

## Homework Assignment #5

### Approximation Algorithms (Winter Semester 2023/24)

#### Exercise 1 – Overpacking in the dual LP for SETCOVER

Consider the dual LP for SETCOVER:

$$\begin{aligned}
 & \text{maximize} && \sum_{u \in U} y_u \\
 & \text{subject to} && \sum_{u \in S} y_u \leq c_S, \quad S \in \mathcal{S} \\
 & && y_u \geq 0, \quad u \in U.
 \end{aligned}$$

Let  $\text{price}(u)$  be the price of element  $u$  as determined by the algorithm GreedySetCover in Lecture #2.

Prove that the solution  $y_u = \text{price}(u)$  is in general not feasible for the dual LP. In particular, find an instance with a set  $S$  that is overpacked by a factor of (approximately)  $H_{|S|}$ , i.e.,  $\sum_{u \in S} y_u \approx H_{|S|} \cdot c_S$ . [4 points]

#### Exercise 2 – Randomized Rounding for SETCOVER

Consider the following randomized algorithm for SETCOVER.

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##### Algorithm 1: RandomizedRounding( $U, \mathcal{S}, c$ )

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Compute an optimal solution  $x$  of the LP relaxation for SETCOVER (as presented in the lecture).  
 $\mathcal{C} \leftarrow \emptyset$   
**foreach**  $S \in \mathcal{S}$  **do**  
 └ Add  $S$  with probability  $x_S$  to  $\mathcal{C}$ .  
**return**  $\mathcal{C}$

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- a) Prove that the expected cost of  $\mathcal{C}$  is exactly OPT. [4 points]
- b) Let  $u \in U$  be an arbitrary element of the ground set. Prove that  $u$  is *not* covered by  $\mathcal{C}$  with probability at most  $1/e$ . [3 points]

*Hint:* Use that, for every  $x \in \mathbb{R}$ , it holds that  $1 + x \leq e^x$ .

### Exercise 3 – Randomized Rounding for SETCOVER with Multiple Rounds

Let  $d$  be a sufficiently large constant. We now run the algorithm RandomizedRounding from Exercise 2 exactly  $\lceil d \cdot \log_2 n \rceil$  times. Let in the following  $\text{OPT}_{\text{relax}}$  be the value of the optimal solution of the relaxed LP for SETCOVER, and let  $\mathcal{C}'$  be the union of all sets selected this way.

- a) Prove that every element  $u \in U$  is *not* selected by  $\mathcal{C}'$  with probability at most  $1/(4n)$ , as long as  $d$  was chosen large enough. **[3 points]**
- b) Prove that the cost of the set  $\mathcal{C}'$  is greater than  $4d \log_2 n \cdot \text{OPT}_{\text{relax}}$  with probability at most  $1/4$ . **[3 points]**
- c) Prove that  $\mathcal{C}'$  is a *feasible* solution of cost at most  $4d \log_2 n \cdot \text{OPT}_{\text{relax}}$  with probability at least  $1/2$ . **[3 points]**