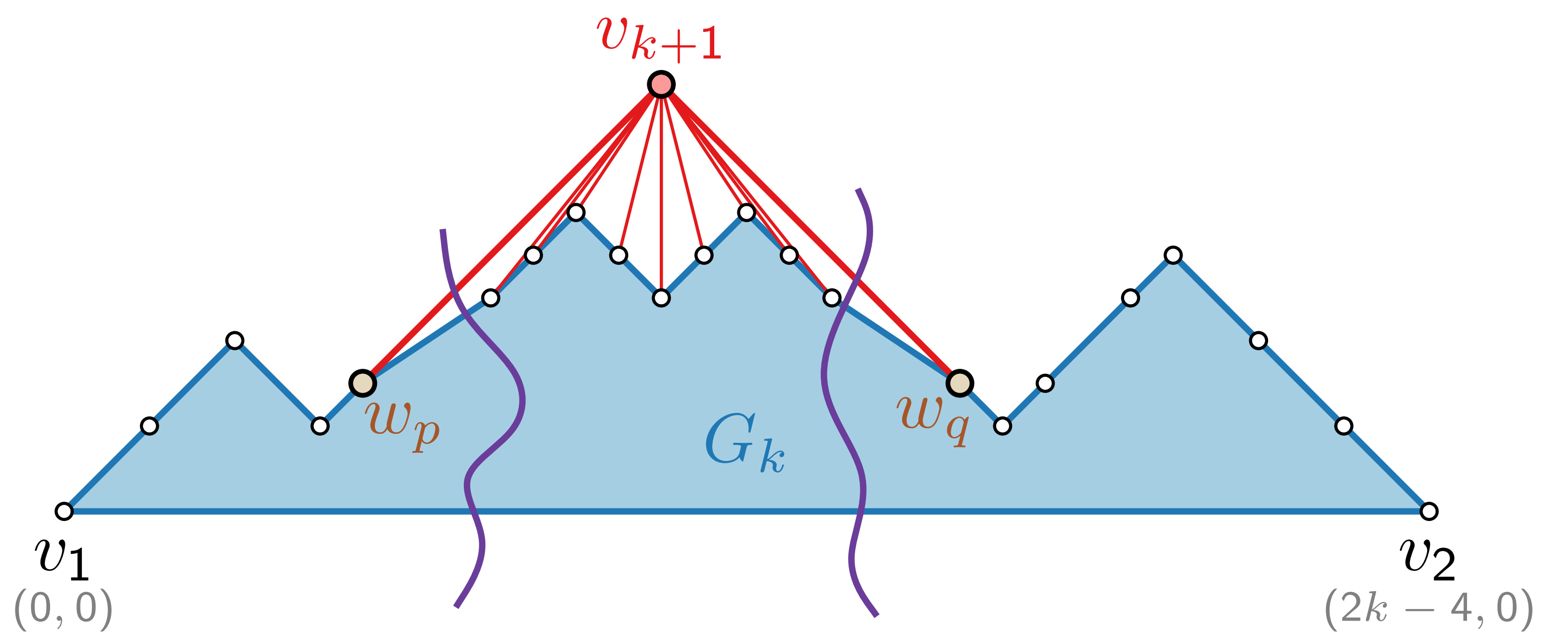
Will v_{k+1} lie on the grid?



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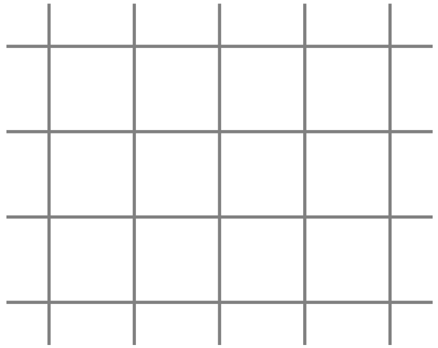

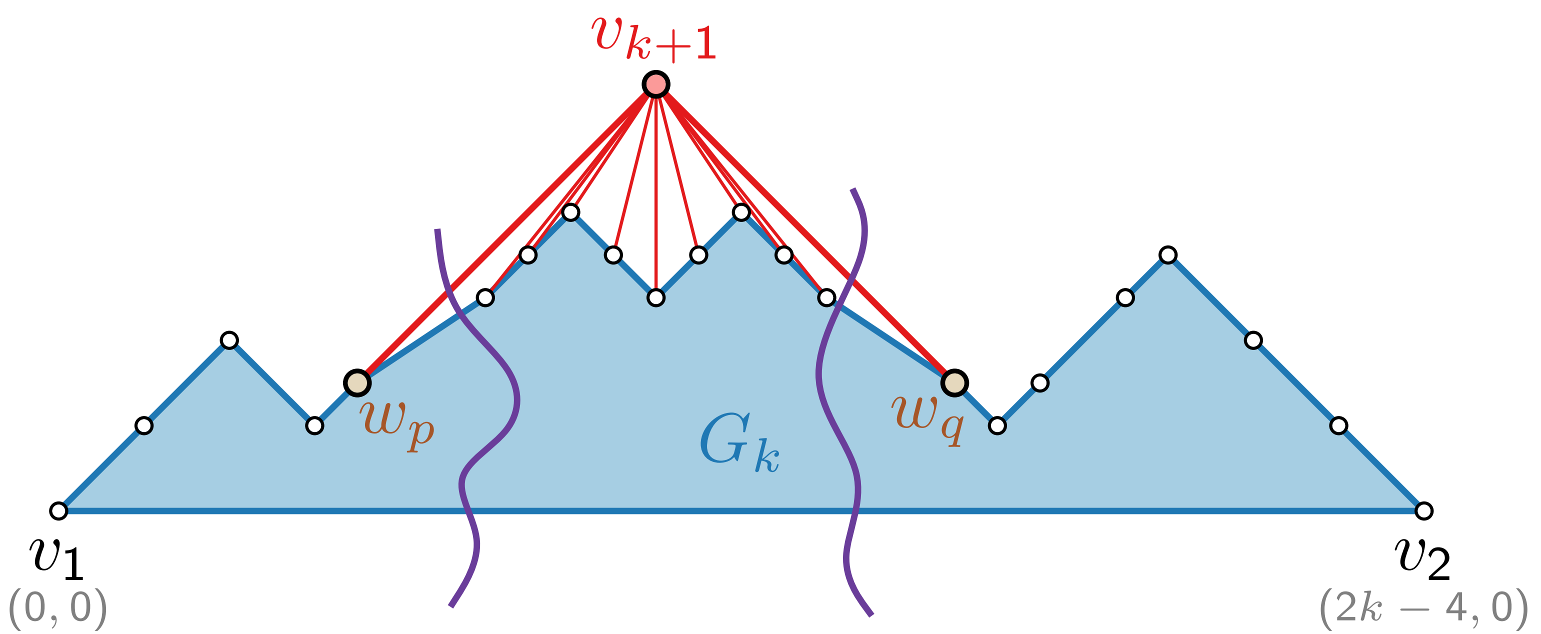

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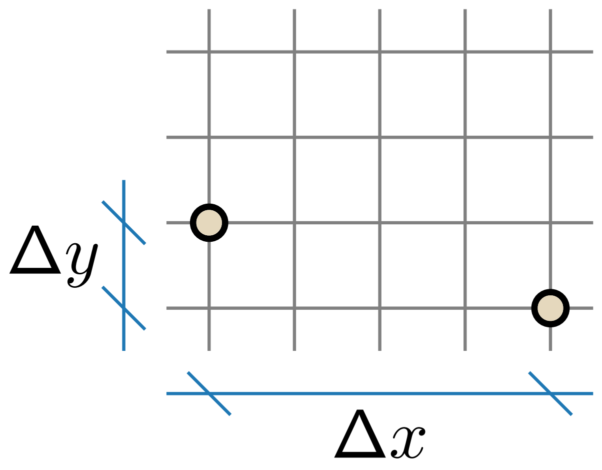

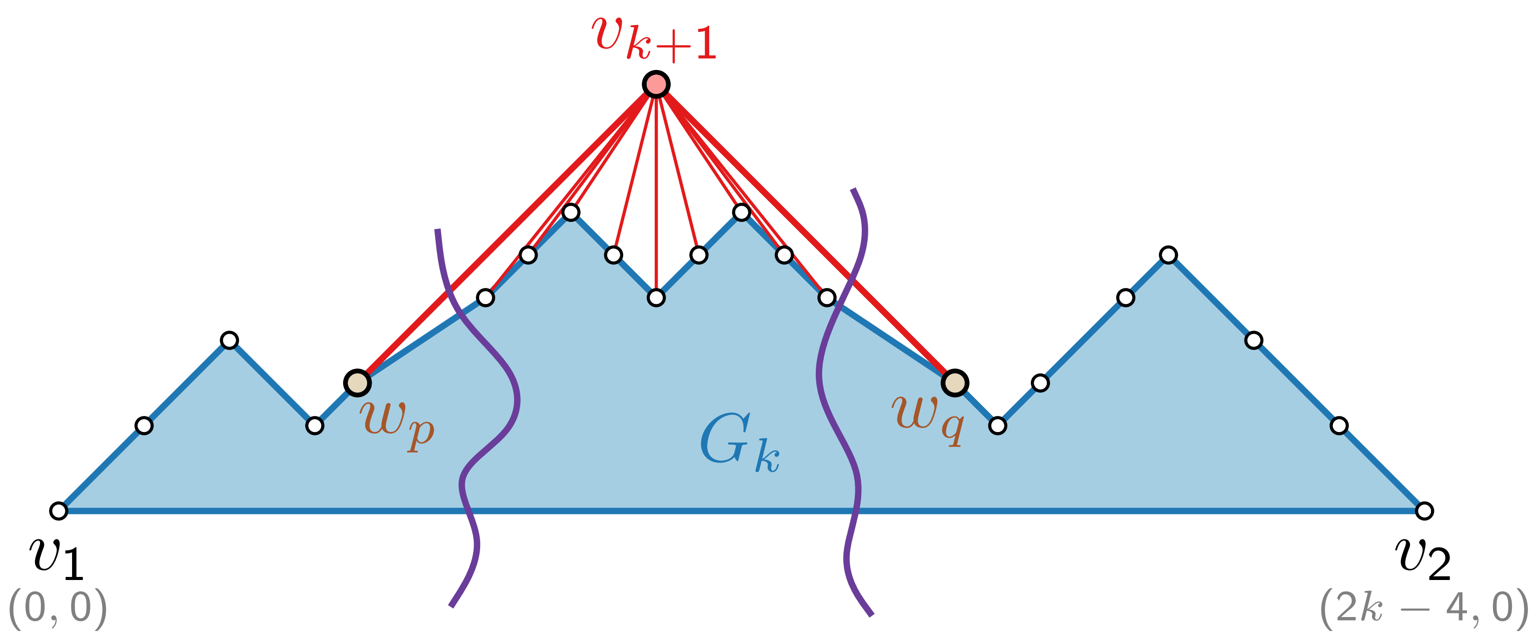

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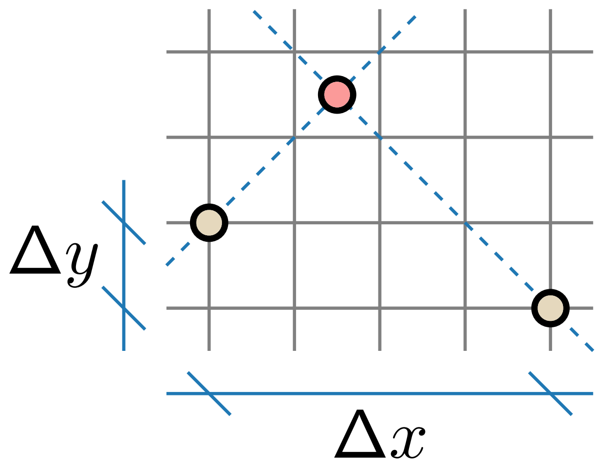

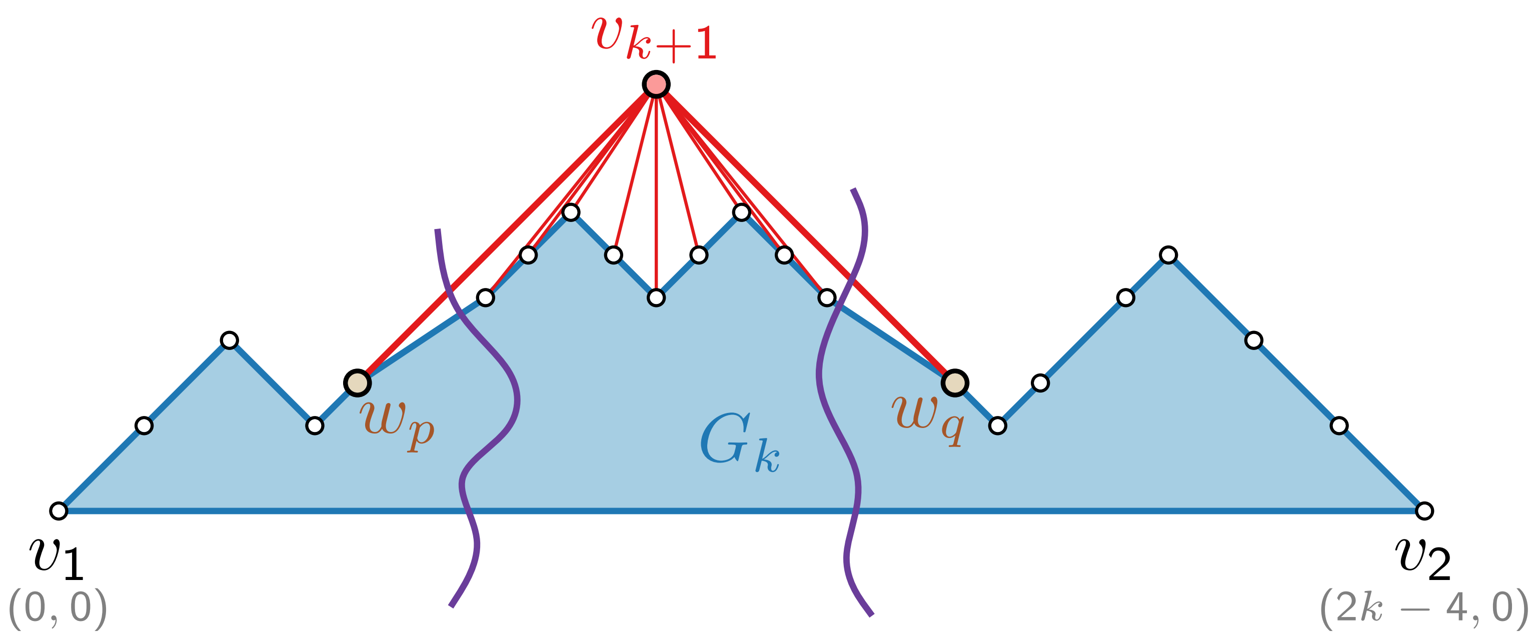

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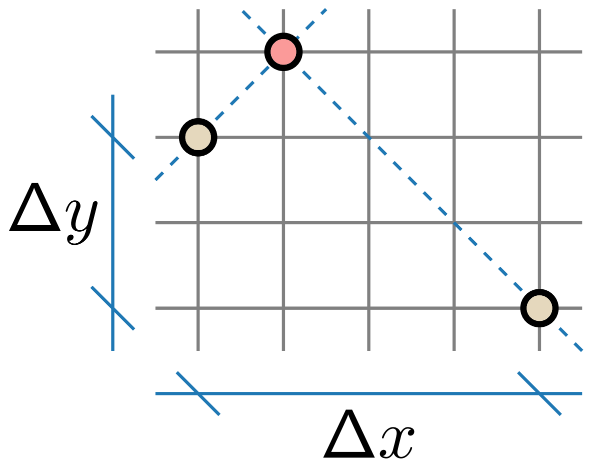

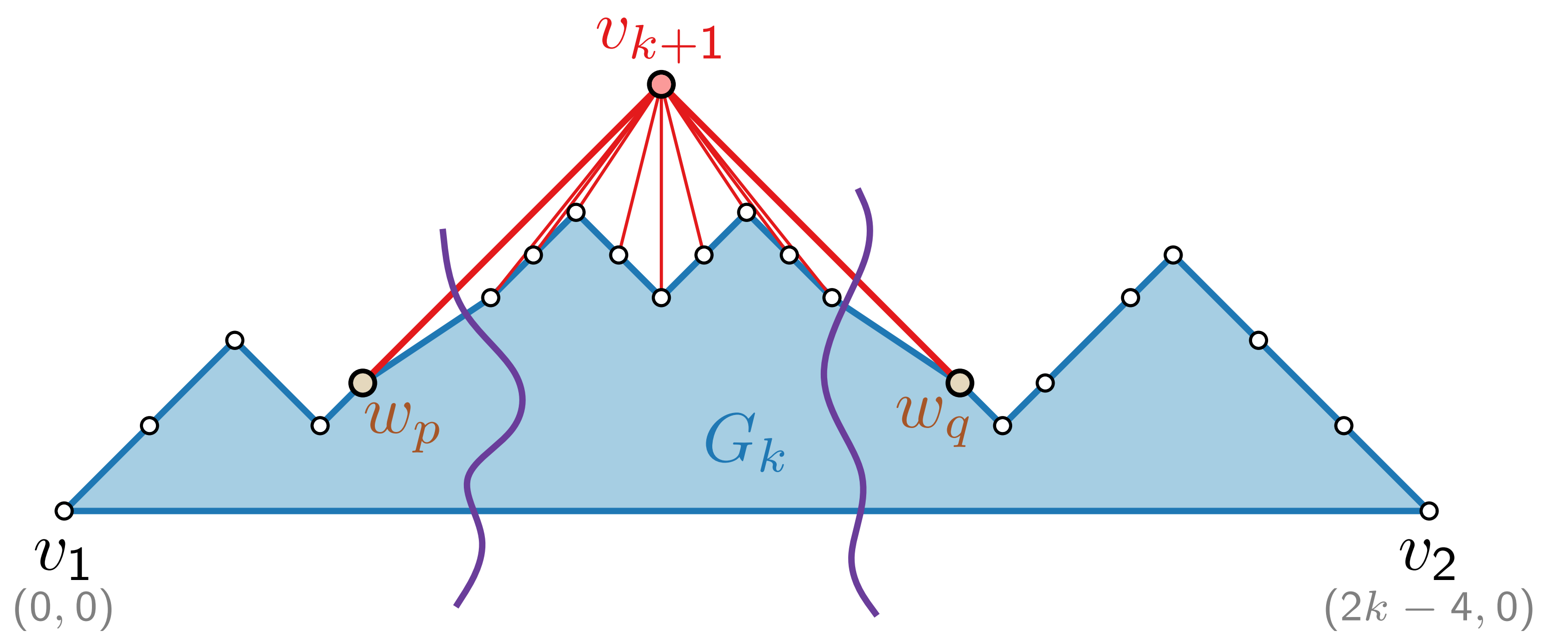

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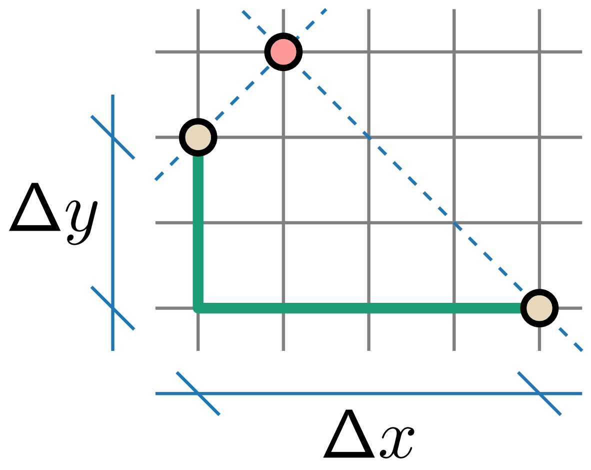

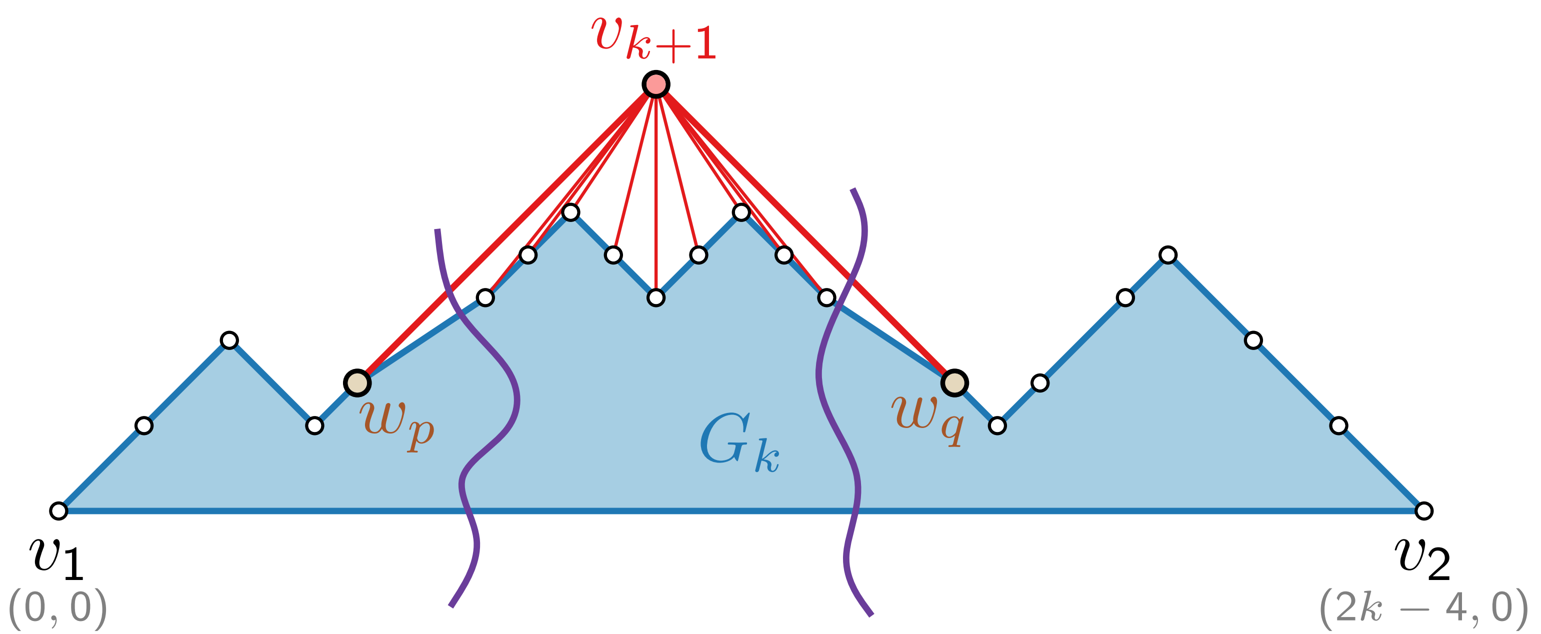

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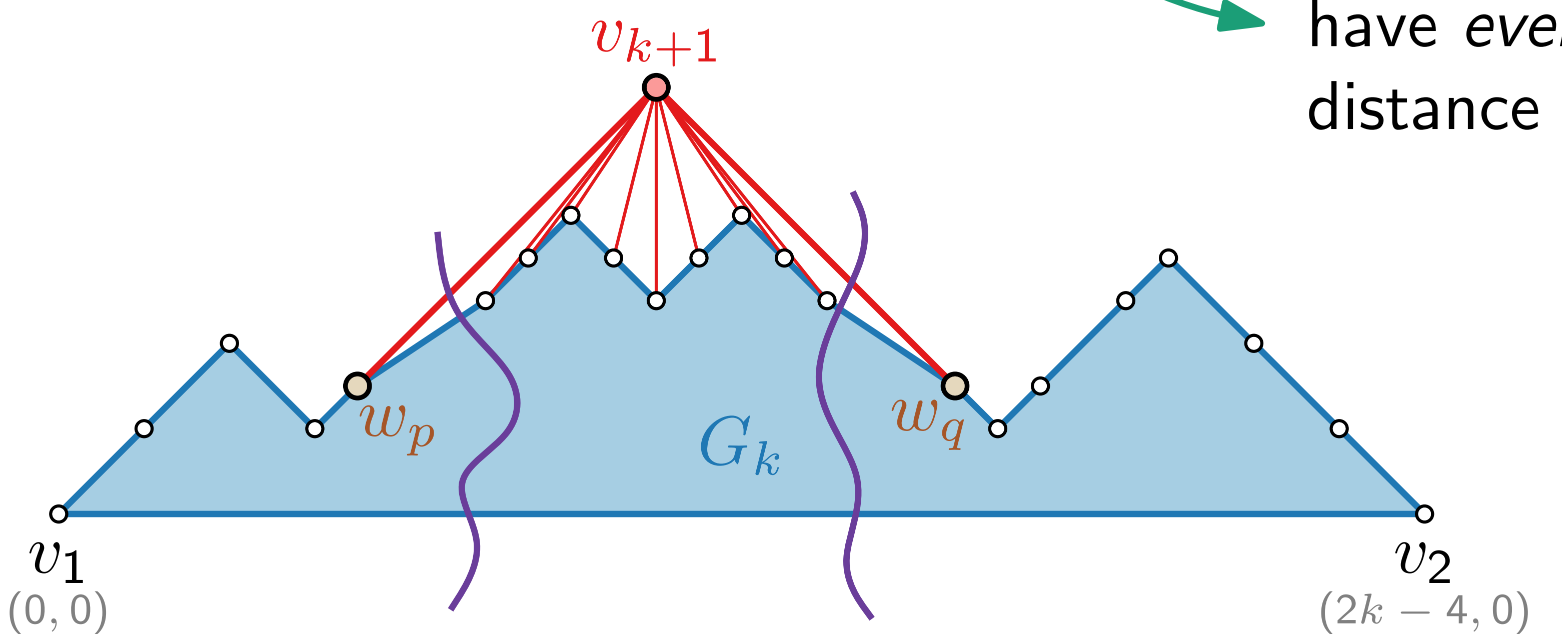

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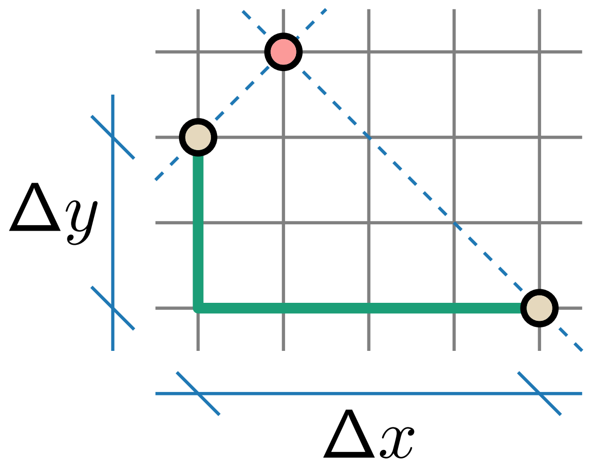

Yes, because w_p and w_q have *even* Manhattan distance $\Delta x + \Delta y$.

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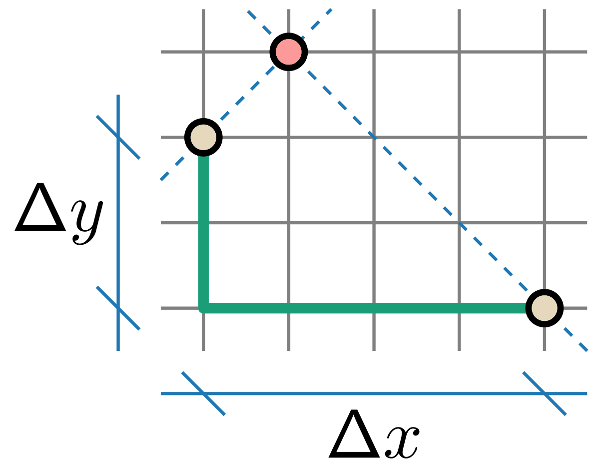
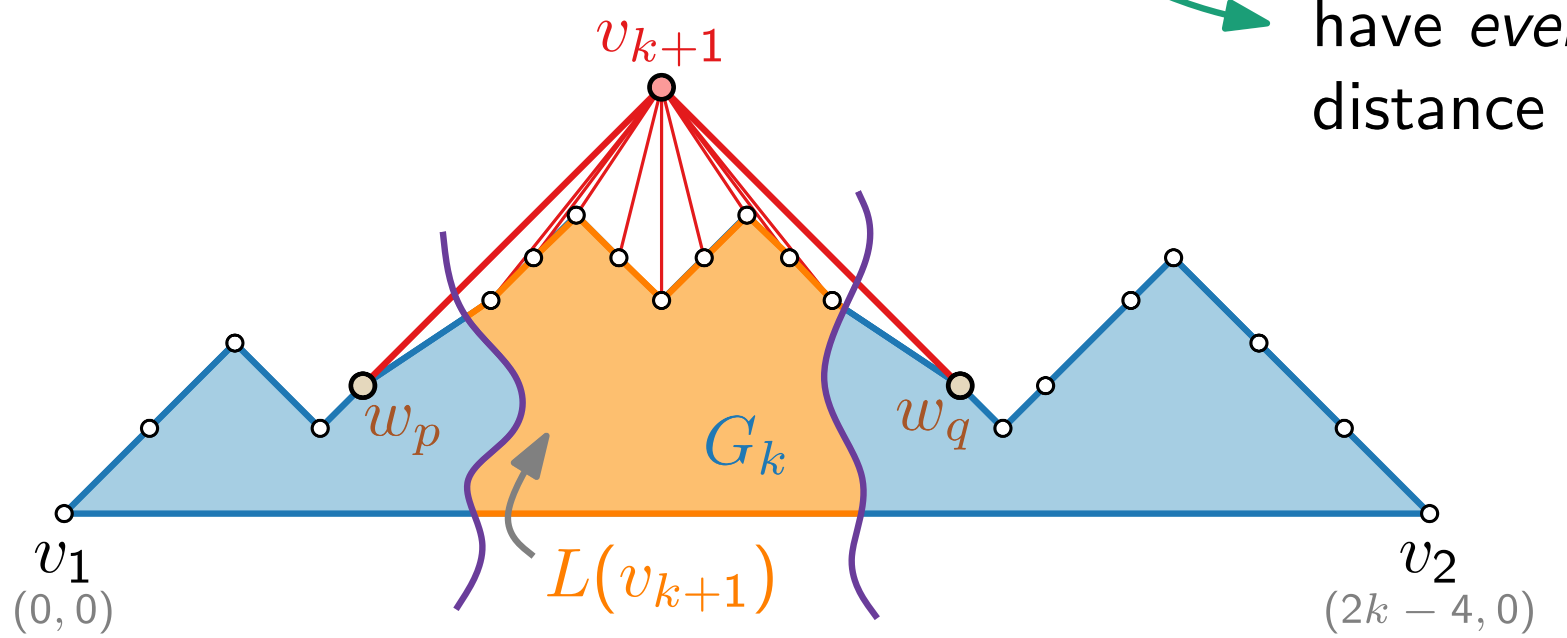
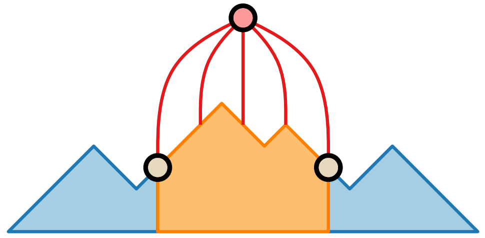


Yes, because w_p and w_q have *even* Manhattan distance $\Delta x + \Delta y$.



Shift Method – Example



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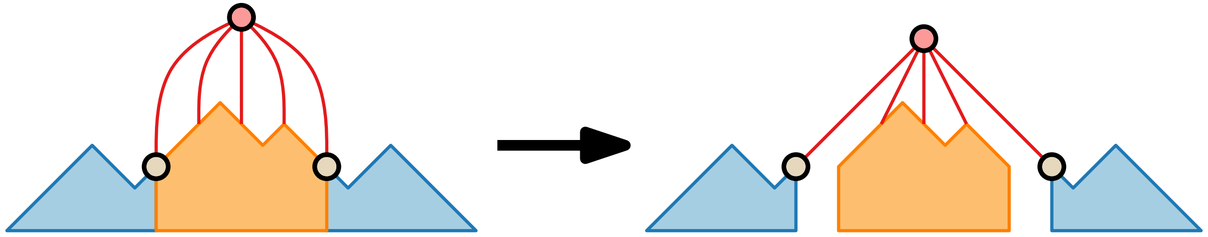

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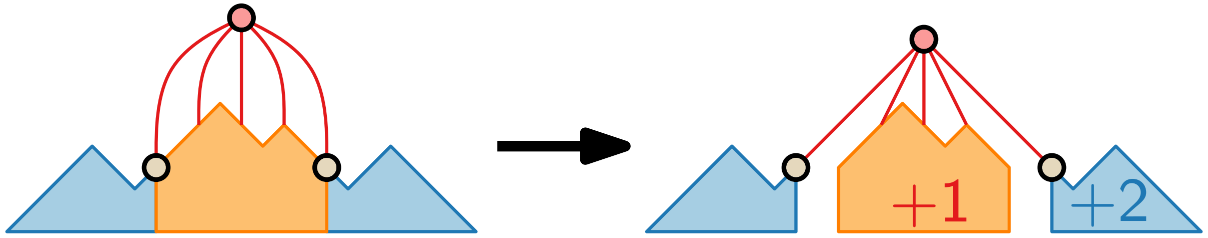

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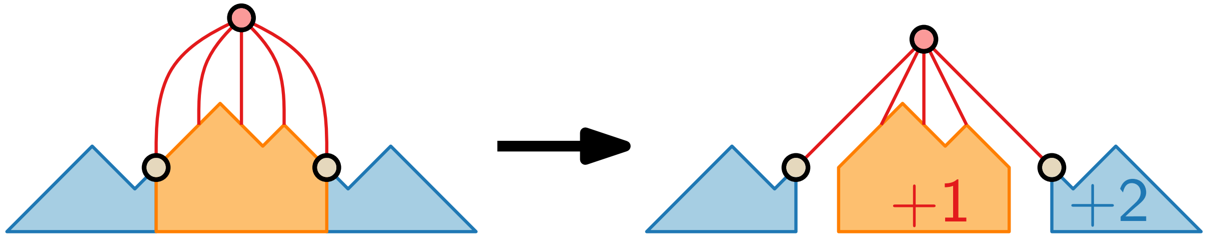

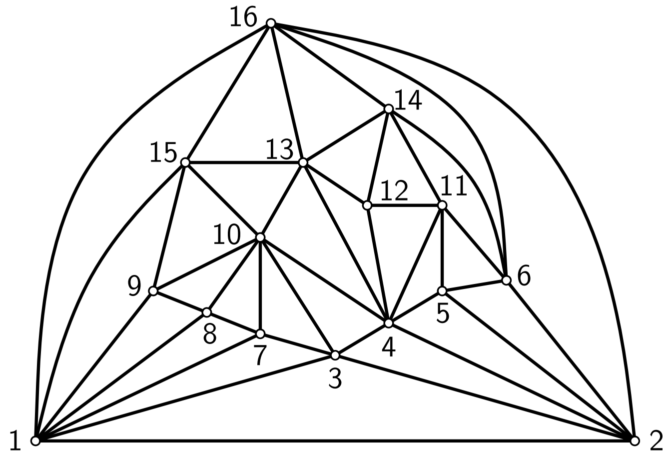

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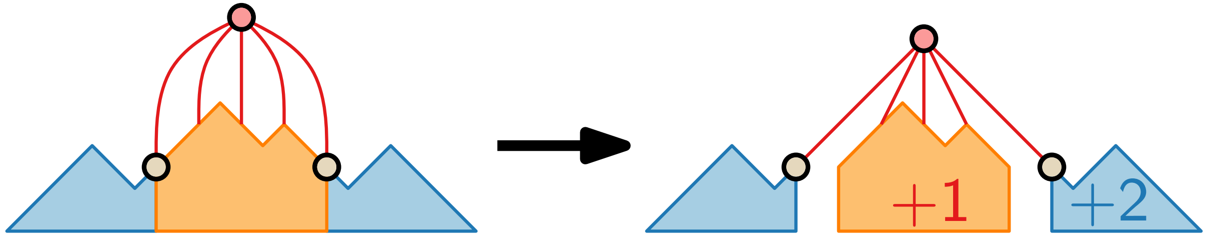

