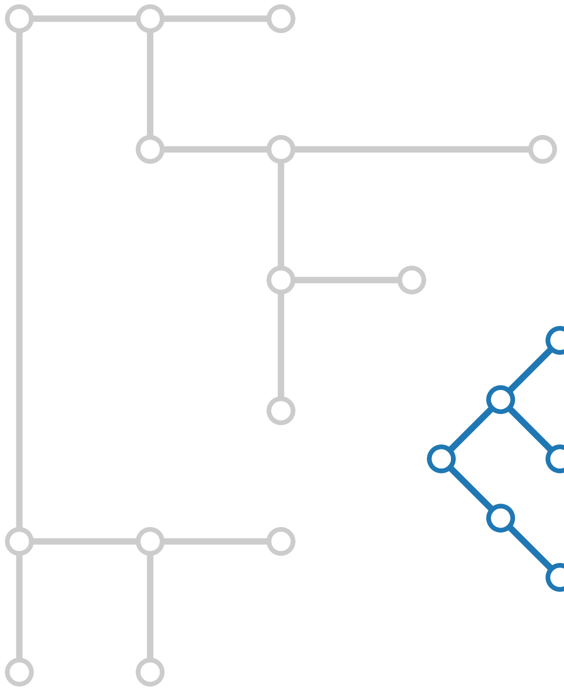
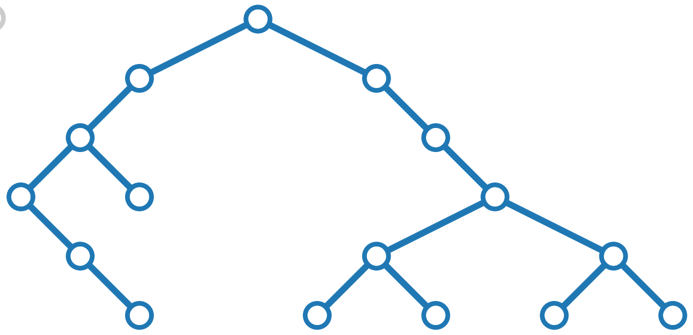
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 1: Drawing Trees

Part I: Layered Drawings





Alexander Wolff

Summer term 2025

(Rooted) Trees

Leaf: vertex of degree 1

Rooted tree: tree with a designated **root**

Ancestor: vertex on the path to the root

Parent: neighbor on the path to the root

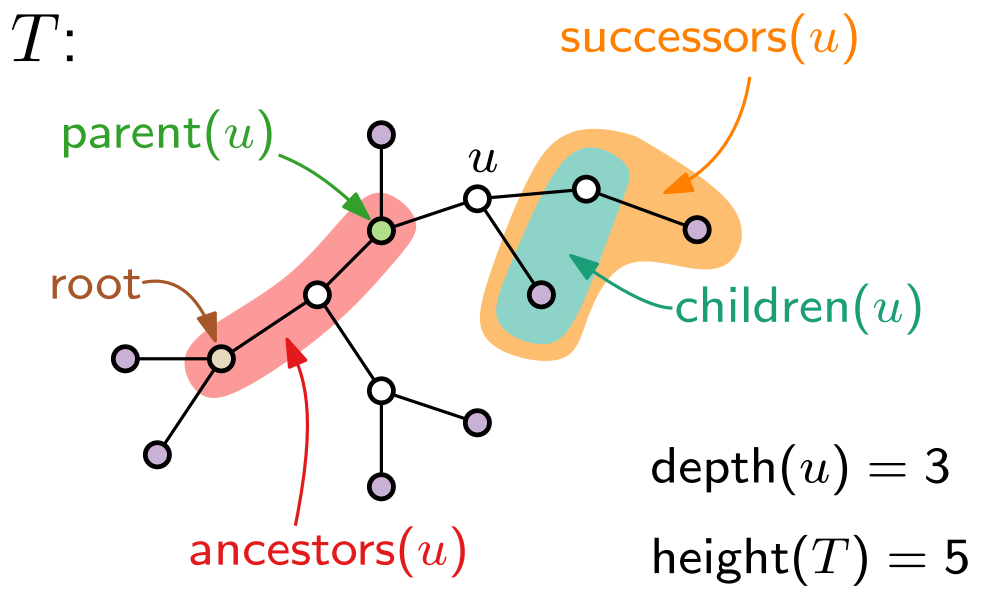
Successor: vertex on the path away from the root

Child: neighbor not on the path to the root

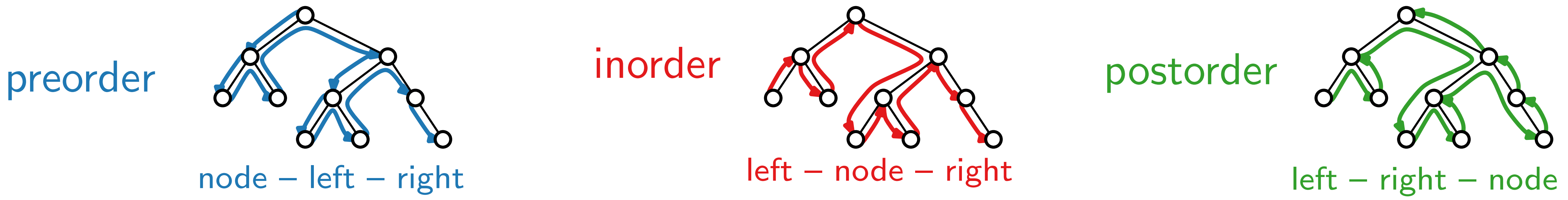
Depth: length of the path to the root

Height: maximum depth of a leaf

Binary Tree: at most two children per vertex (*left* and *right* child)

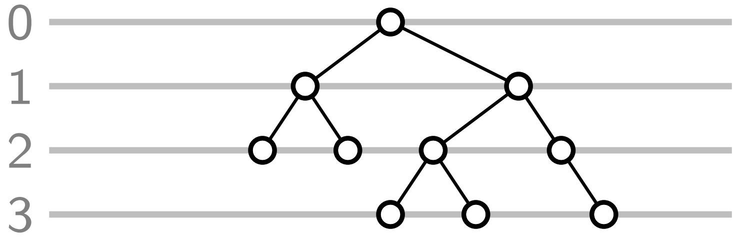


3 types of tree traversals:

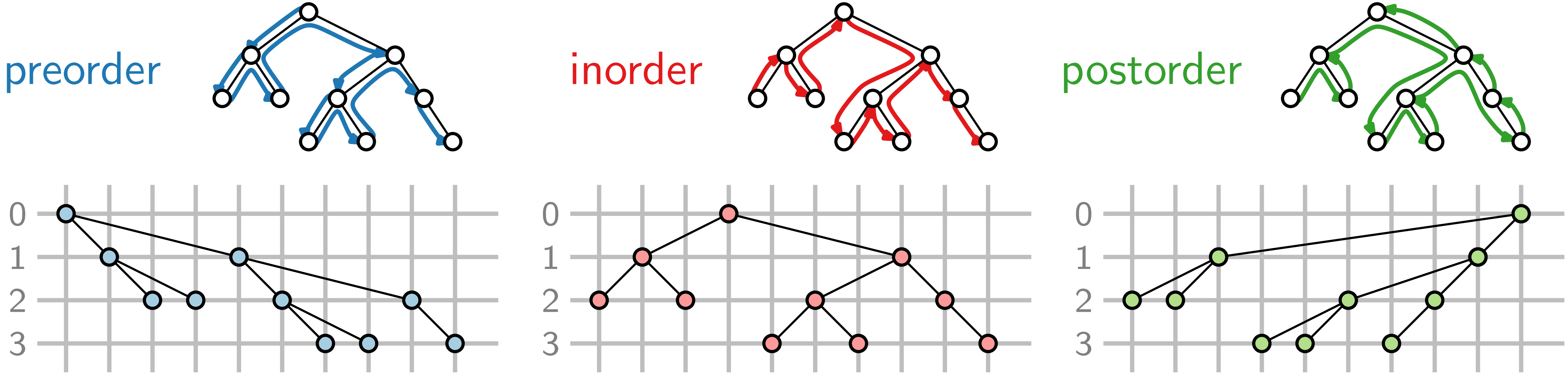


First Grid Layout of Binary Trees

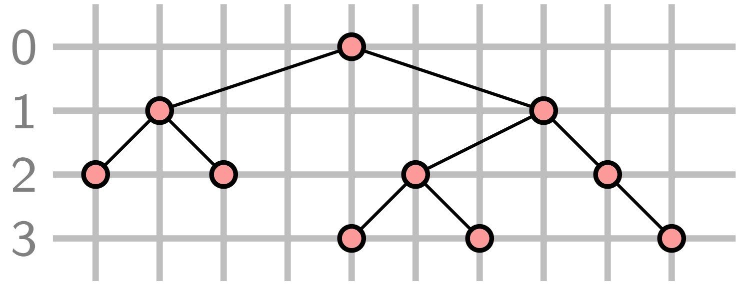
1. Choose y-coordinates: $y(u) = \text{depth}(u)$

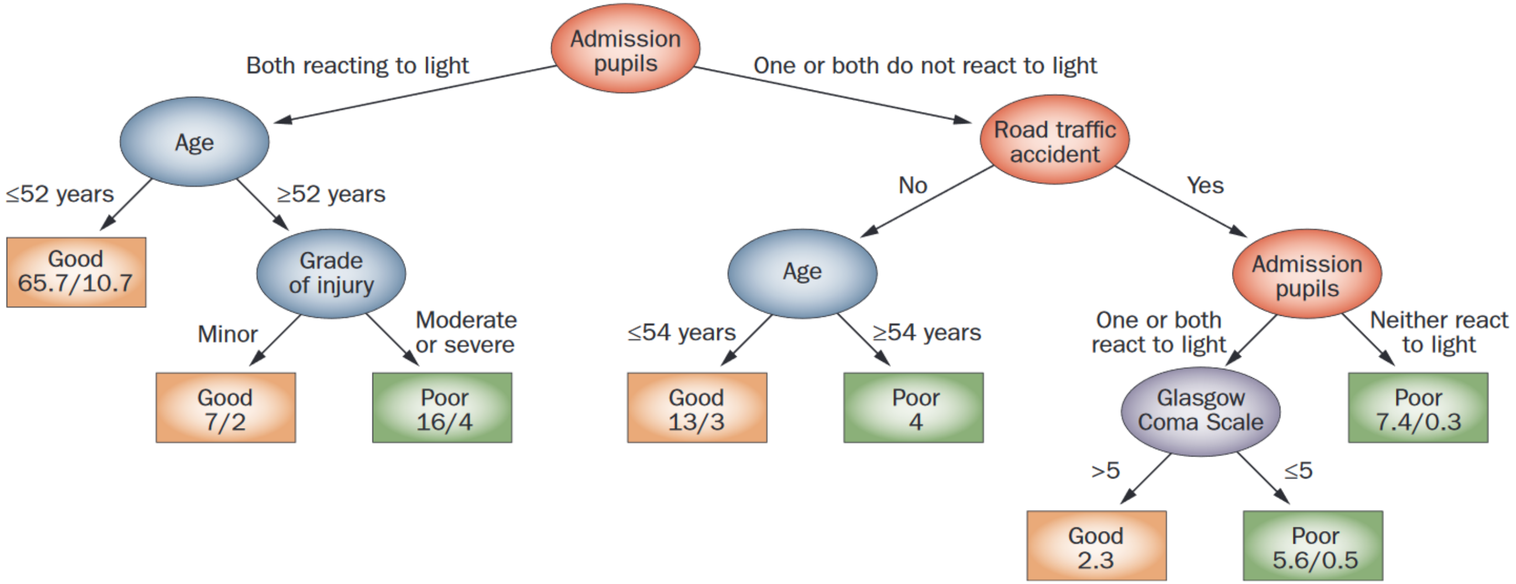


2. Choose x-coordinates:

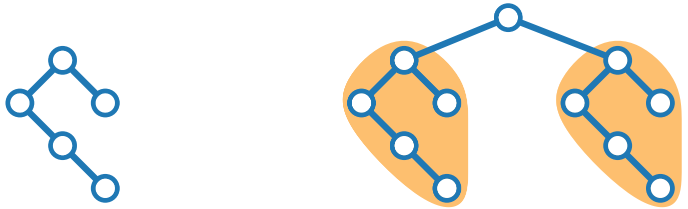


Layered Drawings – Drawing Style





- What are properties of the layout?
- What are the drawing conventions?
- What are aesthetics to optimize?



Drawing conventions

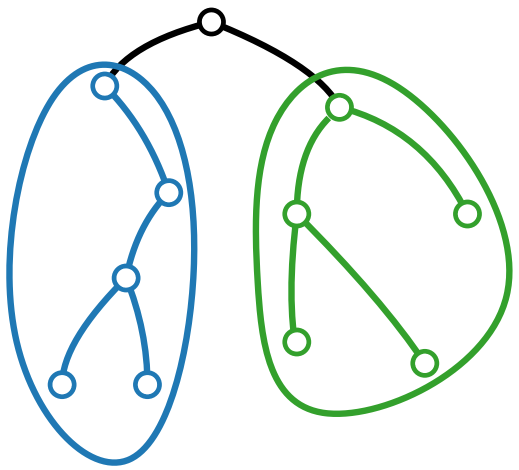
- Vertices lie on layers and have integer coordinates
- Parent centered above children (if there is more than one child)
- Edges are straight-line segments
- Isomorphic subtrees have identical drawings

Drawing aesthetics to optimize

- Area
- Symmetries

Layered Drawings – Algorithm

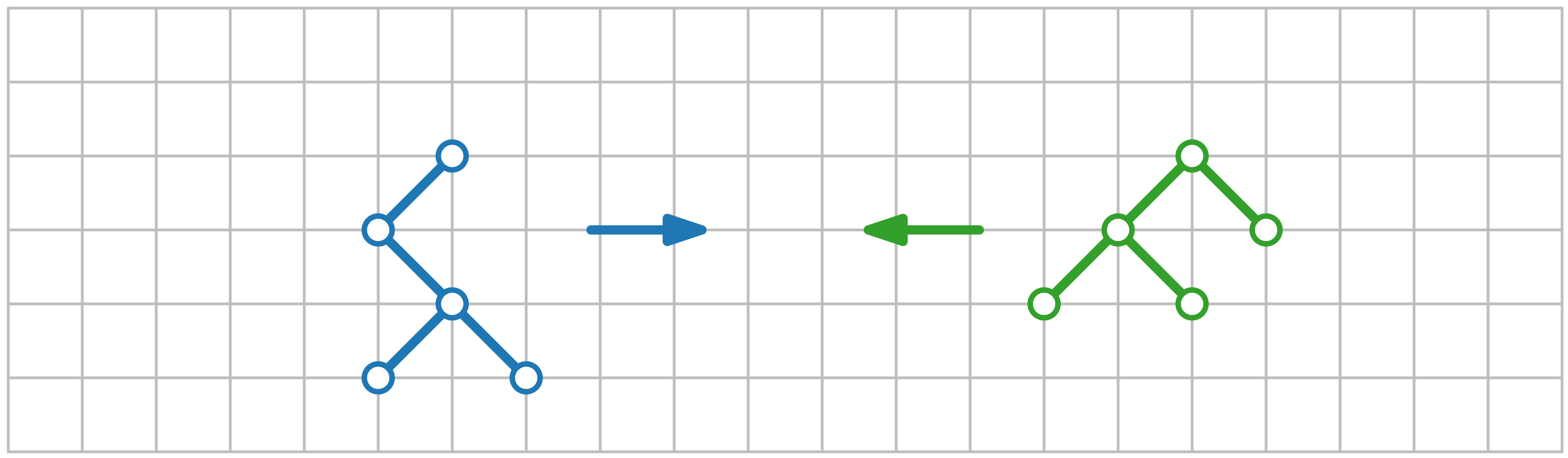
Input: A binary tree T
Output: A layered drawing of T



