Coloring Mixed and Directional Interval Graphs

GD 2022, Tokyo

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UNIVERSITÄT PASSAU

Motivation

Framework for layered graph drawing by Sugiyama, Tagawa, and Toda (1981).

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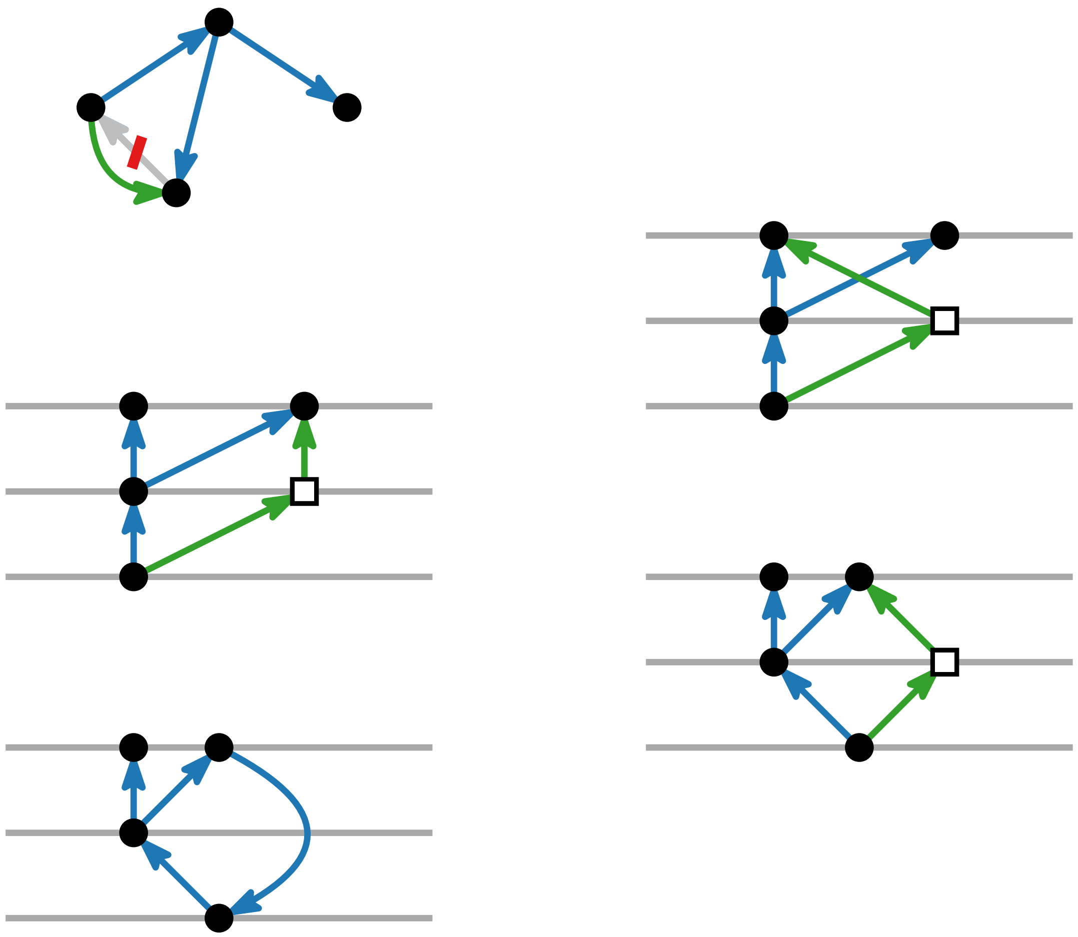
Consists of five phases:

1. cycle elimination
2. layer assignment
3. crossing minimization
4. node placement
5. edge routing



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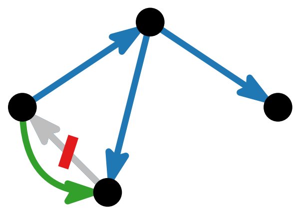
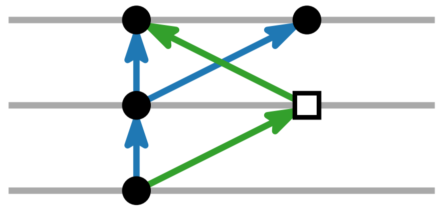
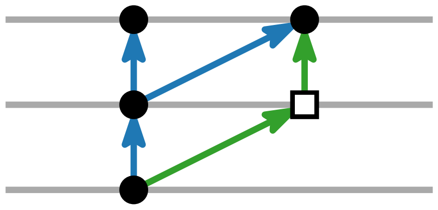


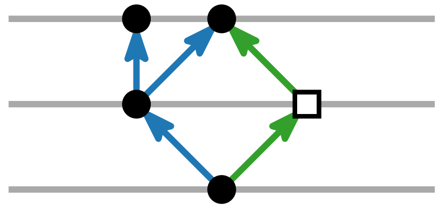
2. layer assignment



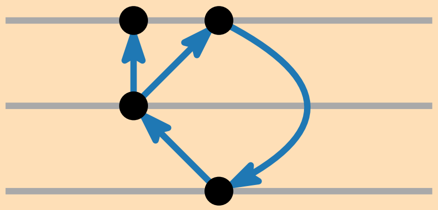
3. crossing minimization



4. node placement



5. edge routing



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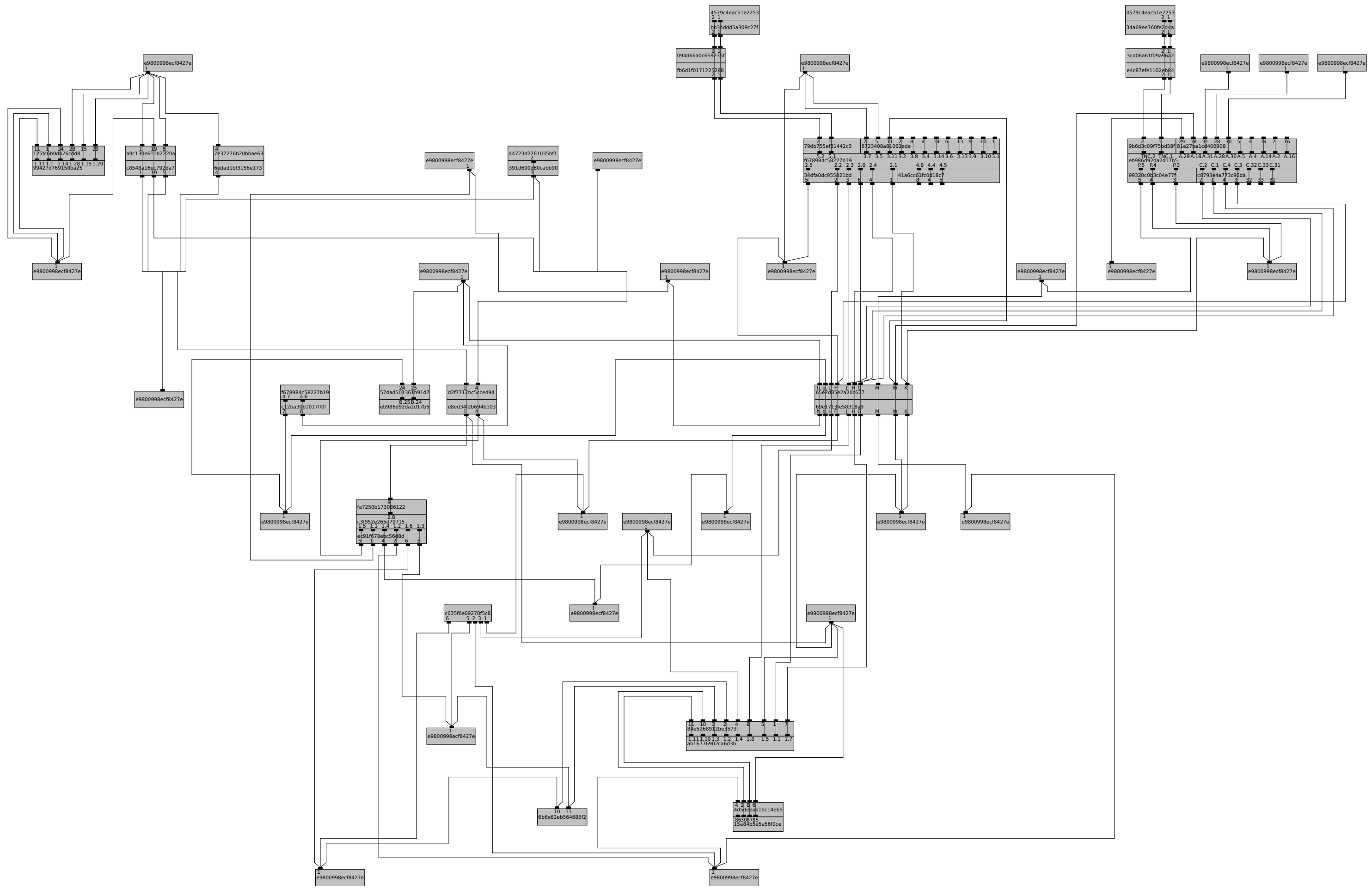
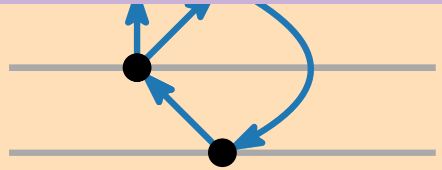


cable plan

[Zink, Walter, Baumeister, Wolff; CGTA'22]



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Motivation – Layered Orthogonal Edge Routing

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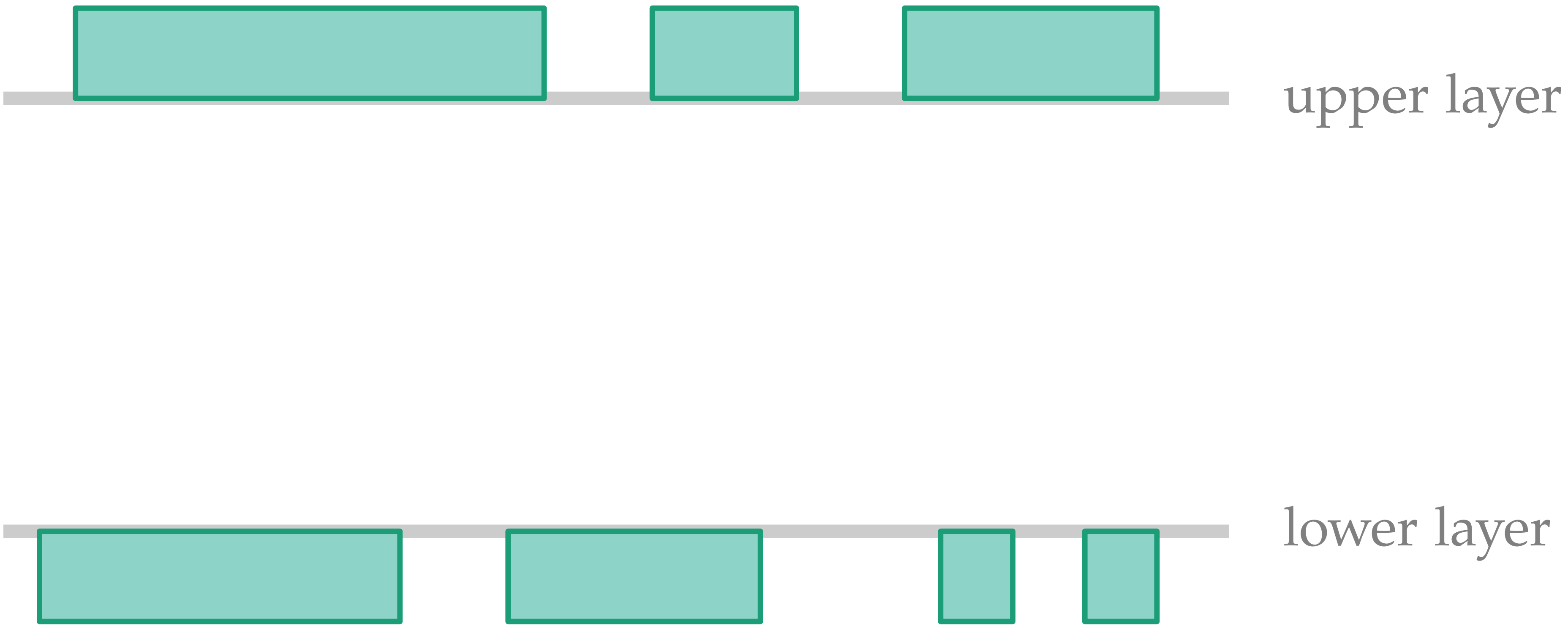
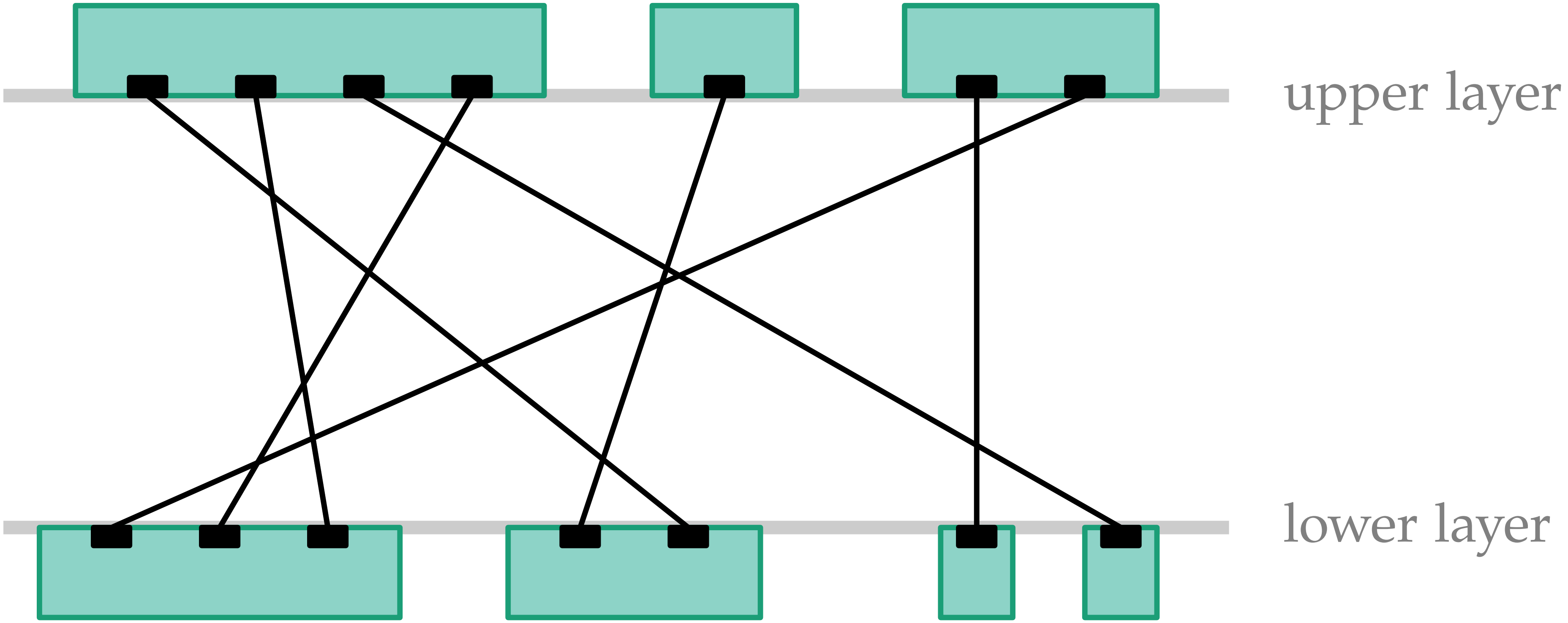


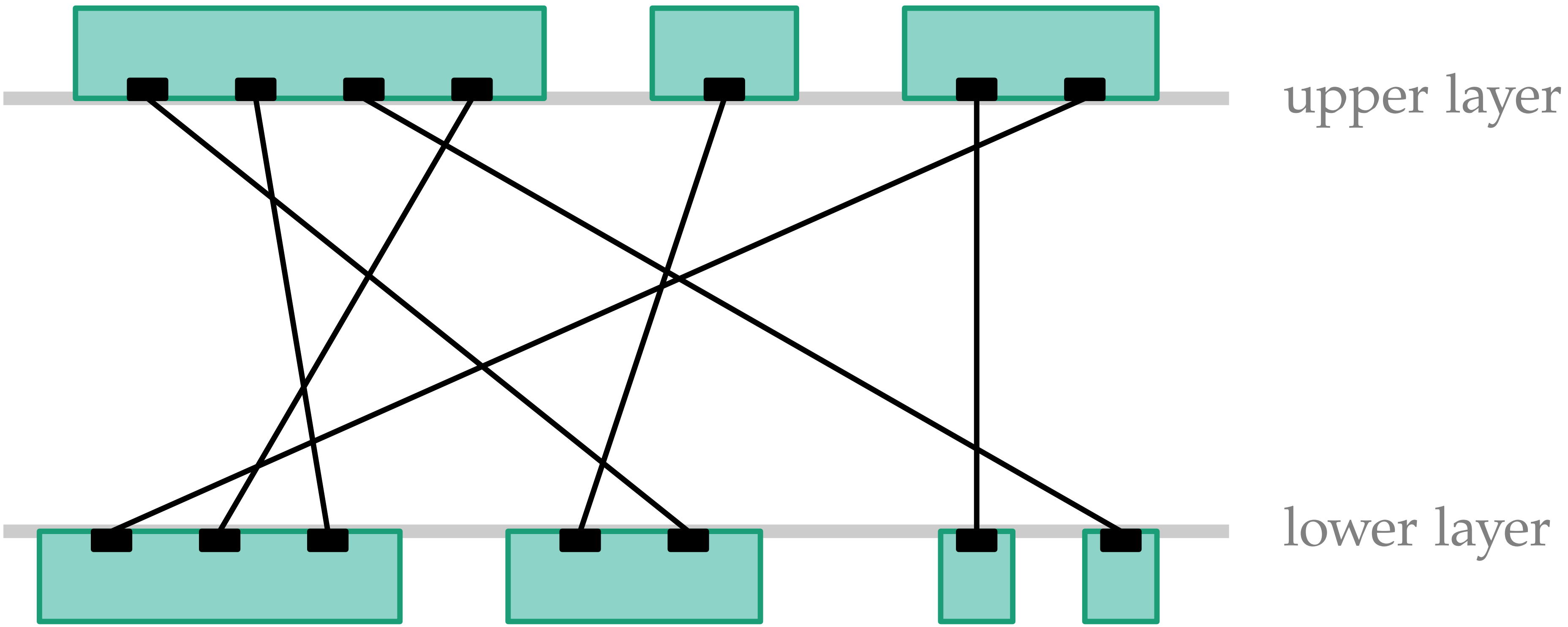
Motivation – Layered Orthogonal Edge Routing

- it suffices to consider each pair of consecutive layers individually
- positions of vertices are fixed
- no two edges share a common end point (vertices have distinct ports)



Motivation – Layered Orthogonal Edge Routing

- draw each edge with at most two vertical and one horizontal line segments



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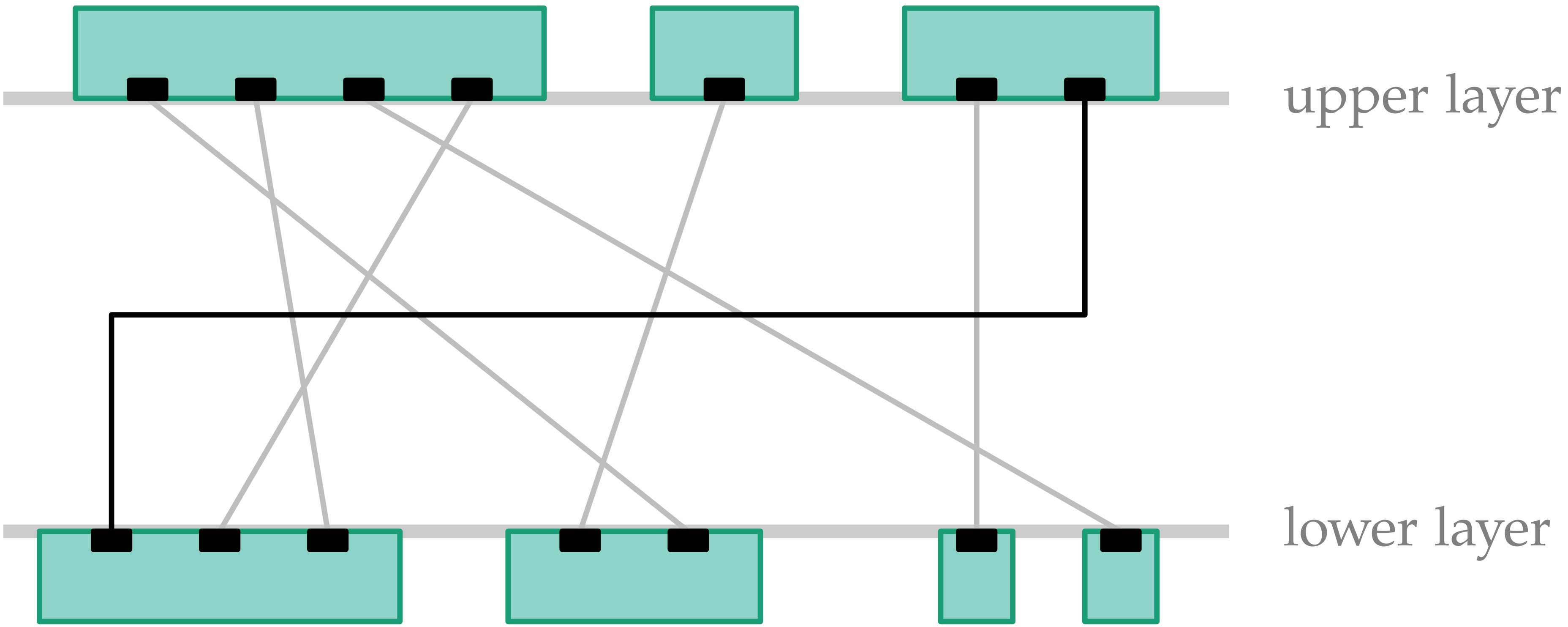

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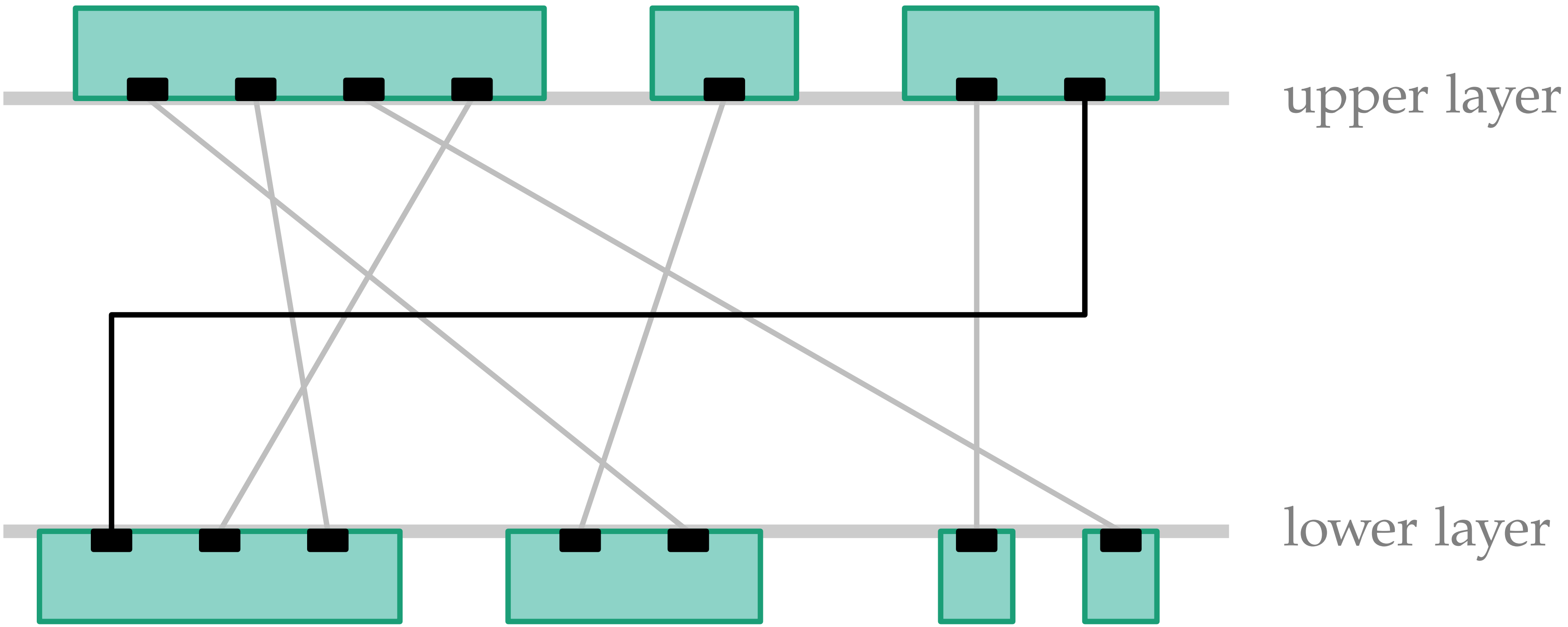
- draw each edge with at most two vertical and one horizontal line segments
- avoid overlaps and double crossings between the same pair of edges



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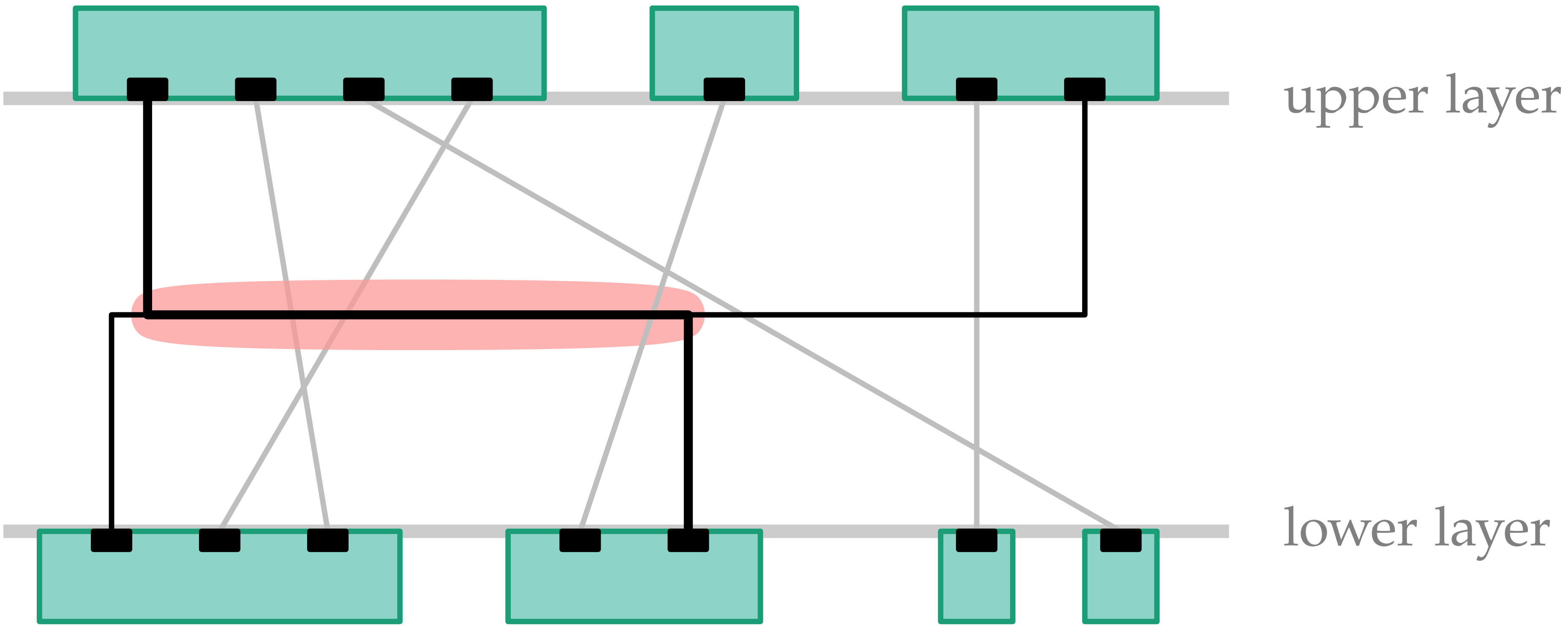

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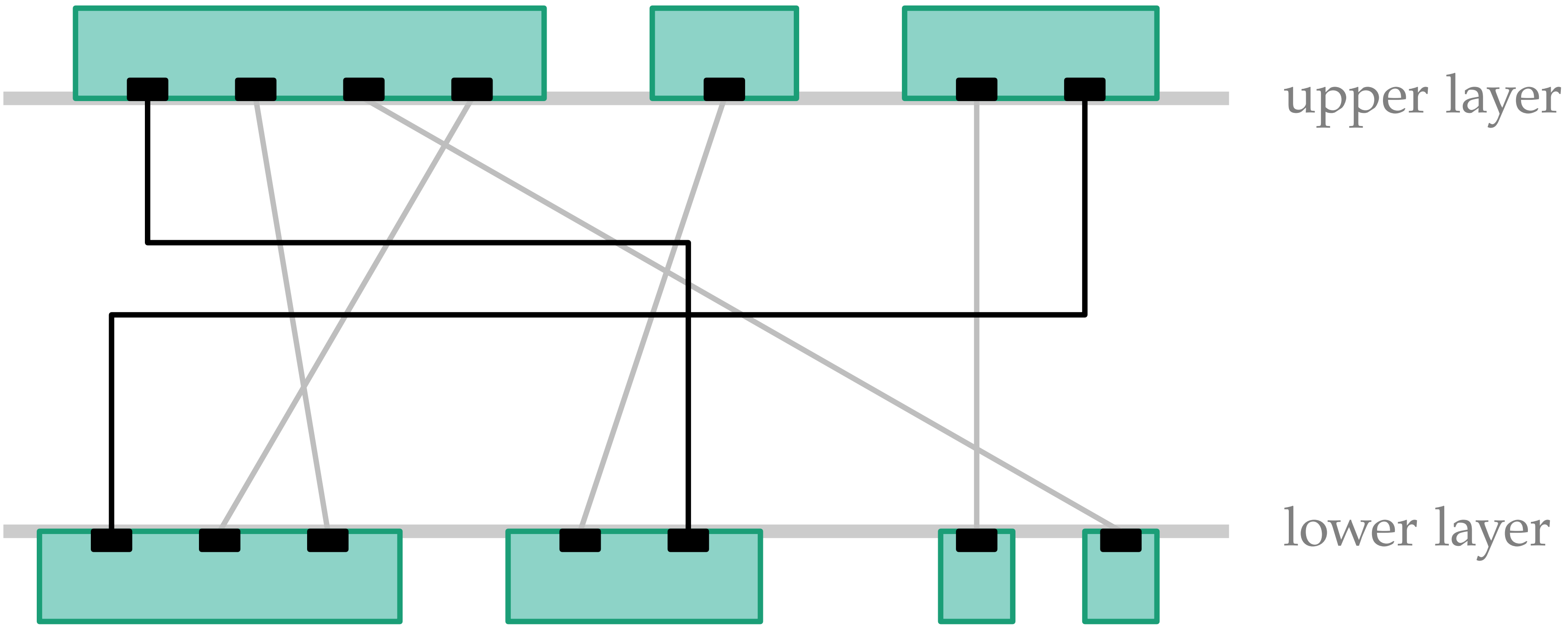

