Coloring and Recognizing Mixed Interval Graphs

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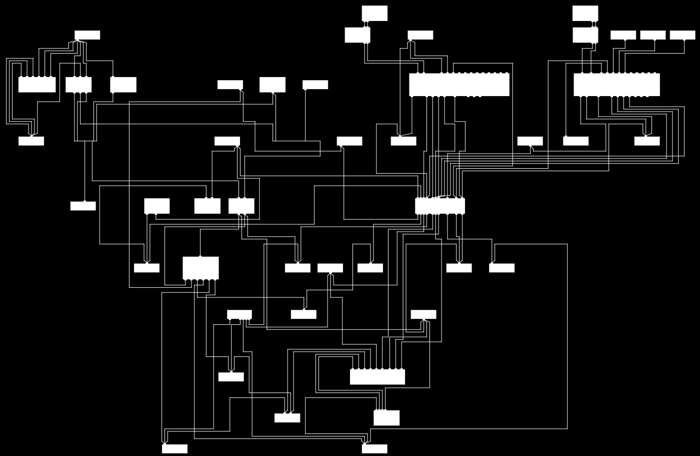
ISAAC 2023 – arXiv: 2303.07960

Previous paper: GD 2022 – arXiv: 2208.14250

Motivation



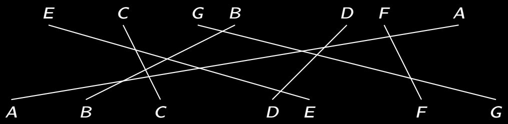
Graph Drawing

- layered drawing
- orthogonal drawing
- Sugiyama's framework:
  - eliminate cycles
  - assign layers
  - minimize crossings
  - place nodes
  - route edges

Motivation

Edge Routing

- two layers
- fixed vertices
- orthogonal edges
- two bends
- no overlaps
- minimize #crossings
- minimize #sub-layers



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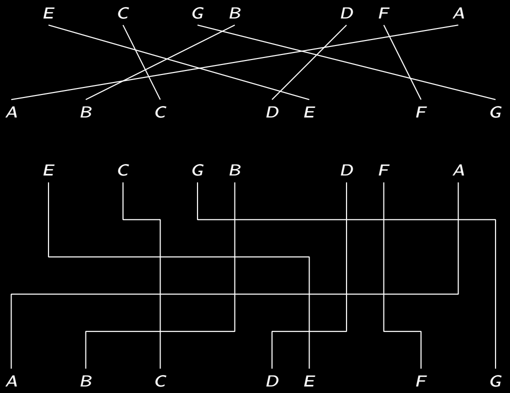
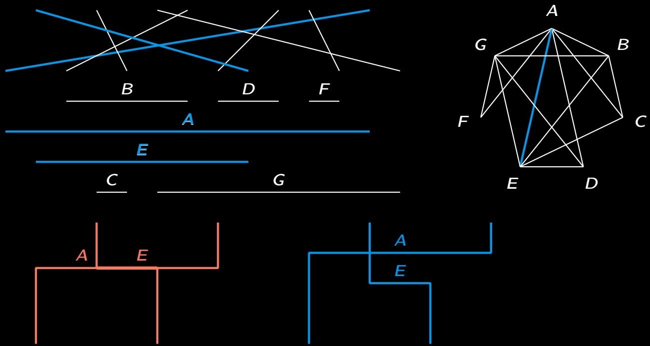