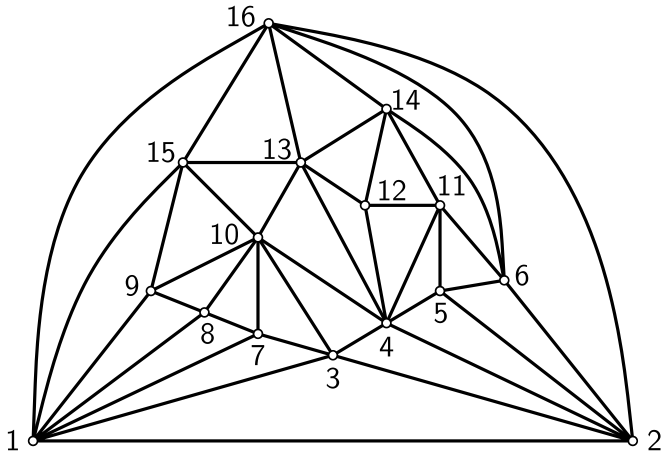

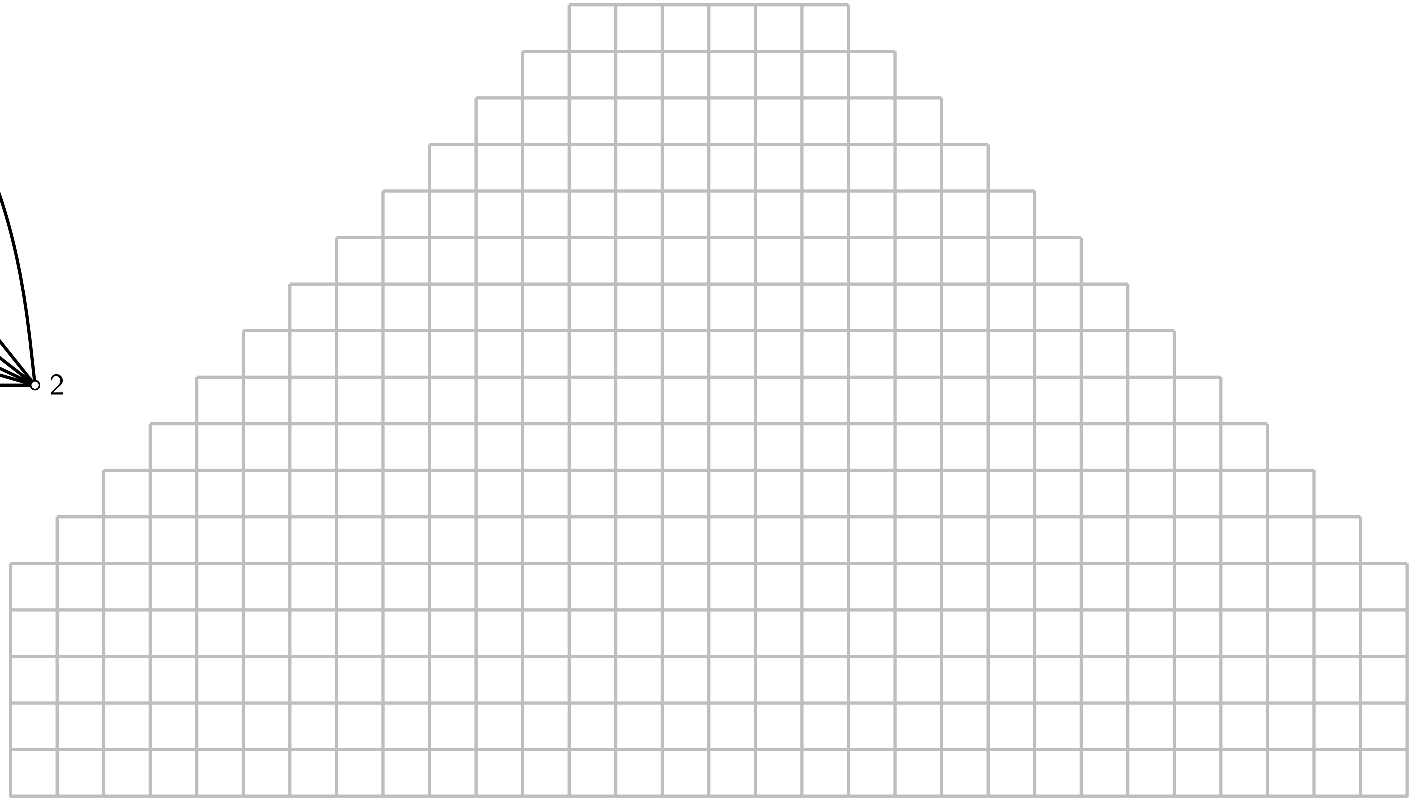

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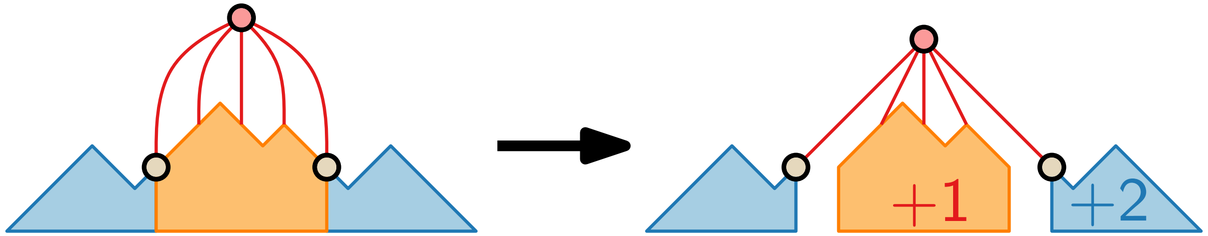

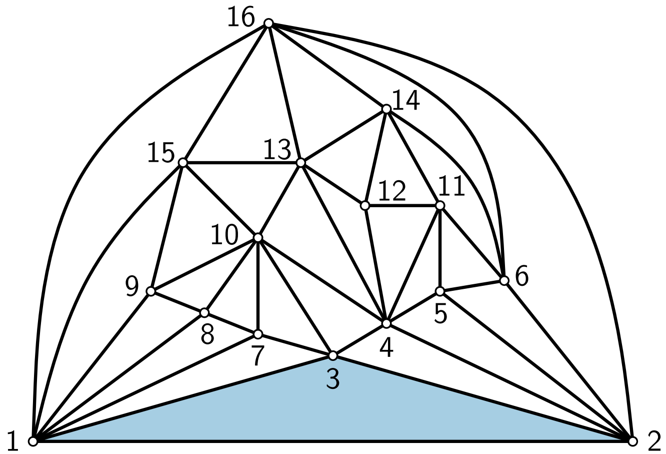

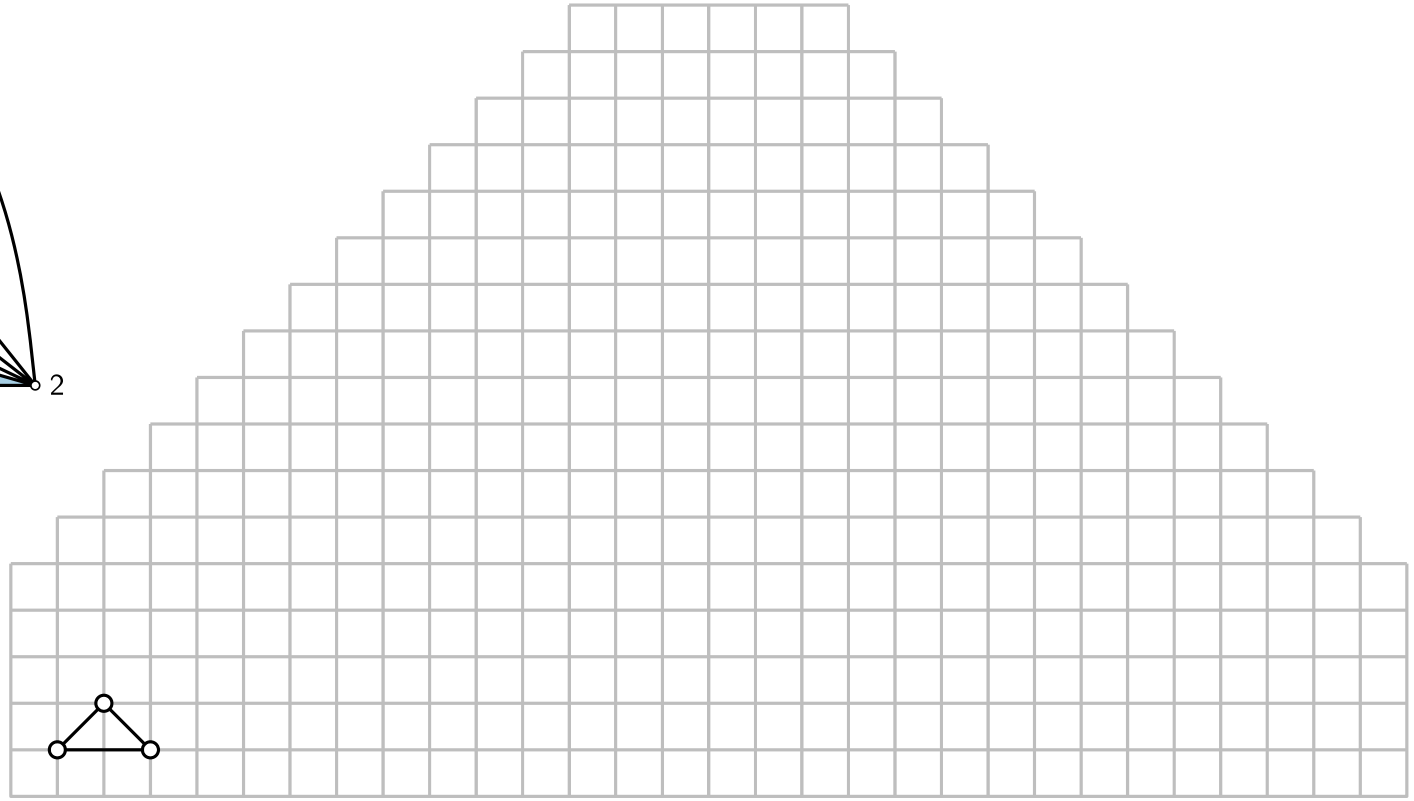

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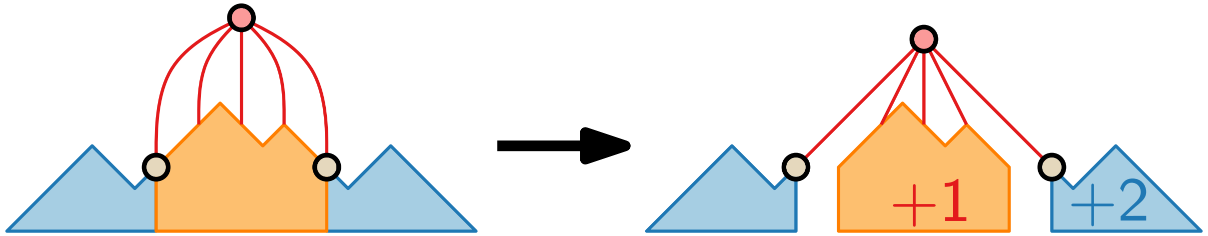

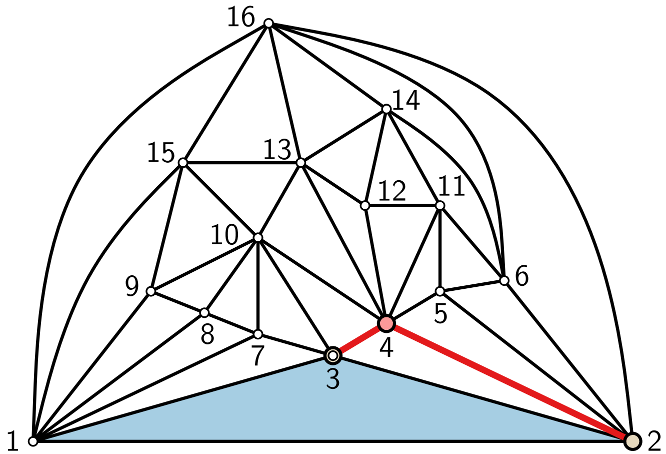

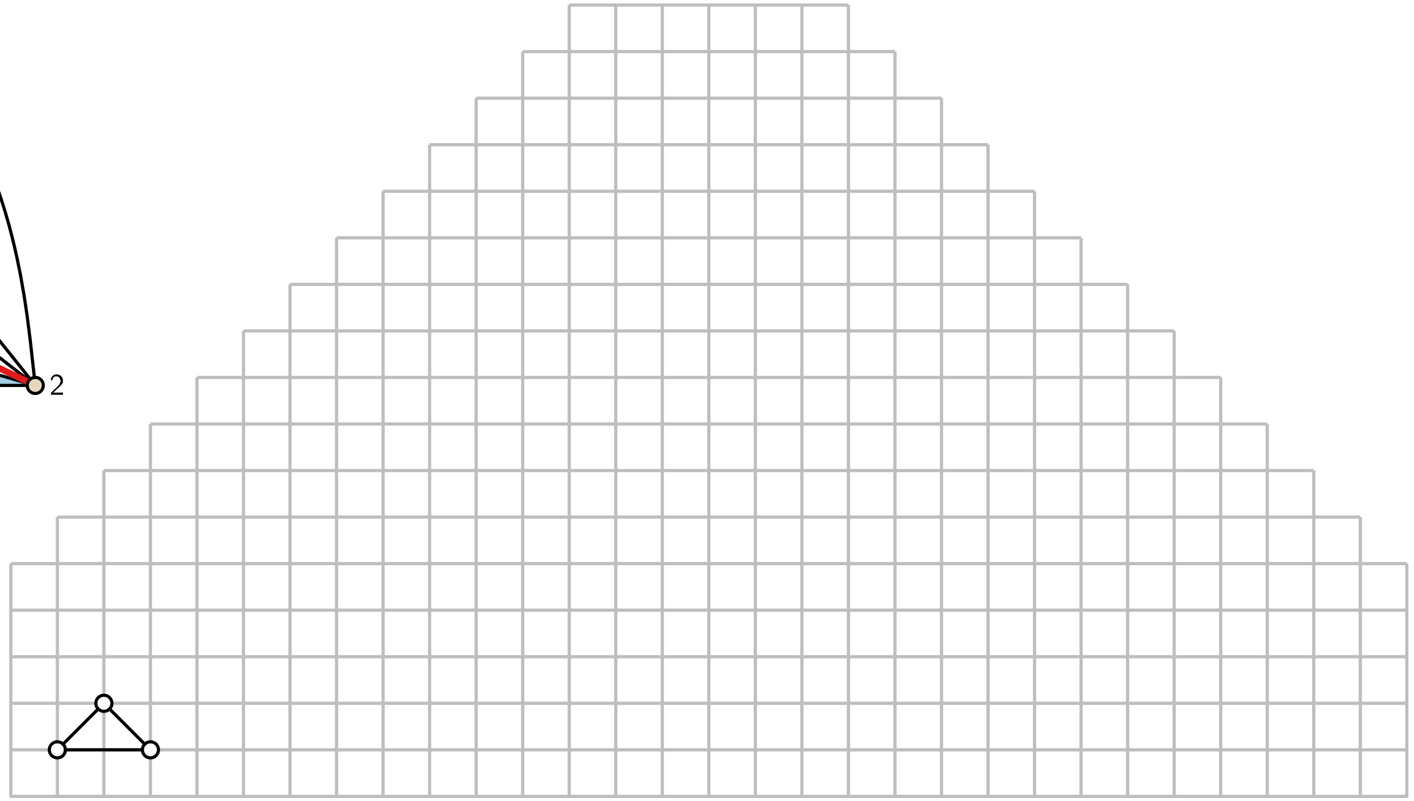

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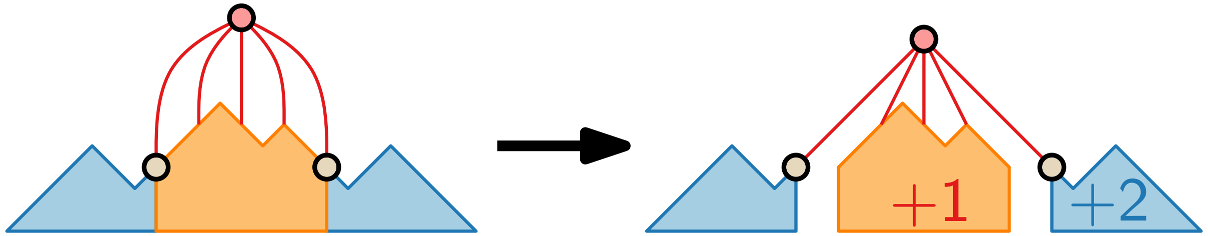

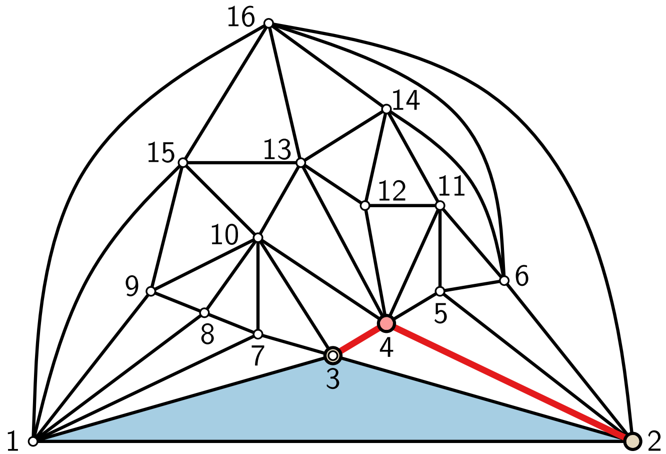

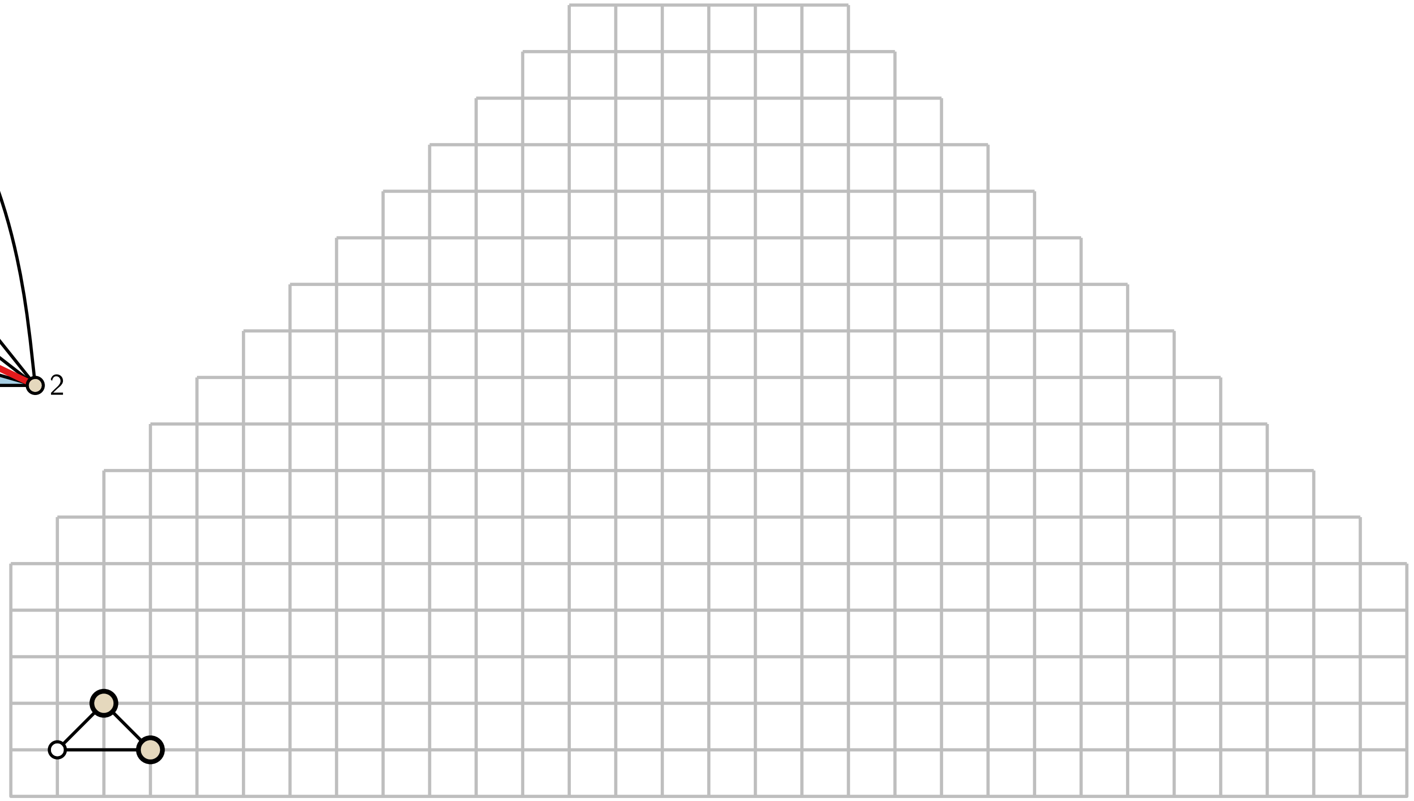

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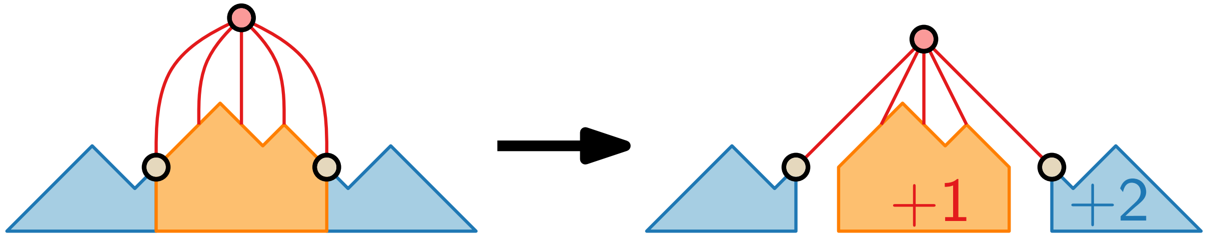

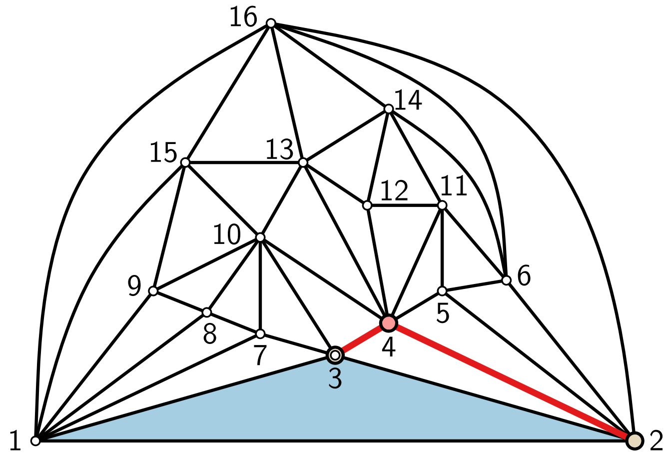

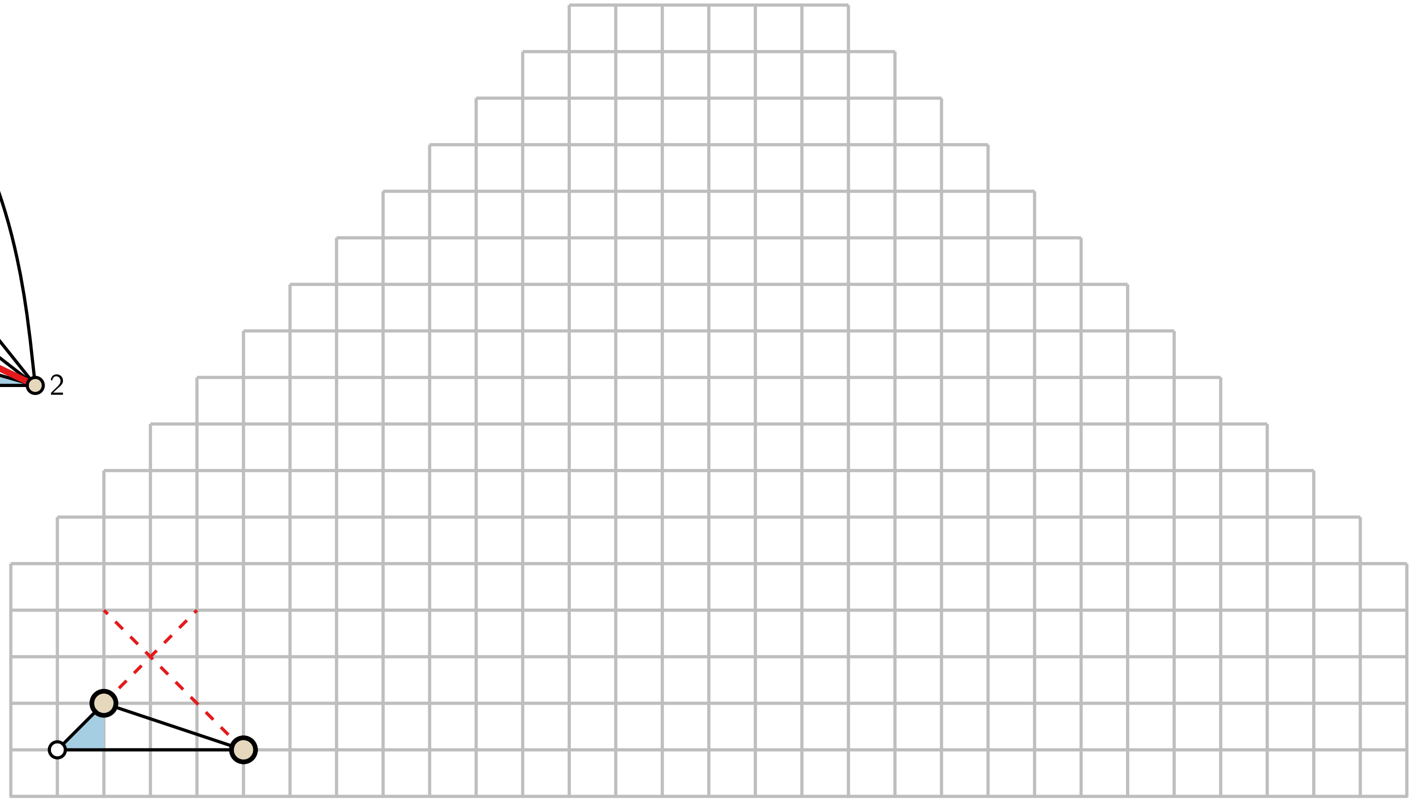

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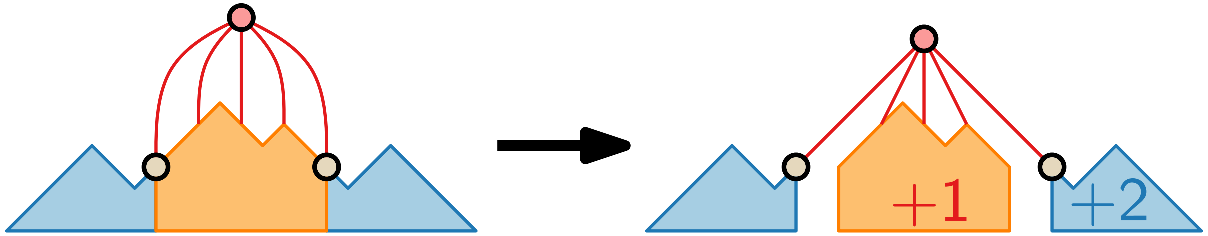

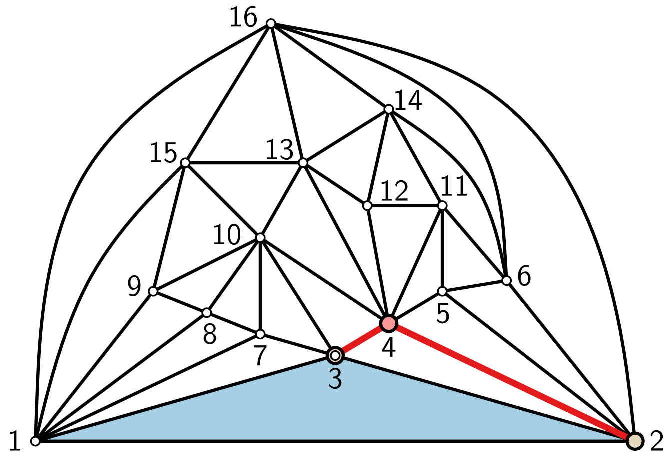

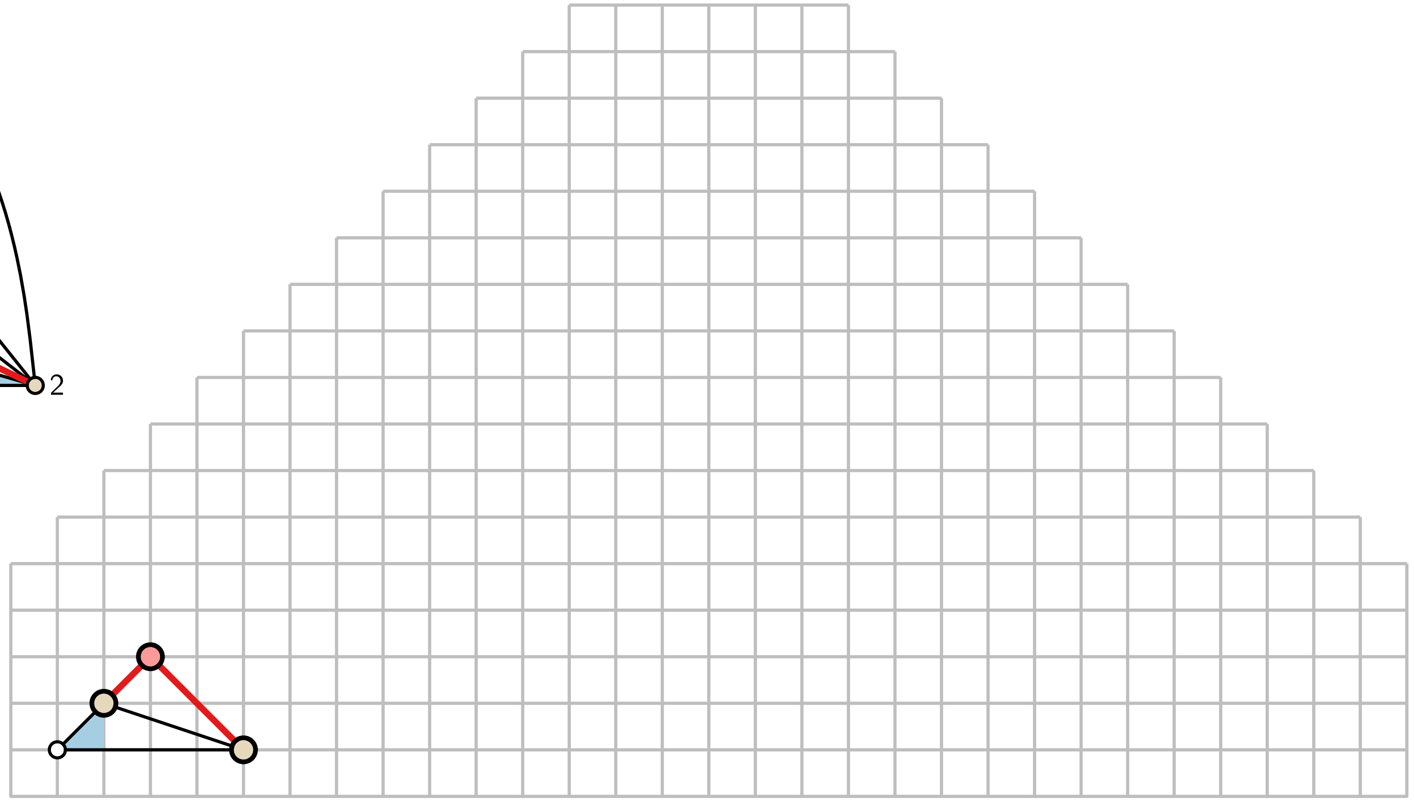

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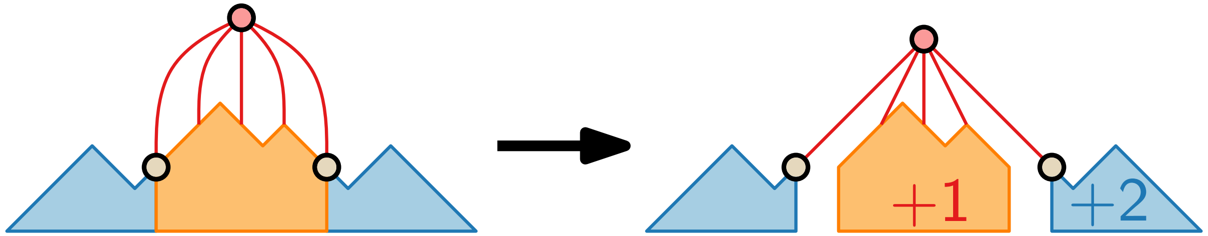

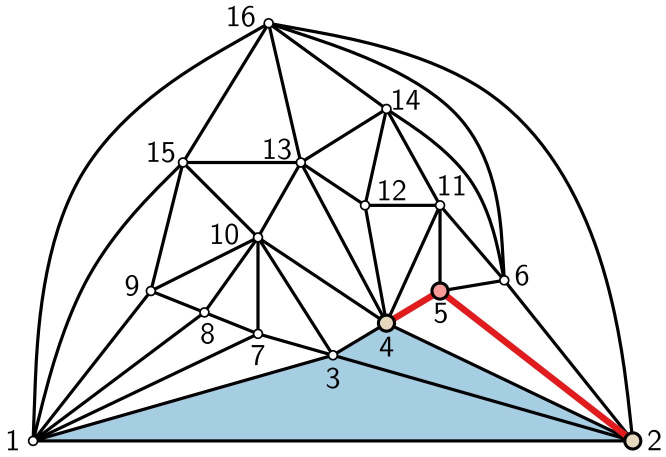

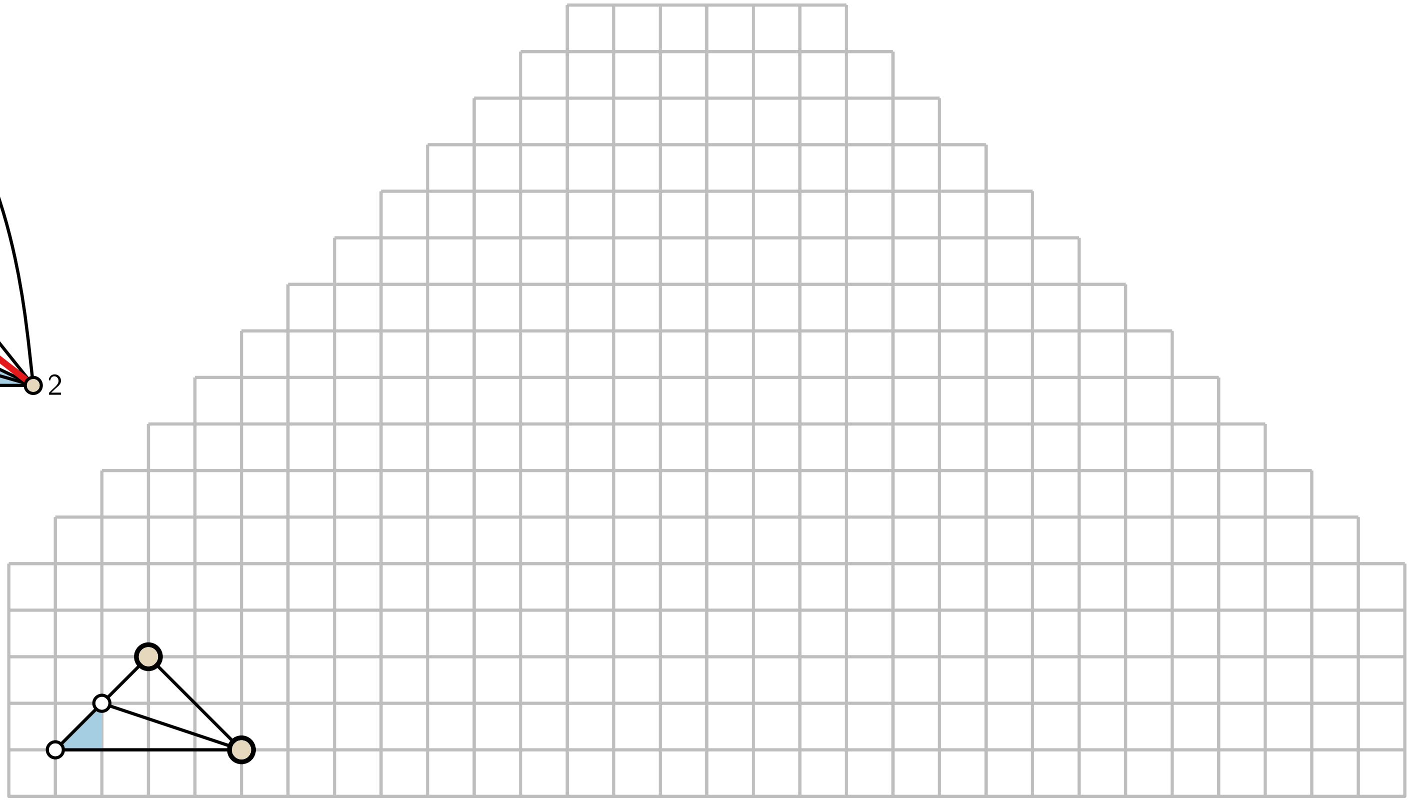

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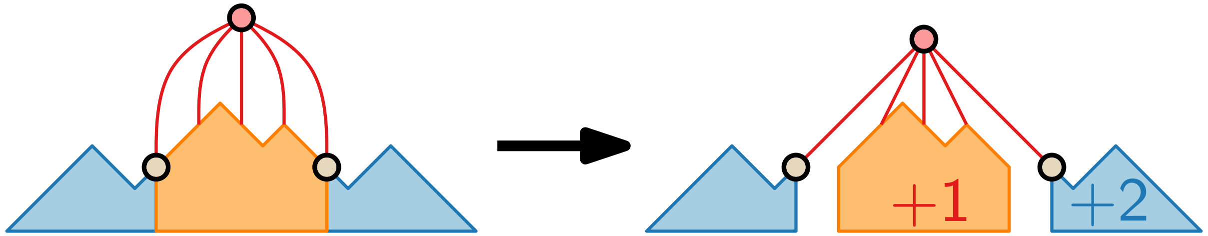