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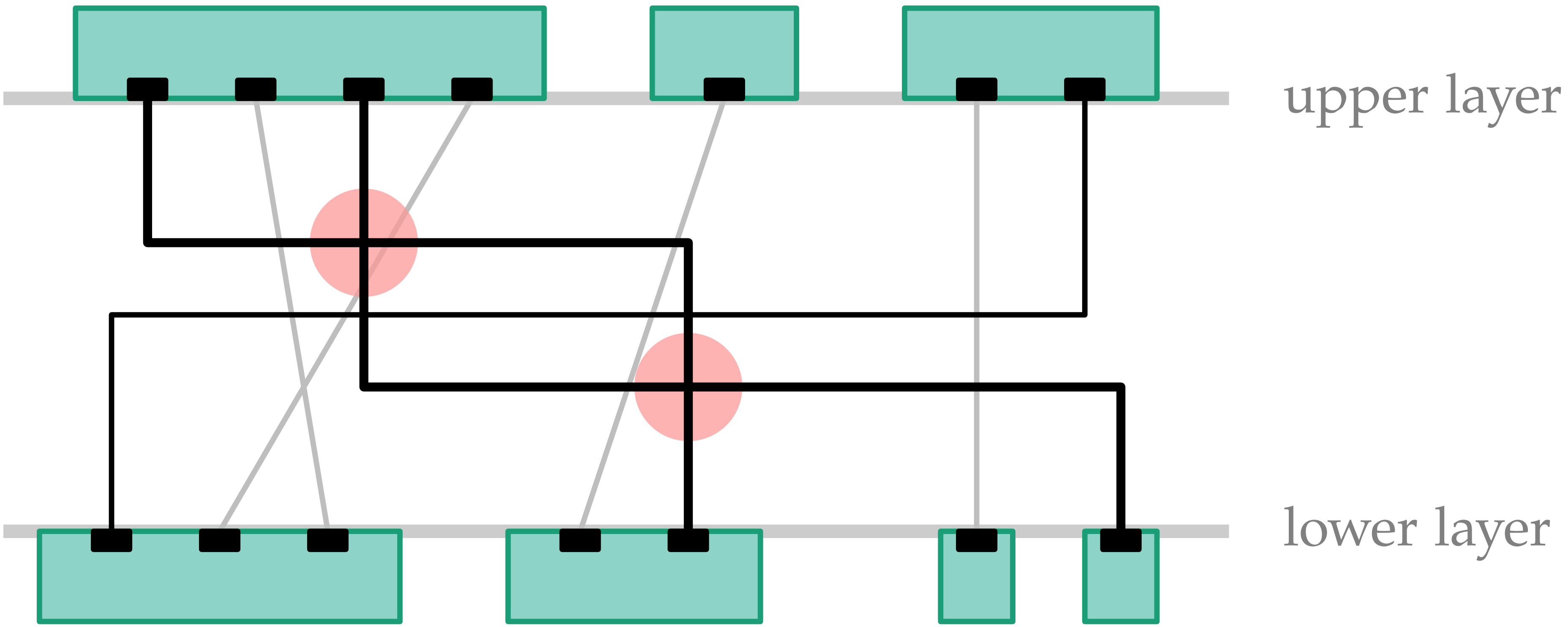

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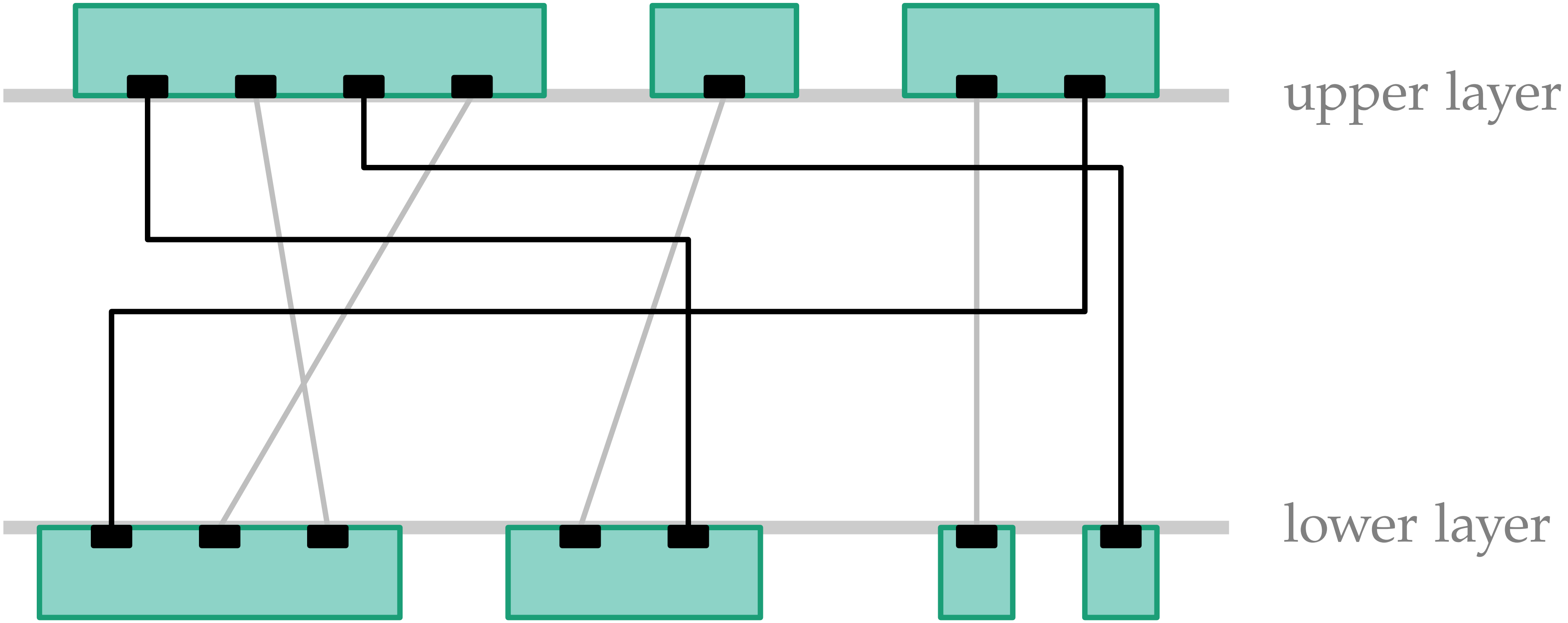

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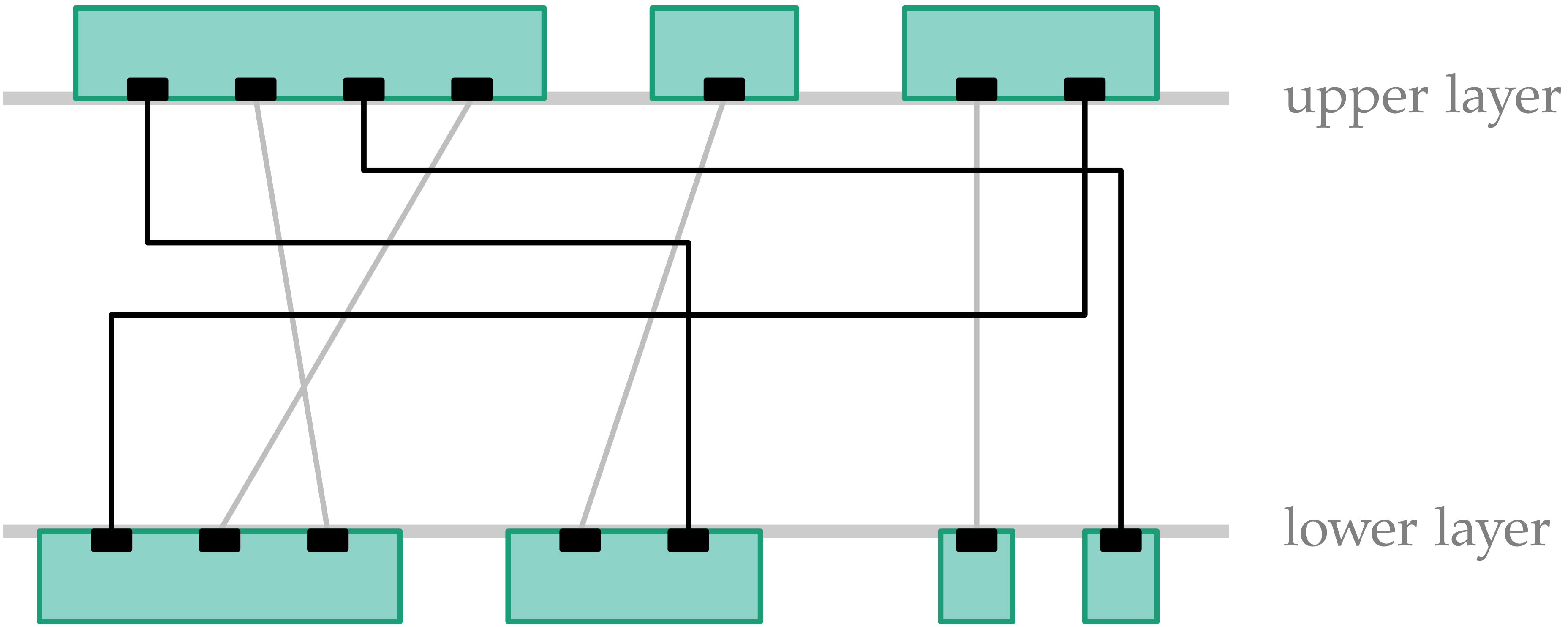
- draw each edge with at most two vertical and one horizontal line segments
- avoid overlaps and double crossings between the same pair of edges
- use as few horizontal intermediate layers (tracks) as possible



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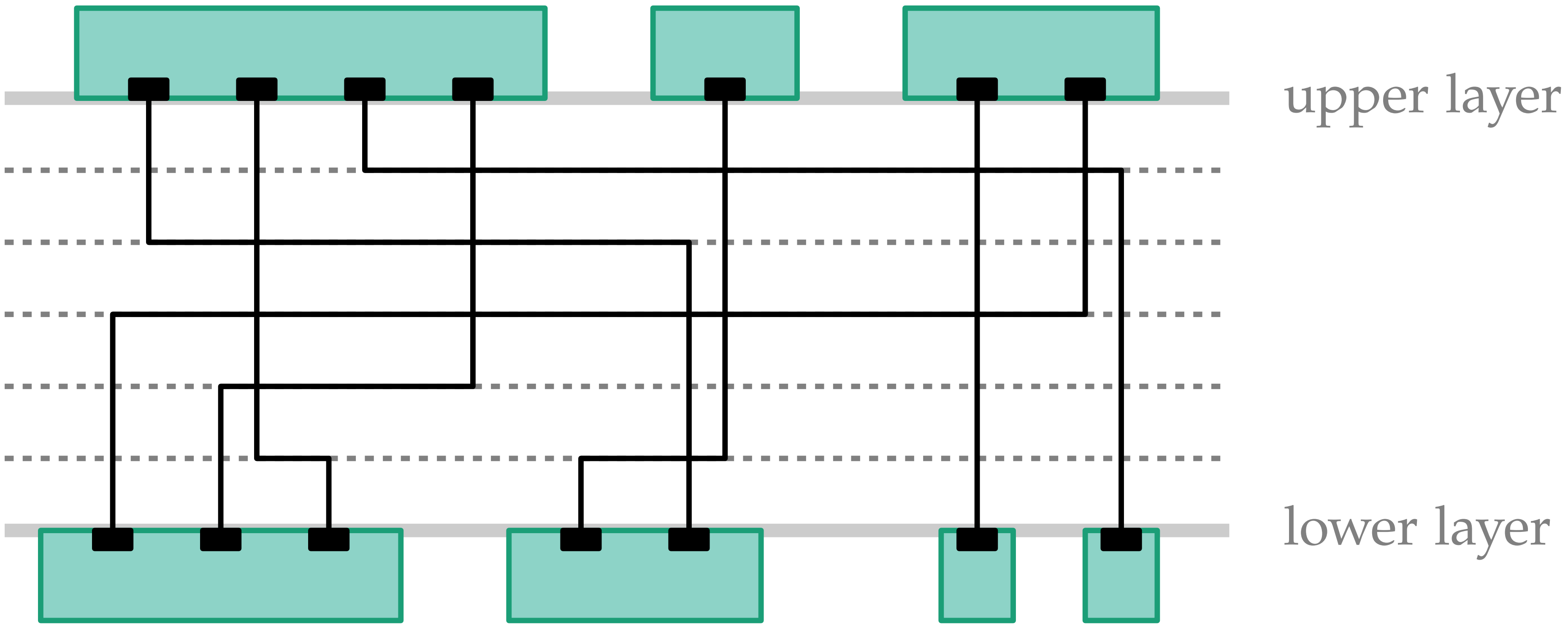

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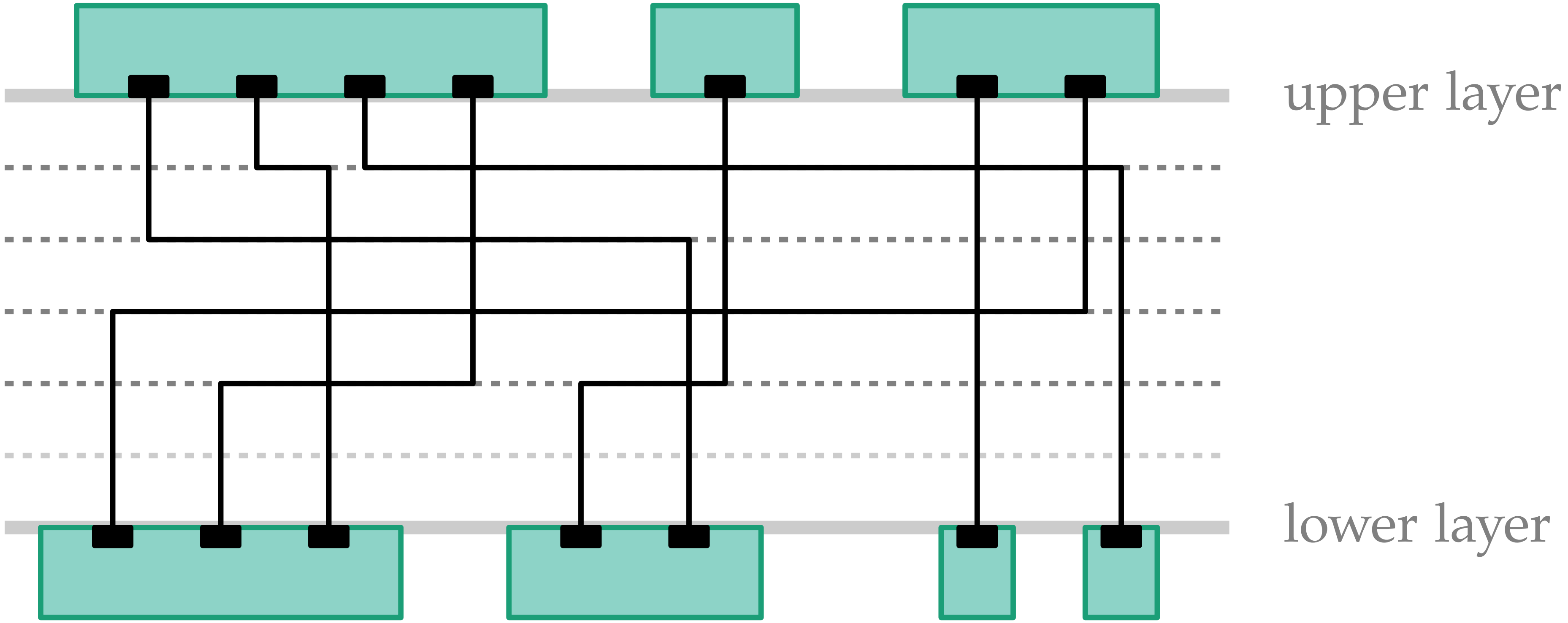

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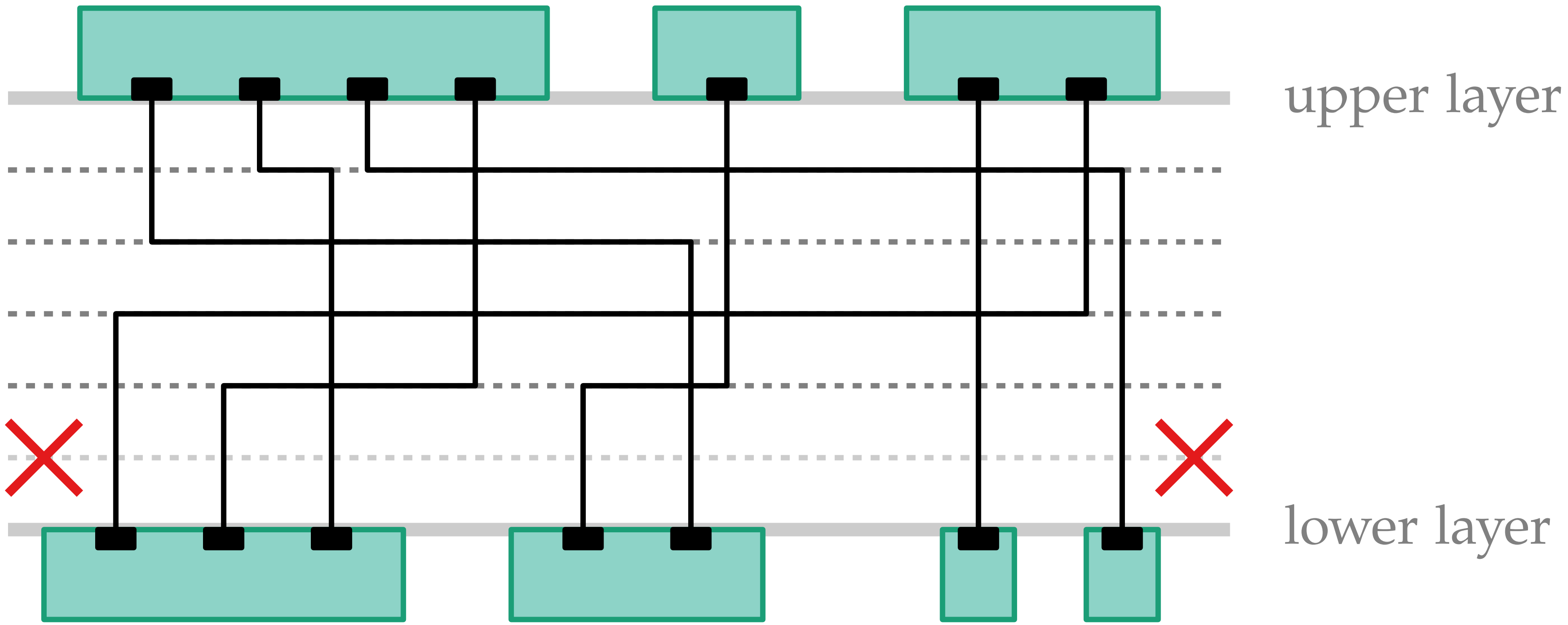
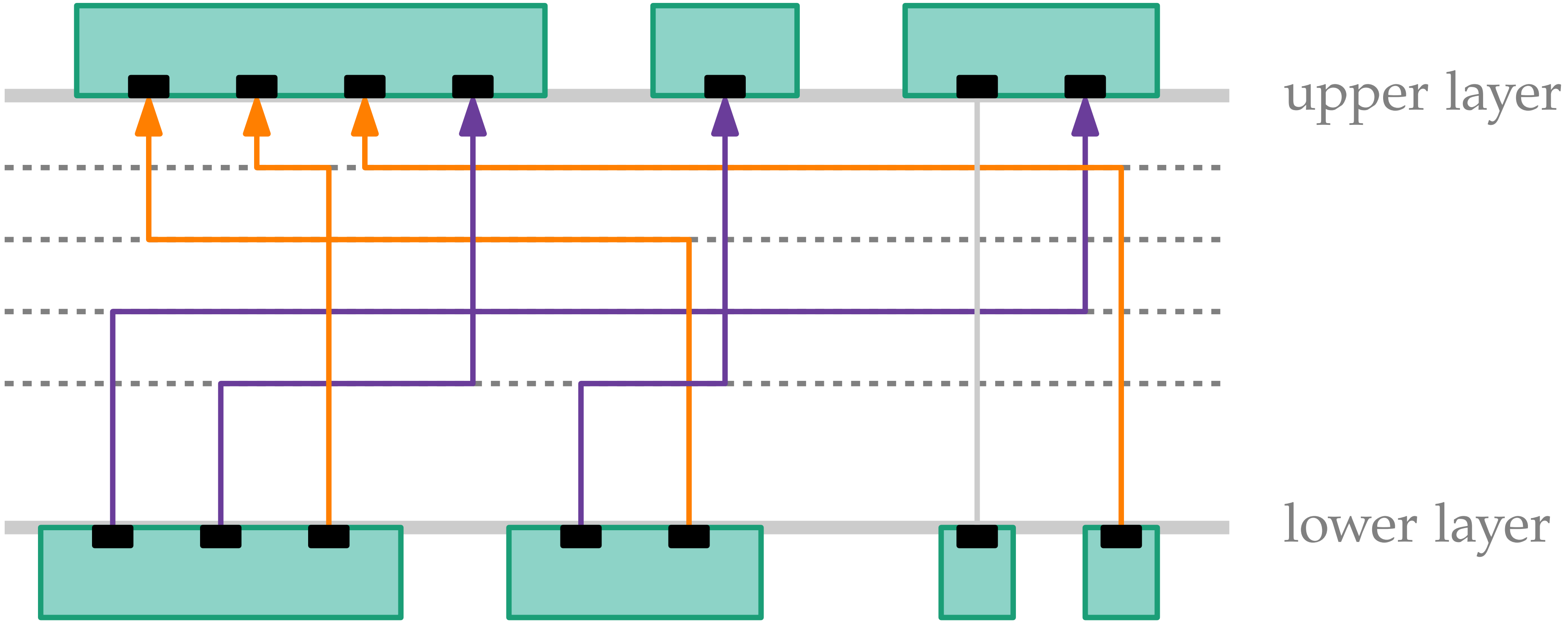


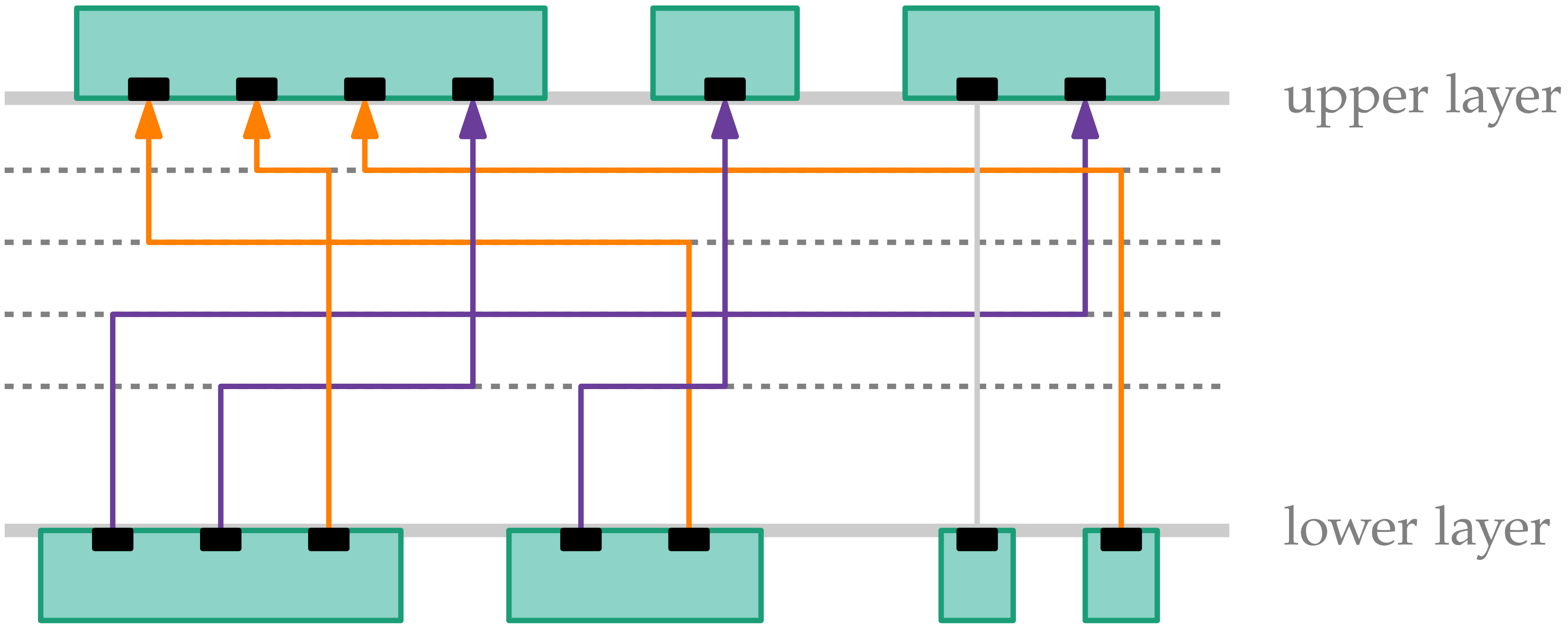
Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges



Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges
- only edges going in the same direction and overlapping partially in x-dimension can cross twice



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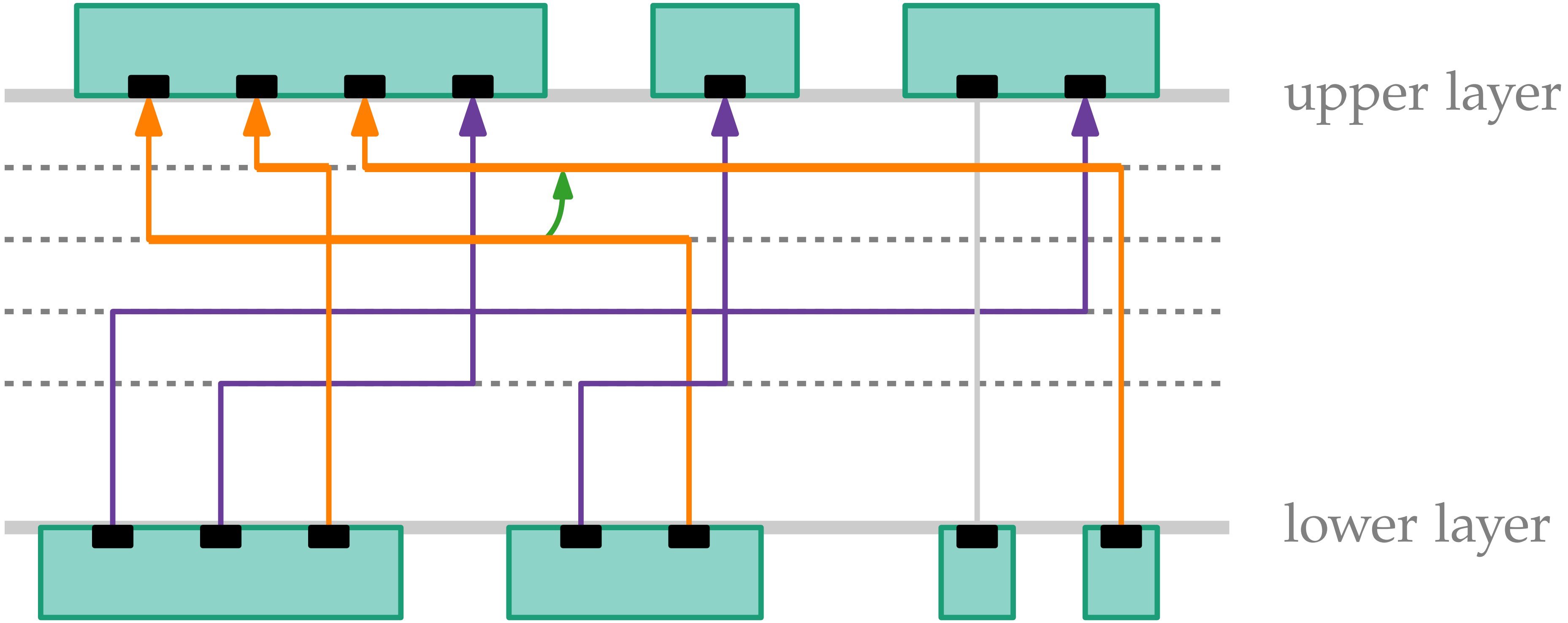
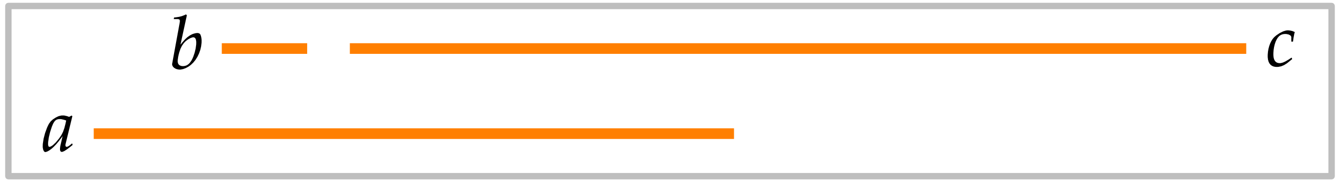
  ⇒ induce a vertical order for the horizontal middle segments



Definition – Directional Interval Graphs

Interval representation: set of intervals



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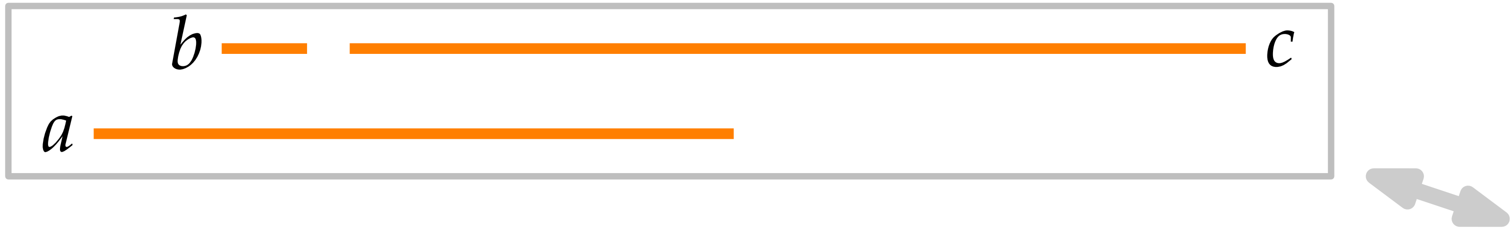

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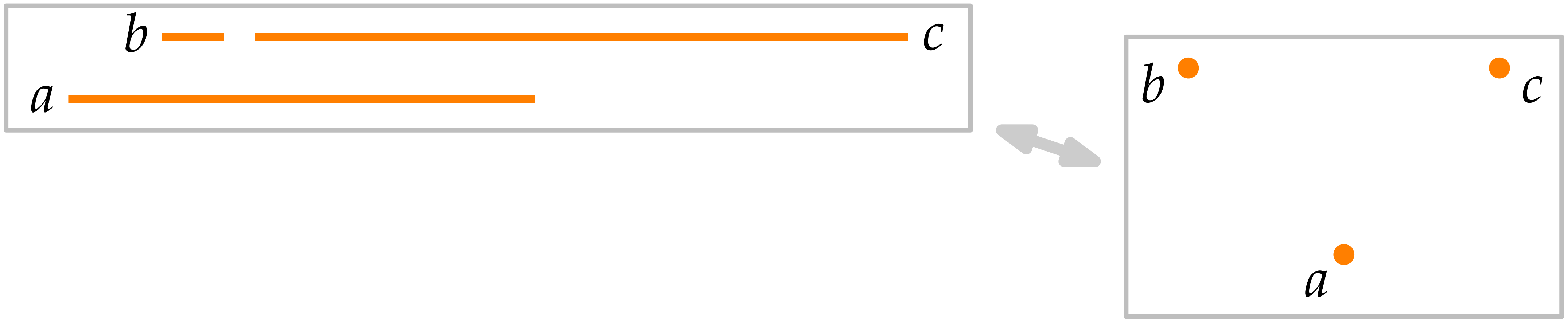

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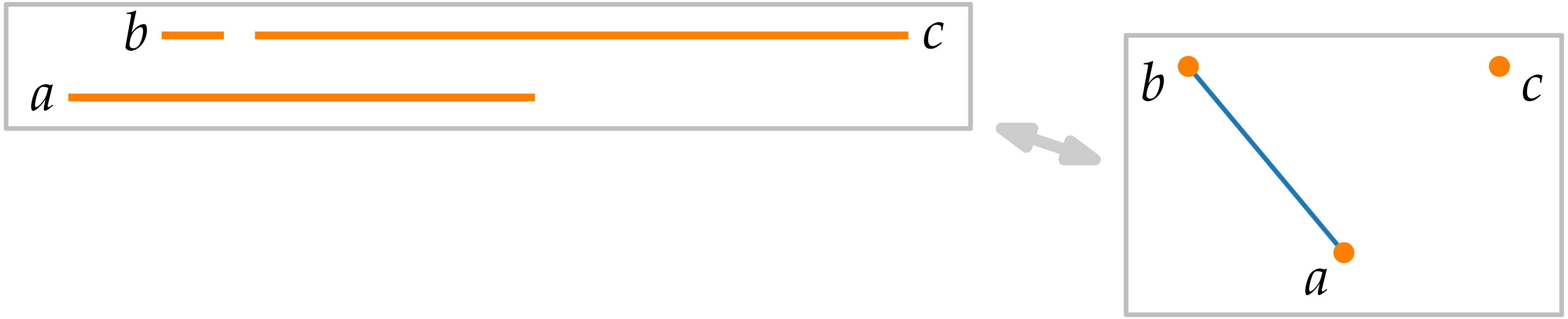