Coloring Mixed Interval Graphs

Mixed Interval Graph

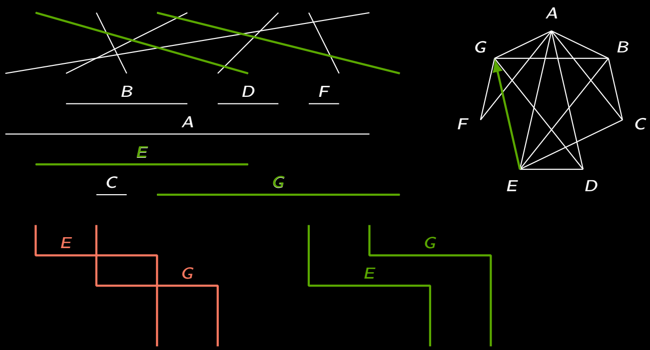
- interval graph



Coloring Mixed Interval Graphs

Mixed Interval Graph

- interval graph
- some edges directed



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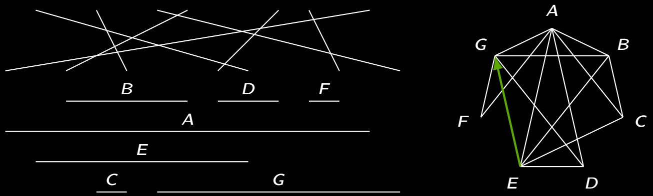

Coloring

- colors: $\mathbb{N}$
- color = sub-layer
- no monochromatic edges: $\{u, w\} \in E \implies c(u) \neq c(w)$
- strictly monotone arcs: $(u, w) \in A \implies c(u) < c(w)$

Coloring Mixed Graphs

Coloring

- any **mixed** graph $G = (V, E, A)$ on input
- minimize number of colors in $c : V \mapsto \mathbb{N}$
- no monochromatic edges: $\{u, w\} \in E \implies c(u) \neq c(w)$
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Observations

- only edges $\implies$ proper coloring (NP-hard)

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- Gallai, Hasse, Roy, Vitaver:
  proper coloring $\approx$ orientation minimizing longest directed path

Coloring Mixed Graphs

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Observations

- only edges $\implies$ proper coloring (NP-hard)
- only arcs $\implies$ longest directed path (easy)
- Gallai, Hasse, Roy, Vitaver:
  proper coloring $\approx$ orientation minimizing longest directed path
- mixed graph coloring $\approx$ partial orientation extension

Theorem

Coloring Mixed Interval Graphs is NP-*hard.*

Geometric Structure

Theorem

Coloring Mixed Interval Graphs is barely NP-*hard.*

Proof.

Coloring Circular Arc Graphs is NP-hard. □

Geometric Structure

Theorem
Coloring Mixed Interval Graphs is NP*-hard.*

Geometric Variants
Interval Graphs + geometric condition
which edges turn into arcs

Geometric Structure

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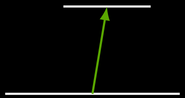
Geometric Variants

Interval Graphs + geometric condition
which edges turn into arcs

Containment Variant



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Interval Graphs + geometric condition which edges turn into arcs

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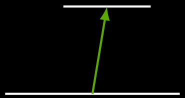
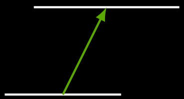


Directional Variant



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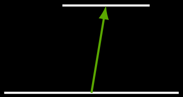

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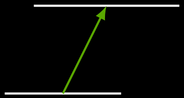
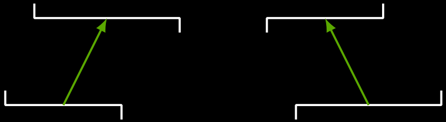


Bidirectional Variant



Summary of Results

Coloring of Mixed Graphs

	general	interval	containment	directional	bidirectional
optimal	NP-hard	NP-hard	NP-hard	✓	NP-hard
approximate	✗	?	2	1	2
recognition	✓	✓	✓	✓	?

New Results

- Containment Mixed Interval Graphs
  - Recognition
  - NP-hardness of coloring
  - $2\omega - 1$ bounds
  - 2-approximation
- Bidirectional Mixed Interval Graphs
  - NP-hardness of coloring
- Mixed Interval Graphs
  - $O(\lambda\omega)$ bounds

Coloring Containment Mixed Interval Graphs: approximation

Theorem

Coloring Containment Mixed Interval Graphs admits a 2-approximation.

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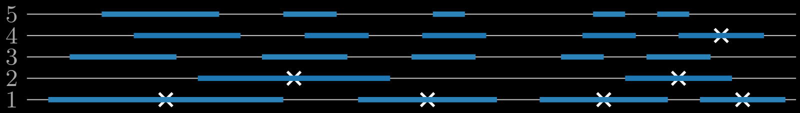

Algortihm

- Select maximal intervals.