Base case: A single vertex 

Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:



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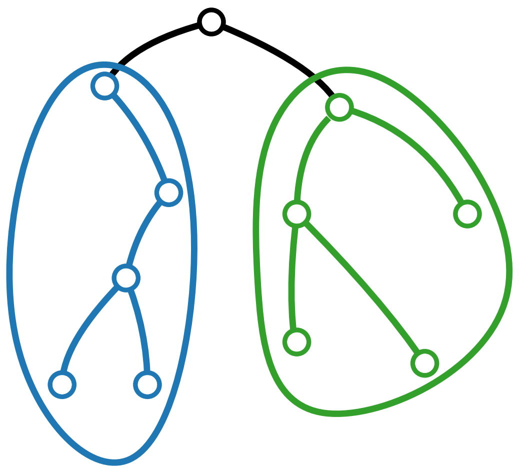

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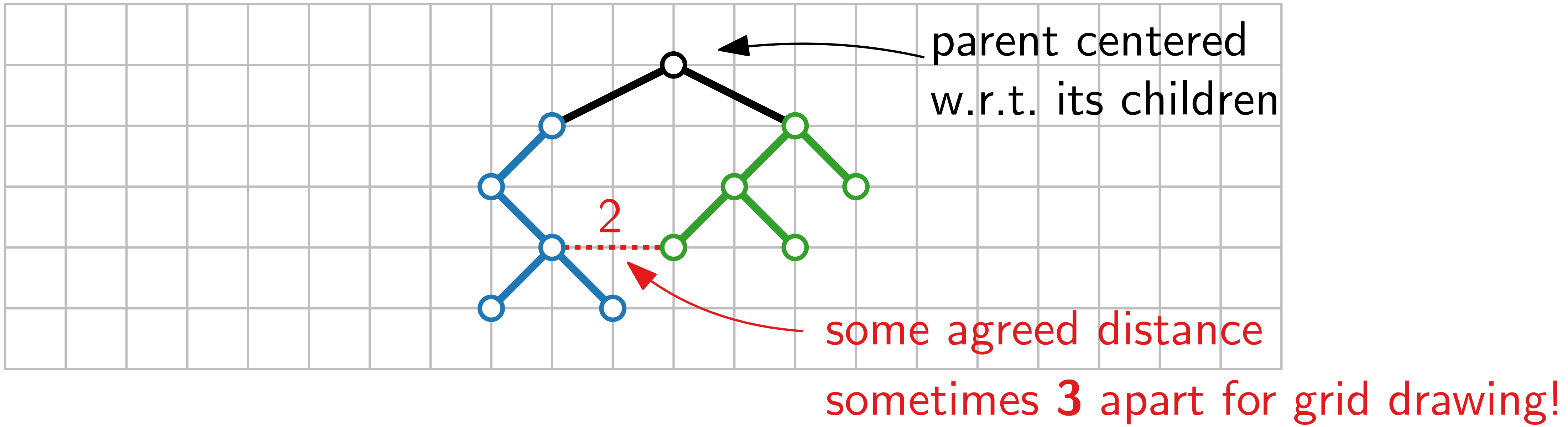


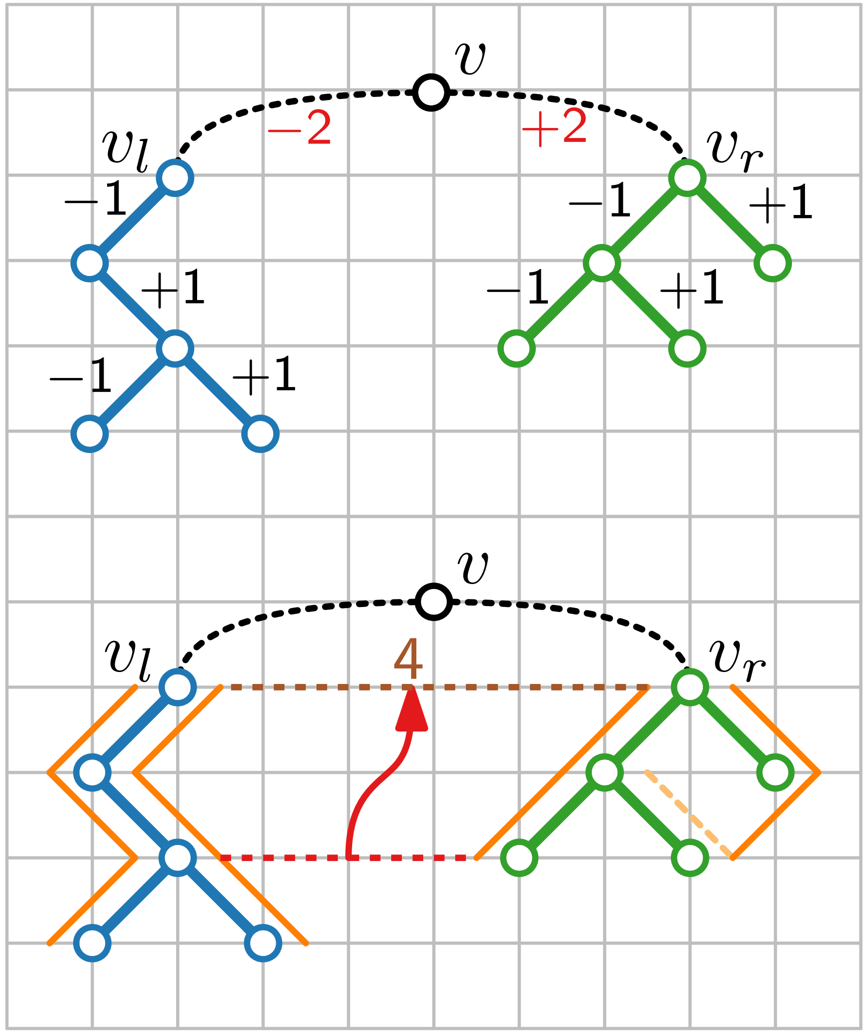
Layered Drawings – Algorithm Details

Phase 1 – postorder traversal:

- For each vertex v, compute horizontal displacement of left child v_l and right child v_r.
- x-offset(v_l) $= -\lceil d_v/2 \rceil$, x-offset(v_r) $= \lceil d_v/2 \rceil$
- At every vertex v store left and right contour of sub-tree $T(v)$.
- A contour is a linked list of vertex coordinates/offsets.
- Find d_v = min. horiz. distance between v_l and v_r.

