Coloring Mixed and Directional Interval Graphs

GD 2022, Tokyo

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Motivation

Framework for layered graph drawing by Sugiyama, Tagawa, and Toda (1981).

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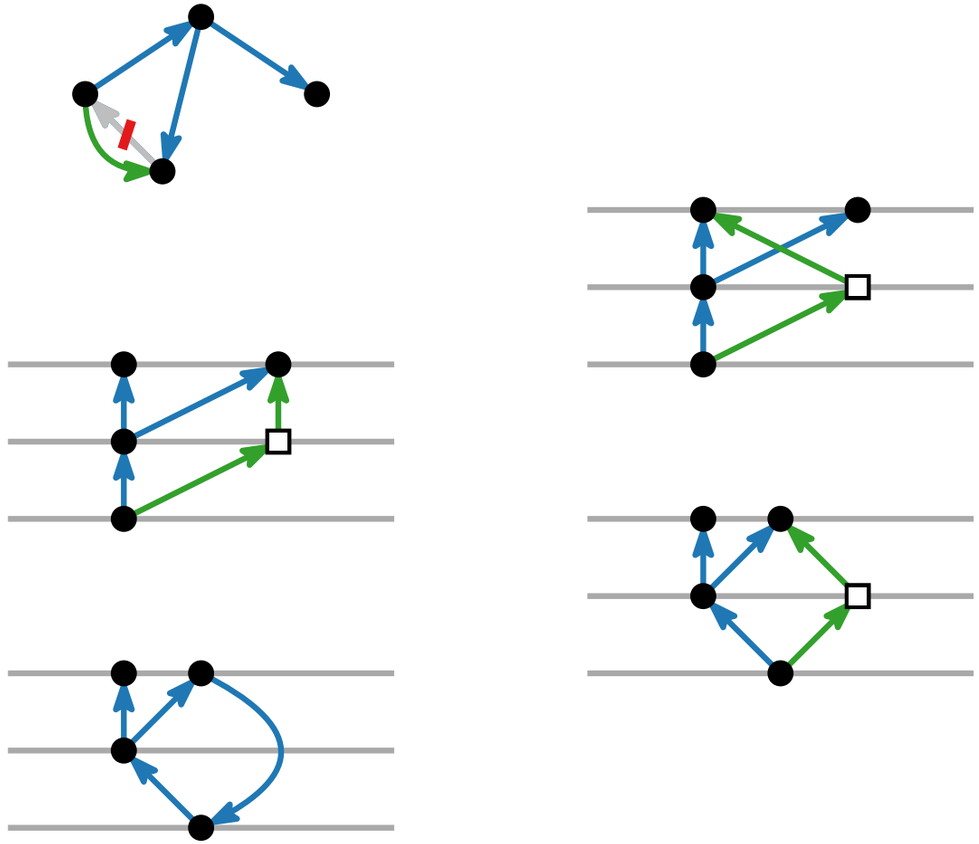
Consists of five phases:

1. cycle elimination
2. layer assignment
3. crossing minimization
4. node placement
5. edge routing



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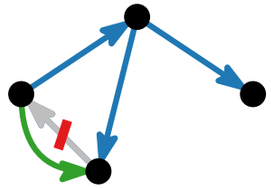
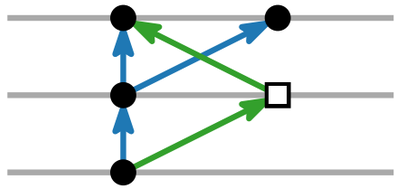
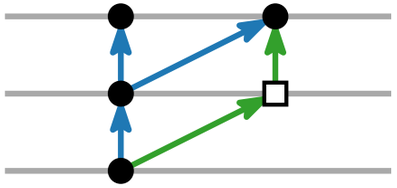


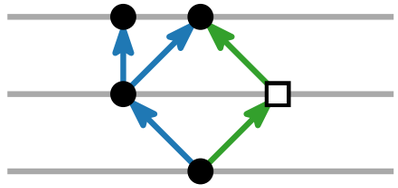
2. layer assignment



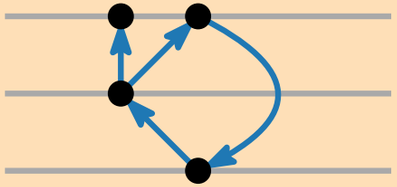
3. crossing minimization



4. node placement



5. edge routing



we want orthogonal edges!

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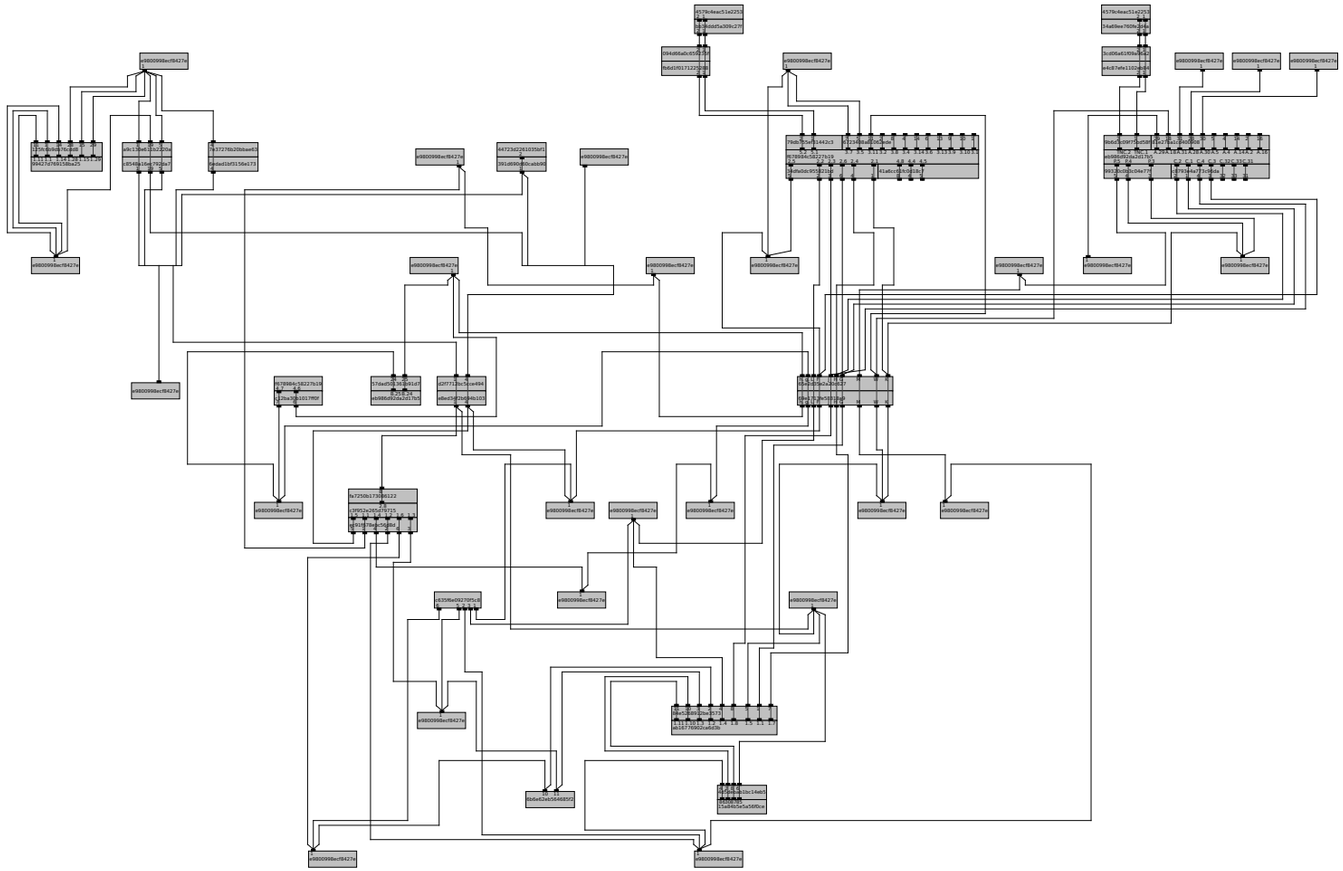
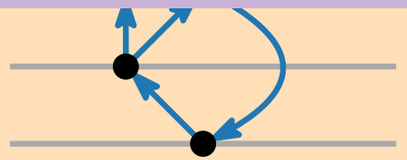


cable plan

[Zink, Walter, Baumeister, Wolff; CGTA'22]



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Motivation – Layered Orthogonal Edge Routing

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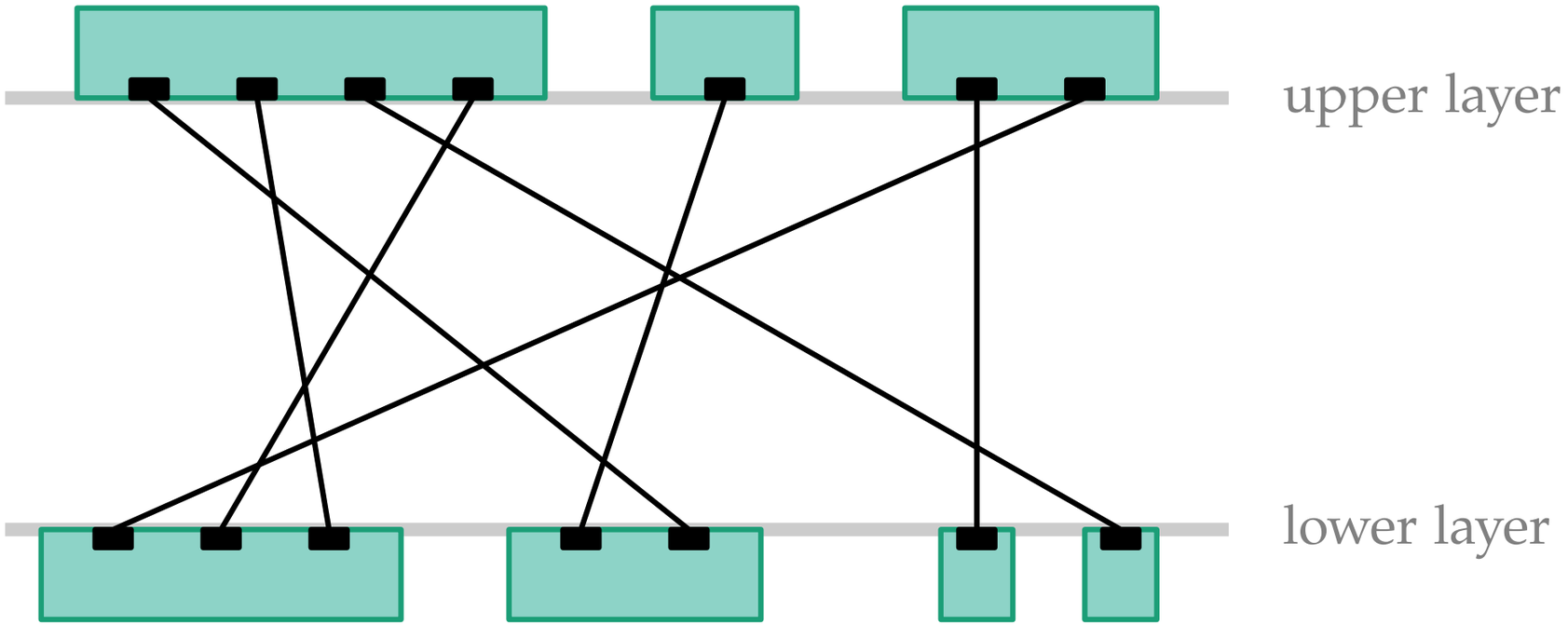
- positions of vertices are fixed

upper layer

lower layer

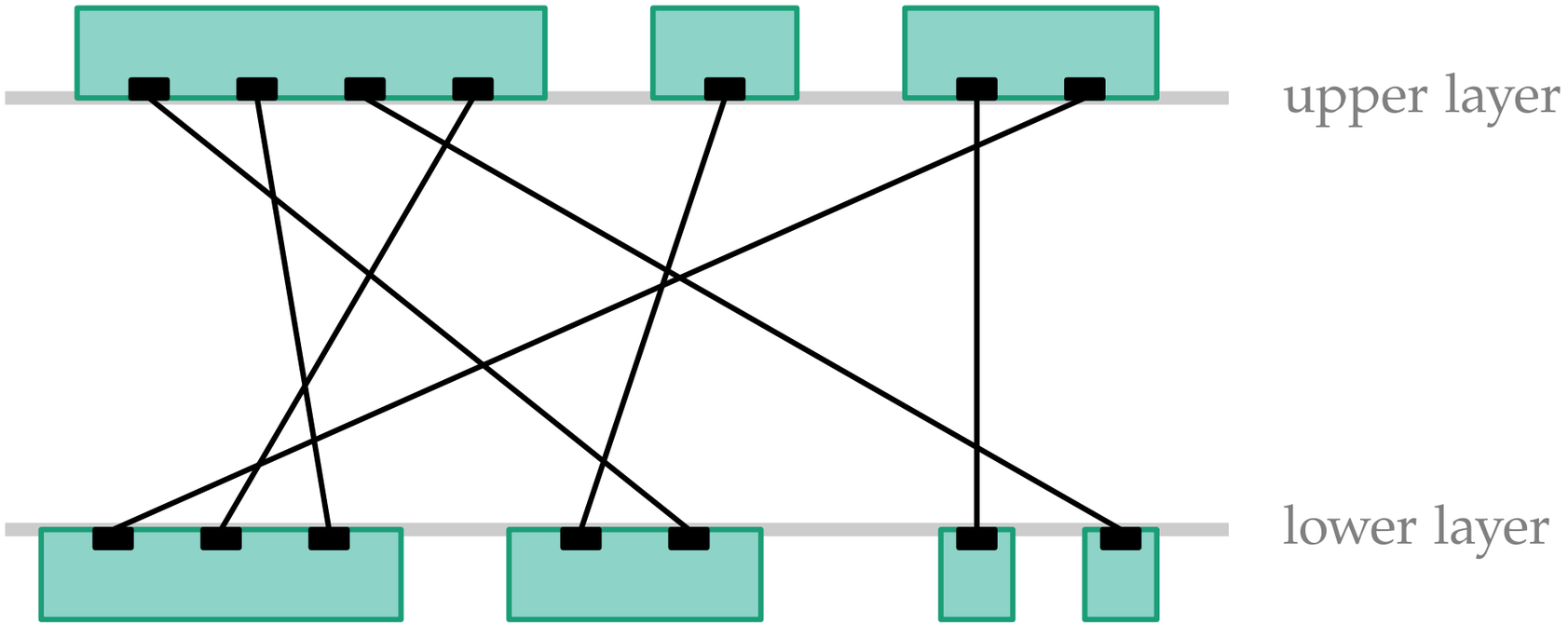
Motivation – Layered Orthogonal Edge Routing

- it suffices to consider each pair of consecutive layers individually
- positions of vertices are fixed
- no two edges share a common end point (vertices have distinct ports)



Motivation – Layered Orthogonal Edge Routing

- draw each edge with at most two vertical and one horizontal line segments



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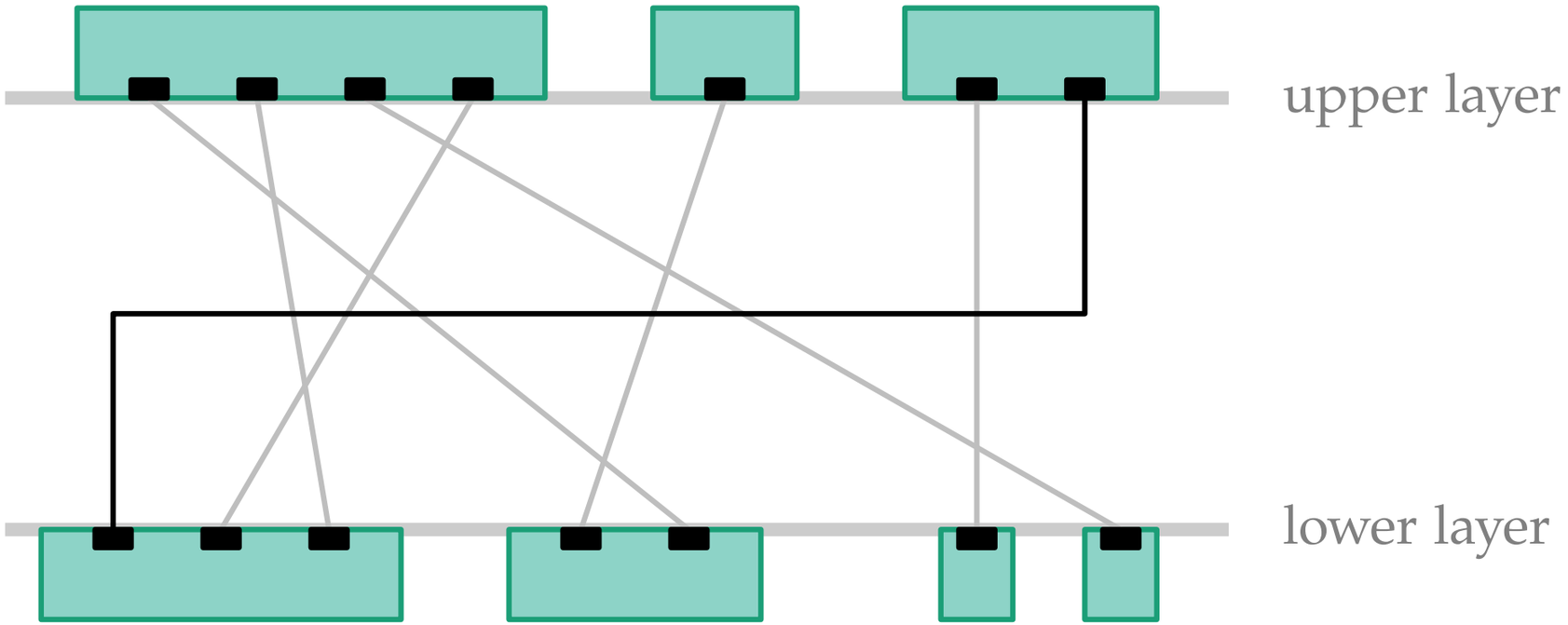

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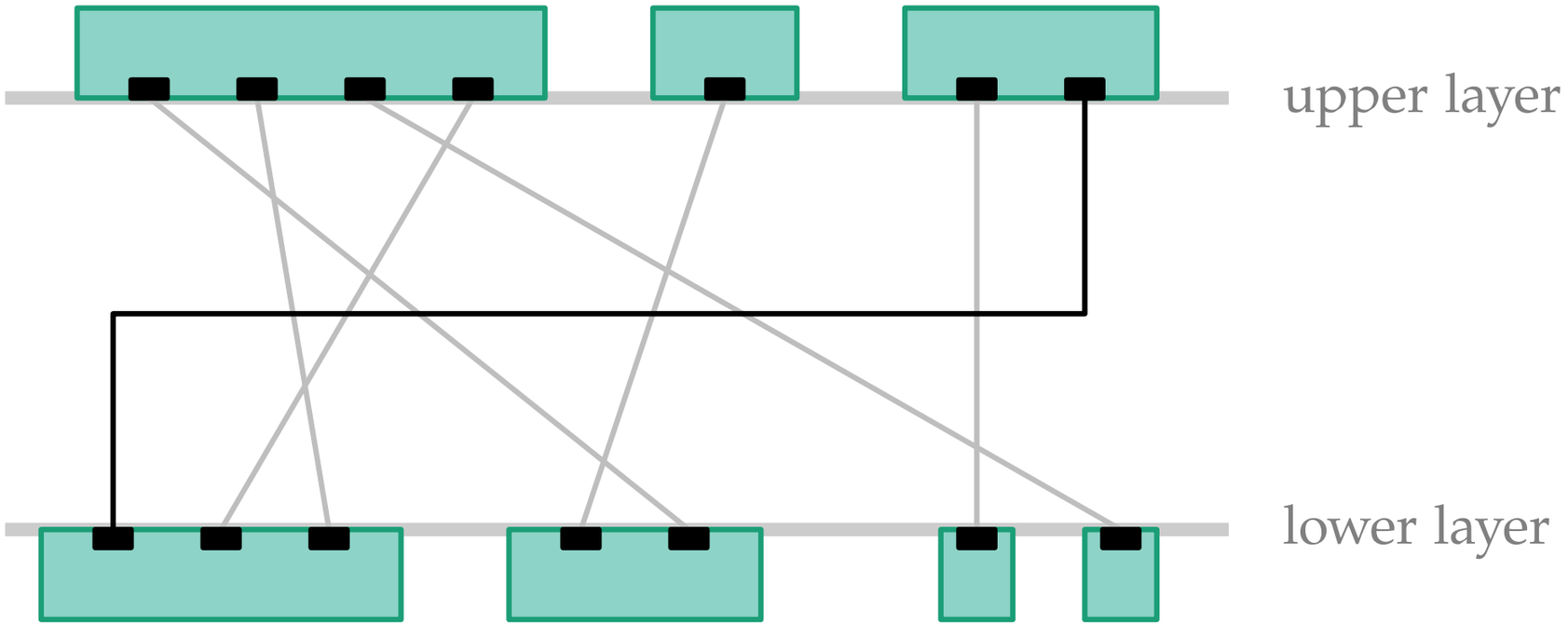
- draw each edge with at most two vertical and one horizontal line segments
- avoid overlaps and double crossings between the same pair of edges



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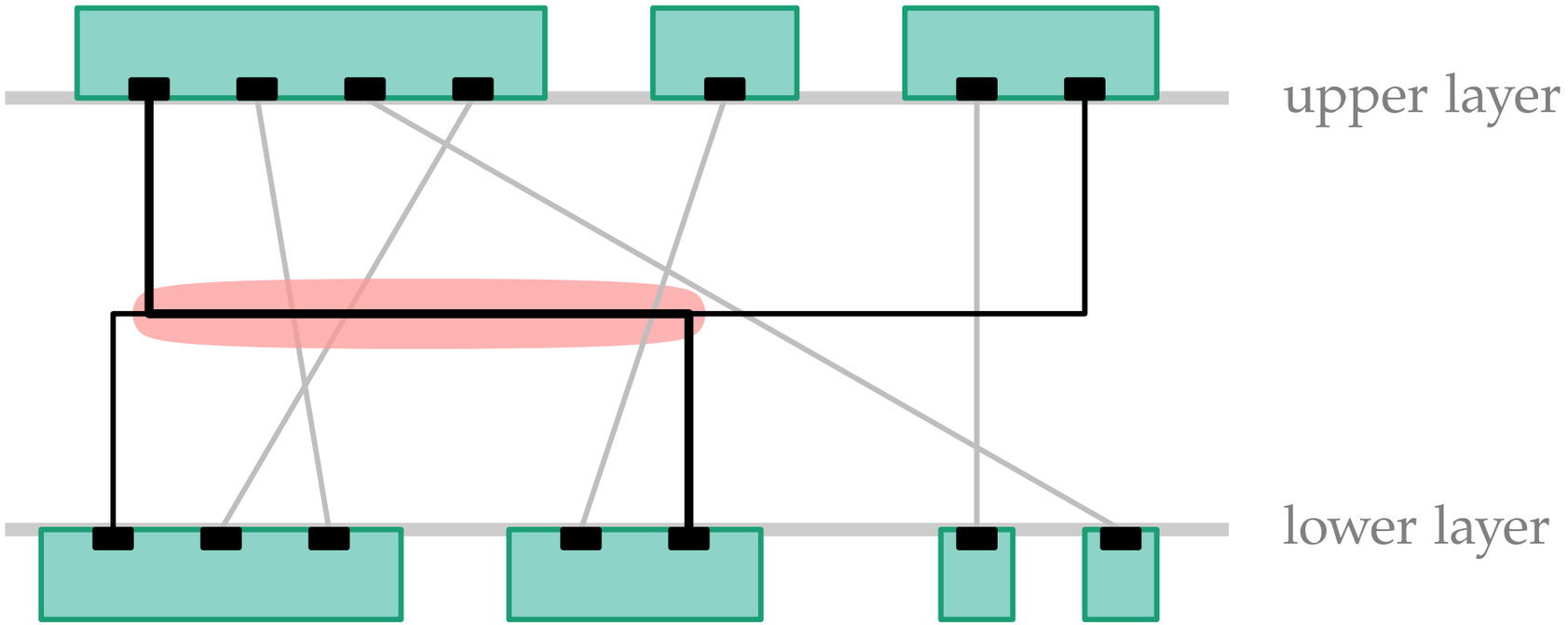

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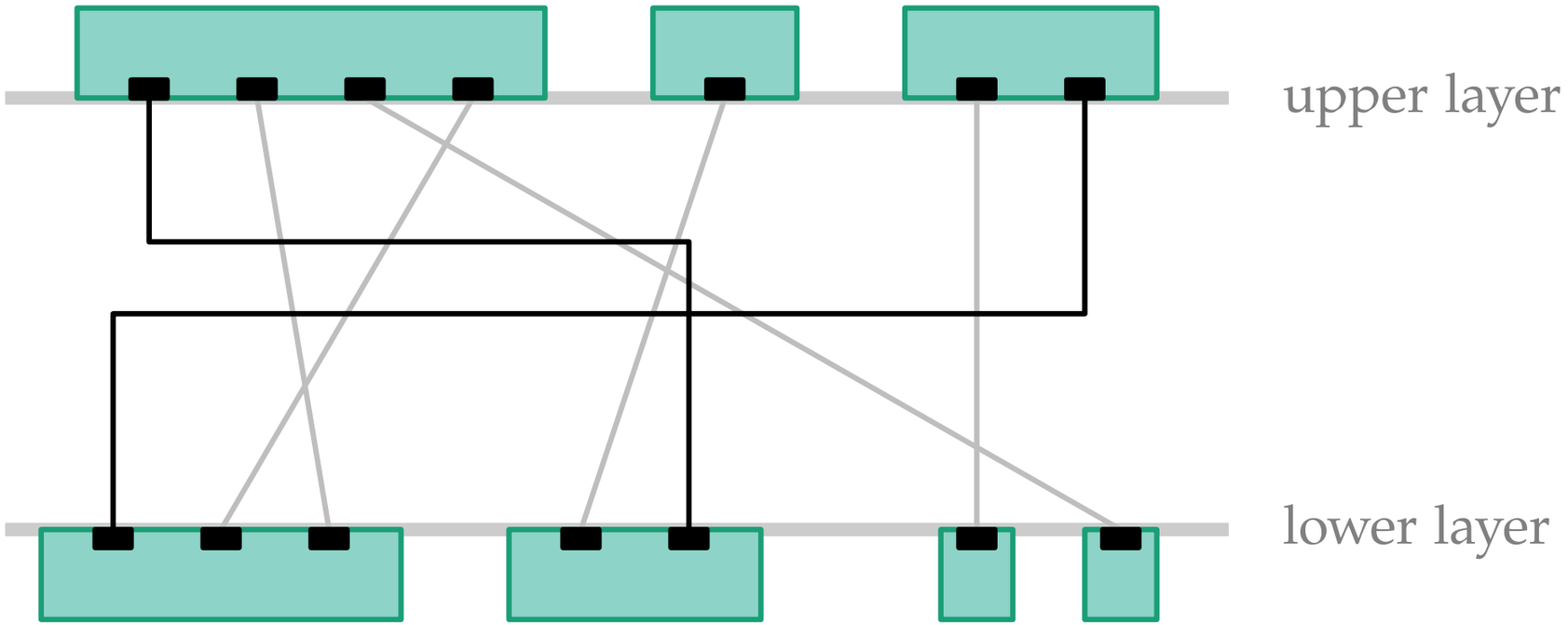

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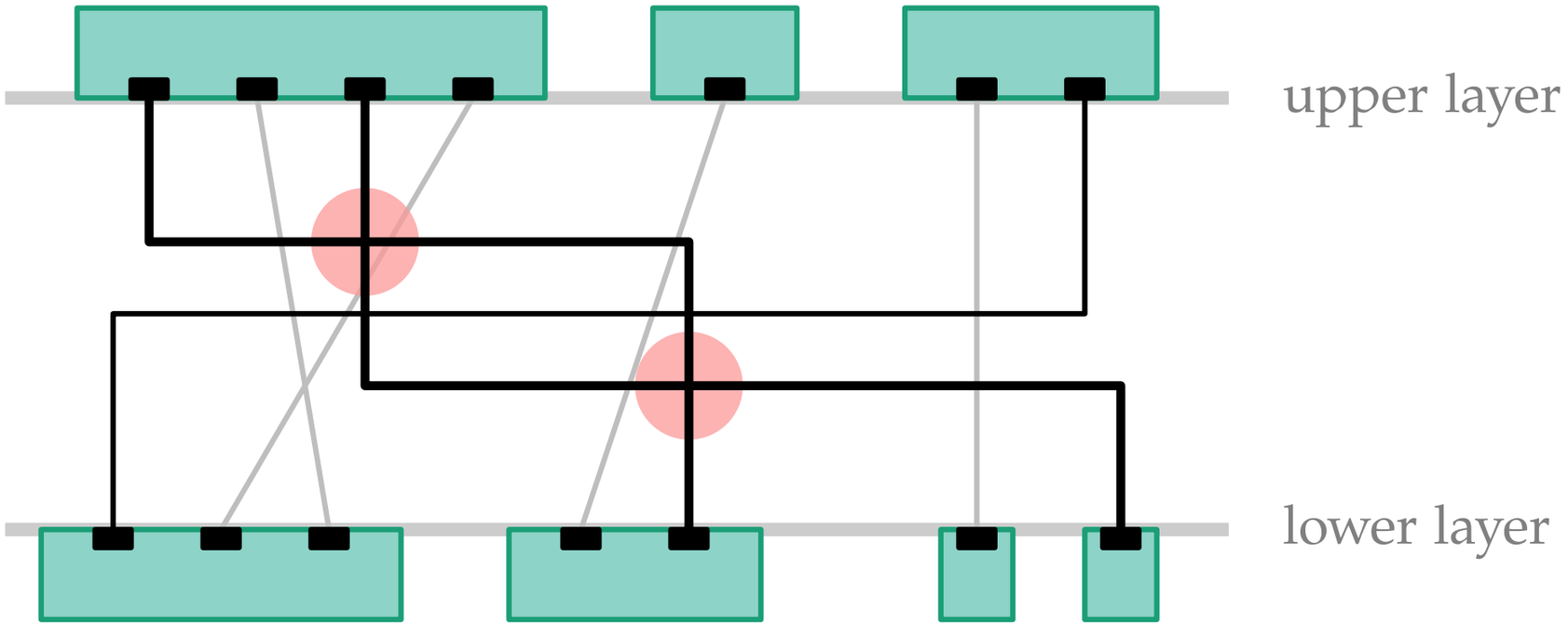

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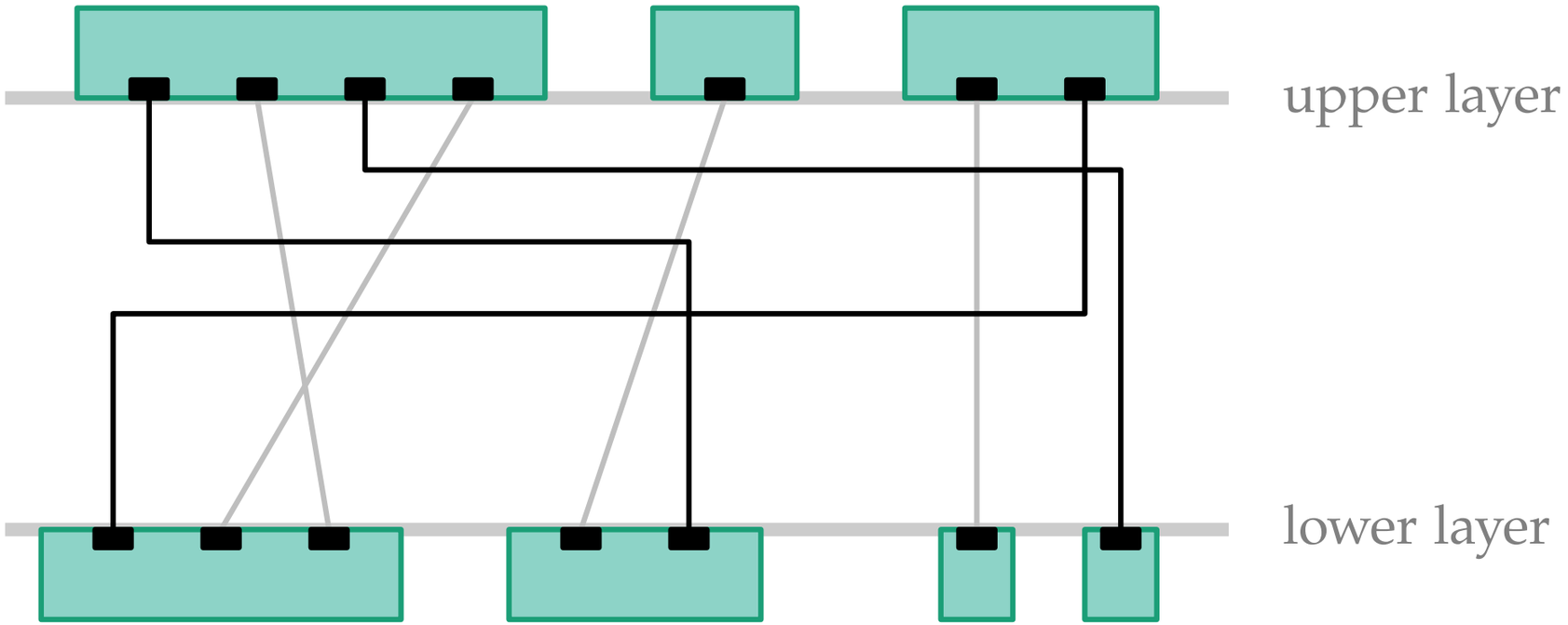
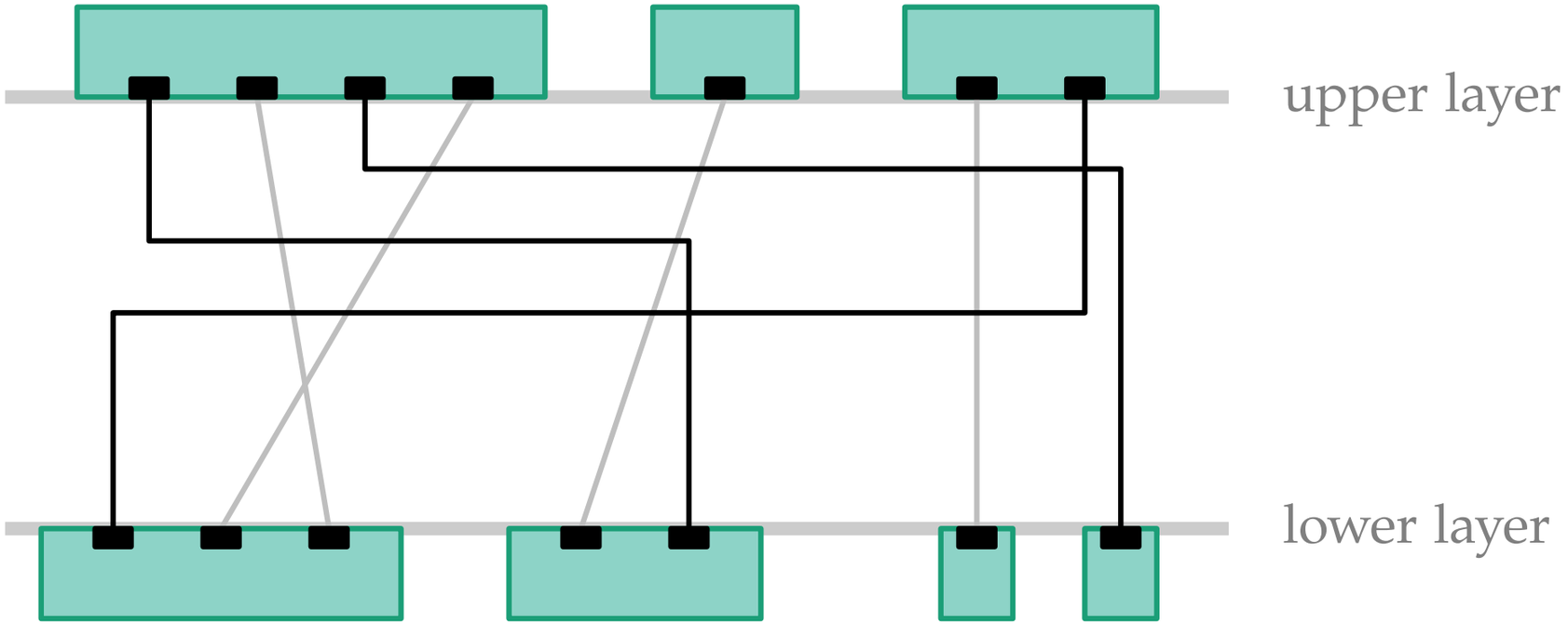