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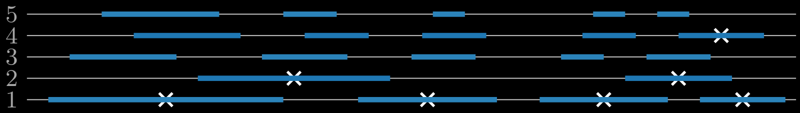

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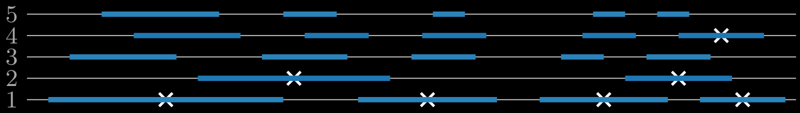

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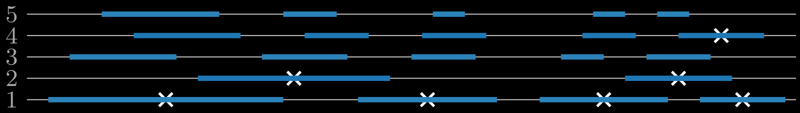

Algortihm

- Select maximal intervals.
- Greedily select minimal subset that covers everything.
- Use colors 1, 2 and decrease clique number.
- Recurse.

Coloring Containment Mixed Interval Graphs: hardness

Theorem
Coloring Containment Mixed Interval Graphs is NP*-hard.*

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3-SAT reduction

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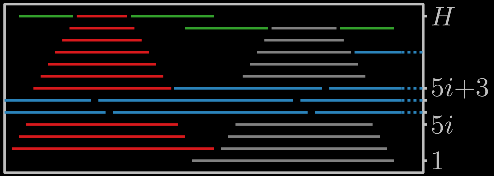

3-SAT reduction

- Variable gadget

Theorem

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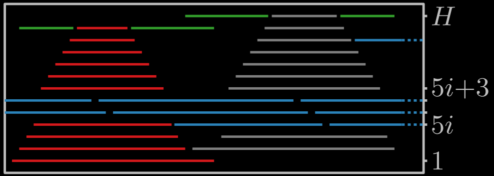

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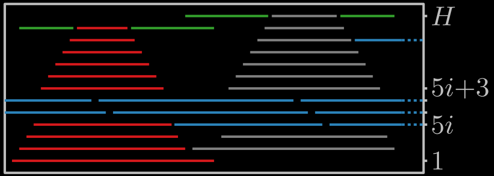

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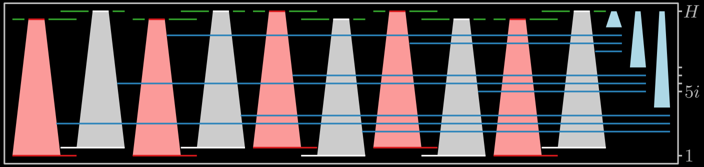

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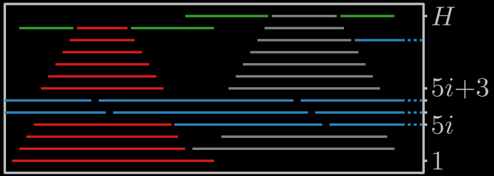

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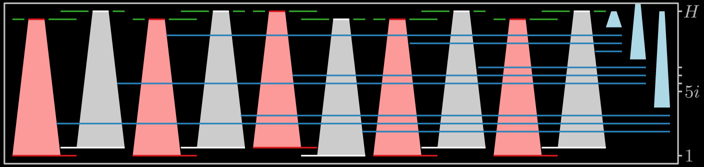

Recognizing Containment Mixed Interval Graphs

Theorem

Containment Mixed Interval Graphs are decidable in P.

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Details

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- Interval Graphs are decidable.
- Arcs give additional information.

