

Homework Assignment #1

Approximation Algorithms (Winter Semester 2024/25)

Exercise 1 – Maximum acyclic subgraph

Devise a factor-1/2 approximation algorithm for the following problem:

Let G be a given directed graph. Find a set of edges $E' \subseteq E(G)$ with maximum cardinality such that the resulting subgraph (V, E') is acyclic.

[5 points]

Exercise 2 – Smallest maximal matching

Give a factor-2 approximation algorithm for the following problem:

Let G be a given undirected graph. Find a maximal matching with smallest cardinality in G .

[5 points]

Exercise 3 – Greedy for MAXIMUM CUT

Let G be a given graph. The MAXIMUM CUT problem asks for a partition of the vertex set $V(G)$ of G into two sets S and $\bar{S}$ such that the number of edges connecting these sets is as large as possible.

Consider the following greedy algorithm for MAXIMUM CUT. For a set $U \subseteq V(G)$ and a vertex $v \in V(G) \setminus U$, let $\deg(v, U)$ denote the number of edges between v and U .

Algorithm 1: Greedy-Max-Cut(G)

```
choose two arbitrary vertices  $v_1, v_2 \in V$ 
 $S \leftarrow \{v_1\}$ 
 $\bar{S} \leftarrow \{v_2\}$ 
foreach  $v \in V \setminus \{v_1, v_2\}$  do
    if  $\deg(v, S) \geq \deg(v, \bar{S})$  then
         $\bar{S} \leftarrow \bar{S} \cup \{v\}$ 
    else
         $S \leftarrow S \cup \{v\}$ 
return Cut ( $S, \bar{S}$ )
```

- a) Let G^n be the undirected graph with vertex set $\{v_1, \dots, v_n\}$ and edge set $\{v_1v_i, v_2v_i : 3 \leq i \leq n\}$. Which result does Greedy-Max-Cut yield, assuming the vertices are handled in the order of their indices? What is the optimal solution? What is the approximation ratio of the algorithm for the graph class $\{G^n : n \in \mathbb{N}, n \geq 3\}$?

[3 points]

- b) Show that the greedy algorithm is a factor- $\frac{1}{2}$ approximation algorithm for arbitrary graphs (with at least two vertices).

Suggestion: The following definition could be helpful in your analysis. Let G be a graph, let S be the vertex set defined by the algorithm above, and let $v \in S$. Let $\text{inter}(v)$ and $\text{intra}(v)$ denote the values $\deg(v, \bar{S})$ and $\deg(v, S)$ *at the point of time* when the algorithm considers v , respectively. Note that these values change during the execution of the algorithm. For $v \in \bar{S}$, $\text{inter}(v)$ and $\text{intra}(v)$ are defined symmetrically.

[7 points]