Introduction to Informatics for Students from all Faculties

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Lecture 02
Digital Logics and Data Representation

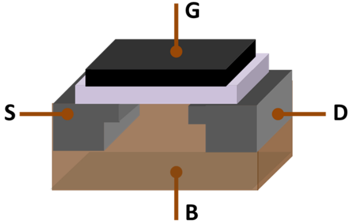
October 29, 2024

Notes:

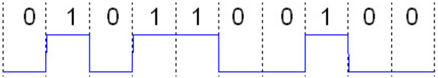
- **Lecture L02: Digital Logics and Data Representation** 29.10.2024
- **Educational objective:** We show how data is represented in a digital computer. We introduce basics of digital logics and show how we can implement arithmetic operations based on logic circuits.
  - Digital Representation of Data
  - Digital Logics and Digital Circuits
  - From Logics to Arithmetics
- **Exercise Sheet 01** due 05.11.2024

Motivation

- **how are data and instructions represented in an electronic digital computer?**
- how can we implement logical operations based on digital electronic circuits?
- how can we perform (arithmetic) operations on digitally encoded data?
- how can we **represent instructions** that can be executed by a programmable computer? → L03
- answering these questions requires basic foundations in terms of **numeral systems** and **digital logics**



schematic view of a transistor



a digital signal

Notes:

Encoding numbers with numeral systems

definition

A **numeral system** is a system that can be used to consistently encode numbers based on a set of symbols that are called **digits**. In a **positional numeral system** the contribution of a digit to the value of the encoded number depends on the **position of the digit**.

example: decimal numeral system

- digits represent different **powers of ten** depending on their position
- we call ten **base** of the decimal number system
- rightmost position represents **power of zero**, i.e. $10^0 = 1$
- powers of ten associated with a position increase from right to left

- value of encoded number is given by the **sum of contributions of individual digits**
- **left-most digit** is called **most-significant digit**
- **right-most digit** is called **least-significant digit**

`MMXXIV`

number 2024 represented in the **Roman numeral system**, where `M` represents thousand, `X` represents ten, `V` represents five and `I` represents one

10^3	10^2	10^1	10^0
2	0	2	4

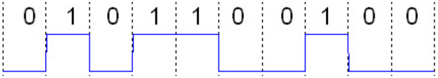
$$2 \cdot 1000 + 2 \cdot 10 + 4 \cdot 1 = 1011$$

number 2024 represented in the **decimal numeral system**, where the value of a digit depends on its position

Notes:

Binary numeral system

- in a digital electronic computer we can use voltage levels to encode two symbols 0 and 1
  - low or no voltage = 0
  - high voltage = 1
- voltage levels can be used to encode binary digits or "bits" (0 or 1) of the **binary numeral system**
- positional encoding analogous to decimal system, but using **base two instead of ten**
- depending on their position **digits represent powers of two**, where the least-significant bit represents $2^0 = 1$
- value of an encoded number is given as the **sum of powers of two**



2^3	2^2	2^1	2^0
1	0	1	1

$$1 \cdot 8 + 1 \cdot 2 + 1 \cdot 1 = 11$$

image credit: Wikipedia, CC-By-SA

Notes:

Powers of two

2^1	2	2^9	512	2^{17}	131,072	2^{25}	33,554,432
2^2	4	2^{10}	1,024	2^{18}	262,144	2^{26}	67,108,864
2^3	8	2^{11}	2,048	2^{19}	524,288	2^{27}	134,217,728
2^4	16	2^{12}	4,096	2^{20}	1,048,576	2^{28}	268,435,456
2^5	32	2^{13}	8,192	2^{21}	2,097,152	2^{29}	536,870,912
2^5	64	2^{14}	16,384	2^{22}	4,194,304	2^{30}	1,073,741,824
2^7	128	2^{15}	32,768	2^{23}	8,388,608	2^{31}	2,147,483,648
2^8	256	2^{16}	65,536	2^{24}	16,777,216	2^{32}	4,294,967,296

rules of thumb for orders of magnitude

$$2^0 = 1 = 10^0$$
$$2^{10} \approx 1,000 = 10^3$$
$$2^{20} \approx 1,000,000 = 10^6$$
$$2^{30} \approx 1,000,000,000 = 10^9$$
$$2^{40} \approx 1,000,000,000,000 = 10^{12}$$
$$\ldots$$

Notes:

Group Exercise 02-01

- Convert the the decimal number 126 into the binary number system.