Motivation – Layered Orthogonal Edge Routing

- draw each edge with at most two vertical and one horizontal line segments
- avoid overlaps and double crossings between the same pair of edges
- use as few horizontal intermediate layers (tracks) as possible



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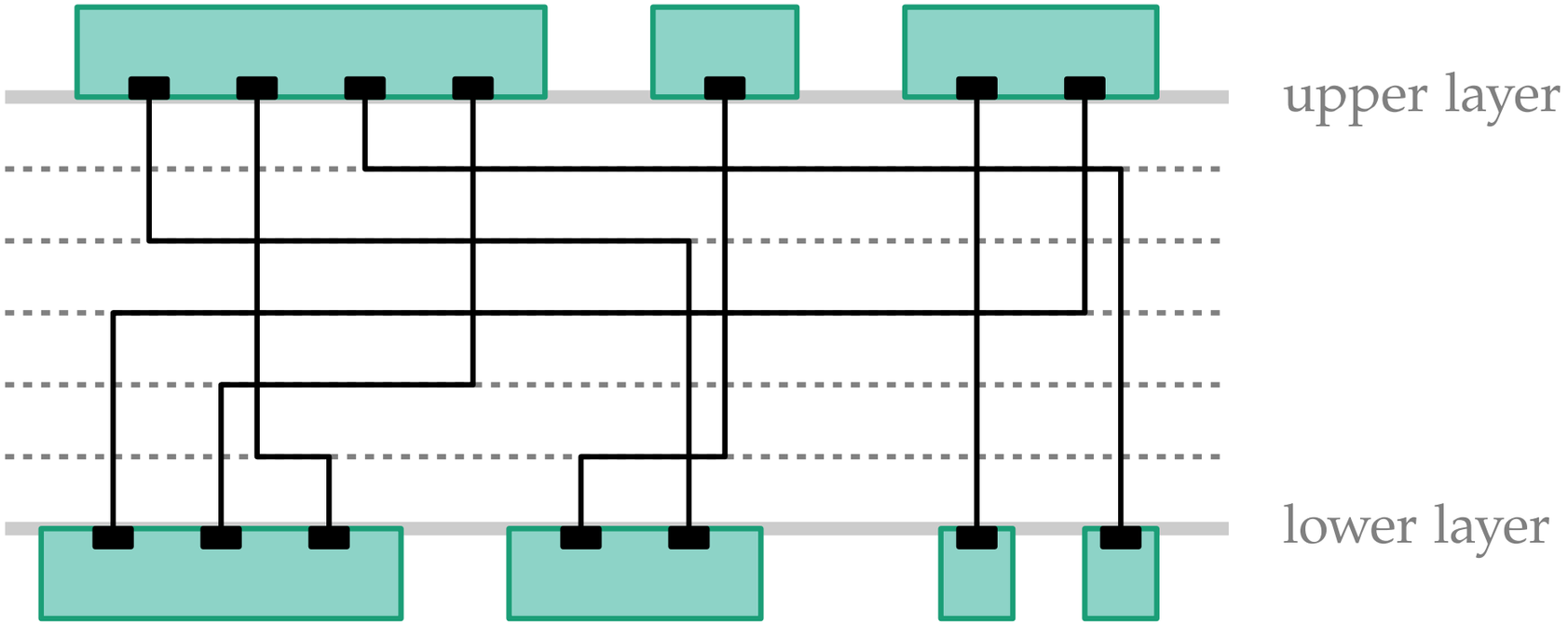

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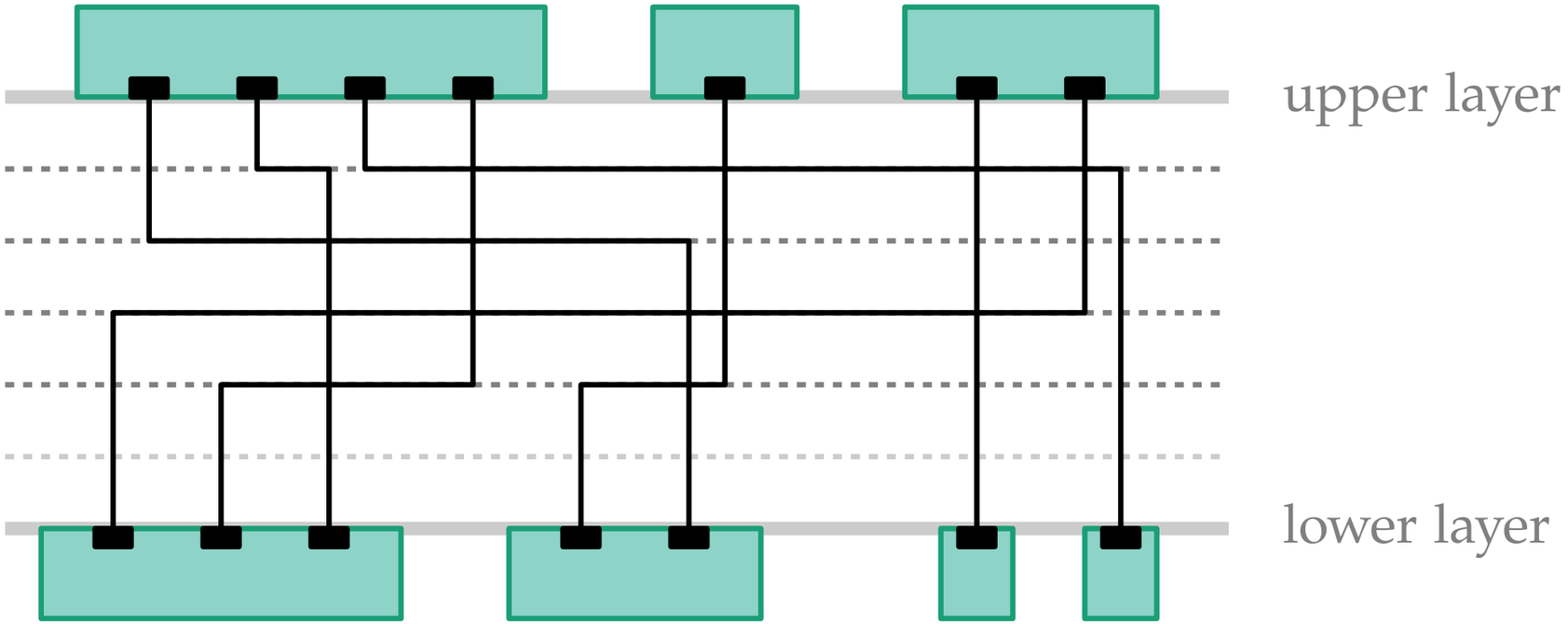

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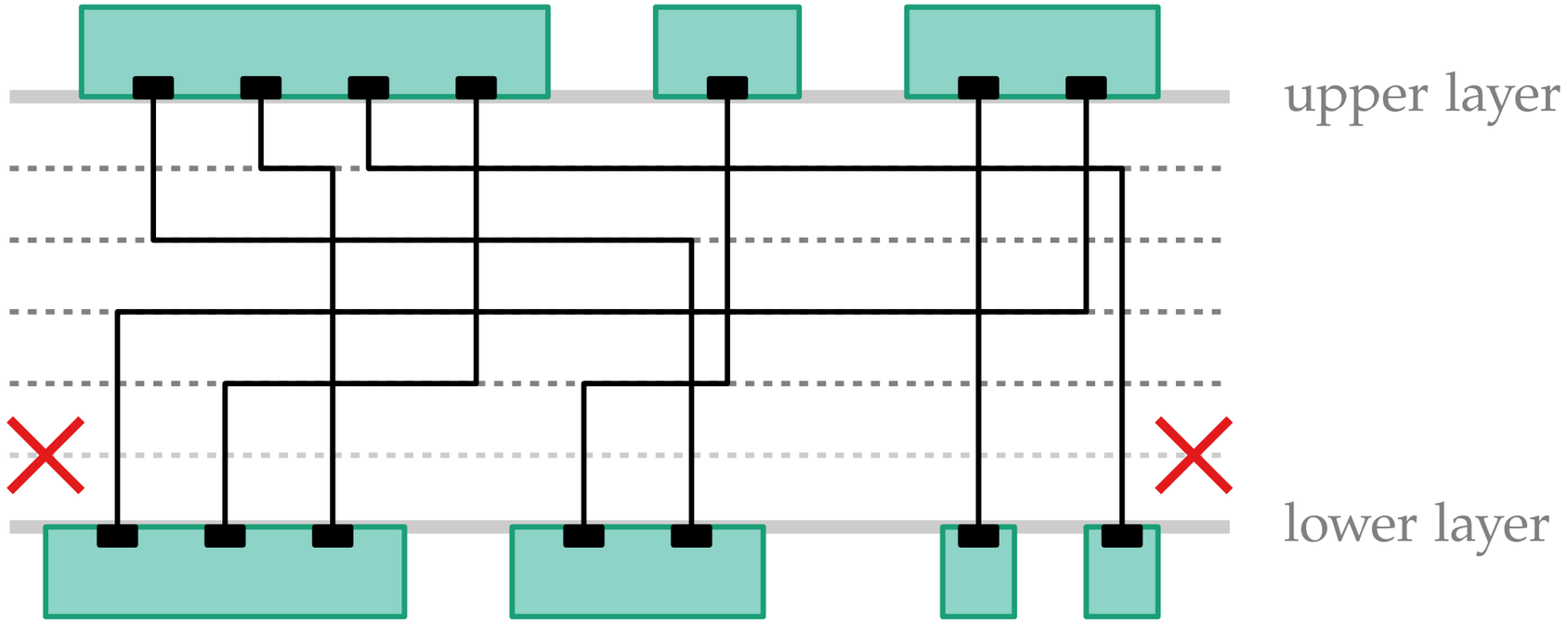
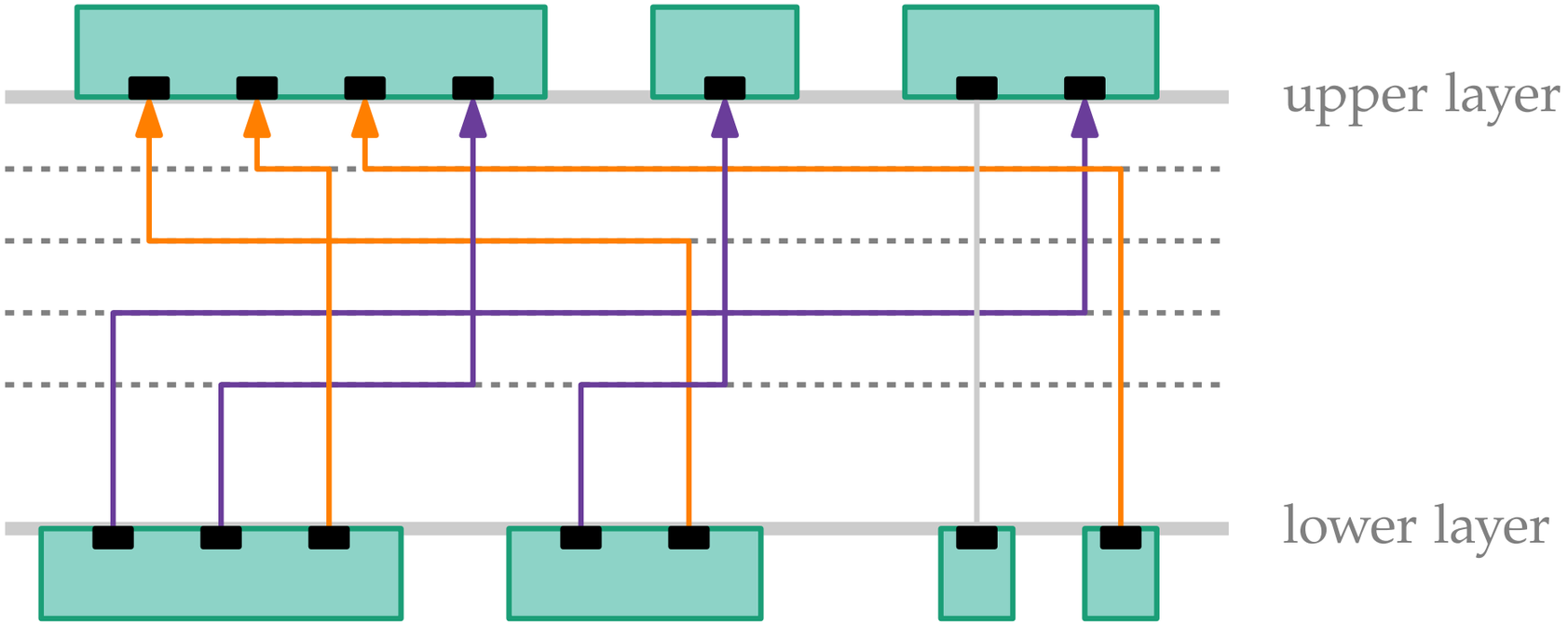


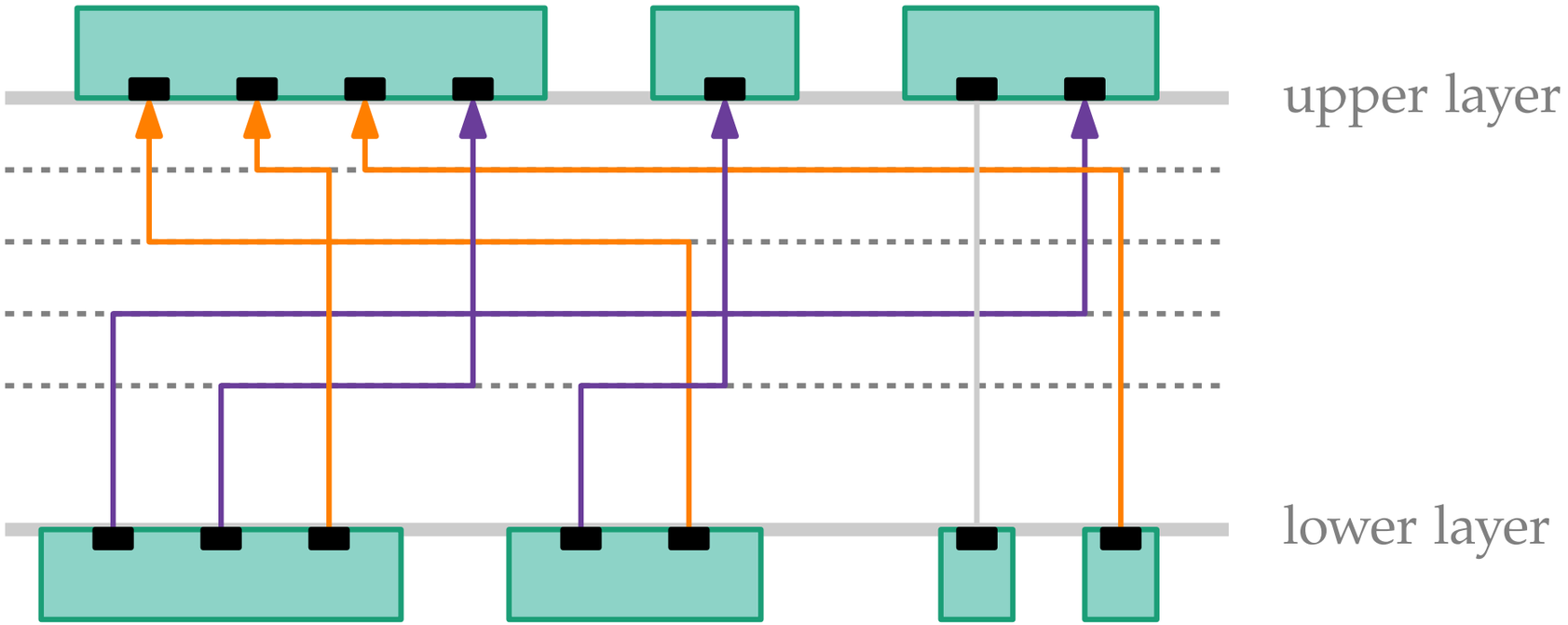
Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges



Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges
- only edges going in the same direction and overlapping partially in x-dimension can cross twice



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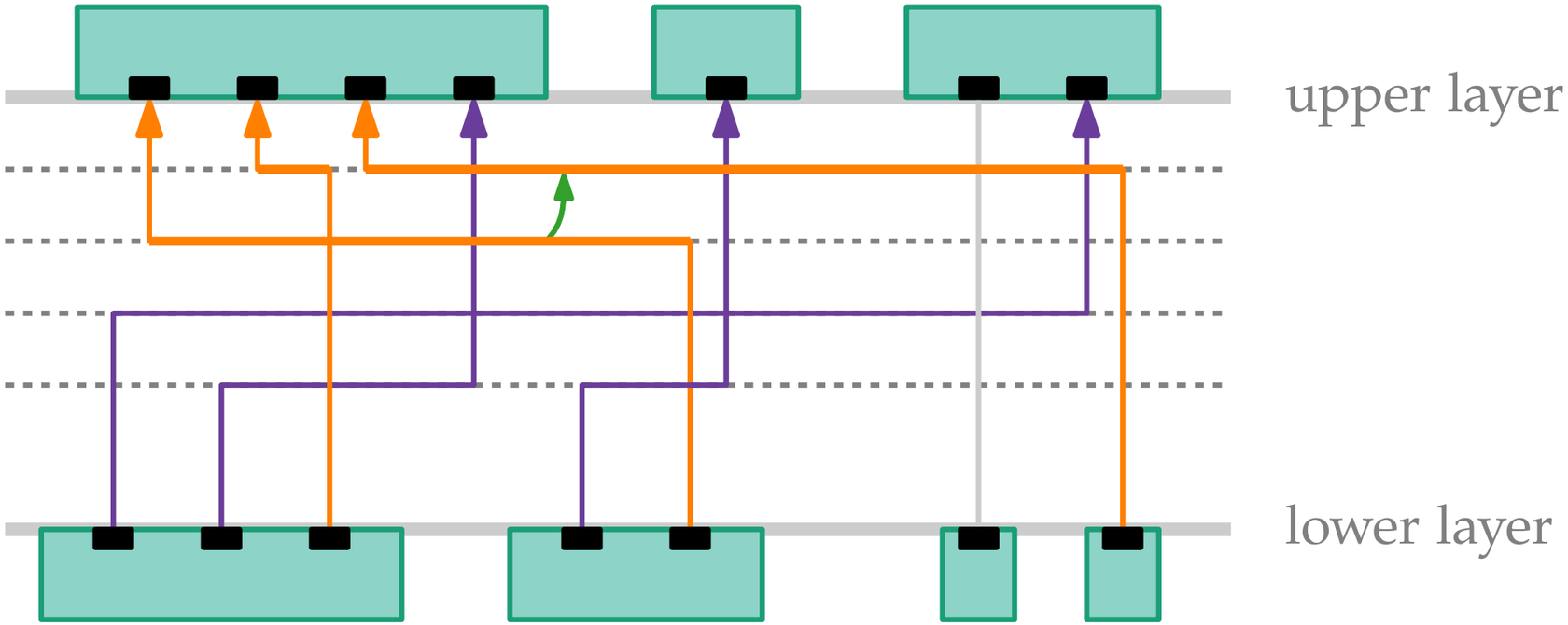
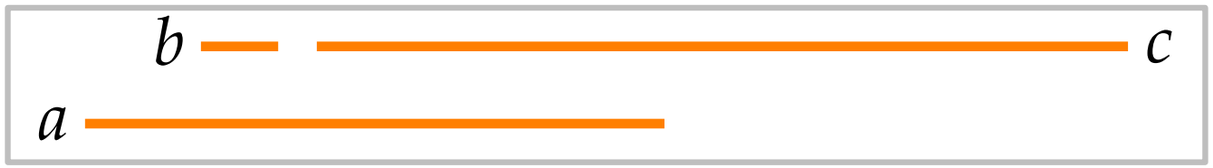
  $\Rightarrow$ induce a vertical order for the horizontal middle segments



Definition – Directional Interval Graphs

Interval representation: set of intervals



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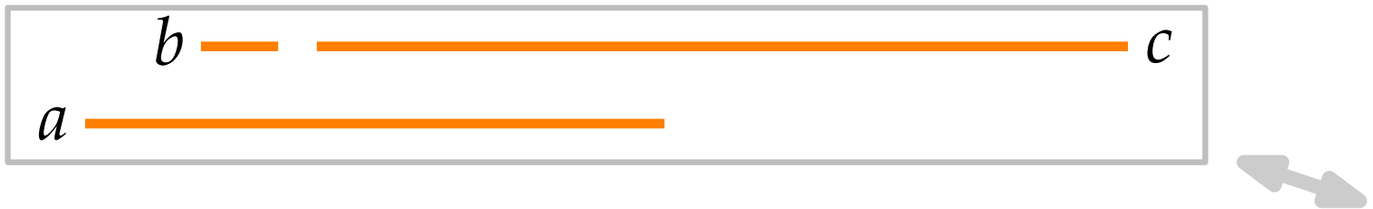

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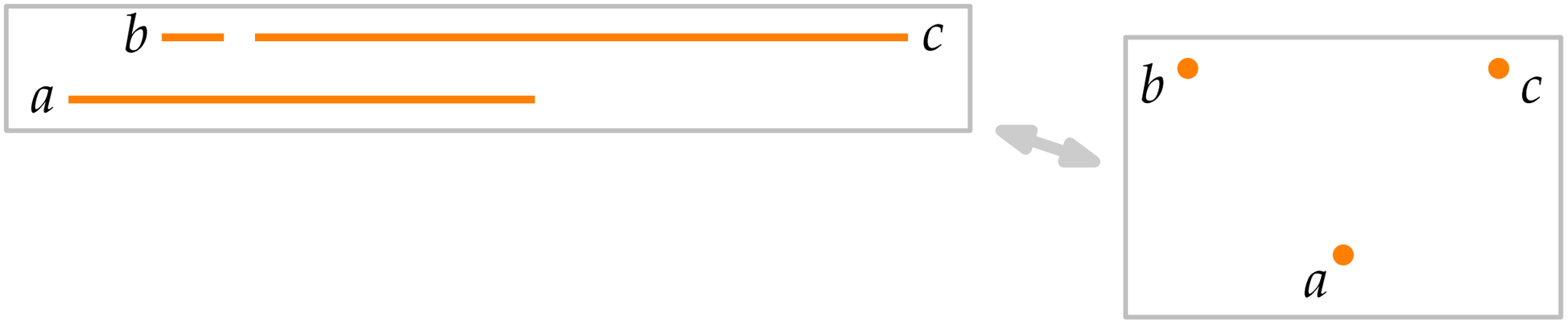

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Directional interval graph:

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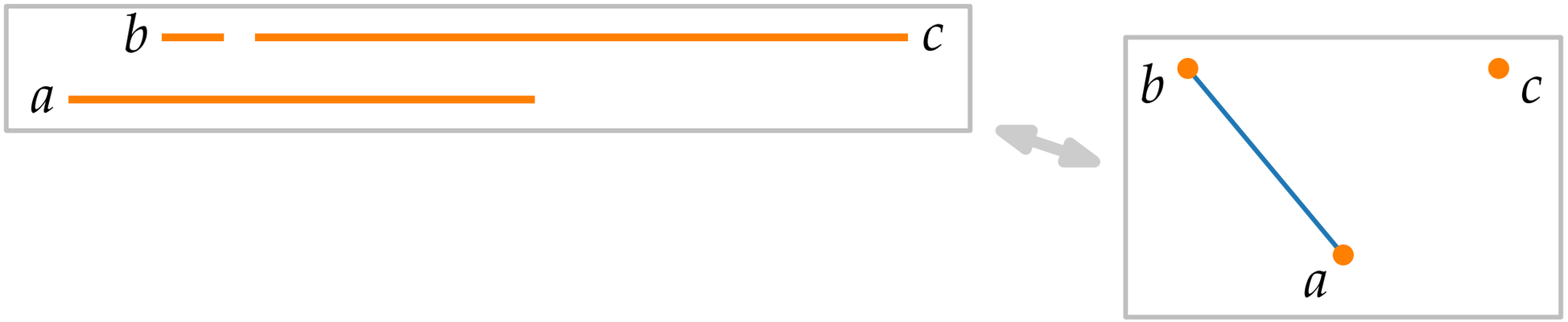

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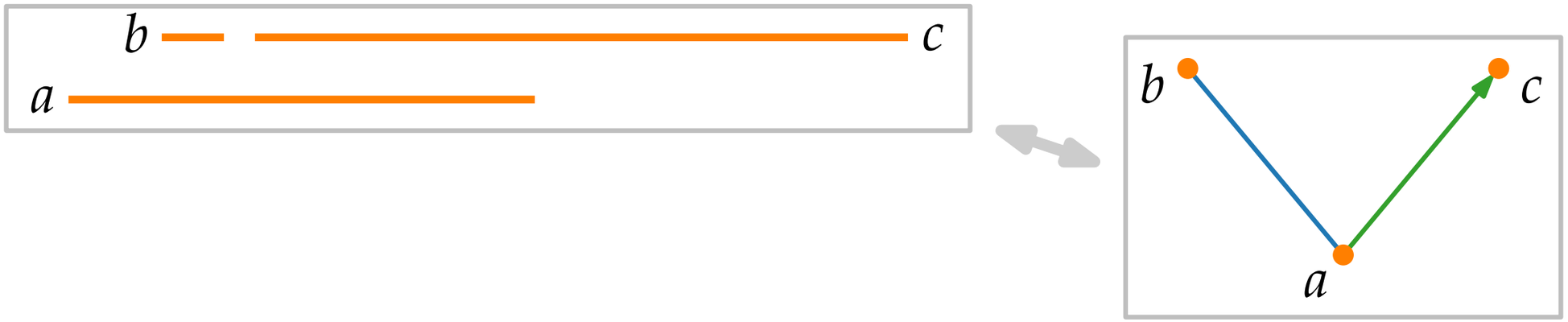

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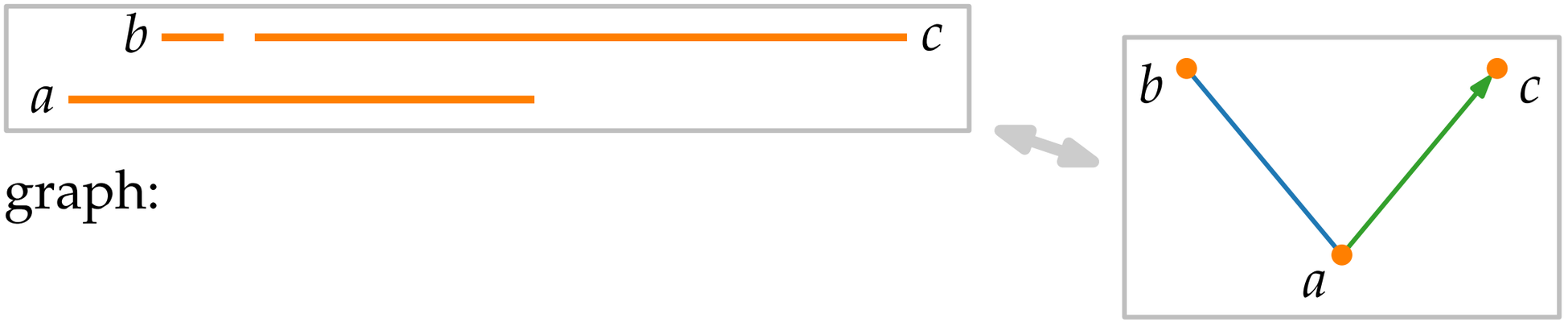

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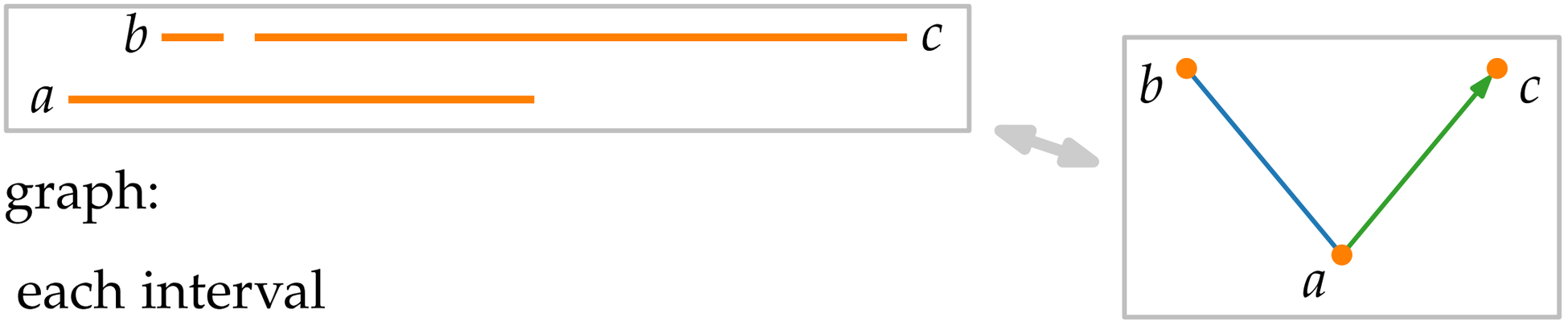

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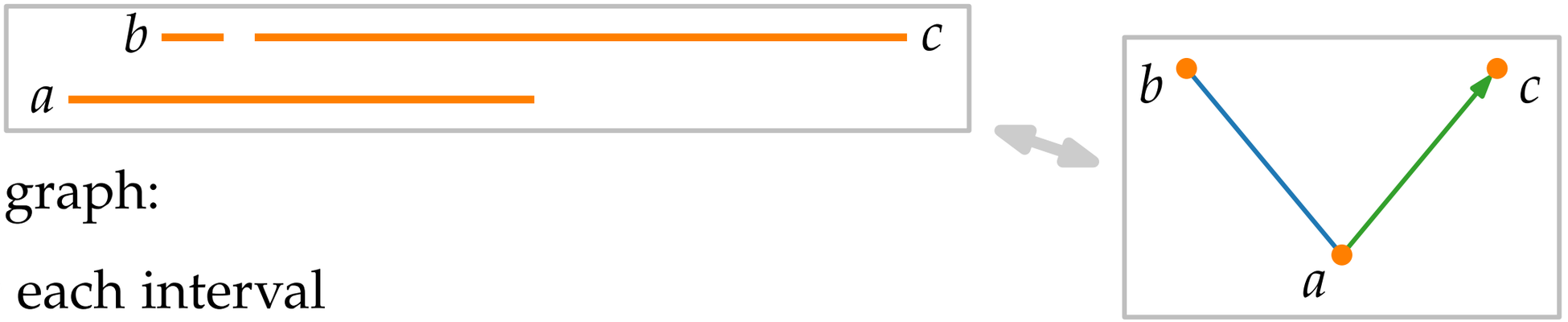

Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:

[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
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Directional interval graphs:

- recognition in $O(n^2)$ time

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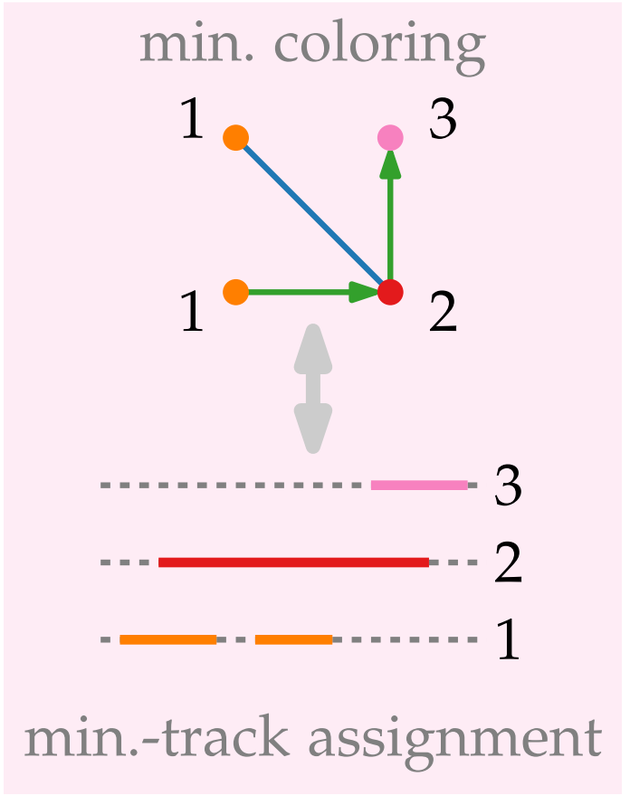

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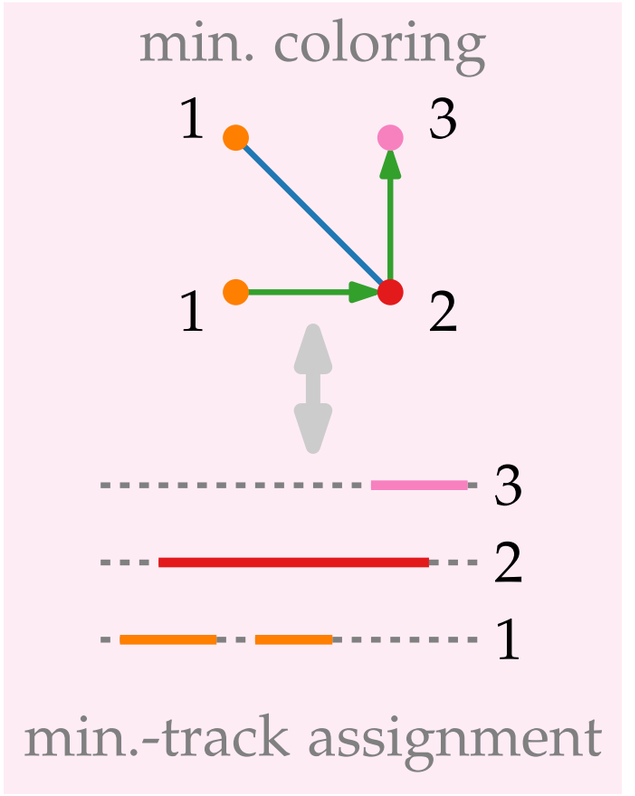

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Interval graphs (no directed edges):

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Directional interval graphs:

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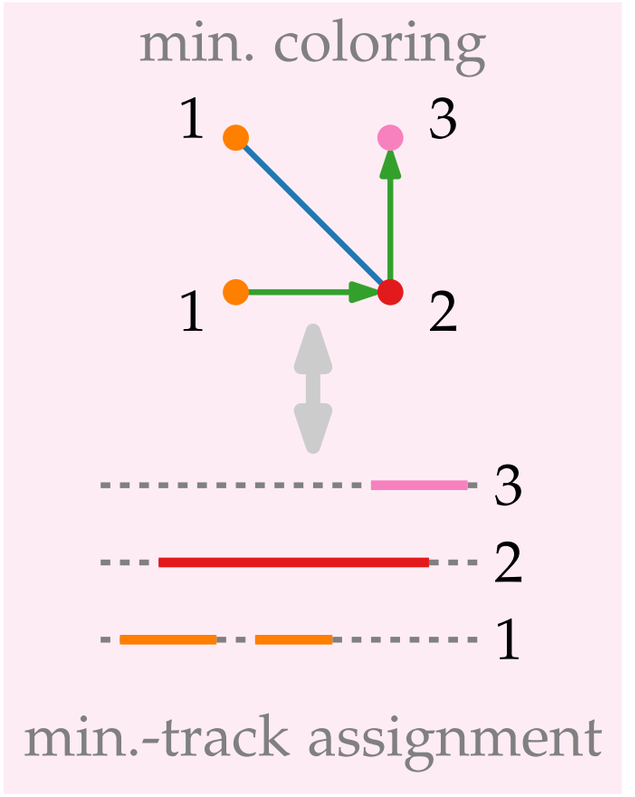
Mixed interval graphs:

- coloring is NP-complete

Directed graphs (only directed edges):

- coloring in linear time using topological sorting



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Directional interval graphs: **our contribution**

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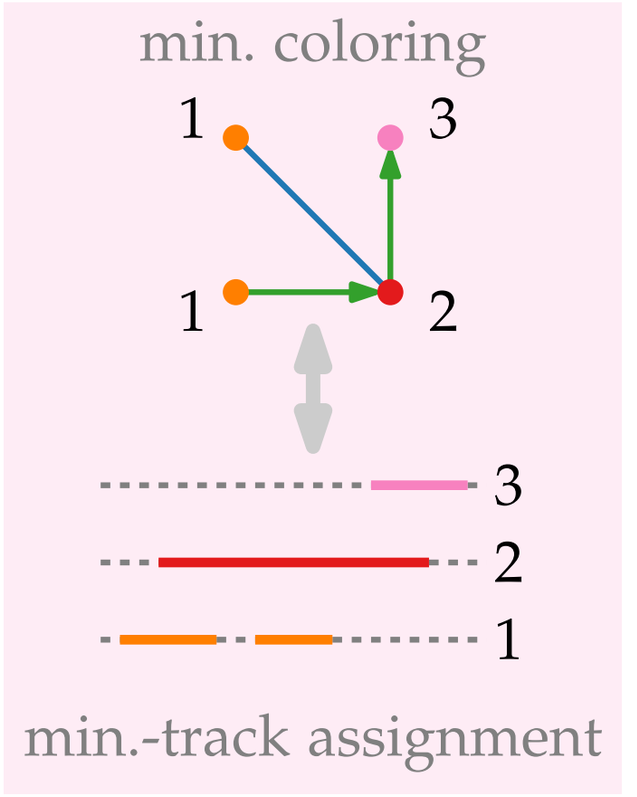
Directed graphs (only directed edges):

- coloring in linear time using topological sorting



$n :=$ # intervals

Coloring Mixed Graphs

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
- ⋆ directed edge uv: $c(u) < c(v)$,
- ⋆ $\max_{v \in V} c(v)$ is minimized.