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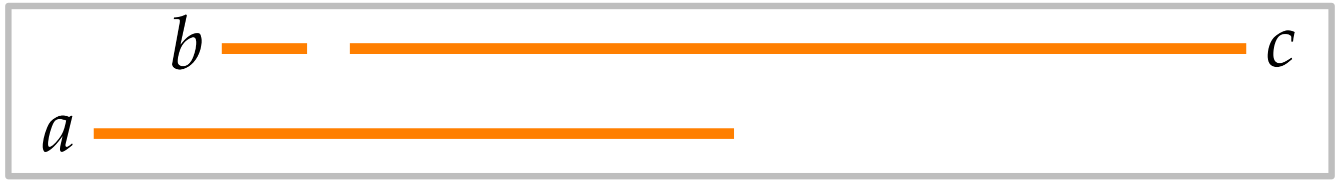

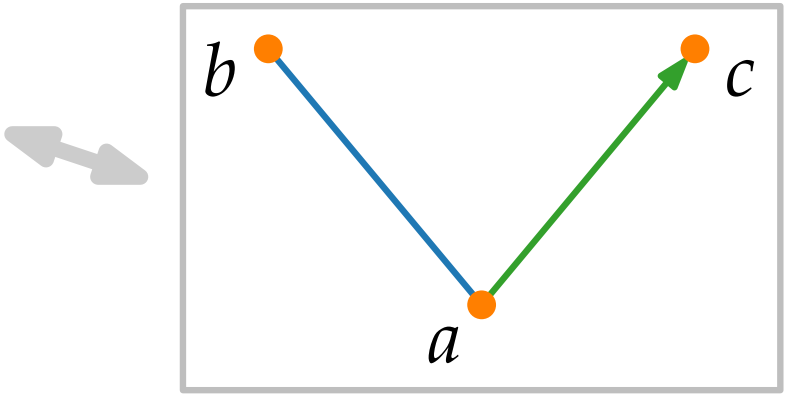

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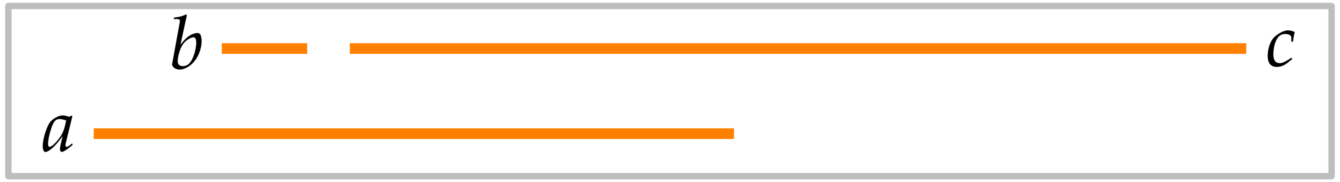

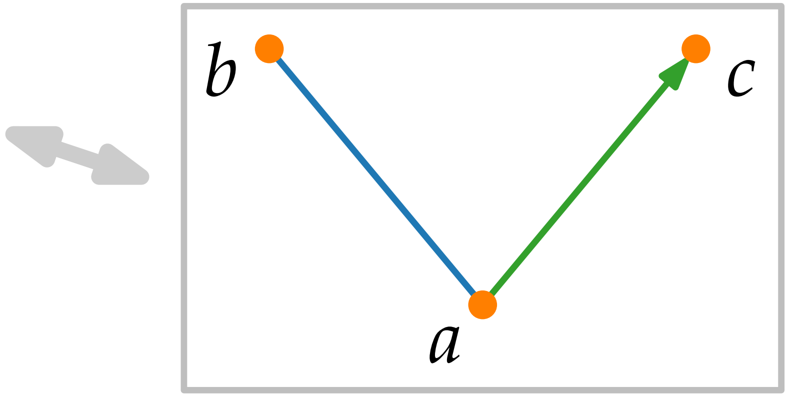

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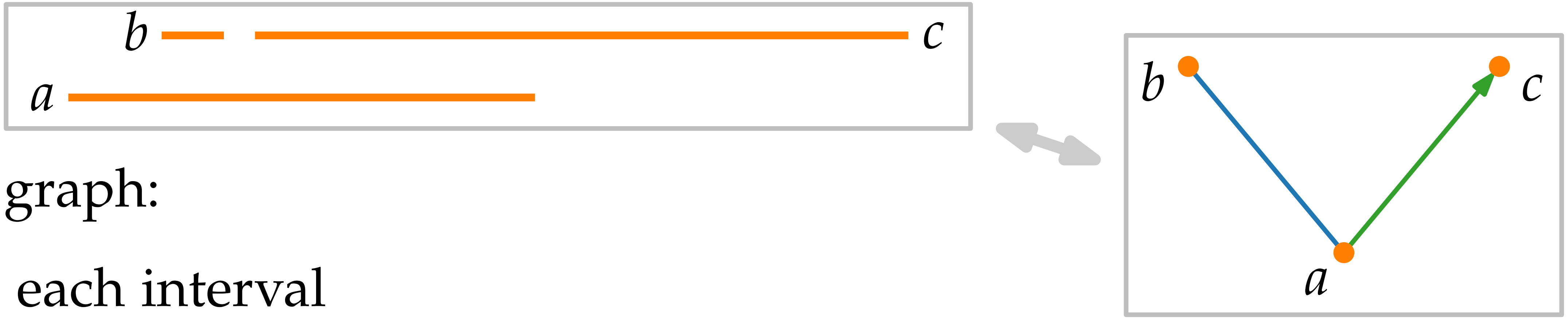

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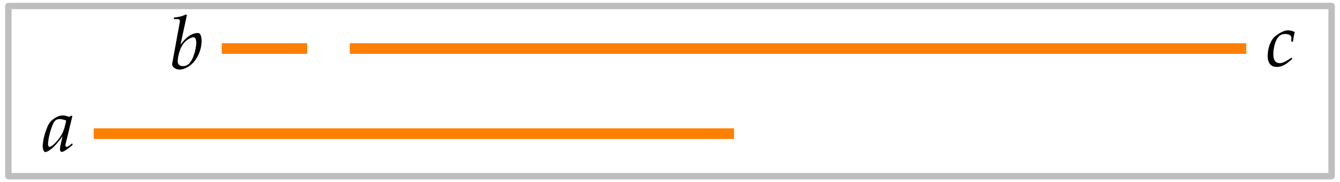

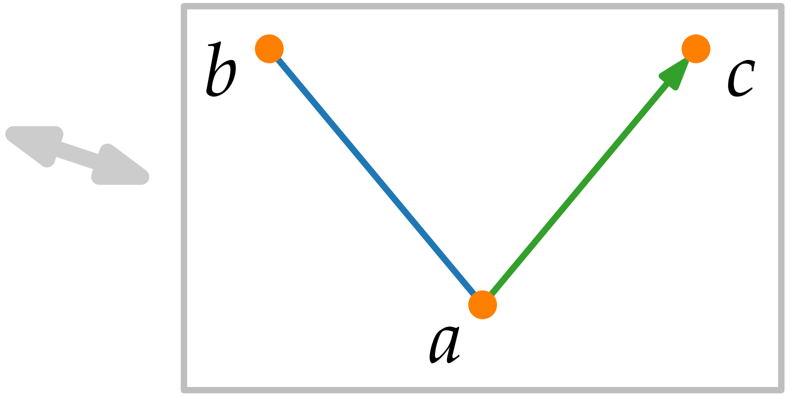

Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:

[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

- ⋆ undirected edge uv: $c(u) \neq c(v)$,
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Directional interval graphs:

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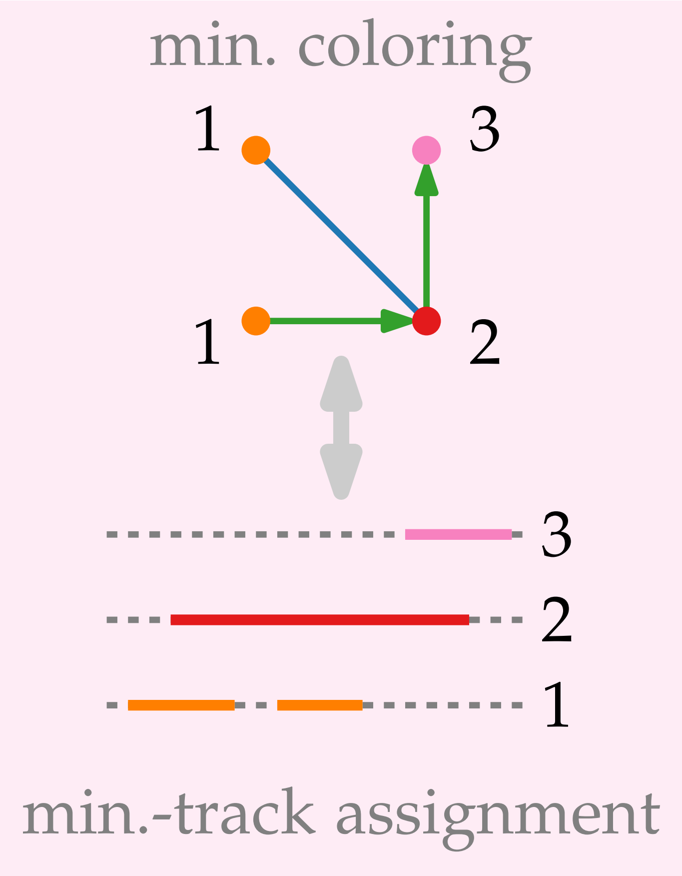

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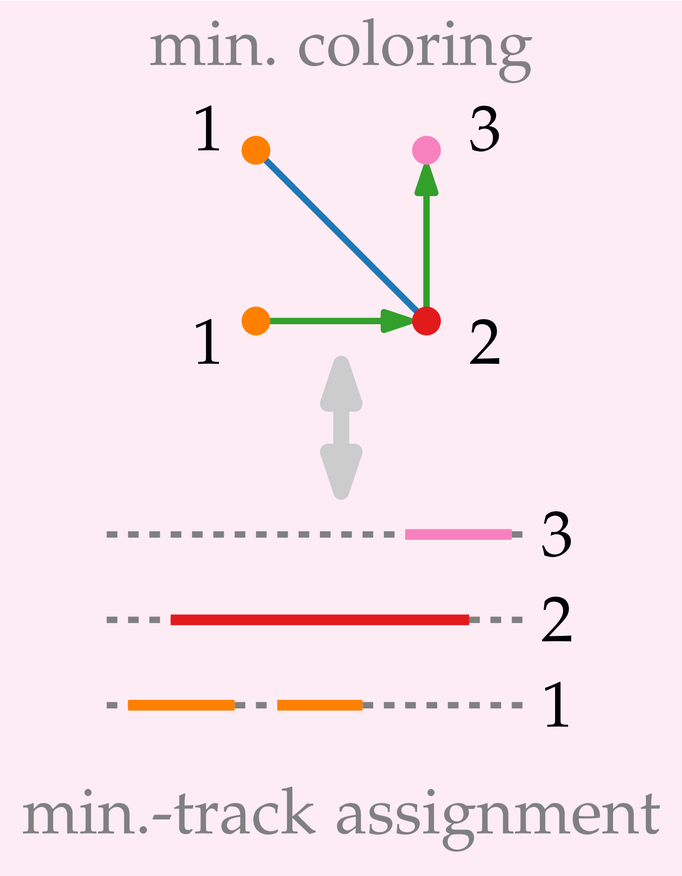

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Mixed interval graphs:

- coloring is NP-complete

Directed graphs (only directed edges):

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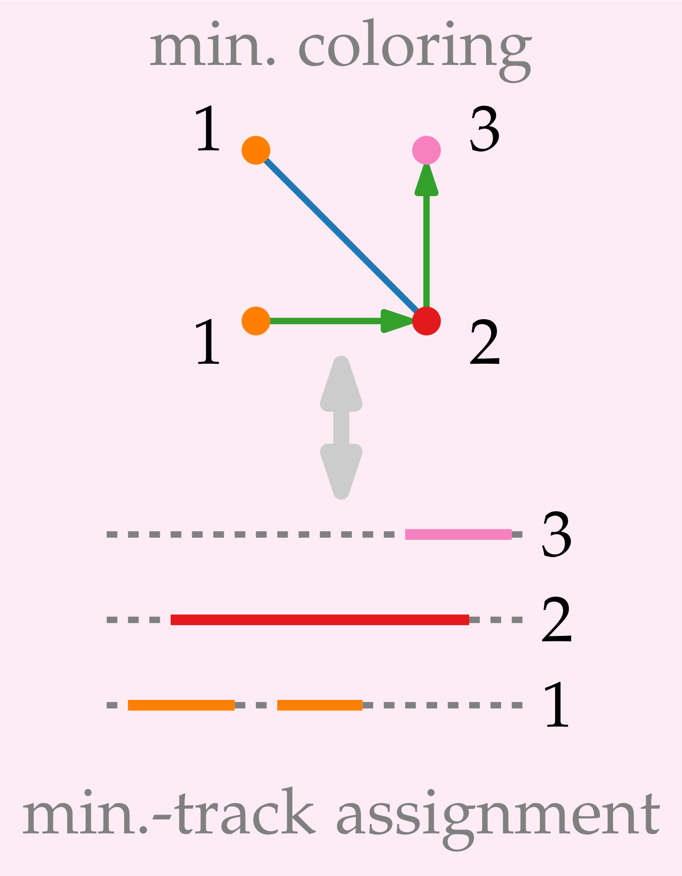

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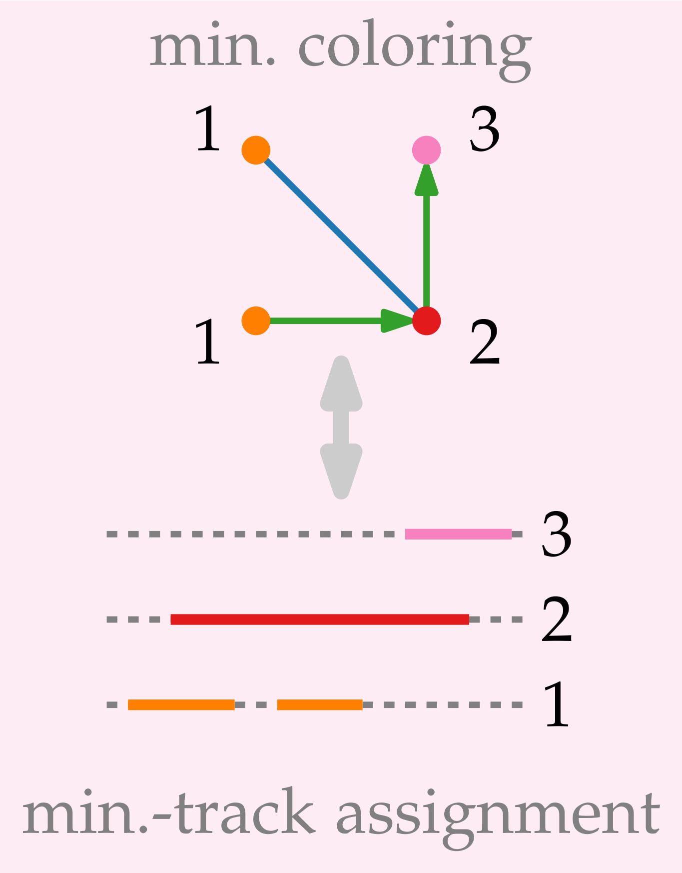

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agenda for this talk

Directed graphs (only directed edges):

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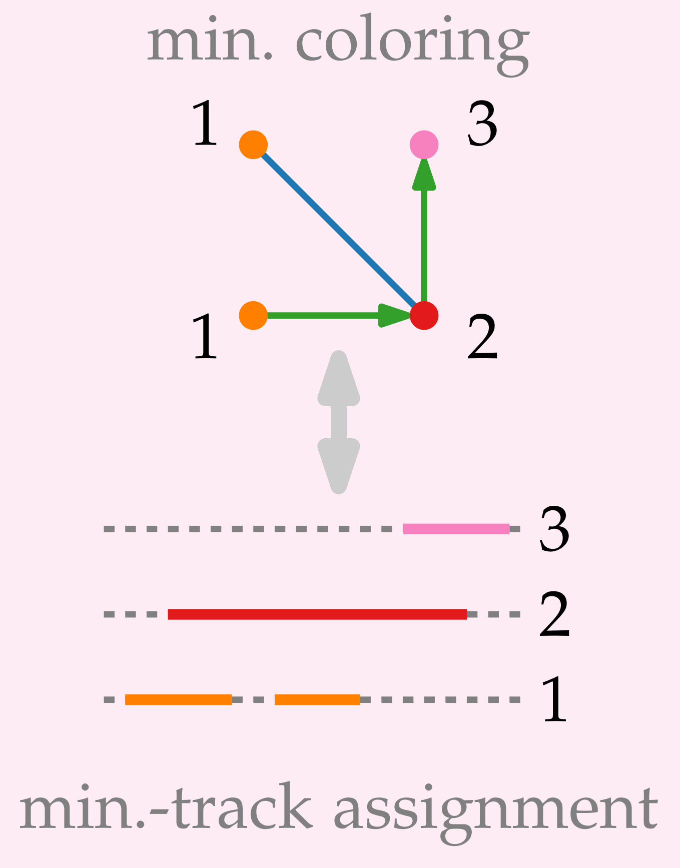

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Given: an interval representation of a directional interval graph G

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`GreedyColoring:`

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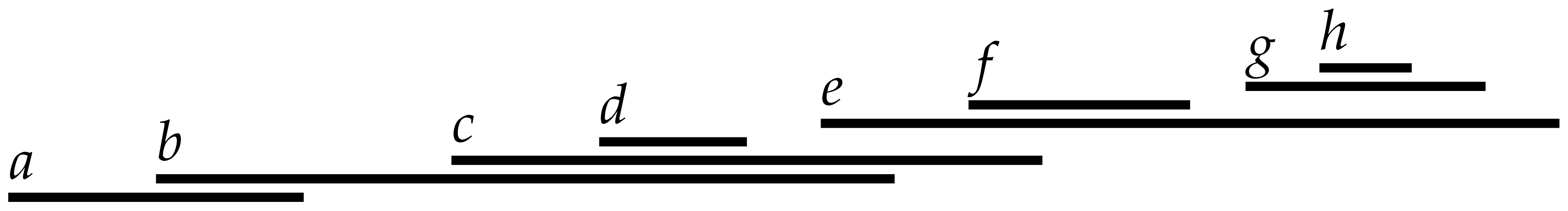

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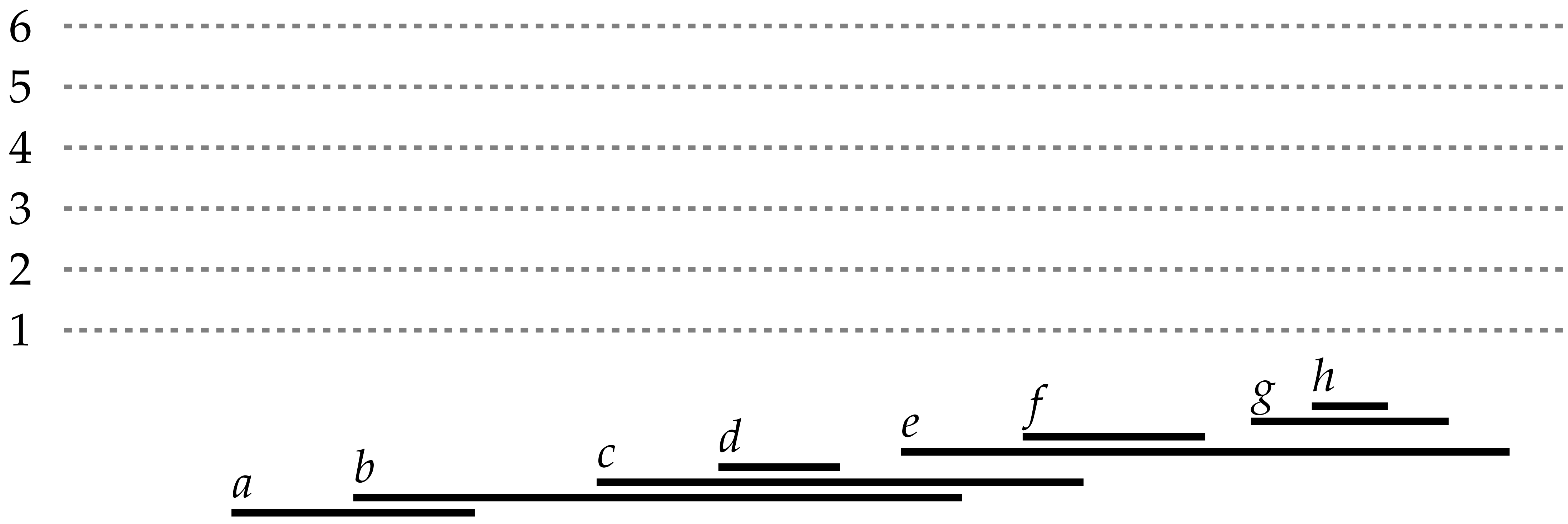

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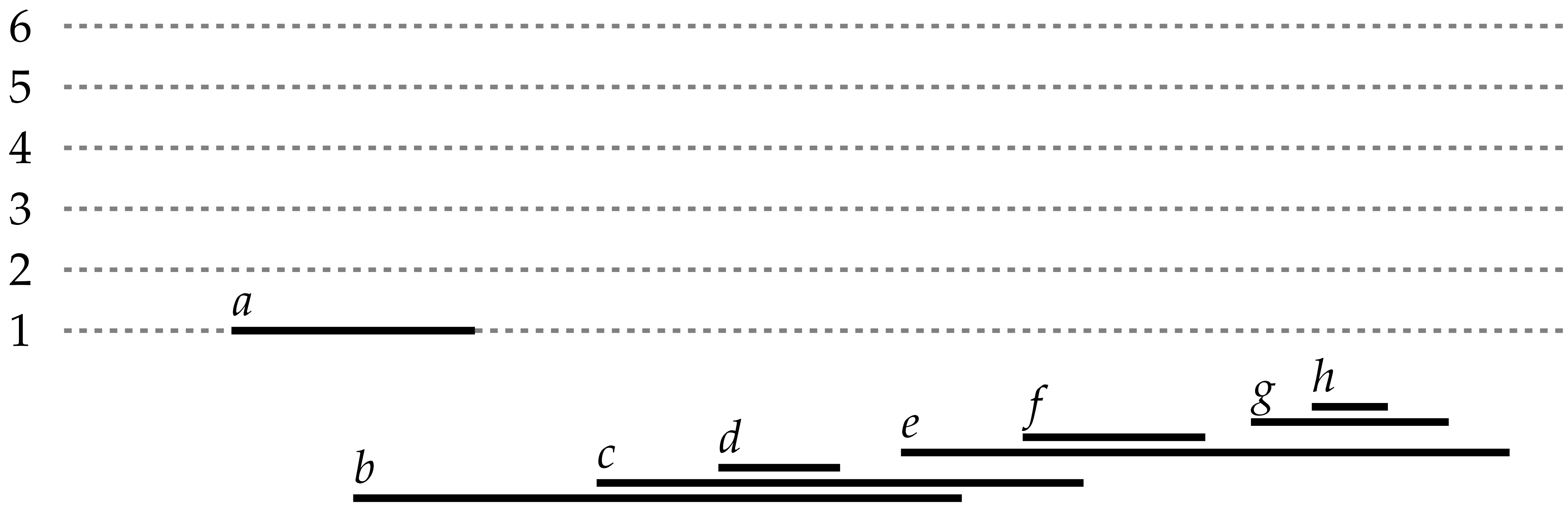

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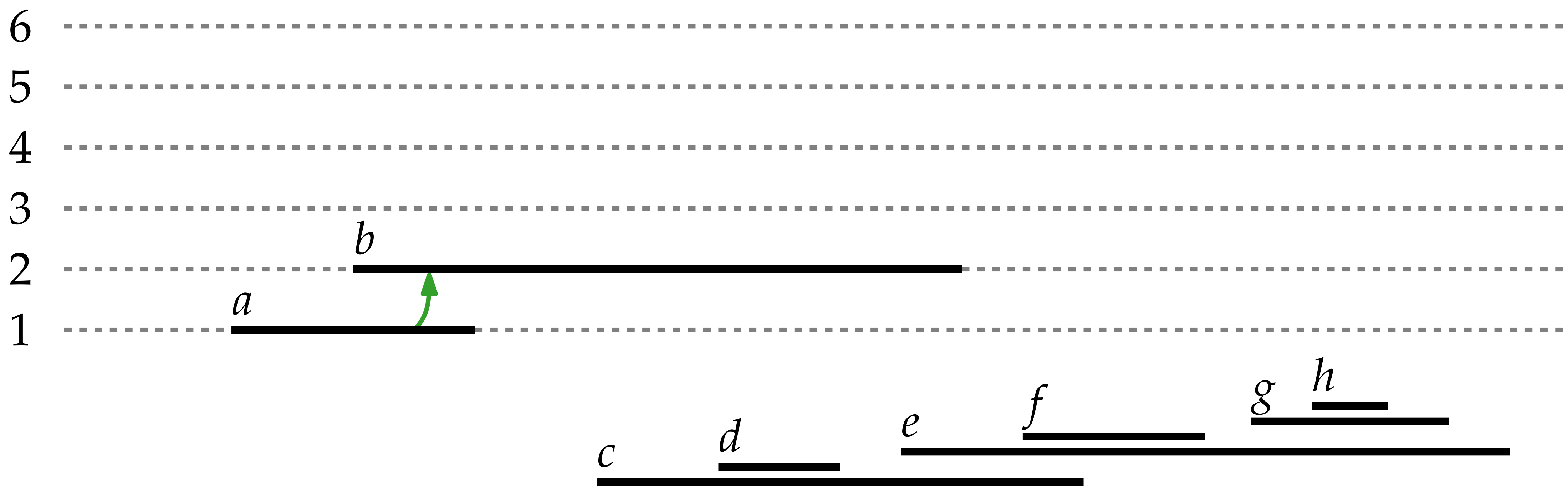

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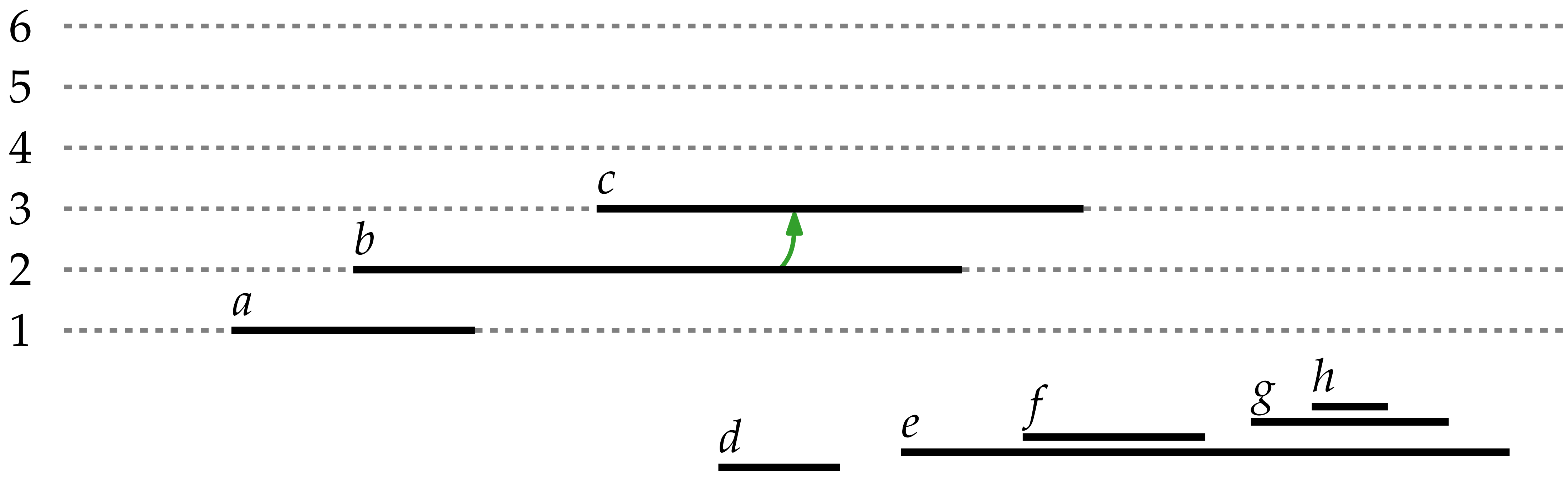

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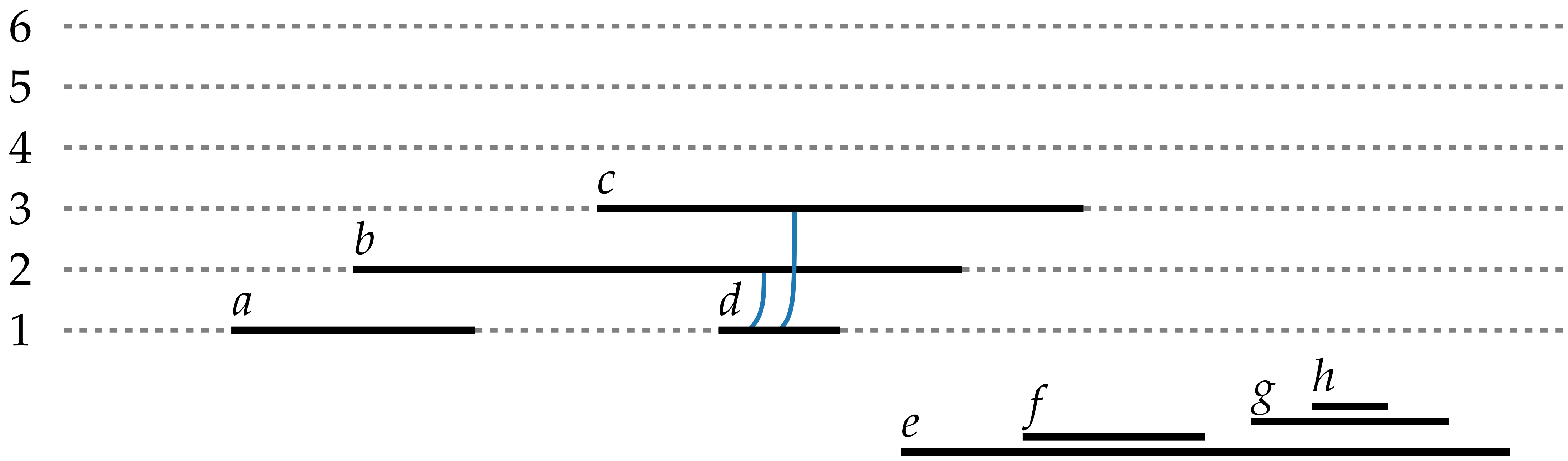

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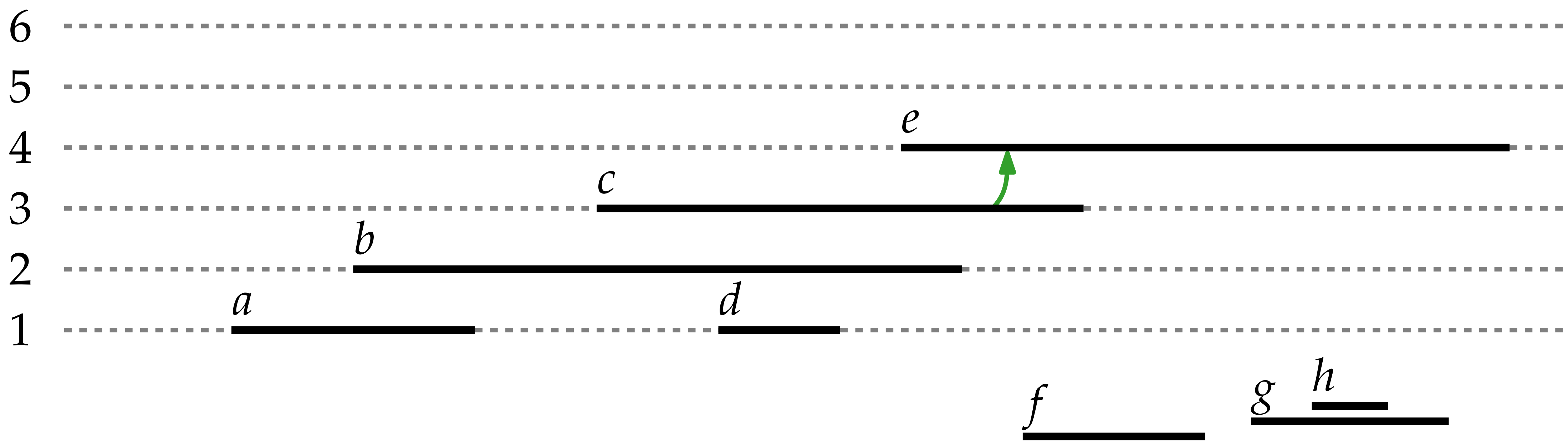

