Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

Würzburg, November 9, 2023

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Homework Assignment #4
Approximation Algorithms (Winter Semester 2023/24)

Exercise 1 – Standard form of LPs

Prove that every linear program can be converted into the following standard form:

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\
\textbf{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad && \forall i = 1, \dots, m, \\
& x_j \geq 0 && \forall j = 1, \dots, n.
\end{aligned}$$

[5 points]

Exercise 2 – Min-s–t-Cut

A matrix A is called *totally unimodular* if every square submatrix has determinant 0, $+1$, or -1. Totally unimodular matrices are a quick way to verify that a linear program has an integral optimum. In particular, if A is totally unimodular and b is integral, then the linear program $\{\min c^\top x \mid Ax \geq b, x \geq 0\}$ has an integral optimum for any c.

Consider the MIN-s–t-CUT problem from the fourth lecture:

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E} c_{uv} d_{uv}\,, \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 \quad && \forall (u,v) \in E \setminus \{(t,s)\}, \\
& p_s - p_t \geq 1, \\
& d_{uv} \geq 0 && \forall (u,v) \in E, \\
& p_u \geq 0 && \forall u \in V.
\end{aligned}$$

Prove that the coefficient matrix A of this LP is totally unimodular and hence the problem has an integral optimum.

Hint: Use induction. **[5 points]**

Exercise 3 – LP relaxation for VERTEXCOVER

Consider the following LP relaxation for VERTEXCOVER on a graph $G = (V, E)$ with vertex weights $c\colon V \to \mathbb{Q}^+$:

$$\begin{aligned} \min \quad & \sum_{v \in V} c(v) x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad uv \in E \\ & x_v \geq 0 \quad v \in V\,. \end{aligned}$$

a) Show that in every extreme point solution of this relaxation, $x_v \in \{0, \frac{1}{2}, 1\}$ holds for all $v \in V$. Derive a factor-2 approximation algorithm for vertex-weighted VERTEXCOVER from this property.

   Hint: Use the fact that a solution is an extreme point solution if and only if it cannot be expressed as a convex combination of two different other solutions: for any solution x where $x_v \notin \{0, \frac{1}{2}, 1\}$ for some v, find two other valid solutions x' and x'' such that $x = \frac{1}{2}(x' + x'')$.

   [6 points]

b) Give a factor-$\frac{3}{2}$ approximation algorithm for planar graphs.

   Hint: Use the fact that for every planar graph, a four-coloring can be calculated in polynomial time. A four-coloring assigns to every vertex one of four colors such that no two neighboring vertices have the same color.

   [4 points]

Please hand in your solutions via WueCampus until 12:00 on Friday, November 17.