Interval graphs (no directed edges):

- coloring in linear time by a greedy algorithm

Directional interval graphs: **our contribution**

- recognition in $O(n^2)$ time ← not in this talk
- coloring in $O(n \log n)$ time by a greedy algorithm

Mixed interval graphs:

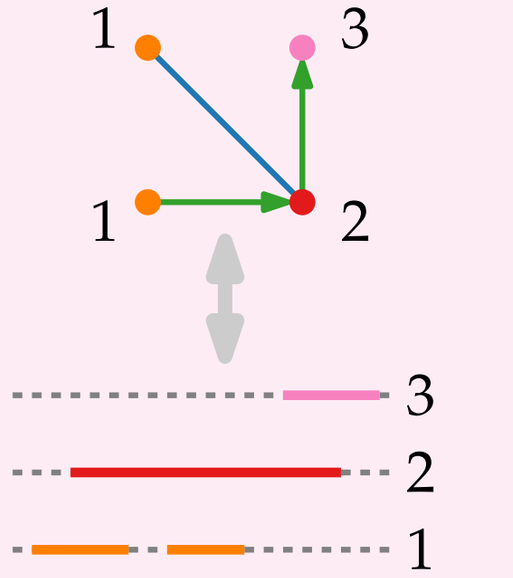
- coloring is NP-complete

agenda for this talk

Directed graphs (only directed edges):

- coloring in linear time using topological sorting

min. coloring



min.-track assignment

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Coloring Directional Interval Graphs

Given: an interval representation of a directional interval graph G

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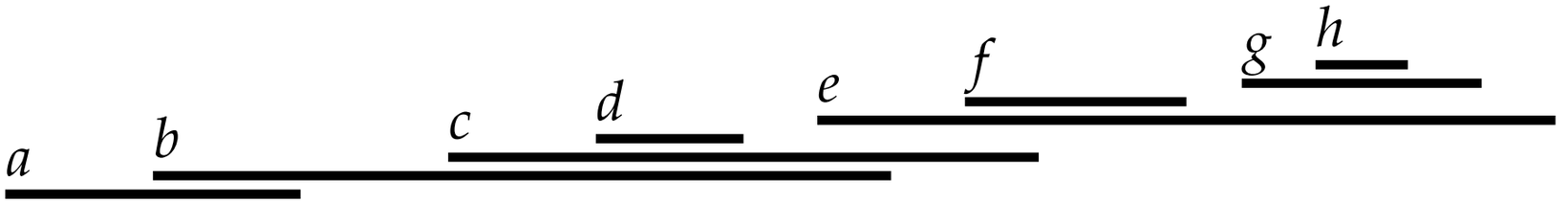

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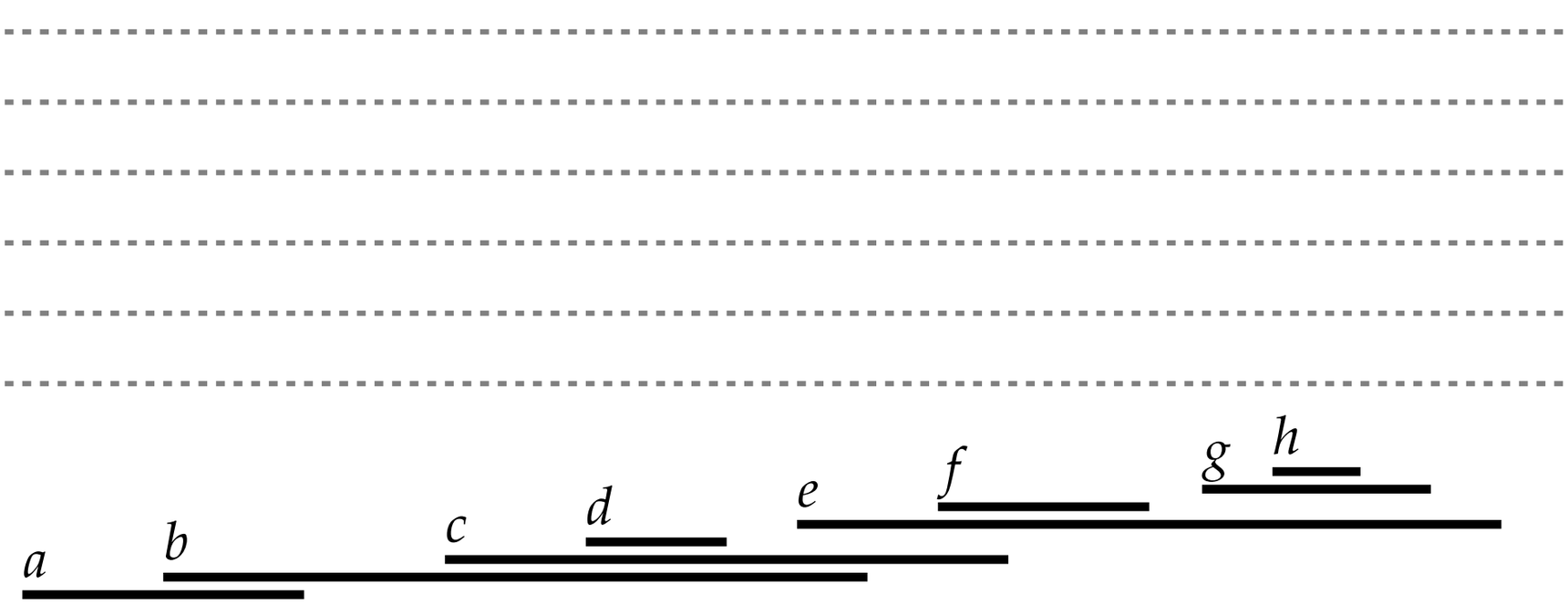

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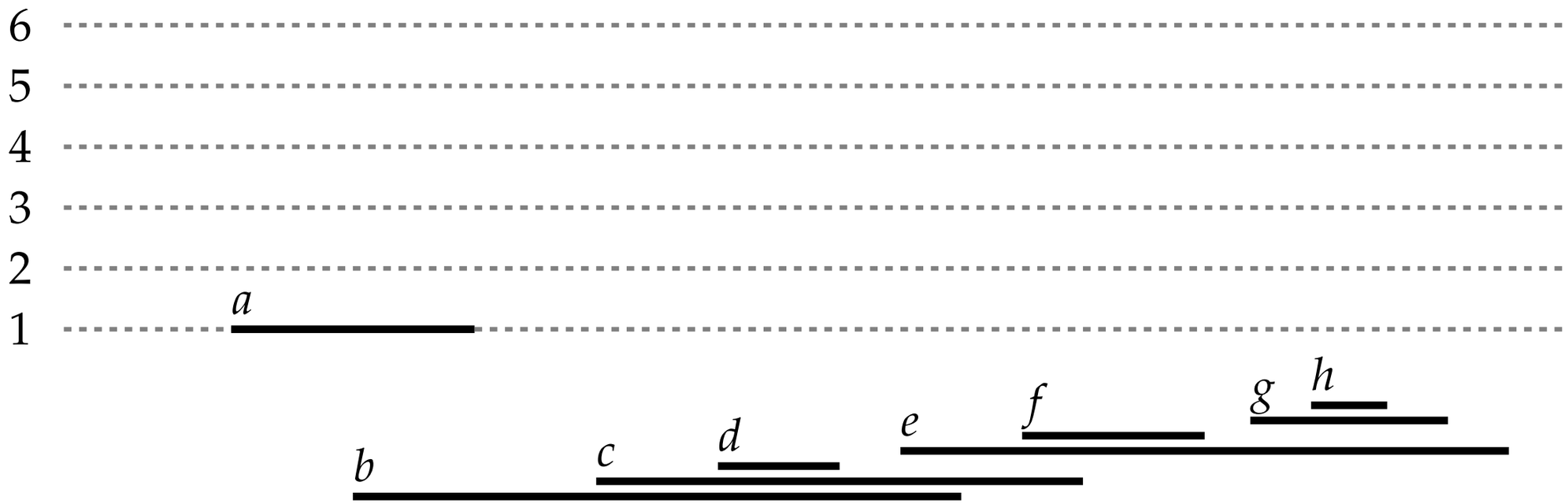

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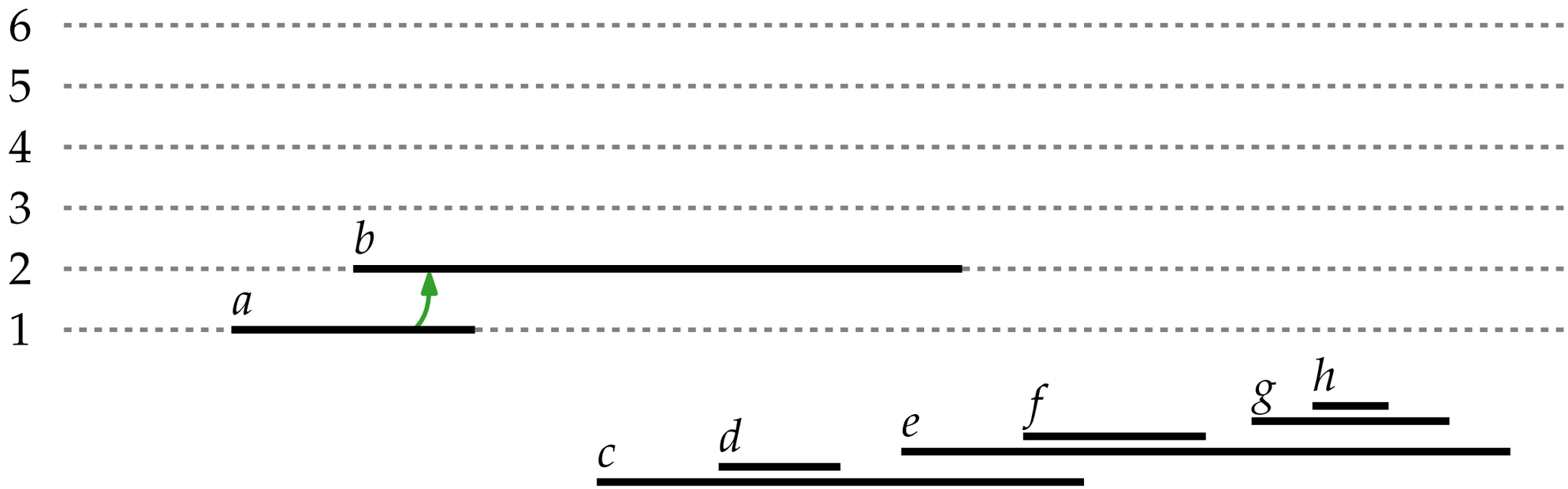

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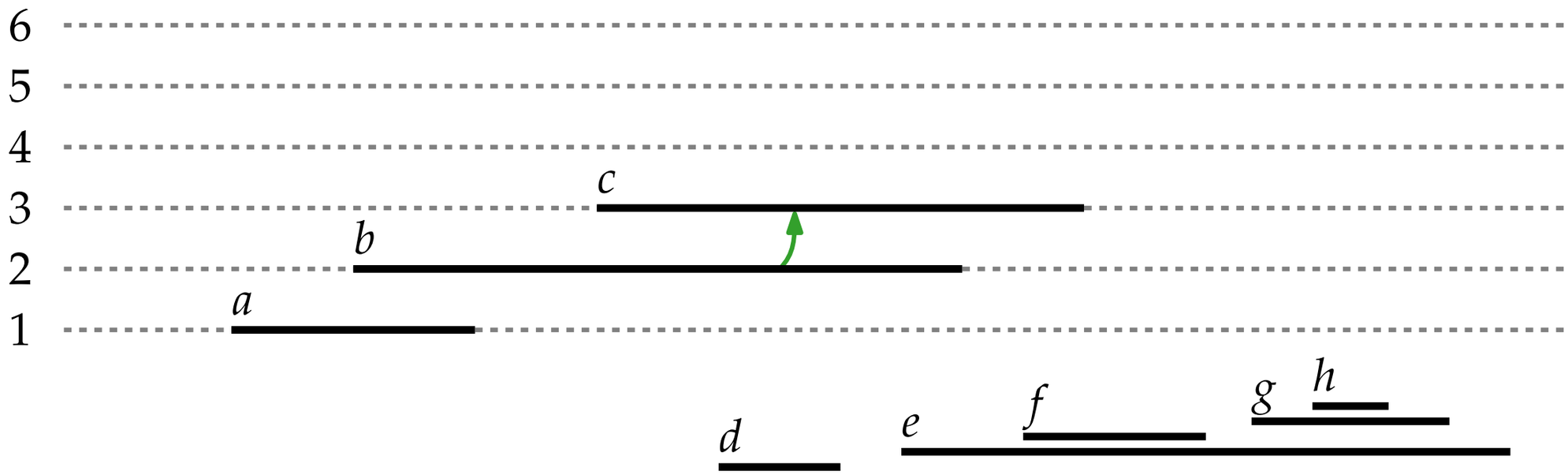

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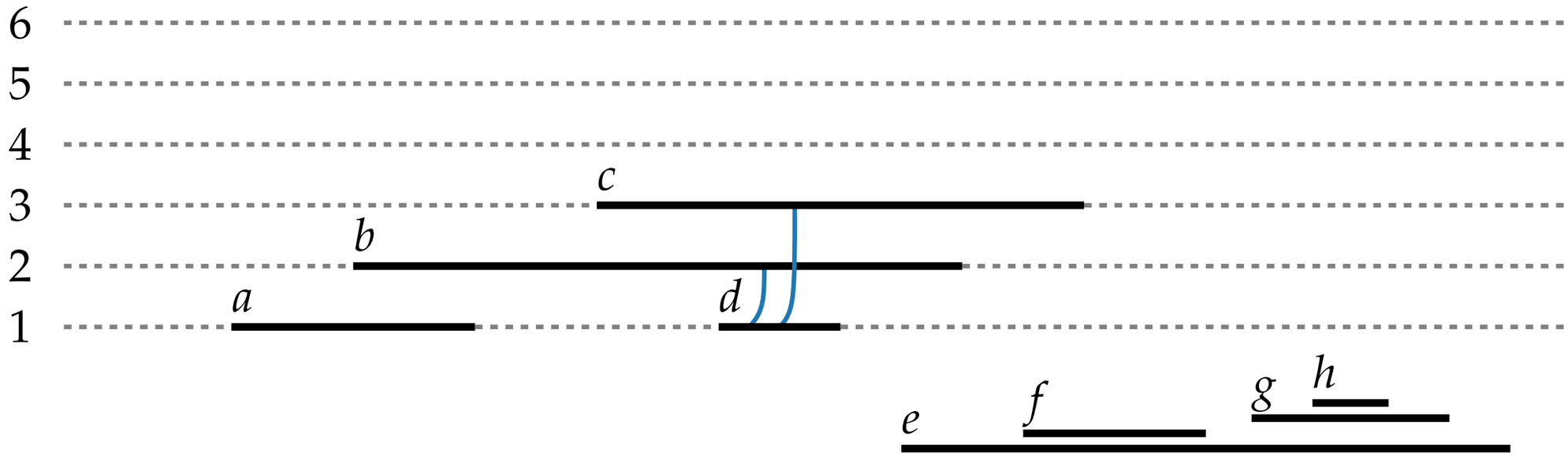

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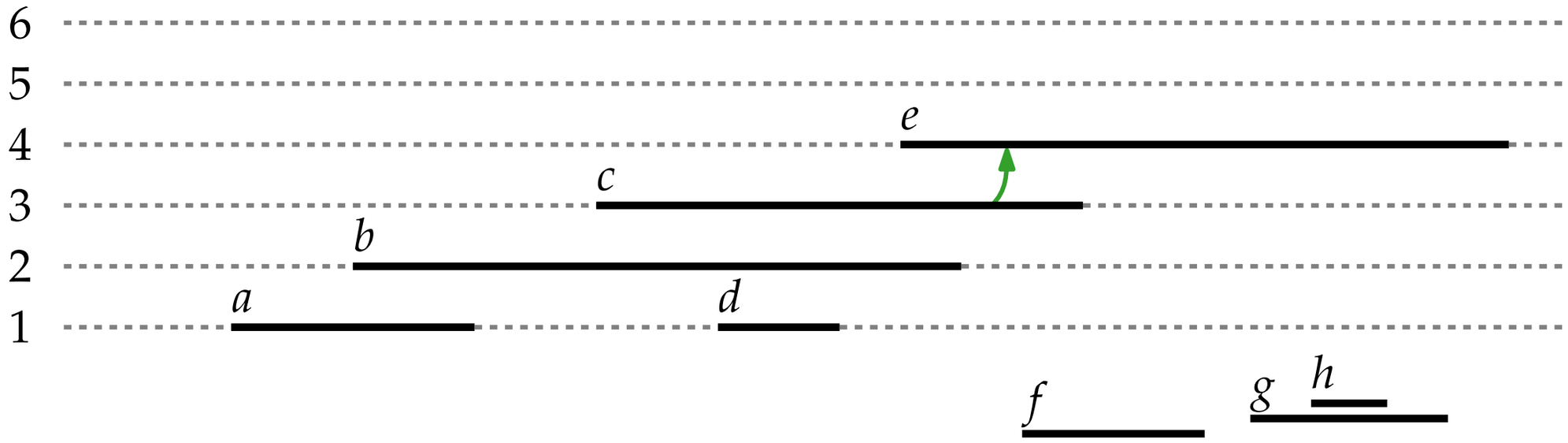

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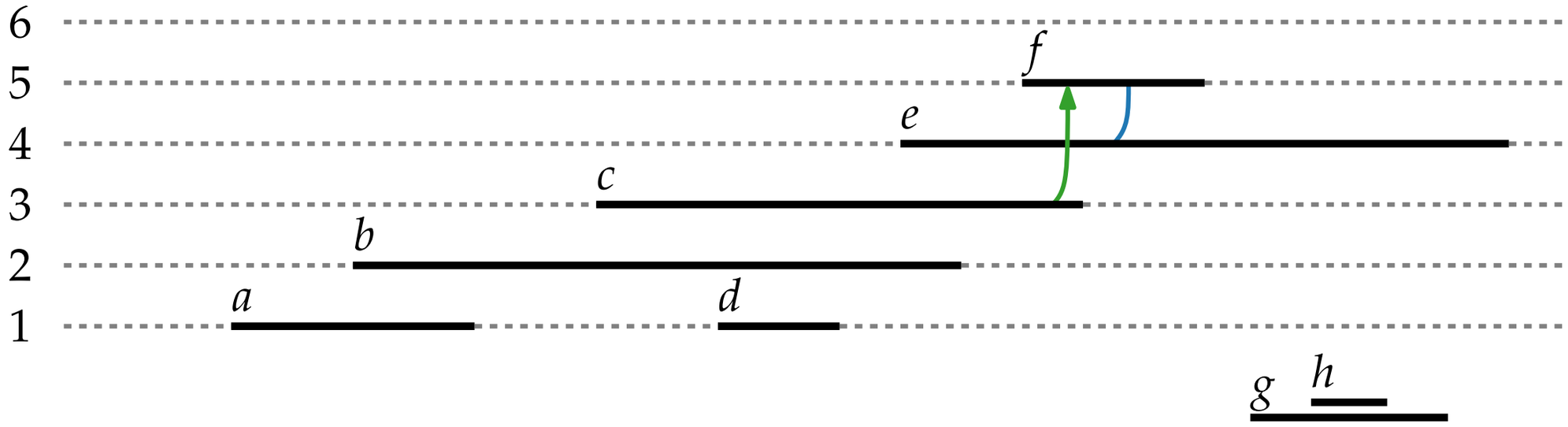

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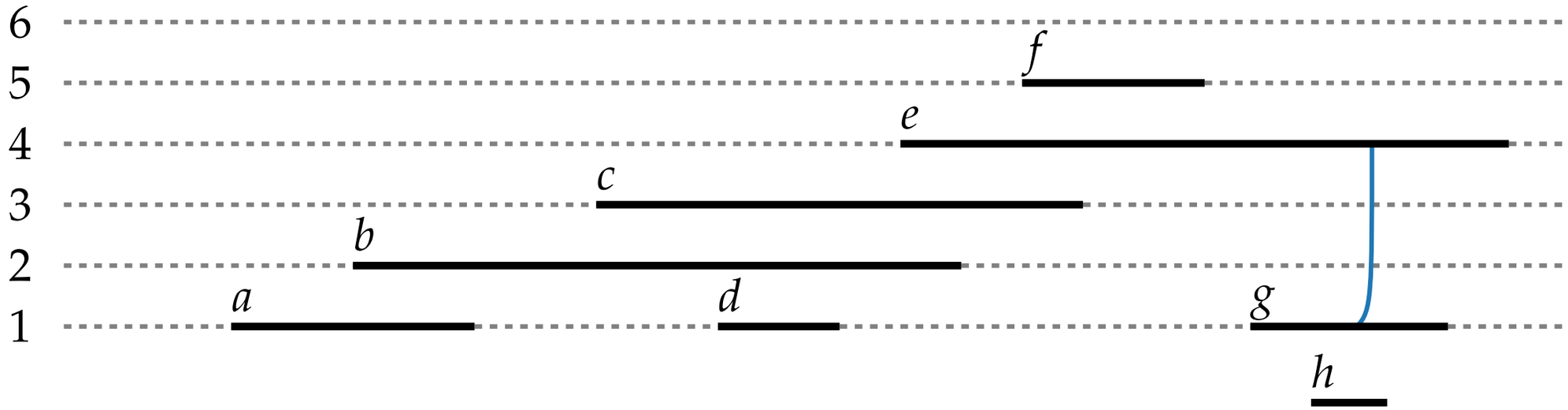

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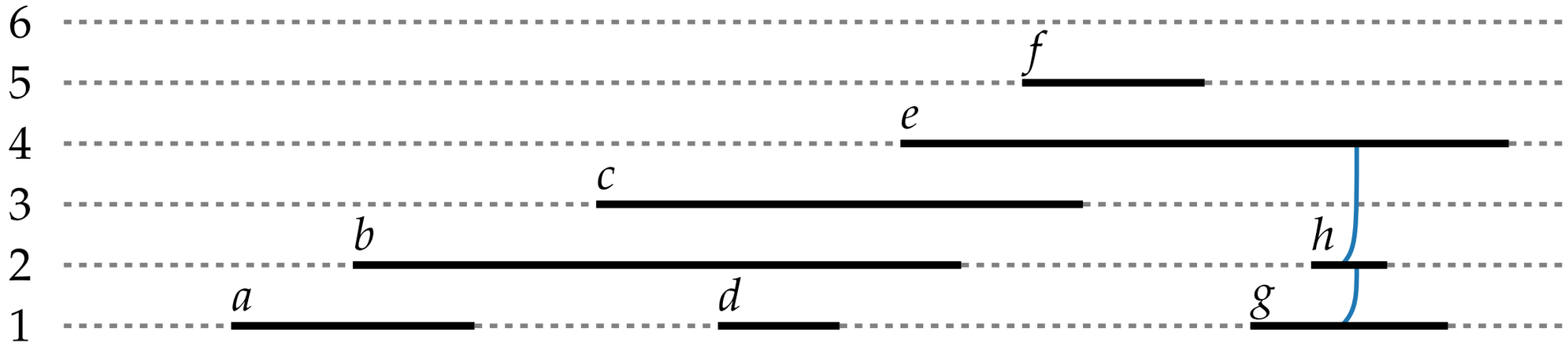

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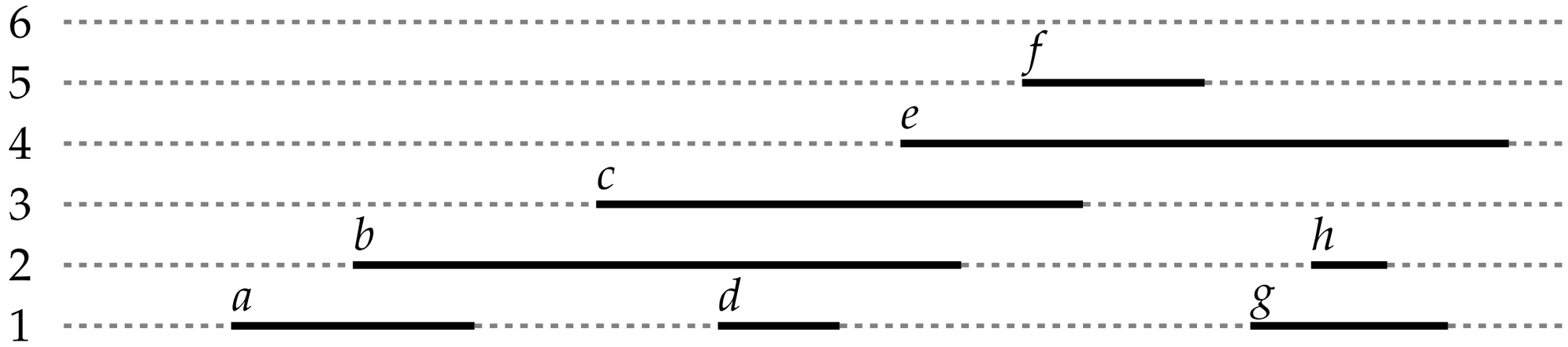

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  $\Rightarrow$ The coloring c uses the minimum number of colors.

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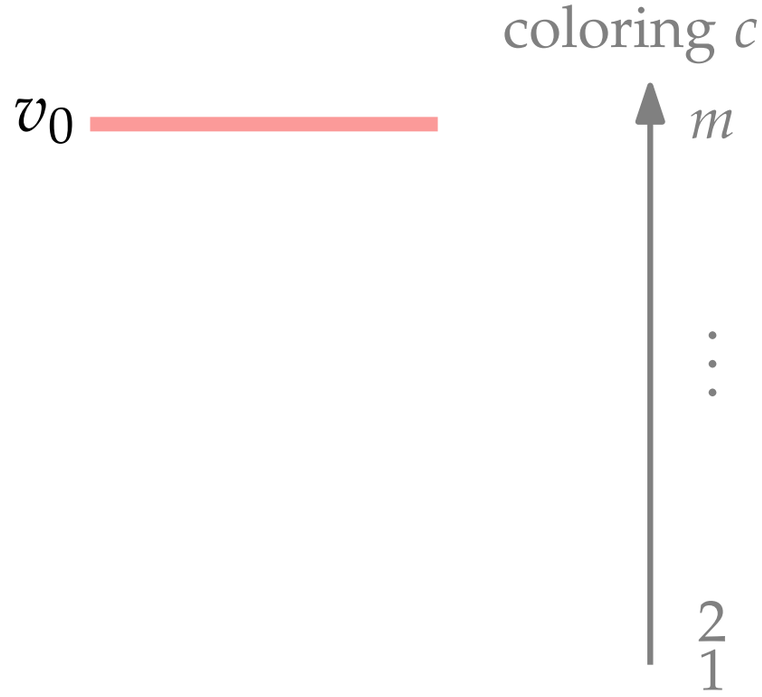

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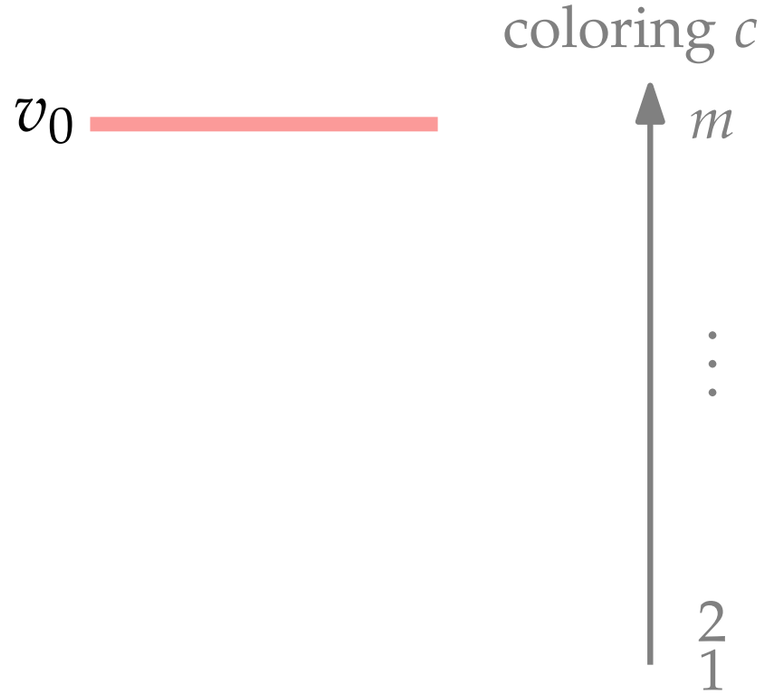

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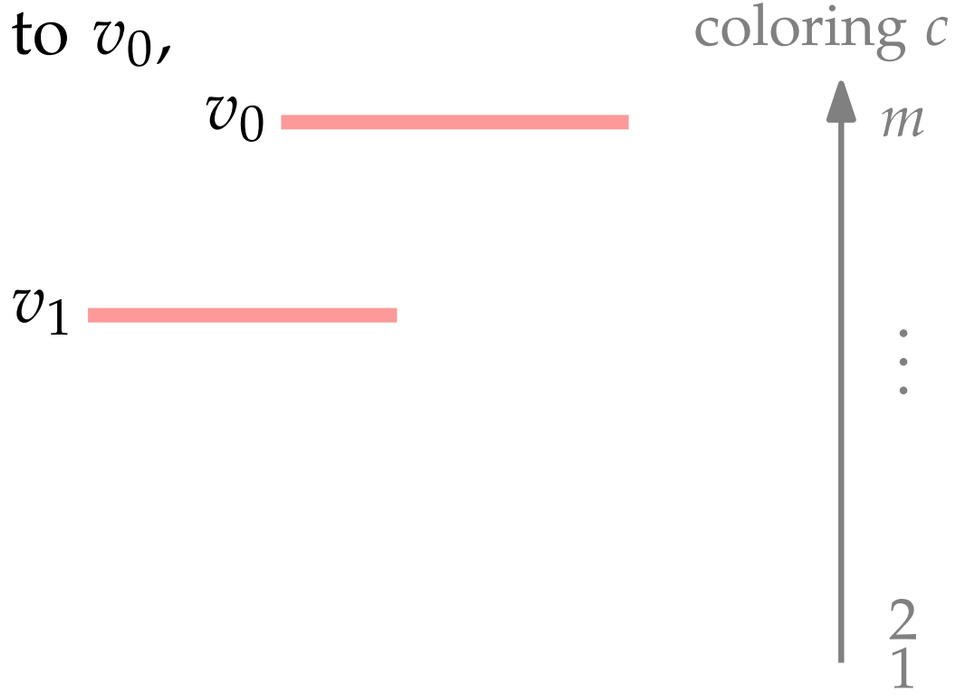