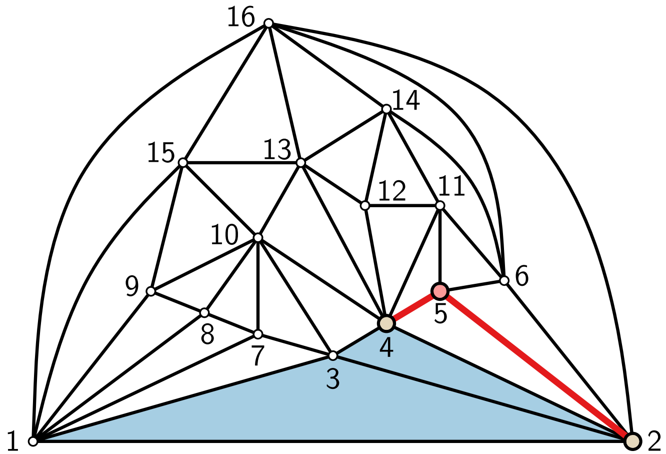

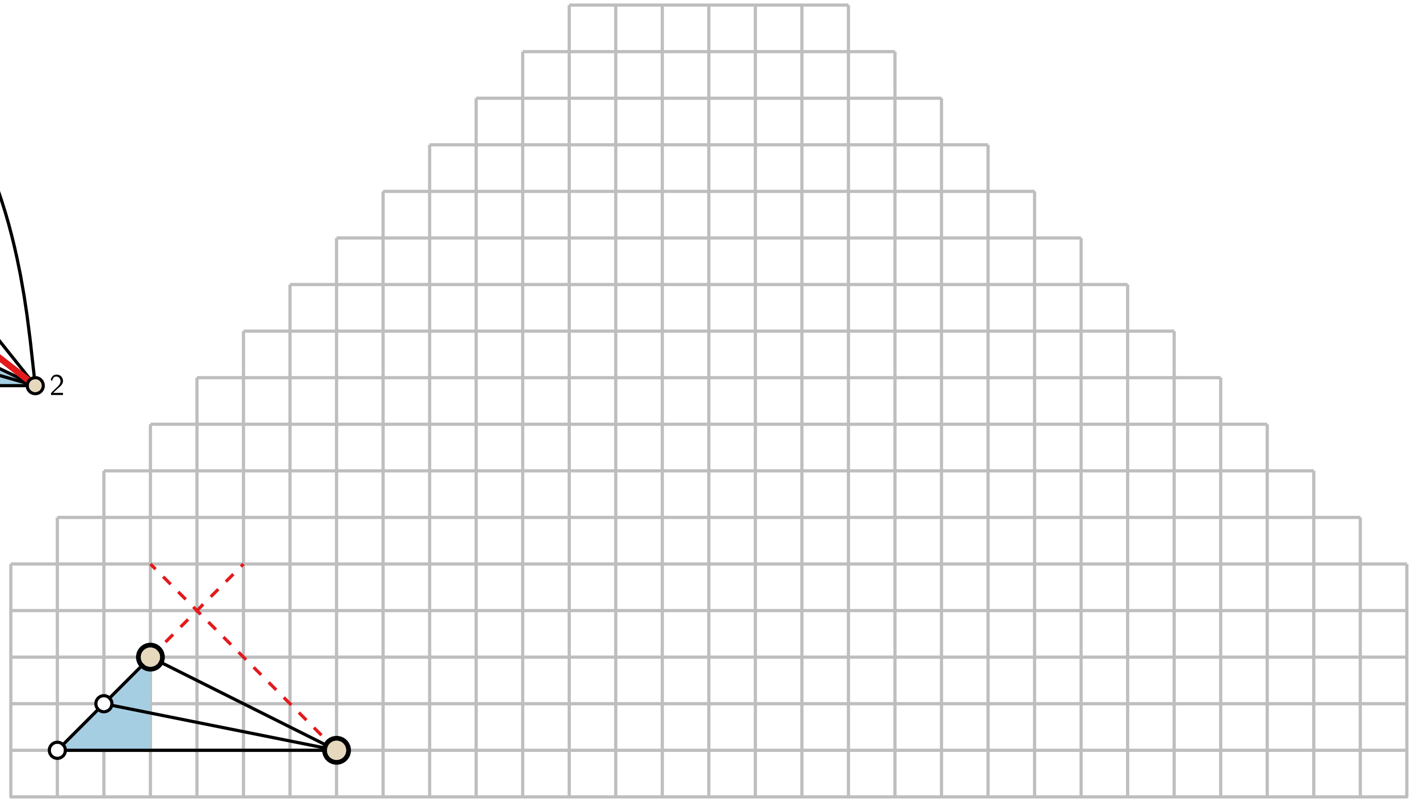

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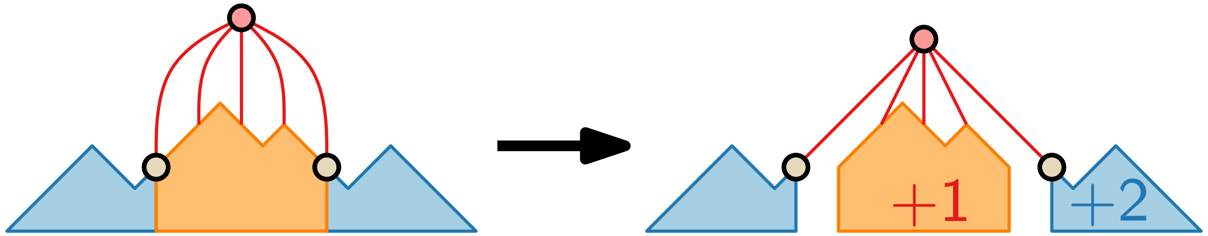

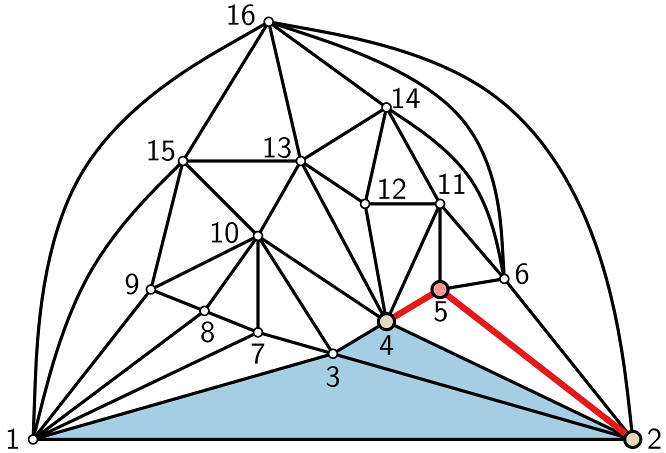

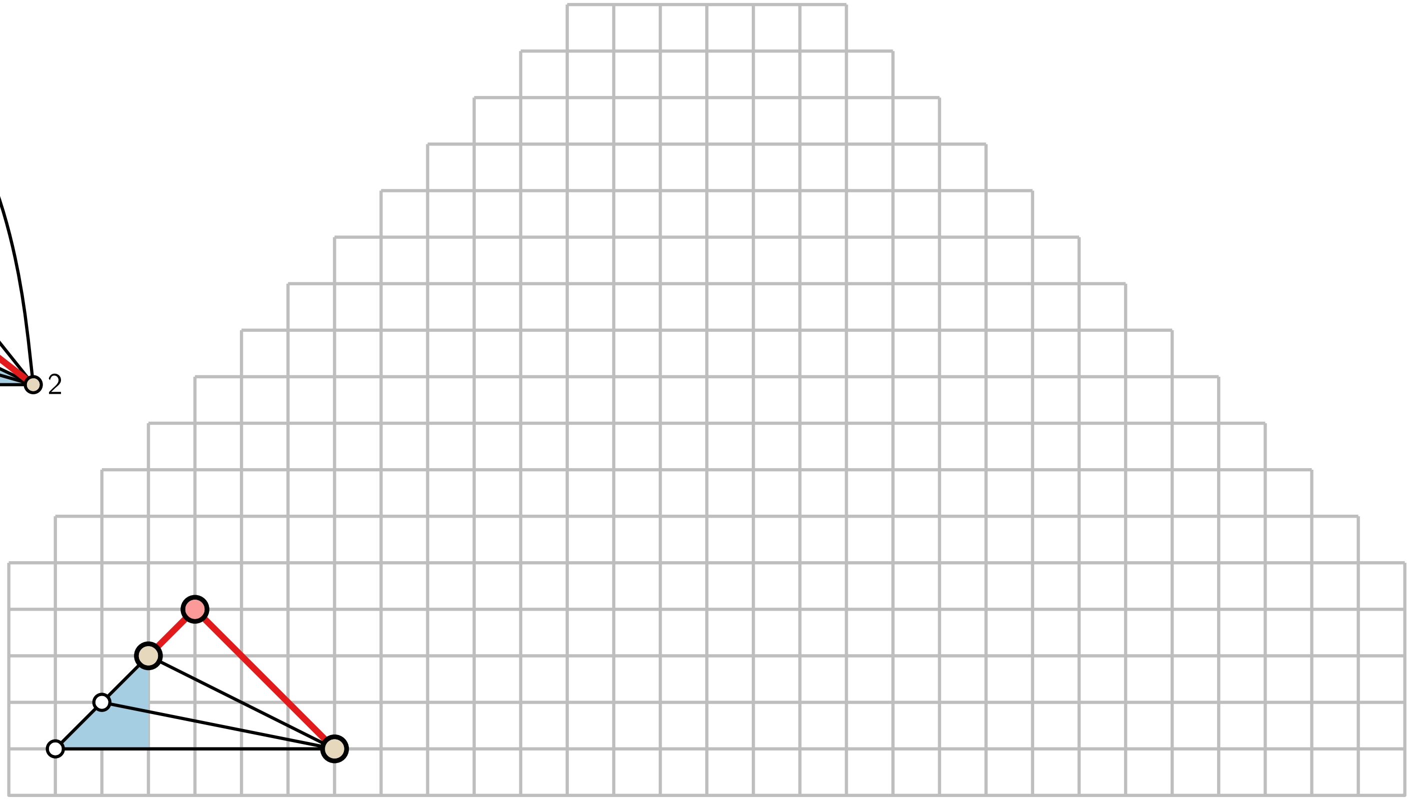

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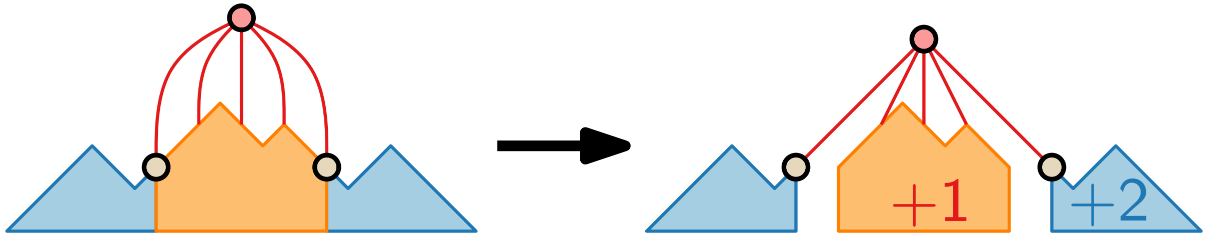

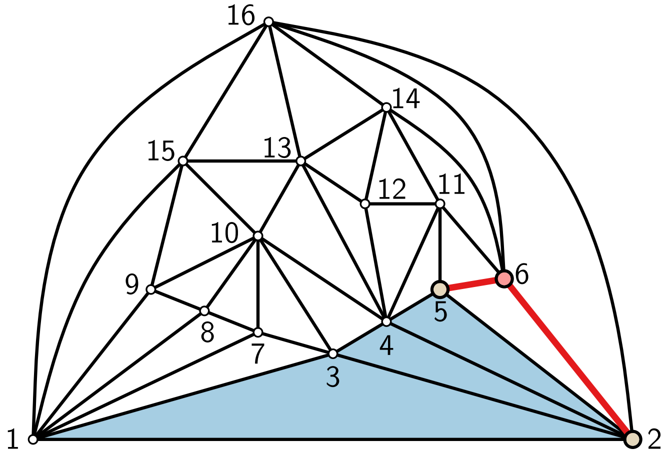

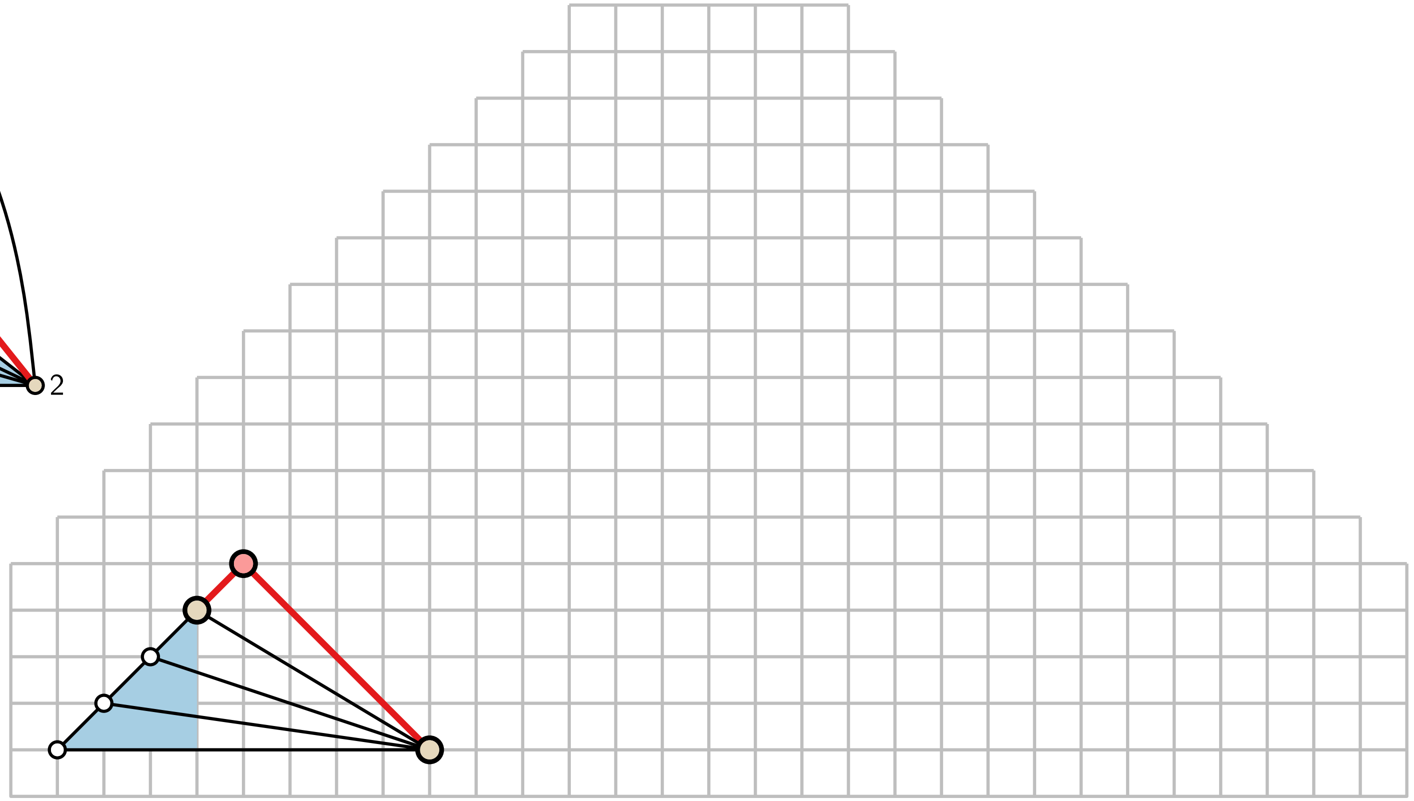

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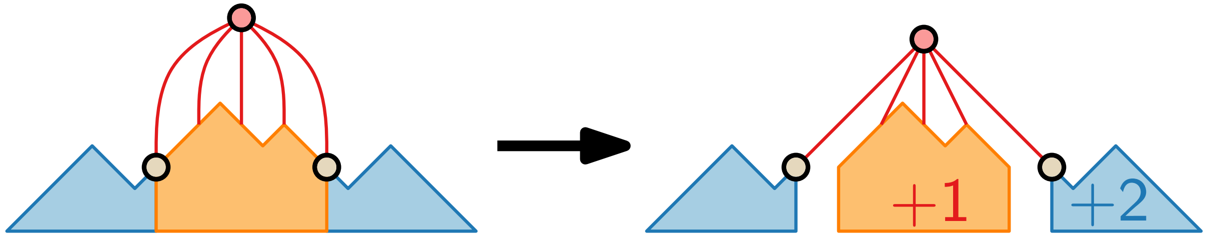

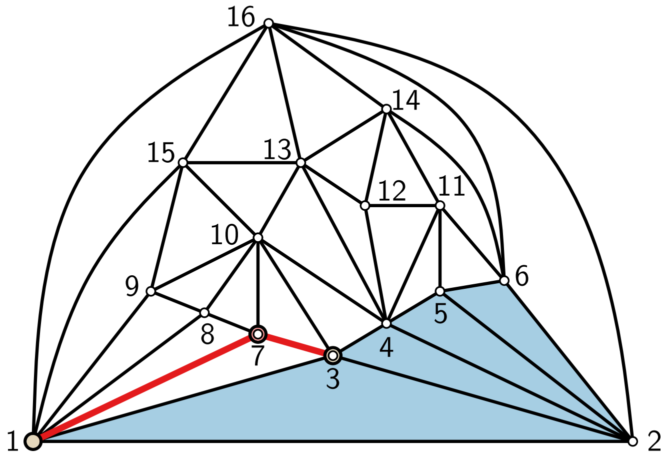

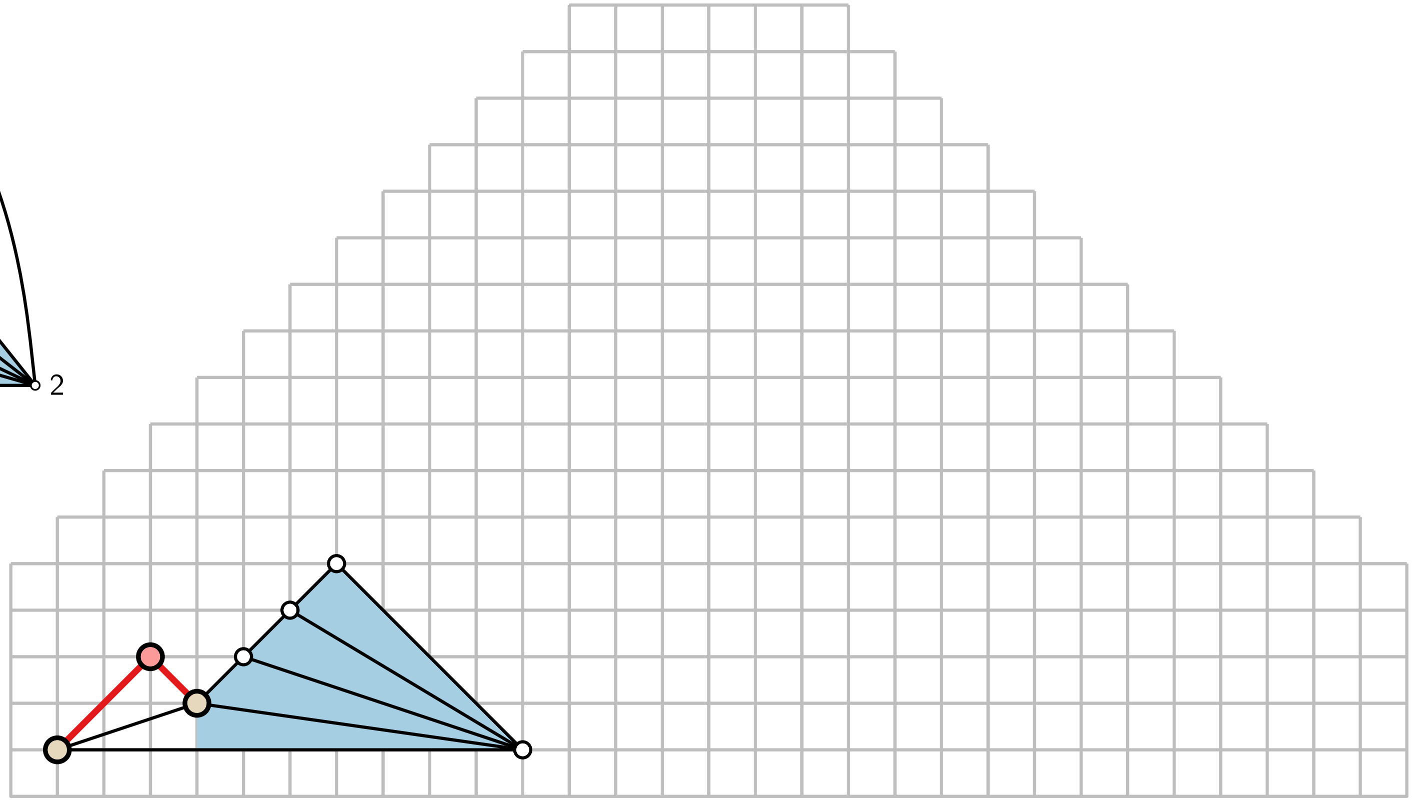

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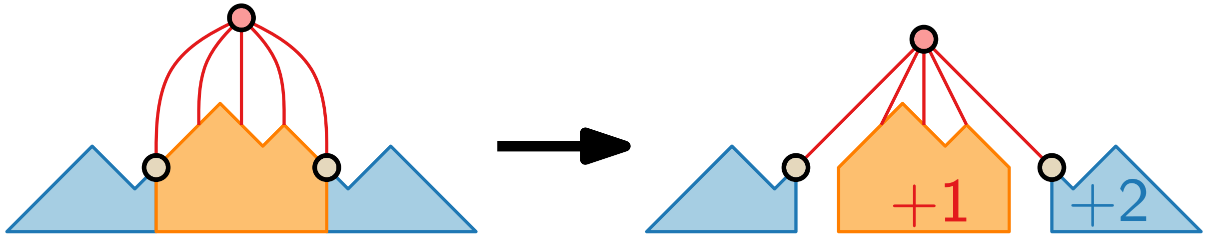

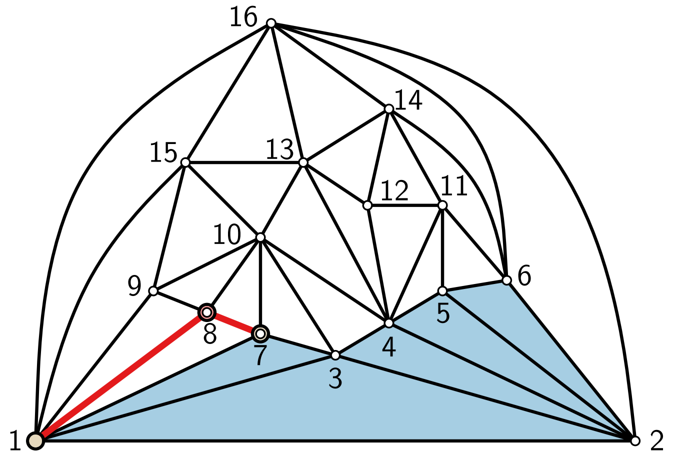

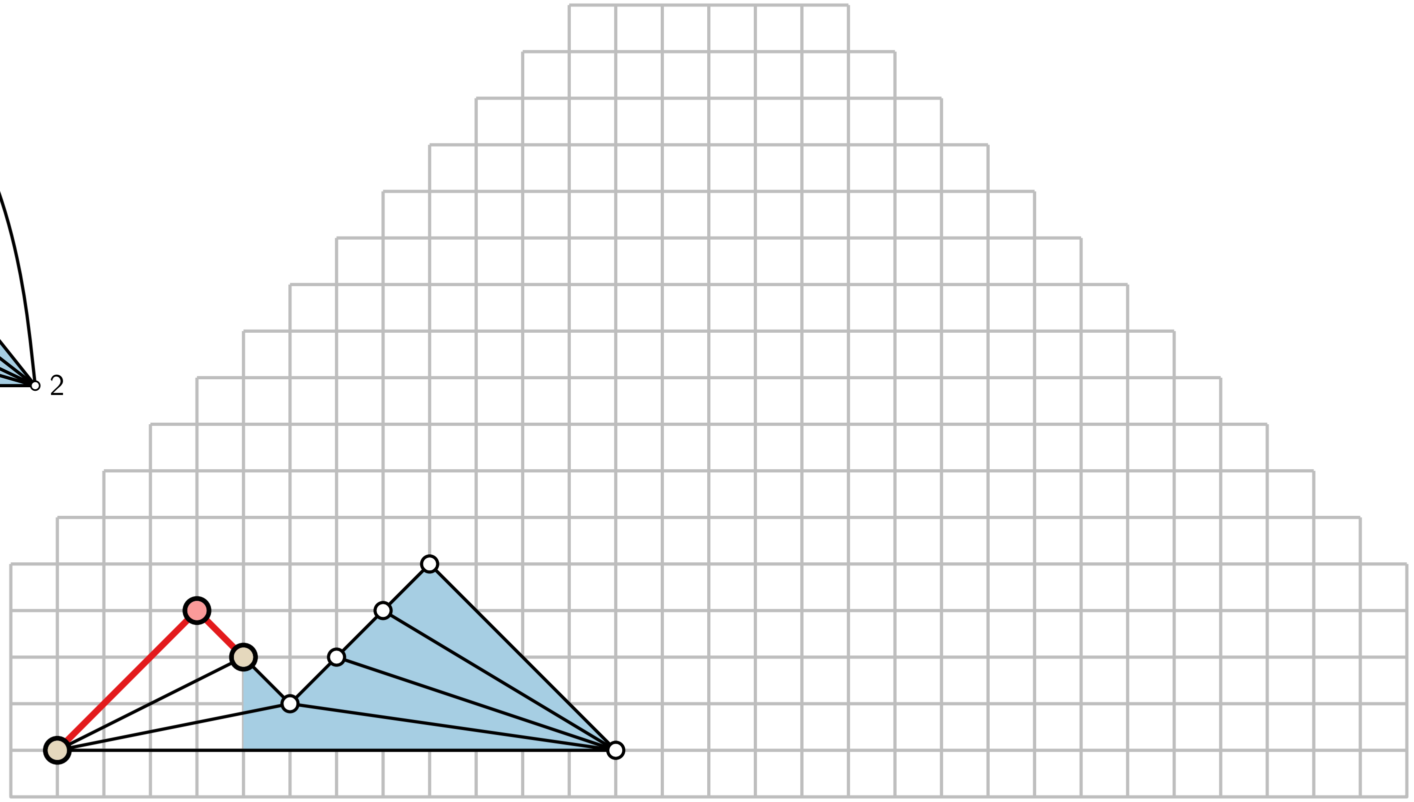

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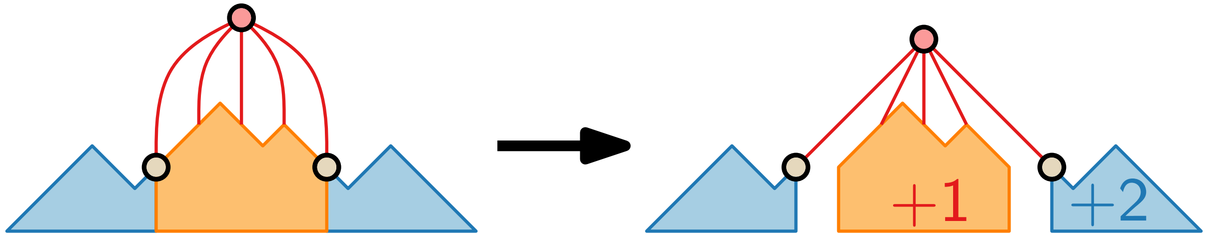

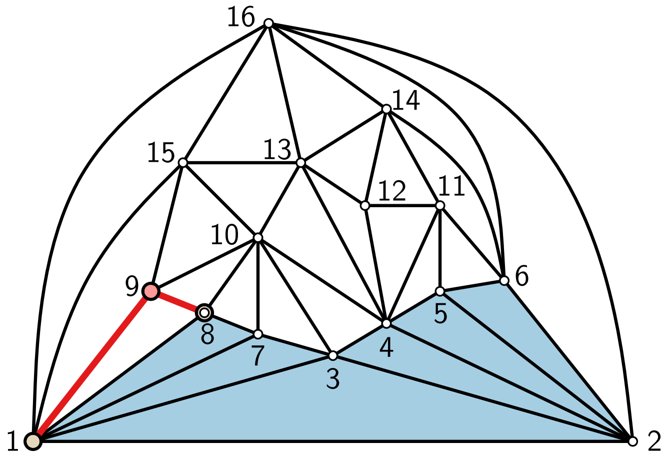

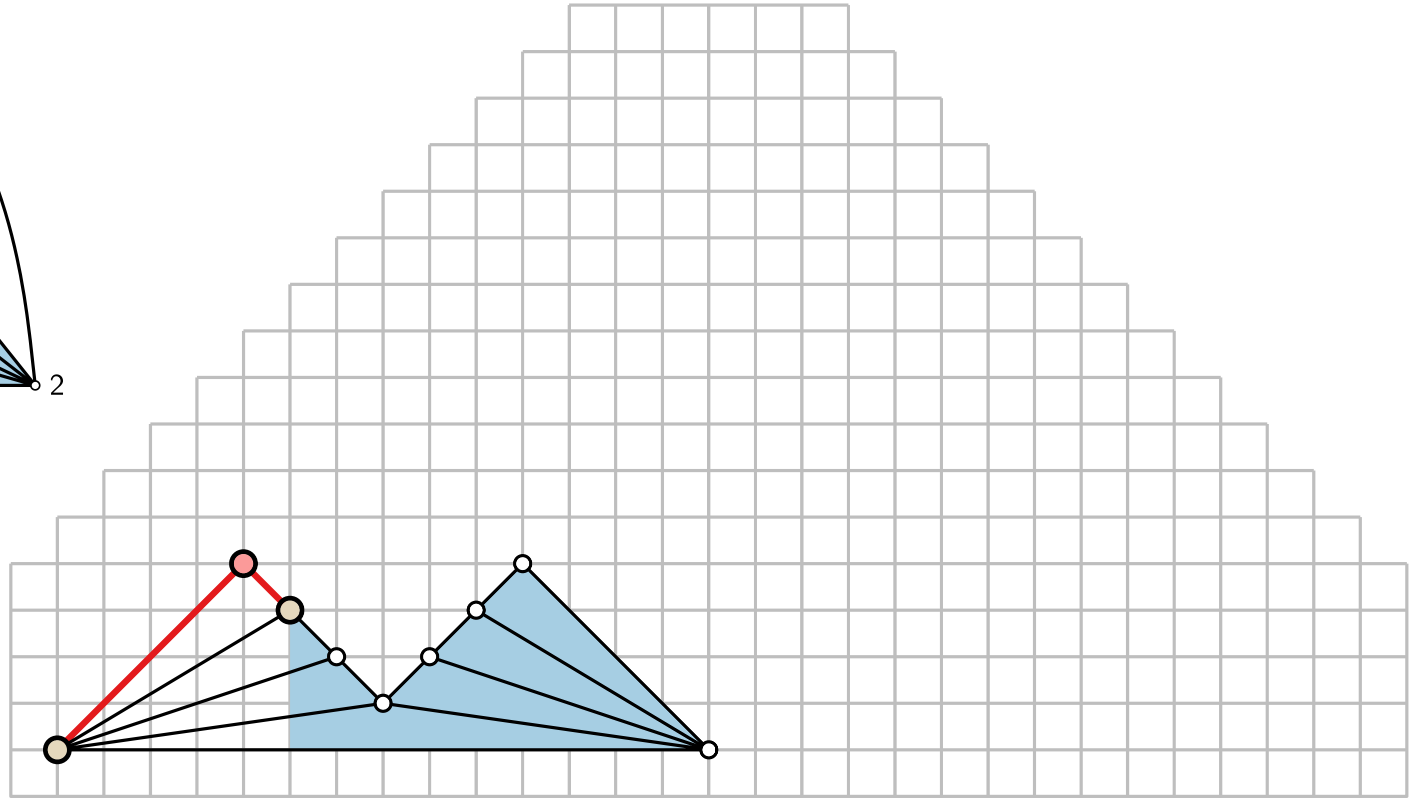

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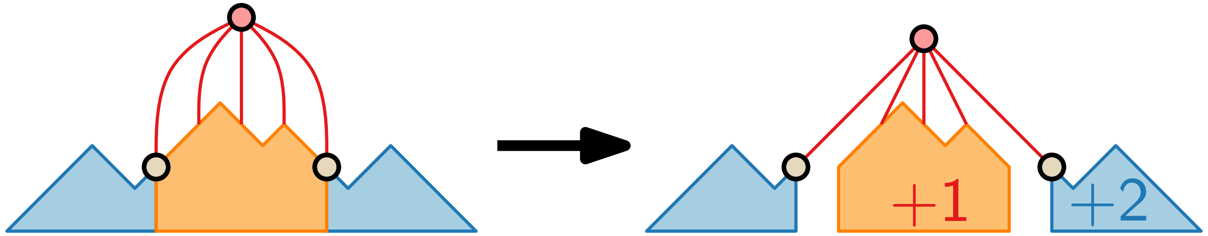

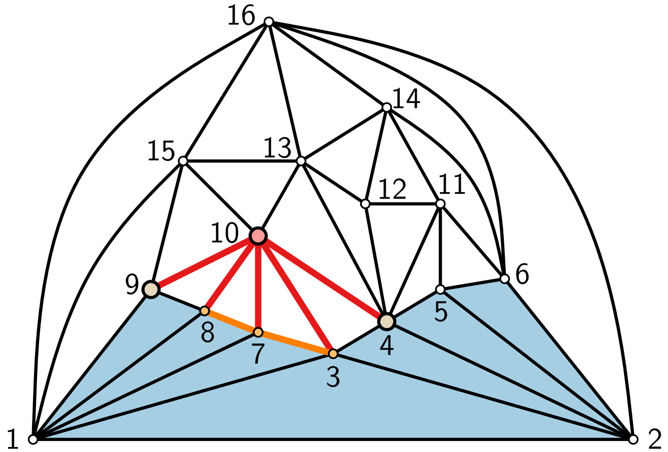

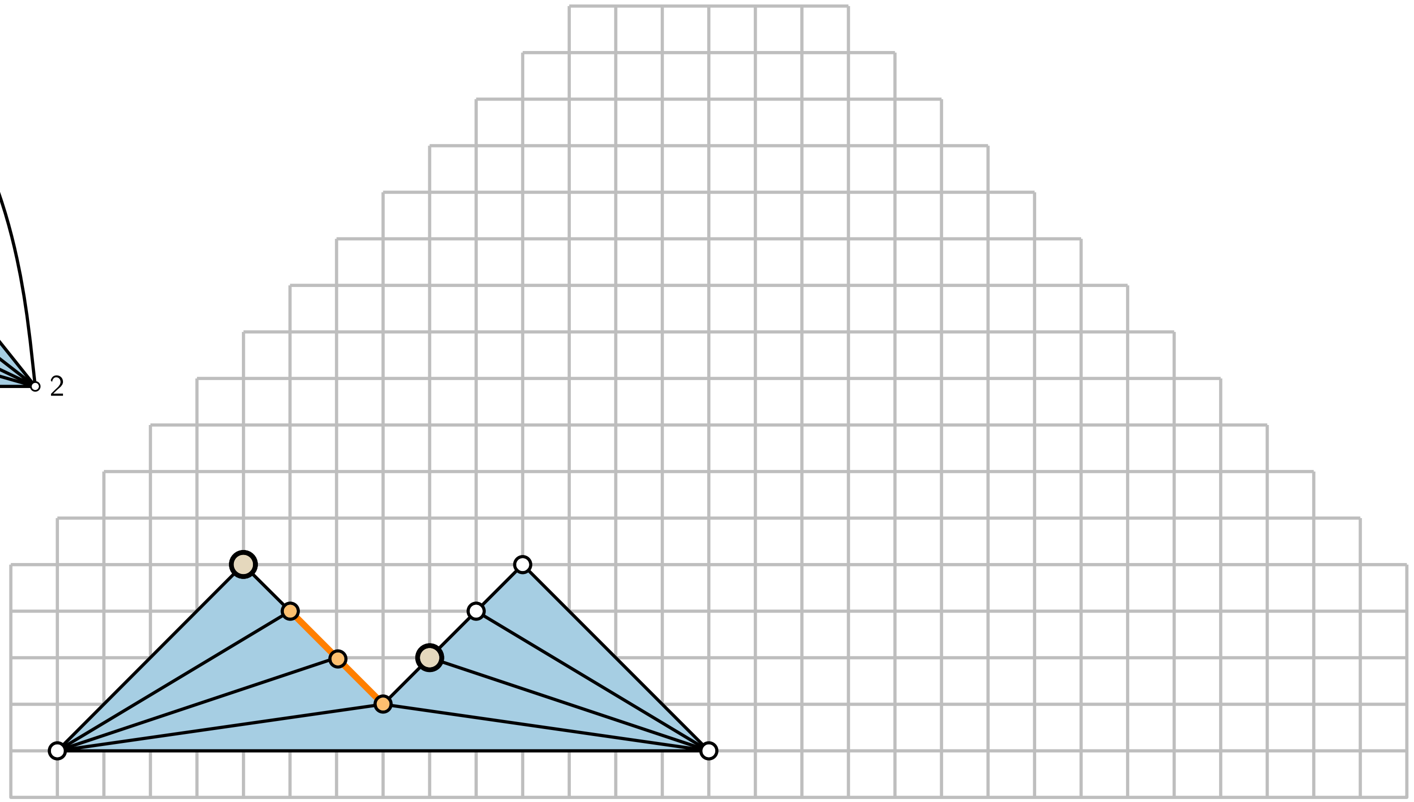

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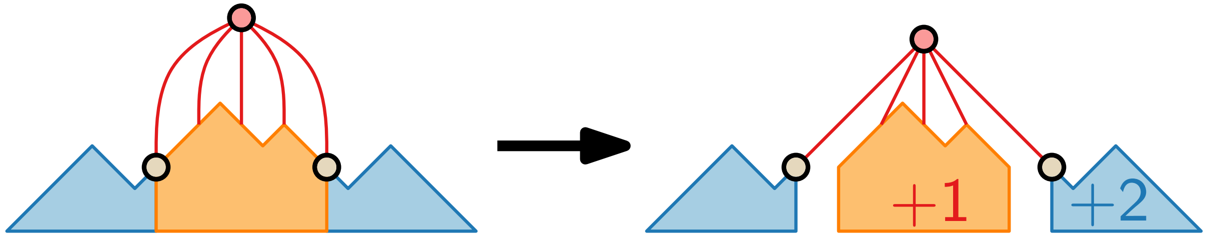

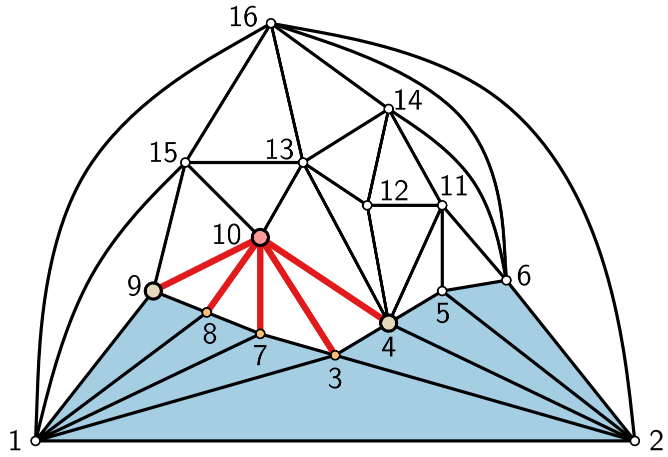

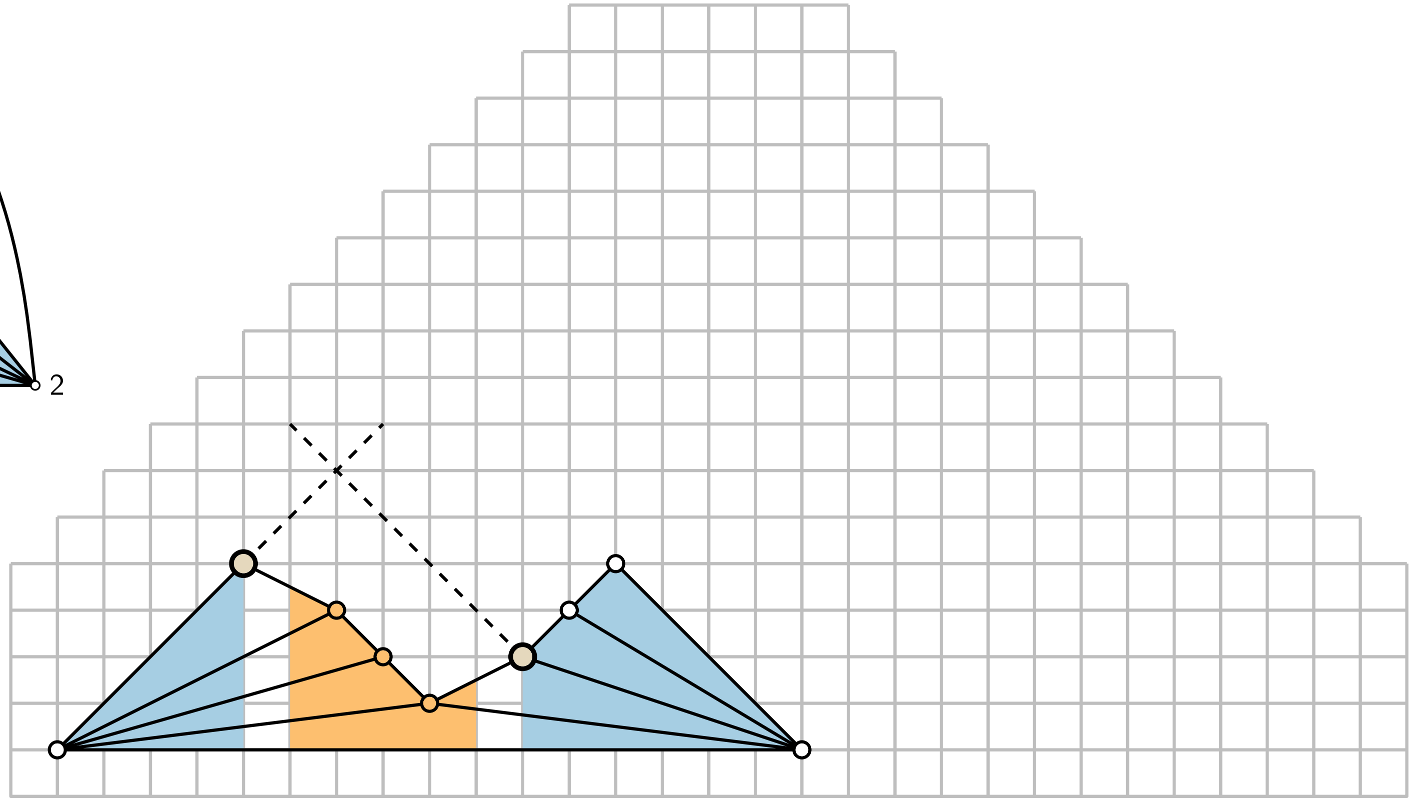

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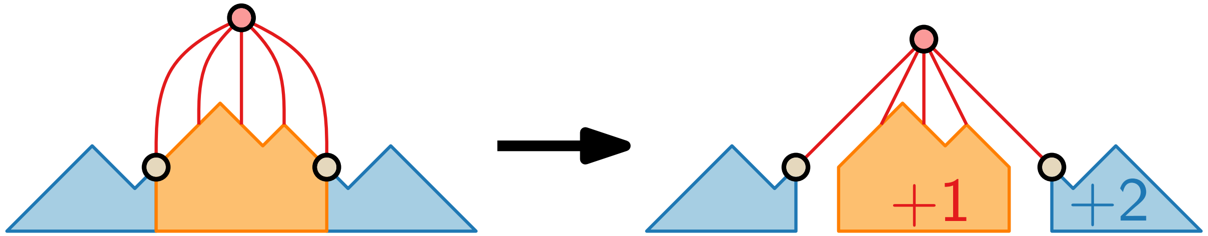

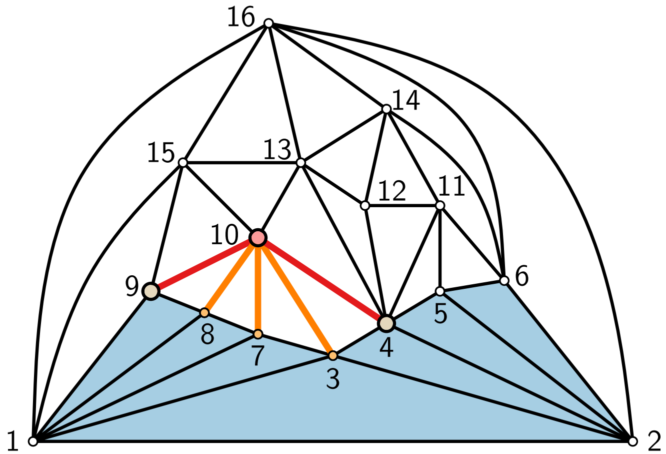

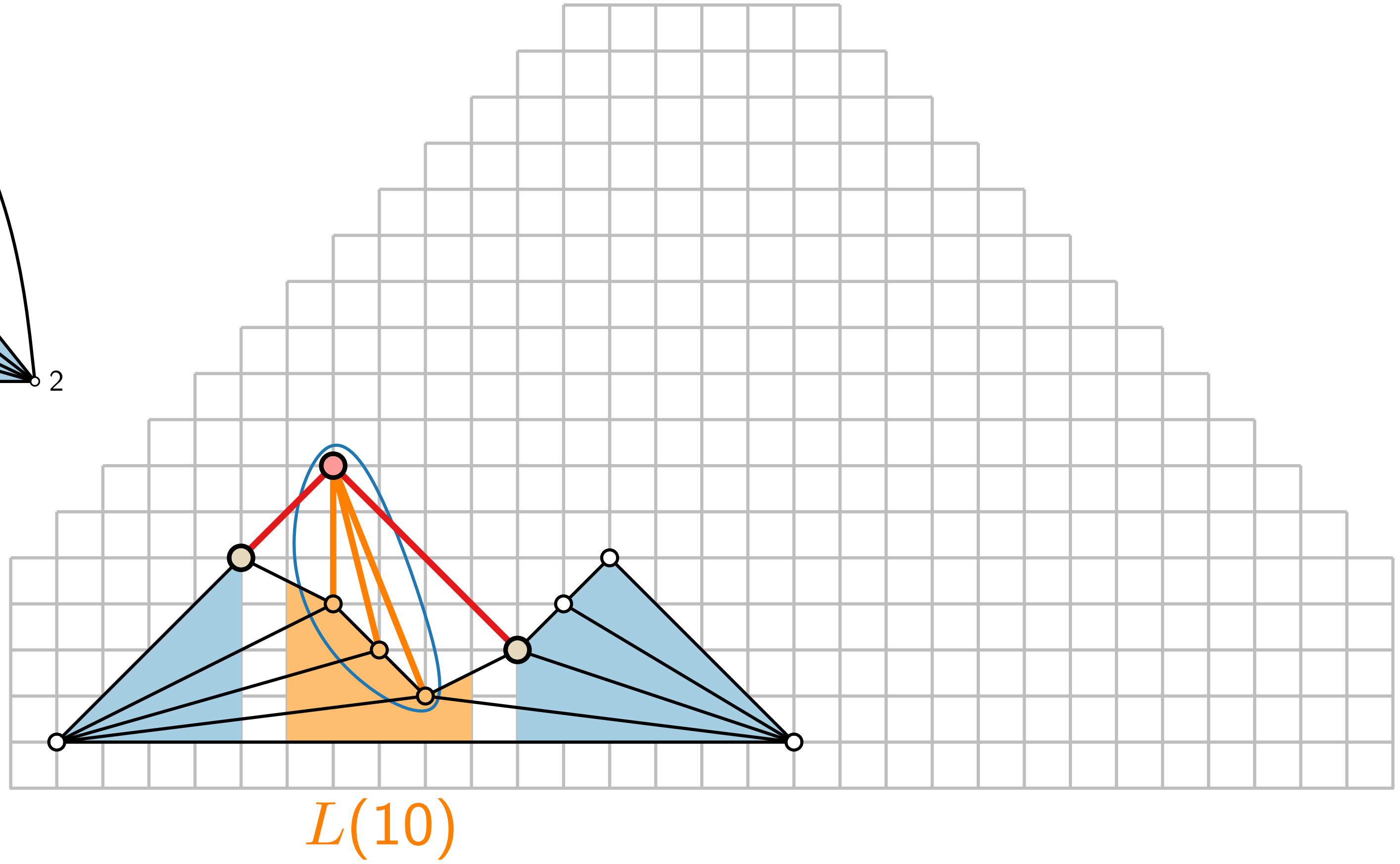

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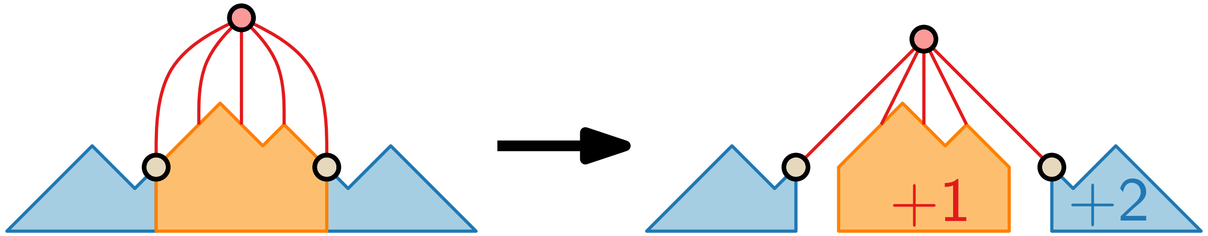

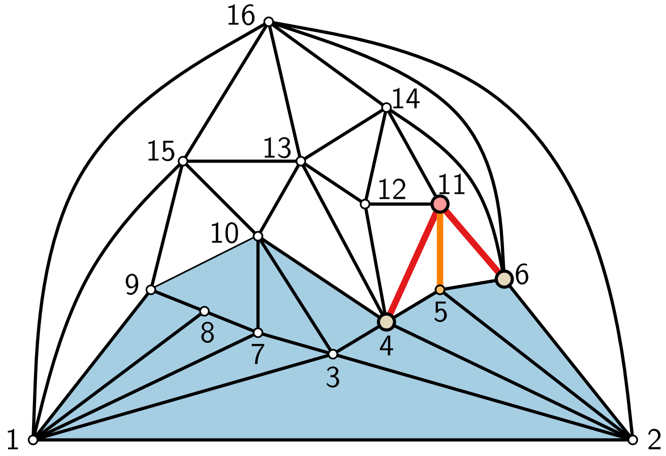

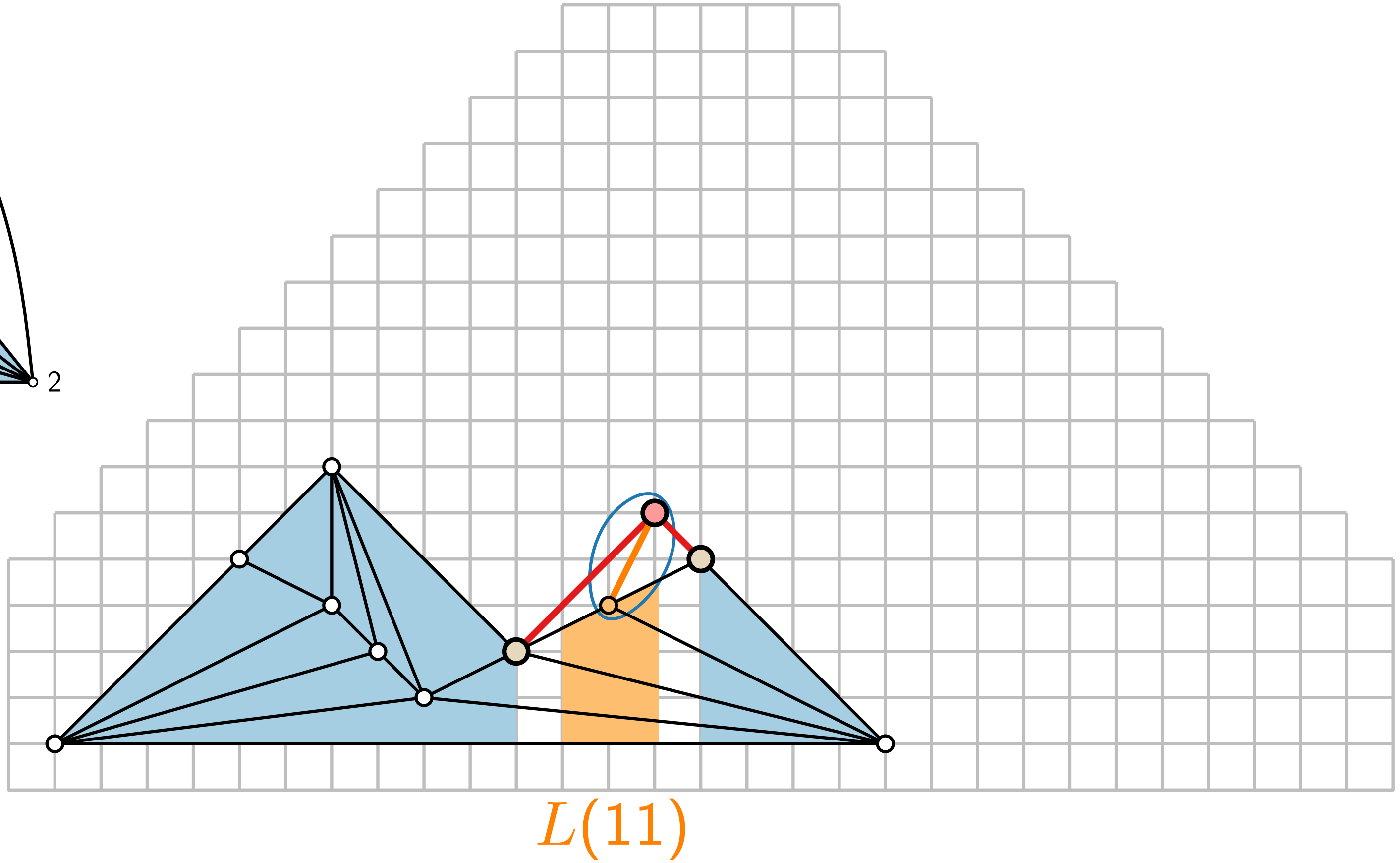

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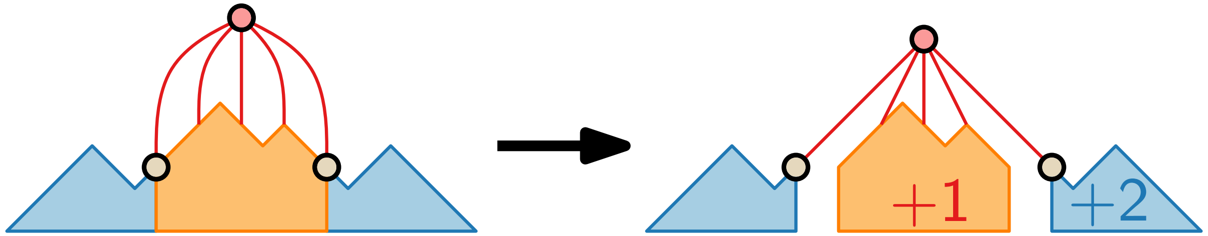

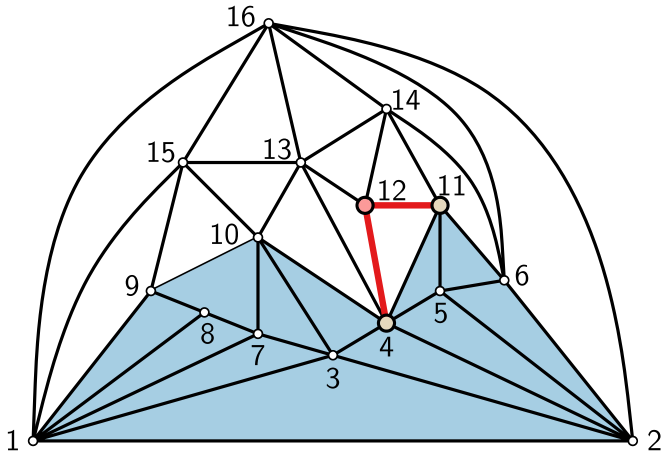

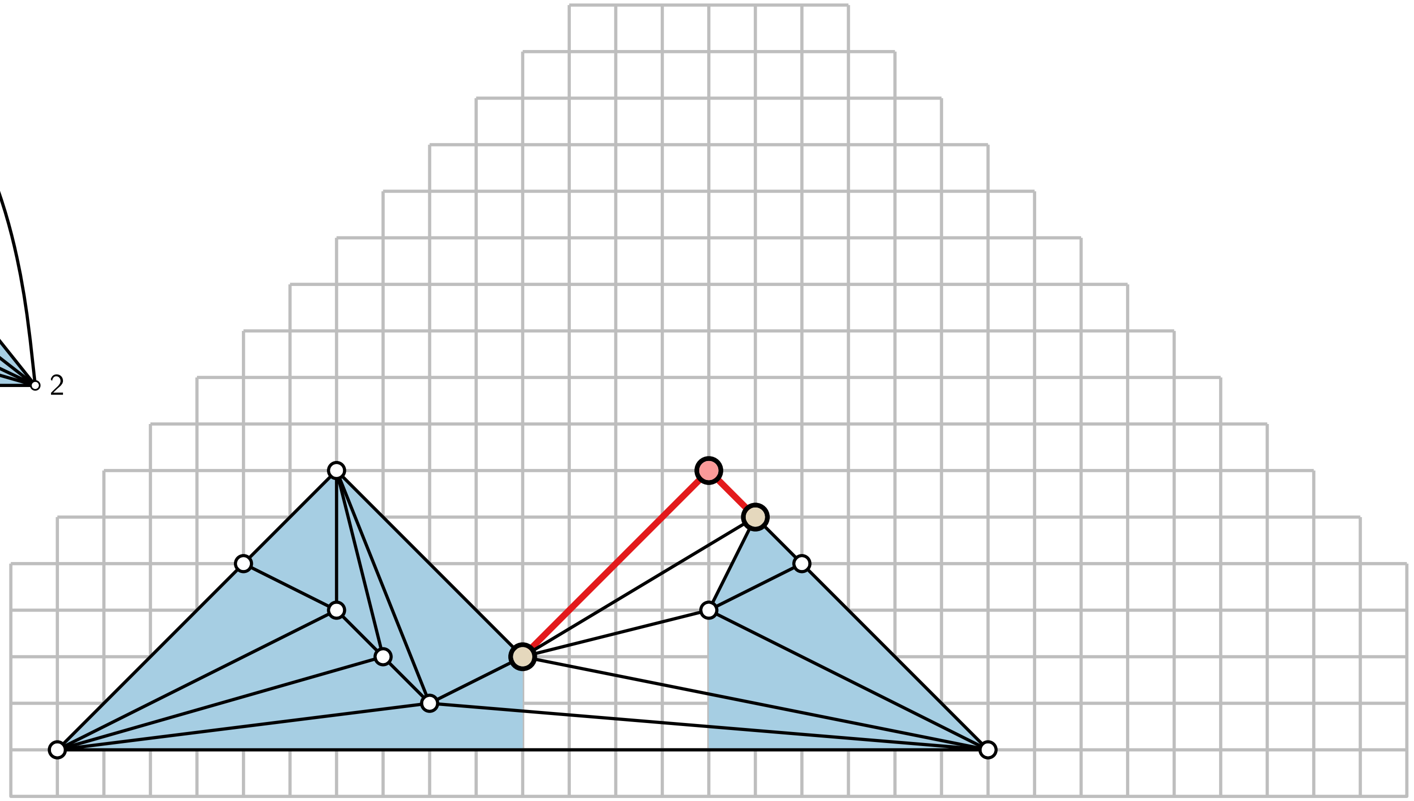

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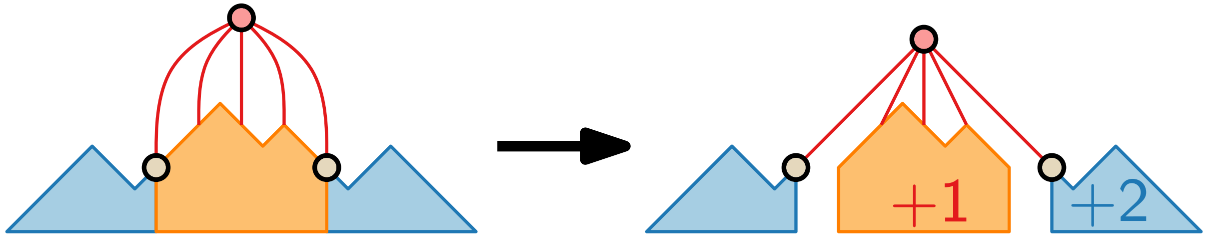

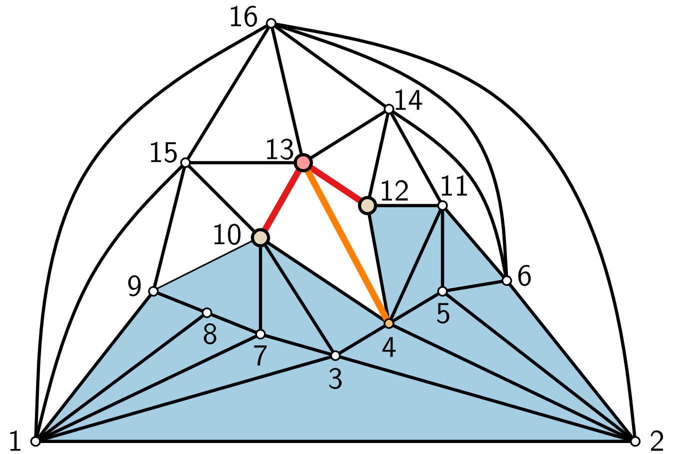

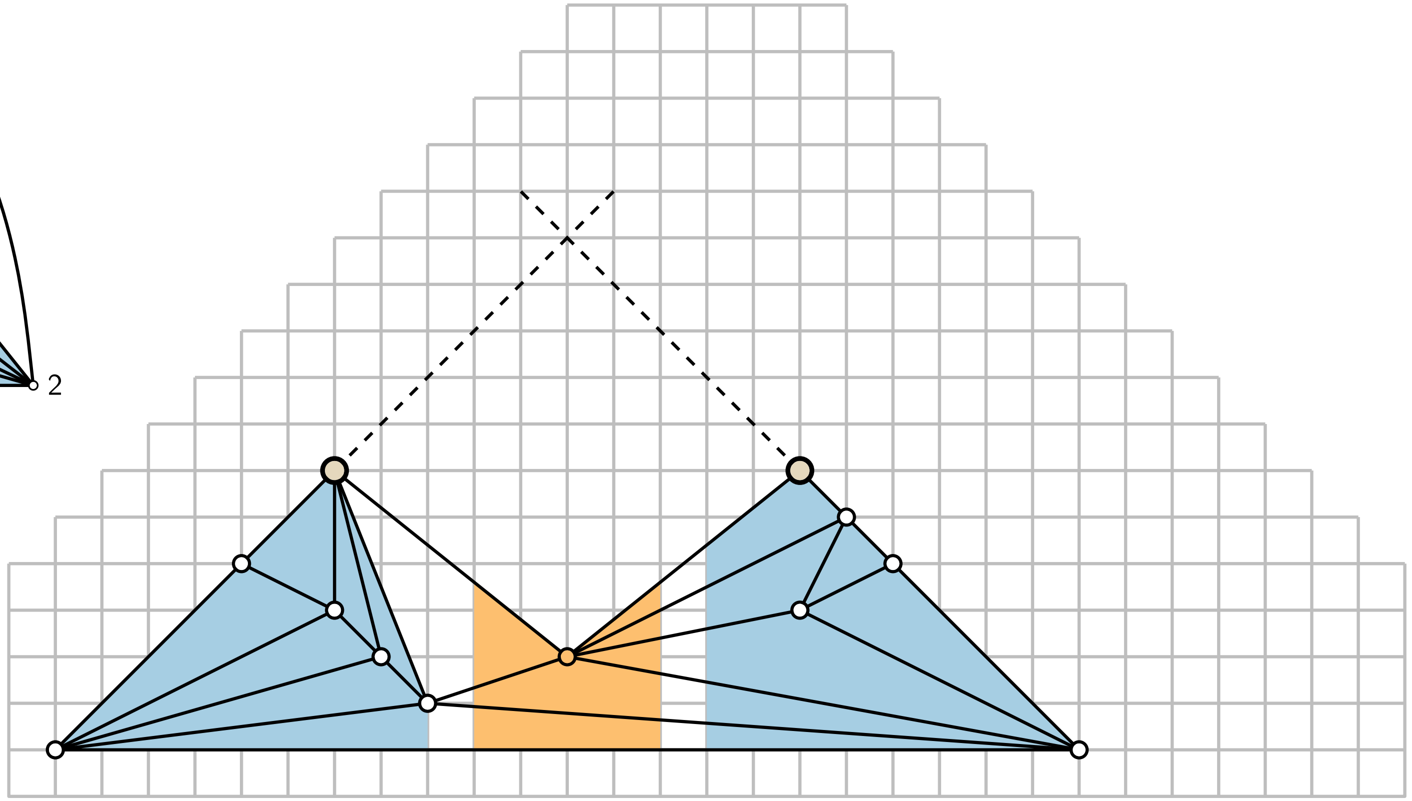

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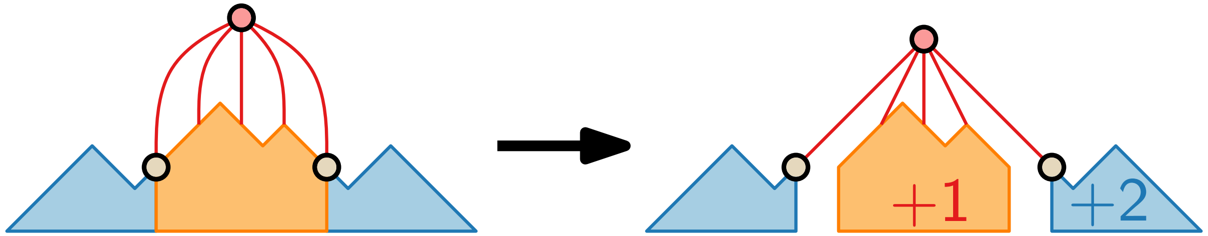

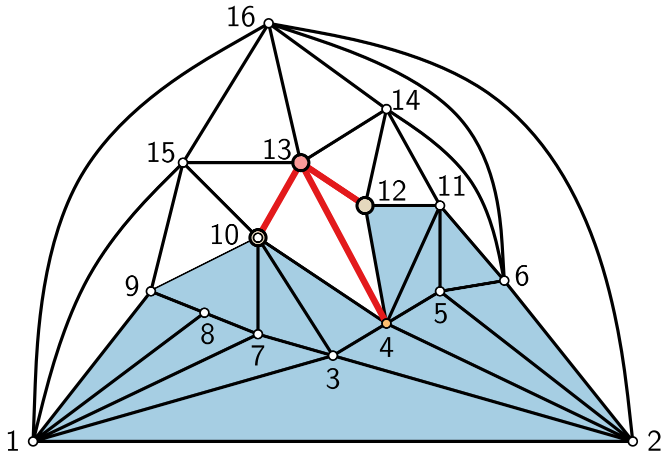

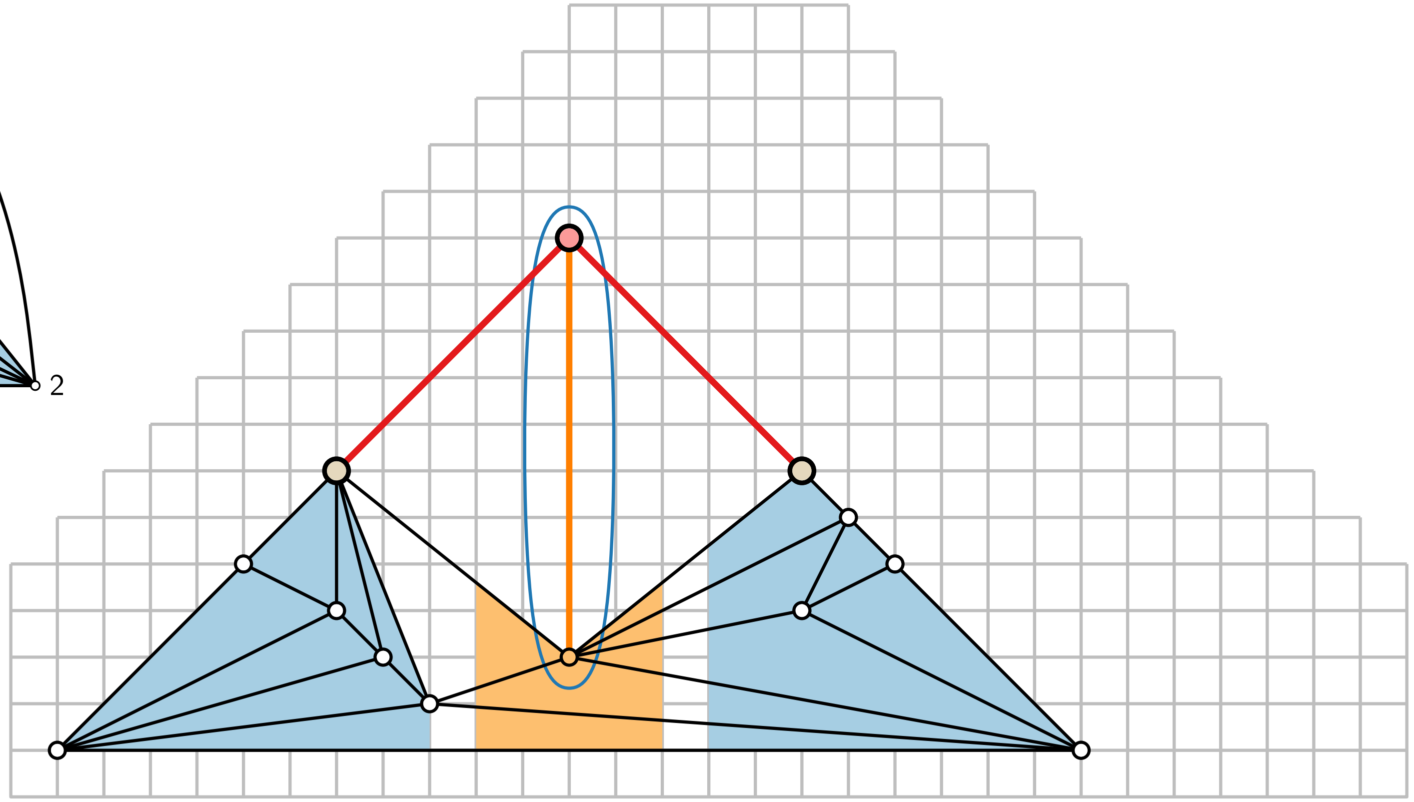

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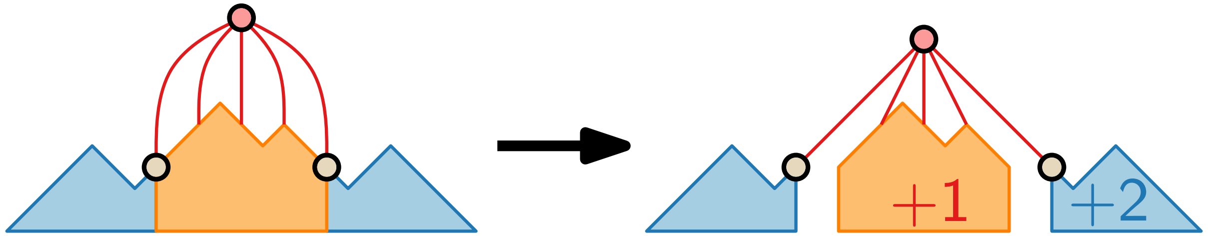

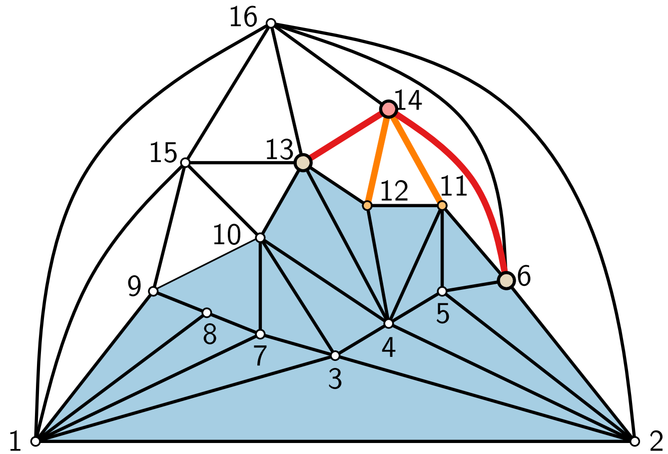

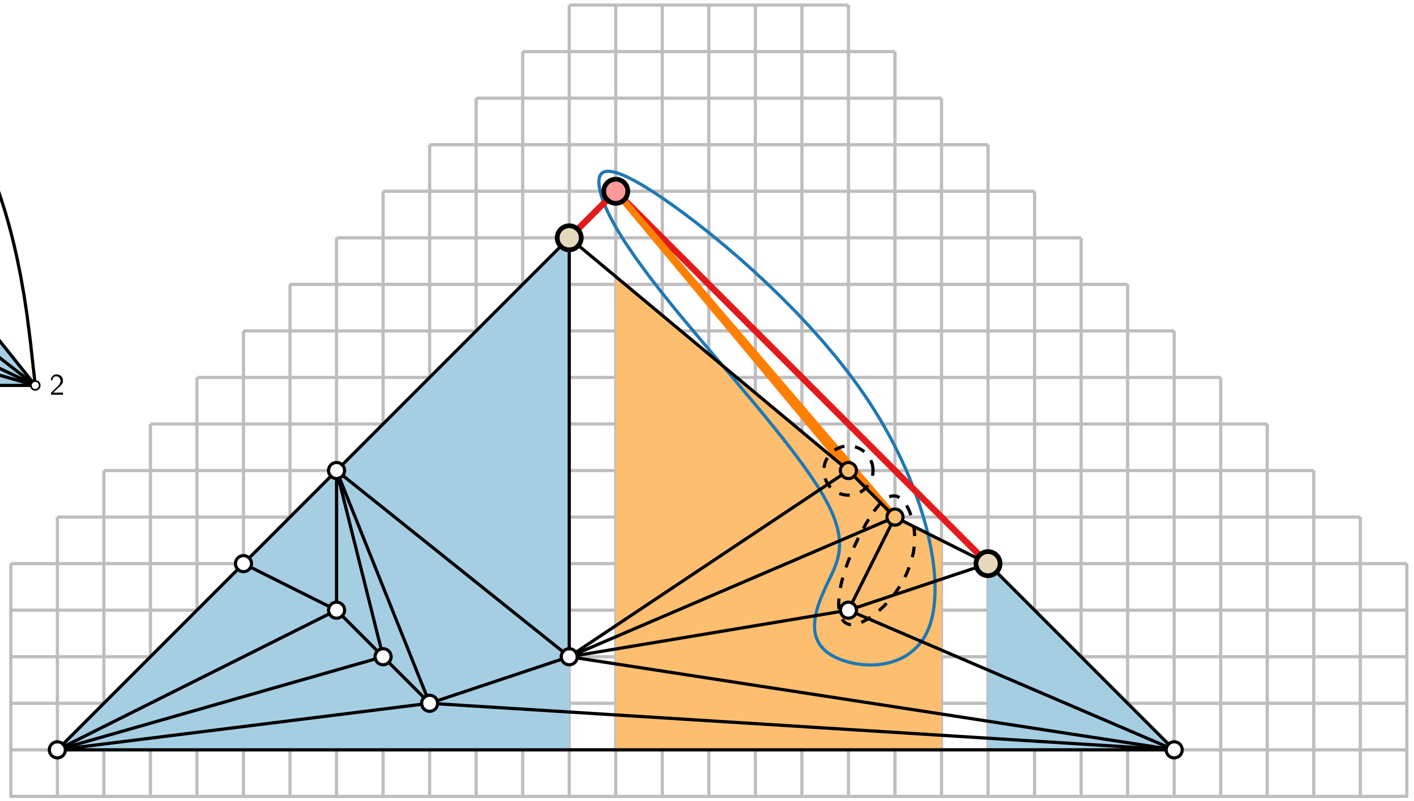

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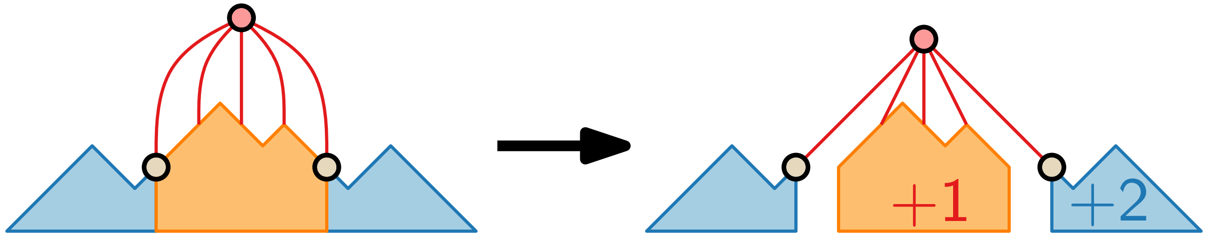

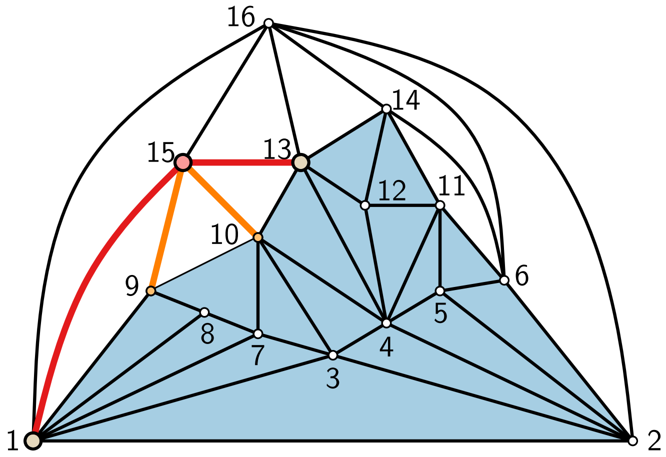

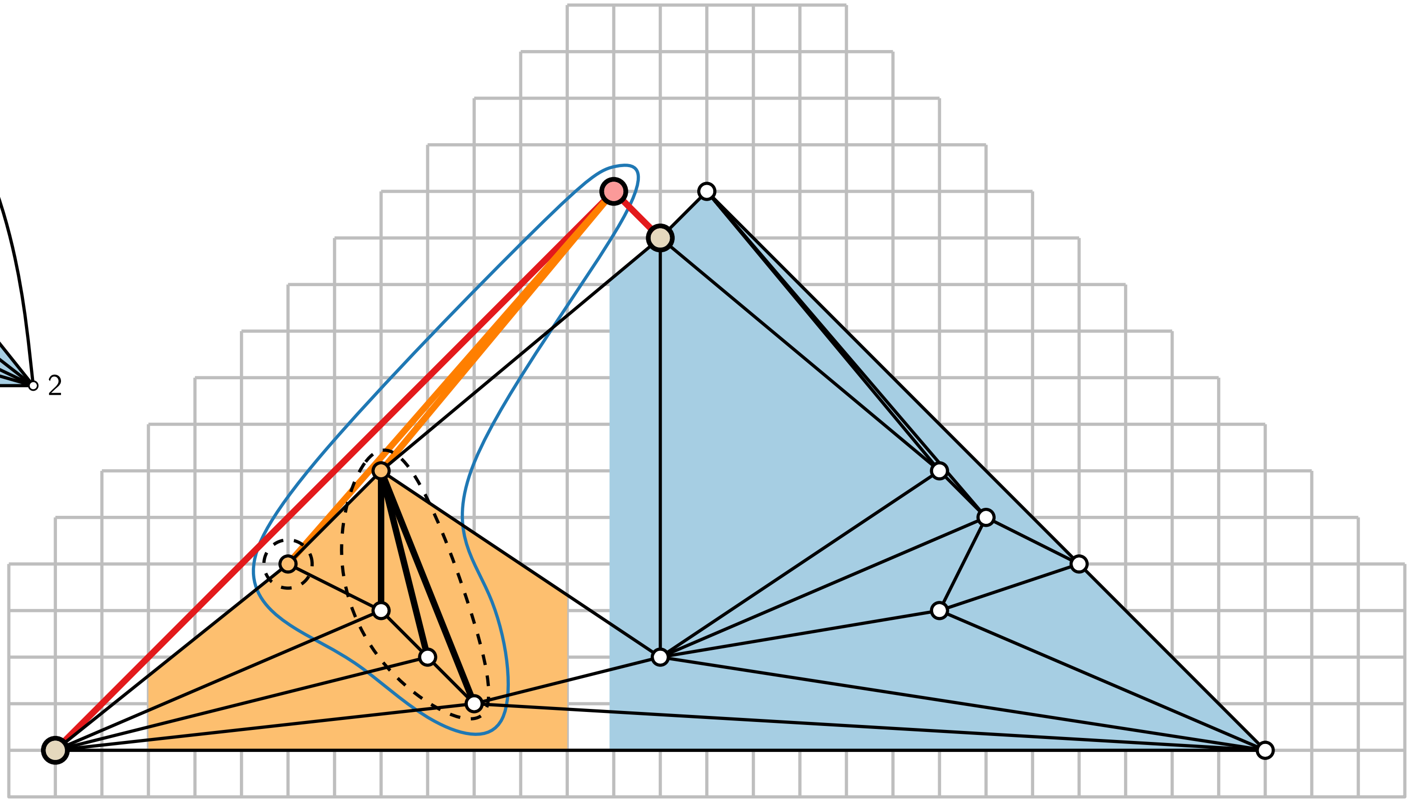

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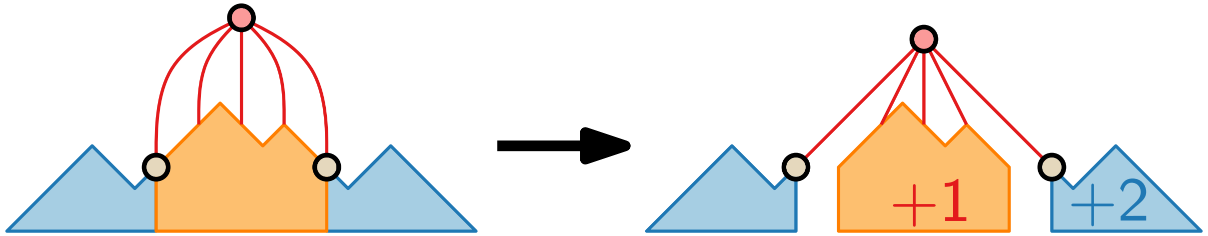
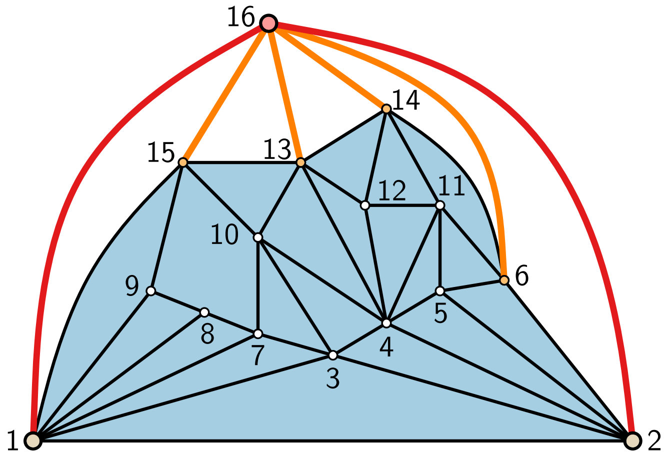
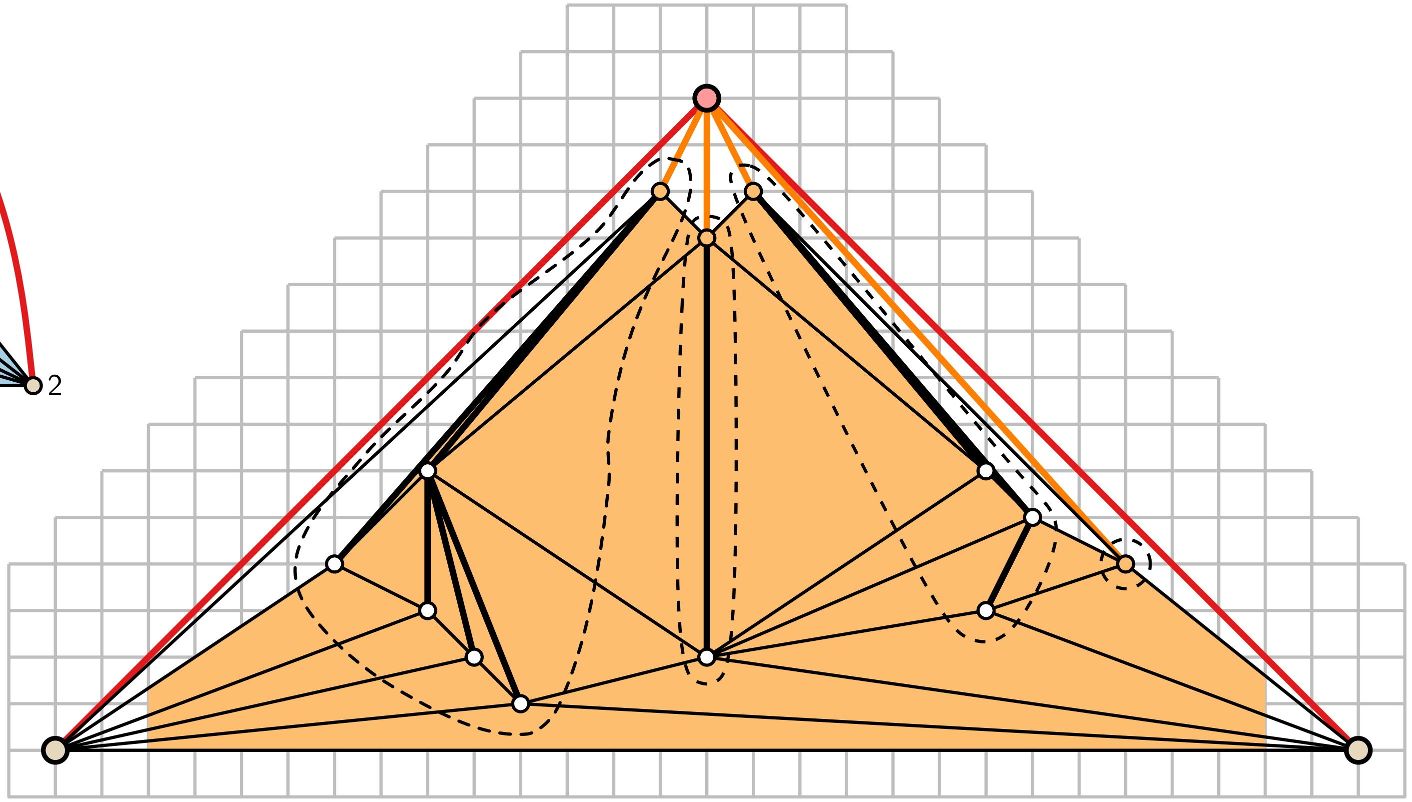
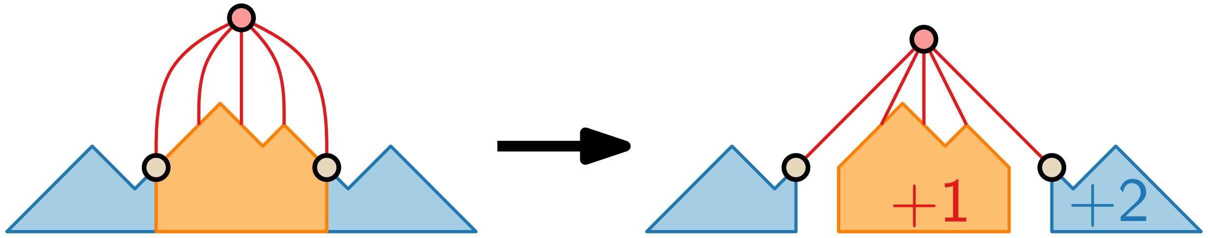
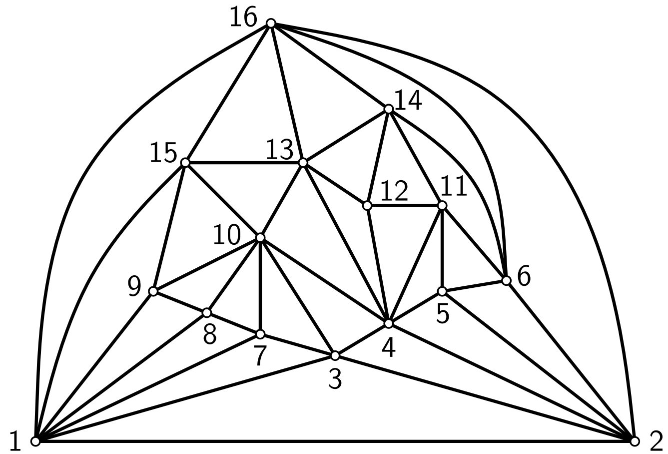


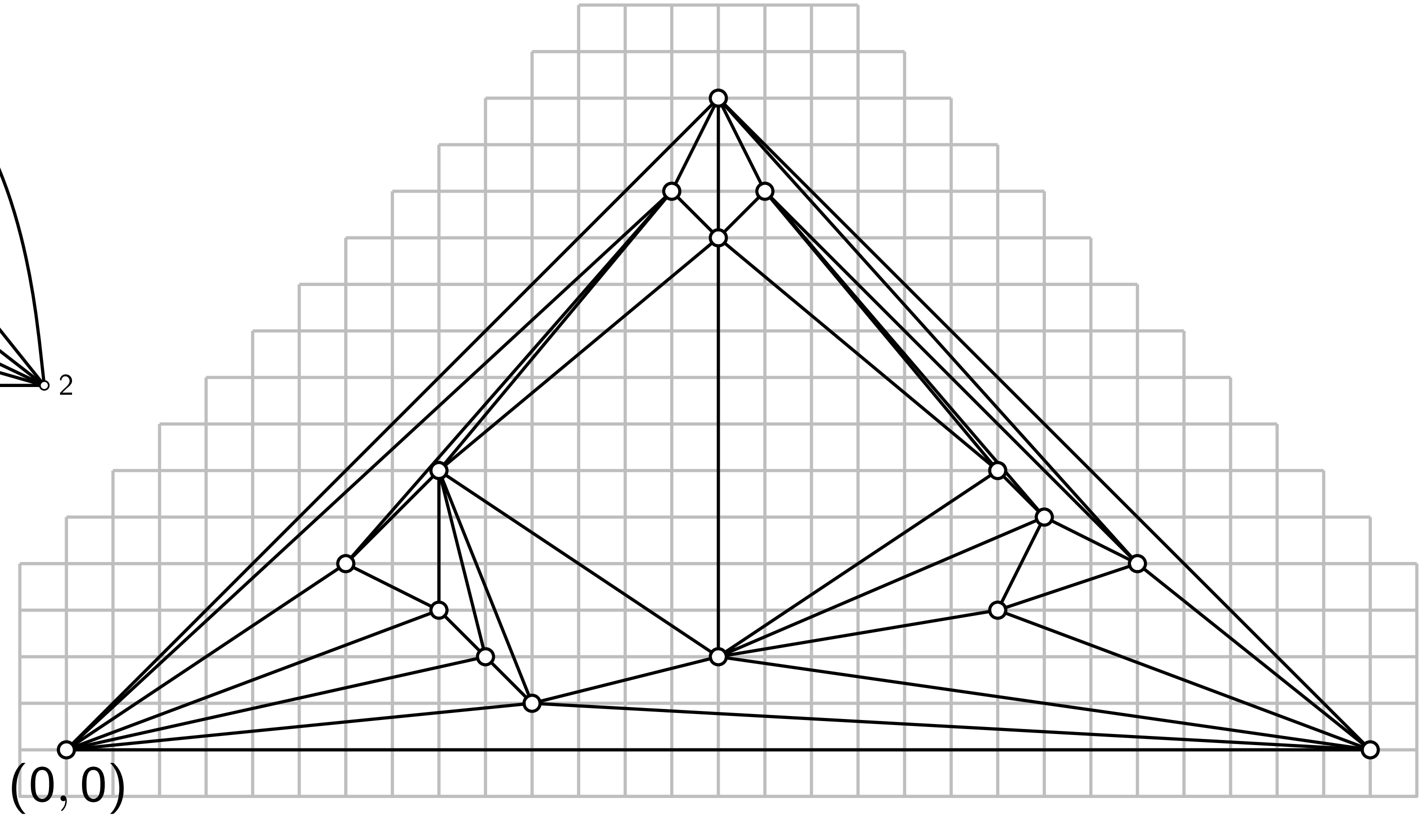




Shift Method – Example







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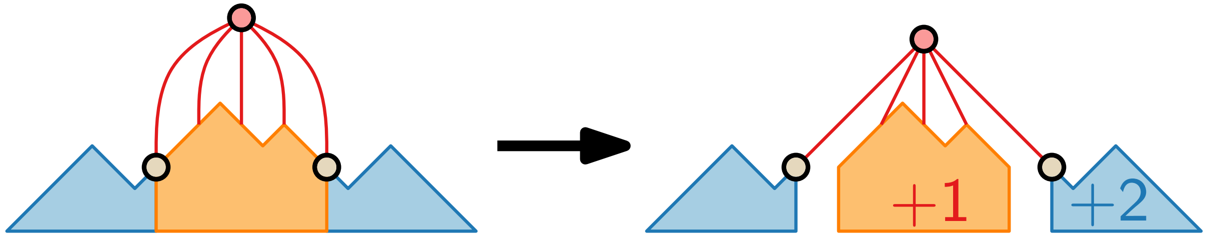

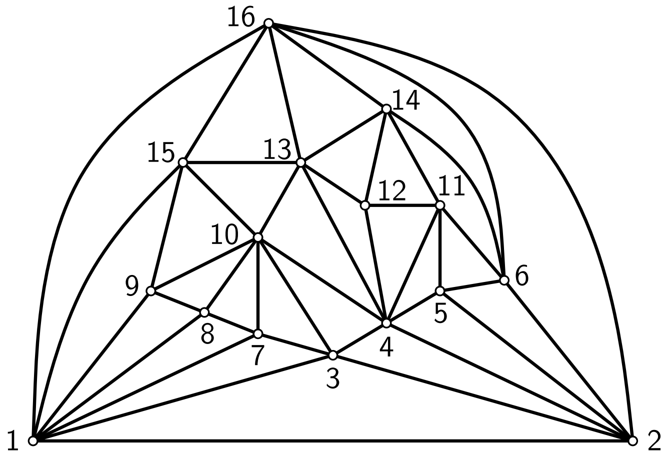

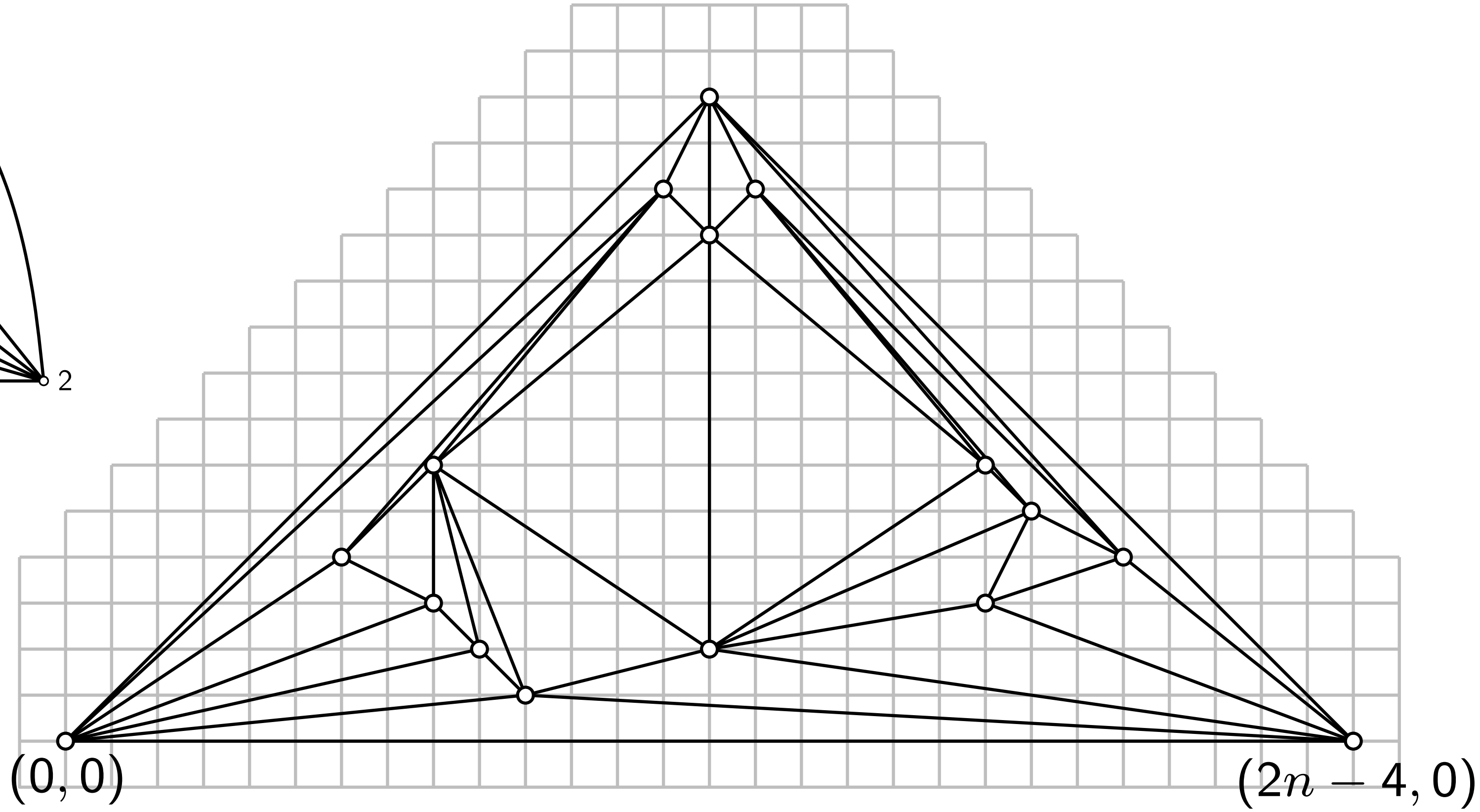

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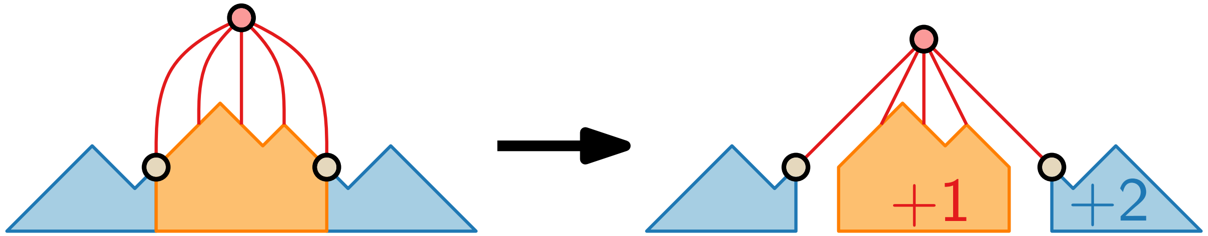
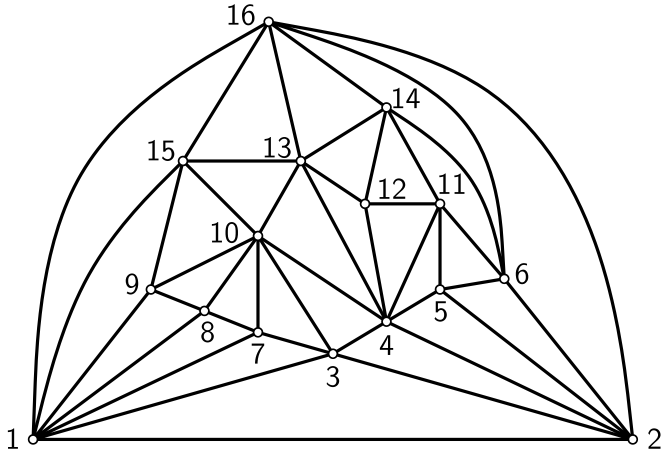
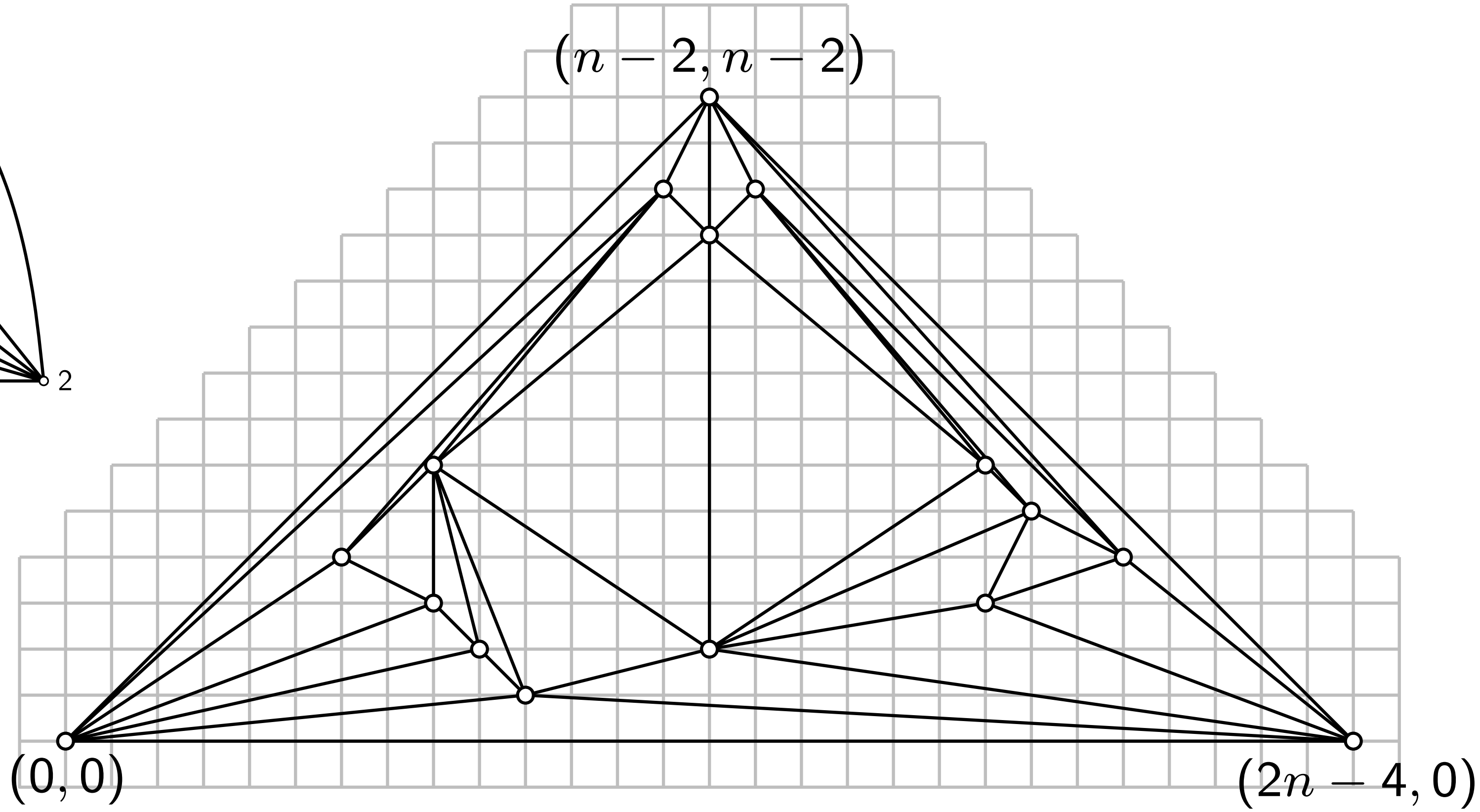
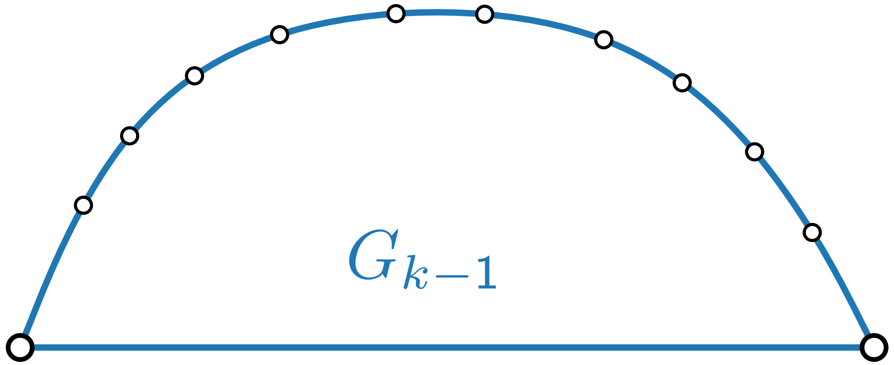






Shift Method – Planarity



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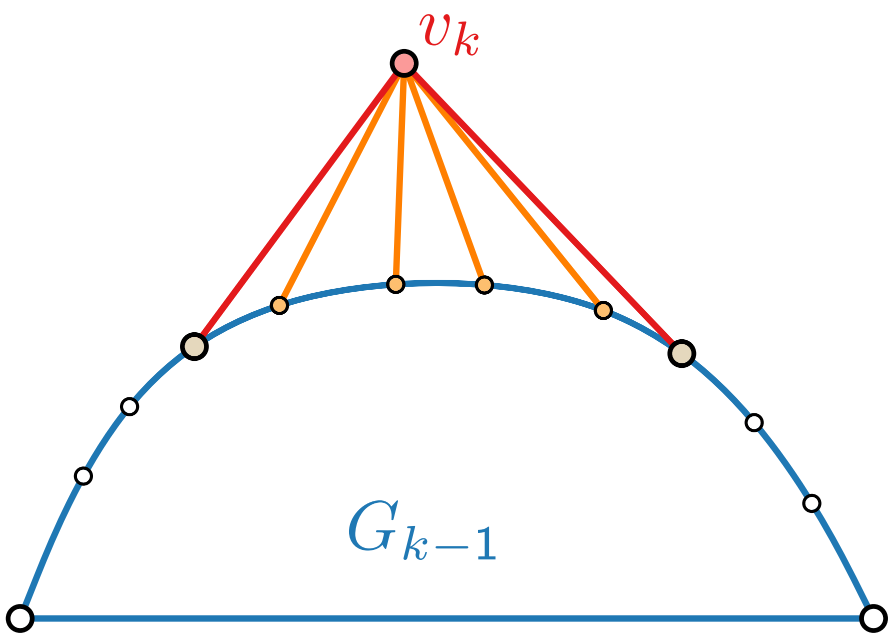

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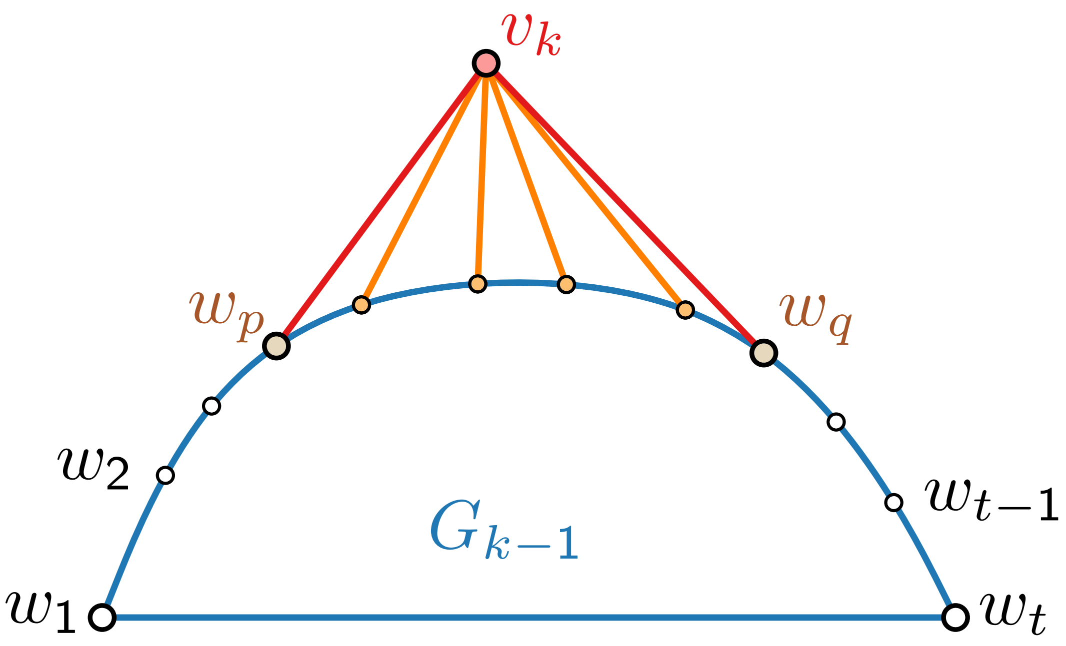

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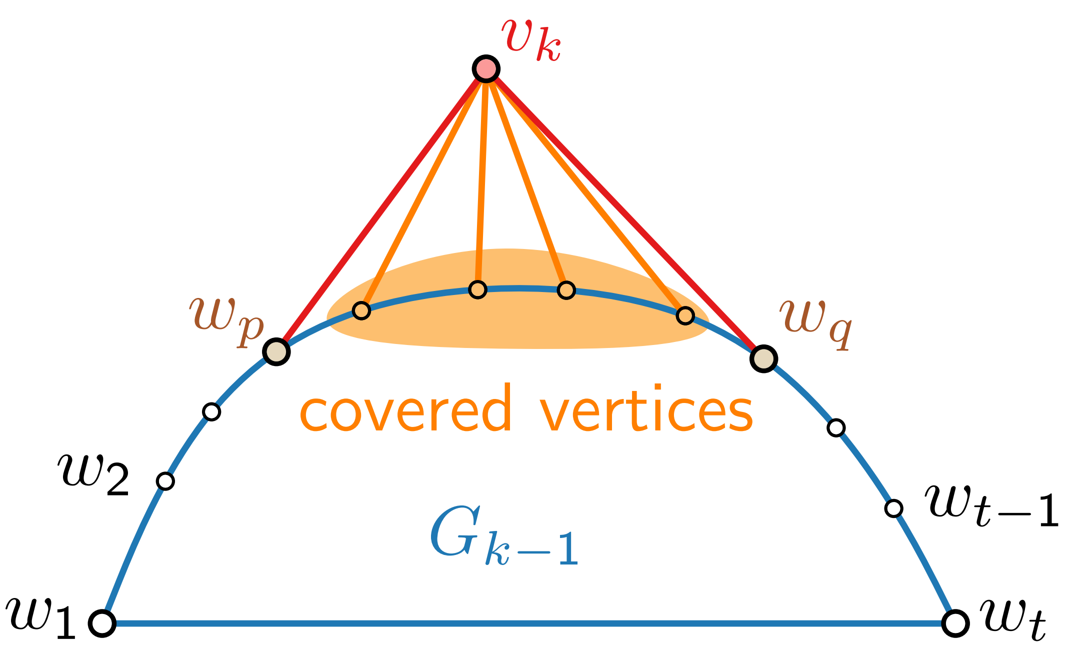
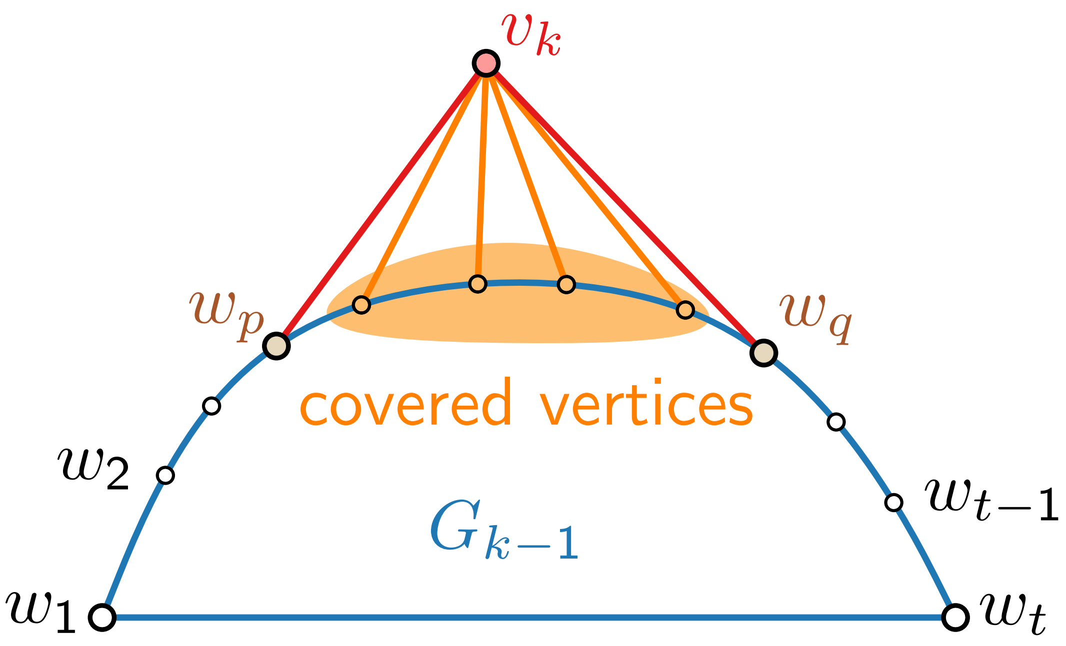


Shift Method – Planarity

Observations.

- Each internal vertex is covered exactly once.



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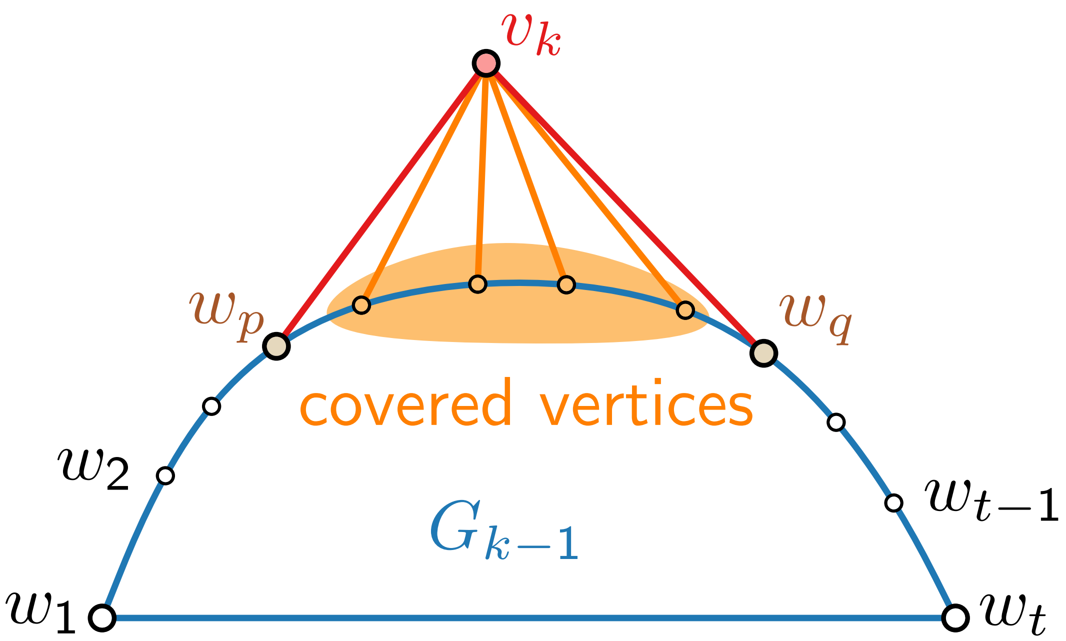
- Covering relation defines a tree in G



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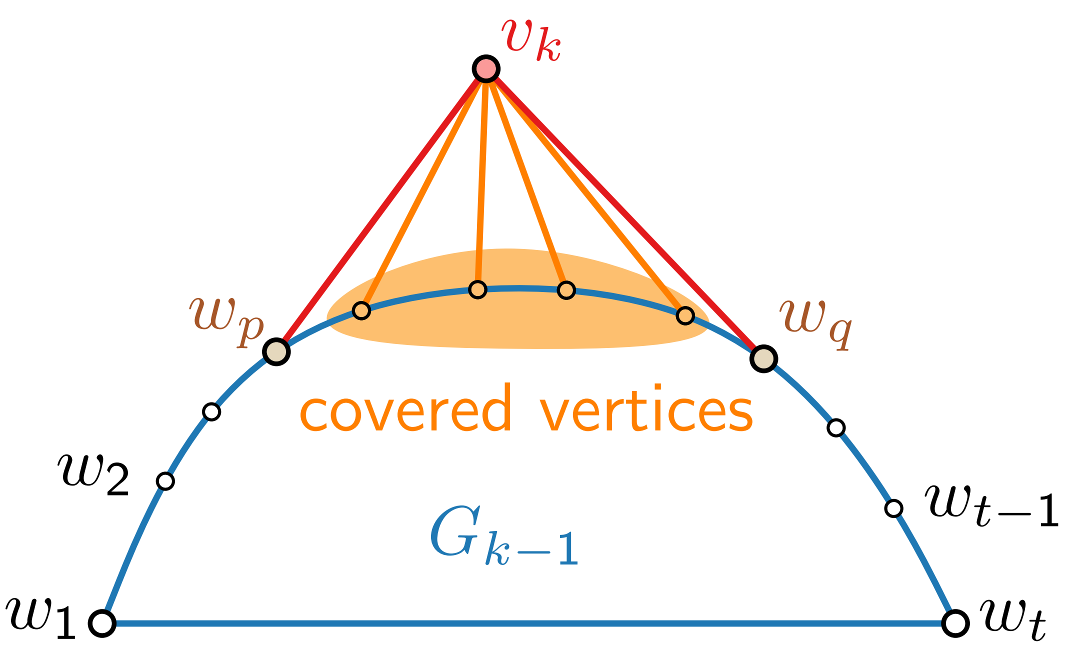

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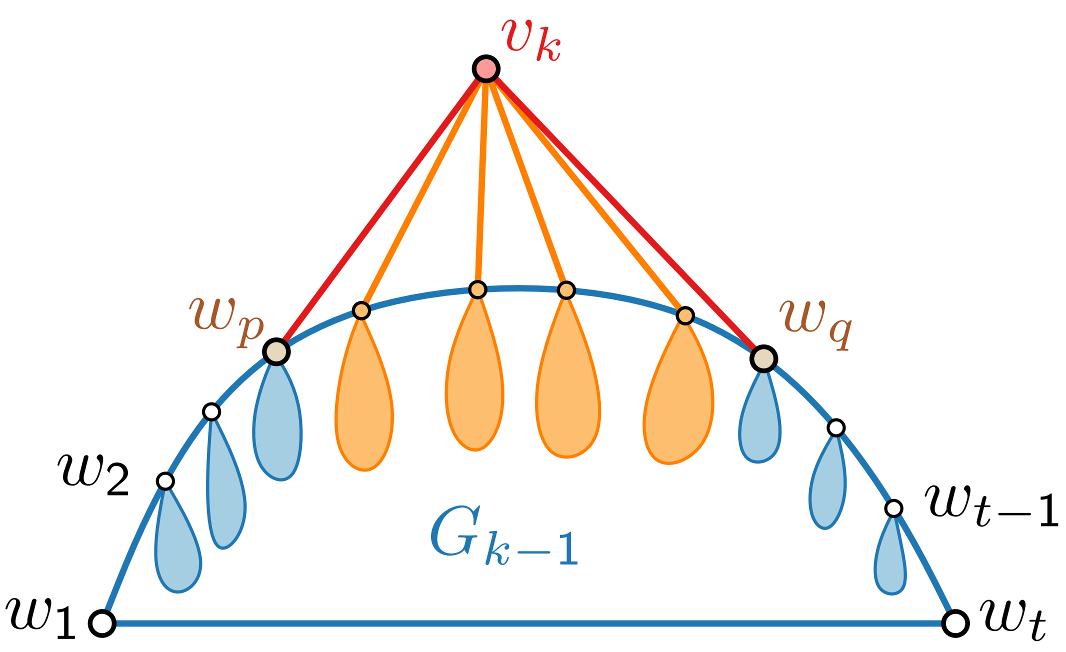

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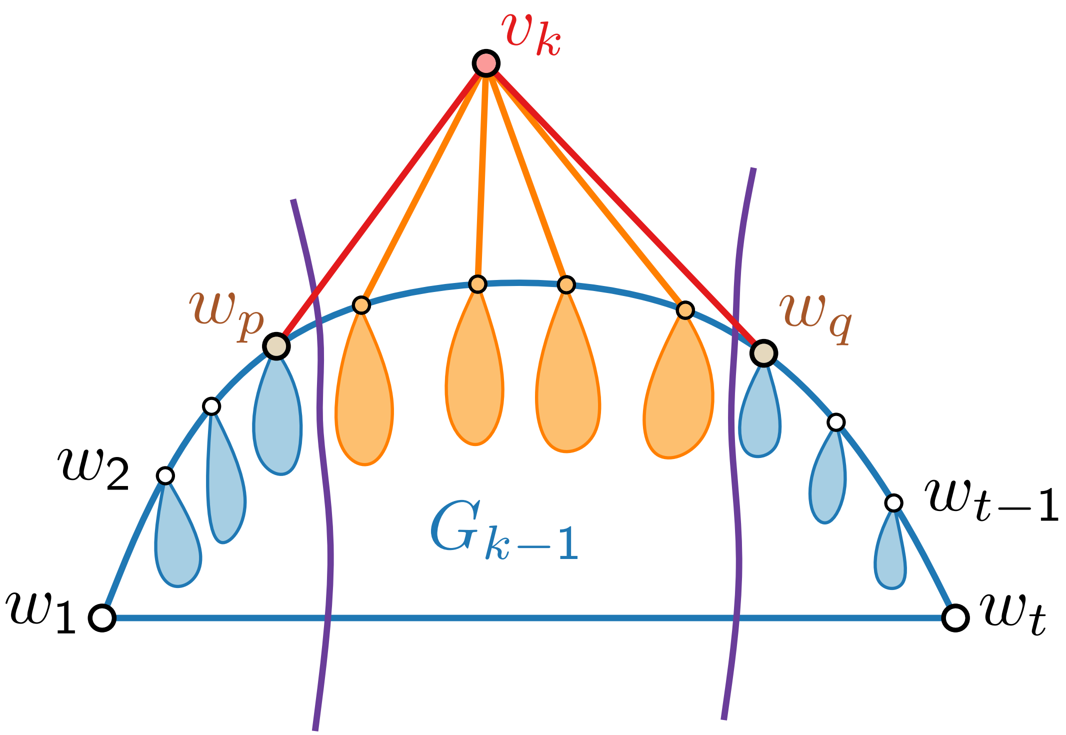

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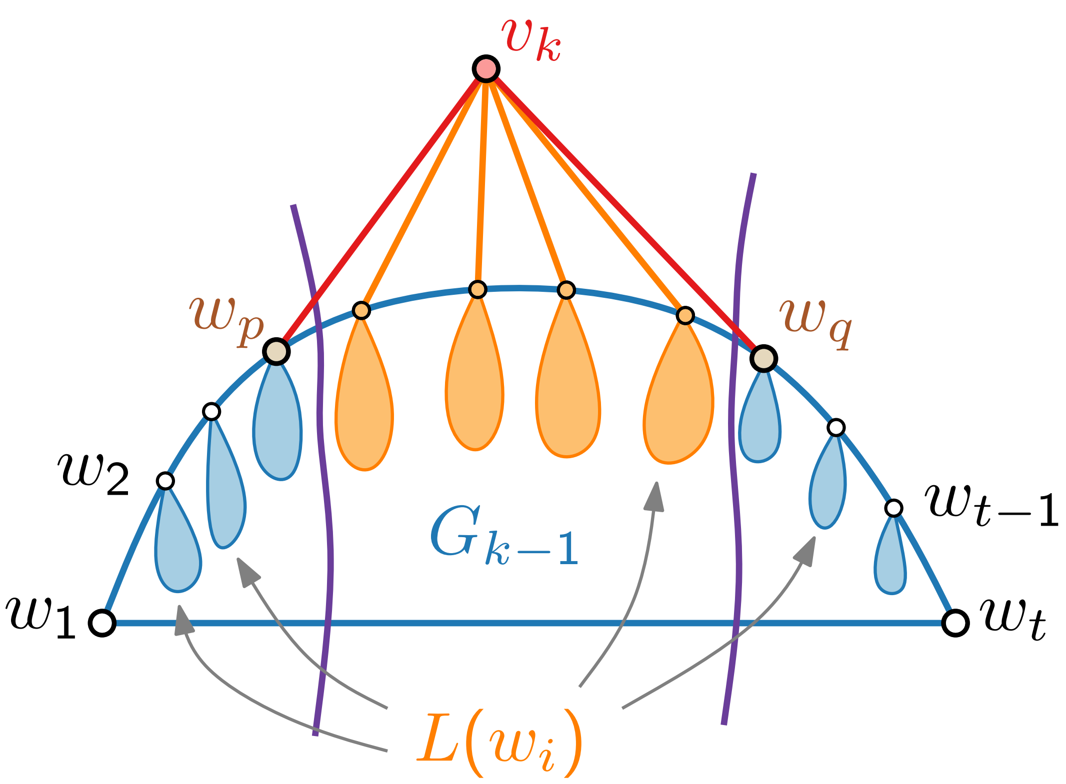

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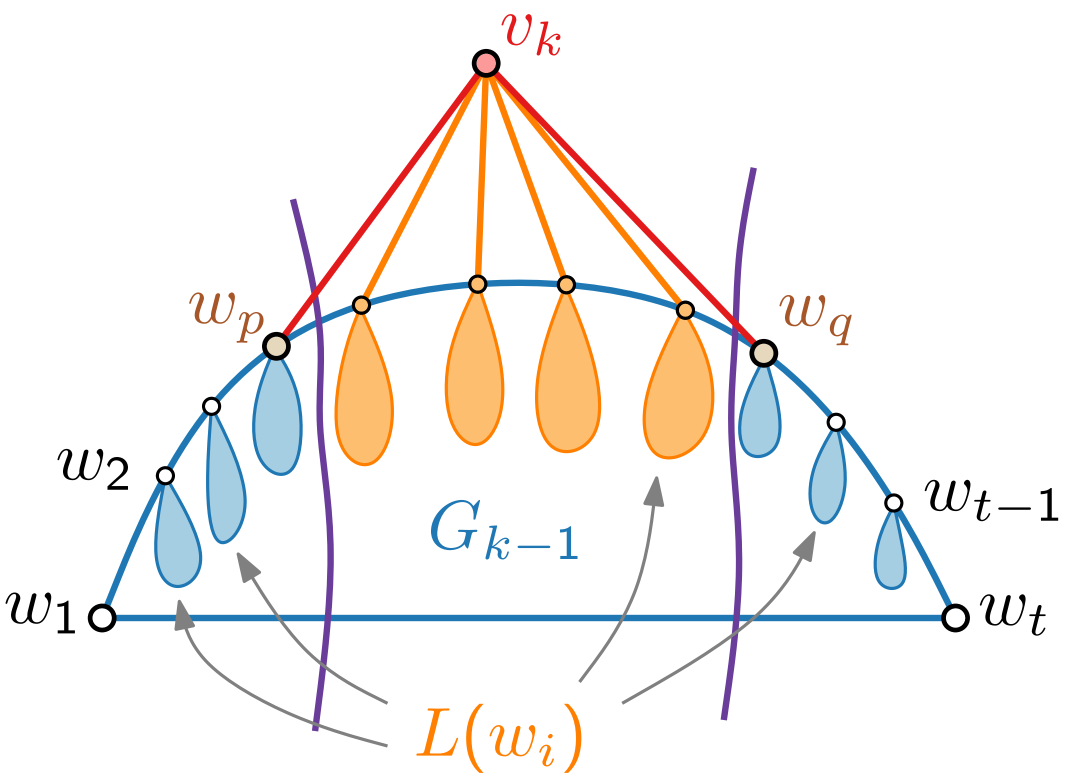

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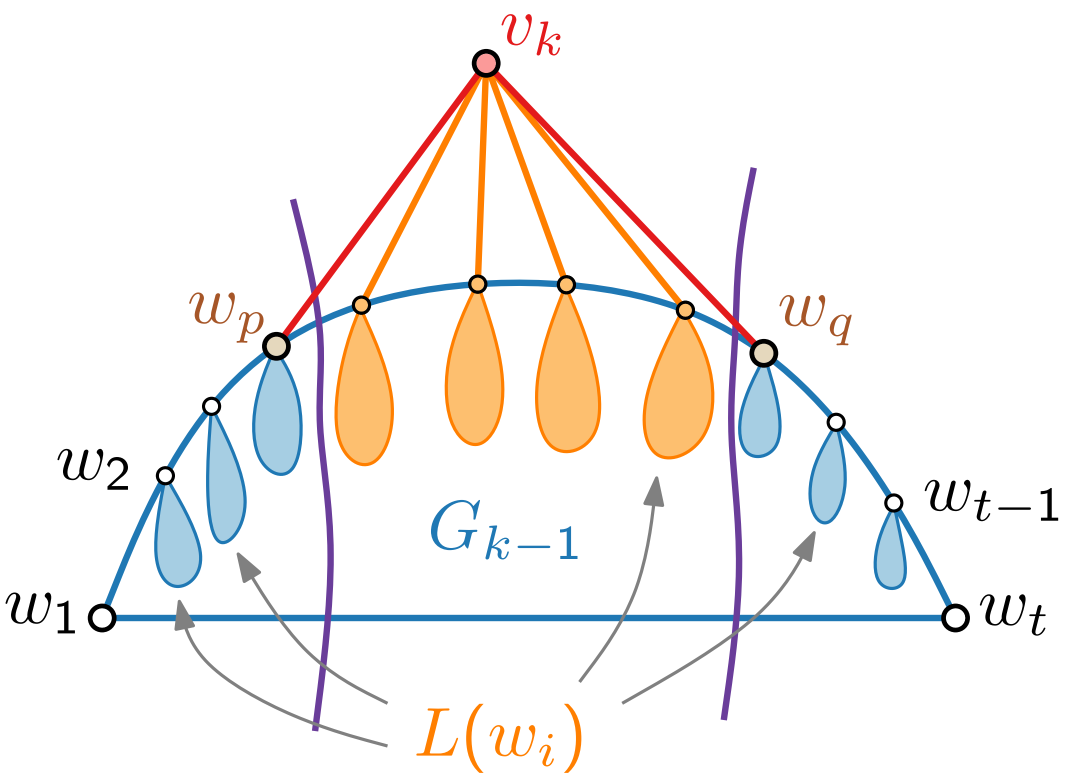

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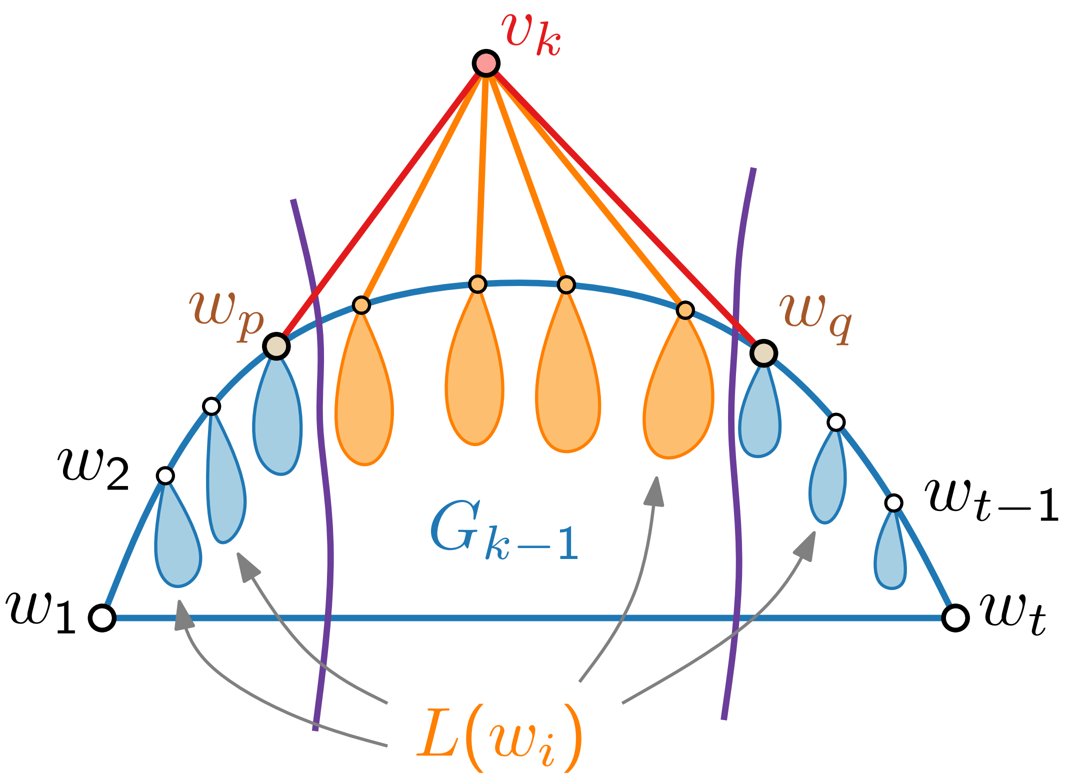

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If G_{k-1} is drawn planar and straight-line, then so is G_k.

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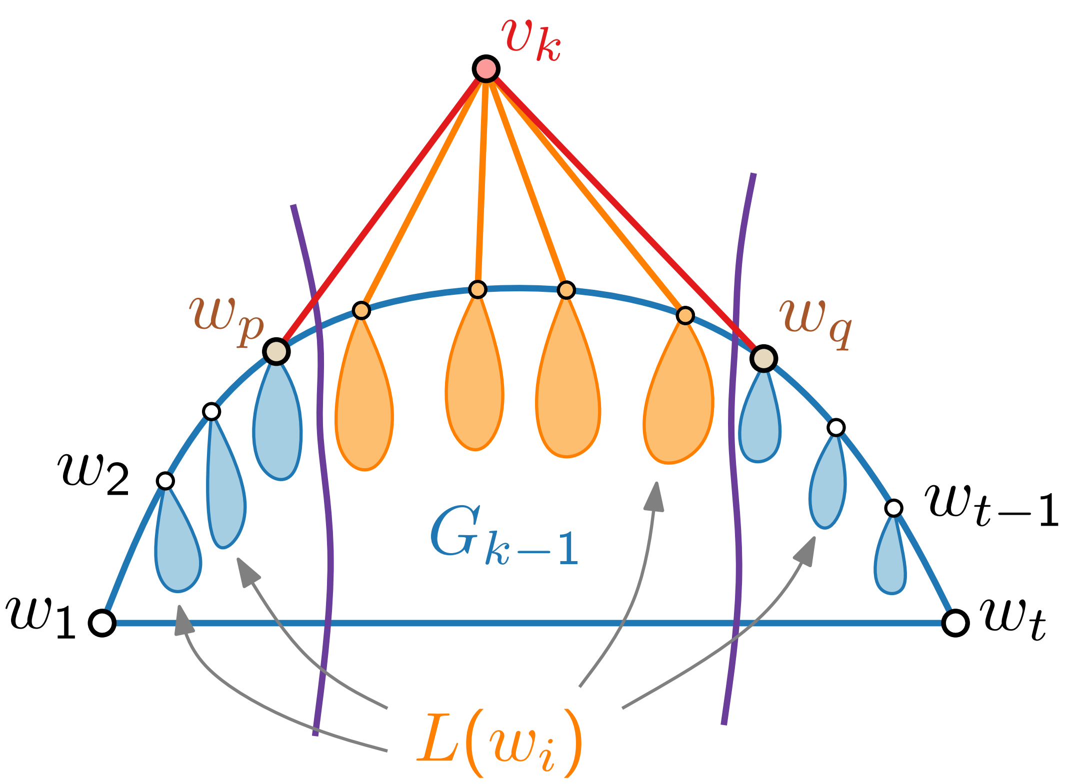

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- New edges don't intersect other edges ($\rightarrow$ invariants).
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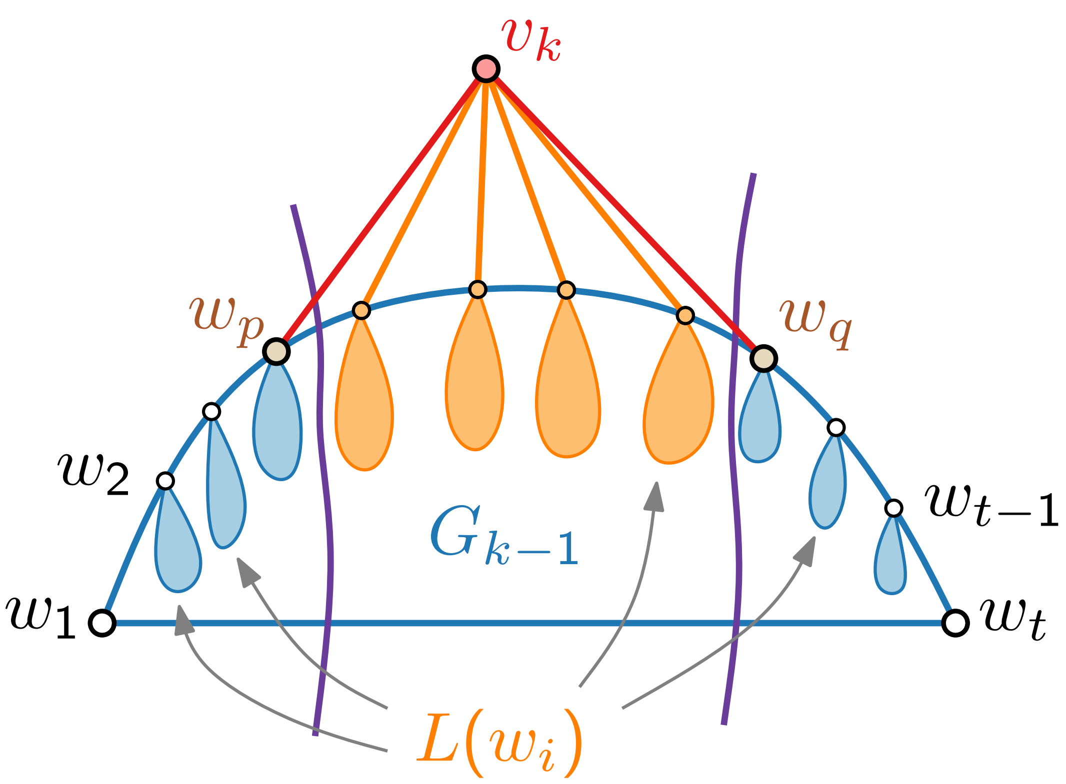

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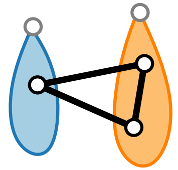

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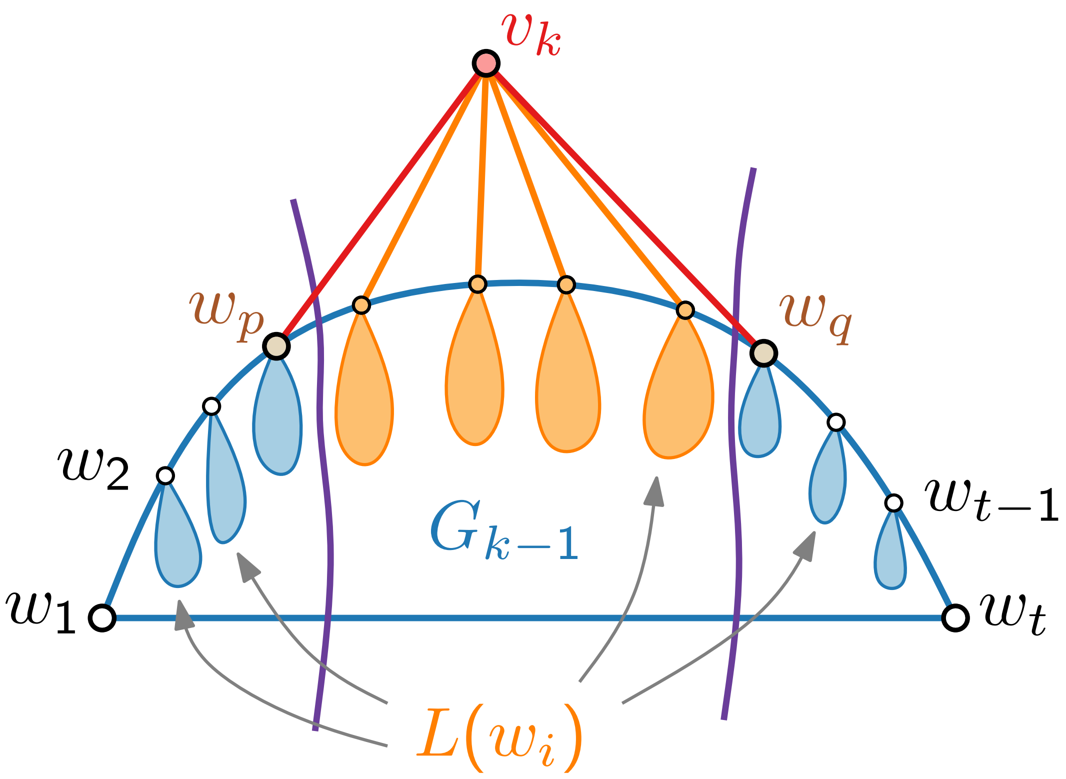

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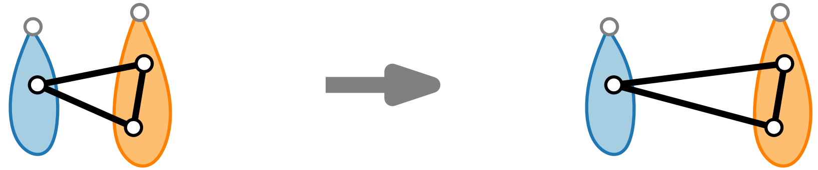

Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod($G, (v_1, v_2, \ldots, v_n)$)

for $k = 1$ to 3 **do**

for $k = 4$ to n **do**

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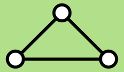

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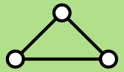
$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

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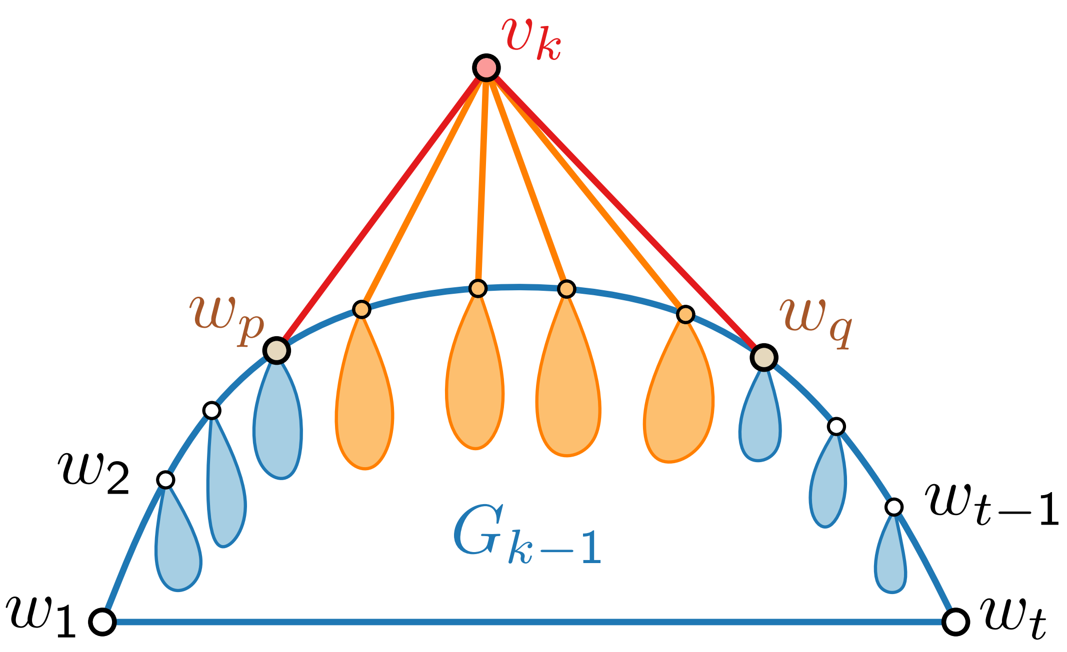

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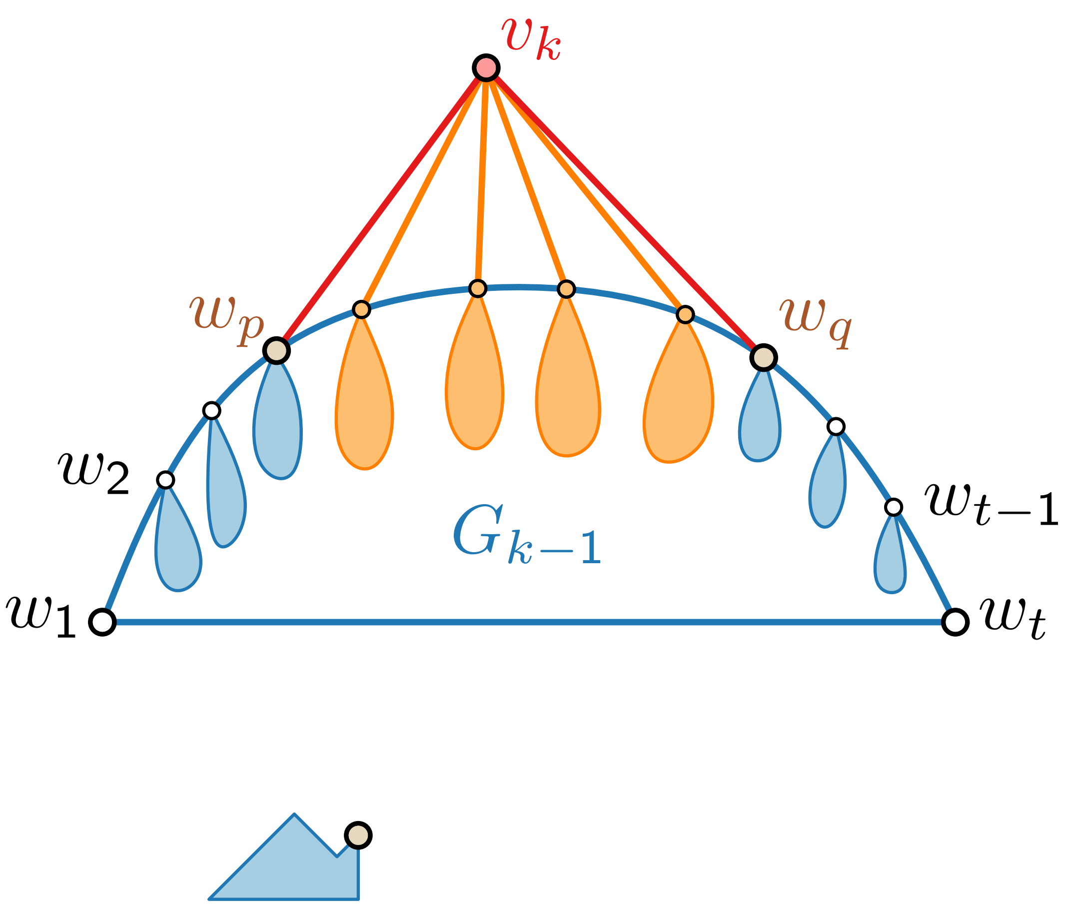

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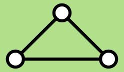

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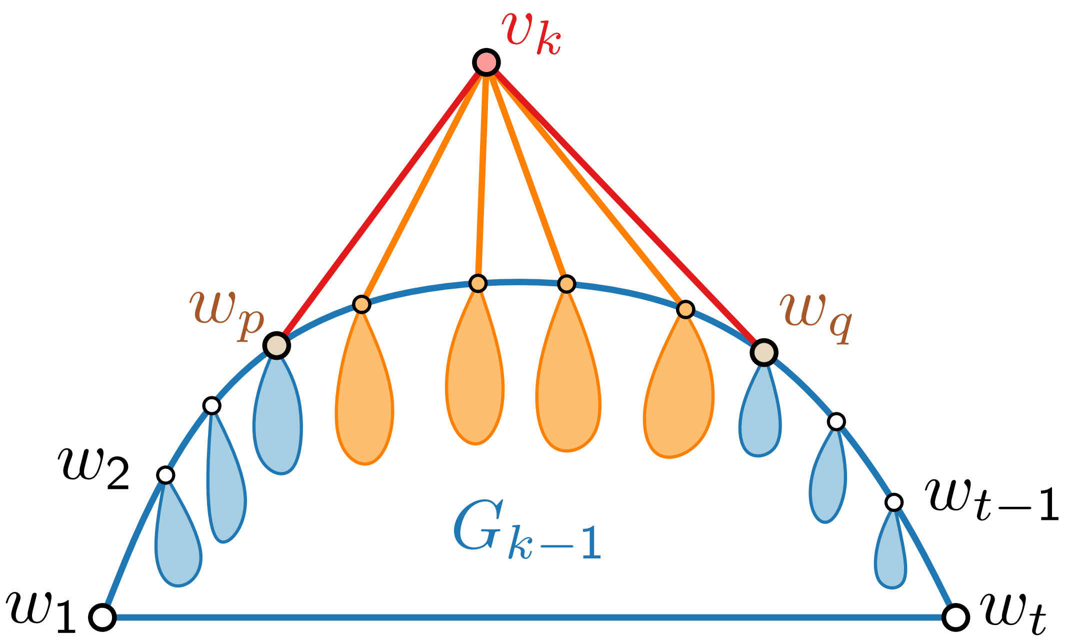

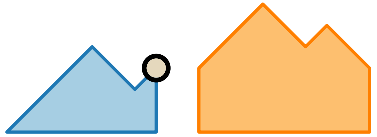

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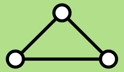
$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

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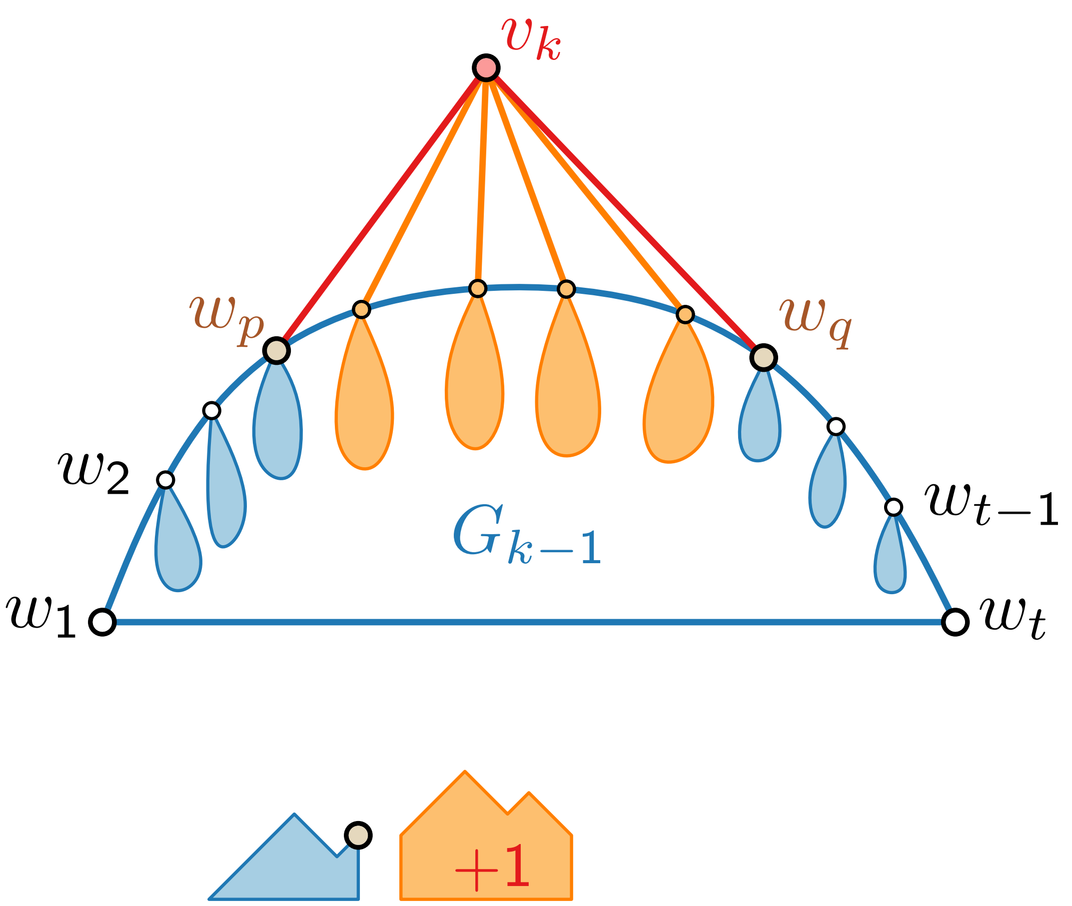

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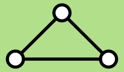

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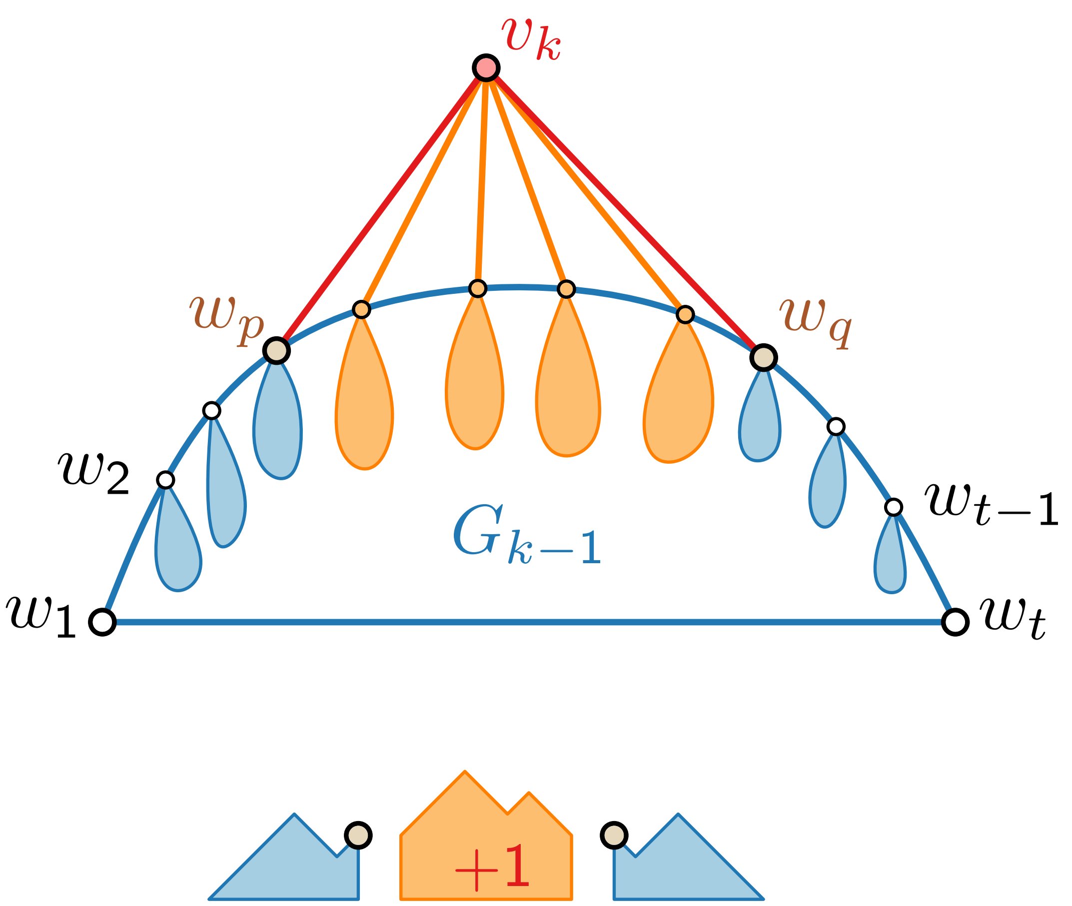

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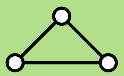
$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

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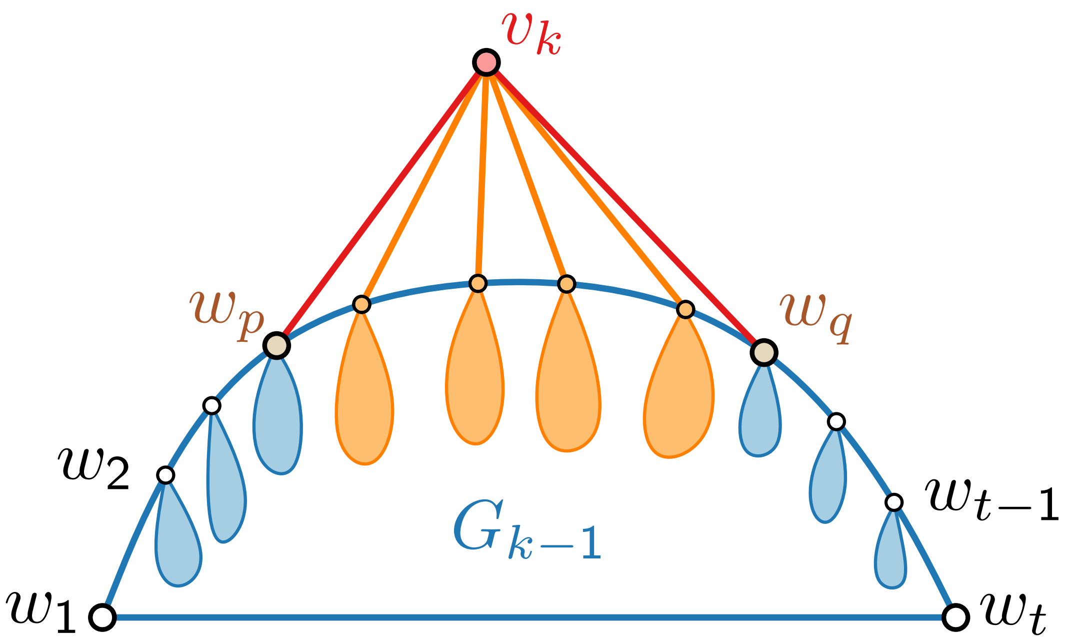

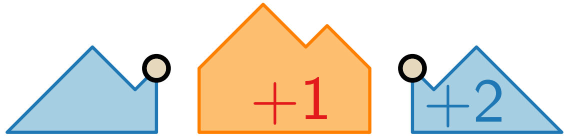

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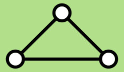

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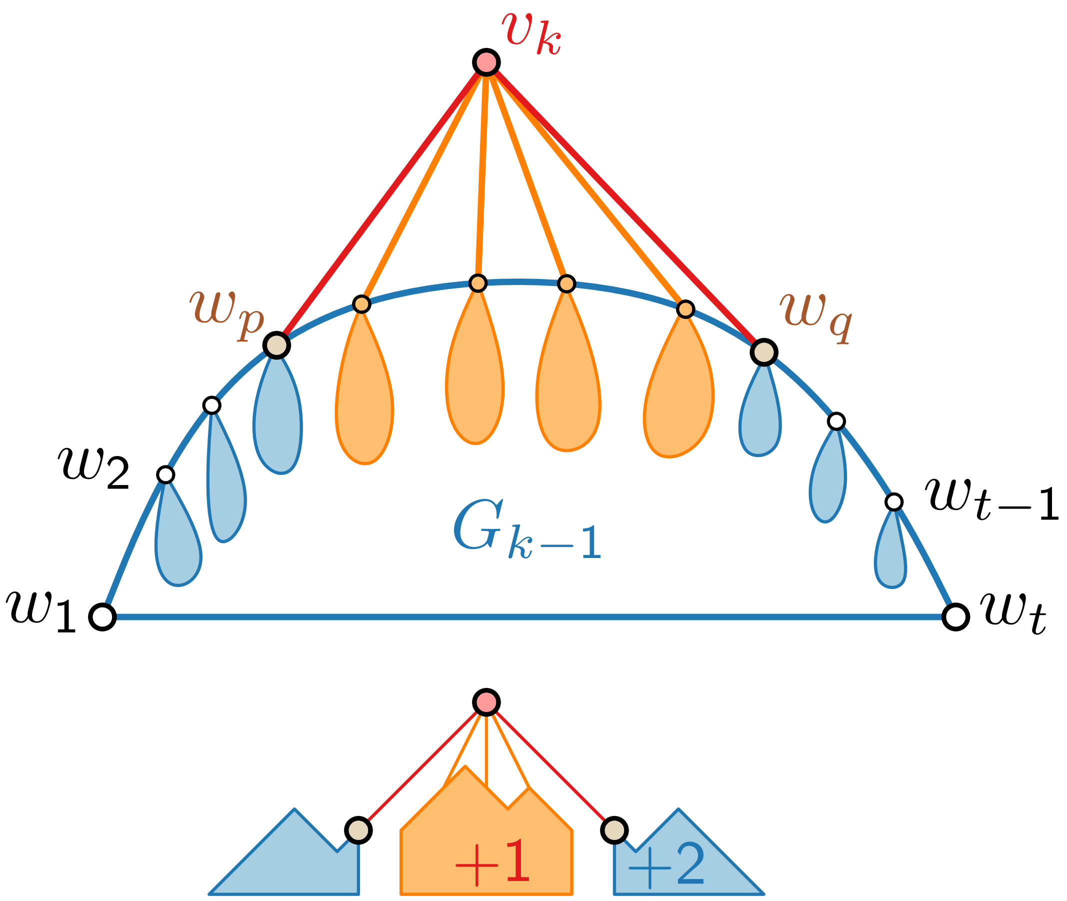
  foreach $v \in \bigcup_{i=q}^{t} L(w_i)$ **do**
    $x(v) \leftarrow x(v) + 2$
  $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$

return $P(v_1), \ldots, P(v_n)$

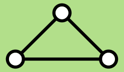


Shift Method – Pseudocode

canonical order of $V(G)$

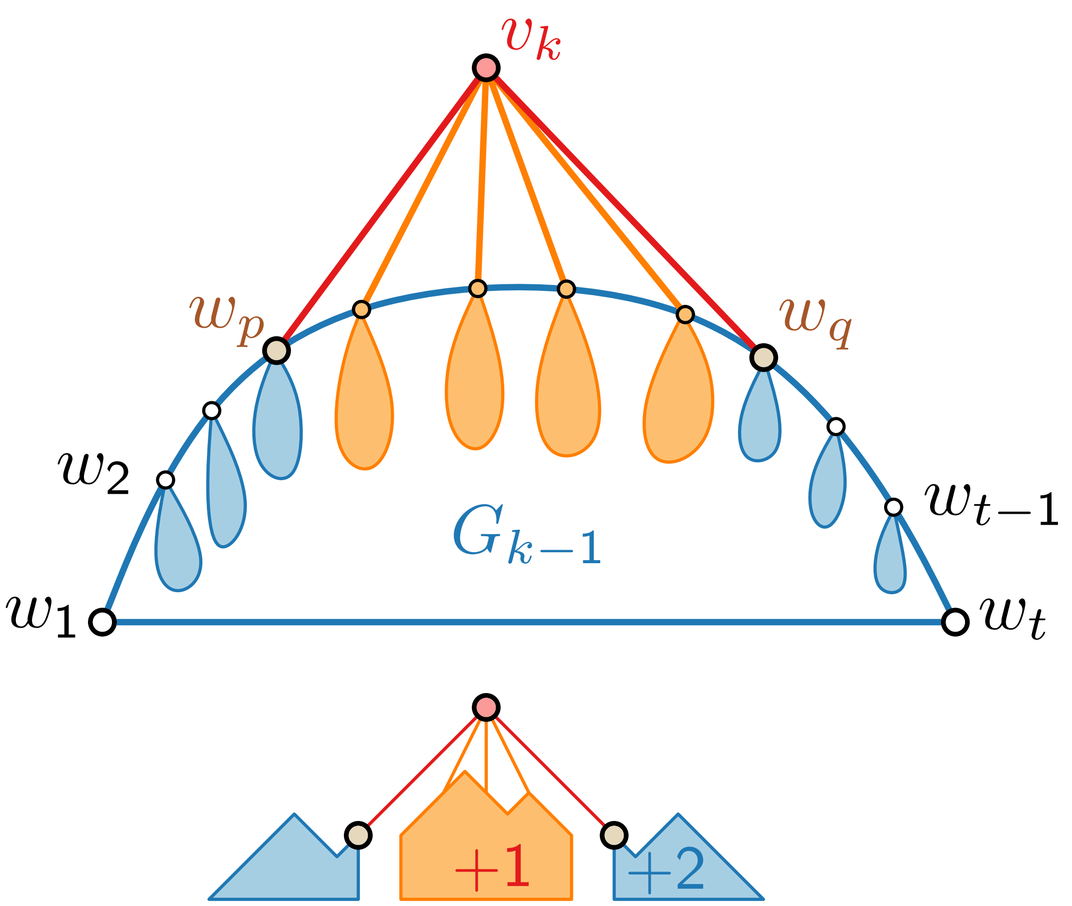
ShiftMethod$(G, (v_1, v_2, \ldots, v_n))$

for $k = 1$ to 3 **do**
- $L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

for $k = 4$ to n **do**
- Let ∂G_{k-1} be $v_1 = w_1, w_2, \ldots, w_{t-1}, w_t = v_2$.
- Let $w_p, \ldots, w_q$ be the neighbors of v_k.
- **foreach** $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do**
  - $x(v) \leftarrow x(v) + 1$
- **foreach** $v \in \bigcup_{i=q}^{t} L(w_i)$ **do**
  - $x(v) \leftarrow x(v) + 2$
- $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$
- $L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$

return $P(v_1), \ldots, P(v_n)$



Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \dots, v_n)$)
- **for** $k = 1$ to 3 **do**
  - $L(v_k) \leftarrow \{v_k\}$
- $P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$
- **for** $k = 4$ to n **do**
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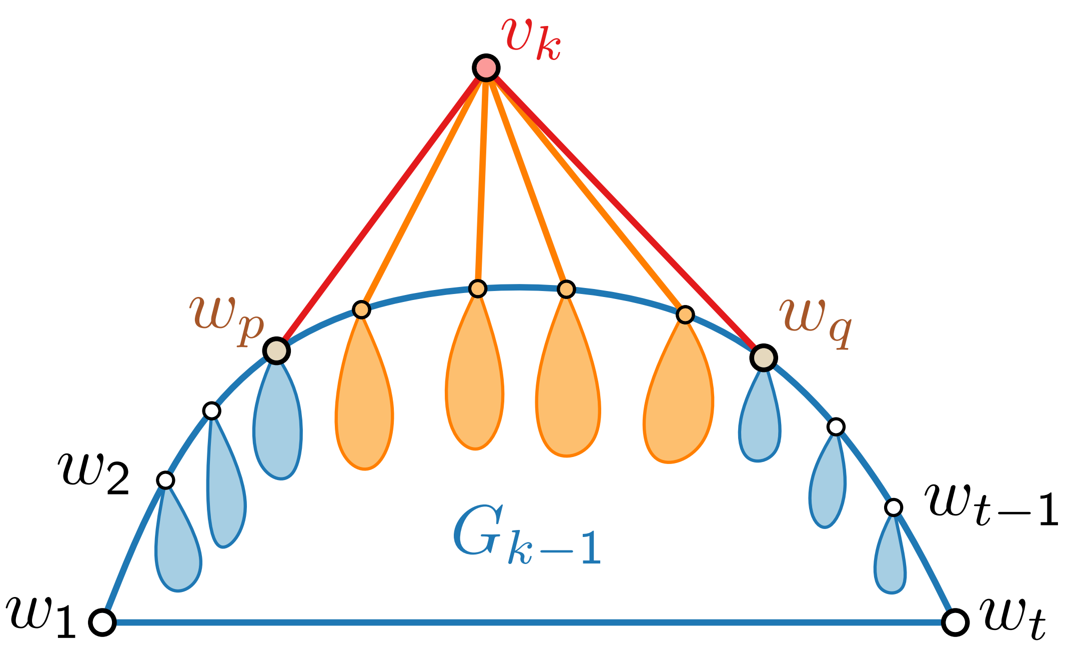
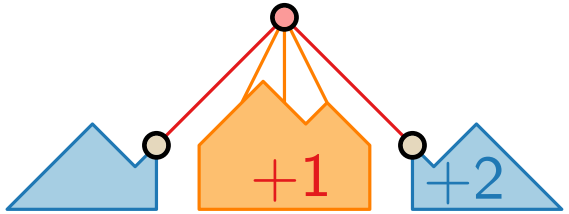
- **return** $P(v_1), \dots, P(v_n)$





Running Time?

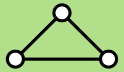
Shift Method – Pseudocode

canonical order of $V(G)$

ShiftMethod(G, $(v_1, v_2, \dots, v_n)$)

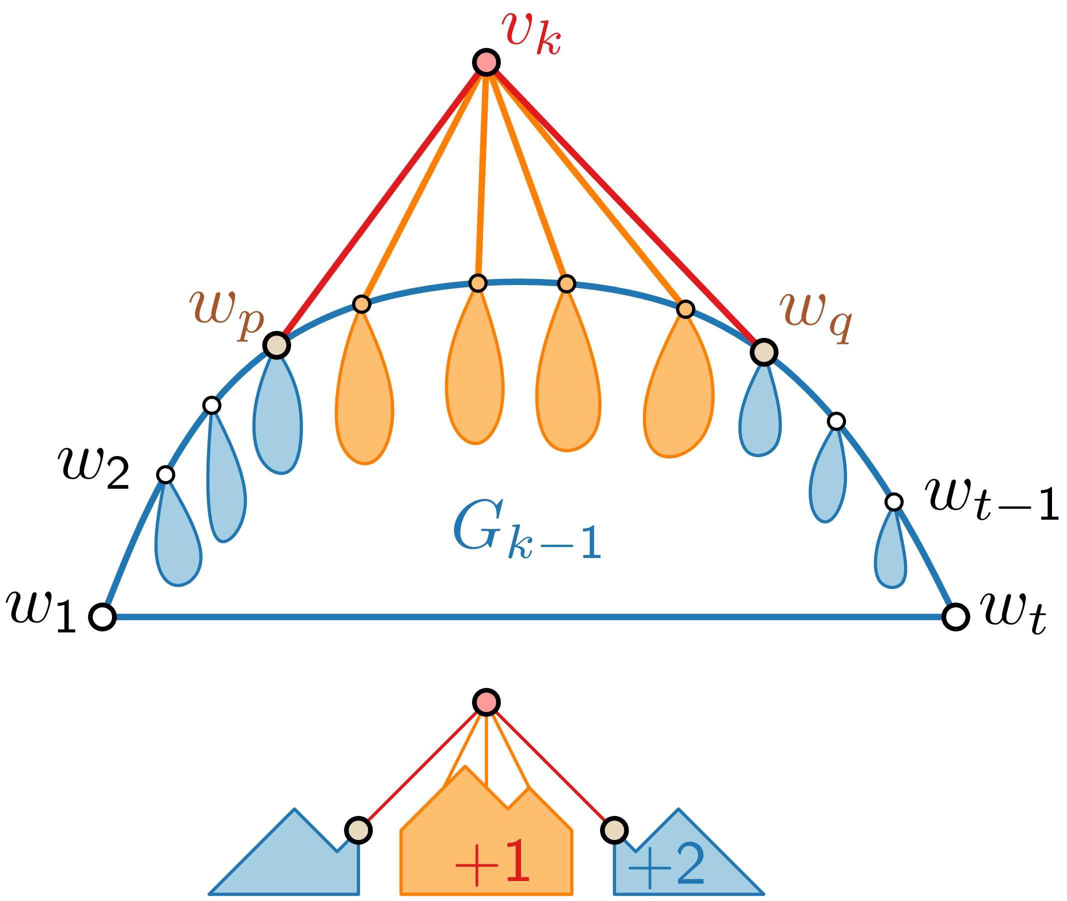
for $k = 1$ to 3 **do**
- $L(v_k) \leftarrow \{v_k\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$



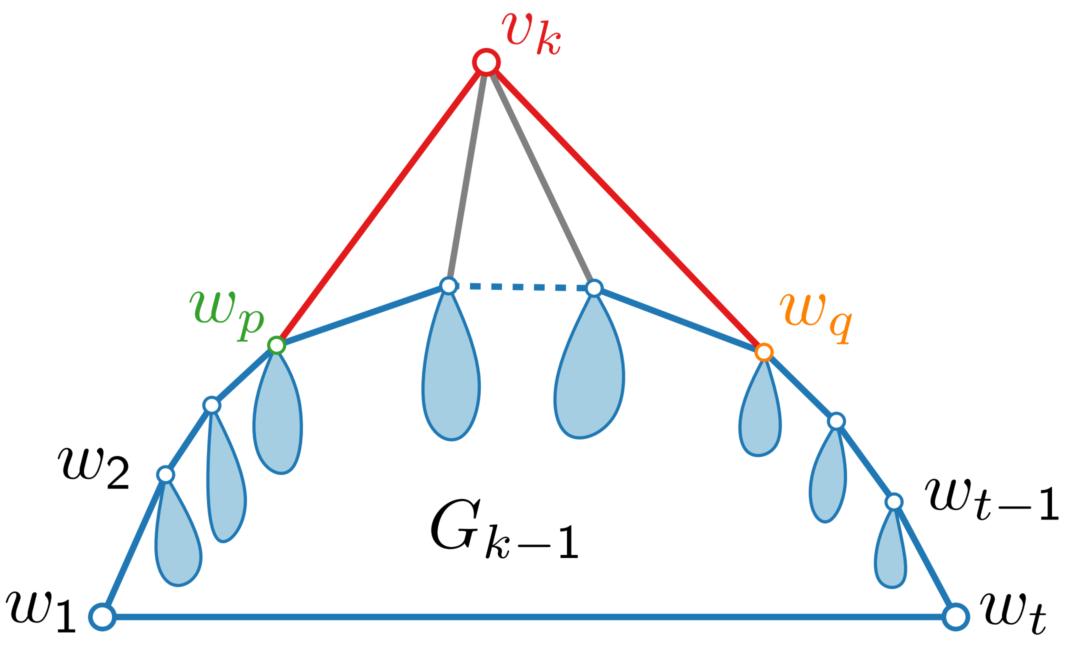
for $k = 4$ to n **do**
- Let ∂G_{k-1} be $v_1 = w_1, w_2, \dots, w_{t-1}, w_t = v_2$.
- Let $w_p, \dots, w_q$ be the neighbors of v_k.
- **foreach** $v \in \bigcup_{i=p+1}^{q-1} L(w_i)$ **do** // *$\mathcal{O}(n^2)$ in total*
  - $x(v) \leftarrow x(v) + 1$
- **foreach** $v \in \bigcup_{i=q}^{t} L(w_i)$ **do** // *$\mathcal{O}(n^2)$ in total*
  - $x(v) \leftarrow x(v) + 2$
- $P(v_k) \leftarrow$ intersection of slope-± 1 diagonals through $P(w_p)$ and $P(w_q)$
- $L(v_k) \leftarrow \bigcup_{i=p+1}^{q-1} L(w_i) \cup \{v_k\}$

return $P(v_1), \dots, P(v_n)$



Running Time?

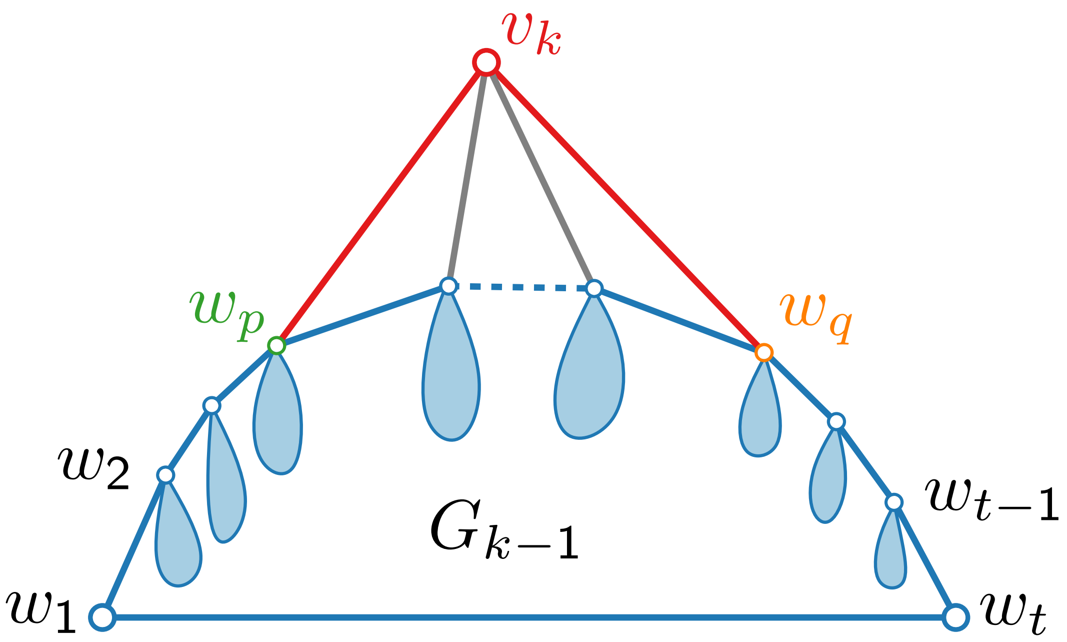
Shift Method – Linear-Time Implementation



Shift Method – Linear-Time Implementation

Idea 1.

To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$



Shift Method – Linear-Time Implementation

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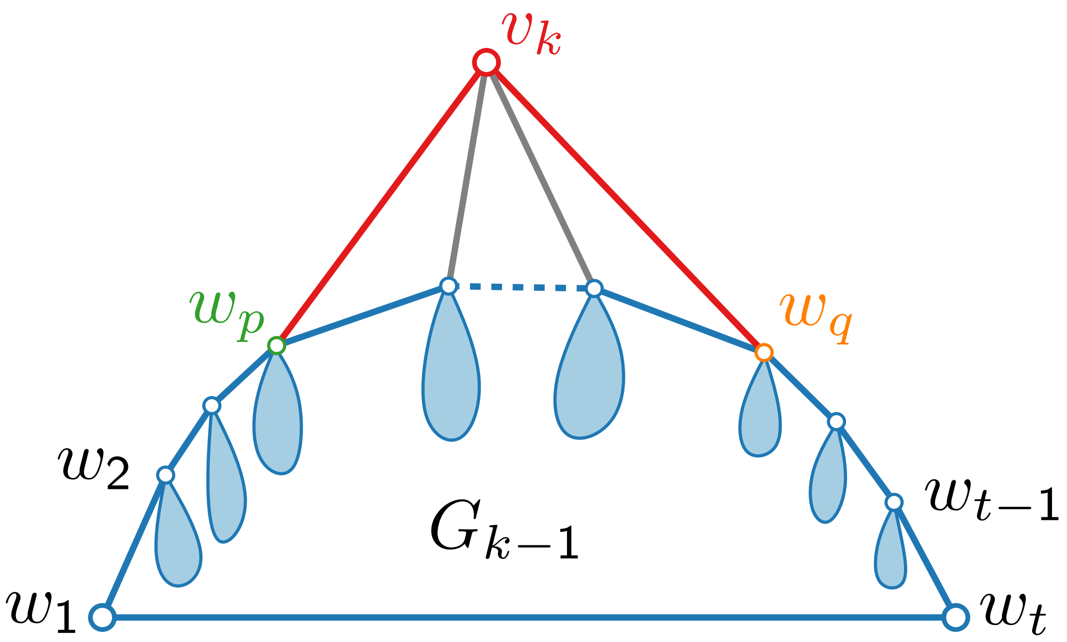

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

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To compute $x(v_k)$ and $y(v_k)$,
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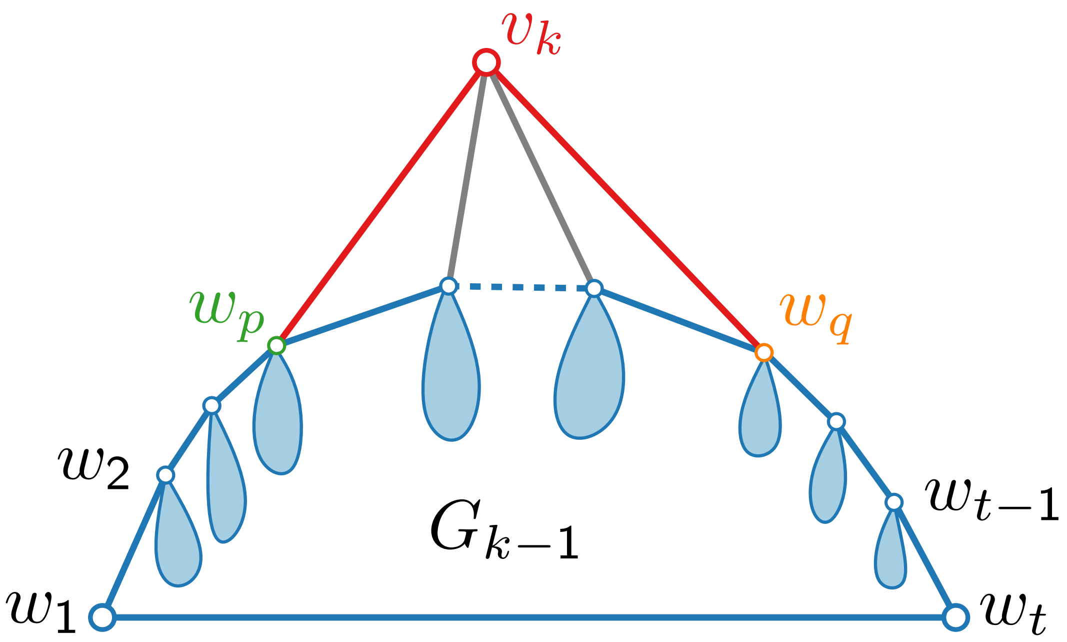


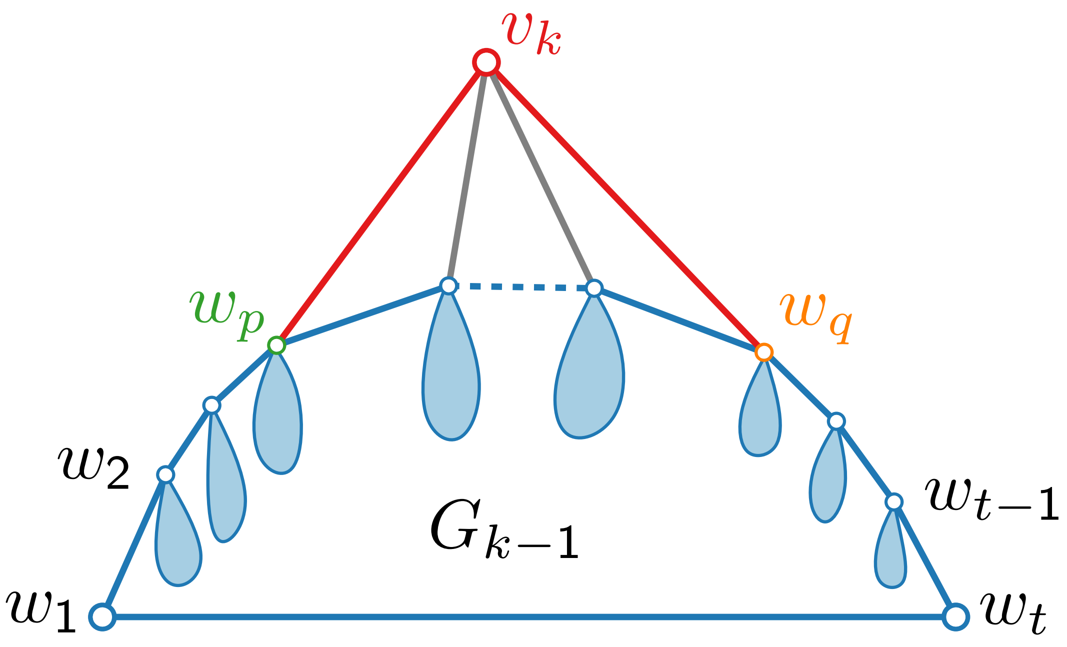
$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

Shift Method – Linear-Time Implementation

Idea 1.
To compute $x(v_k)$ and $y(v_k)$,
we need only $y(w_p)$, $y(w_q)$, and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates,
we store, for each vertex within a specific spanning tree,
the x-distance to its parent (v_1 is the root).



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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Shift Method – Linear-Time Implementation

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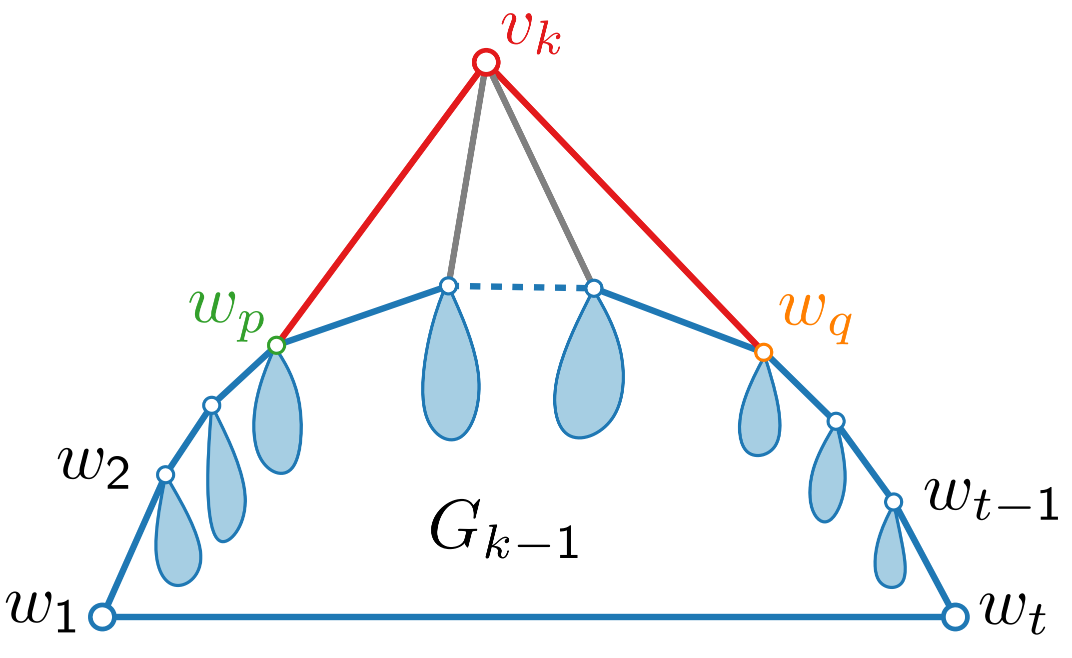

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(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear-Time Implementation

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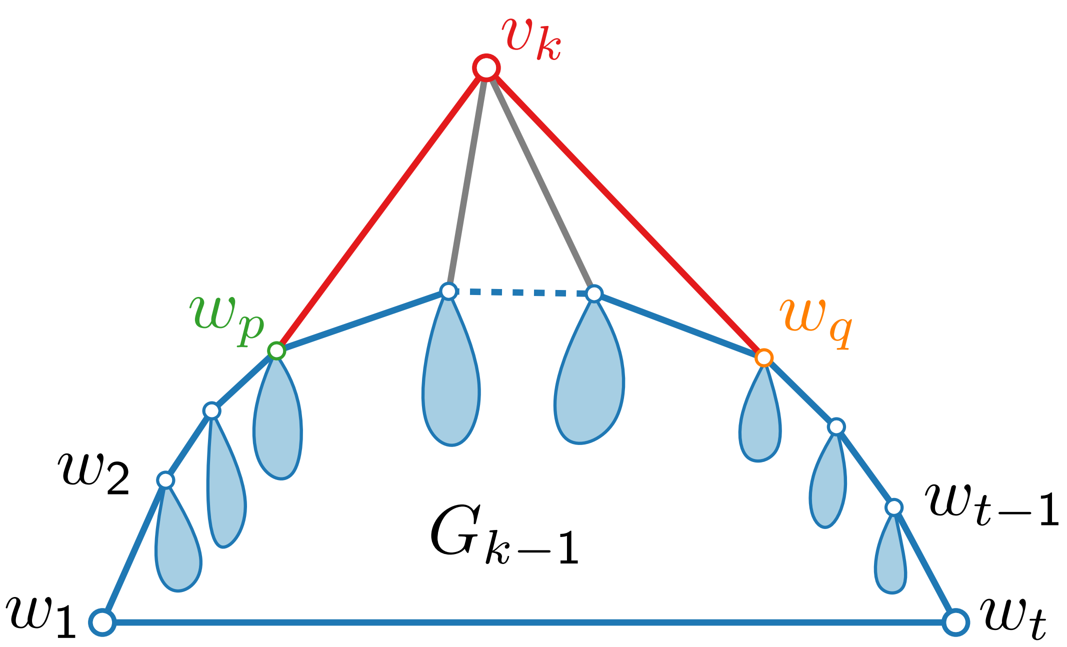

Idea 2.
Instead of storing explicit x-coordinates,
we store, for each vertex within a specific spanning tree,
the x-distance to its parent (v_1 is the root).

After an x-distance is computed for each v_k,
use preorder traversal to compute all x-coordinates.

$$(1)\quad x(v_k) = \tfrac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$$

$$(2)\quad y(v_k) = \tfrac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$$

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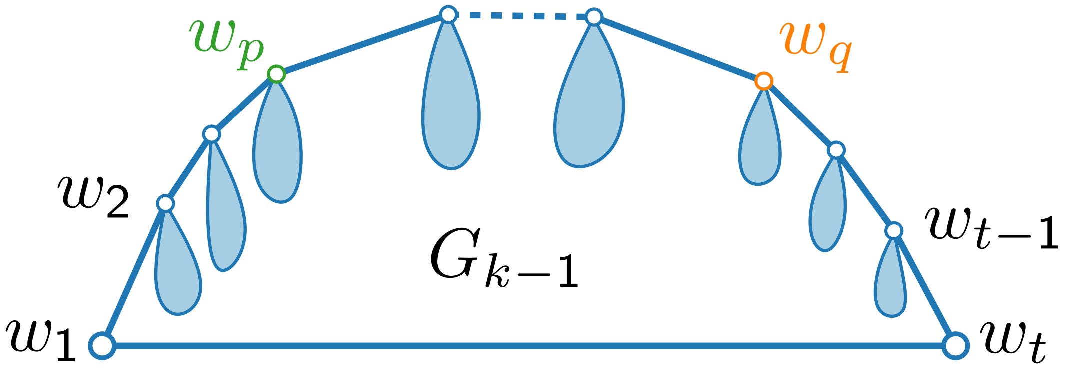
Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



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Shift Method – Linear-Time Implementation

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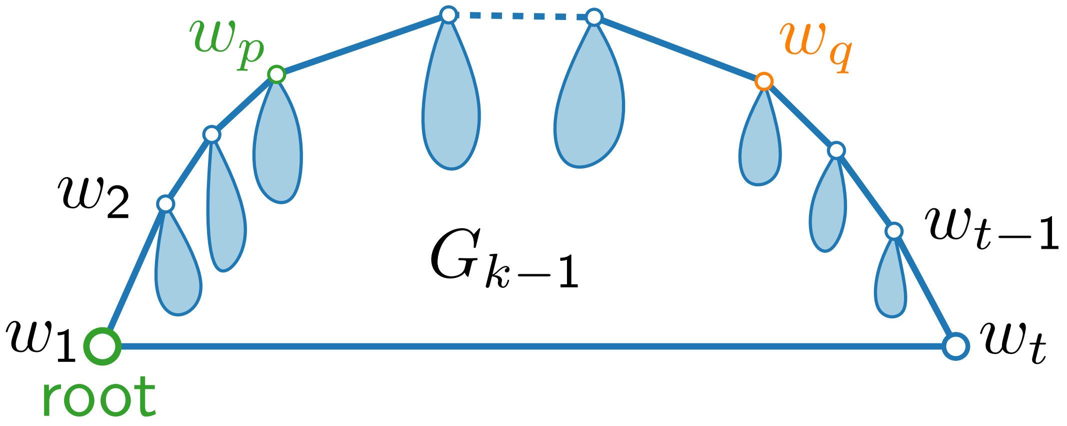

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Shift Method – Linear-Time Implementation

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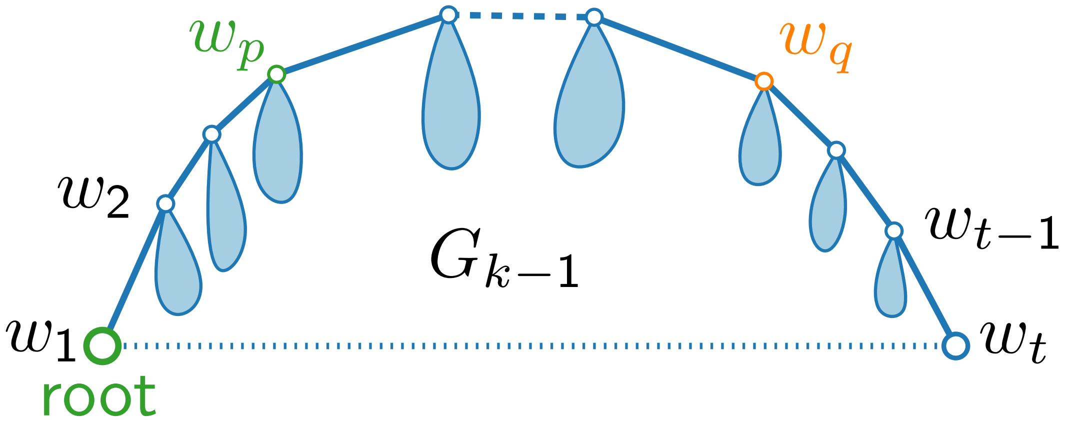

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Shift Method – Linear-Time Implementation

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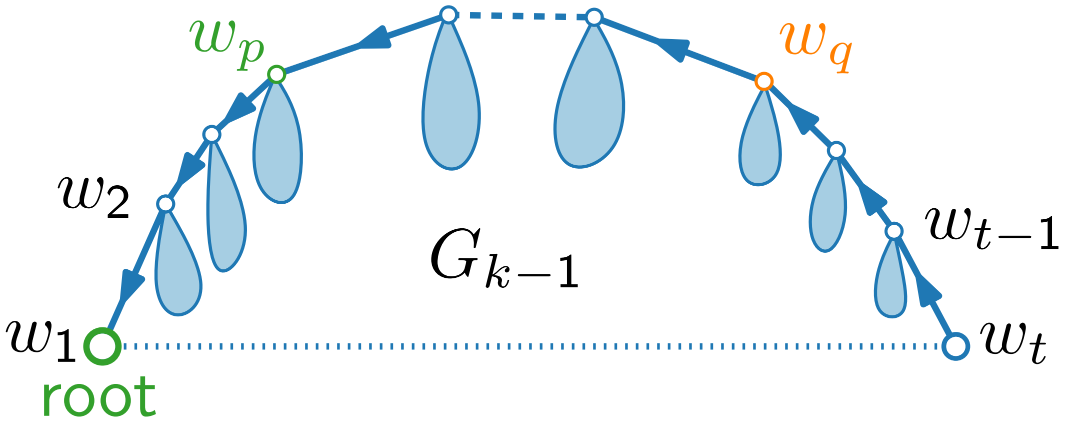

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Shift Method – Linear-Time Implementation

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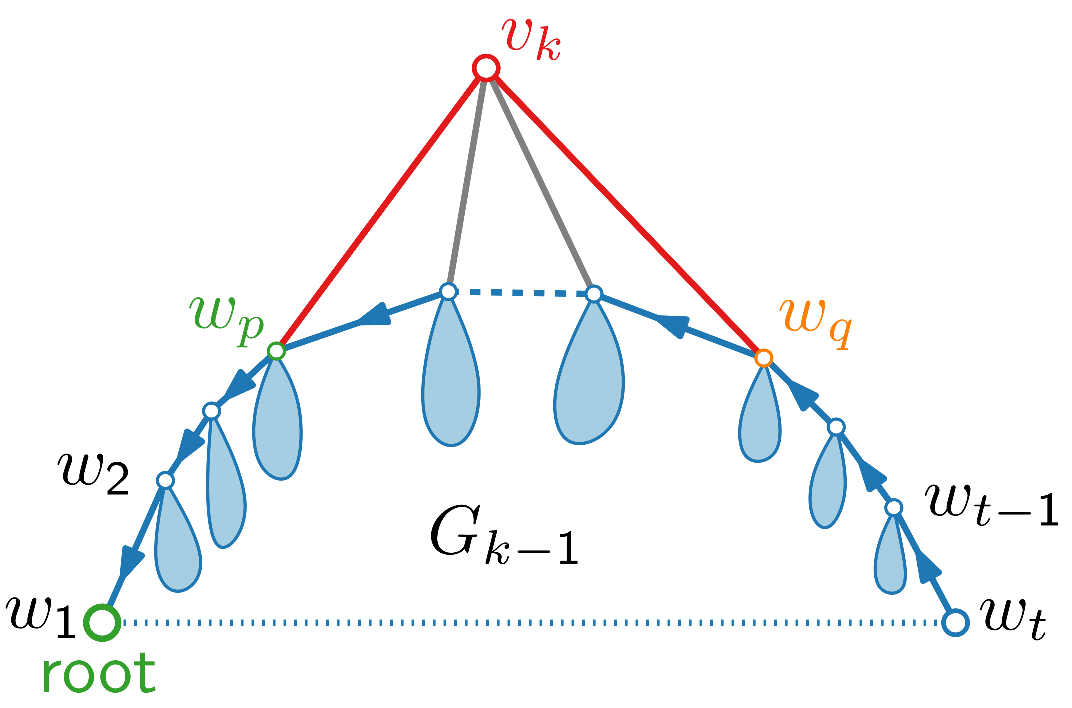

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Shift Method – Linear-Time Implementation

Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
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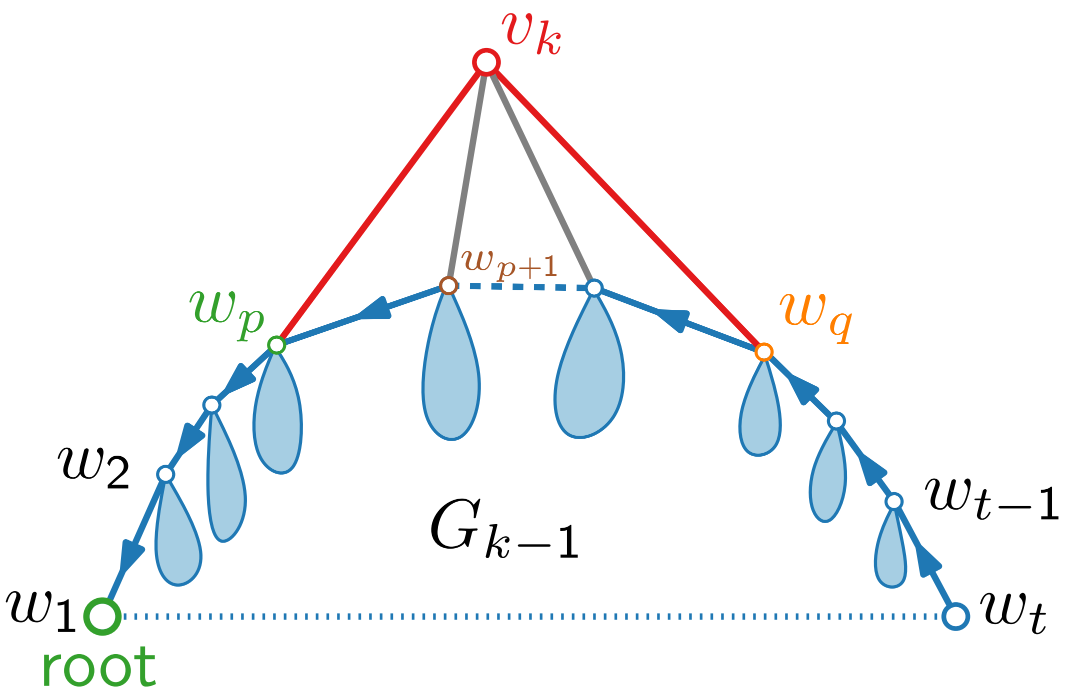
Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++



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Shift Method – Linear-Time Implementation

Relative x-distance tree.

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- x-offset $\Delta_x(v)$ from parent
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Calculations.

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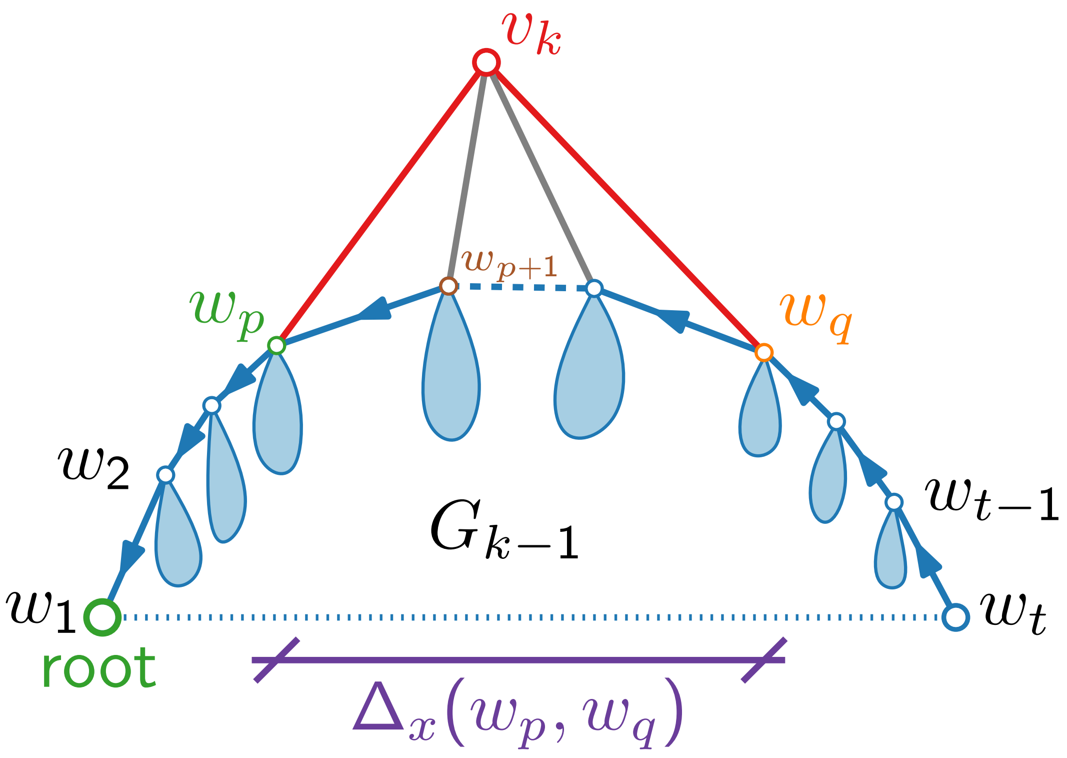

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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Shift Method – Linear-Time Implementation

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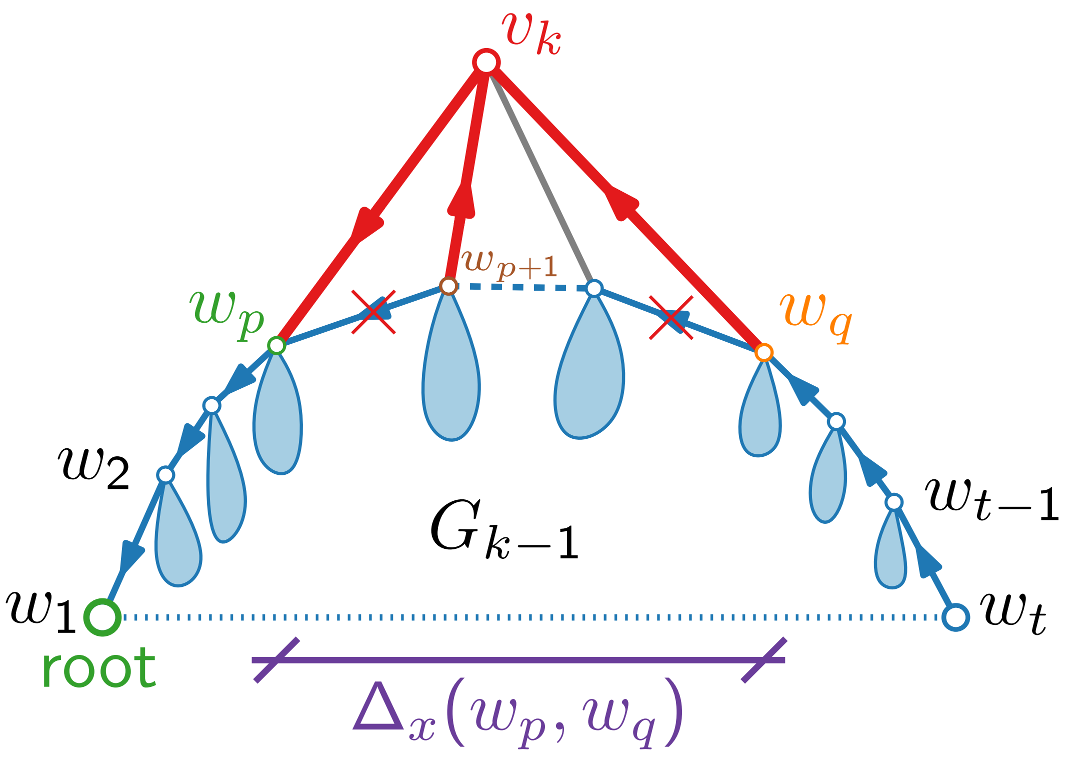
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$



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Shift Method – Linear-Time Implementation

Relative x-distance tree.

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- x-offset $\Delta_x(v)$ from parent
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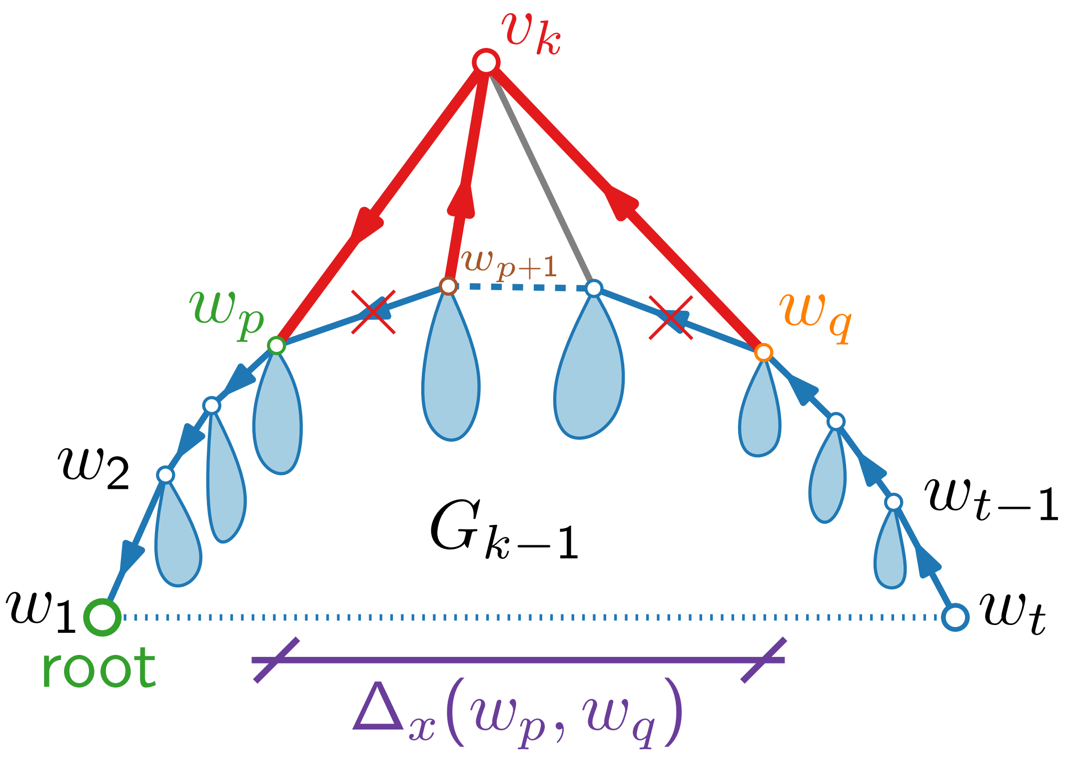
Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)



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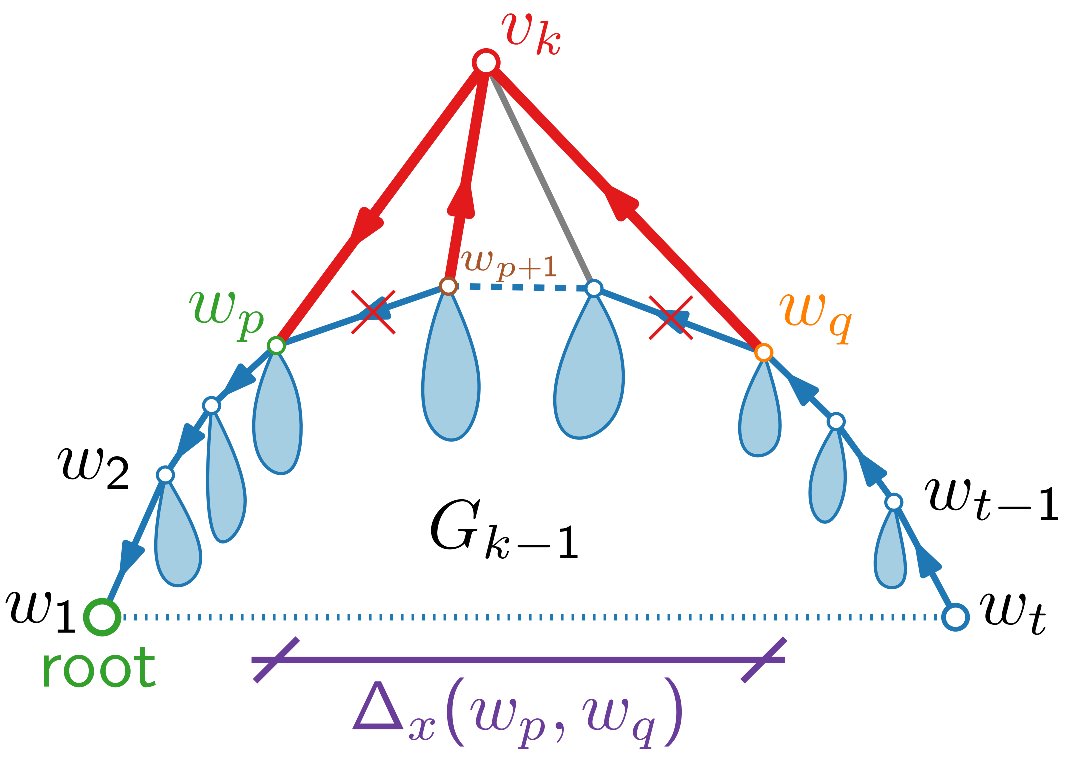
Shift Method – Linear-Time Implementation



Relative x-distance tree.

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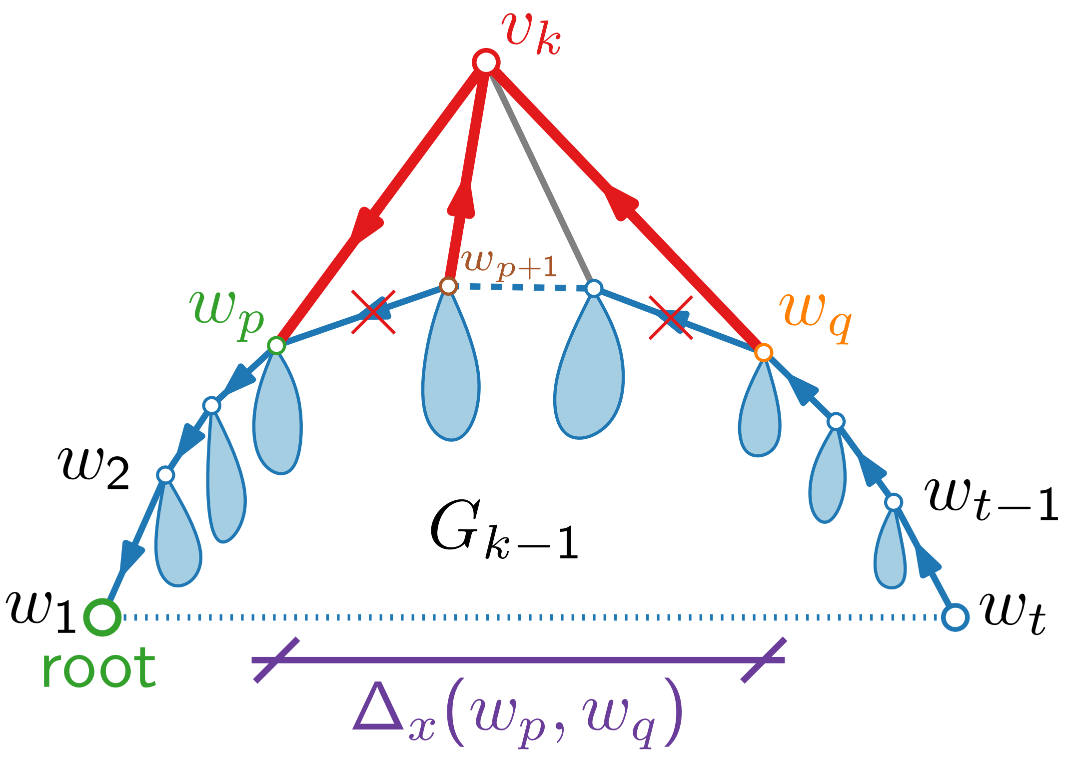
Shift Method – Linear-Time Implementation



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Calculations.

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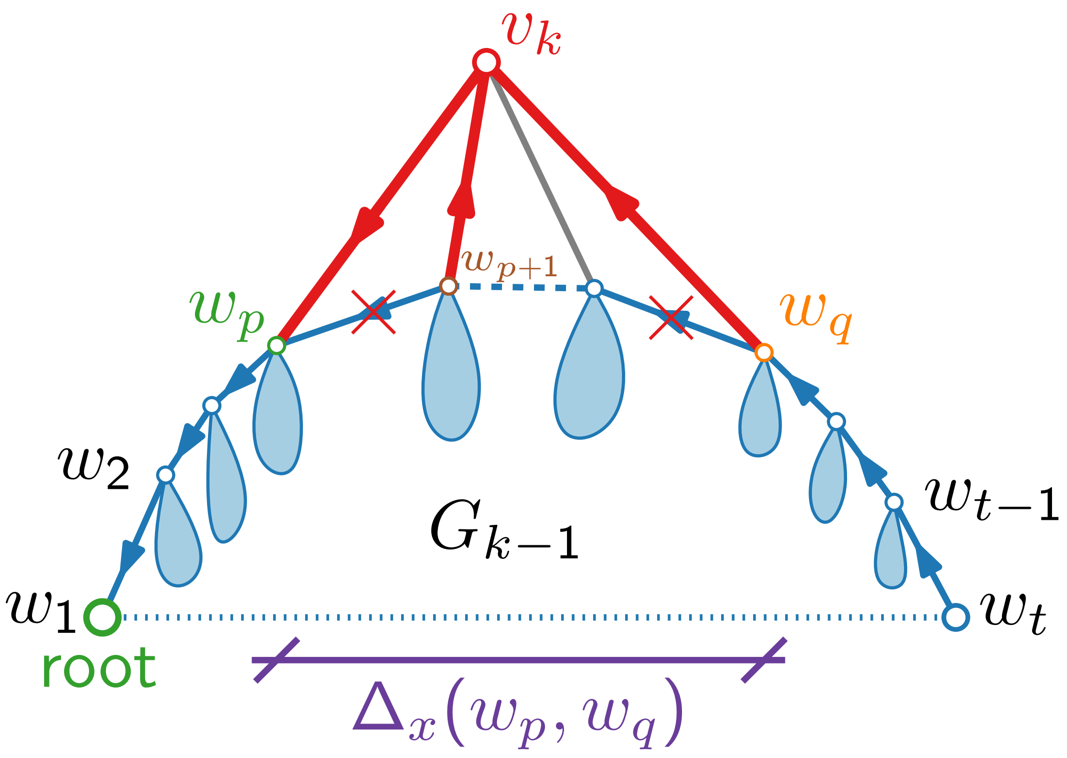
Shift Method – Linear-Time Implementation



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- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$

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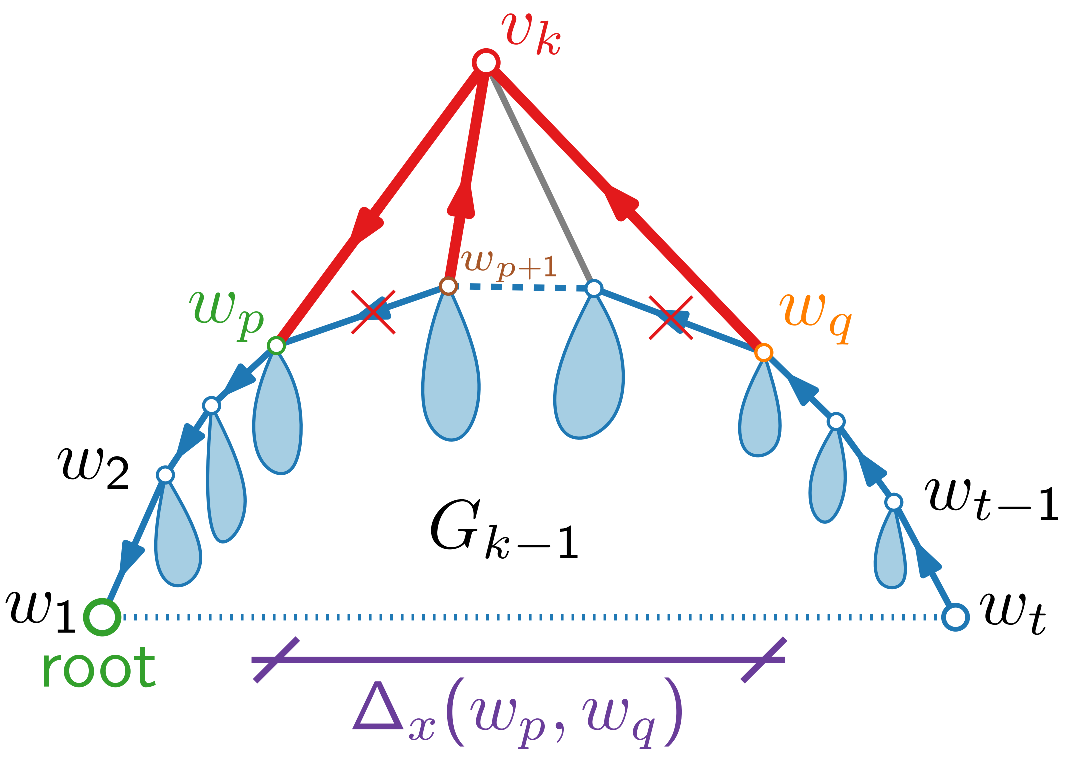
Shift Method – Linear-Time Implementation



Relative x-distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
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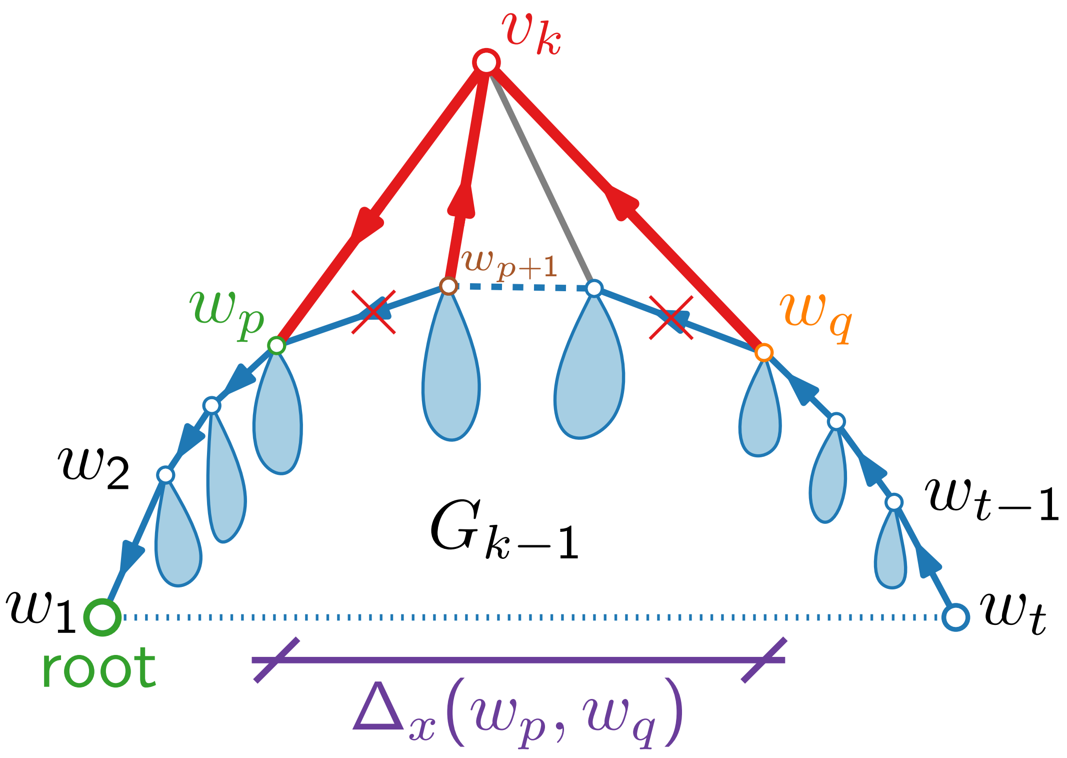

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takes ? time

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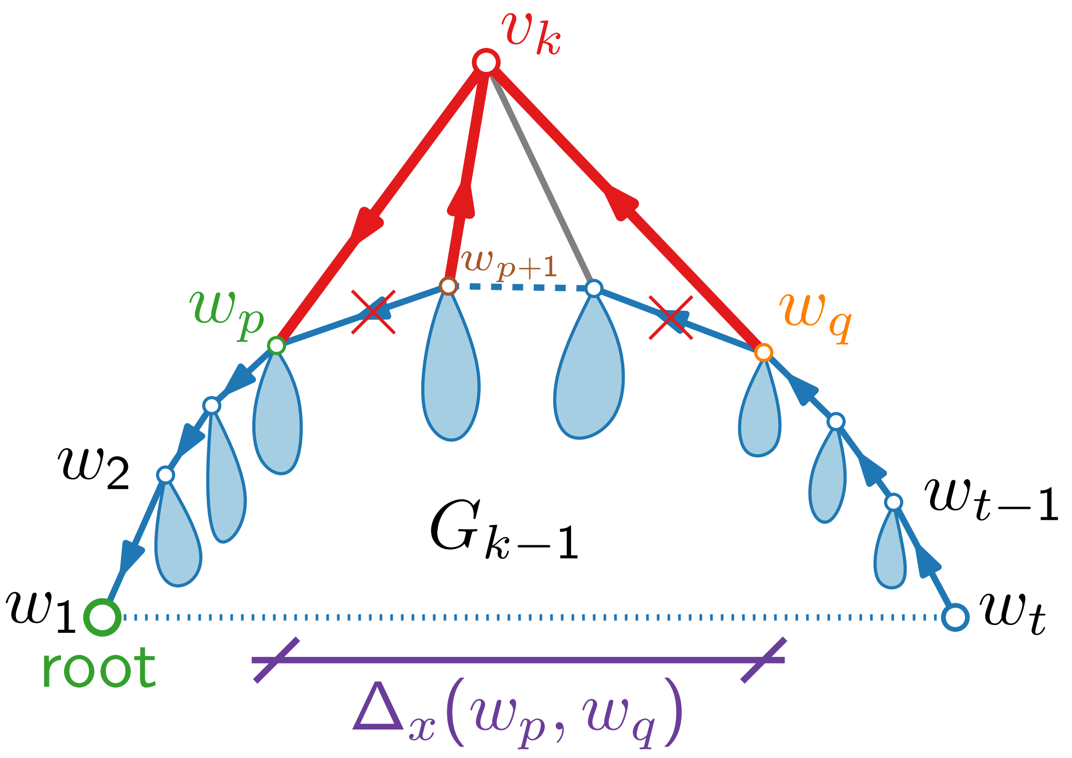

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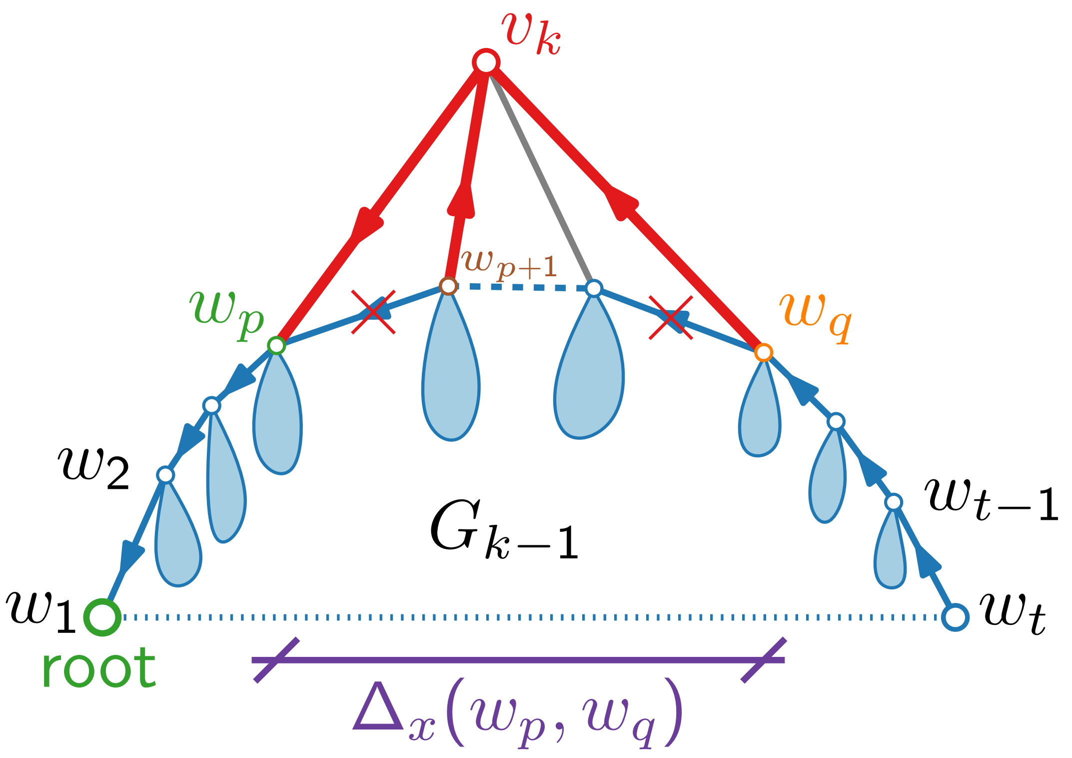
takes $\mathcal{O}(n)$ time in total ☺



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Discussion

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- The linear-time implementation was later proposed by Chrobak and Payne.
- Although we are guaranteed to get a very small grid, only straight-line edges, and no edge crossings, the resulting drawings are not always visually pleasing: the drawings tend to have very small angles and a big variance in the size of the triangular faces.
- A quite different approach yielding similar results is by Schnyder ($\rightarrow$ next lecture).

Literature

- [PGD Ch. 4.2] for detailed explanation of the shift method
- [de Fraysseix, Pach, Pollack 1990] "How to draw a planar graph on a grid"
  – original paper introducing the shift method
- [Chrobak, Payne 1995] "A linear-time algorithm for drawing a planar graph on a grid"
  – original paper on how to implement the shift method in linear time