- Convert the binary number 10010101 into the decimal number system.

Notes:

Hexadecimal numbers

- long binary numbers are **difficult to read and memorize for humans**
- **hexadecimal numeral system** (base 16) yields human-friendly representation of large (binary) numbers
- to distinguish 16 numbers from 0 to 15 with single digit, we extend the symbols $0, \ldots, 9$ by letters $A, \ldots F$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F

- **prefix $0x$** used to denote hexadecimal number, e.g. 0x10 = 16
- sice $2^4 = 16$, each hexadecimal digit corresponds to **four bits**, i.e. **easy to convert binary/hexadecimal numbers**

0101101011110011

	16^3	16^2	16^1	16^0
0x	5	A	F	3

$$5 \cdot 4096 + 10 \cdot 256 + 15 \cdot 16 + 3 \cdot 1 = 23283$$

0x	5	A	F	3
	0101	1010	1111	0011

Notes:

Group Exercise 02-01

Convert the following decimal numbers into the **hexadecimal and binary numeral system**.

- 3
- 4
- 19
- 64
- 255

Notes:

Group Exercise 02-02

Convert the following hexadecimal numbers into the **decimal and the binary numeral system.**

- 0x10
- 0xF0
- 0xAA
- 0x0100
- 0xFFFF

Notes:

Encoding text

- how can we **encode text in a digital computer**?
- idea: use numbers to encode text characters and represent each number by group of bits

ASCII encoding

American Standard Code for Information Interchange (ASCII) defines a 7-bit encoding of 128 different characters.

- code table maps numbers 0 to 127 (represented by 7 bits) to character and vice-versa

example

Binary ASCII-encoded text “Jurist”

1001010 1110101 1110010 1101001 1110011 1110100

Hexadecimal ASCII-encoded text “Jurist”

0x 4A 75 72 59 73 74

USASCII code chart

<table>
<tr><th colspan="5">b7 b6 b5 →</th><th>0 0 0</th><th>0 0 1</th><th>0 1 0</th><th>0 1 1</th><th>1 0 0</th><th>1 0 1</th><th>1 1 0</th><th>1 1 1</th></tr>
<tr><th>Bits b4</th><th>b3</th><th>b2</th><th>b1</th><th>Column → Row ↓</th><th>0</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>NUL</td><td>DLE</td><td>SP</td><td>0</td><td>@</td><td>P</td><td>`</td><td>p</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td>1</td><td>1</td><td>SOH</td><td>DC1</td><td>!</td><td>1</td><td>A</td><td>Q</td><td>a</td><td>q</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>0</td><td>2</td><td>STX</td><td>DC2</td><td>"</td><td>2</td><td>B</td><td>R</td><td>b</td><td>r</td></tr>
<tr><td>0</td><td>0</td><td>1</td><td>1</td><td>3</td><td>ETX</td><td>DC3</td><td>#</td><td>3</td><td>C</td><td>S</td><td>c</td><td>s</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>0</td><td>4</td><td>EOT</td><td>DC4</td><td>$</td><td>4</td><td>D</td><td>T</td><td>d</td><td>t</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>5</td><td>ENQ</td><td>NAK</td><td>%</td><td>5</td><td>E</td><td>U</td><td>e</td><td>u</td></tr>
<tr><td>0</td><td>1</td><td>1</td><td>0</td><td>6</td><td>ACK</td><td>SYN</td><td>&</td><td>6</td><td>F</td><td>V</td><td>f</td><td>v</td></tr>
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<tr><td>1</td><td>0</td><td>0</td><td>0</td><td>8</td><td>BS</td><td>CAN</td><td>(</td><td>8</td><td>H</td><td>X</td><td>h</td><td>x</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>1</td><td>9</td><td>HT</td><td>EM</td><td>)</td><td>9</td><td>I</td><td>Y</td><td>i</td><td>y</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>0</td><td>10</td><td>LF</td><td>SUB</td><td>*</td><td>:</td><td>J</td><td>Z</td><td>j</td><td>z</td></tr>
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<tr><td>1</td><td>1</td><td>0</td><td>1</td><td>13</td><td>CR</td><td>GS</td><td>-</td><td>=</td><td>M</td><td>]</td><td>m</td><td>}</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>0</td><td>14</td><td>SO</td><td>RS</td><td>.</td><td>></td><td>N</td><td>^</td><td>n</td><td>~</td></tr>
<tr><td>1</td><td>1</td><td>1</td><td>1</td><td>15</td><td>SI</td><td>US</td><td>/</td><td>?</td><td>O</td><td>_</td><td>o</td><td>DEL</td></tr>
</table>