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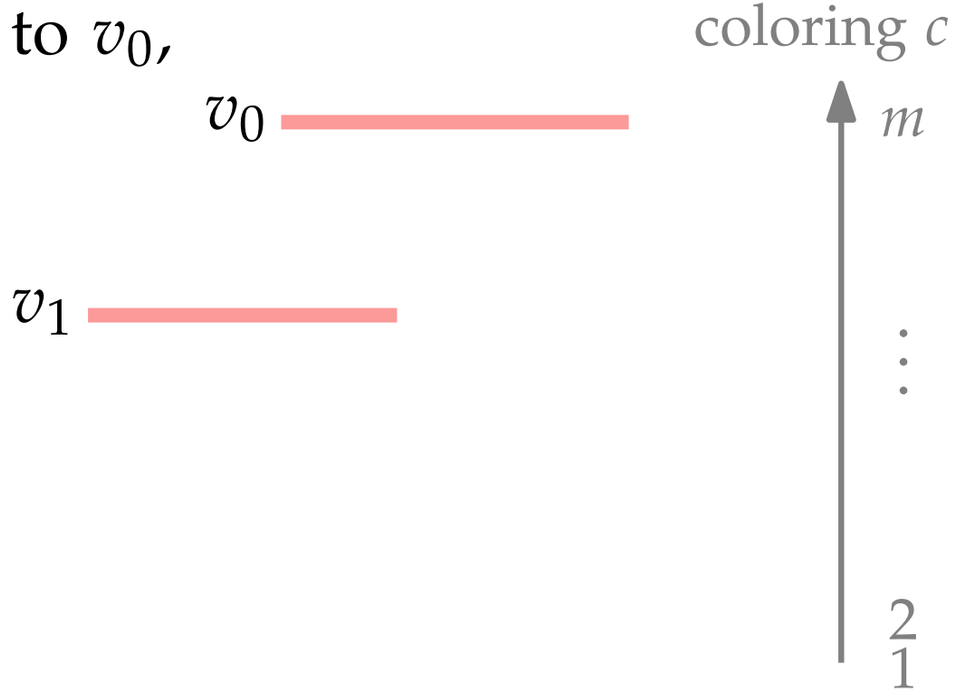

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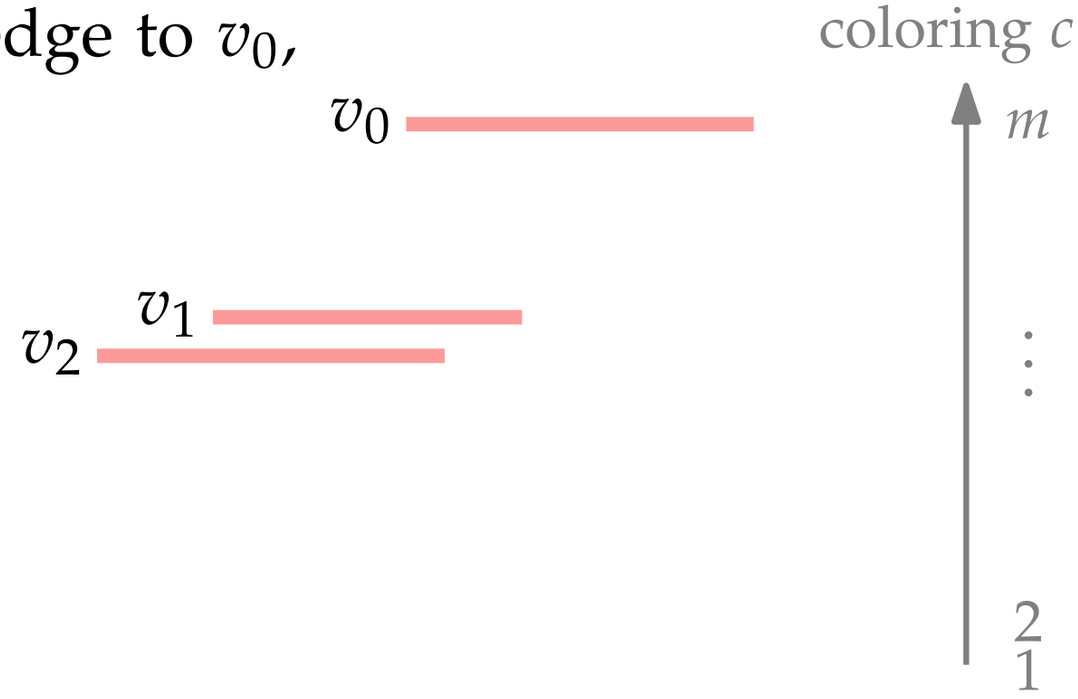

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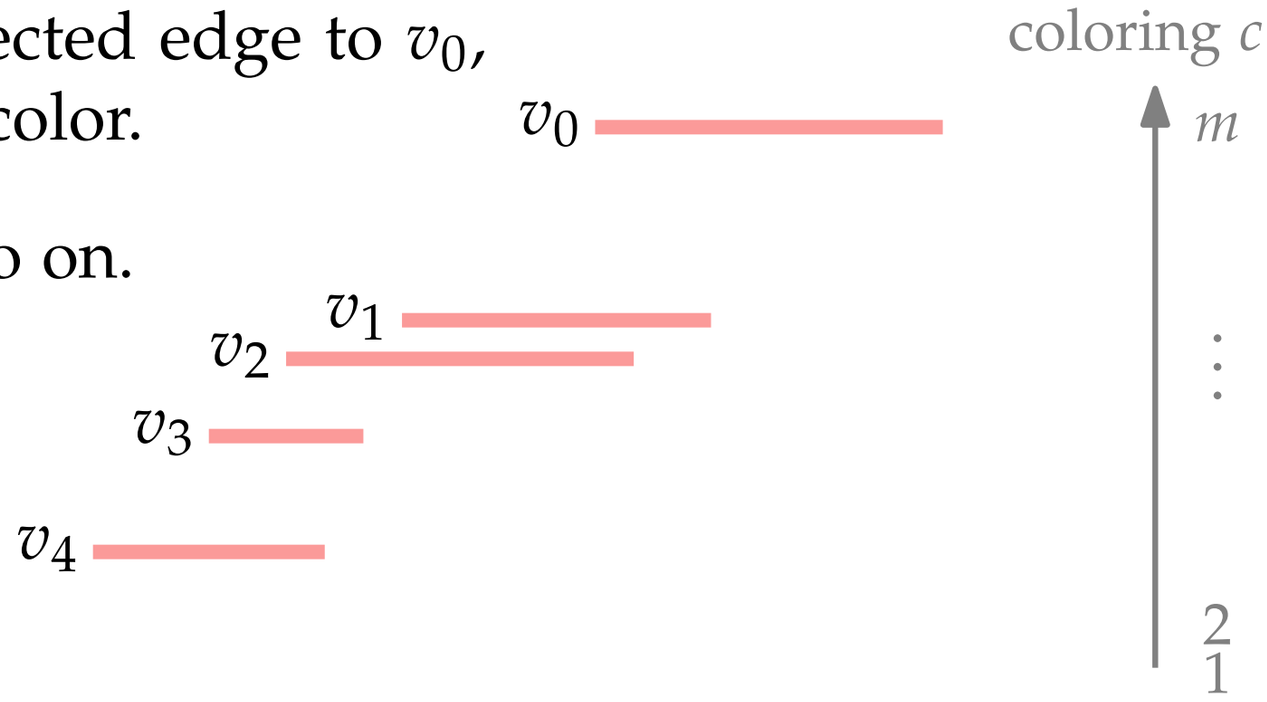

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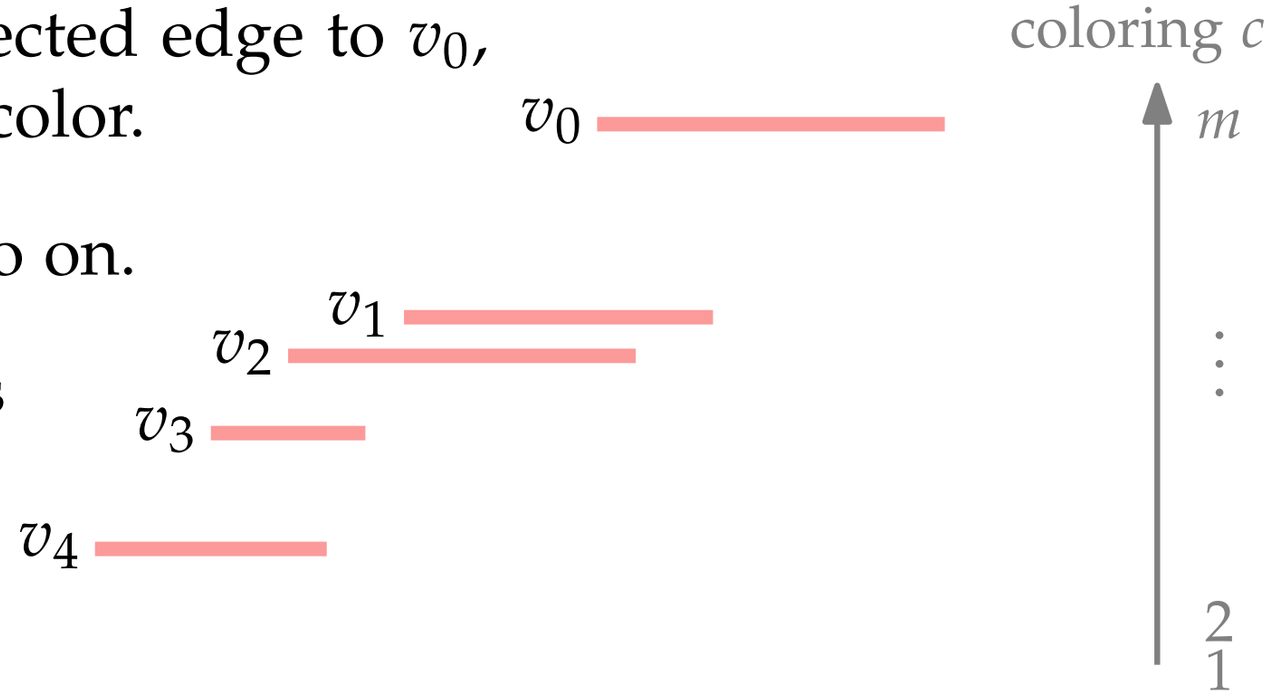

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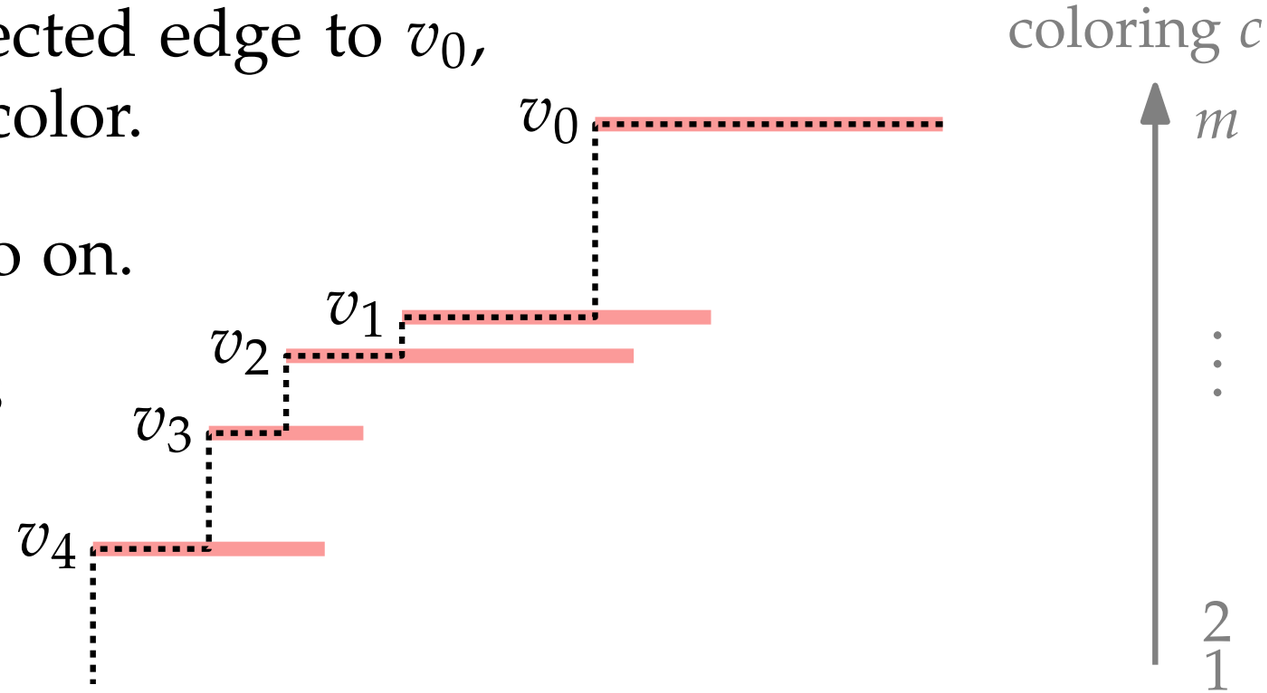

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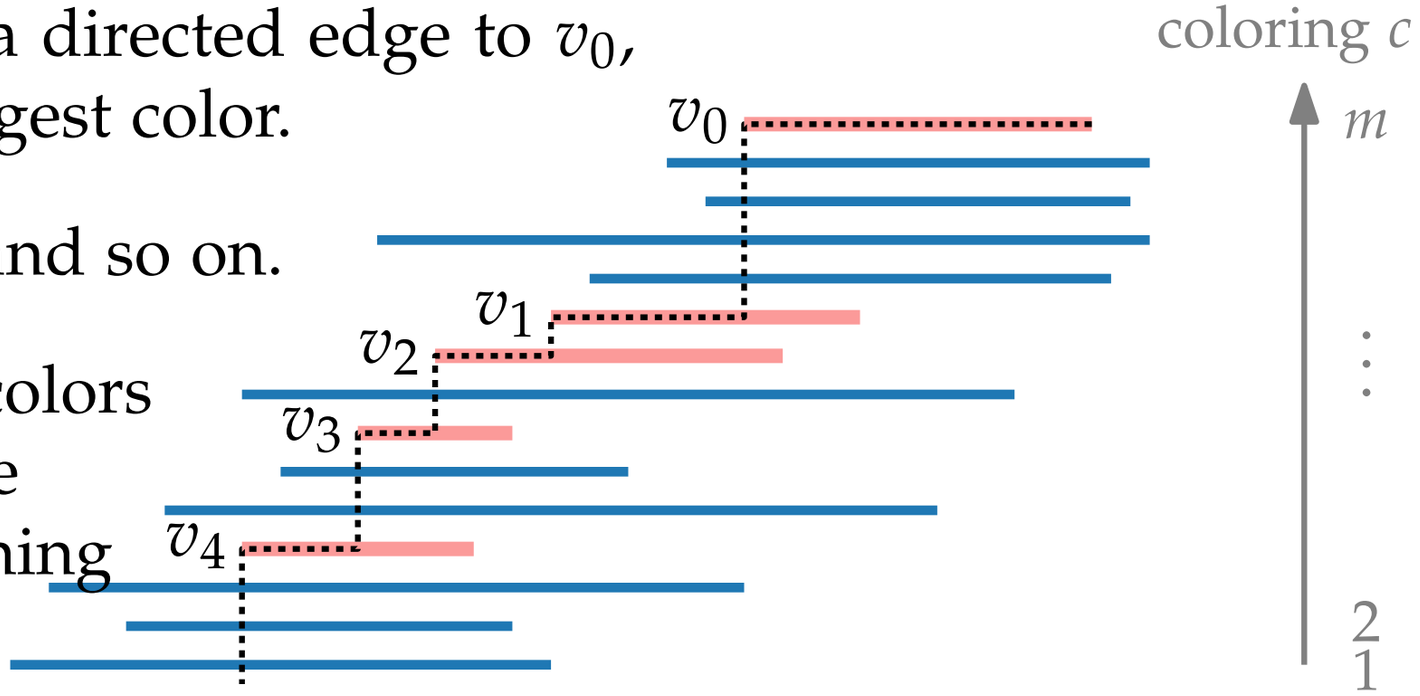

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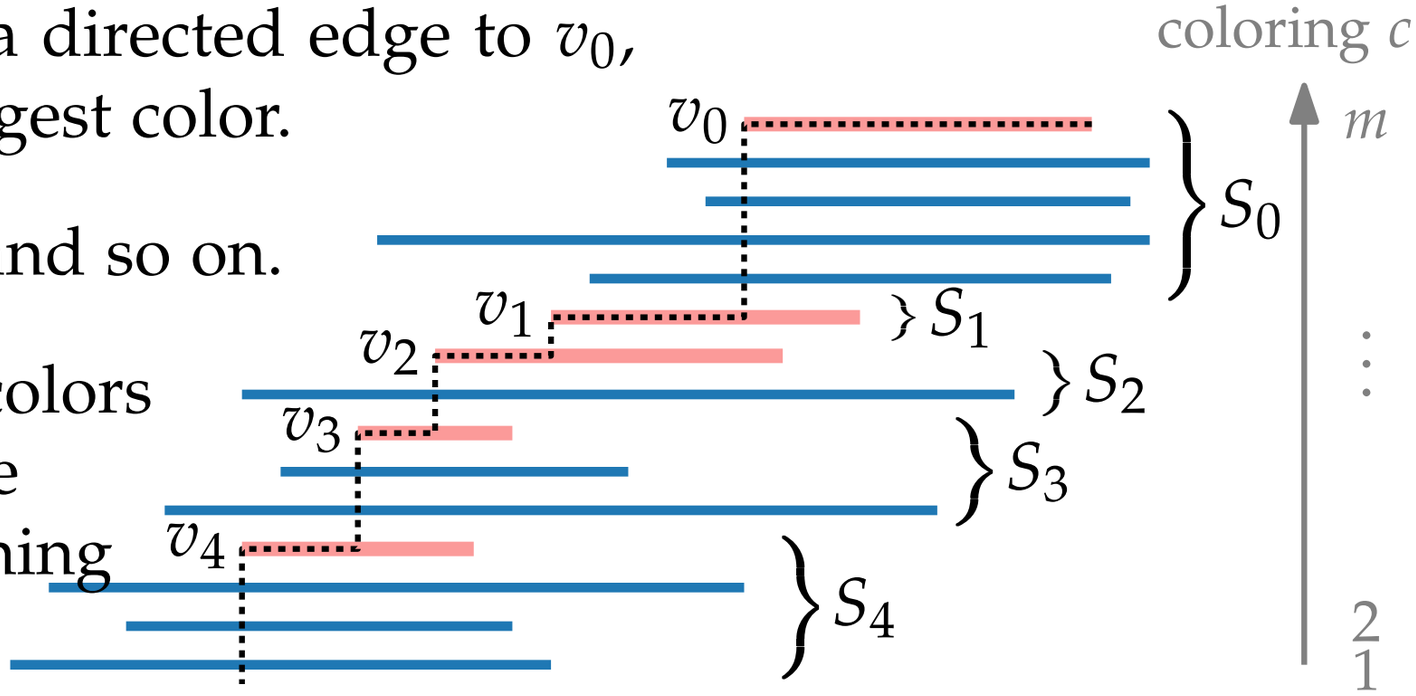

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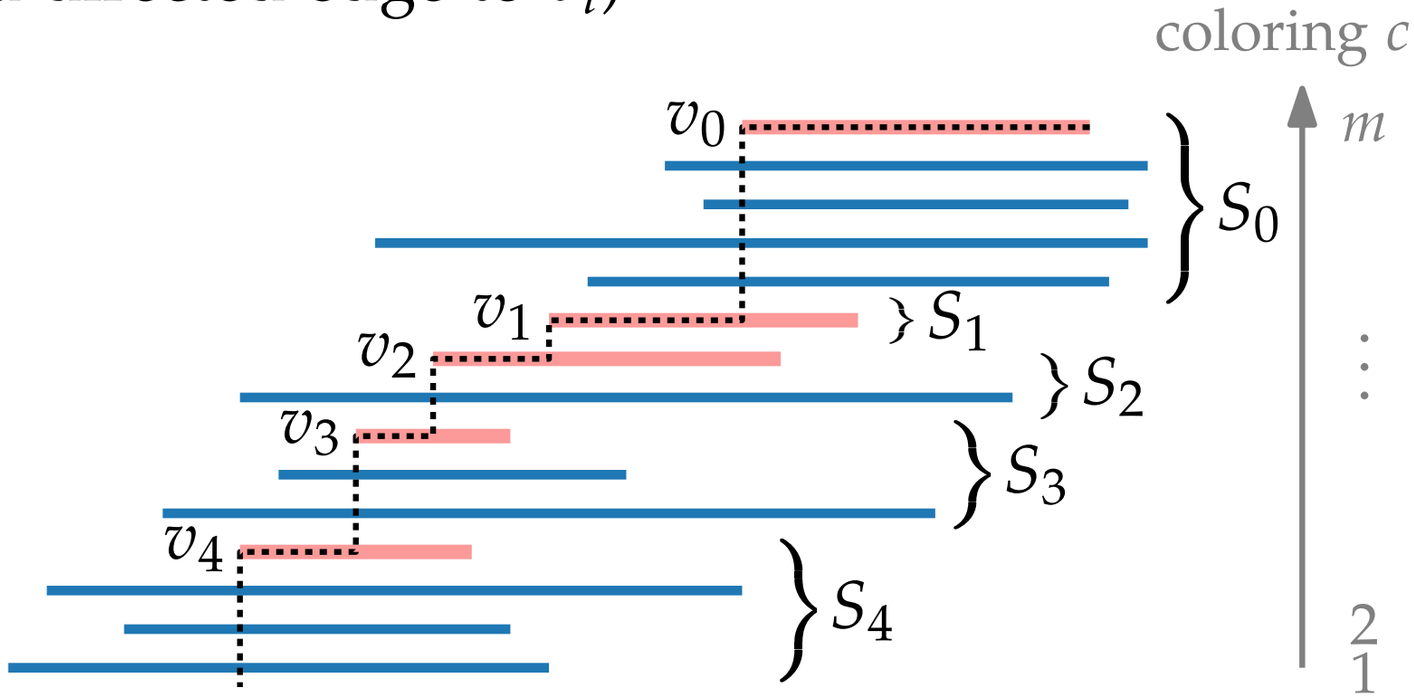

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- Hence, for every *step* S_i, all intervals contain v_i. (otherwise they would have a directed edge to v_i)
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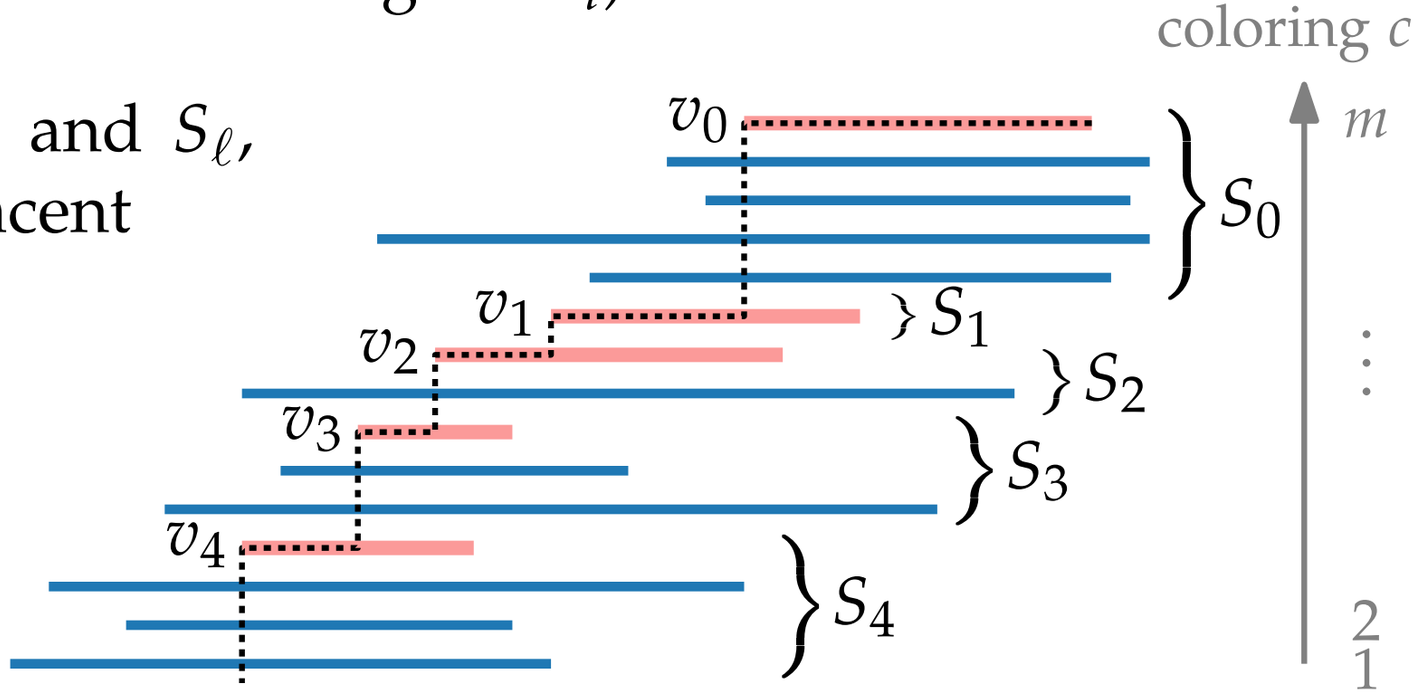

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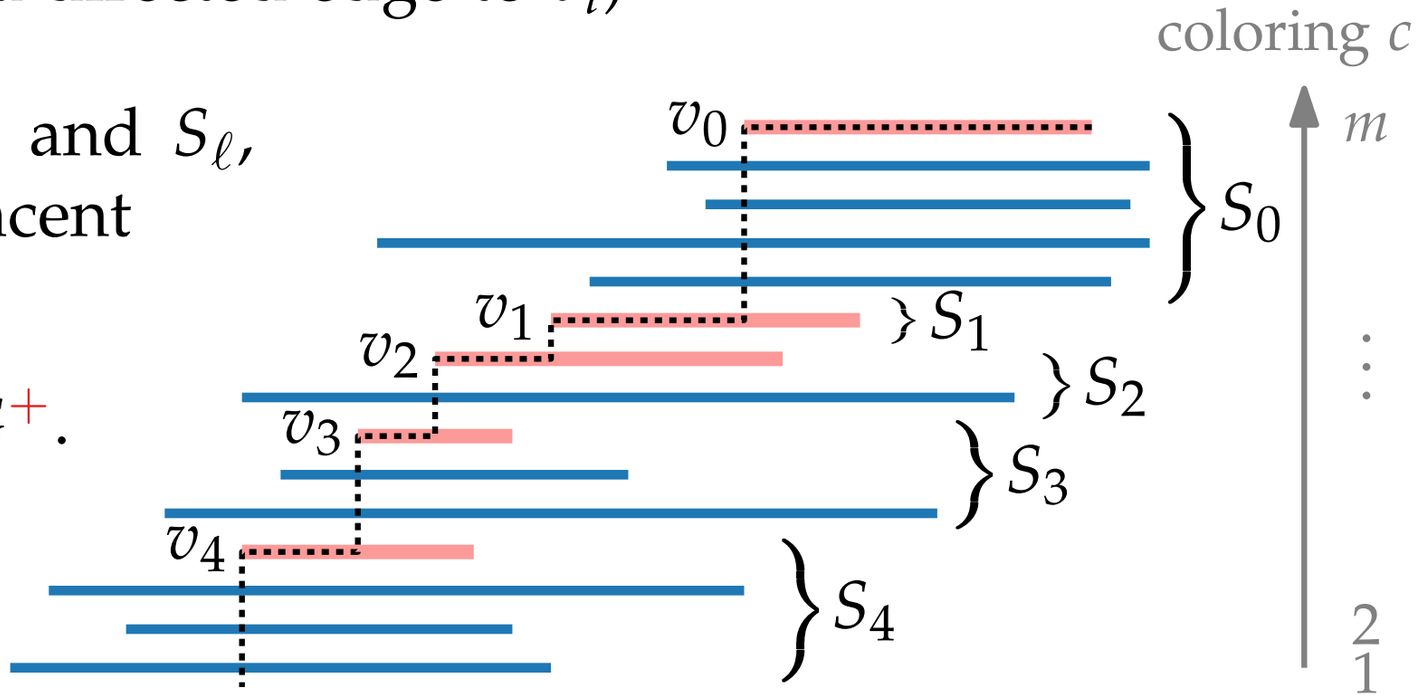

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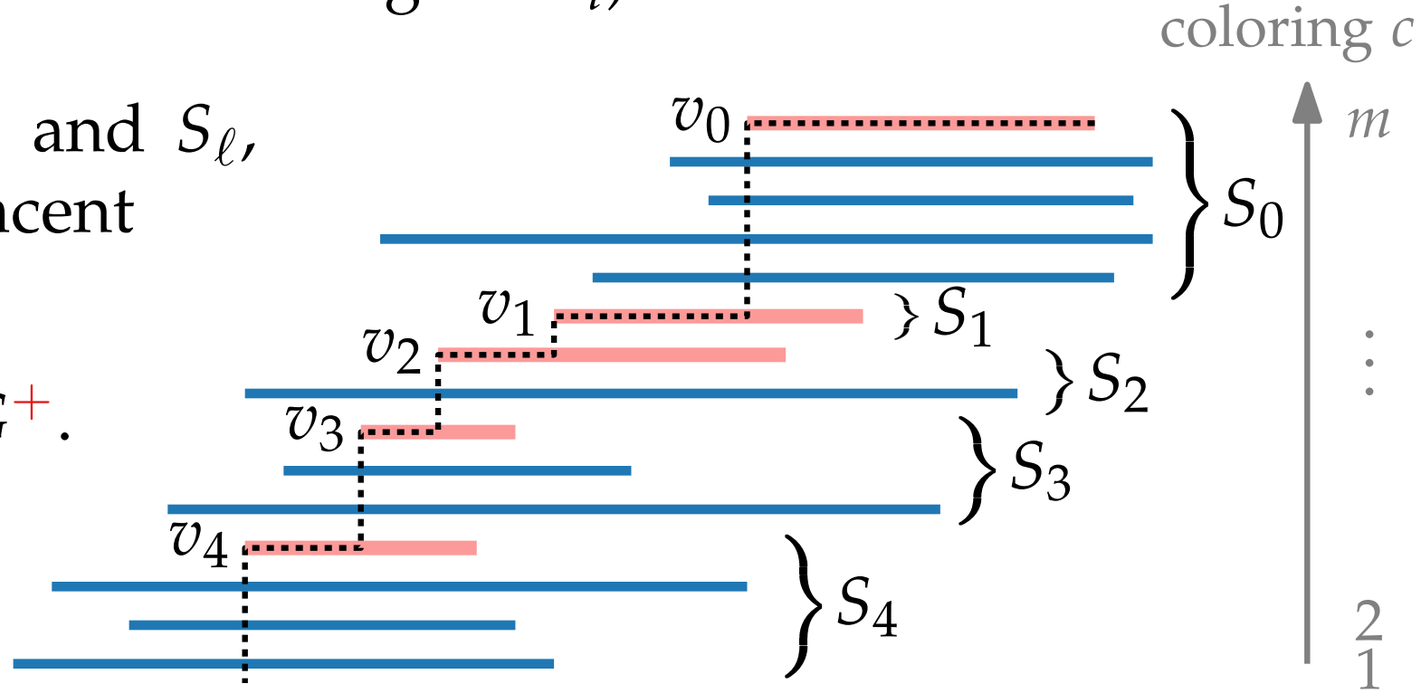

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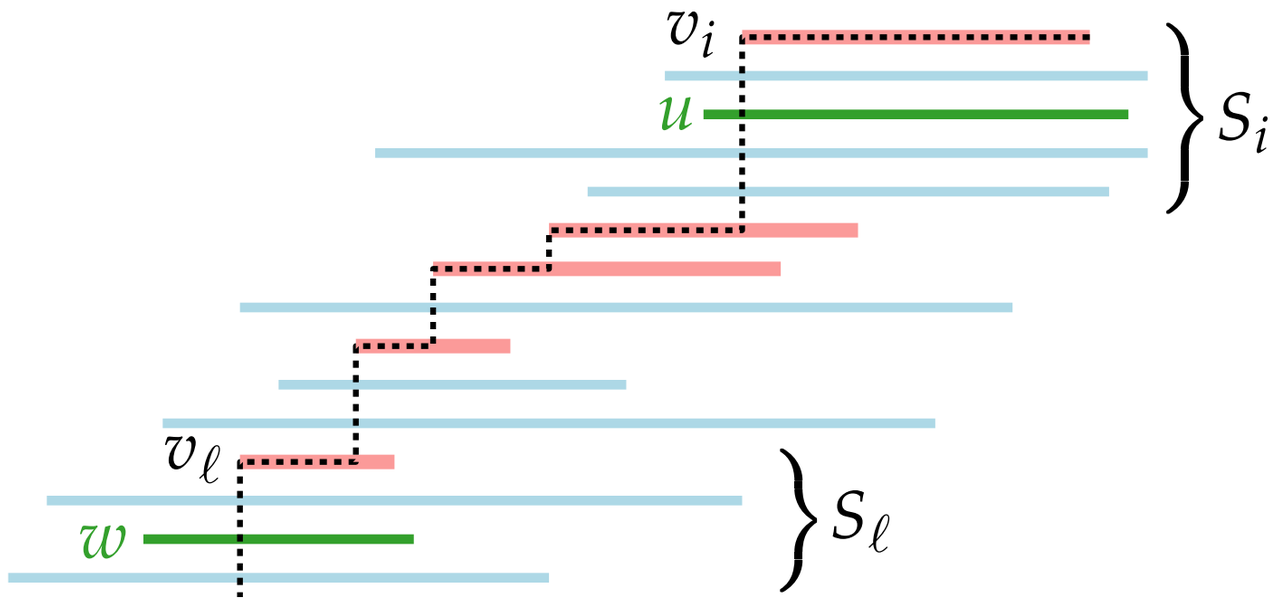

