Coloring Mixed and Directional Interval Graphs

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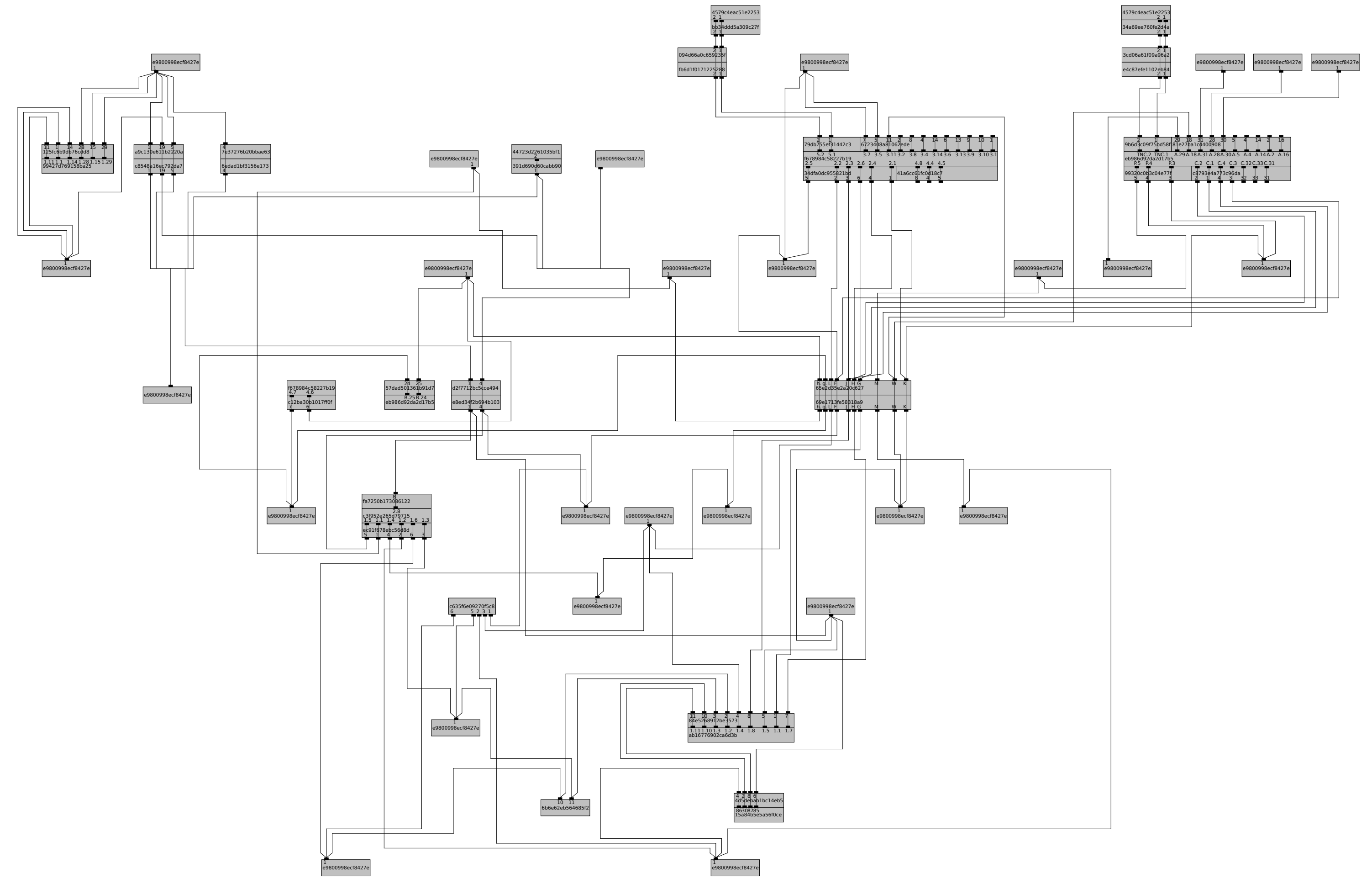
Motivation

Framework for layered g

Input: directed graph G

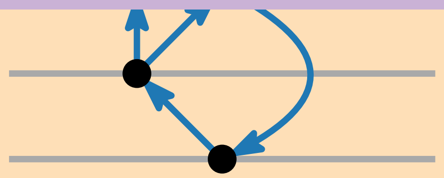
Consists of five phases:

1. cycle elimination
2. layer assignment
3. crossing minimiza
4. node placement
5. edge routing



cable plan

[Zink, Walter, Baumeister, Wolff; CGTA'22]

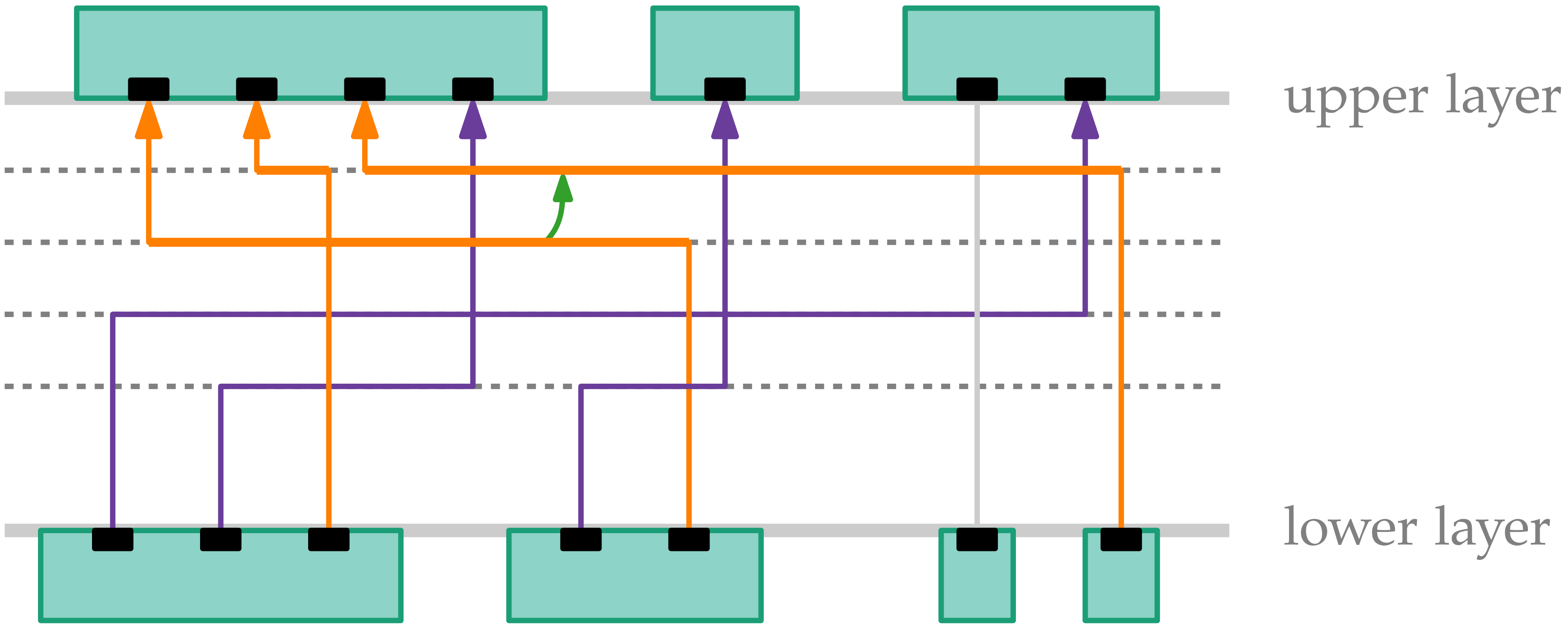


we want orthogonal edges!

Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges
- only edges going in the same direction and overlapping partially in x-dimension can cross twice