Phase 2 – preorder traversal:

- Compute x- and y-coordinates



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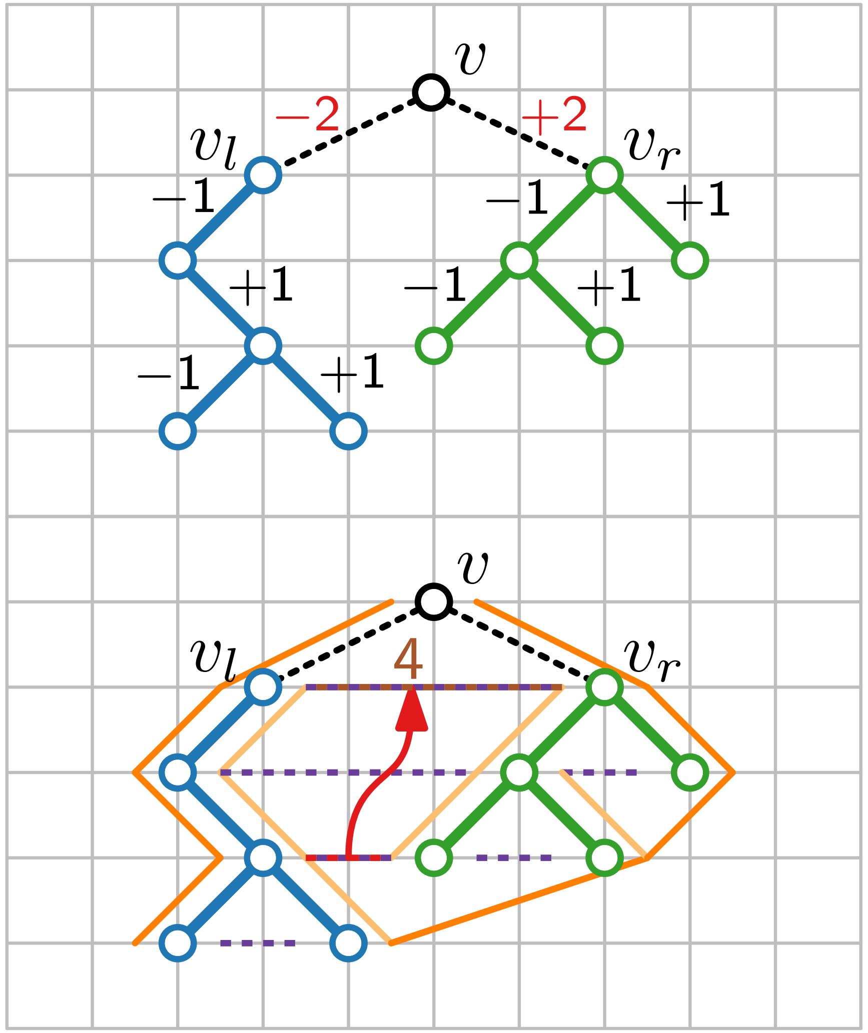


Phase 2 – preorder traversal:

- Compute x- and y-coordinates

Runtime?

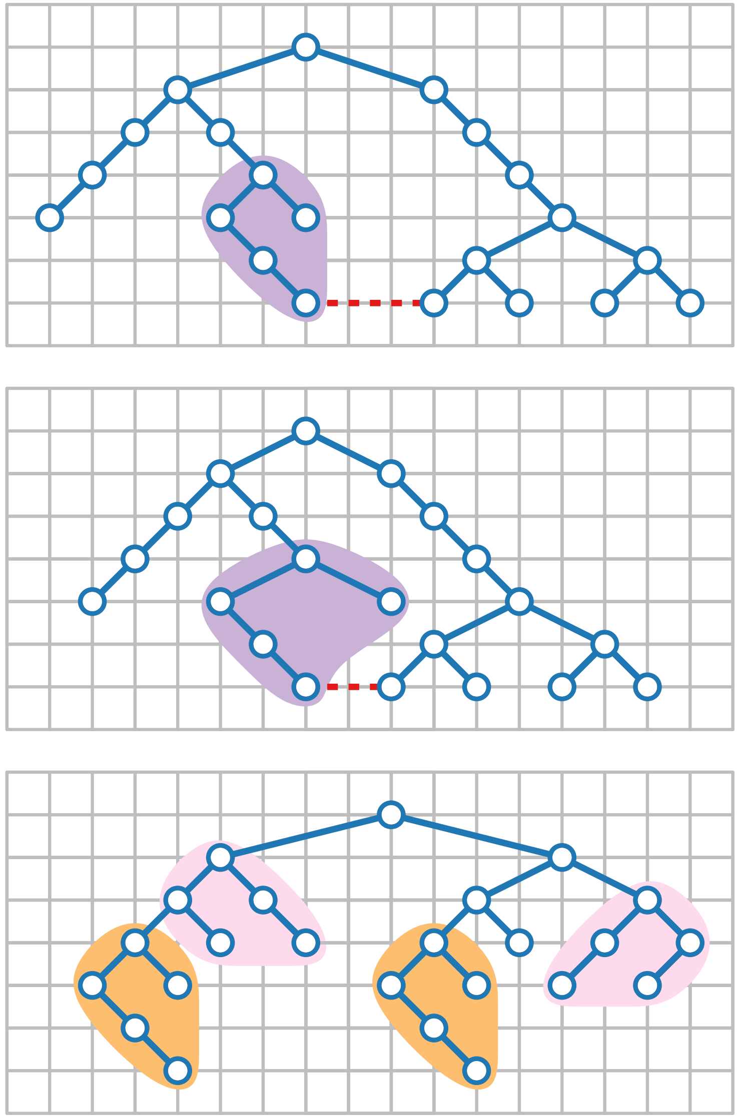
- How often do we take a step along a contour? *in total $\mathcal{O}(n)$ times!* *where $n = \#$ vertices*

Layered Drawings – Result

Theorem. [Reingold & Tilford '81]

Let T be a binary tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time such that:

- Γ is planar, straight-line and strictly downward
- Γ is layered: y-coordinate of vertex v is $-\text{depth}(v)$
- Horizontal and vertical distances are at least 1
- Each vertex with > 1 child is centered w.r.t. its children
- Area of Γ is in $\mathcal{O}(n^2)$ – but not optimal! ← NP-hard
- Simply isomorphic subtrees have congruent drawings, up to translation
- Axially isomorphic subtrees have congruent drawings, up to translation and reflection



Layered Drawings – Result

Theorem. [Reingold & Tilford '81]

Let T be a ~~binary~~ rooted tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time such that:

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- Γ is layered: y-coordinate of vertex v is $-\text{depth}(v)$
- Horizontal and vertical distances are at least 1
- Each vertex with > 1 child is centered w.r.t. its children
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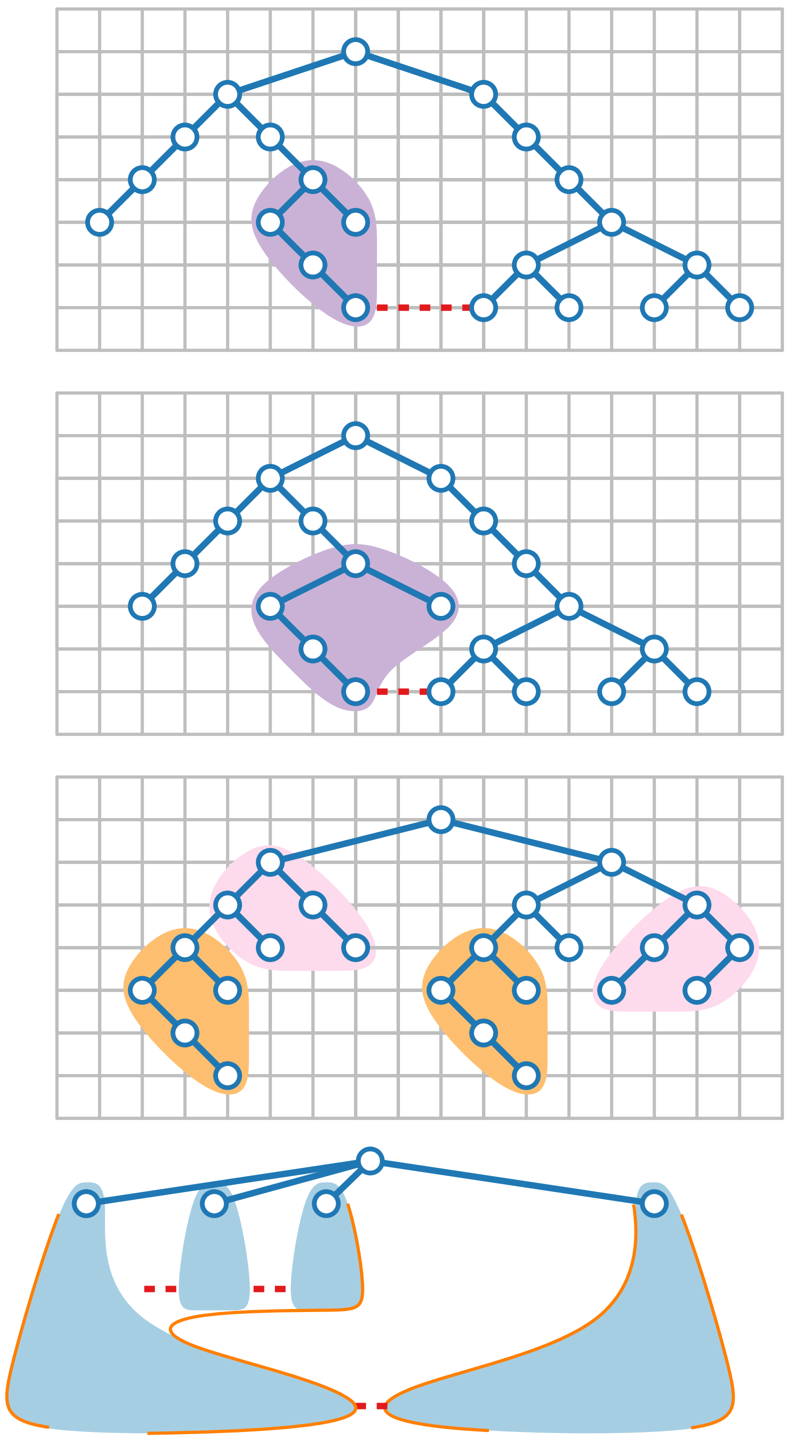
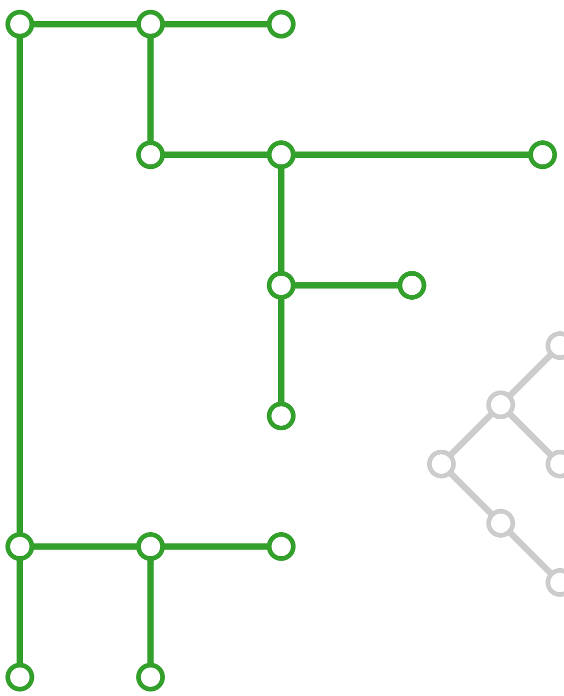


extension to non-binary rooted trees

Visualization of Graphs

Lecture 1: Drawing Trees

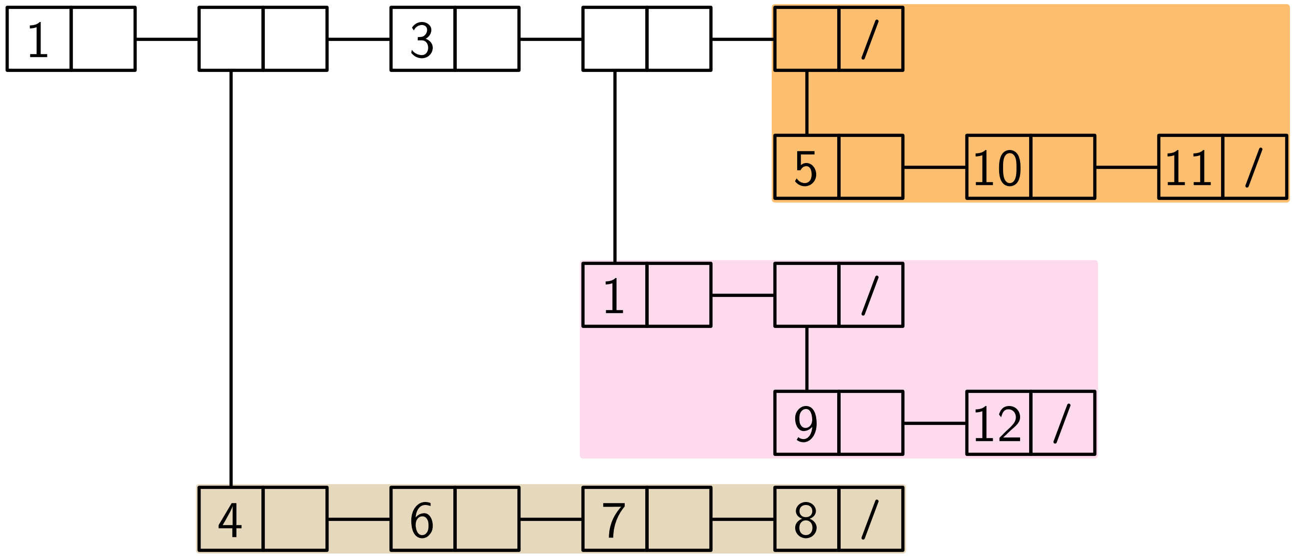
Part II: HV-Drawings



HV-Drawings – Drawing Style

Applications

- Cons cell diagram in LISP
- *Cons* (constructs) are memory objects that hold two values or pointers to values



Source: after gajon.org/trees-linked-lists-common-lisp/

Drawing conventions

- Children are vertically or horizontally aligned with their parent
- The bounding boxes of the subtrees of the children are disjoint
- Edges are strictly down- or rightwards

Drawing aesthetics to optimize

- Height, width, area

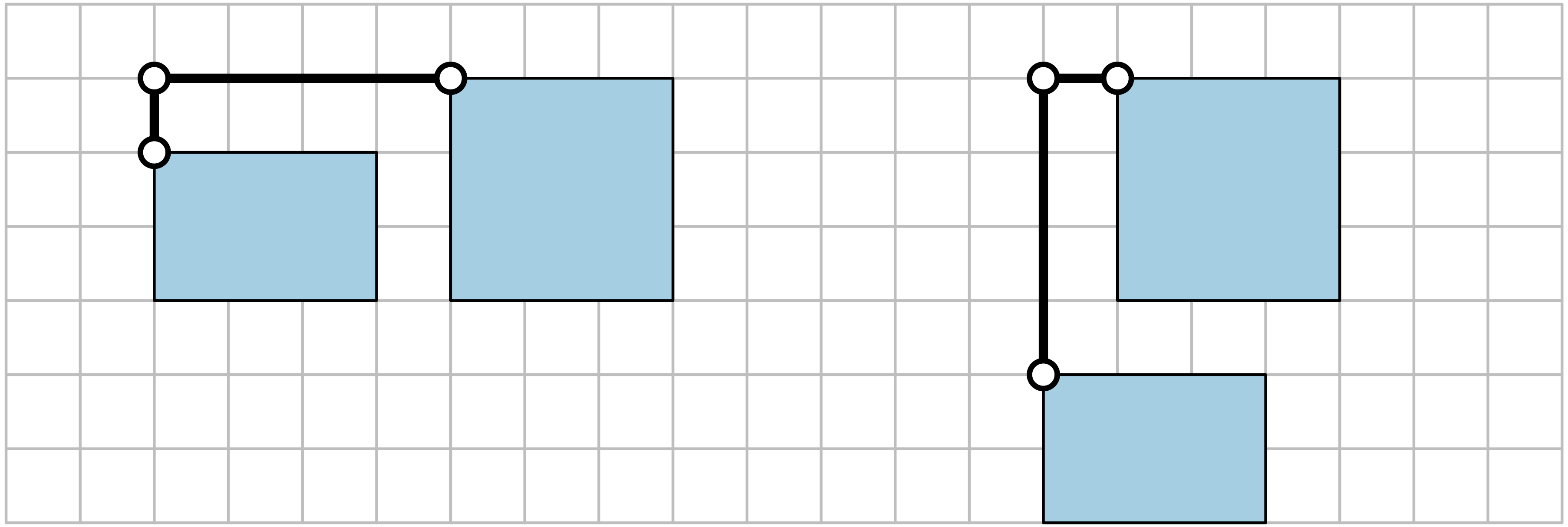
HV-Drawings – Algorithm

Input: A binary tree T
Output: An HV-drawing of T

Base case: 

Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer: horizontal combination vertical combination

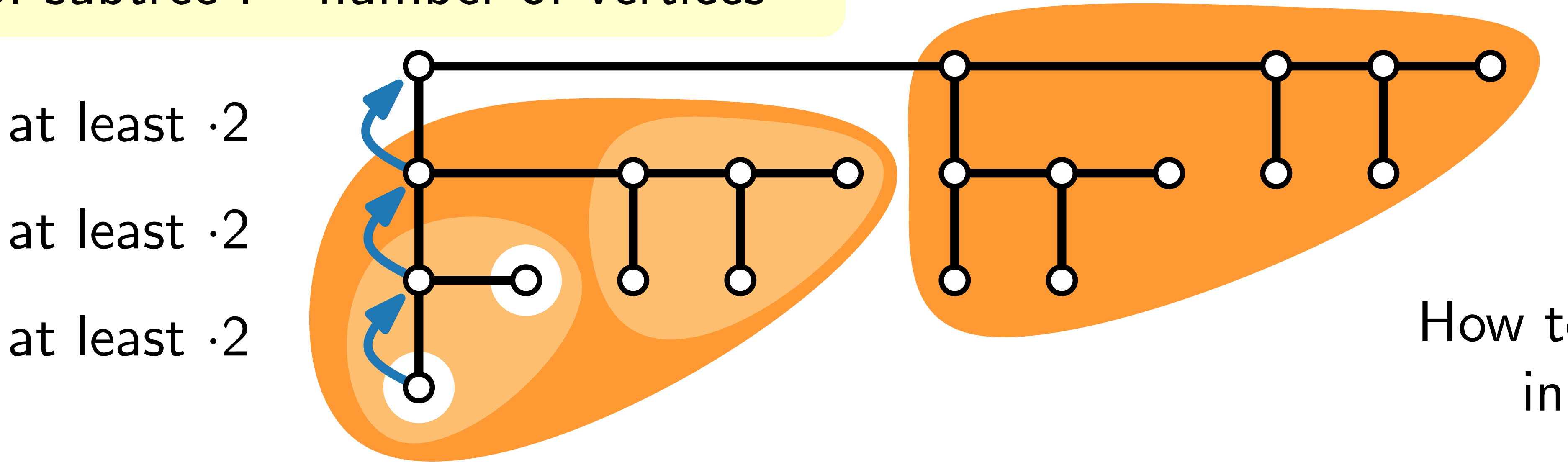


HV-Drawings – Right-Heavy HV-Layout

Right-heavy approach

- Always apply horizontal combination
- Place the larger subtree to the right ← *This can change the embedding!*

Size of subtree := number of vertices



How to implement this in linear time?

Lemma. Let T be a binary tree. The drawing constructed by the right-heavy approach has

- width at most $n - 1$ and
- height at most $\log_2 n$.

HV-Drawings – Result

Theorem.
Let T be a binary tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is an HV-drawing (planar, orthogonal, strictly right-/downward)
- Width is at most $n - 1$
- Height is at most $\log_2 n$
- Area is in $\mathcal{O}(n \log n)$ ← worst-case optimal [exercise]
- Simply and axially isomorphic subtrees have congruent drawings up to translation

HV-Drawings – Result

Theorem.
rooted
Let T be a ~~binary~~ tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

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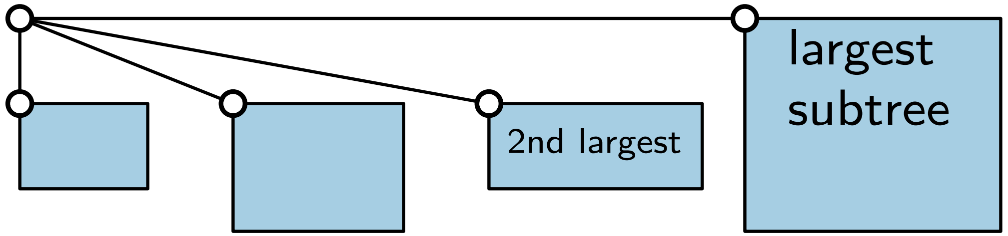
General rooted tree



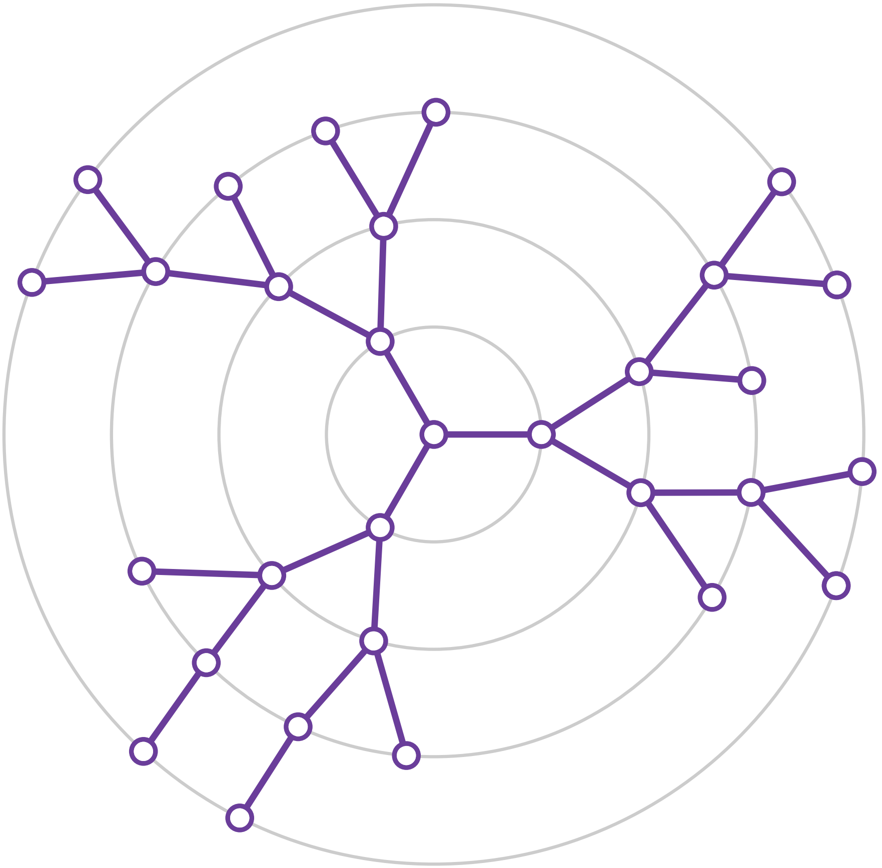
Optimal area?

Not with divide & conquer approach, but can be computed with Dynamic Programming.

