Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

Würzburg, October 20, 2023

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Homework Assignment #1
Approximation Algorithms (Winter Semester 2023/24)

Exercise 1 – Maximum acyclic subgraph

Devise a factor-1/2 approximation algorithm for the following problem:

Let G be a given directed graph. Find a set of edges $E' \subseteq E(G)$ with maximum cardinality such that the resulting subgraph (V, E') is acyclic.

[5 points]

Exercise 2 – Smallest maximal matching

Give a factor-2 approximation algorithm for the following problem:

Let G be a given undirected graph. Find a maximal matching with smallest cardinality in G.

[5 points]

Exercise 3 – Greedy for MAXIMUM CUT

Let G be a given graph. The MAXIMUM CUT problem asks for a partition of the vertex set $V(G)$ of G into two sets S and $\bar{S}$ such that the number of edges connecting these sets is as large as possible.

Consider the following greedy algorithm for MAXIMUM CUT. For a set $U \subseteq V(G)$ and a vertex $v \in V(G) \setminus U$, let $\deg(v, U)$ denote the number of edges between v and U.

Algorithm 1: Greedy-Max-Cut(G)

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choose two arbitrary vertices v1, v2 ∈ V
S ← {v1}
S̄ ← {v2}
foreach v ∈ V \ {v1, v2} do
    if deg(v, S) ≥ deg(v, S̄) then
        S̄ ← S̄ ∪ {v}
    else
        S ← S ∪ {v}
return Cut (S, S̄)
```

a) Let G^n be the undirected graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{v_1 v_i, v_2 v_i : 3 \leq i \leq n\}$. Which result does Greedy-Max-Cut yield, assuming the vertices are handled in the order of their indices? What is the optimal solution? What is the approximation ratio of the algorithm for the graph class $\{G^n : n \in \mathbb{N}, n \geq 3\}$?

[3 points]

b) Show that the greedy algorithm is a factor-$\frac{1}{2}$ approximation algorithm for arbitrary graphs (with at least two vertices).

Suggestion: The following definition could be helpful in your analysis. Let G be a graph, let S be the vertex set defined by the algorithm above, and let $v \in S$. Let $\text{inter}(v)$ and $\text{intra}(v)$ denote the values $\deg(v, \bar{S})$ and $\deg(v, S)$ *at the point of time* when the algorithm considers v, respectively. Note that these values change during the execution of the algorithm. For $v \in \bar{S}$, $\text{inter}(v)$ and $\text{intra}(v)$ are defined symmetrically.

[7 points]

Please hand in your solutions on WueCampus until 12:00 on Friday, October 27.