

Homework Assignment #5

Approximation Algorithms (Winter Semester 2023/24)

Exercise 1 – Overpacking in the dual LP for SETCOVER

Consider the dual LP for SETCOVER:

$$\begin{array}{ll}\text{maximize} & \sum_{u \in \mathcal{U}} y_u \\ \text{subject to} & \sum_{u \in S} y_u \leq c_S, \quad S \in \mathcal{S} \\ & y_u \geq 0, \quad u \in \mathcal{U}.\end{array}$$

Let $\text{price}(u)$ be the price of element u as determined by the algorithm GreedySetCover in Lecture #2.

Prove that the solution $y_u = \text{price}(u)$ is in general not feasible for the dual LP. In particular, find an instance with a set S that is overpacked by a factor of (approximately) $H_{|S|}$, i.e., $\sum_{u \in S} y_u \approx H_{|S|} \cdot c_S$.
[4 points]

Exercise 2 – Randomized Rounding for SETCOVER

Consider the following randomized algorithm for SETCOVER.

Algorithm 1: RandomizedRounding($\mathcal{U}, \mathcal{S}, c$)

Compute an optimal solution x of the LP relaxation for SETCOVER (as presented in the lecture).

$\mathcal{C} \leftarrow \emptyset$

foreach $S \in \mathcal{S}$ **do**

 Add S with probability x_S to \mathcal{C} .

return \mathcal{C}

a) Prove that the expected cost of \mathcal{C} is exactly OPT. [4 points]

b) Let $u \in \mathcal{U}$ be an arbitrary element of the ground set. Prove that u is *not* covered by \mathcal{C} with probability at most $1/e$. [3 points]

Hint: Use that, for every $x \in \mathbb{R}$, it holds that $1 + x \leq e^x$.

Exercise 3 – Randomized Rounding for SETCOVER with Multiple Rounds

Let d be a sufficiently large constant. We now run the algorithm RandomizedRounding from Exercise 2 exactly $\lceil d \cdot \log_2 n \rceil$ times. Let in the following $\text{OPT}_{\text{relax}}$ be the value of the optimal solution of the relaxed LP for SETCOVER, and let \mathcal{C}' be the union of all sets selected this way.

- a) Prove that every element $u \in \mathcal{U}$ is *not* selected by \mathcal{C}' with probability at most $1/(4n)$, as long as d was chosen large enough. **[3 points]**
- b) Prove that the cost of the set \mathcal{C}' is greater than $4d \log_2 n \cdot \text{OPT}_{\text{relax}}$ with probability at most $1/4$. **[3 points]**
- c) Prove that \mathcal{C}' is a *feasible* solution of cost at most $4d \log_2 n \cdot \text{OPT}_{\text{relax}}$ with probability at least $1/2$. **[3 points]**