Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

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Homework Assignment #2
Approximation Algorithms (Winter Semester 2024/25)

Exercise 1 – MAXIMUM k-COVERAGE

In this exercise, we consider the MAXIMUM k-COVERAGE problem, which is a variant of the SET COVER problem introduced in the lecture. Let U be a set, let $\mathcal{S}$ be a set of subsets of U such that $U = \bigcup \mathcal{S}$, and let $k \leq |\mathcal{S}|$ be a natural number. The problem MAXIMUM k-COVERAGE asks for a subset $\mathcal{S}' \subseteq \mathcal{S}$ with $|S'| = k$ such that the number $|\bigcup \mathcal{S}'|$ of elements covered by $\mathcal{S}'$ is maximized.

For this problem, we consider a greedy algorithm that iteratively performs the following step: In each iteration, select a set from $\mathcal{S}$ that contains the largest number of uncovered elements. The algorithm terminates after at most k iterations or when U is covered. Let $\ell \leq k$ be the number of iterations that the algorithm performs.

In the following, we analyze the quality of this algorithm. For $i \in \{1, \dots, \ell\}$, let ALG_i be the number of elements that are covered by the sets that have been selected in the steps up to and including step i. We set $\mathrm{ALG}_0 = 0$.

a) Show that $\mathrm{ALG}_1 \geq \frac{1}{k} \cdot \mathrm{OPT}$. In other words, show that the greedy algorithm covers at least OPT/k elements with the set selected in the first step. **[4 points]**

b) Show by induction that for each $i \in \{0, \dots, \ell\}$, the following holds: $\mathrm{OPT} - \mathrm{ALG}_i \leq \left(1 - \frac{1}{k}\right)^i \cdot \mathrm{OPT}$.

   Hint: Use the fact that, at the beginning of iteration i, there exist at least $\mathrm{OPT} - \mathrm{ALG}_{i-1}$ uncovered elements in the optimal solution. **[8 points]**

c) Show that the greedy algorithm has approximation ratio $\left(1 - \frac{1}{e}\right) \approx 0.63$.

   Hint: Use that, for any positive integer j, it holds that $\left(1 - \frac{1}{j}\right)^j \leq \frac{1}{e}$. **[3 points]**

Exercise 2 – Factor-h approximation for unweighted SET COVER

Let $(U, \mathcal{S})$ be an unweighted SET COVER instance in which each element from U is contained in at most h sets from $\mathcal{S}$. We refer to h as the *frequency*. (For frequency $h = 2$, we obtain VERTEXCOVER.)

Consider the following iterative algorithm:

Algorithm 1: SimpleGreedySetCover(U, $\mathcal{S}$)

```
S' ← ∅
while U ≠ ∅ do
    mark an arbitrary element u ∈ U
    R ← {S ∈ S | u ∈ S}
    S' ← S' ∪ R
    U ← U \ ⋃R   // remove all elements from U that are covered by R
return S'
```

Show that this algorithm yields a factor-h approximation and that the algorithm generalizes the factor-2 approximation algorithm for VERTEX COVER from the lecture.

Hint: Consider the set U_m of elements marked by the algorithm and find a relation of $|U_m|$ and OPT.

[5 points]