ASCII code table from printer handbook, 1971

image credit: public domain

Notes:

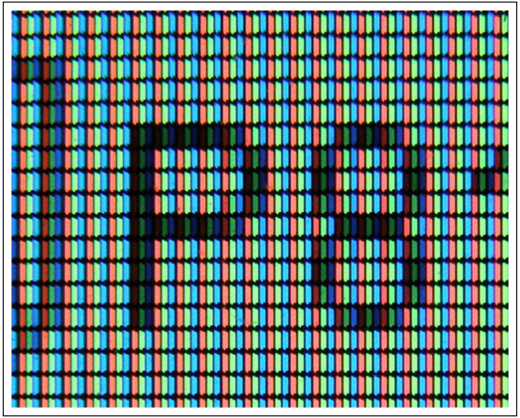
Encoding images

- how can we represent **image data** in a digital computer?

digital images

A digital (raster) image is a picture that is composed of a **rectangular arrangement of pixels**, where each pixel either represents a brightness and (possibly a color value).

- idea: use numbers to represent birghtness (and colors) of pixels
  - **grayscale pixels**: 8 bits encoding 255 brightness levels from black (0) to white (255)
  - **color pixels**: 3×8 bits encoding 255 brightness levels of red (R), green (G), blue (B)
- image can be digitally encoded by **sequence of bits**, where groups of 8 or 24 bits represent grayscale or color pixels in rectangular grid

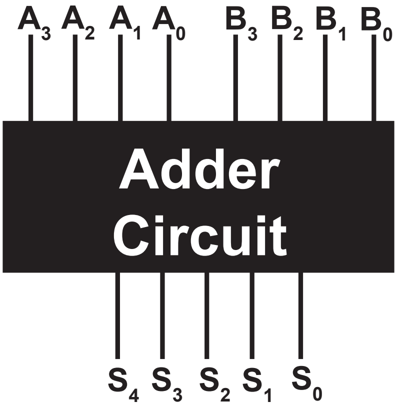


Closeup of pixels (consisting of a red, green, and blue subpixel) on a liquid crystal display (LCD) laptop screen

Notes:

From binary numbers to arithmetics

- we introduced the **digital representation of numbers, text, and images** by bits
- how can a digital computer perform **arithmetic operations** like addition, multiplication, etc.?
- as example, let us consider **addition of two binary numbers with 4 bits each**
- given **input of 8 bits** (i.e. binary numbers A and B with 4 bits each), we must compute **5 bits of output** that encode the sum $A + B$

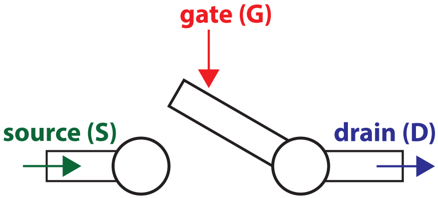


Notes:

- Why do we need five bits for the sum of two four bit binary numbers?
- The largest sum that we can have for two four bit binary numbers is the sum of $1111_b = 15$ and $1111_b = 15$ which is $30 = 1110_b$. This requires 5 bits.

Digital logics

- **digital logics** is the basis for any (arithmetic) operation of a digital computer
- all functions of a digital computer can eventually be reduced to simple logical operations implemented by electronic switches (e.g. transistors)
- consider one bit representing voltage levels of source (S), gate (G), and drain (D)
- considering S and G as inputs, the output D represents a **logical AND operation**



S	G	D
0	0	0
0	1	0
1	0	0
1	1	1

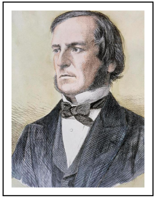
Notes:

Boolean Logic

- formal treatment of logics pioneered by mathematician **George Boole**
- Boole studied **logical operations** in analogy to arithmetic operations like $+$, $-$, $\times$ → G Boole, 1854
- basic logical operations `AND`, `OR` and `NOT` can be mapped to **arithmetic operations** on numbers 0 and 1 representing `False` and `True`

basic operations in Boolean logics

- A AND $B \equiv A \times B$
- A OR $B \equiv A + B - A \times B$
- NOT $A \equiv 1 - A$



