Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

Würzburg, November 15, 2024

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Homework Assignment #4
Approximation Algorithms (Winter Semester 2024/25)

Exercise 1 – Standard form of LPs

Prove that every linear program can be converted into the following standard form:

$$\begin{aligned} \textbf{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \textbf{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \forall i \in \{1, \ldots, m\} \\ & x_j \geq 0 \quad \forall j \in \{1, \ldots, n\}. \end{aligned}$$

[5 points]

Exercise 2 – Minumum s–t Cut

A matrix A is called *totally unimodular* if every square submatrix has determinant 0, $+1$, or -1. If A is totally unimodular and b is integral, then the linear program $\{\min c^\top x \mid Ax \geq b, x \geq 0\}$ has an integral extreme point solution for any cost vector c.

Consider the MIN-s–t-CUT problem from lecture #4:

$$\begin{aligned} \textbf{minimize} \quad & \sum_{(u,v) \in E} c_{uv} d_{uv} \\ \textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}, \\ & p_s - p_t \geq 1, \\ & d_{uv} \geq 0 \quad \forall (u,v) \in E, \\ & p_u \geq 0 \quad \forall u \in V. \end{aligned}$$

Prove that the coefficient matrix A of this LP is totally unimodular and hence the problem has an integral extreme point solution.

Hint: Use induction. **[5 points]**

Exercise 3 – Dual program of VERTEXCOVER

Write a linear program for VERTEXCOVER and determine its dual. How can you interpret the dual linear program (with integer variables)? How does the dual and its intepretation change if you consider WEIGHTEDVERTEXCOVER? **[5 points]**

Exercise 4 – VERTEXCOVER for planar graphs

Give a factor-1.5 approximation algorithm for (weighted) VERTEXCOVER, restricted to planar graphs.
[5 points]

Hint: Use the facts that (a) the vertex cover polytope is half-integral and (b) for every planar graph, a 4-coloring can be computed in polynomial time. A 4-coloring assigns to every vertex one of four colors such that no two neighboring vertices have the same color.

Please hand in your solutions via WueCampus by 10:00 on Friday, November 12.