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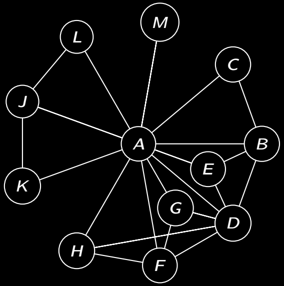
- Arcs give additional constraints.

Recognizing Containment Mixed Interval Graphs: PQ-trees



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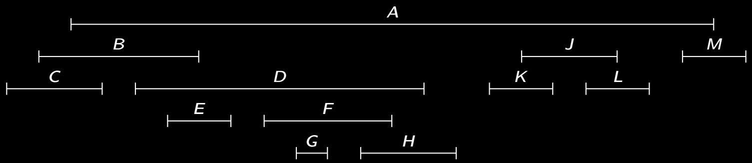

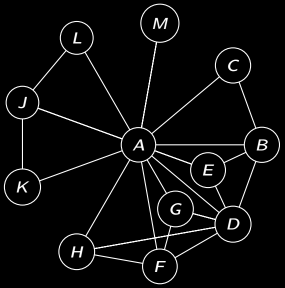

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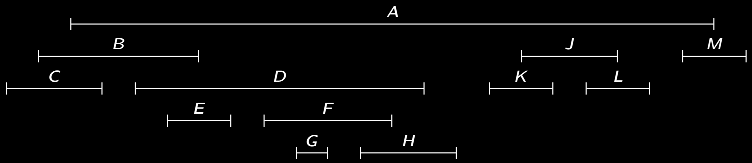
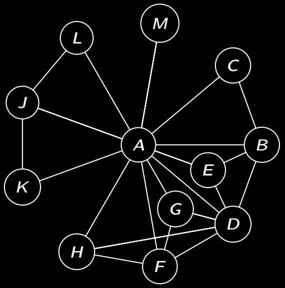
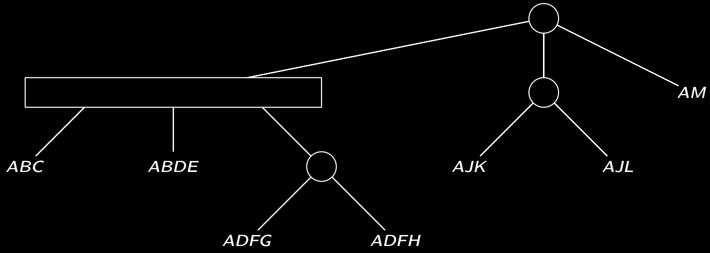
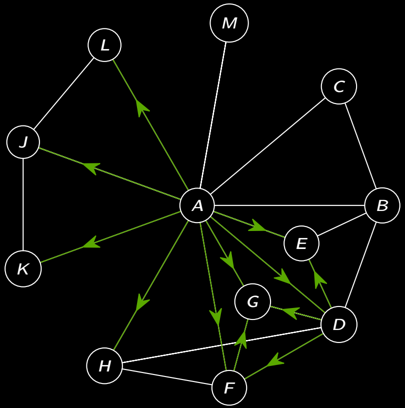




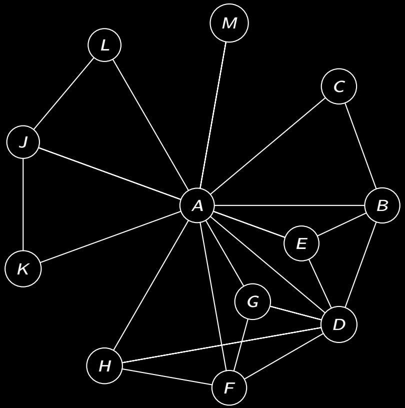




Details

- INPUT: mixed graph $G = (V, E, A)$
- OUTPUT: interval representation

Recognizing Containment Mixed Interval Graphs: algorithm



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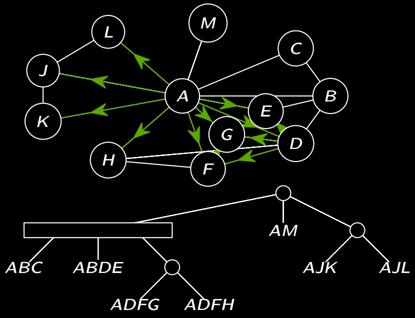