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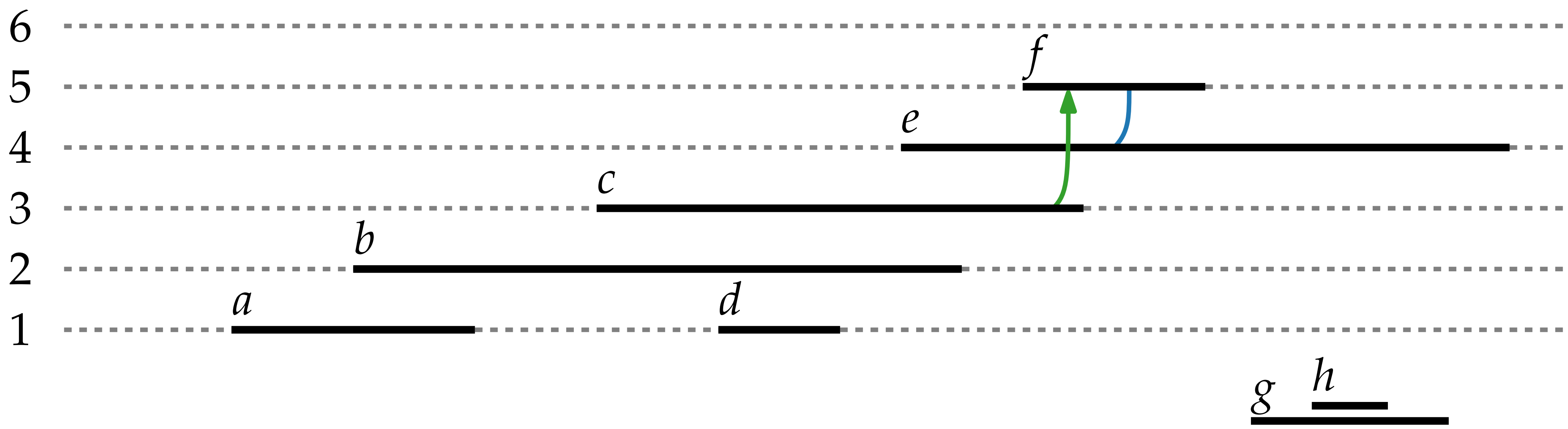

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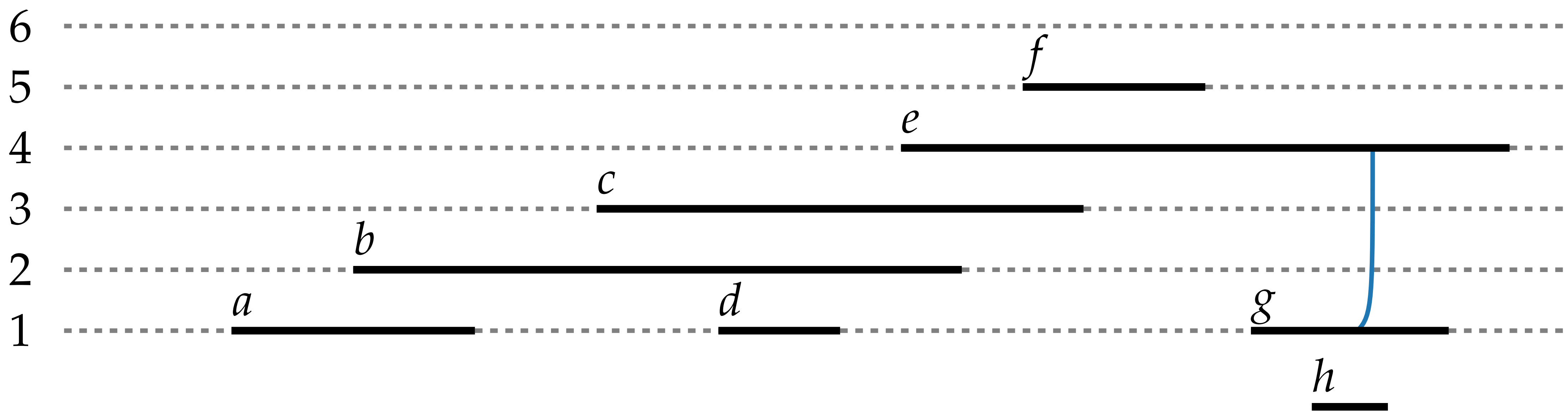

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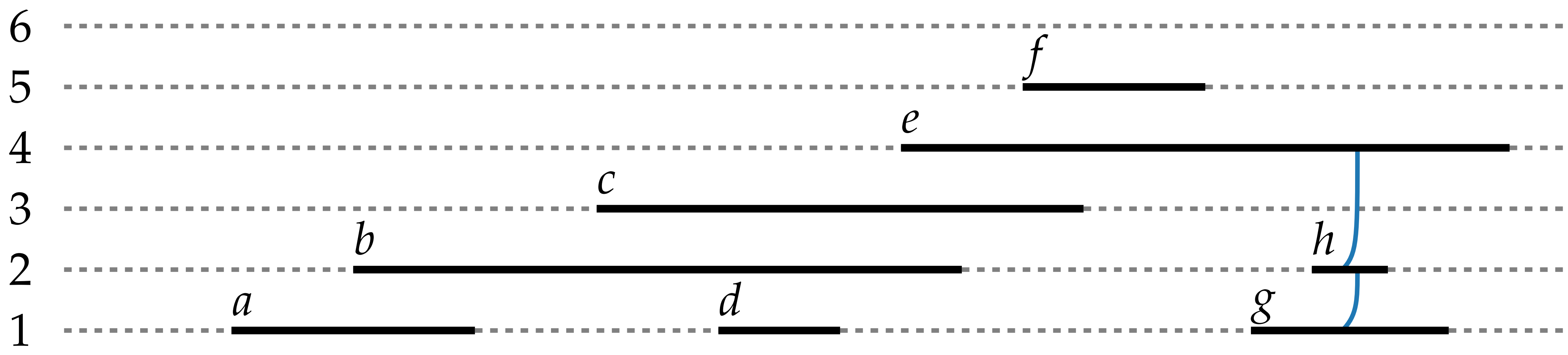

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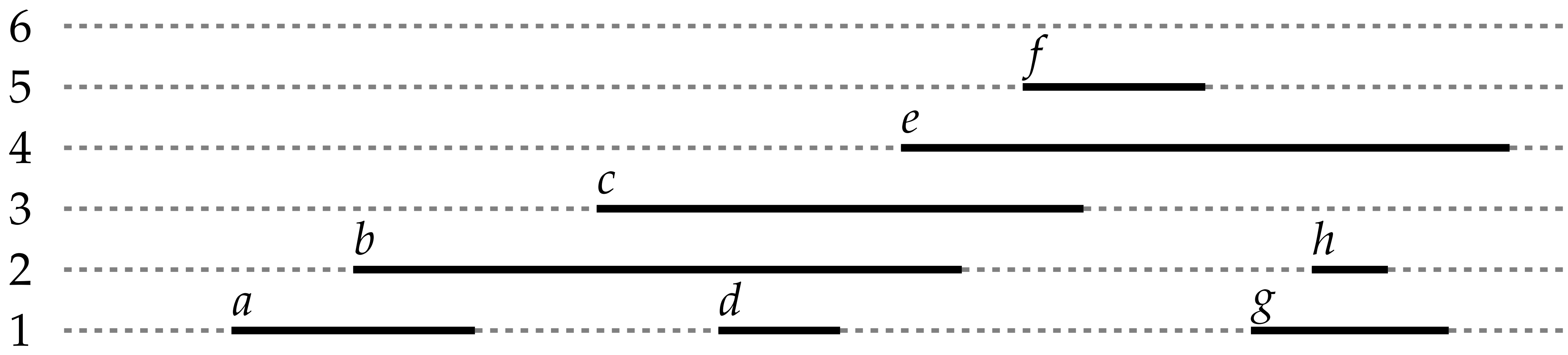

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  $\Rightarrow$ the coloring c uses the minimum number of colors

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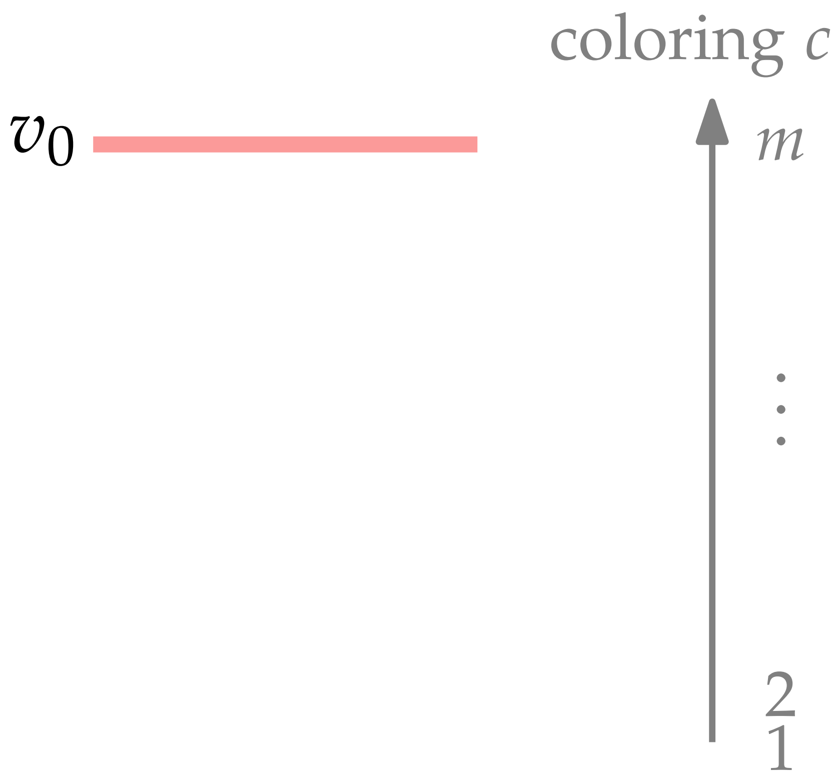

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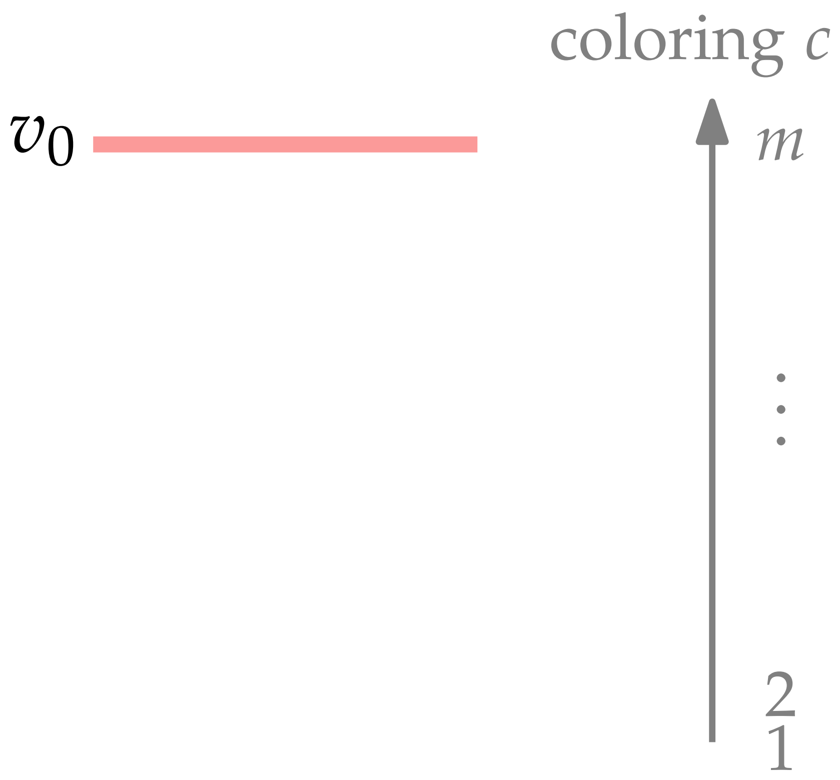

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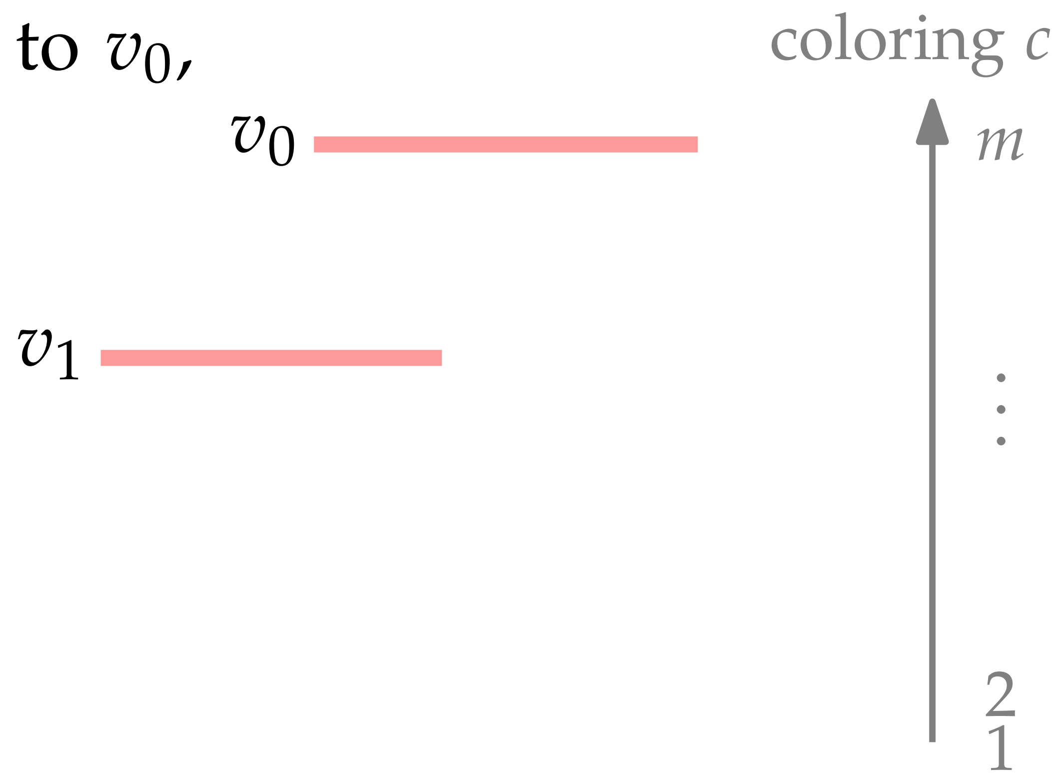

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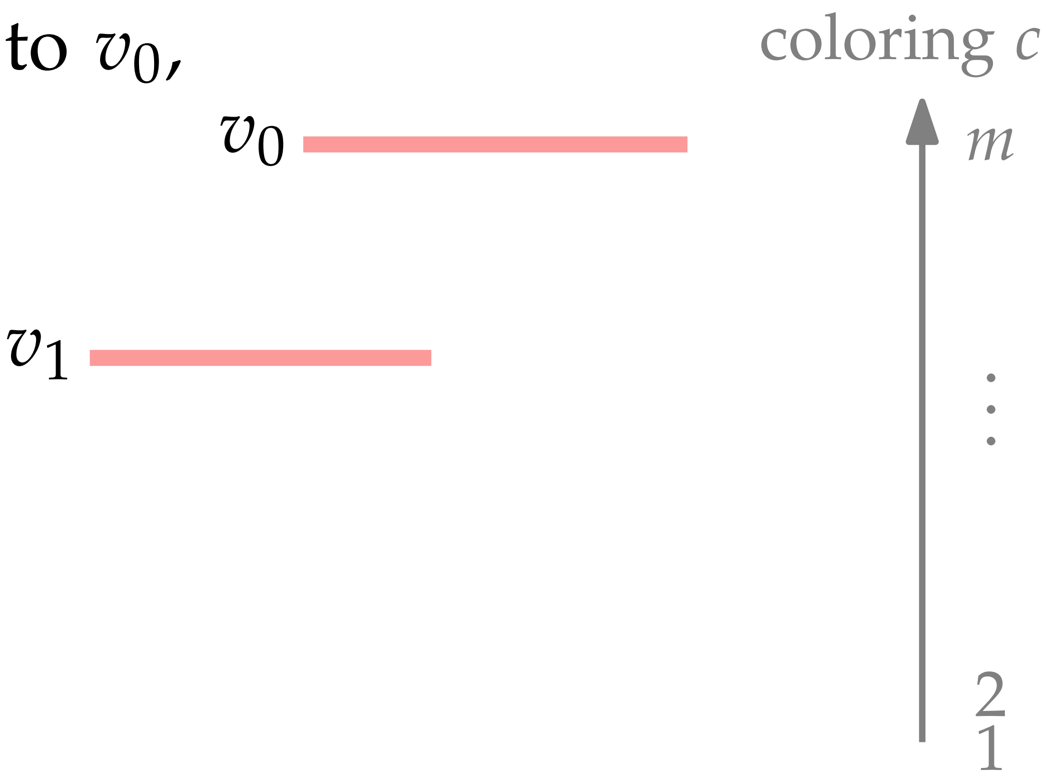

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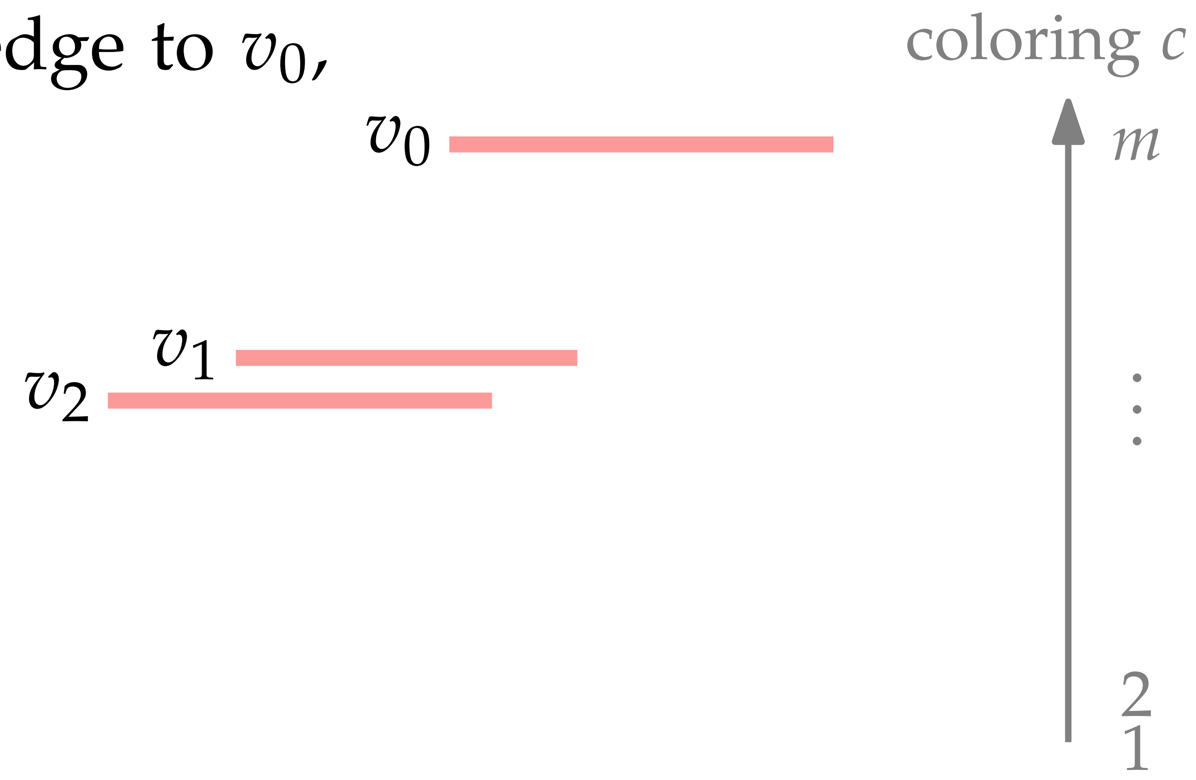

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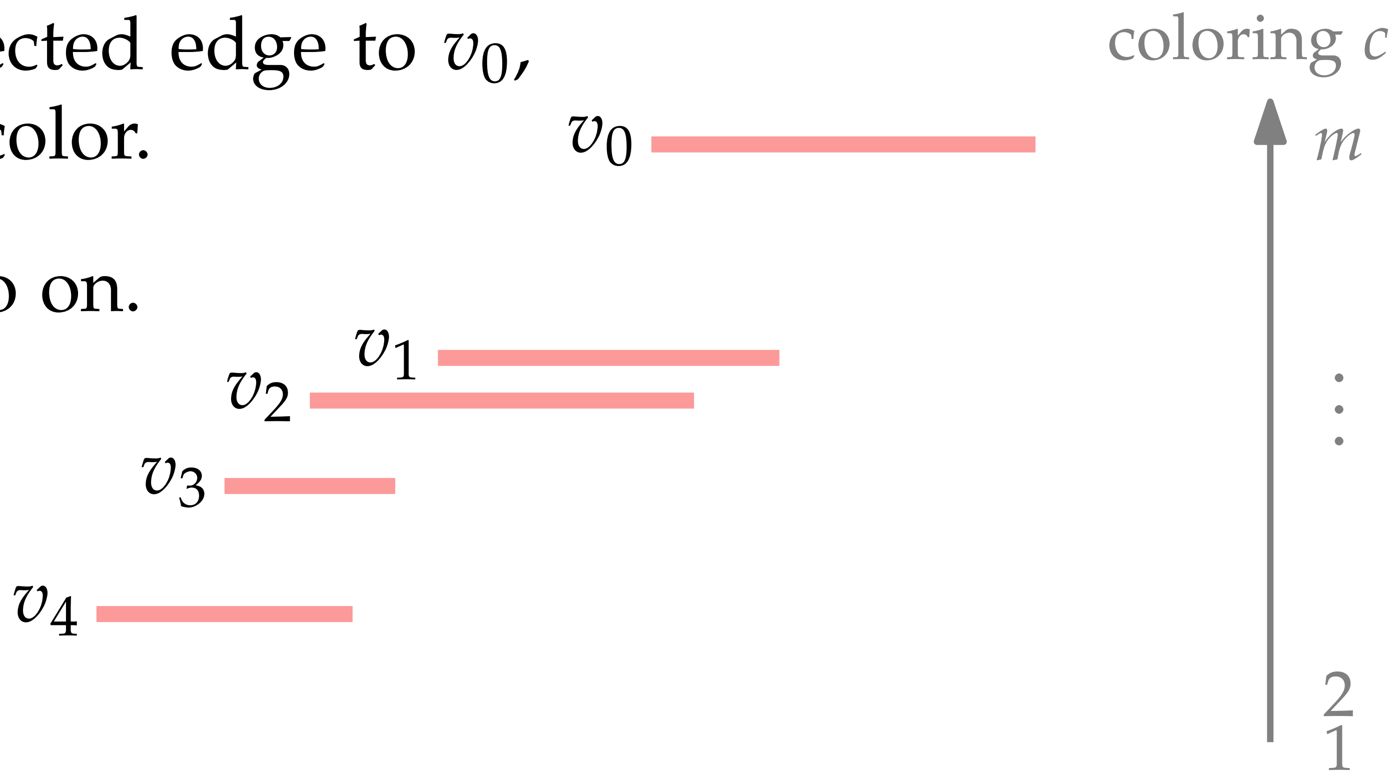

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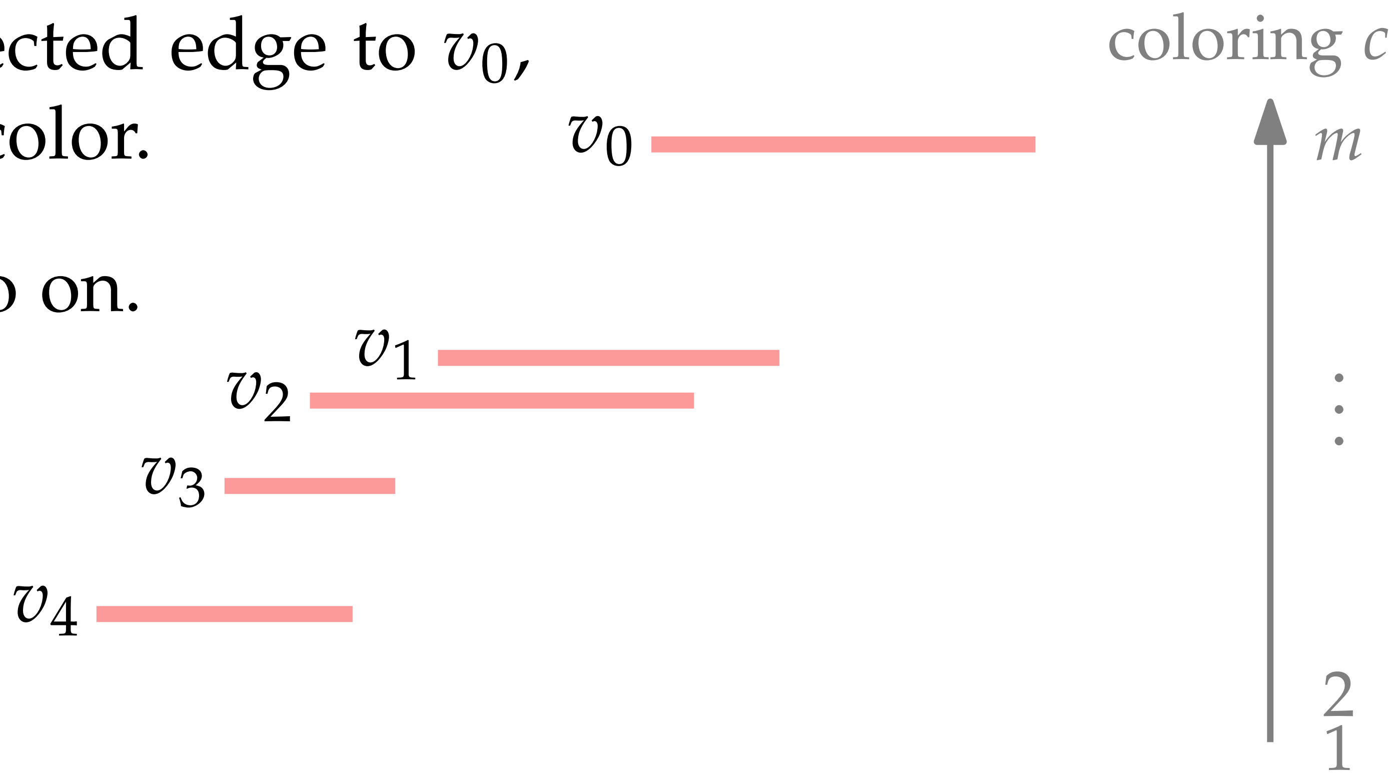

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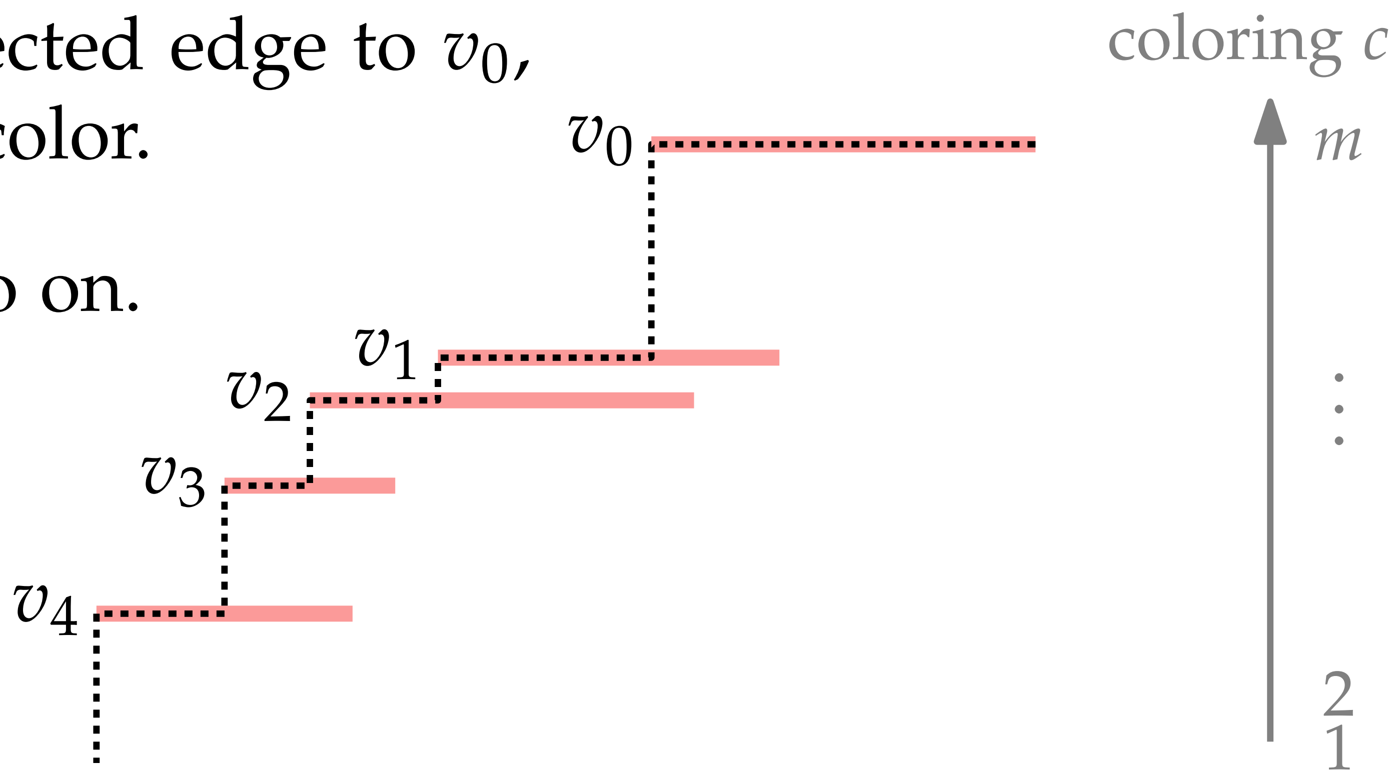