  ⇒ induce a vertical order for the horizontal middle segments

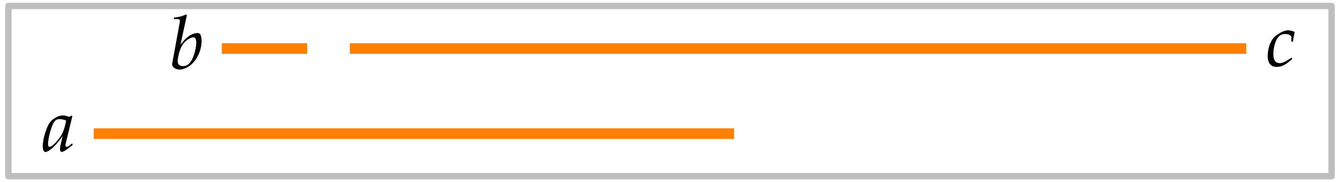


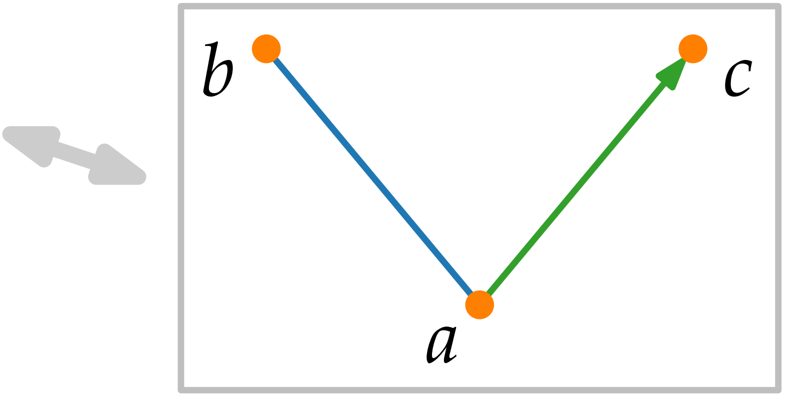
Definition – Directional Interval Graphs

Interval representation: set of intervals

Directional interval graph:

- vertex for each interval
- undirected edge if one interval contains another
- directed edge (towards the right interval) if the intervals overlap partially





Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
- ⋆ directed edge uv: $c(u) < c(v)$,
- ⋆ $\max_{v \in V} c(v)$ is minimized.

Interval graphs (no directed edges):

- coloring in linear time by a greedy algorithm

Directional interval graphs: **our contribution**

- recognition in $O(n^2)$ time ← not in this talk
- coloring in $O(n \log n)$ time by a greedy algorithm

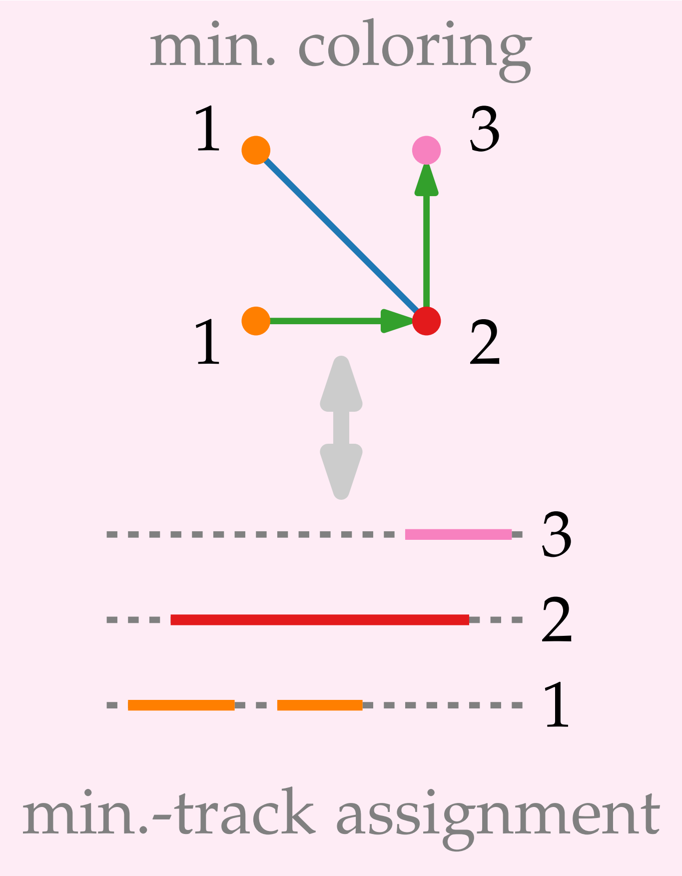
Mixed interval graphs:

- coloring is NP-complete

agenda for this talk

Directed graphs (only directed edges):

- coloring in linear time using topological sorting



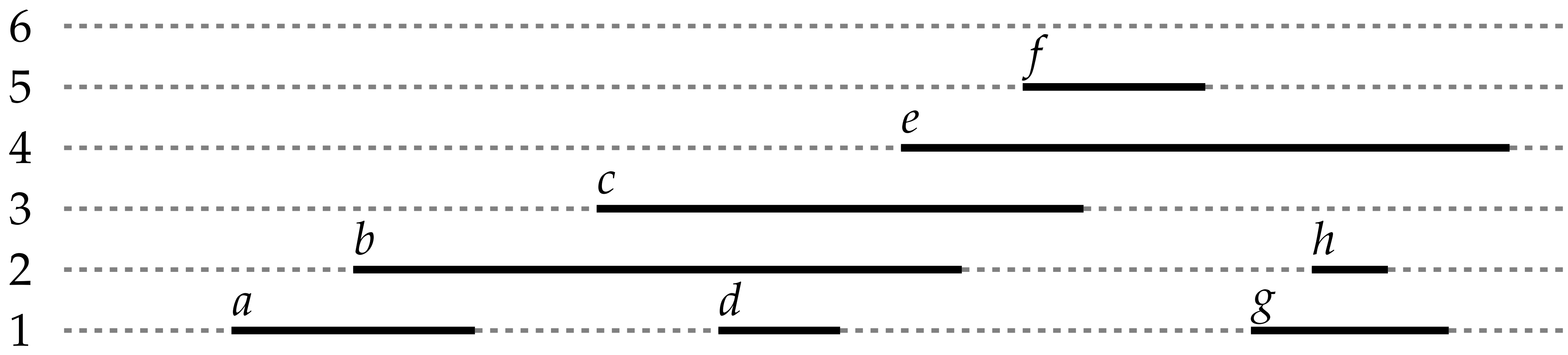
$n :=$ # intervals

Coloring Directional Interval Graphs

Given: an interval representation of a directional interval graph G

`GreedyColoring:`

1. sort all intervals by left endpoint
2. for each interval, assign the smallest available color respecting incident edges



Coloring Directional Interval Graphs

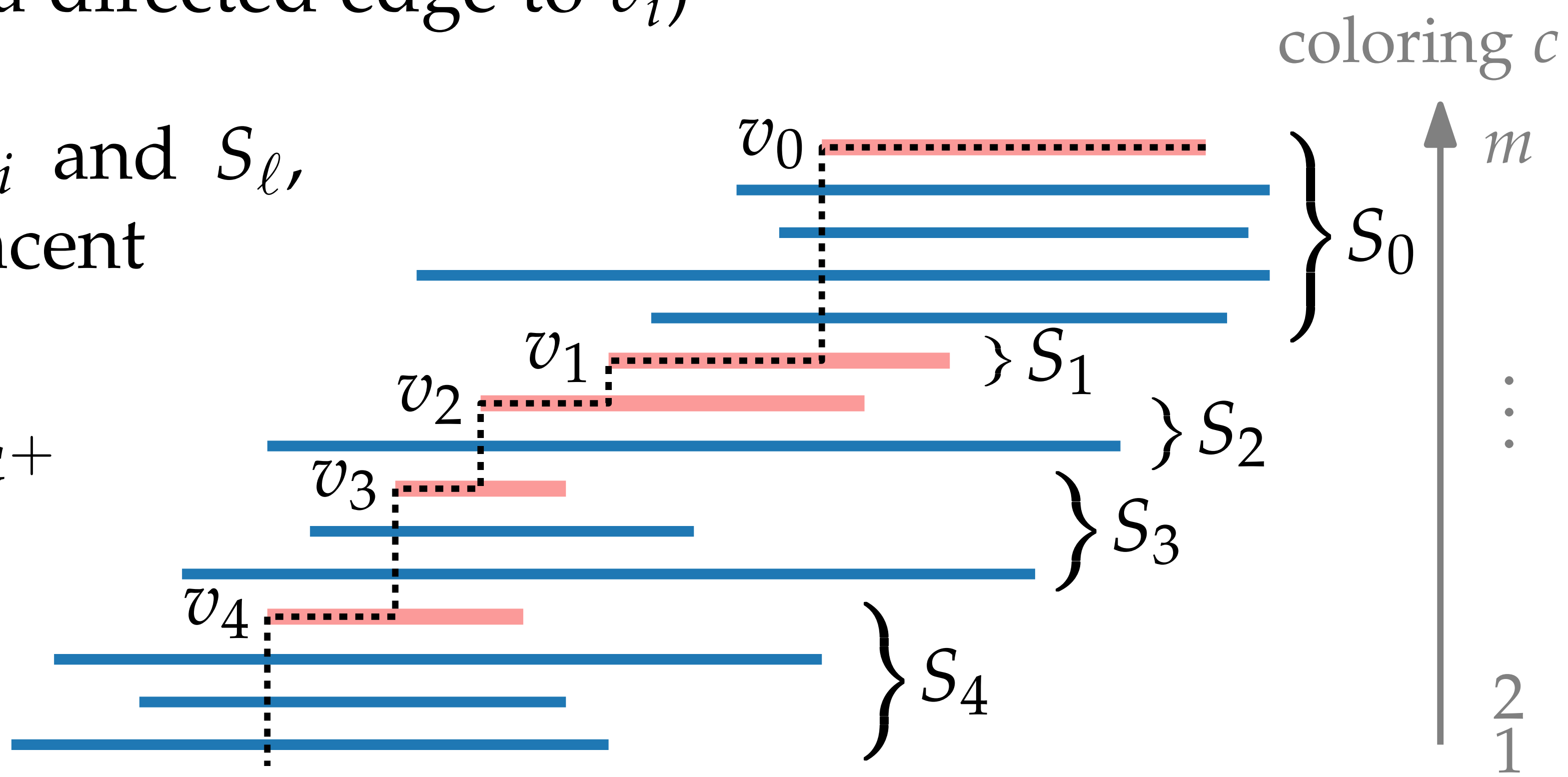
Theorem 1:
A coloring c computed by `GreedyColoring` has the minimum number of colors.