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- INPUT: mixed graph $G = (V, E, A)$
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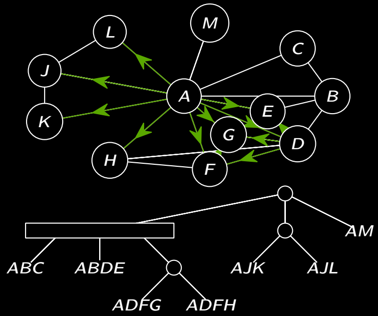
- STEP 1: turn arcs into edges. Check that G is an Interval Graph
- STEP 2: find a *good* rotation of PQ-tree



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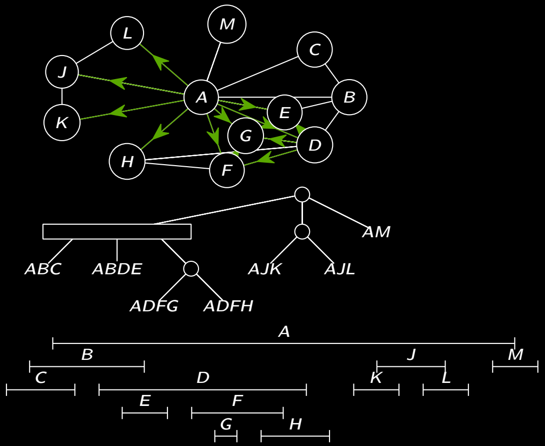

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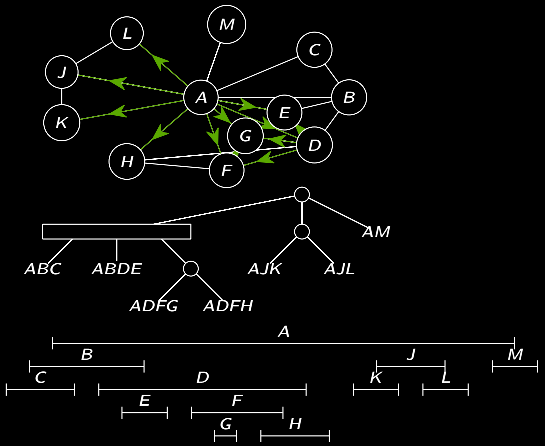

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Summary of Results

	general	interval	containment	directional	bidirectional
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approximate	✗	?	2	1	2
recognition	✓	✓	✓	✓	?

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Open Problems

- Coloring Mixed Interval Graphs: approximation

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Open Problems

- Coloring Mixed Interval Graphs: approximation
- Coloring Containment/Bidirectional Mixed Interval Graphs: better approximation

Summary of Results

	general	interval	containment	directional	bidirectional
optimal	NP-hard	NP-hard	NP-hard	✔	NP-hard
approximate	✖	?	2	1	2
recognition	✔	✔	✔	✔	?

Open Problems

- Coloring Mixed Interval Graphs: approximation
- Coloring Containment/Bidirectional Mixed Interval Graphs: better approximation
- Recognizing Bidirectional Mixed Interval Graphs

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	general	interval	containment	directional	bidirectional
optimal	NP-hard	NP-hard	NP-hard	✓	NP-hard
approximate	✗	?	2	1	2
recognition	✓	✓	✓	✓	?

Open Problems

- Coloring Mixed Interval Graphs: approximation
- Coloring Containment/Bidirectional Mixed Interval Graphs: better approximation
- Recognizing Bidirectional Mixed Interval Graphs
- Coloring Mixed Graphs

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recognition	✓	✓	✓	✓	?

Open Problems

- Coloring Mixed Interval Graphs: approximation
- Coloring Containment/Bidirectional Mixed Interval Graphs: better approximation
- Recognizing Bidirectional Mixed Interval Graphs
- Coloring Mixed Graphs
- Geometric Mixed Graphs