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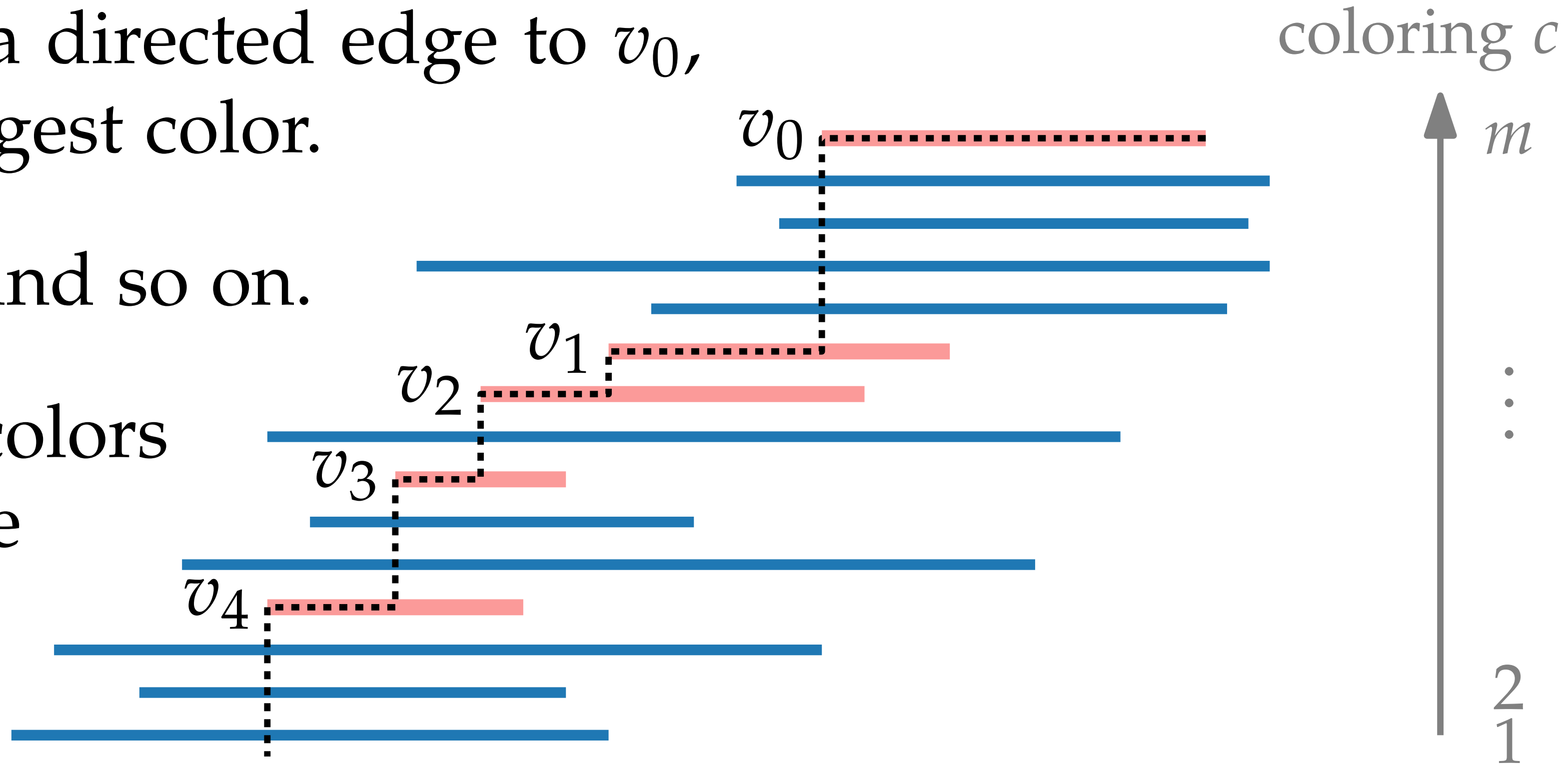

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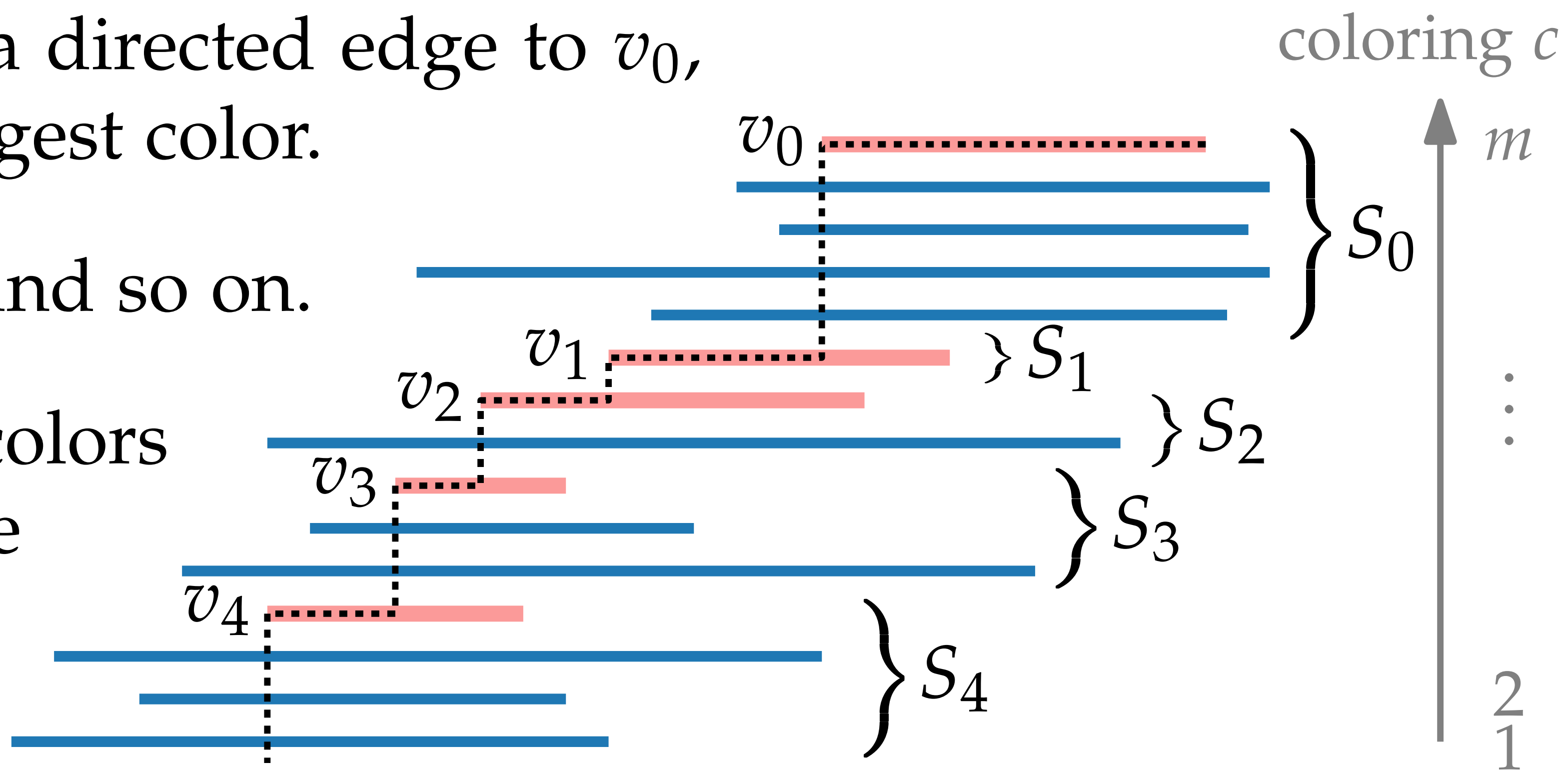

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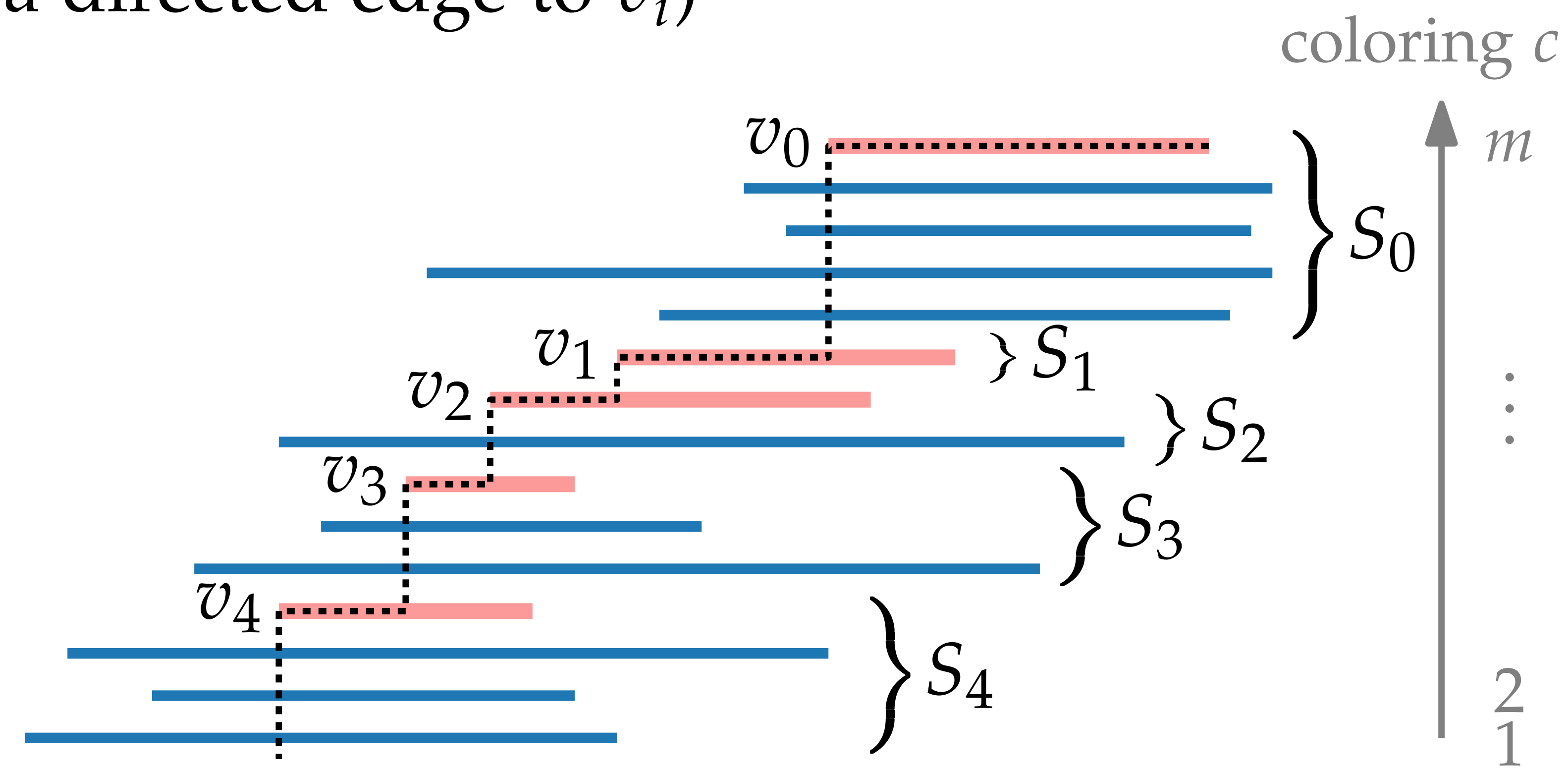

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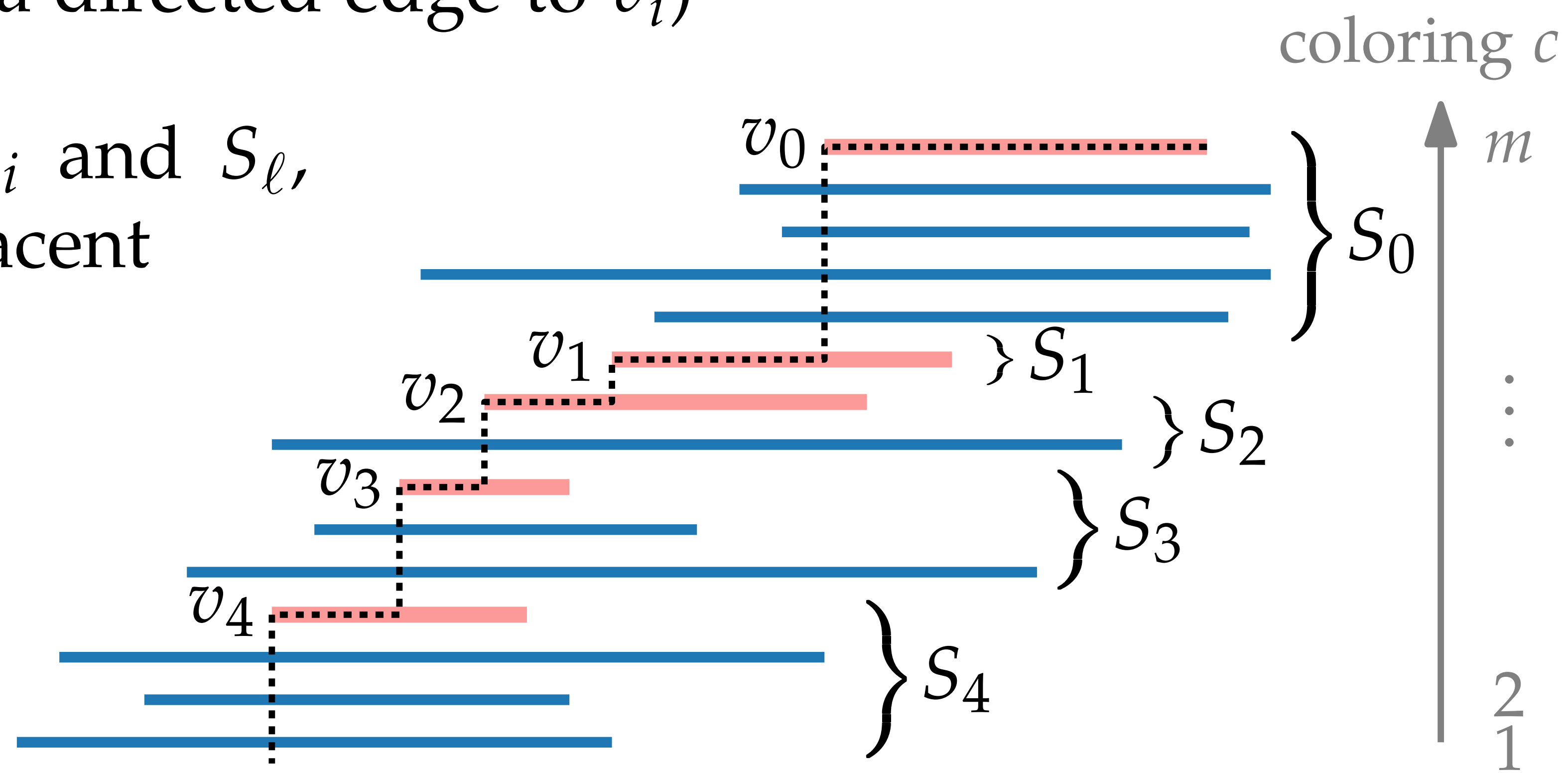

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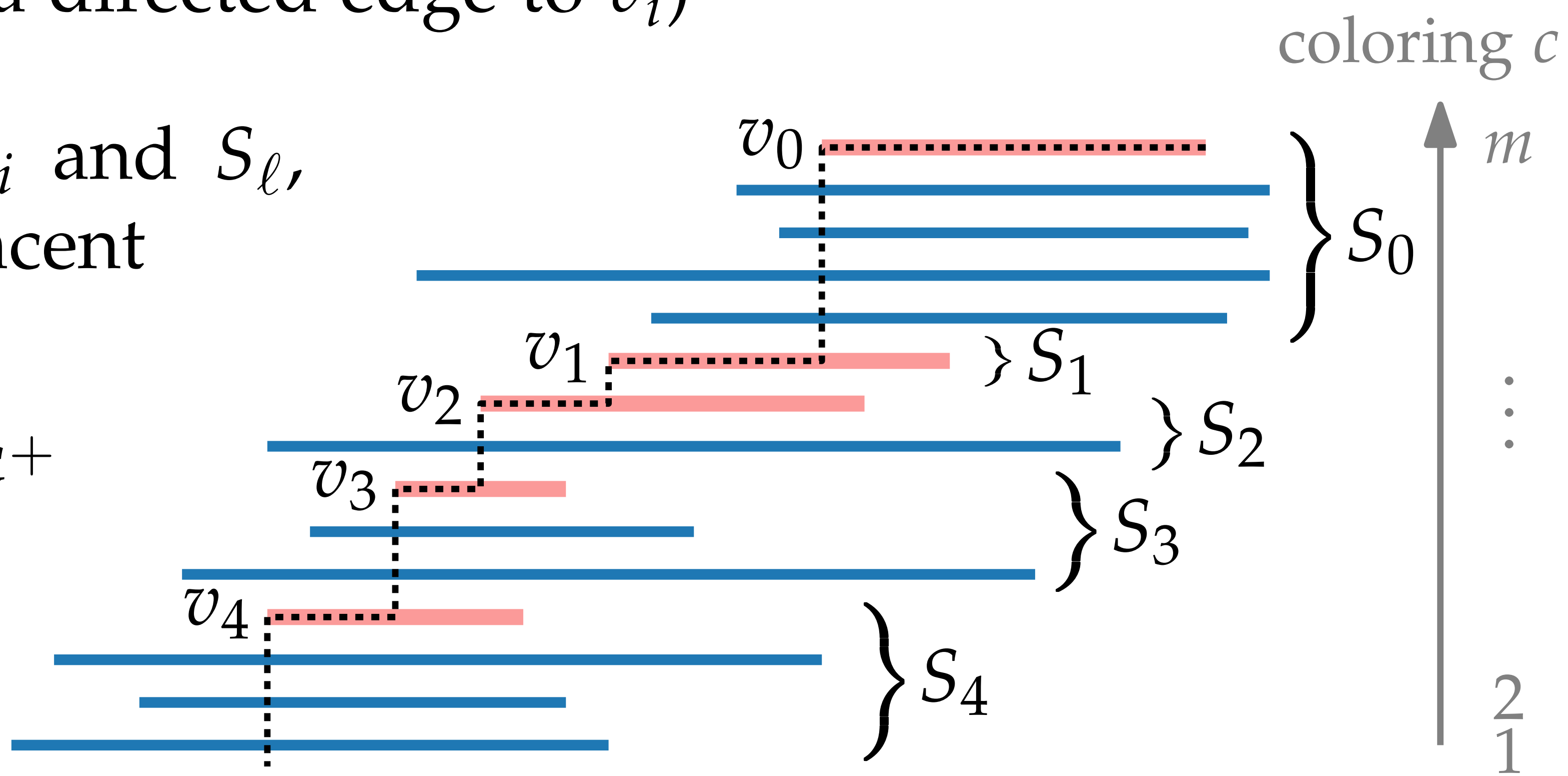

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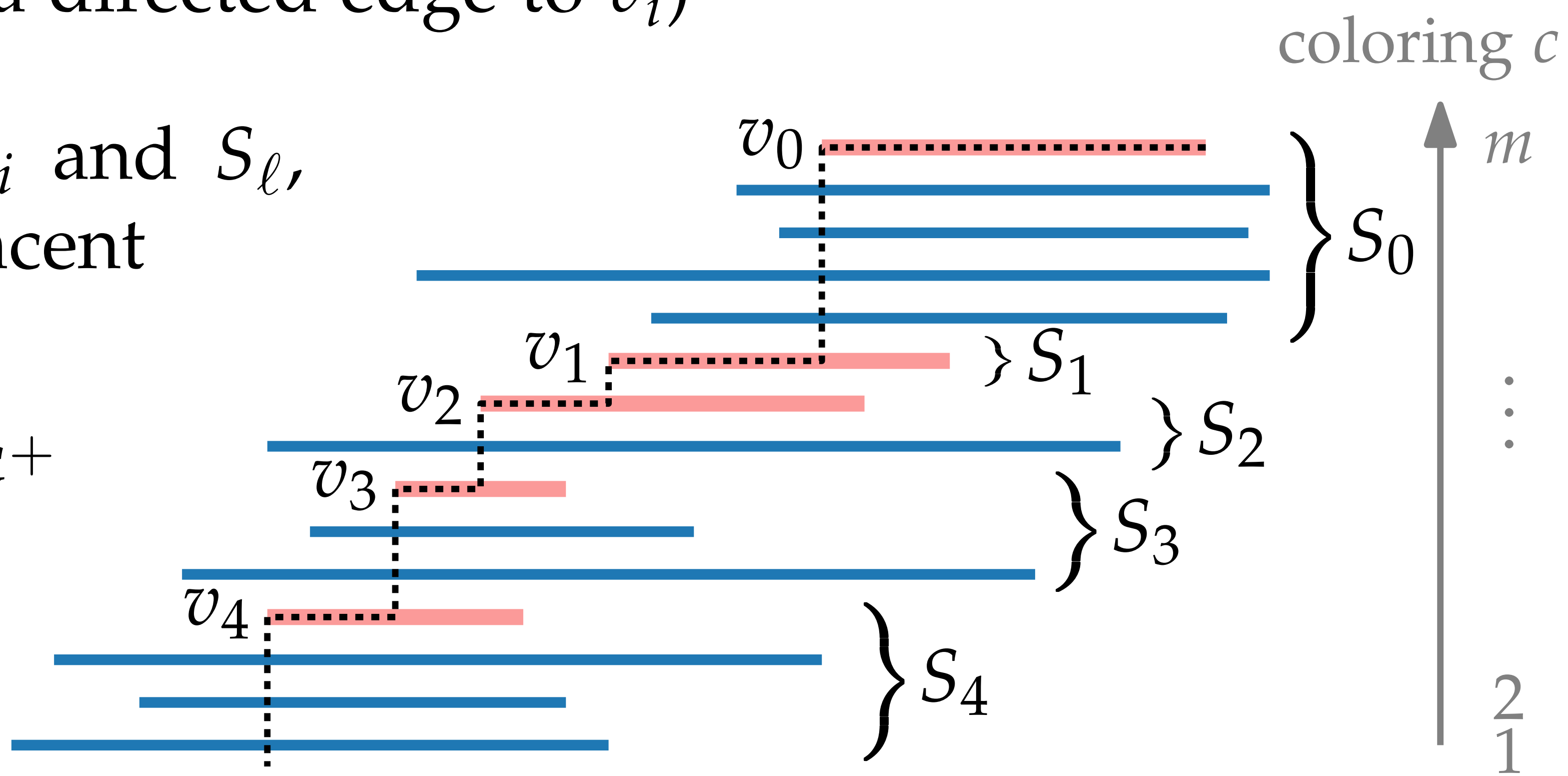
  $\Rightarrow$ S alone requires m colors in G $\square$



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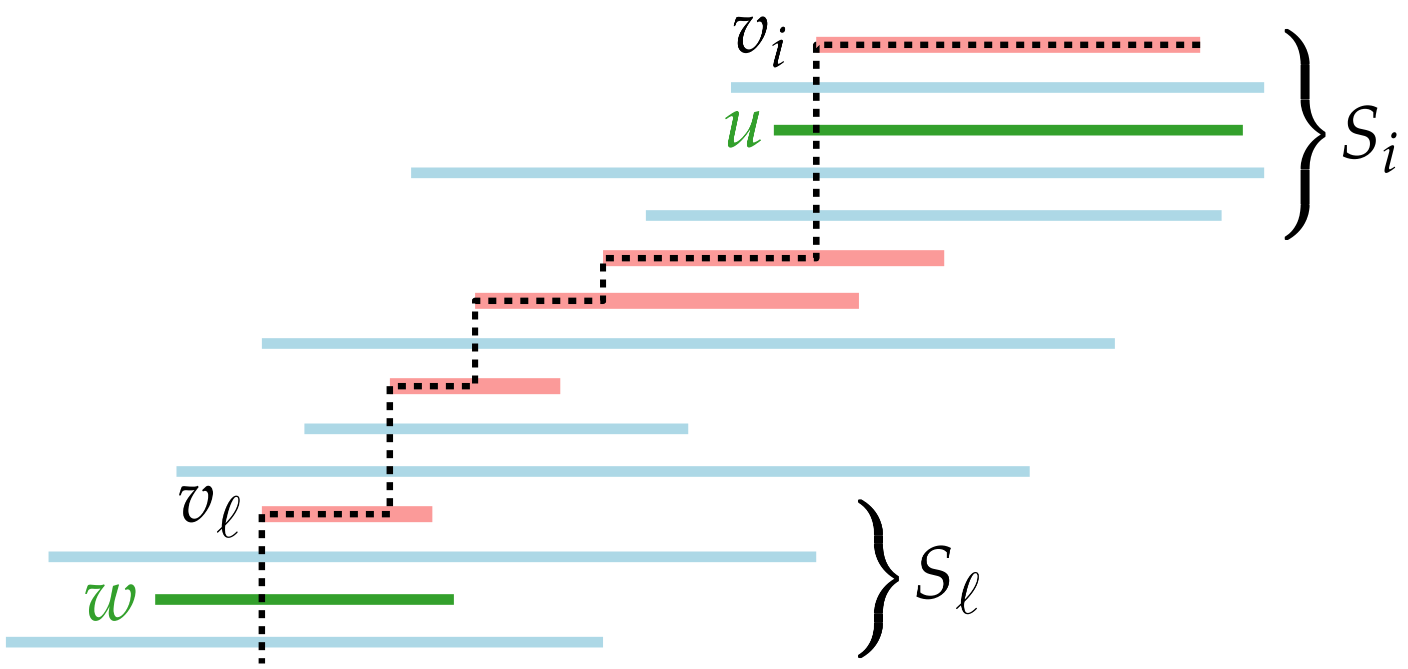

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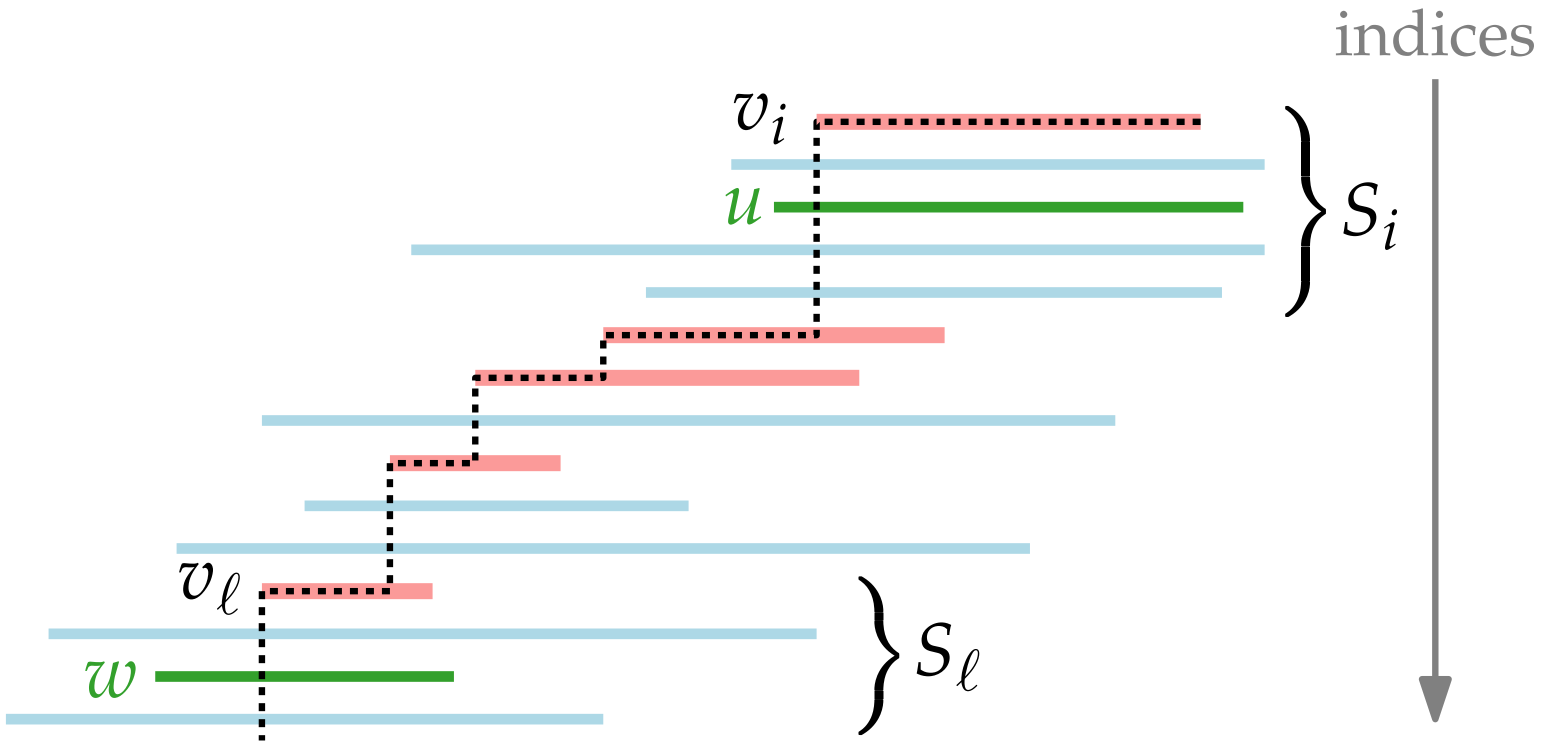

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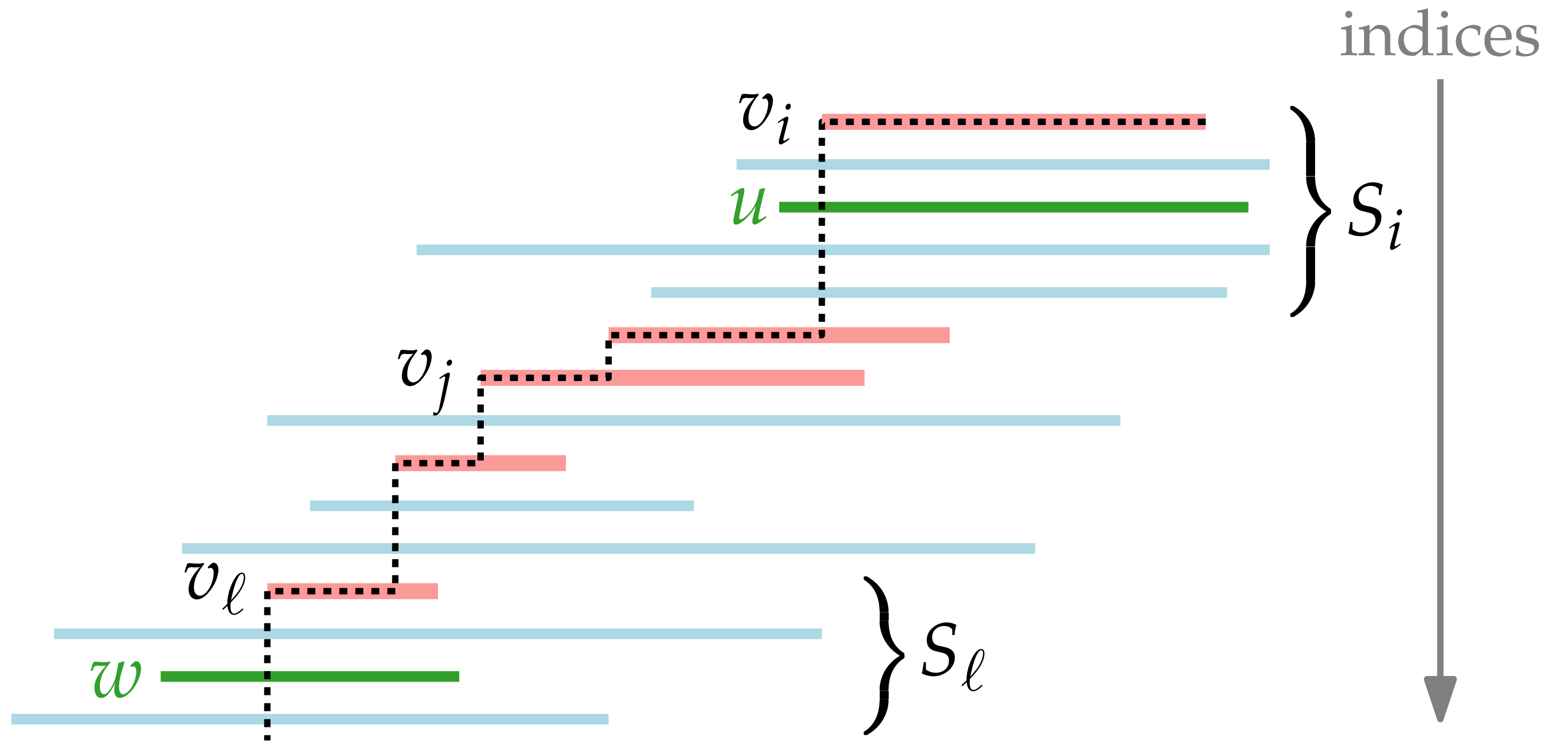