George Boole (1815 – 1864)

image credit: Wikimedia Commons, public domain

Notes:

Truth tables

- **truth tables** define output of logical operations
- each row in the truth table is one **possible combination of inputs**
- we use 0 and 1 to represent logical values `False` and `True`

AND

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

OR

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

NOT

A	NOT A
0	1
1	0

XOR

A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

basic operations in Boolean logics

- $A \text{ AND } B \equiv A \times B$
- $A \text{ OR } B \equiv A + B - A \times B$
- $\text{NOT } A \equiv 1 - A$

Notes:

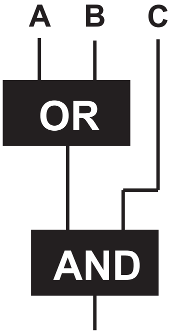
Boolean functions and digital circuits

- basic Boolean logical operations NOT, AND, OR and XOR are defined for one or two inputs, respectively
- we can use these as building blocks of more complex **Boolean functions** with more than one or two inputs
- in formular notation we use brackets to **determine order in which operations are executed**

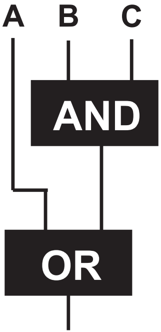
examples

- A OR NOT B
- [A OR B] AND C
- A OR [B AND C]

- Boolean functions can be represented as **digital circuits** where **logic gates** represent Boolean operations



logical circuit implementing Boolean function [A OR B] AND C



logical circuit implementing Boolean function A OR [B AND C]

image credit:

Notes:

Group Exercise 02-03

- Give the truth table for the Boolean function A OR $[B$ AND NOT $C]$ with three inputs A, B, C.

Notes:

Group Exercise 02-03

- Give a formula for the Boolean function represented by the following truth table.

A	B	C	?
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Notes:

Group Exercise 02-03

- Show that the **logical operation `XOR`** (with two inputs A and B) can be constructed as Boolean function that only uses `AND`, `OR` and `NOT` operations.

Notes:

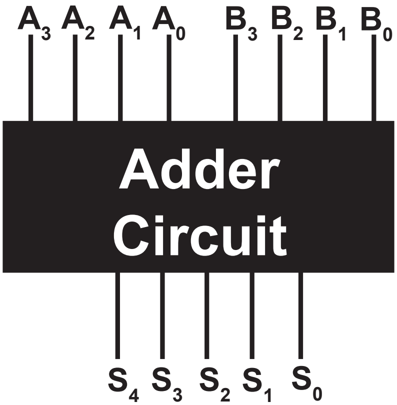
Group Exercise 02-03

- Show that the **logical operation `OR`** (with two inputs A and B) can be constructed as Boolean function that only uses `AND` and `NOT` operations.

Notes:

From Digital Circuits to Arithmetics

- we have seen how complex Boolean functions can be implemented based on the basic logical operations `AND`, `OR`, and `NOT`
- assume that we have digital inputs that represent two numbers A and B in the binary numeral system
- how can we perform **arithmetic operations** like addition or multiplication?
- idea: specify **Boolean functions** that give the correct digital output for each combination of digital inputs



Notes:

Adding decimal numbers

- let us consider the **pencil-and-paper algorithm** to **add numbers** in the decimal numeral system

1	2	5	7
2_1	9	3	2
4	1	8	9

pecil-and-paper algorithm to add two numbers

step 1 start at right-most position

step 2 add digits at current position

step 3 write last digit of sum below current position

step 4 for sums ≥ 10 additionally **carry over** 1 to position on the left

step 5 move one position to left and go to step 2

- allows to reduce addition of numbers with any number of digits to step-wise addition of individual digits

open questions

- how can we apply this to binary numbers ?
- how can we map addition to logical operations?