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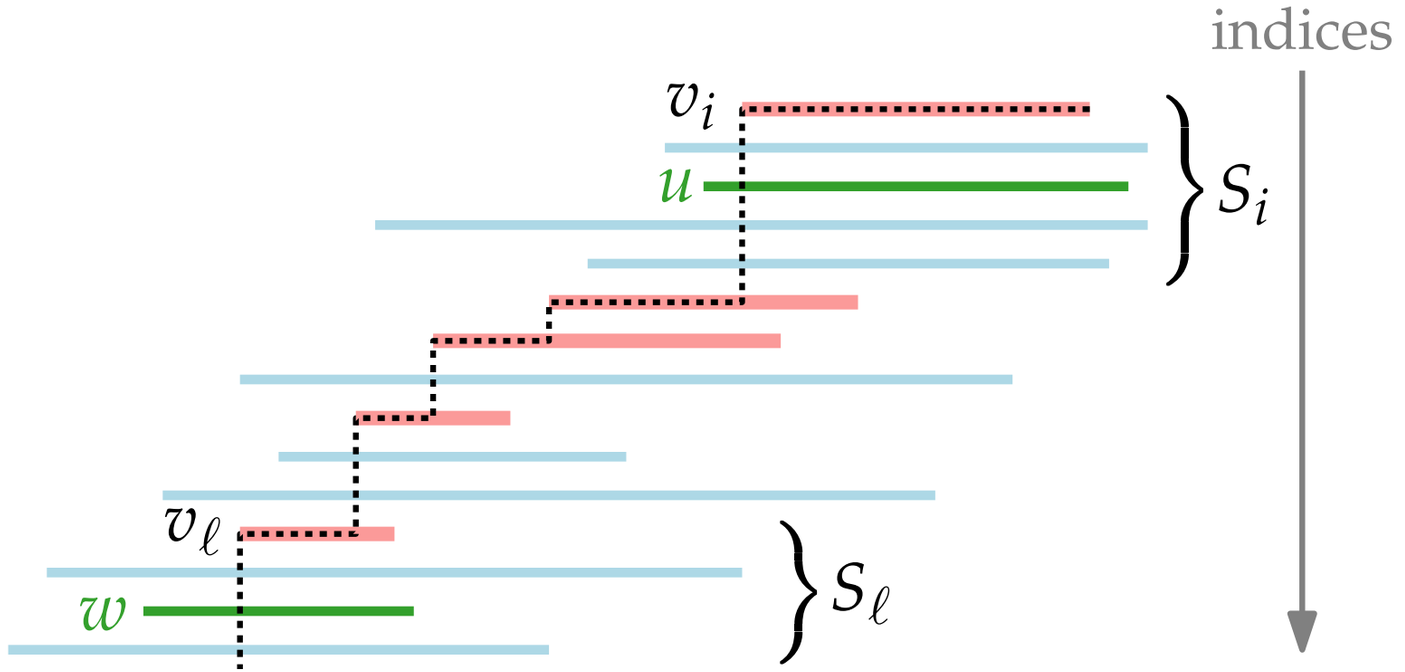

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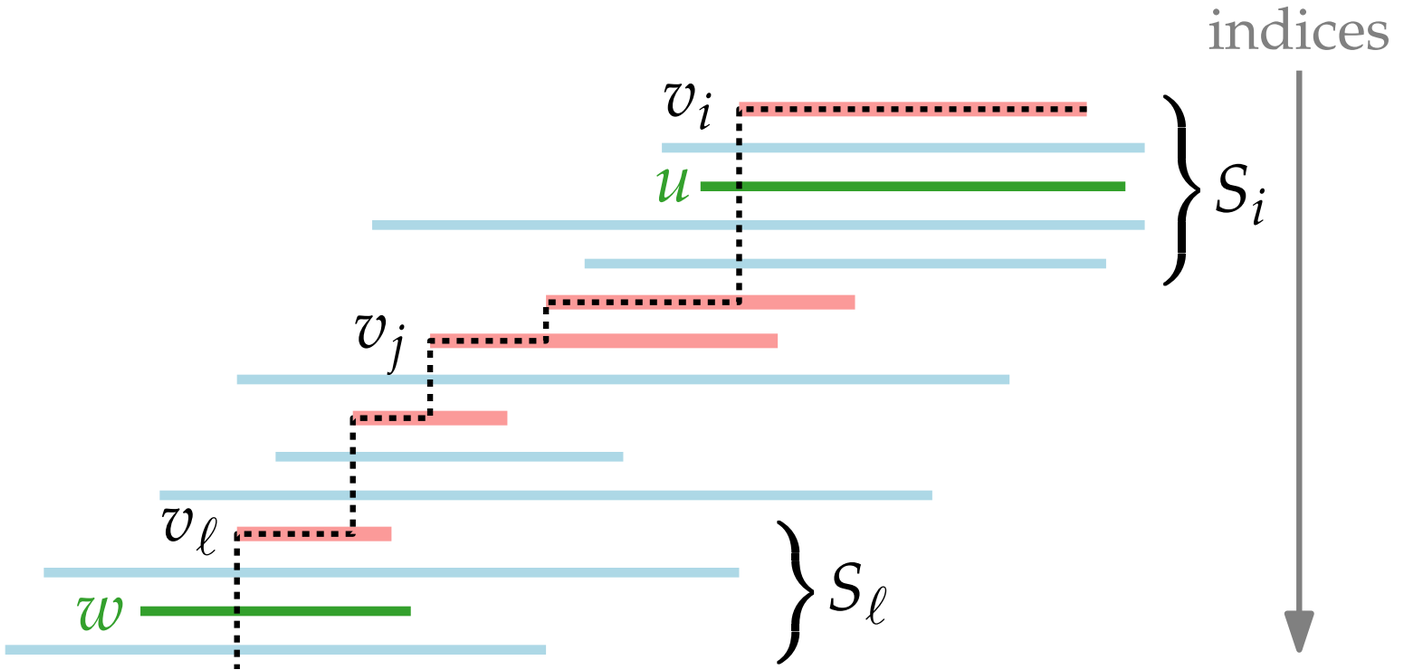

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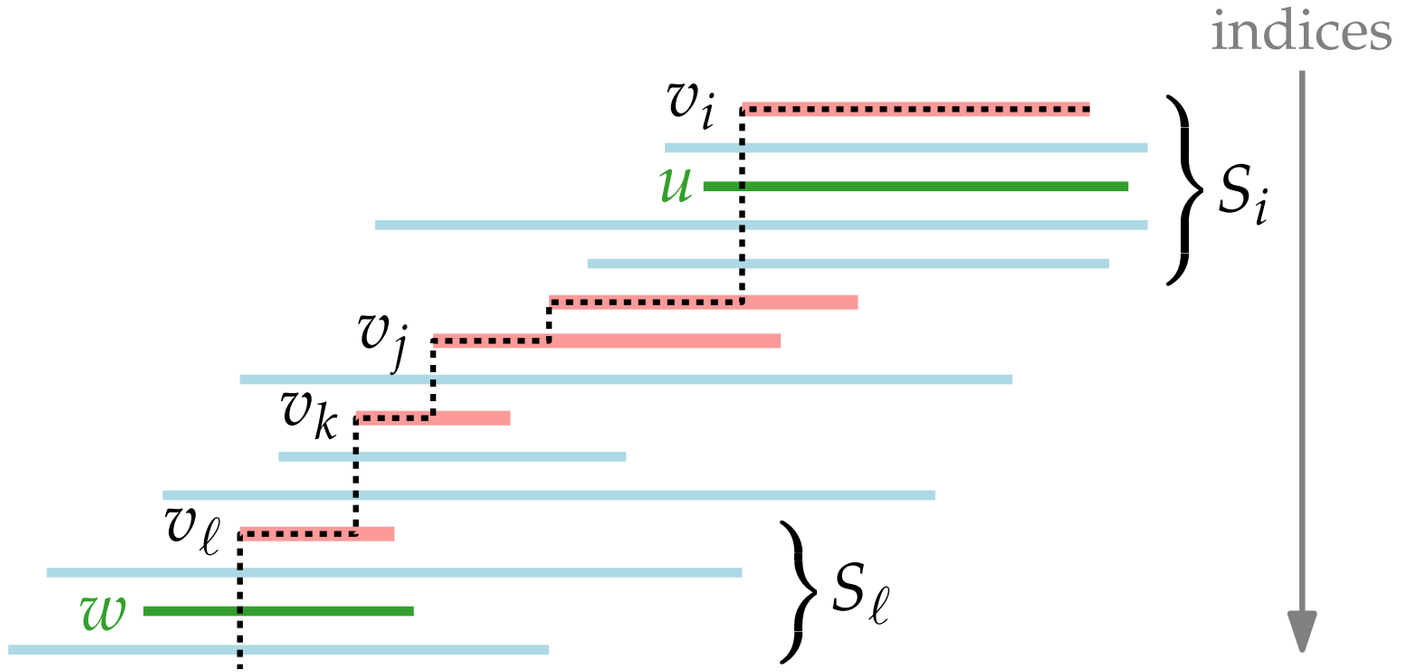

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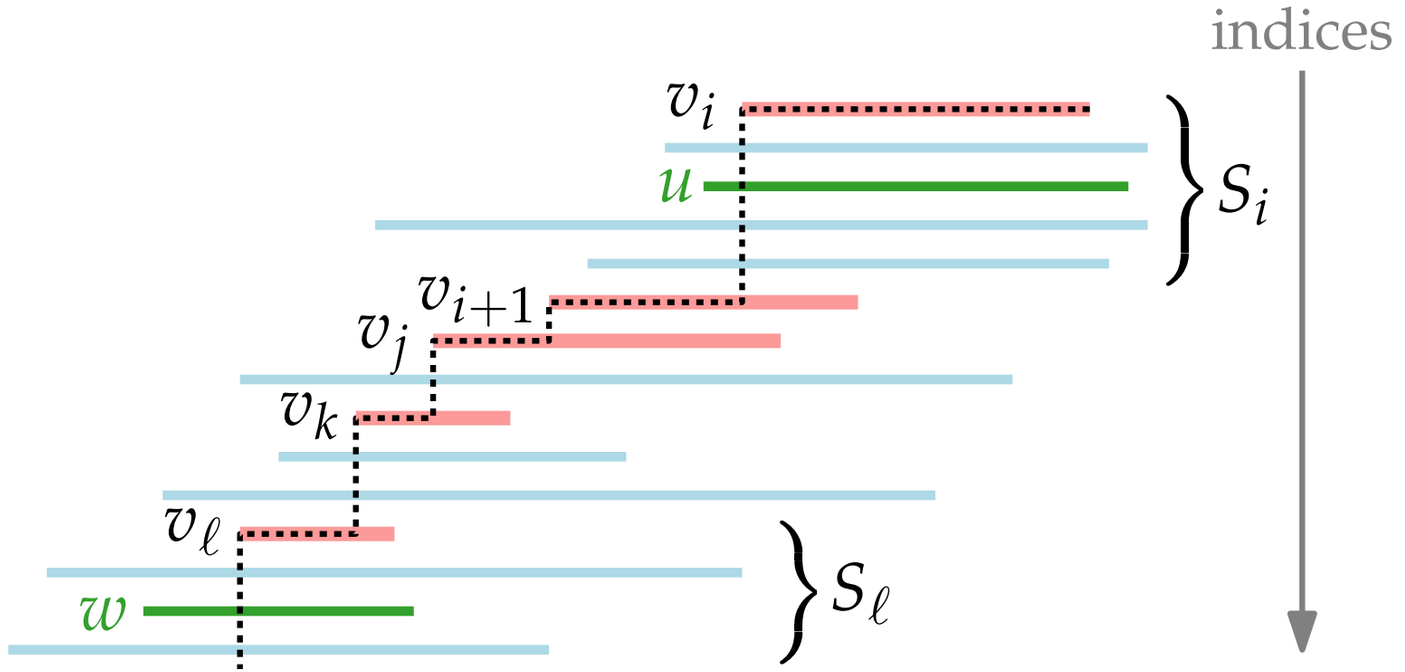

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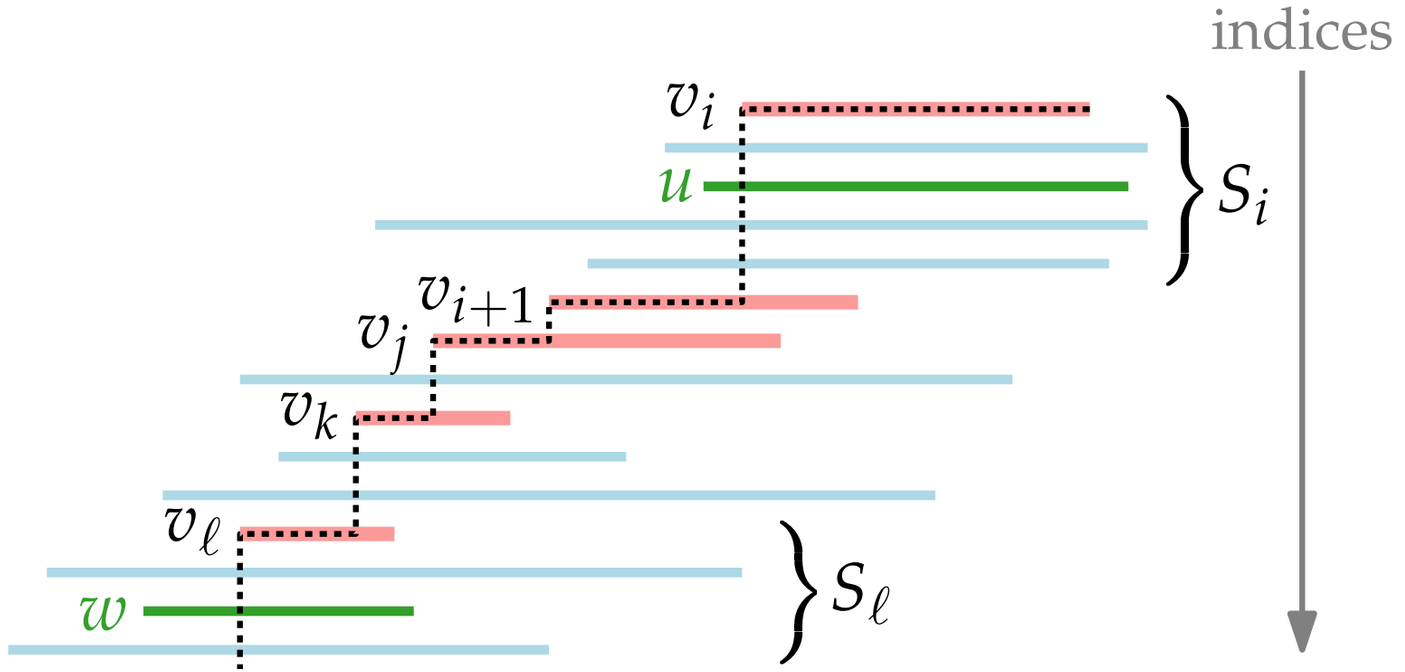

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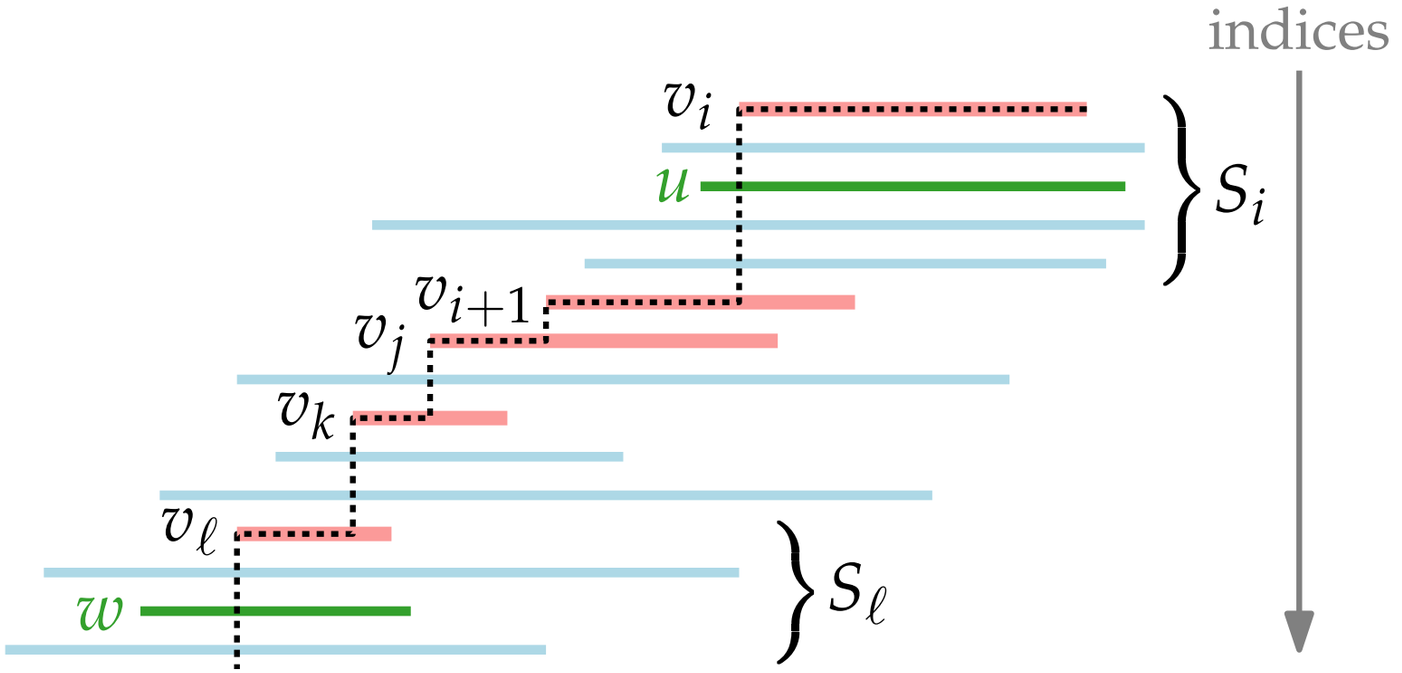

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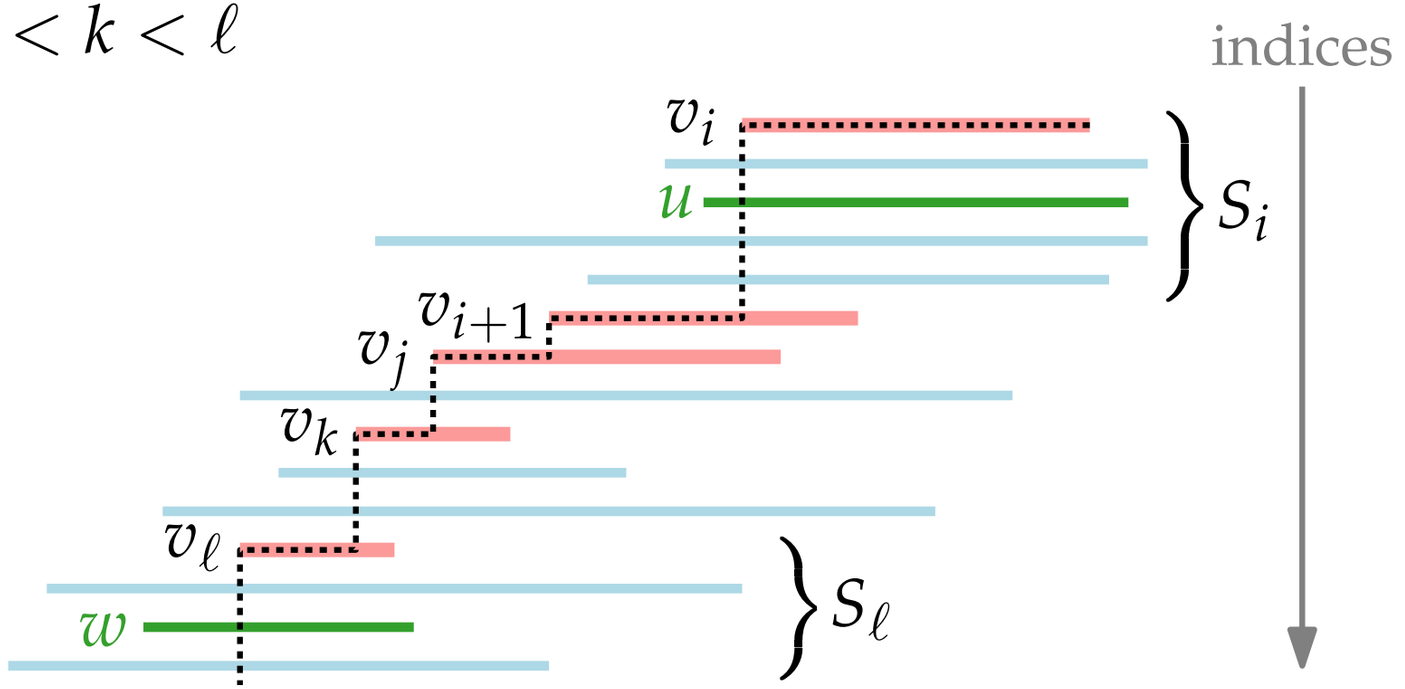

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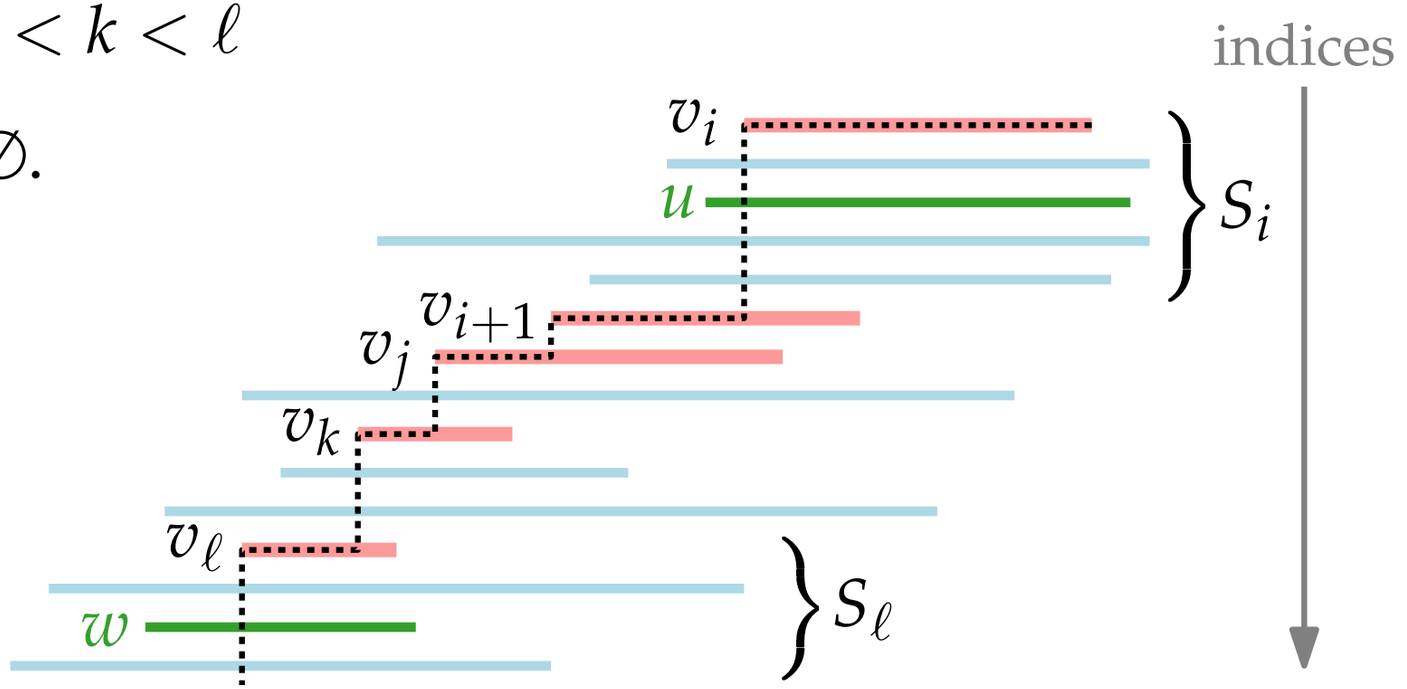

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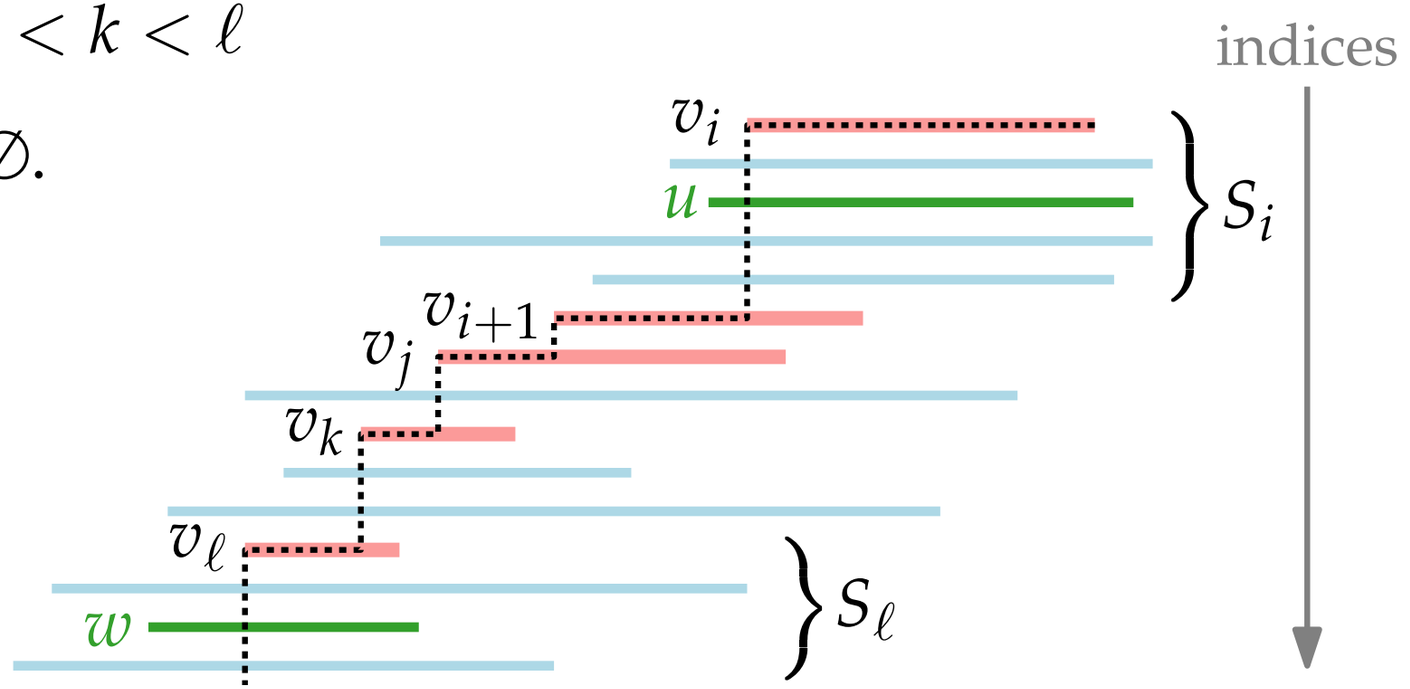

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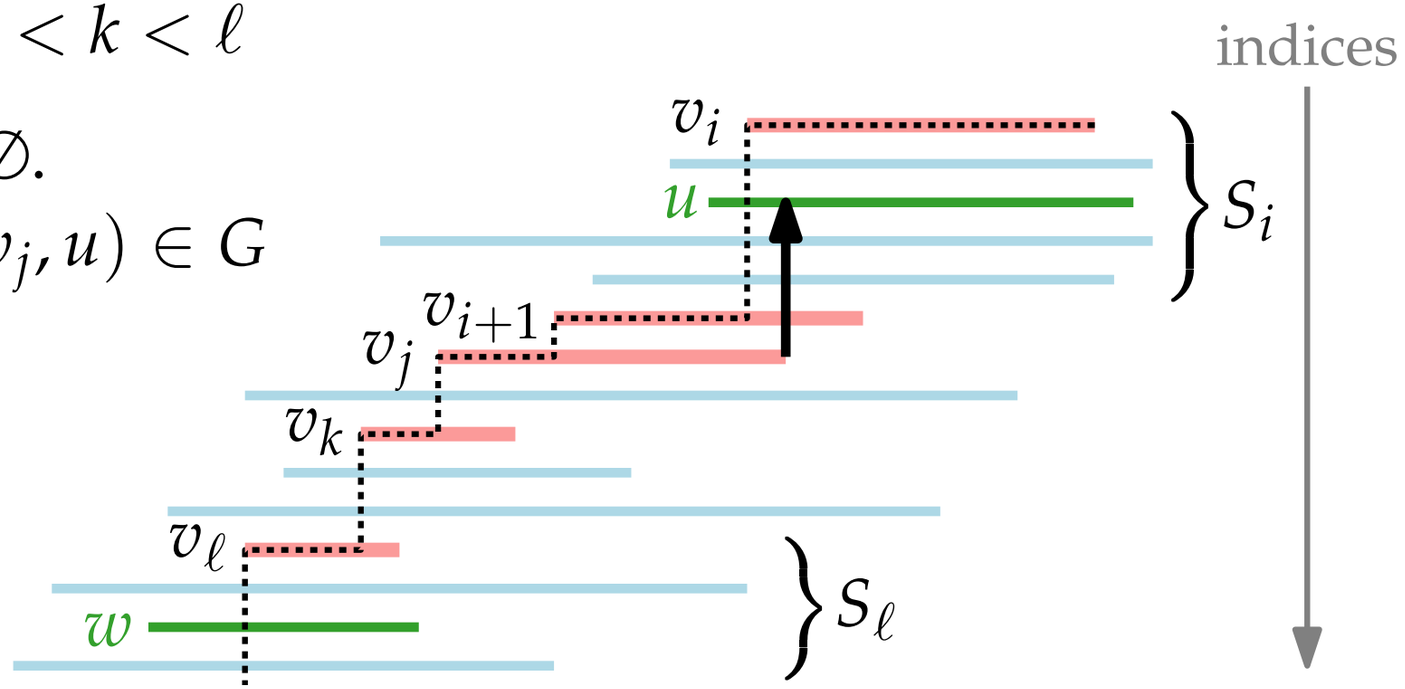

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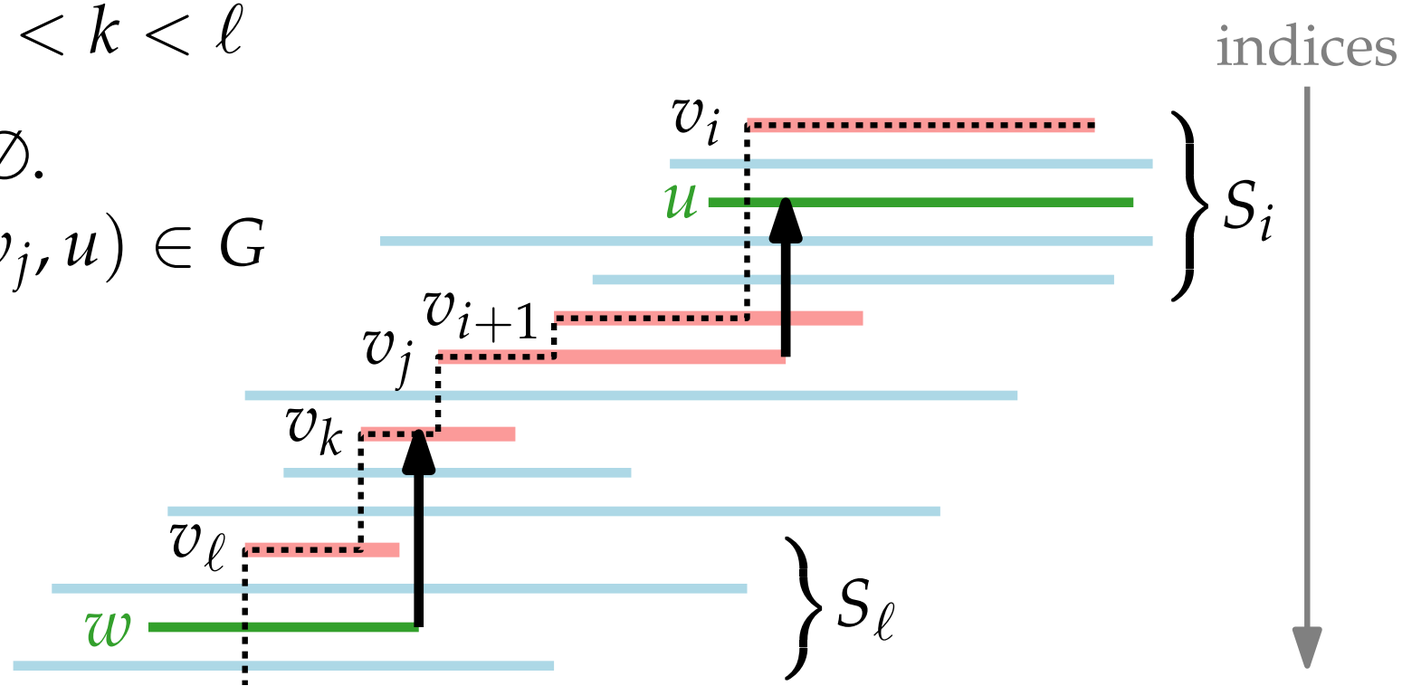

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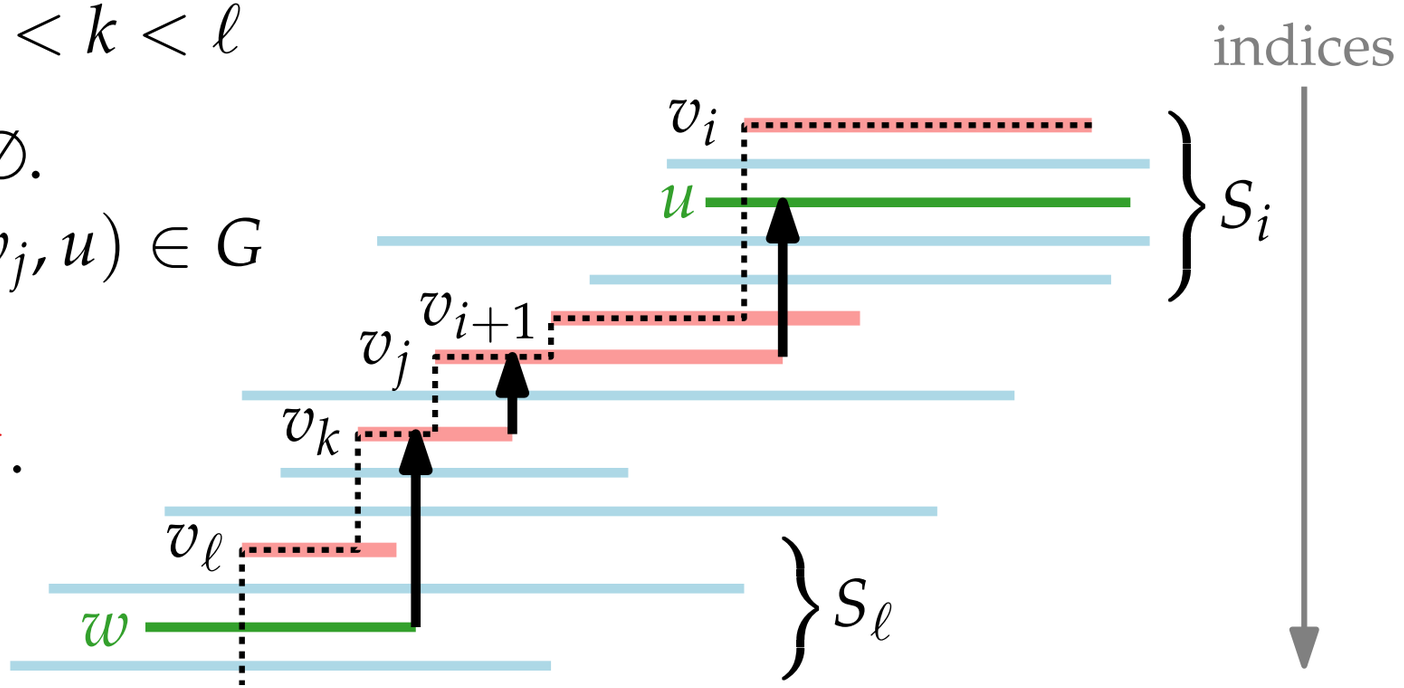

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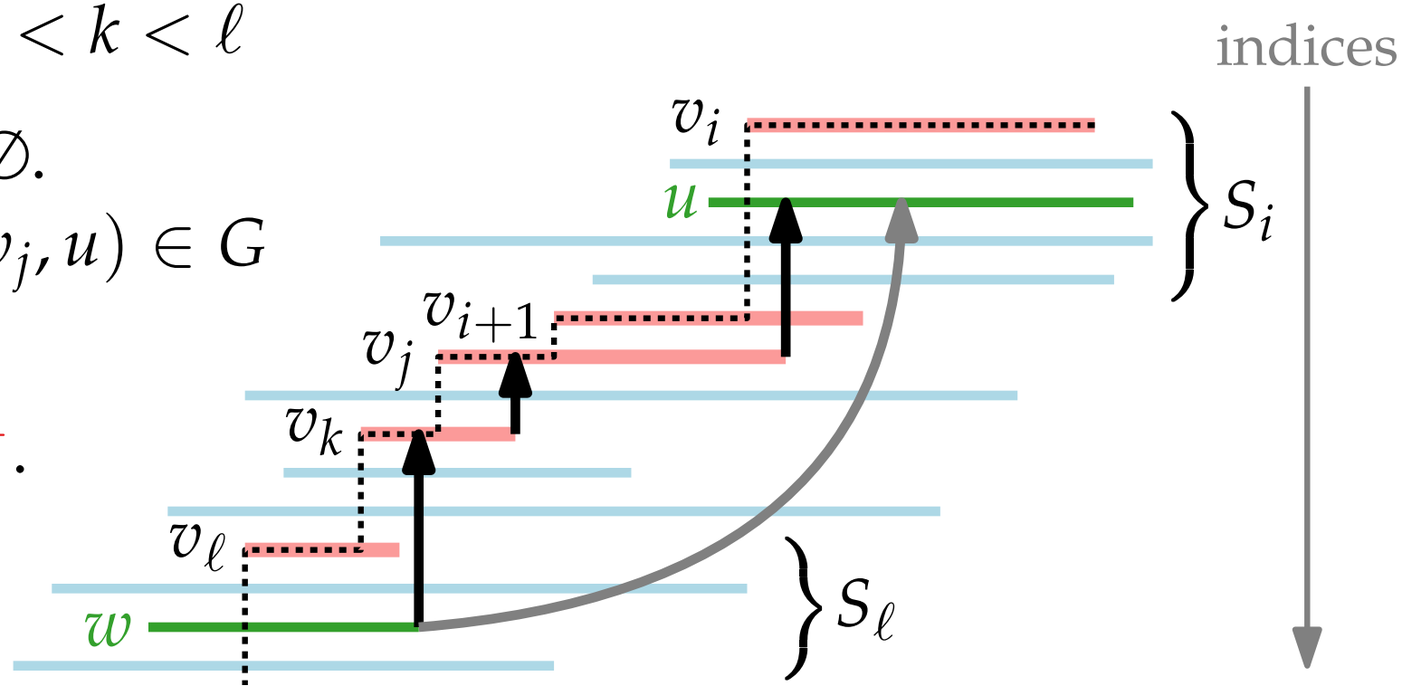

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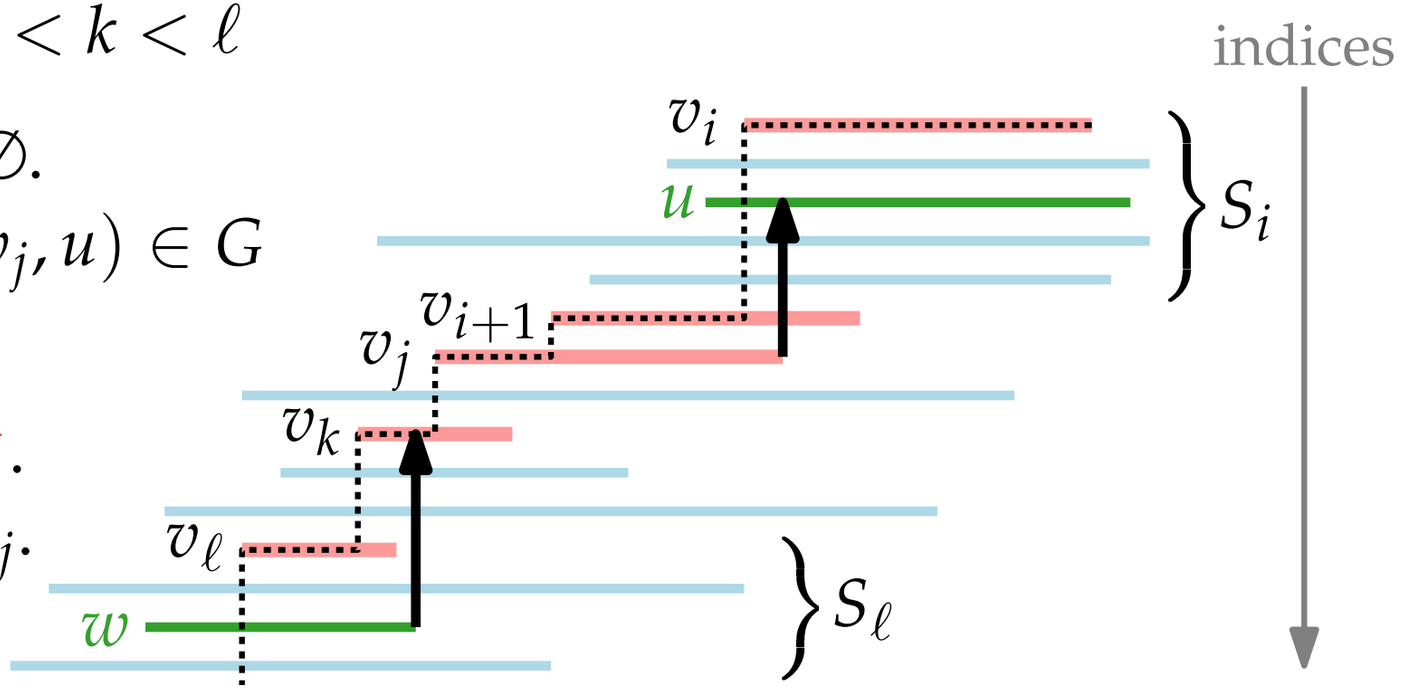

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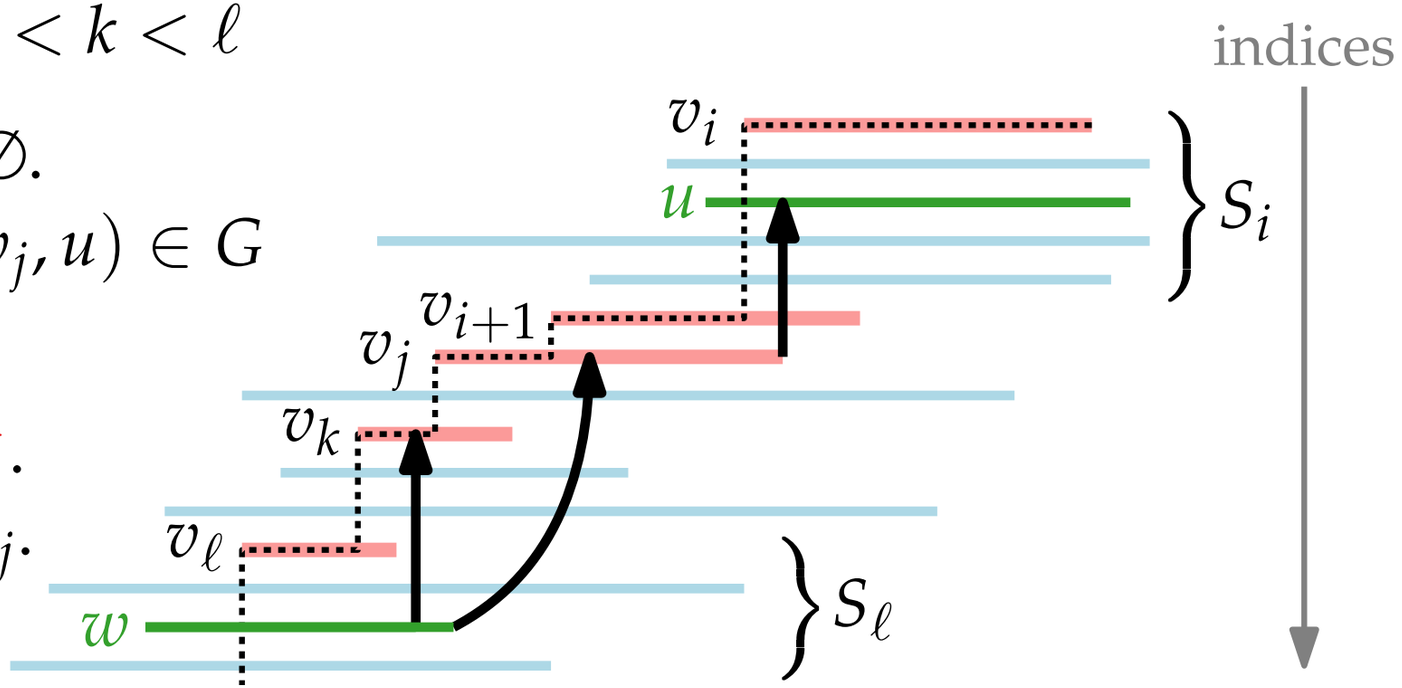

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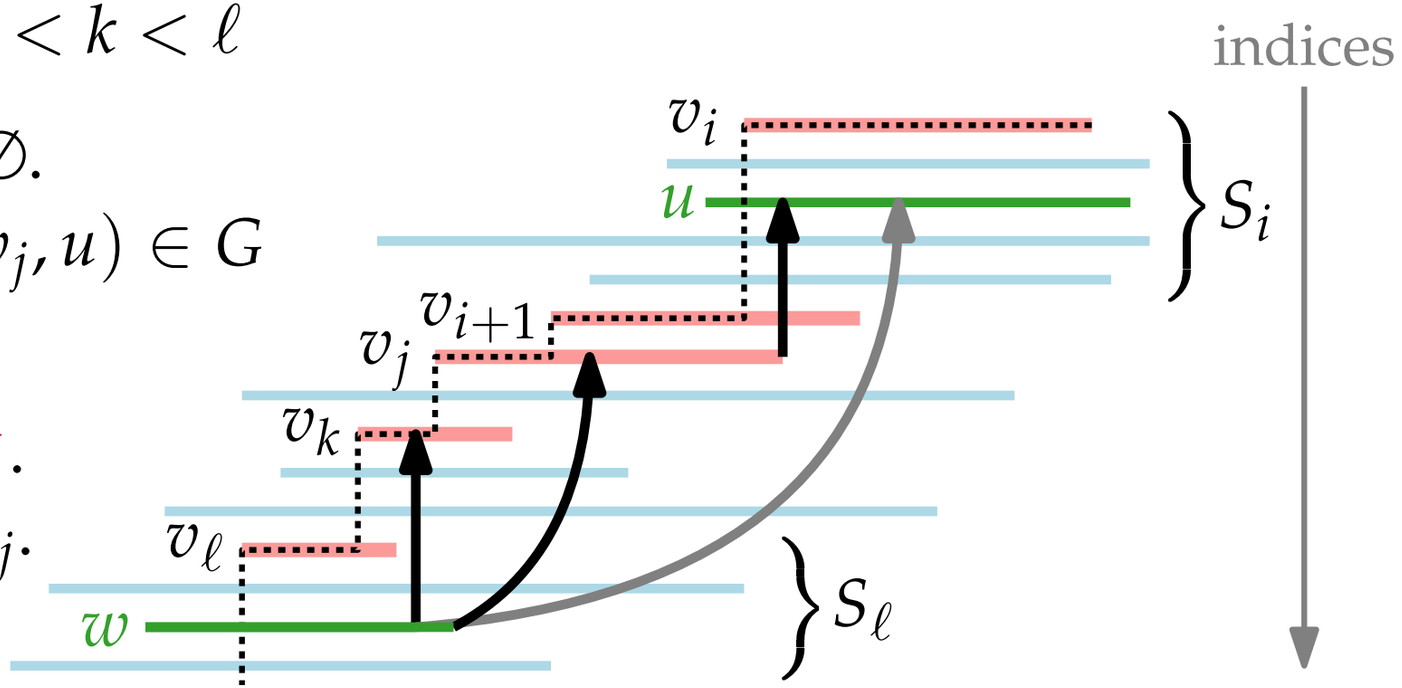

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Find a graph coloring $c\colon V \to \mathbb{N}$ such that:
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

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- recognition in $O(n^2)$ time
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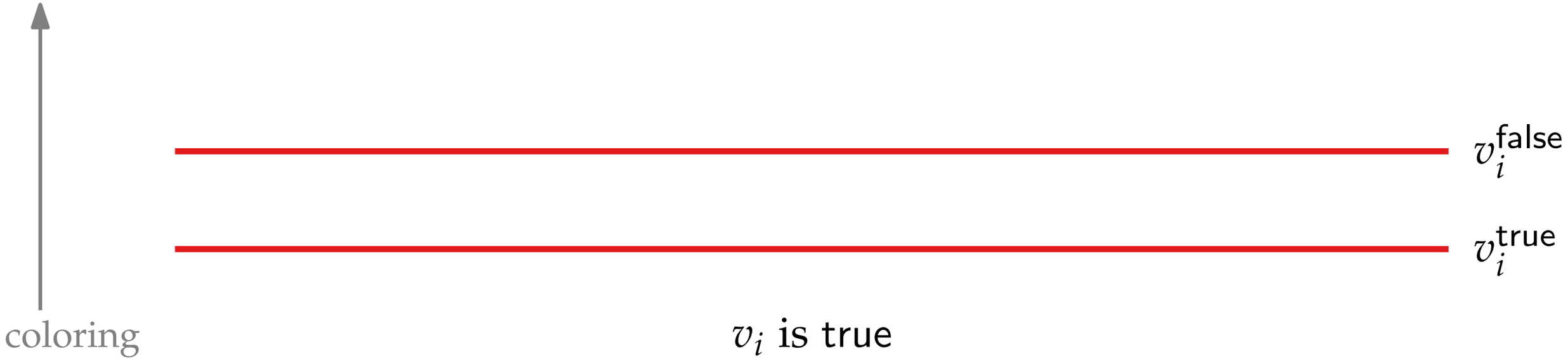

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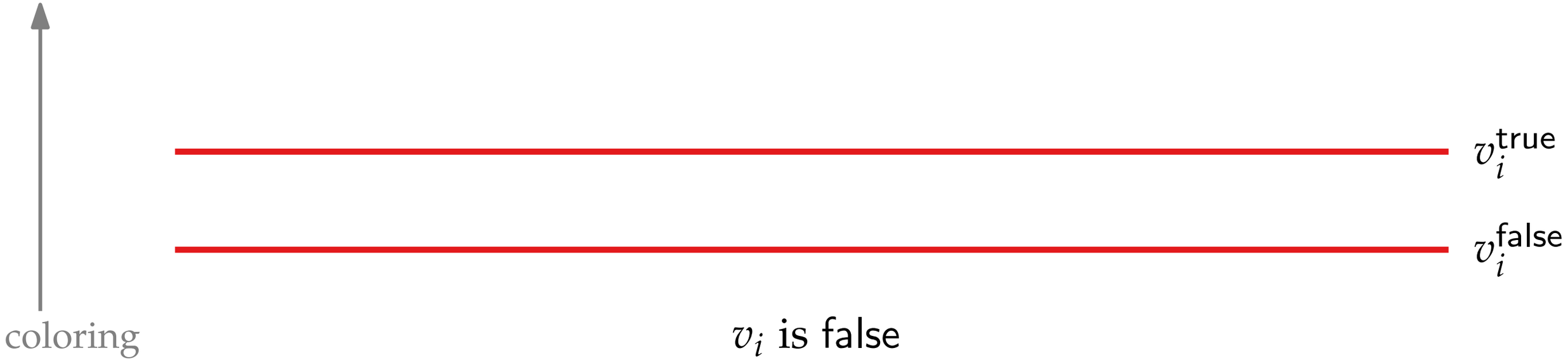

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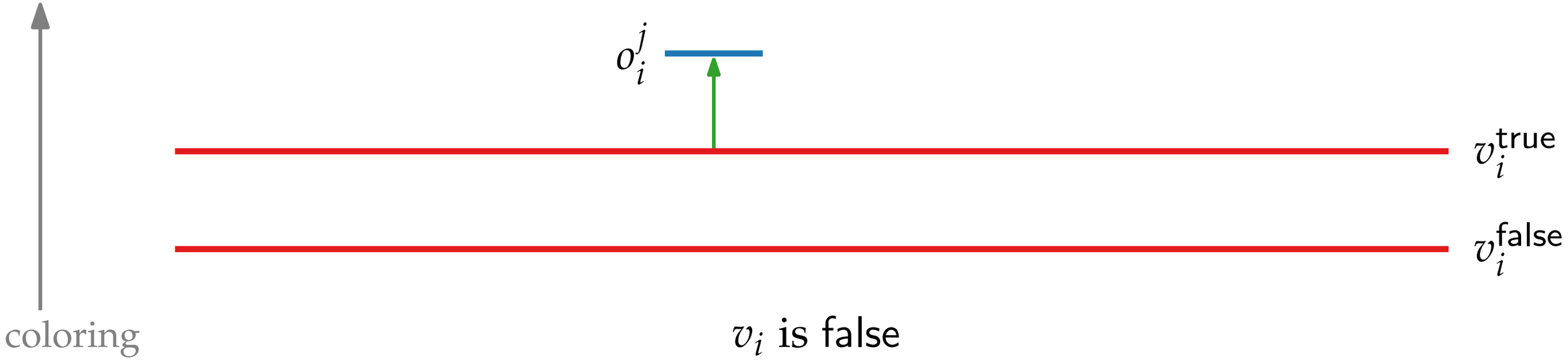

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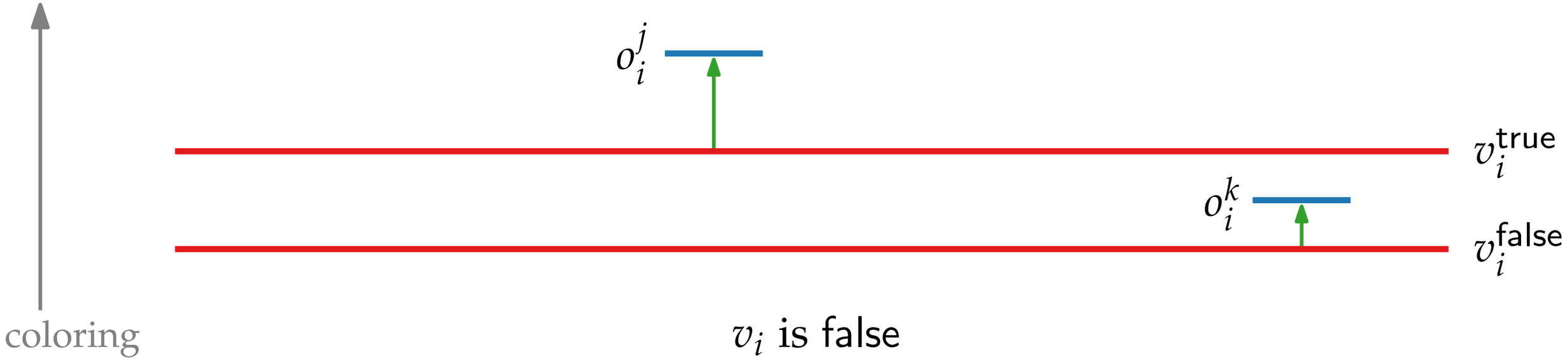

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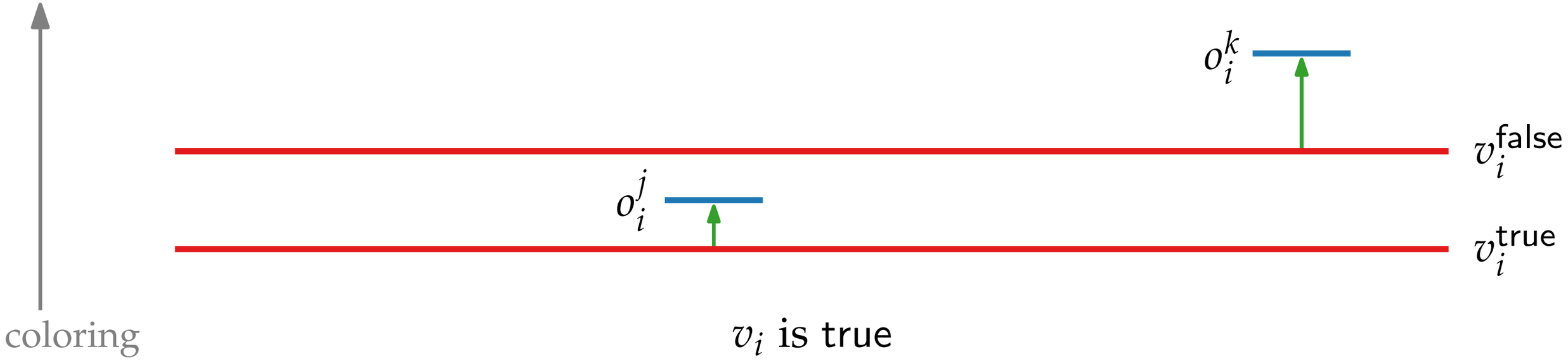

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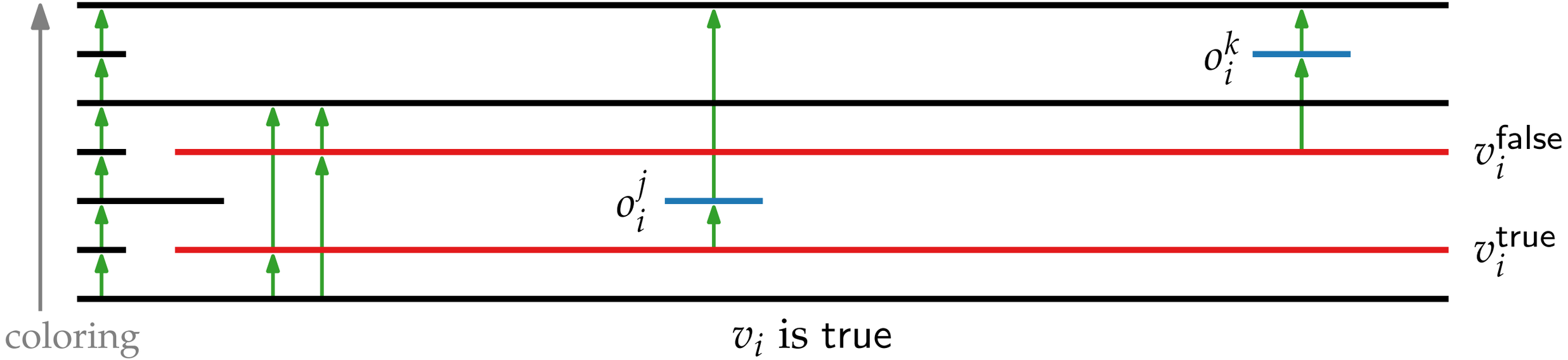

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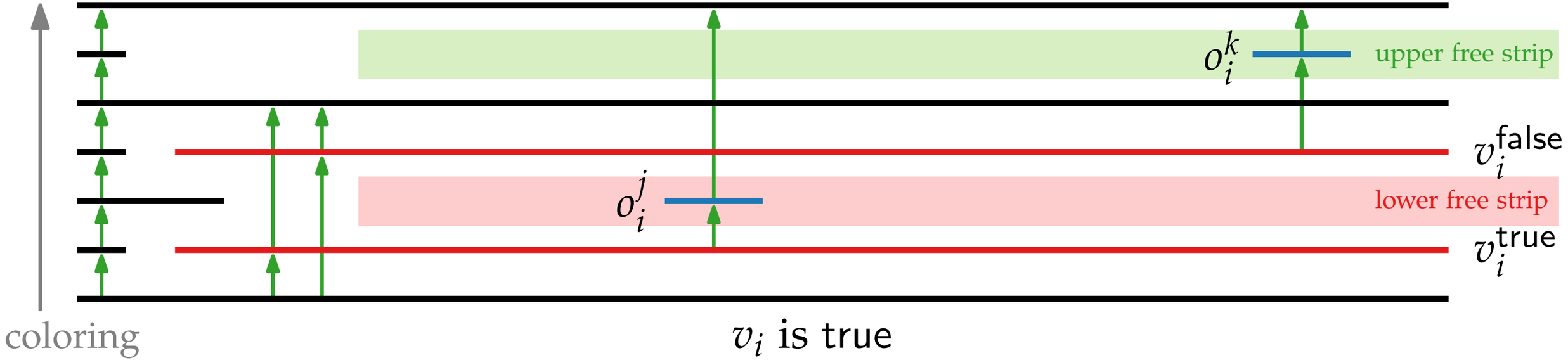

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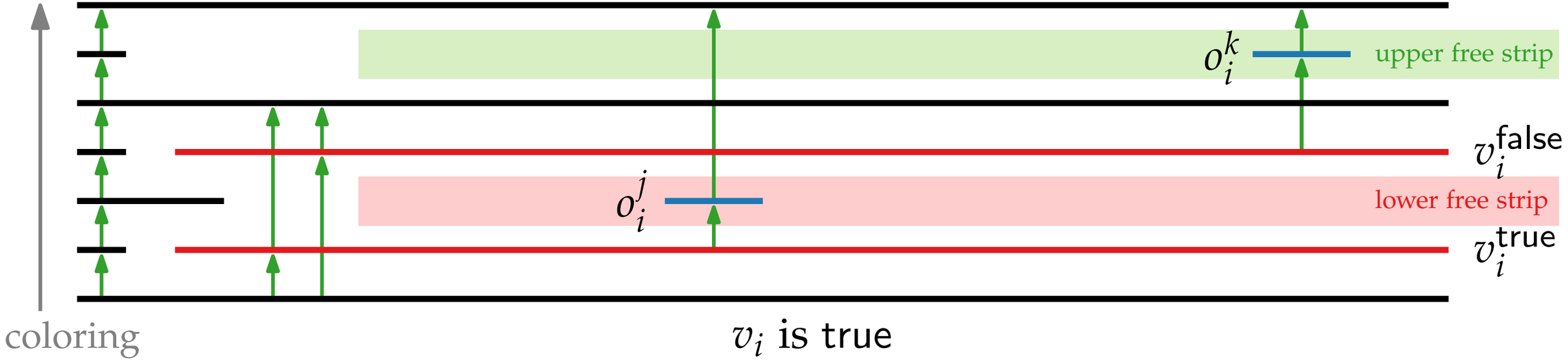

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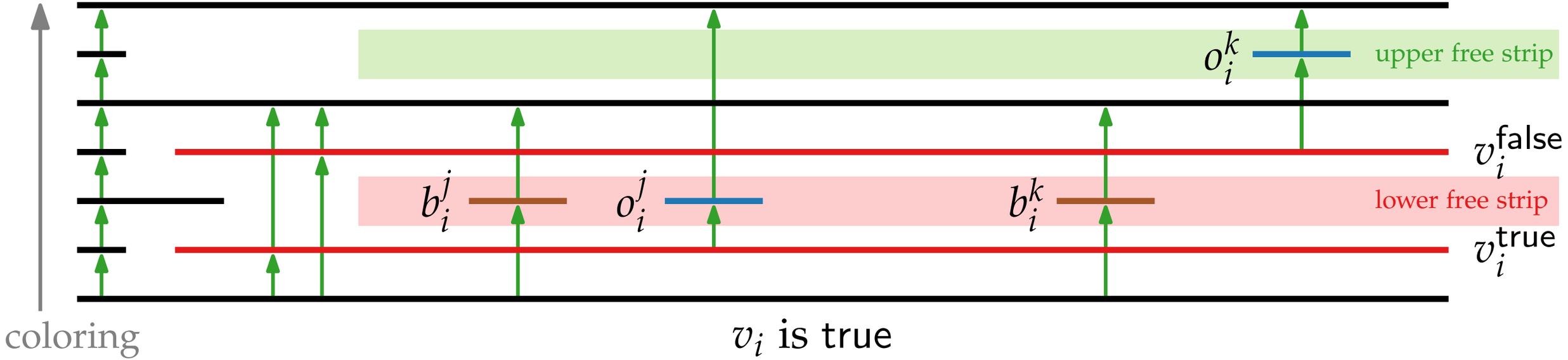

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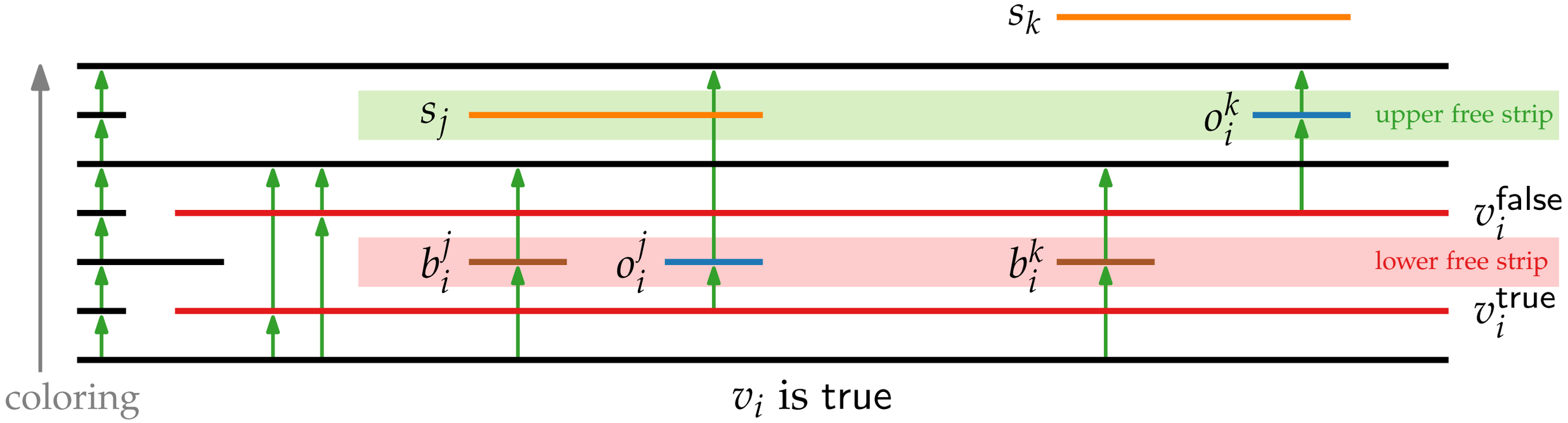

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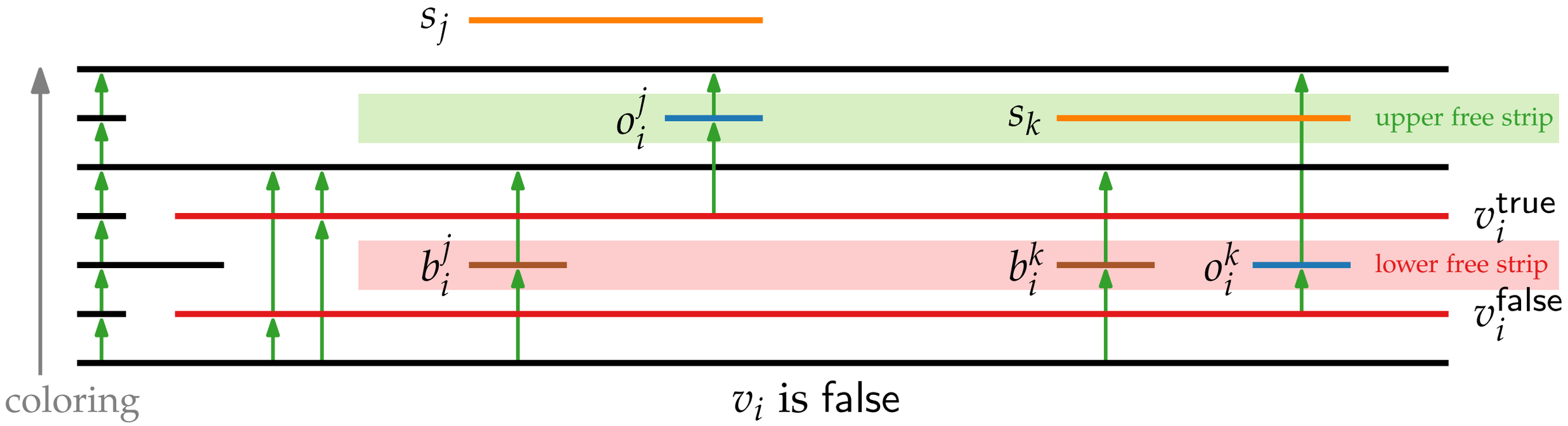

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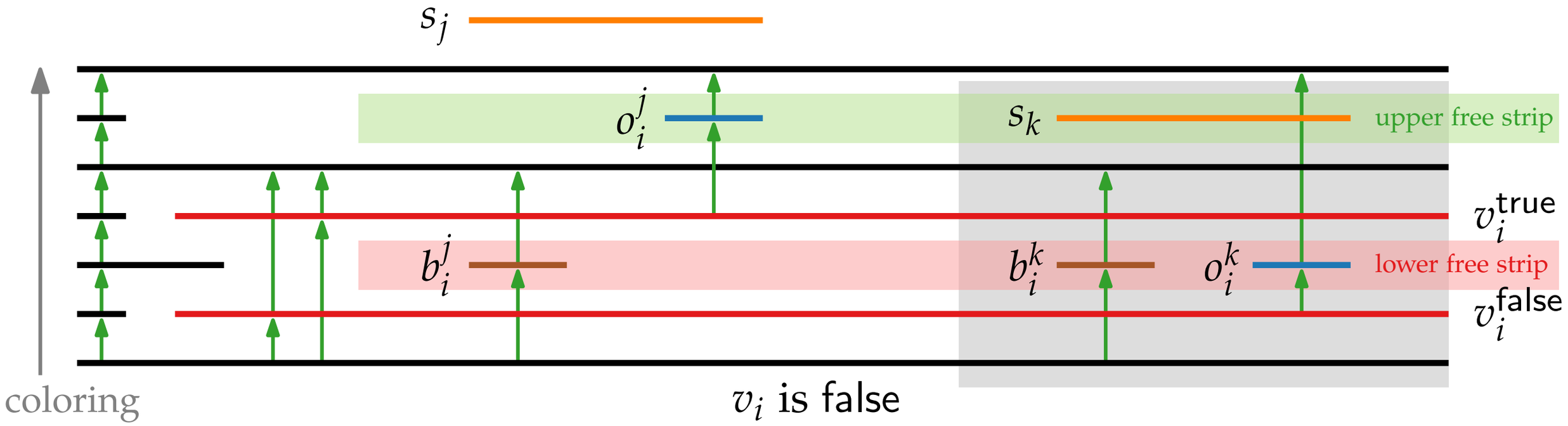

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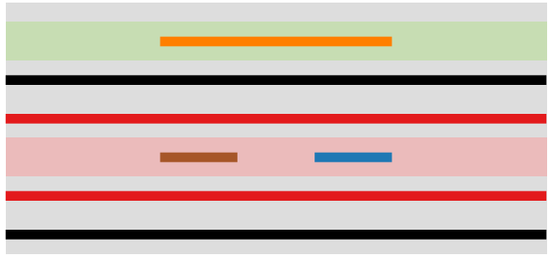

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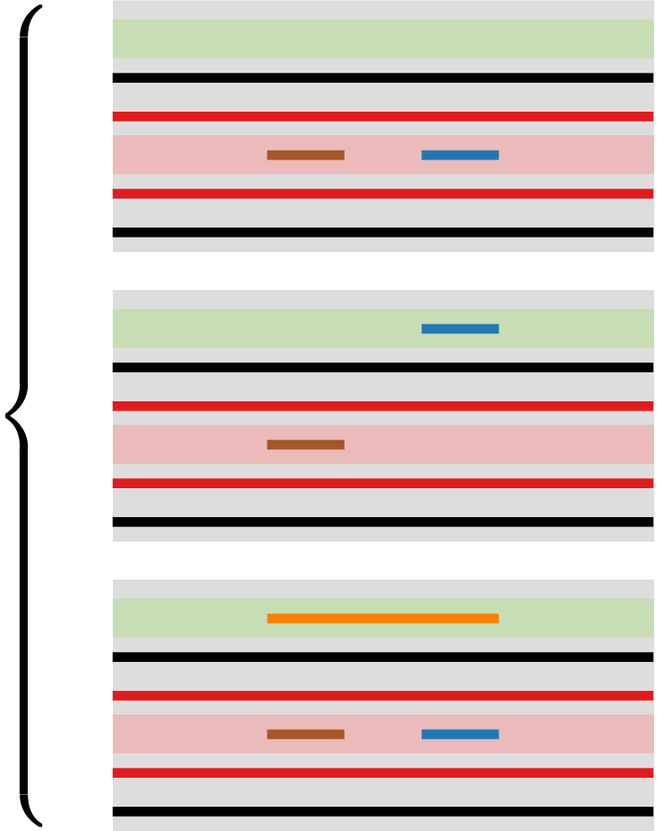

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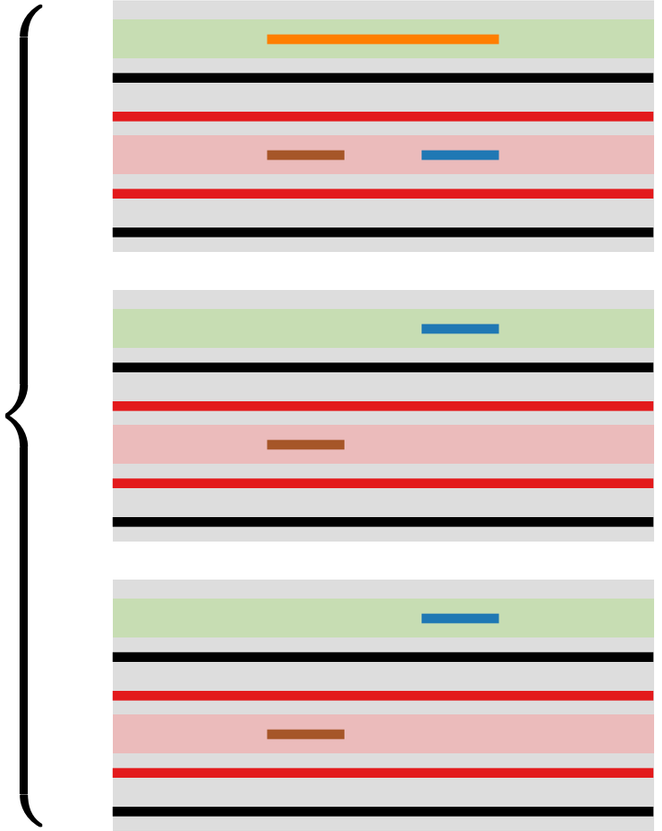

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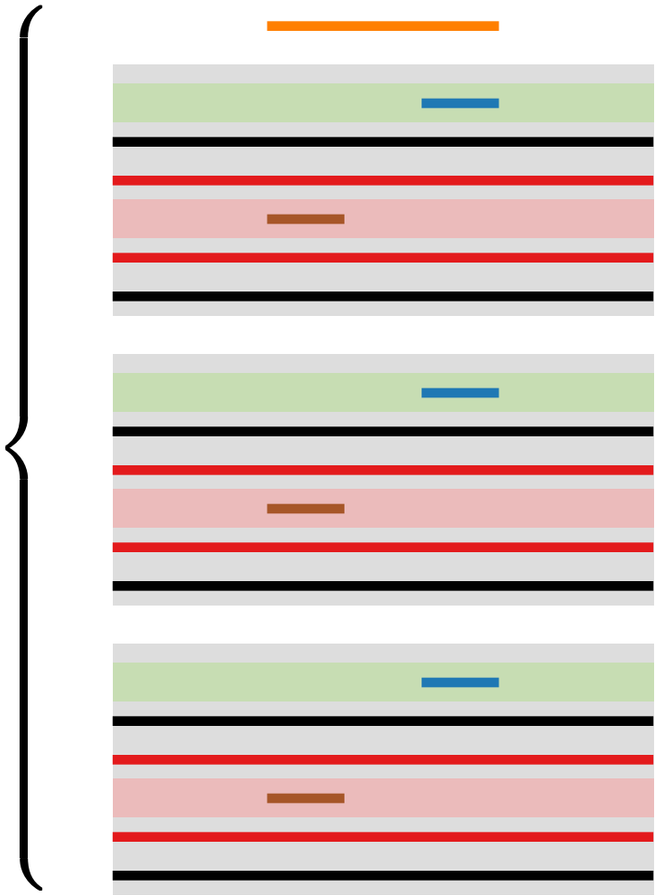

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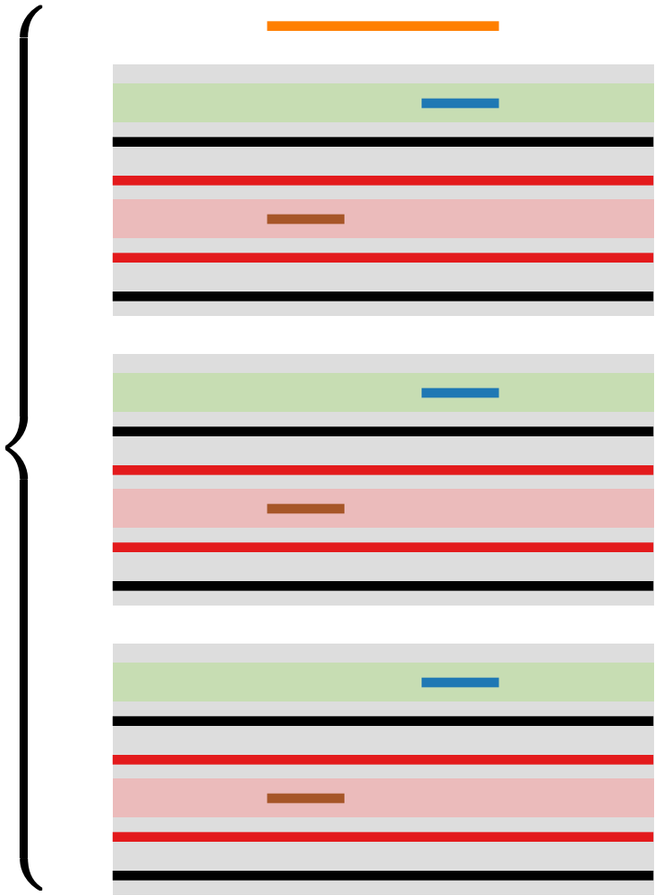

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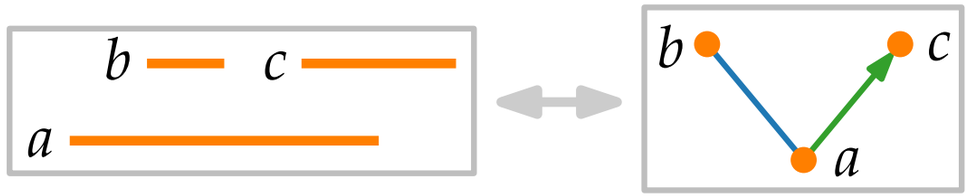
Φ is satisfiable
$\Leftrightarrow$ G_Φ admits a coloring with $6n$ colors. $\square$

Conclusion and Open Problems



- We have introduced the natural concept of directional interval graphs.

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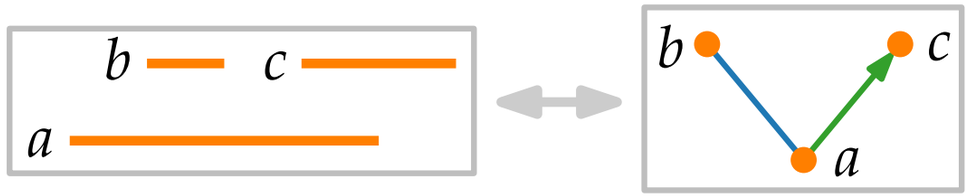

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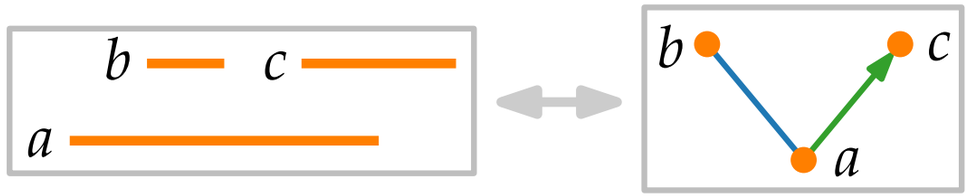

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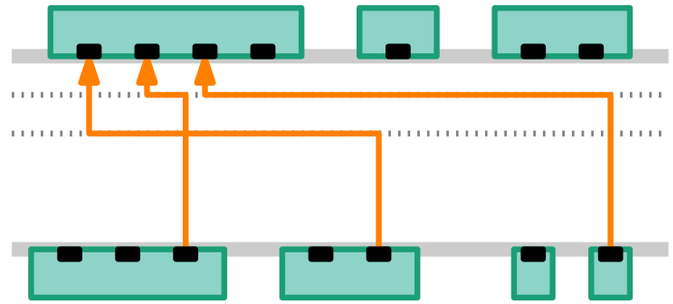

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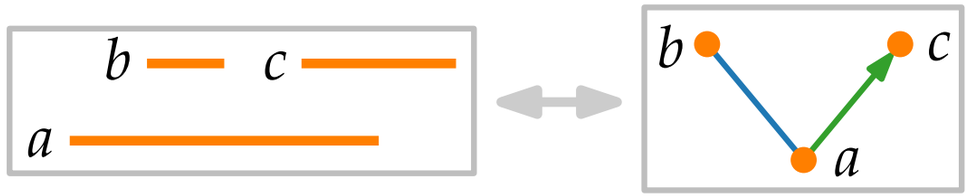

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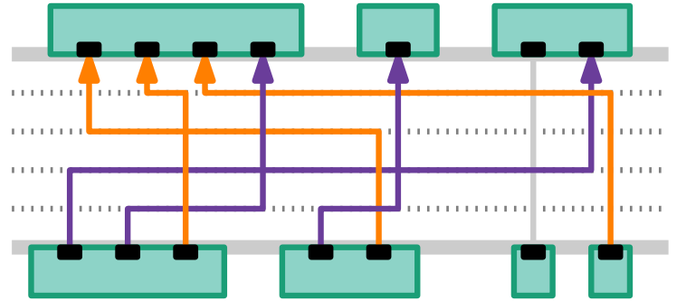

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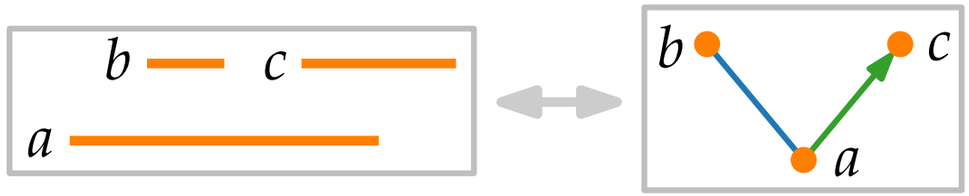

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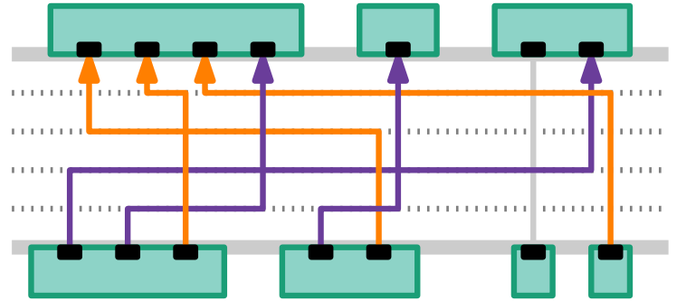

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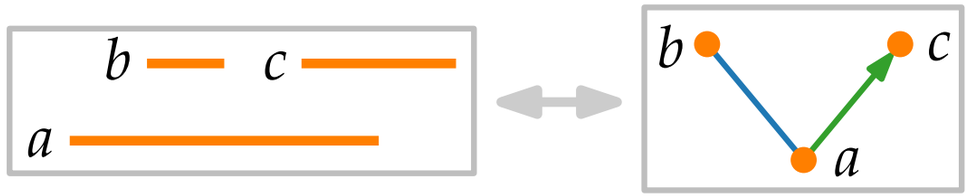

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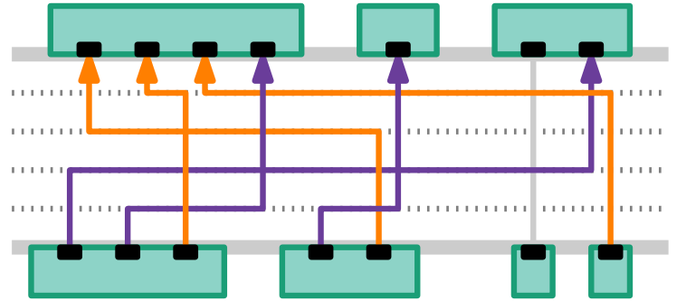

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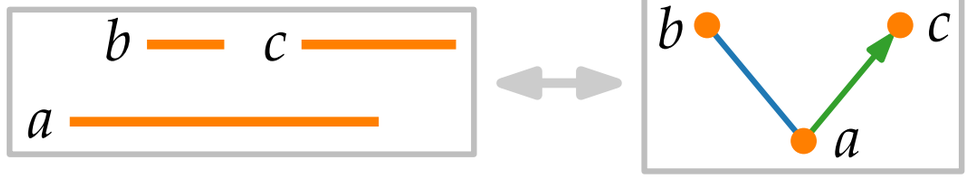

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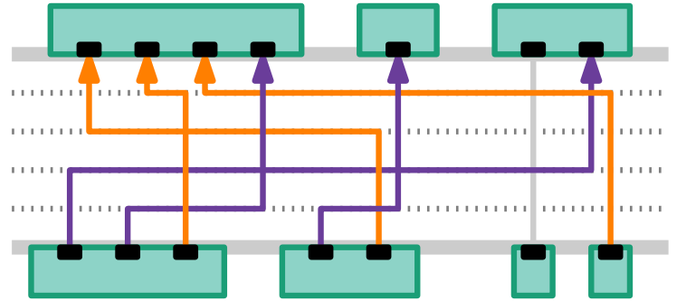

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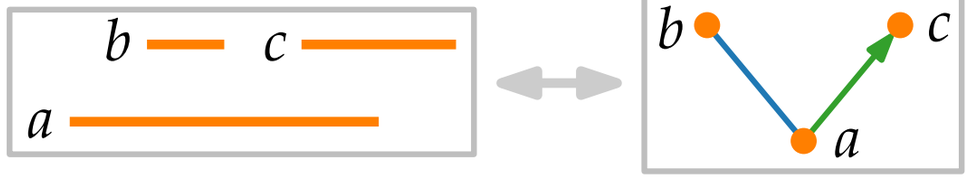

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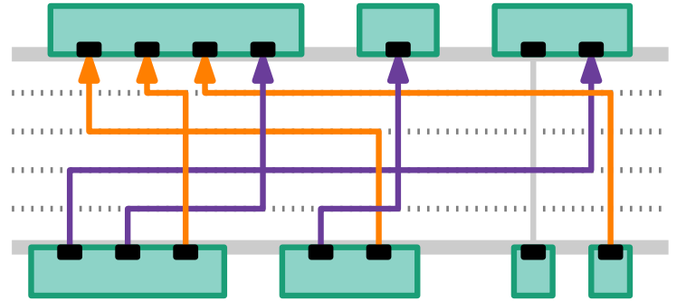

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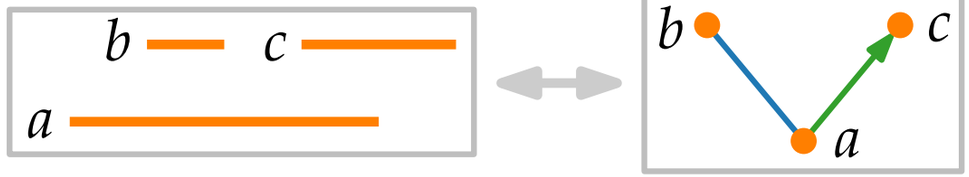

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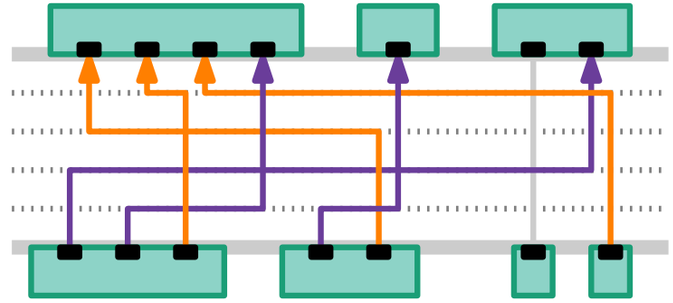

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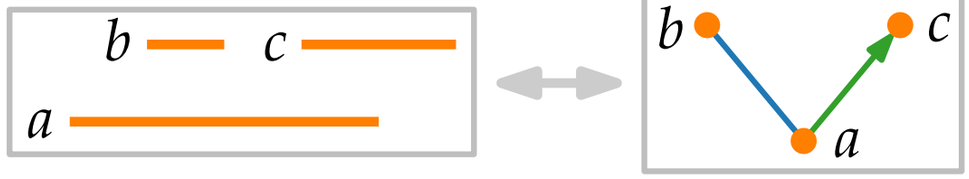

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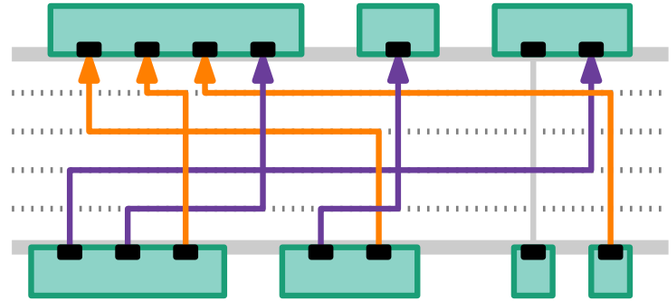

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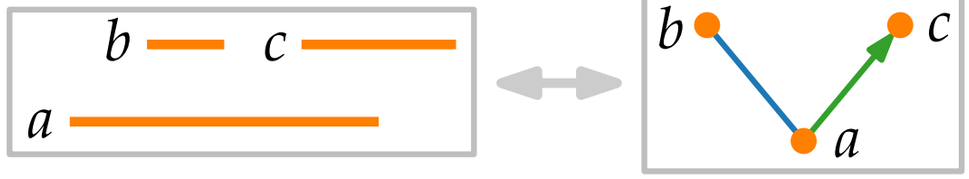

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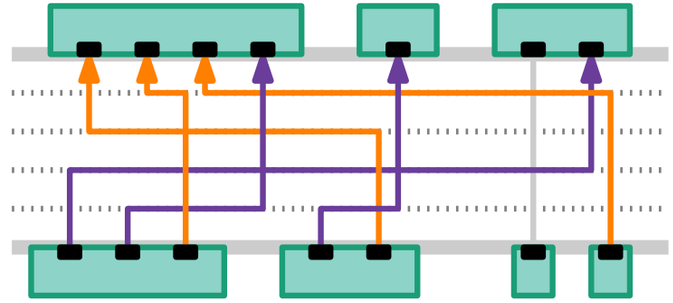

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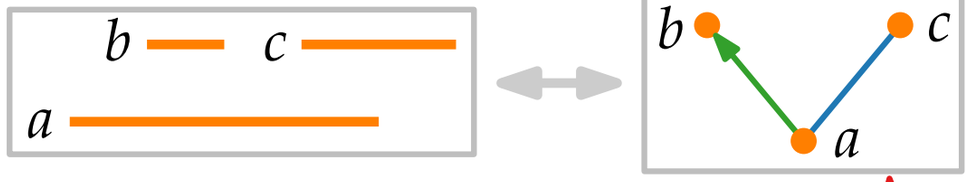

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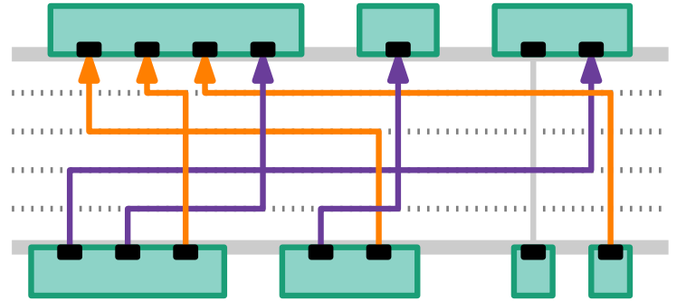

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Coloring and Recognizing Mixed Interval Graphs

ISAAC 2023, Kyoto

Grzegorz Gutowski — Uniwersytet Jagielloński Kraków

Konstanty Szaniawski — Uniwersytet Jagielloński Kraków

Felix Klesen — Julius-Maximilians-Universität Würzburg

Paweł Rzążewski — Warsaw University of Technology

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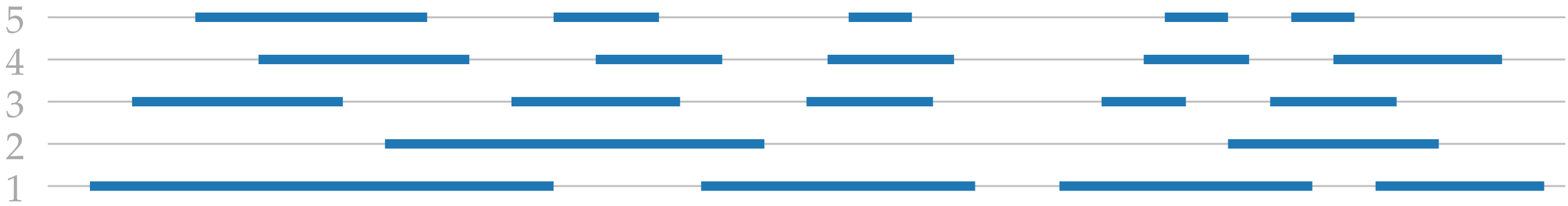
Johannes Zink — Warsaw University of Technology

Some Observation about Interval Containment Graphs



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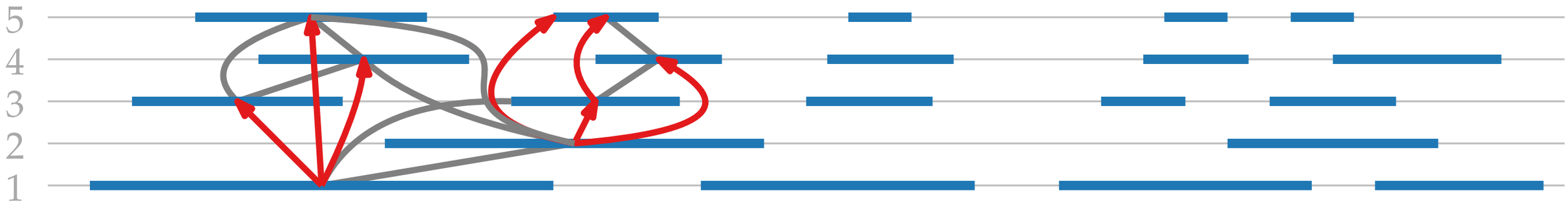

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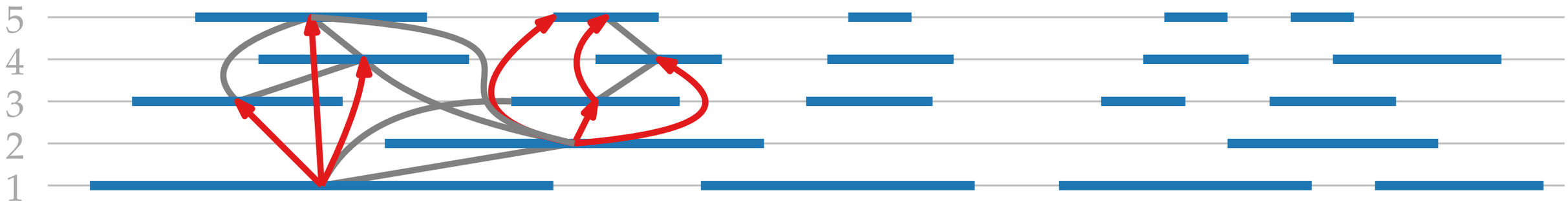

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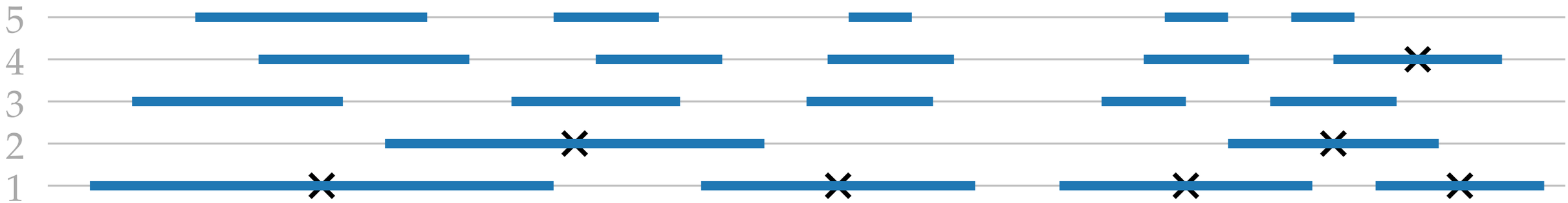

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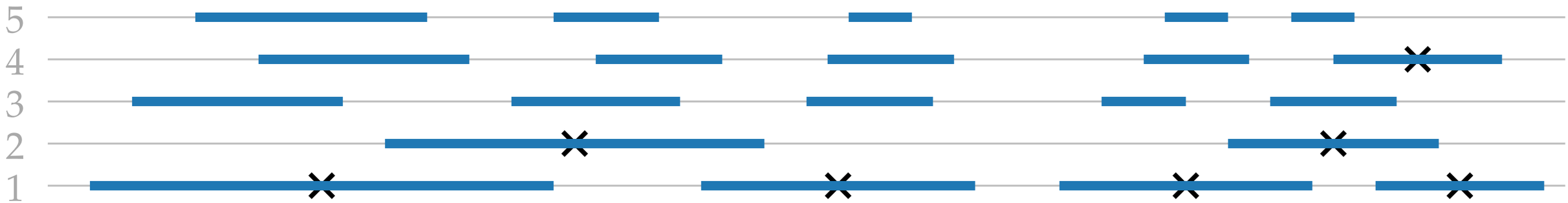

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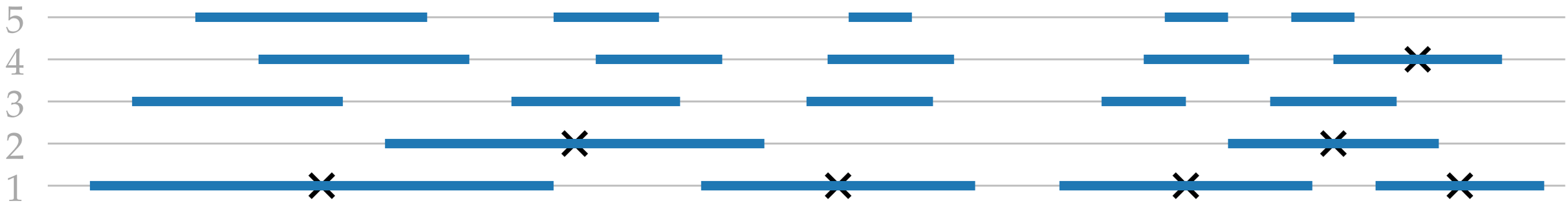

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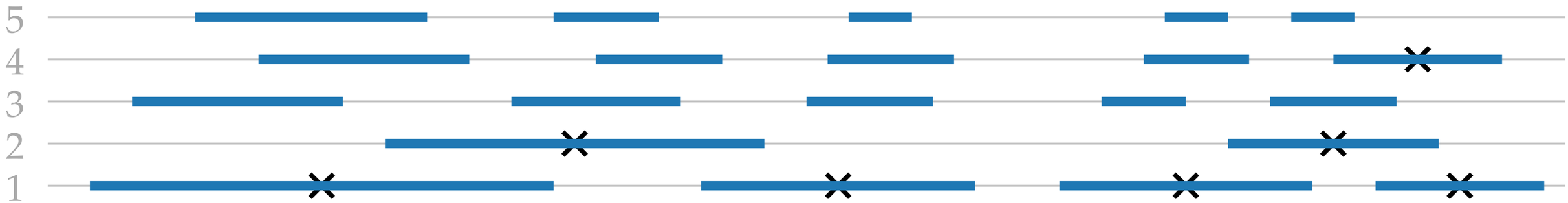

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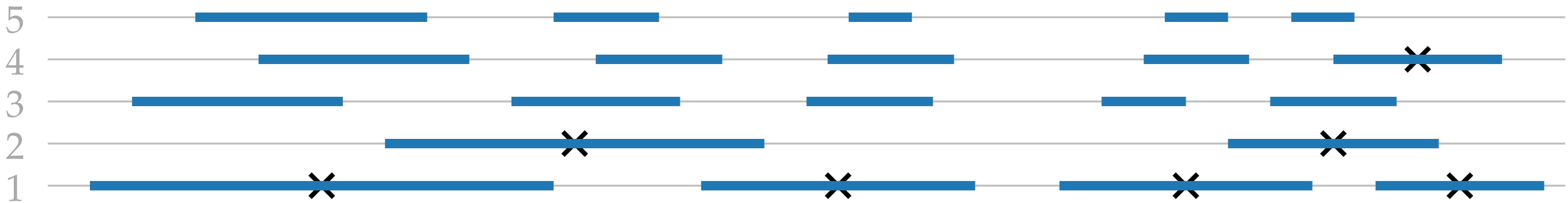

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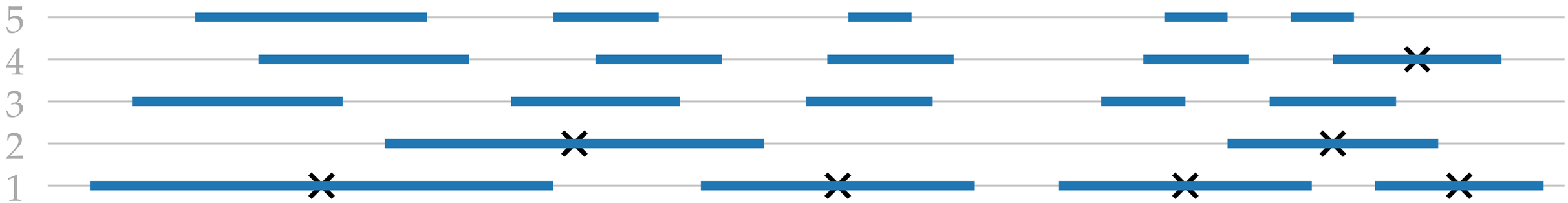

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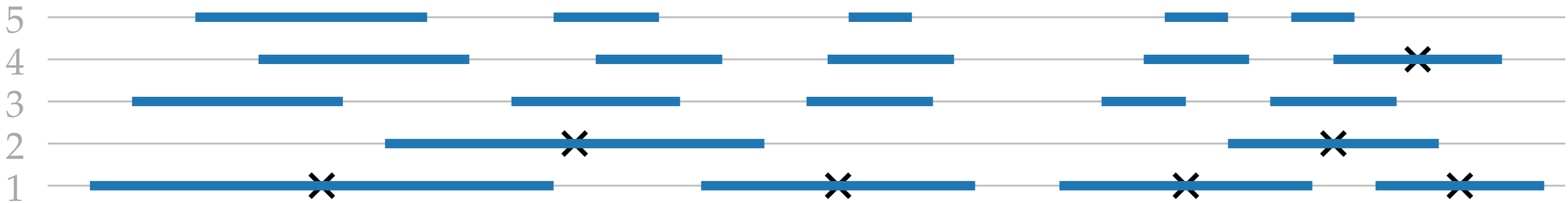

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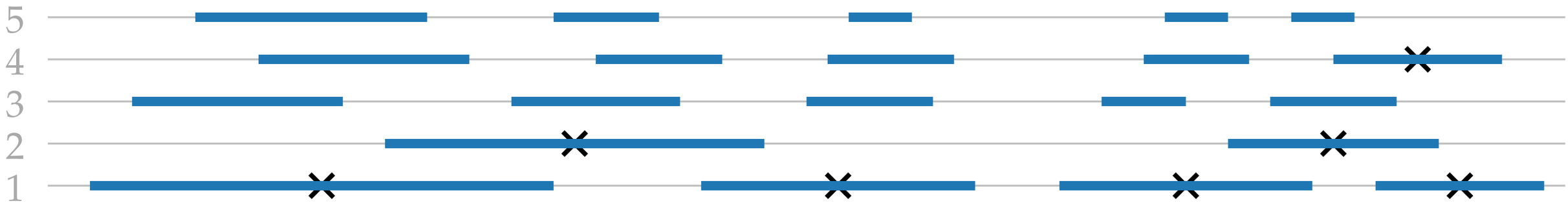

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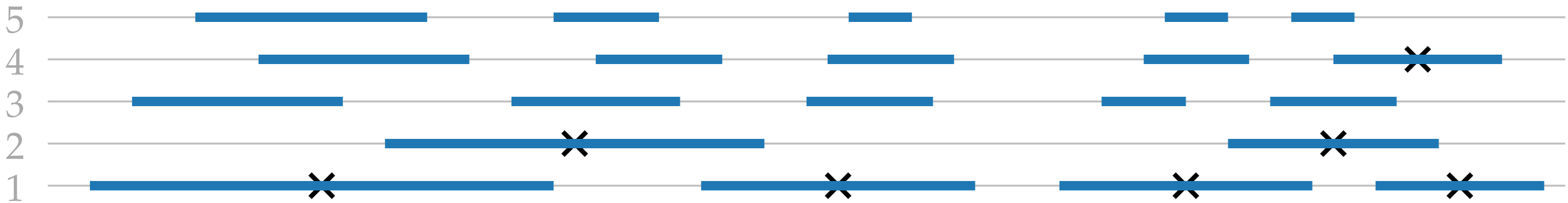

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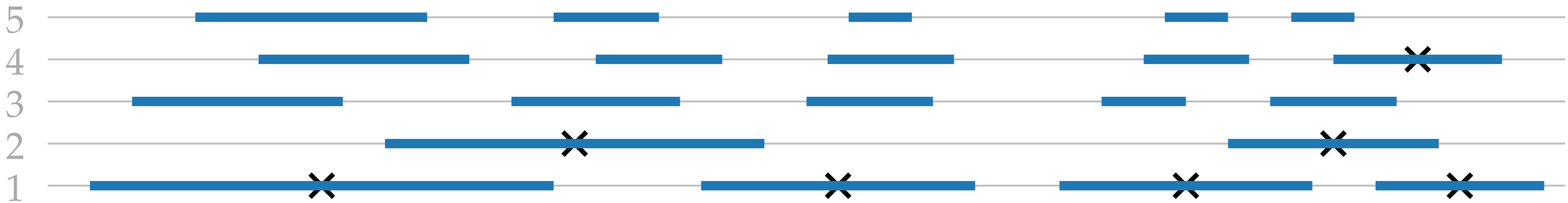

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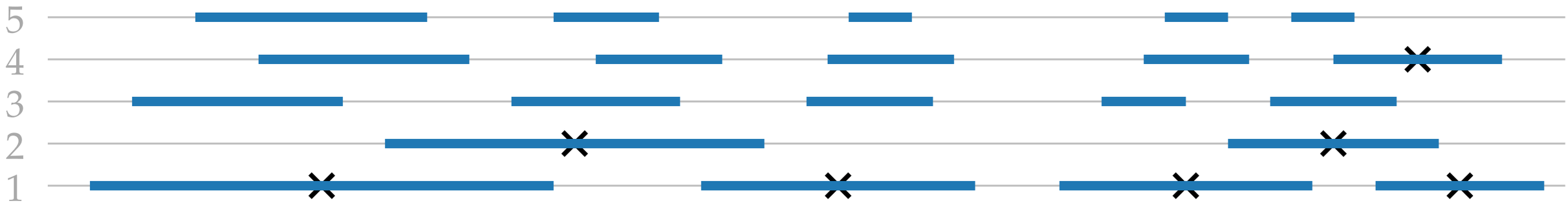
Some Observation about Interval Containment Graphs



Let $\mathcal{I}$ be a set of intervals. Let $G = \mathcal{C}[\mathcal{I}]$ be the containment graph induced by $\mathcal{I}$.
Let $M(\mathcal{I})$ be the set of inclusion-wise maximum elements in $\mathcal{I}$.

Then $\mathcal{C}[M(\mathcal{I})]$ is a *proper* interval graph – no interval contains another interval.

Also note that $\bigcup M(\mathcal{I}) = \bigcup \mathcal{I}$.

Let R be an inclusion-wise minimal subset of $M(\mathcal{I})$ such that $\bigcup R = \bigcup \mathcal{I}$.

Claim. $\mathcal{C}[R]$ is an undirected linear forest.

Proof. $\mathcal{C}[R]$ is proper $\Rightarrow$ contains no induced $K_{1,3}$ and no induced C_ℓ for $\ell \geq 4$.
It remains to show that $\mathcal{C}[R]$ contains no triangle.

A 2-Approximation Algorithm for Coloring

Theorem. For any set $\mathcal{I}$ of intervals,
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With f_1 and f_2, we construct a proper coloring f of G using colors $\{1, \ldots, 2\omega - 1\}$.

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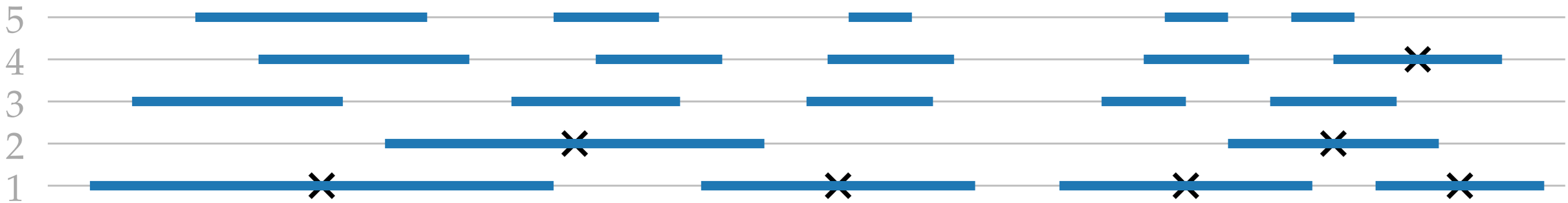

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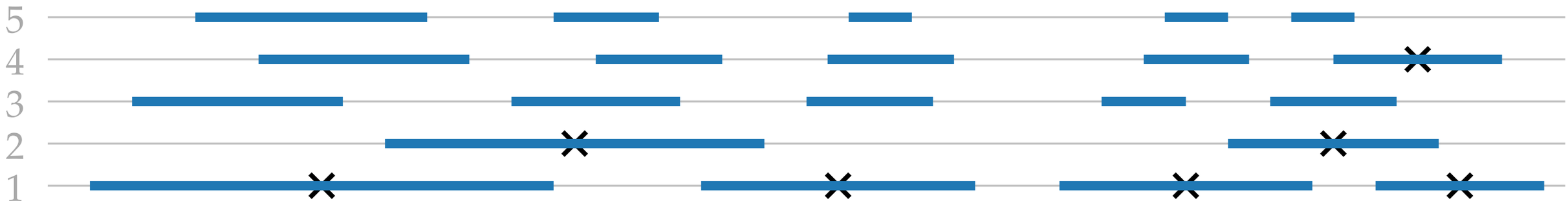

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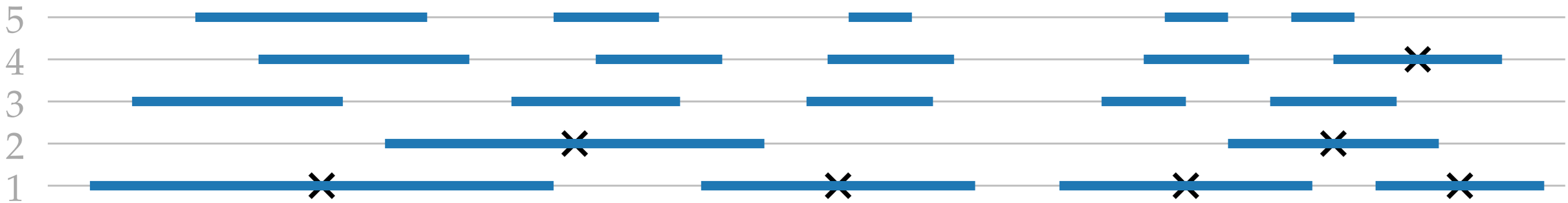

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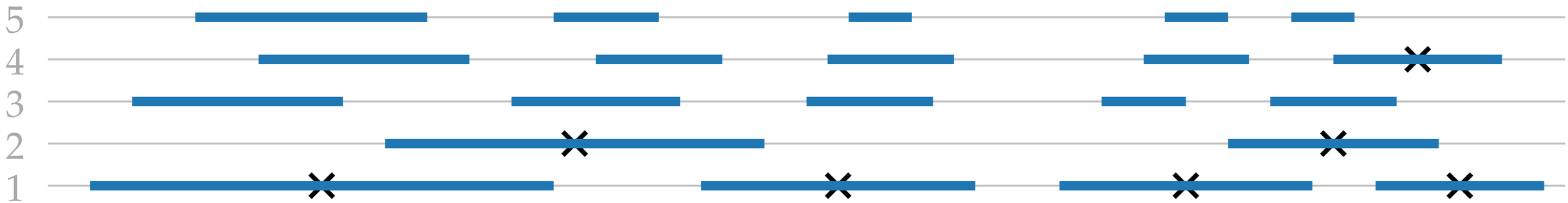

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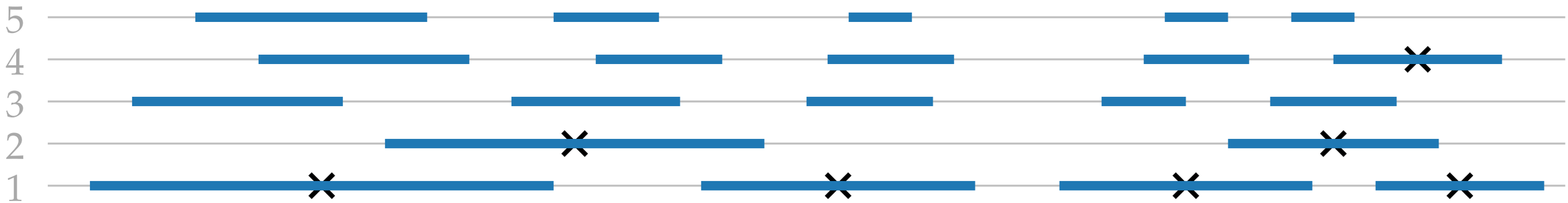

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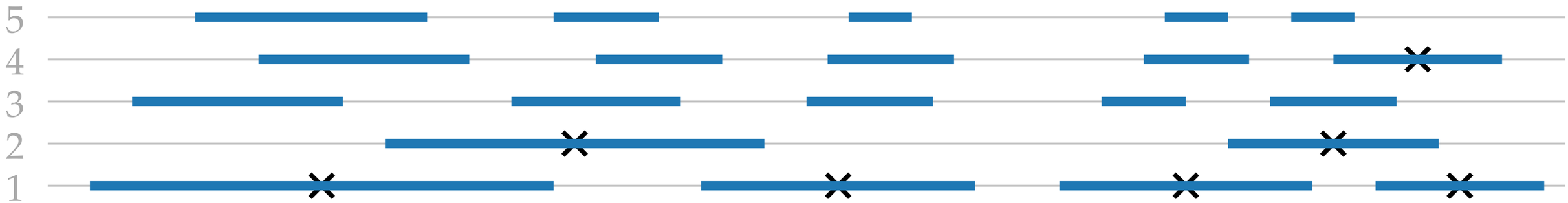

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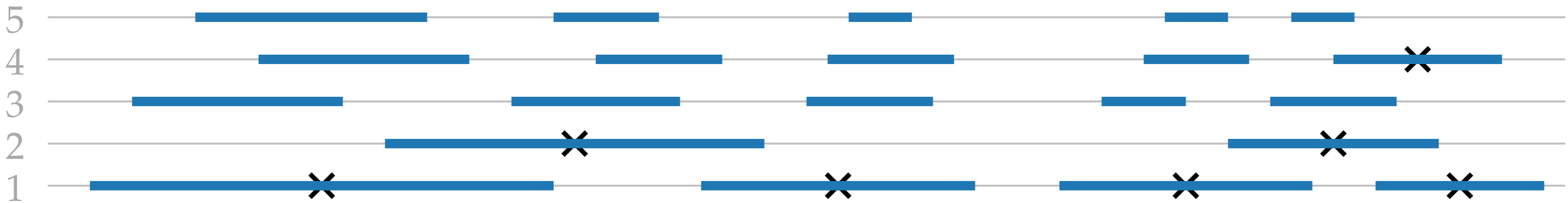

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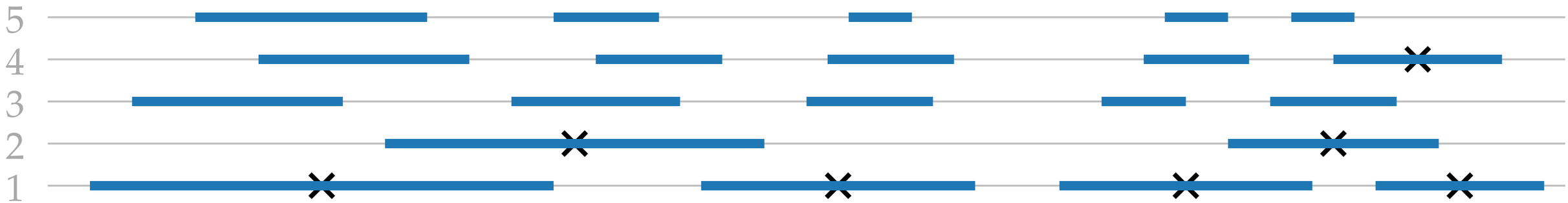

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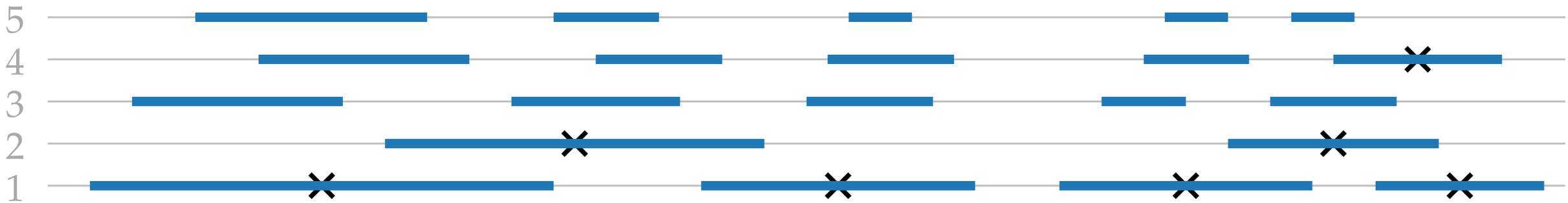

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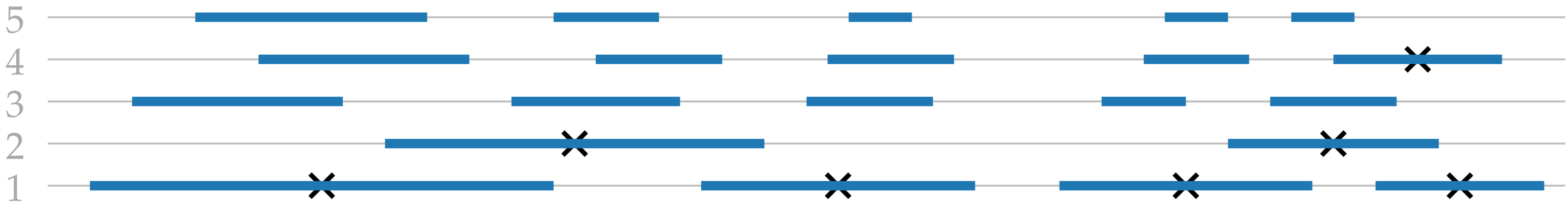

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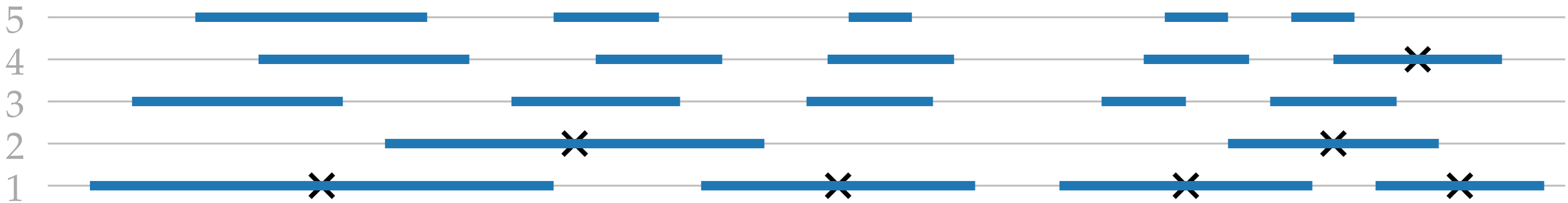

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Corollary. There is a 2-approximation for coloring interval containment graphs properly. Given n intervals, the algorithm runs in $O(n \log n)$ time.

A Lower Bound Example

Proposition. There is an infinite family $(\mathcal{I}_n)_{n\geq 1}$ of sets of intervals with $|\mathcal{I}_n| = 3 \cdot 2^{n-1} - 2$, $\chi(\mathcal{C}[\mathcal{I}_n]) = 2n - 1$, and $\omega(\mathcal{C}[\mathcal{I}_n]) = n$.

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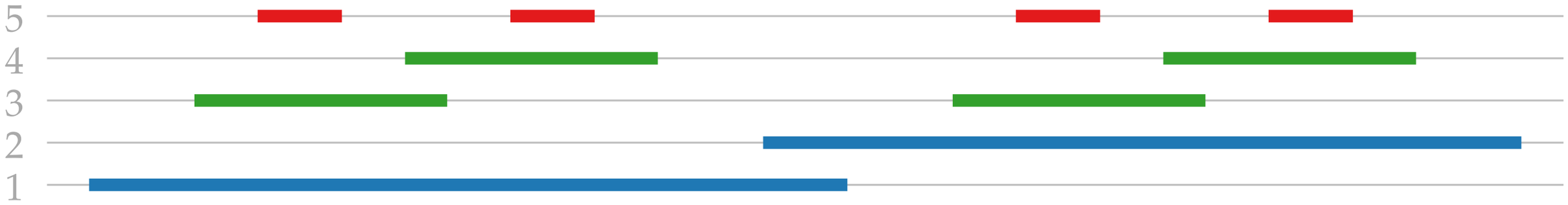

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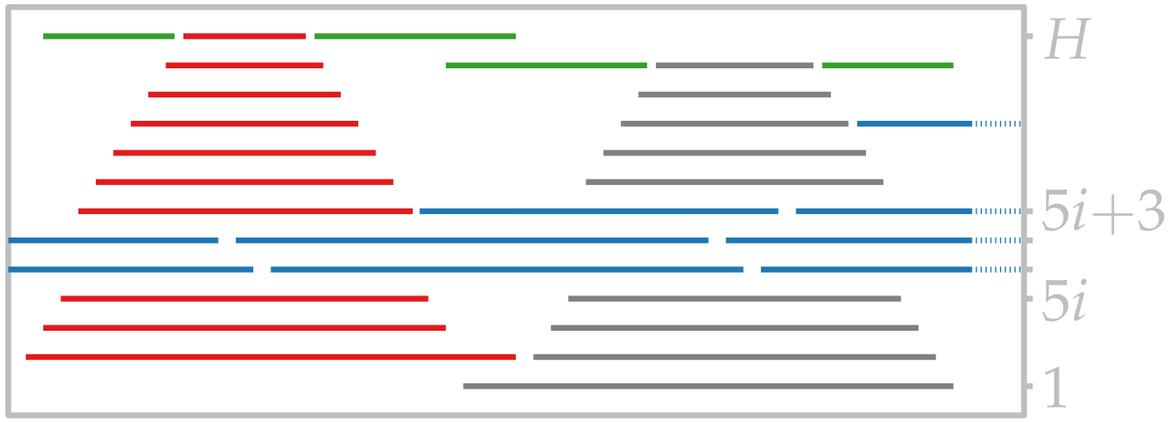

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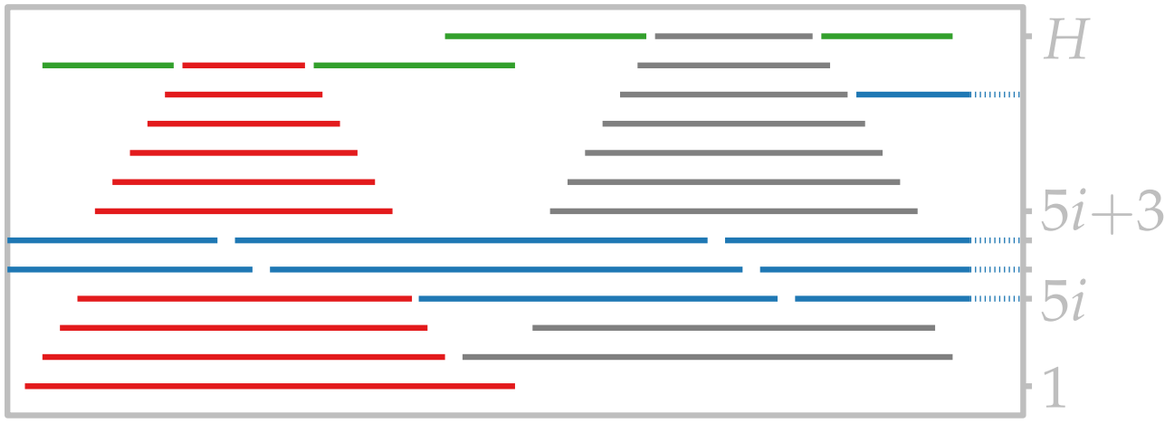

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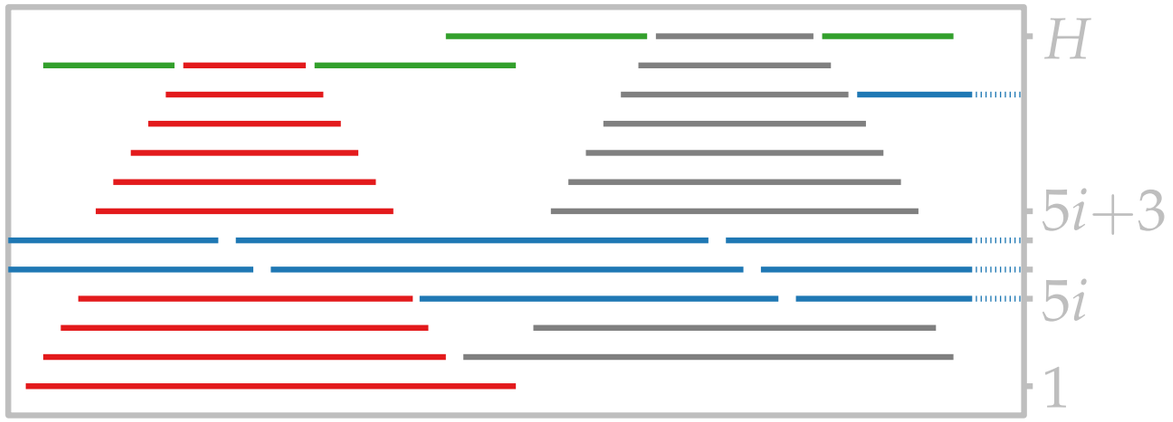

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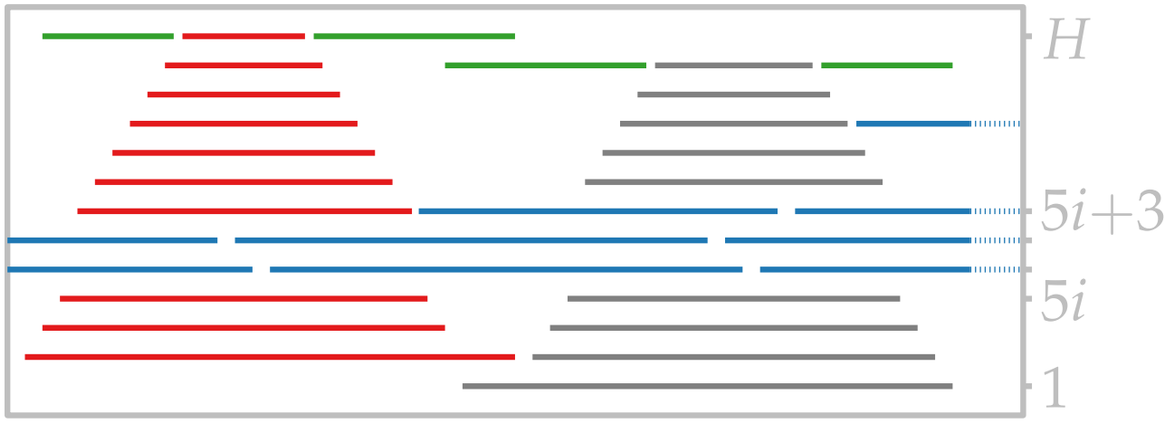

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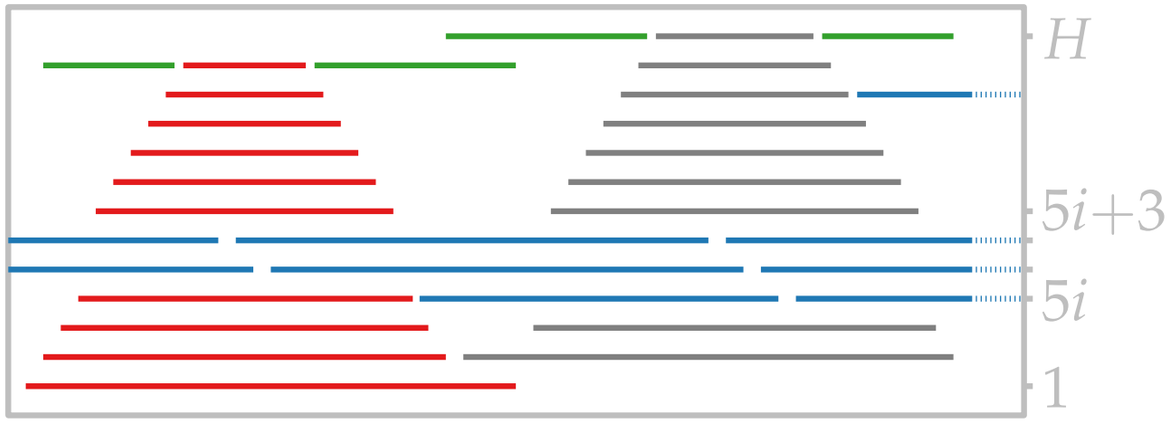

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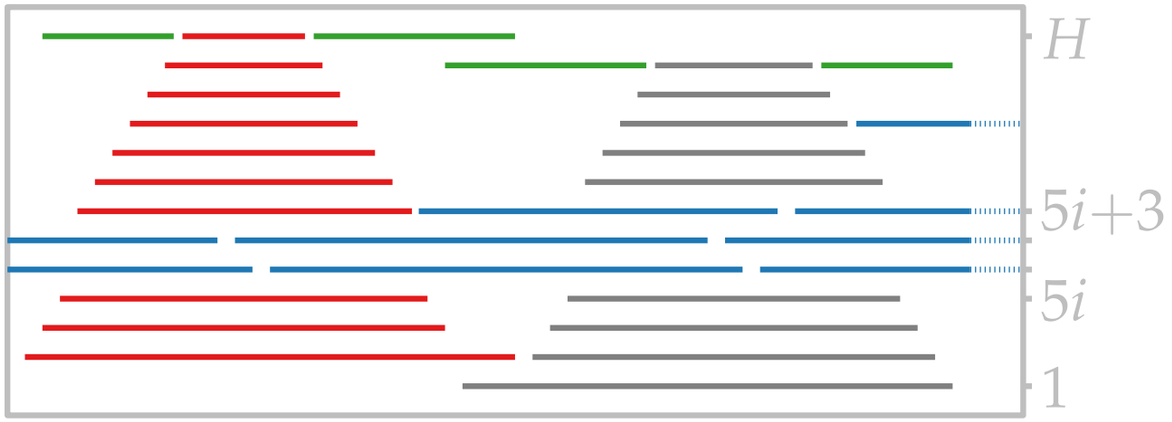
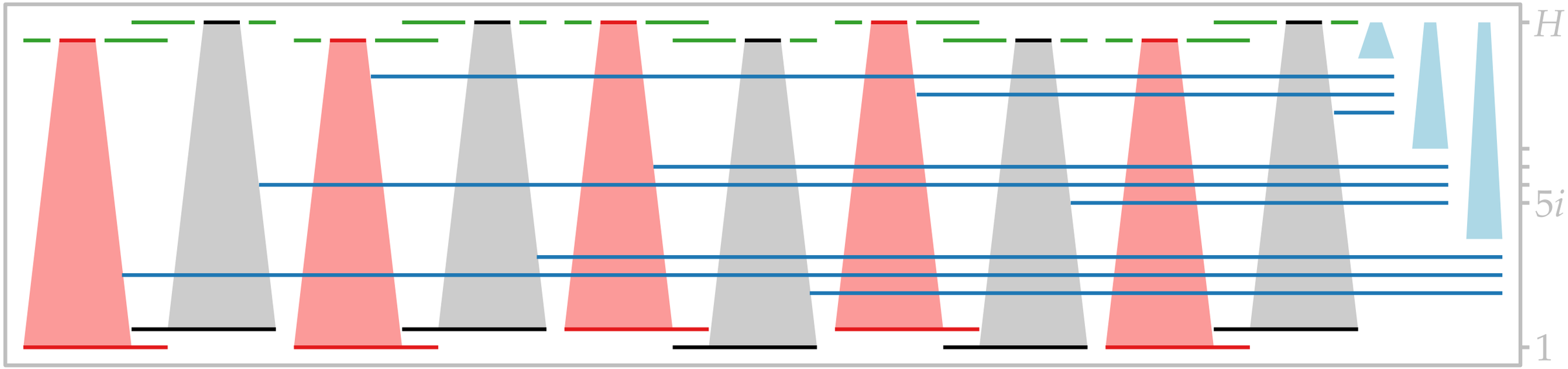


x true

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be an instance of 3-SAT with variables $\{x_1, x_2, \ldots, x_n\}$, and let $H = 5m + 1$. We construct a set $\mathcal{I}_\varphi$ of intervals.

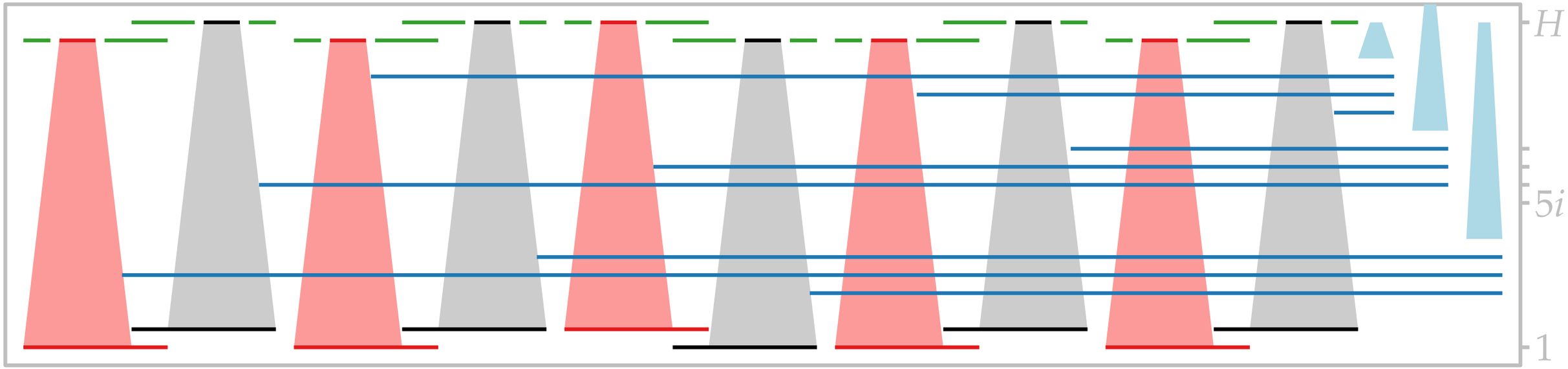
Clause Gadget



x_1 false x_2 false x_3 true x_4 true x_5 false

Example for $(\neg x_2 \vee \neg x_4 \vee x_5) \wedge (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3)$.

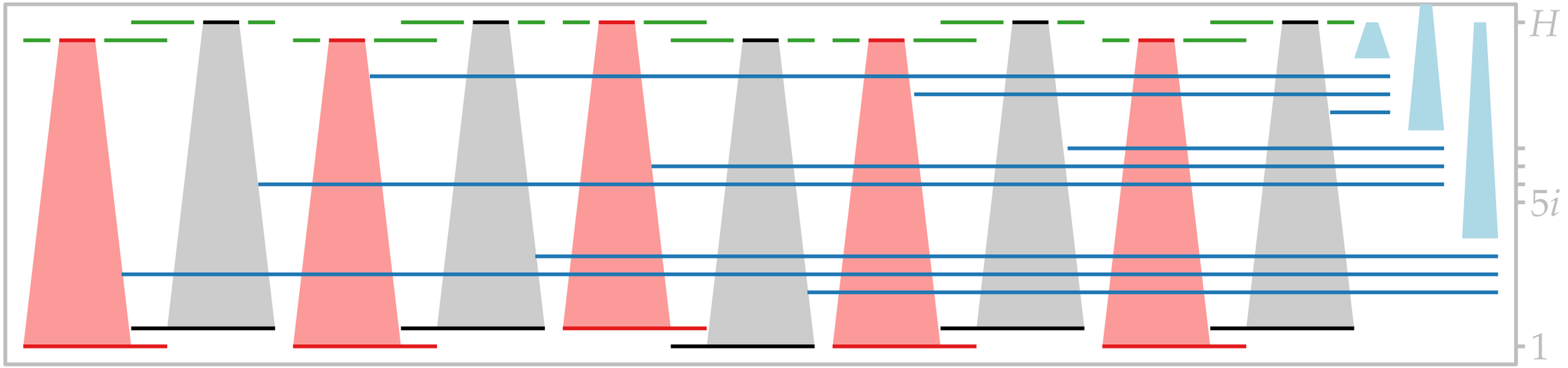
Clause Gadget



x_1 false x_2 false x_3 true x_4 false x_5 false

Example for $(\neg x_2 \vee \neg x_4 \vee x_5) \wedge (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3)$.

Clause Gadget



x_1 false x_2 false x_3 true x_4 false x_5 false

Example for $(\neg x_2 \vee \neg x_4 \vee x_5) \wedge (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3)$.

The graph $\mathcal{C}[\mathcal{I}_\varphi]$ admits a proper coloring with H colors $\Leftrightarrow$ φ is satisfiable. $\square$

Summary

Mixed interval graph class	Coloring complexity	lower bound	upper bound	approximation	Recognition
containment	NP-hard	$2\omega-1$	$2\omega-1$	2	$O(nm)$
directional	$O(n \log n)$			1	$O(n^2)$
bidirectional	NP-hard			2	open
general	NP-hard	$(\lambda+2)\omega/2$	$(\lambda+1)\omega$	$\min\{\omega, \lambda+1\}$	$O(n+m)$ [LB79]

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