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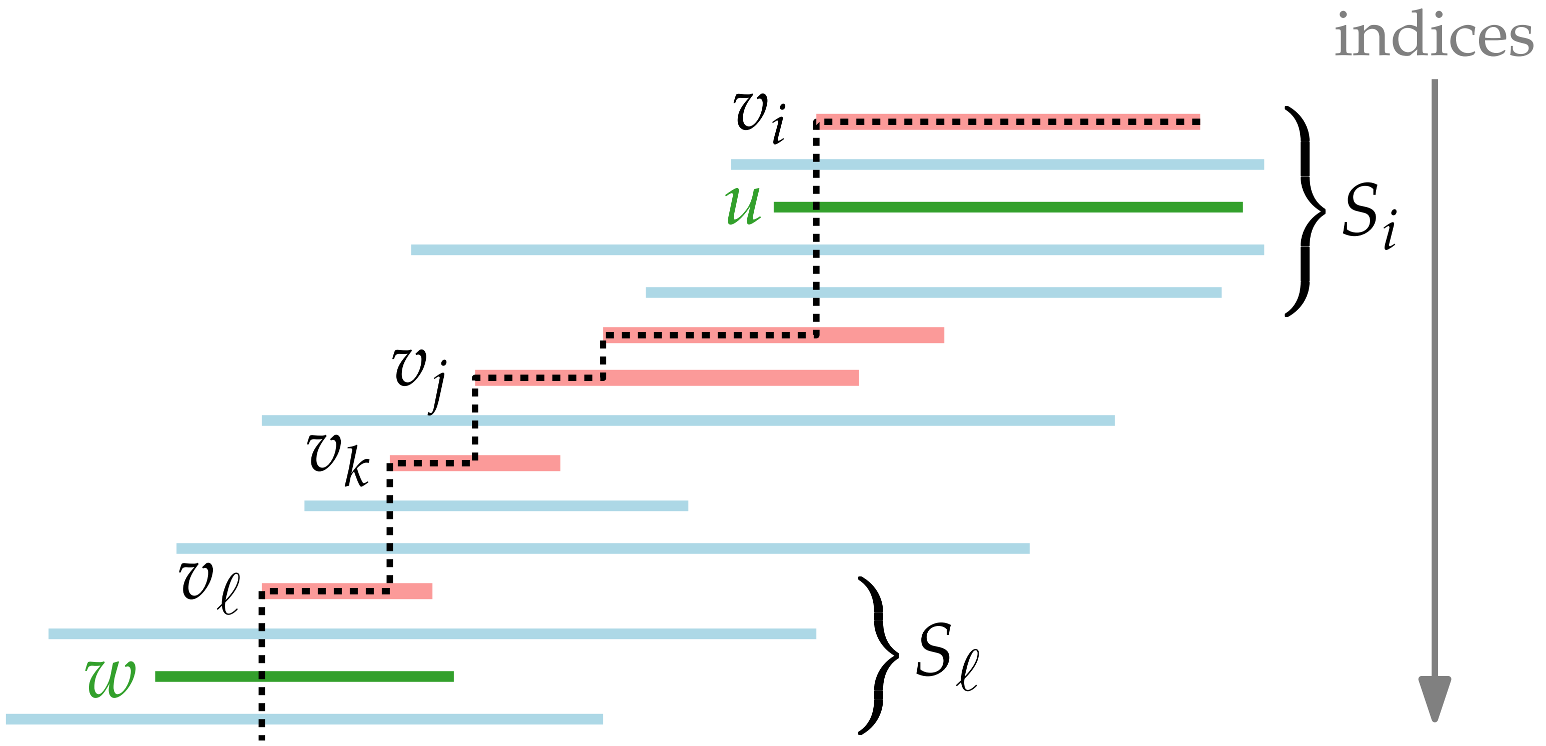

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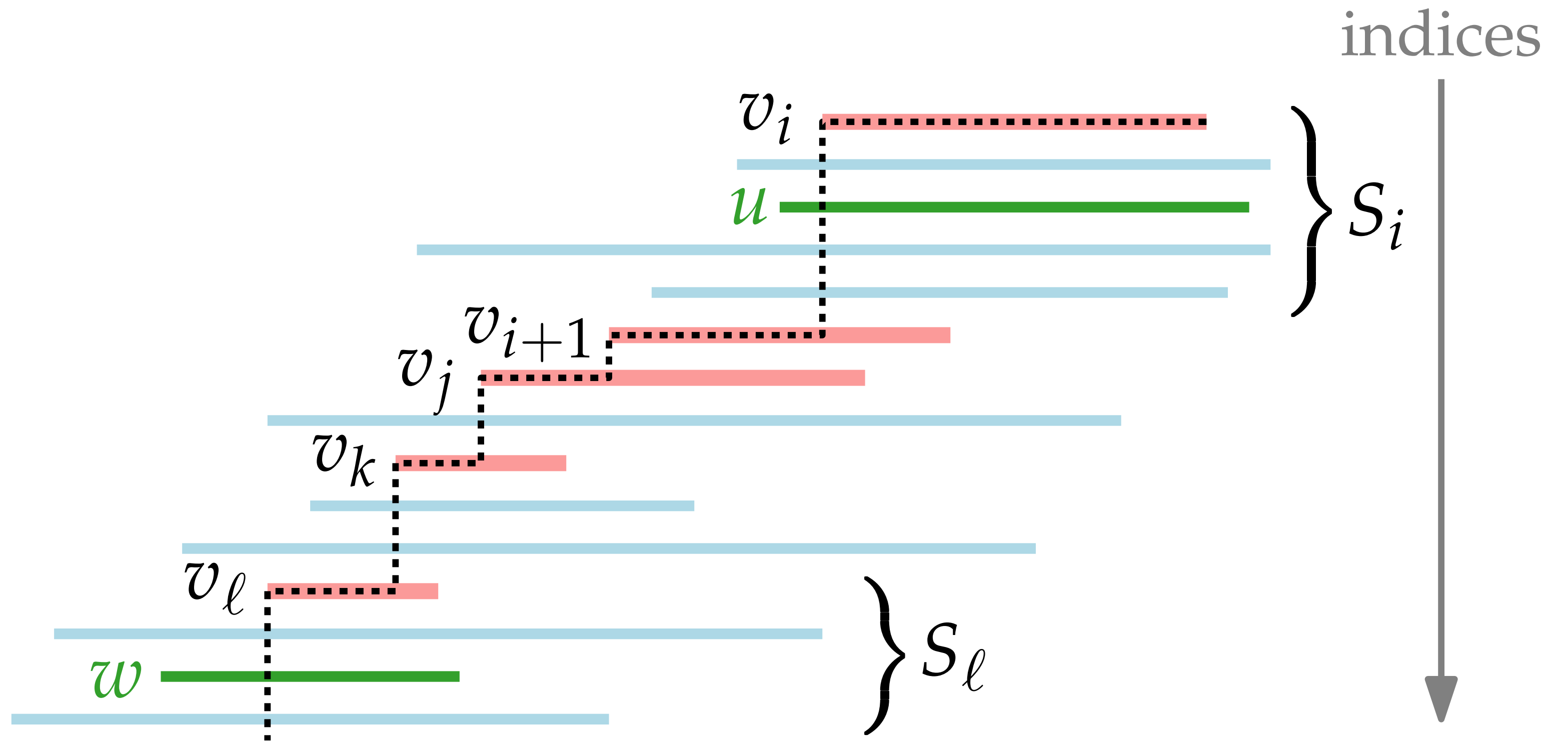

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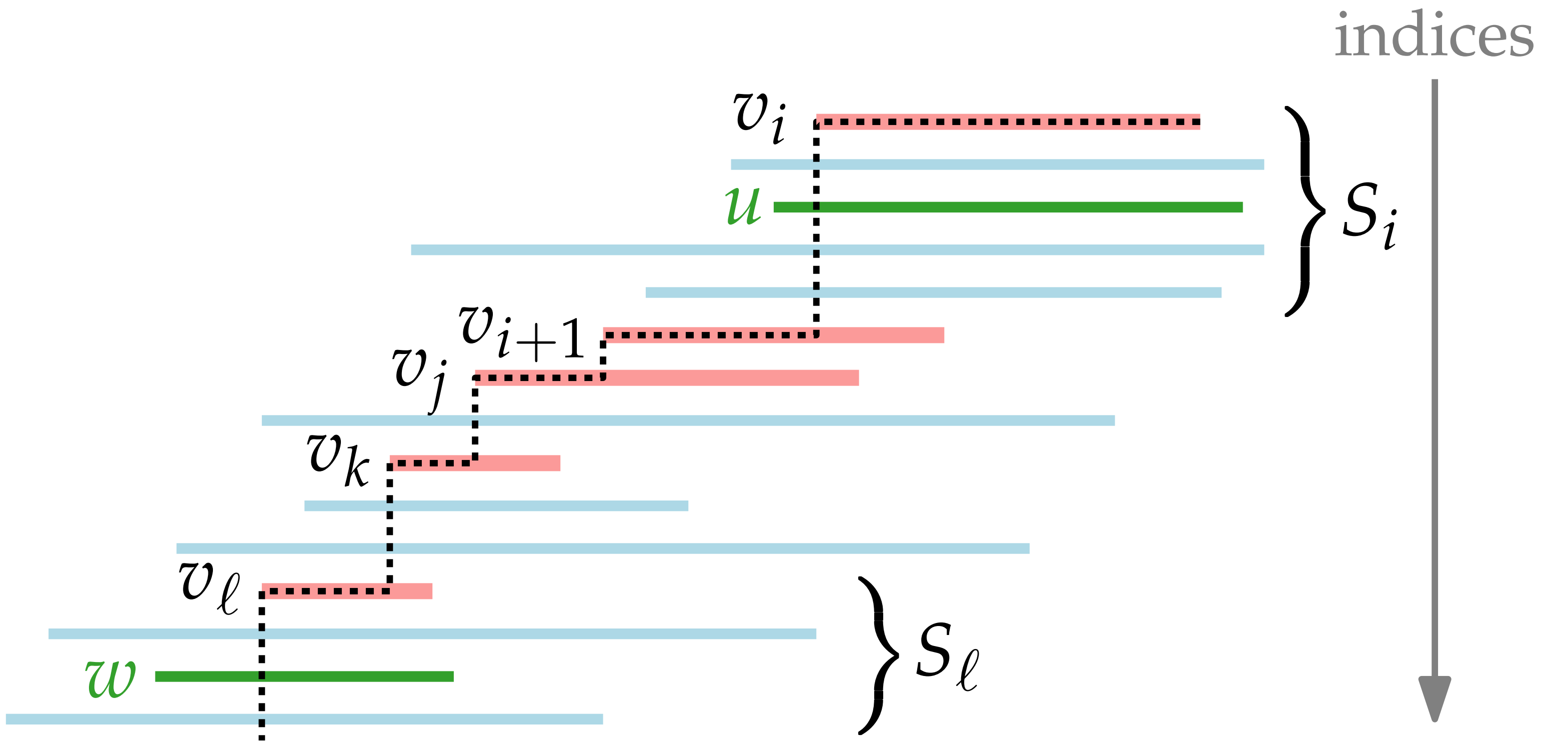

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$$u \cap v_{i+1} \neq \emptyset \qquad \Longrightarrow_{u \cap w = \emptyset}$$
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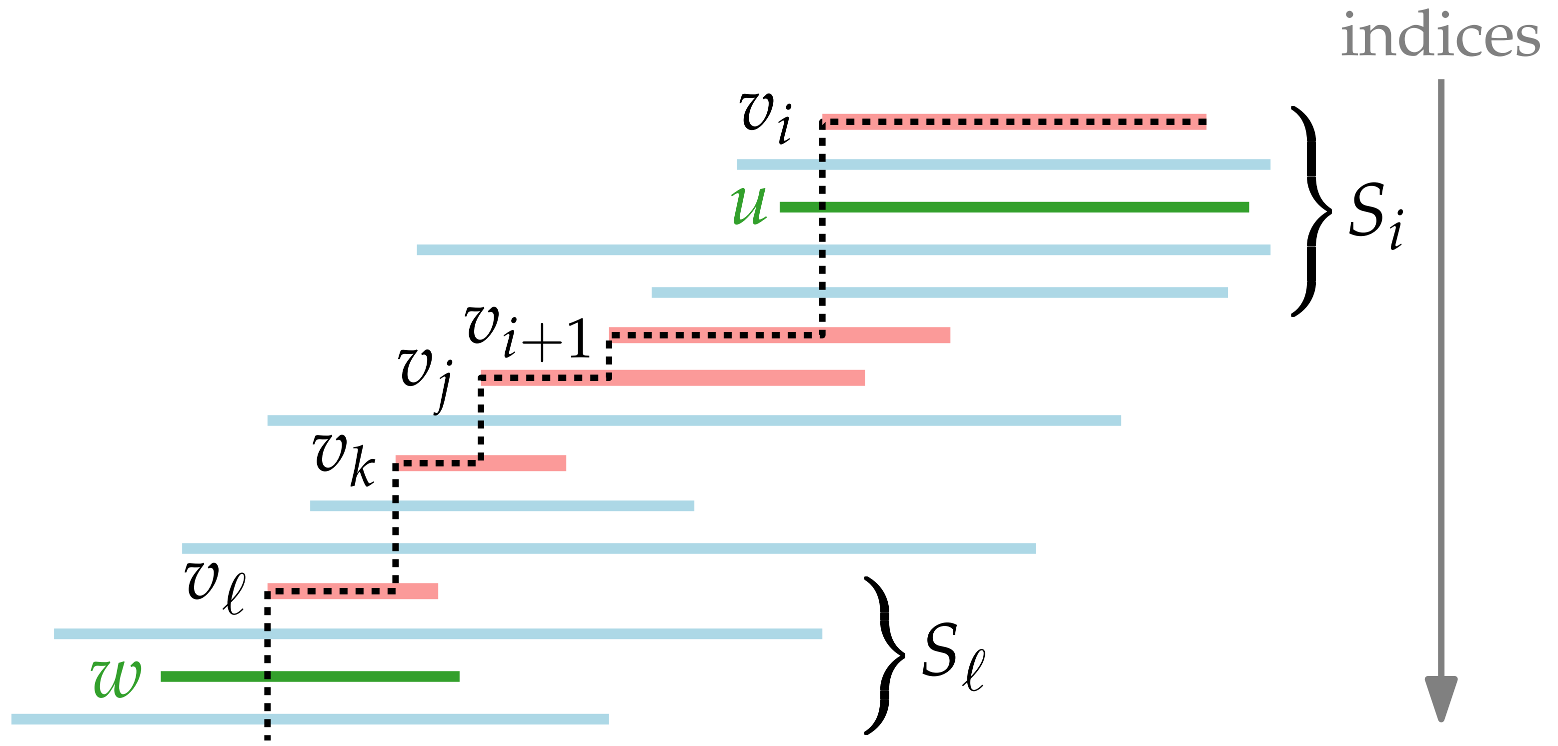

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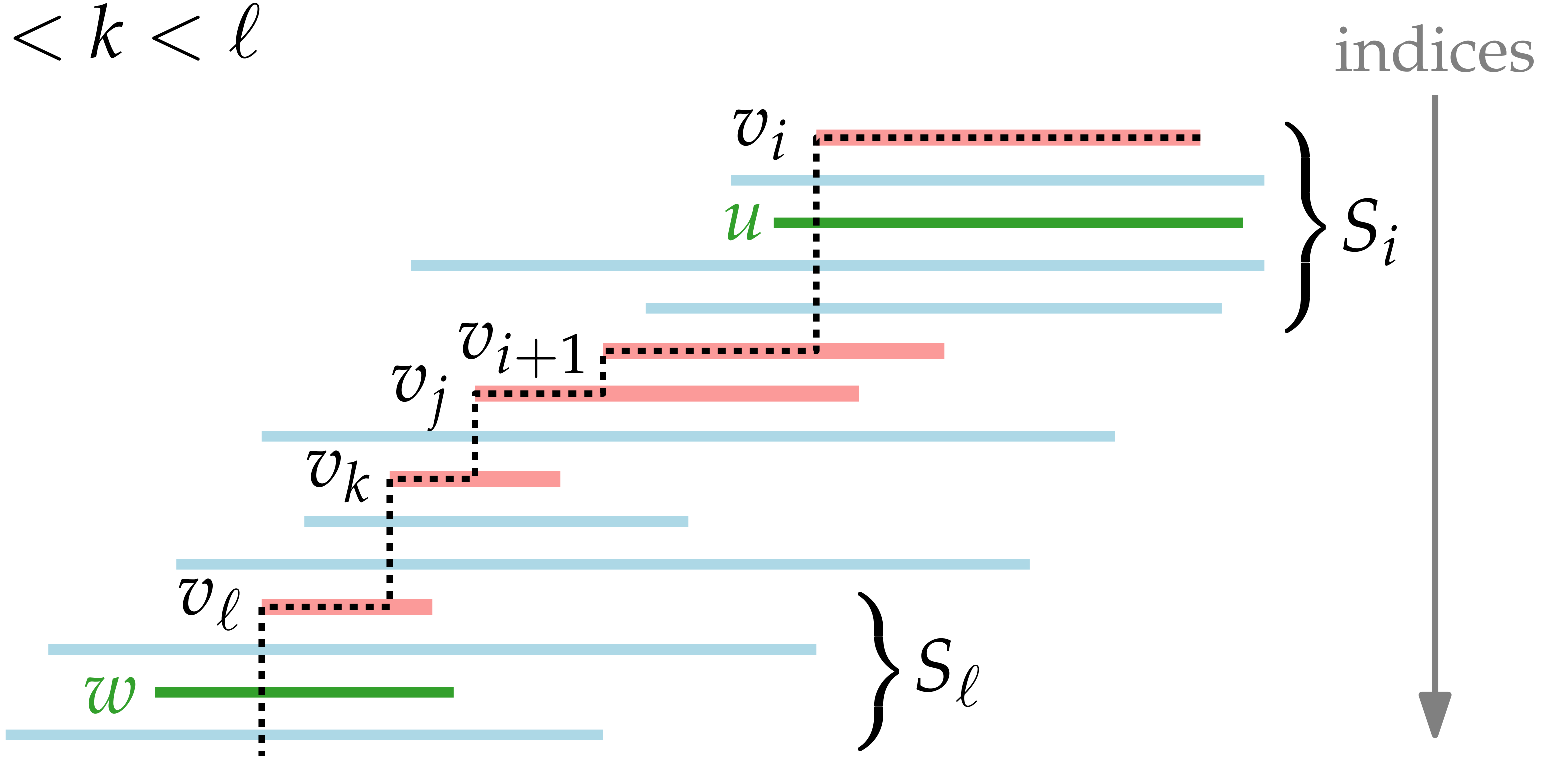

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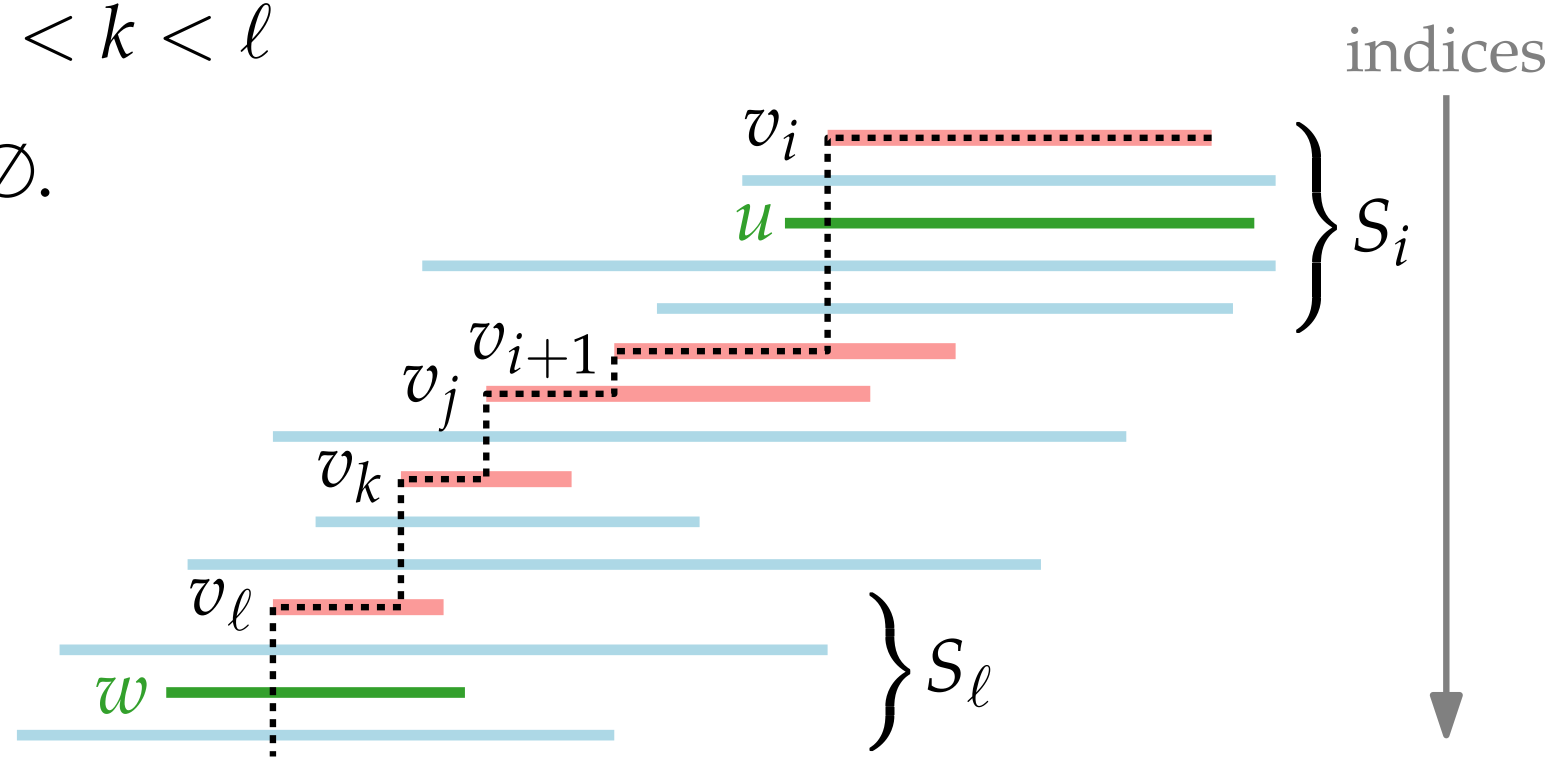

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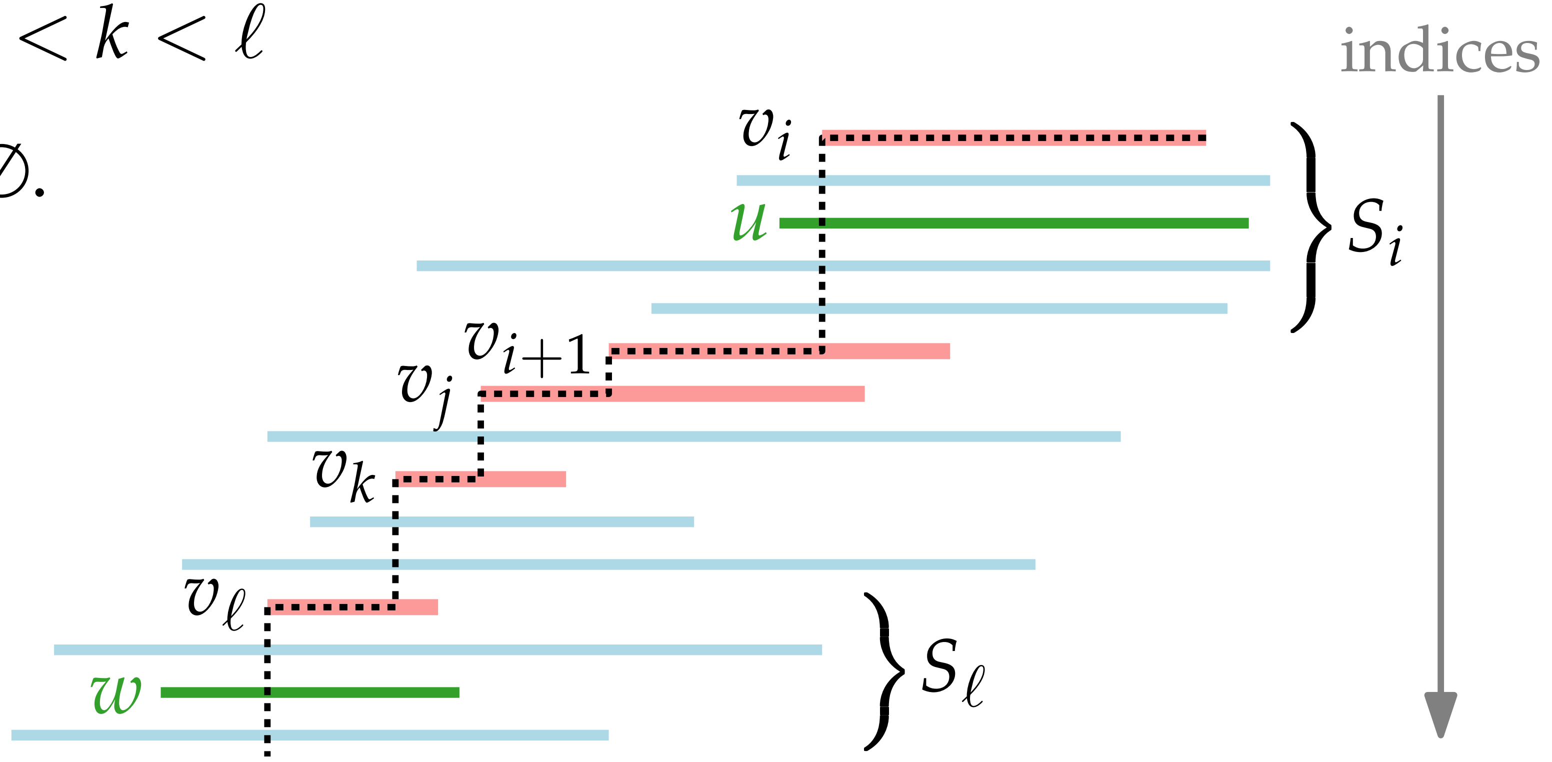
$\Rightarrow$ u and v_j overlap



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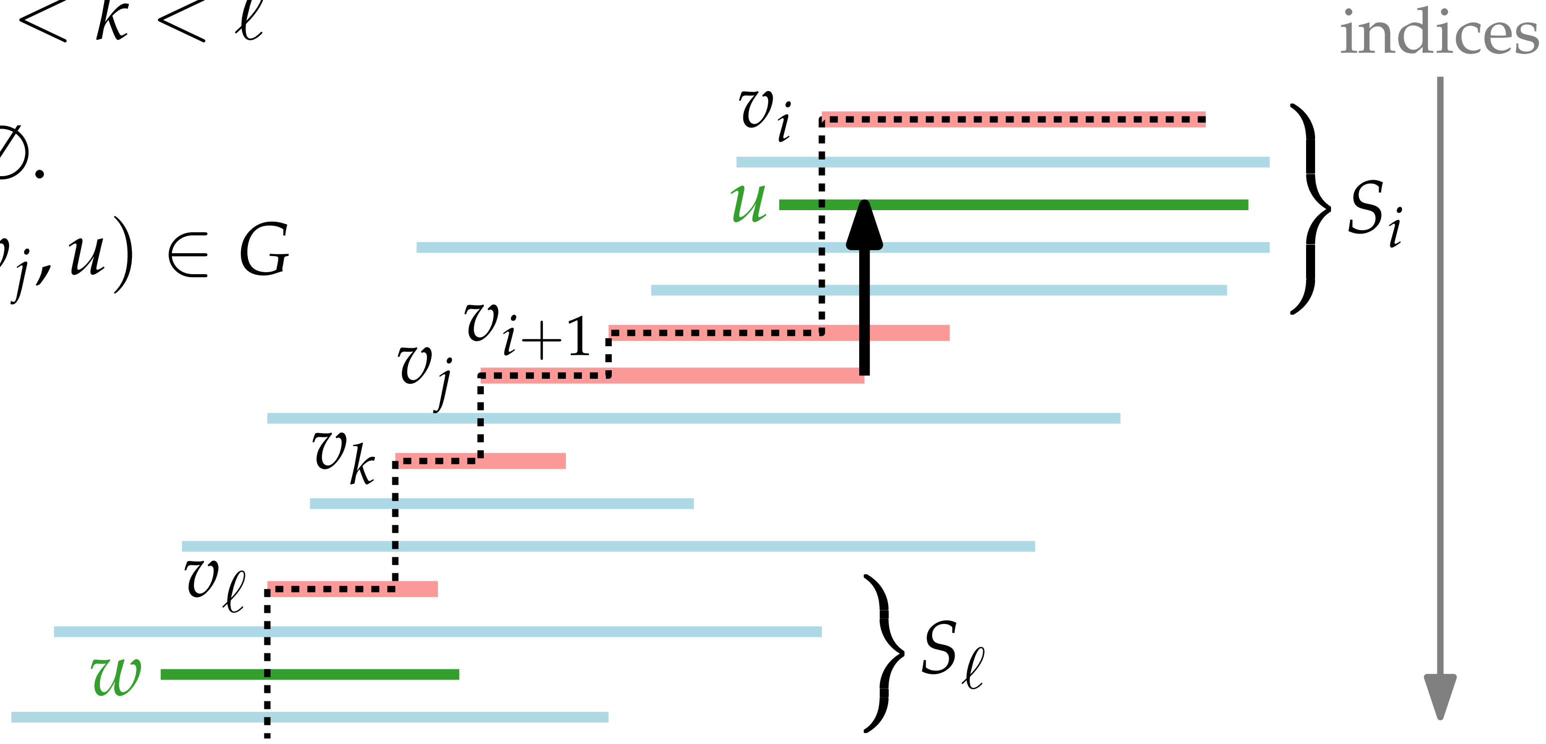
By definition, $u \cap v_{j+1} = \emptyset$.
$\Rightarrow u$ and v_j overlap $\Rightarrow (v_j, u) \in G$



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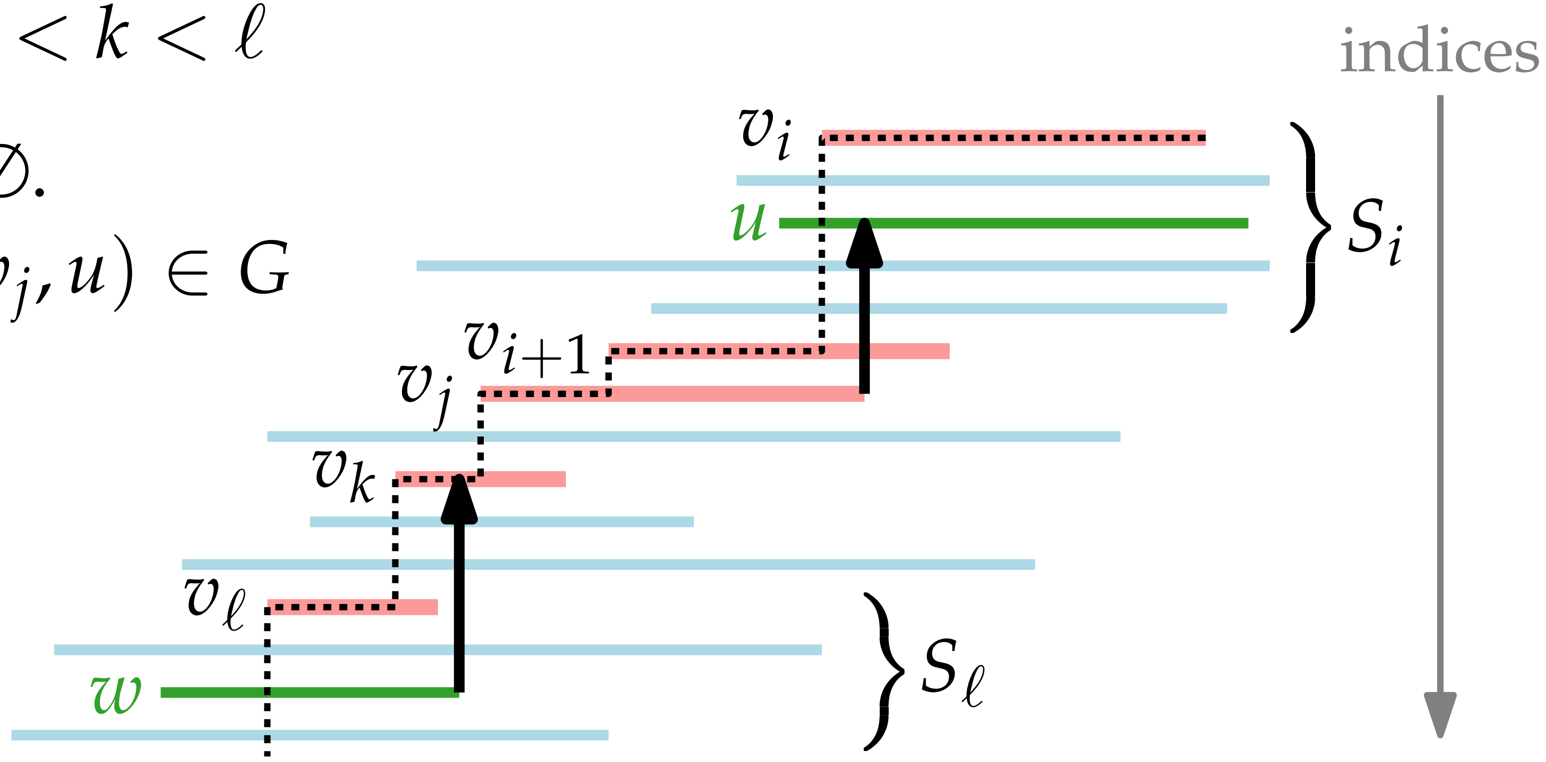