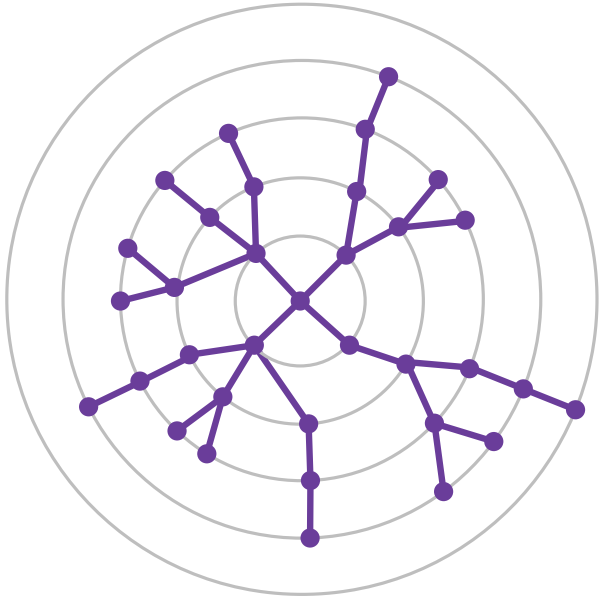
Visualization of Graphs

Lecture 1:
Drawing Trees

Part III:
Radial Layouts



Radial Layouts – Drawing Style



Drawing conventions

- Vertices lie on circular layers according to their depth
- Drawing is planar

Drawing aesthetics to optimize

- Balanced distribution of the vertices

How can an algorithm optimize the distribution of the vertices?

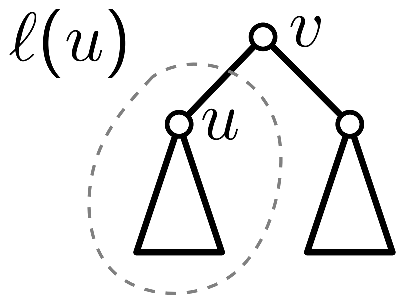
Radial Layouts – Algorithm Attempt

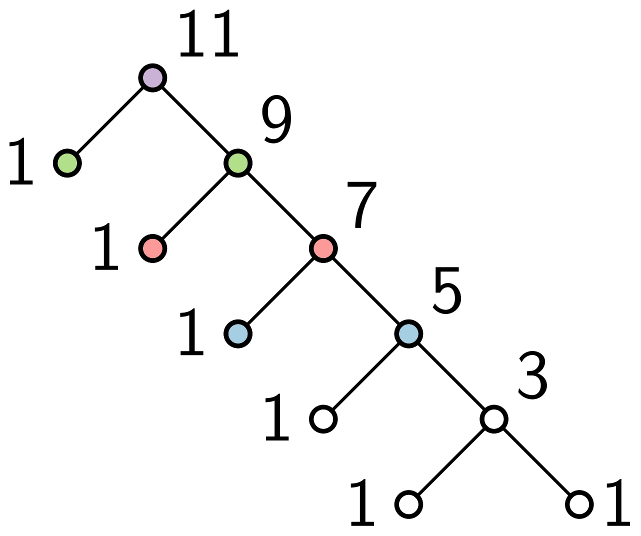
Idea

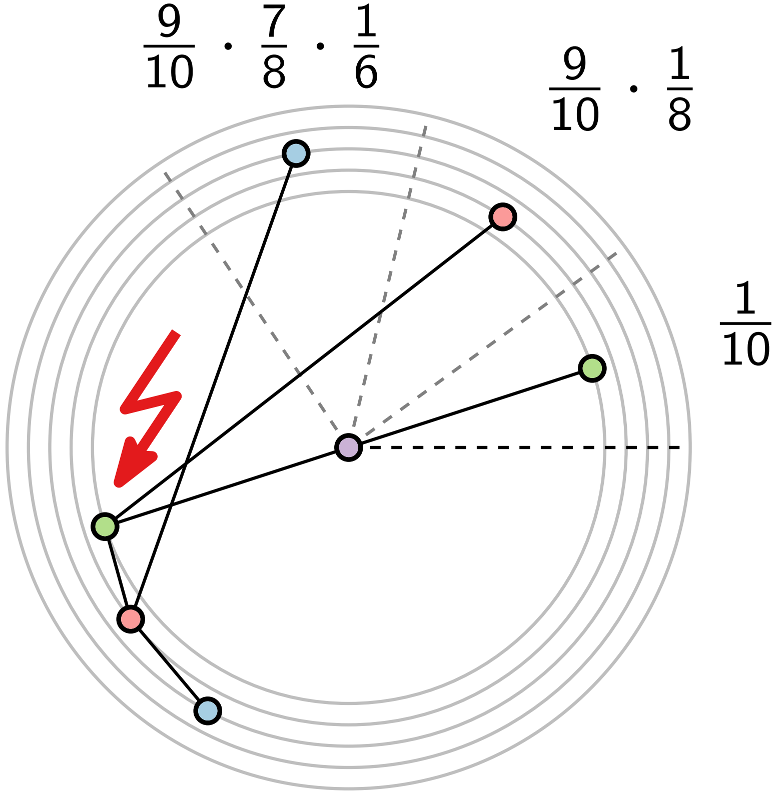
- Reserve area corresponding to size $\ell(u)$ of $T(u)$:

$$\tau_u = \frac{\ell(u)}{\ell(v) - 1}$$

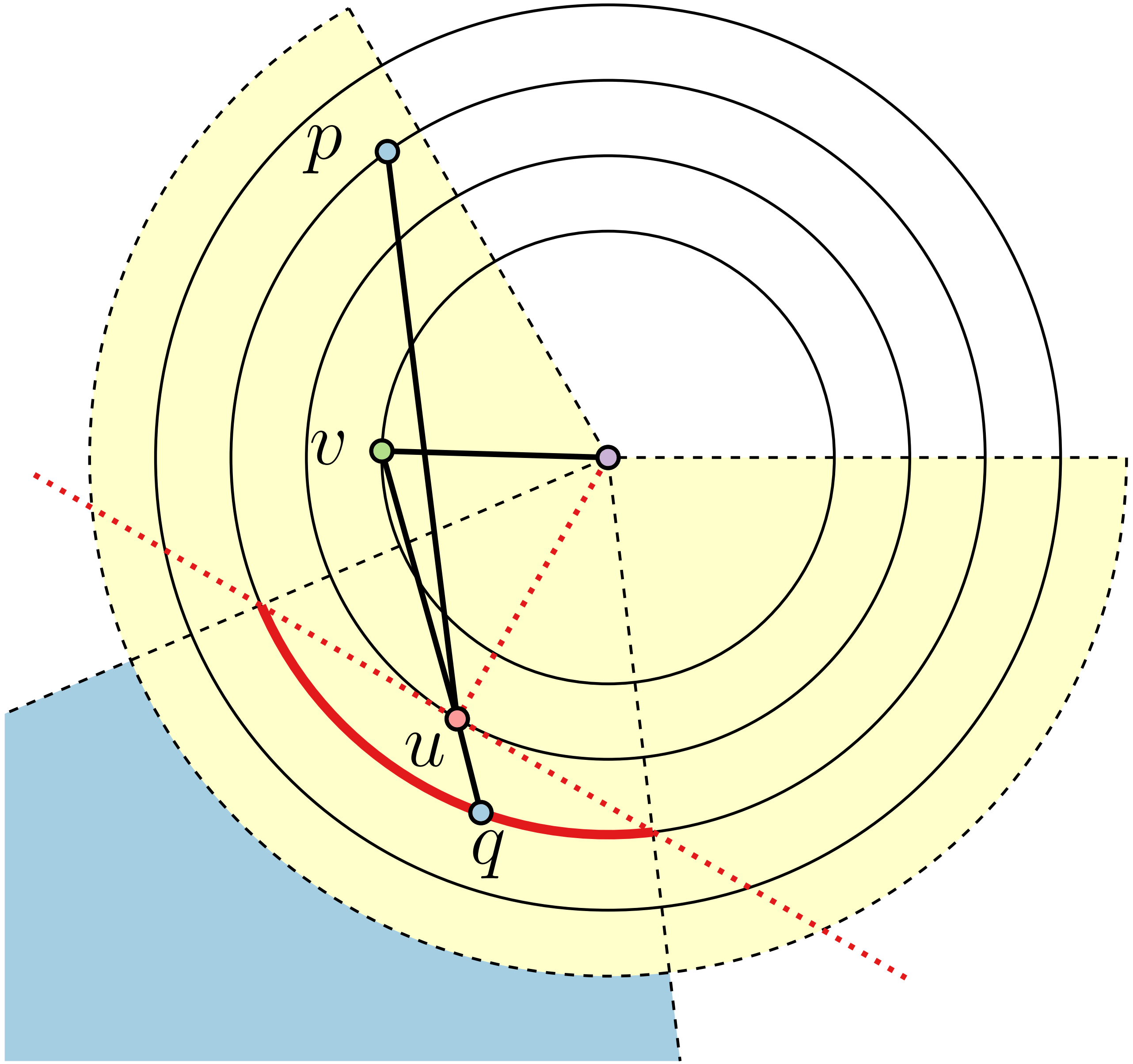
- Place u in the middle of its area



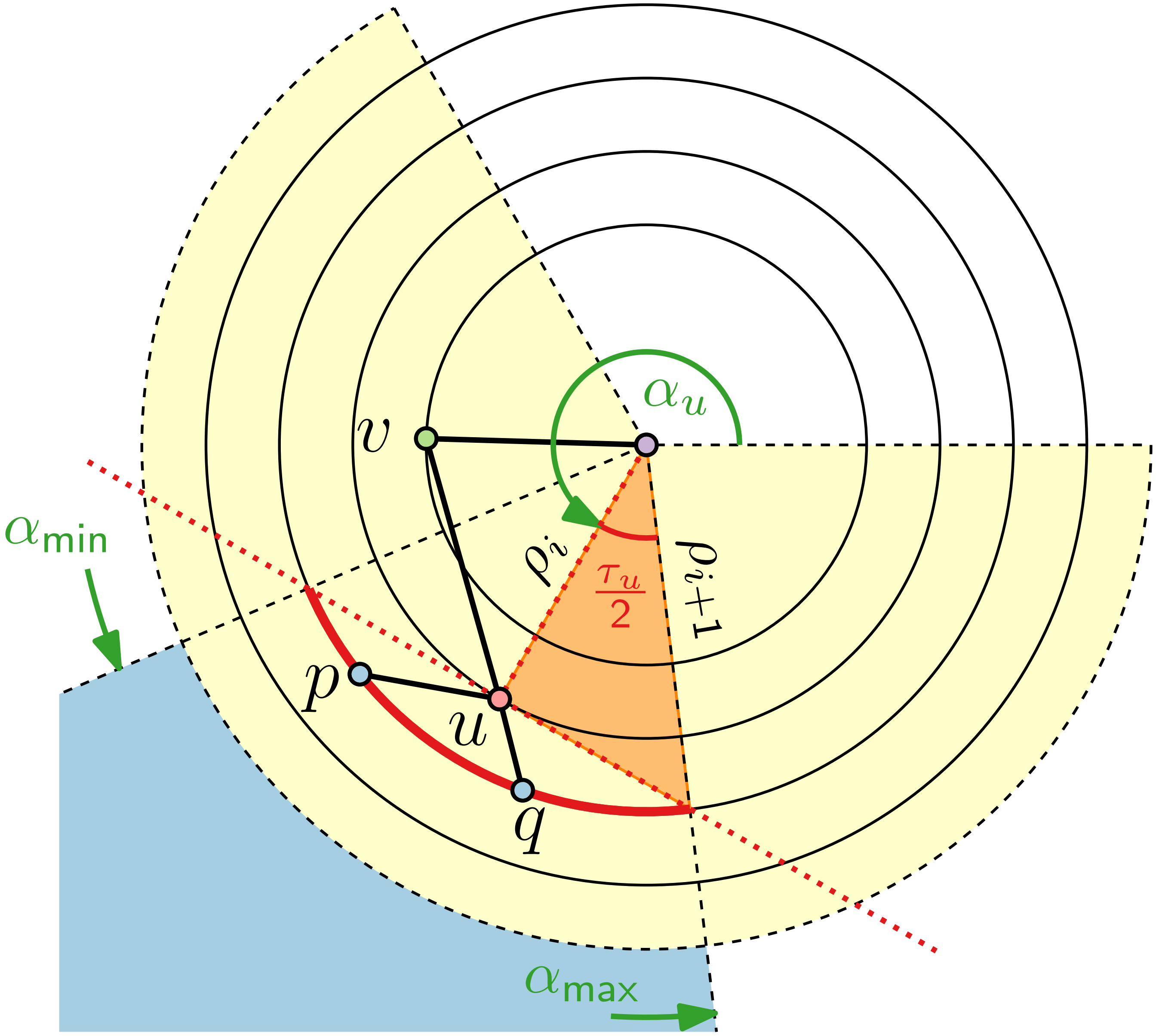




Radial Layouts – How To Avoid Crossings



Radial Layouts – How To Avoid Crossings



- τ_u – angle of the wedge corresponding to vertex u
- $\ell(u)$ – number of nodes in the subtree rooted at u
- ρ_i – radius of layer i
- $\cos(\tau_u/2) = \rho_i/\rho_{i+1}$
- $\tau_u = \min\left\{\frac{\ell(u)}{\ell(v)-1}\cdot\tau_v,\ 2\arccos\frac{\rho_i}{\rho_{i+1}}\right\}$
- Alternative:

  $\alpha_{\min} = \alpha_u - \arccos\frac{\rho_i}{\rho_{i+1}}$

  $\alpha_{\max} = \alpha_u + \arccos\frac{\rho_i}{\rho_{i+1}}$

Radial Layouts – Pseudocode

```
RadialTreeLayout(tree T, root r ∈ T, radii ρ_1 < ··· < ρ_k)
    postorder(r)
    preorder(r, 0, 0, 2π)
    return (d_v, α_v)_{v ∈ V(T)}
    // vertex positions in polar coordinates
```

```
postorder(vertex v)
    ℓ(v) ← 1
    foreach child w of v do
        postorder(w)
        ℓ(v) ← ℓ(v) + ℓ(w)
```

```
preorder(vertex v, t, α_min, α_max)
    d_v ← max{0, ρ_t}                      // output
    α_v ← (α_min + α_max)/2
    if t > 0 then
        α_min ← max{α_min, α_v − arccos(ρ_t / ρ_{t+1})}
        α_max ← min{α_max, α_v + arccos(ρ_t / ρ_{t+1})}
    left ← α_min
    foreach child w of v do
        right ← left + ℓ(w)/(ℓ(v)−1) · (α_max − α_min)
        preorder(w, t + 1, left, right)
        left ← right
```

Runtime? $\mathcal{O}(n)$

Correctness? ✓

Radial Layouts – Result

Theorem.
Let T be a rooted tree with n vertices. The algorithm RadialTreeLayout constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is a radial, crossing-free drawing,
- vertices lie on circles according to their depth, and
- the area of Γ is quadratic in max-degree(T) $\times$ height(T) (see [GD Ch. 3.1.3] for the details).

Literature

- [GD, Chapter 3] divide and conquer methods for rooted trees and series-parallel graphs
- [Reingold, Tilford '81] "Tidier Drawings of Trees"
  – original paper for level-based layout algo
- [Reingold, Supowit '83] "The complexity of drawing trees nicely"
  – linear program and NP-hardness proof for area minimization
- `treevis.net` – compendium of drawing methods for trees