Proof sketch:

- Clearly, for each $S_i \setminus \{v_i\}$, all intervals contain v_i. (otherwise they would have a directed edge to v_i)
- **Claim:** for any two steps S_i and S_ℓ, every pair of intervals is adjacent in the transitive closure G^+.

  $\Rightarrow$ $S = \bigcup S_i$ is a clique in G^+

  $\Rightarrow$ S alone requires m colors in G $\square$



Proof of the Claim

Claim: Any two intervals $u \in S_i$ and $w \in S_\ell$ are adjacent in G^+.

Proof. W.l.o.g., $u \cap w = \emptyset$ and $i < \ell$.

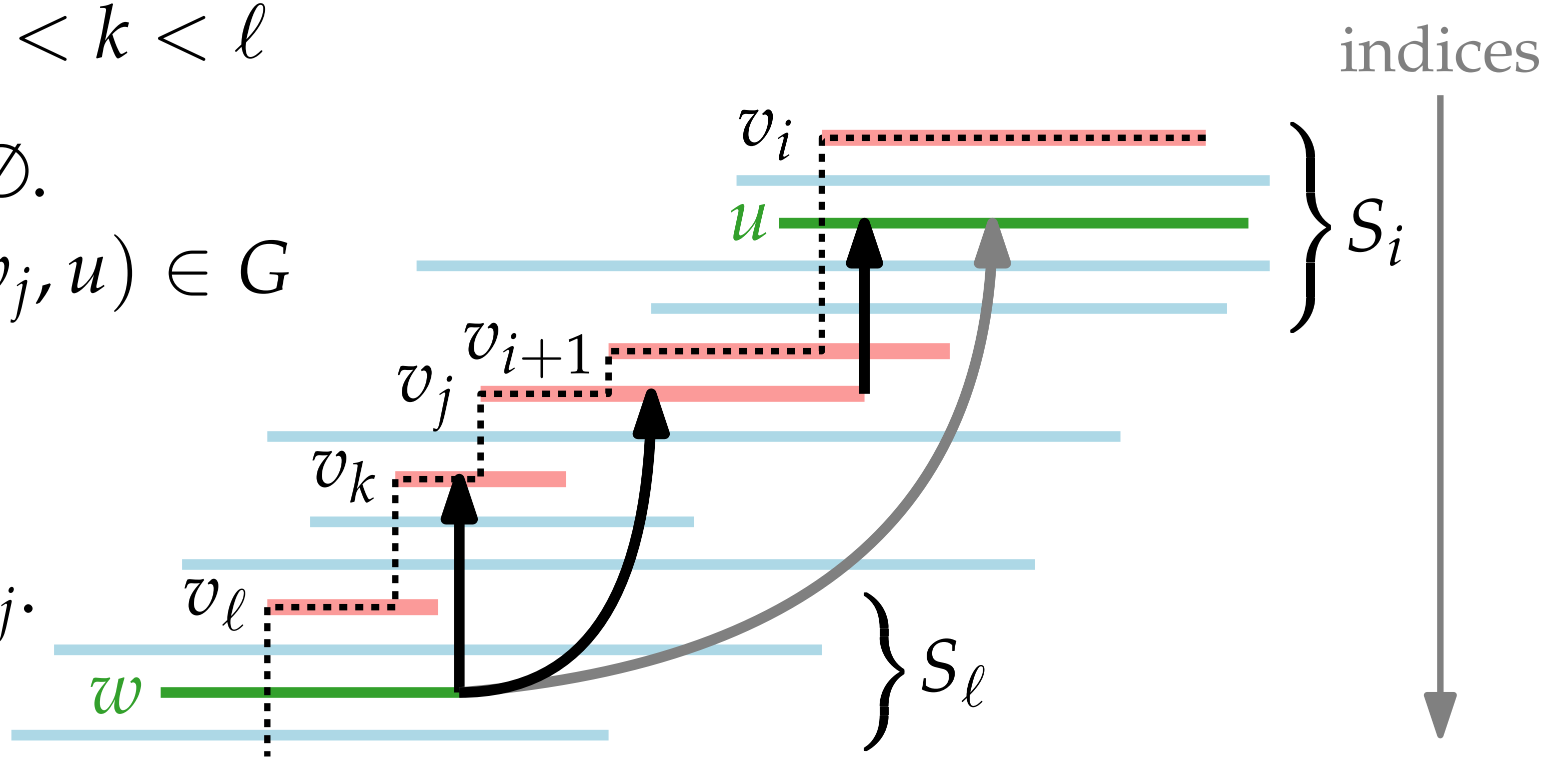
Let j be the largest index s.t. $v_j \cap u \neq \emptyset$.
Let k be the smallest index s.t. $v_k \cap w \neq \emptyset$.

$$\begin{matrix} u \cap v_{i+1} \neq \emptyset \\ w \cap v_{\ell-1} \neq \emptyset \end{matrix} \quad \underset{u \cap w = \emptyset}{\Longrightarrow} \quad \begin{matrix} i < j < \ell \\ i < k < \ell \end{matrix}$$

By definition, $u \cap v_{j+1} = \emptyset$.
$\Rightarrow$ u and v_j overlap $\Rightarrow (v_j, u) \in G$
Similarly, $(w, v_k) \in G$.

If $j < k$, then $(v_k, v_j) \in G$.
If $j \geq k$, then w overlaps v_j.

Transitivity $\Rightarrow$ claim.



Overview

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

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Interval graphs (no directed edges):

- coloring in linear time by a greedy algorithm

Directional interval graphs: **our contribution**

- recognition in $O(n^2)$ time
- coloring in $O(n \log n)$ time by a greedy algorithm

Mixed interval graphs:

- coloring is NP-complete

Directed graphs (only directed edges):

- coloring in linear time using topological sorting

$n :=$ # intervals

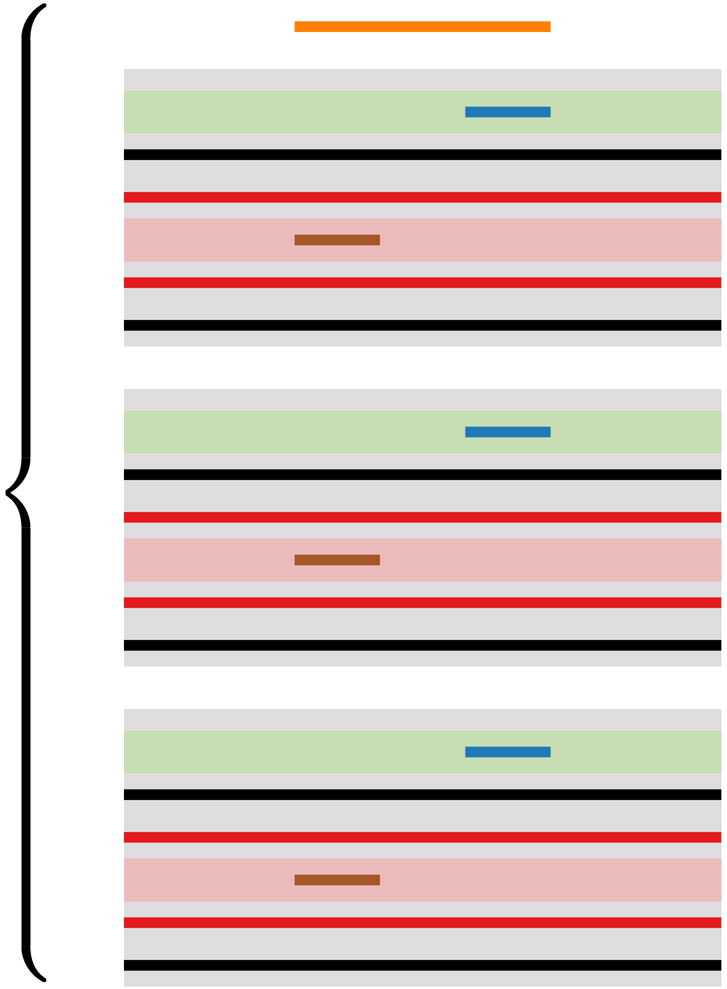
Coloring Mixed Interval Graphs

Theorem 2:
Deciding whether a mixed interval graph admits a k-coloring is NP-complete.

Proof sketch:

clause gadget:

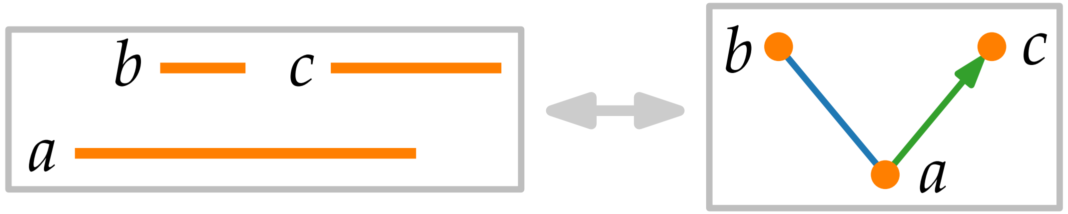
$6n + 1$ colors
($n :=$ # variables)



Φ is satisfiable $\Leftrightarrow$ G_Φ admits a coloring with $6n$ colors

□

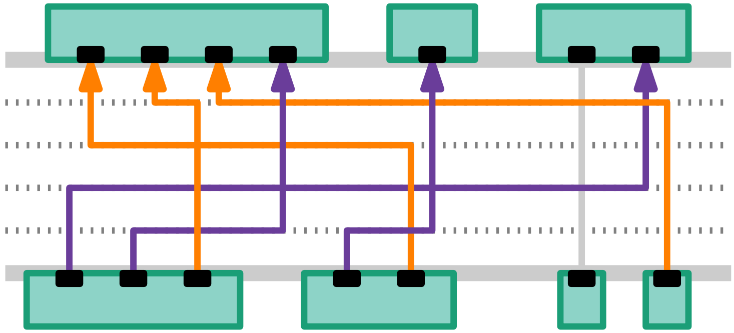
Conclusion and Open Problems



- We have introduced the natural concept of directional interval graphs.
- A simple greedy algorithm colors these graphs optimally in $O(n \log n)$ time.
  $n :=$ # vertices
- In layered graph drawing, this corresponds to routing “left-going” edges orthogonally to the fewest horizontal tracks. (Symmetrically “right-going”.)

⇒ Combining the drawings of left-going and right-going edges yields a 2-approximation for the number of tracks. (bidirectional interval graphs)

can we do better?



- In our paper, we present a constructive $O(n^2)$-time algorithm for recognizing directional interval graphs, which is based on PQ-trees.

  bidirectional?

- For the more general case of mixed interval graphs, coloring is NP-hard.
  (Remark: NP-hardness requires both directed and undirected edges.)