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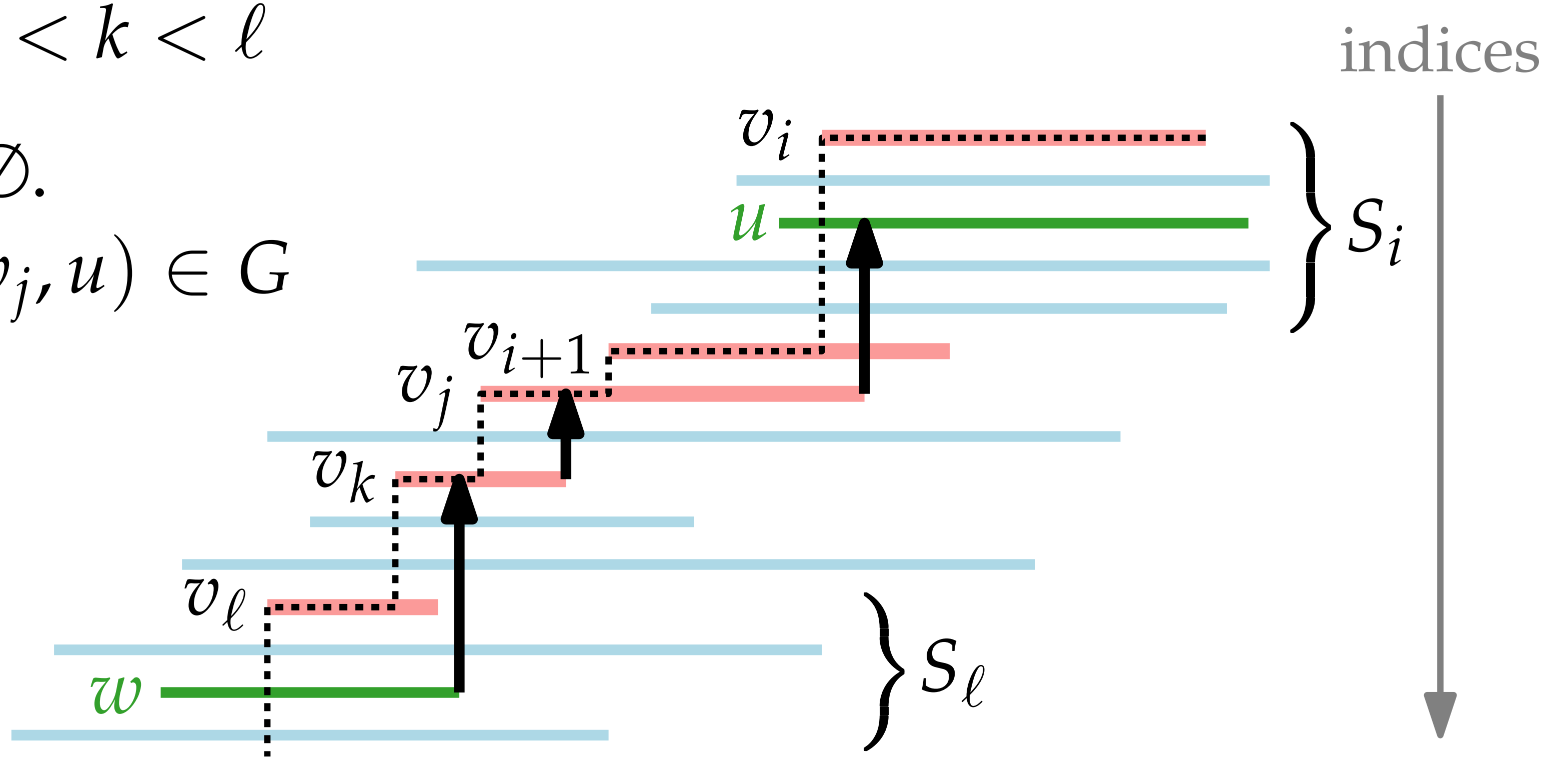

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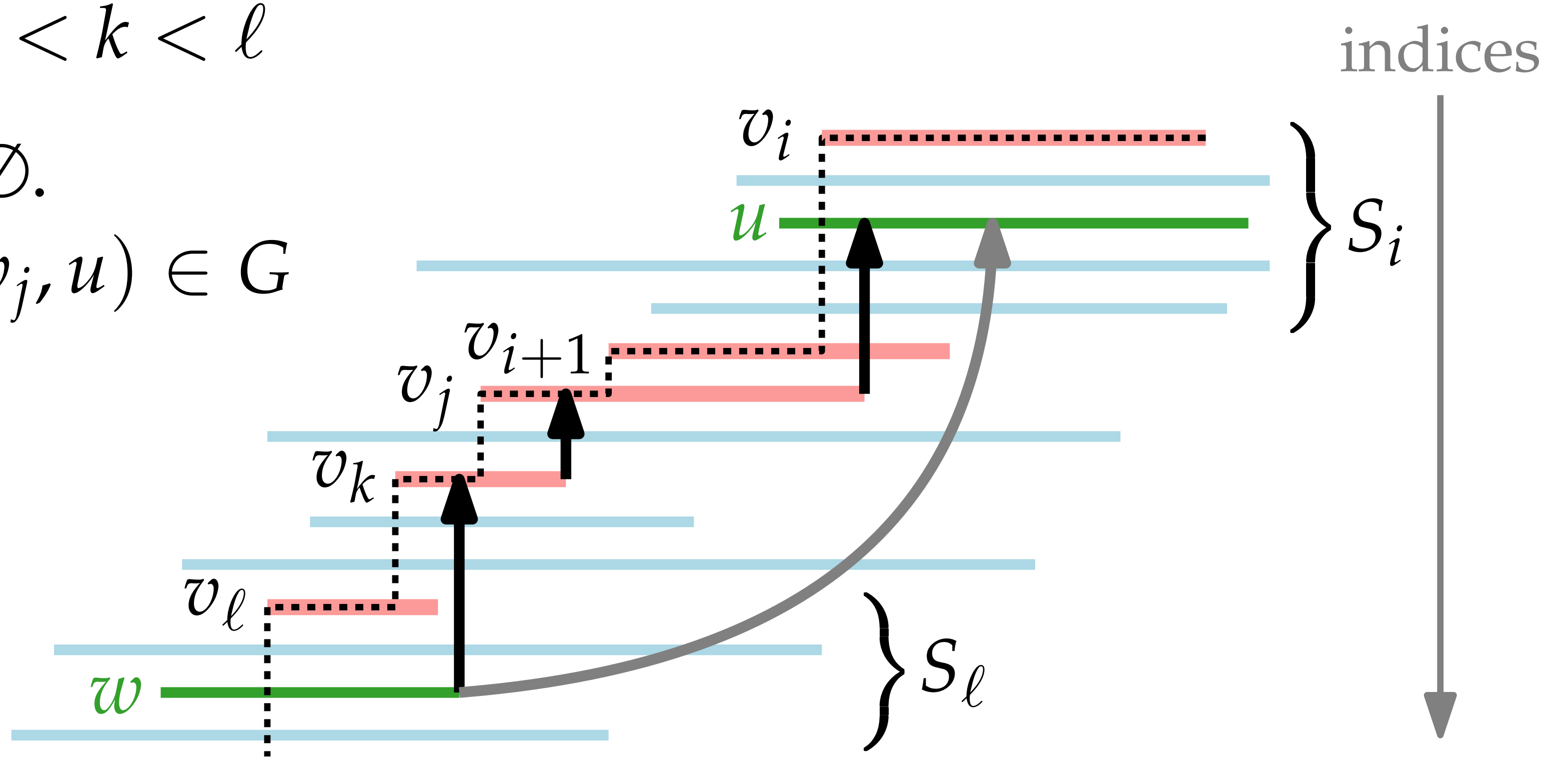

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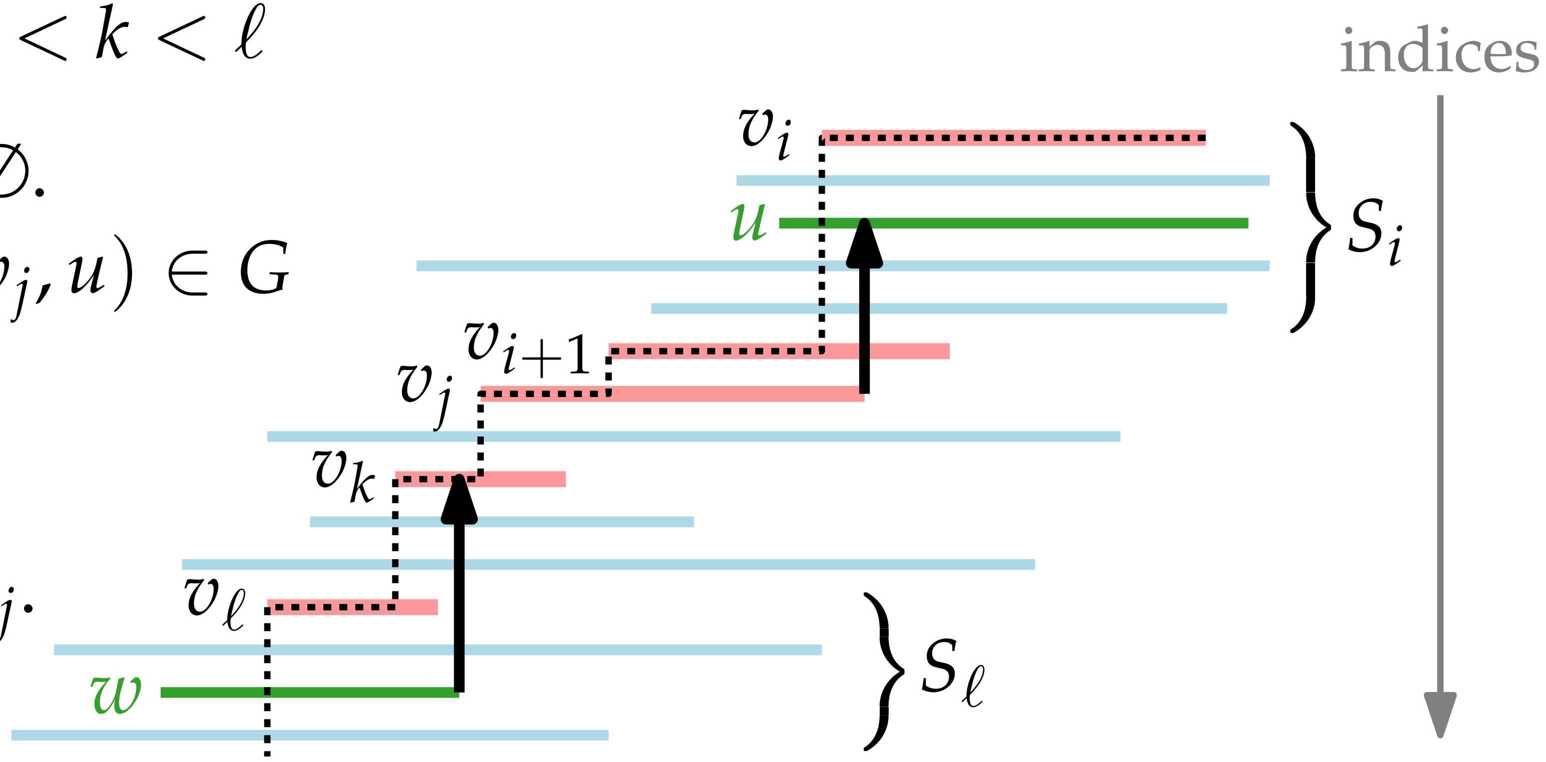

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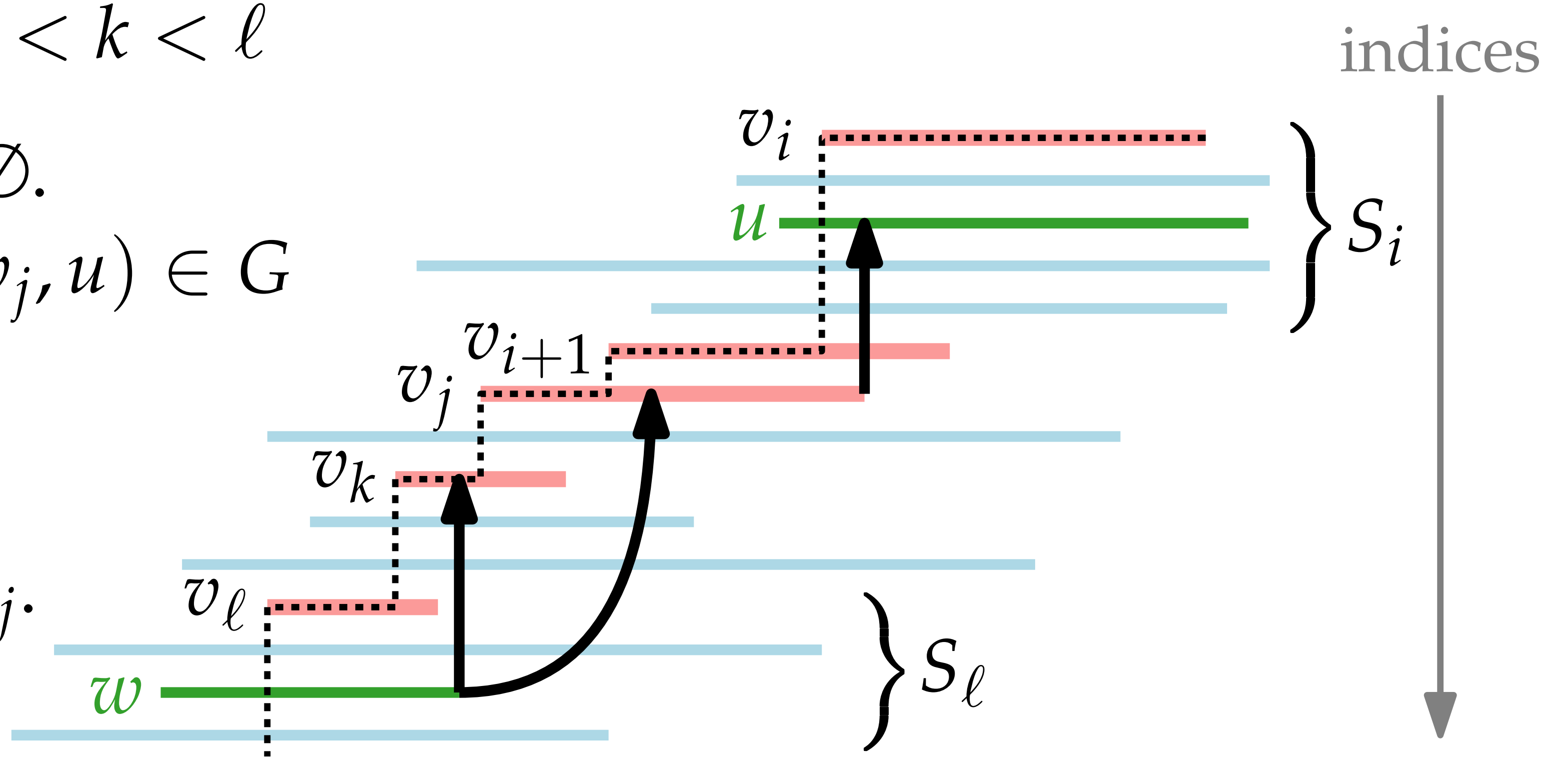

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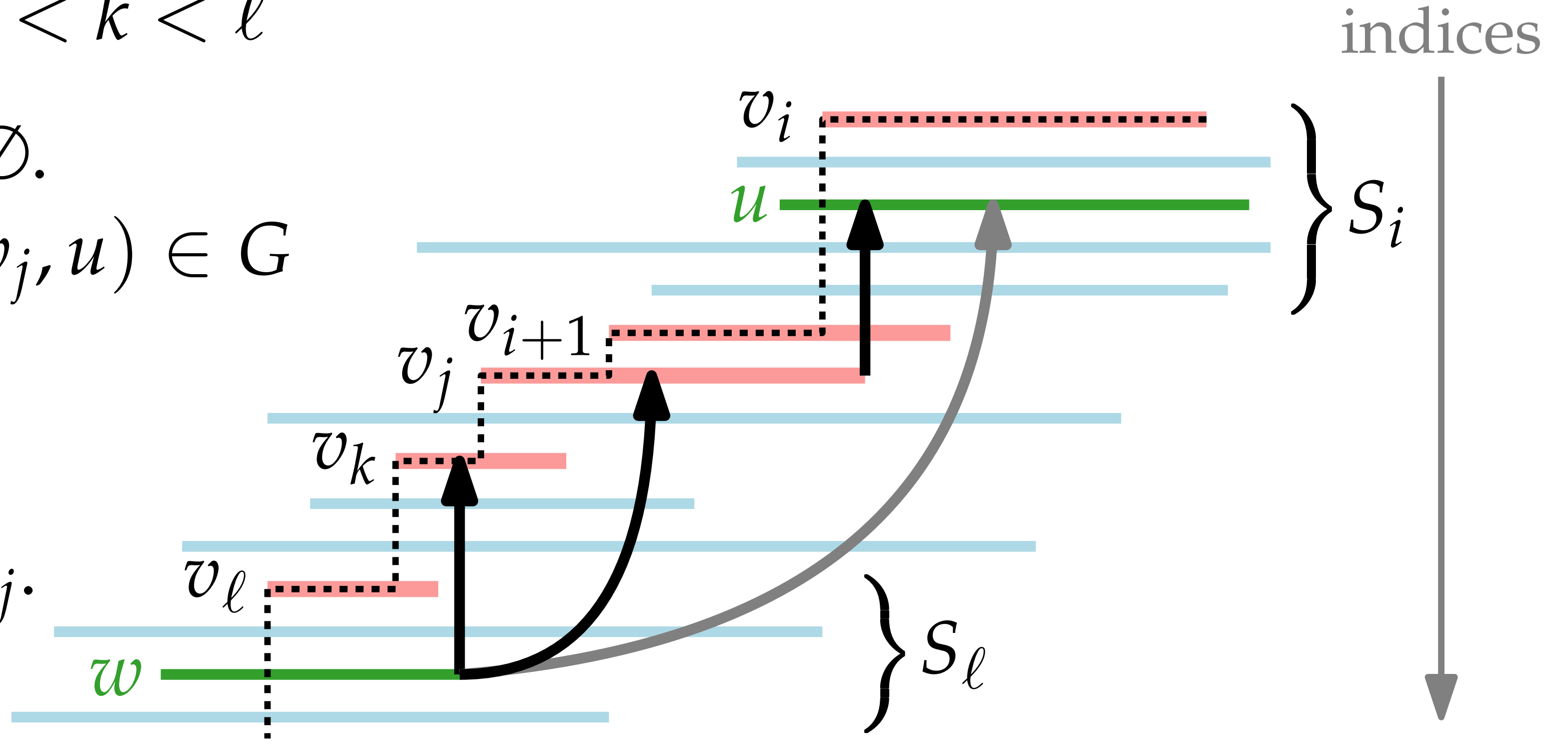

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Find a graph coloring $c\colon V \to \mathbb{N}$ such that:
[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

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- recognition in $O(n^2)$ time
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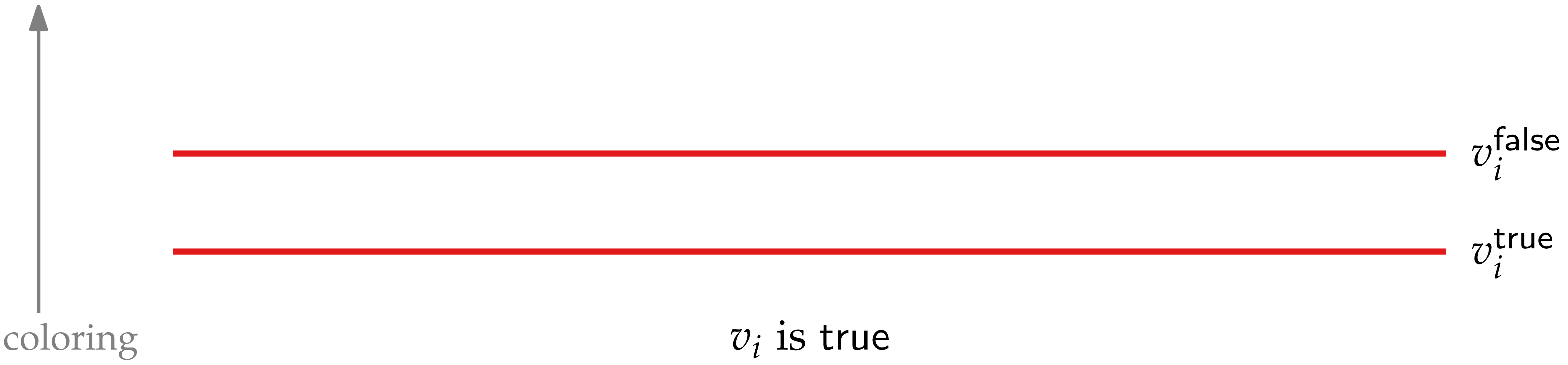

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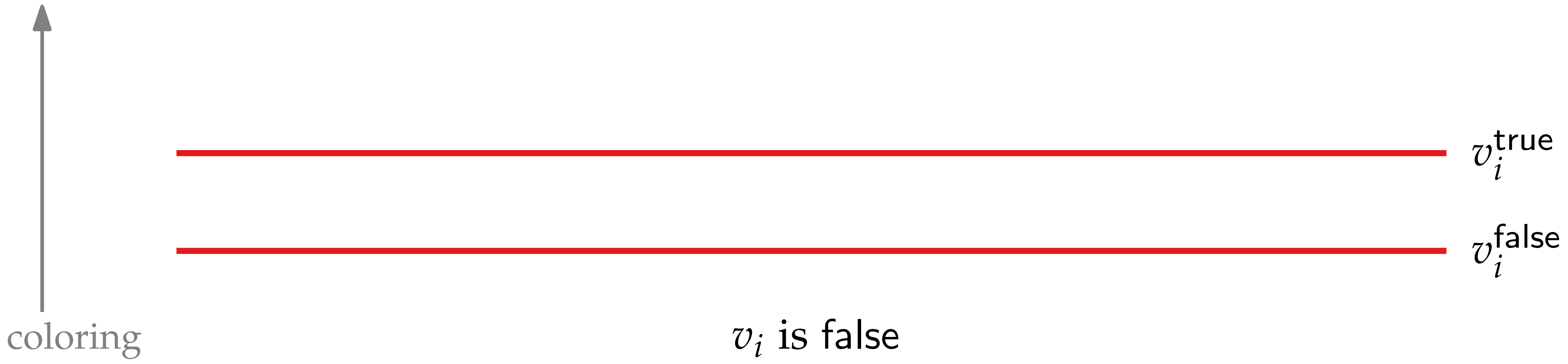

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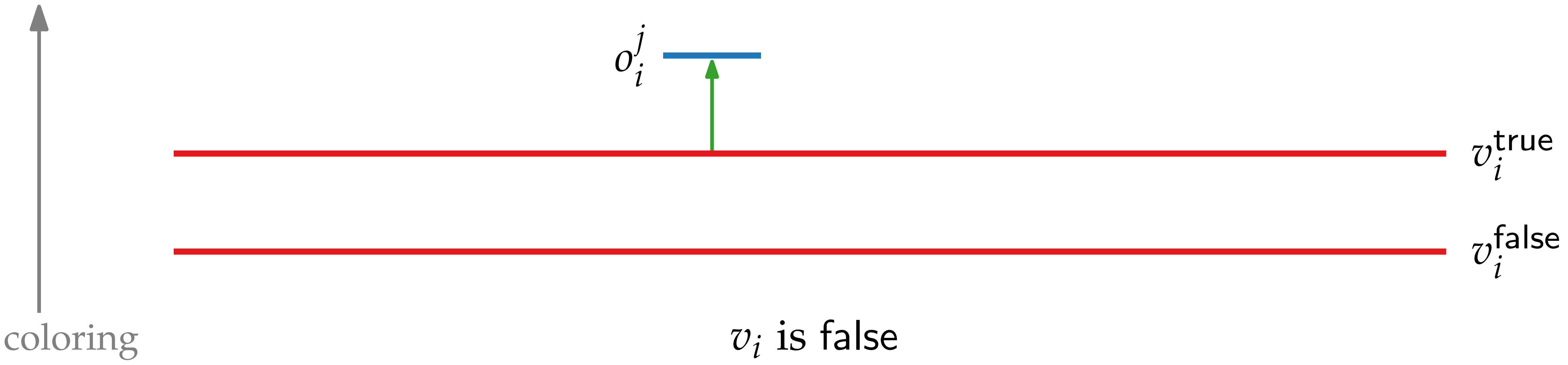

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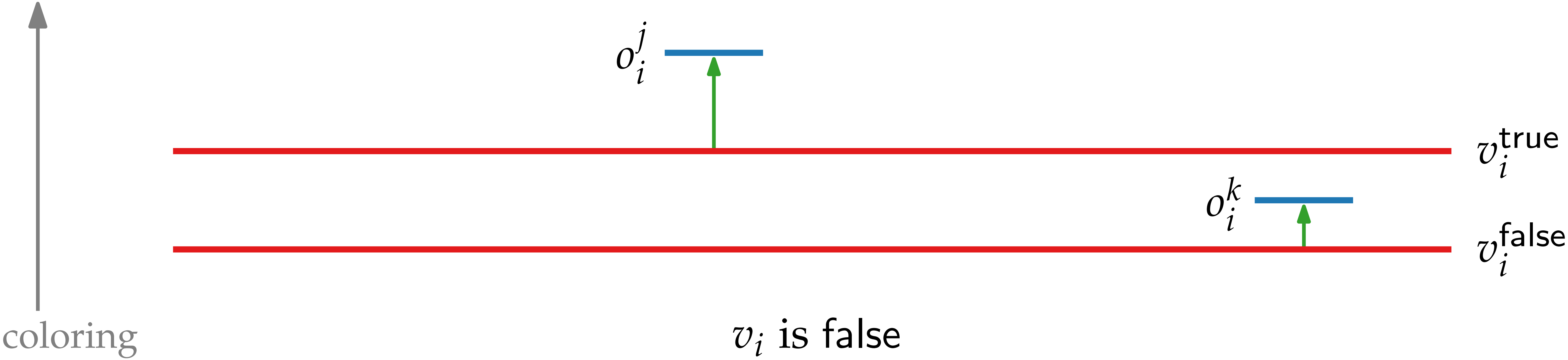

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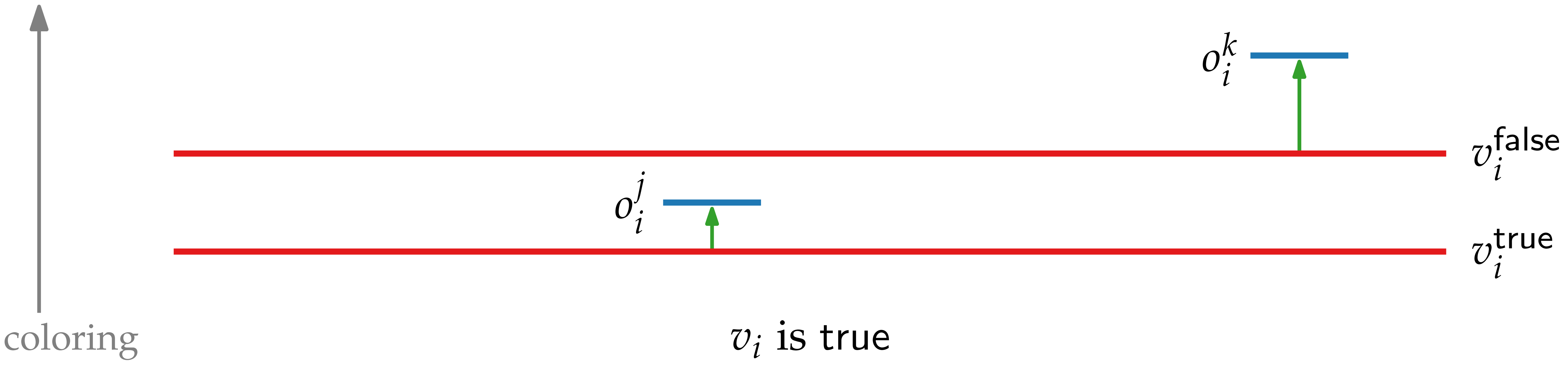

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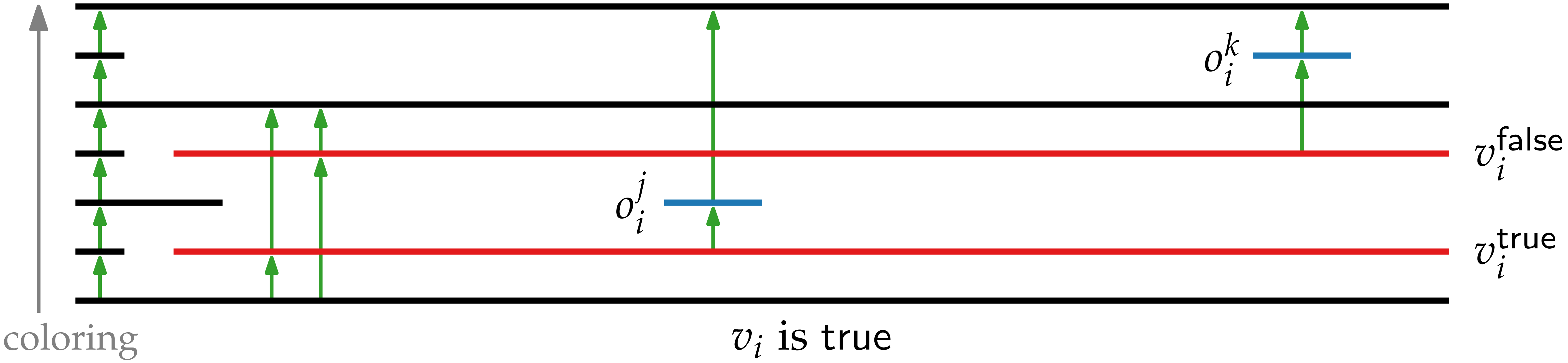

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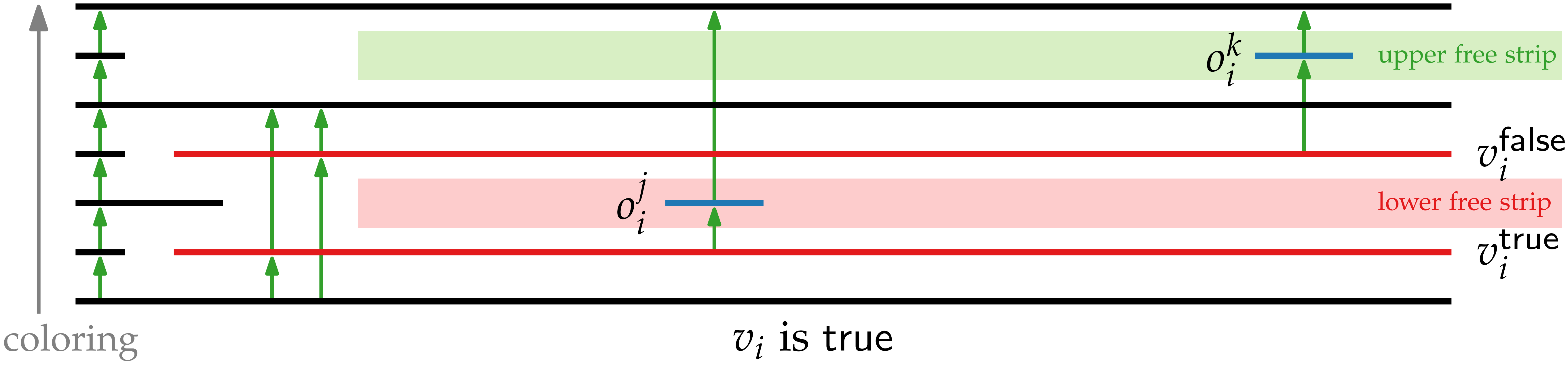

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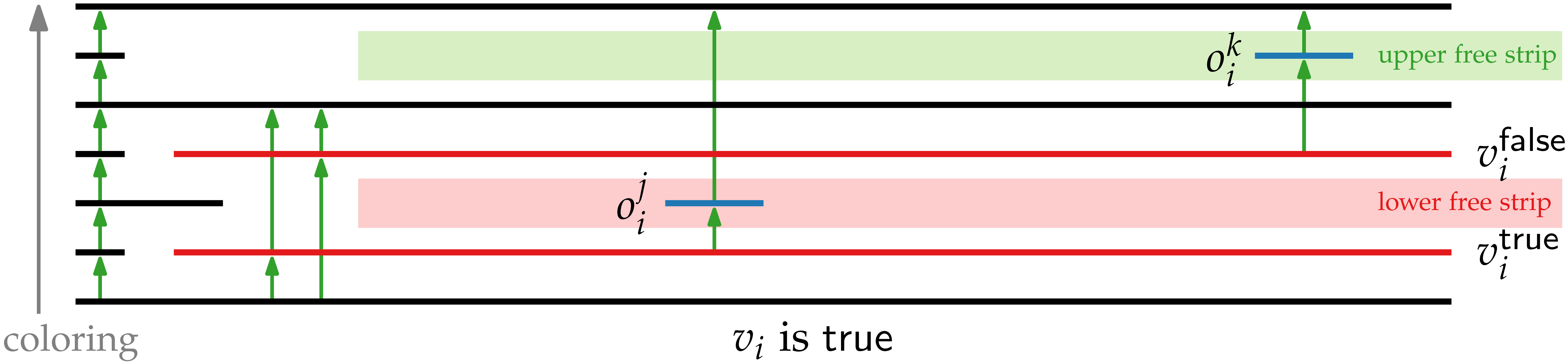

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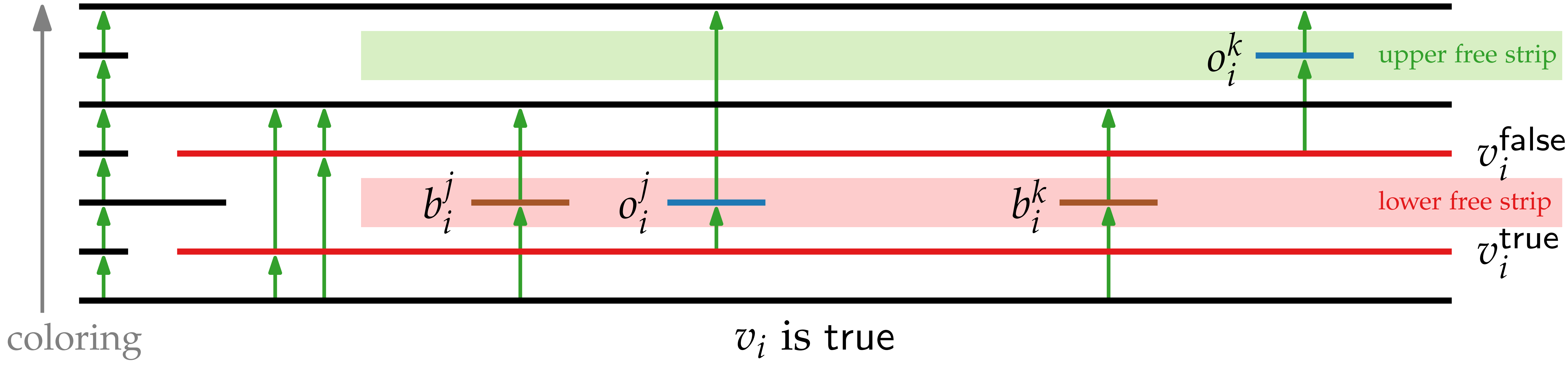

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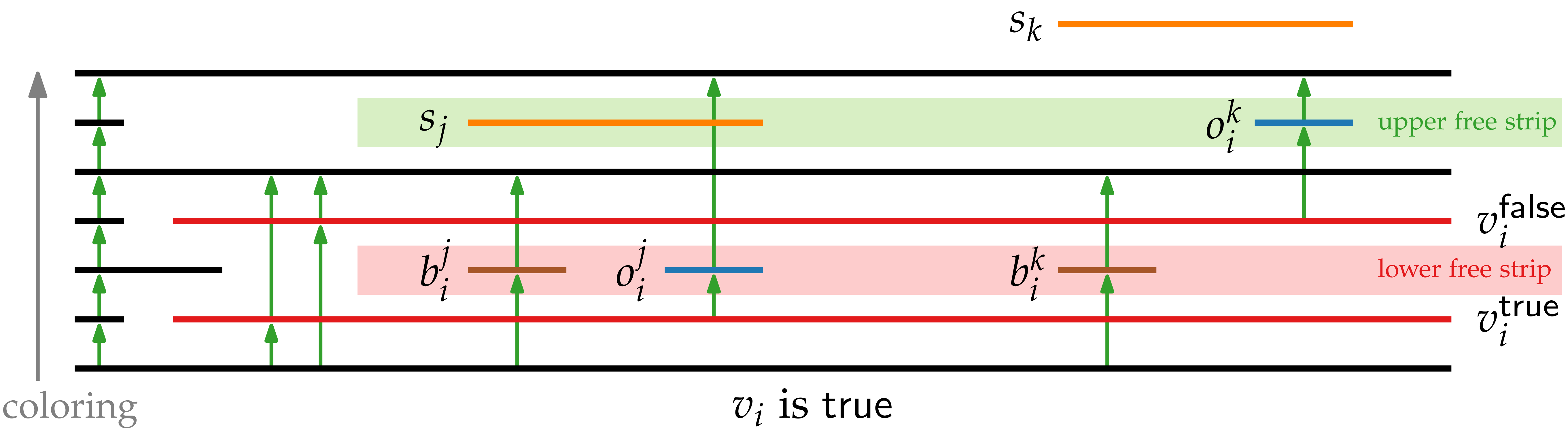

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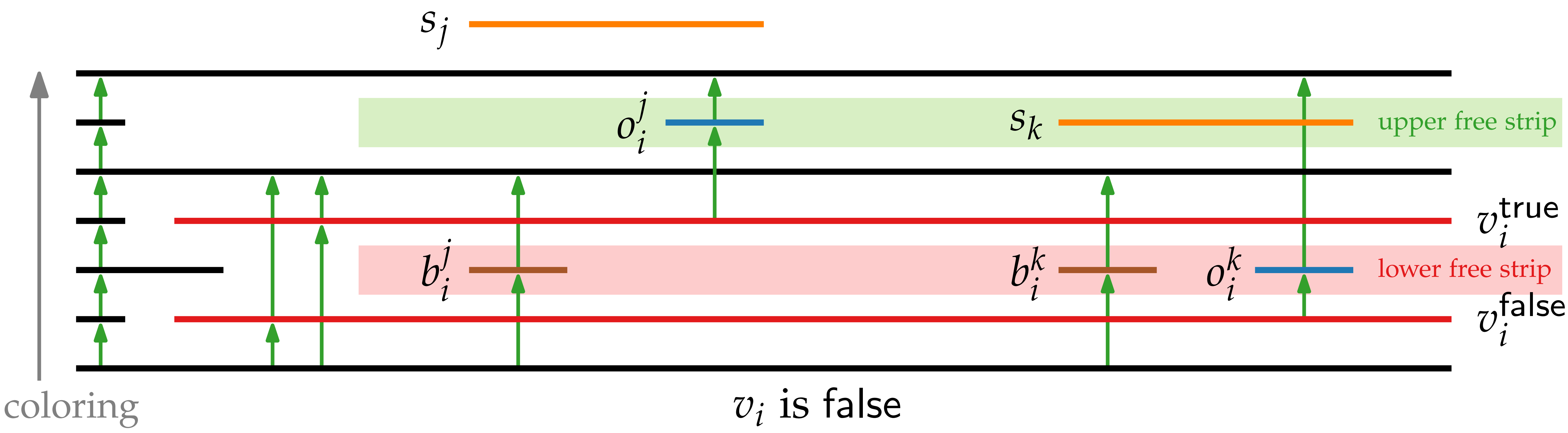

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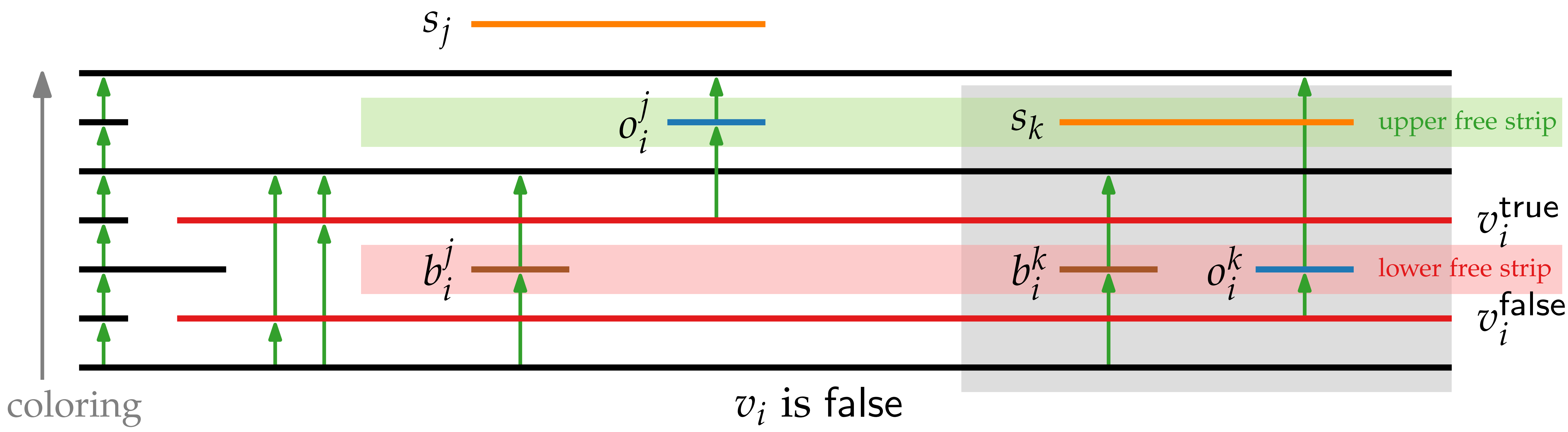

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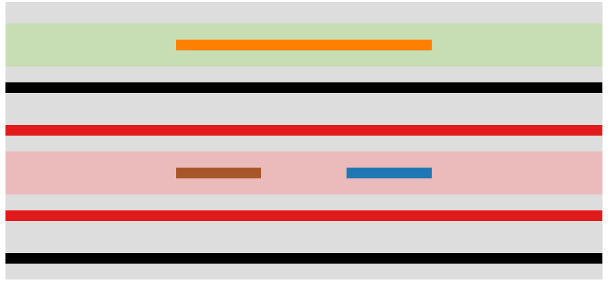

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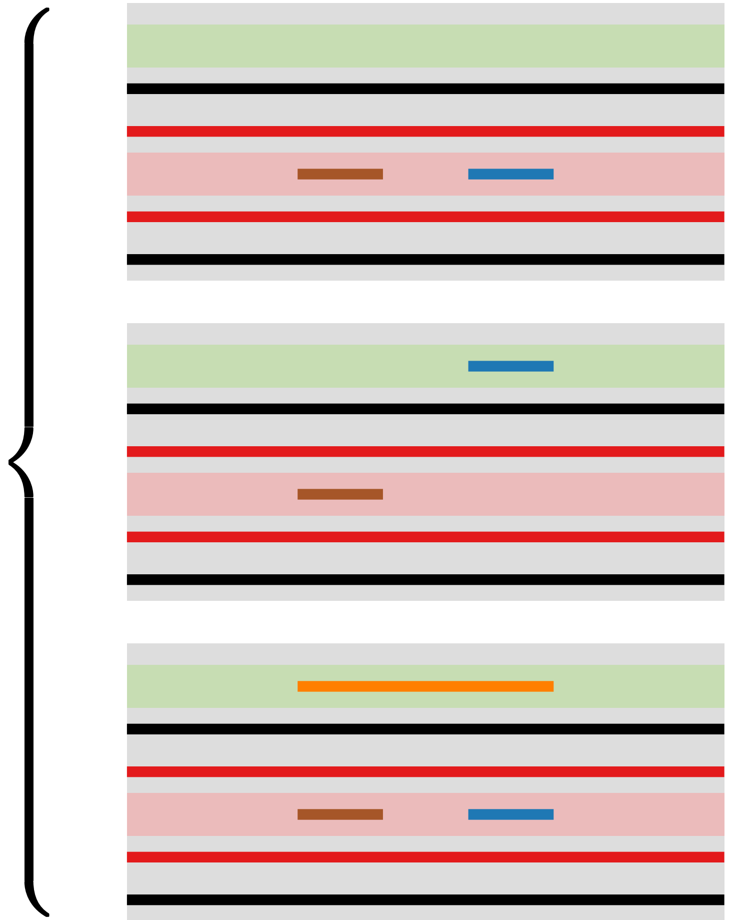

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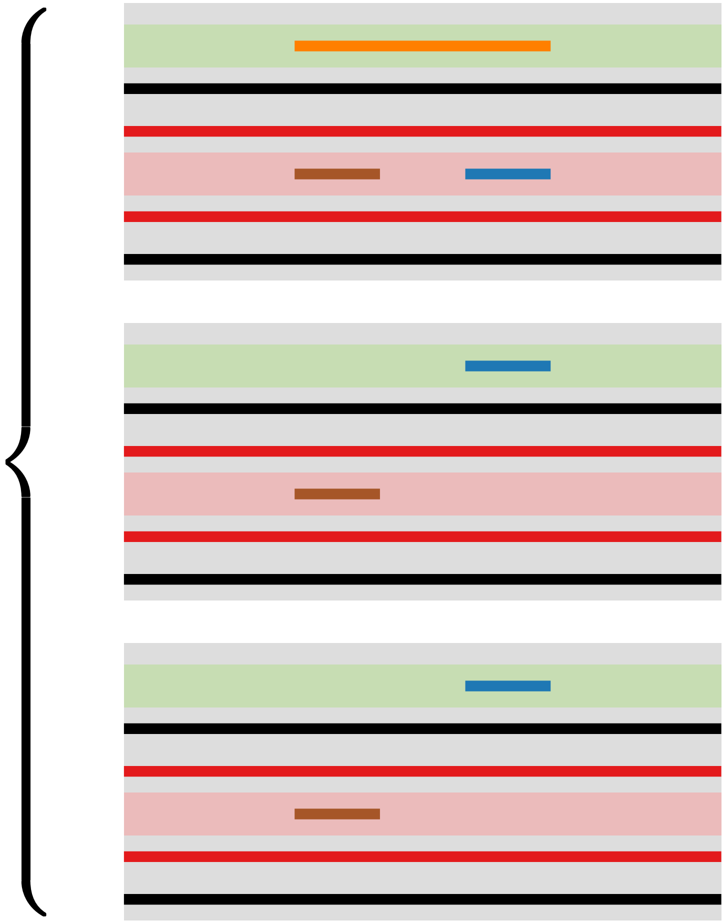

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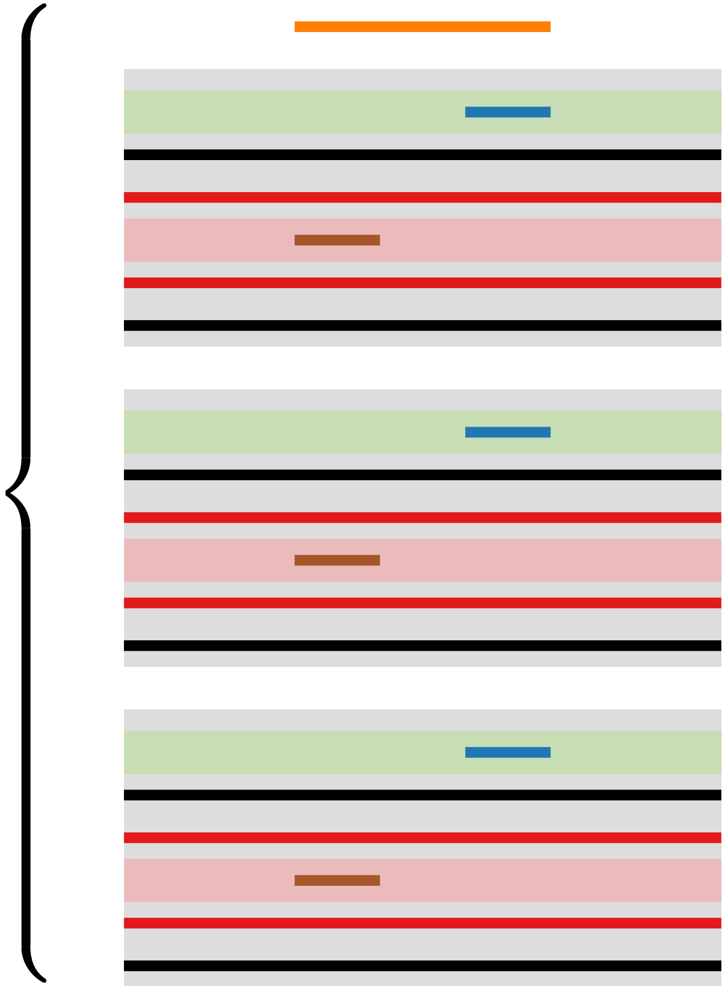

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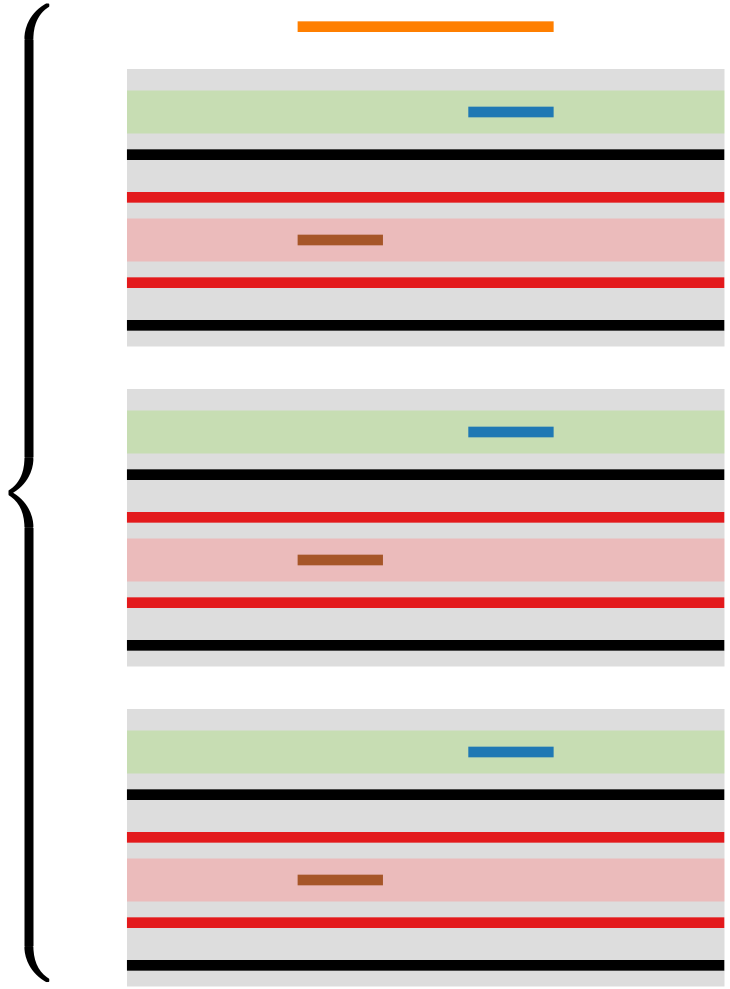
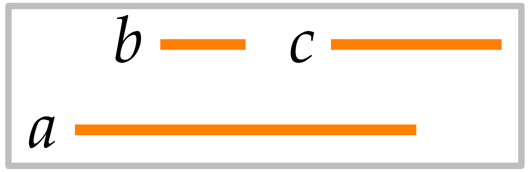
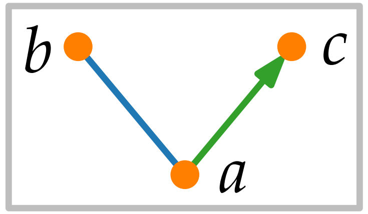


Φ is satisfiable $\Leftrightarrow$ G_Φ admits a coloring with $6n$ colors

$\square$

Conclusion and Open Problems





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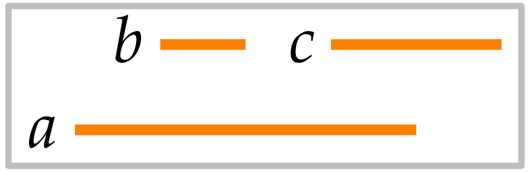

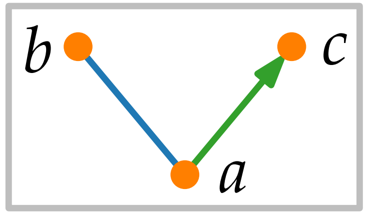

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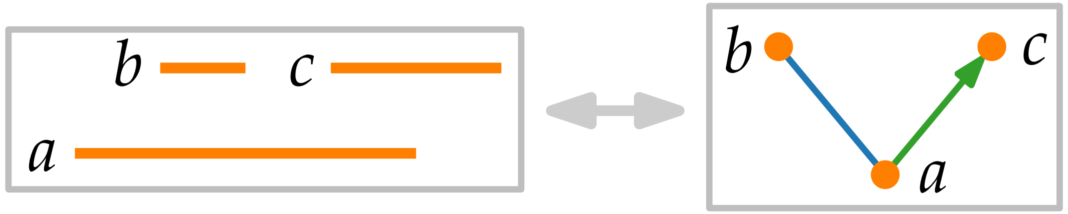

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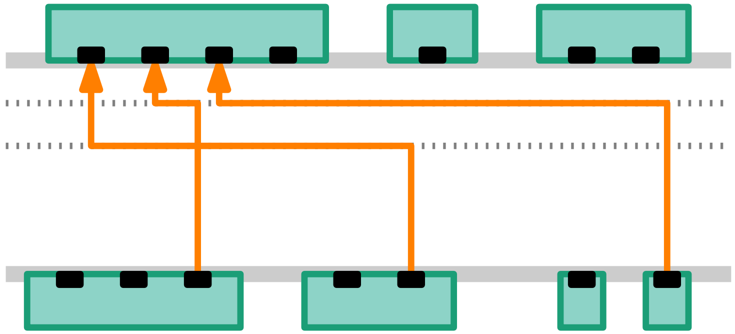

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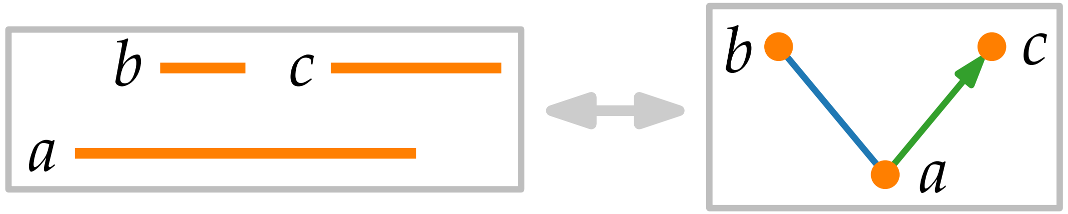

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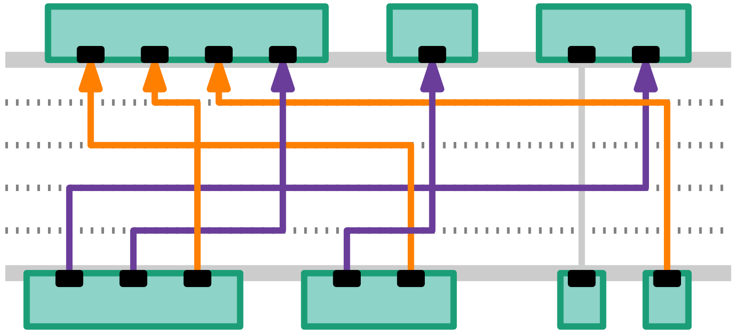

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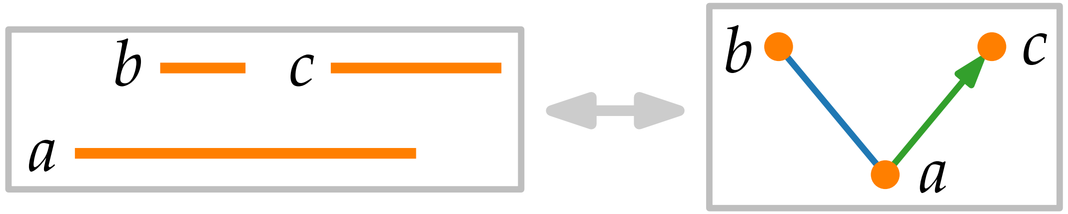

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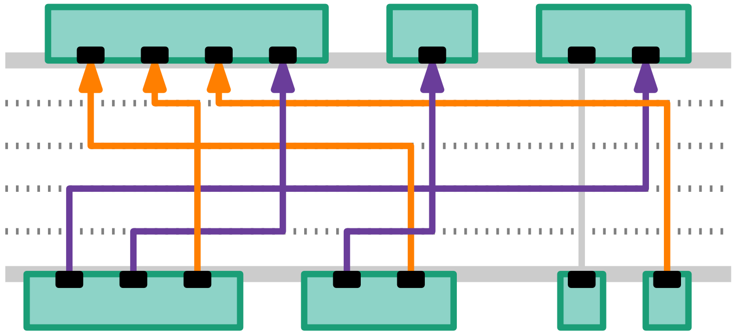

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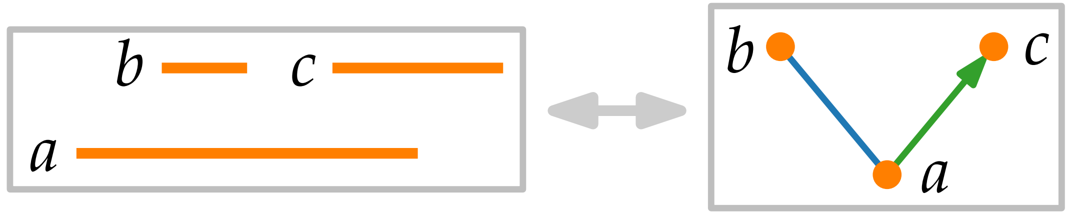

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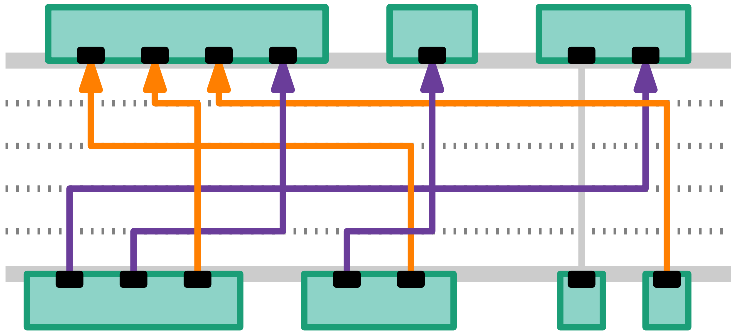

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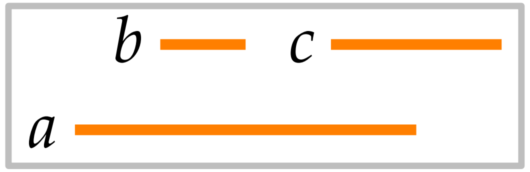

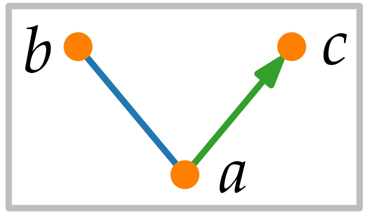

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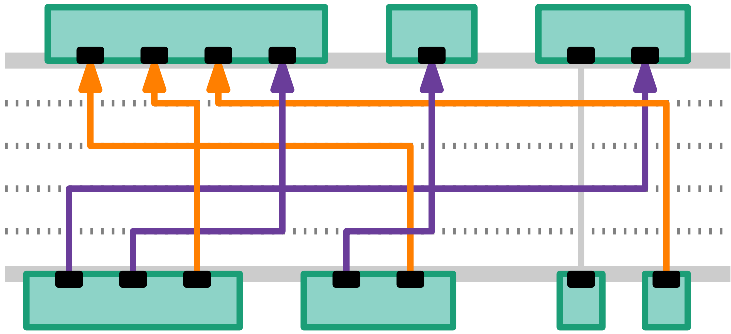

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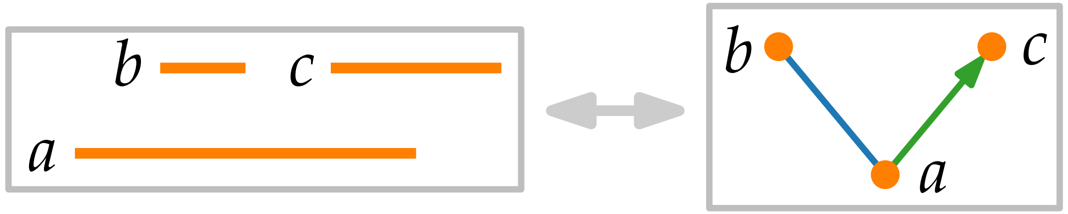
- For the more general case of mixed interval graphs, coloring is NP-hard.
  (Remark: NP-hardness requires both directed and undirected edges.)

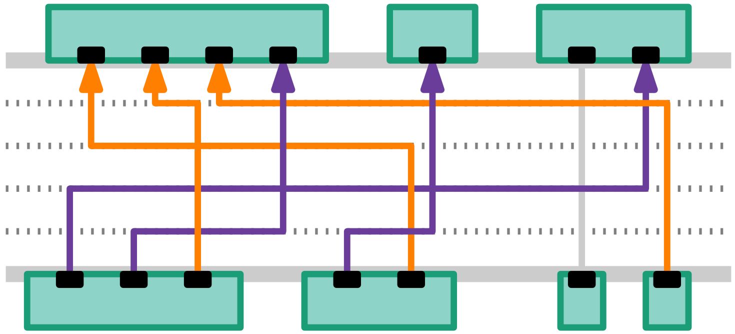
Conclusion and Open Problems



- We have introduced the natural concept of directional interval graphs.
- A simple greedy algorithm colors these graphs optimally in $O(n \log n)$ time.

  $n :=$ # vertices
- In layered graph drawing, this corresponds to routing "left-going" edges orthogonally to the fewest horizontal tracks. (Symmetrically "right-going".)

⇒ Combining the drawings of left-going and right-going edges yields a 2-approximation for the number of tracks. (bidirectional interval graphs)



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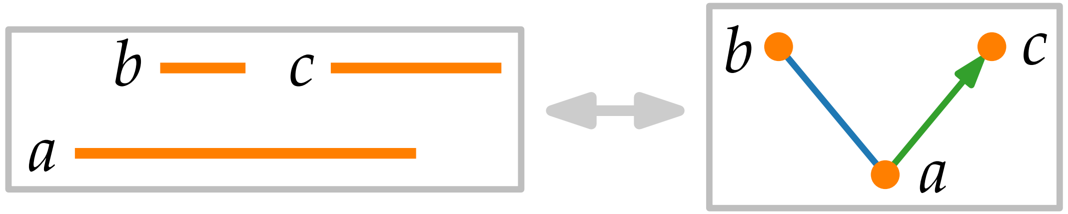

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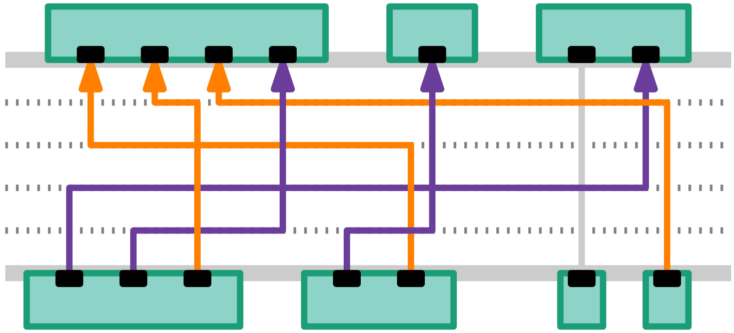

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