Notes:

Adding binary numbers

- how does pencil-and-paper addition work for **binary numbers**?
- we can apply the same algorithm but we only have binary digits (bits) 0 and 1

0	1	1	0
0_1	1	0	1
1	0	1	1

pecil-and-paper algorithm to add two numbers

step 1 start at right-most position

step 2 add digits at current position

step 3 write last digit of sum below current position

step 4 for sums ≥ 2 additionally **carry over** 1 to position on the left

step 5 move one position to left and go to step 2

- a **carry bit** of one is created whenever the sum is larger than base two of the binary numeral system

Notes:

Half adder circuit

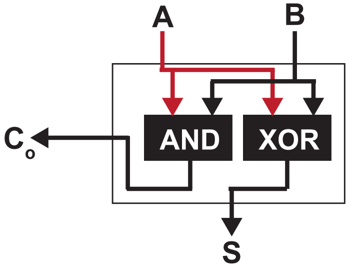
- we can write down the truth table of a Boolean function that generates the **sum** S for two binary digits A and B
- sum bit can be generated by a single XOR operation on A and B
- we can further write a Boolean function that generates the **carry-over bit** C_o
- carry bit can be generated by a single AND operation on A and B
- we call the resulting digital circuit a **half adder**
- is this enough to add two binary numbers with **any number of digits**?

A	B	S
0	0	0
0	1	1
1	0	1
1	1	0

$$S = A \ \texttt{XOR} \ B$$

A	B	C_o
0	0	0
0	1	0
1	0	0
1	1	1

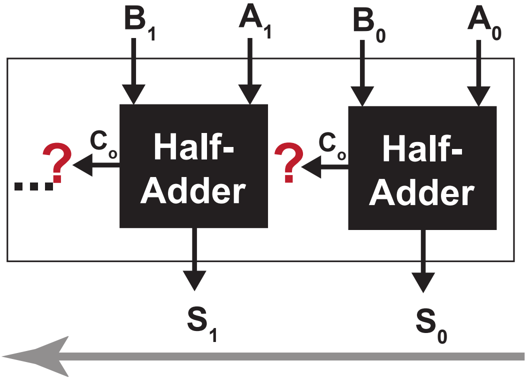
$$C_o = A \ \texttt{AND} \ B$$



Notes:

Adding numbers with more than one bit?

Can we connect multiple half adder circuits to add numbers A and B that consist of more than one bit?



problem

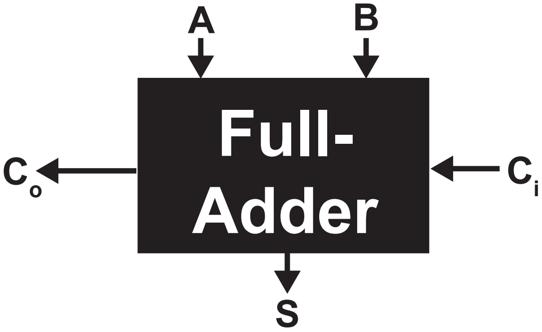
Half adder produces carry-out bit for next position but does not have input that allows to consider carry bit from previous position!

Notes:

Full adder circuit

- paper-and-pencil algorithm requires to **include carry bit** from the right when summing digits for a given position
- need Boolean function that takes two bits A and B and a “carry-in” bit C_i as input and generates sum S and “carry-out” bit C_o
- resulting digital circuit is called **full adder**
- **carry-out output** generated for one position is used as **carry-in input** for next position to the left
- this digital circuit design implementes our simple pen-and-pencil method

0	1	1	0
0_1	1	0	1
1	0	1	1



Notes:

Group Exercise 02-04

- Write down truth tables for the sum and carry-out bit of a full-adder.
- Give formulas for Boolean functions for the sum and the carry-out bit of a full-adder.

Notes:

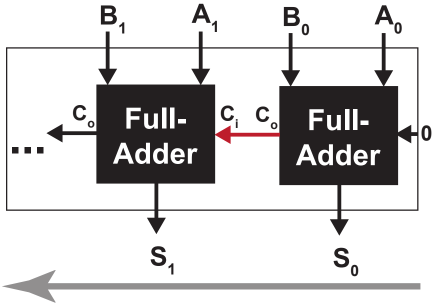
Carry-Ripple Adder

- we can build a full adder based on logical operations `AND`, `OR` and `XOR`.
- we can connect multiple full adders to logical circuit that adds binary numbers with **any number of digits**

idea for adding two 4 bit numbers

- use sequence of 4 full adders, one for each pair of digits of inputs A and B
- each full adder generates one bit of sum S
- connect carry-out of each full adder with carry-in of next full adder in sequence
- last carry-out is **most-significant bit** of the sum
- we can set **carry-in of first full adder** to zero

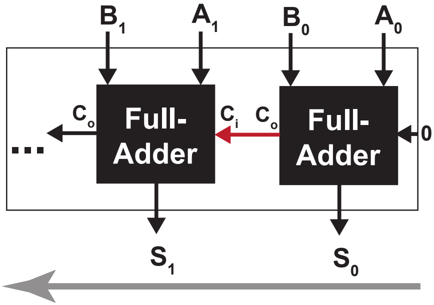
- we call this design a **carry-ripple adder**



Notes:

Computation time of Carry-Ripple Adder

- what is the **computation time** of a carry-ripple adder?
- after setting input bits, we must wait until all carry bits have “rippled” through sequence of full adders
- “‘gate delay” until sum is computed corresponds to number of full adders in sequence
- let us assume that it takes a full adder **one nanosecond** ($= 10^{-9}$ = 1 billionth of a second) to compute the sum of two bits
- adding two 32 bit numbers with a carry-ripple adder then takes 32 nanoseconds
- in practice we use other designs like the **carry-lookahead or carry-select adder** that speed up the computation time

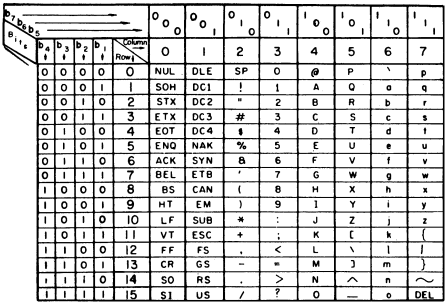


Notes:

Conclusion

- we have shown how we can **digitally represent data like numbers, text, or images**
- we introduced **basic operations in Boolean logics** like `AND`, `OR`, `NOT`, and `XOR`
- we expressed **Boolean functions** given in a truth table using basic Boolean operations
- we demonstrated how to implement arithmetic operations on binary representations of numbers using **digital circuits**
- digital logics is the **foundation of all digital computers and technology**

USASCII code chart



$b_7 b_6 b_5$ →					0 0 0	0 0 1	0 1 0	0 1 1	1 0 0	1 0 1	1 1 0	1 1 1
b_4	b_3	b_2	b_1	Row ↓ / Column →	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	'	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	|
1	1	0	1	13	CR	GS	−	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	—	o	DEL

ASCII code table

image credit: public domain

Notes:

Self-study questions

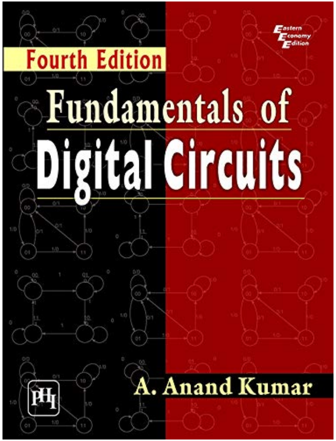
1. What is a bit?
2. Give and example for a positional and a non-positional numeral system to represent numbers.
3. Convert the decimal number 42 into the binary numeral system.
4. Convert the binary number 101010 into the decimal numeral system.
5. Convert the hexadecimal number $0x2A$ into the decimal and the binary numeral system.
6. Explain how you can use electronic switches to implement the basic logical operations AND, OR, and NOT.
7. Use a truth table to explain the difference between the OR and the XOR operation.
8. Give the truth table for the Boolean formula $[A \text{ OR } \text{ NOT } B] \text{ AND } C$.
9. Use a truth table to show that the logical operator X OR Y corresponds to NOT (NOT X AND NOT Y).
10. Give the truth table for the outputs of a half and a full adder.
11. Explain the difference between a half and a full adder.
12. What is the largest output that a carry-select adder for two four-bit inputs and a carry-in input can produce?
13. Draw a diagram of the digital circuit implementation of a full adder.

Notes:

Literature

reading list

- A Anand Kumar: **Fundamentals of Digital Circuits**, PHI Learning, 2016
- K Fricke: **Digitaltechnik**, Springer Vieweg, 2018
- U Schöningh: **Logik für Informatiker**, Spektrum, 2005
- C Meinel, M Mundhenk: **Mathematische Grundlagen der Informatik**, BG Teubner, 2000



Notes: