Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

Würzburg, November 22, 2024

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Homework Assignment #5
Approximation Algorithms (Winter Semester 2024/25)

Exercise 1 – Overpacking in the dual LP for SetCover

Consider the dual LP for SetCover:

$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \sum_{u \in S} y_u \leq c_S \quad \text{for every } S \in \mathcal{S} \\ & y_u \geq 0 \quad \text{for every } u \in U. \end{aligned}$$

Let $\text{price}(u)$ be the price of element u as determined by the algorithm GreedySetCover in Lecture #2.

Prove that the solution $y_u = \text{price}(u)$ is in general not feasible for the dual LP. In particular, find an instance with a set S that is overpacked by a factor of (approximately) $H_{|S|}$, i.e., $\sum_{u \in S} y_u \approx H_{|S|} \cdot c_S$. **[4 points]**

Exercise 2 – Randomized Rounding for SetCover

Consider the following randomized algorithm for SetCover.

Algorithm 1: RandomizedRounding$(U, \mathcal{S}, c)$

```
Compute an optimal solution x of the LP relaxation for SetCover (as presented in the lecture).
C ← ∅
foreach S ∈ 𝒮 do
    add S with probability x_S to C
return C
```

a) Prove that the expected cost of $\mathcal{C}$ is exactly OPT_{LP}. **[4 points]**

b) Let $u \in U$ be an arbitrary element of the ground set. Prove that u is *not* covered by $\mathcal{C}$ with probability at most $1/e$. **[3 points]**

   Hint: Use that, for every $x \in \mathbb{R}$, it holds that $1 + x \leq e^x$.

Exercise 3 – Randomized Rounding for SETCOVER with Multiple Rounds

Let d be a sufficiently large constant. We now run the algorithm RandomizedRounding from Exercise 2 exactly $\lceil d \cdot \ln n \rceil$ times, where $n = |U|$. In the following, let $\mathrm{OPT}_{\mathrm{LP}}$ be the value of the optimal solution of the LP relaxation for the SETCOVER ILP, and let $\mathcal{C}'$ be the union of all sets selected this way.

a) Prove that every element $u \in U$ is *not* covered by $\mathcal{C}'$ with probability at most $1/(4n)$, as long as d was chosen large enough. **[3 points]**

b) Prove that the cost of the set $\mathcal{C}'$ is greater than $4\lceil d \ln n \rceil \cdot \mathrm{OPT}_{\mathrm{LP}}$ with probability at most $1/4$. **[3 points]**

c) Prove that $\mathcal{C}'$ is a *feasible* solution of cost at most $4\lceil d \ln n \rceil \cdot \mathrm{OPT}_{\mathrm{LP}}$ with probability at least $1/2$. **[3 points]**

Please hand in your solutions via WueCampus by 10:00 on Friday, November 29.