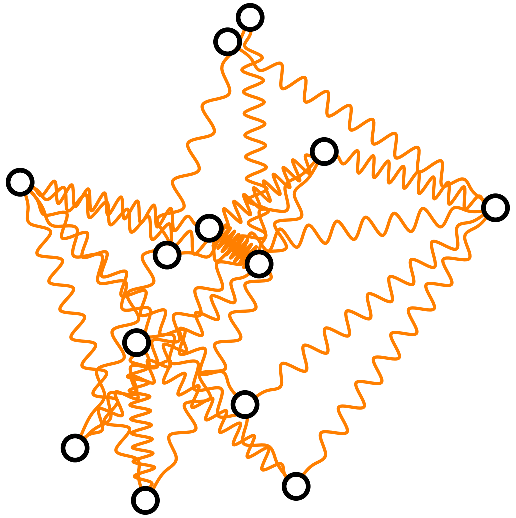
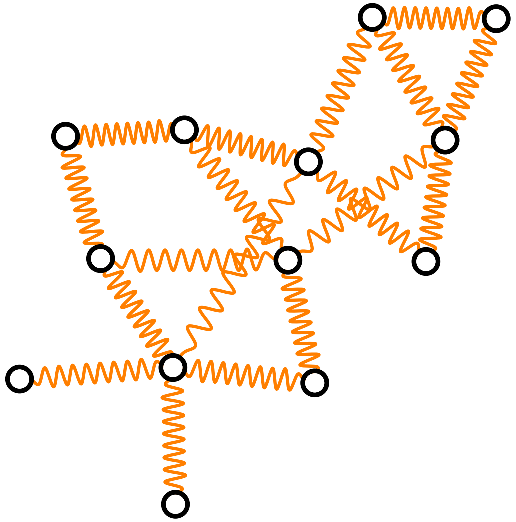
Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part I:
Spring Embedders

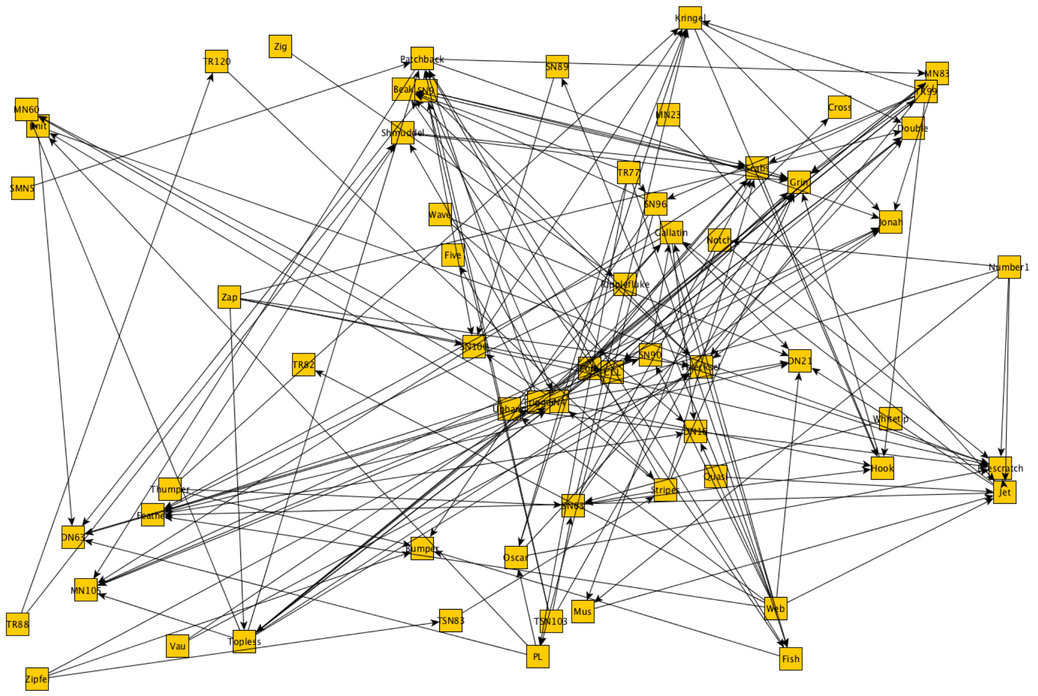


Alexander Wolff

Summer term 2025

General Layout Problem

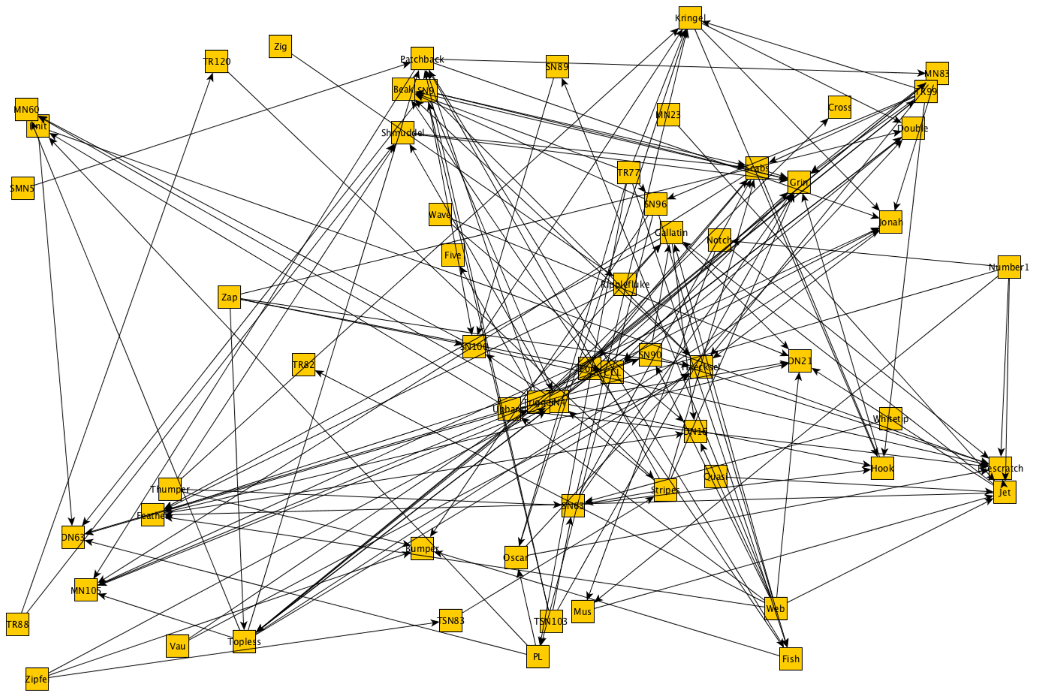
Input: Graph G

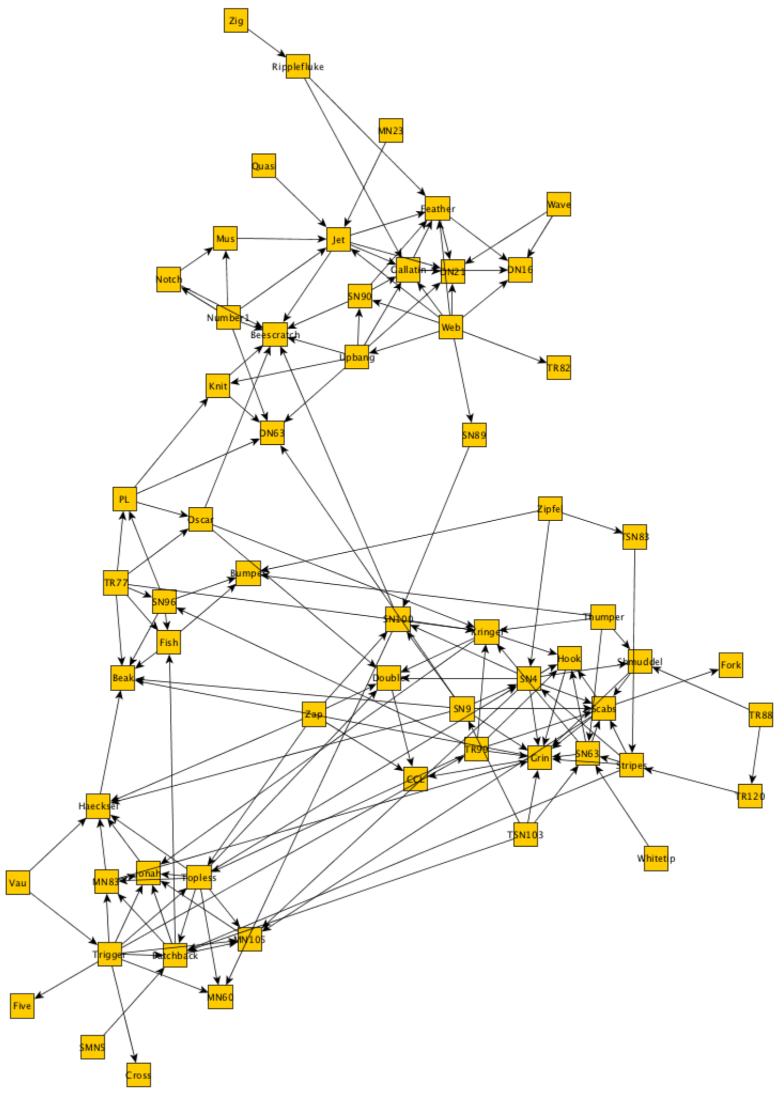


General Layout Problem

Input: Graph G

Output: Clear and readable straight-line drawing of G





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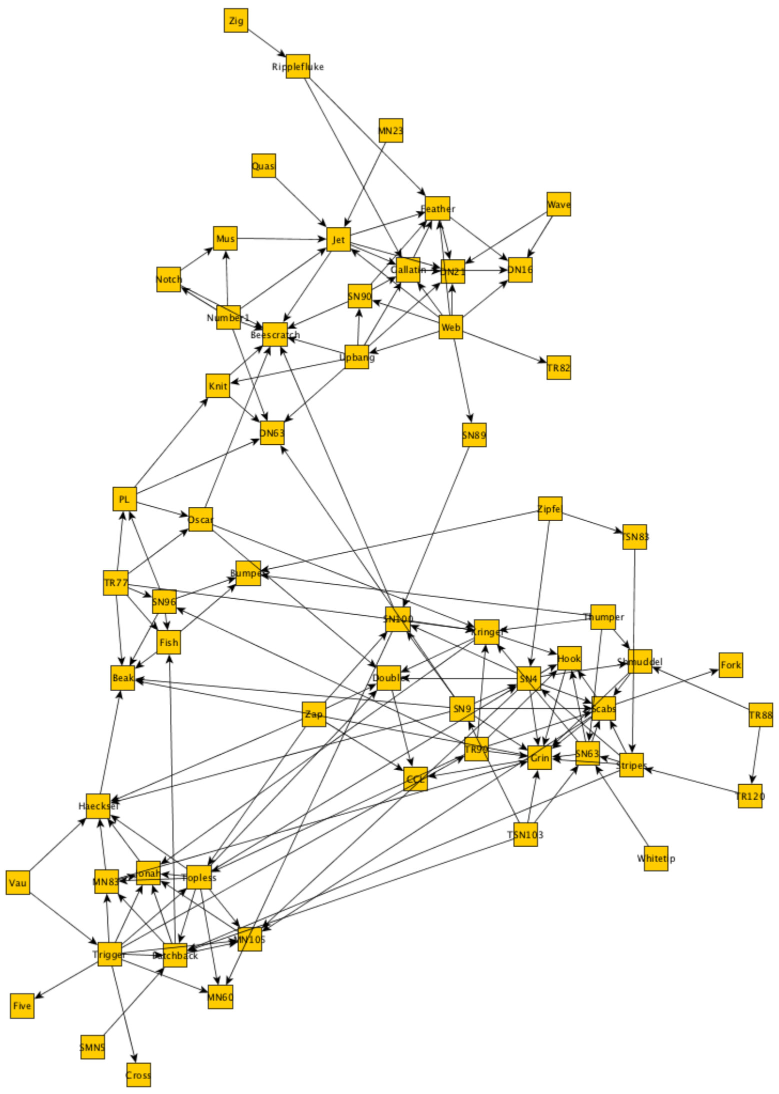
Drawing aesthetics to optimize:



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Drawing aesthetics to optimize:

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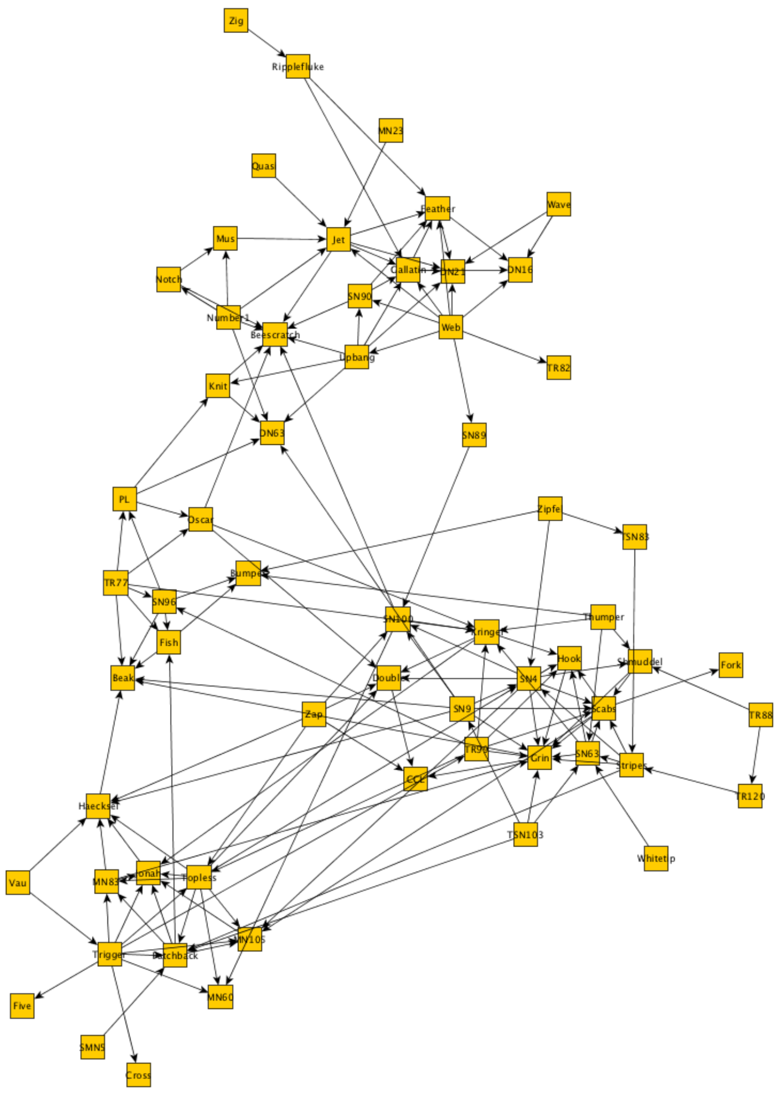

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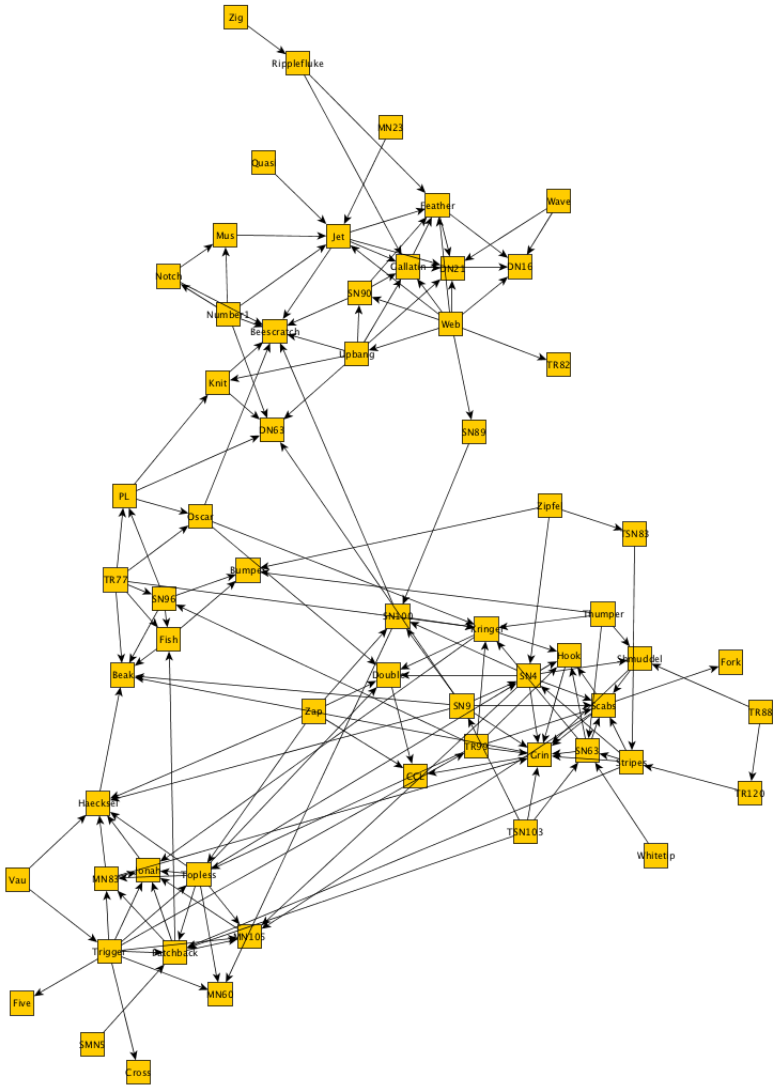

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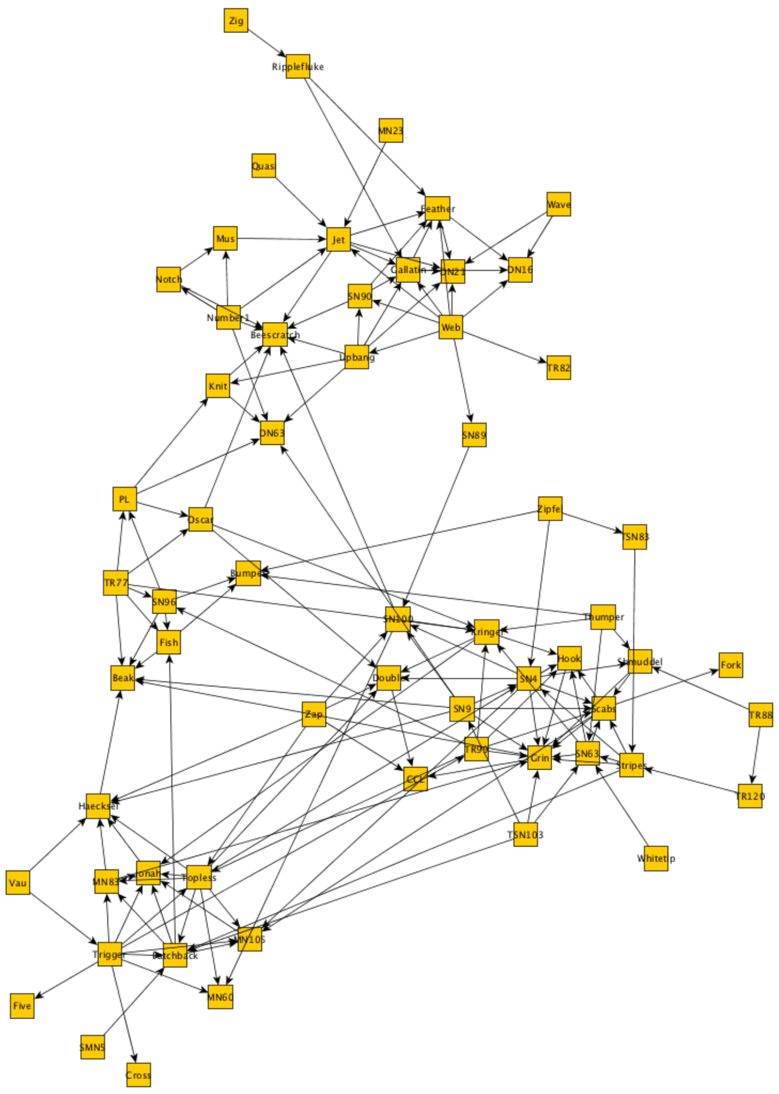
Drawing aesthetics to optimize:

- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**



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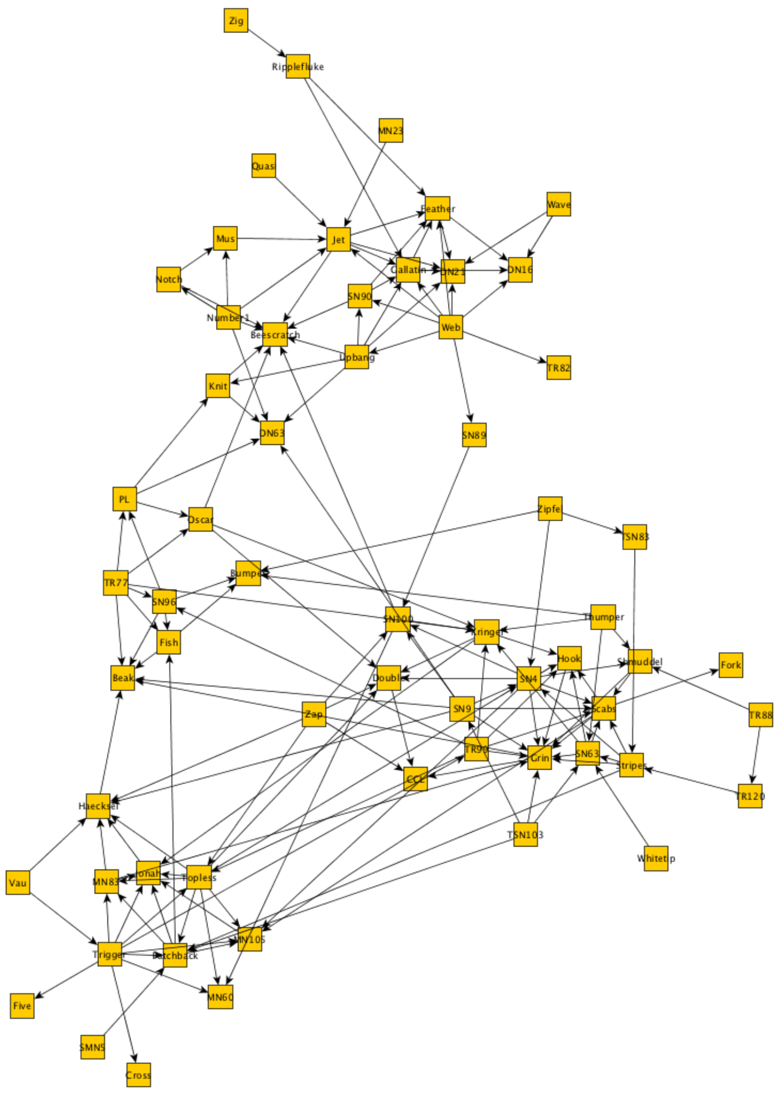

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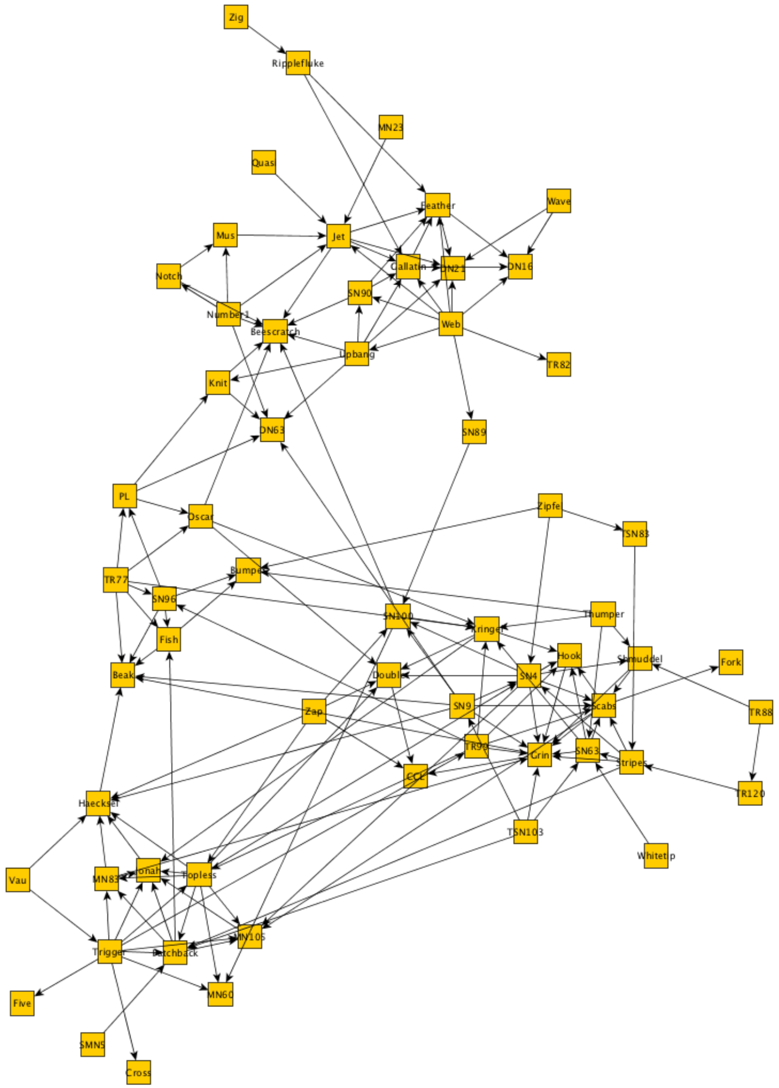

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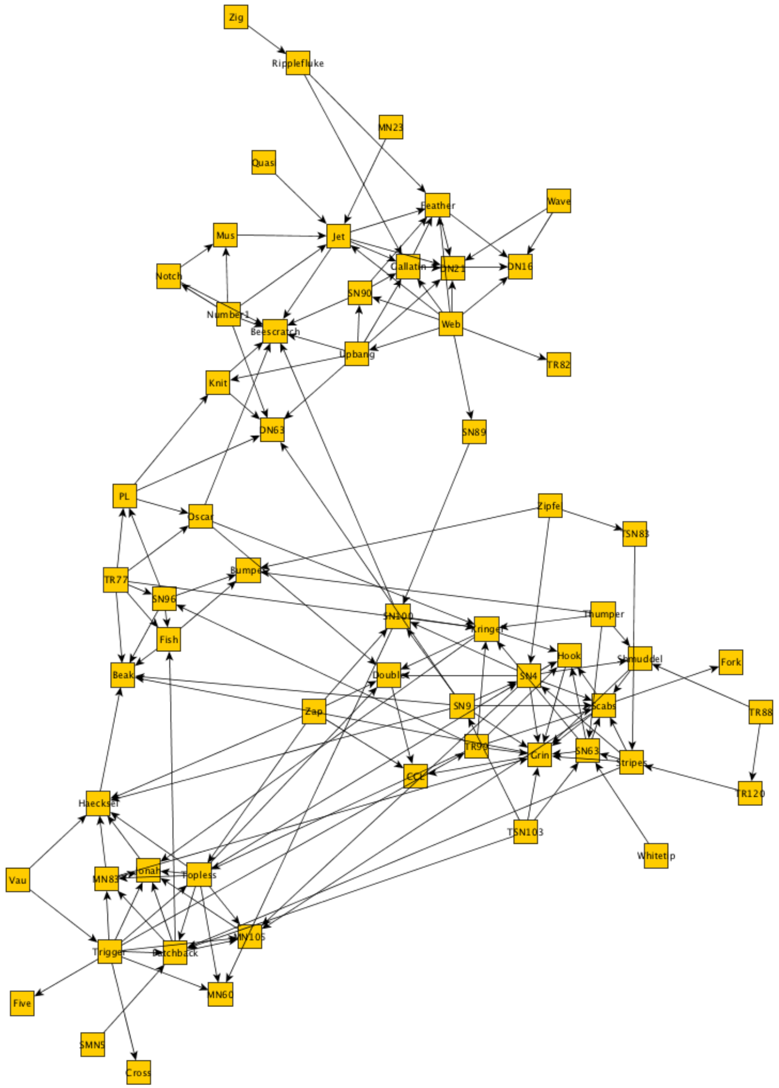
- nodes distributed evenly



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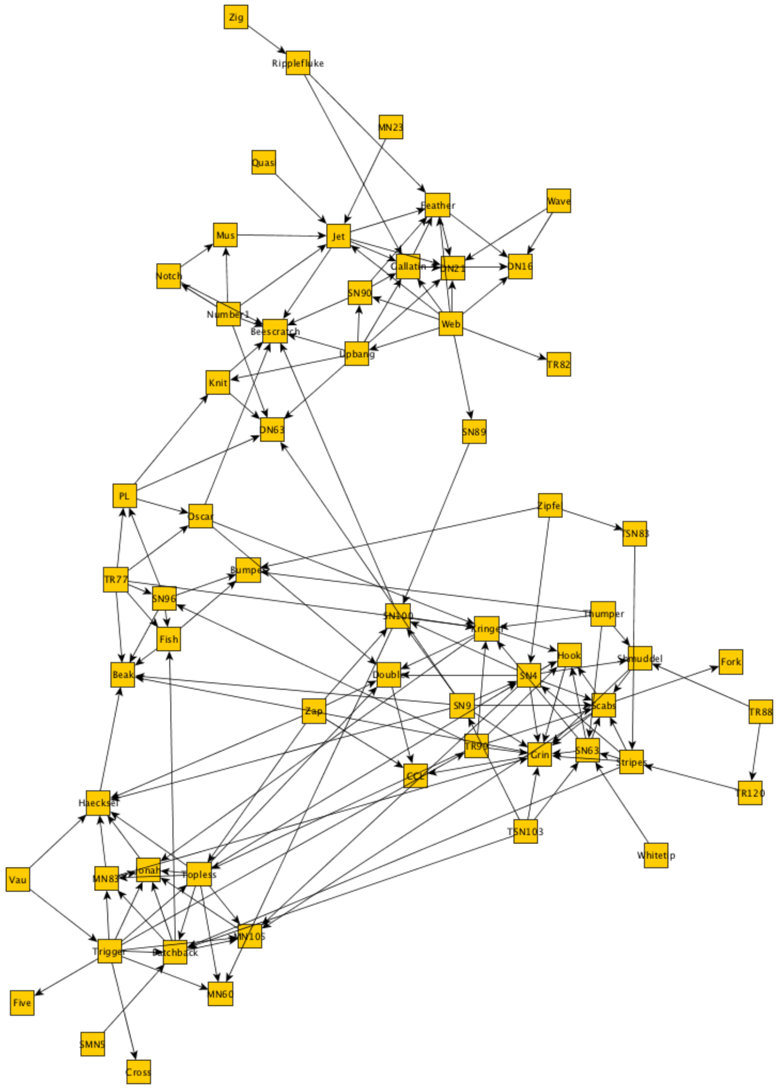

Optimization criteria partially contradict each other.

Fixed Edge Lengths?

Input: Graph G, required length $\ell(e)$ for each edge $e \in E(G)$.

Output: Drawing of G that realizes the given edge lengths.

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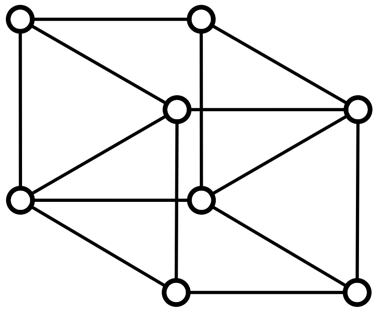

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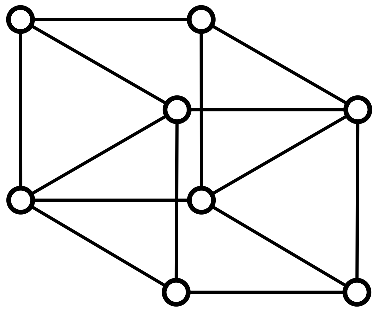

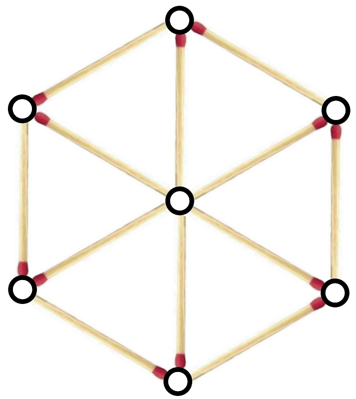

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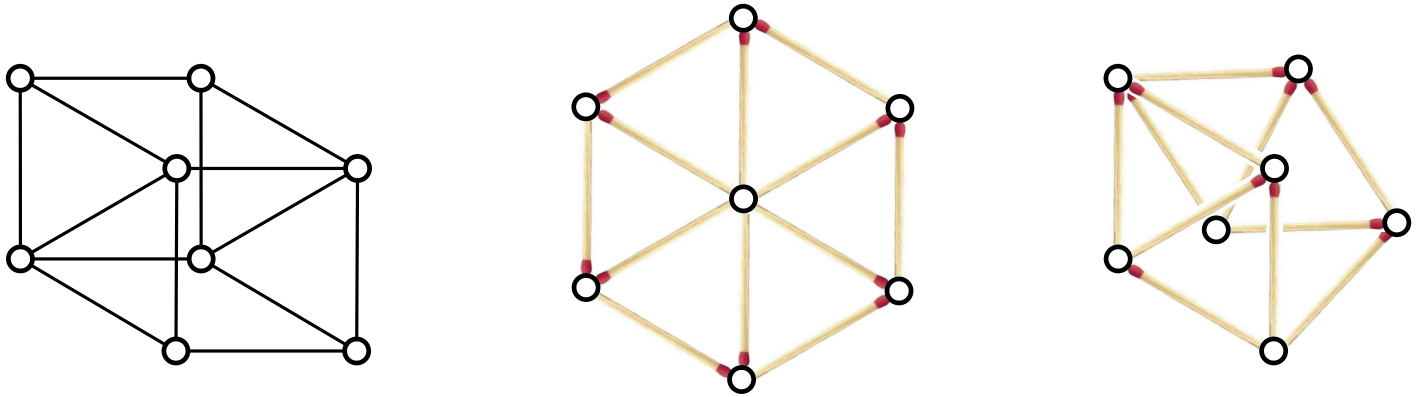

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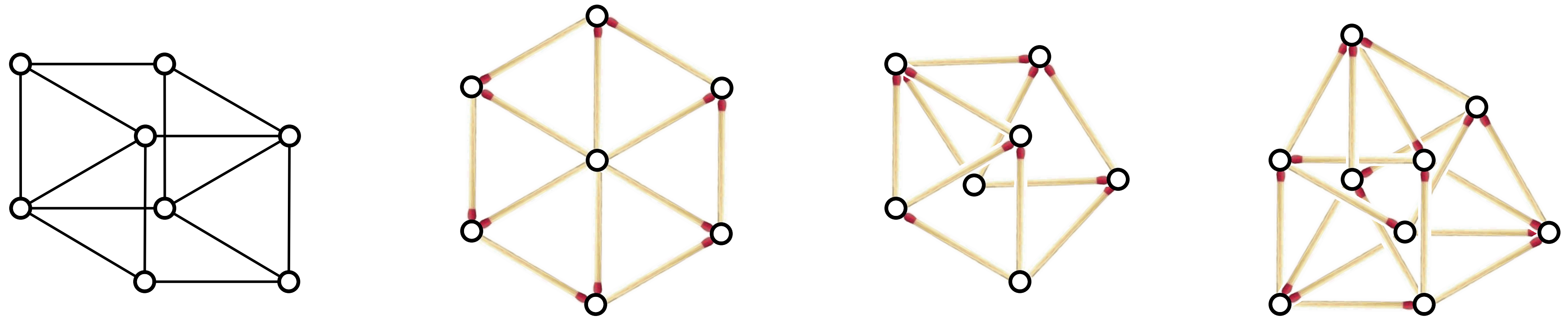

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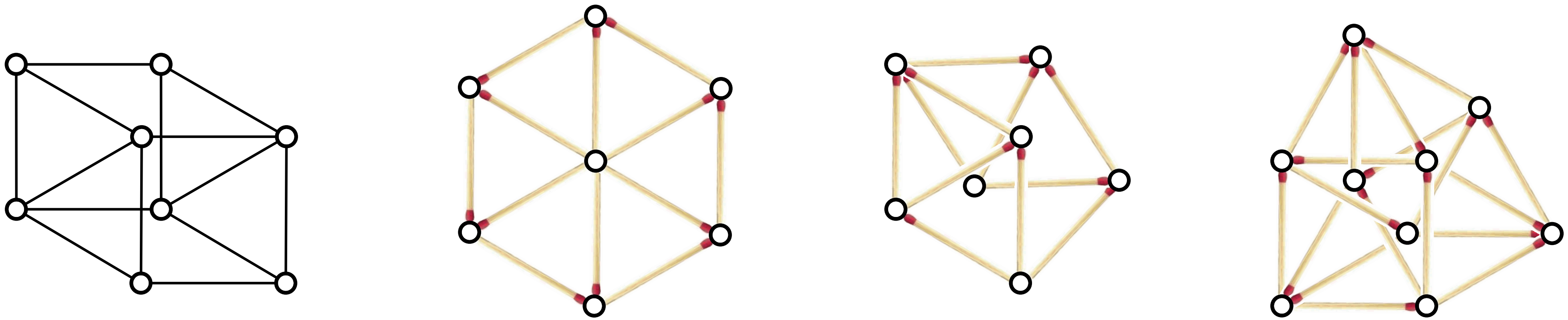

NP-hard for

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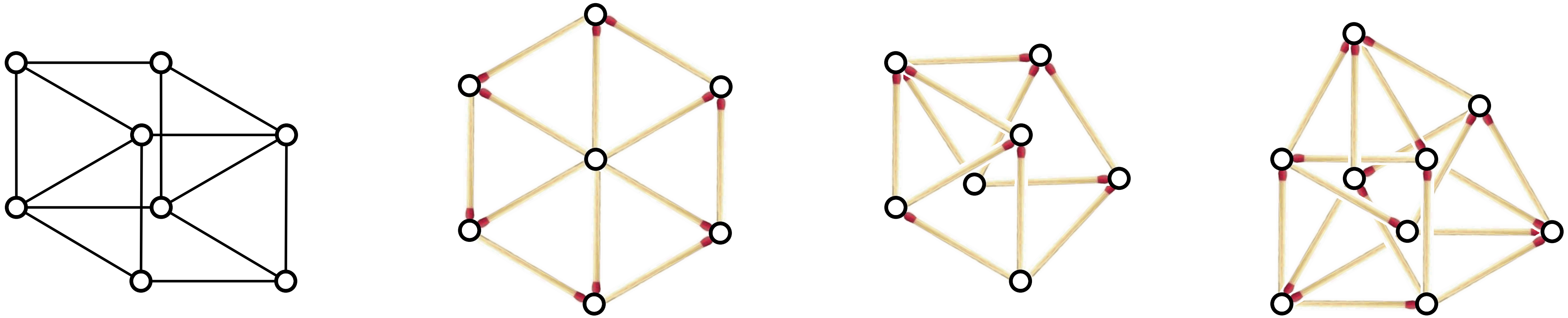

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- uniform edge lengths in any dimension [Johnson '82]

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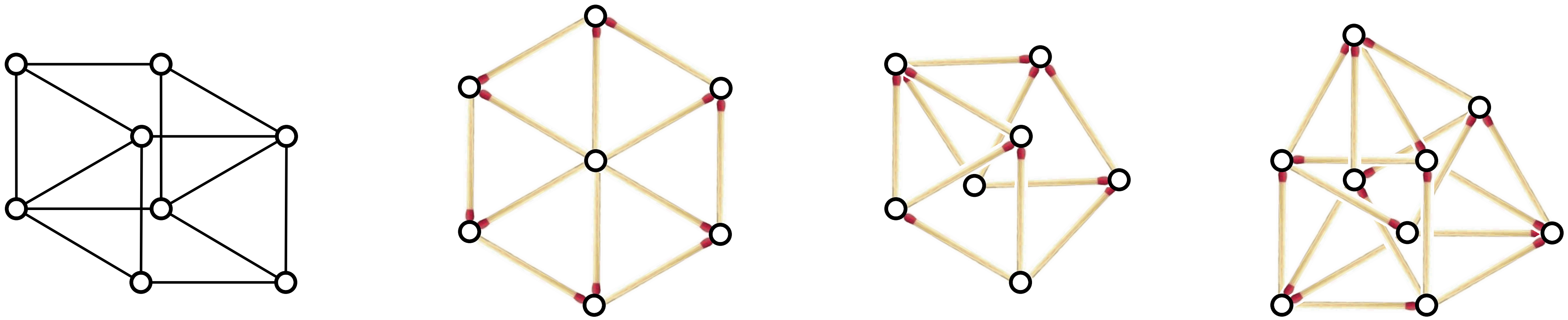

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- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]

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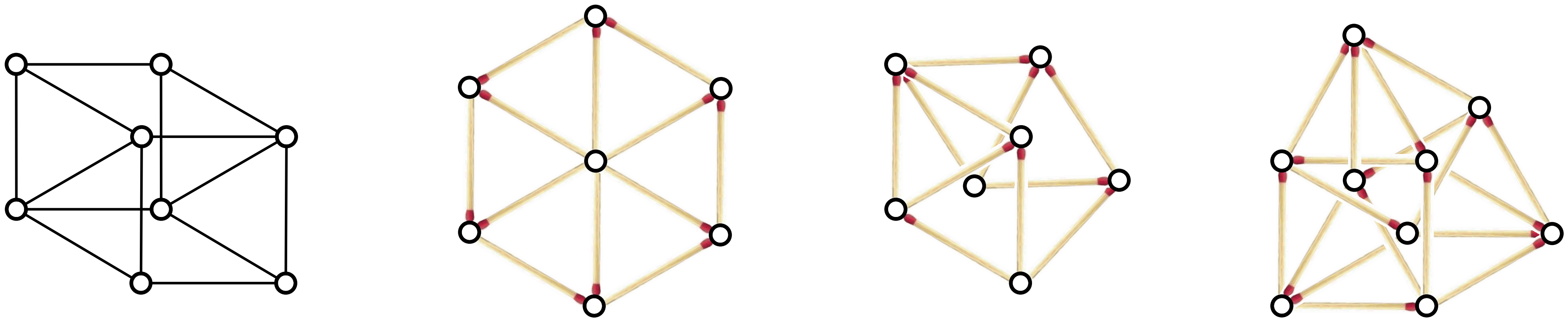

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths in $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

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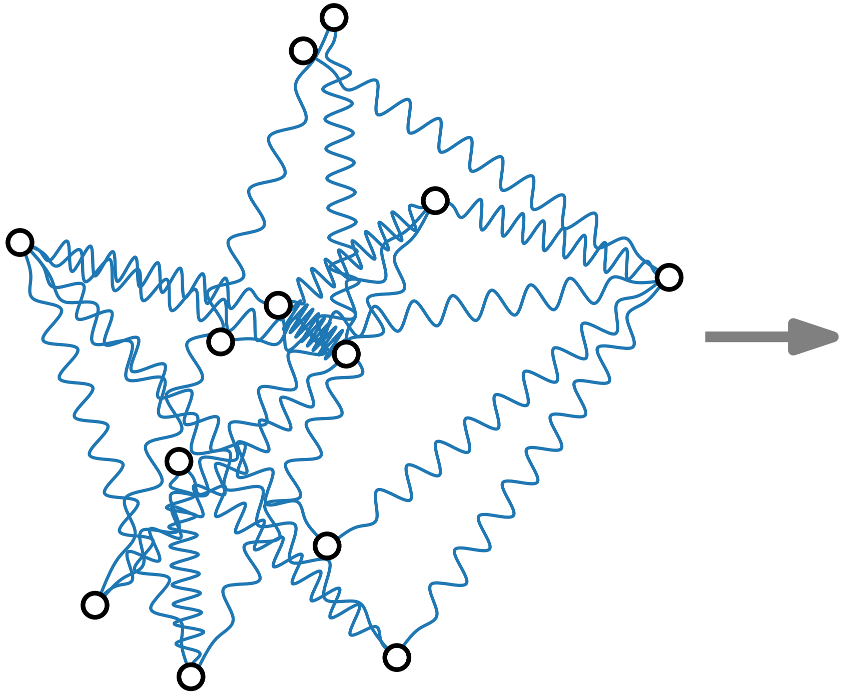

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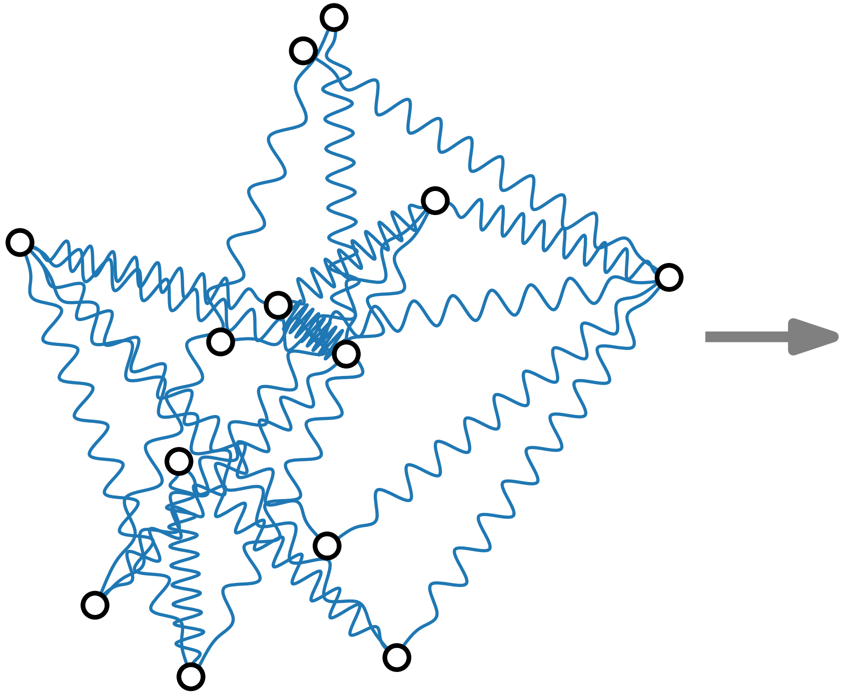

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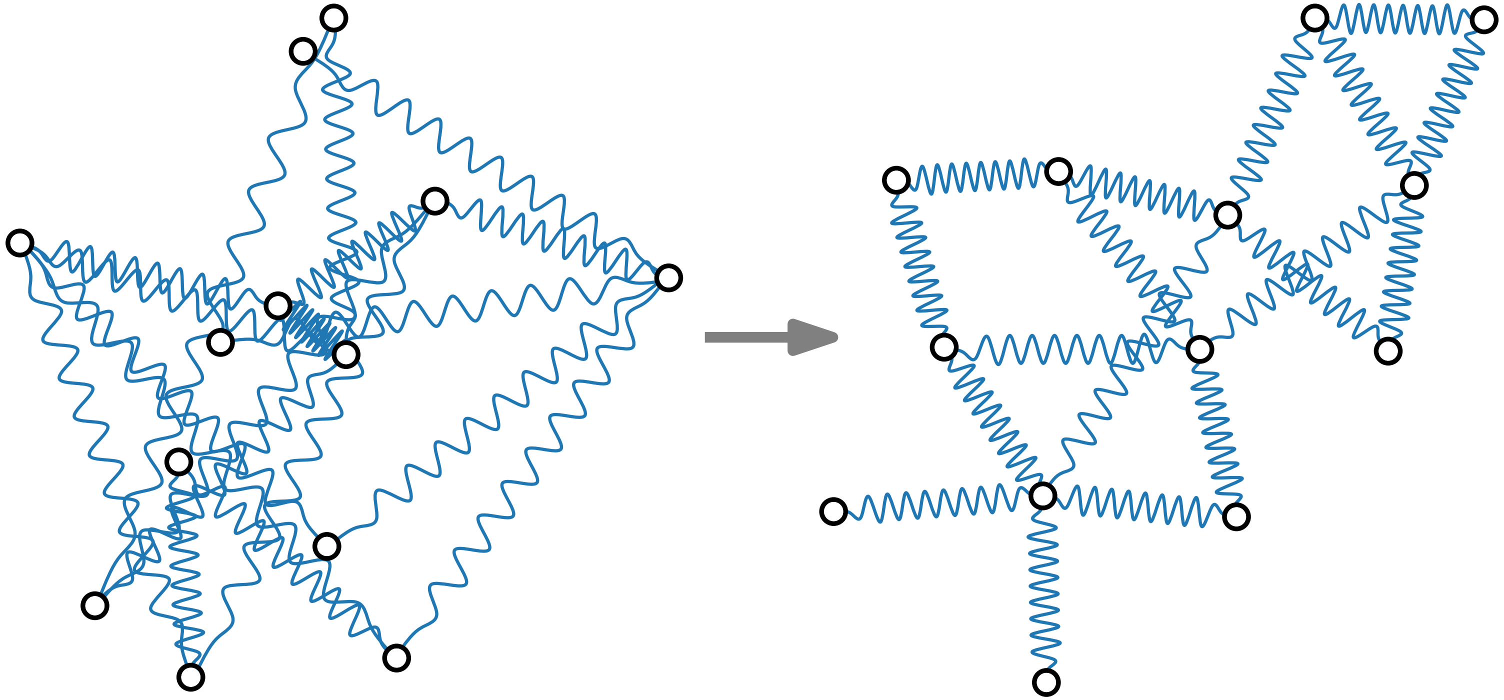

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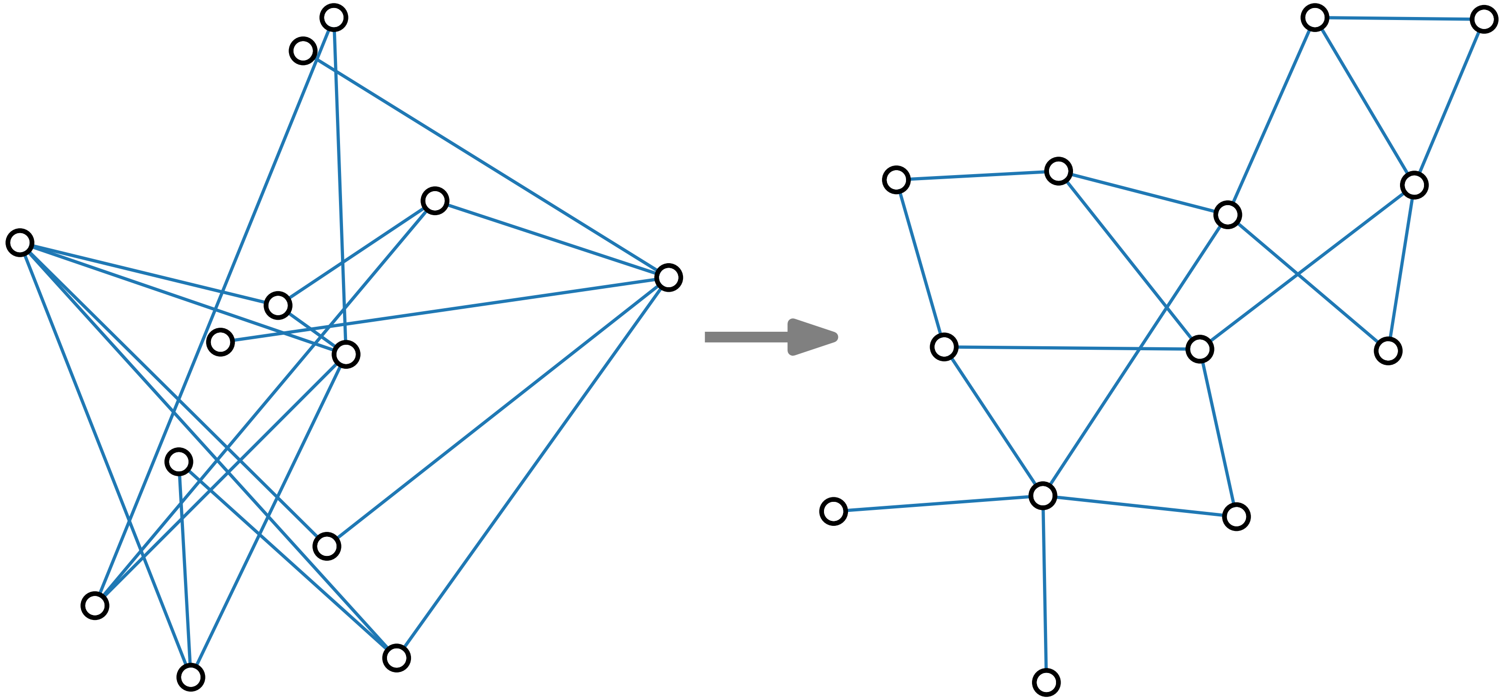

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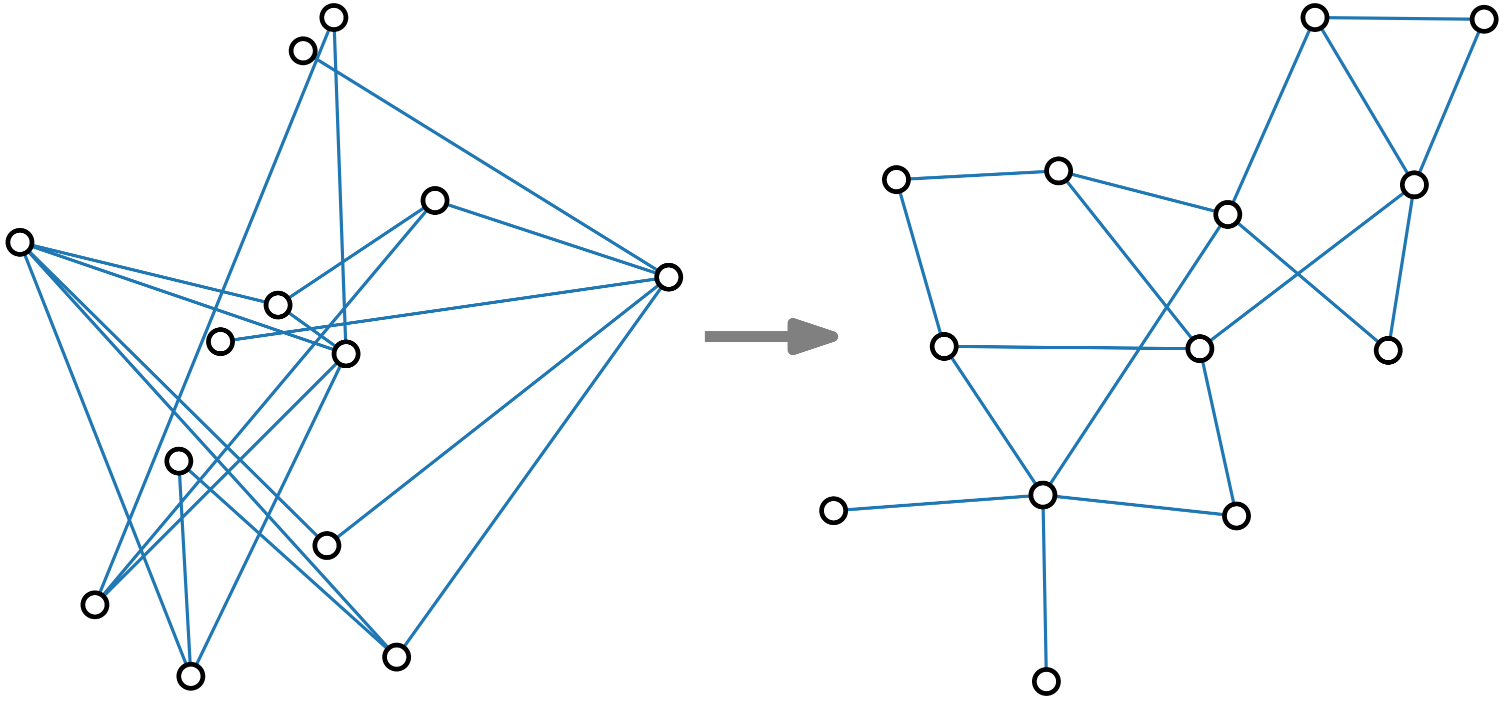
Attractive forces.



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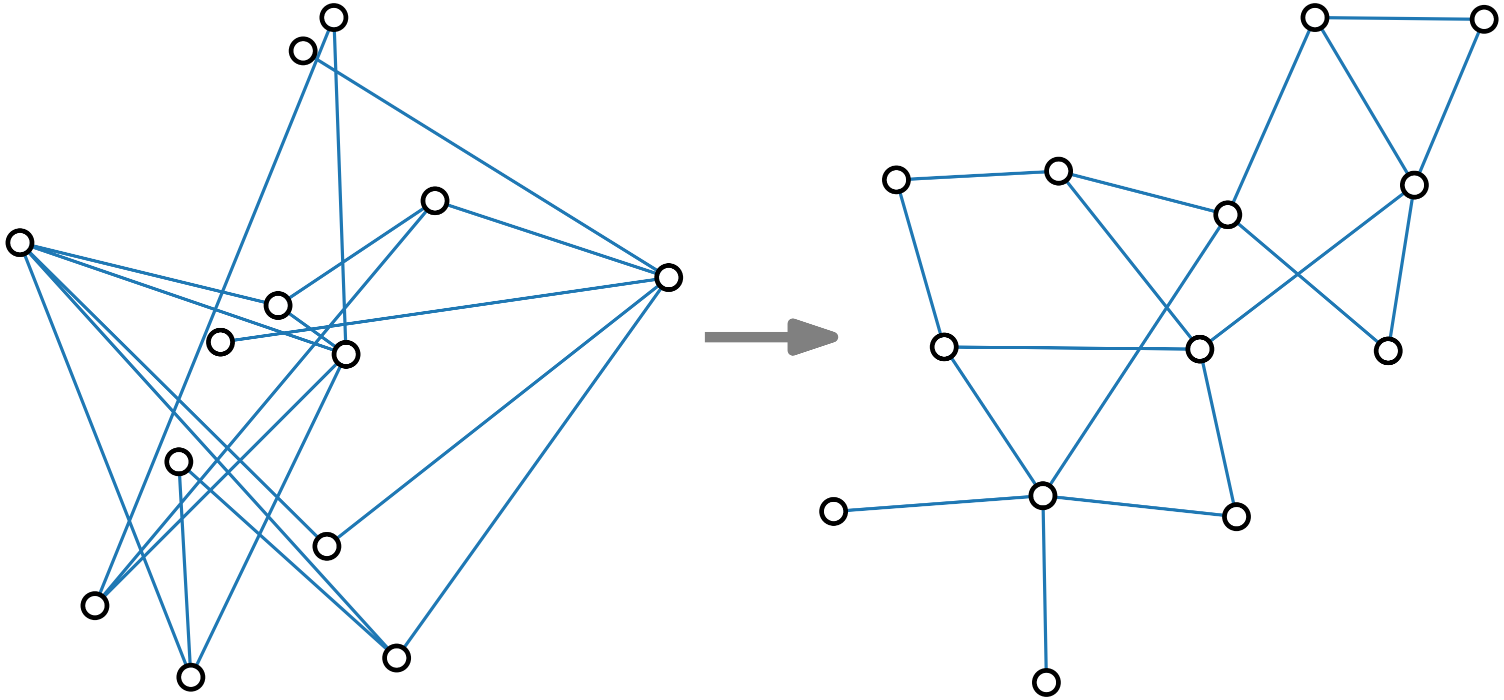

Attractive forces.

pairs $\{u, v\}$ of adjacent vertices:

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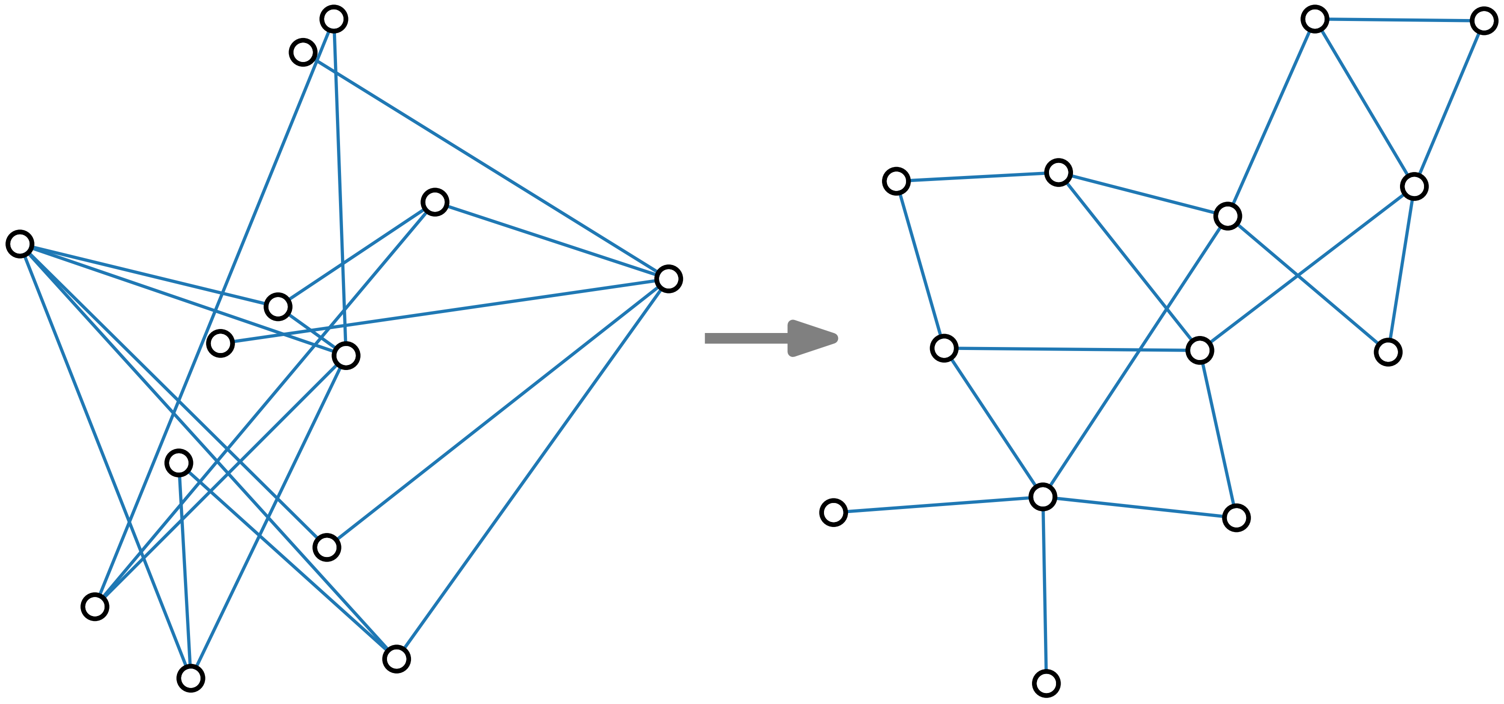

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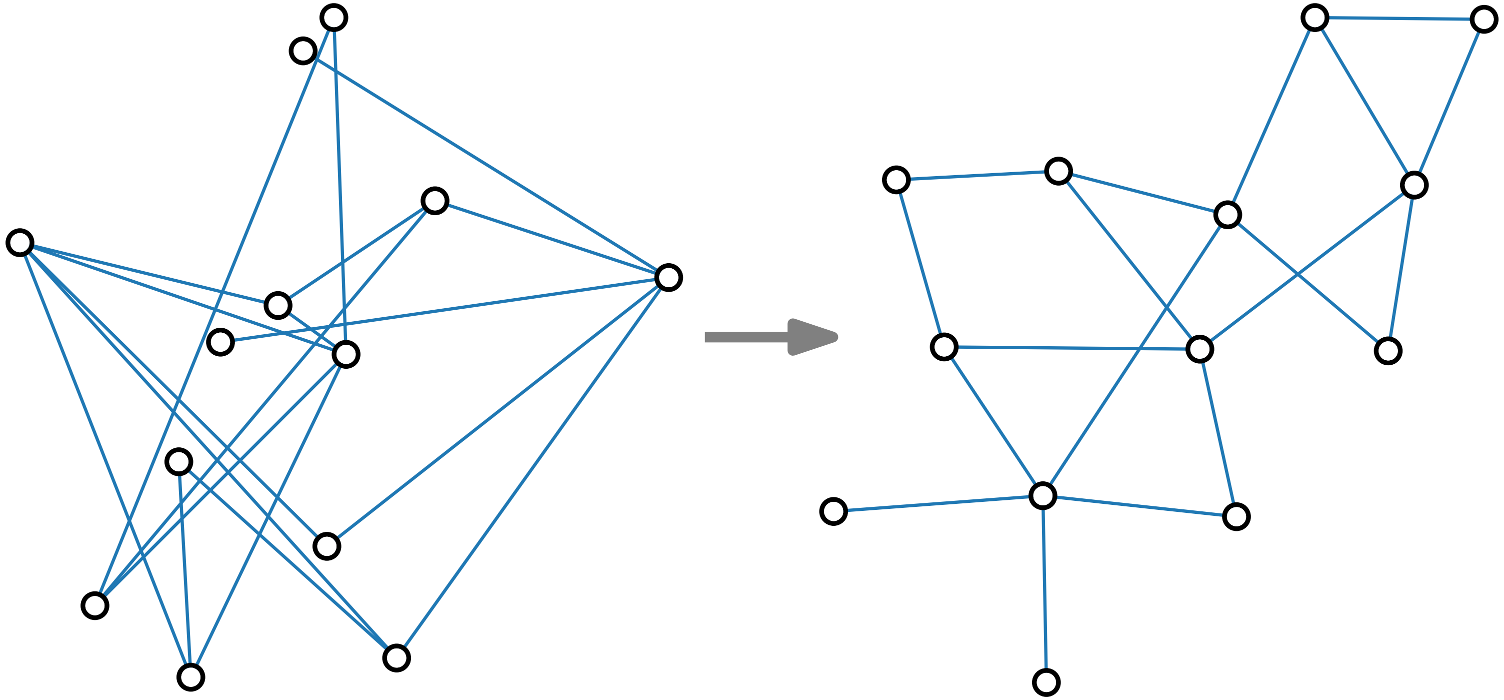

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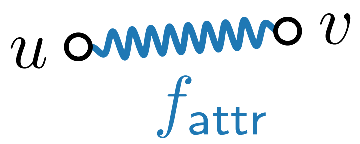

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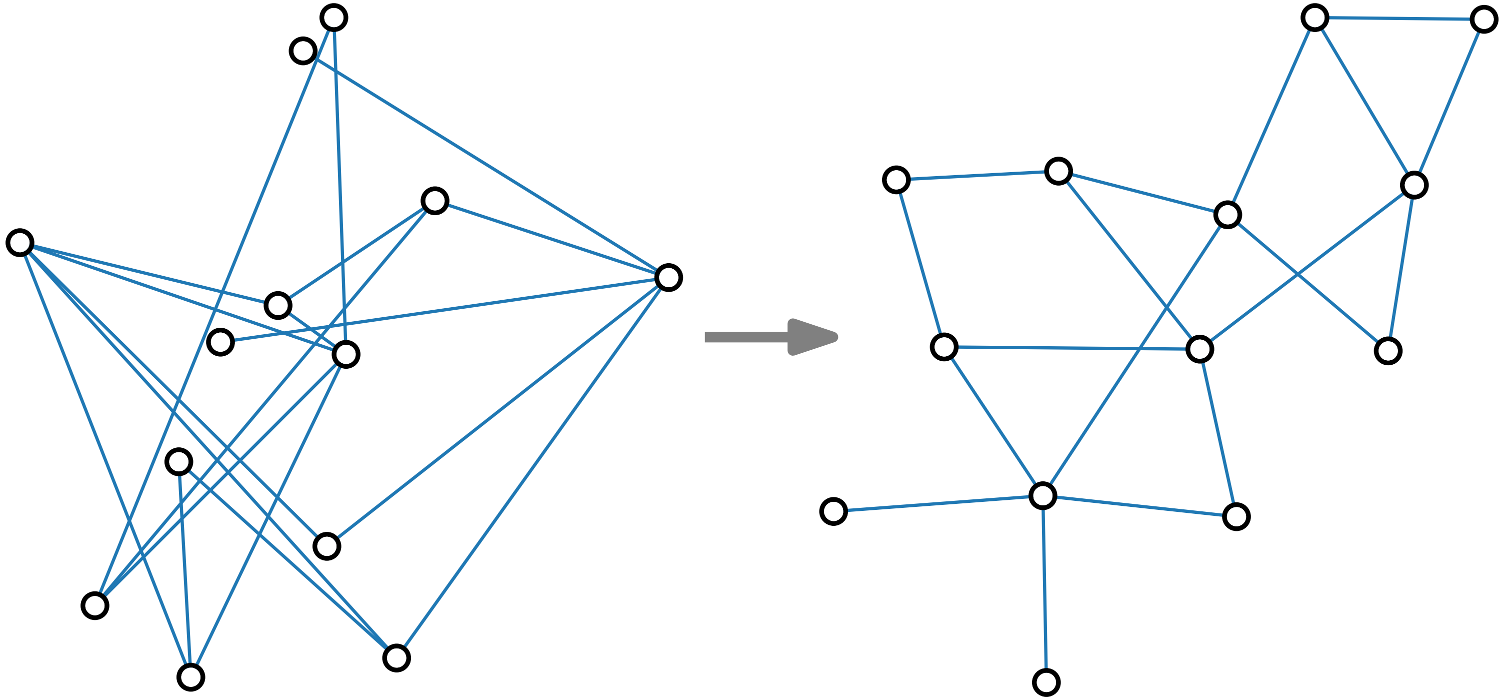

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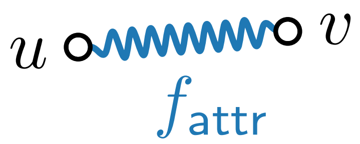

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any pair $\{x, y\}$ of vertices:

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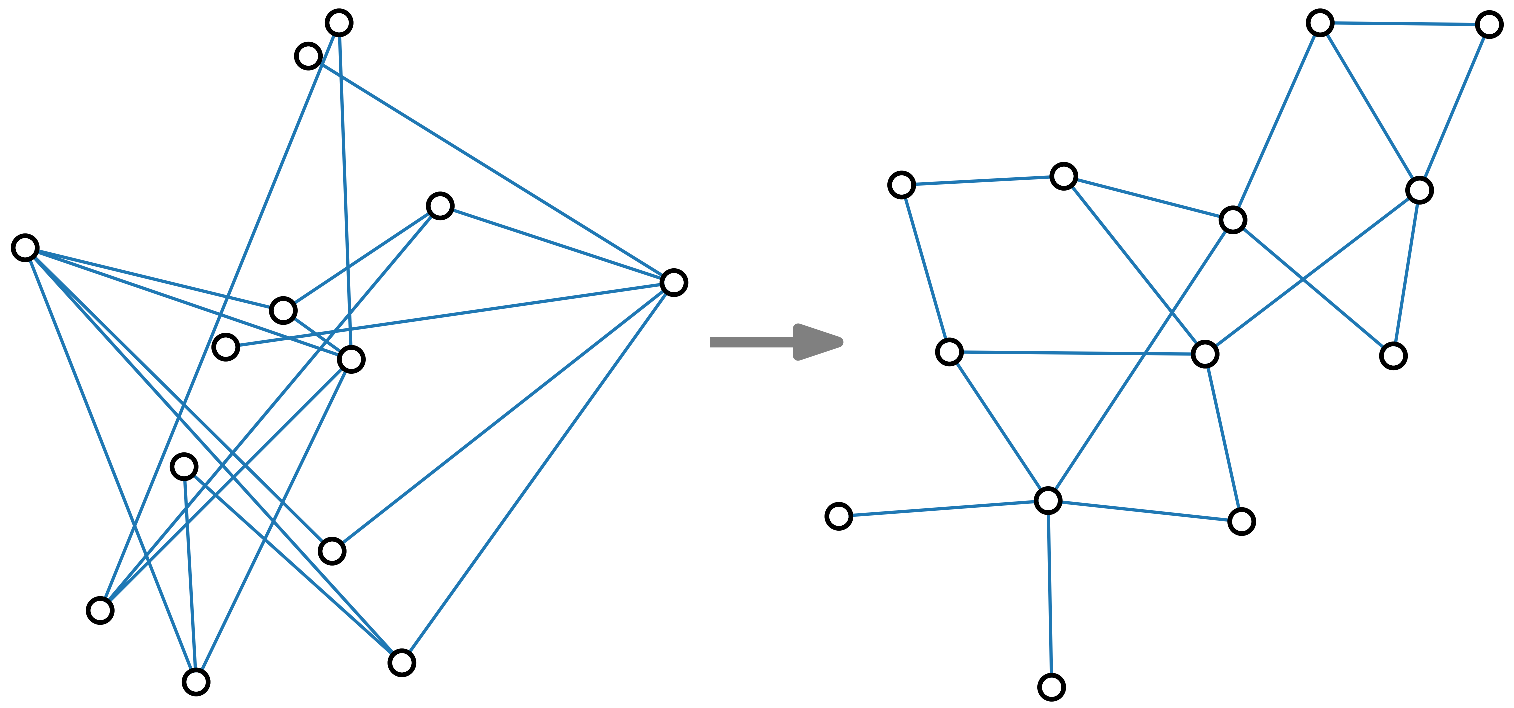

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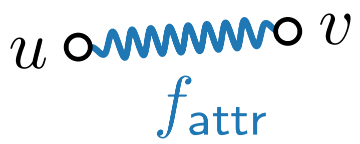

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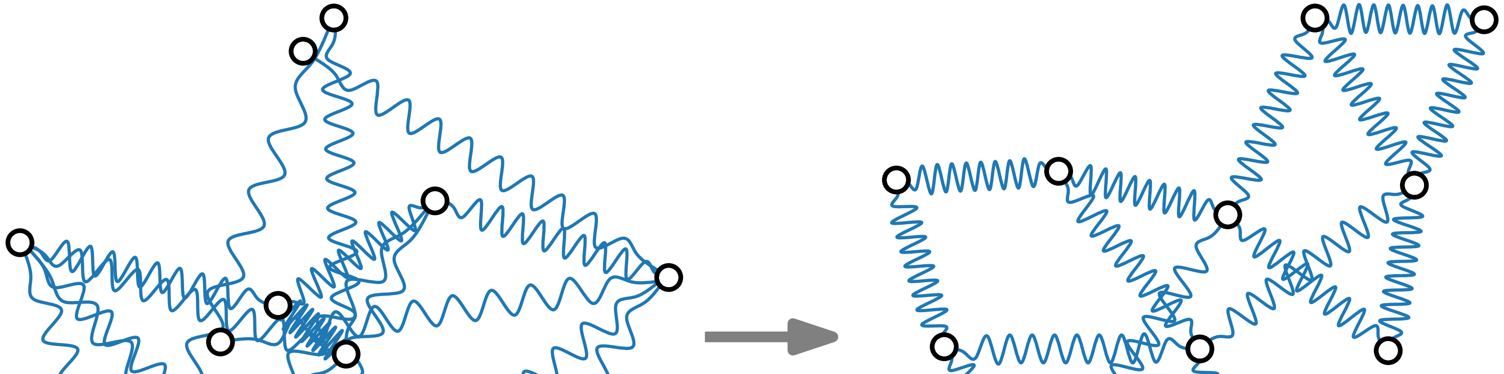

So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

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Force-Directed Algorithms

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

return p

Force-Directed Algorithms

initial layout; may be randomly chosen positions

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initial layout; may be randomly chosen positions

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

threshold

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initial layout; may be randomly chosen positions

max # iterations

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threshold (assume $F_v(0) = \infty$)

$t \leftarrow 1$
while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
    $t \leftarrow t + 1$
return p

end layout

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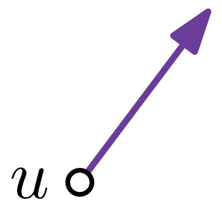
```
t ← 1
while t ≤ K and max_{v∈V(G)} ||F_v(t − 1)|| > ε do
    foreach u ∈ V(G) do
        F_u(t) ←
    t ← t + 1
return p
```

end layout



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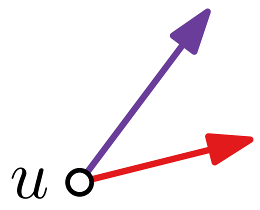

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vertices adjacent to u

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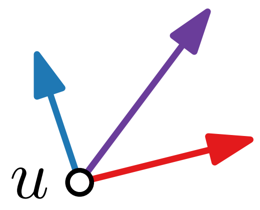

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    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
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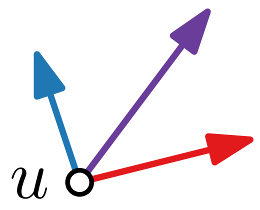

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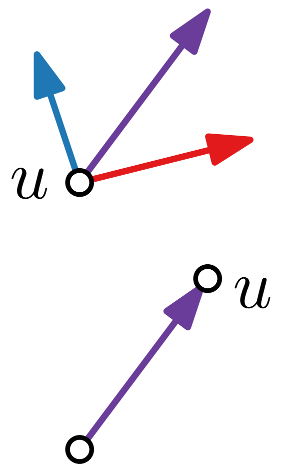

Force-Directed Algorithms

initial layout; may be randomly chosen positions

max # iterations

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threshold (assume $F_v(0) = \infty$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

vertices adjacent to u

  foreach $u \in V(G)$ **do**

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cooling factor

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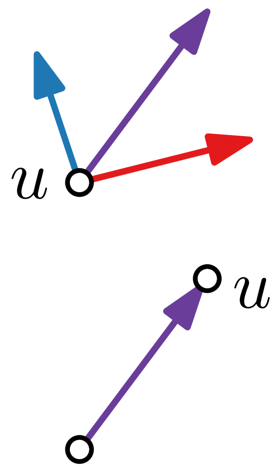

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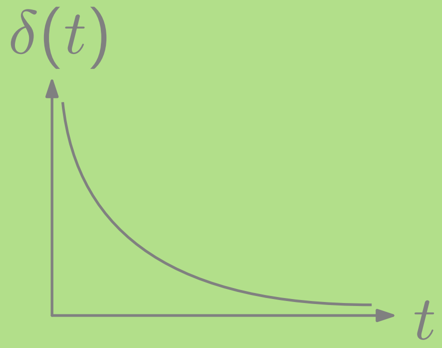

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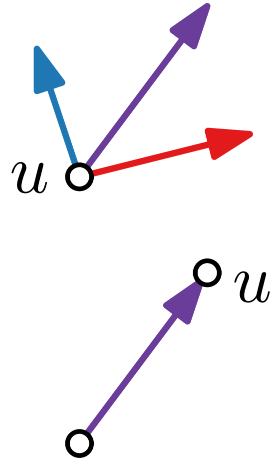

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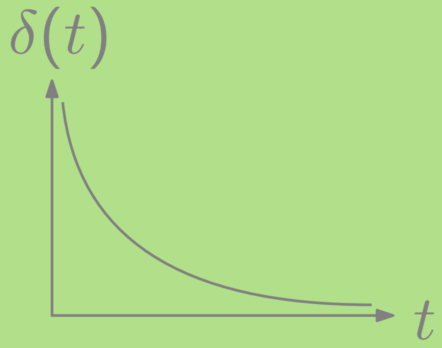

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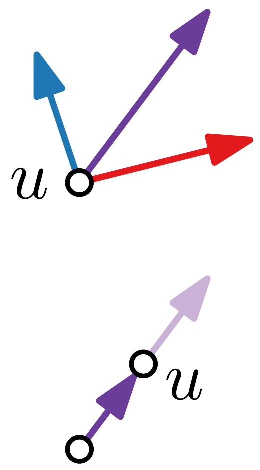

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- **Repulsive forces**

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- **Resulting displacement vector**

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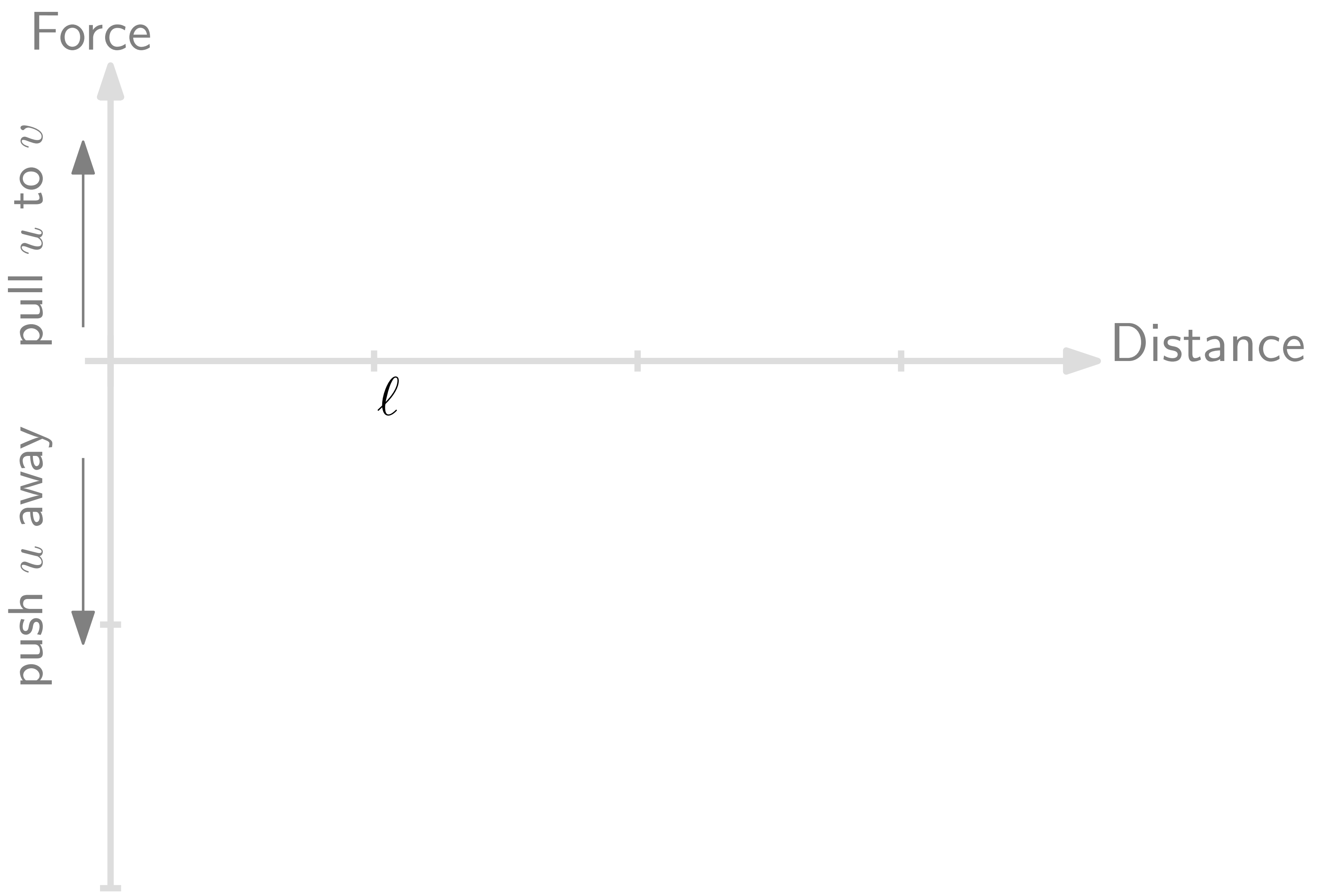

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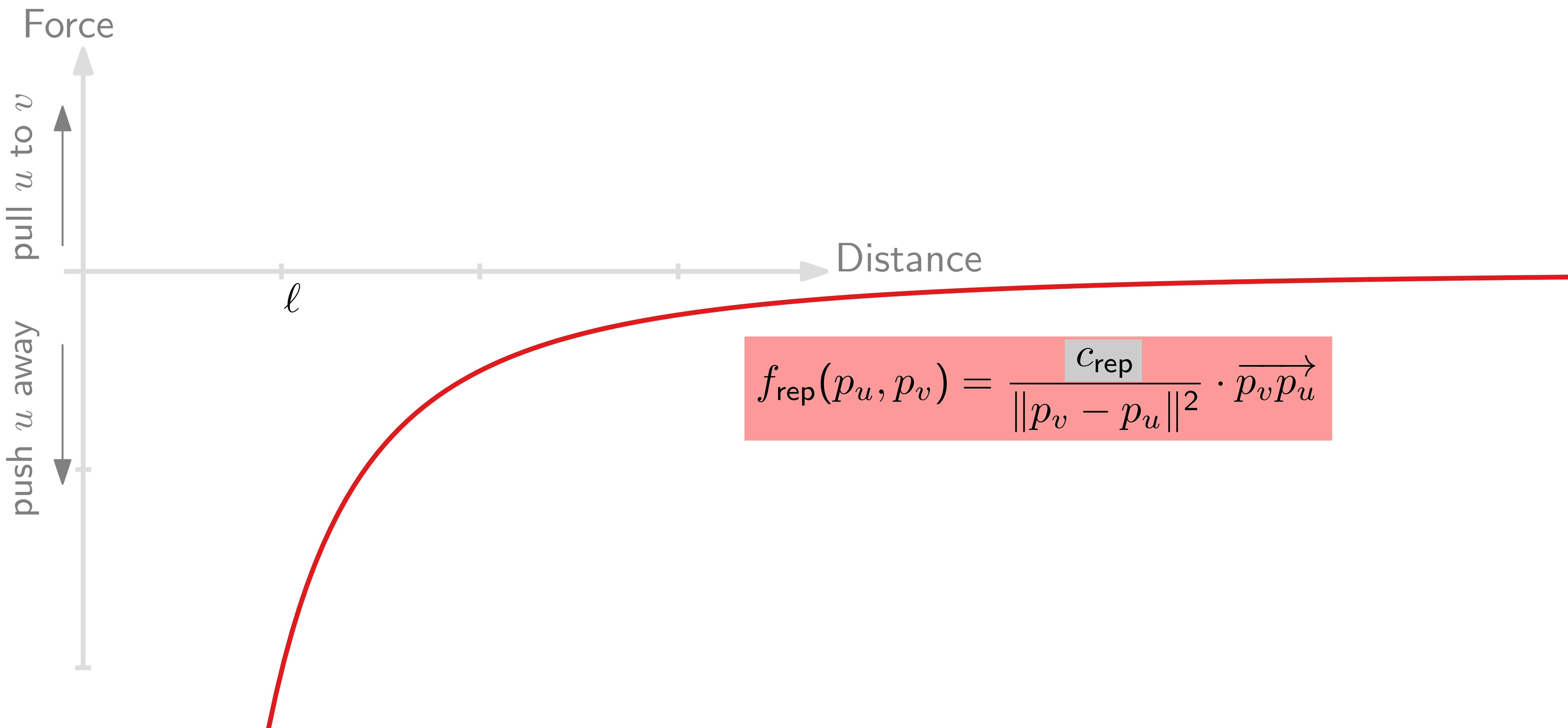

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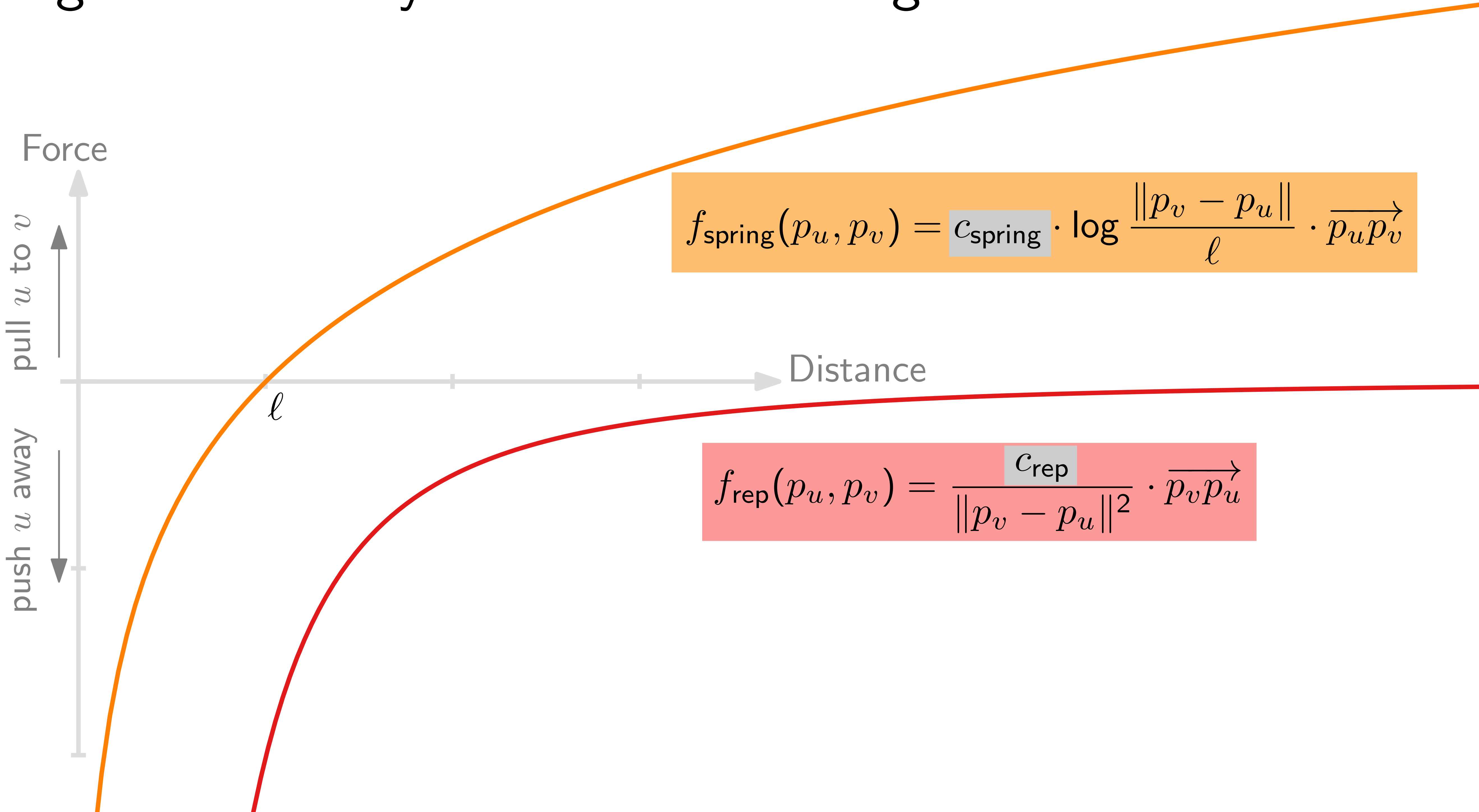

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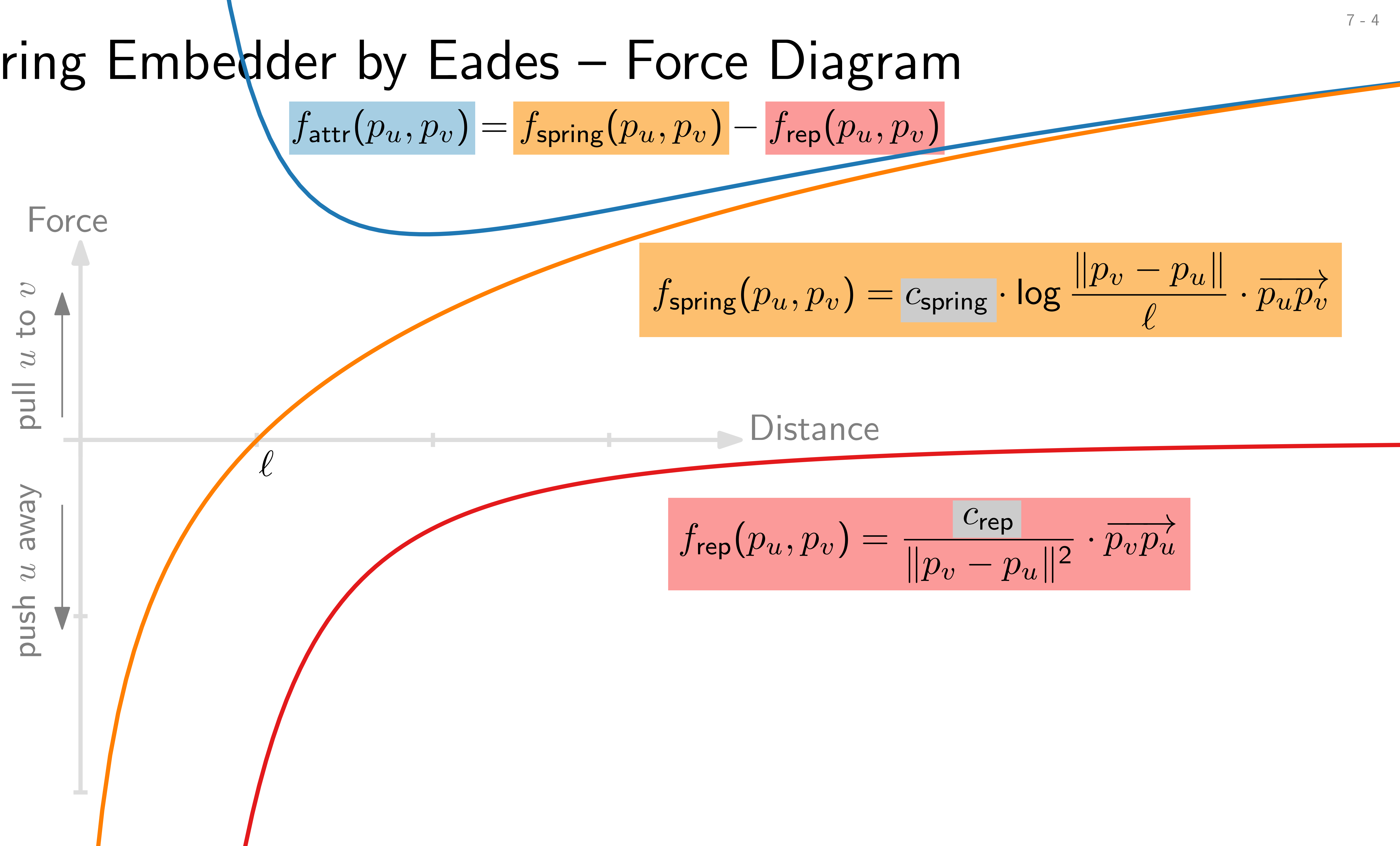

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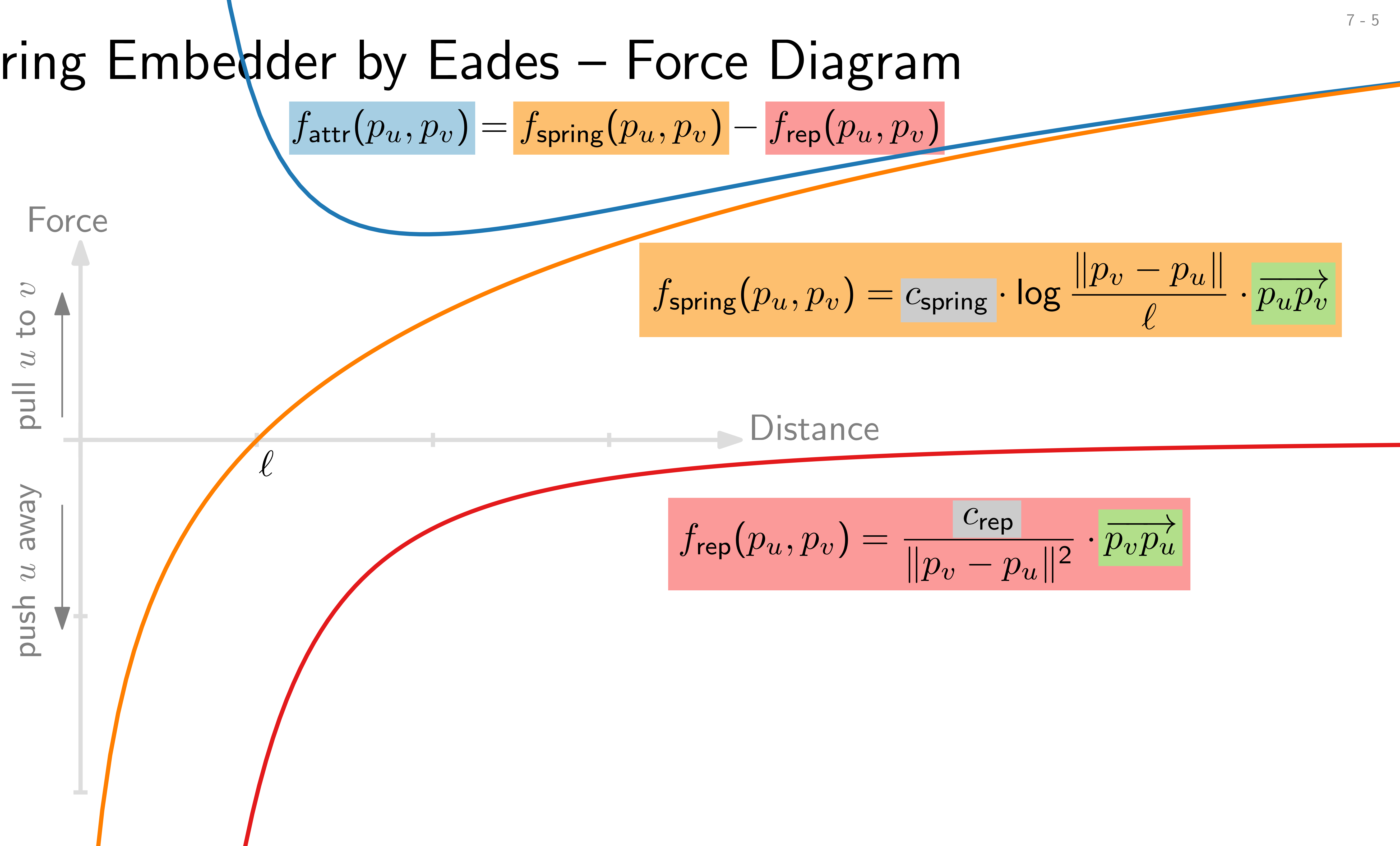

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Variant by Fruchterman & Reingold

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repulsion constant (e.g., 2.0)

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  $t \leftarrow 1$
  while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
    foreach $u \in V(G)$ **do**
      $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Variant by Fruchterman & Reingold

```
ForceDirected(graph G, p = (p_v)_{v∈V}, ε > 0, K ∈ ℕ)
  t ← 1
  while t ≤ K and max_{v∈V(G)} ‖F_v(t − 1)‖ > ε do
      foreach u ∈ V(G) do
          F_u(t) ← Σ_{v∈V(G)} f_rep(p_u, p_v) + Σ_{v∈Adj[u]} f_attr(p_u, p_v)
      foreach u ∈ V(G) do
          p_u ← p_u + δ(t) · F_u(t)
      t ← t + 1
  return p
```

- **Repulsive forces**

$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

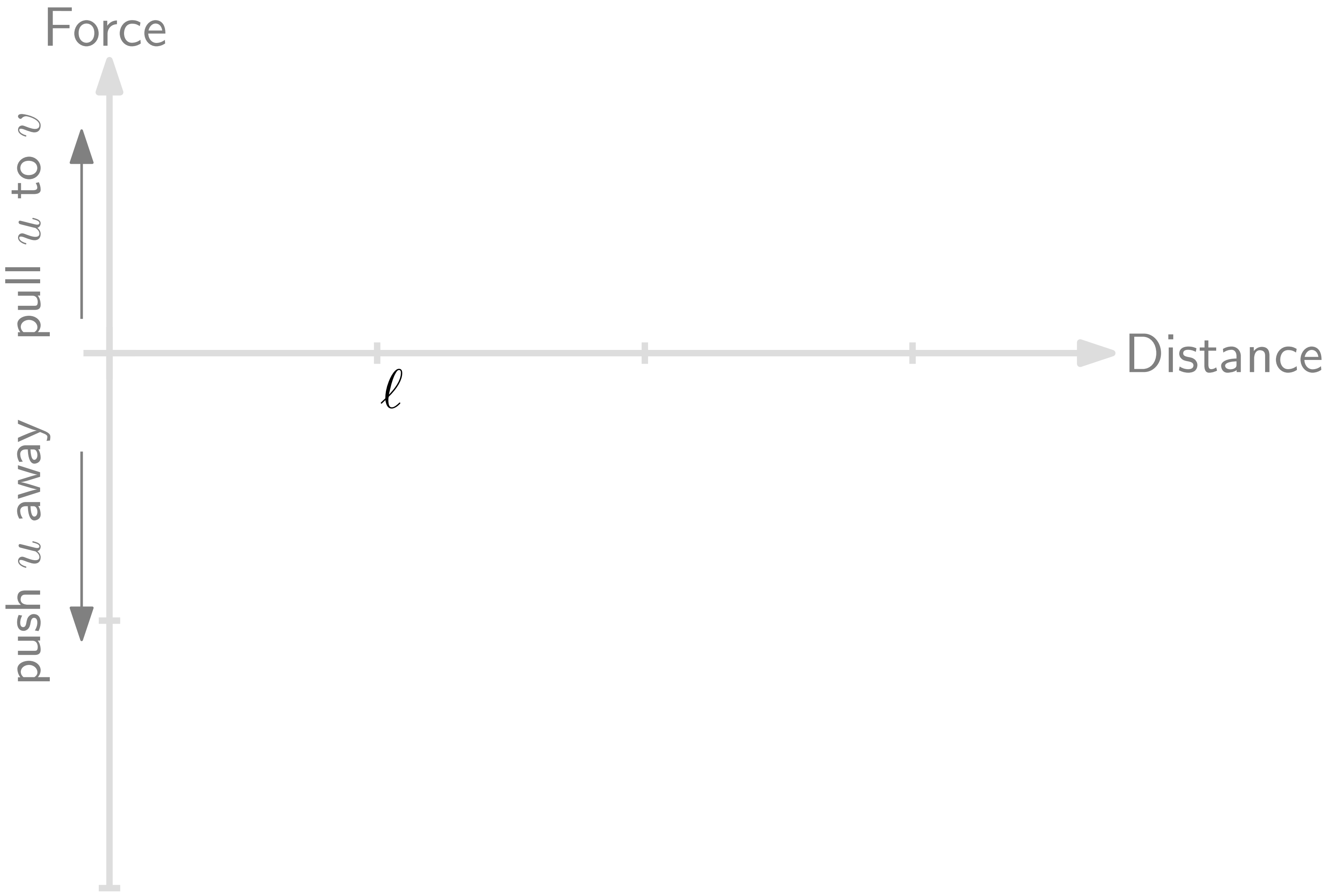
- **Resulting displacement vector**

$$F_u = \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$$

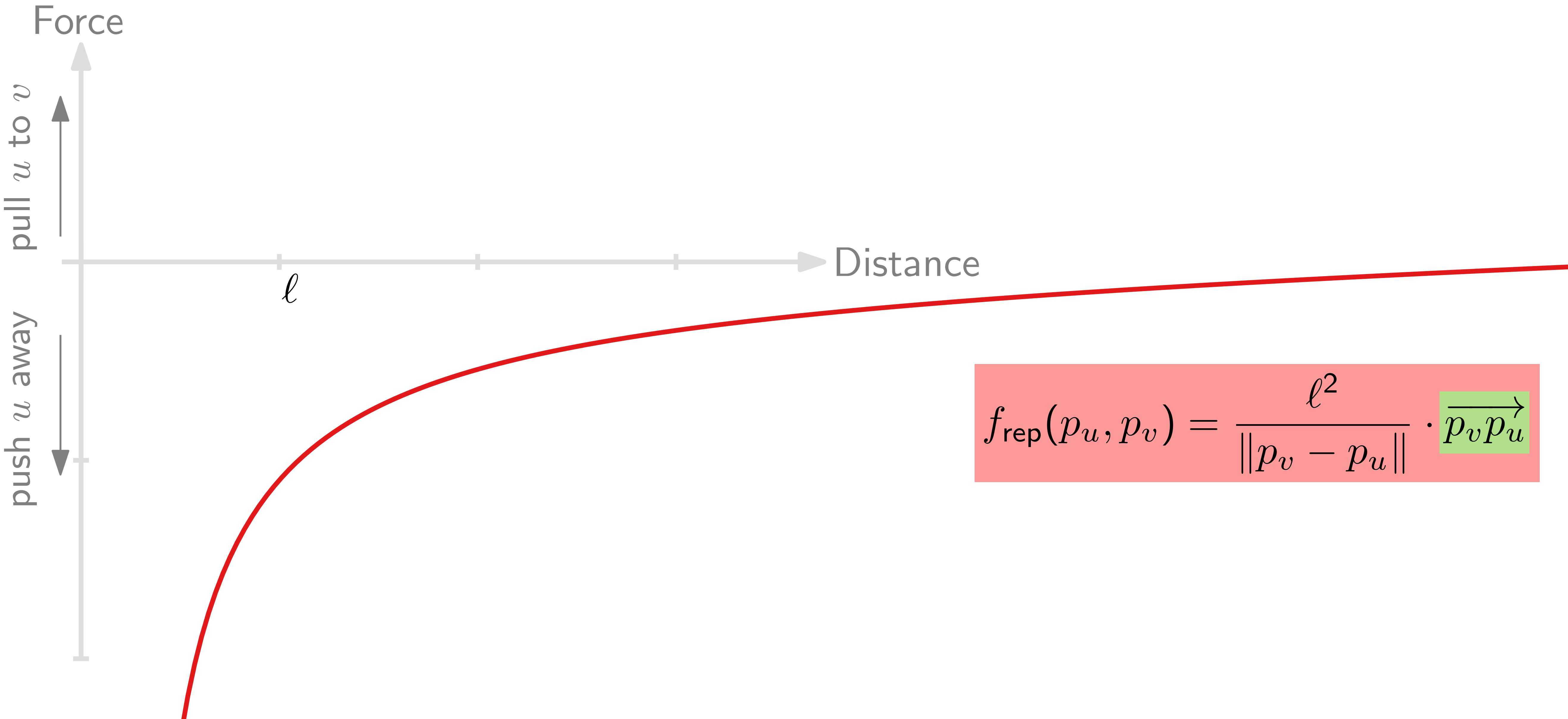
Notation.

- $\|p_u - p_v\|$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram

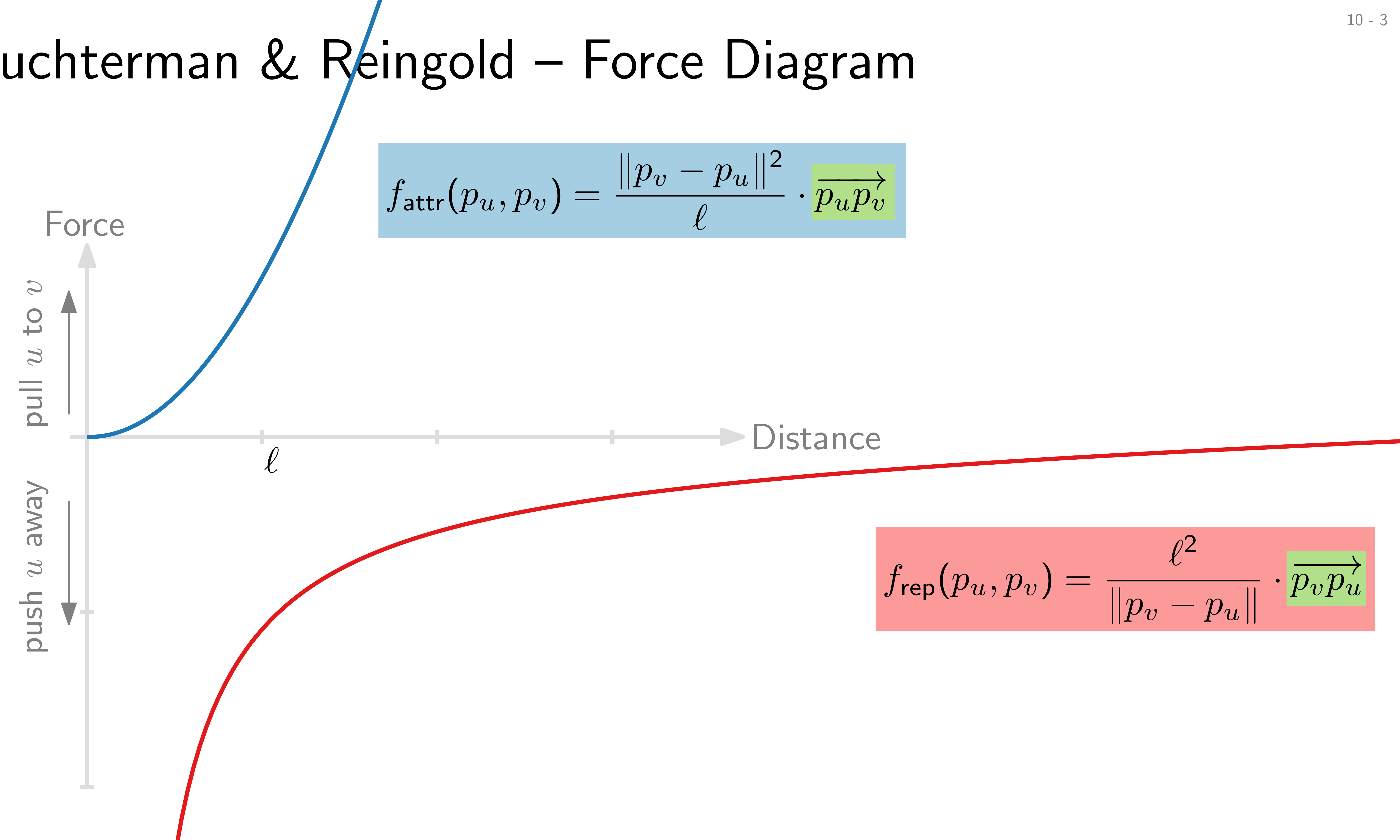


Fruchterman & Reingold – Force Diagram



$$f_{\mathsf{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram



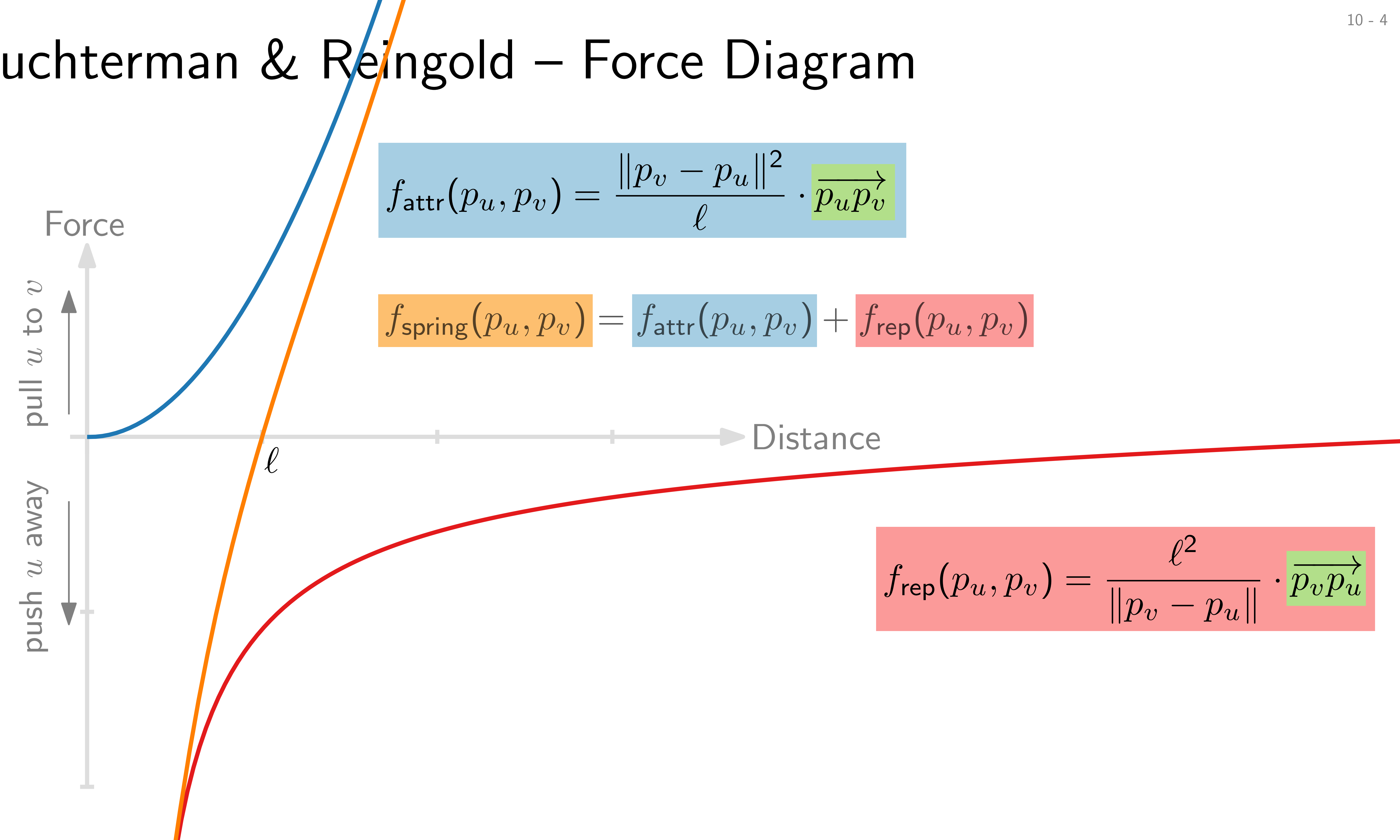
$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_v - p_u\|^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

$$f_{\text{spring}}(p_u, p_v) = f_{\text{attr}}(p_u, p_v) + f_{\text{rep}}(p_u, p_v)$$



$$f_{\text{rep}}(p_u, p_v) = \frac{\ell^2}{\|p_v - p_u\|} \cdot \overrightarrow{p_v p_u}$$

Adaptability

degree of vertex u, i.e., $|\text{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
- Set $f_{\text{attr}}^{\text{new}}(p_u, p_v) = f_{\text{attr}}^{\text{old}}(p_u, p_v)/\Phi(u)$

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \mathrm{deg}(u)/2$
- Set $f_{\mathrm{attr}}^{\mathrm{new}}(p_u, p_v) = f_{\mathrm{attr}}^{\mathrm{old}}(p_u, p_v)/\Phi(u)$

Gravitation.

- Define centroid $\sigma = 1/|V(G)| \cdot \sum_{v \in V(G)} p_v$
- Add force $f_{\mathrm{grav}}(v) = c_{\mathrm{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma}$

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \mathrm{deg}(u)/2$
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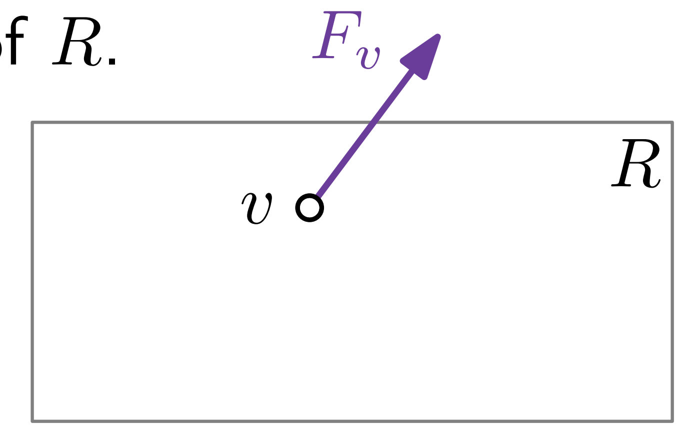
- Add force $f_{\mathrm{grav}}(v) = c_{\mathrm{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v \sigma}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \deg(u)/2$
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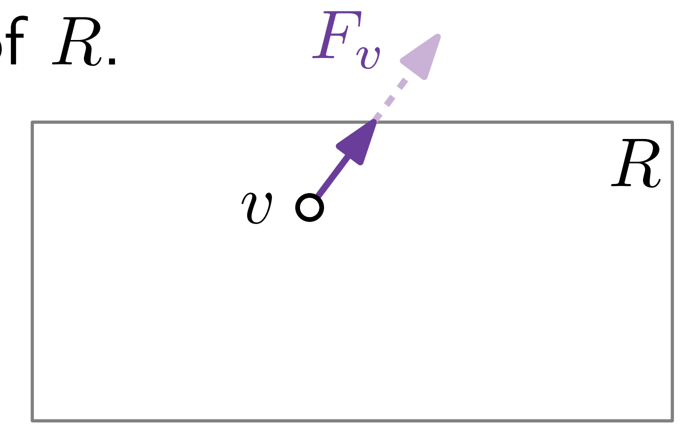

Adaptability

degree of vertex u, i.e., $|\text{Adj}[u]|$

Inertia. ("Trägheit")

- Define vertex mass $\Phi(u) = 1 + \text{deg}(u)/2$
- Set $f_{\text{attr}}^{\text{new}}(p_u, p_v) = f_{\text{attr}}^{\text{old}}(p_u, p_v)/\Phi(u)$

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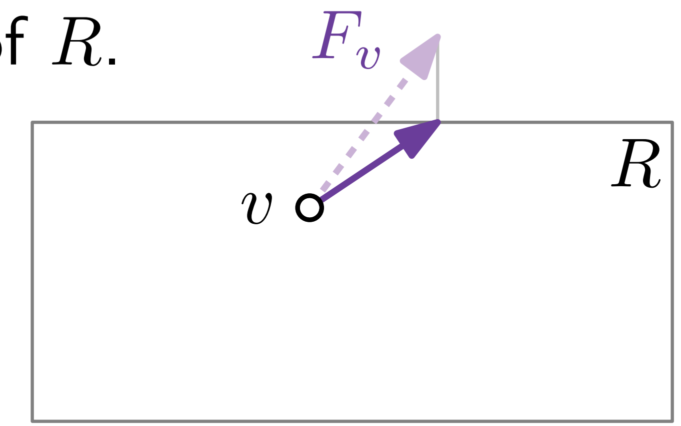

Adaptability

degree of vertex u, i.e., $|\mathrm{Adj}[u]|$

Inertia. (“Trägheit”)

- Define vertex mass $\Phi(u) = 1 + \mathrm{deg}(u)/2$
- Set $f_{\mathrm{attr}}^{\mathrm{new}}(p_u, p_v) = f_{\mathrm{attr}}^{\mathrm{old}}(p_u, p_v)/\Phi(u)$

Gravitation.

- Define centroid $\sigma = 1/|V(G)| \cdot \sum_{v \in V(G)} p_v$
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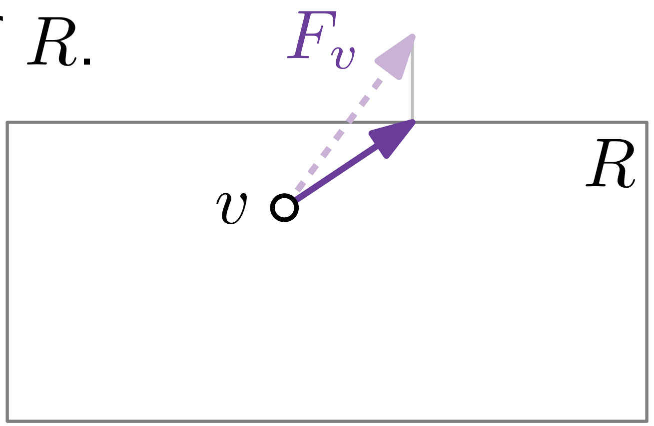
If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speed-ups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected(graph G, $p = (p_v)_{v \in V(G)}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

  foreach $u \in V(G)$ **do**

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  foreach $u \in V(G)$ **do**

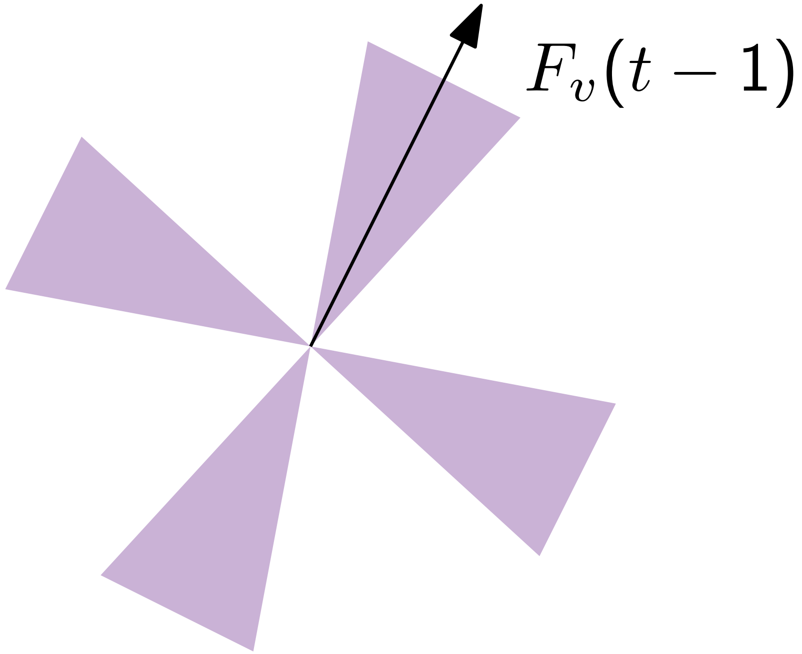
    $p_u \leftarrow p_u + \underset{\delta_v(t)}{\cancel{\delta(t)}} \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

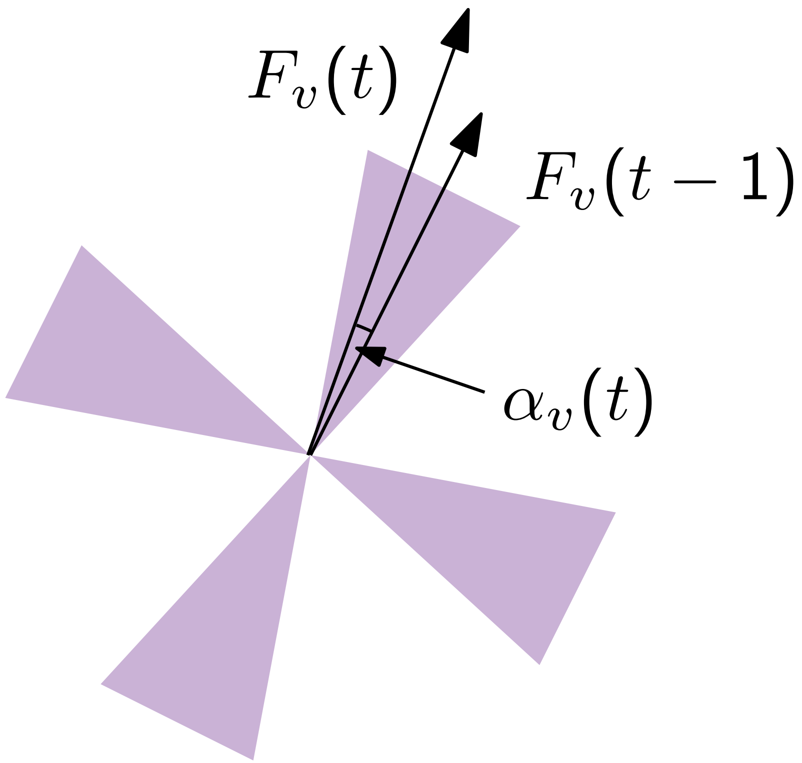
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



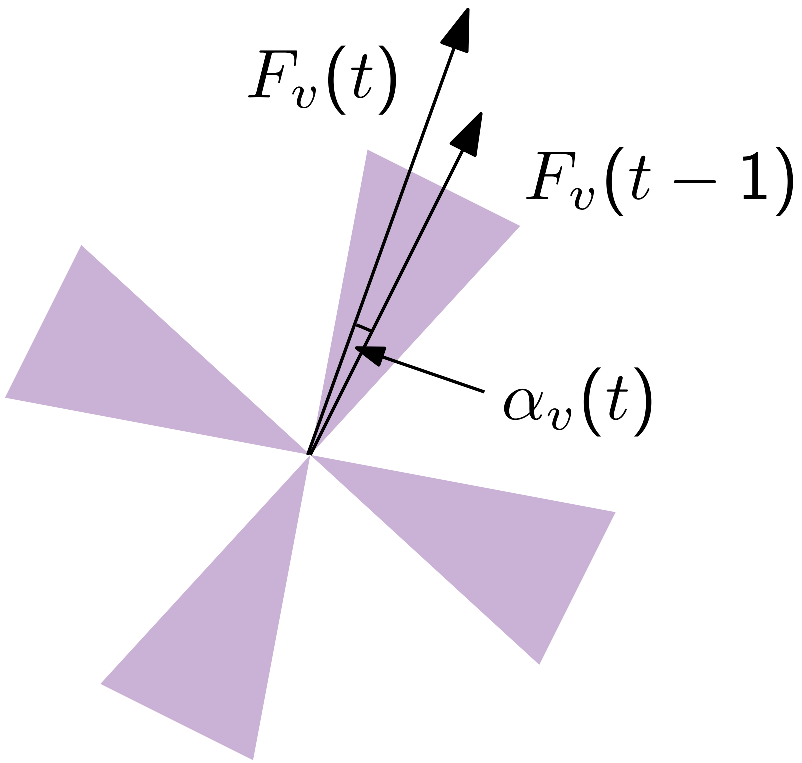
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

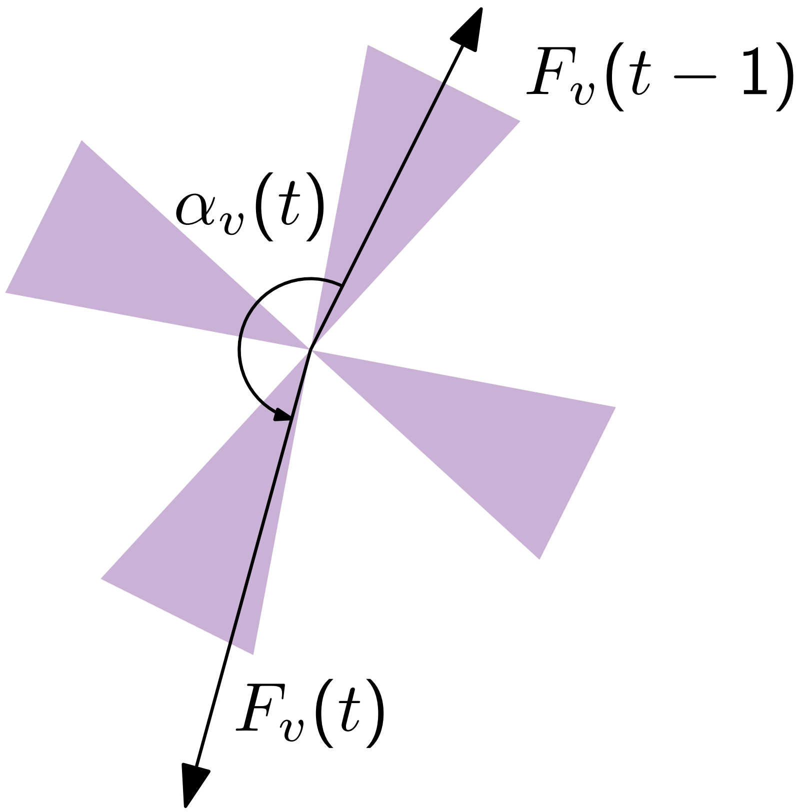


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

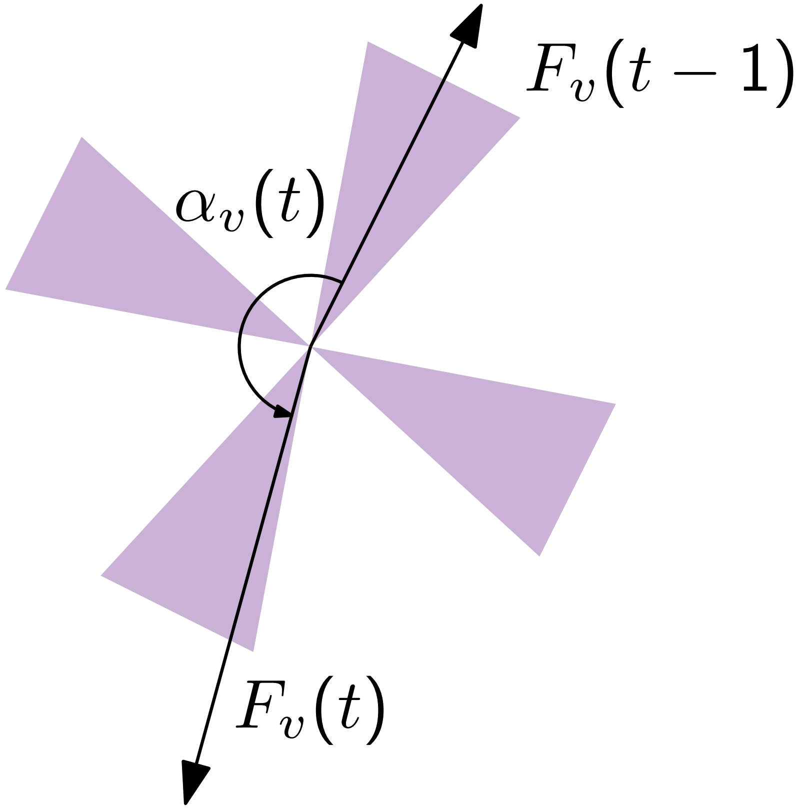
[Frick, Ludwig, Mehldau '95]



Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

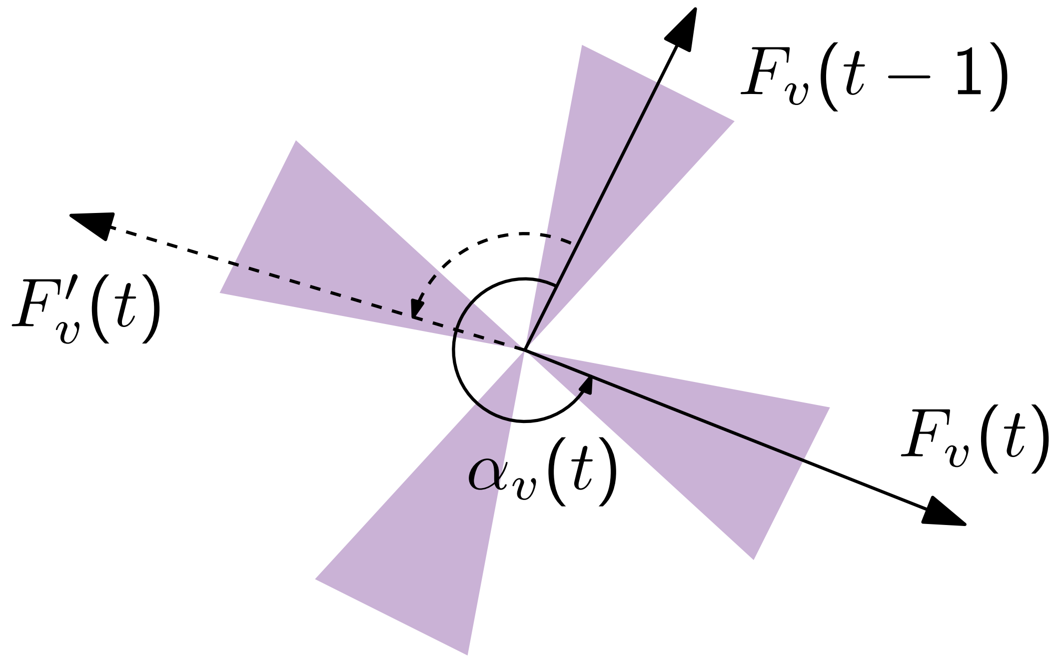


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

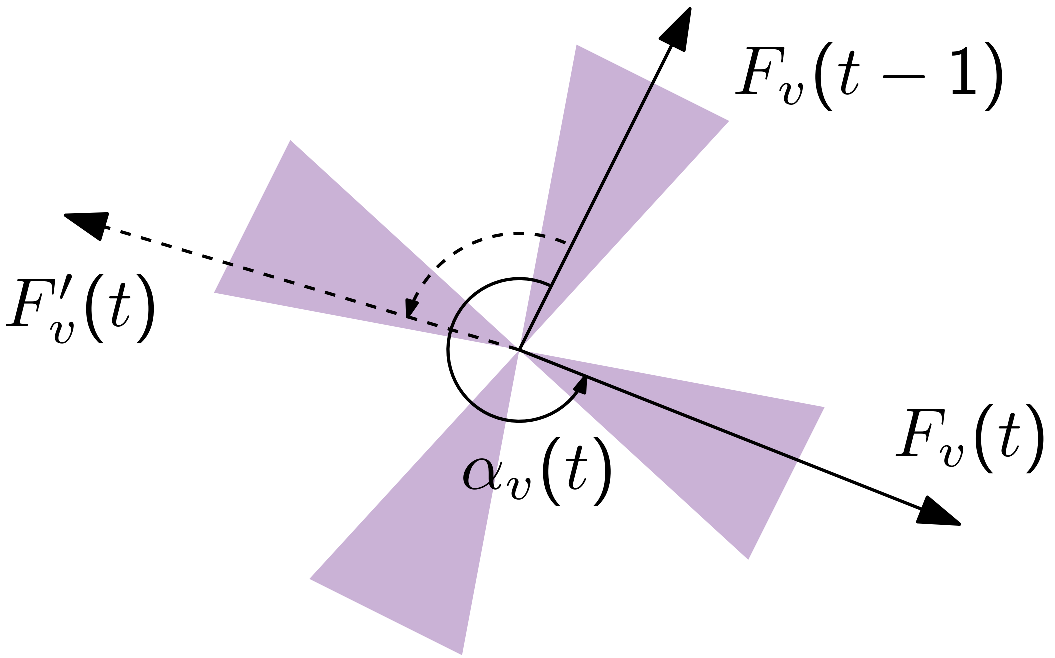


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Oscillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

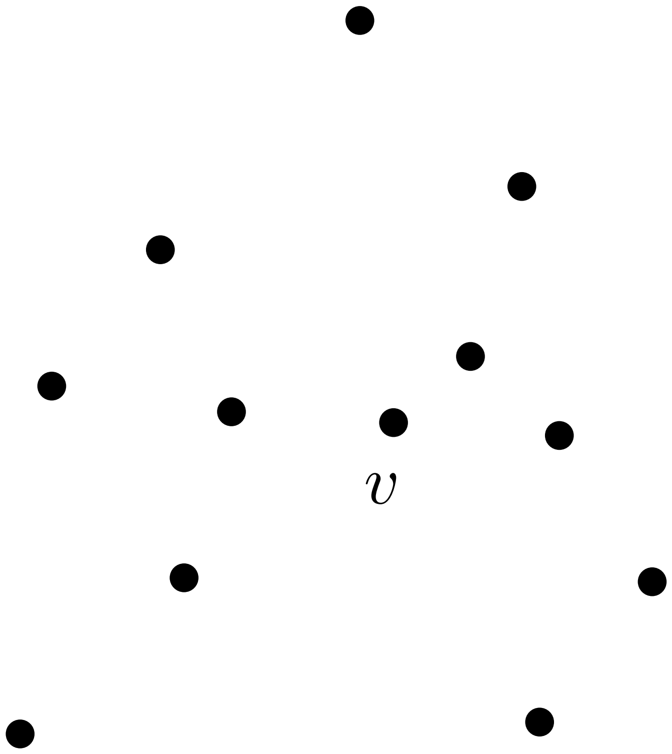
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

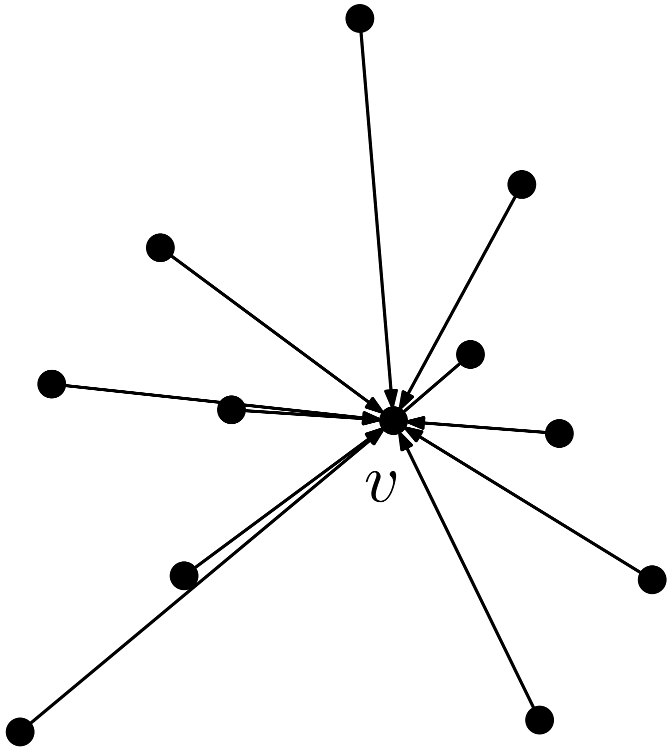
Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



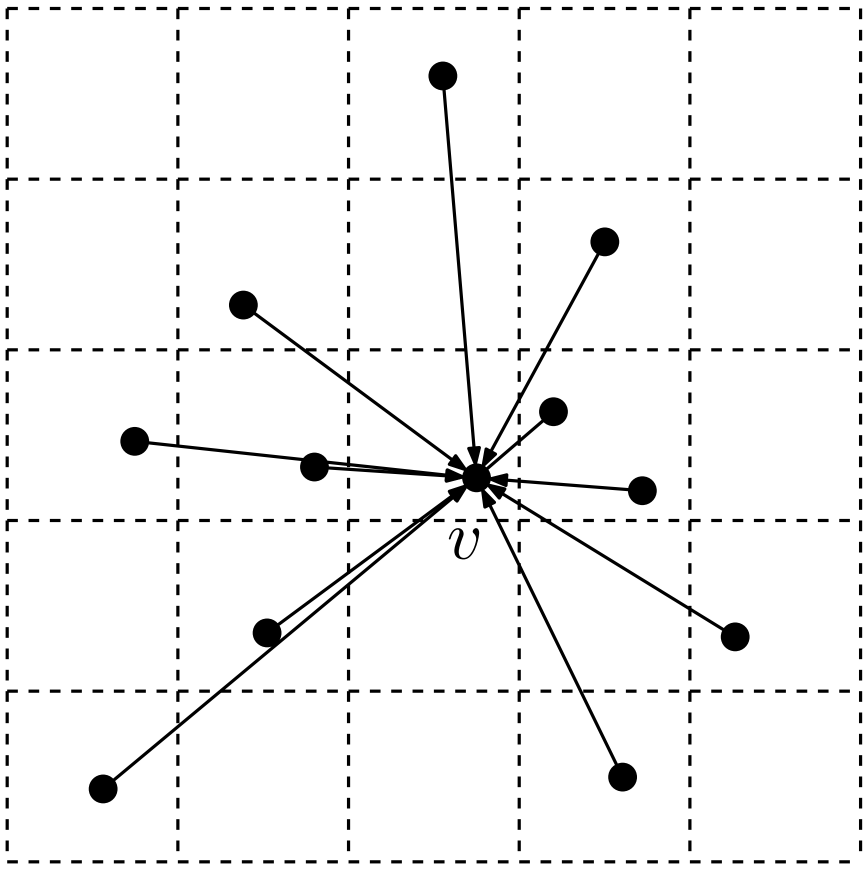
Speeding up “Convergence” via Grids

[Fruchterman & Reingold ’91]



Speeding up "Convergence" via Grids

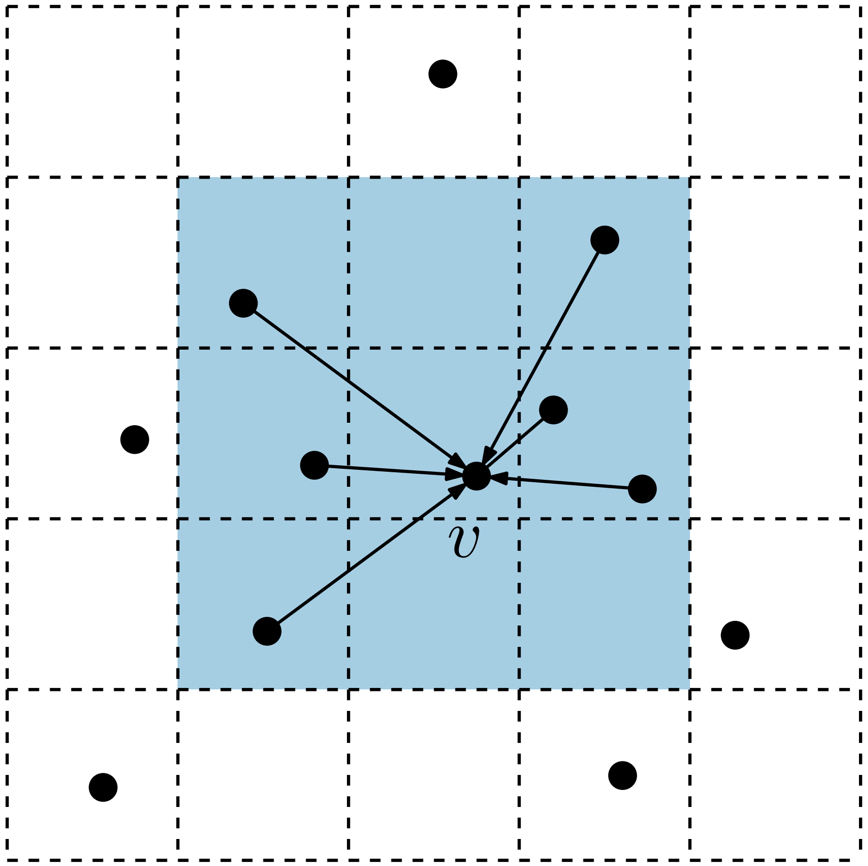
[Fruchterman & Reingold '91]



- divide plane into a grid

Speeding up "Convergence" via Grids

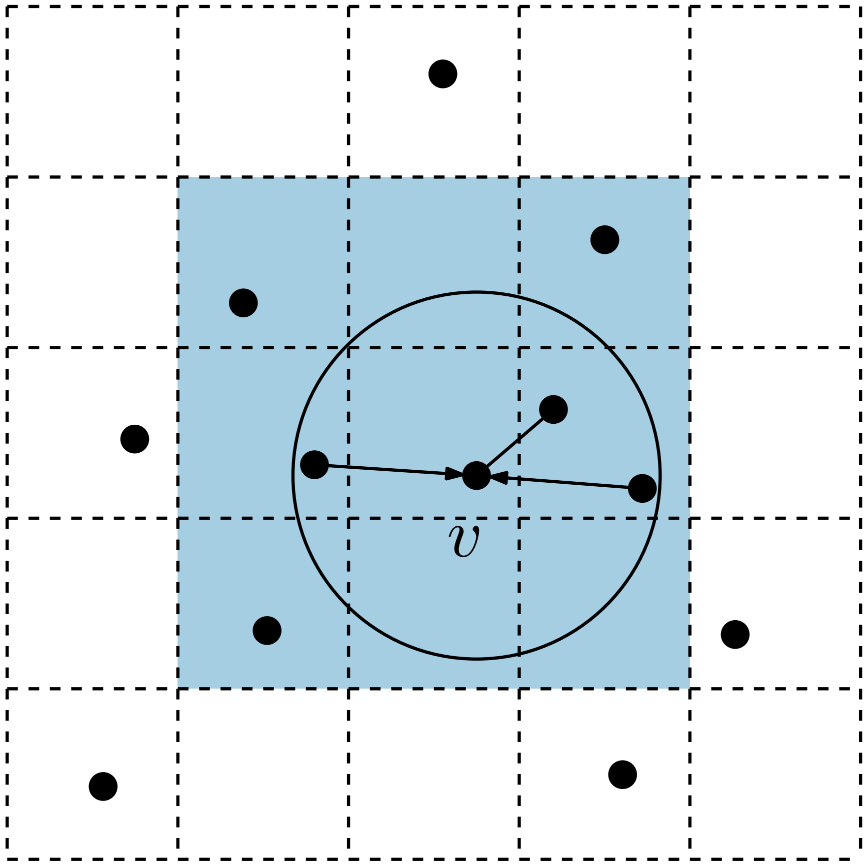
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells

Speeding up "Convergence" via Grids

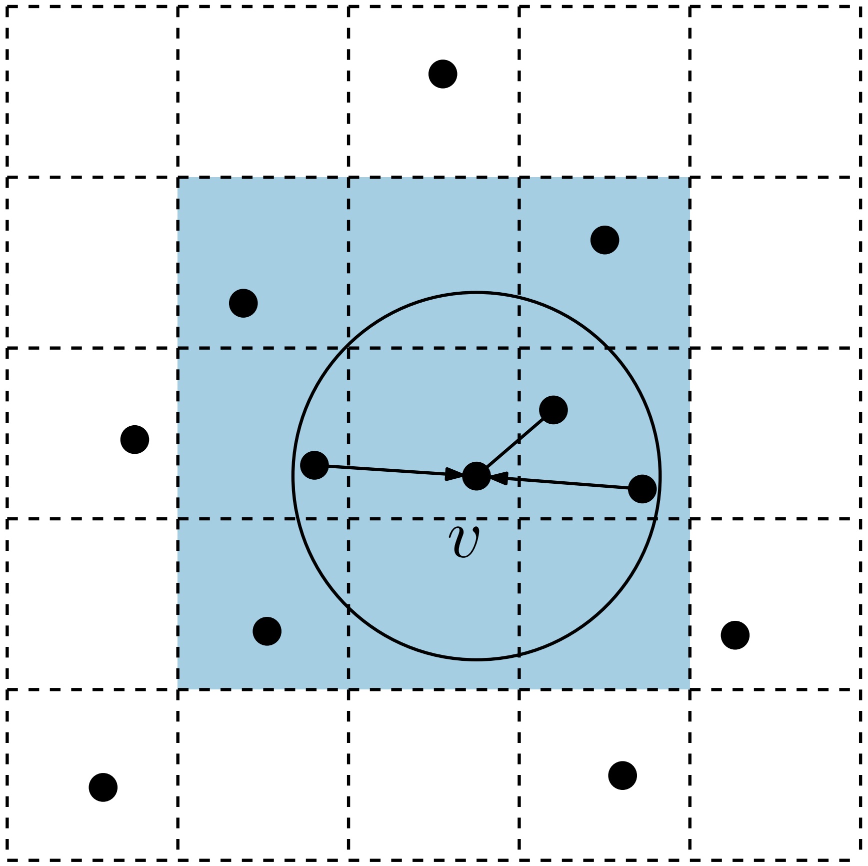
[Fruchterman & Reingold '91]



- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



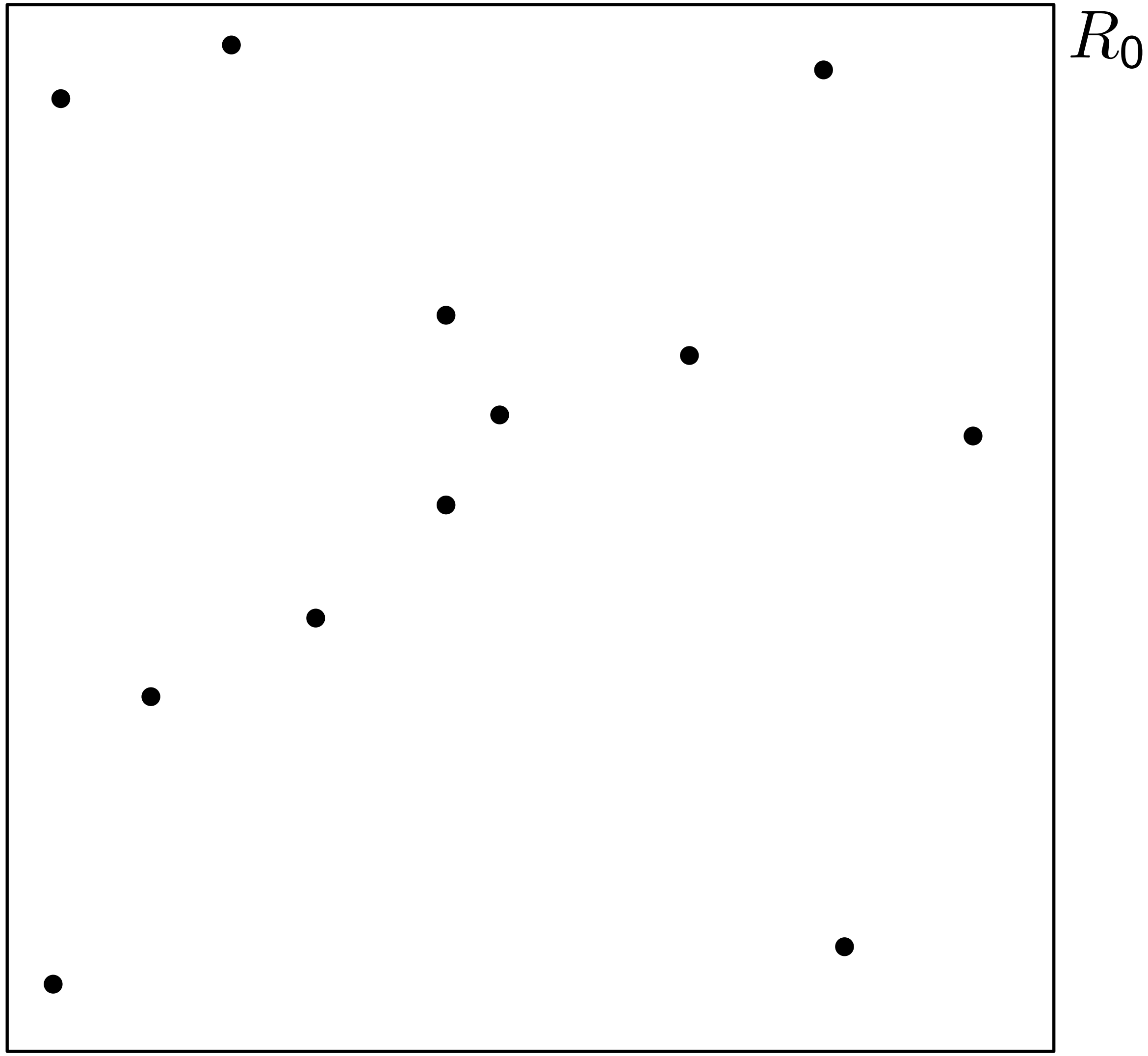
- divide plane into a grid
- consider repulsive forces only to vertices in neighboring cells
- and only if the distance is less than some threshold

Discussion.

- good idea to improve actual runtime
- asymptotic runtime does not improve
- might introduce oscillation and thus a quality loss

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

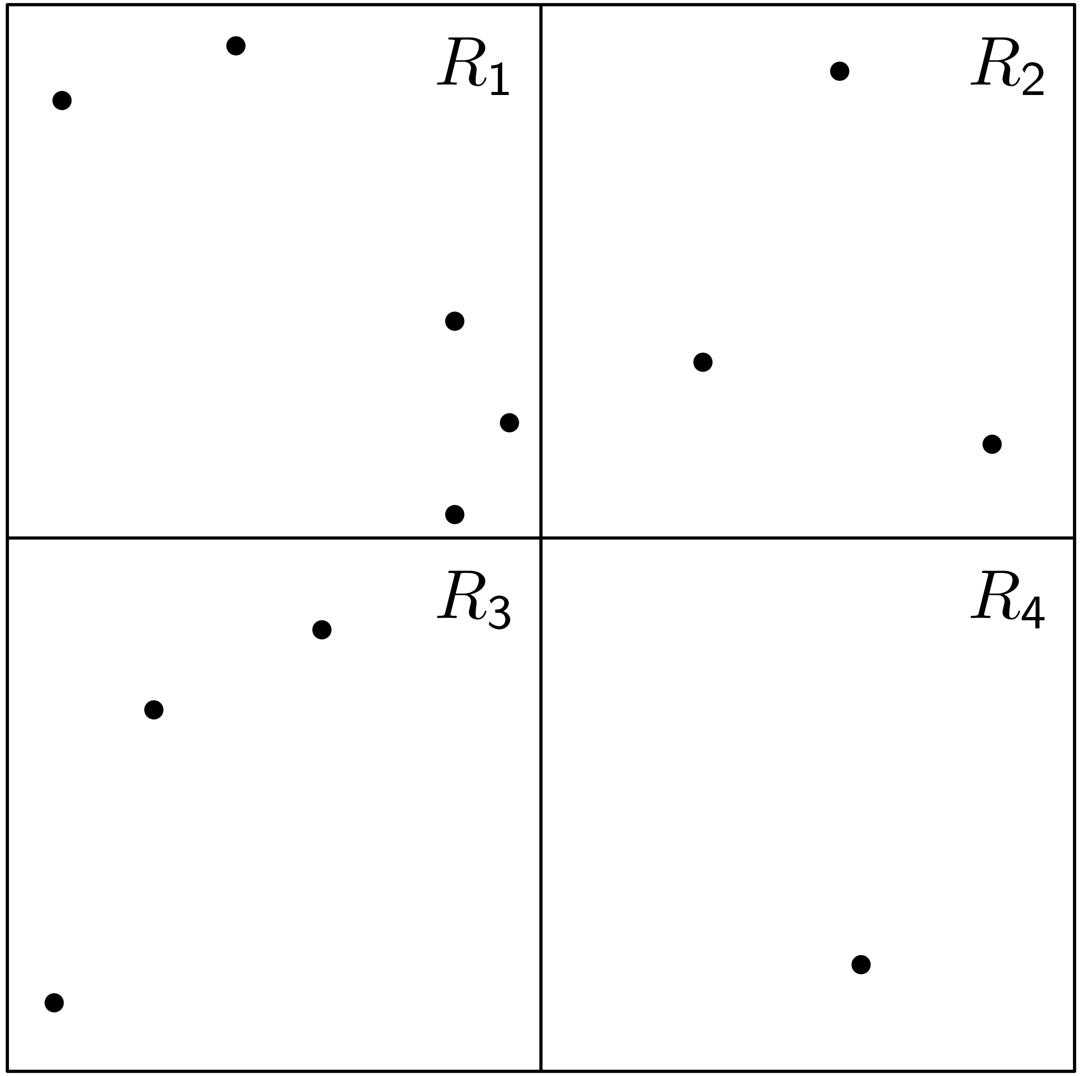


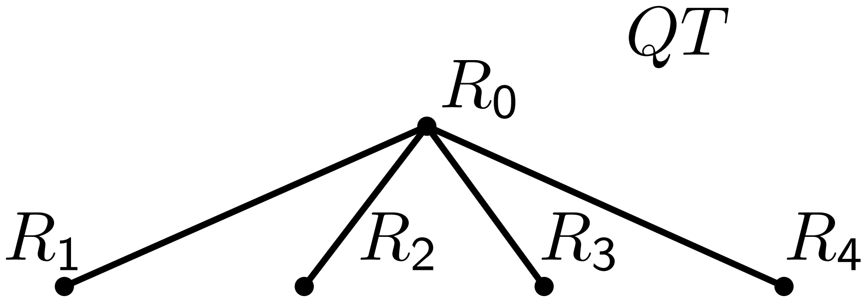
QT

R_0

Speeding up Repulsive-Force Computation with Quad Trees

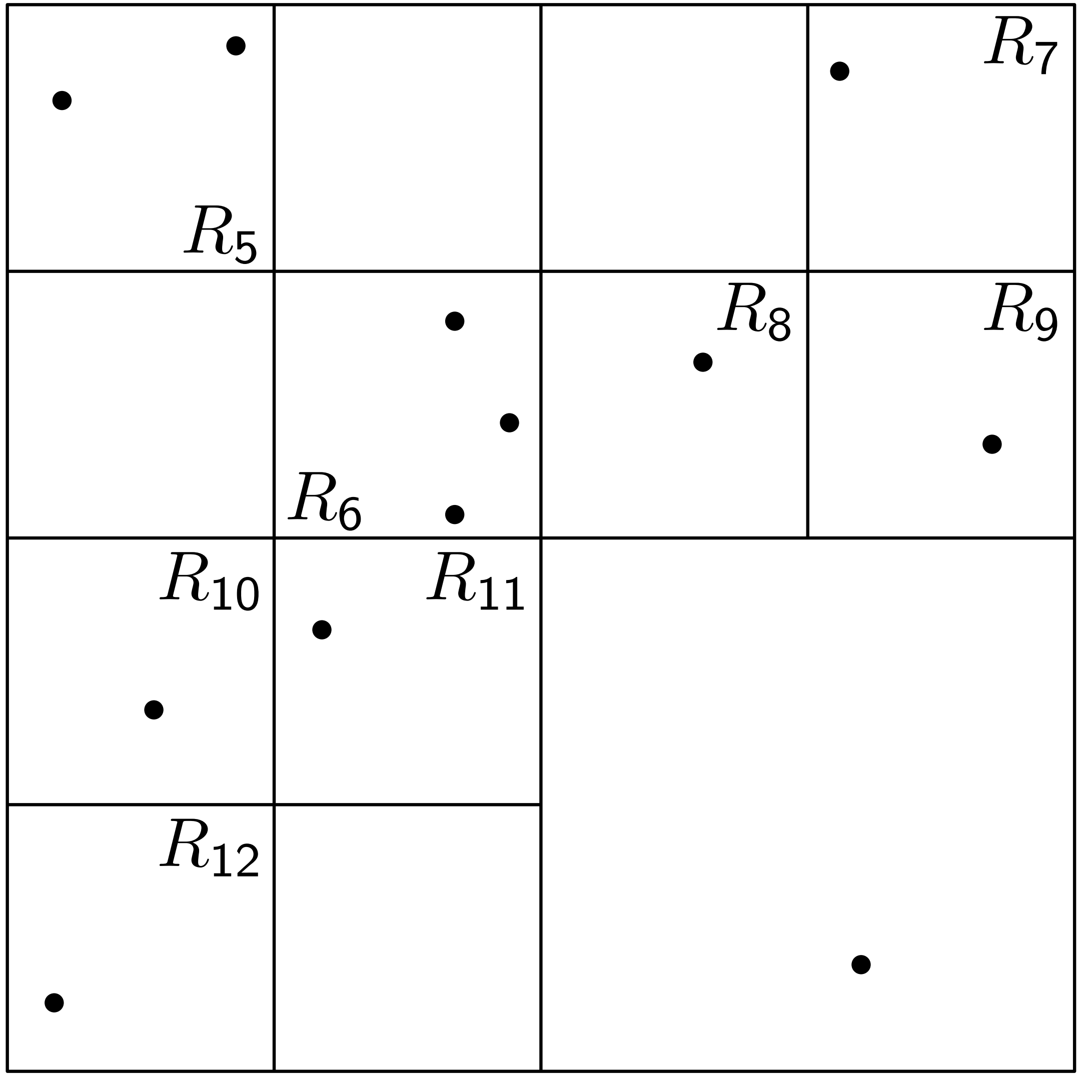
[Barnes, Hut '86]

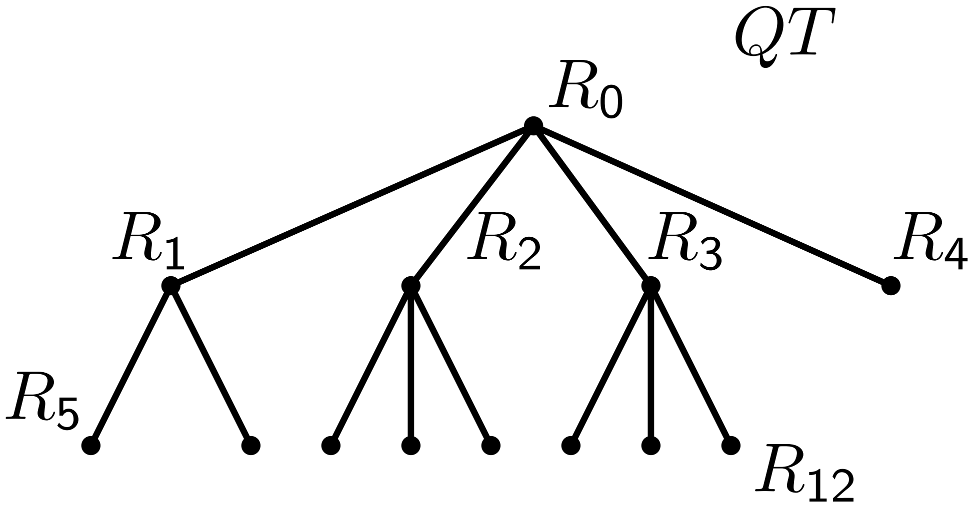




Speeding up Repulsive-Force Computation with Quad Trees

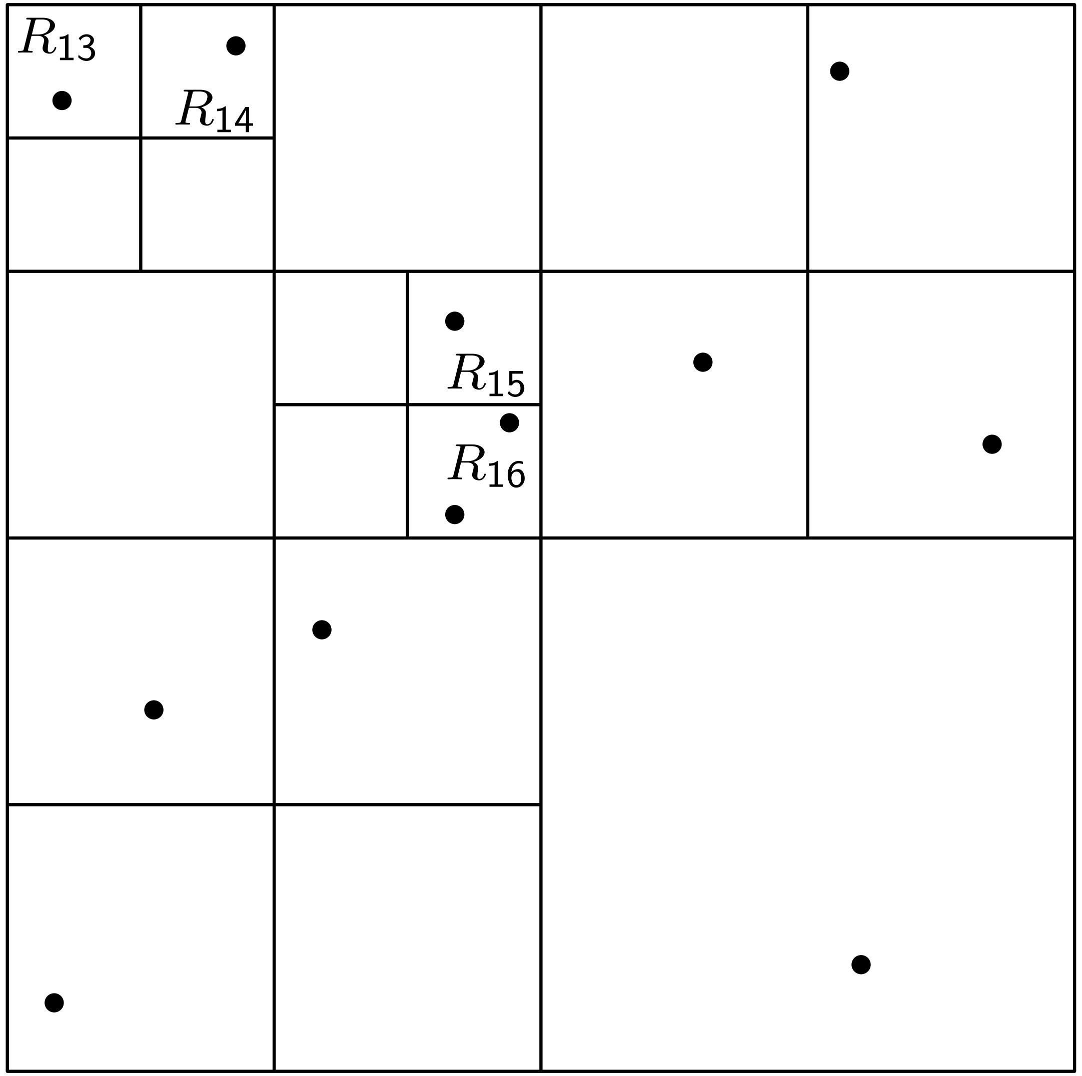
[Barnes, Hut '86]

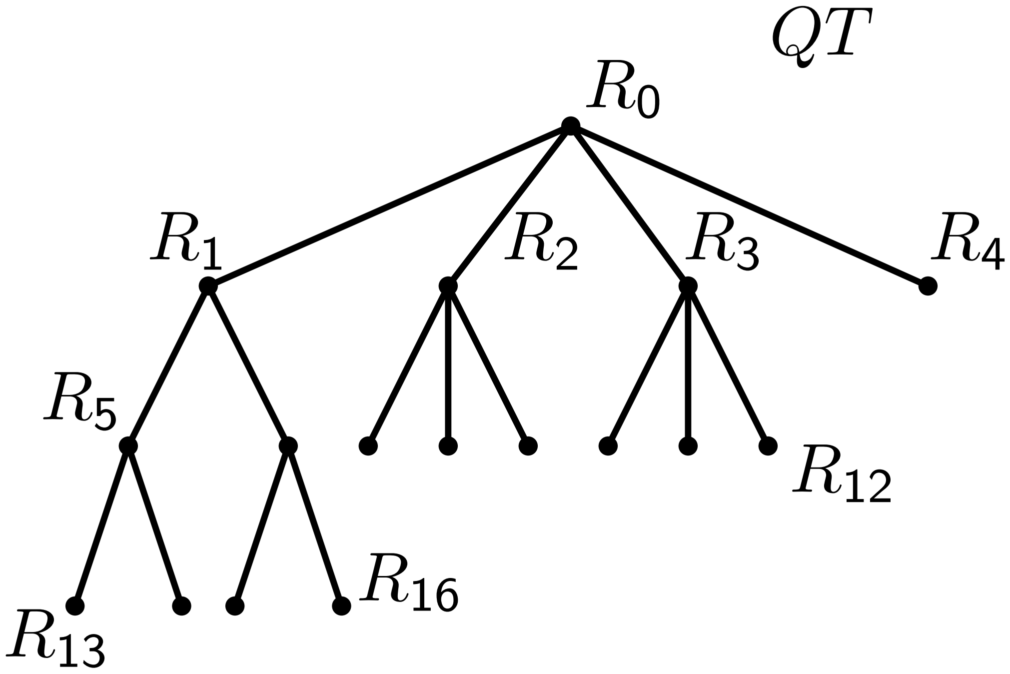




Speeding up Repulsive-Force Computation with Quad Trees

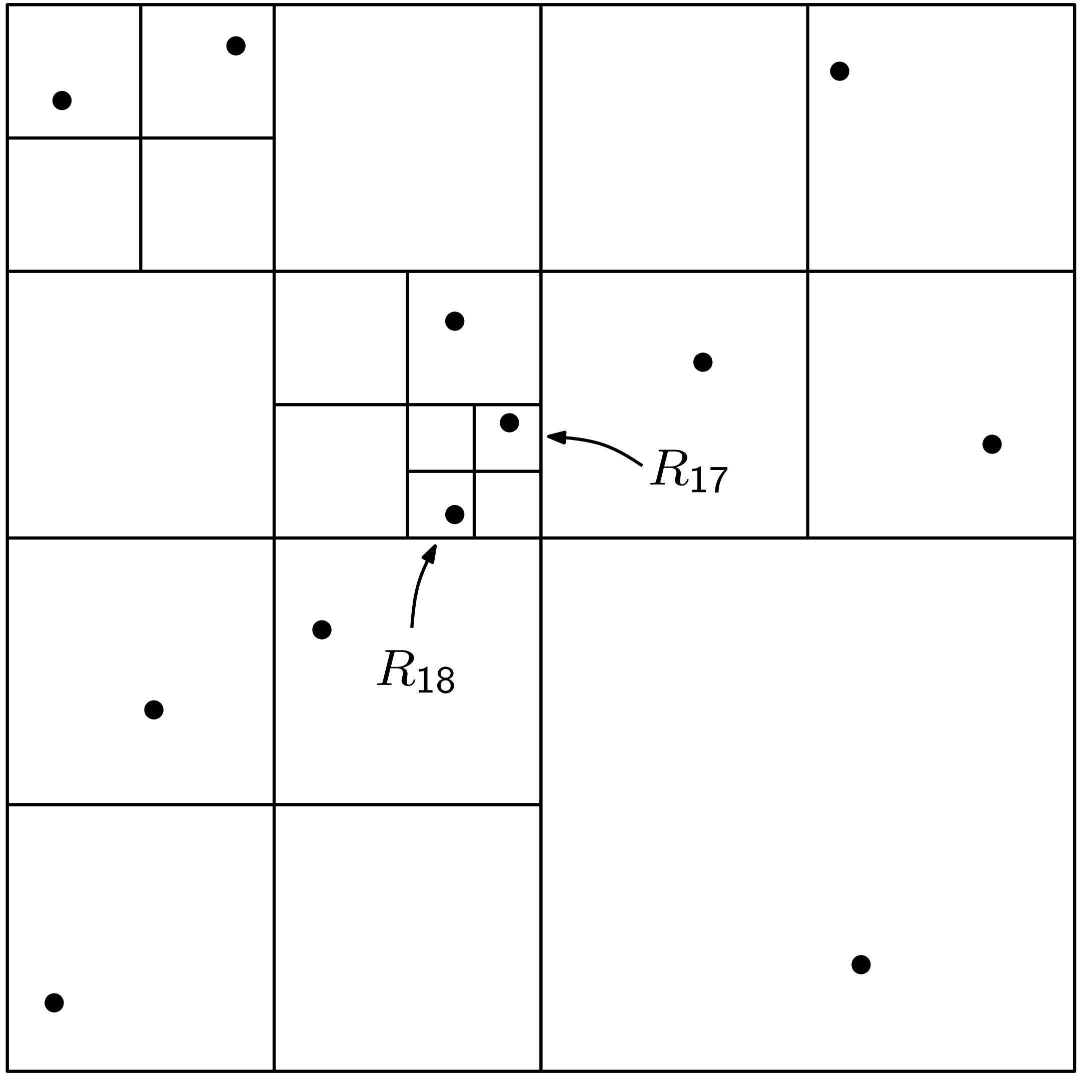
[Barnes, Hut '86]

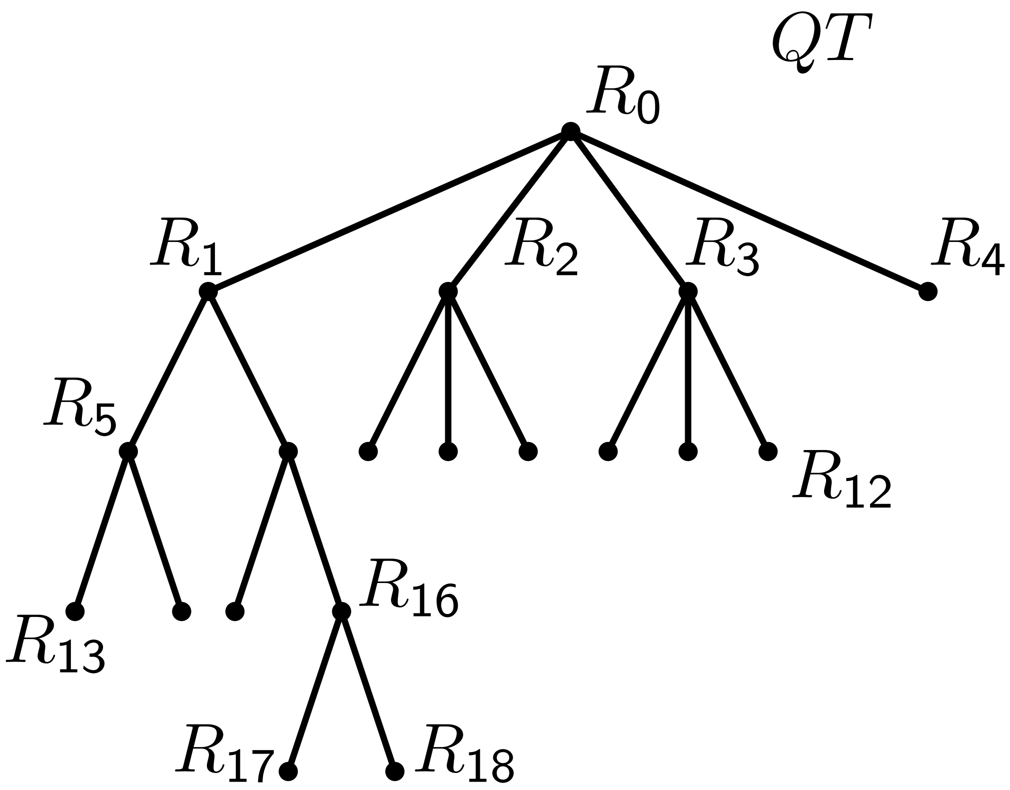




Speeding up Repulsive-Force Computation with Quad Trees

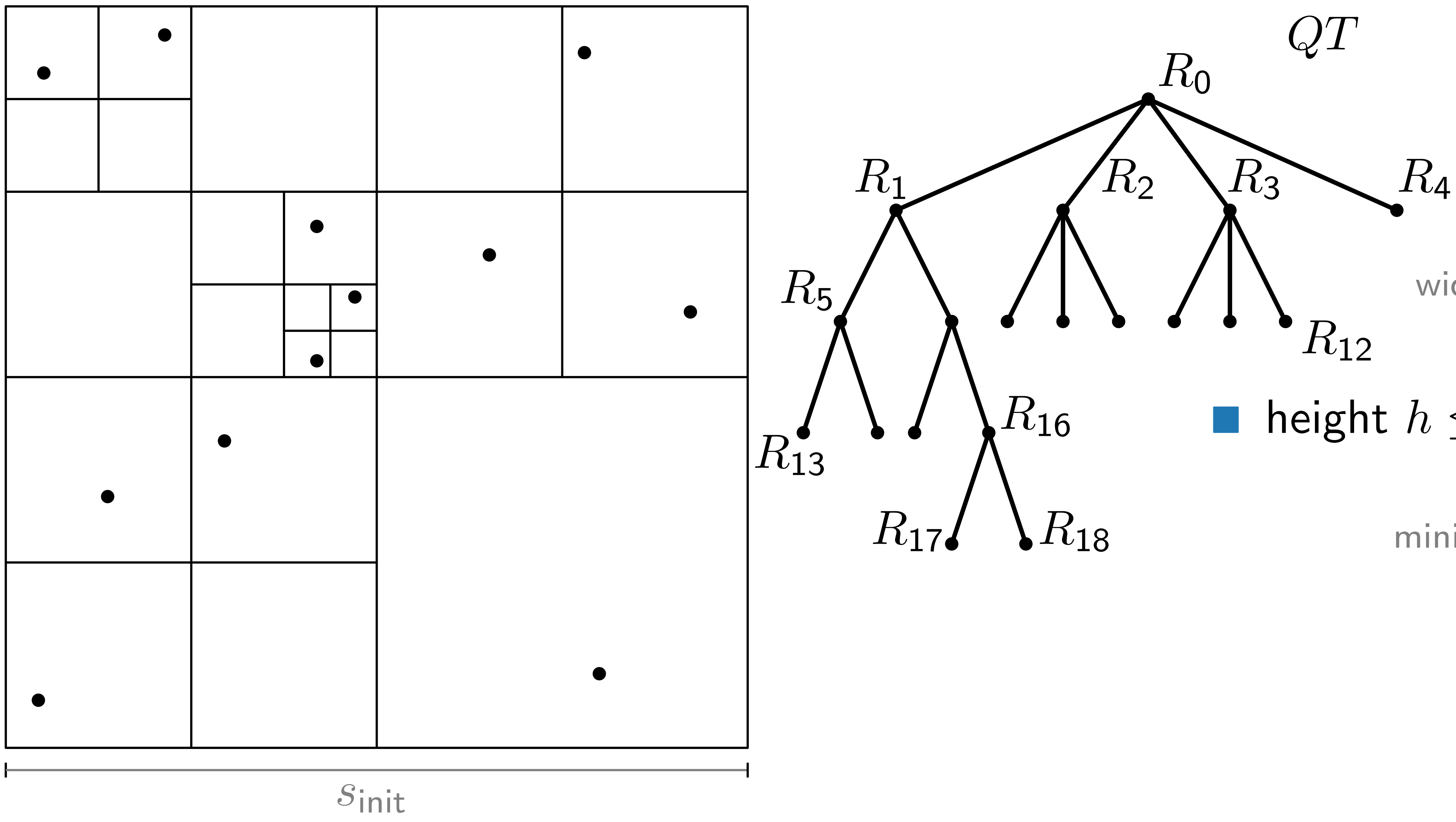
[Barnes, Hut '86]





Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]



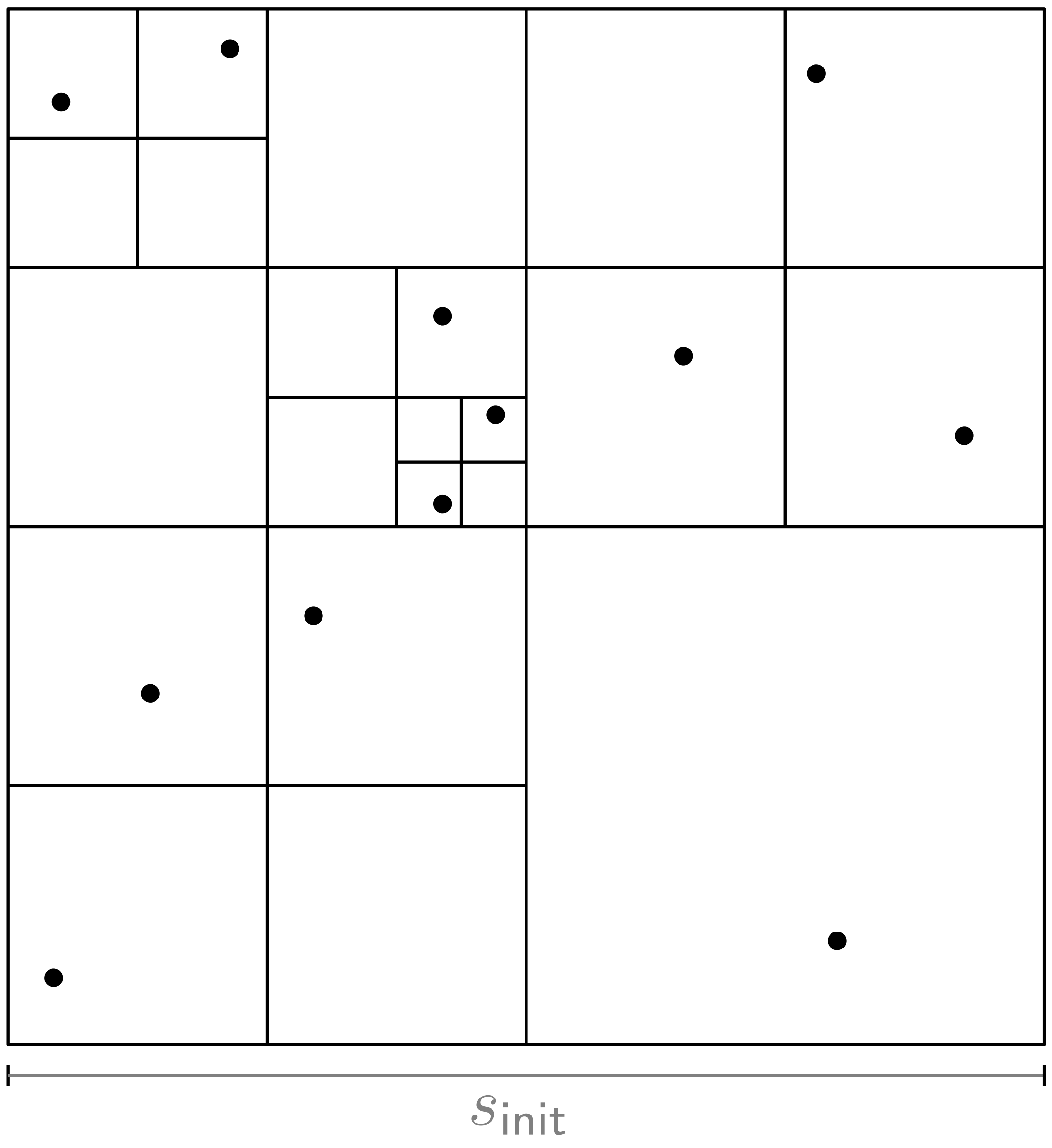
- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\text{min}}}\right) + \frac{3}{2}$

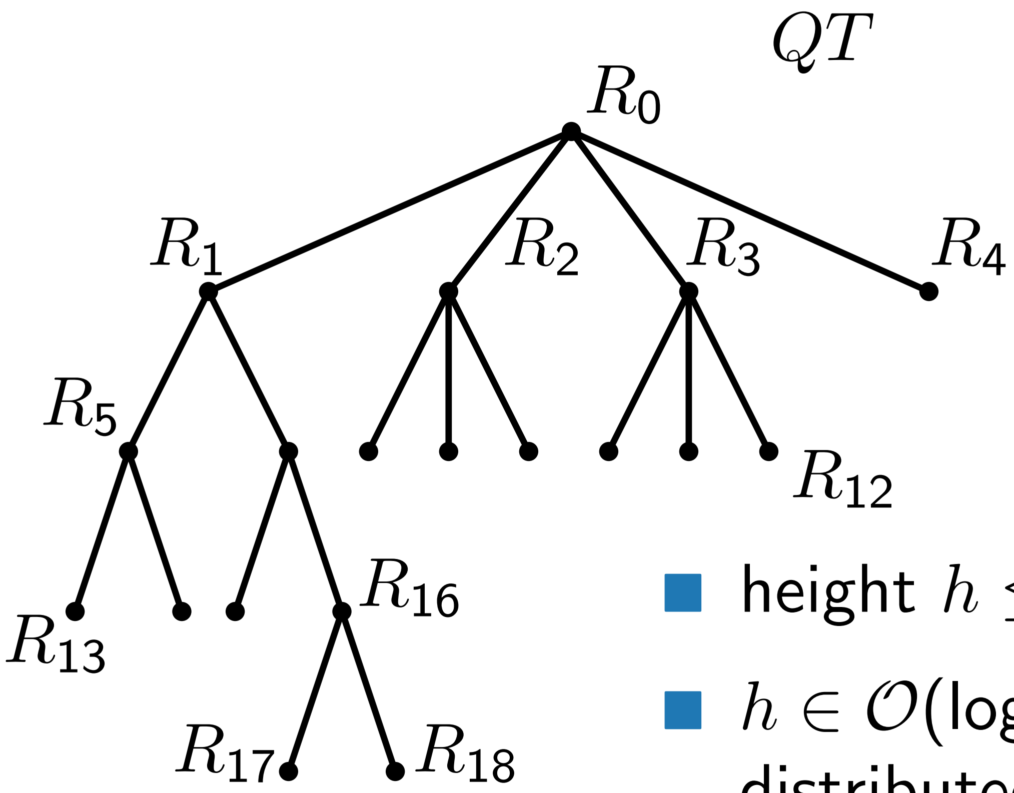
width/height of outer square

minimum distance btw. two pts.

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

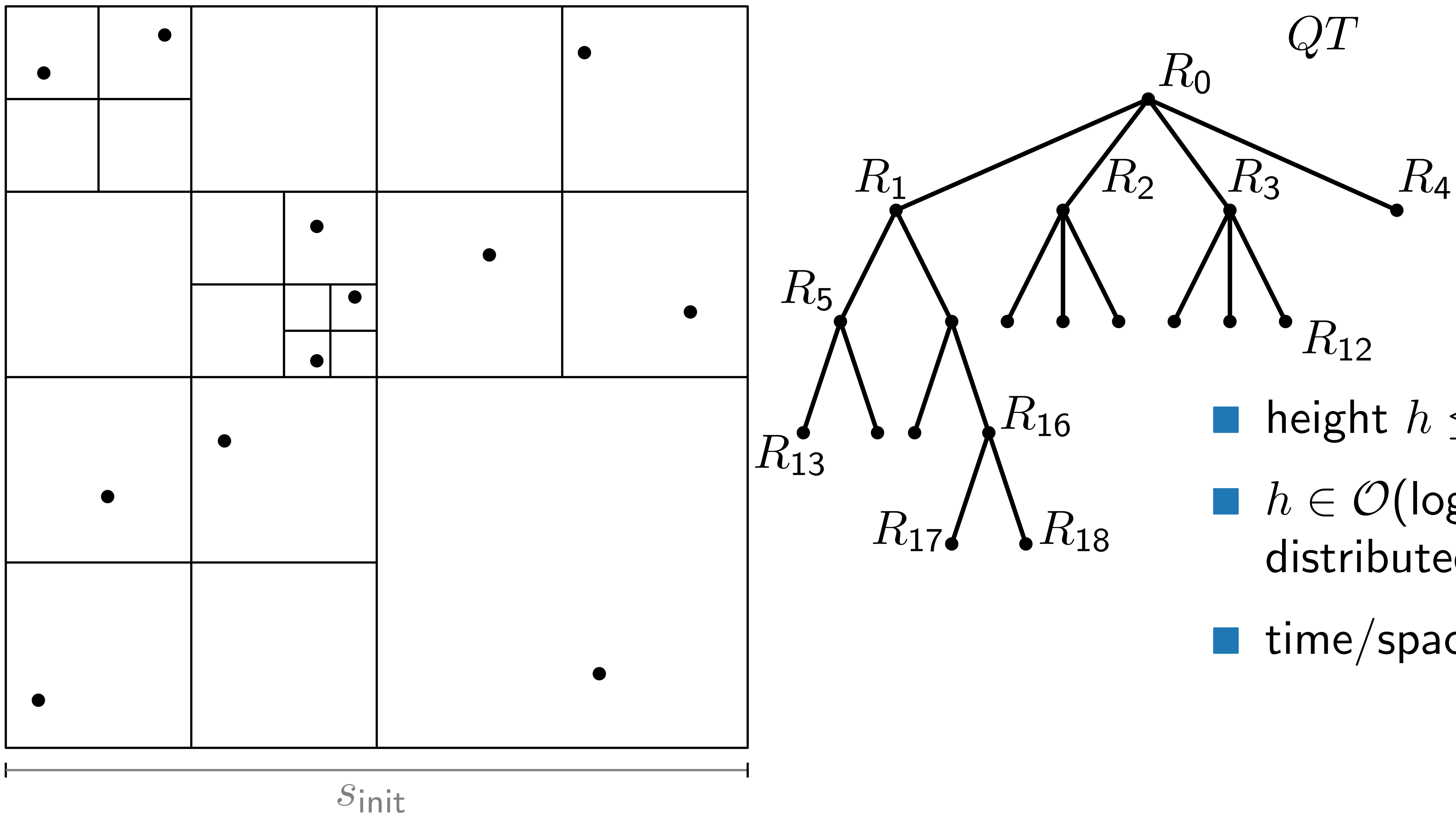




- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box

Speeding up Repulsive-Force Computation with Quad Trees

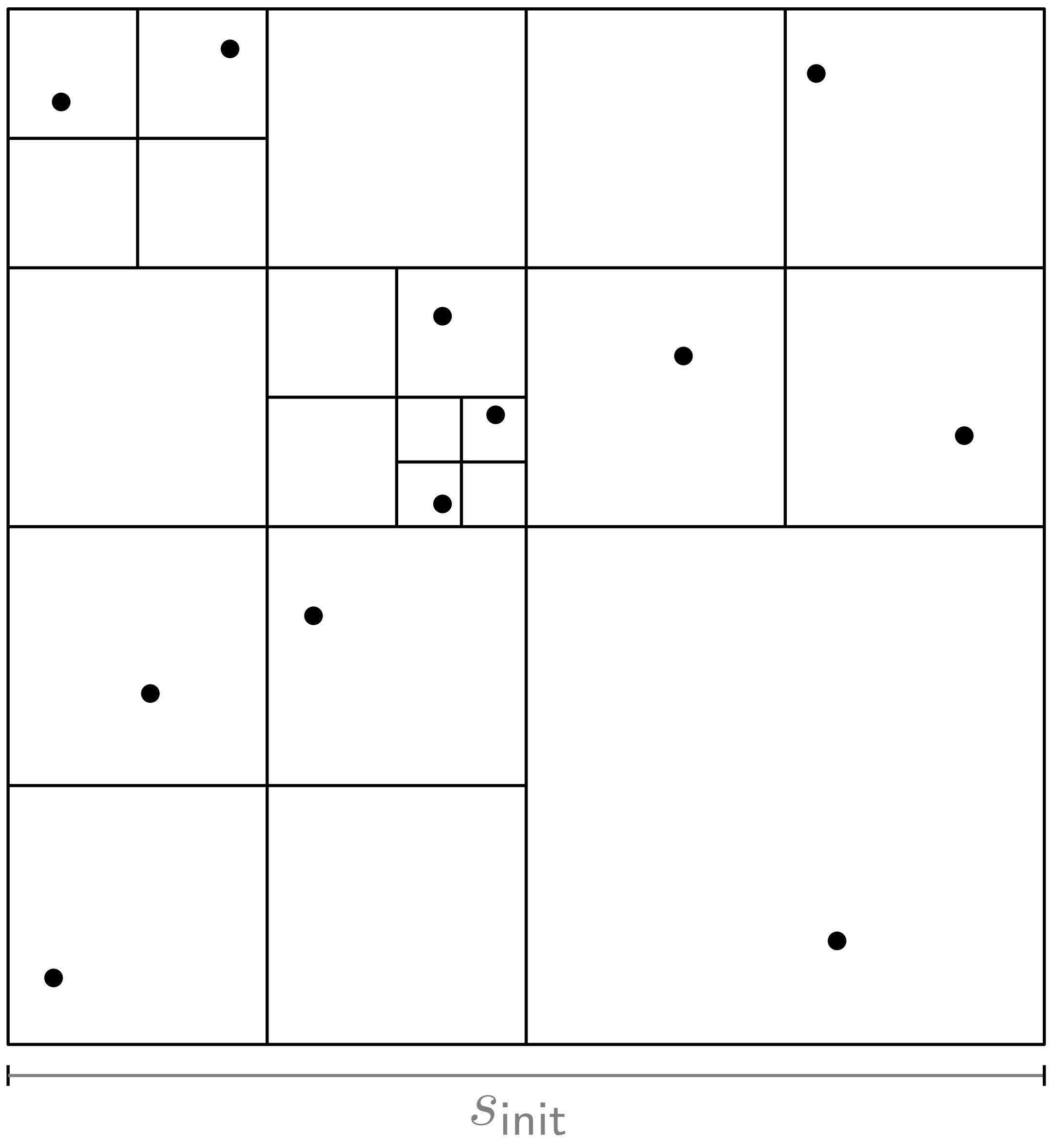
[Barnes, Hut '86]

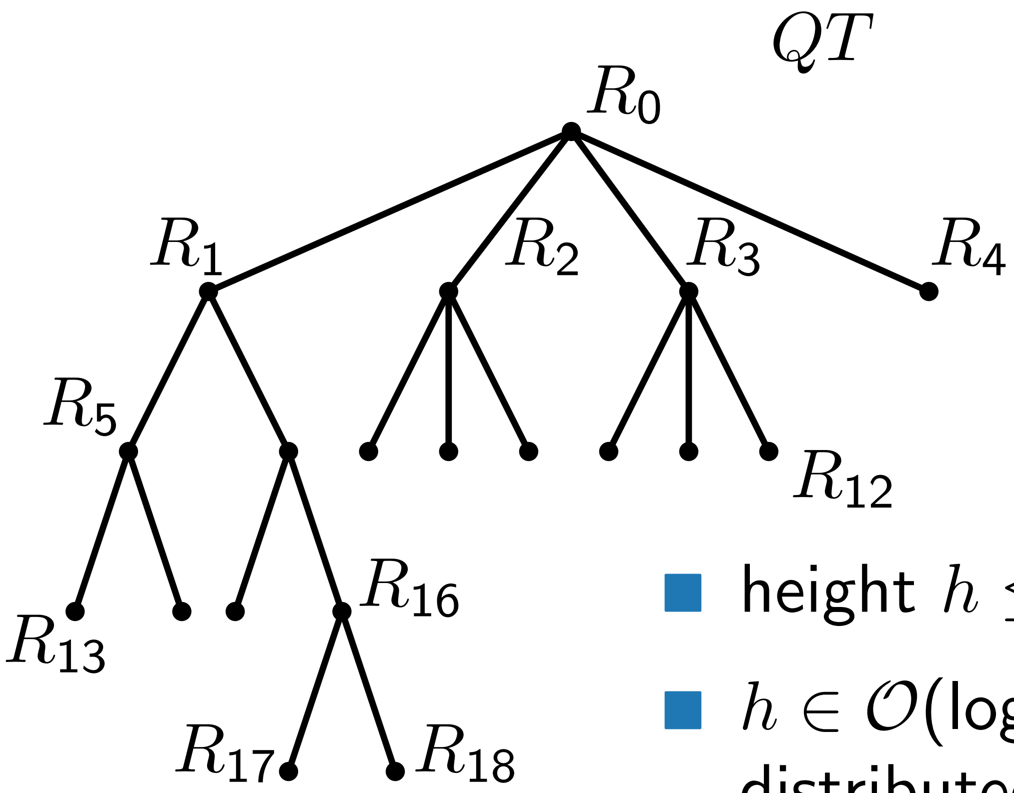


- height $h \leq \log_2\left(\frac{s_{\text{init}}}{d_{\text{min}}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$

Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]

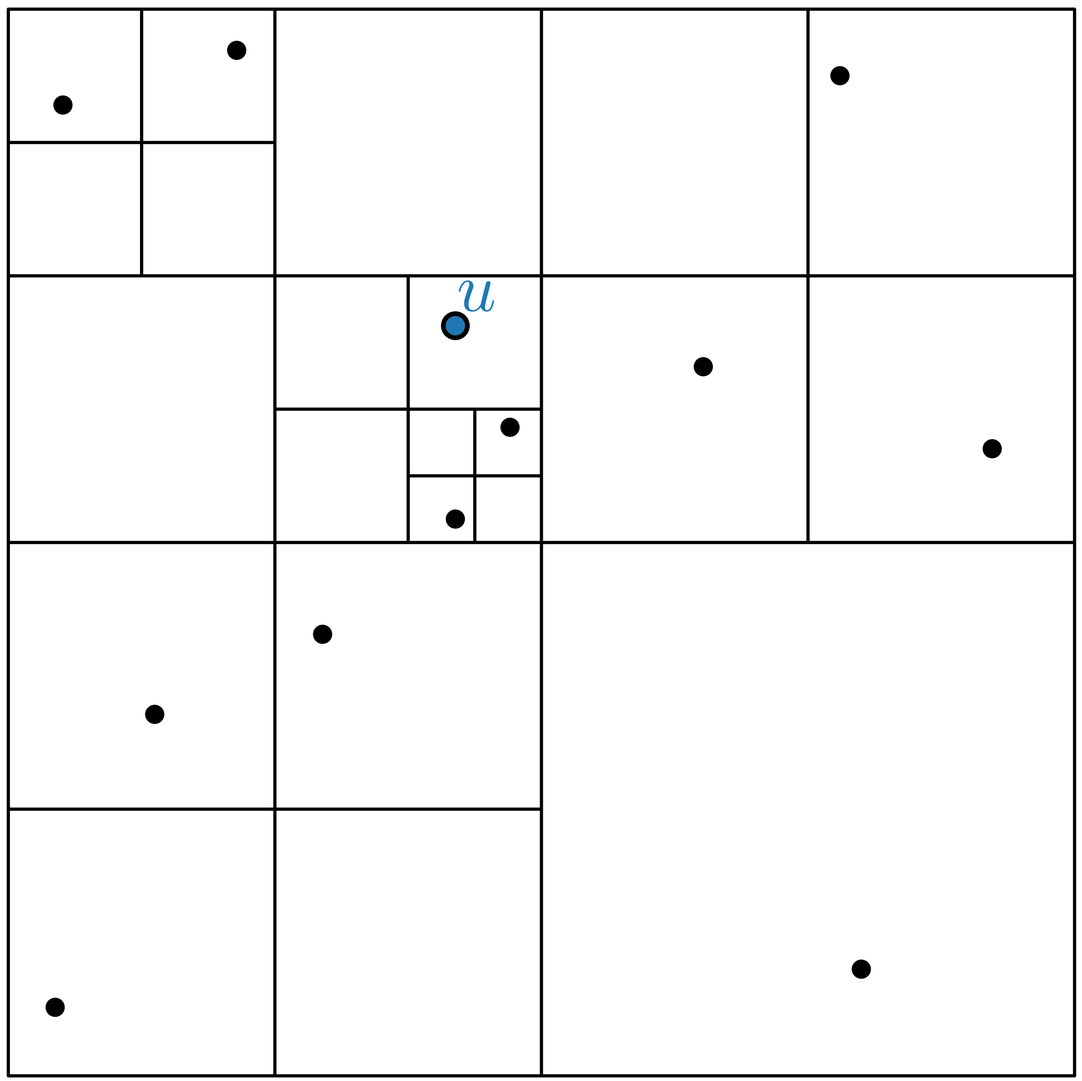


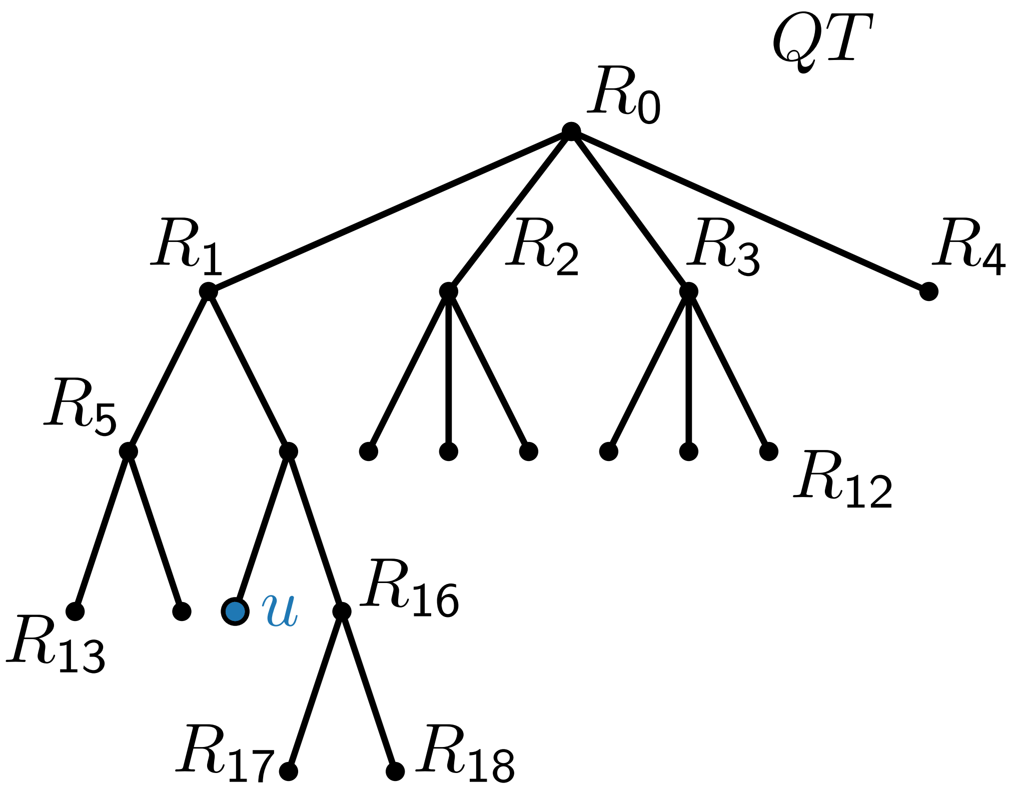


- height $h \le \log_2\left(\frac{s_{\text{init}}}{d_{\min}}\right) + \frac{3}{2}$
- $h \in \mathcal{O}(\log n)$ if vertices evenly distributed in the initial box
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time

Speeding up Repulsive-Force Computation with Quad Trees

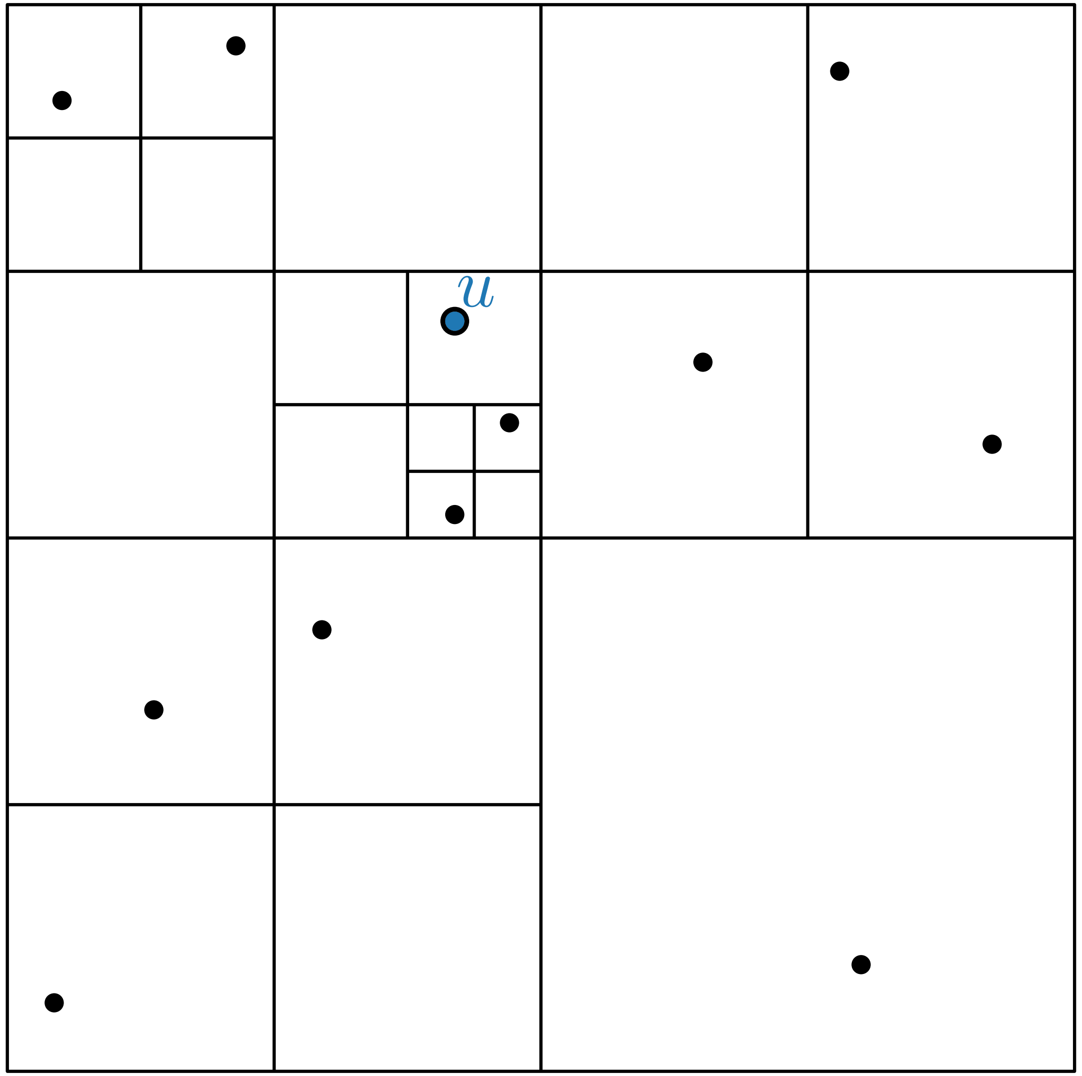
[Barnes, Hut '86]

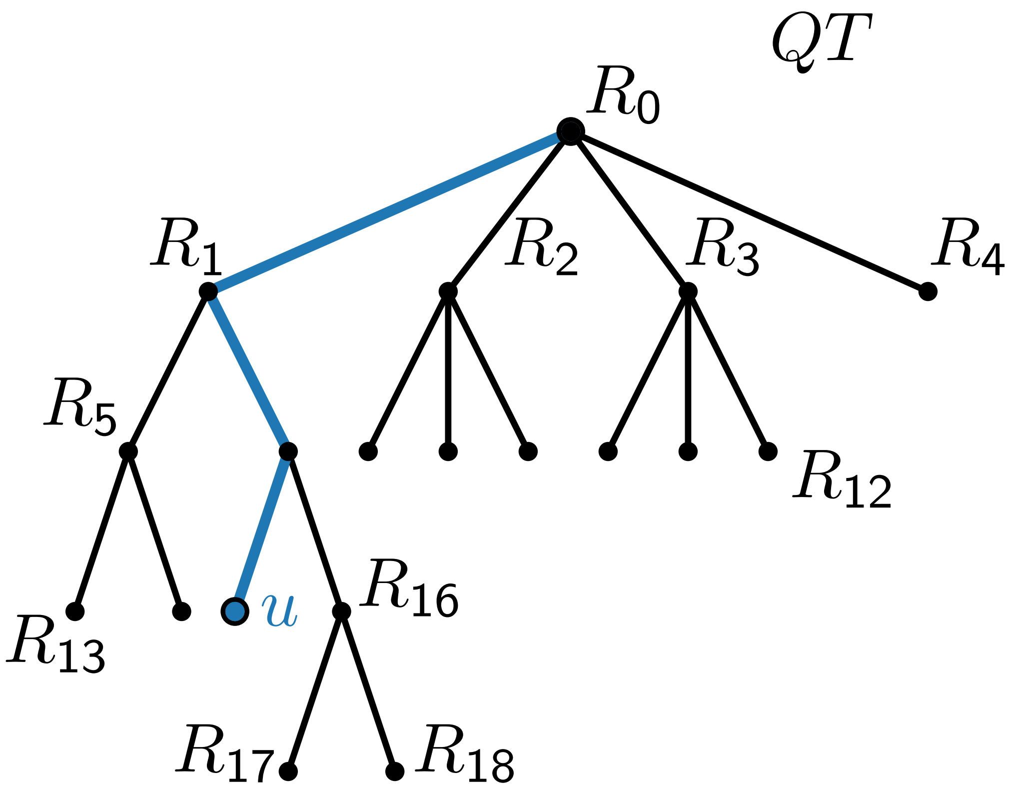




Speeding up Repulsive-Force Computation with Quad Trees

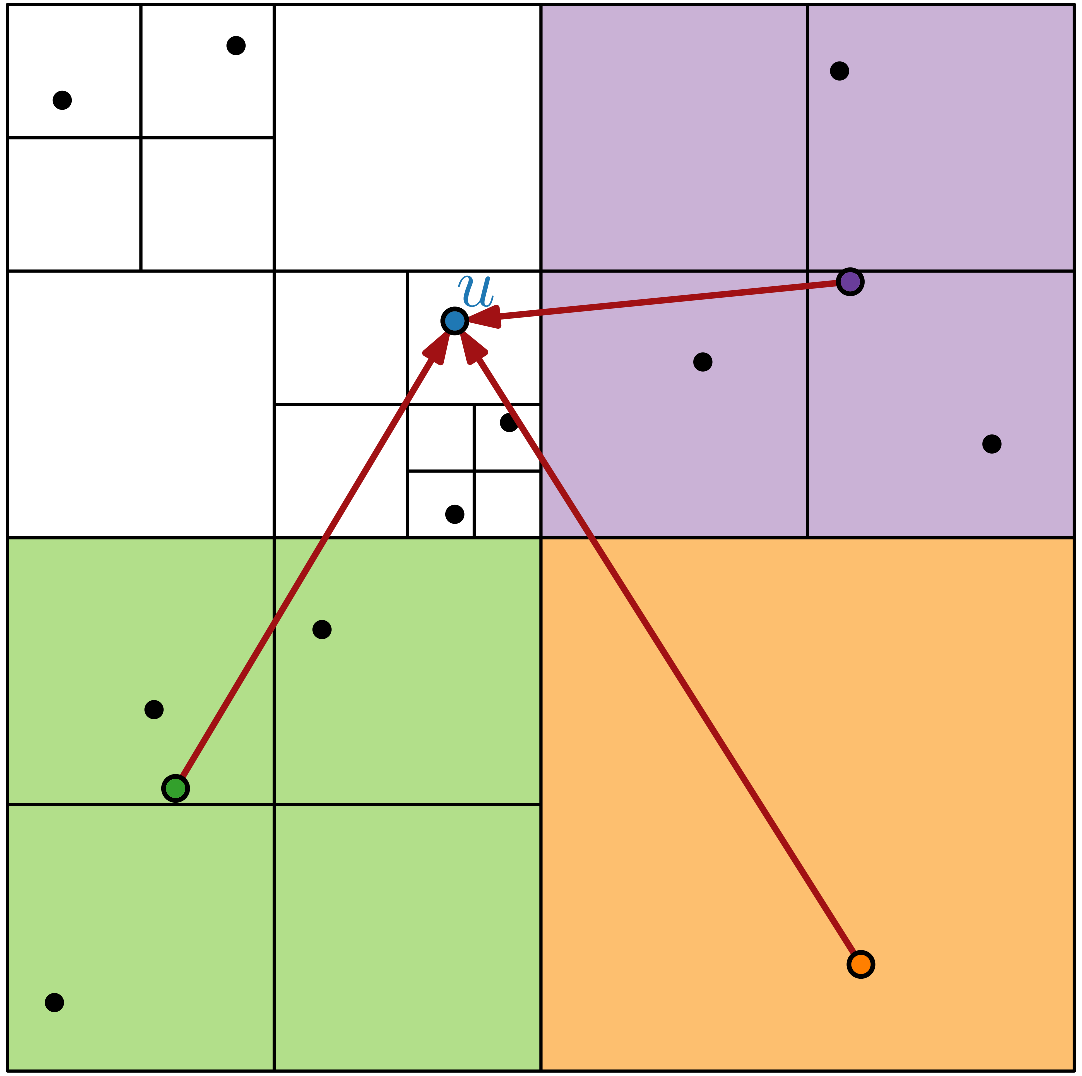
[Barnes, Hut '86]

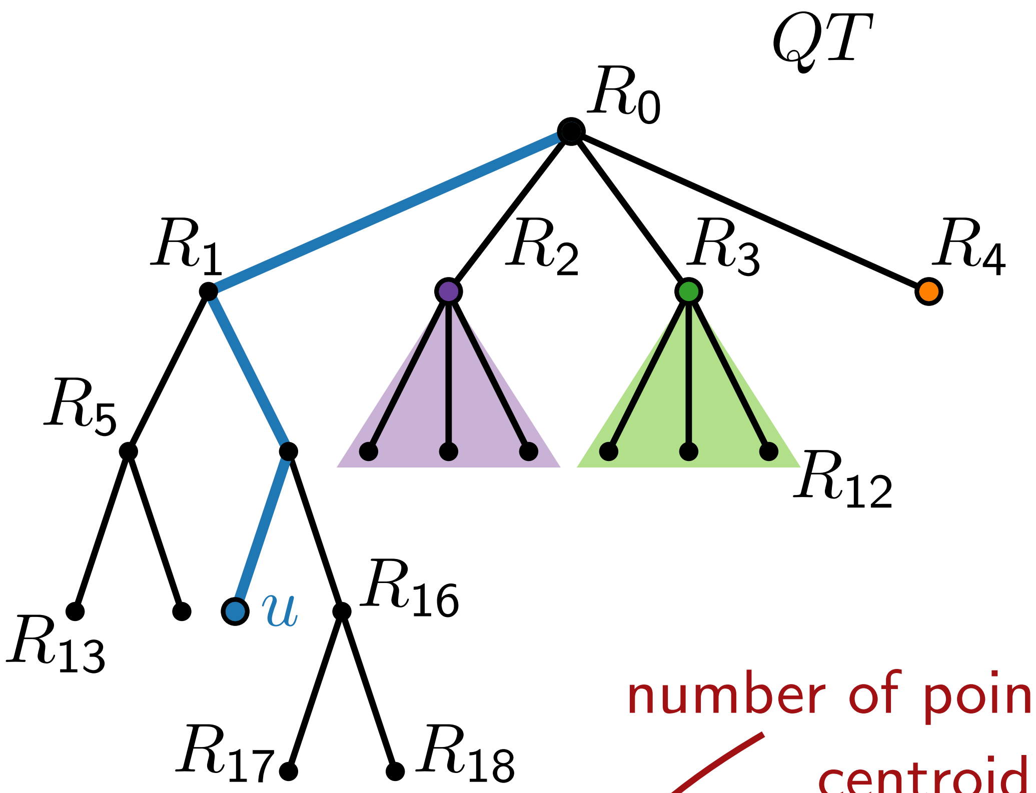




Speeding up Repulsive-Force Computation with Quad Trees

[Barnes, Hut '86]





number of points in the subtree R_i

centroid of R_i (pre-computed)

$$f_{\text{rep}}(R_i, p_u) = \boxed{|R_i|} \cdot f_{\text{rep}}(\boxed{\sigma_{R_i}}, p_u)$$

for each child R_i of a vertex on path from root to u.

Speeding up Repulsive-Force Computation with Quad Trees

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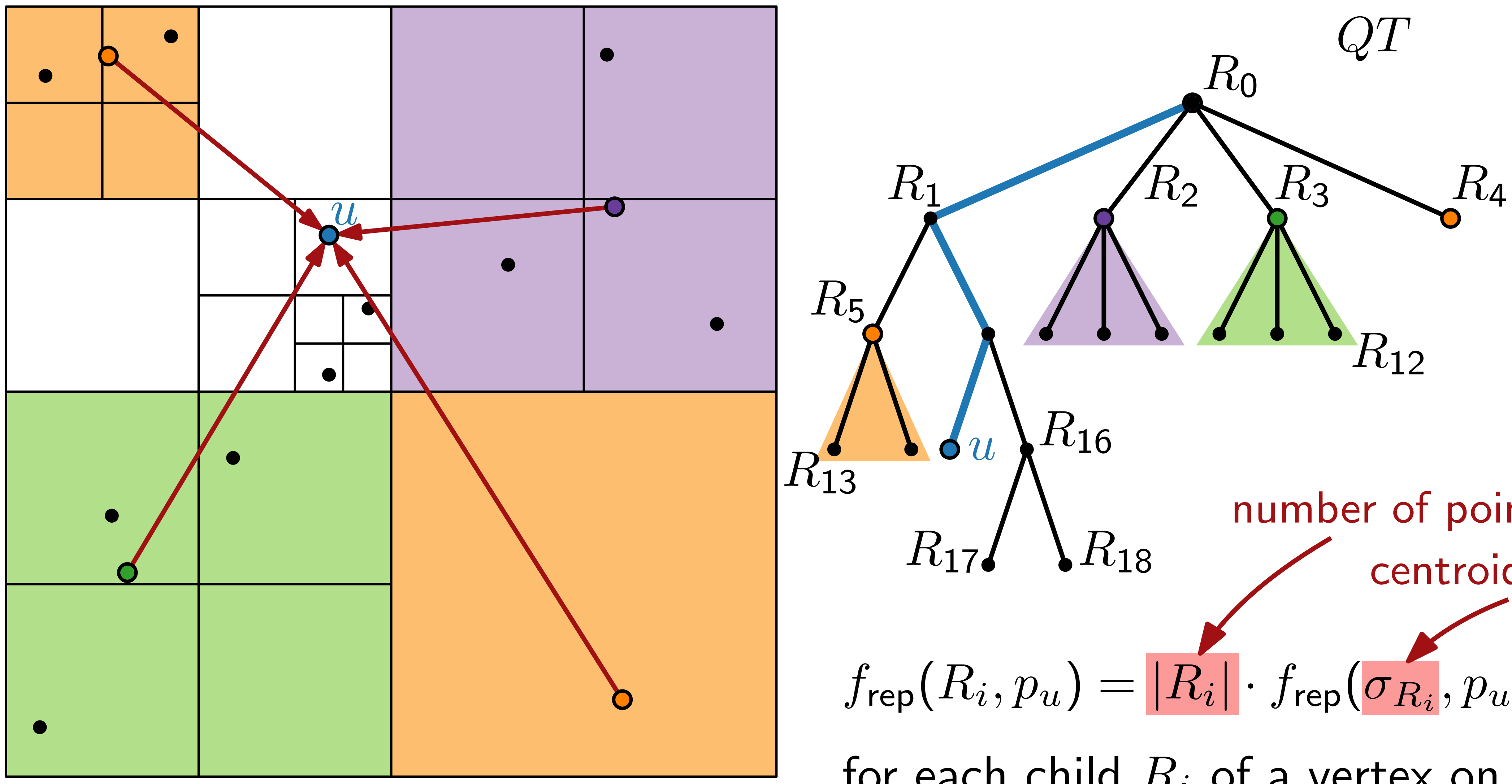

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Speeding up Repulsive-Force Computation with Quad Trees

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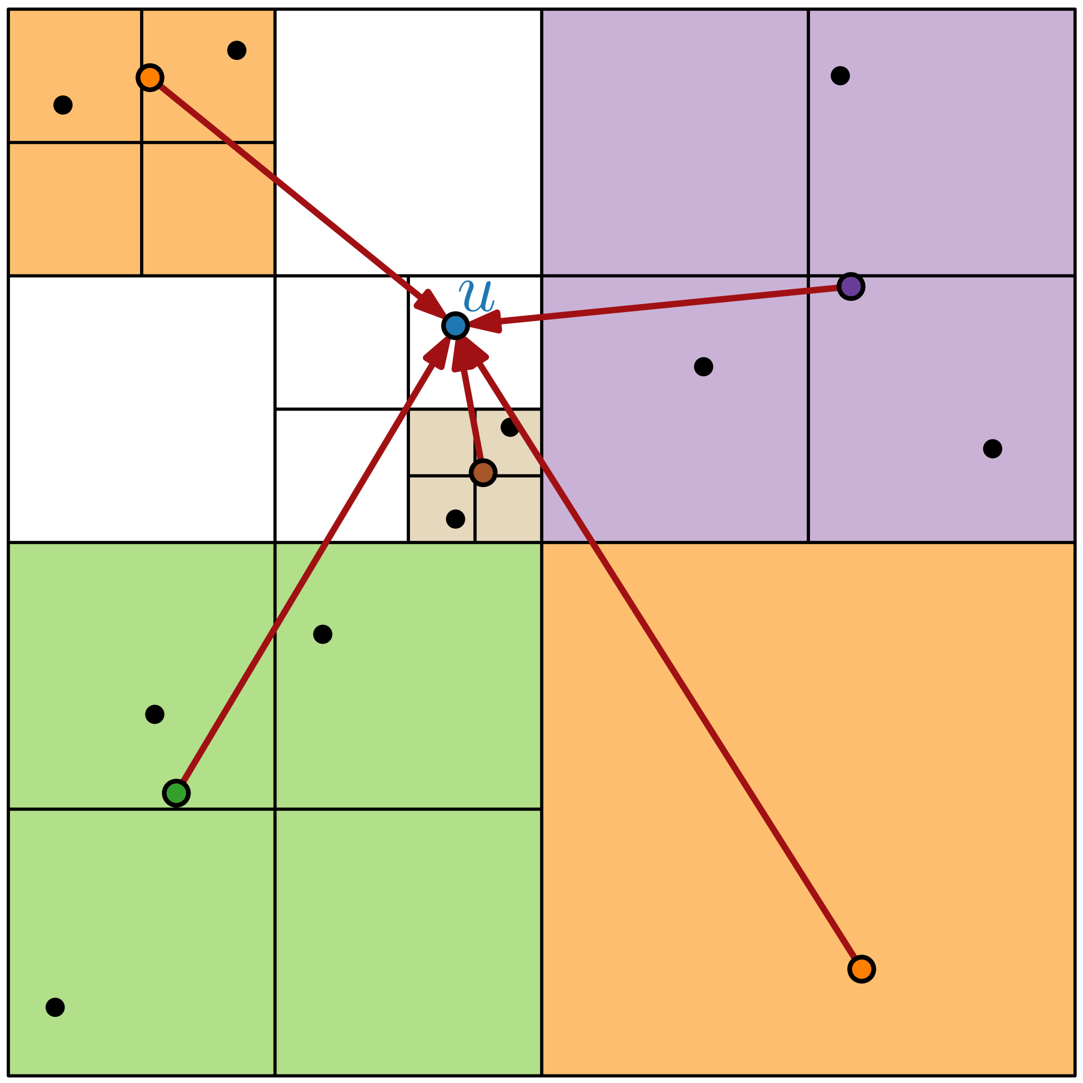

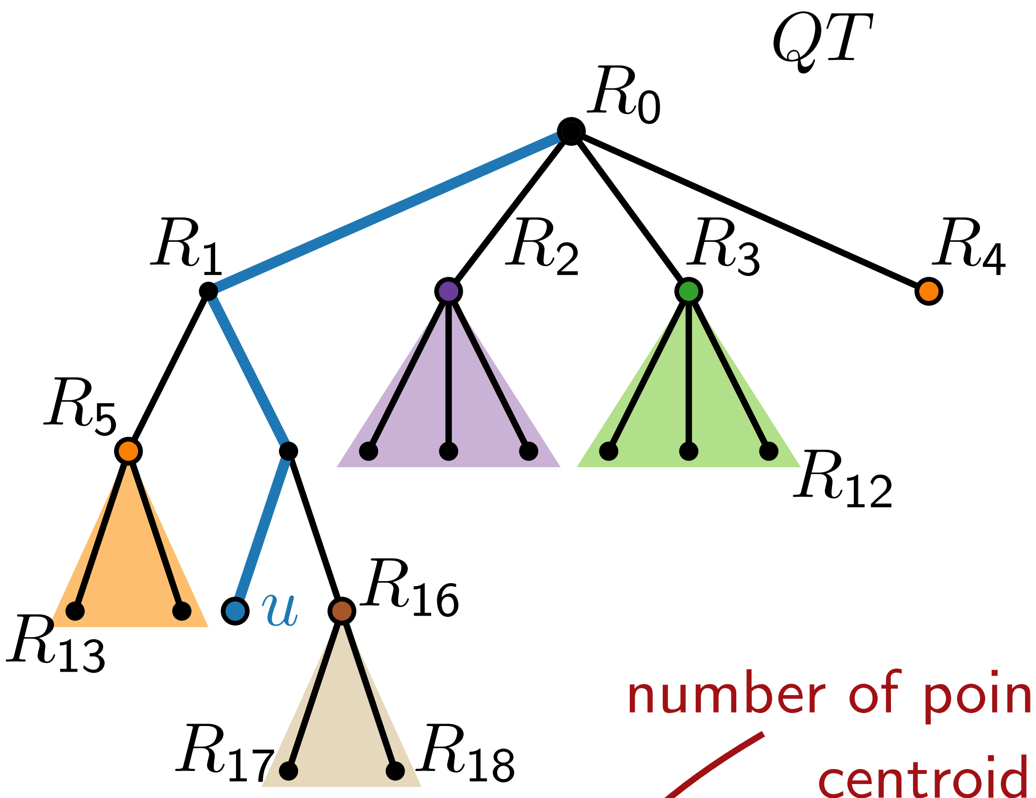

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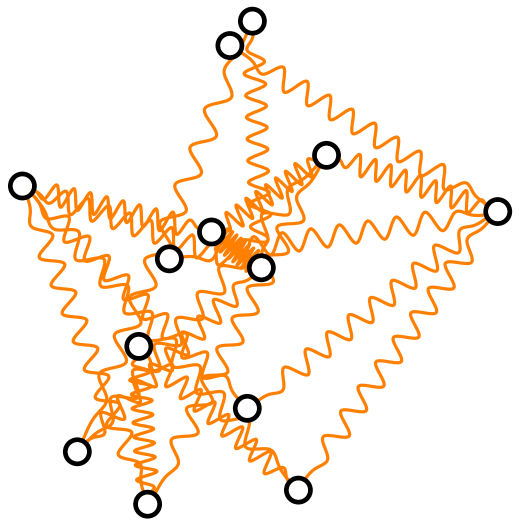
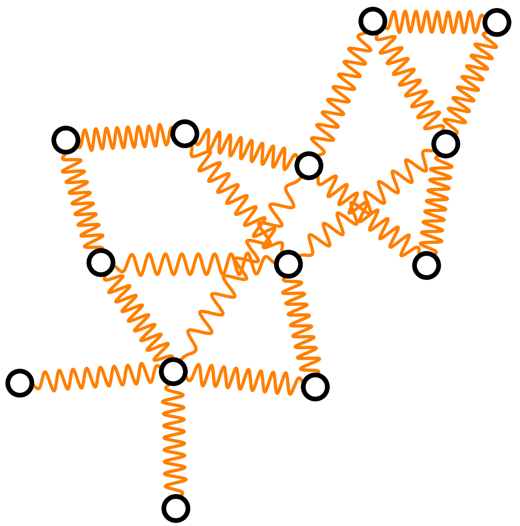
Visualization of Graphs

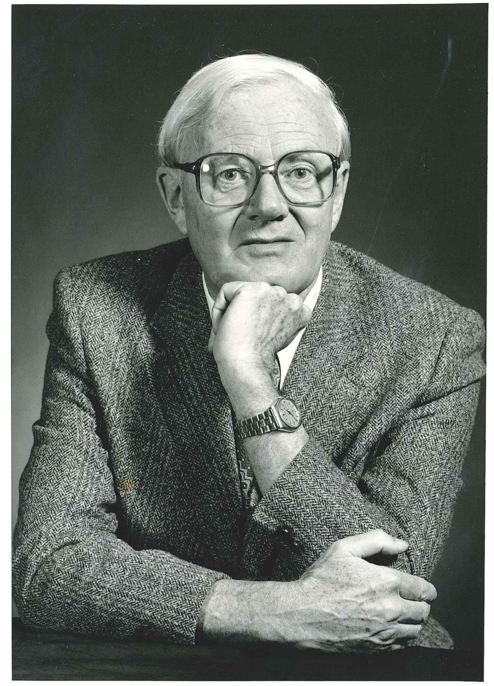
Lecture 2:
Force-Directed Drawing Algorithms



Part II:
Tutte Embeddings



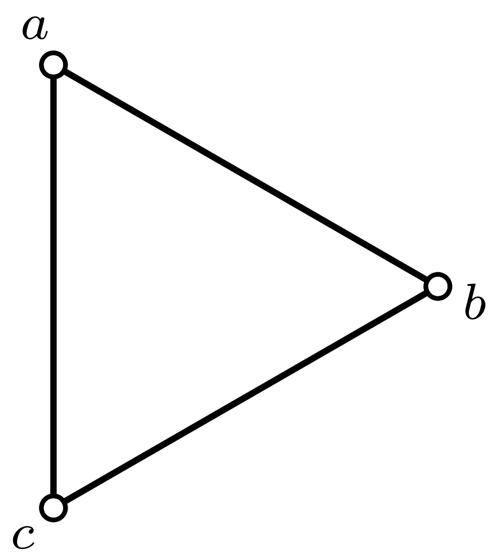
Idea

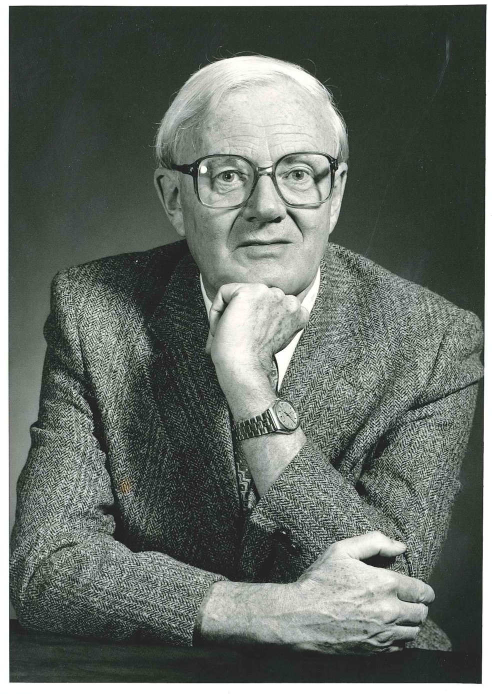


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c)

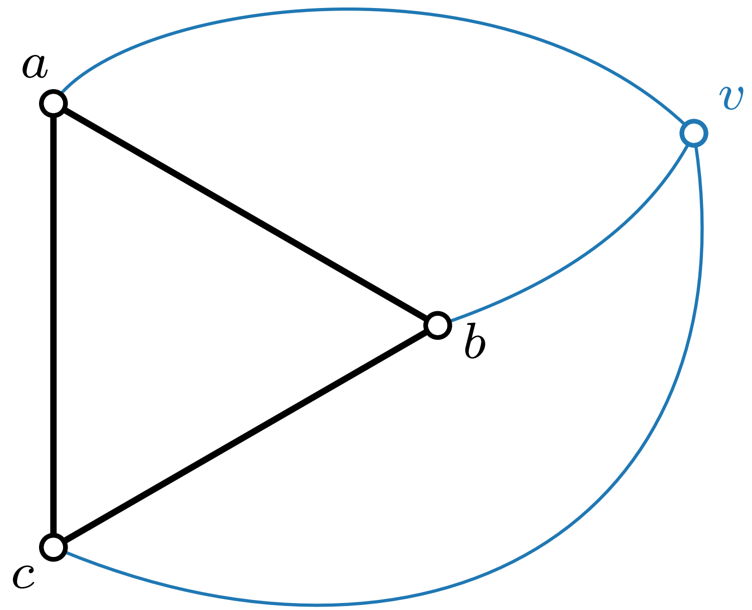


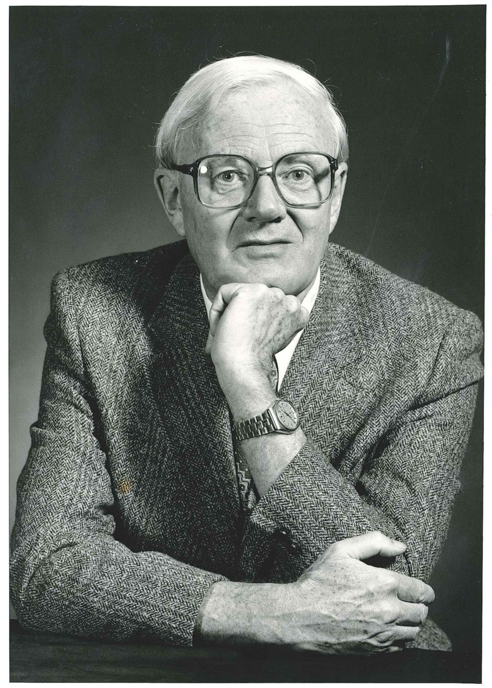


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with a common neighbor v



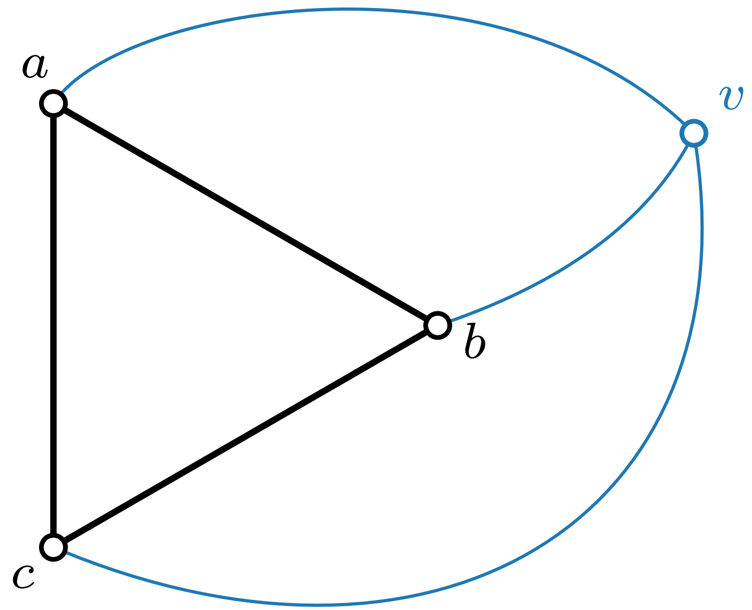


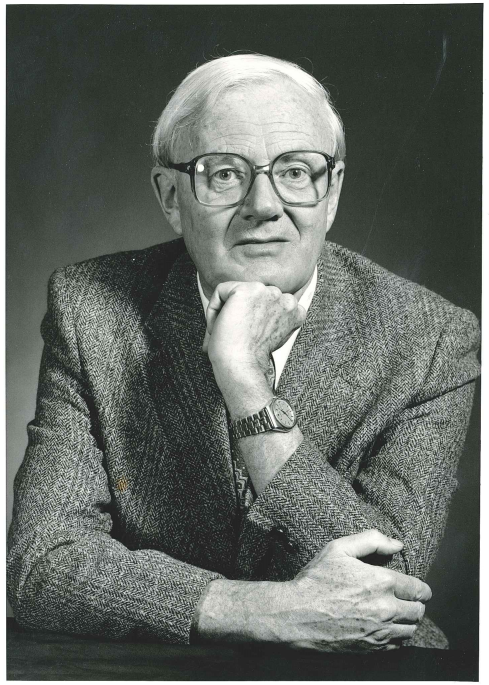
William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?





William T. Tutte
1917 – 2002

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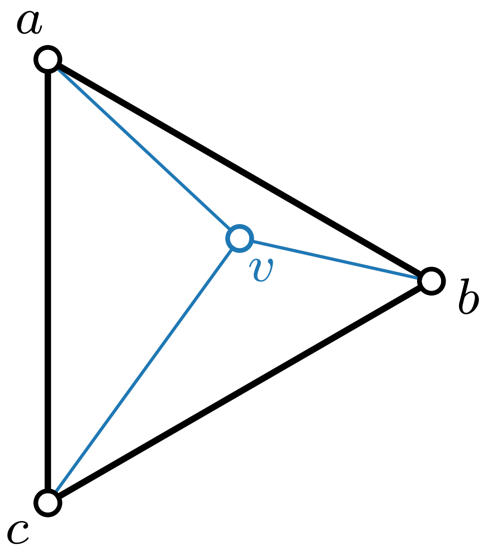
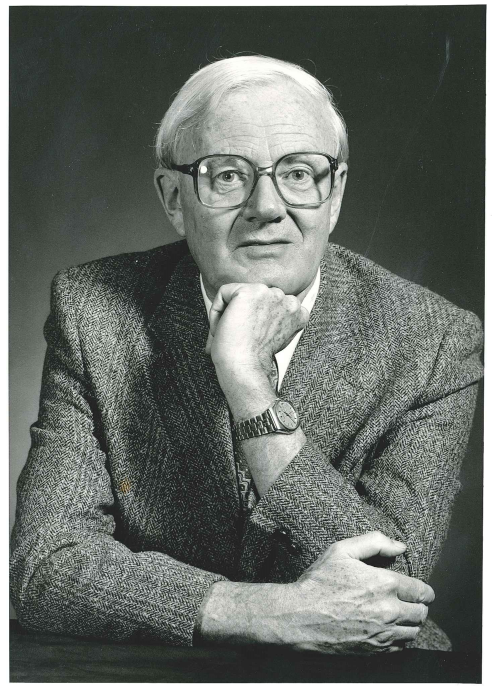
Where would you place v?





William T. Tutte
1917 – 2002

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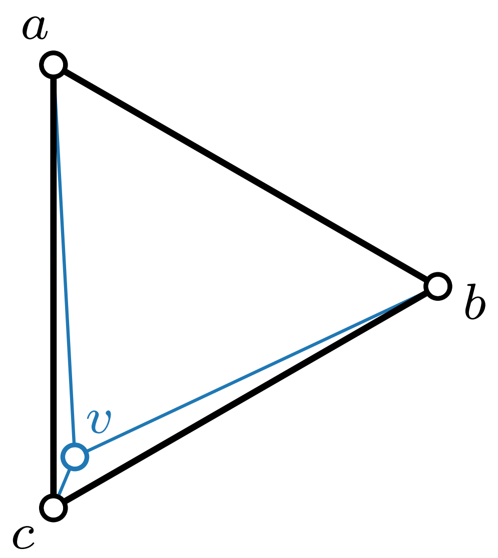

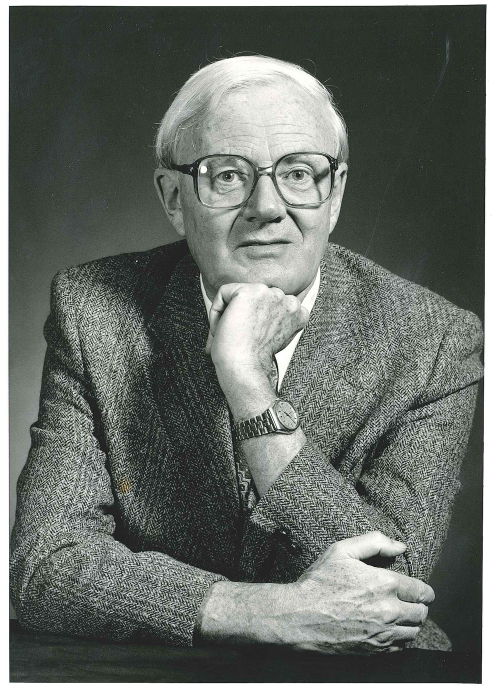

William T. Tutte
1917 – 2002

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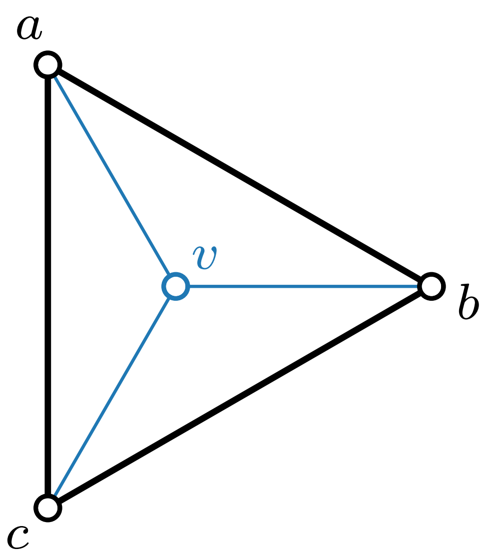

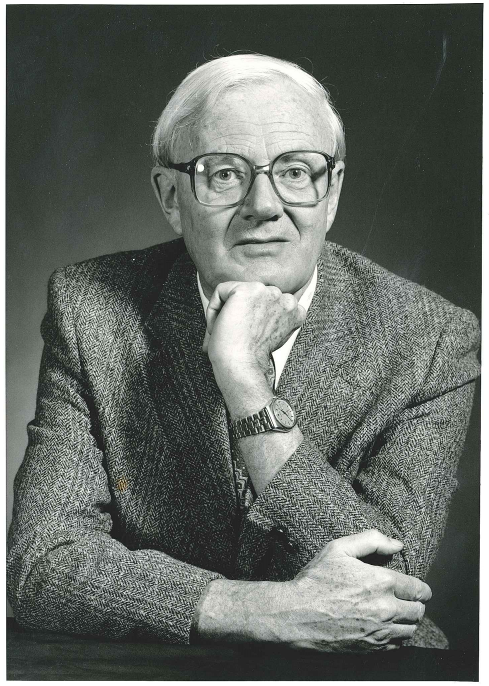

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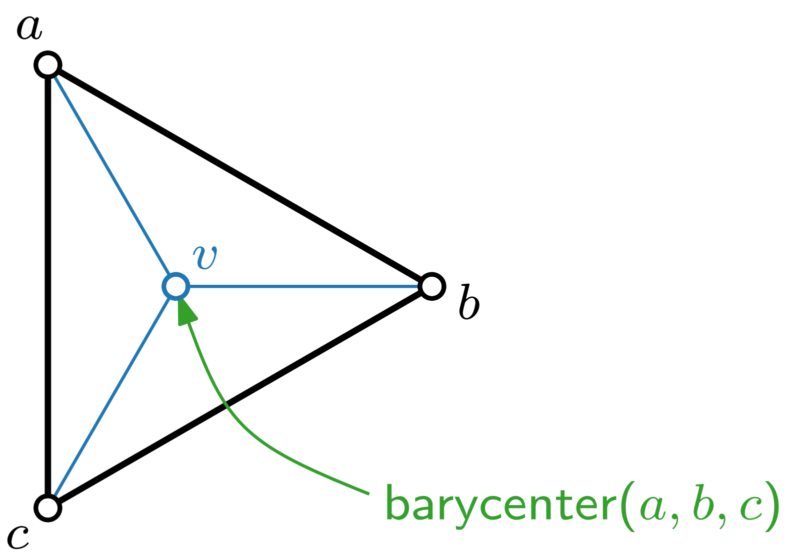

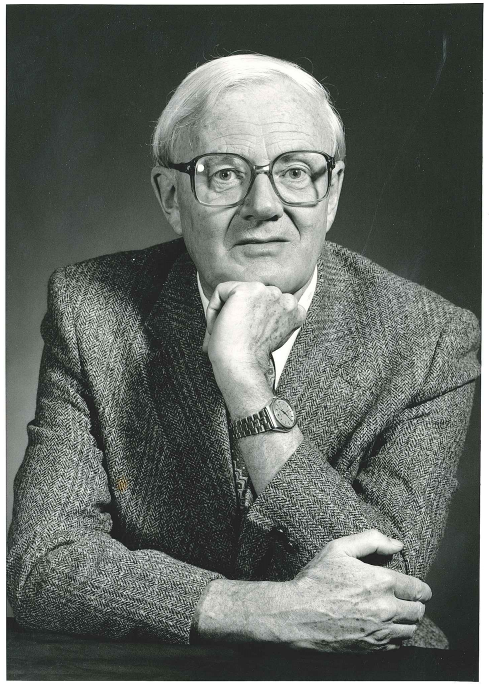

William T. Tutte
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Idea

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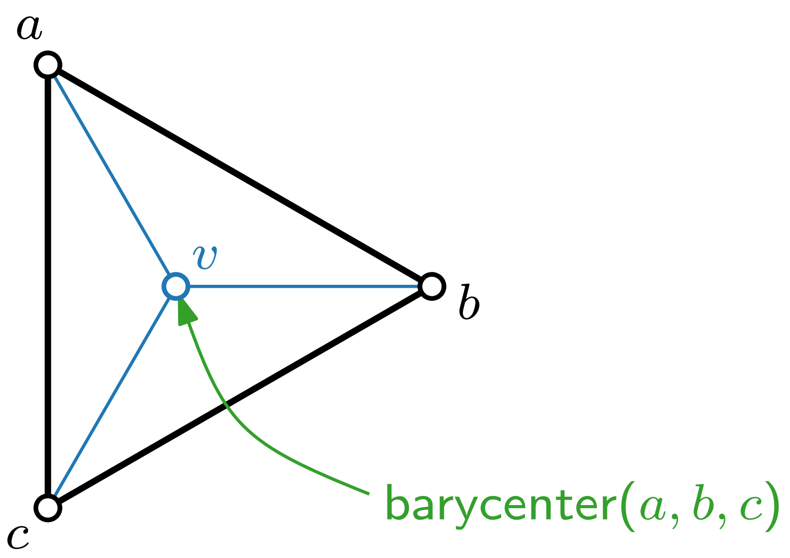
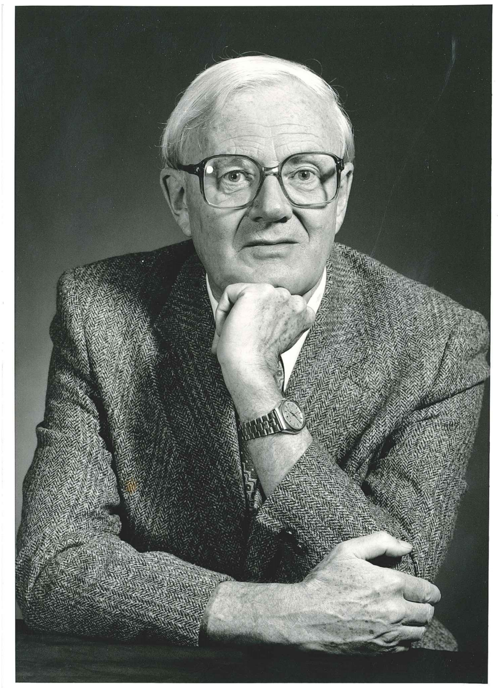




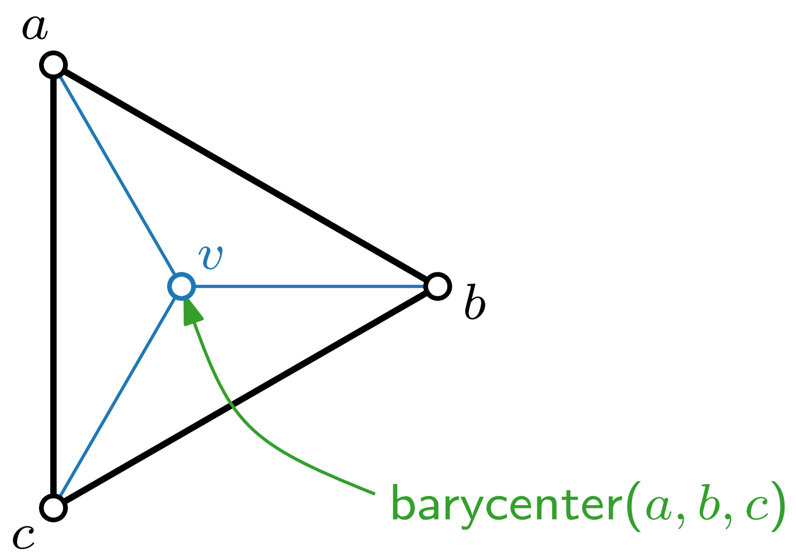
William T. Tutte
1917 – 2002

barycenter$(x_1, \ldots, x_k) =$?

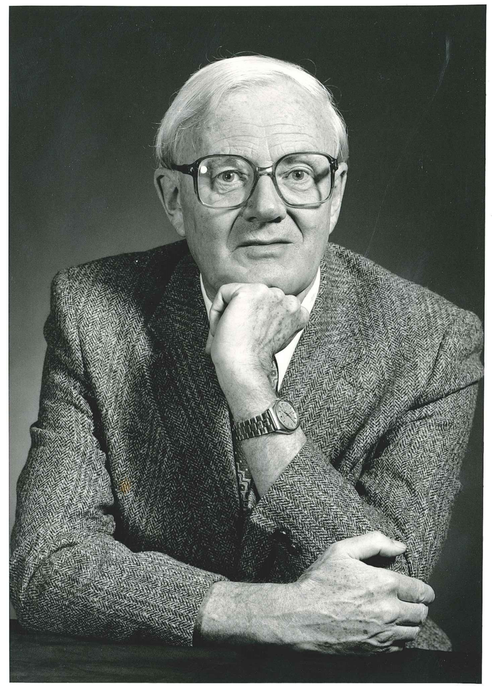
Idea

Consider a fixed triangle (a, b, c) with a common neighbor v

Where would you place v?



barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$



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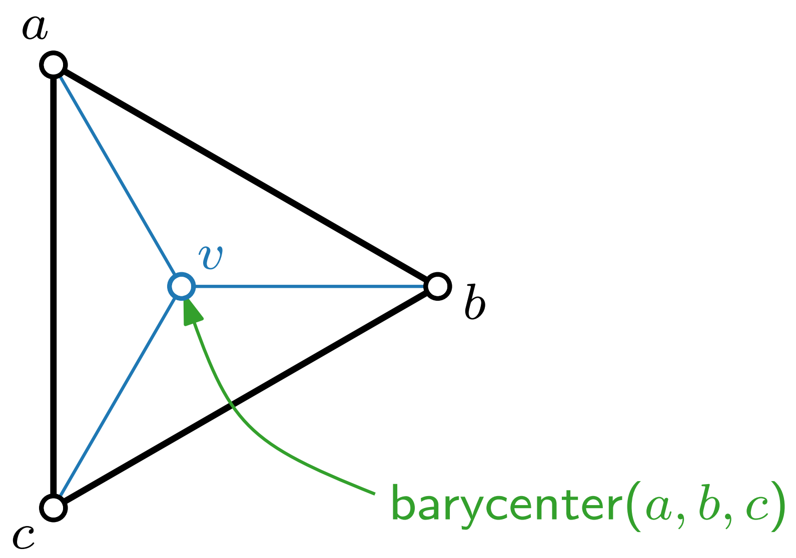

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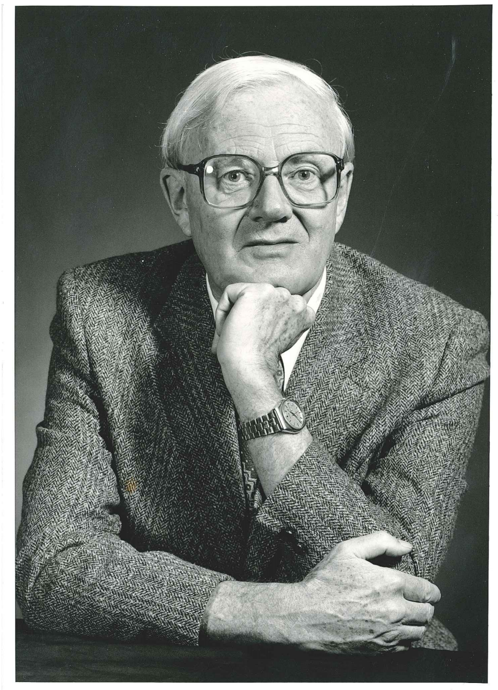

William T. Tutte
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Idea.
Repeatedly place every vertex at barycenter of neighbors.

Tutte's Forces

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
  while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
    foreach $u \in V(G)$ **do**
      $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\mathsf{rep}}(p_u, p_v) + \sum_{v \in \mathsf{Adj}[u]} f_{\mathsf{attr}}(p_u, p_v)$
    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

Tutte's Forces

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  foreach $u \in V(G)$ **do**

    $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot F_u(t)$

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return p

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$
$= \sum_{v \in \text{Adj}[u]} p_v /$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

```
t ← 1
while t ≤ K and max_{v∈V(G)} ‖F_v(t − 1)‖ > ε do
    foreach u ∈ V(G) do
        F_u(t) ← Σ_{v∈V(G)} f_rep(p_u, p_v) + Σ_{v∈Adj[u]} f_attr(p_u, p_v)
    foreach u ∈ V(G) do
        p_u ← p_u + ~~δ(t)~~ 1 · F_u(t)
    t ← t + 1
return p
```

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \textstyle\sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u)$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

- $t \leftarrow 1$
- **while** $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
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$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

Tutte's Forces

Goal.

$p_u = \text{barycenter}(\text{Adj}[u])$
$= \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$\boxed{F_u(t)} = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u$
$= \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u)$
$$= \sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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  foreach $u \in V(G)$ **do**
    $p_u \leftarrow p_u + \cancel{\delta(t)}\, 1 \cdot \boxed{F_u(t)}$
  $t \leftarrow t + 1$
return p

barycenter$(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u])$$
$$= \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u$$
$$= \textstyle\sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u)$$
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- $t \leftarrow 1$
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$$\text{barycenter}(x_1, \ldots, x_k) = \textstyle\sum_{i=1}^{k} x_i / k$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

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Tutte's Forces

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- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
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$\text{barycenter}(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i / k$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u)/\deg(u) = \boxed{\displaystyle\sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)
  $t \leftarrow 1$
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    foreach $u \in V(G)$ **do**
      $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

barycenter$(x_1, \dots, x_k) = \sum_{i=1}^{k} x_i / k$

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$\boxed{F_u(t)} = \sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \boxed{\sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**

  foreach $u \in V(G)$ **do**

    $\boxed{F_u(t)} \leftarrow \boxed{\sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v)} + \boxed{\sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)}$

  foreach $u \in V(G)$ **do**

    $p_u \leftarrow p_u + \cancel{\delta(t)}\ 1 \cdot \boxed{F_u(t)}$

  $t \leftarrow t + 1$

return p

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

Global minimum: $p_u = (0, 0)\ \forall u \in V(G)$

- **Repulsive forces** $\boxed{f_{\text{rep}}(p_u, p_v) = 0}$
- **Attractive forces**

$$\boxed{f_{\text{attr}}(p_u, p_v) = \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}}$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

Tutte's Forces

Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

$$F_u(t) = \textstyle\sum_{v \in \text{Adj}[u]} p_v / \deg(u) - p_u = \sum_{v \in \text{Adj}[u]} (p_v - p_u) / \deg(u) = \displaystyle\sum_{v \in \text{Adj}[u]} \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

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    - $F_u(t) \leftarrow \sum_{v \in V(G)} f_{\text{rep}}(p_u, p_v) + \sum_{v \in \text{Adj}[u]} f_{\text{attr}}(p_u, p_v)$
  - **foreach** $u \in V(G)$ **do**
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$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$

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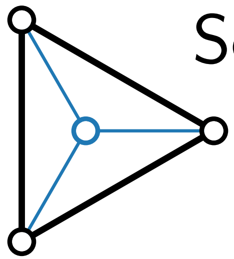

Solution: fix coordinates of outer face!

- **Repulsive forces** $f_{\text{rep}}(p_u, p_v) = 0$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v}$$

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

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Goal.

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

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- **Repulsive forces** $\boxed{f_{\text{rep}}(p_u, p_v) = 0}$
- **Attractive forces**

$$f_{\text{attr}}(p_u, p_v) = \begin{cases} 0 & \text{if } u \text{ fixed,} \\ \frac{\|p_u - p_v\|}{\deg(u)} \overrightarrow{p_u p_v} & \text{otherwise.} \end{cases}$$

ForceDirected(graph G, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

> $t \leftarrow 1$
> **while** $t \leq K$ **and** $\max_{v \in V(G)} \|F_v(t-1)\| > \varepsilon$ **do**
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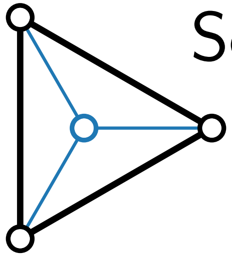

Solution: fix coordinates of outer face! ☺

$\overrightarrow{p_u p_v}$ = unit vector pointing from u to v

$\|p_u - p_v\|$ = Euclidean distance between u and v

System of Linear Equations

Goal.

$p_u = \mathsf{barycenter}(\mathsf{Adj}[u]) \ = \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u =$ barycenter(Adj$[u]$) $= \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) \; = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u)$
$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u)$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

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$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v$$
$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$
$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

System of Linear Equations

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$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$$Ax = b$$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0}$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

$Ax = b \quad Ay = b$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u =$ barycenter$(\mathsf{Adj}[u])$ $= \sum_{v \in \mathsf{Adj}[u]} p_v / \deg(u)$

$$Ax = b \qquad Ay = b \qquad b = (0)_n$$

Two systems of linear equations:

$$x_u = \sum_{v \in \mathsf{Adj}[u]} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{v \in \mathsf{Adj}[u]} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{v \in \mathsf{Adj}[u]} x_v = 0$$

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System of Linear Equations

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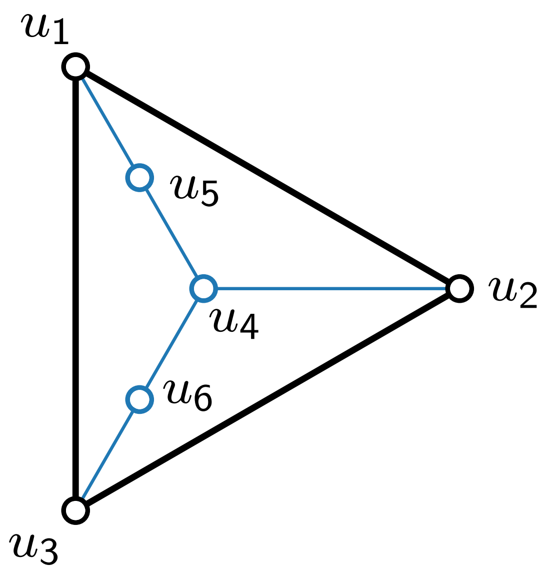

System of Linear Equations

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$$Ax = b \quad Ay = b \quad b = (0)_n$$

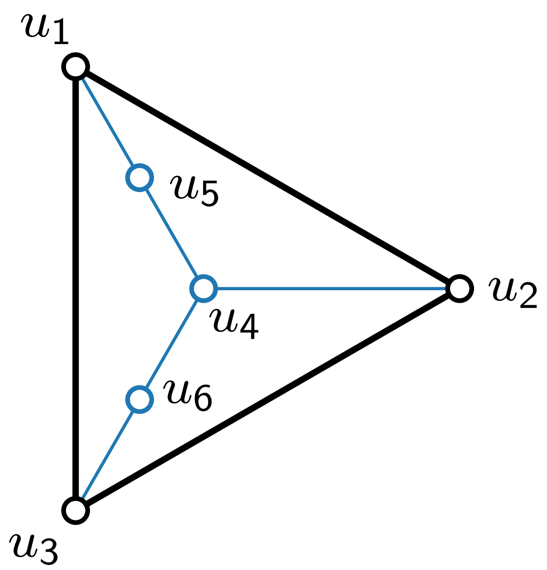
$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \iff \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \iff \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \iff \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \iff \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$

A



$$\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

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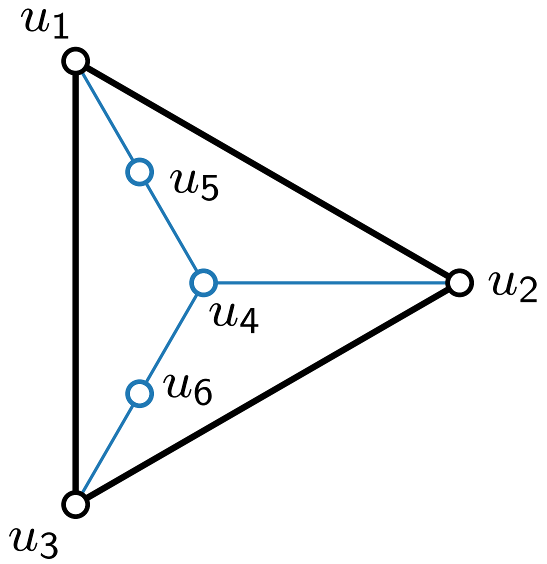
A



$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{pmatrix} \\ \\ \\ \\ \\ \\ \end{pmatrix}$$

System of Linear Equations

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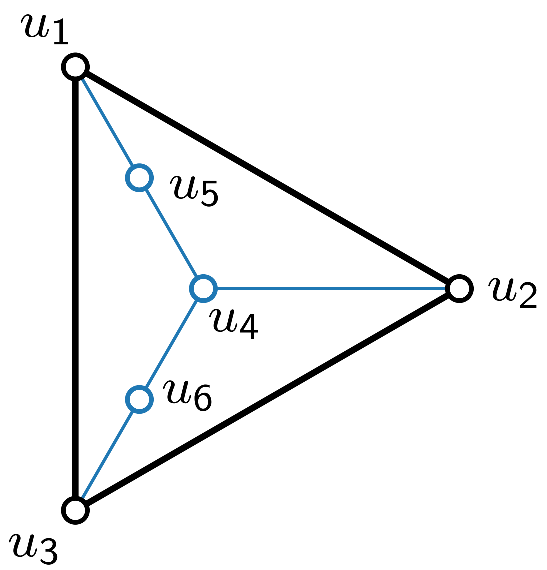

$$A = \begin{array}{c|c} & u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \\ \hline \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

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Two systems of linear equations:

$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0}$

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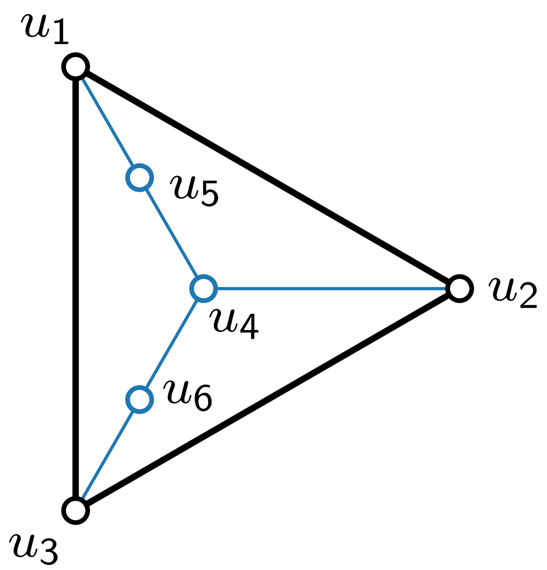

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

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Two systems of linear equations:

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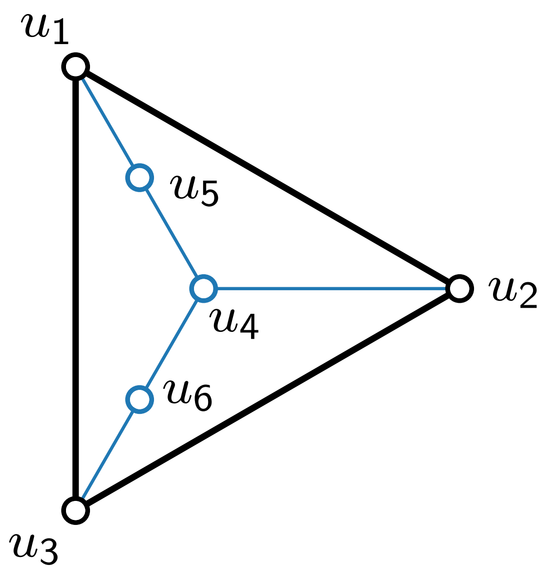
$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$



$$A: \quad \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

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Two systems of linear equations:

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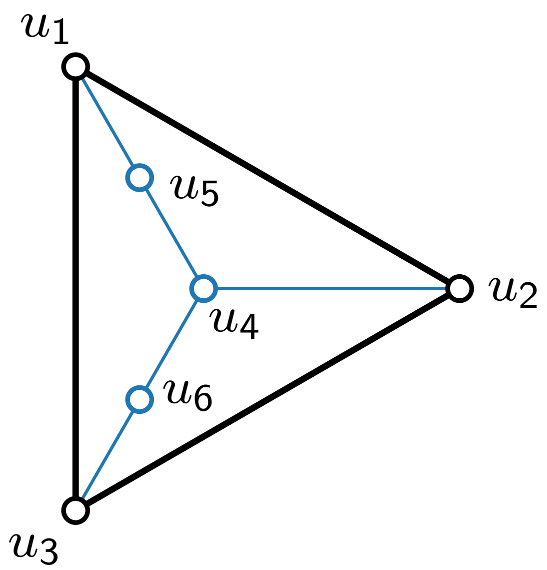
$$y_u = \textstyle\sum_{v \in \mathsf{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \textstyle\sum_{v \in \mathsf{Adj}[u]} y_v \Leftrightarrow \boxed{\deg(u) \cdot y_u - \textstyle\sum_{v \in \mathsf{Adj}[u]} y_v = 0}$$



$$A \quad \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

System of Linear Equations

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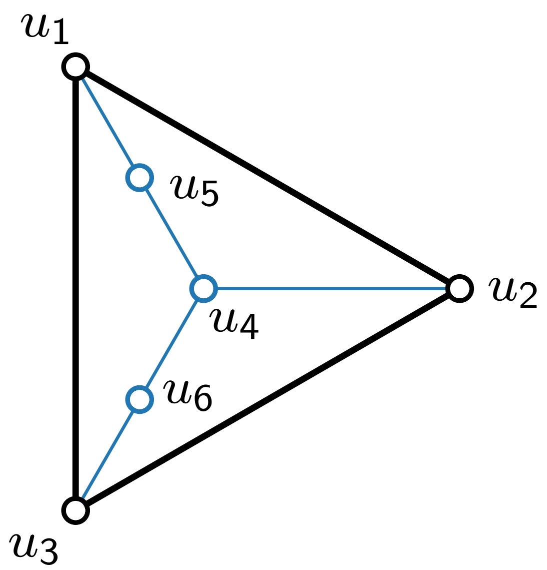

$$A:\quad \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

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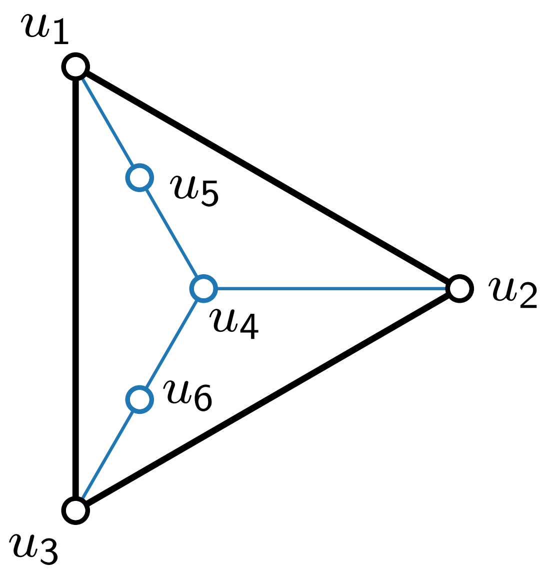
Two systems of linear equations:

$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$

$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$



$$A \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

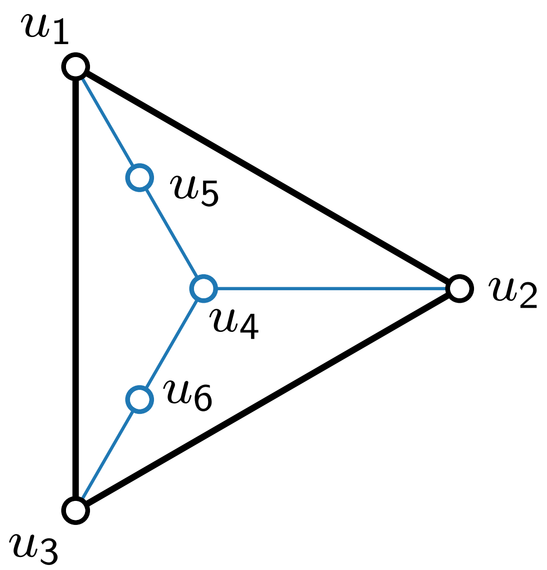
$$Ax = b \quad Ay = b \quad b = (0)_n$$

$$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$$

Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

$$y_u = \sum_{v \in \text{Adj}[u]} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{v \in \text{Adj}[u]} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{v \in \text{Adj}[u]} y_v = 0$$



$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\text{Adj}[u]) = \sum_{v \in \text{Adj}[u]} p_v / \deg(u)$

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Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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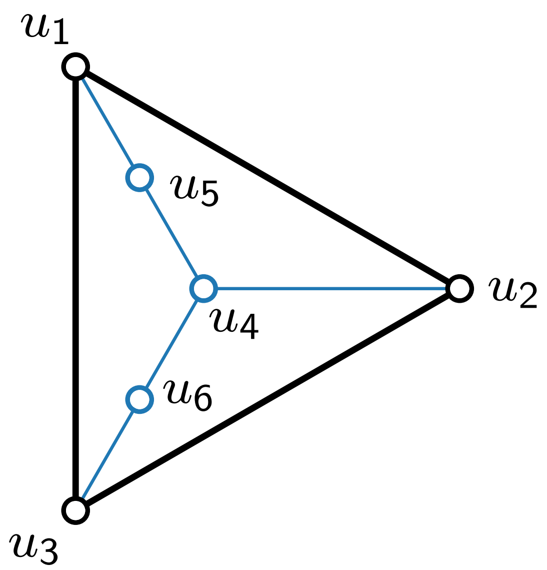

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & 3 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

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Two systems of linear equations:

$$x_u = \sum_{v \in \text{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \text{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \text{Adj}[u]} x_v = 0$$

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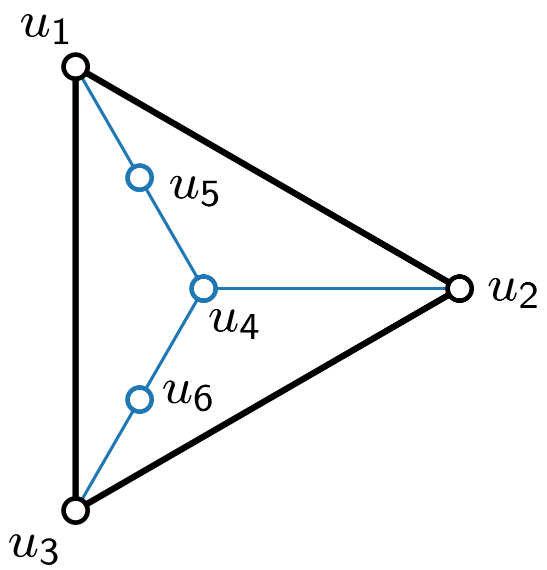

$$A \quad \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

System of Linear Equations

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Two systems of linear equations:

$$x_u = \sum_{v \in \mathrm{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathrm{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \mathrm{Adj}[u]} x_v = 0$$

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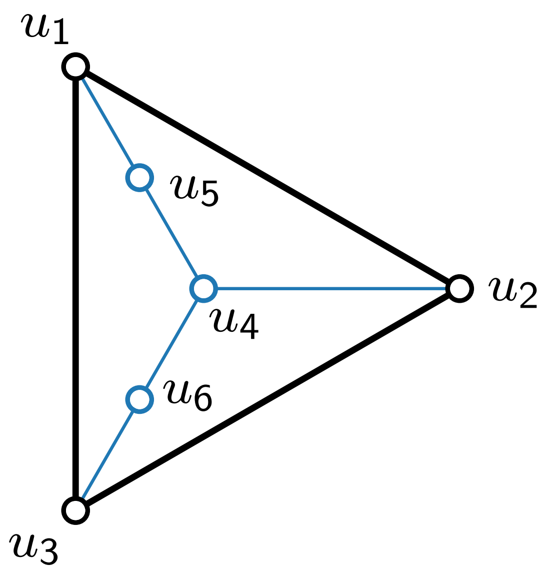

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \multicolumn{6}{c}{\begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}} \end{array}$$

System of Linear Equations

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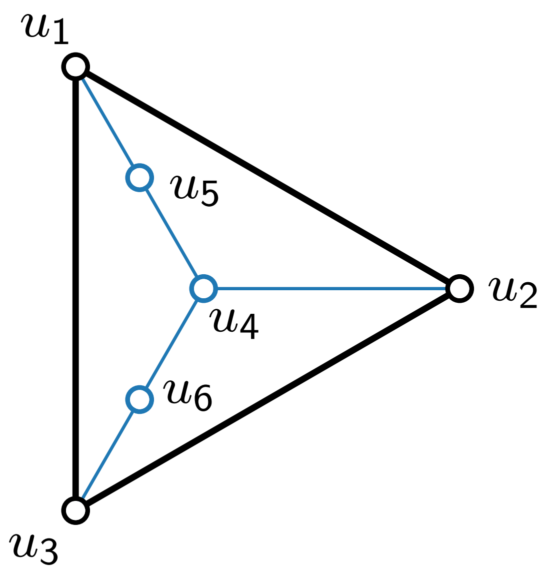

$$A \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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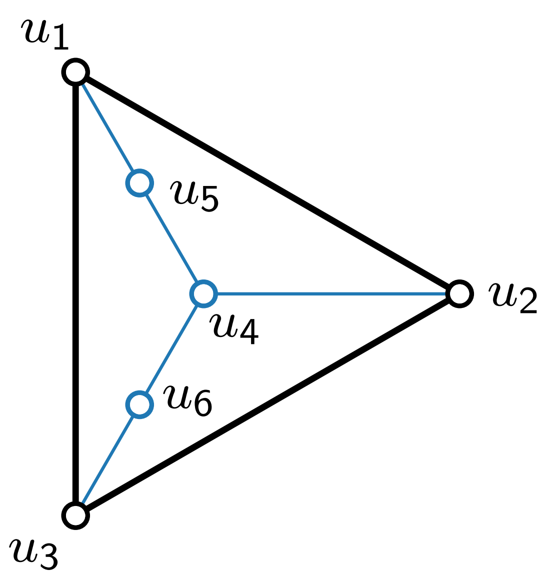

$$A$$

$$\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ & & & & & \end{pmatrix}
\end{array}$$

System of Linear Equations

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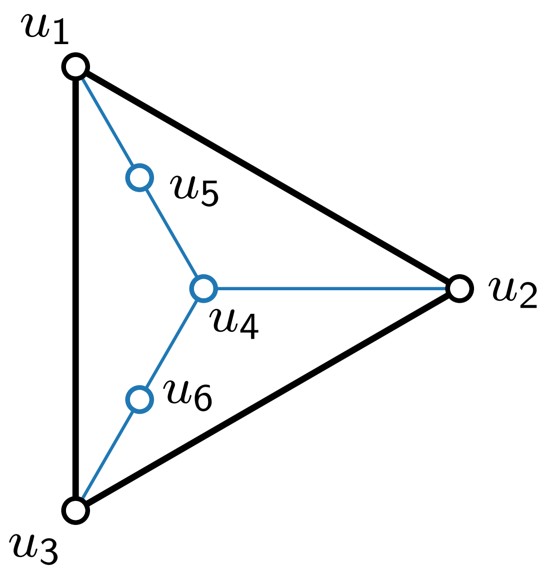

$$A$$

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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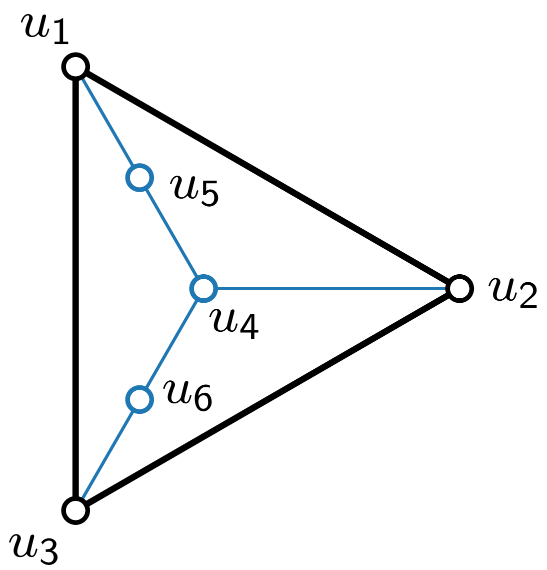

$$A = \begin{array}{c|c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

$A_{ii} = \deg(u_i)$

System of Linear Equations

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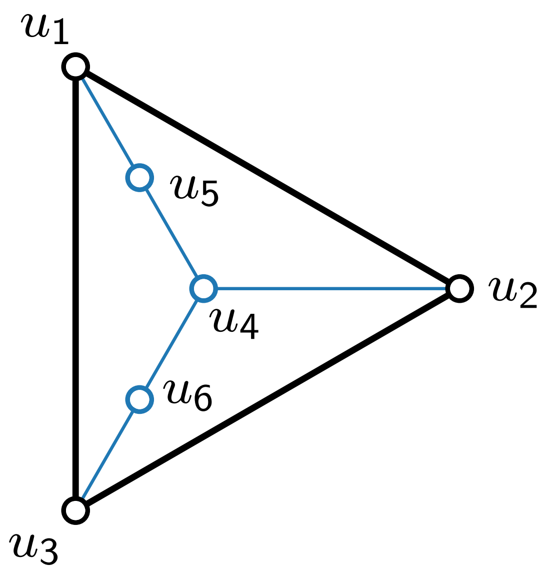

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

System of Linear Equations

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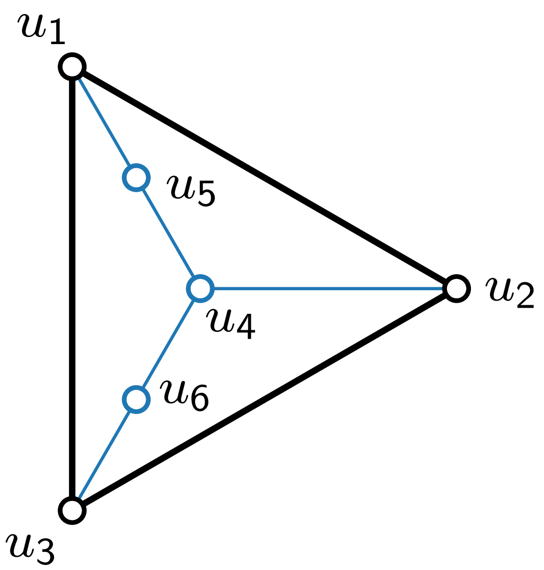

$$A = \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

Laplacian matrix of G

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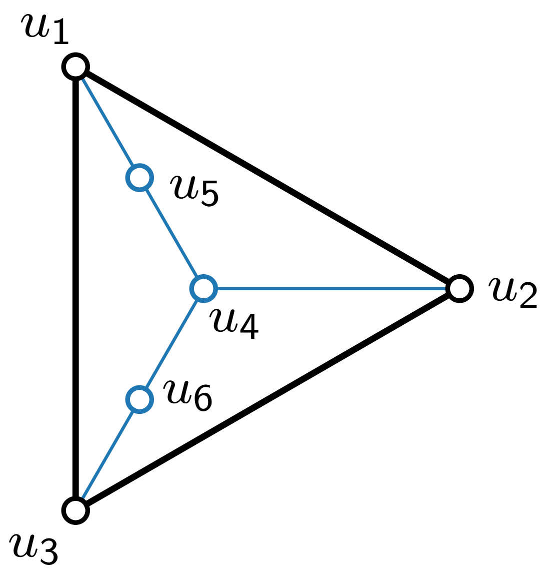

$$A = \begin{array}{c|c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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unique solution

System of Linear Equations

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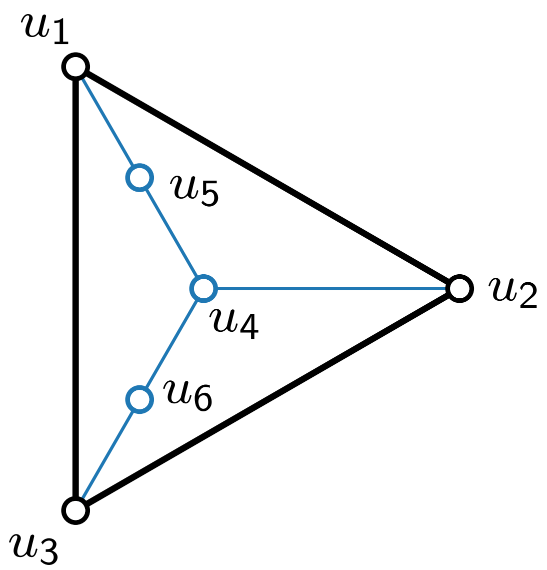

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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variables, constraints, $\det(A) =$

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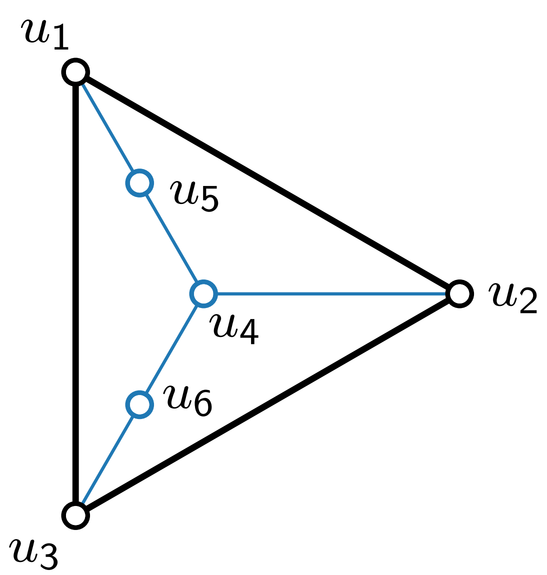

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} \; A$$

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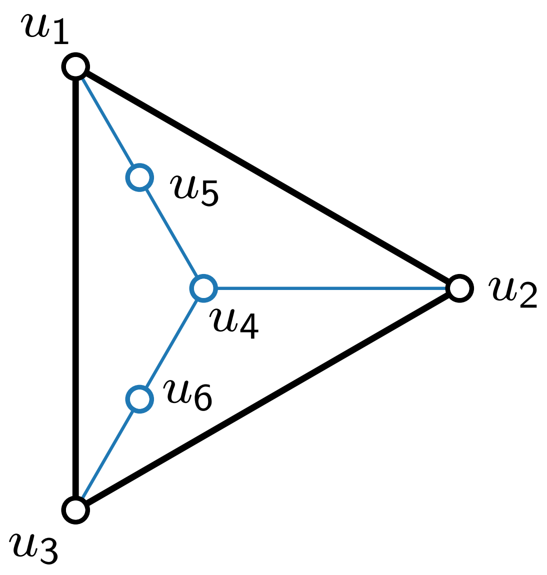

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Laplacian matrix of G

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n variables, n constraints, $\det(A) =$

unique solution

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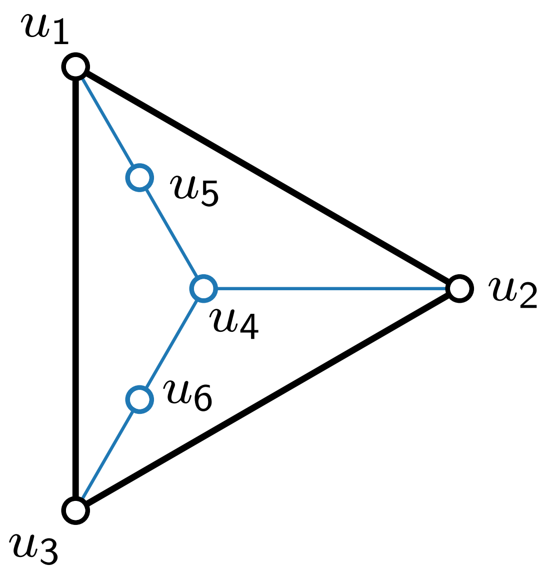

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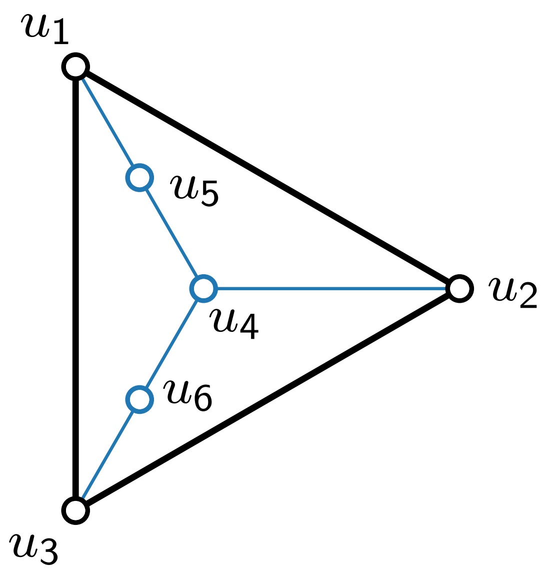

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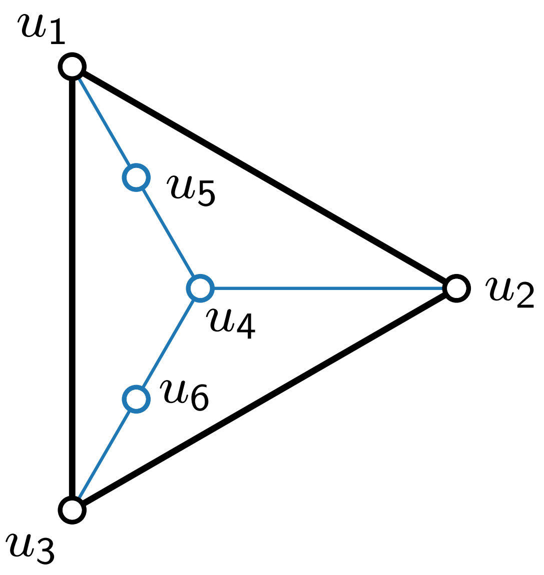

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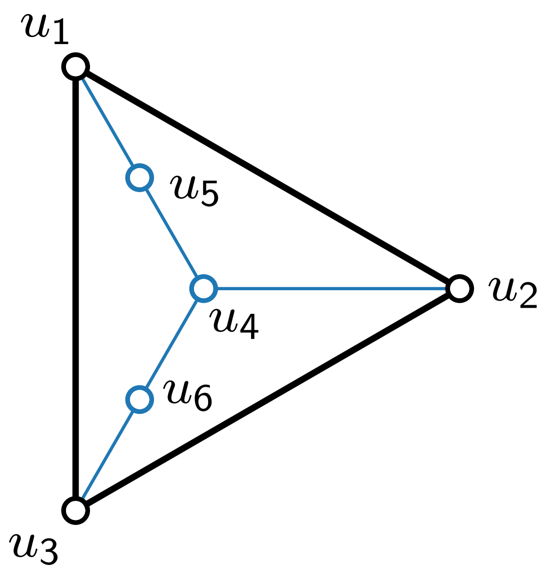

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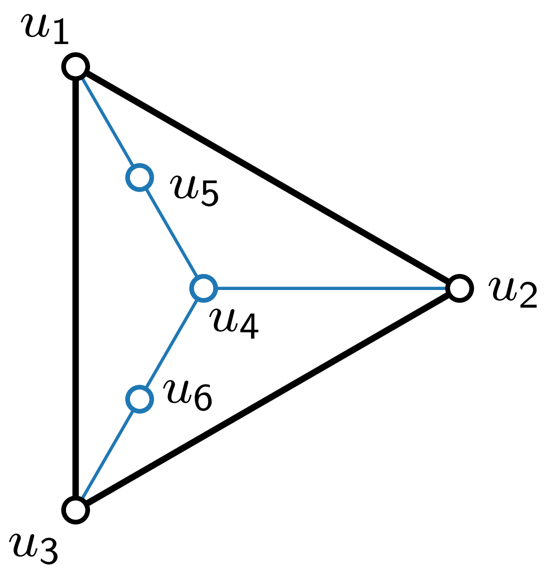

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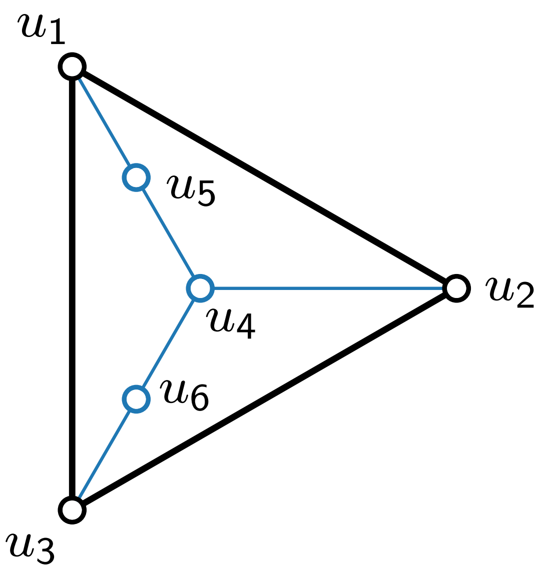

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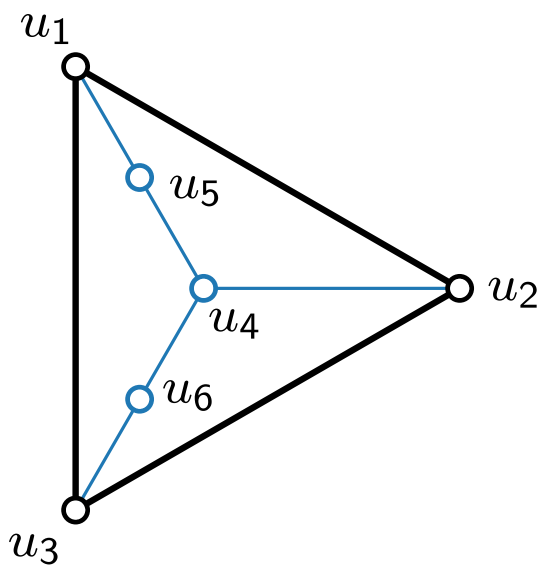

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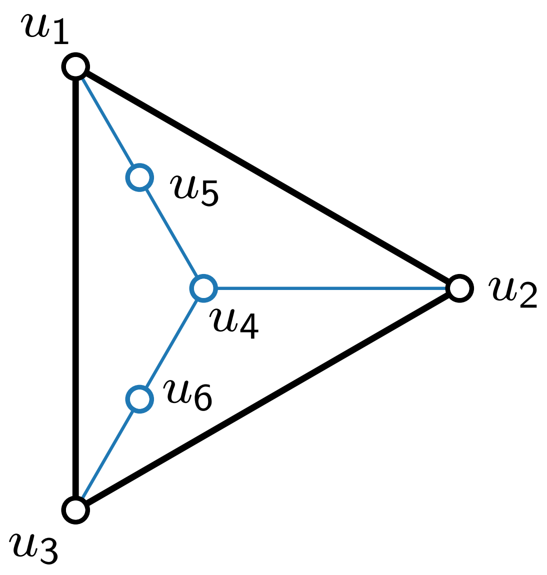

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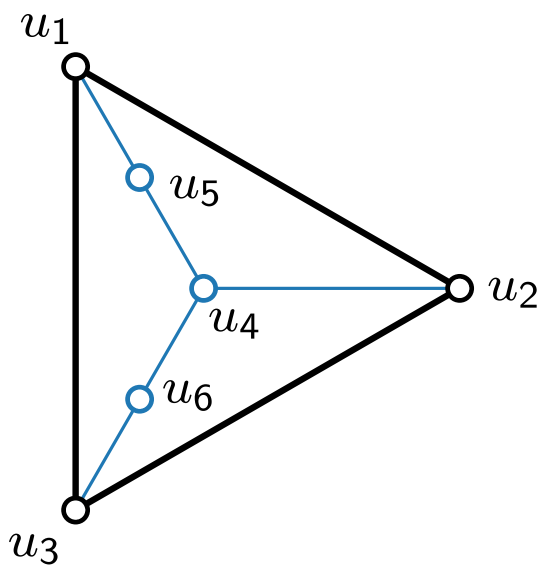

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Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

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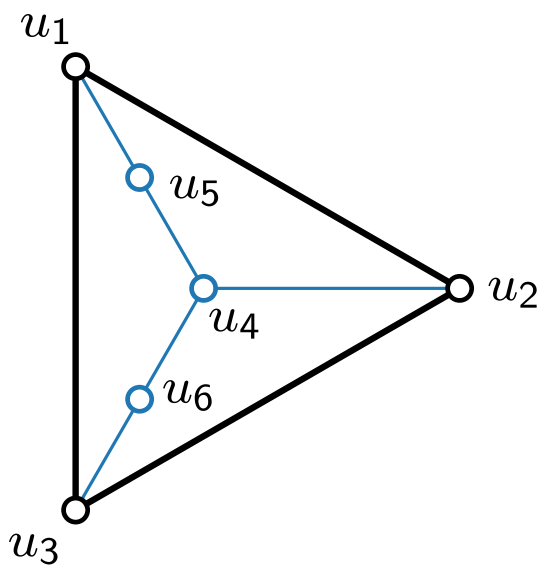

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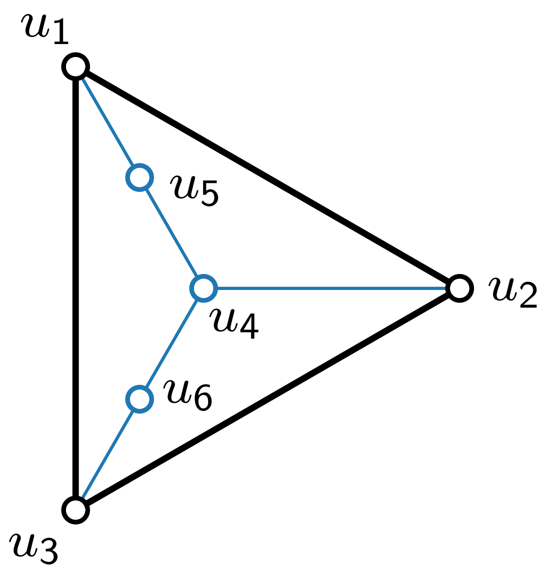

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$A_{ii} = \deg(u_i)$

$$A_{ij, i \neq j} = \begin{cases} -1 & u_i u_j \in E \\ 0 & u_i u_j \notin E \end{cases}$$

k variables, k constraints, $\det(A) > 0$

$k = \#$free vertices

$\Rightarrow$ unique solution ☺

Solution:
1. No need to change fixed vertices.
2. Constraints that depend on fixed vertices are constant and can be moved into b.

System of Linear Equations

solve two systems of linear equations

Theorem. **= Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$

$p_u =$ barycenter(Adj[u]) $=$

$$x_u = \sum_{v \in \mathrm{Adj}[u]} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{v \in \mathrm{Adj}[u]} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{v \in \mathrm{Adj}[u]} x_v = 0$$

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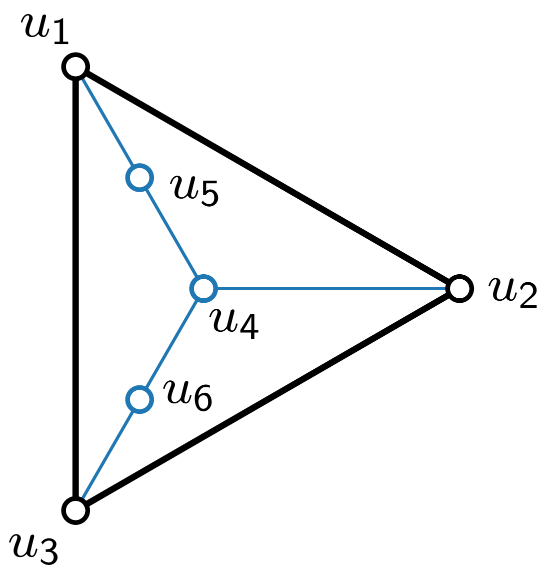

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & 0 & 0 & -1 & -1 & 0 & 2 \end{array}$$

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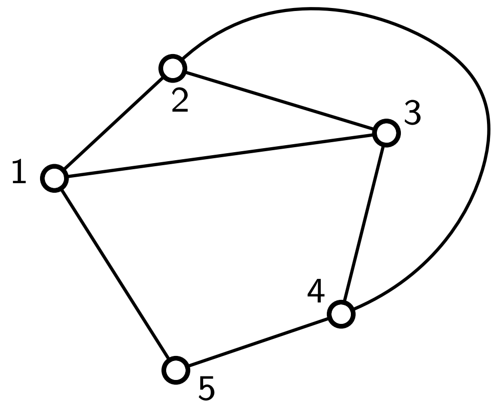
G **connected**: $\exists\, u$–v path for every vertex pair $\{u, v\}$.



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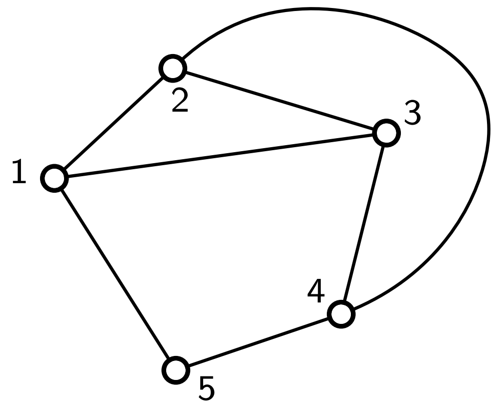

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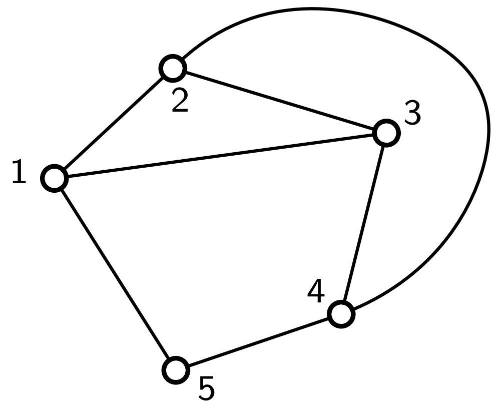

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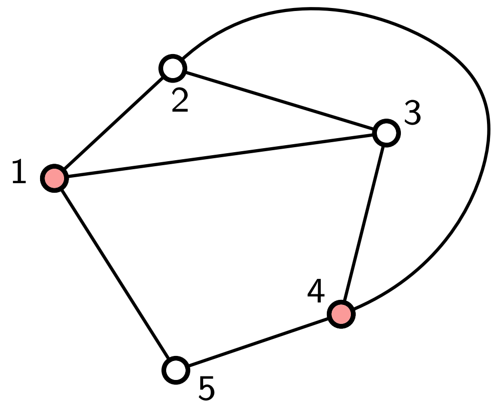

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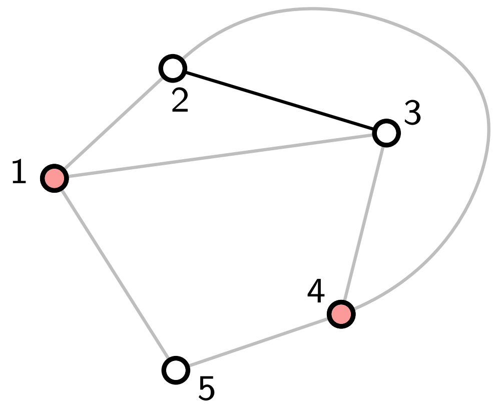

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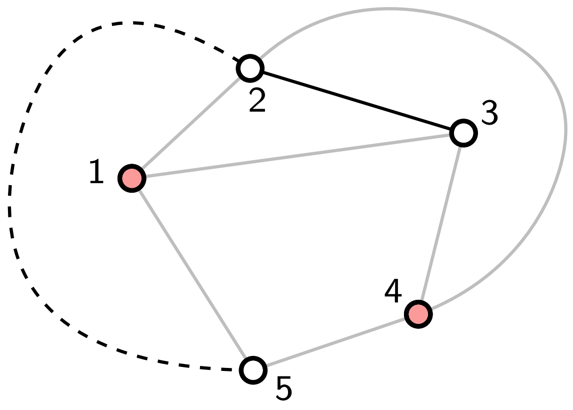

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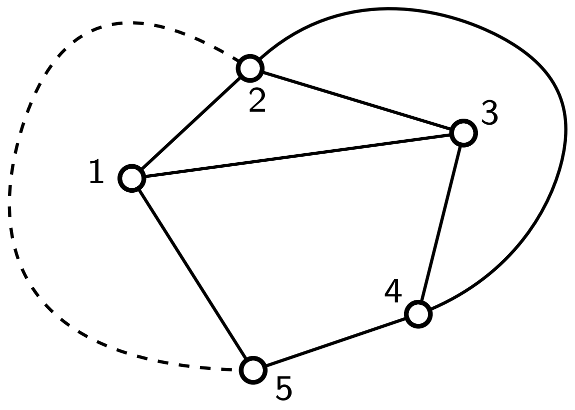

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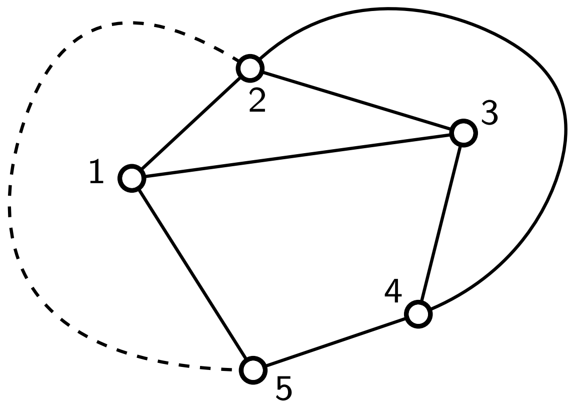

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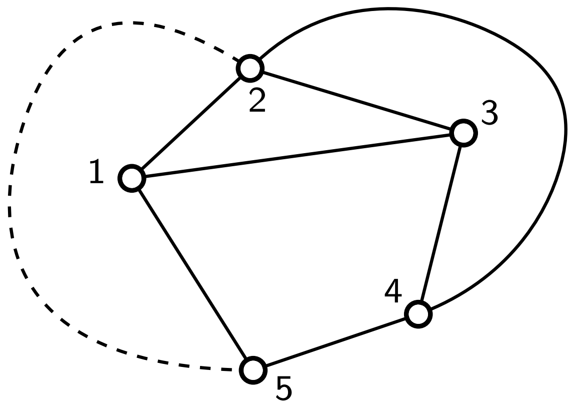

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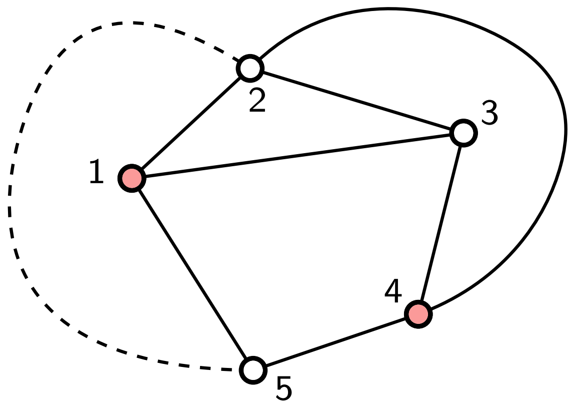

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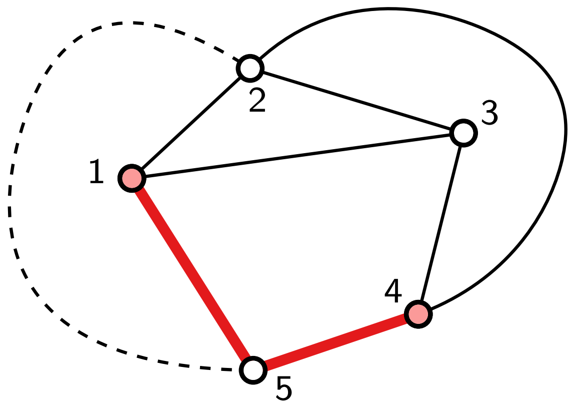

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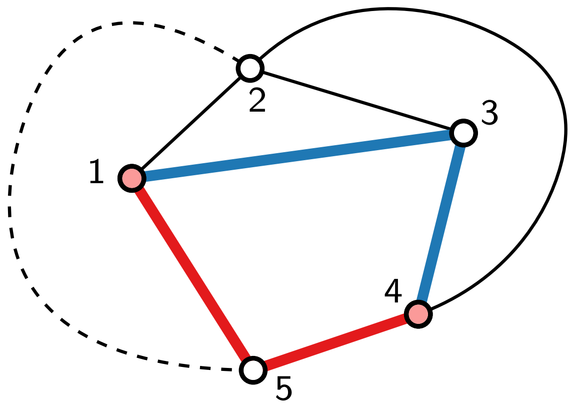

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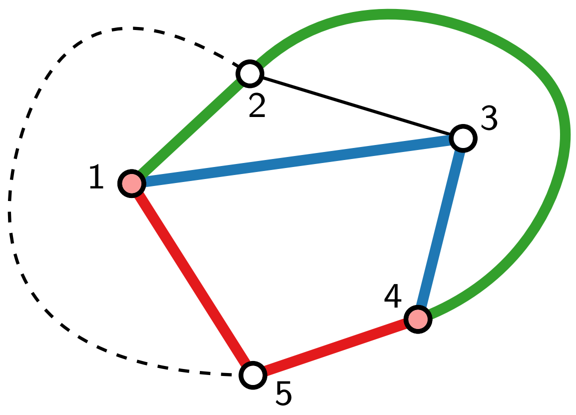

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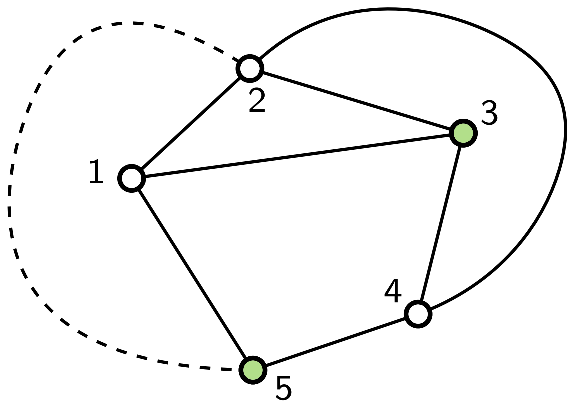

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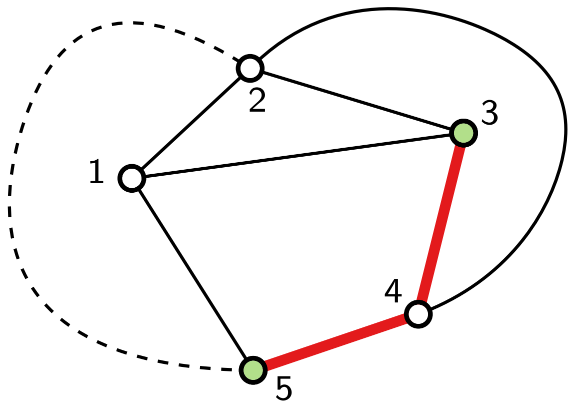

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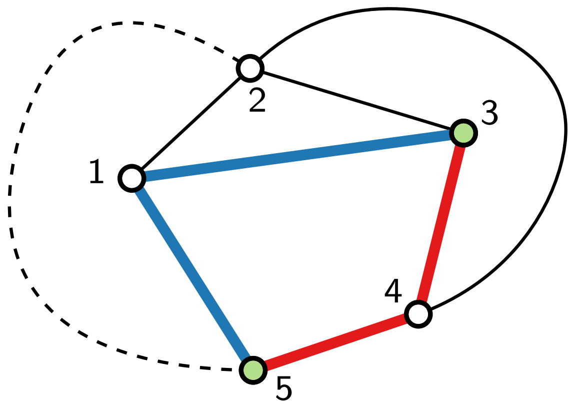

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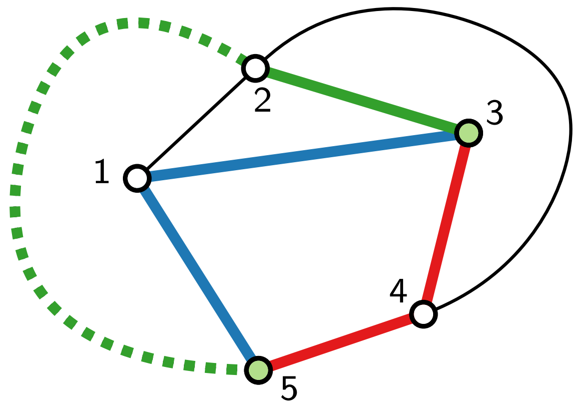

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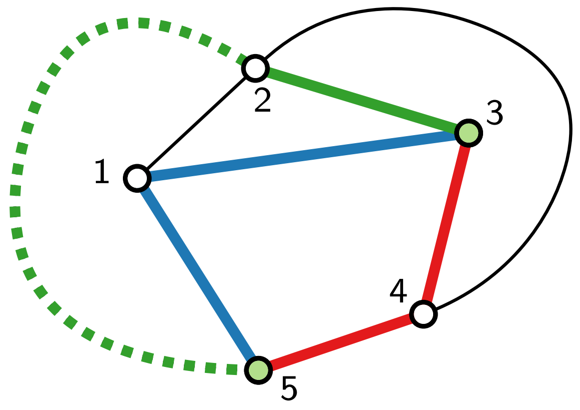

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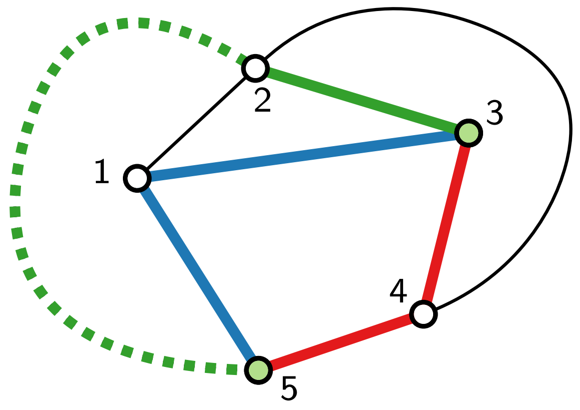

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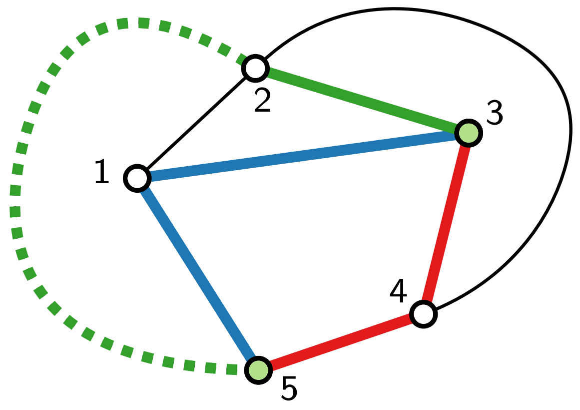

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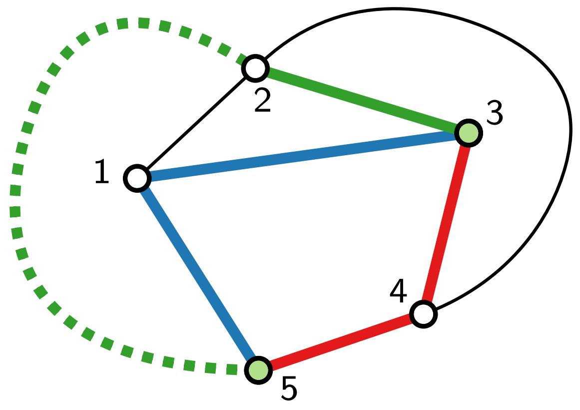

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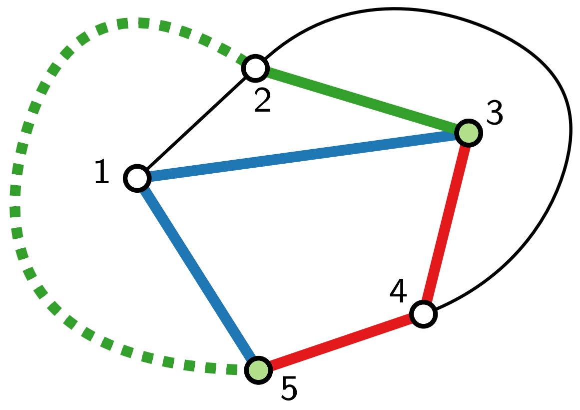

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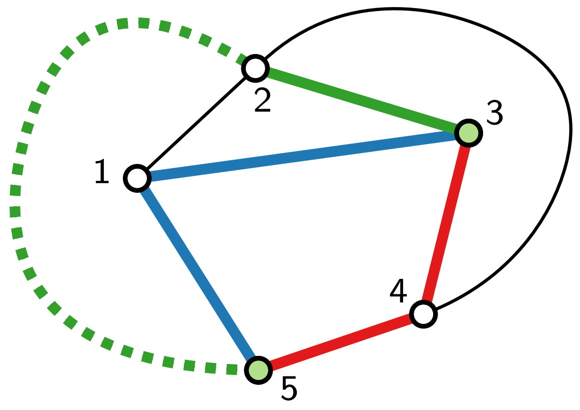

Γ_1

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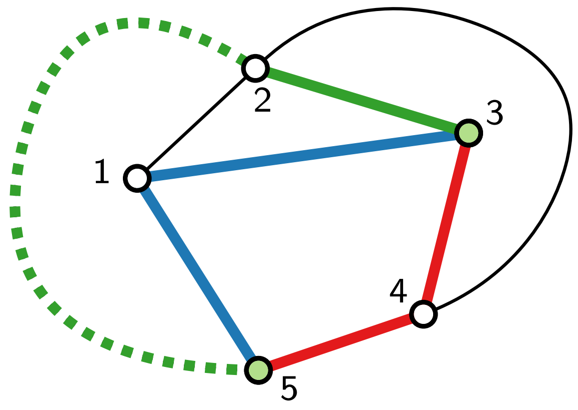

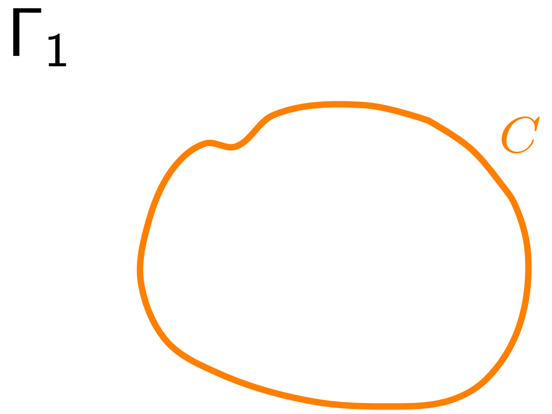

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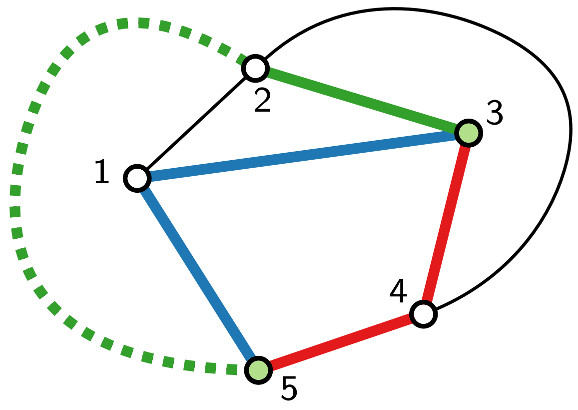

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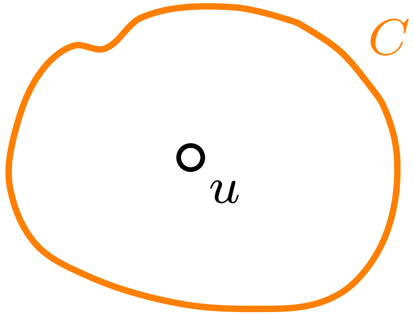

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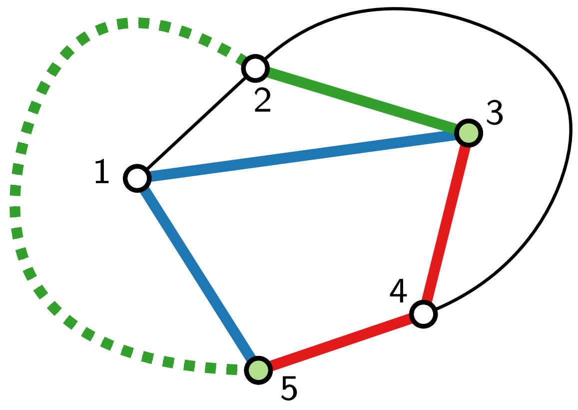

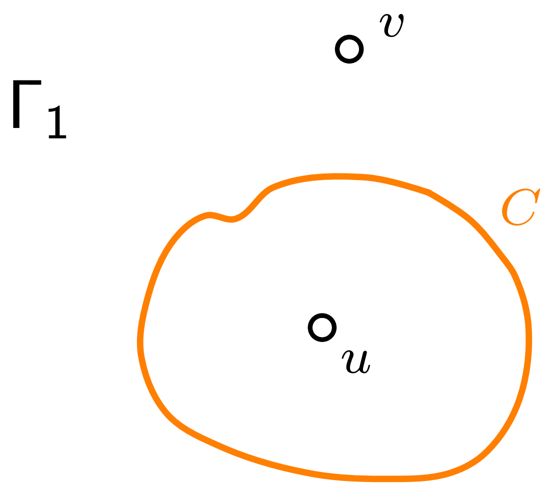

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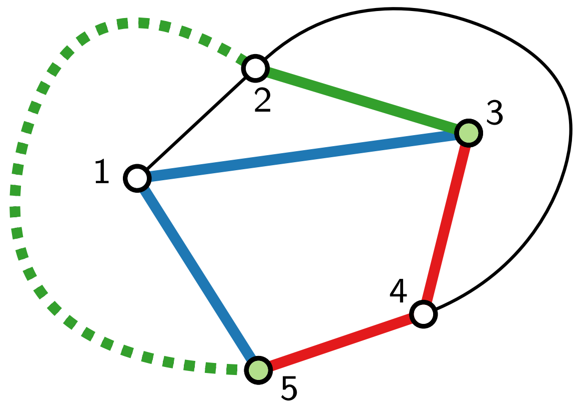

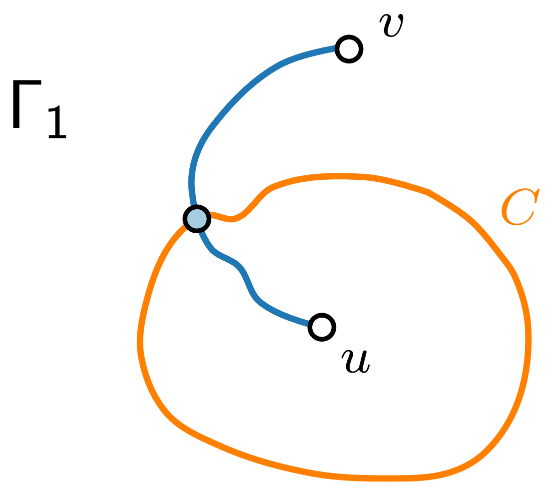

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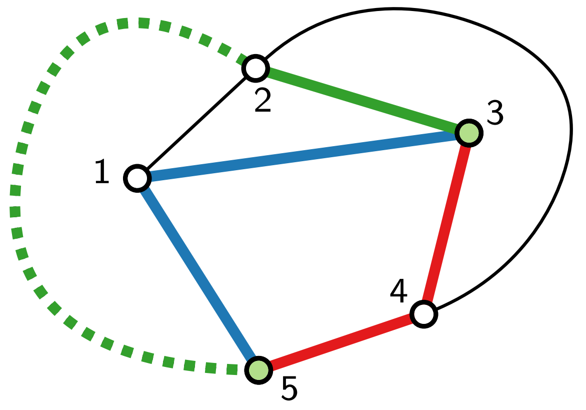

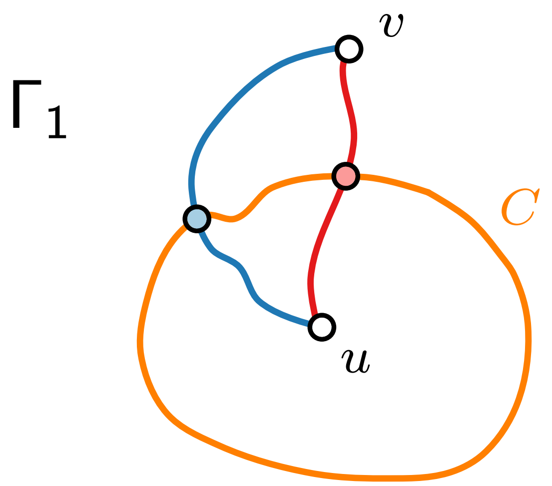

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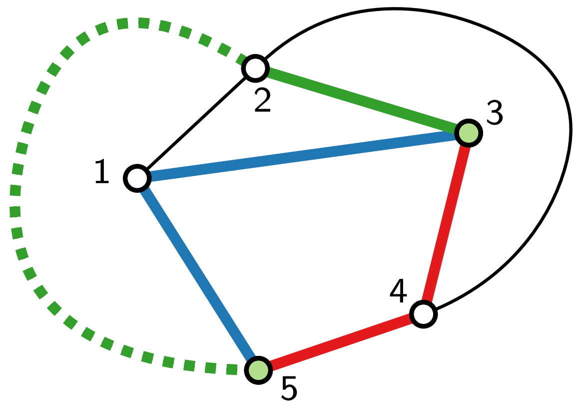

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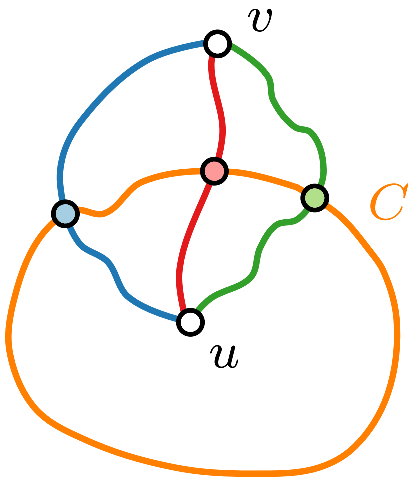

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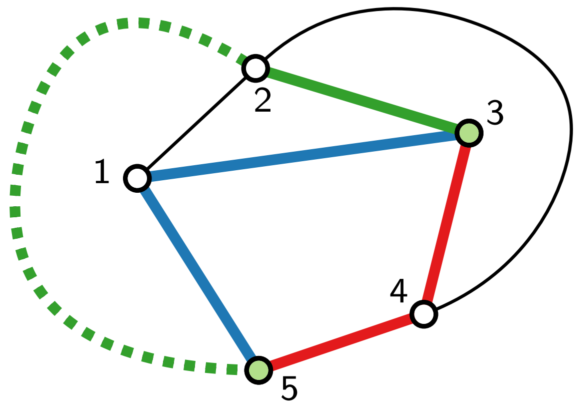

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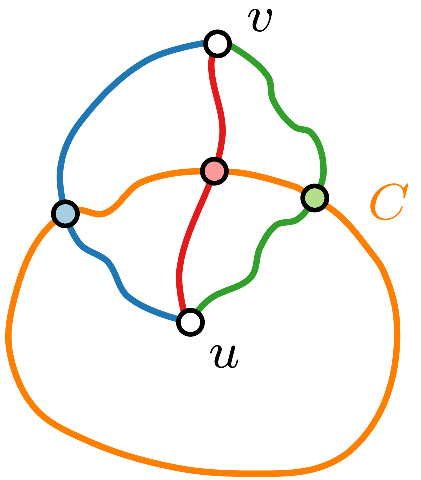

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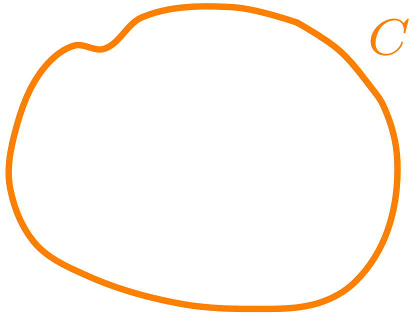

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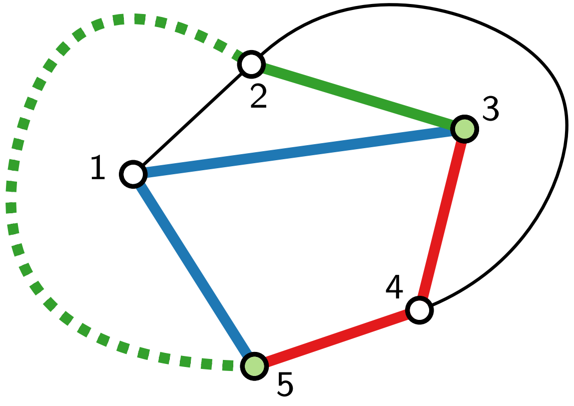

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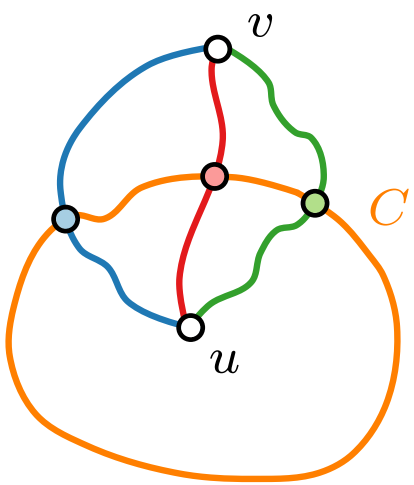

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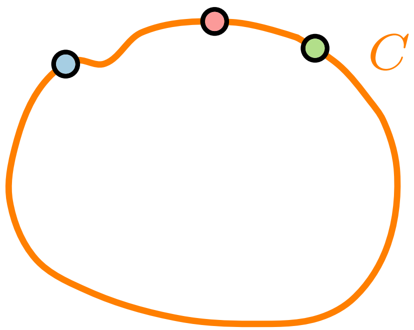

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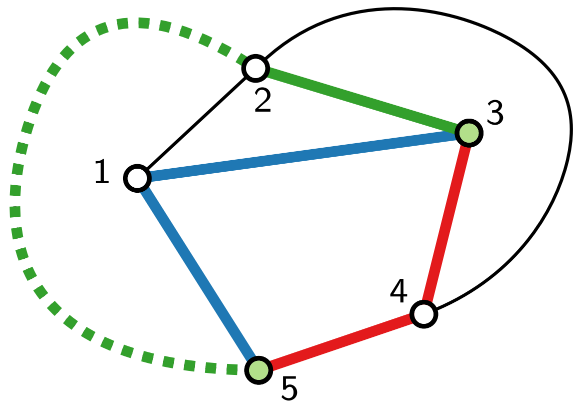

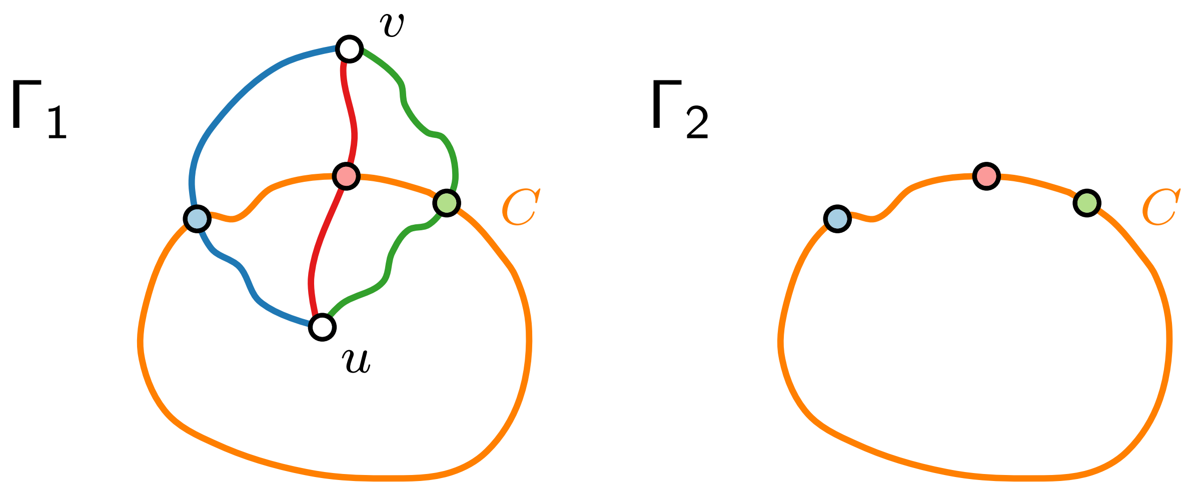

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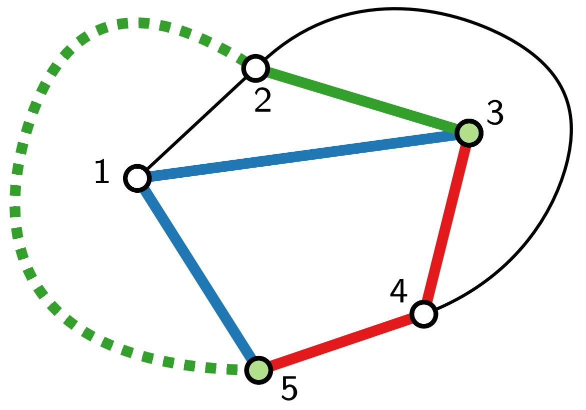

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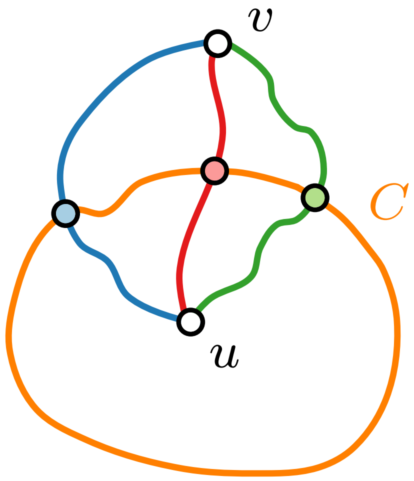
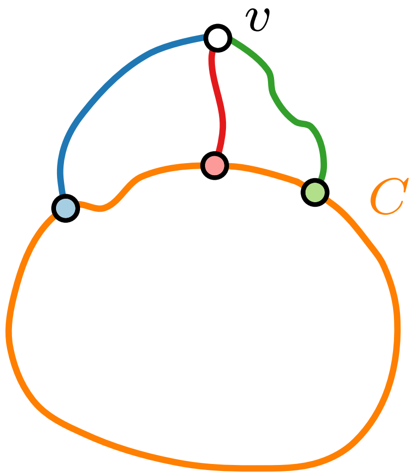


Γ_2



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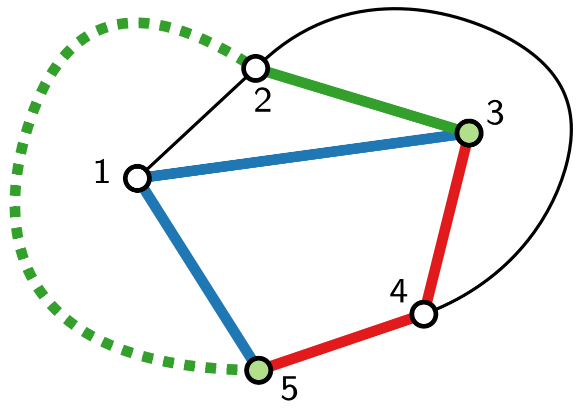

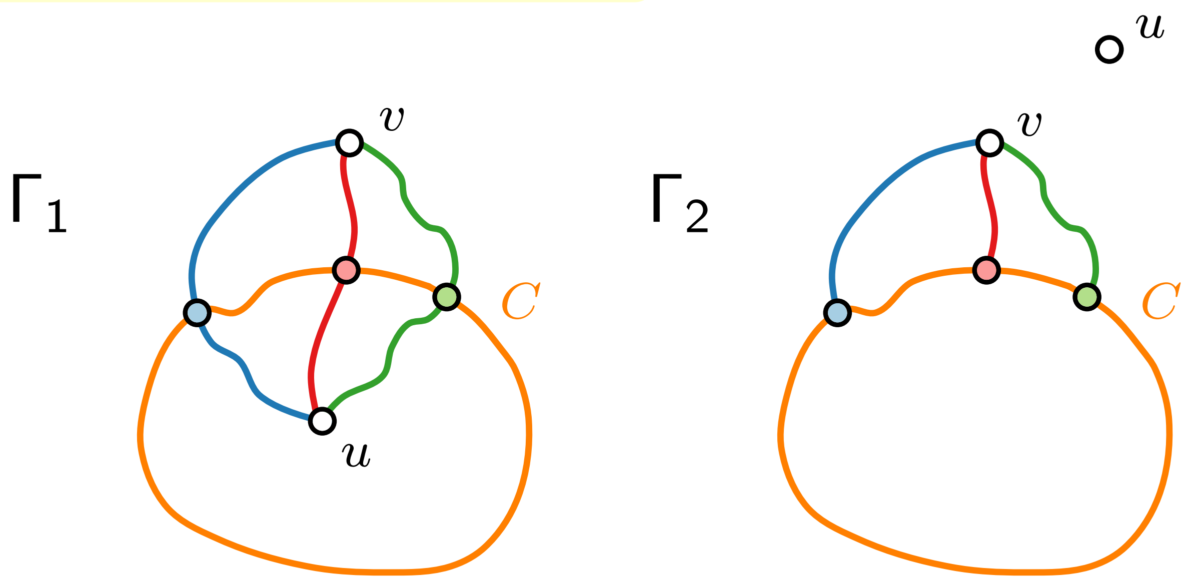

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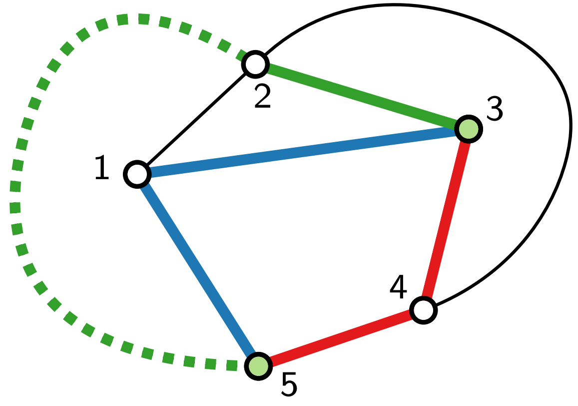

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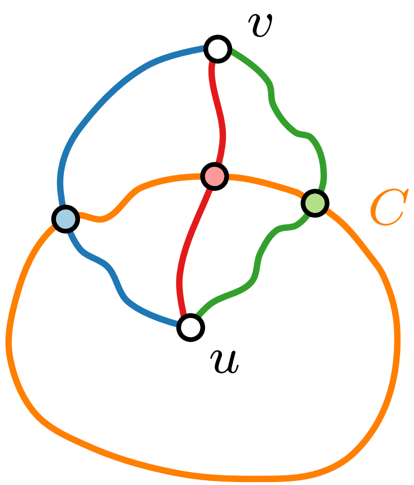

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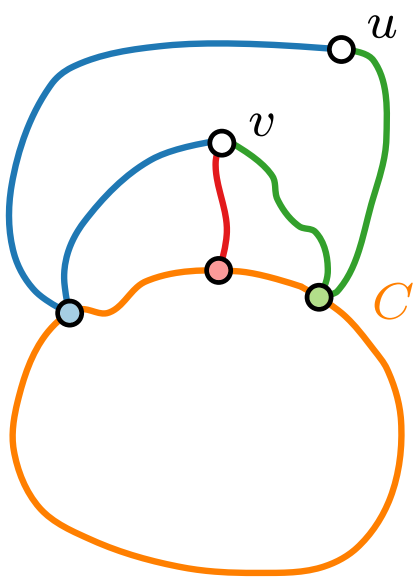

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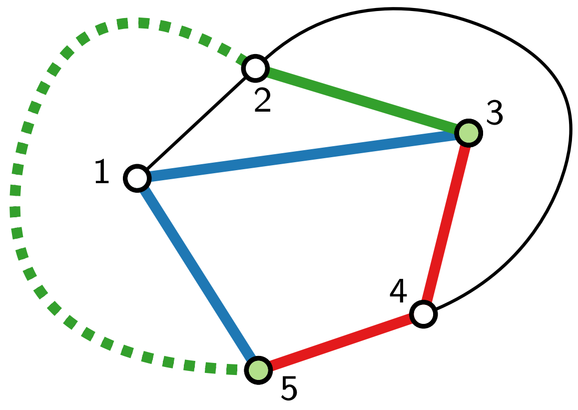

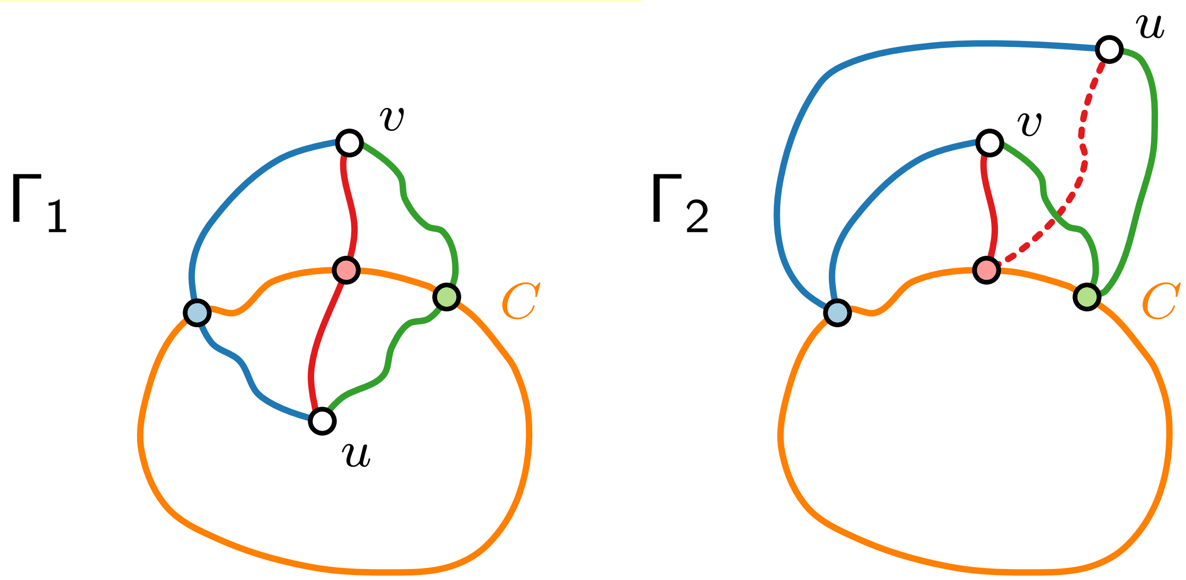

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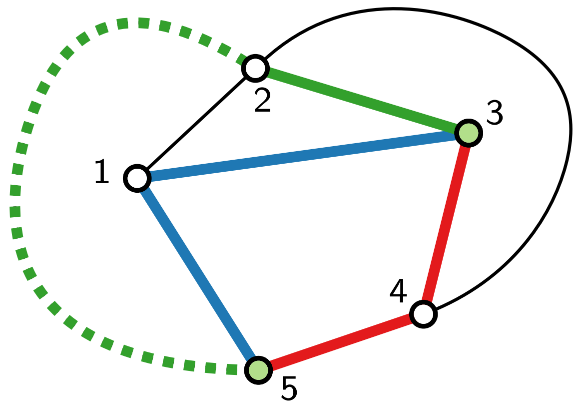

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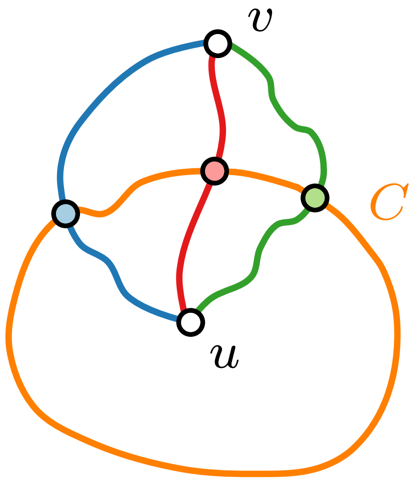

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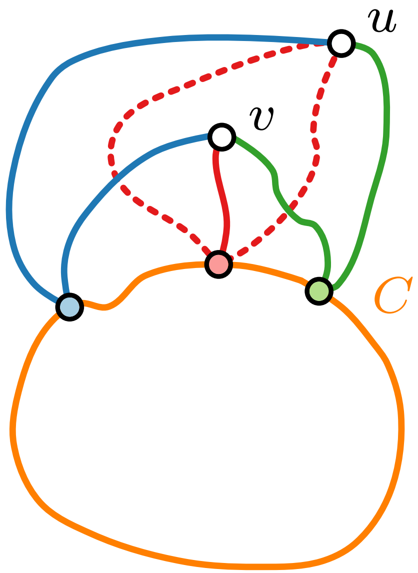

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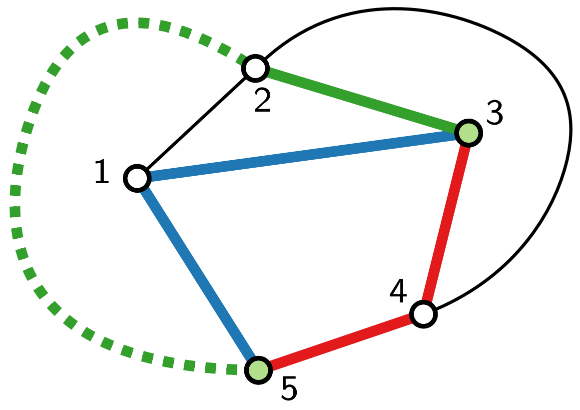

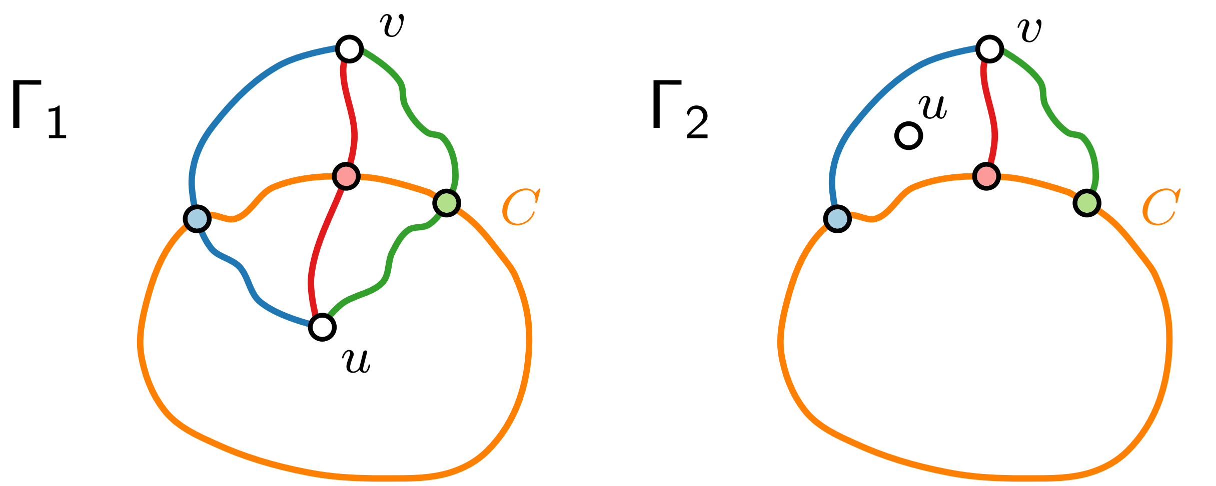

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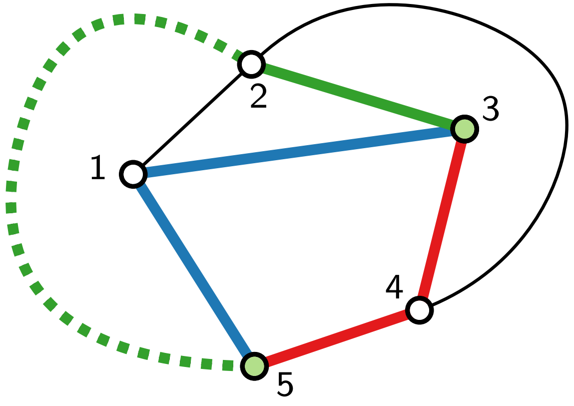

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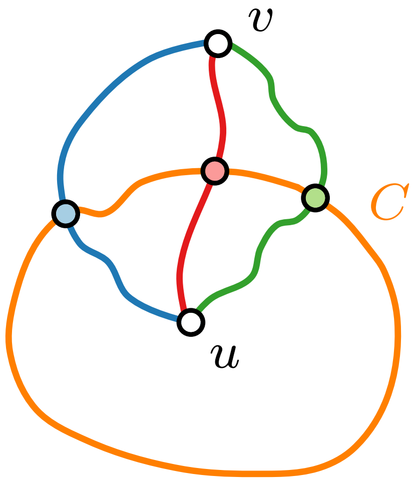

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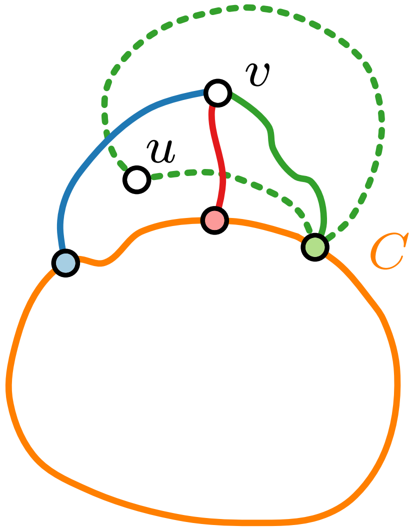

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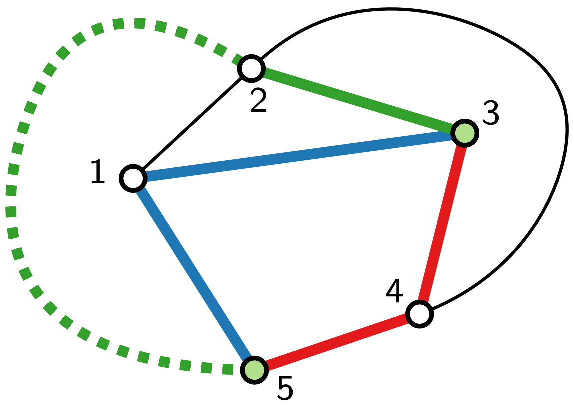
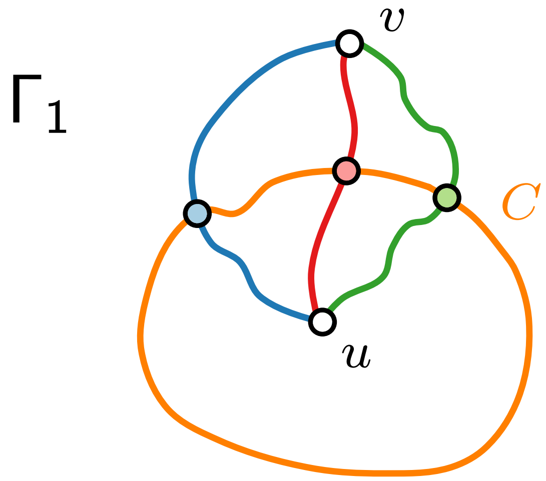
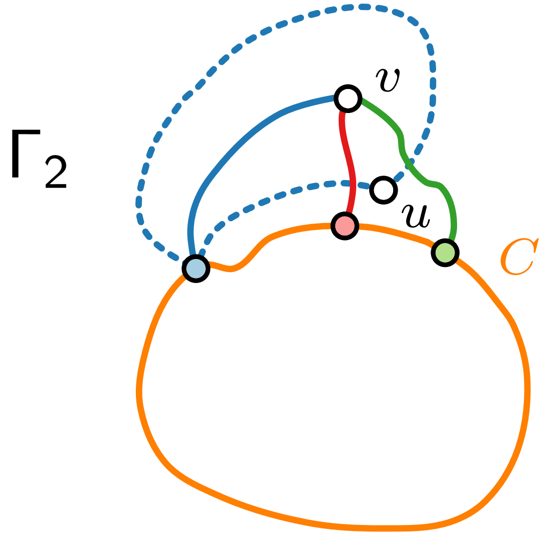


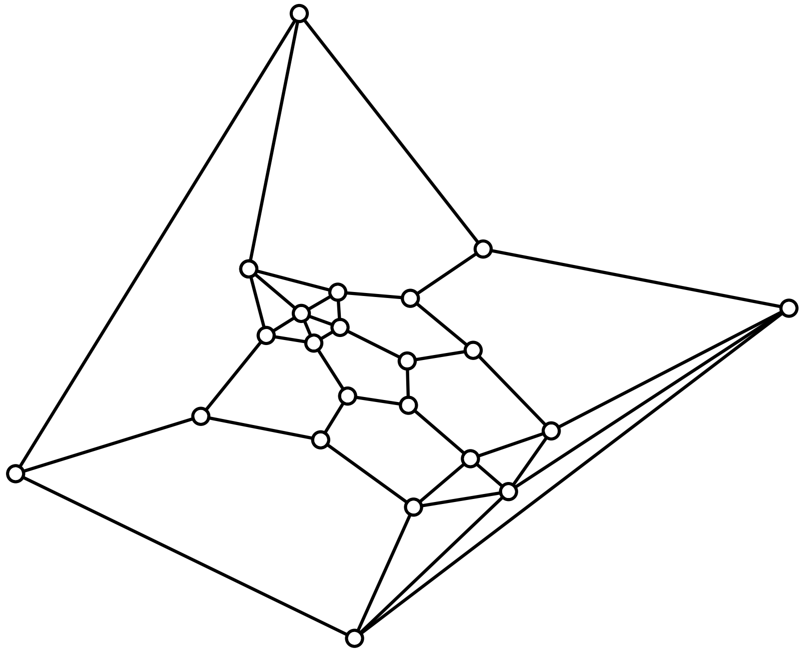




Tutte's Theorem

Theorem. [Tutte 1963]

Let G be a 3-connected planar graph,



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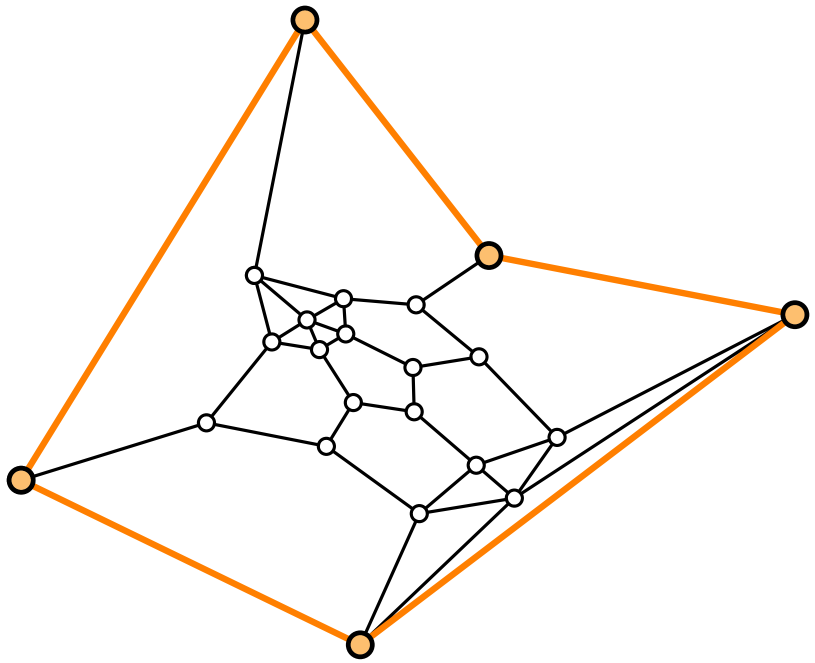
Theorem. [Tutte 1963]

Let G be a 3-connected planar graph, and let C be a face of its unique embedding.



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Theorem. [Tutte 1963]

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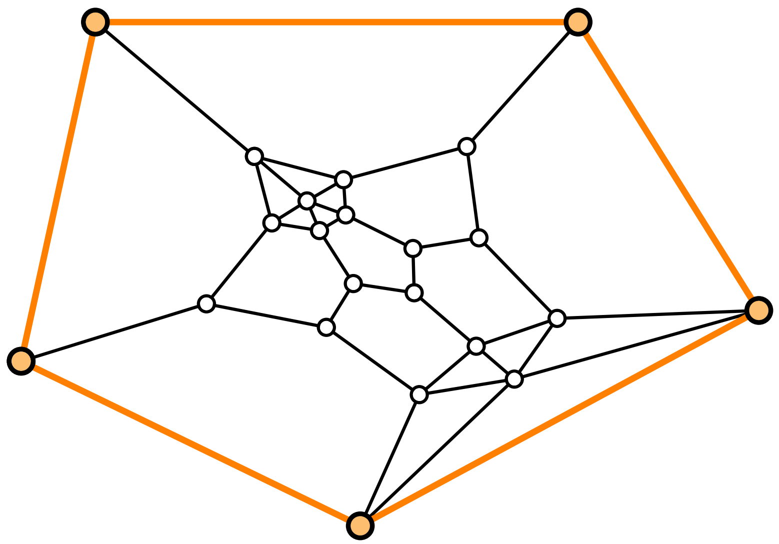

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Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and
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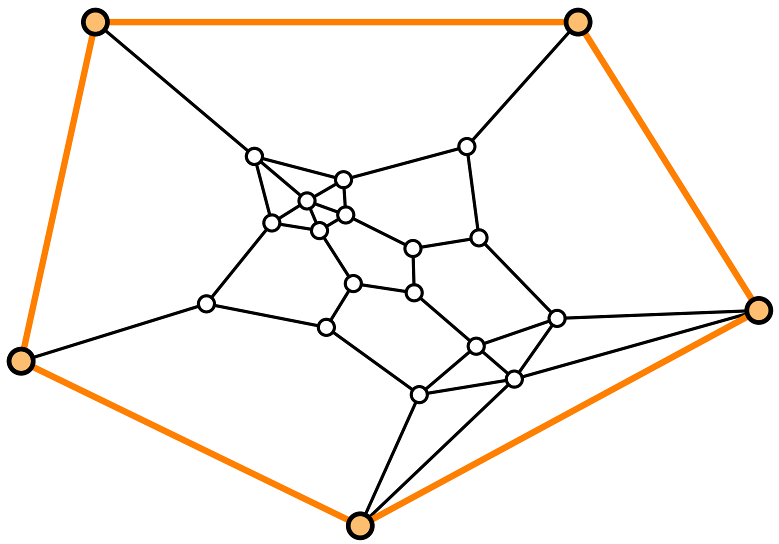
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar



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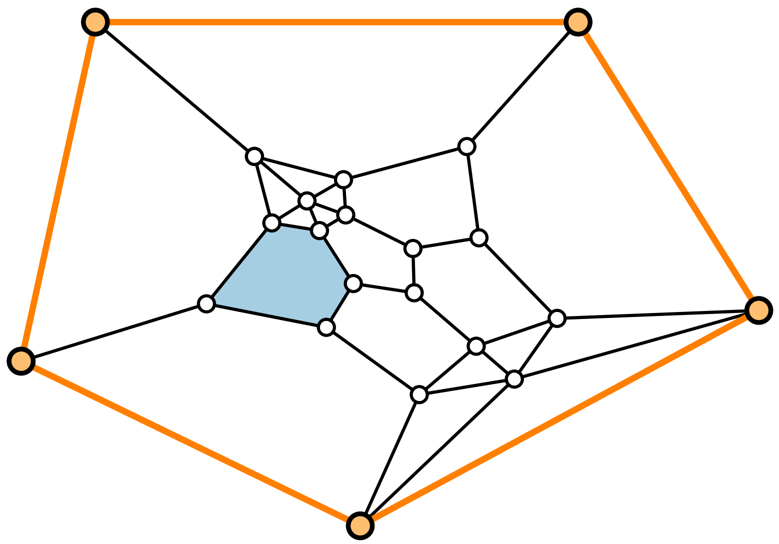
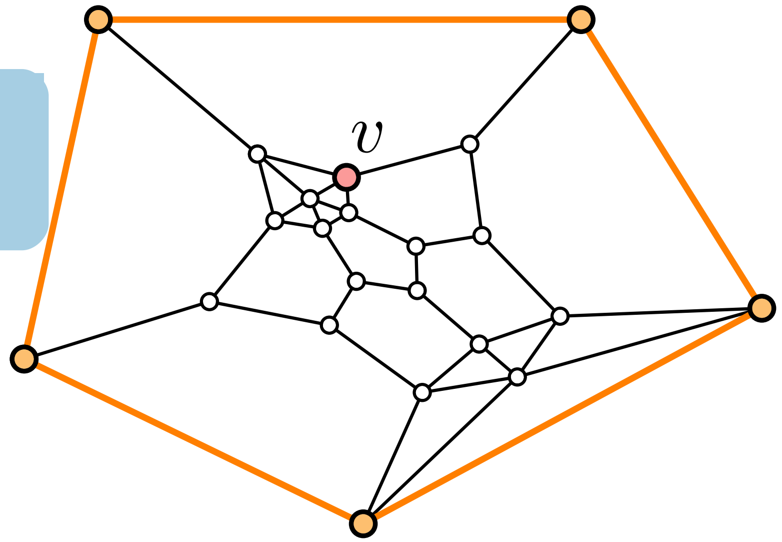


Properties of Tutte Drawings

Property 1. Let $v \in V(G)$ be free



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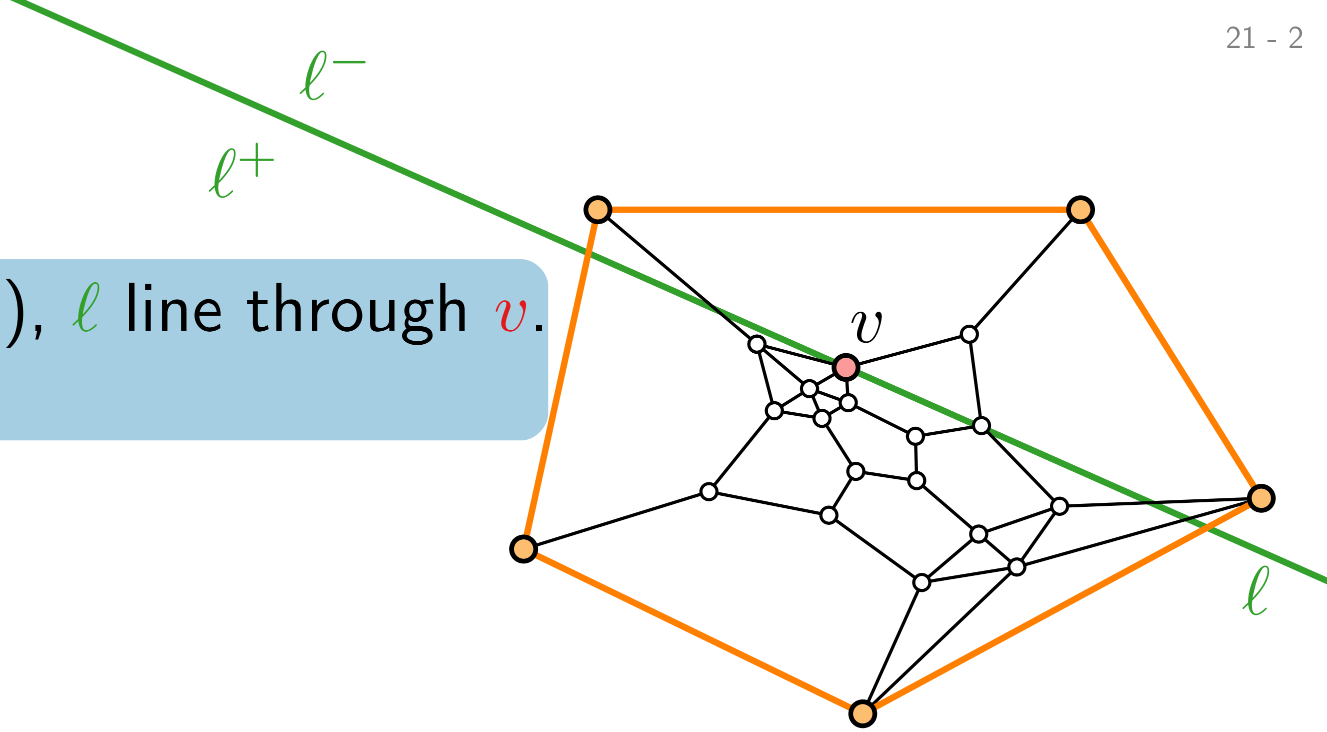

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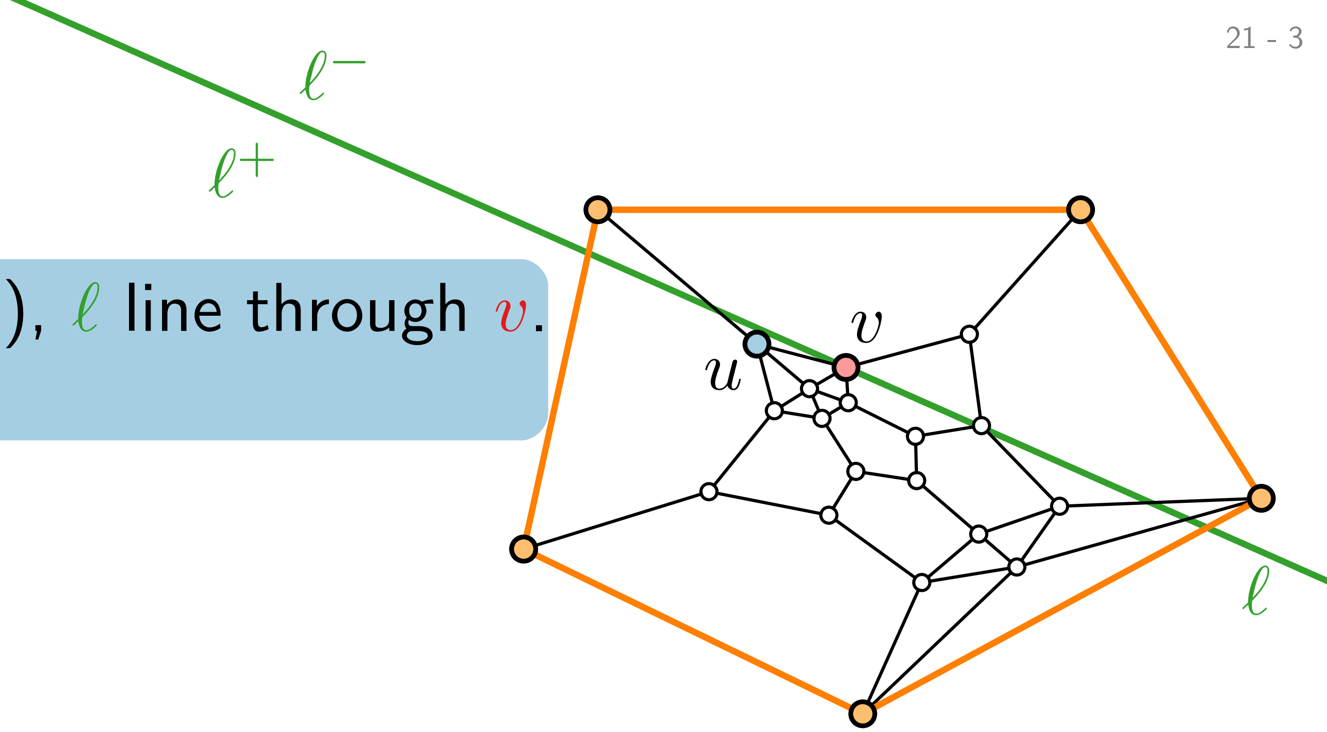
$\exists u \in \text{Adj}[v] \cap \ell^+ \Rightarrow$



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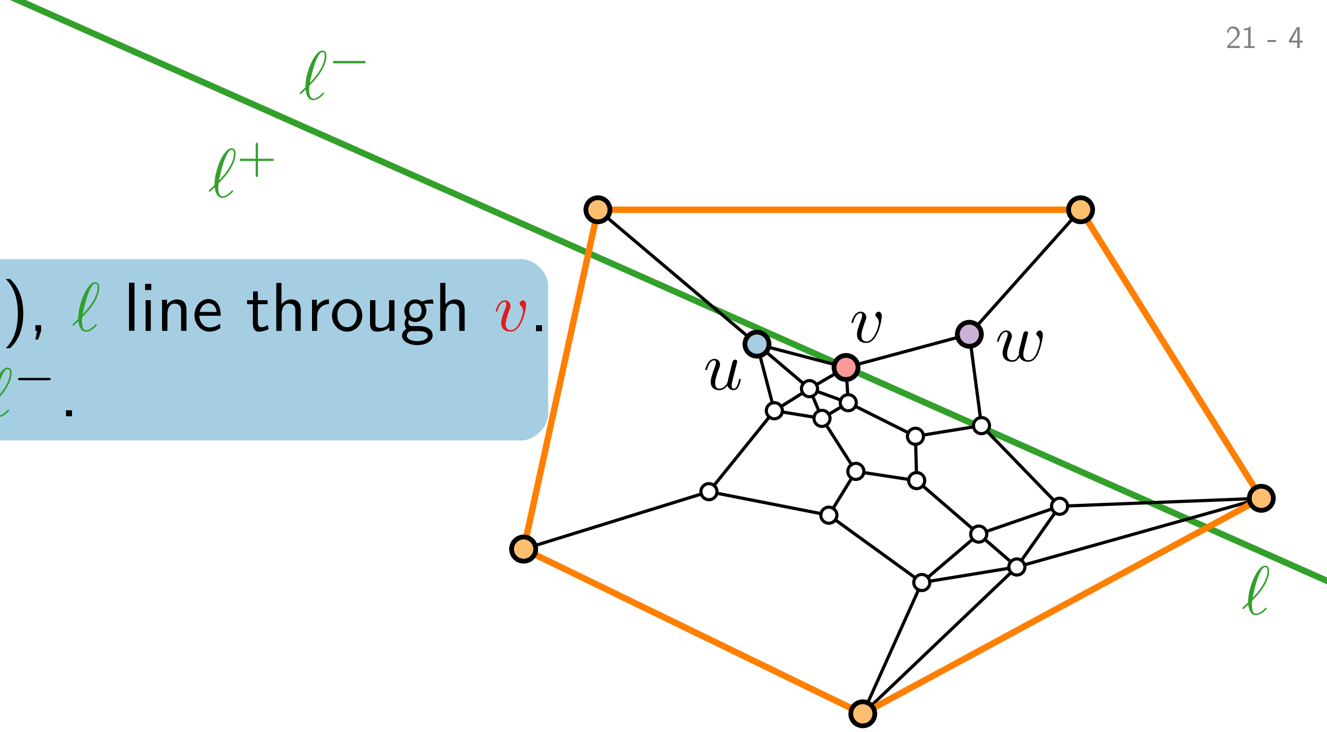

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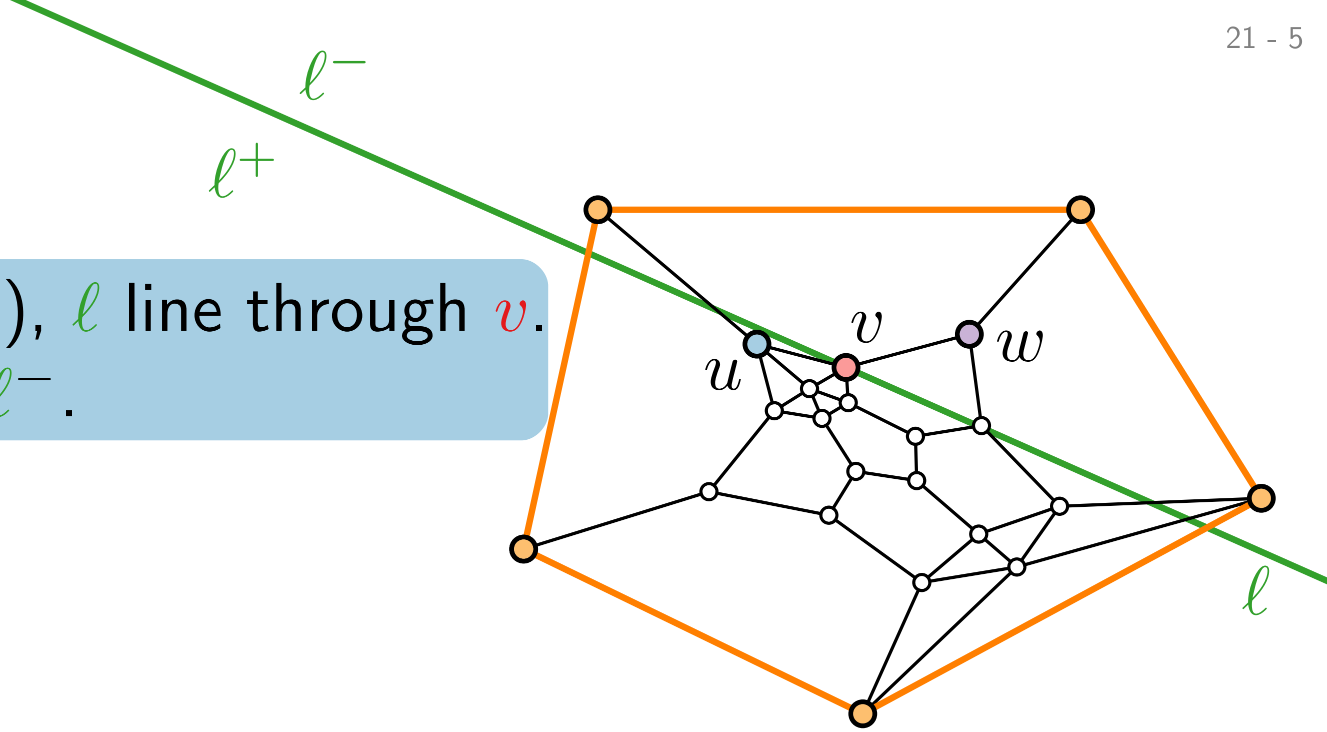

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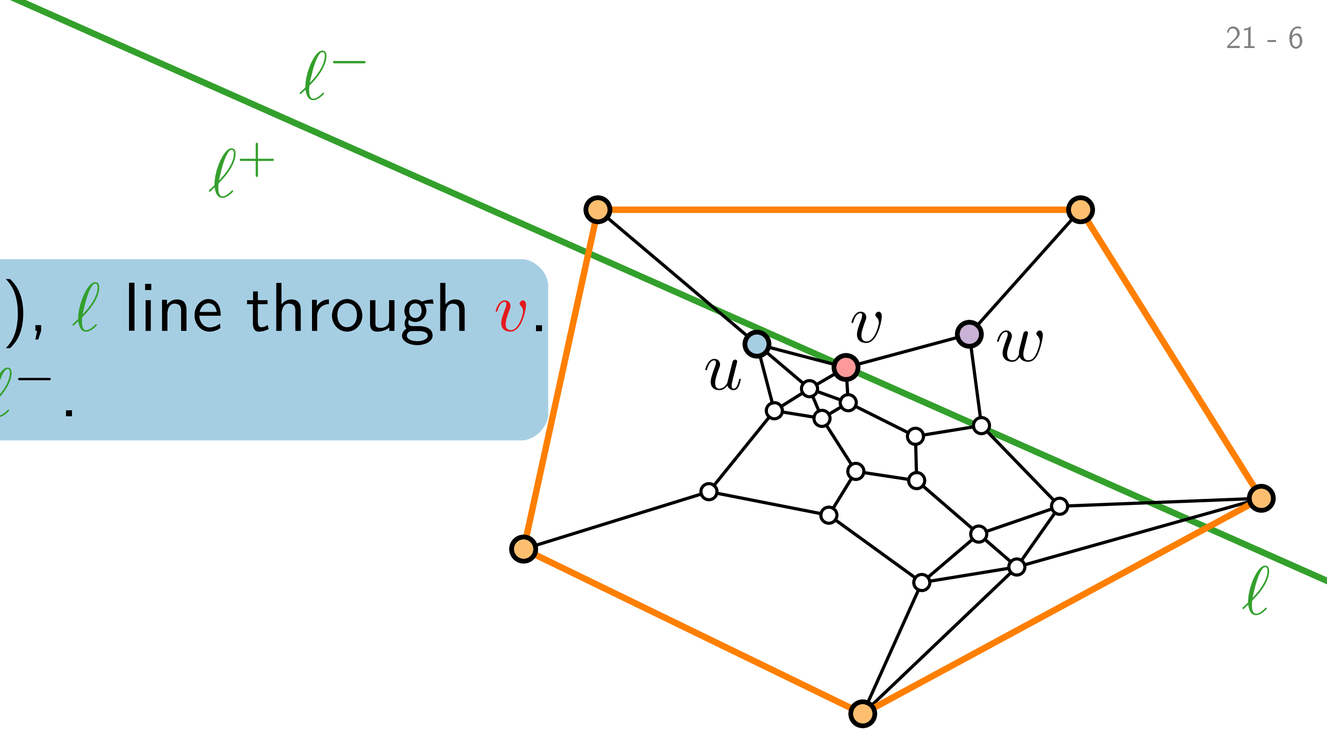

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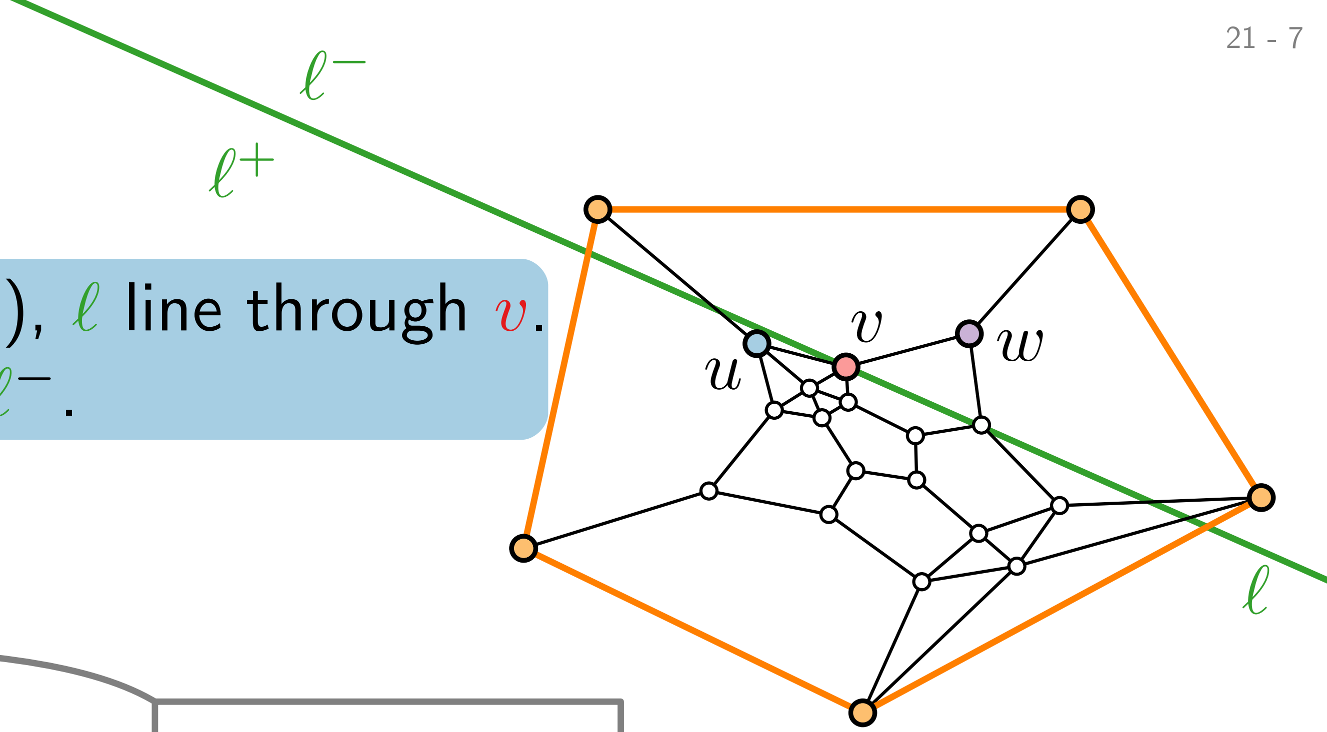

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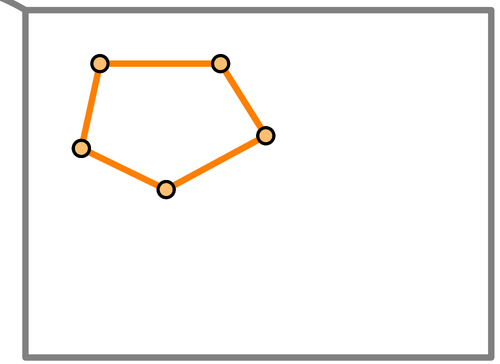

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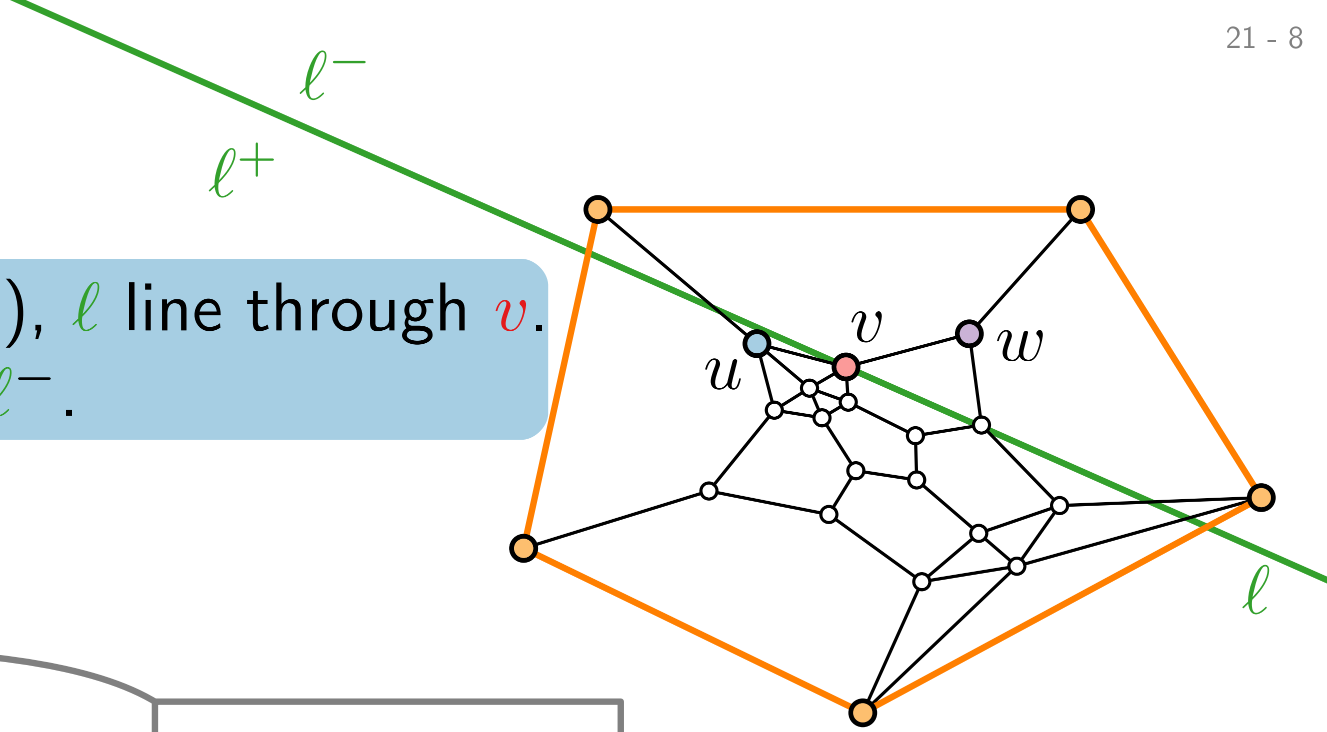

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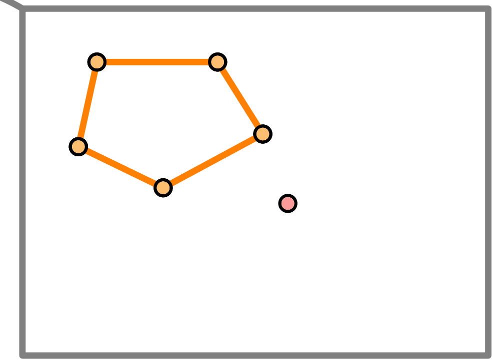

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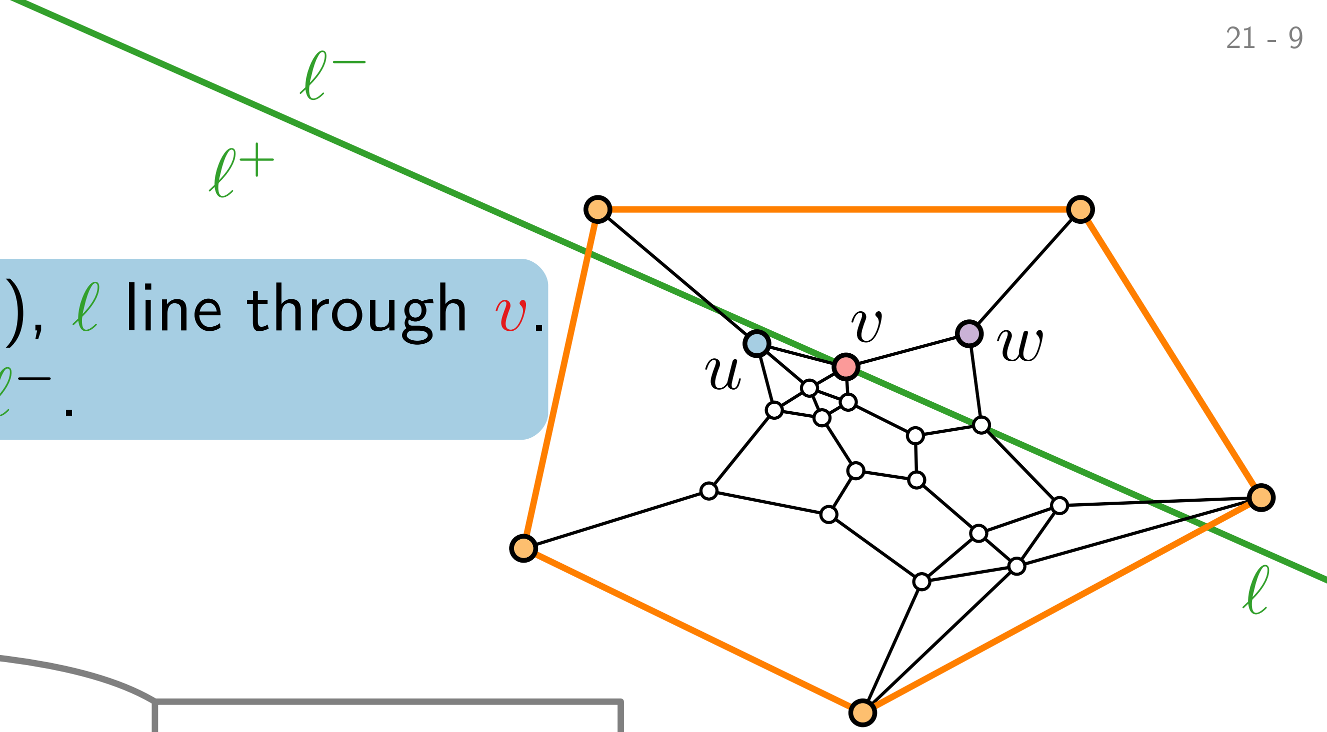

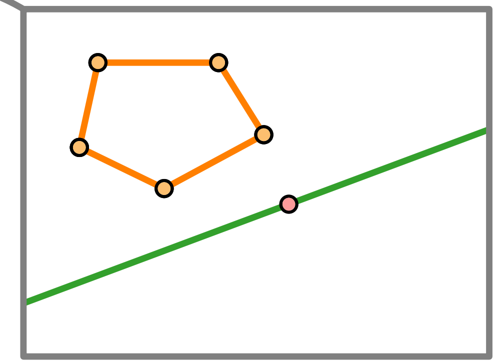

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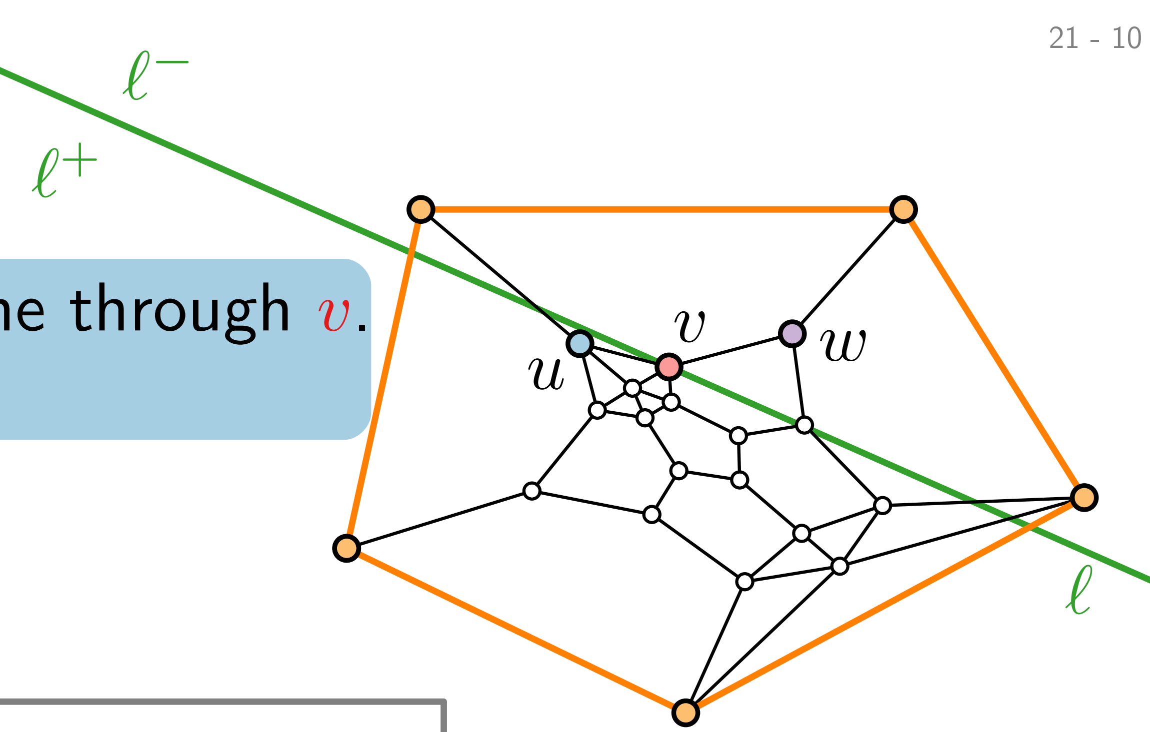

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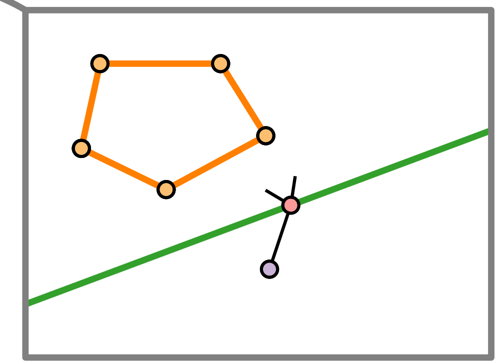

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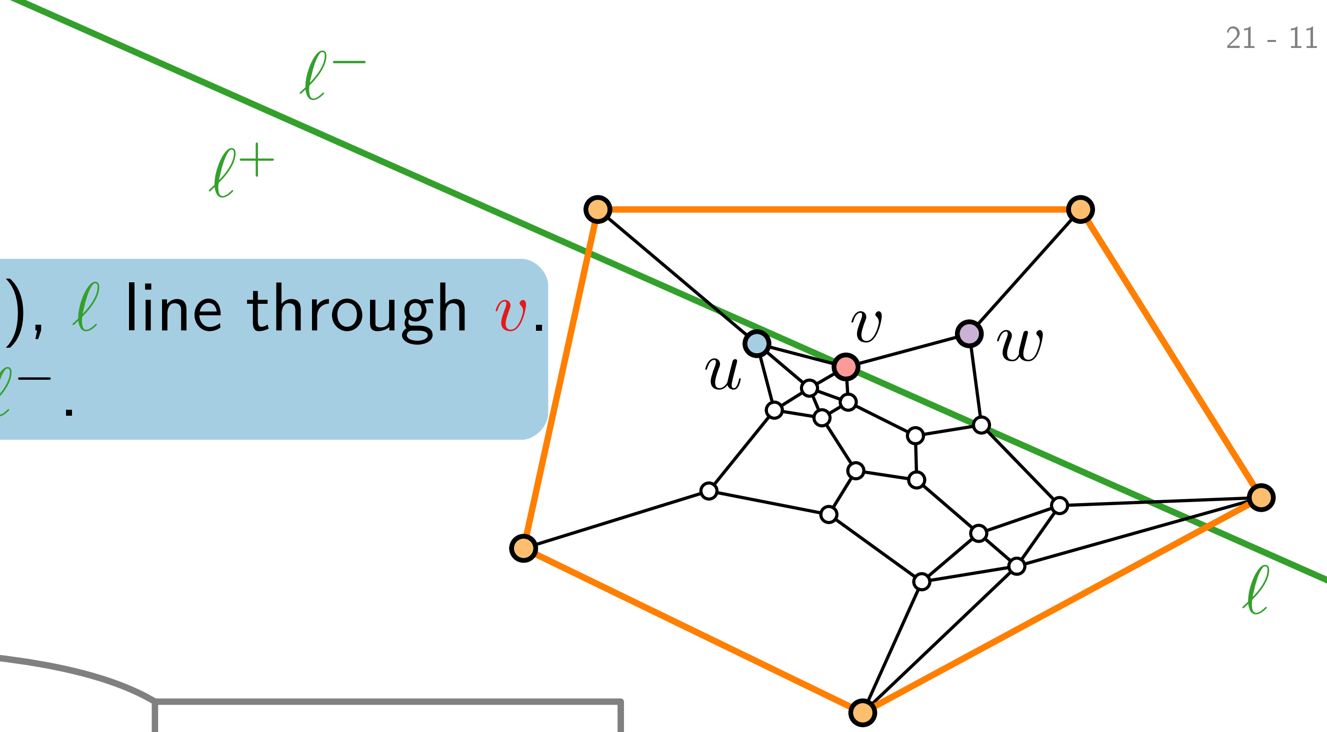

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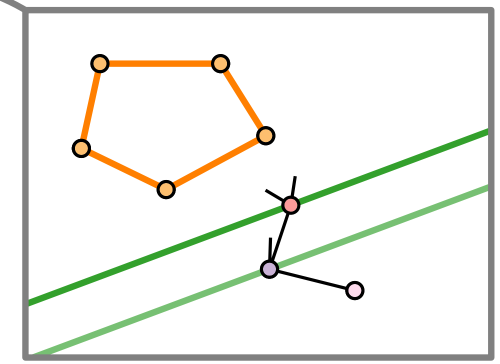

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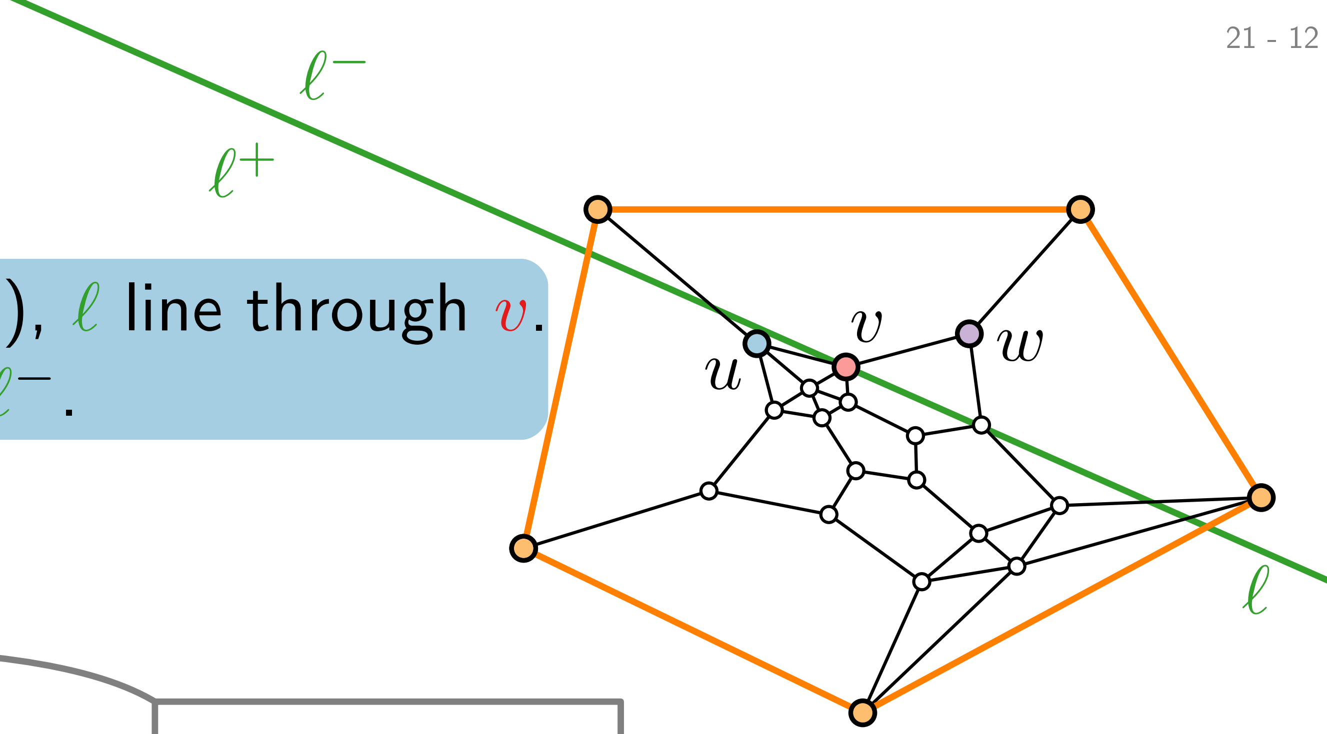

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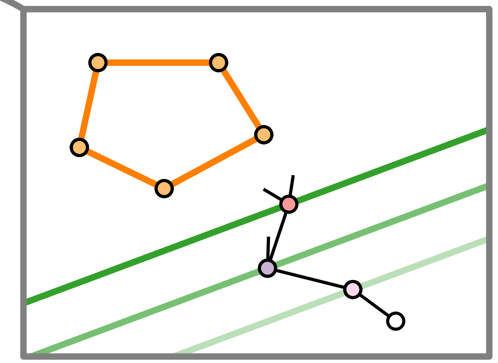

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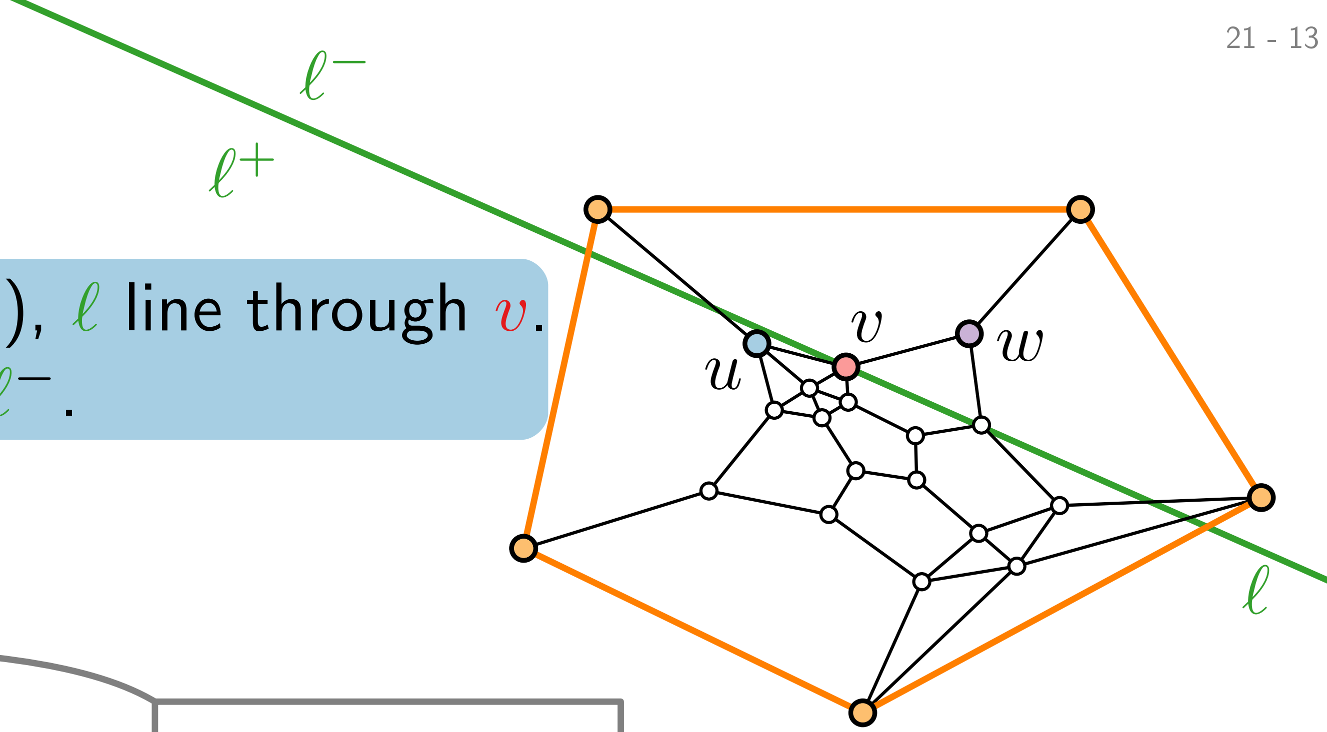

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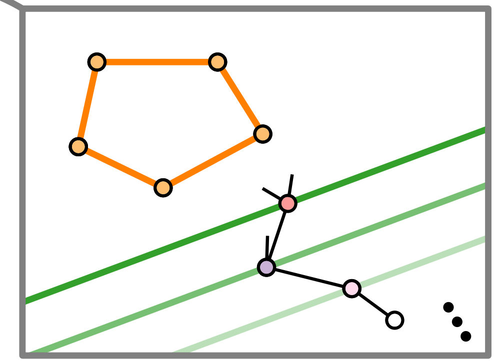

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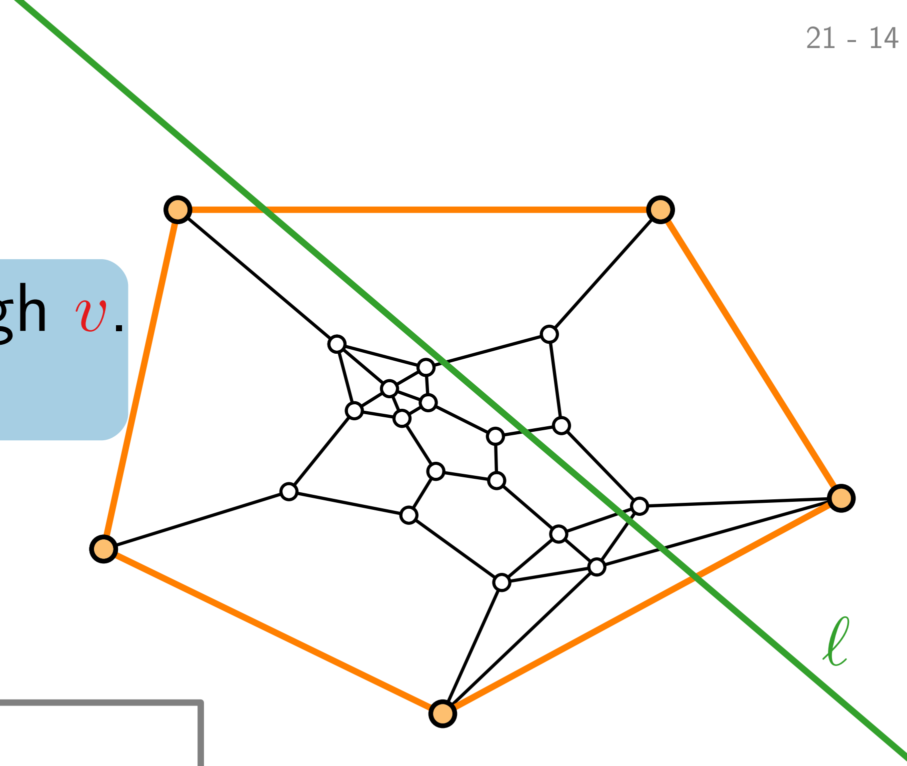

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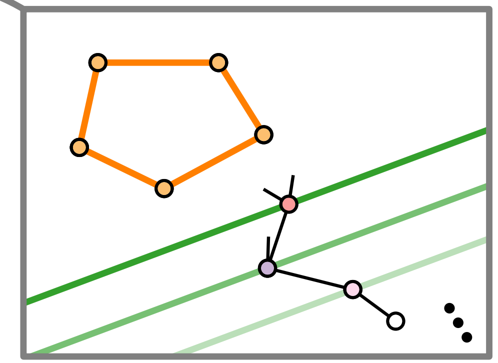

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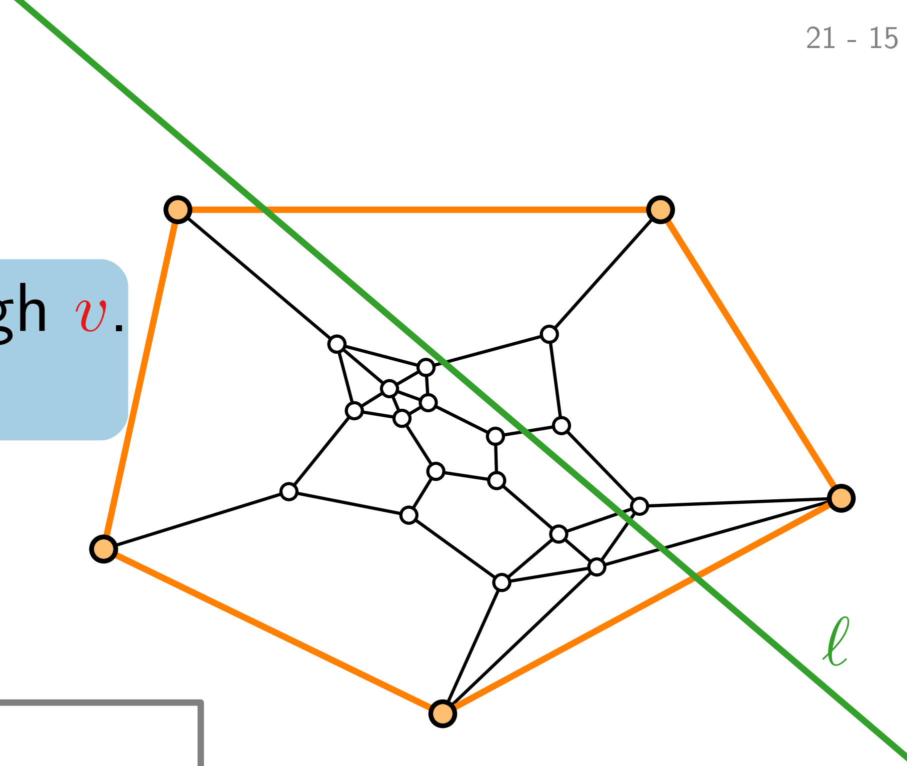
Properties of Tutte Drawings



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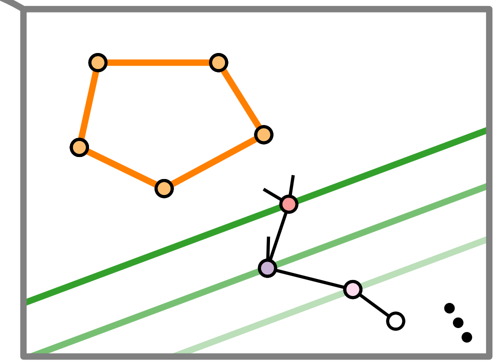

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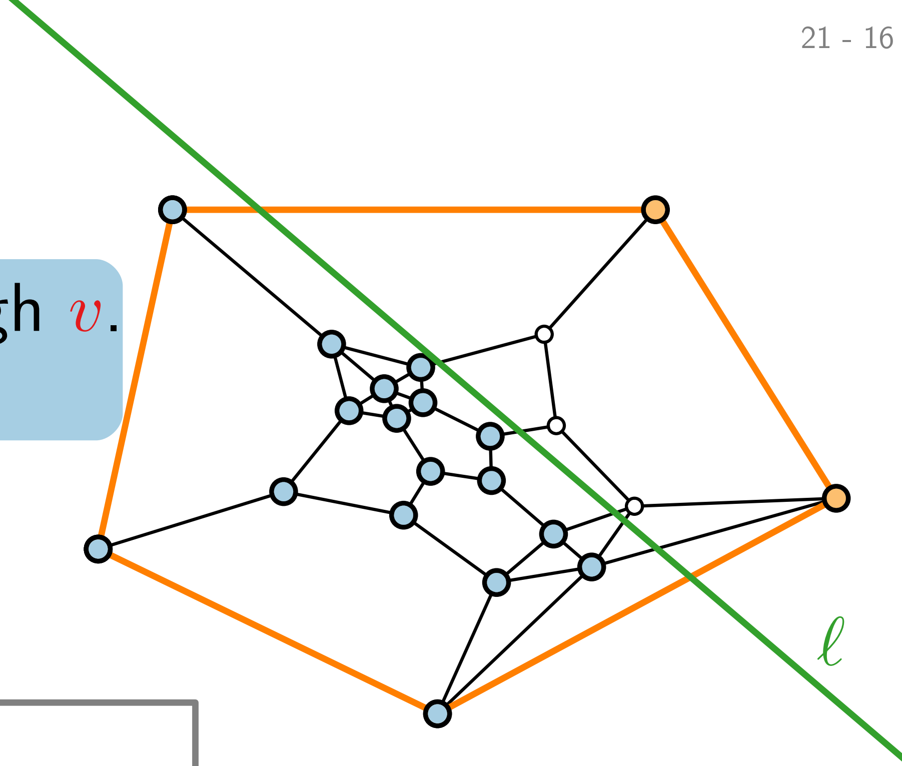

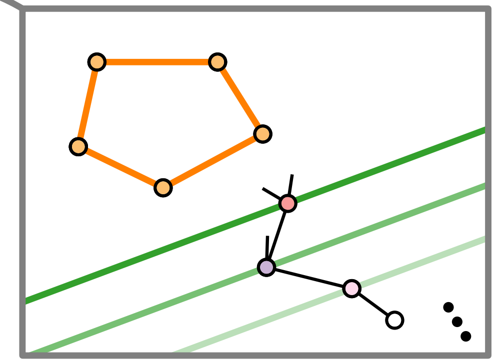

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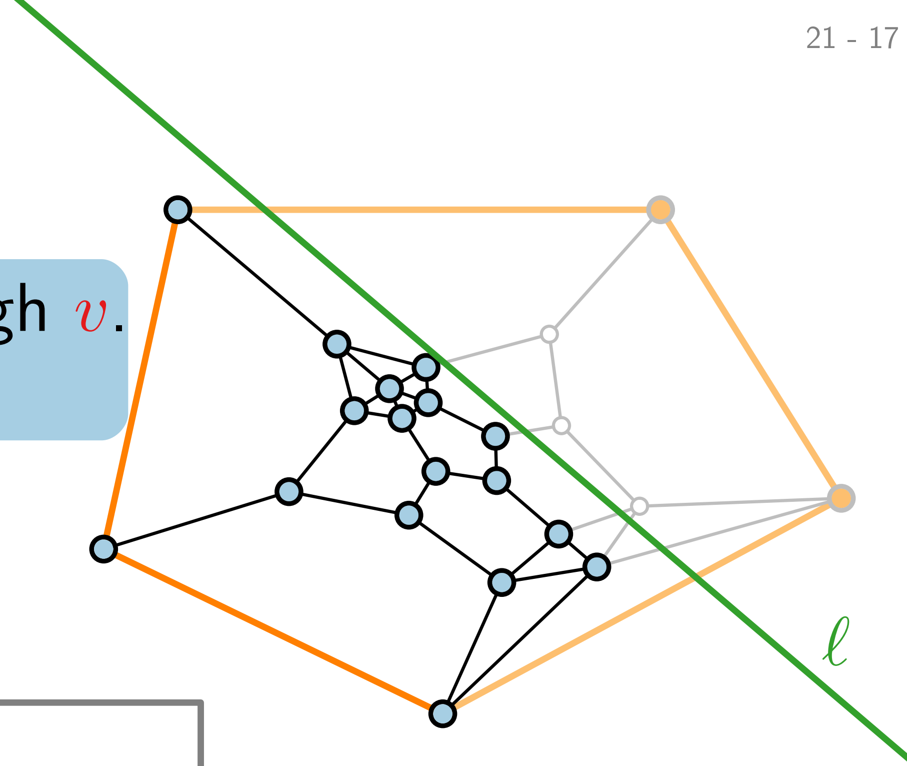

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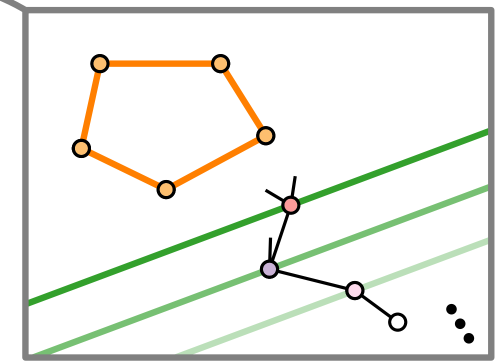

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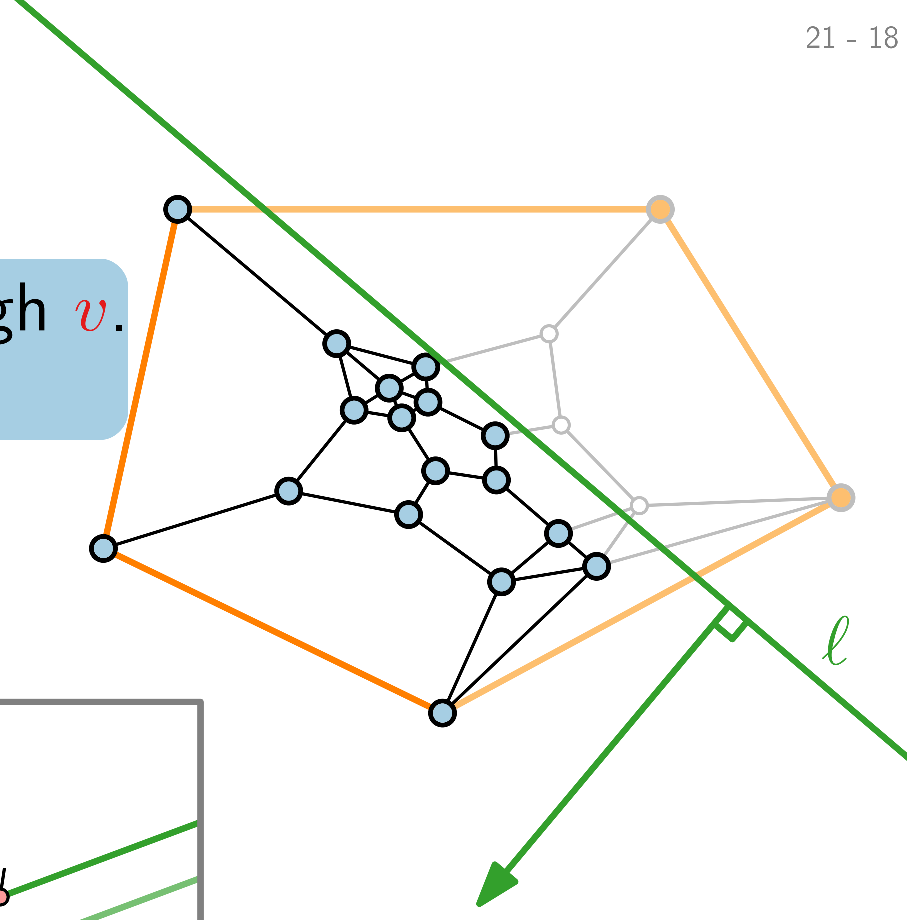

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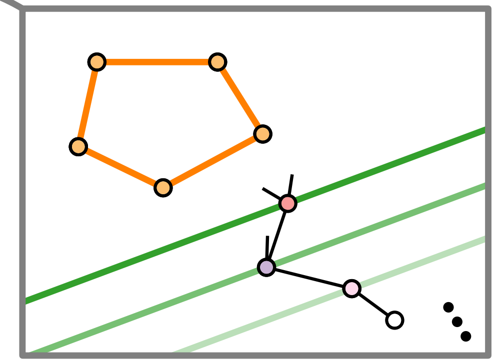

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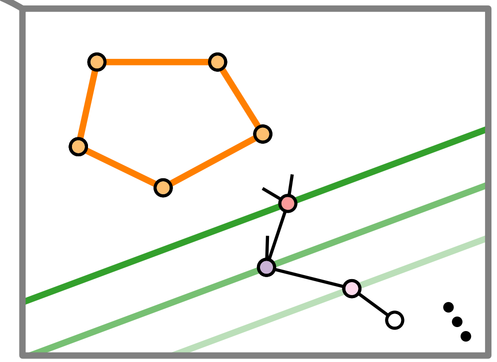

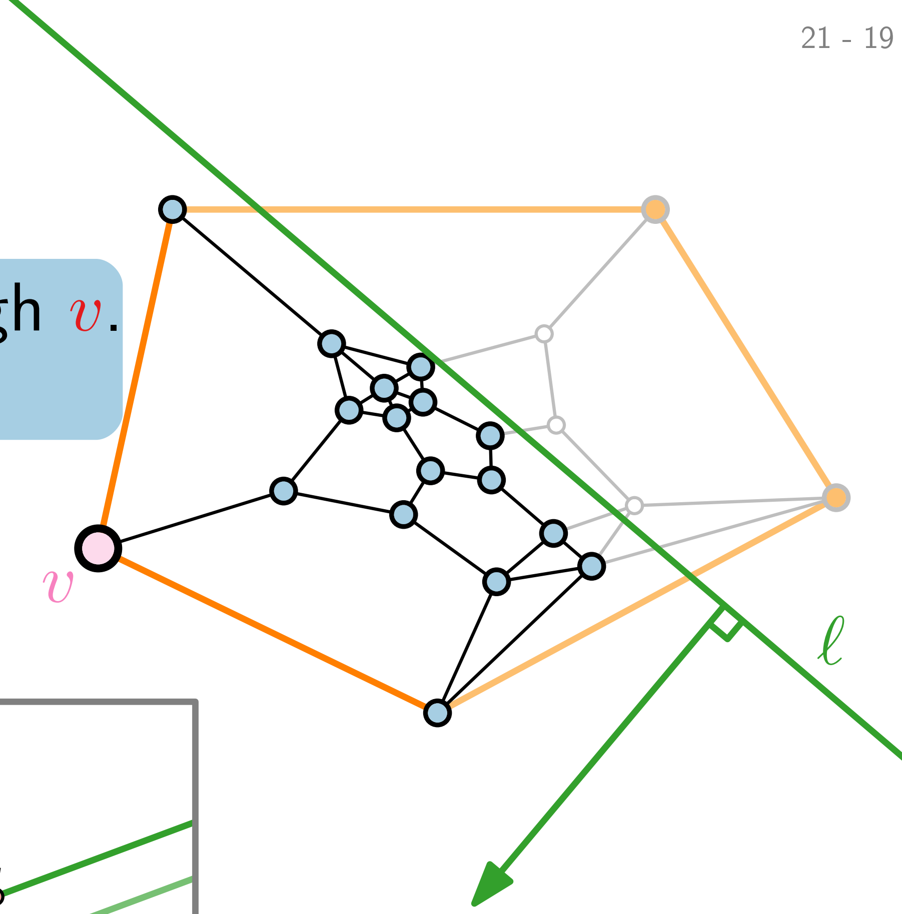

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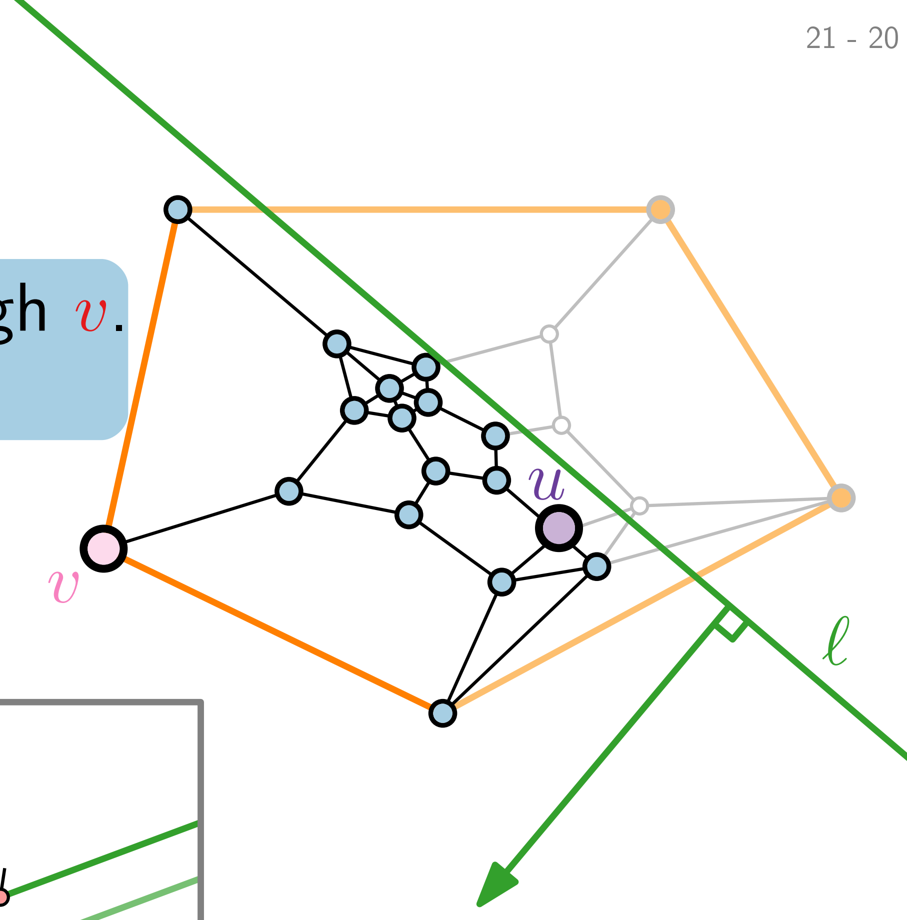

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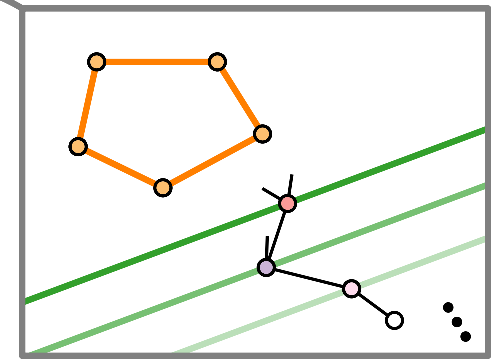

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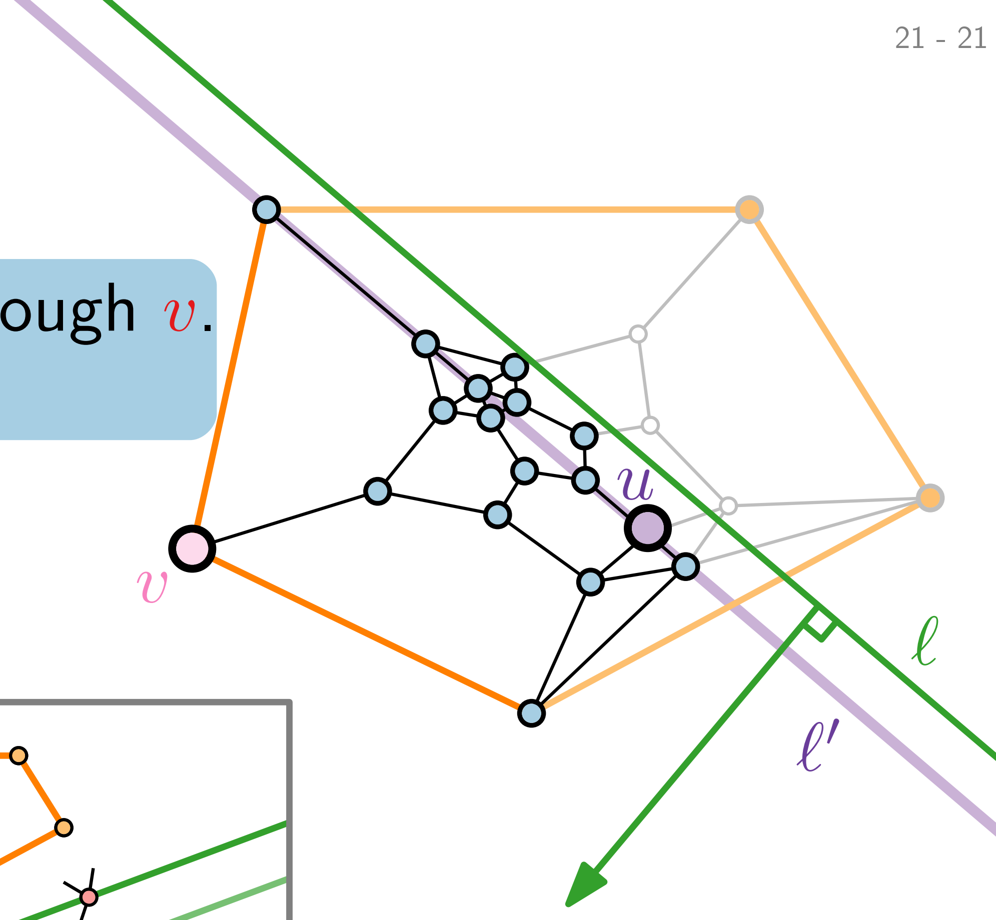

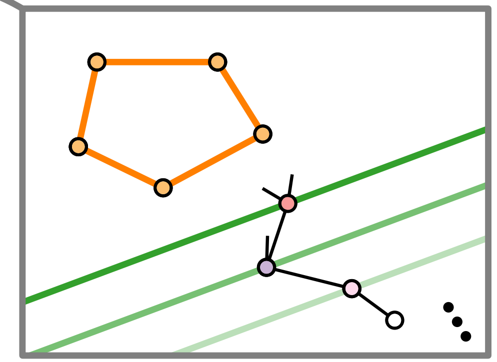

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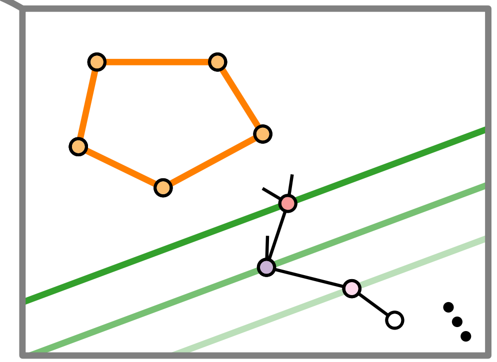

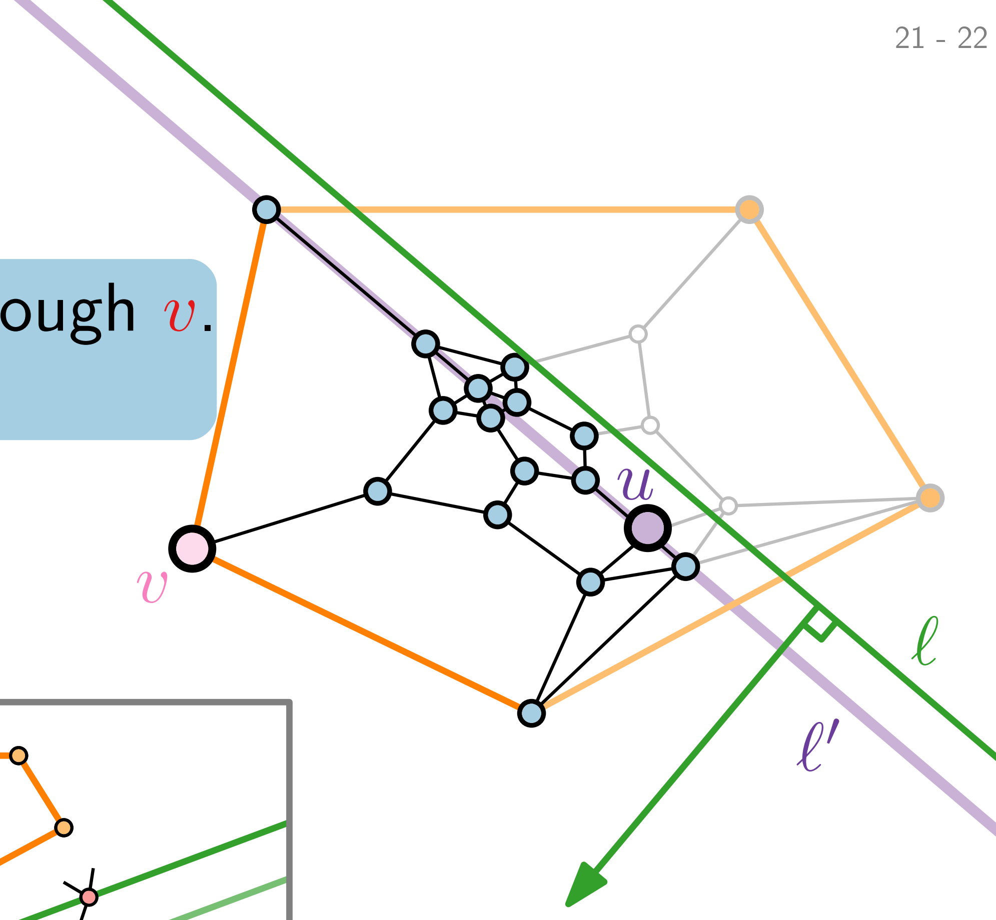

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G connected, v not on ℓ'

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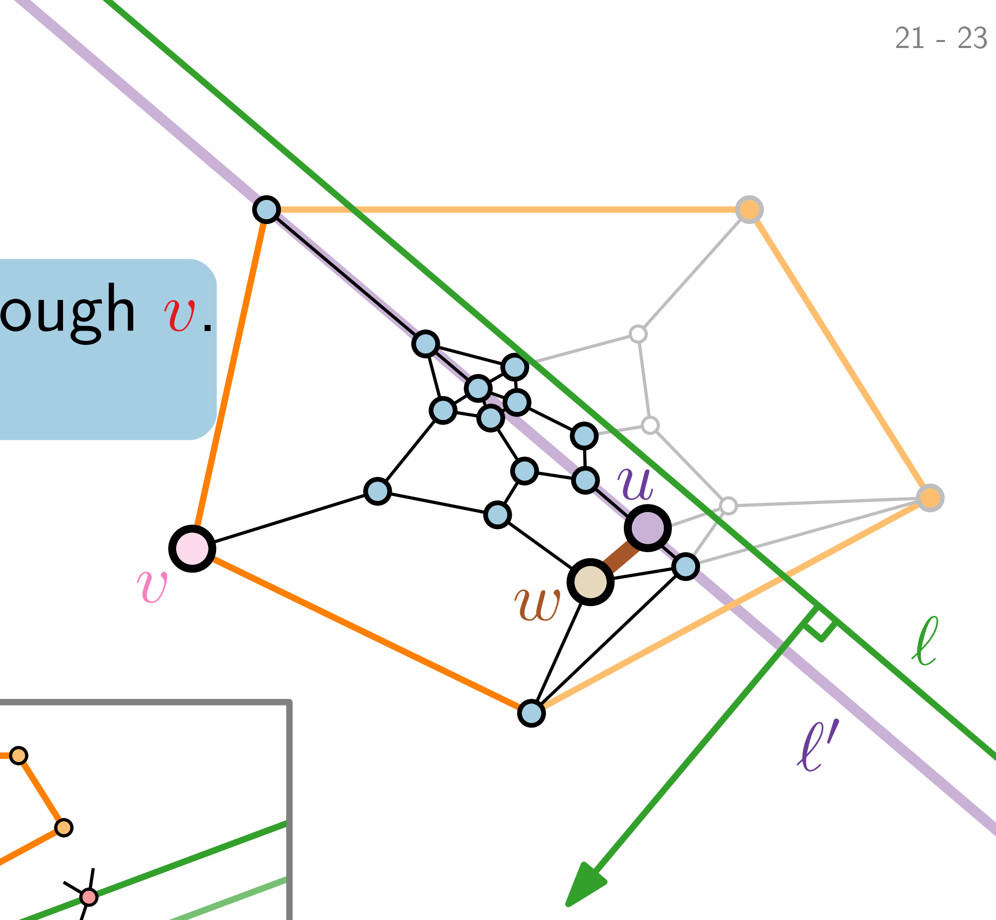

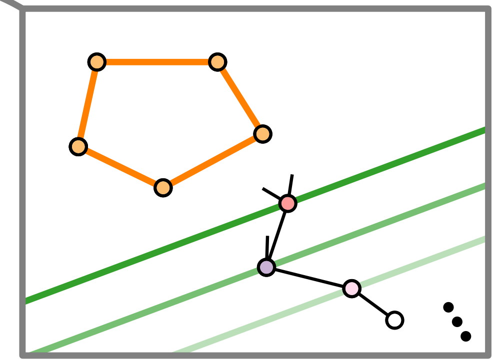

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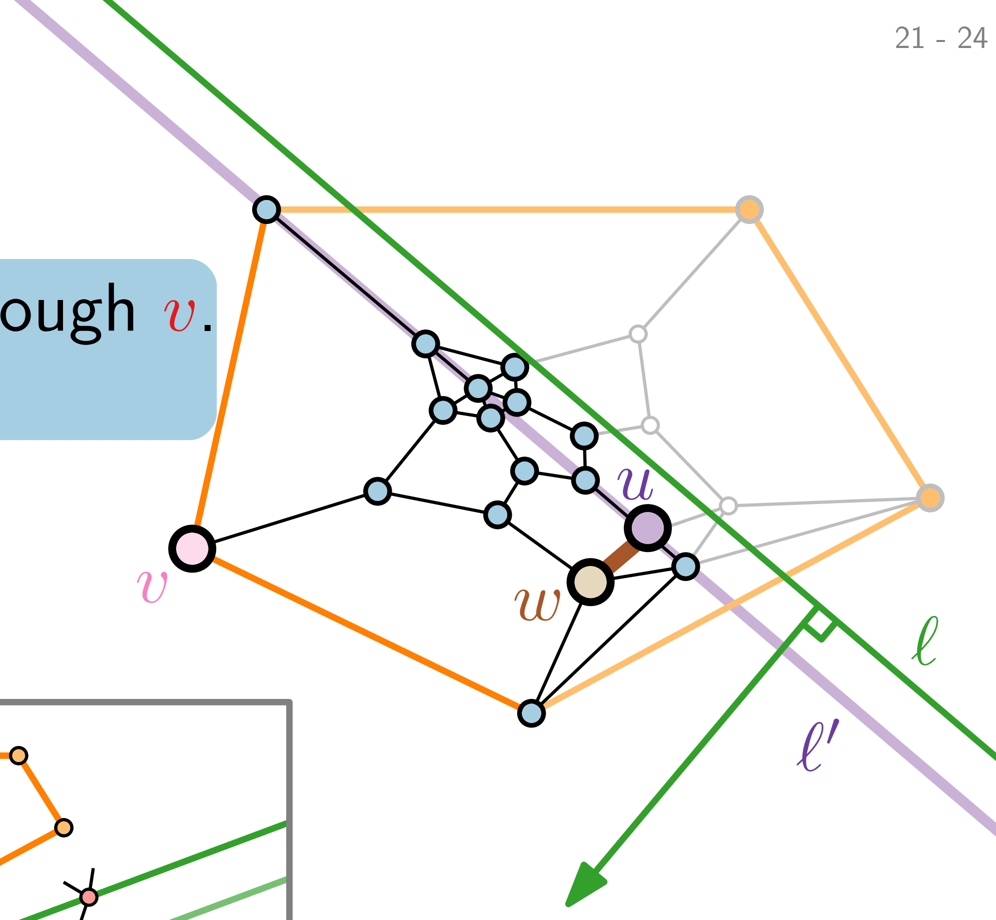

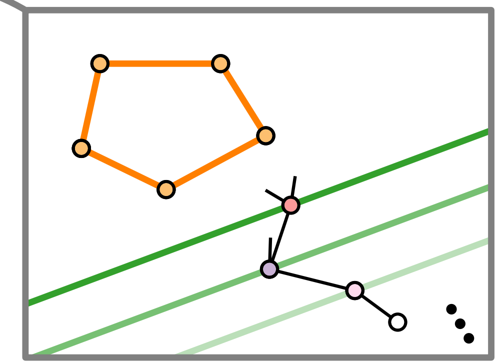

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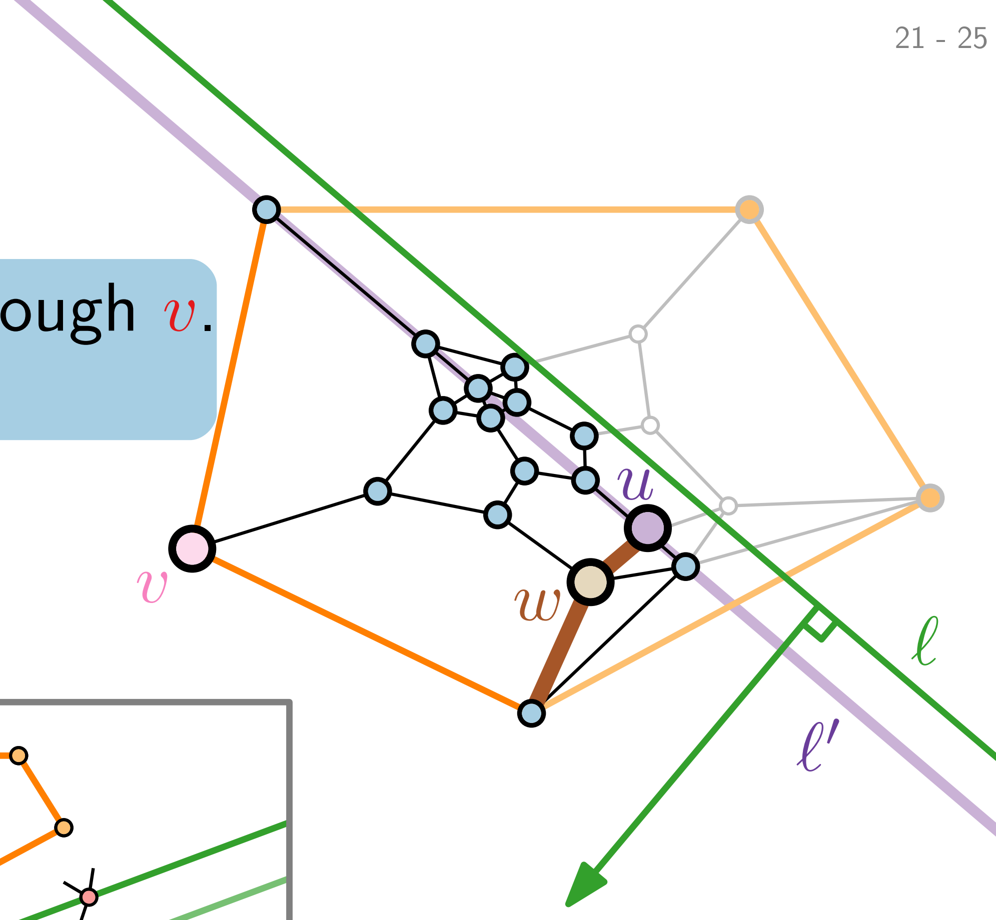

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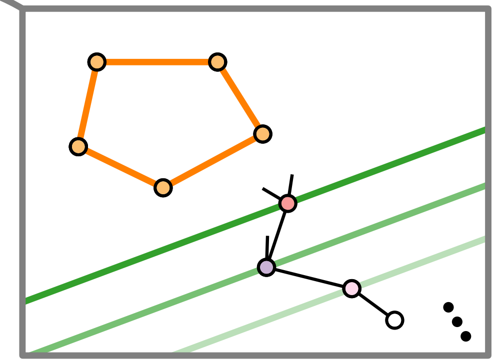

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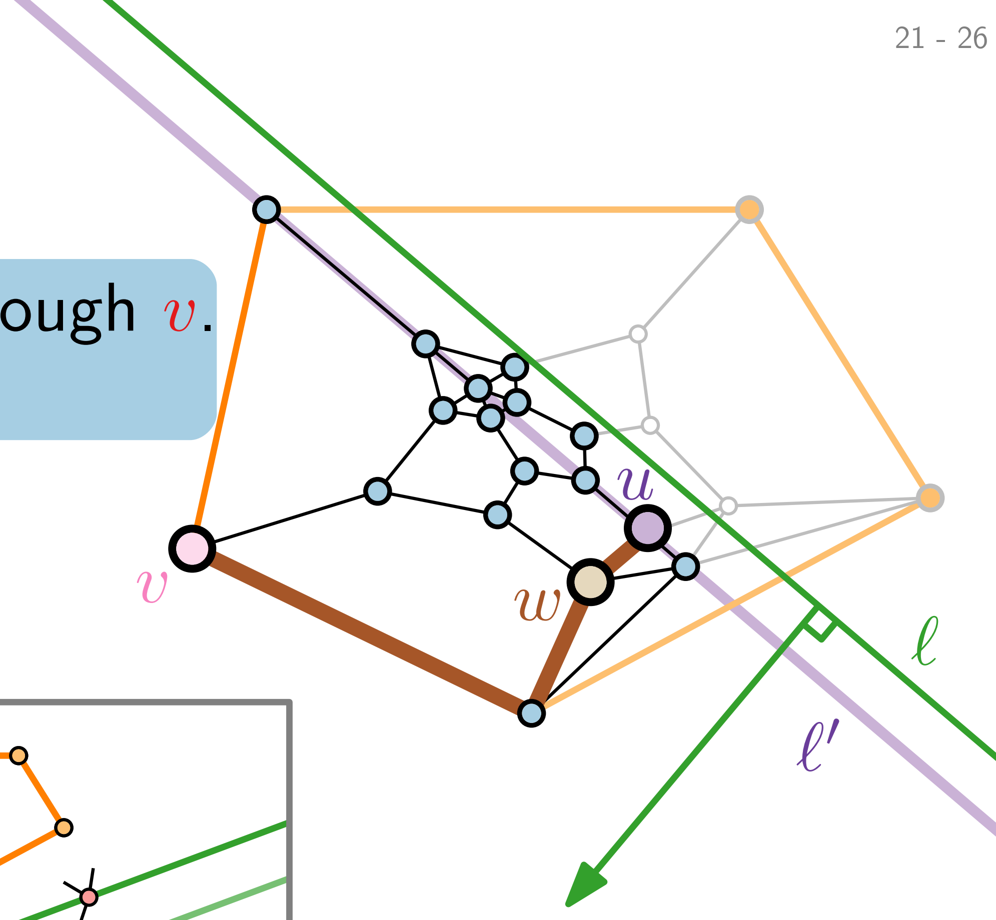

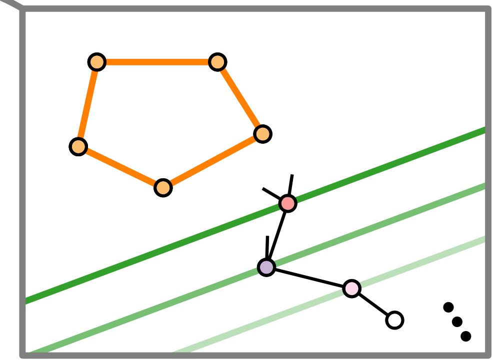

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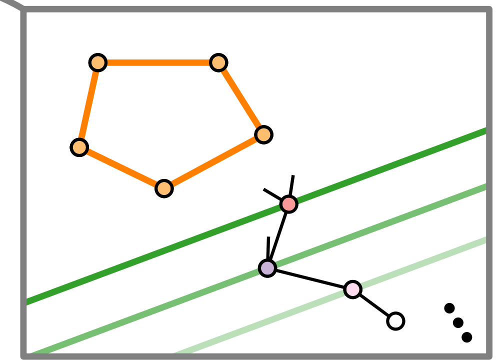

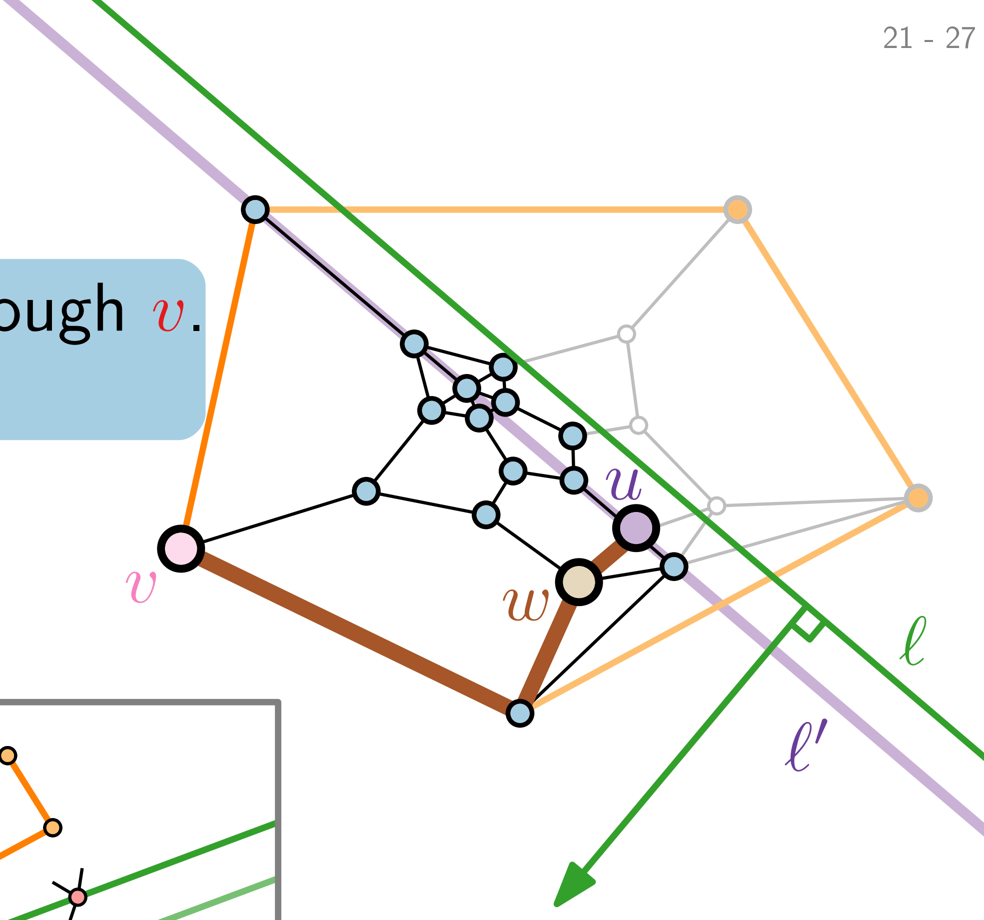

Choose v farthest away from ℓ.

Pick any vertex $u \in V_\ell$, ℓ' parallel to ℓ through u

G connected, v not on ℓ' $\Rightarrow \exists$ neighbor $w \in V_\ell$ of u on the same side of ℓ' as v.

Move ℓ' onto w and repeat $\Rightarrow \exists$ path from u to C, and along C to v.

Property 4. No vertex is collinear with all its neighbors.

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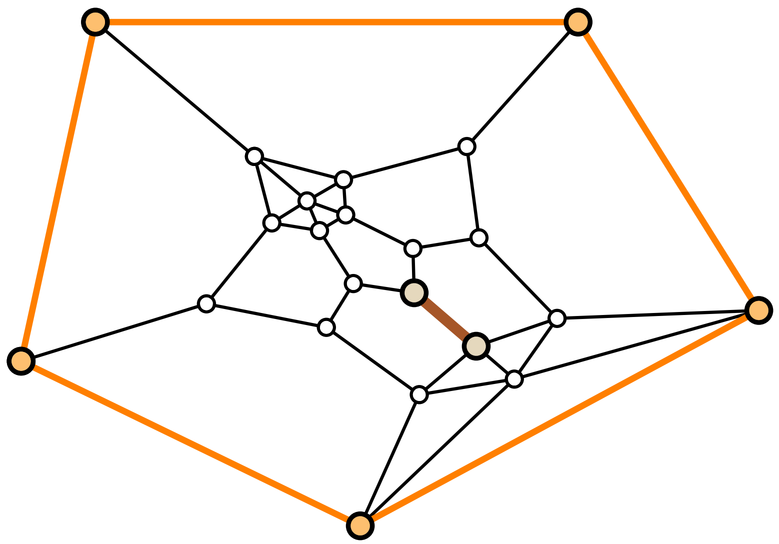
Lemma. Let uv be a non-boundary edge,



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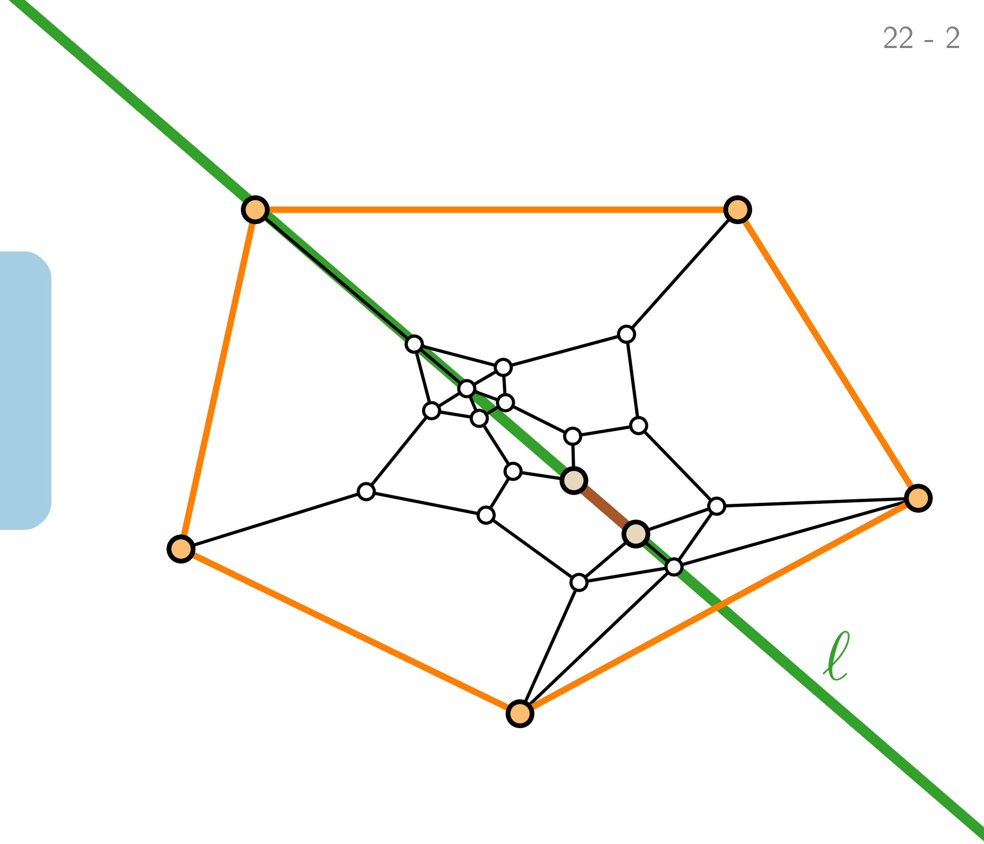

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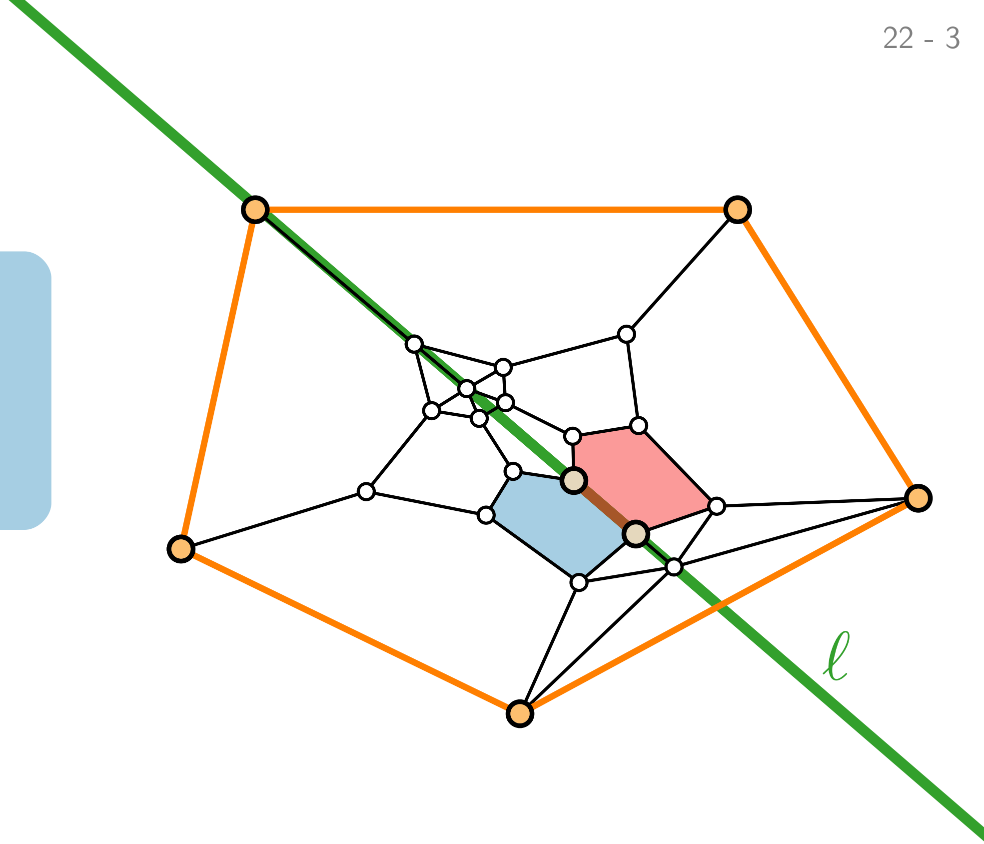

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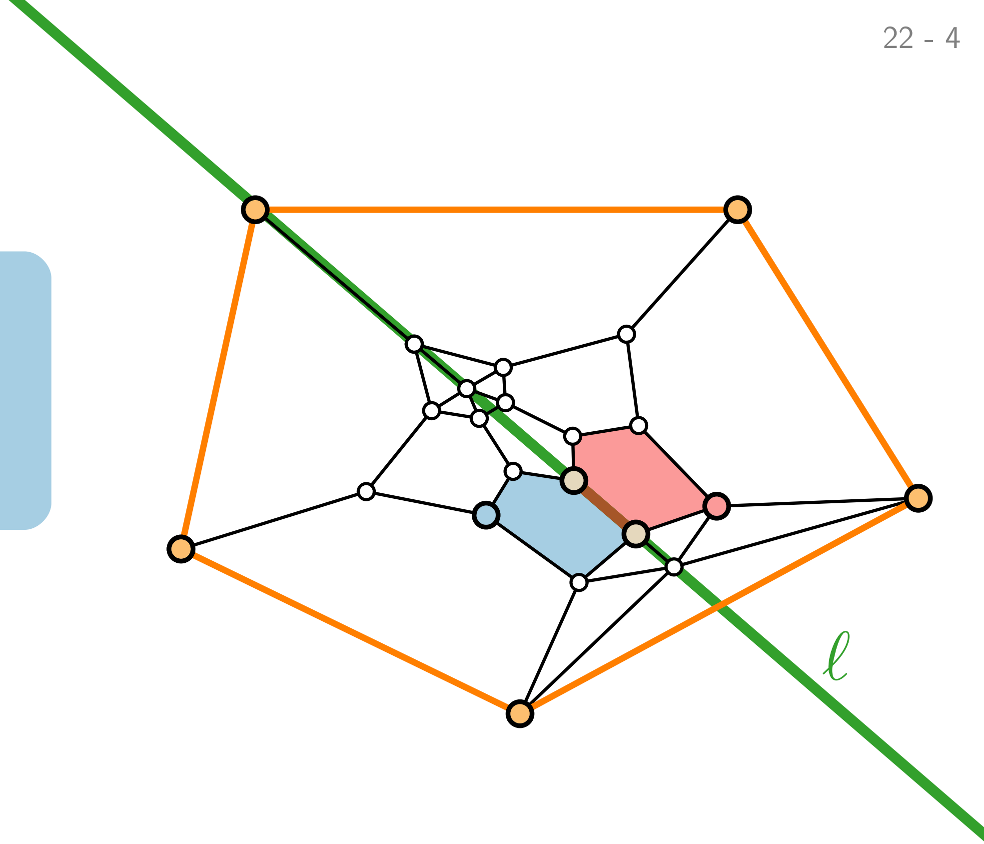

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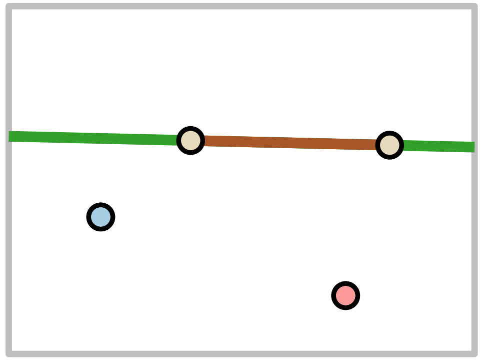

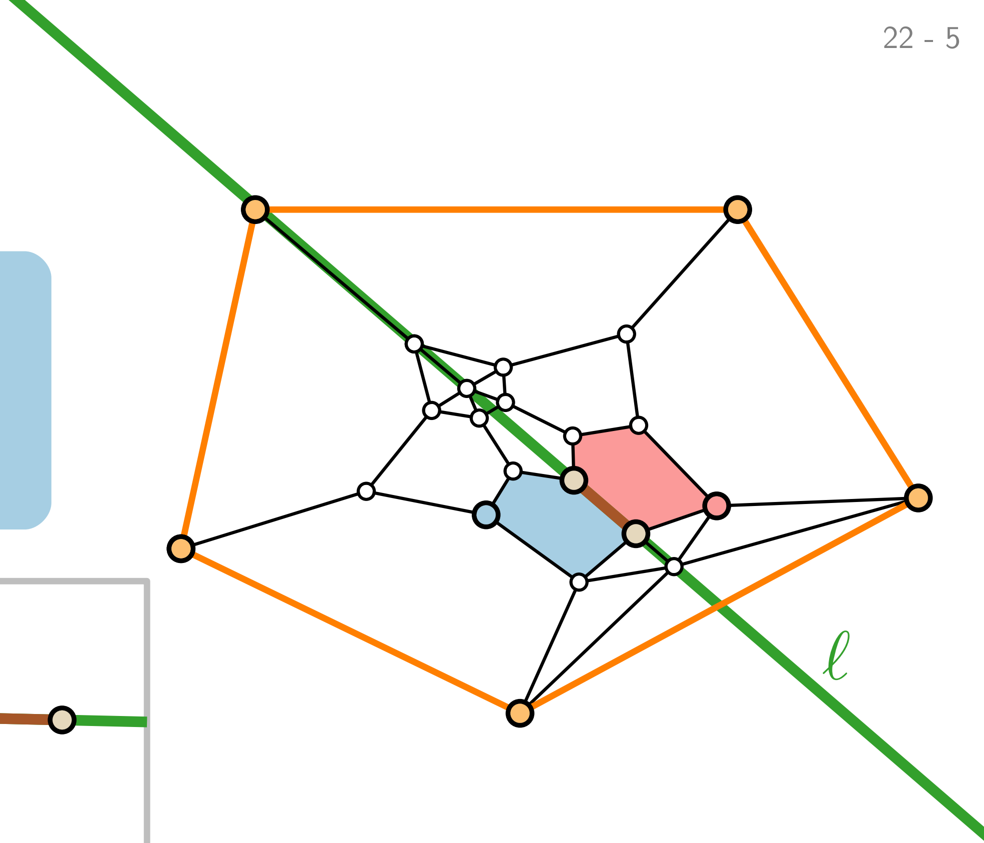

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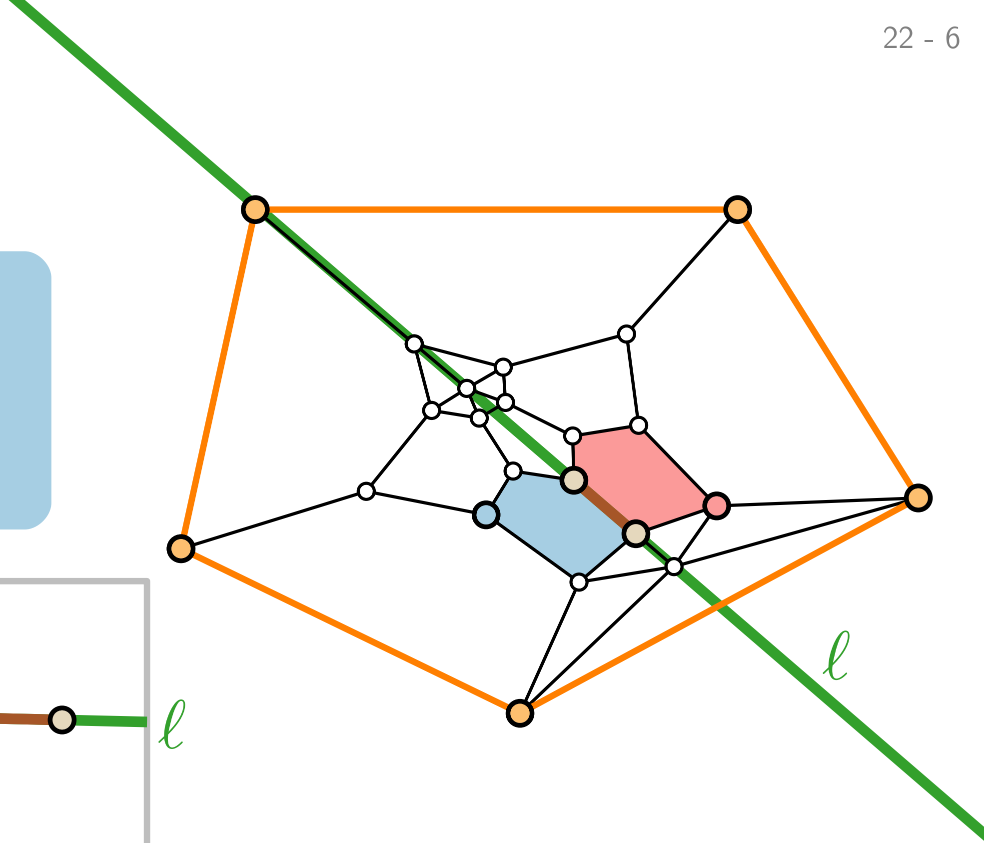

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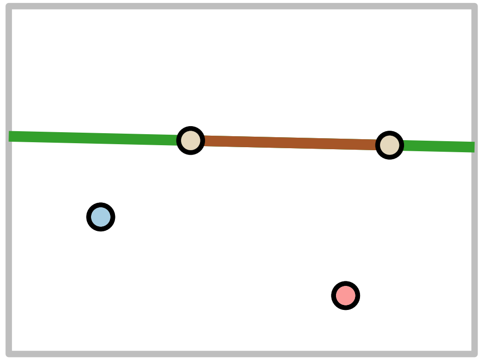

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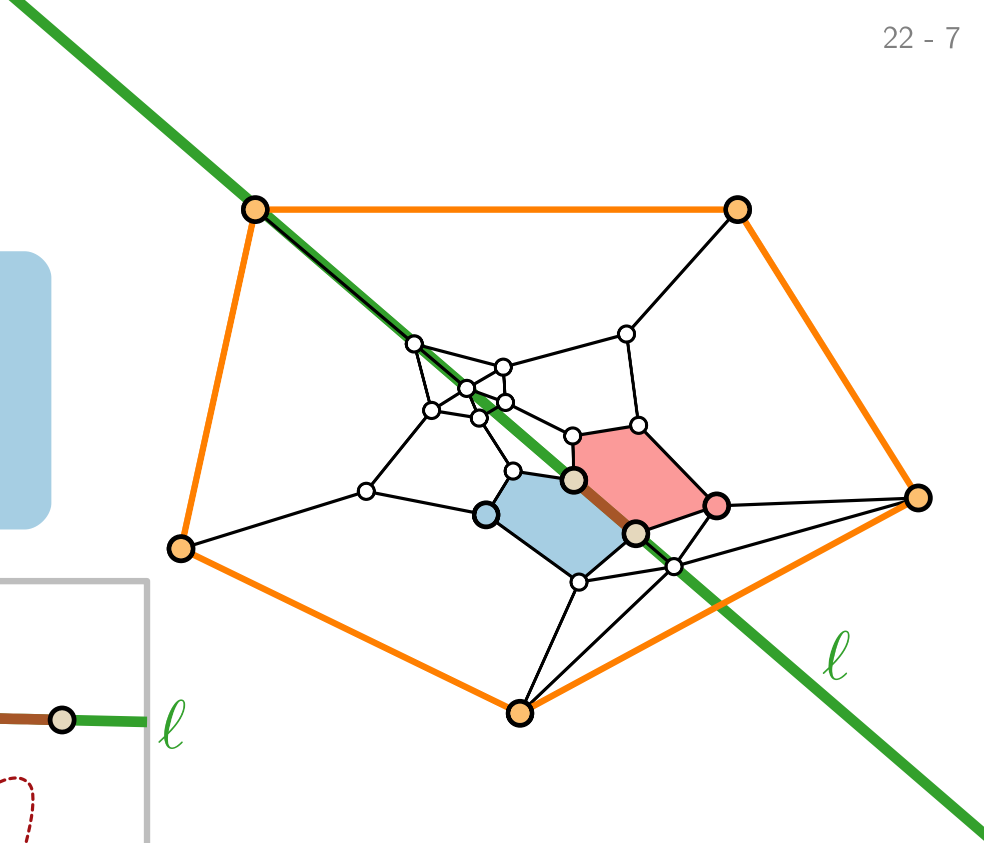

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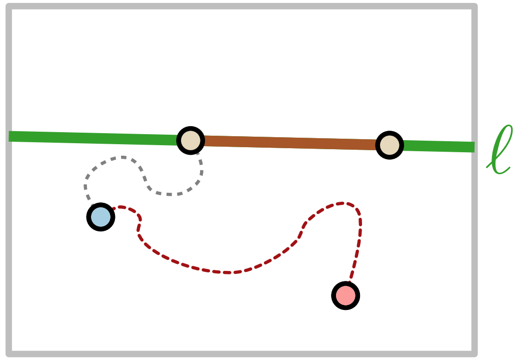

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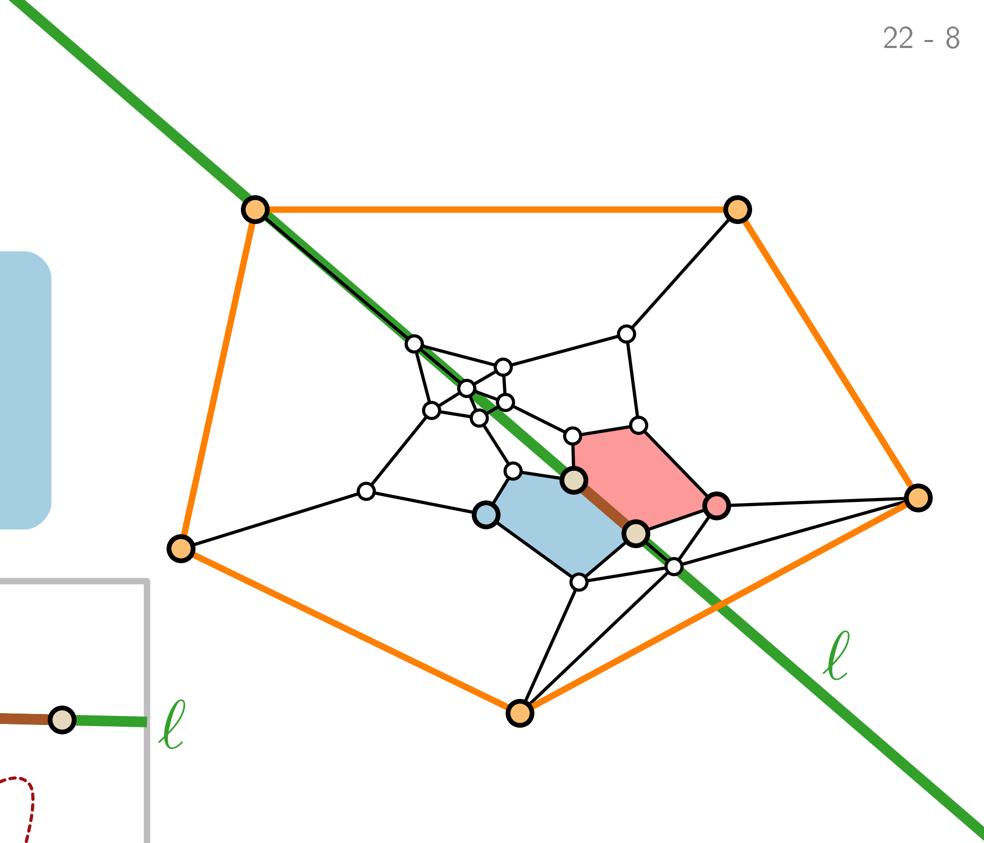

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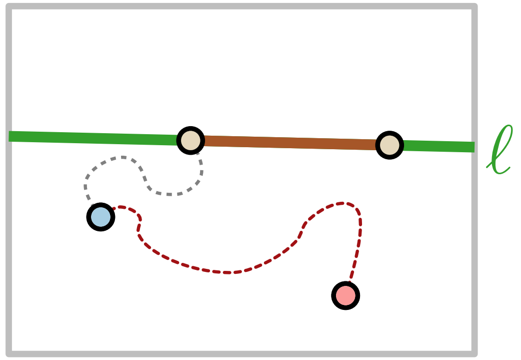

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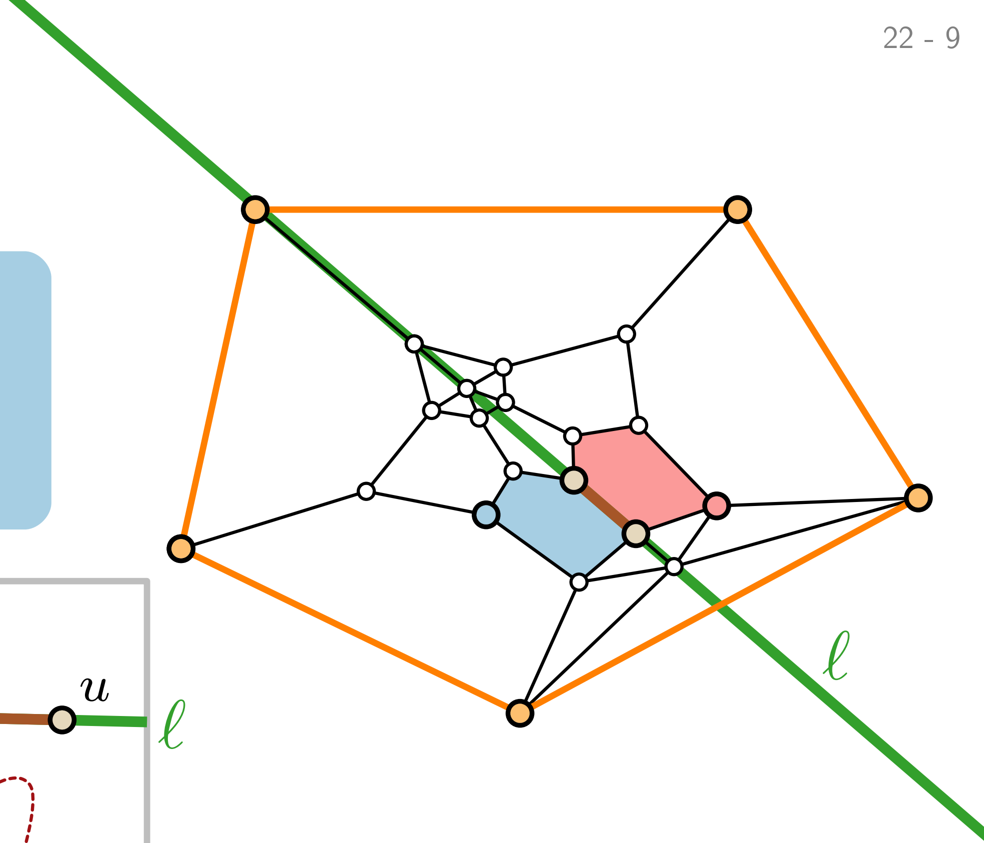

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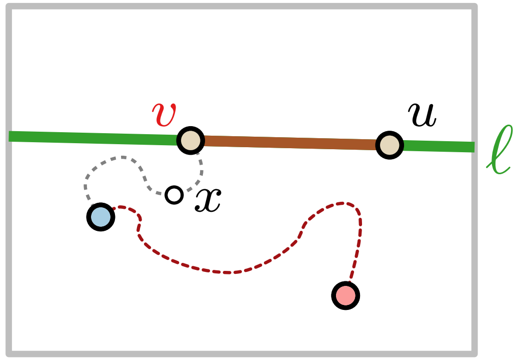

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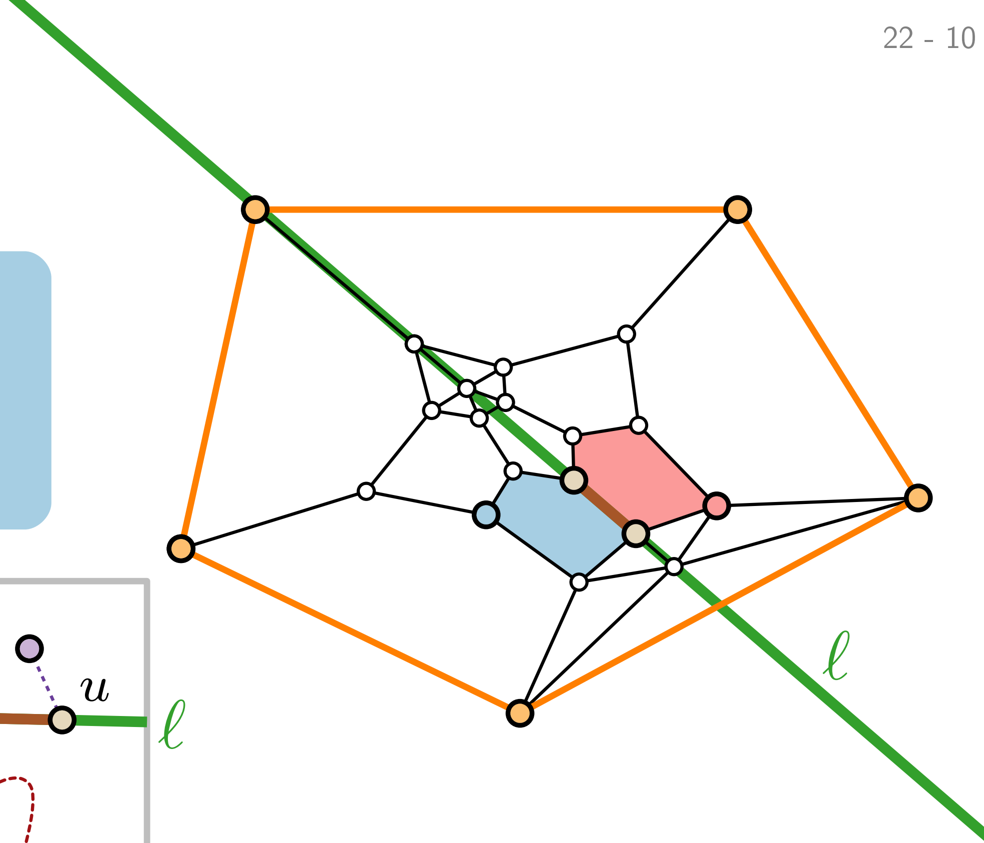

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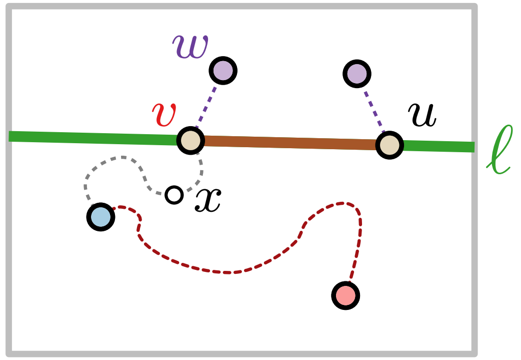

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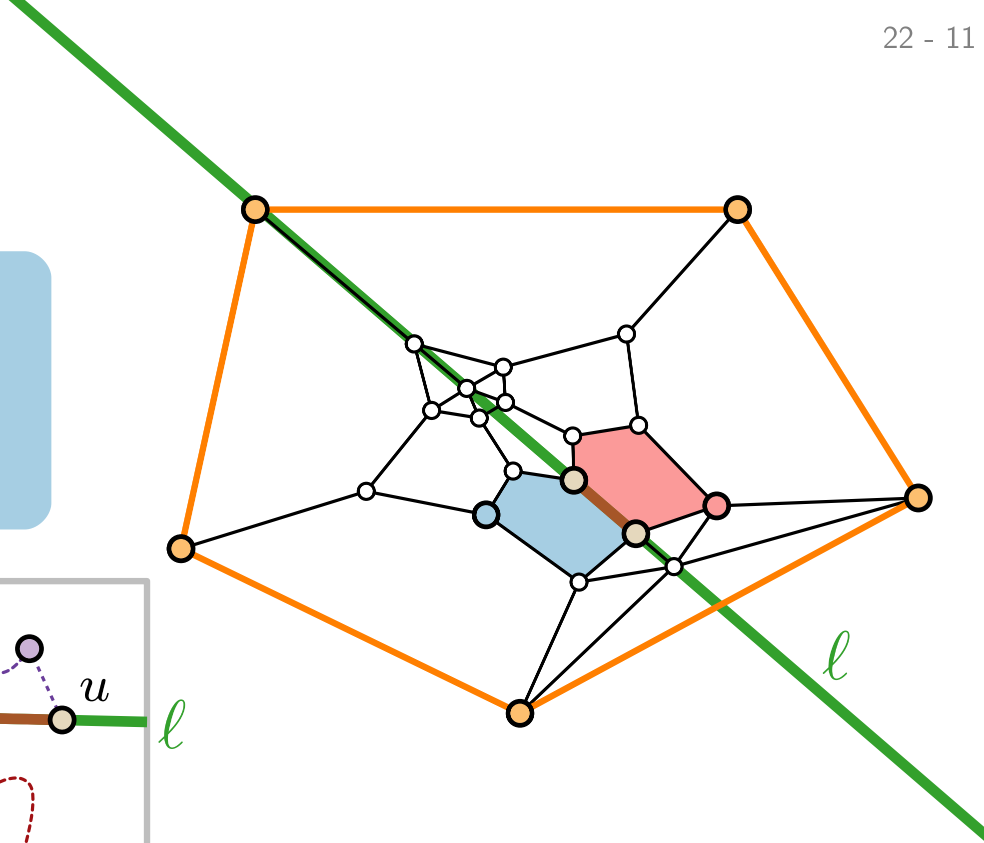

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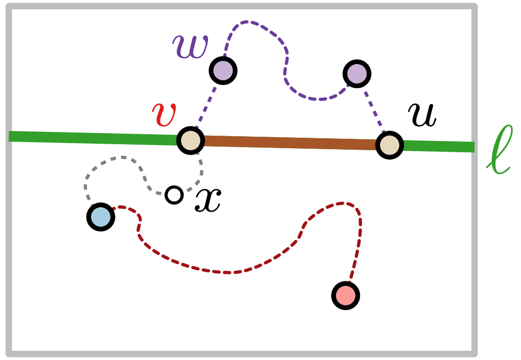

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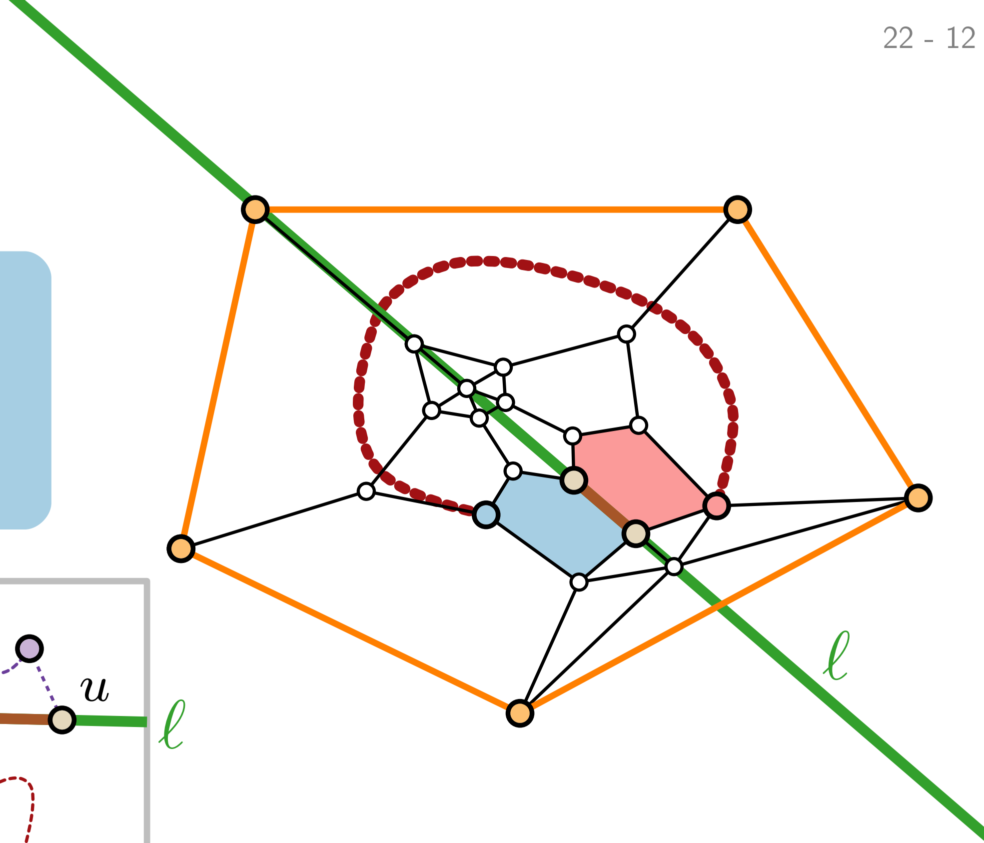

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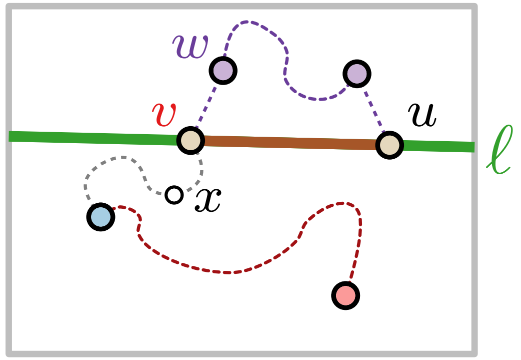

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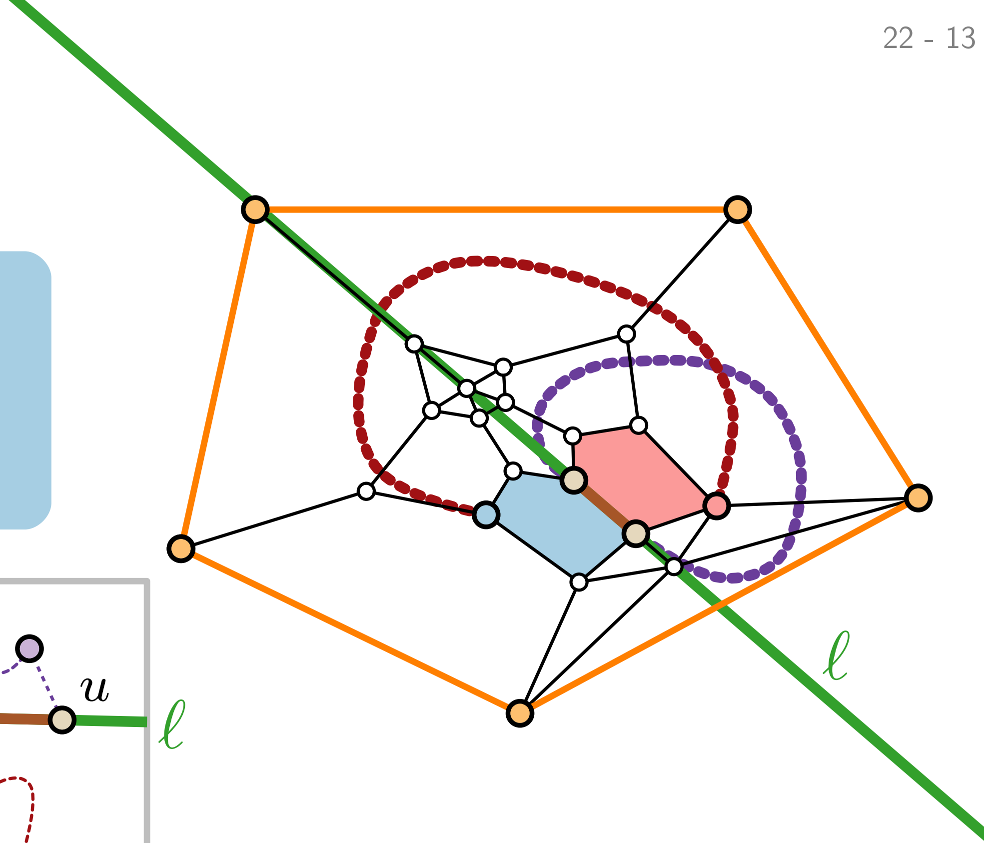

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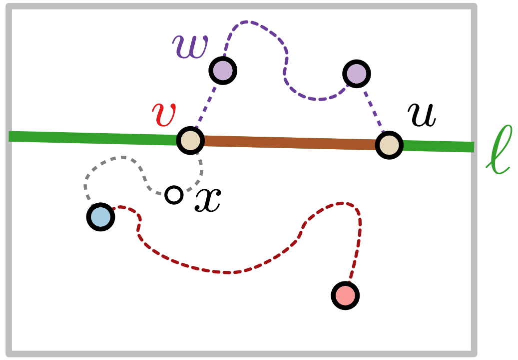

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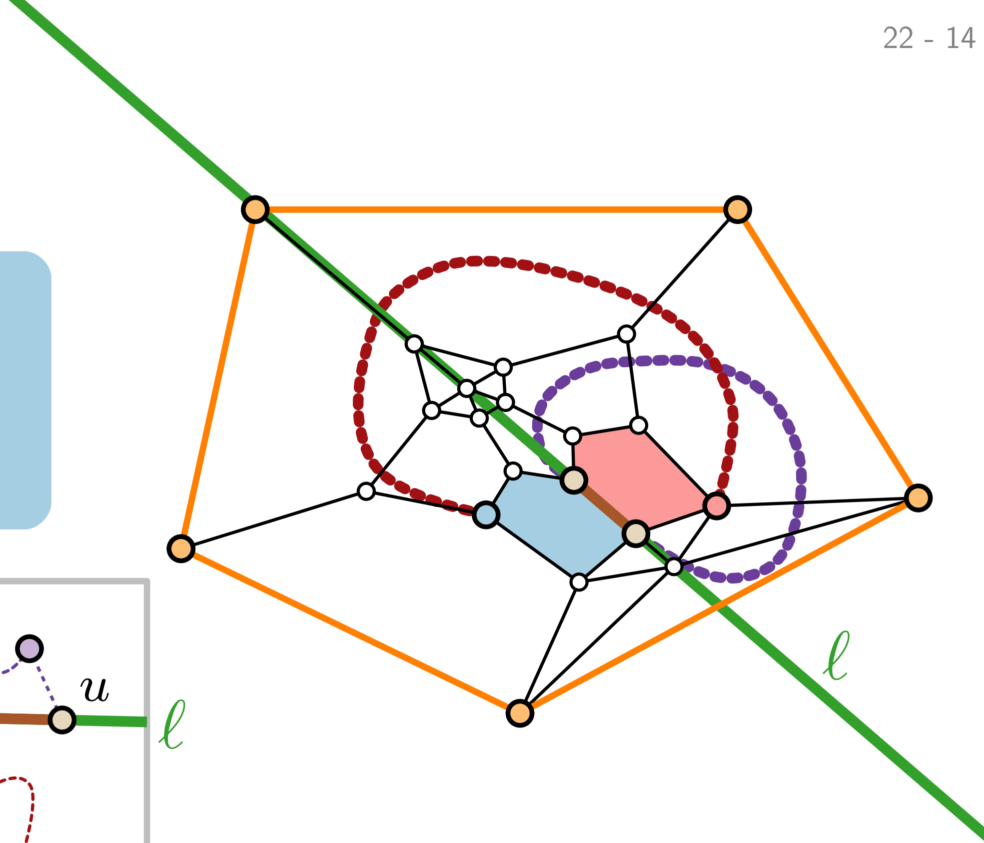

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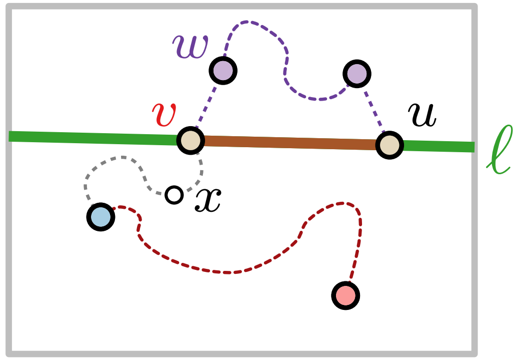

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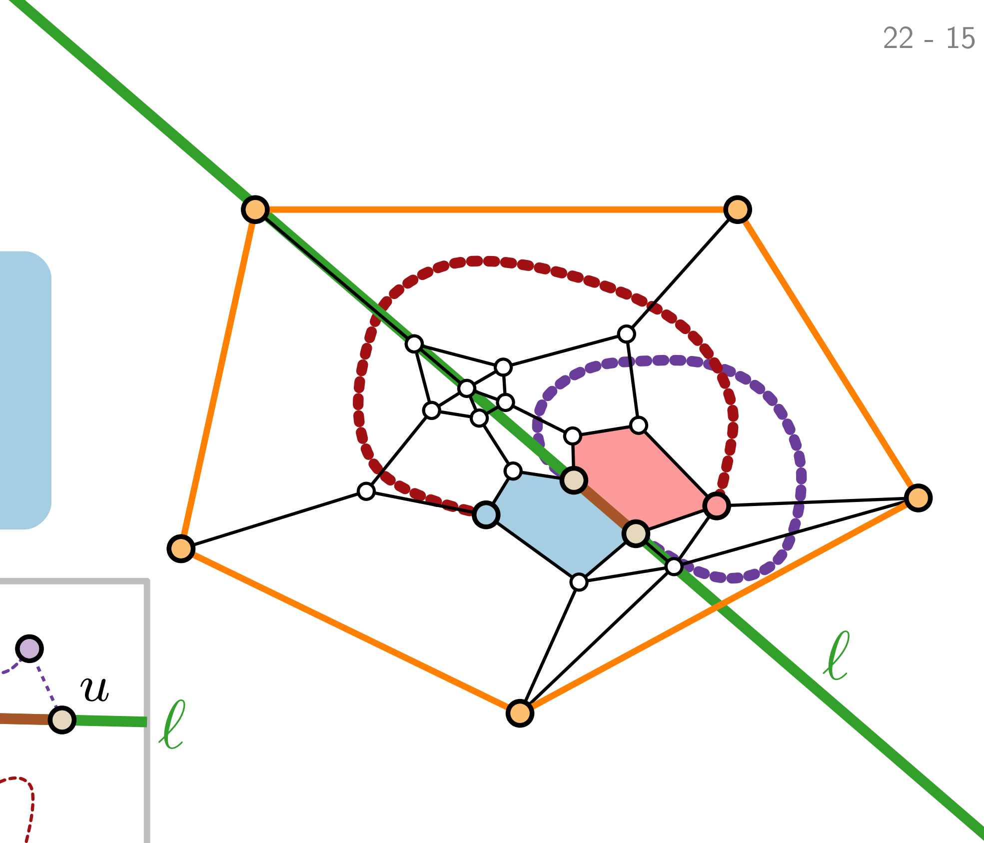
Proof of Tutte's Theorem



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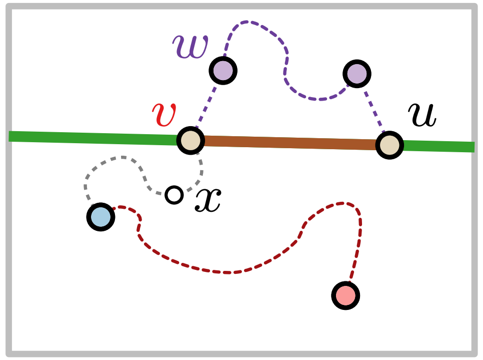

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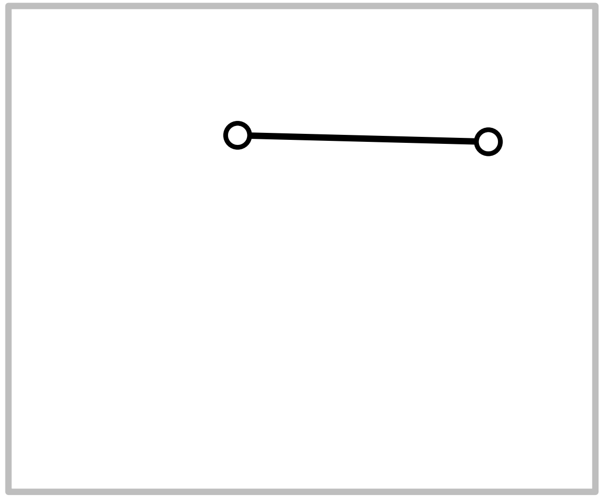

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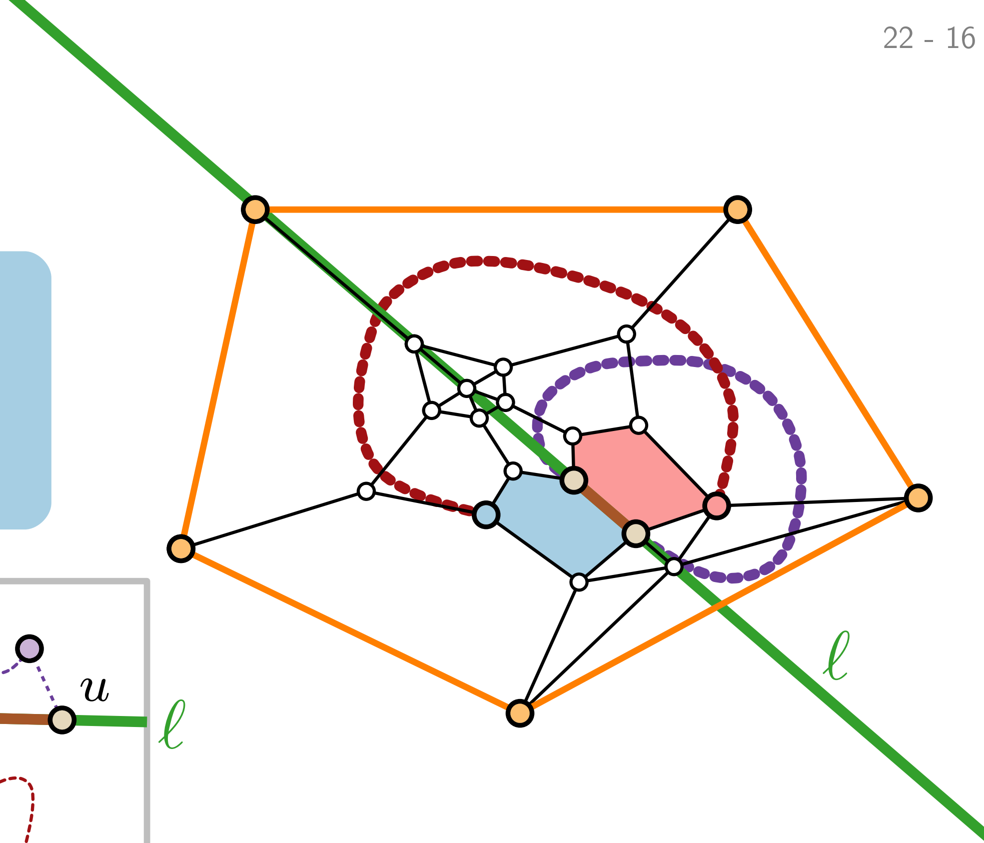

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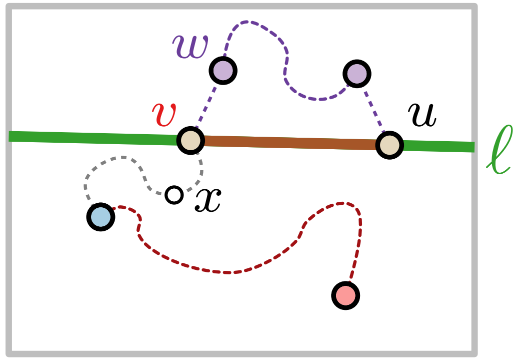

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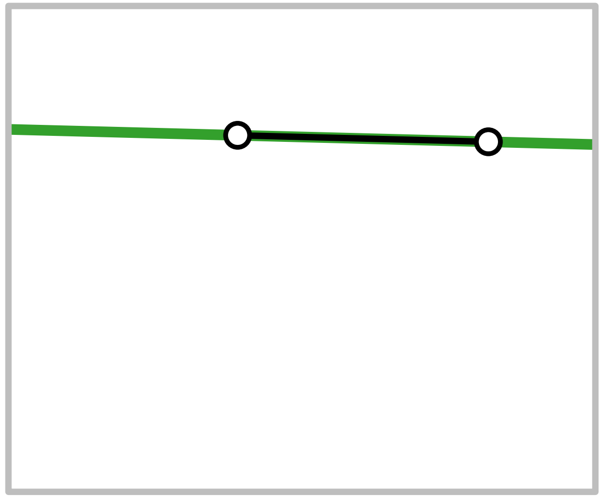

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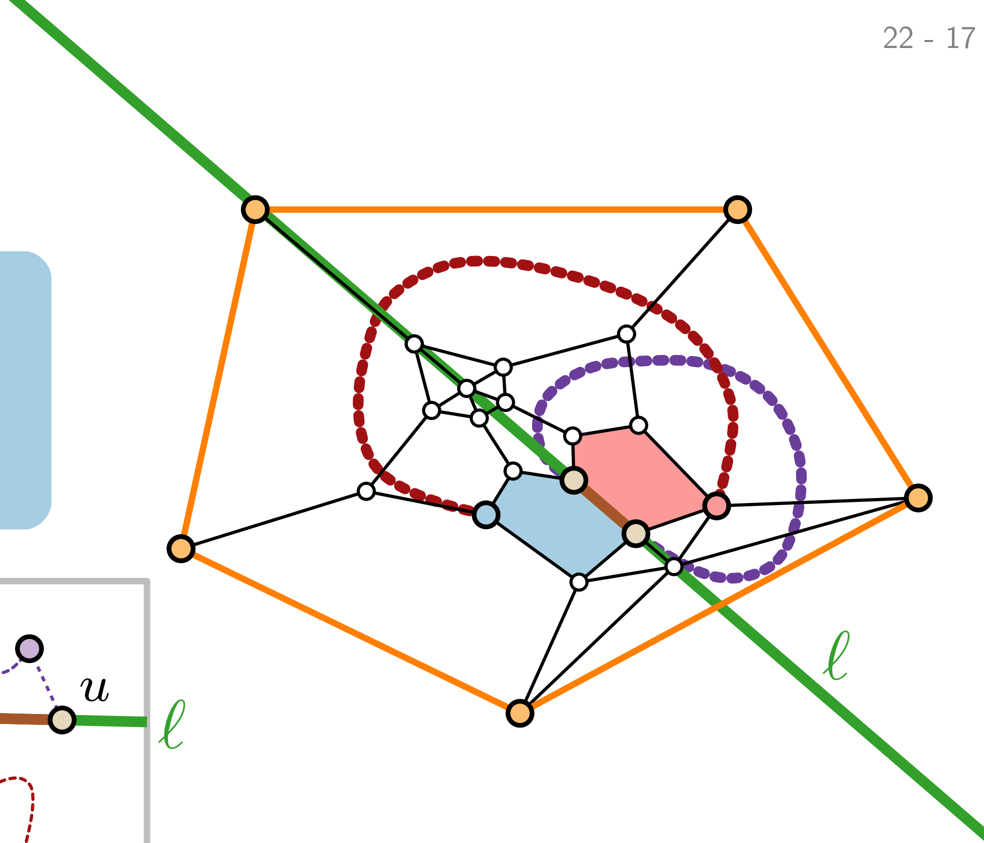

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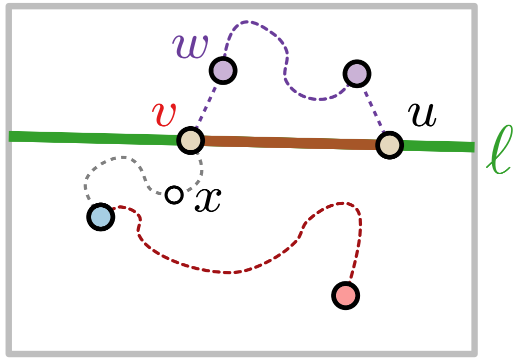

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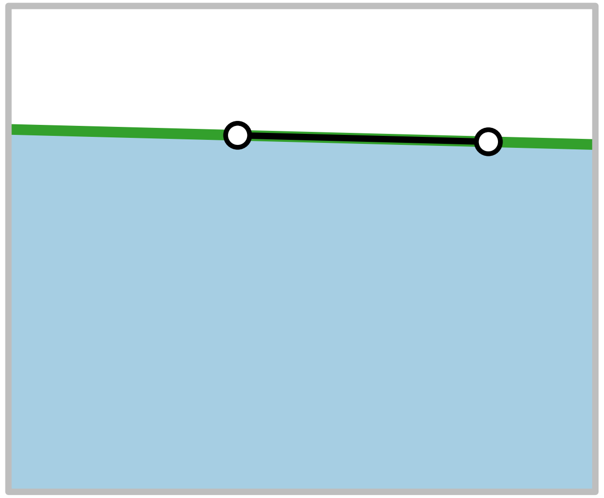

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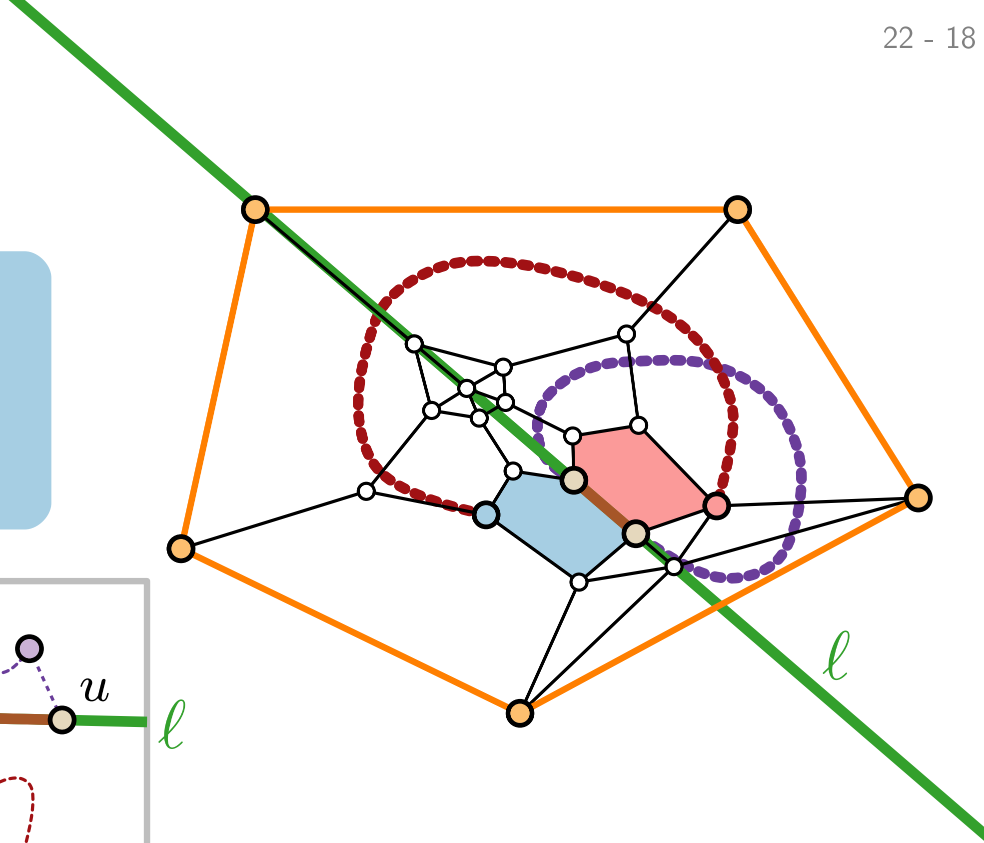

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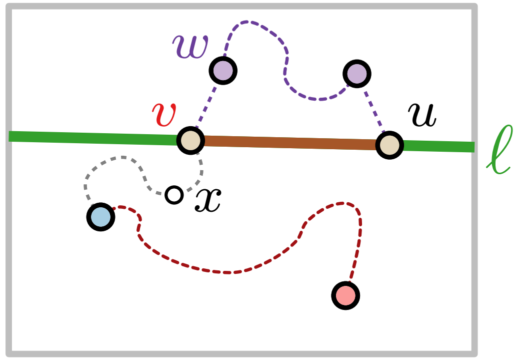

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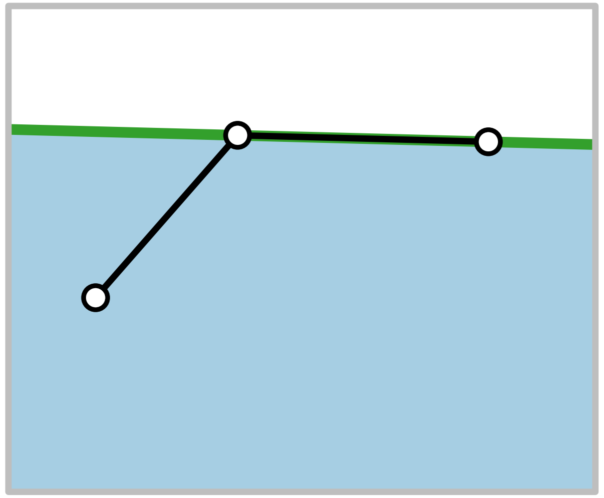

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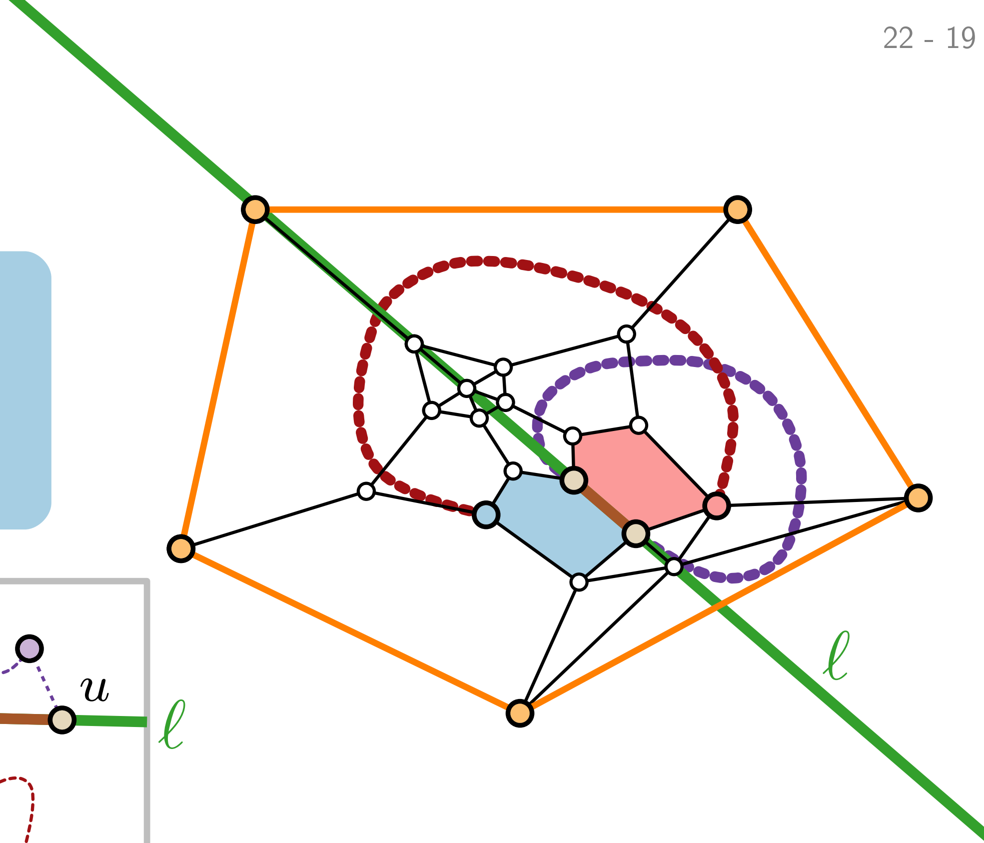

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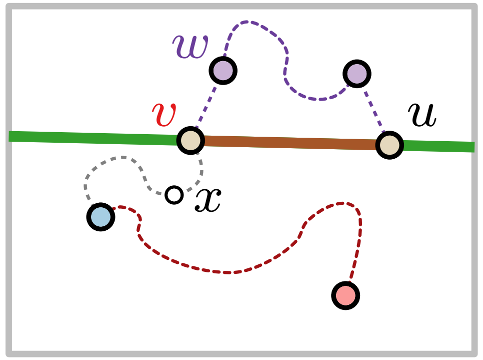

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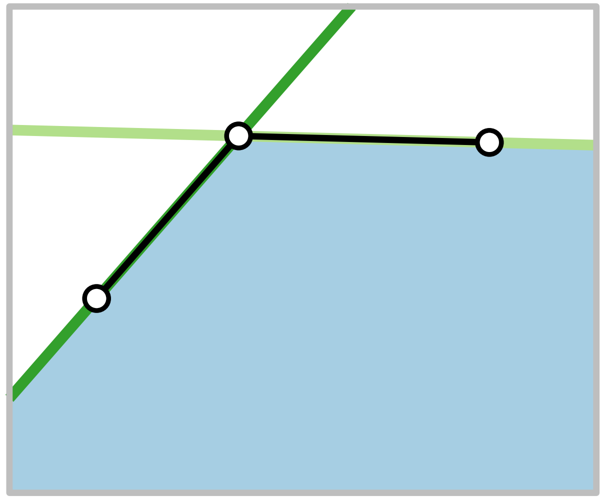

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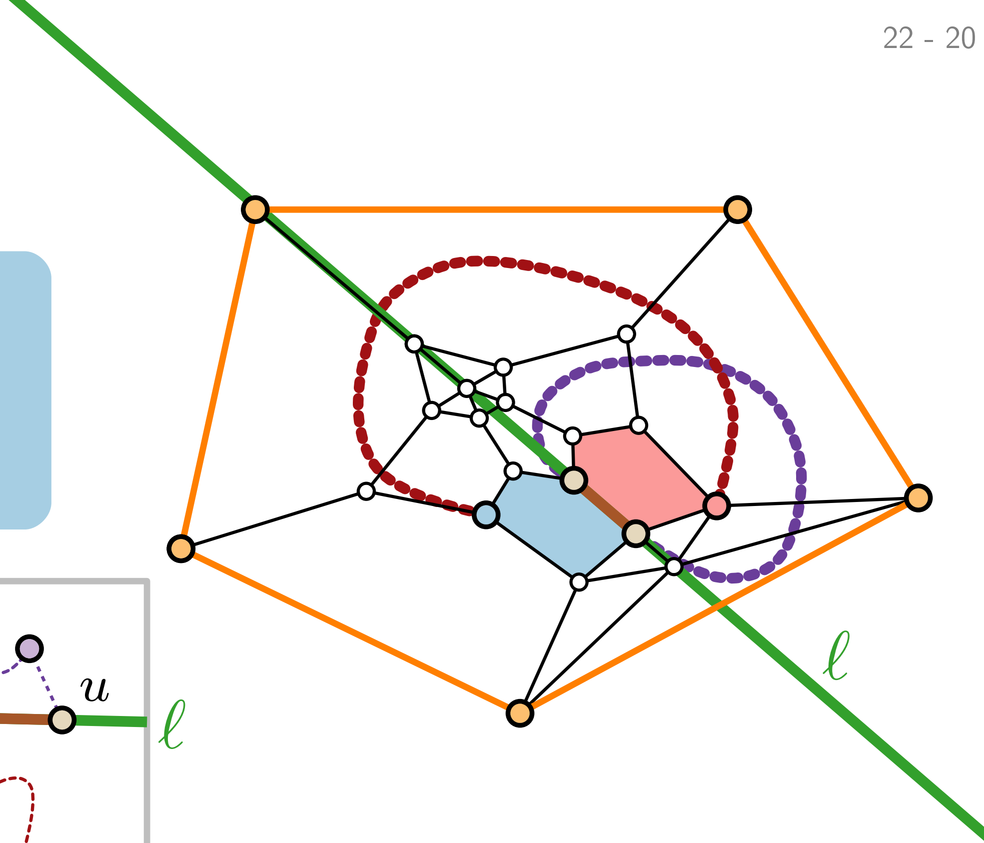

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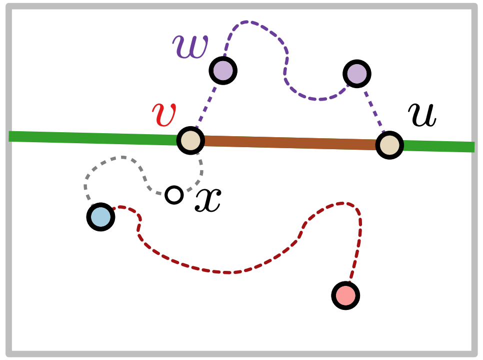

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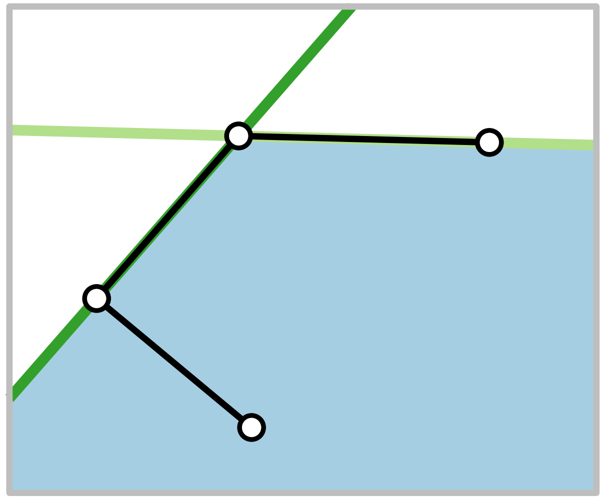

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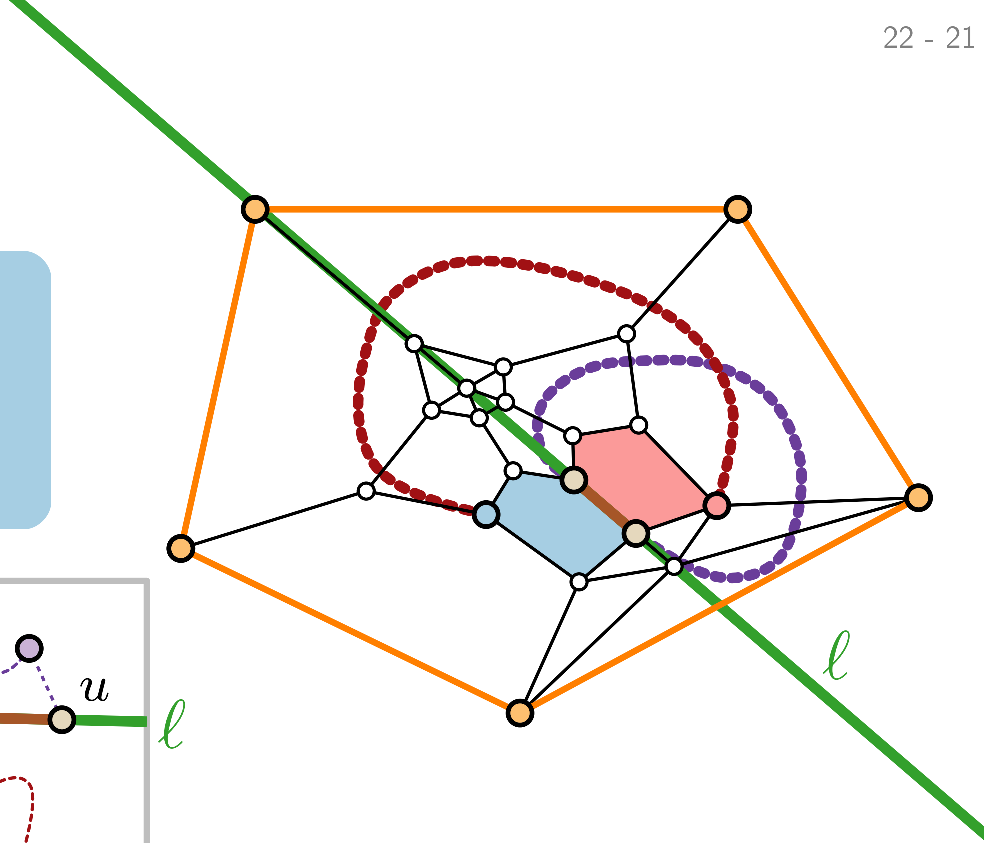

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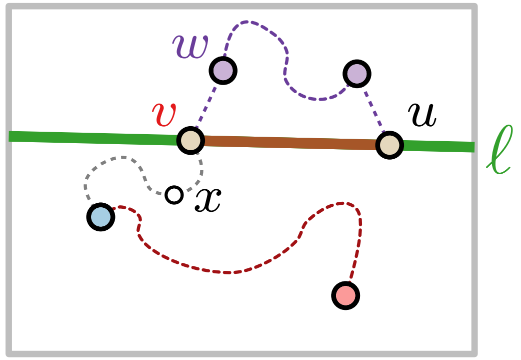

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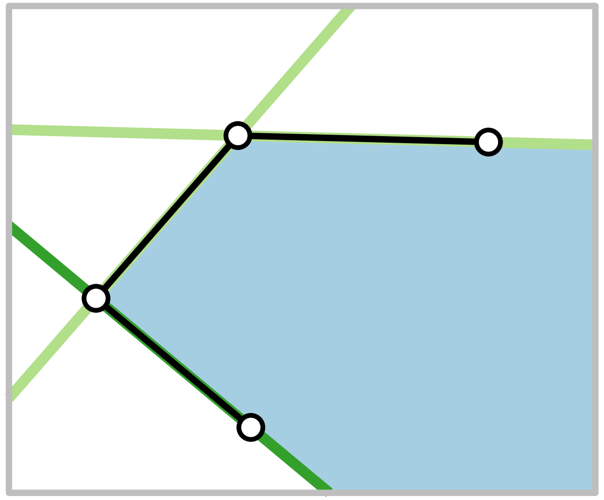

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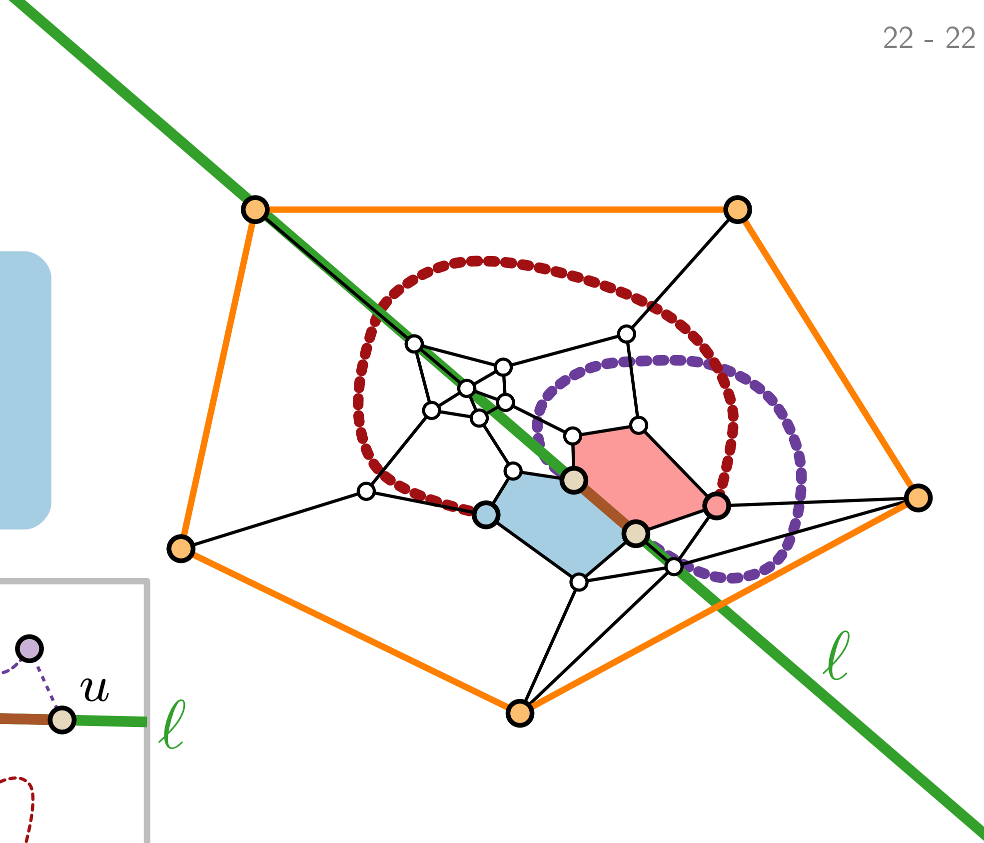

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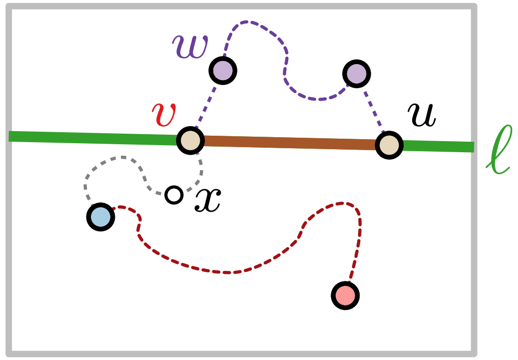

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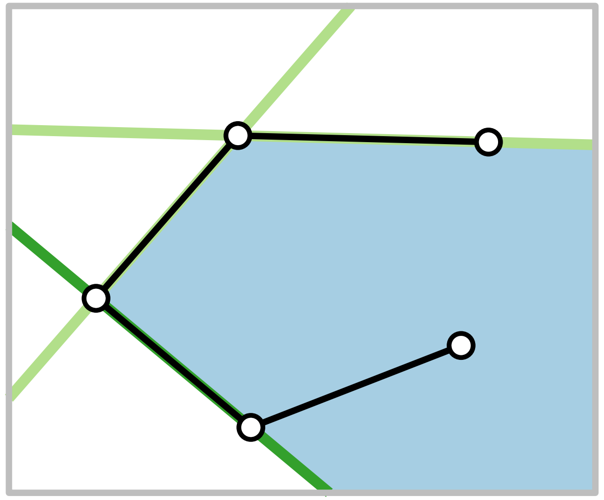

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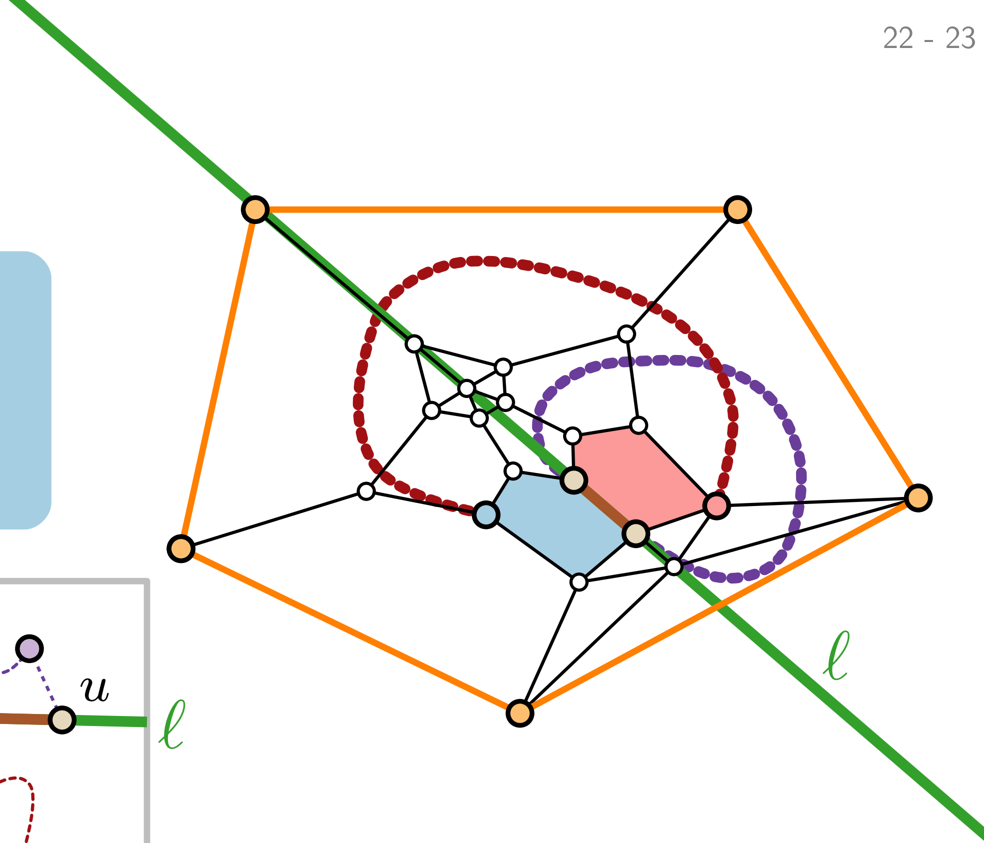

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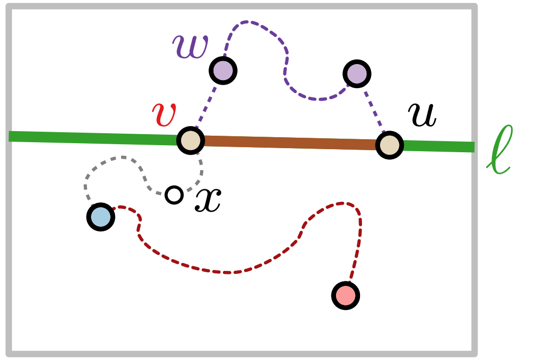

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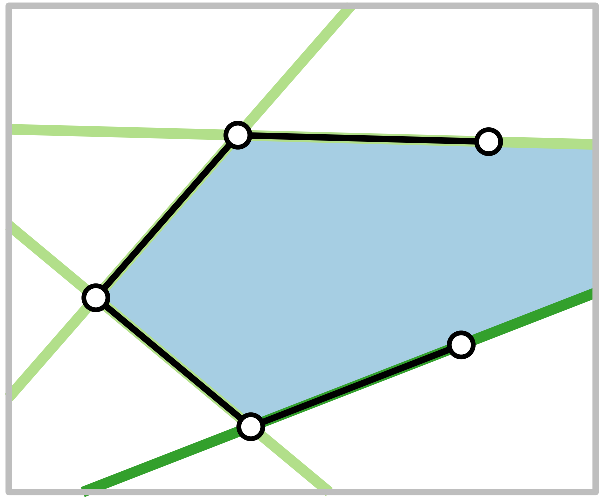

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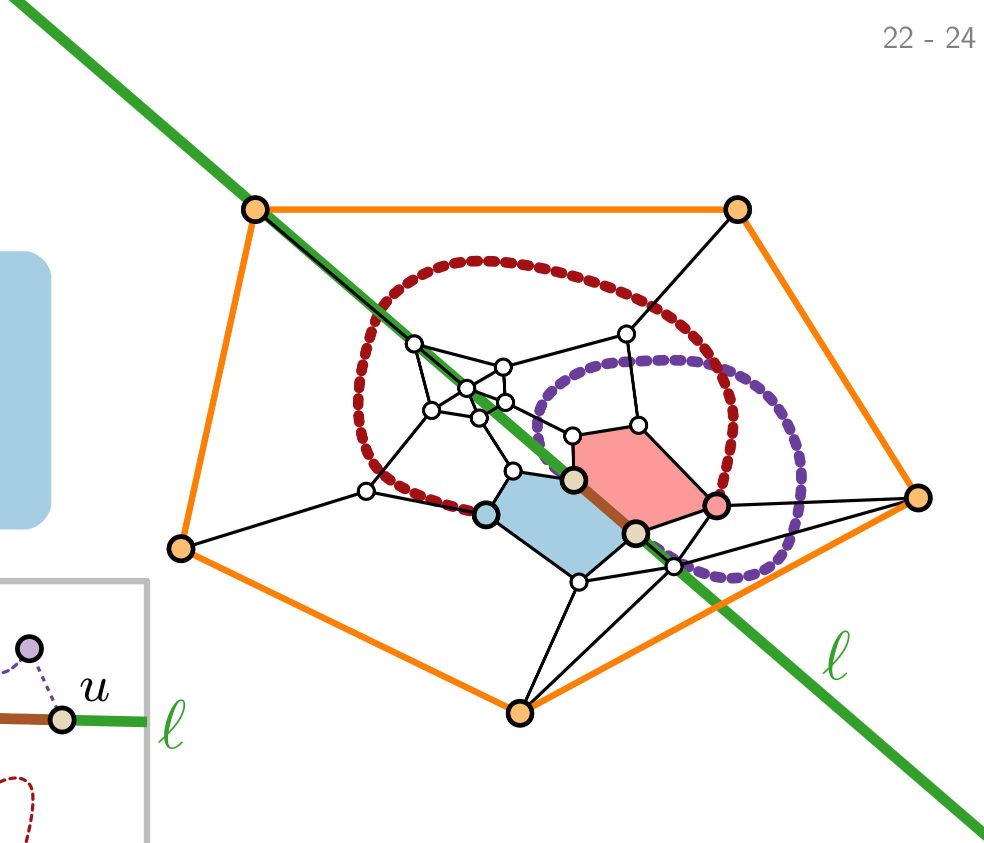

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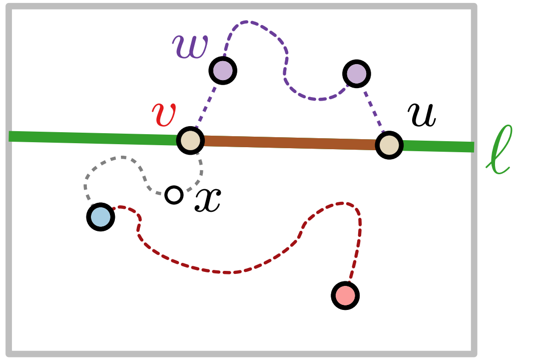

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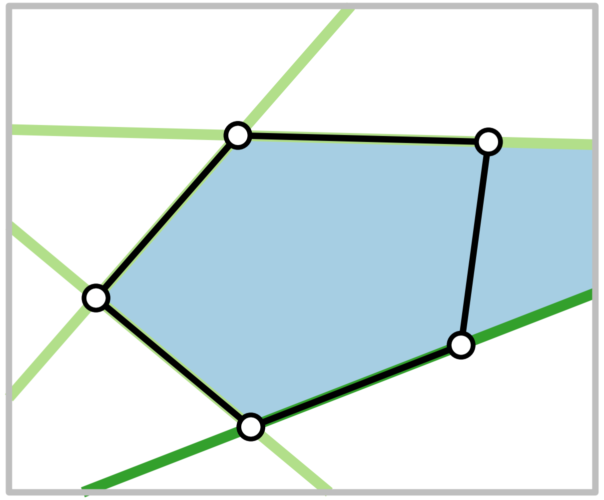

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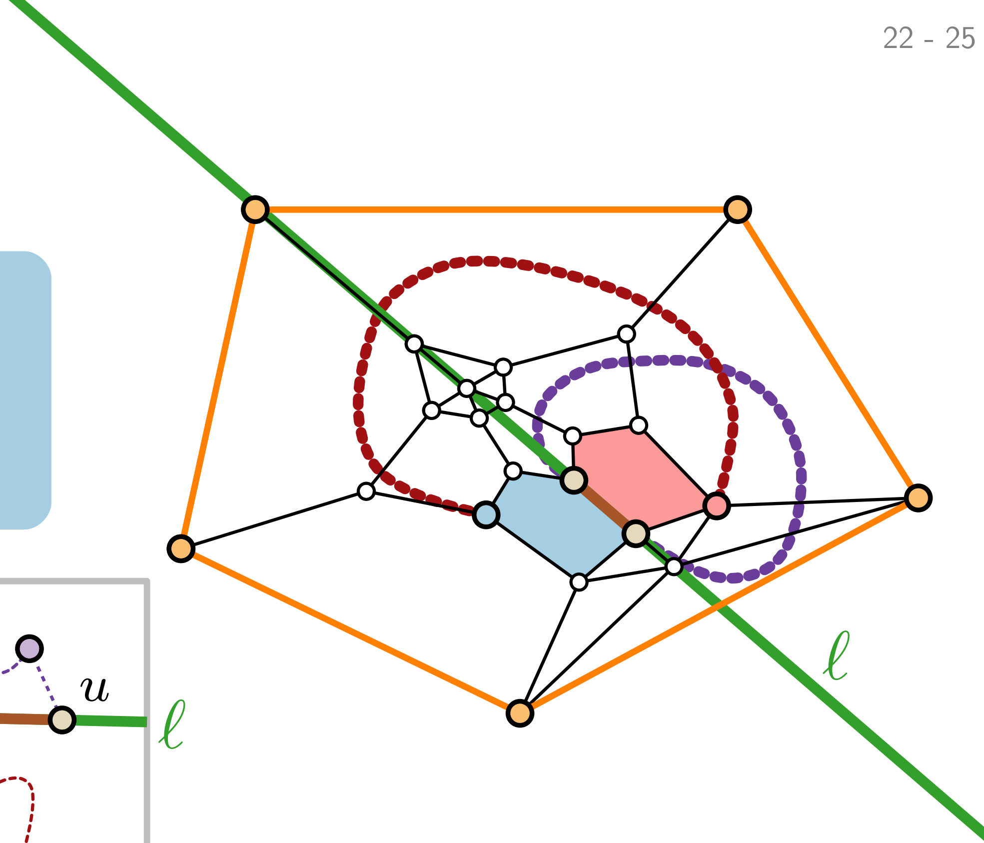

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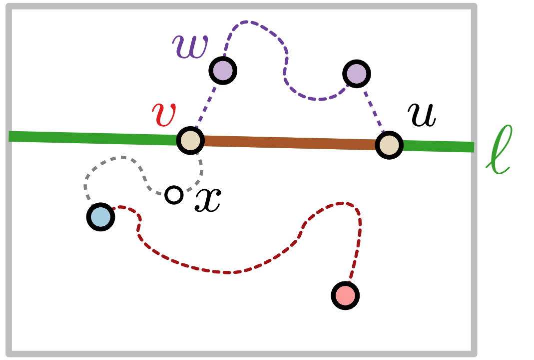

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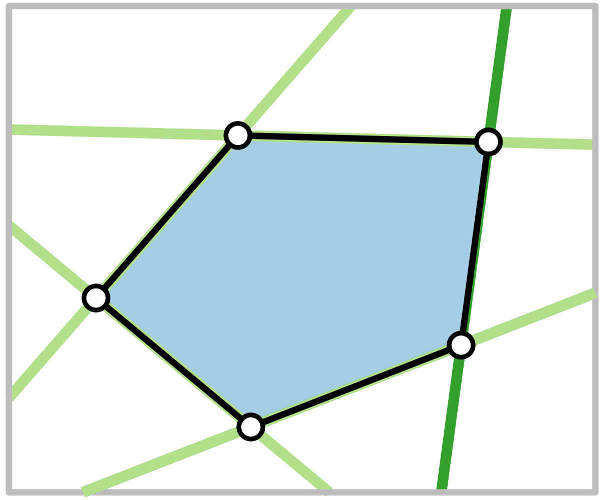

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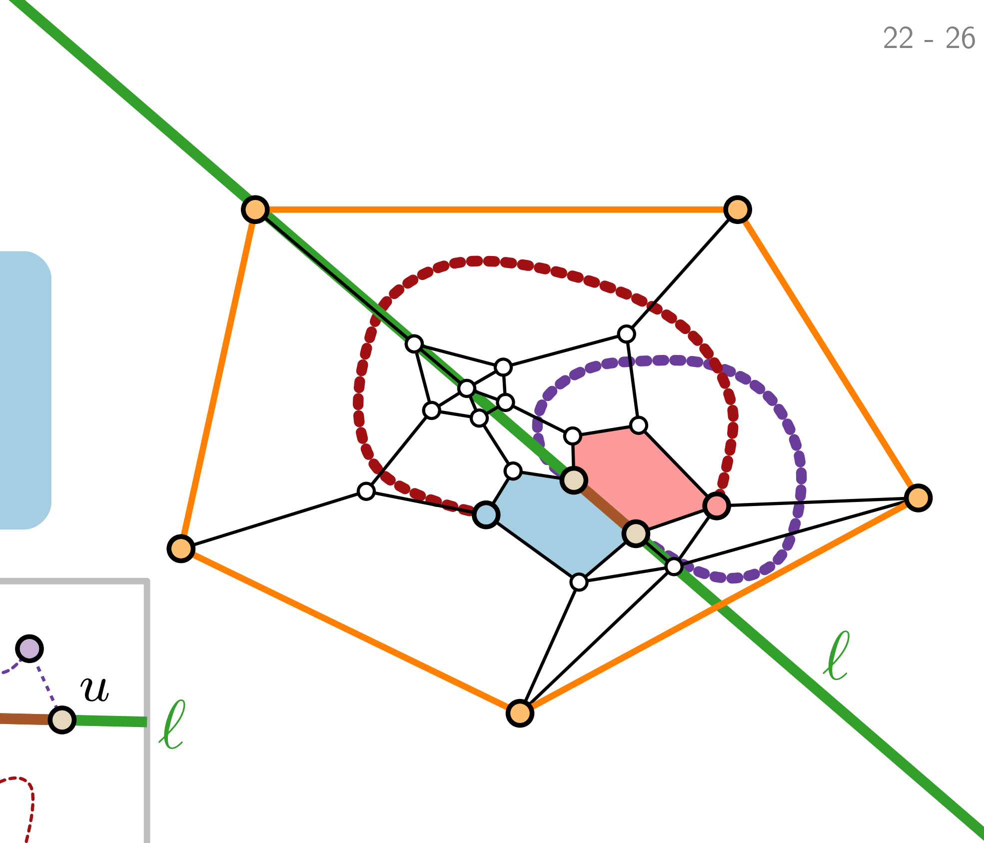

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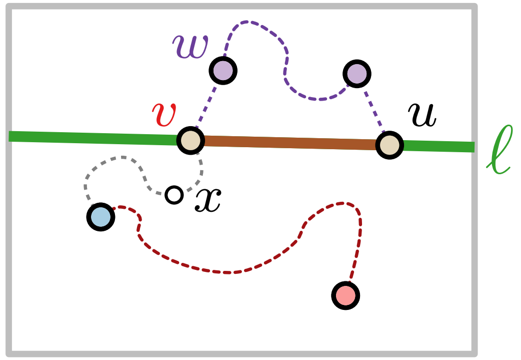

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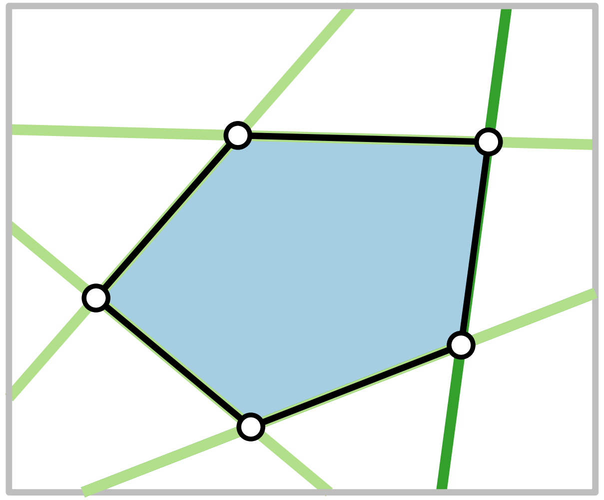

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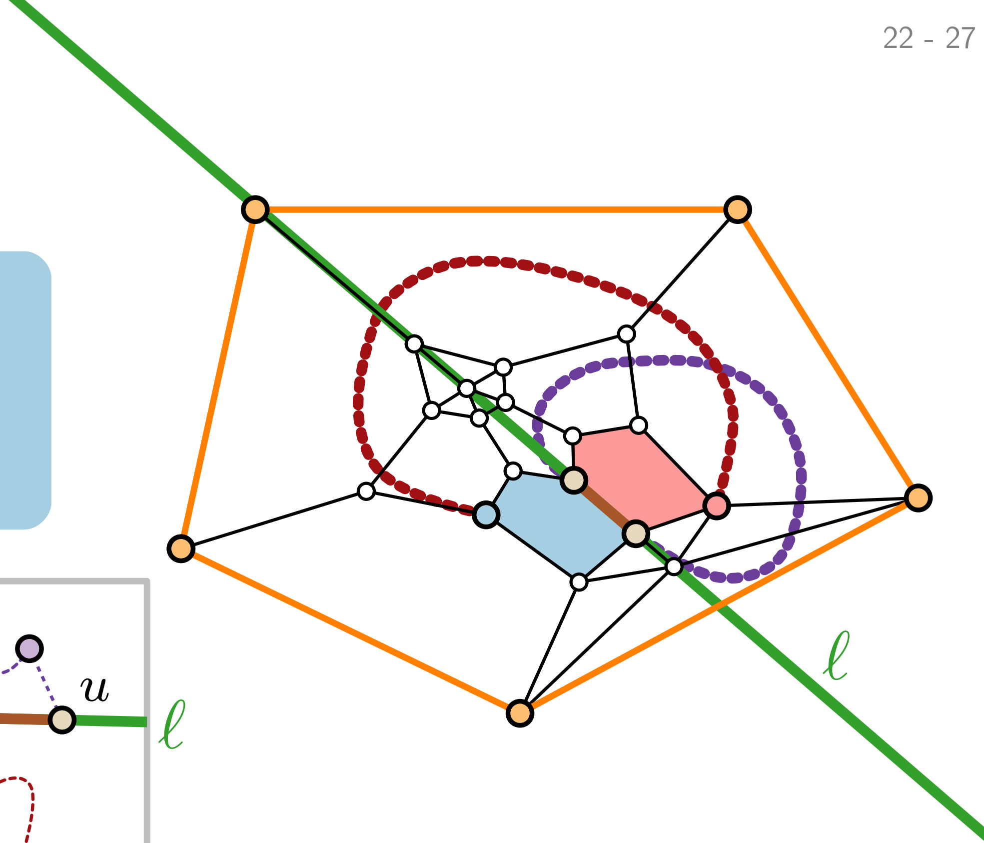

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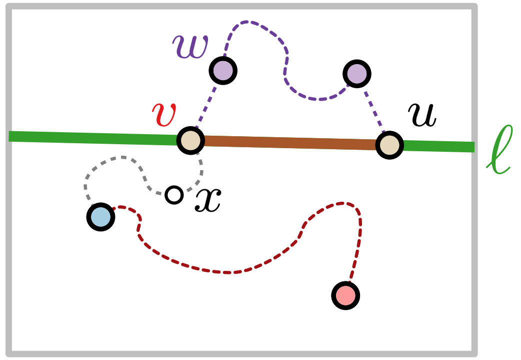

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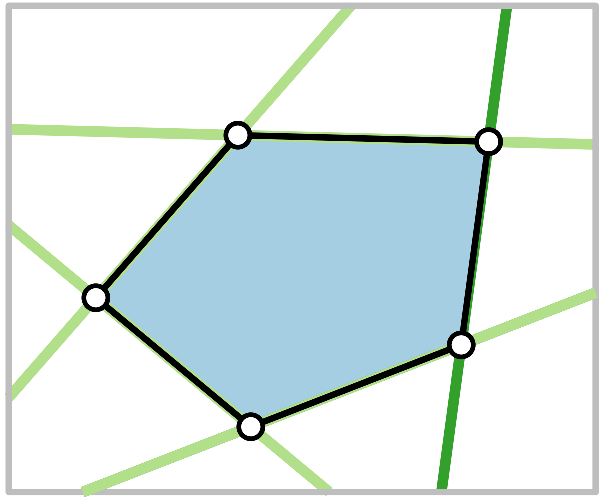

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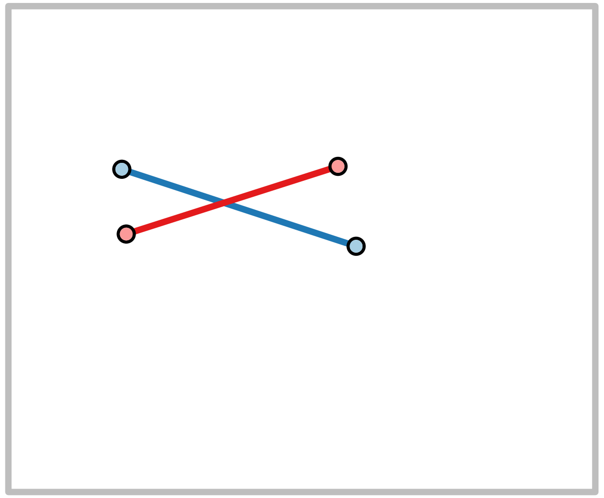

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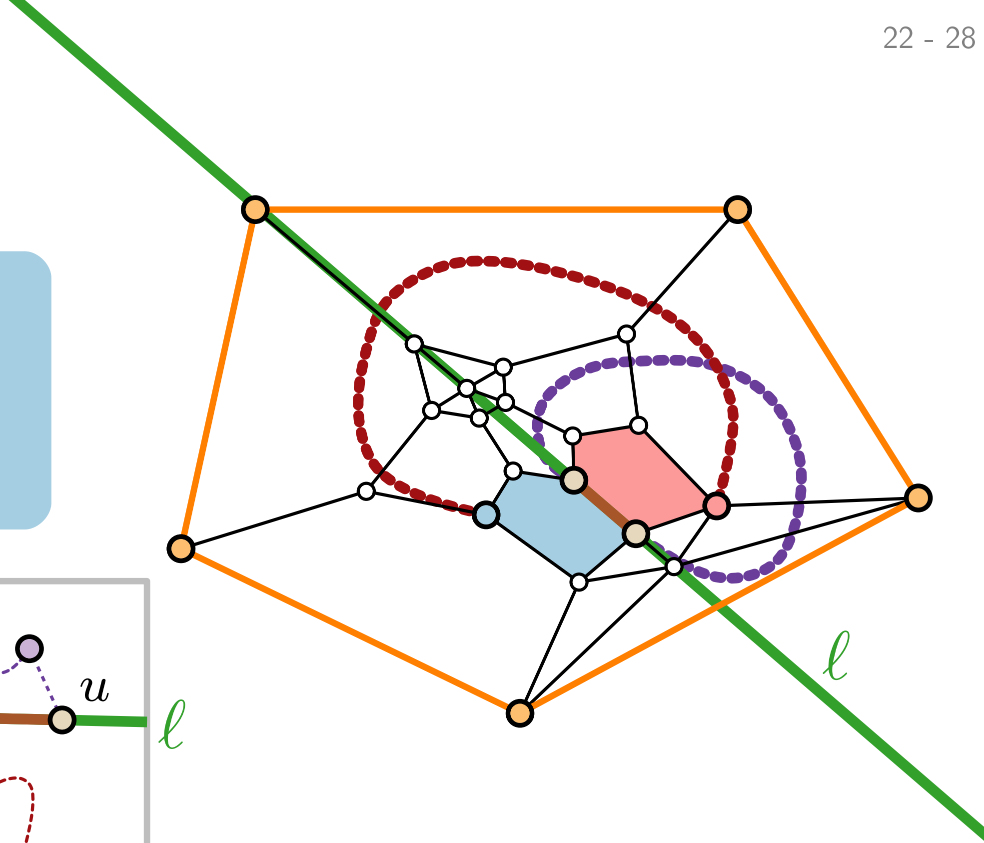

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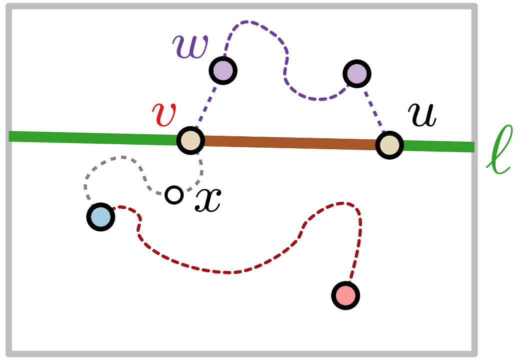

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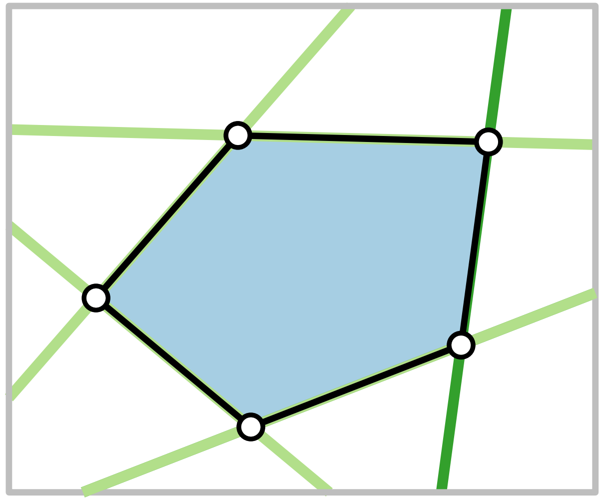

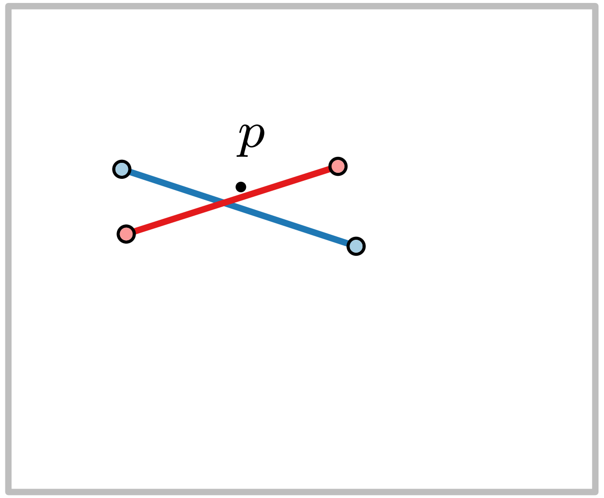

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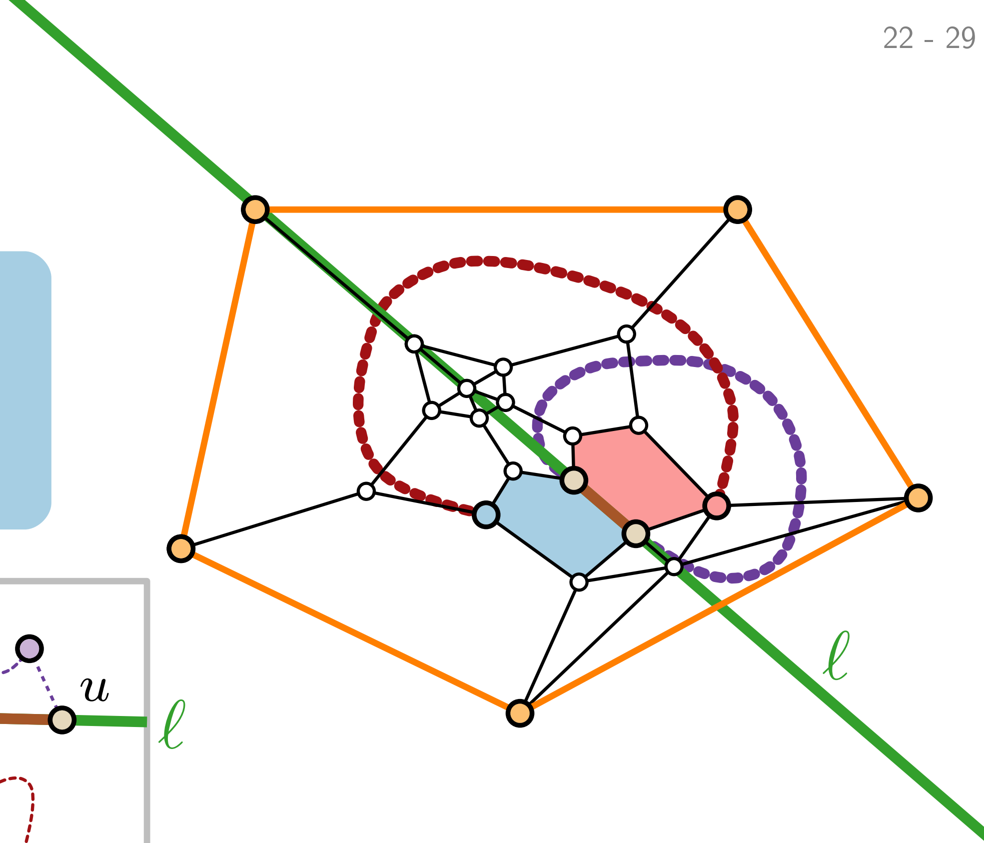

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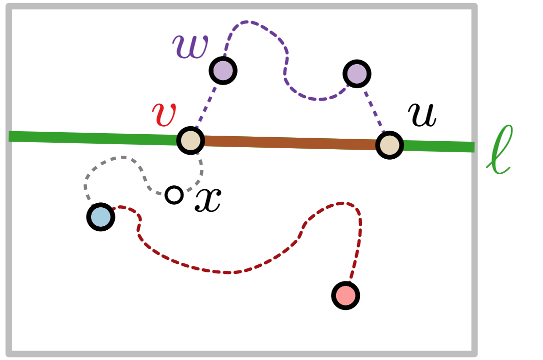

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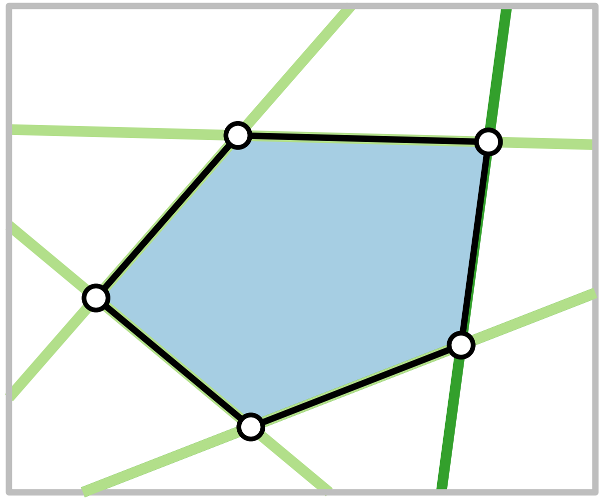

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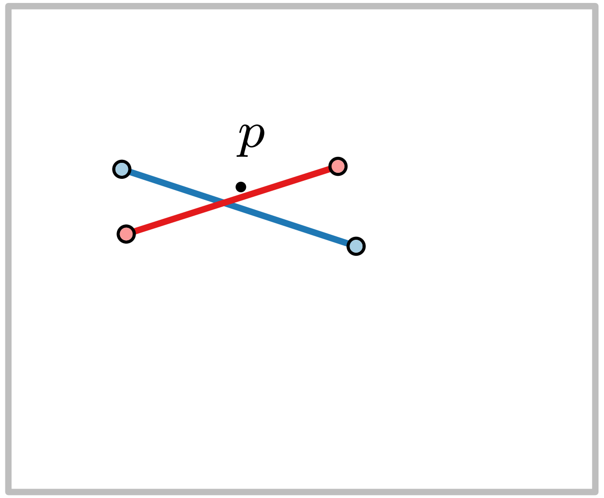

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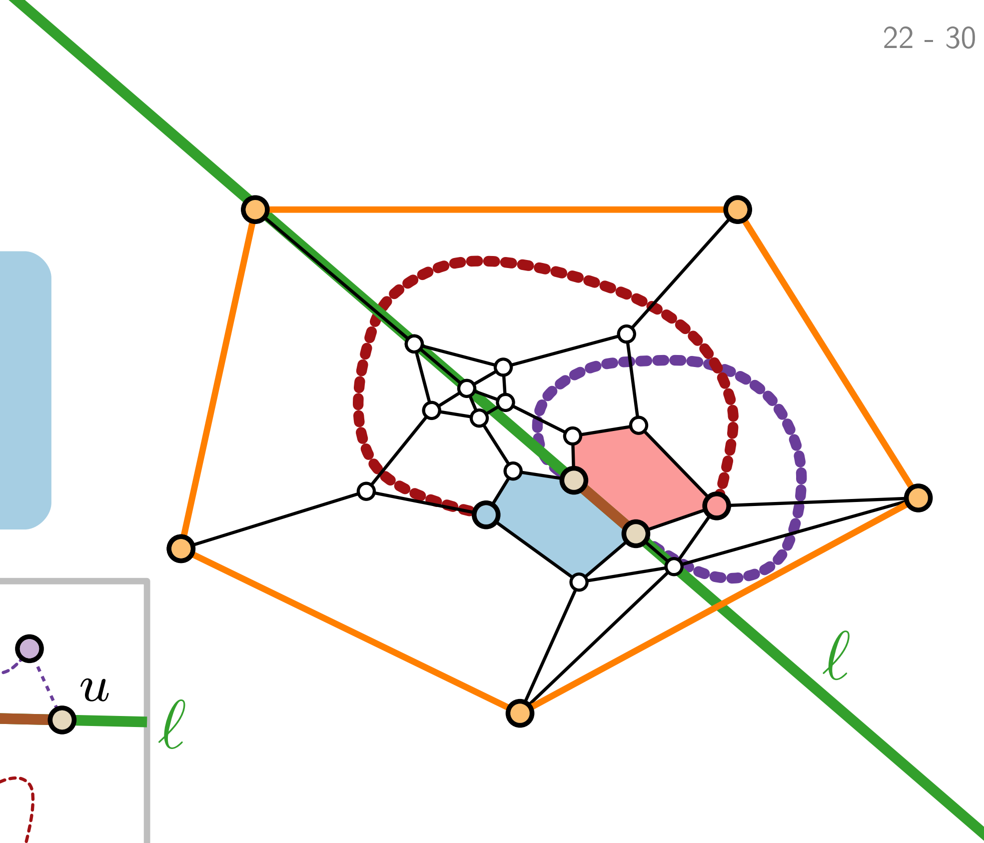

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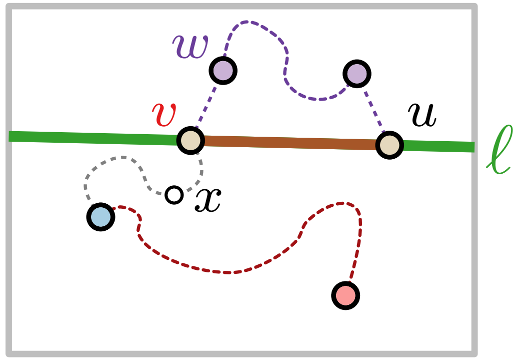

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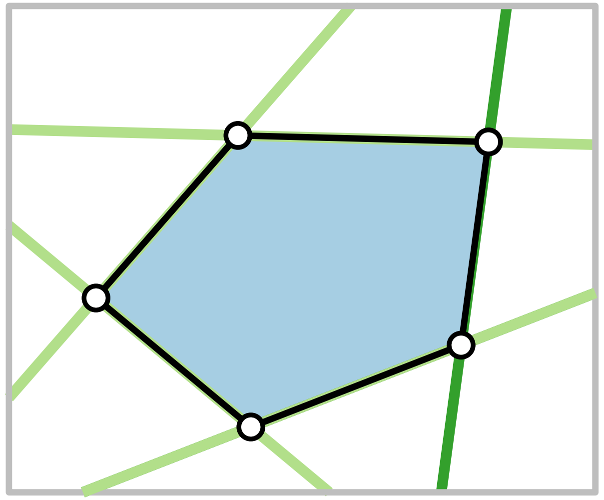

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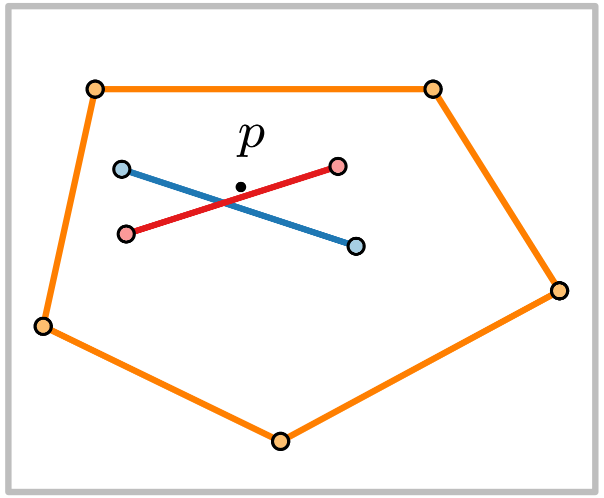

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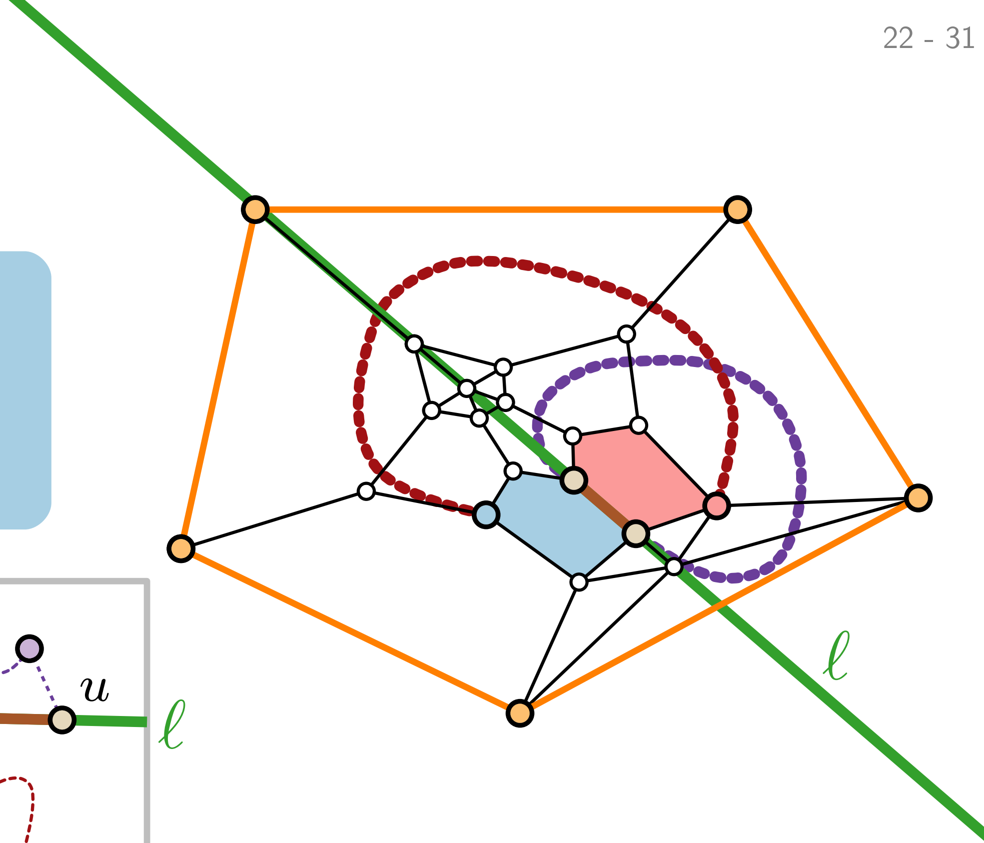

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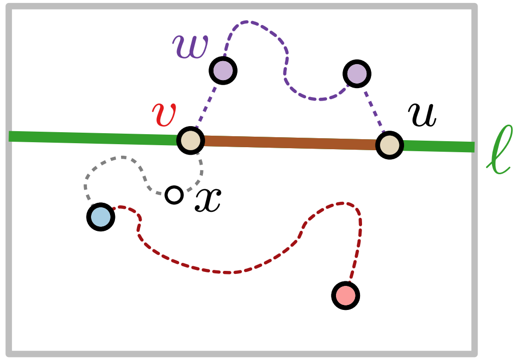

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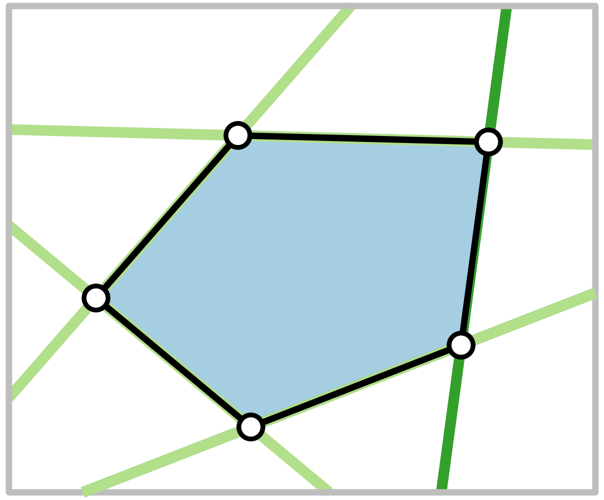

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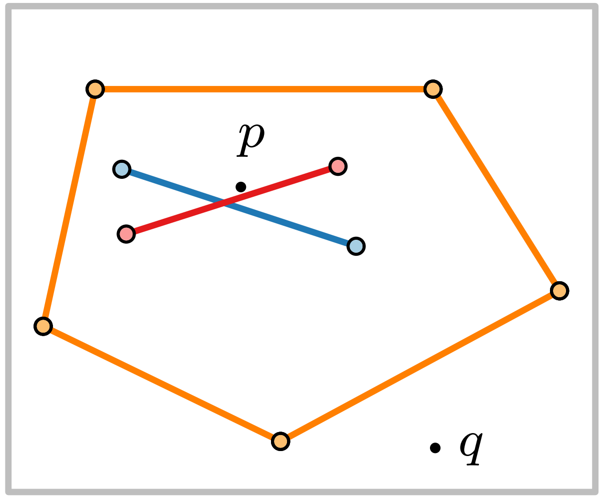

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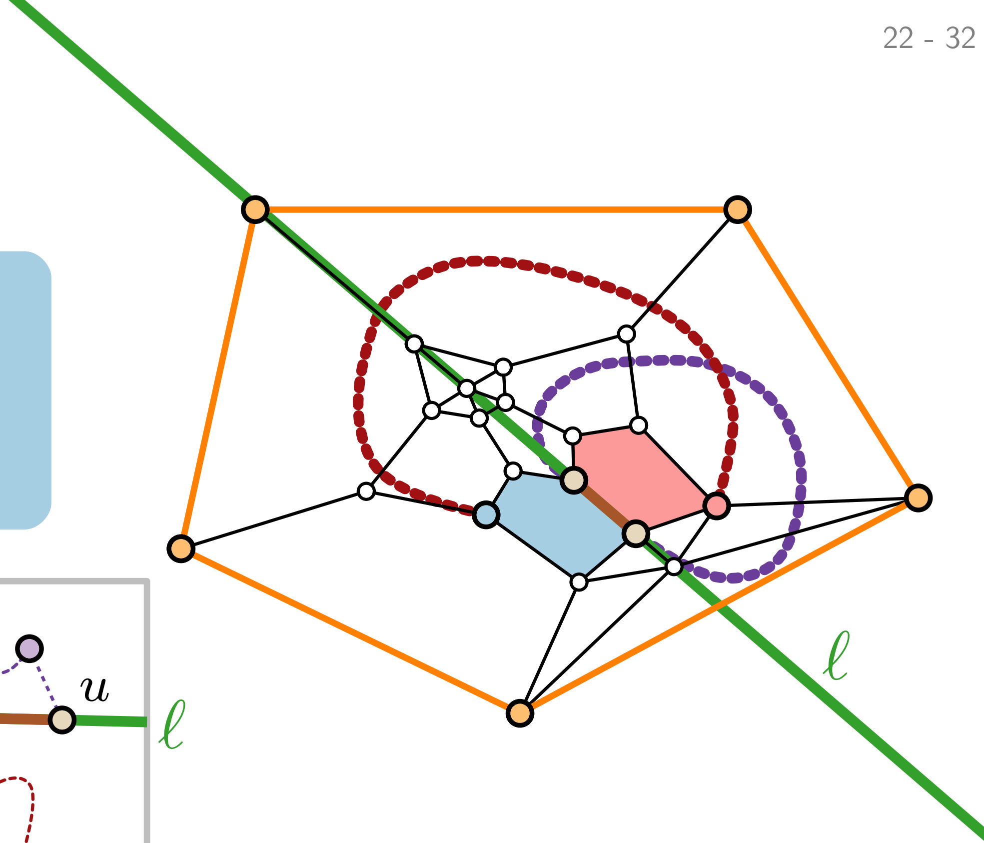

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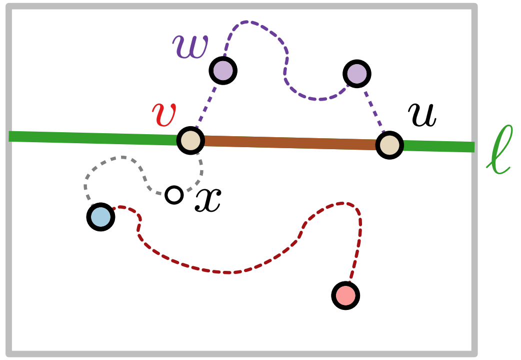

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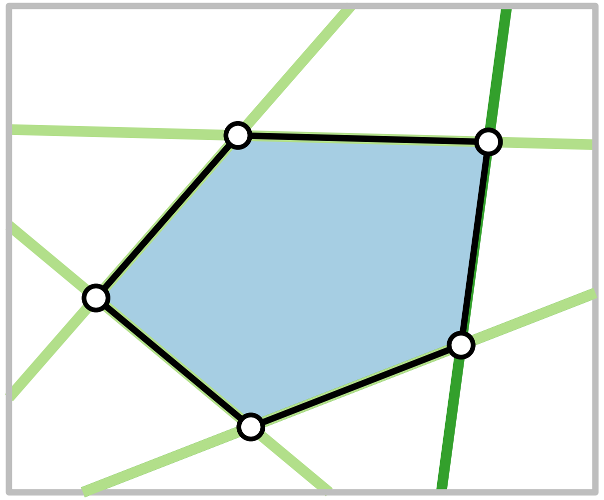

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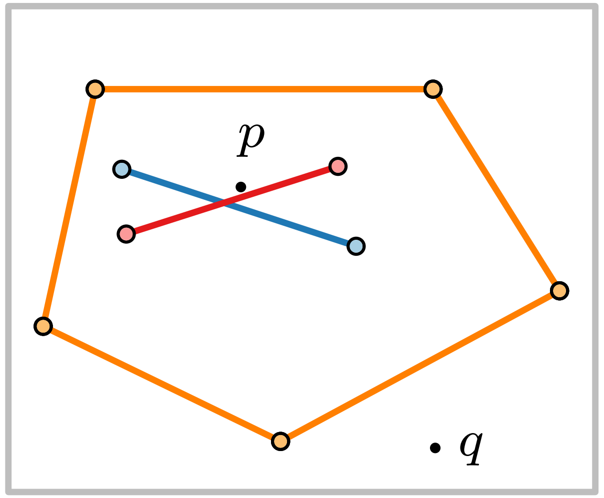

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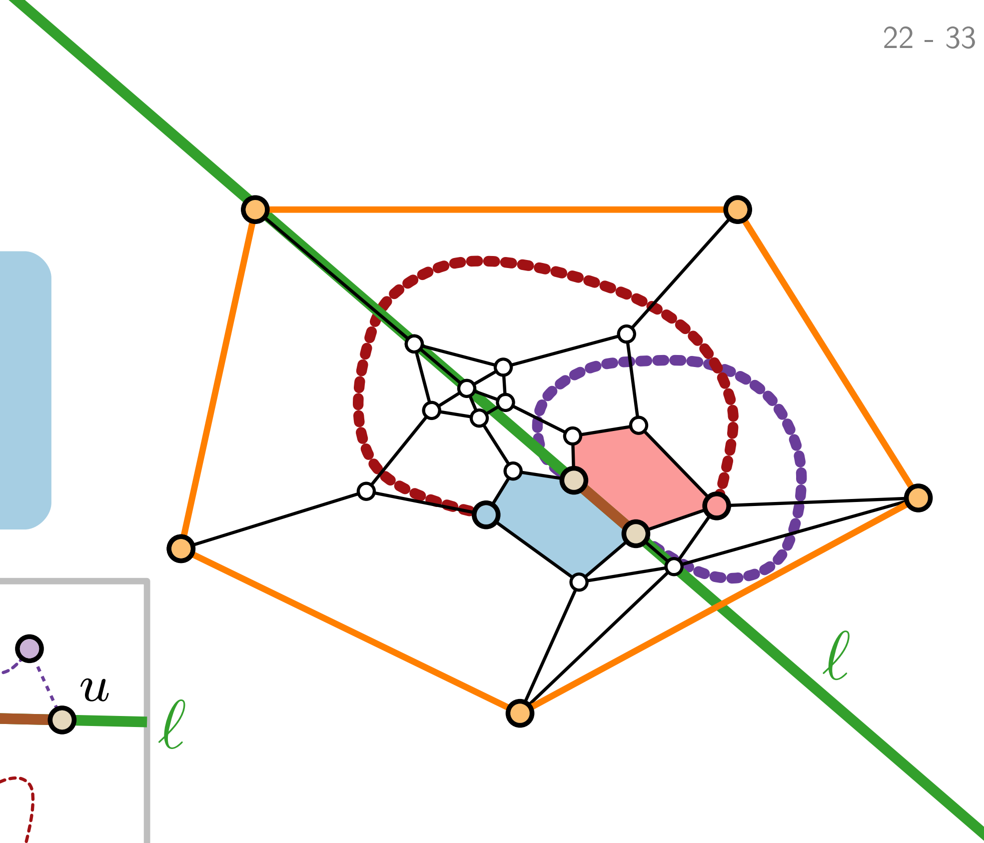

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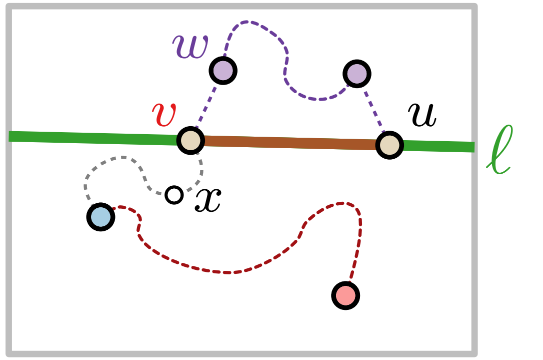

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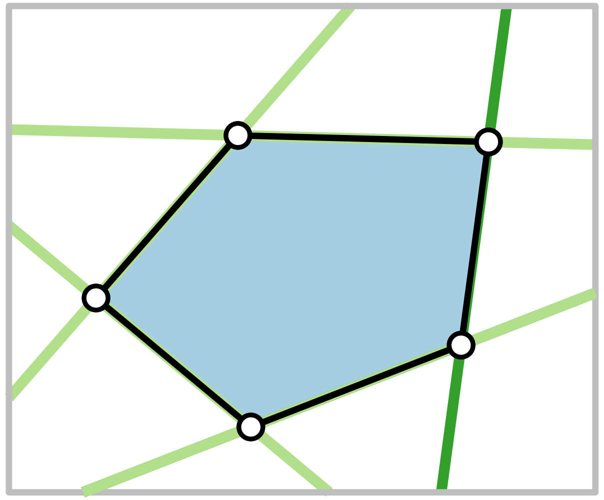

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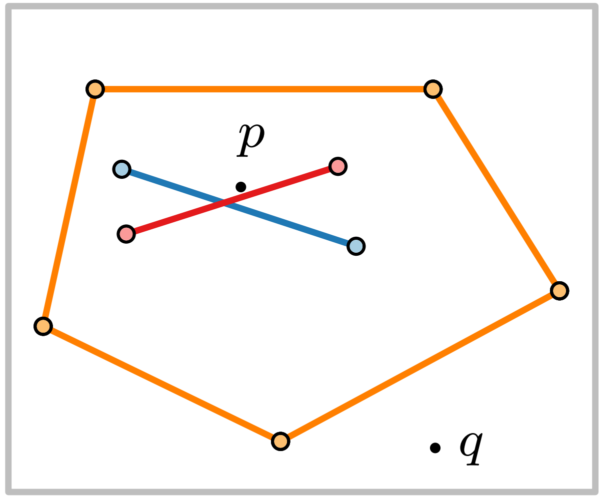

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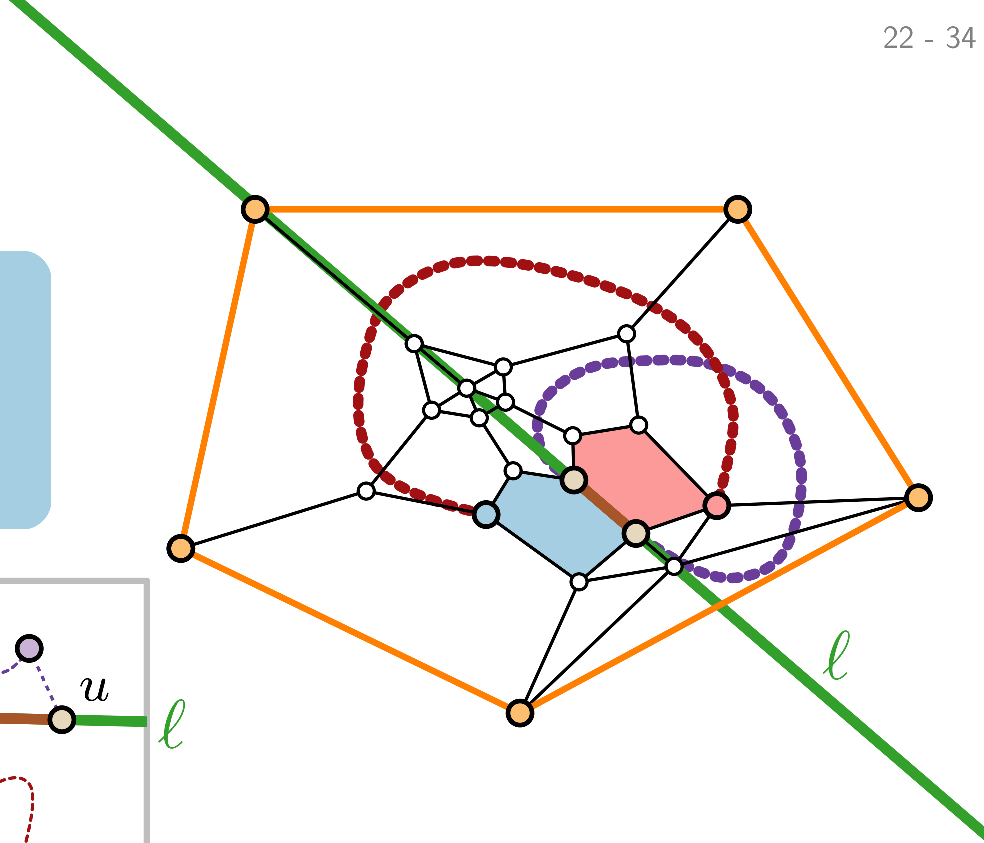

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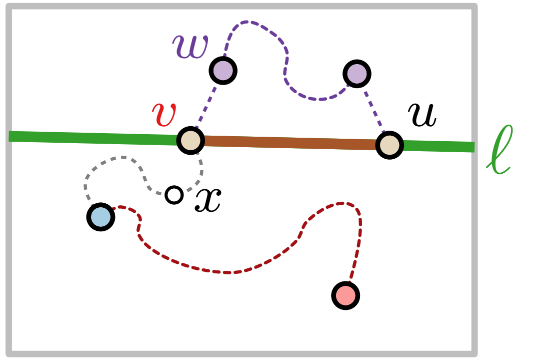

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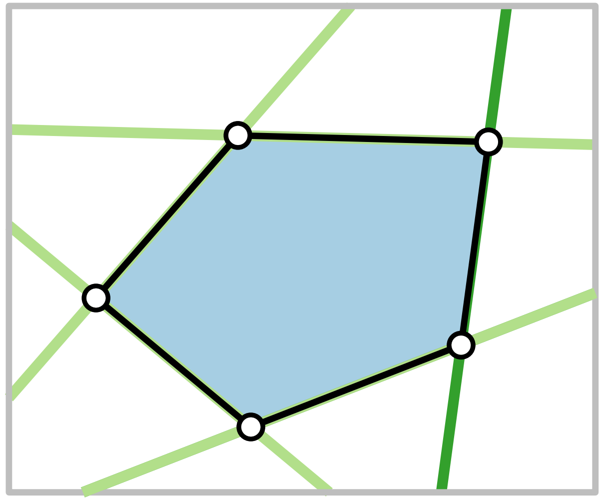

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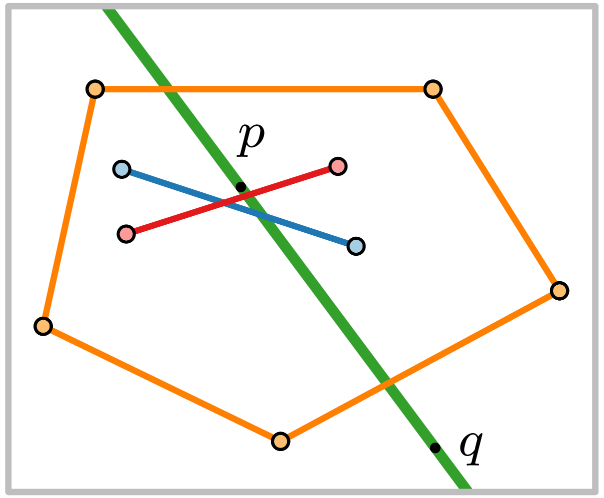

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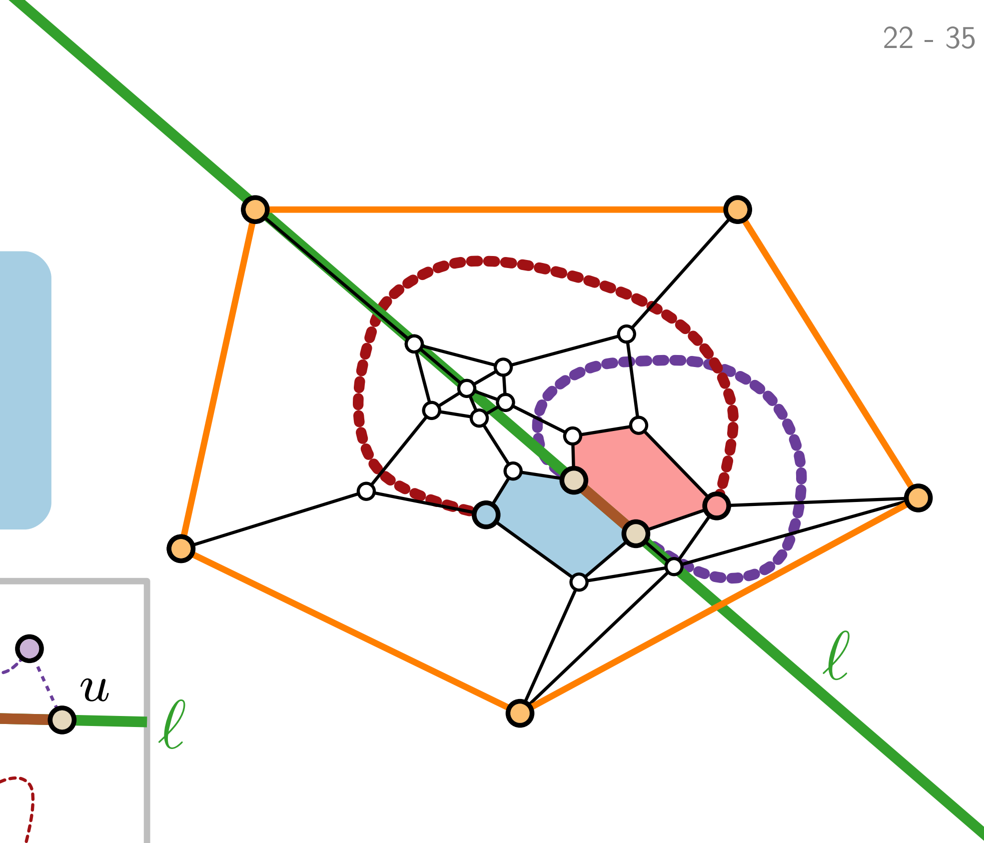

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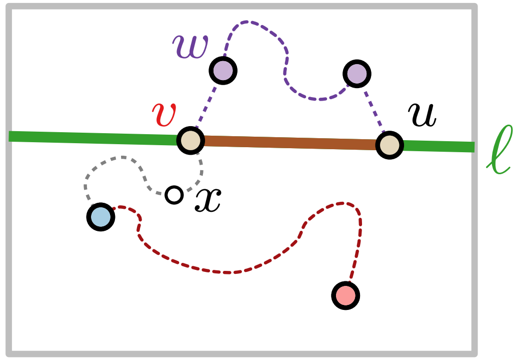

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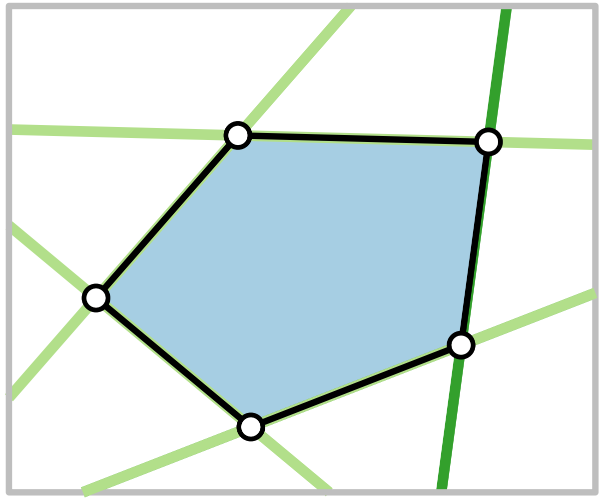

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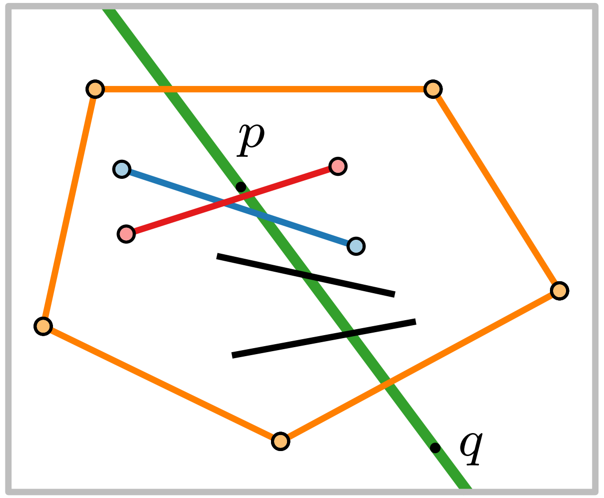

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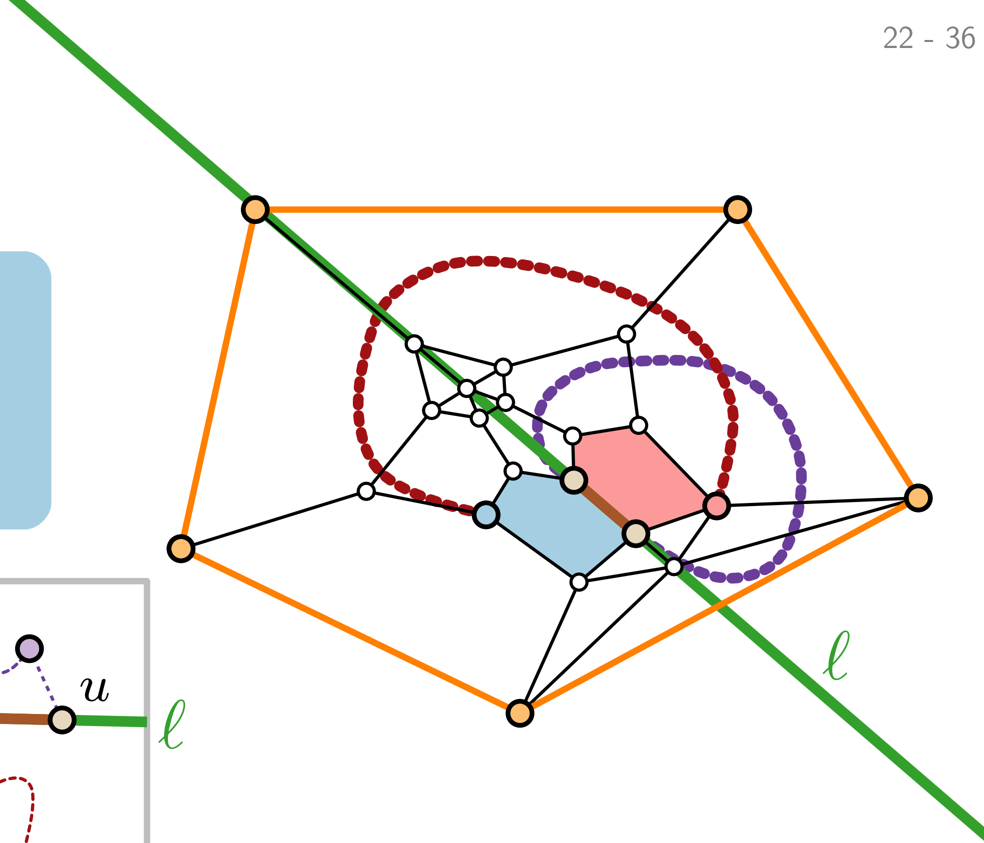

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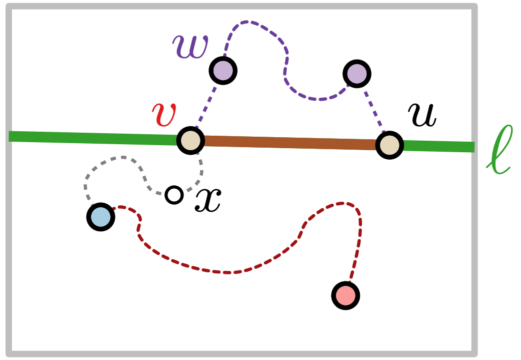

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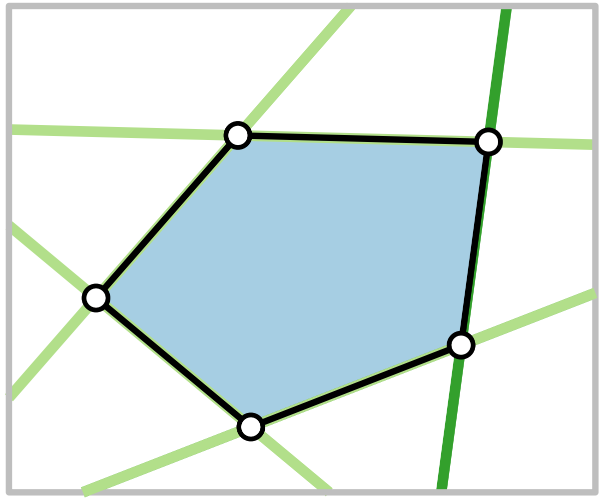

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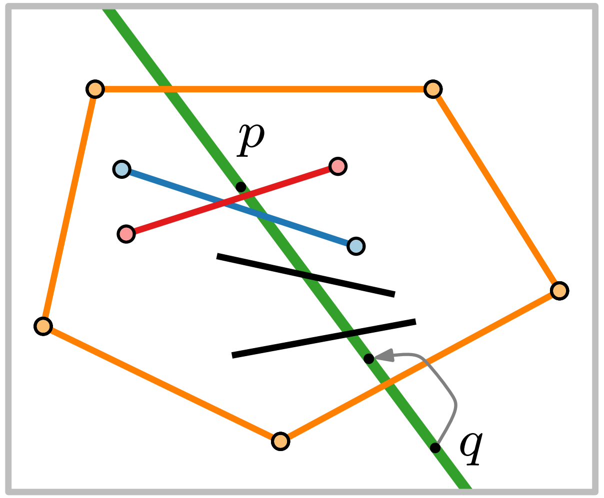

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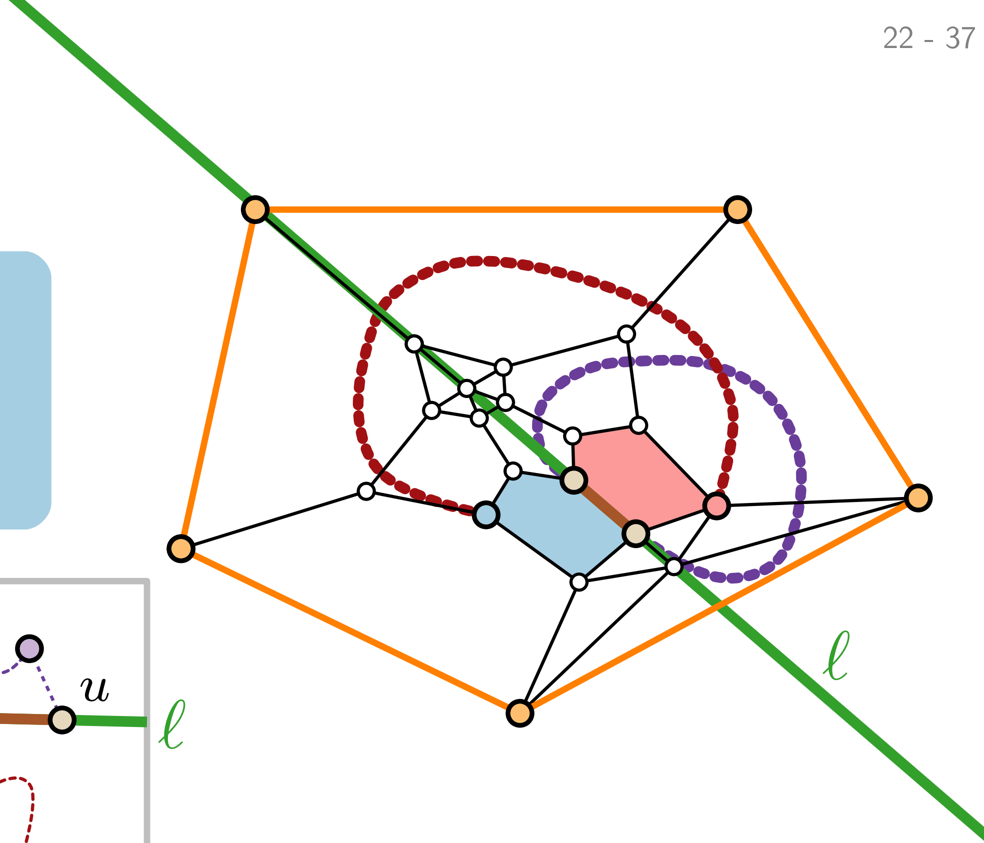

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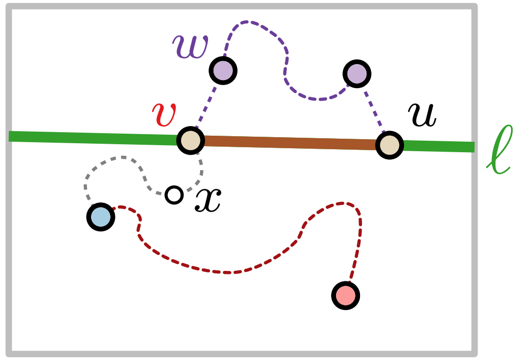

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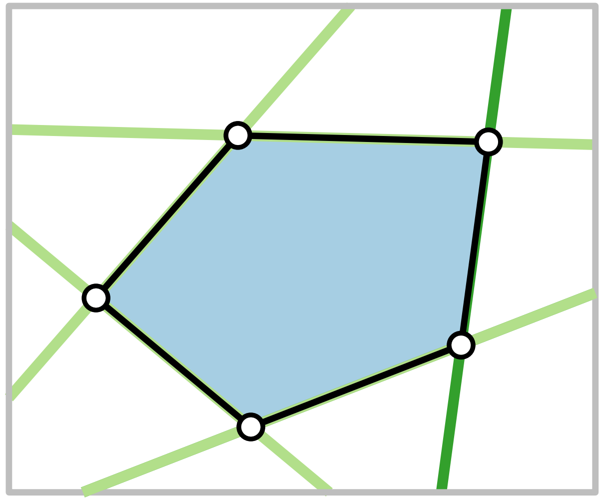

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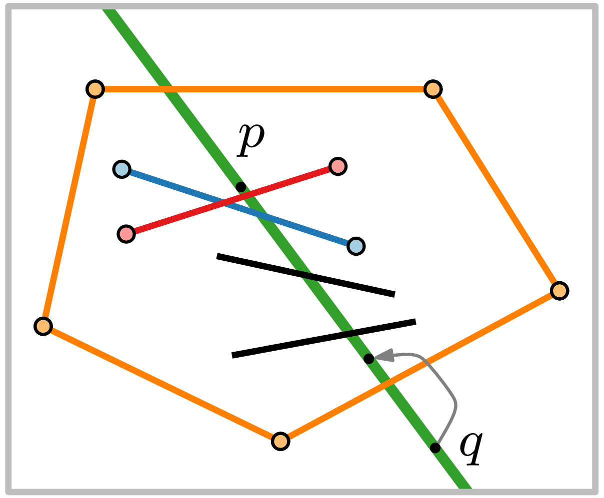

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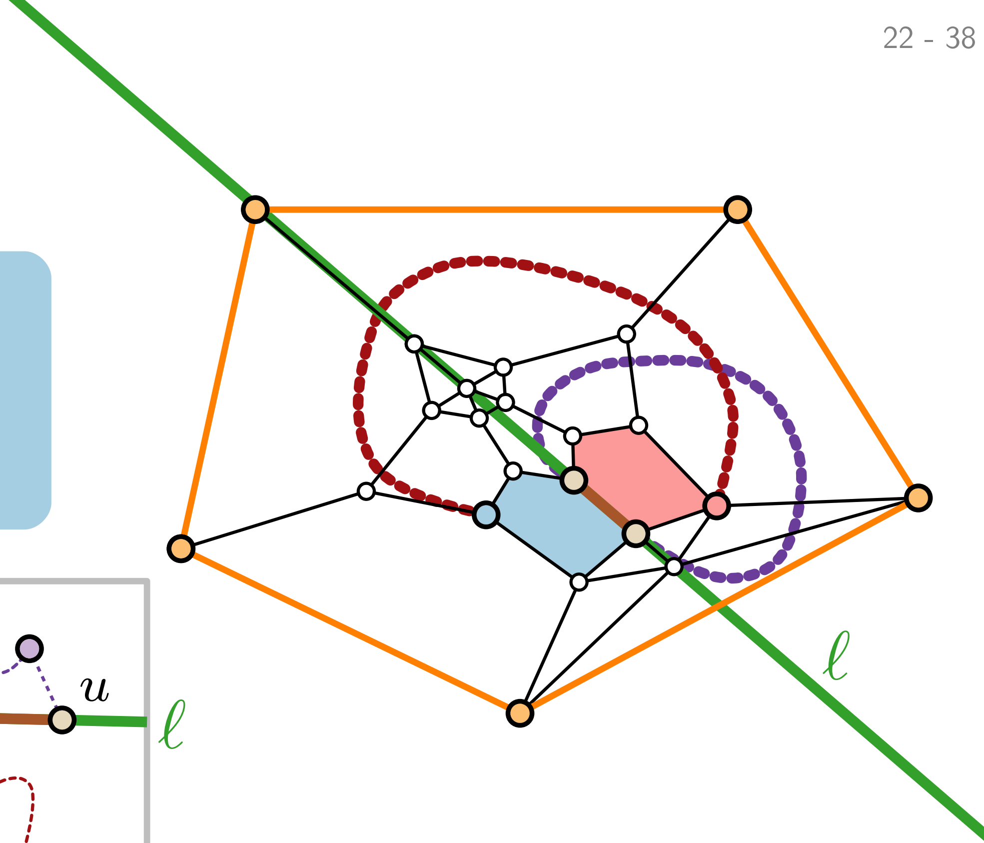

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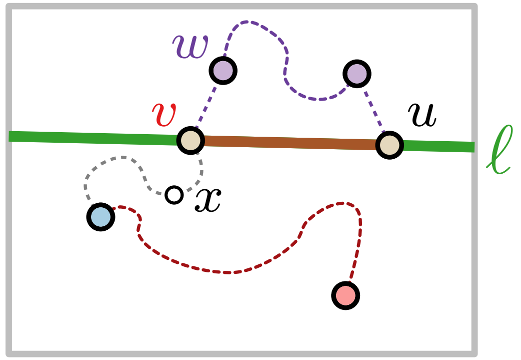

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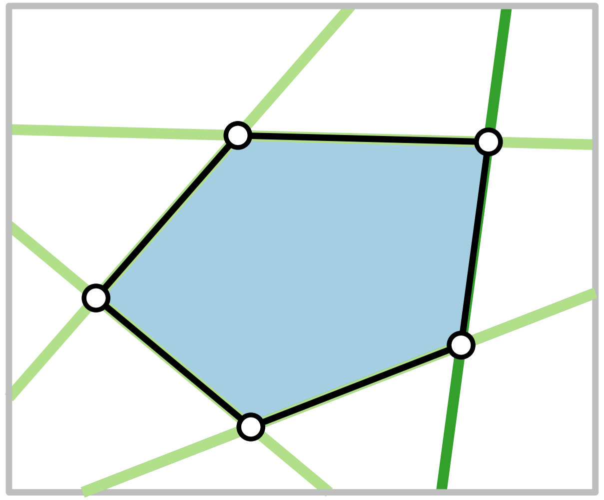

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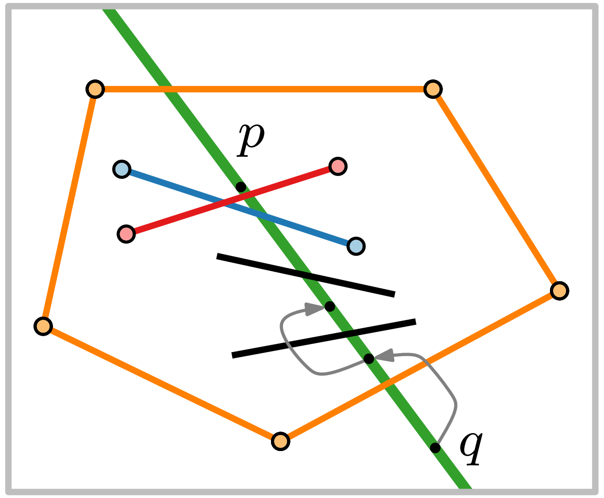

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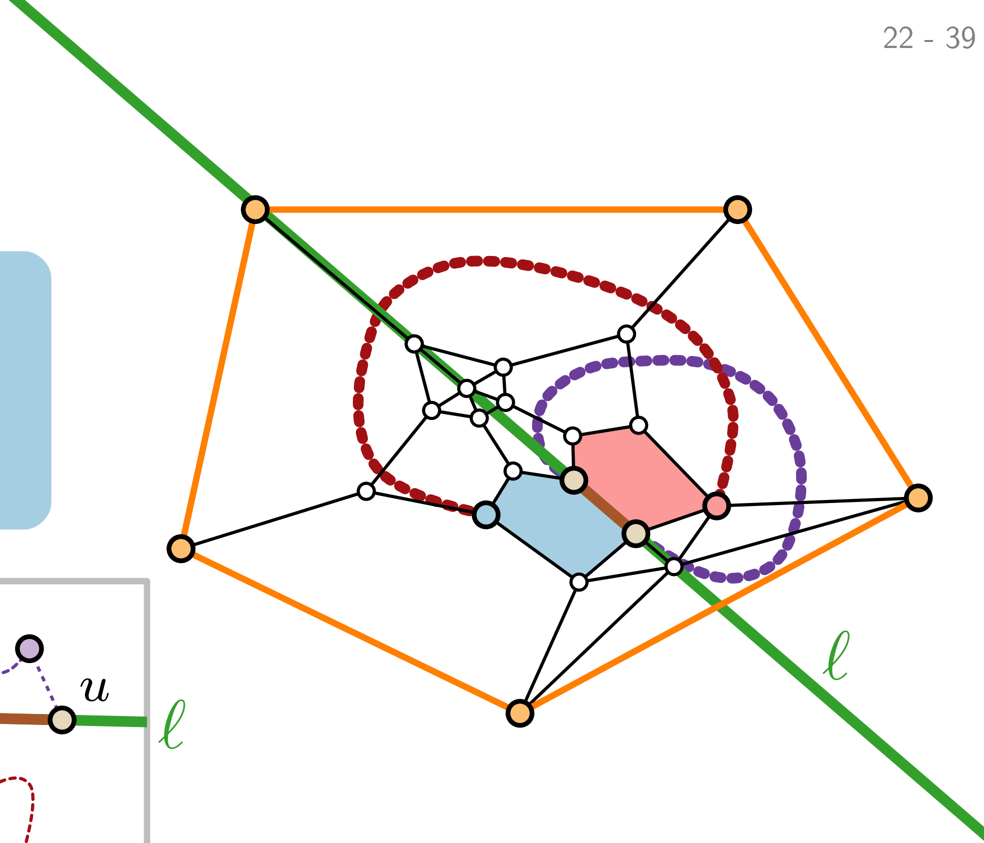

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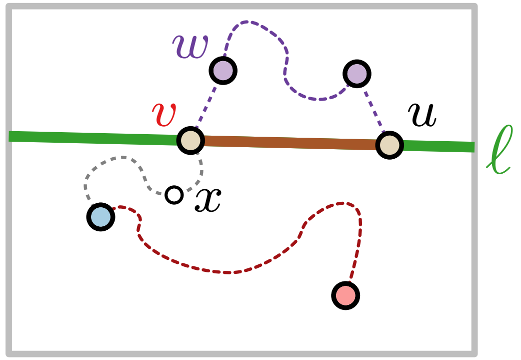

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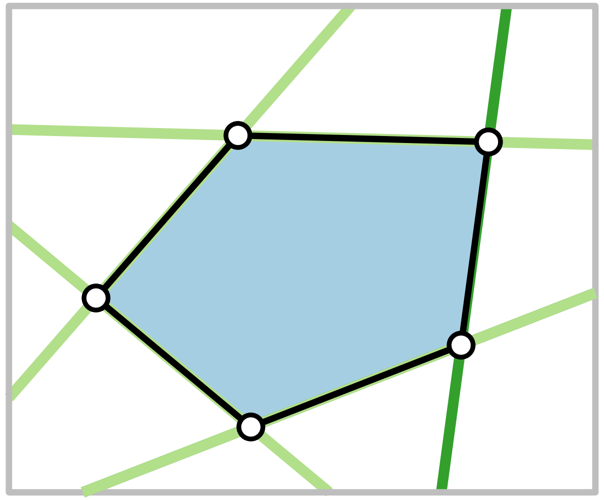

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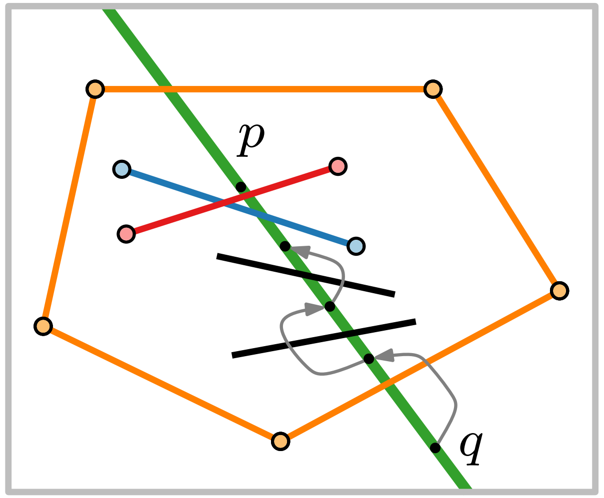

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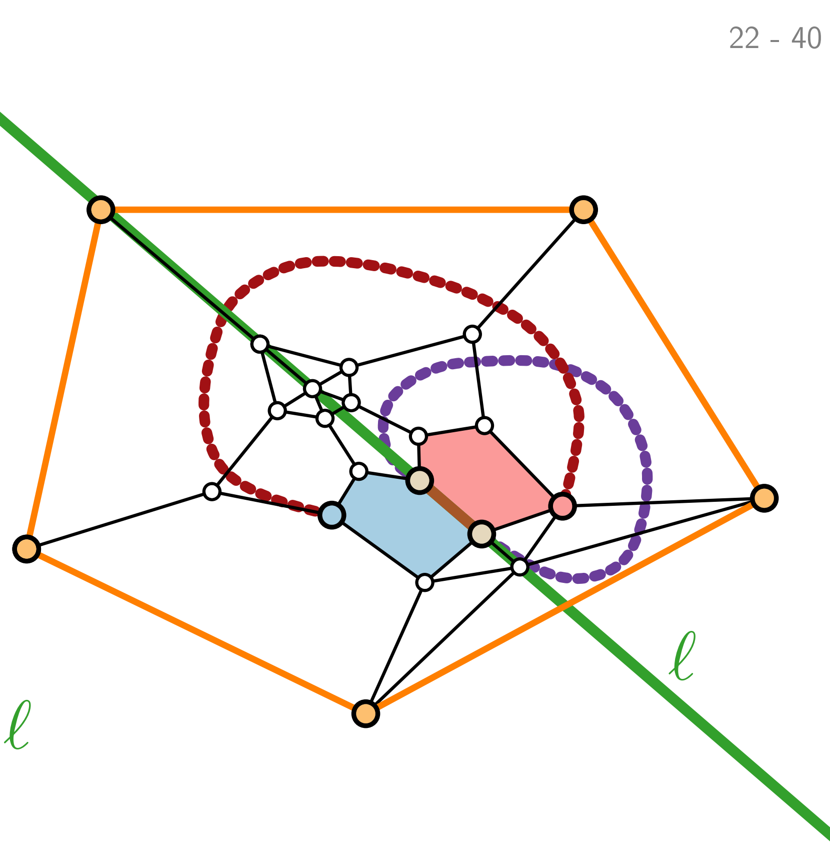

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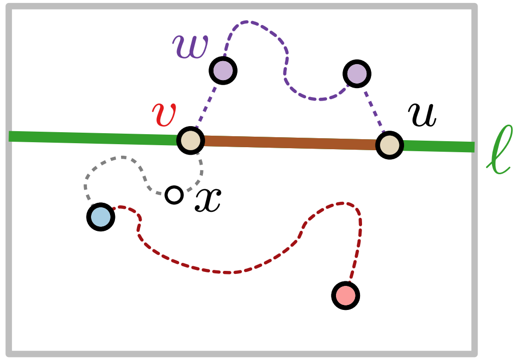

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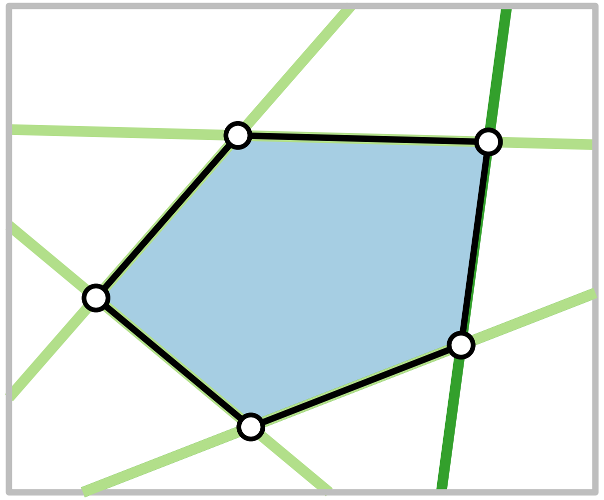

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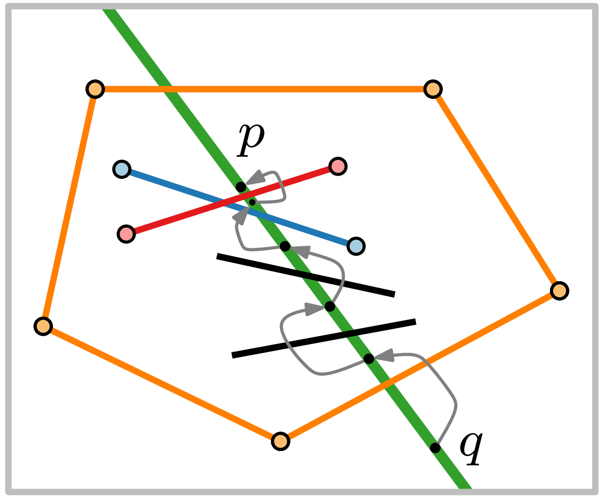

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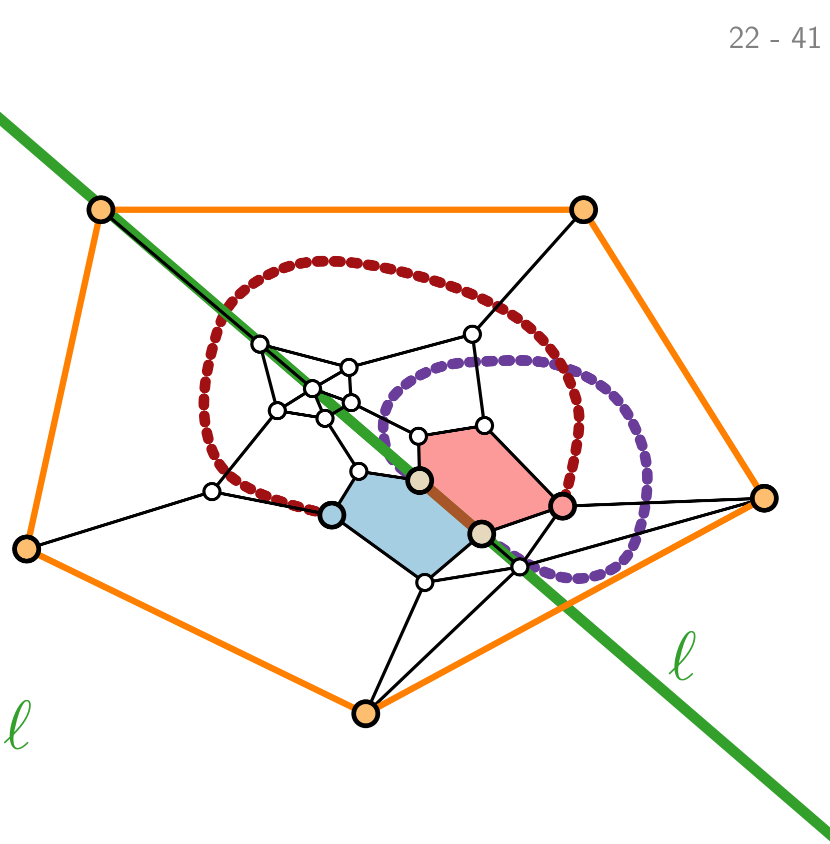

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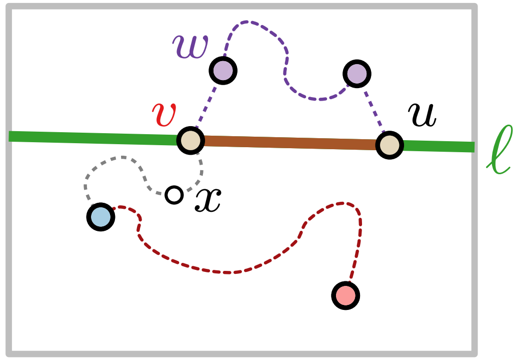

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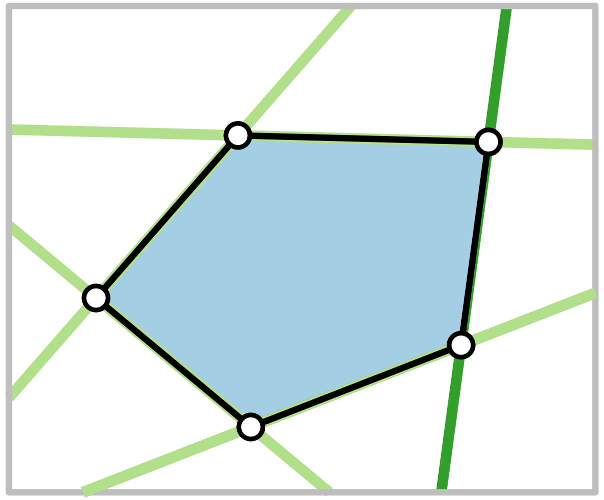

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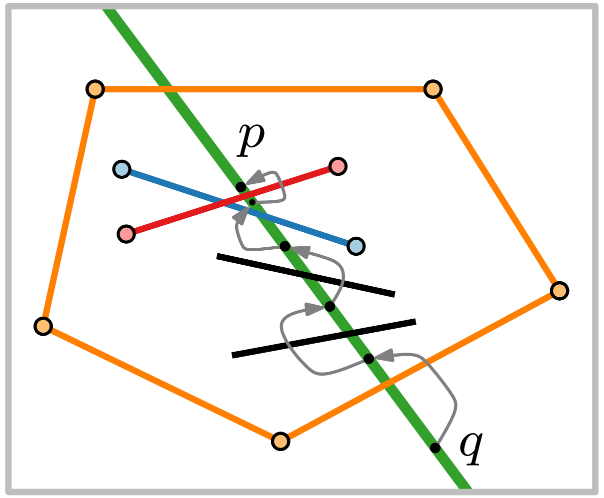

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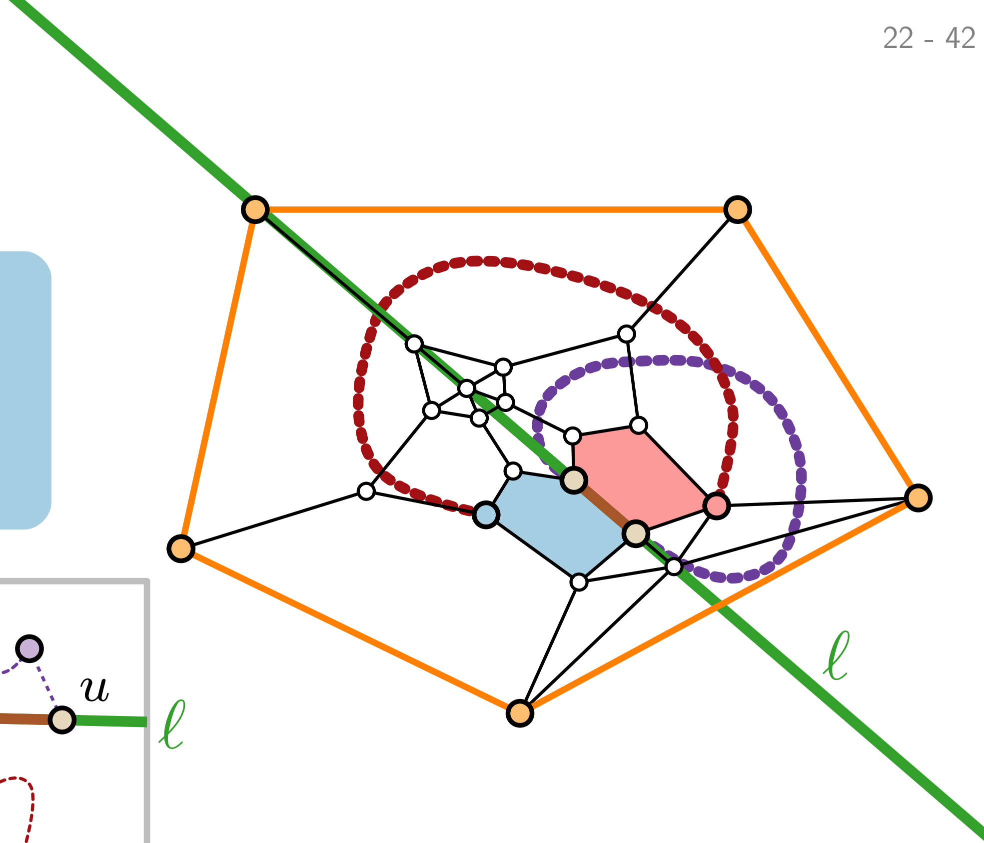

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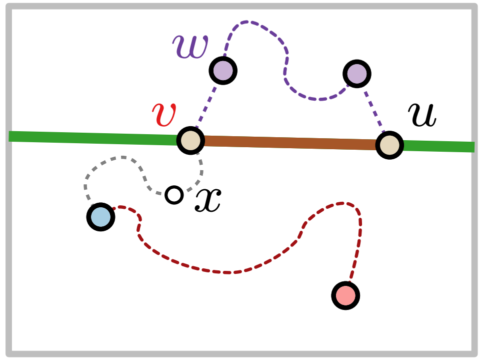

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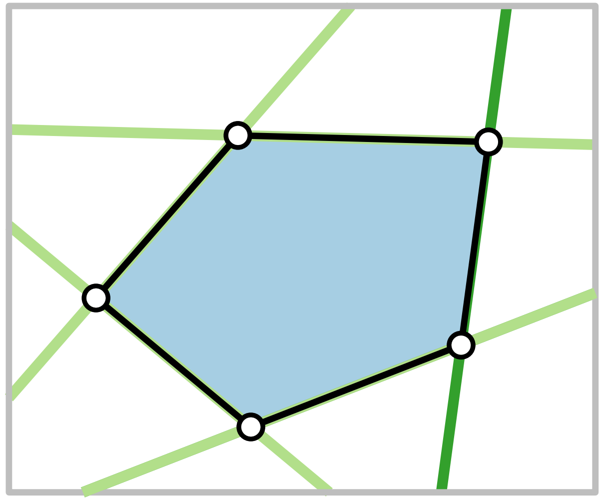

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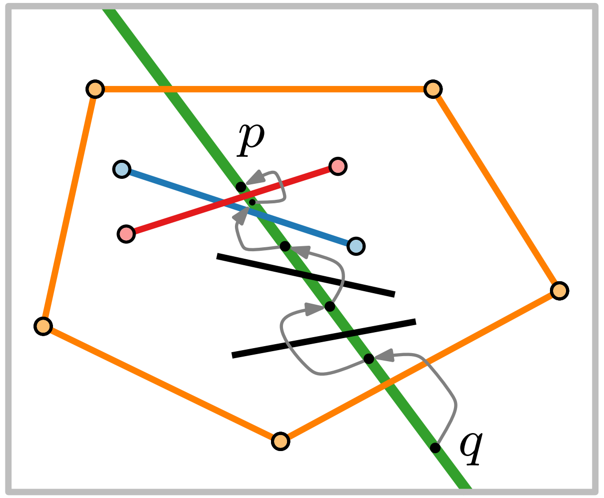

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- In practice, Tutte drawings are hardly used because the inner parts often become tiny.

Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph