Approximation Algorithms

Lecture 10: Minimum-Degree Spanning Tree via Local Search

Part I: Minimum-Degree Spanning Tree

Alexander Wolff

Winter 2023/24

Minimum-Degree Spanning Tree

Given: A connected graph G.

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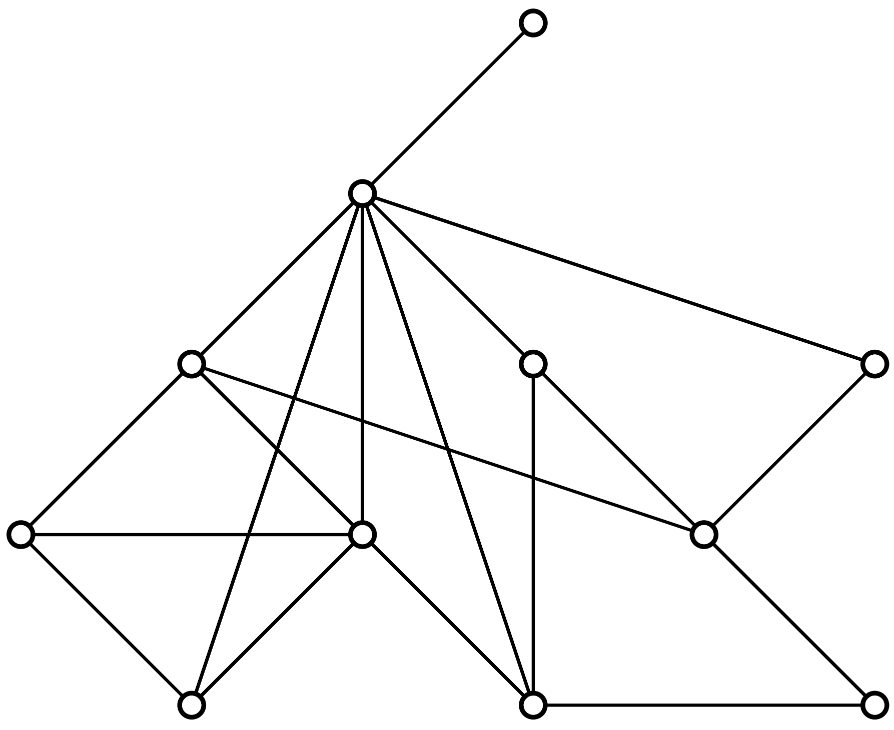
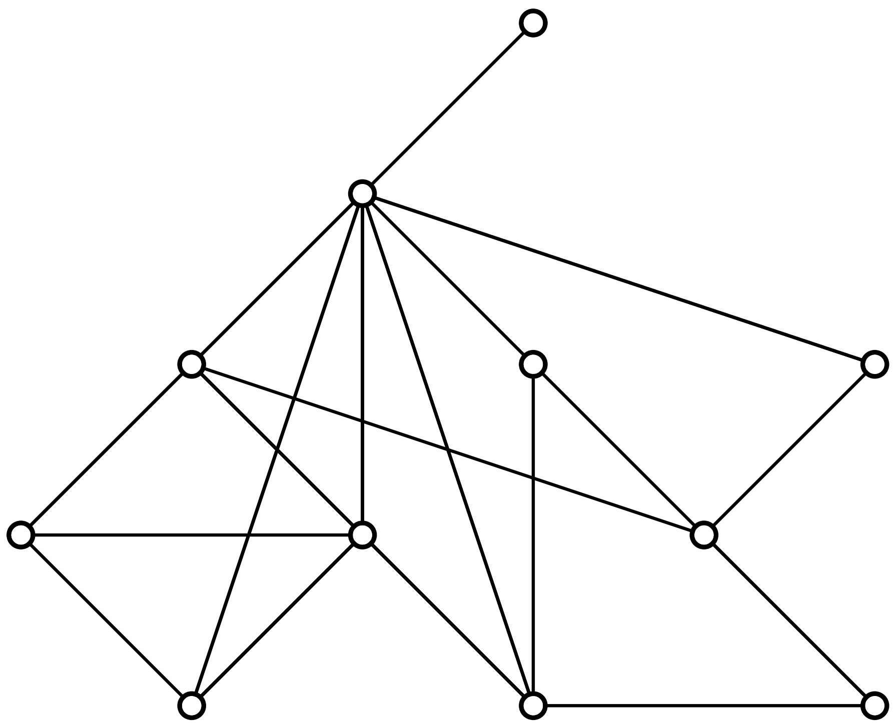


Minimum-Degree Spanning Tree

Given: A connected graph G.

Task: Find a spanning tree T that has the smallest maximum degree $\Delta(T)$ among all spanning trees of G.



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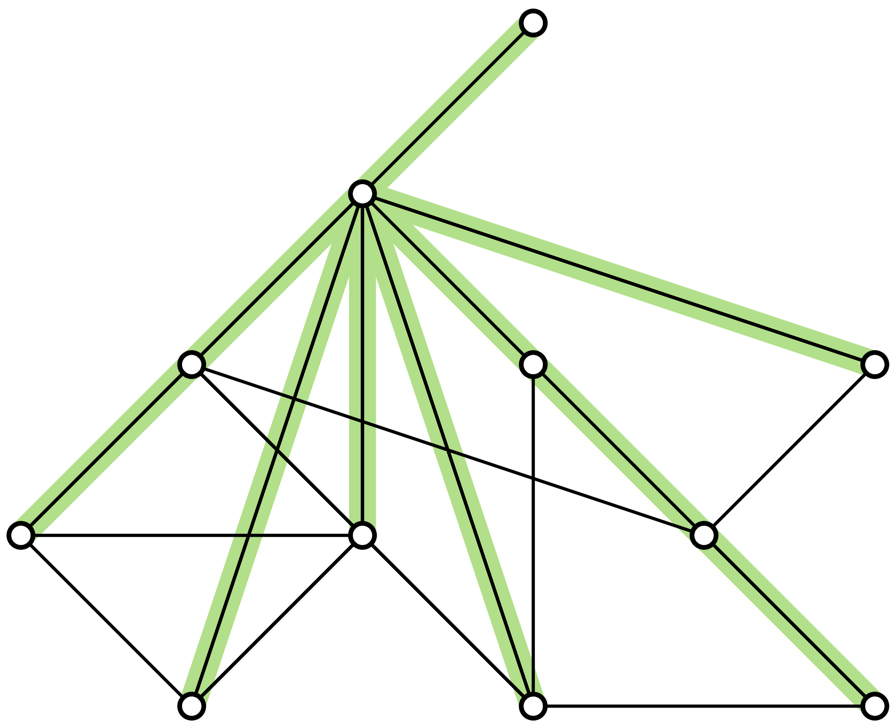

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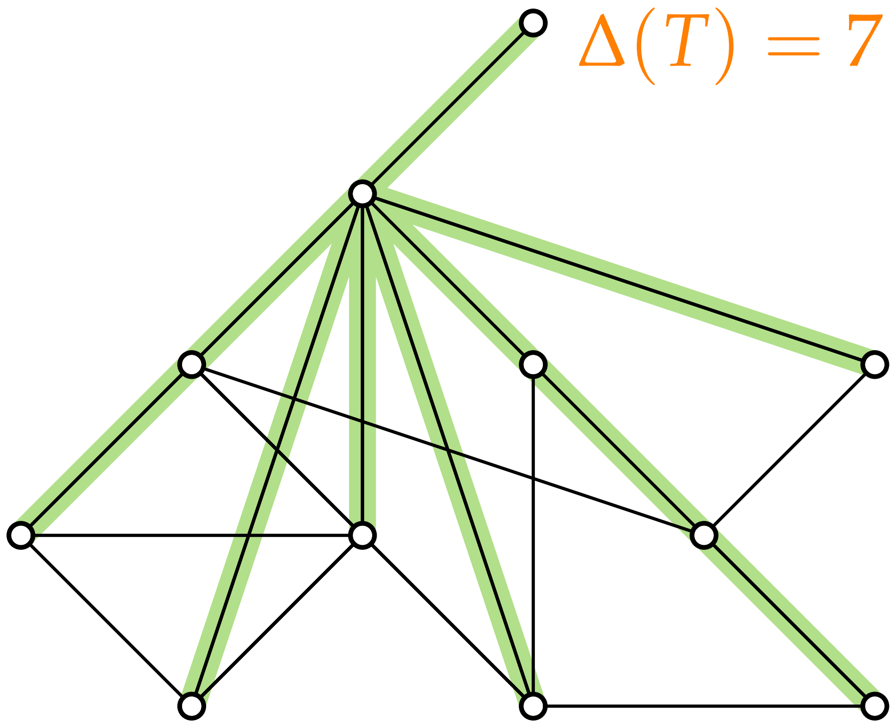

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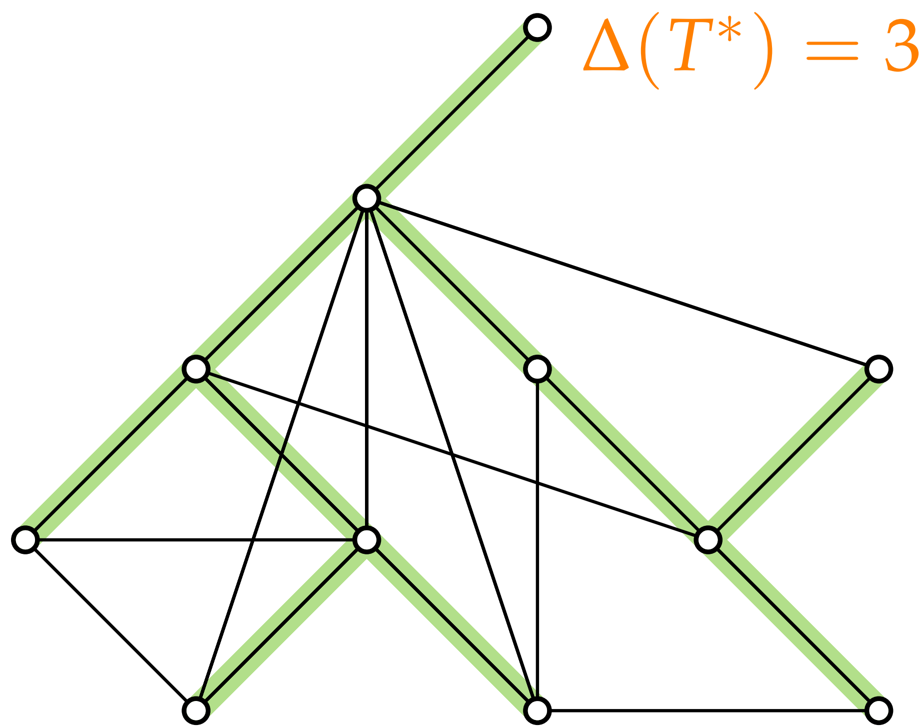

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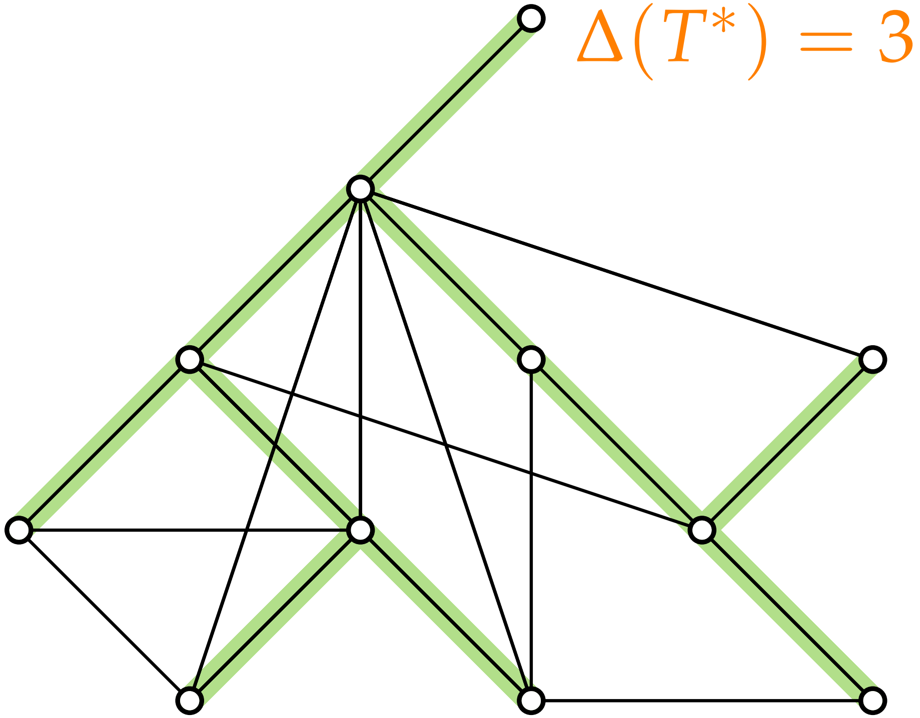
NP-hard. 



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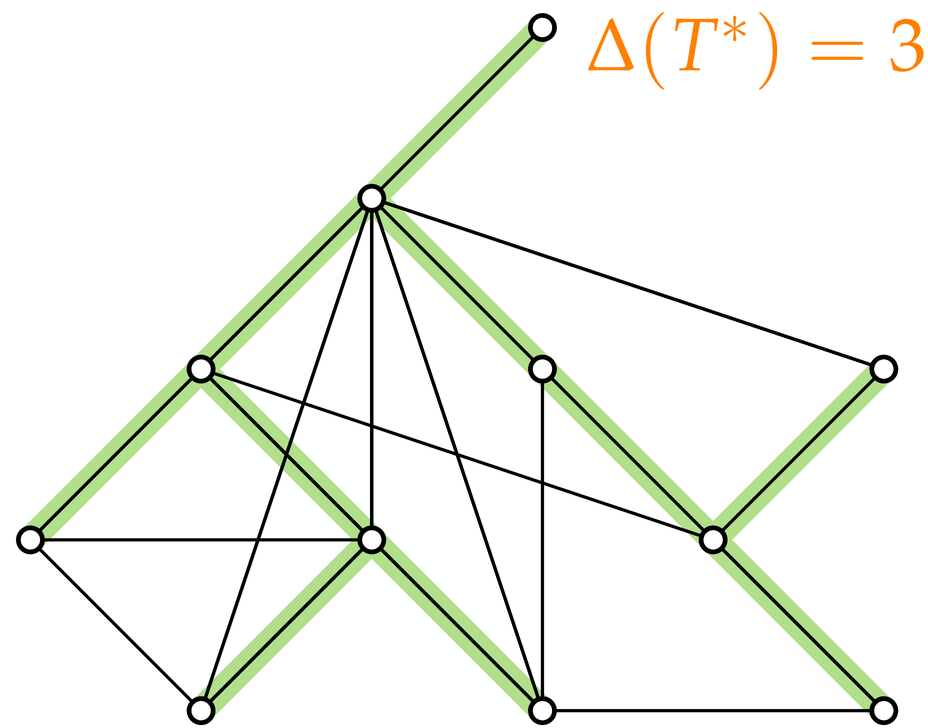
NP-hard. ☹

Why?



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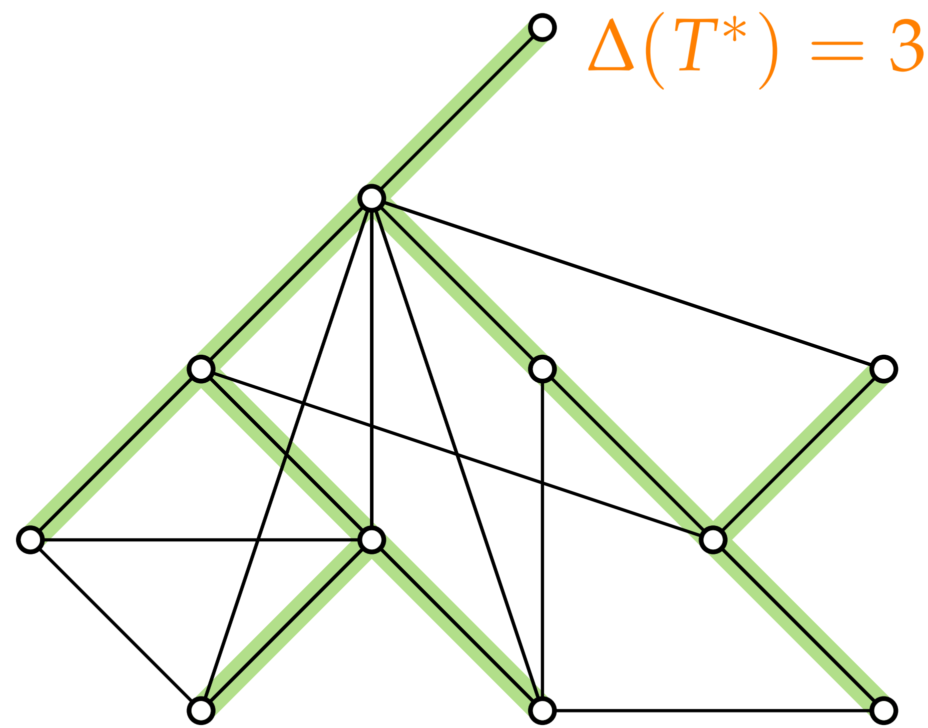
Task: Find a spanning tree T that has the smallest maximum degree $\Delta(T)$ among all spanning trees of G.

NP-hard.

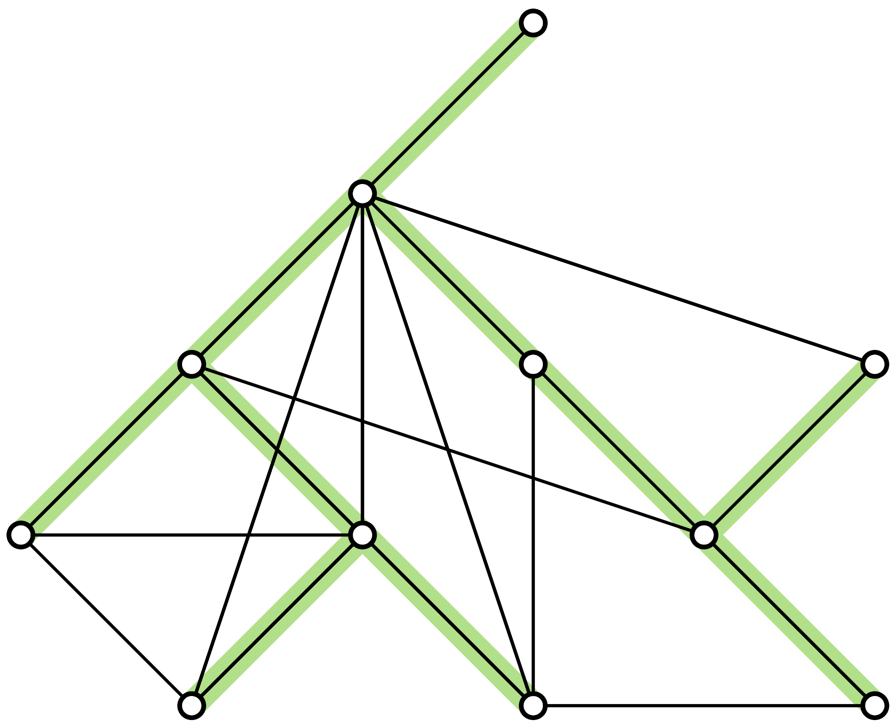
Why?

Special case of Hamiltonian Path!



Warm-up

Obs. 1. A spanning tree T has...



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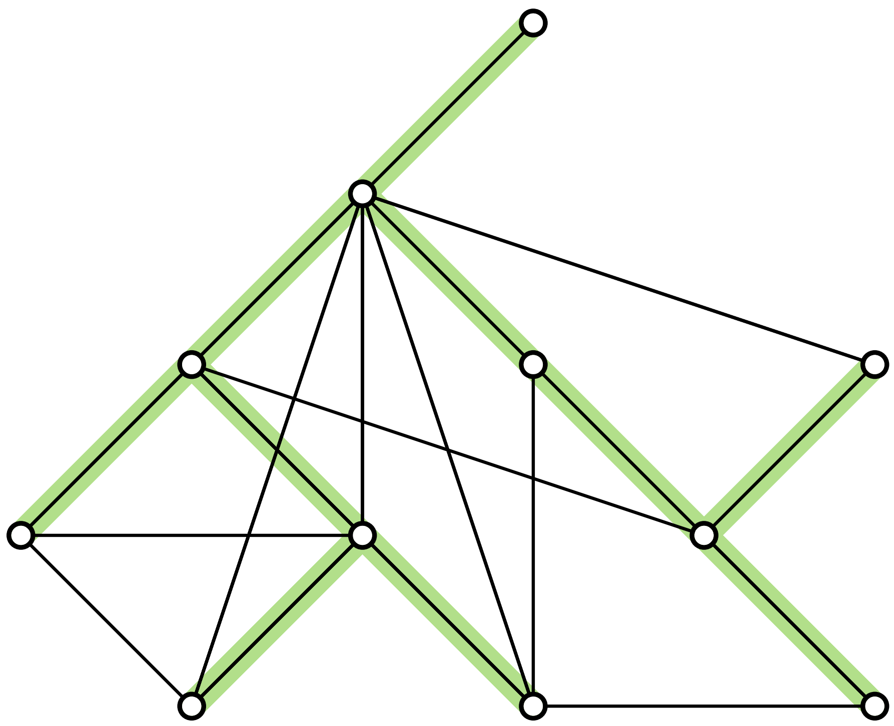
Obs. 1. A spanning tree T has...

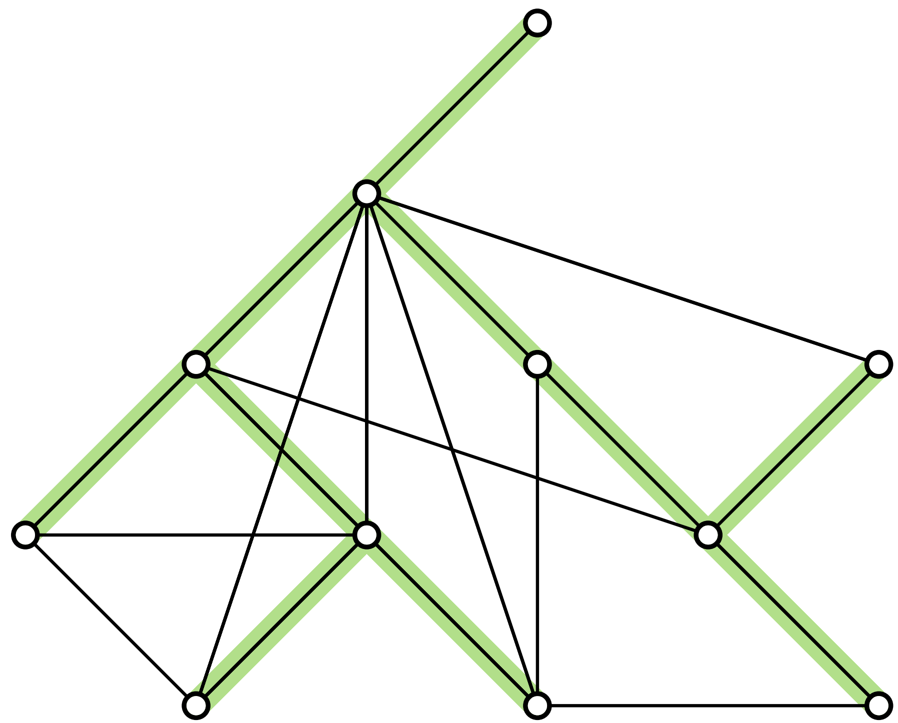
- n vertices and ? edges,



Warm-up

Obs. 1. A spanning tree T has...

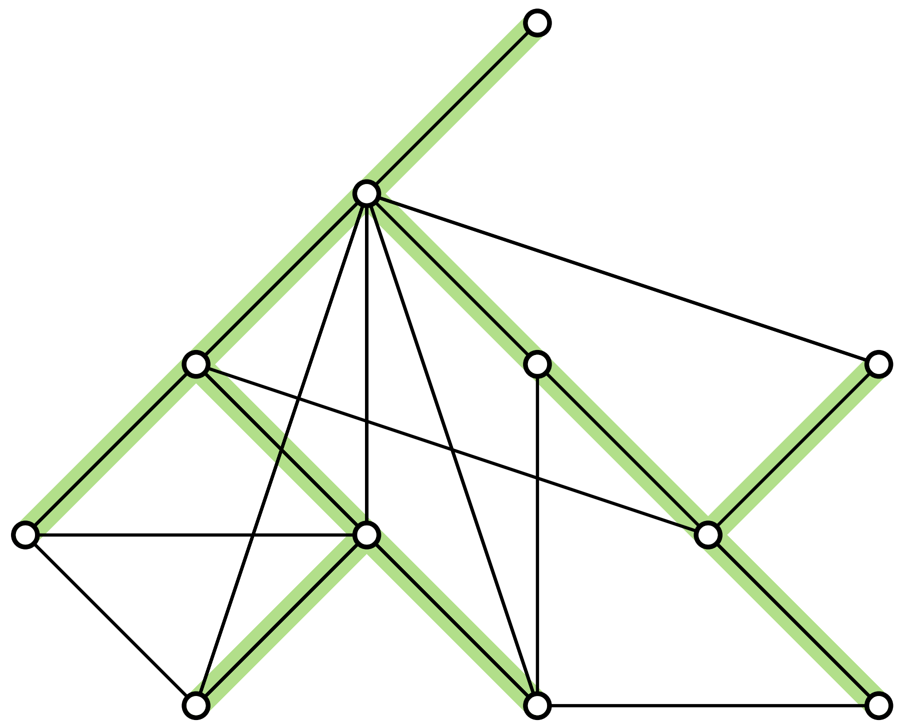
- n vertices and ? edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) =$?



Warm-up

Obs. 1. A spanning tree T has...

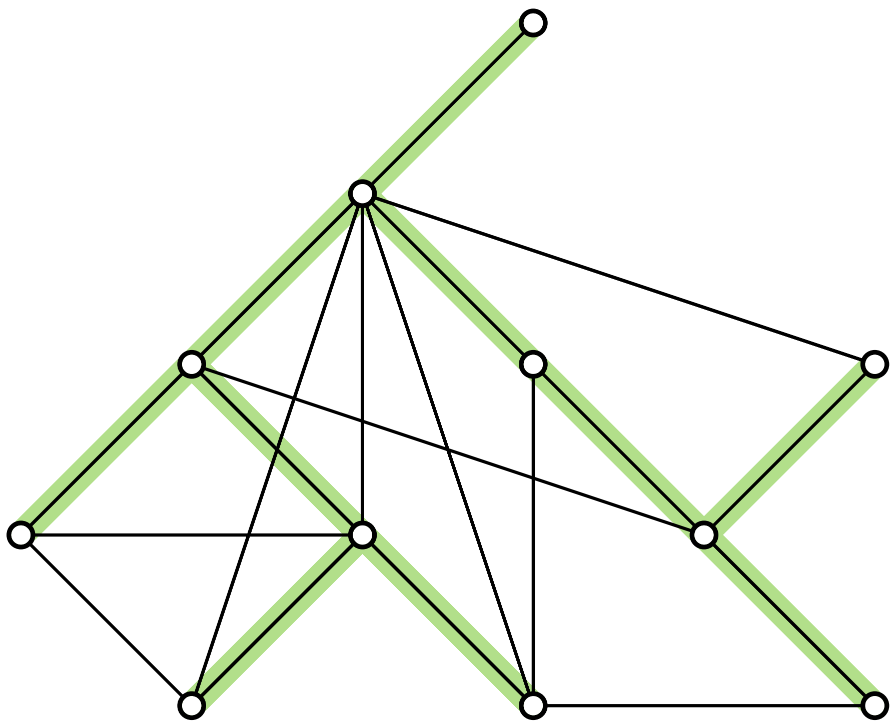
- n vertices and ? edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) =$?
- average degree ?



Warm-up

Obs. 1. A spanning tree T has...

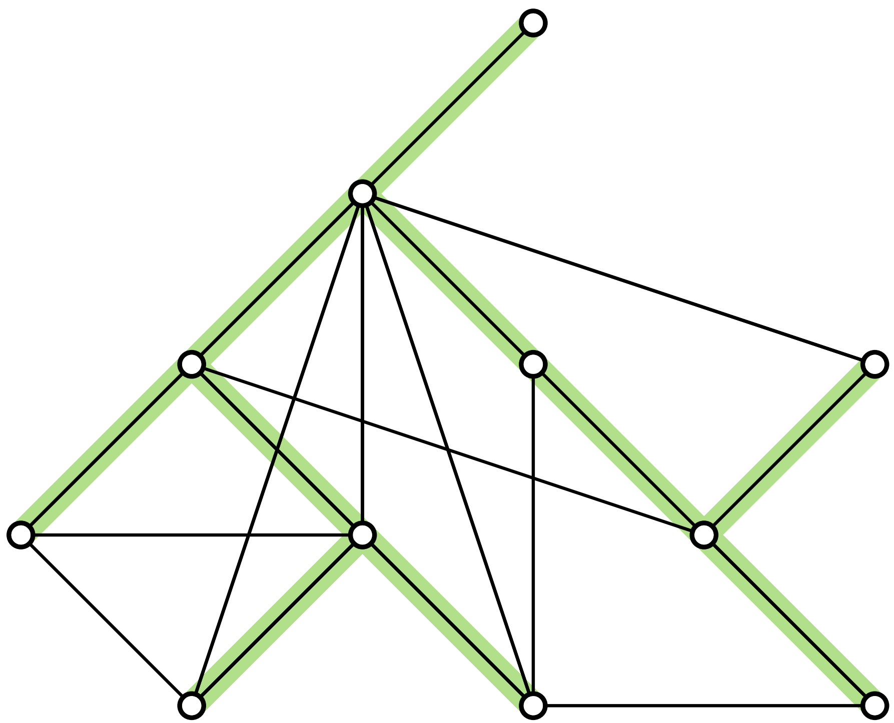
- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) =$?
- average degree ?



Warm-up

Obs. 1. A spanning tree T has...

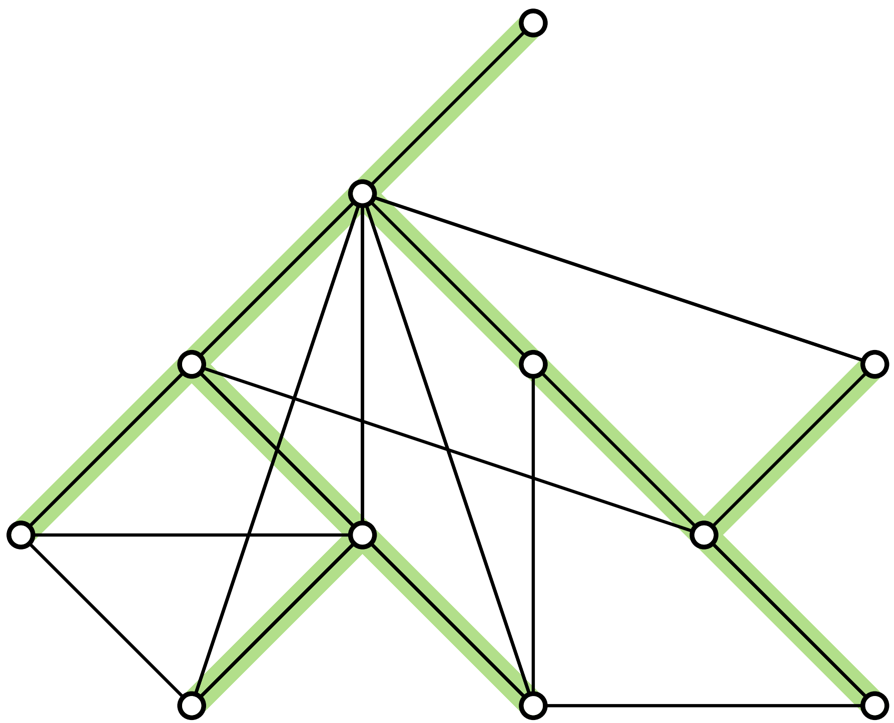
- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n - 2$,
- average degree ?



Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n-2$,
- average degree < 2.

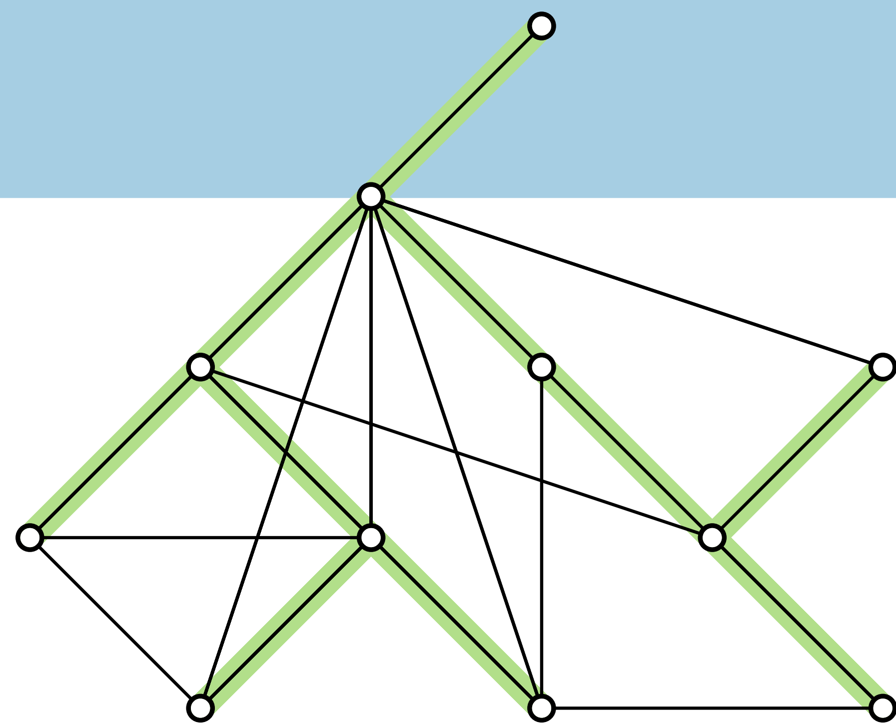


Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n - 2$,
- average degree < 2.

Obs. 2. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq$?



Warm-up

Obs. 1. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n-2$,
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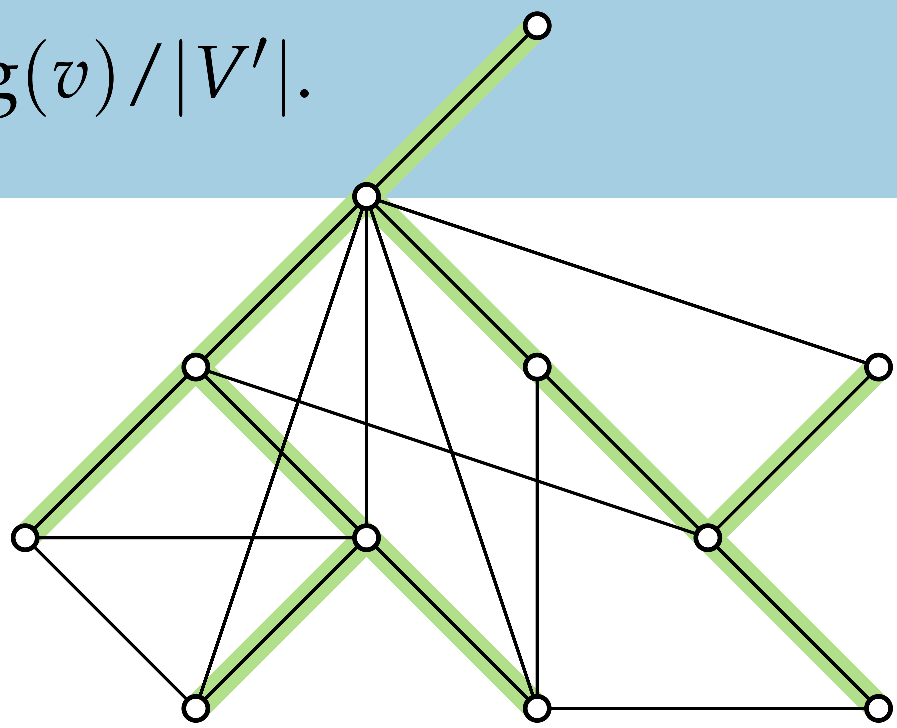
Obs. 2. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq \sum_{v \in V'} \deg(v) / |V'|$.



Warm-up

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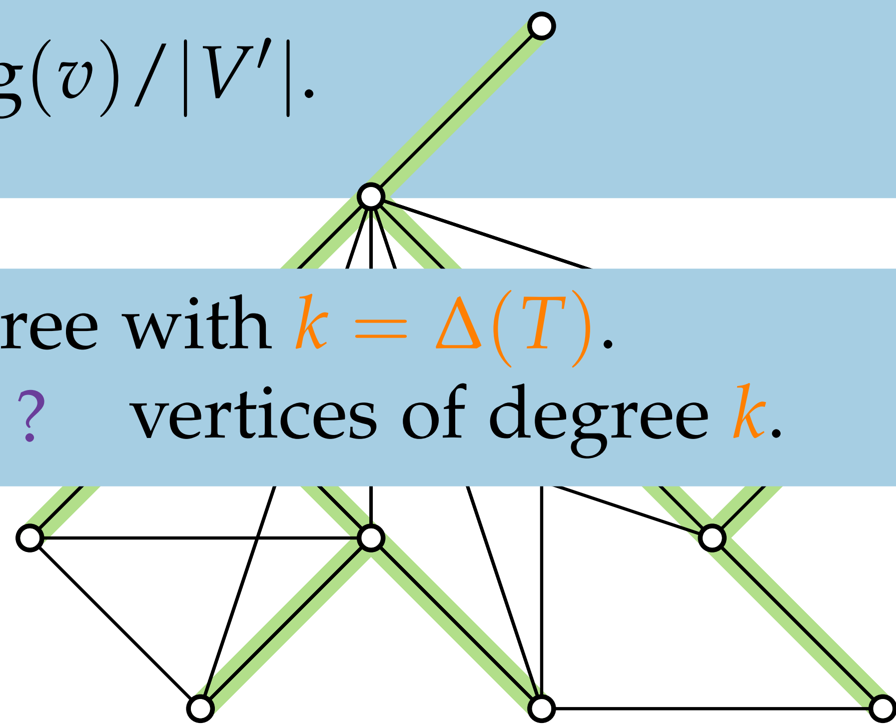

Obs. 3. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most ? vertices of degree k.

Warm-up

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- n vertices and $n-1$ edges,
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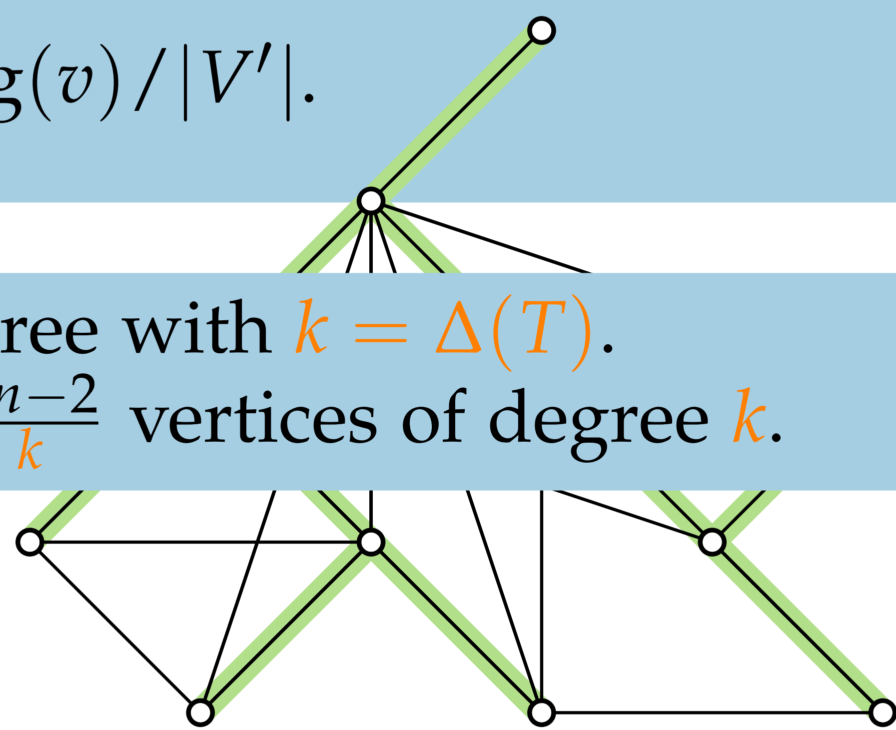
Then $\Delta(G) \geq \sum_{v \in V'} \deg(v) / |V'|$.

Obs. 3. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most $\frac{2n-2}{k}$ vertices of degree k.



Approximation Algorithms

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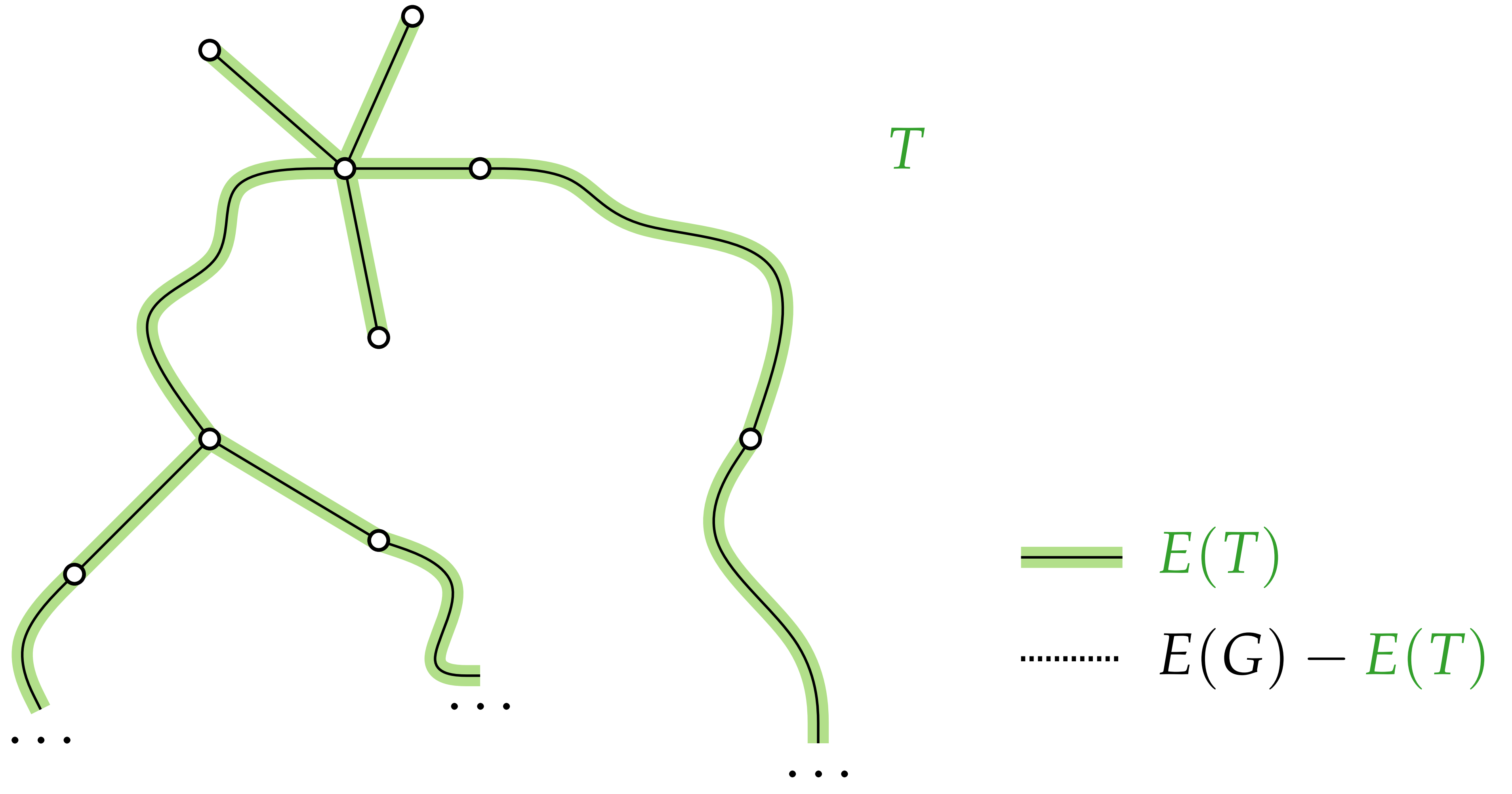
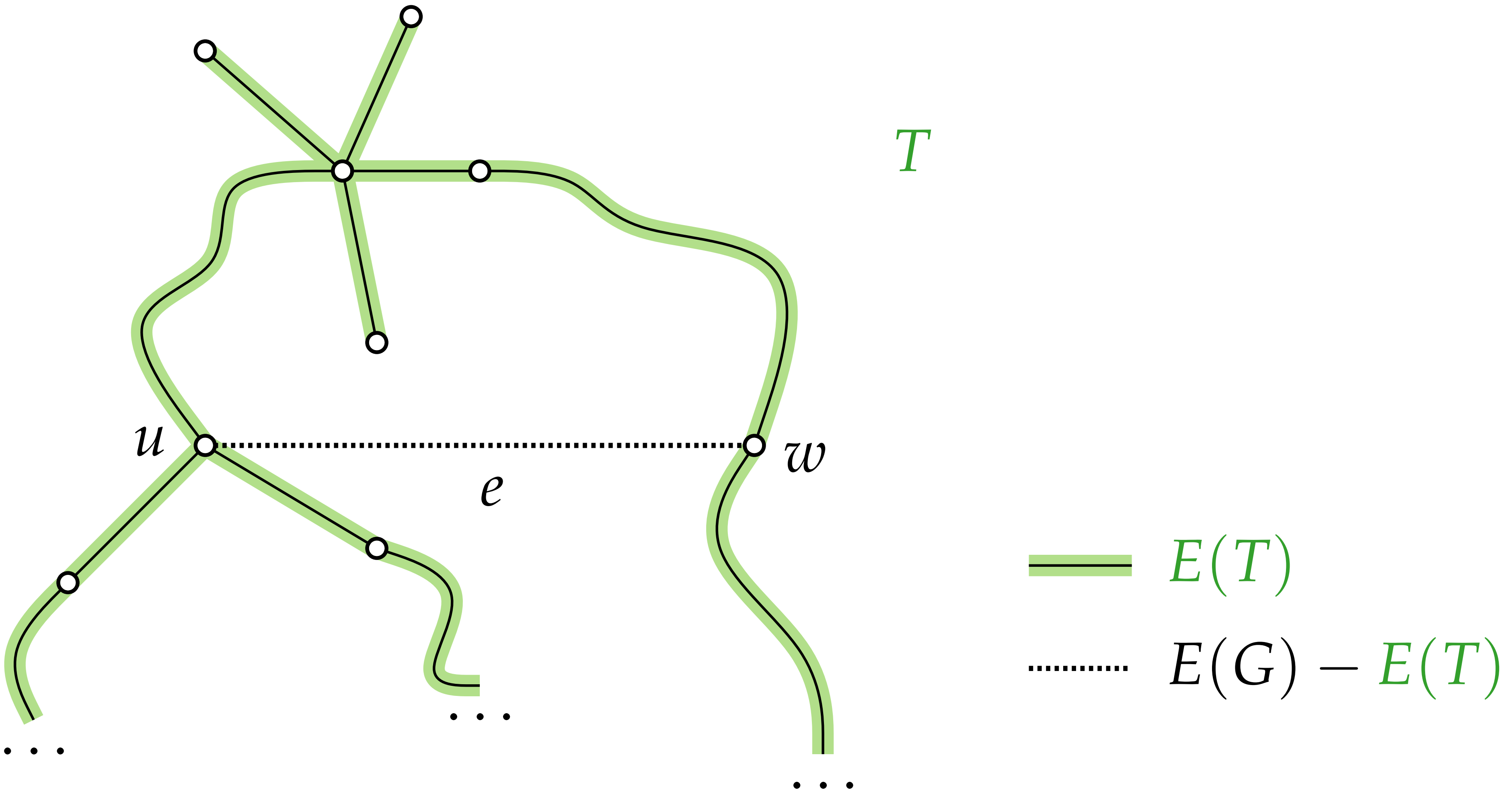
Minimum-Degree Spanning Tree
via Local Search

Part II:
Edge Flips and Local Search

Edge Flips



Edge Flips



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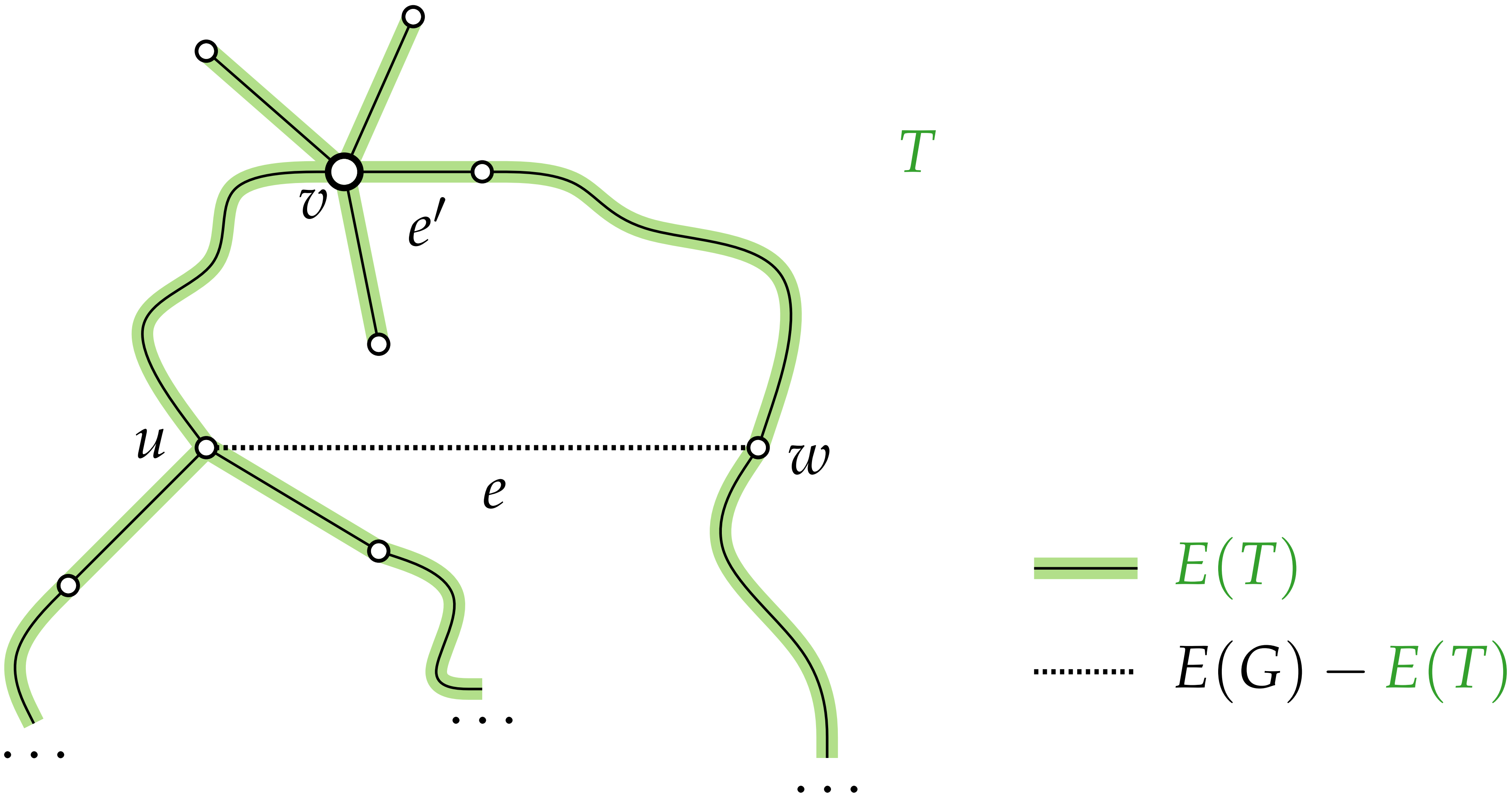

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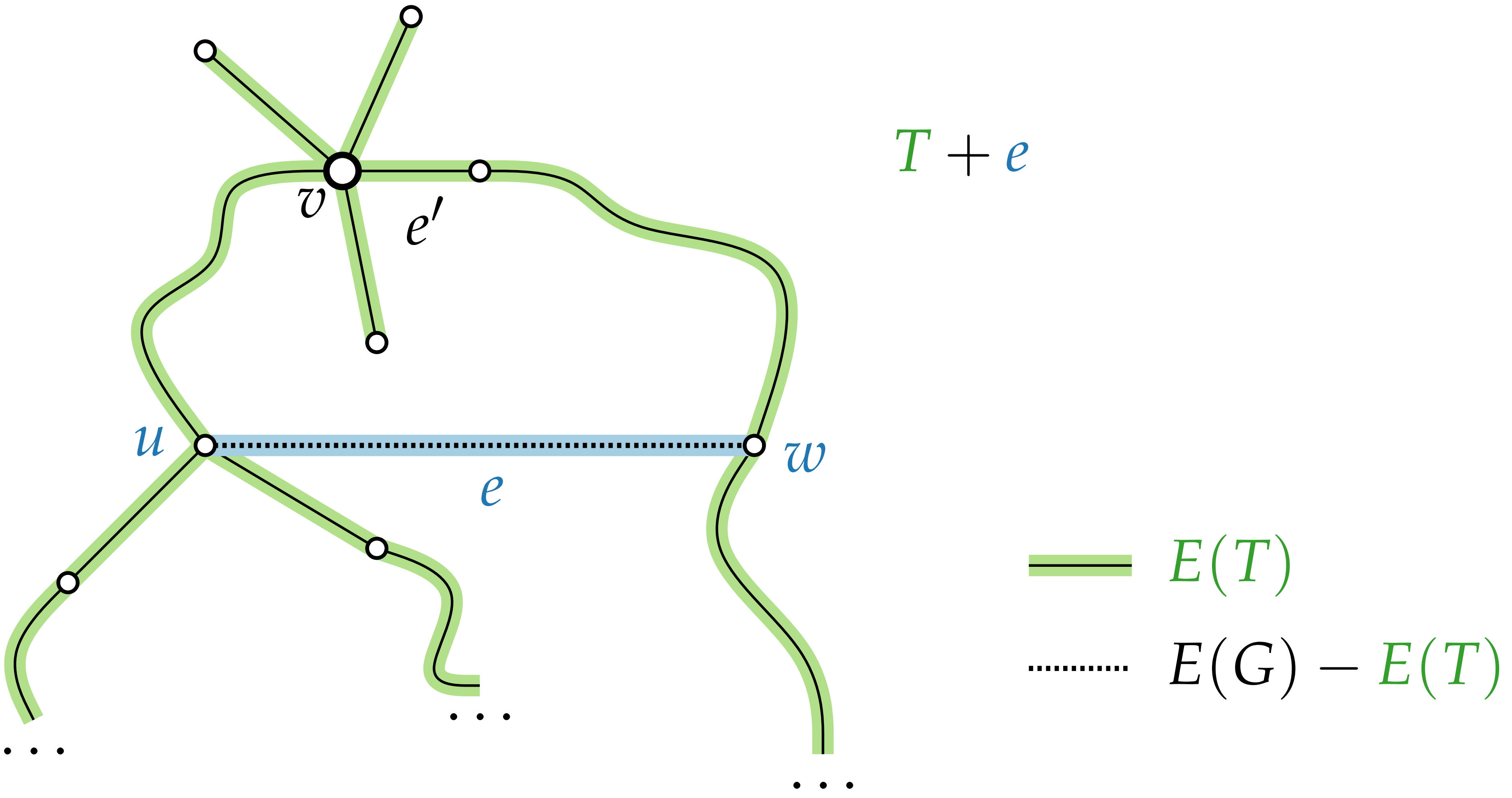

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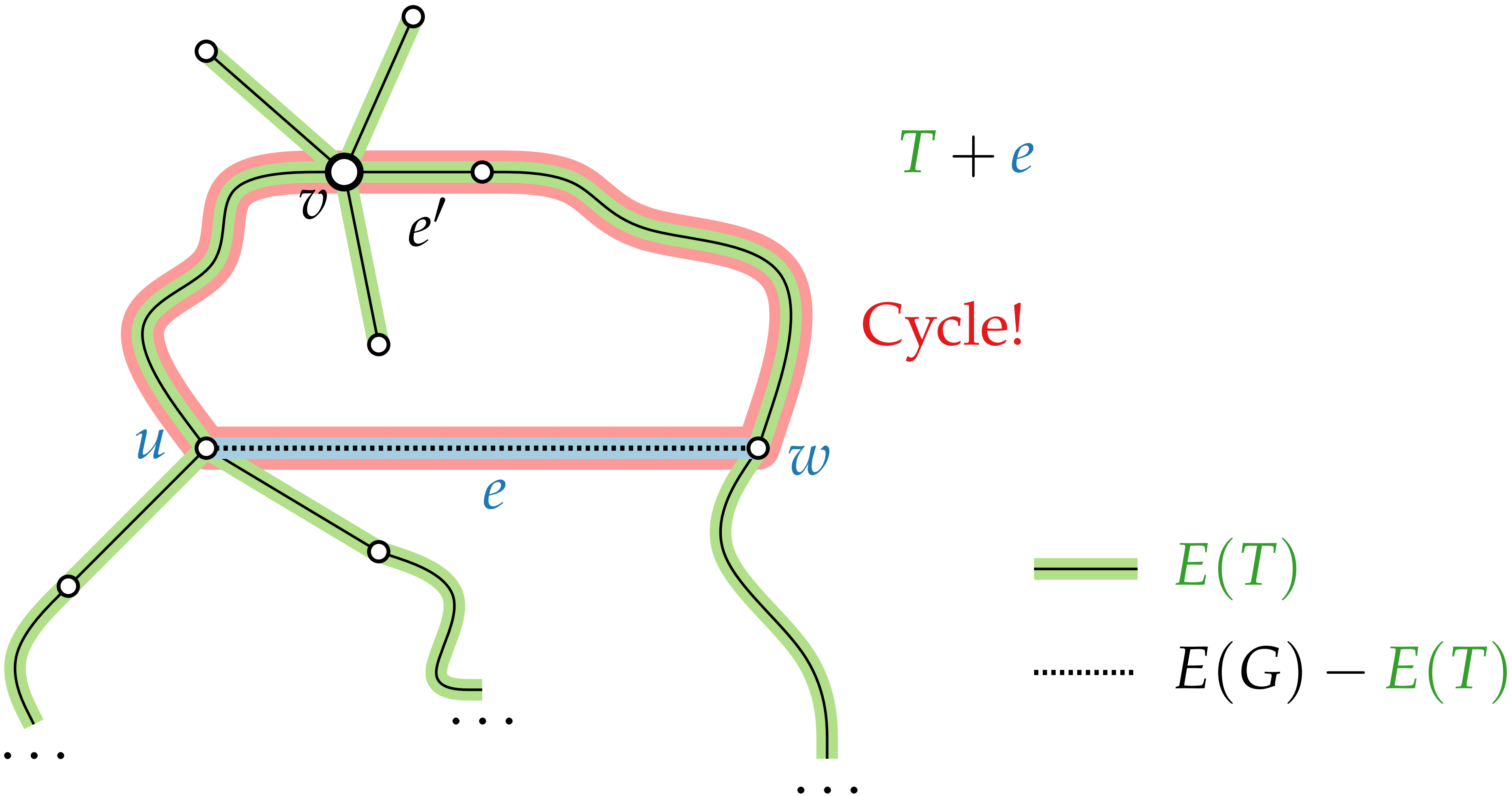

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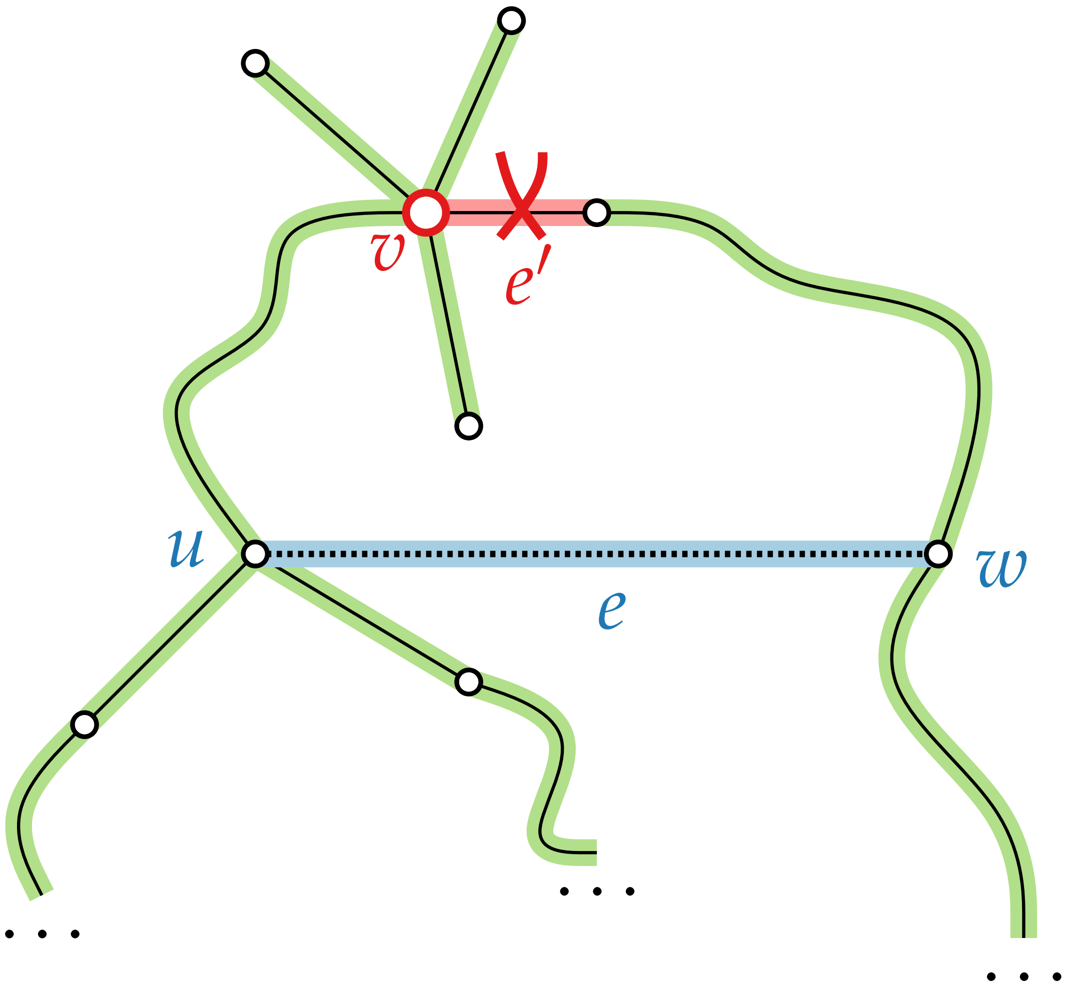
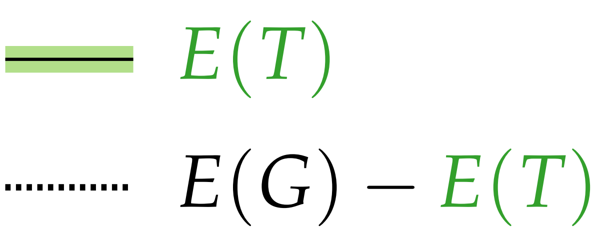


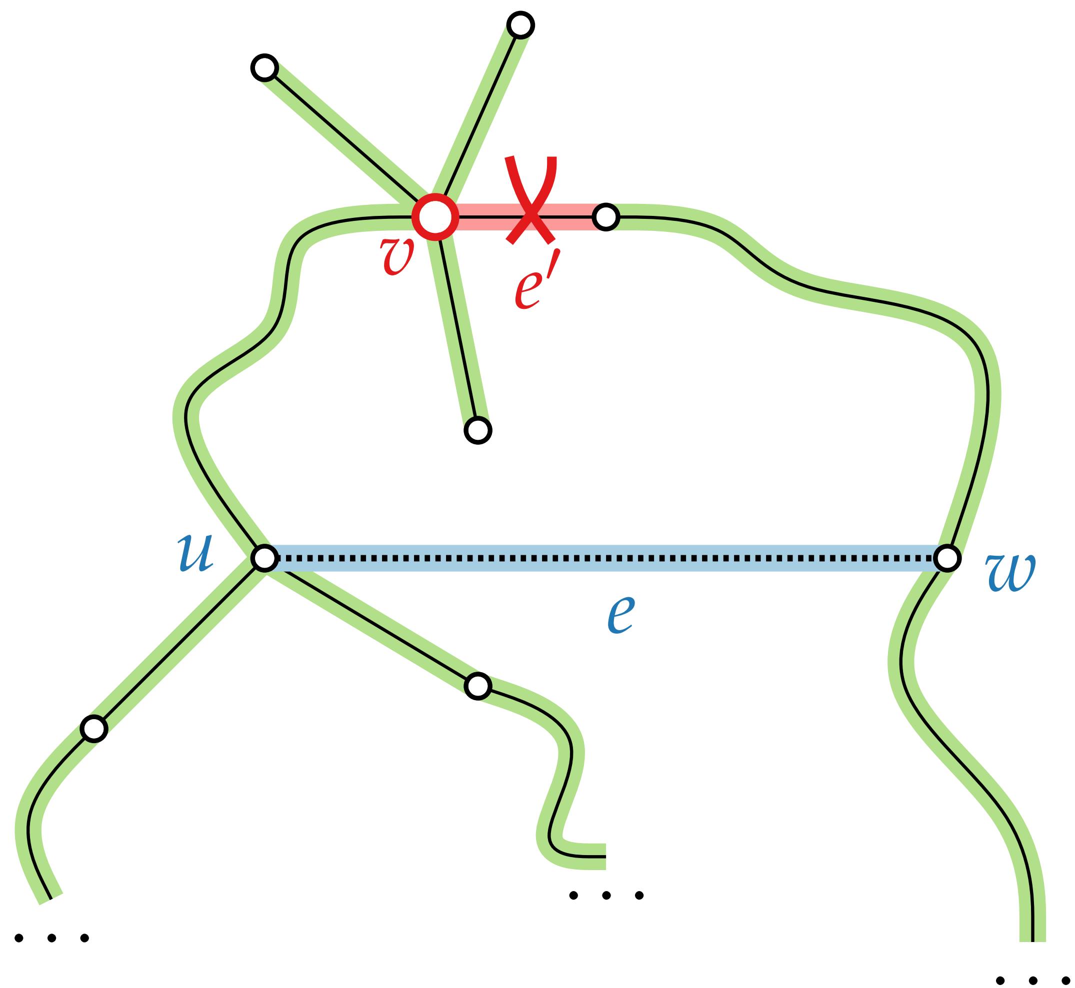
$T + e - e'$

is a new spanning tree



Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) >$



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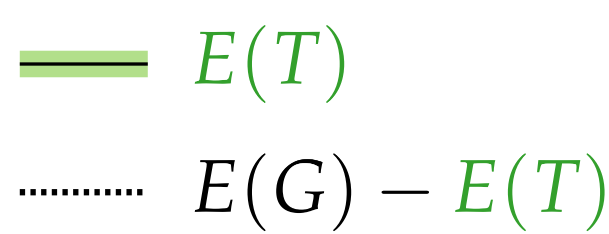
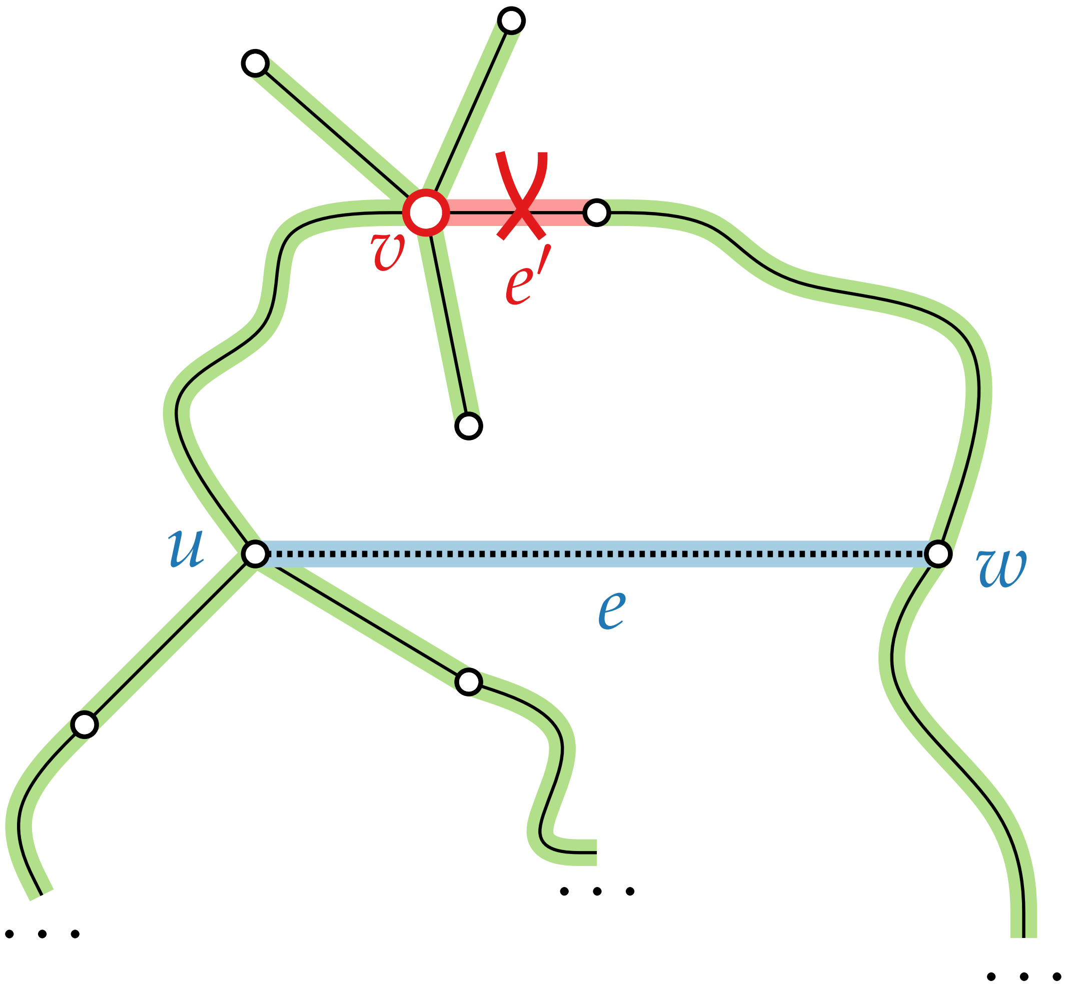


Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) > \max\{\deg_T(u), \deg_T(w)\} + 1$.



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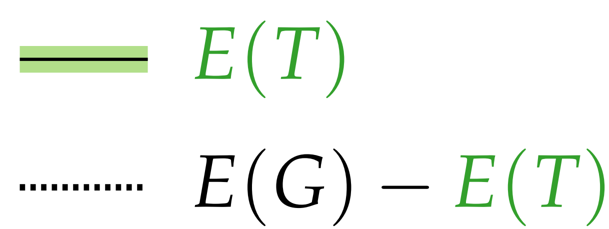

Local Search

```
MinDegSpanningTreeLocalSearch(graph G)
  T ← any spanning tree of G
  while ∃ improving flip in T for a vertex v
          with deg_T(v) ≥ Δ(T) − ℓ do
    do the improving flip
  return T
```

Local Search

MinDegSpanningTreeLocalSearch(graph G)
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  while $\exists$ improving flip in T for a vertex v
      with $\deg_T(v) \geq \Delta(T) - \ell$ **do**
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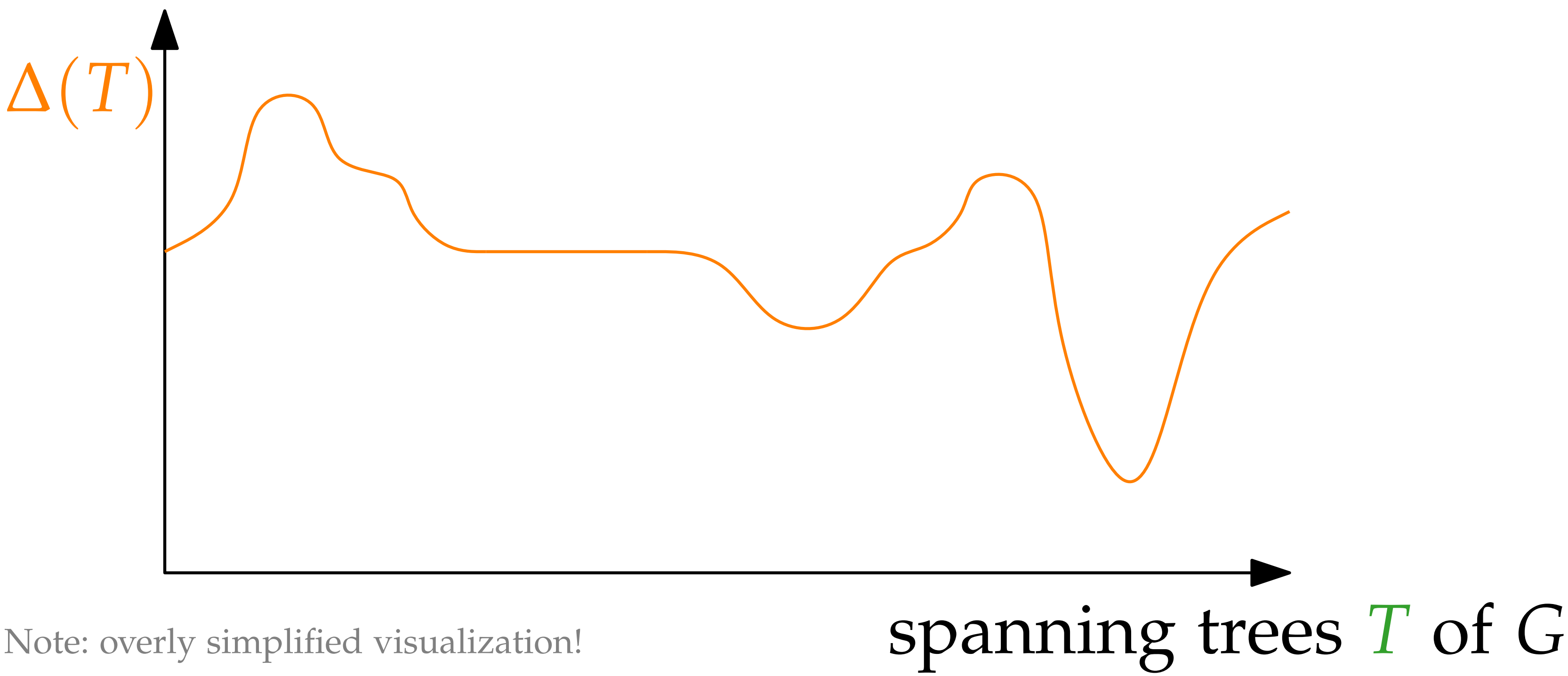

Note: overly simplified visualization!

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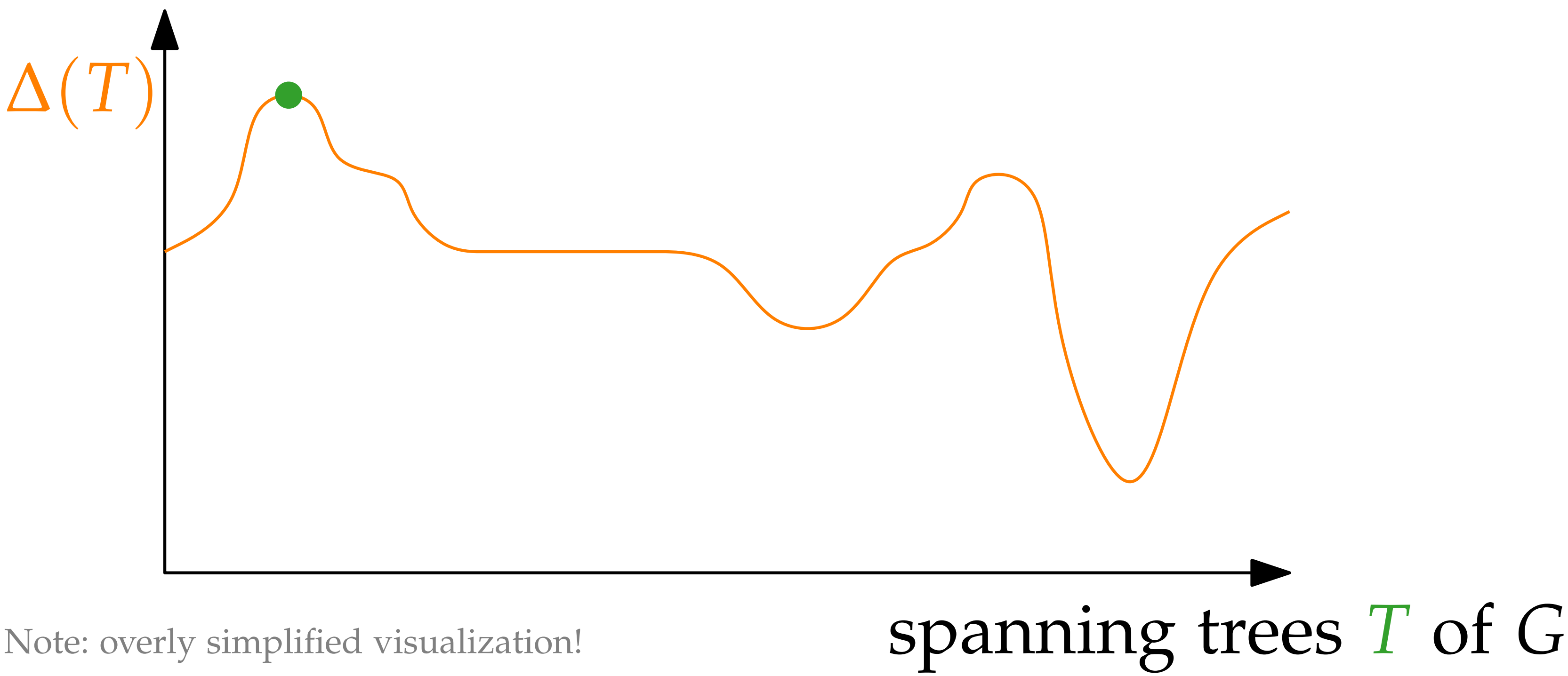

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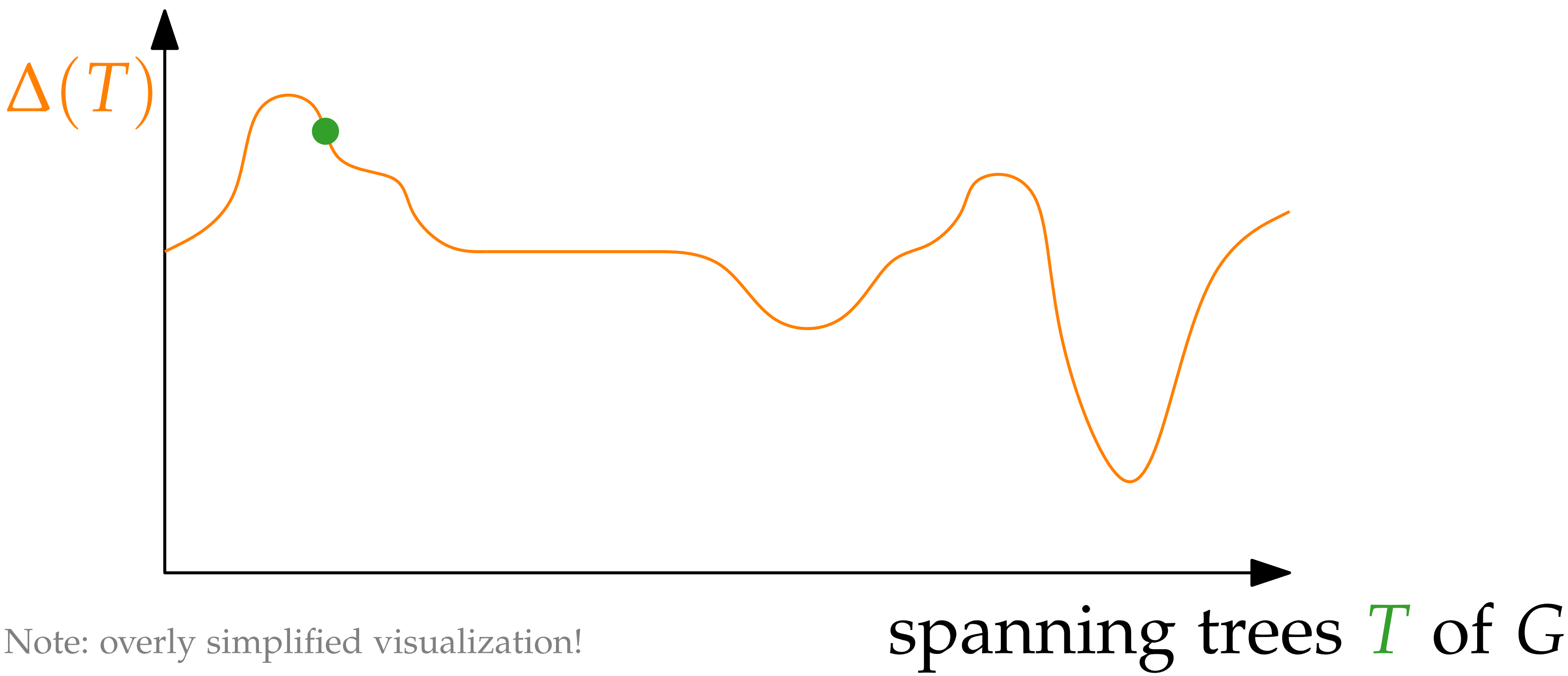

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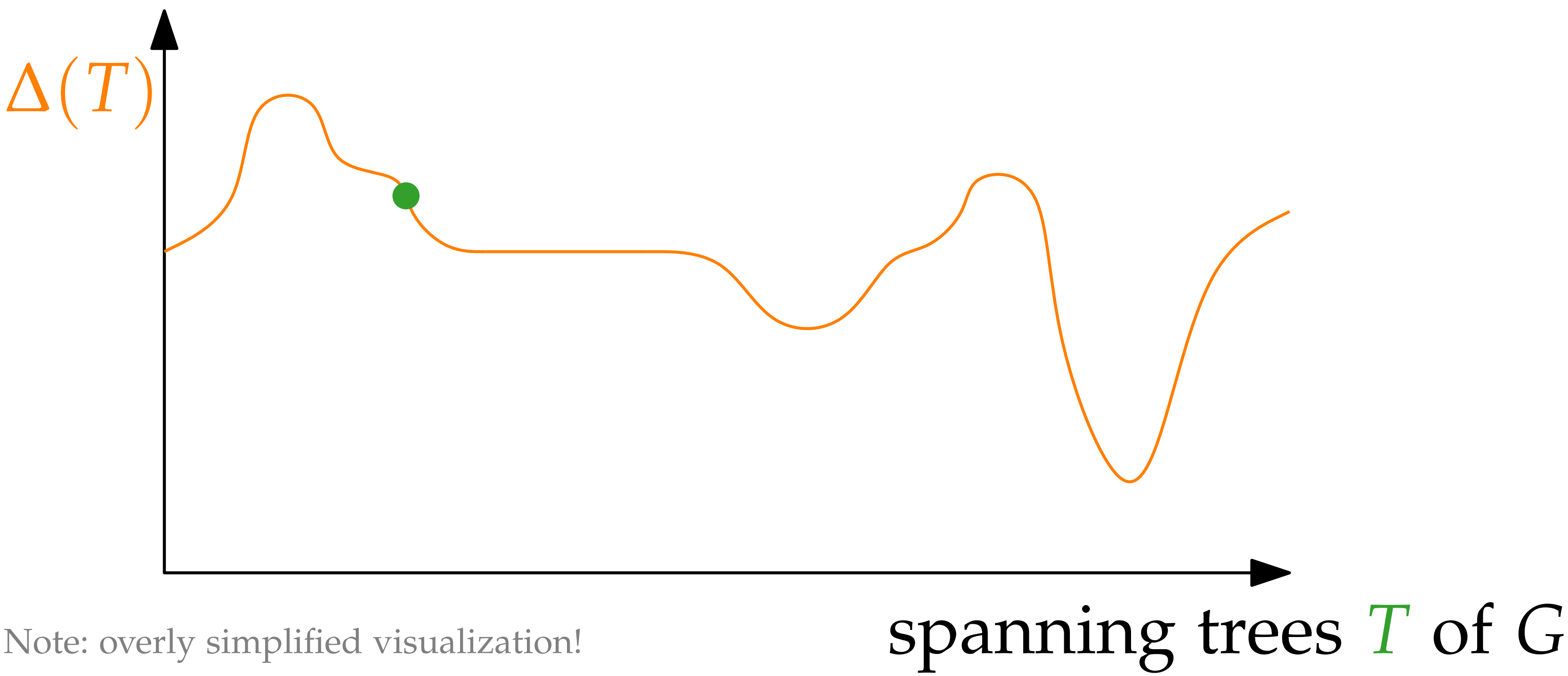

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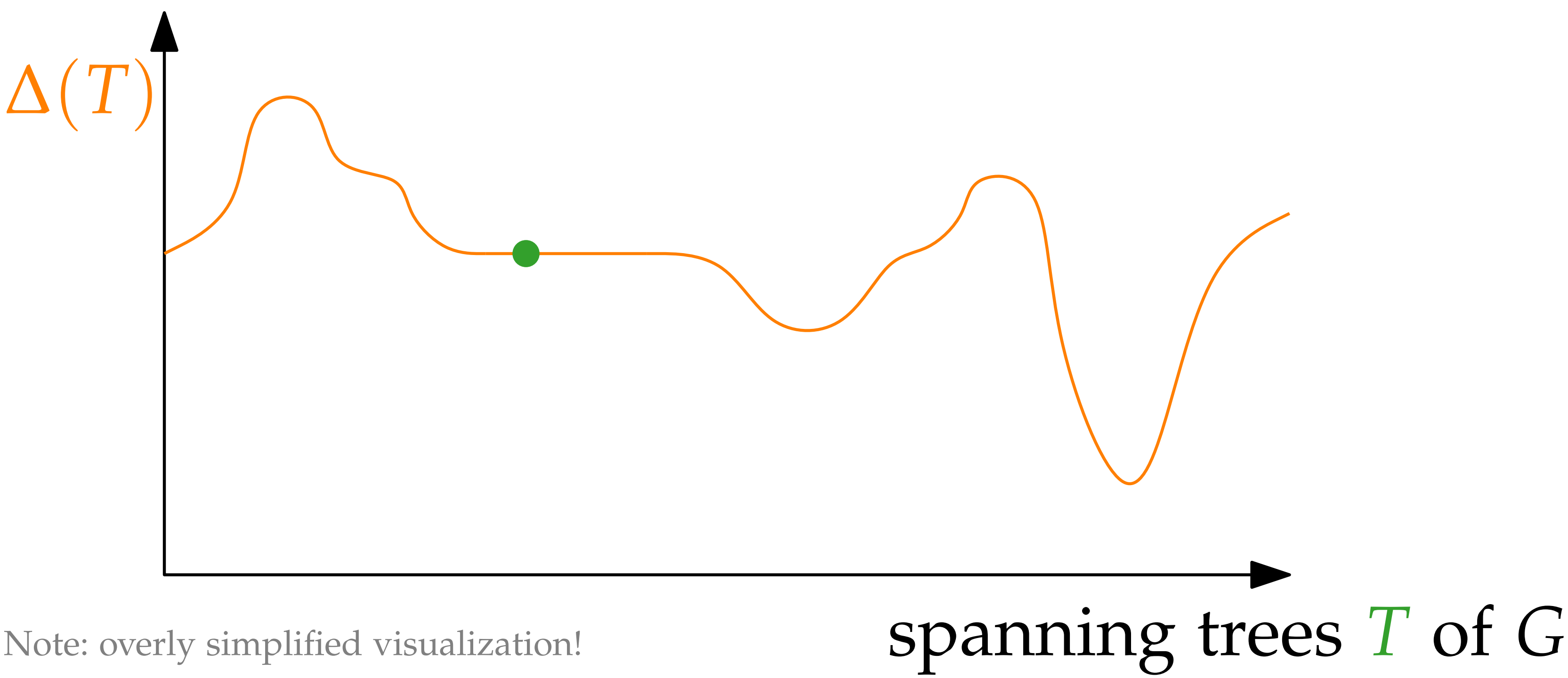

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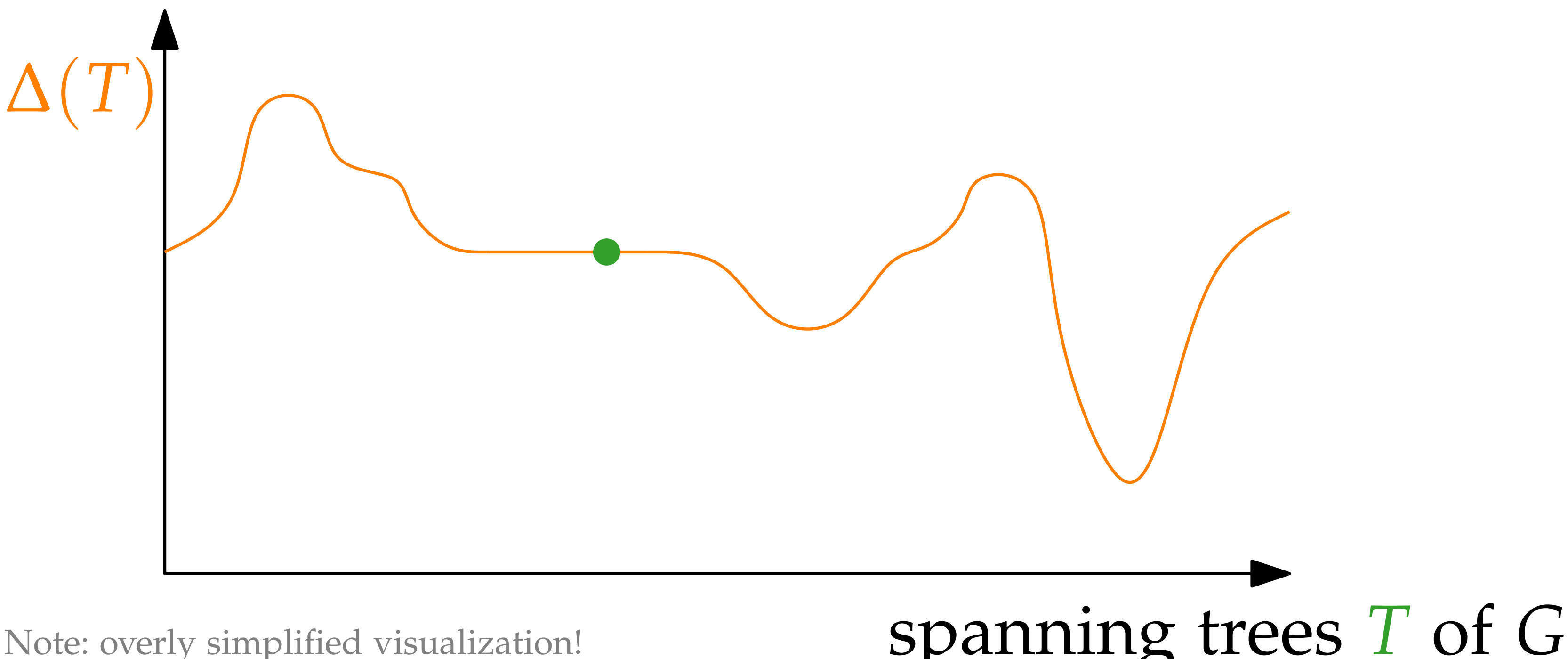

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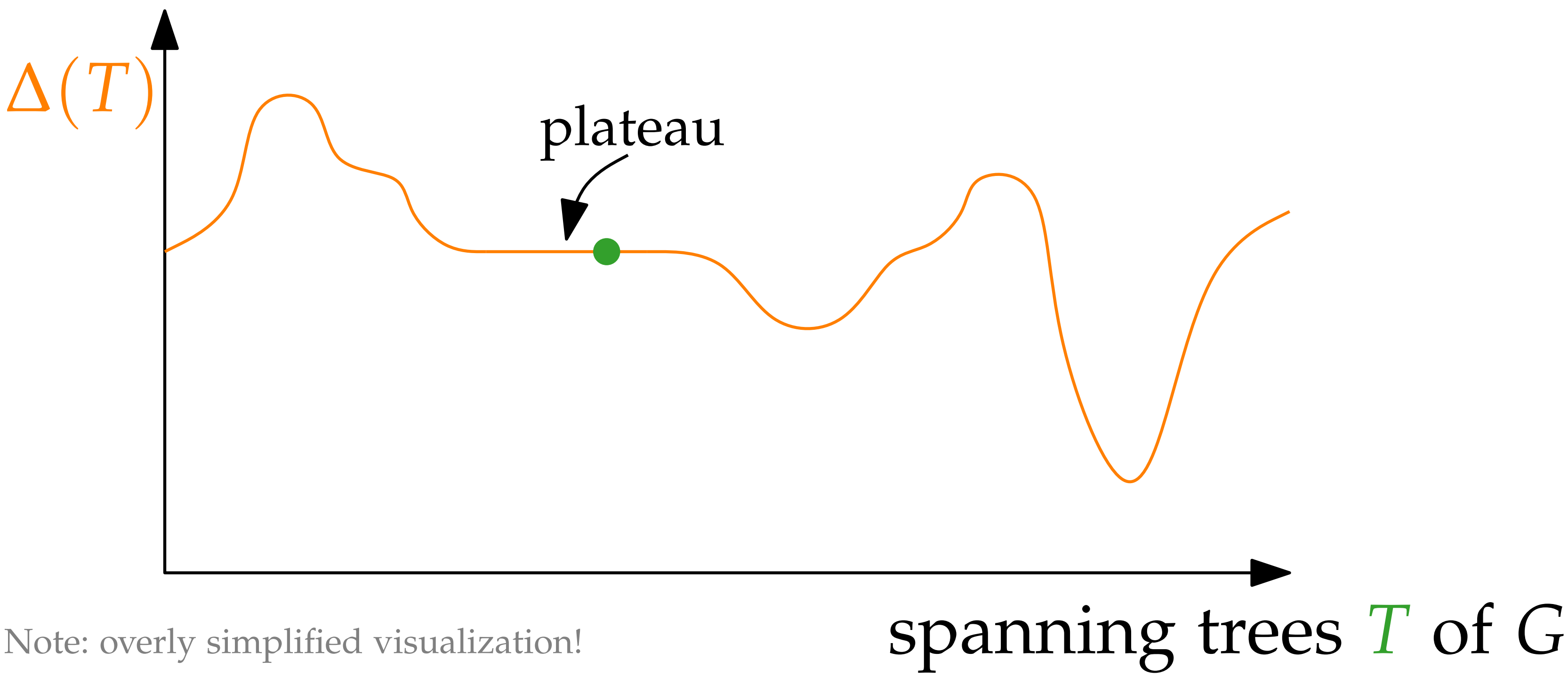

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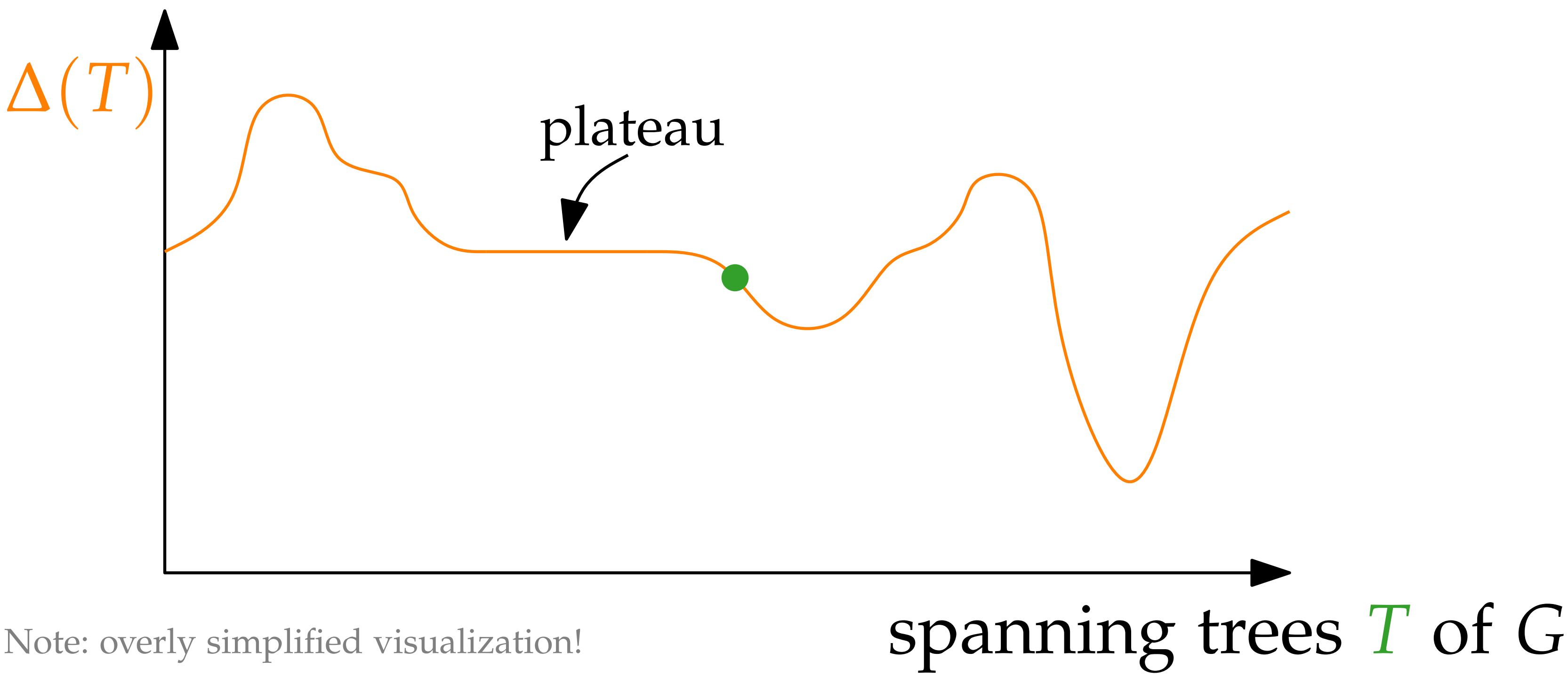

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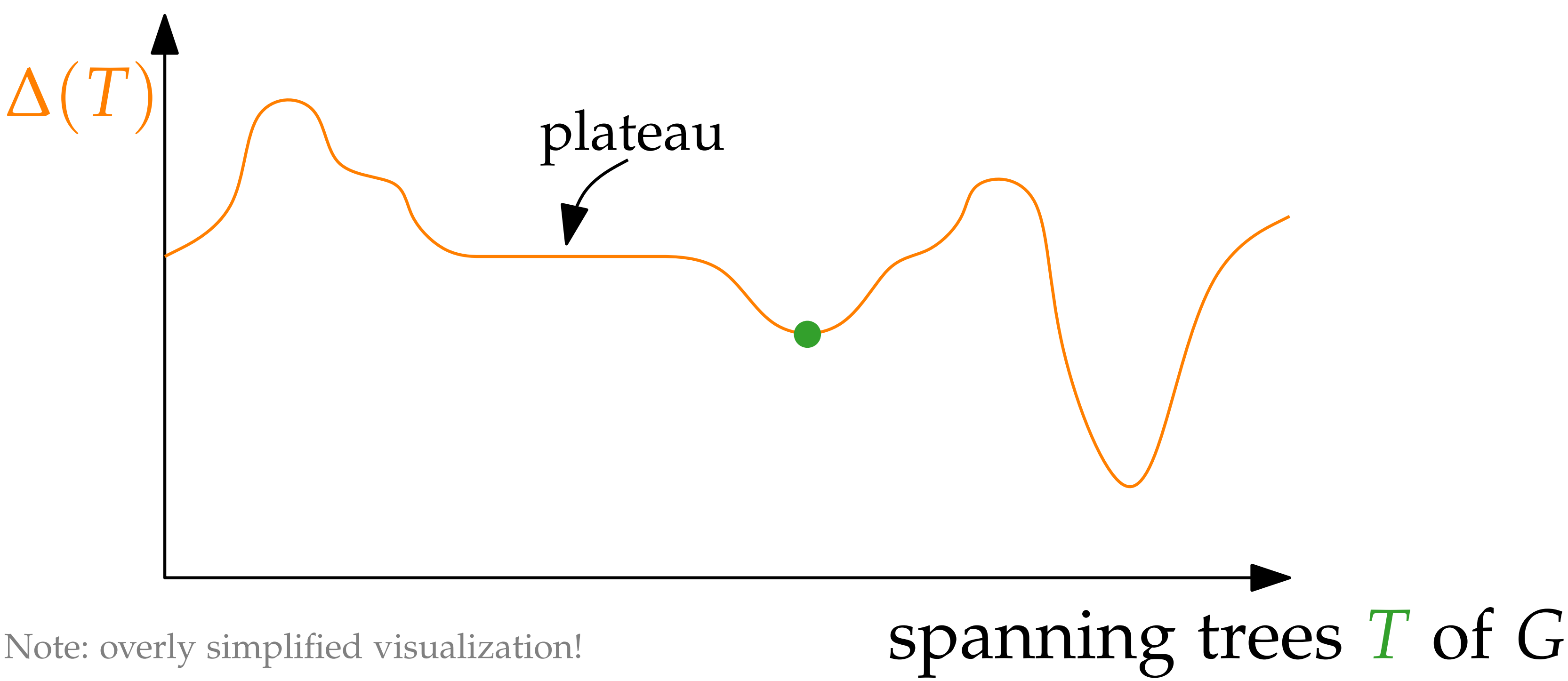

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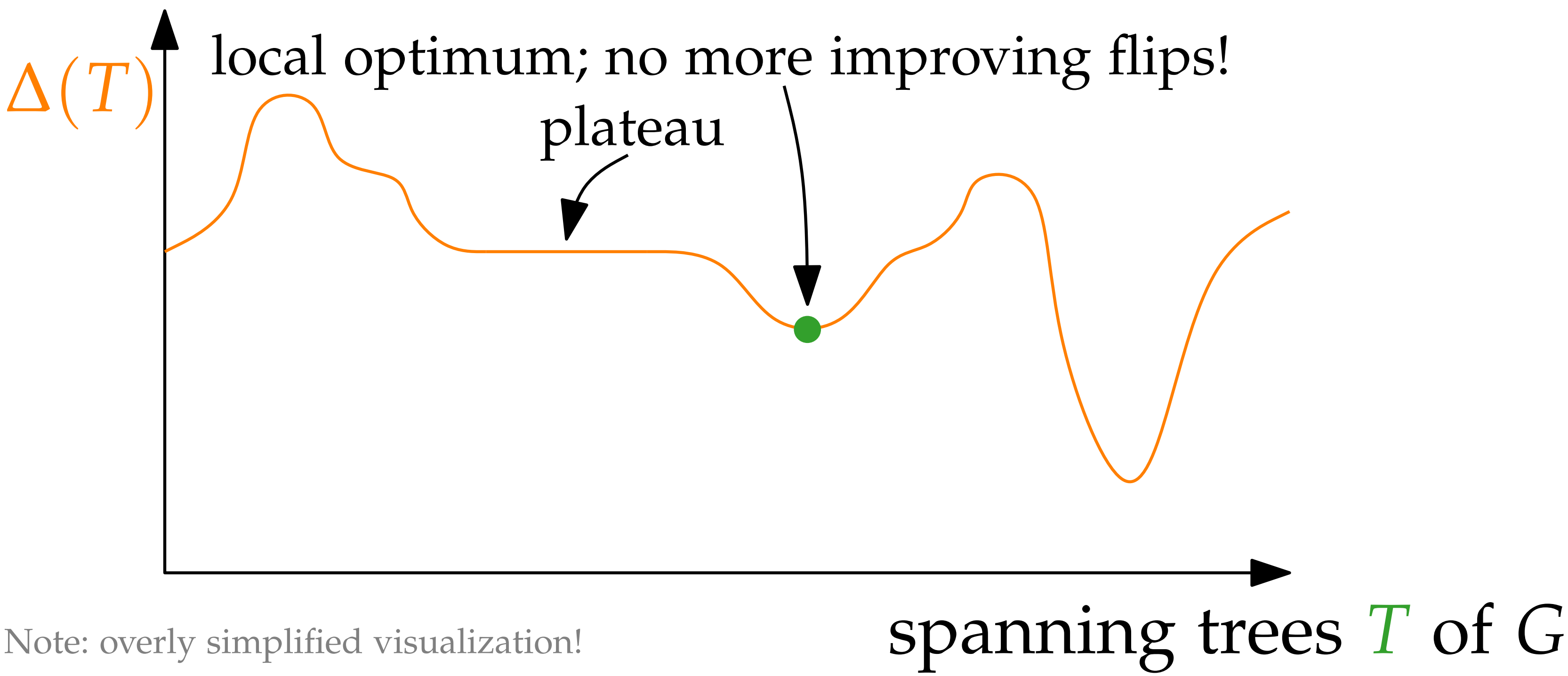

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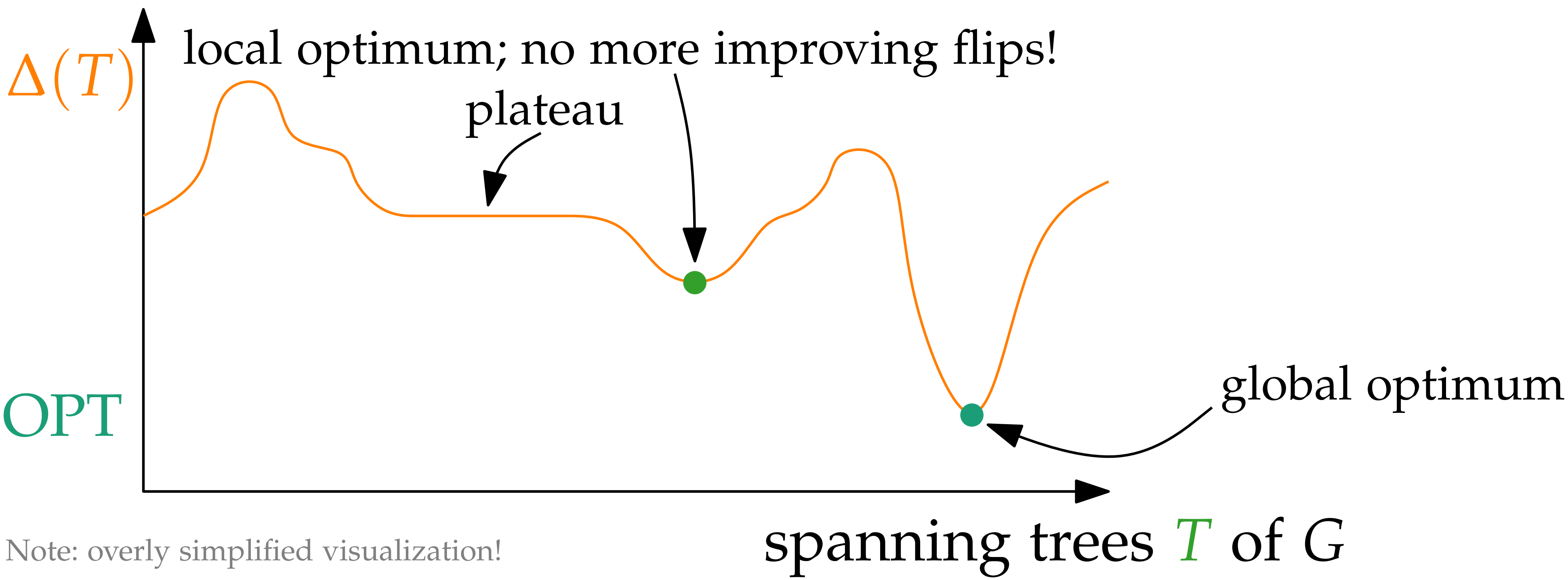

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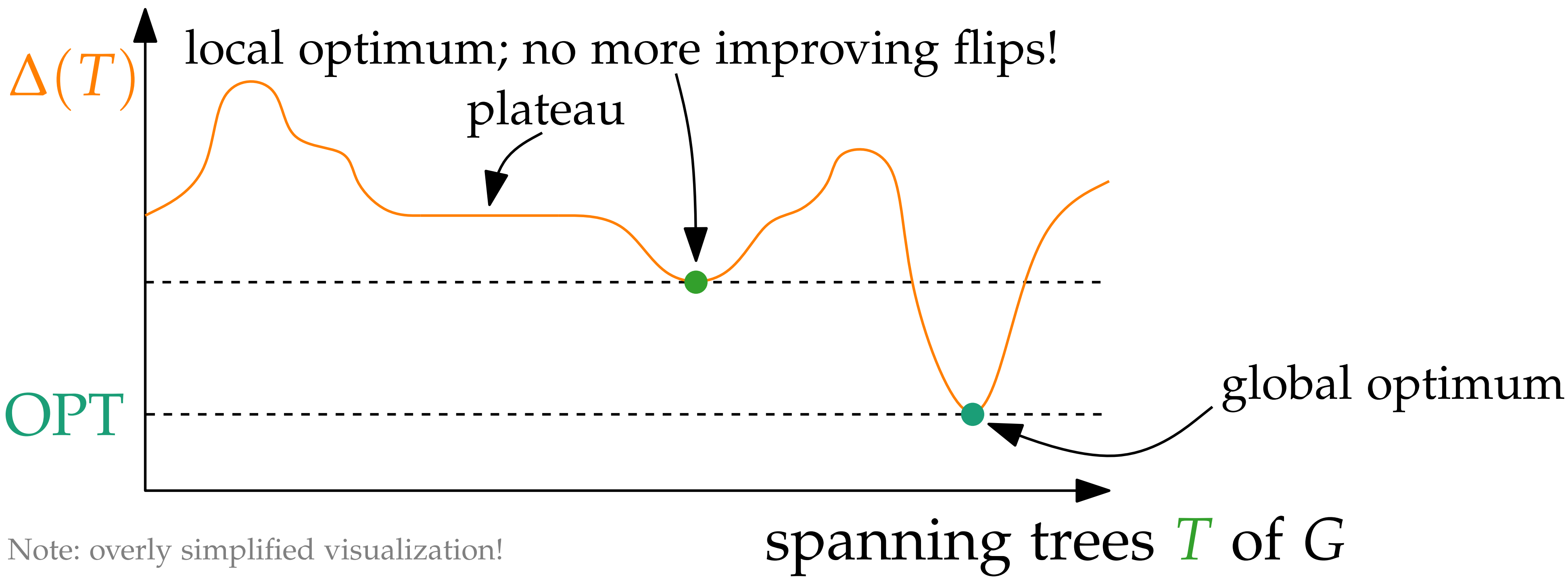

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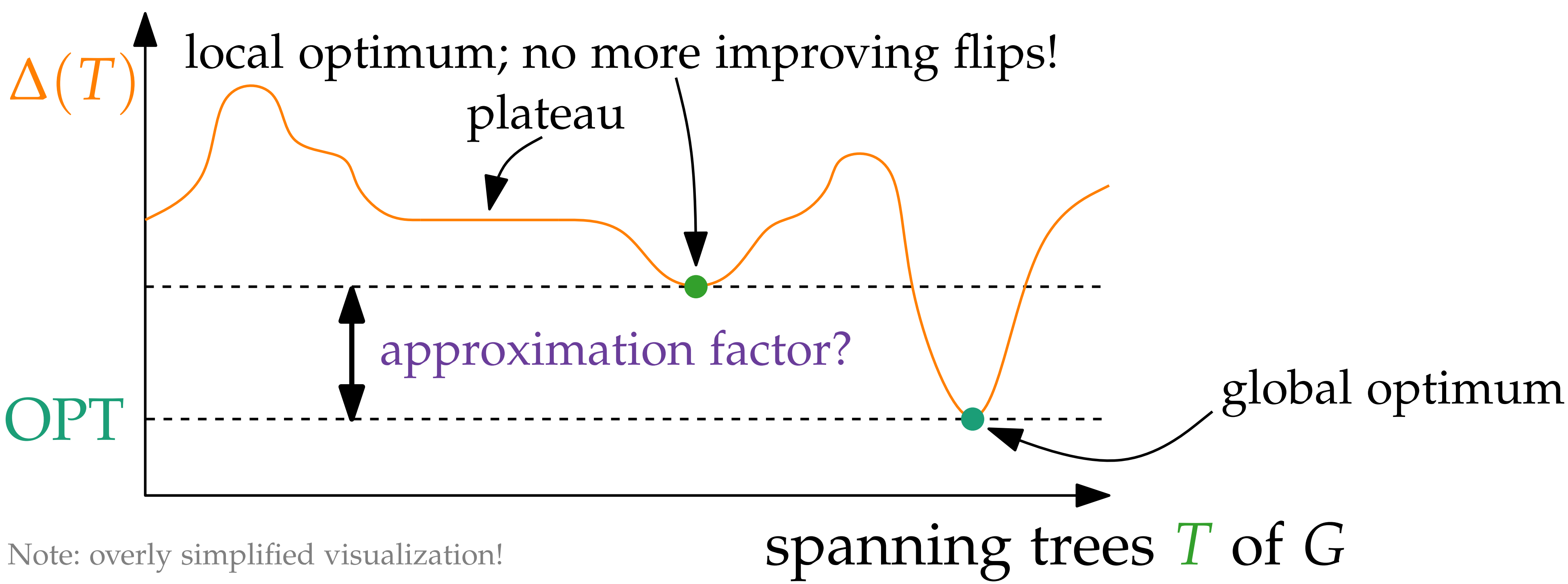

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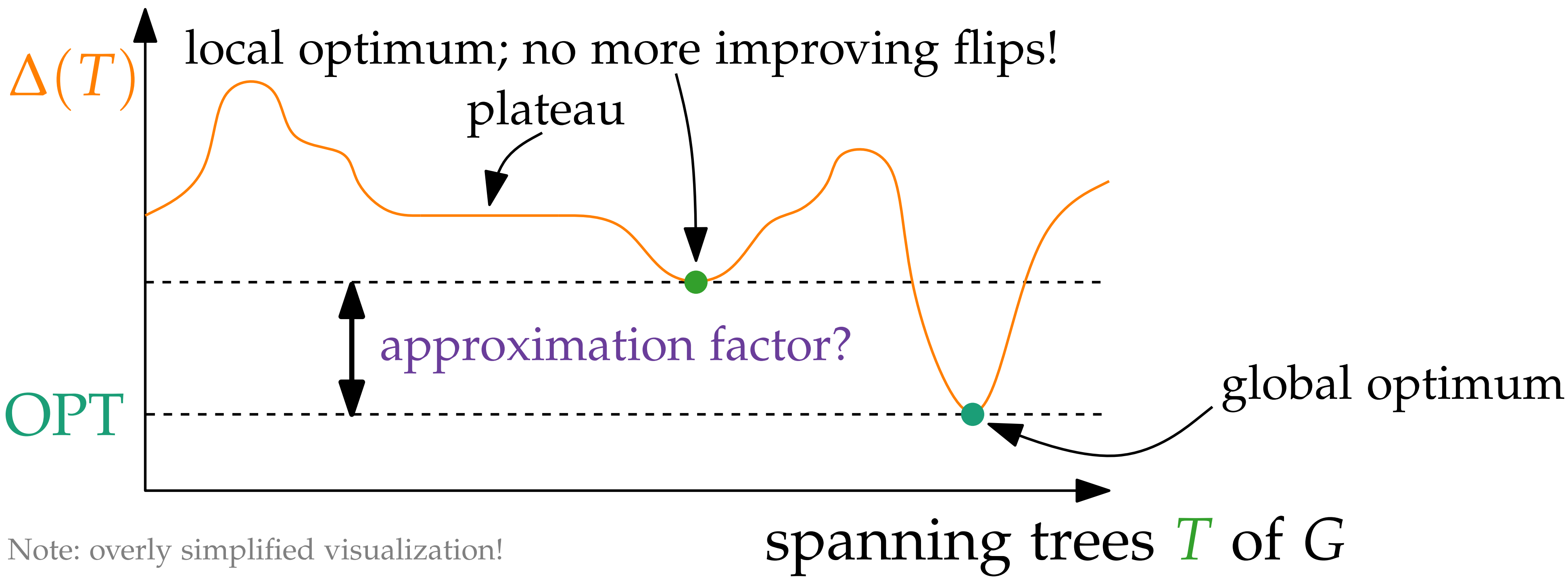
- Termination?



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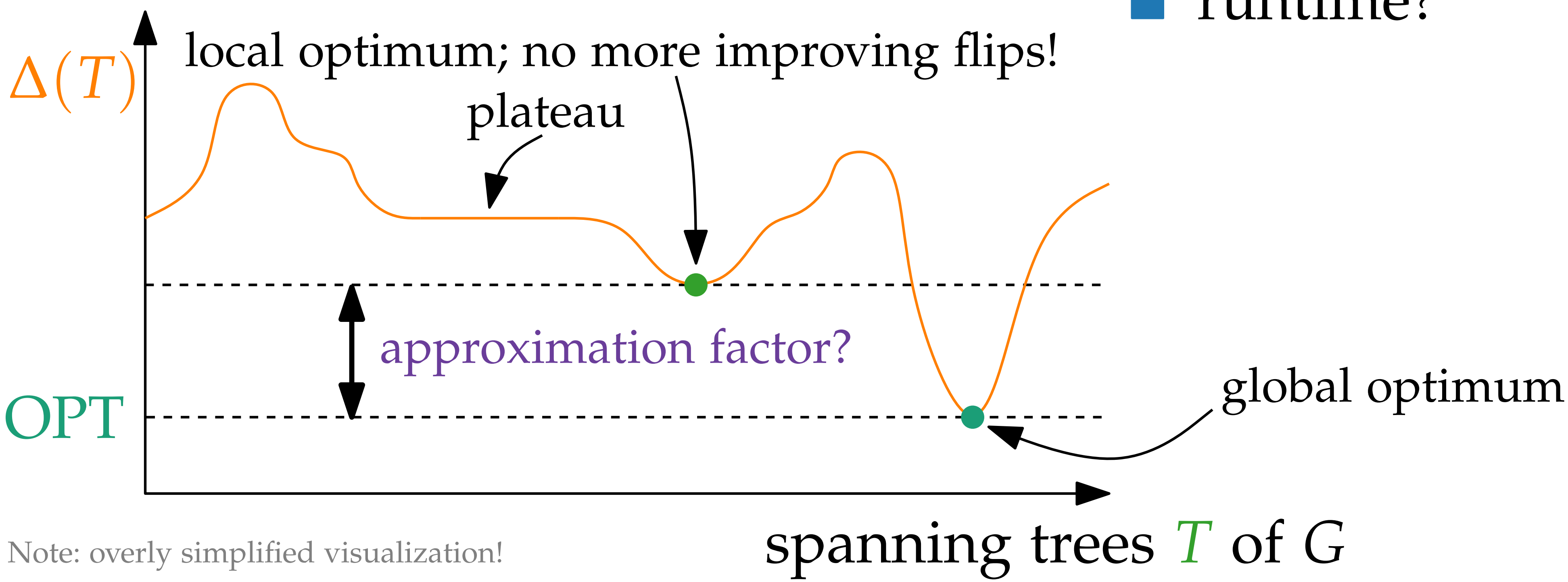
- Termination?
- runtime?



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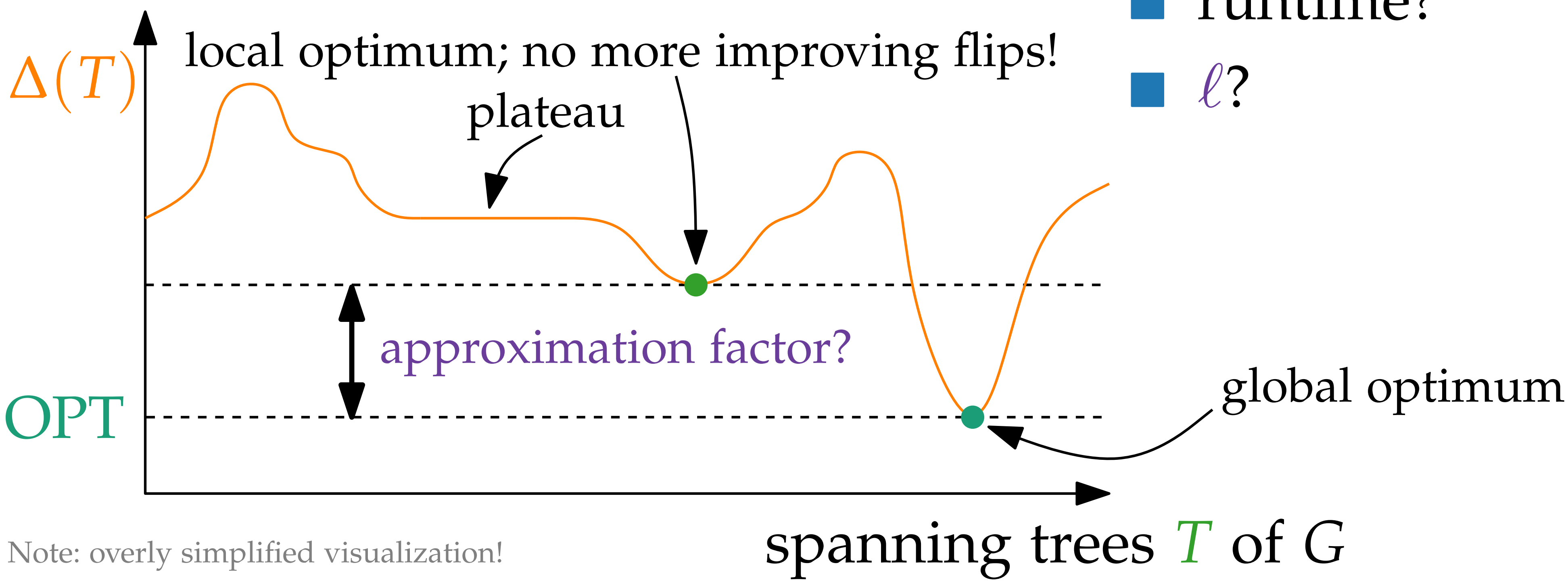
- Termination?
- runtime?
- ℓ?



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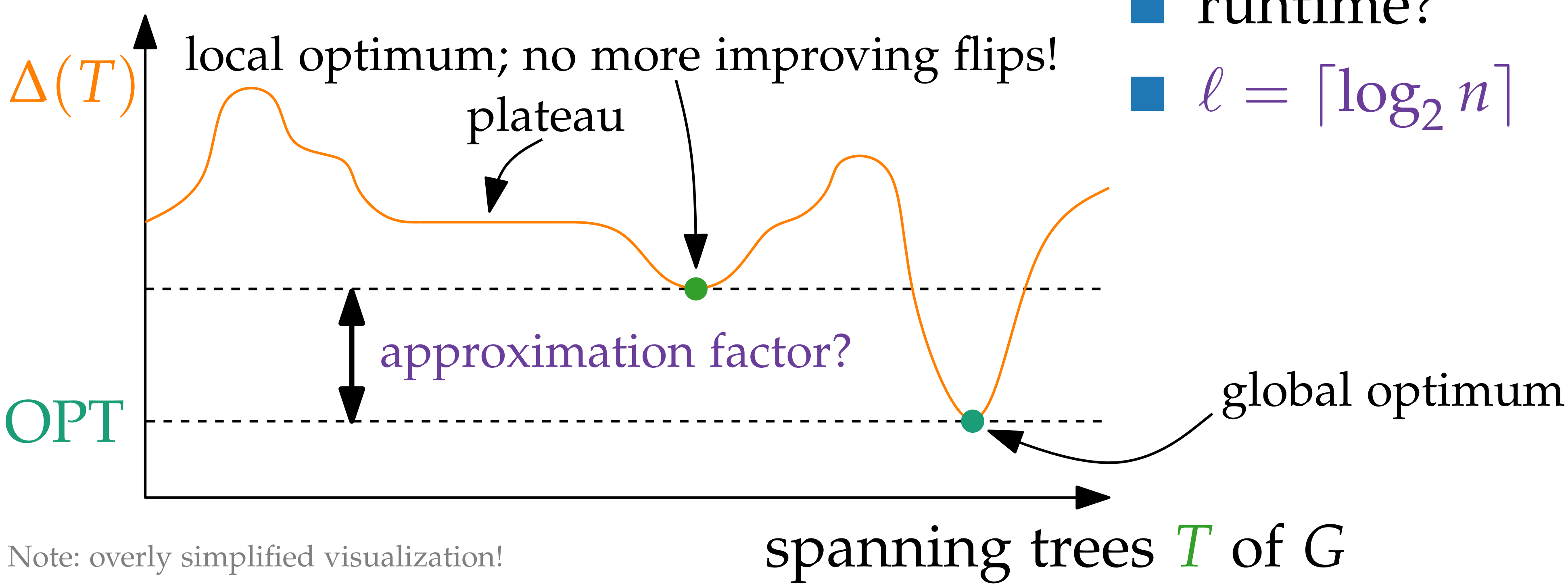
- Termination?
- runtime?
- $\ell = \lceil \log_2 n \rceil$



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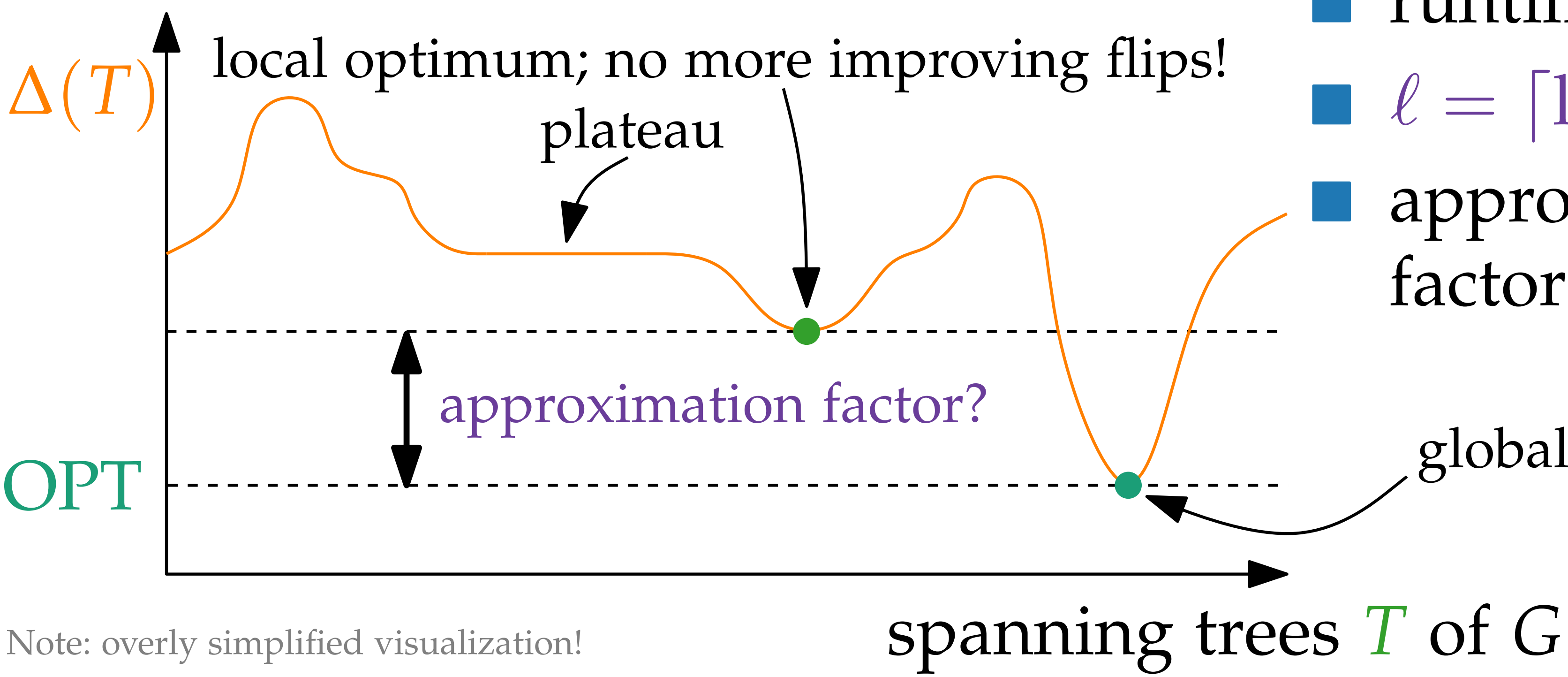
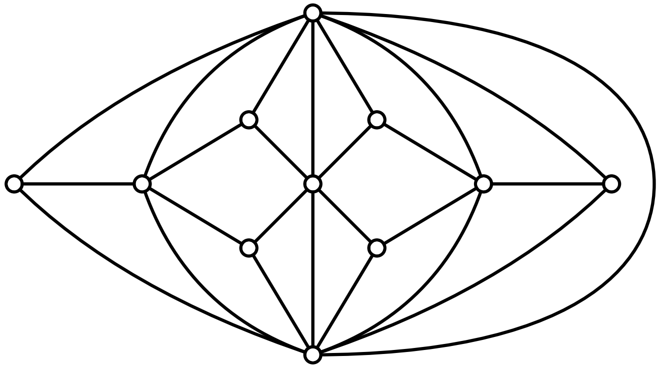
- Termination?
- runtime?
- $\ell = \lceil \log_2 n \rceil$
- approximation factor?

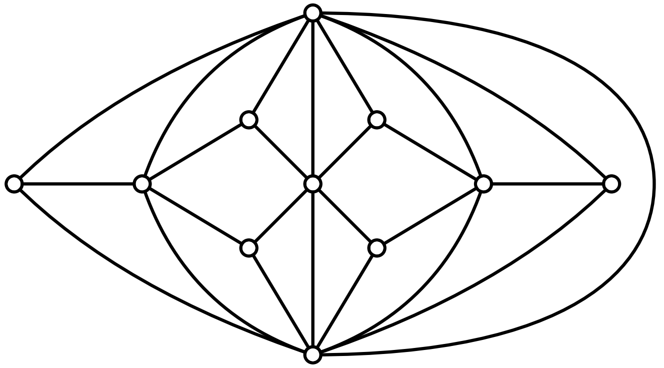


Note: overly simplified visualization!

Example

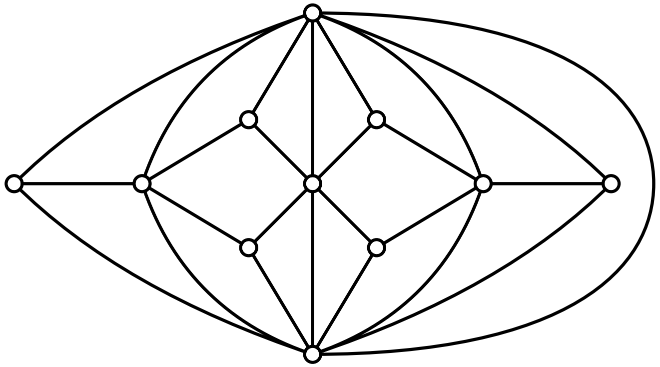


Example

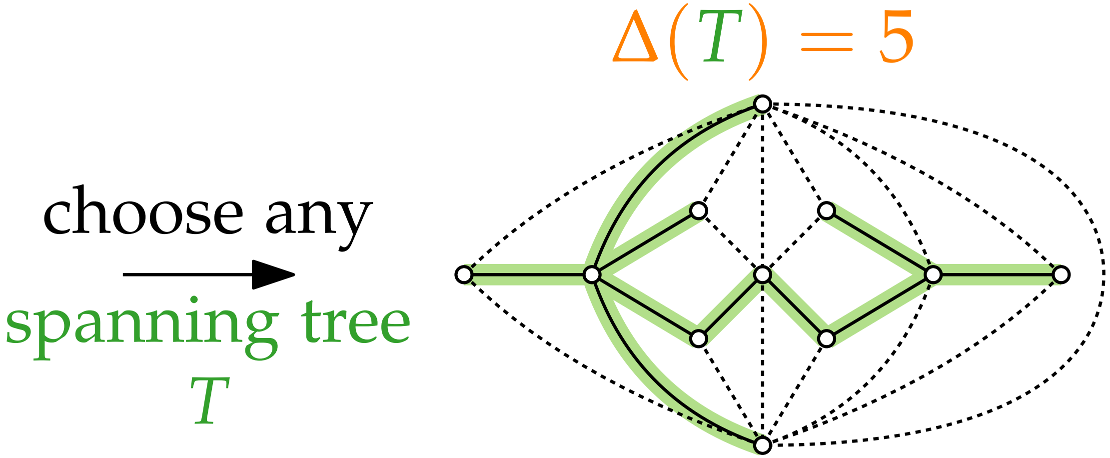


Goldner–Harary graph (minus two edges)

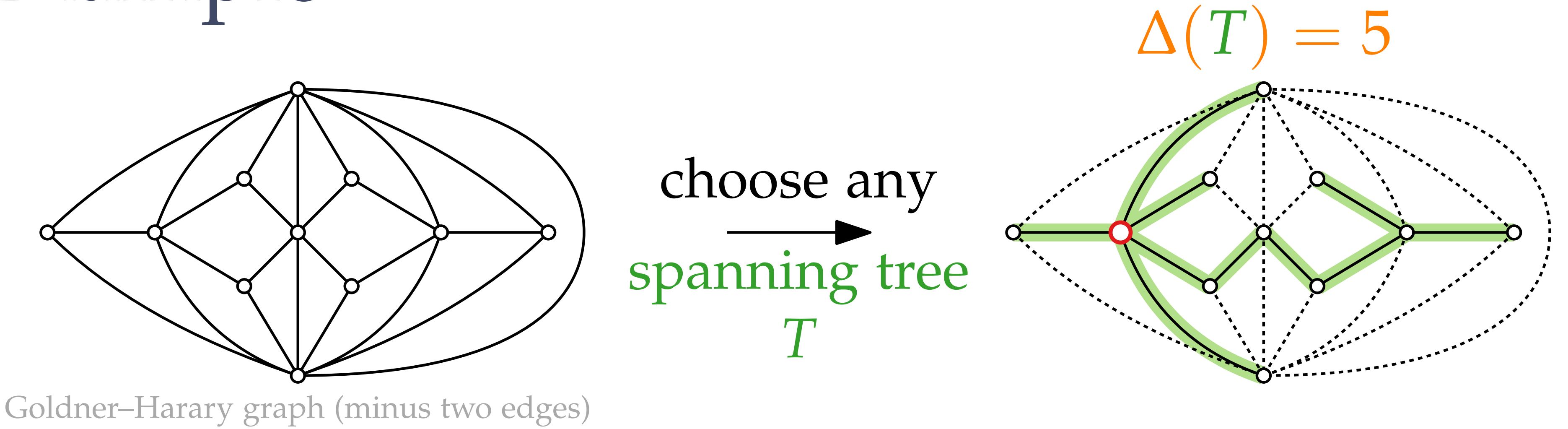
Example



Goldner–Harary graph (minus two edges)

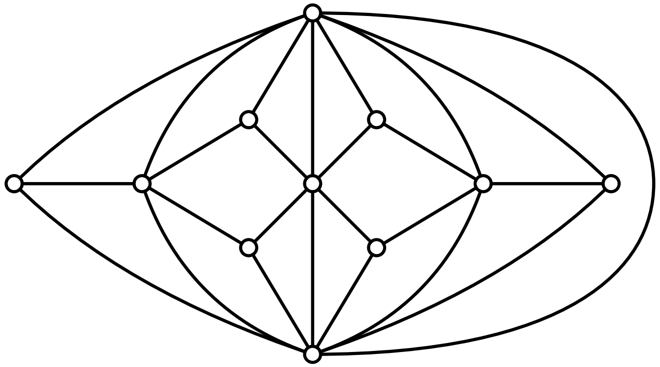


Example

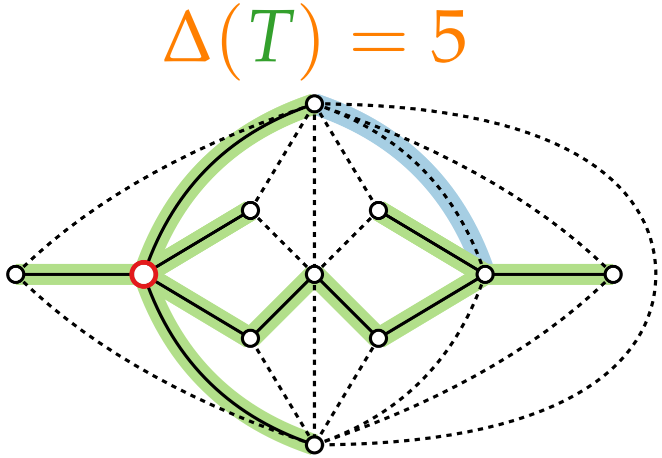


$\Delta(T) = 5$
choose any
spanning tree
T
Goldner–Harary graph (minus two edges)

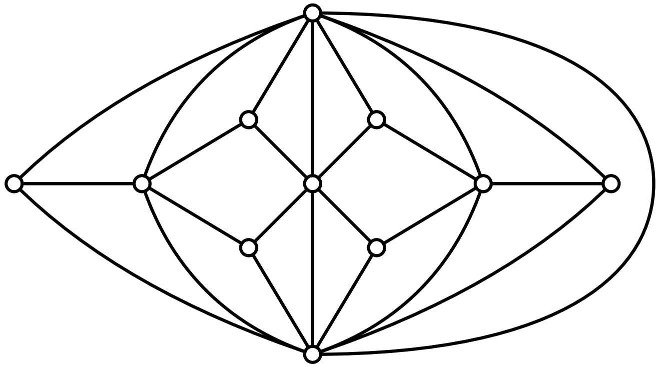
Example


Goldner–Harary graph (minus two edges)

choose any
spanning tree
T



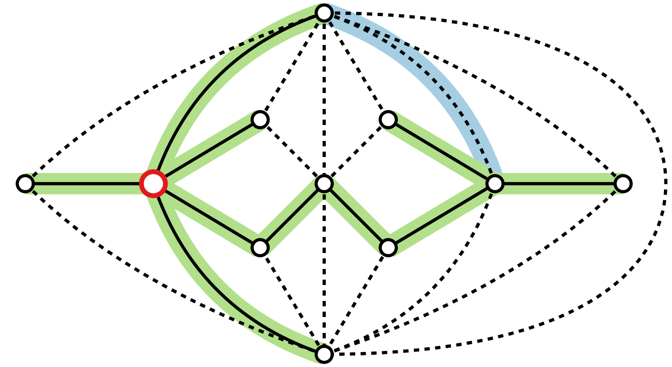
Example



Goldner–Harary graph (minus two edges)

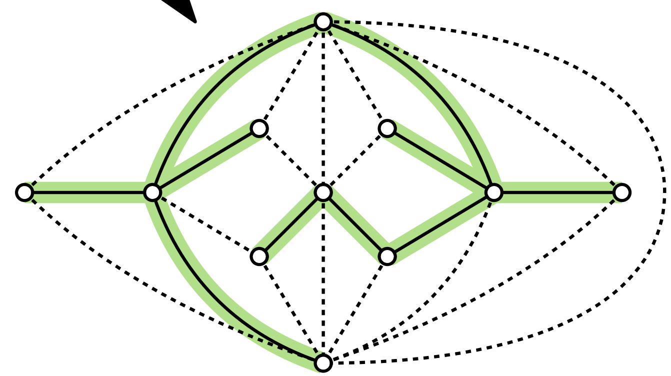
choose any → spanning tree T

$\Delta(T) = 5$

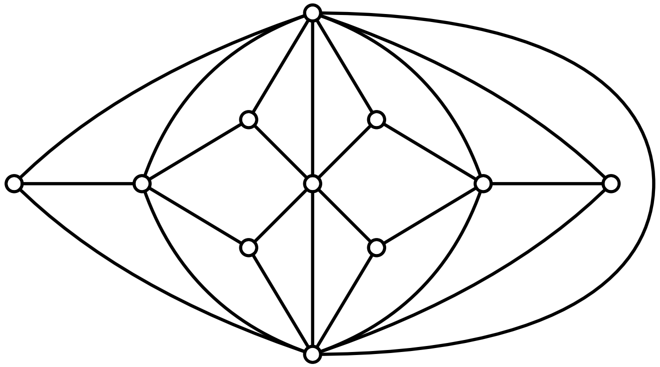


improving flip

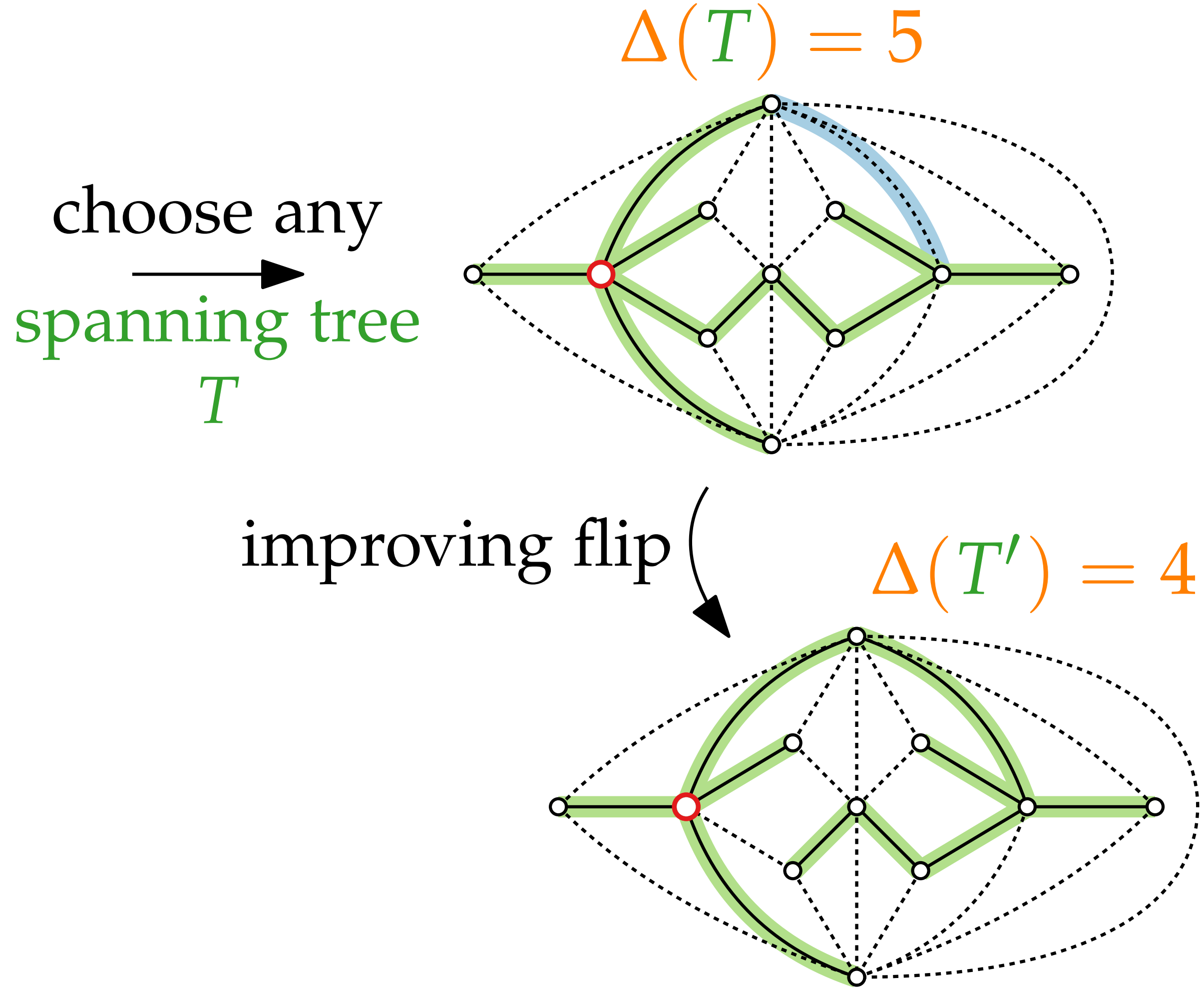
$\Delta(T') = 4$



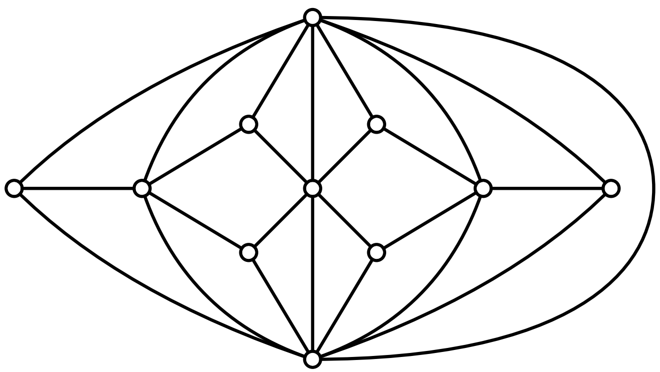
Example



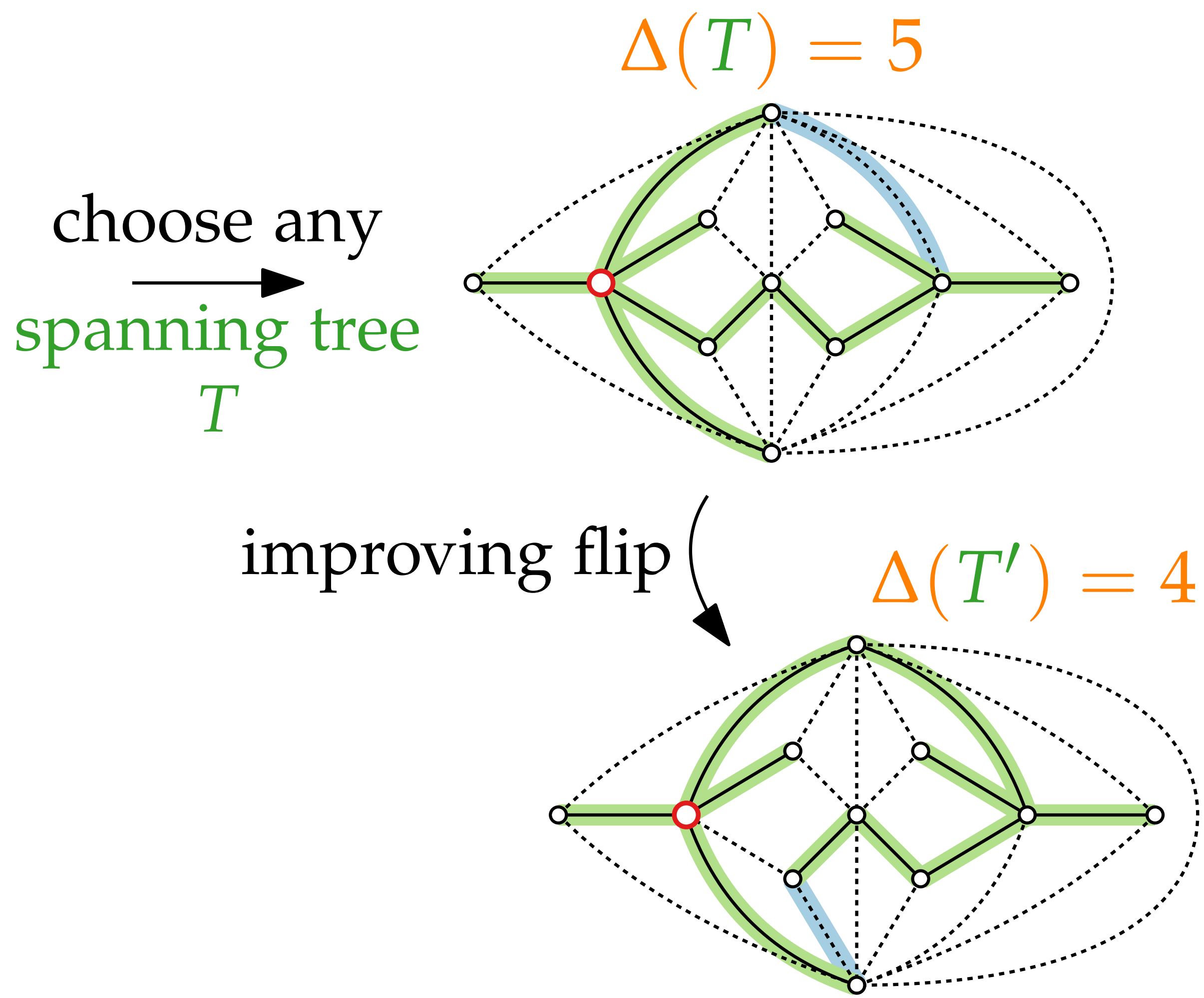
Goldner–Harary graph (minus two edges)



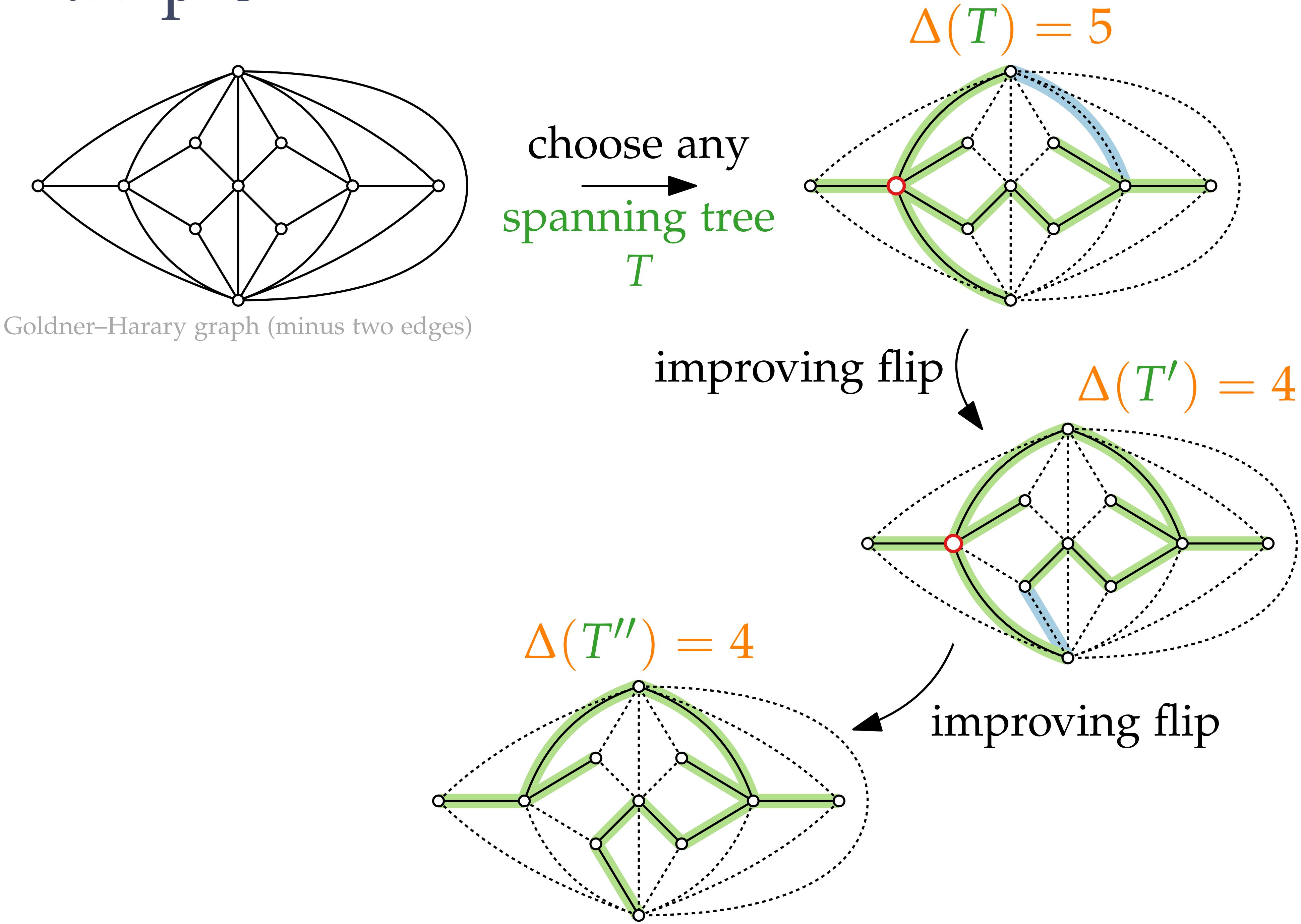
Example



Goldner–Harary graph (minus two edges)

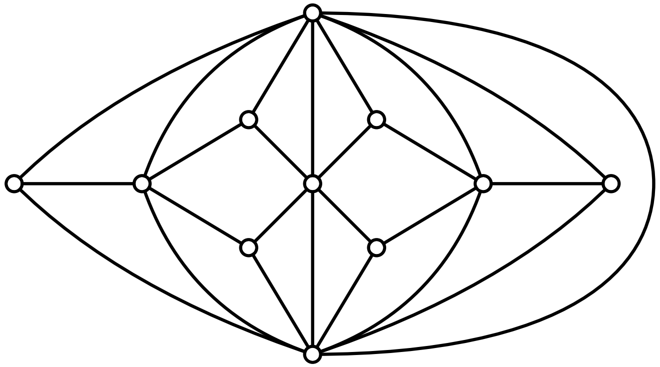


Example

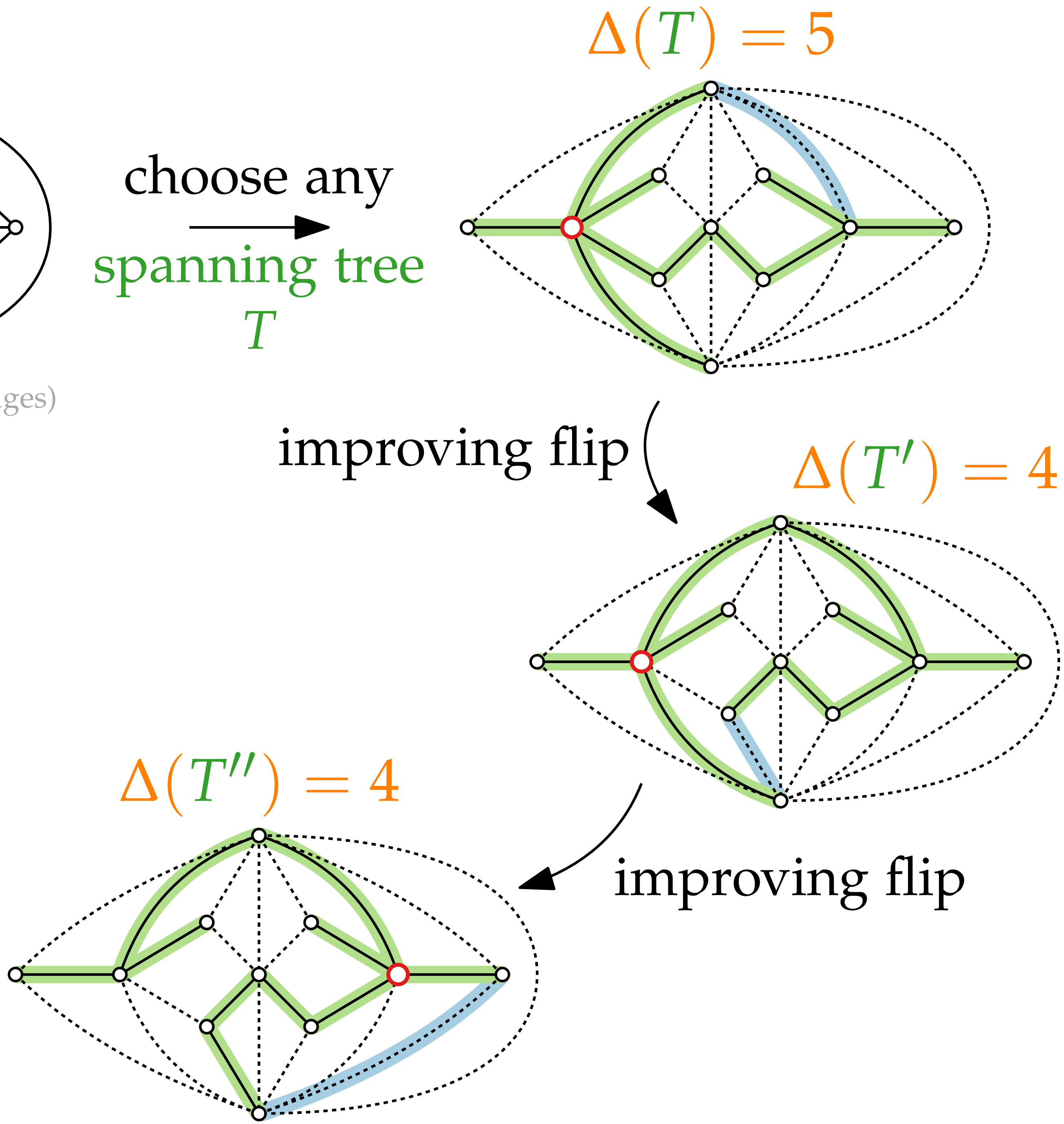


Goldner–Harary graph (minus two edges)
choose any
spanning tree
T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$

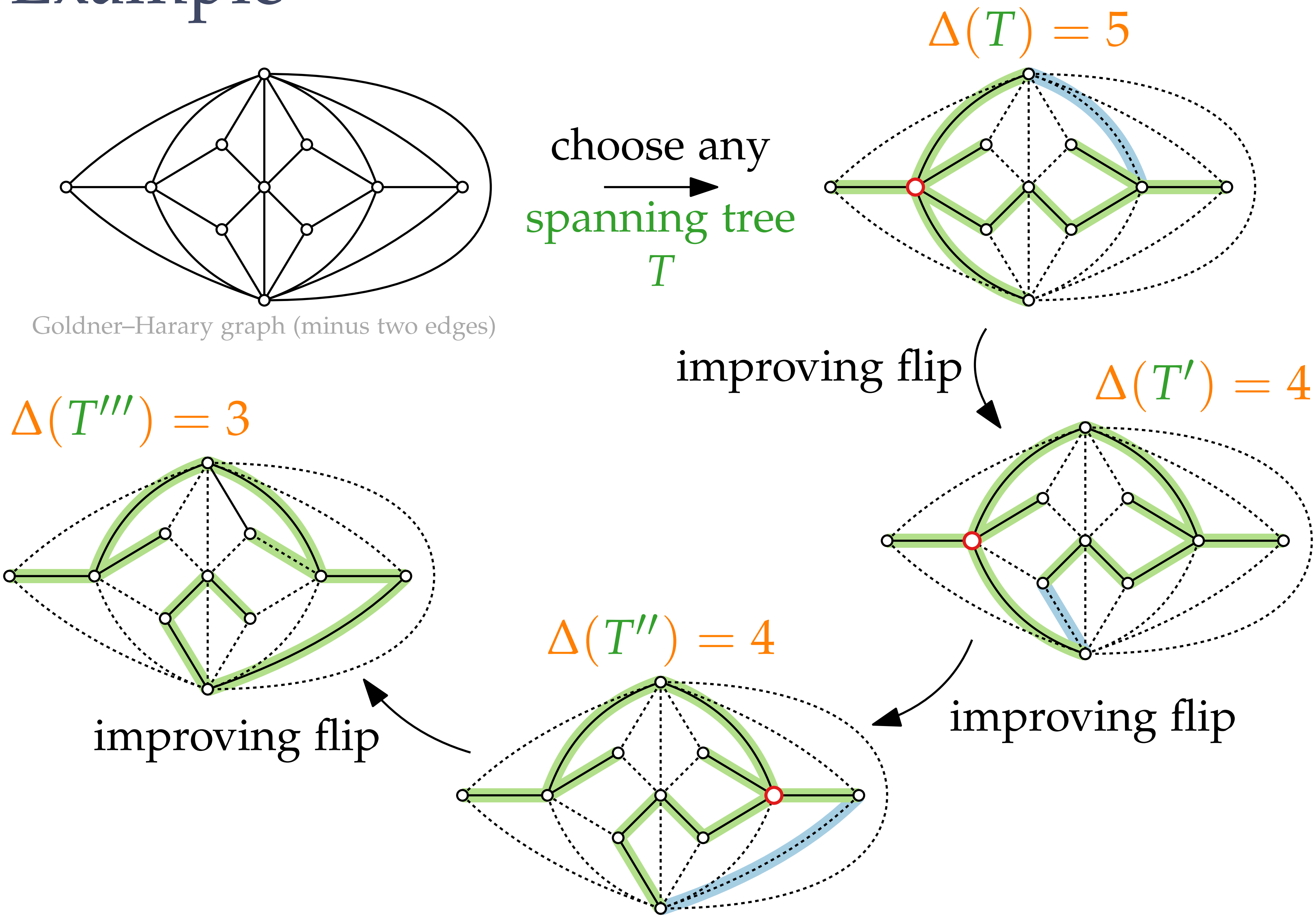
Example



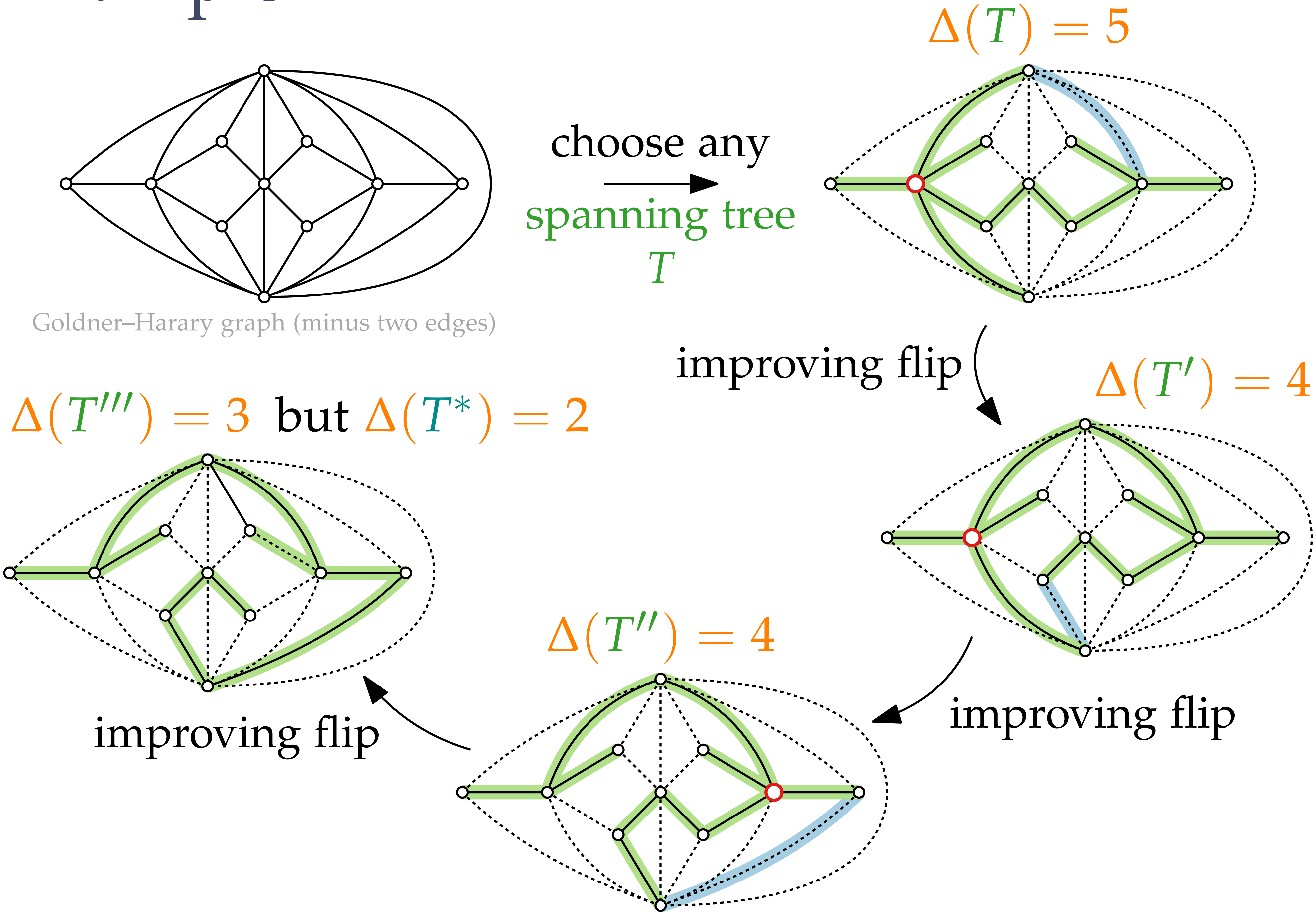
Goldner–Harary graph (minus two edges)



Example

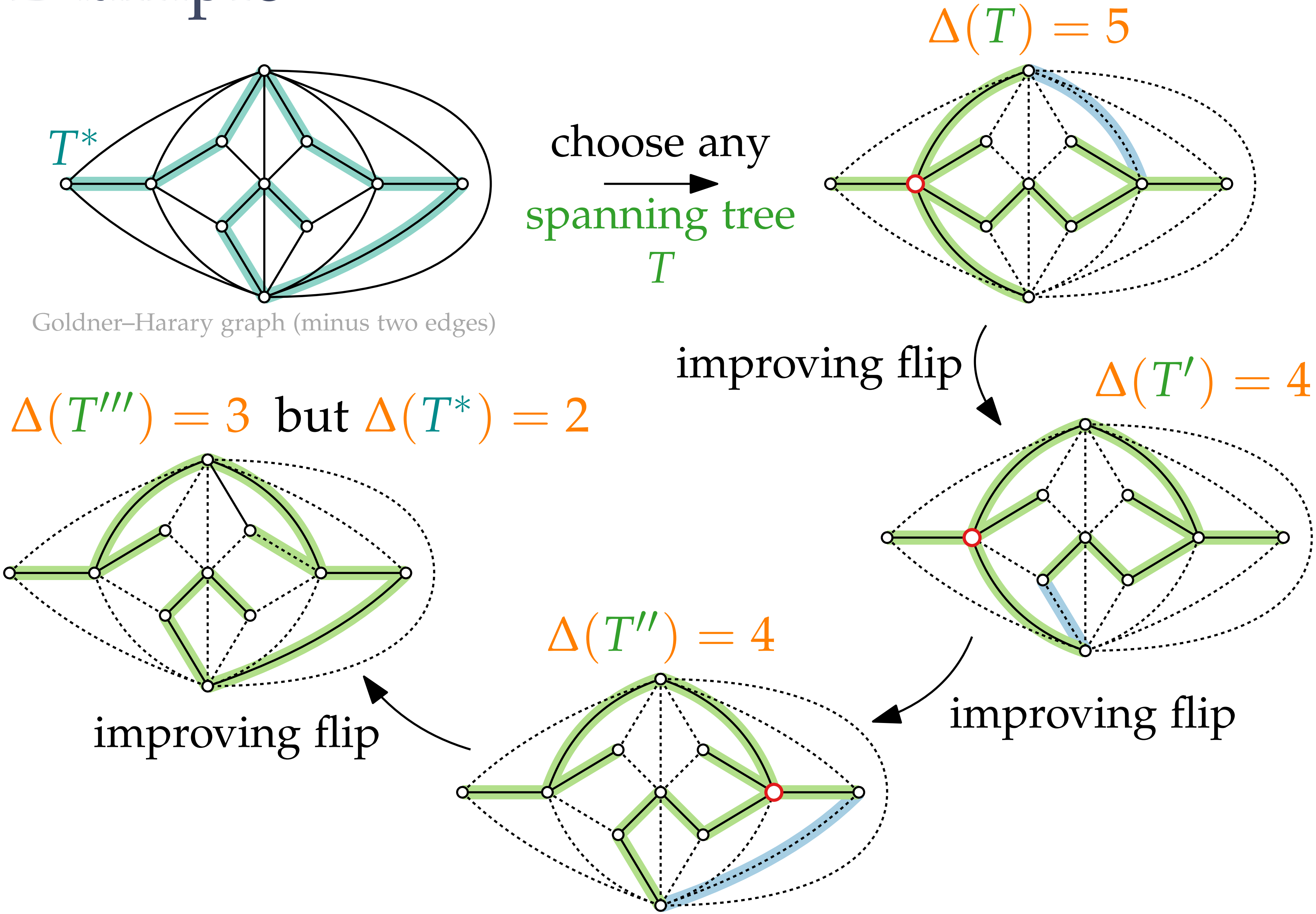

Goldner–Harary graph (minus two edges)
choose any
spanning tree
T
$\Delta(T) = 5$
improving flip
$\Delta(T') = 4$
improving flip
$\Delta(T'') = 4$
improving flip
$\Delta(T''') = 3$

Example



Goldner–Harary graph (minus two edges)
choose any
spanning tree
T
Δ(T) = 5
improving flip
Δ(T′) = 4
improving flip
Δ(T″) = 4
improving flip
Δ(T‴) = 3 but Δ(T*) = 2

Example

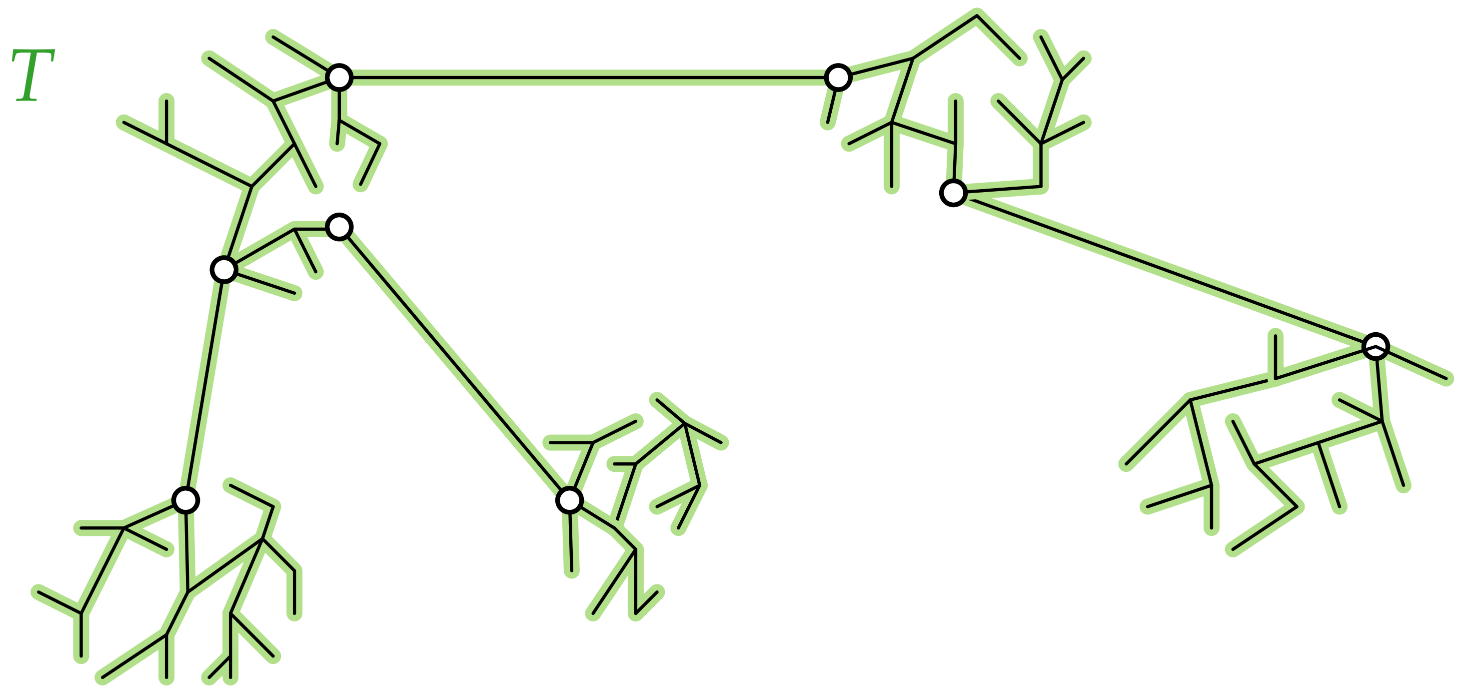

$\Delta(T) = 5$
T^*
choose any
spanning tree
T
Goldner–Harary graph (minus two edges)
improving flip
$\Delta(T') = 4$
$\Delta(T''') = 3$ but $\Delta(T^*) = 2$
improving flip
$\Delta(T'') = 4$
improving flip

Approximation Algorithms

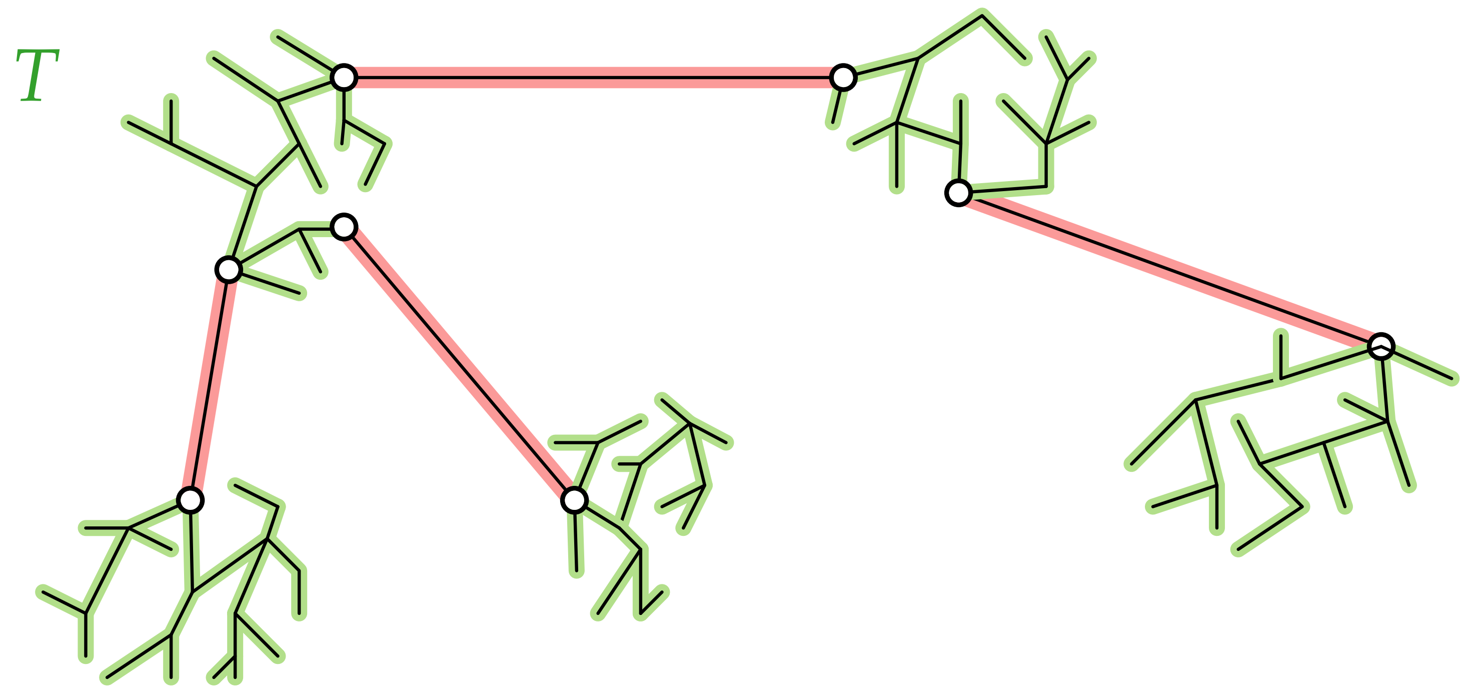
Lecture 10:
Minimum-Degree Spanning Tree via Local Search

Part III:
Lower Bound

Decomposition

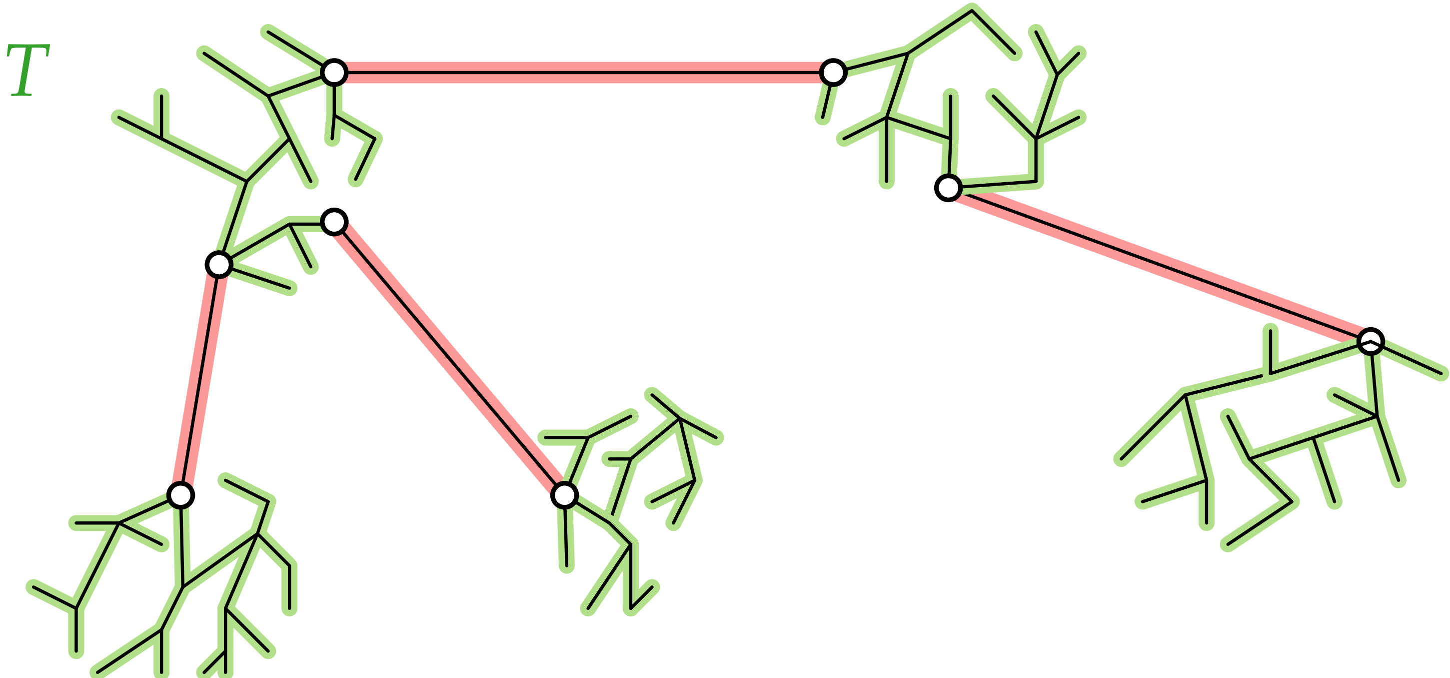


Decomposition



Decomposition

- Removing k edges decomposes T into $k+1$ components



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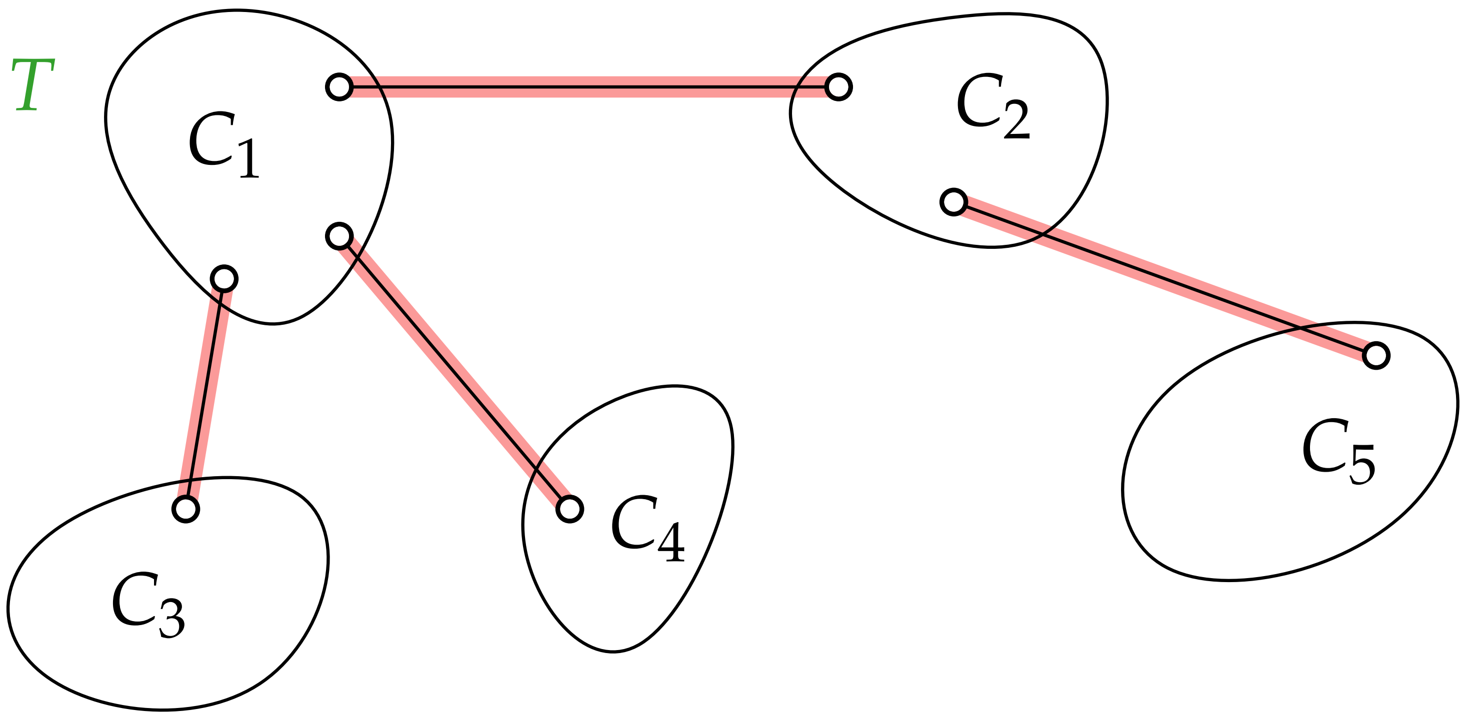
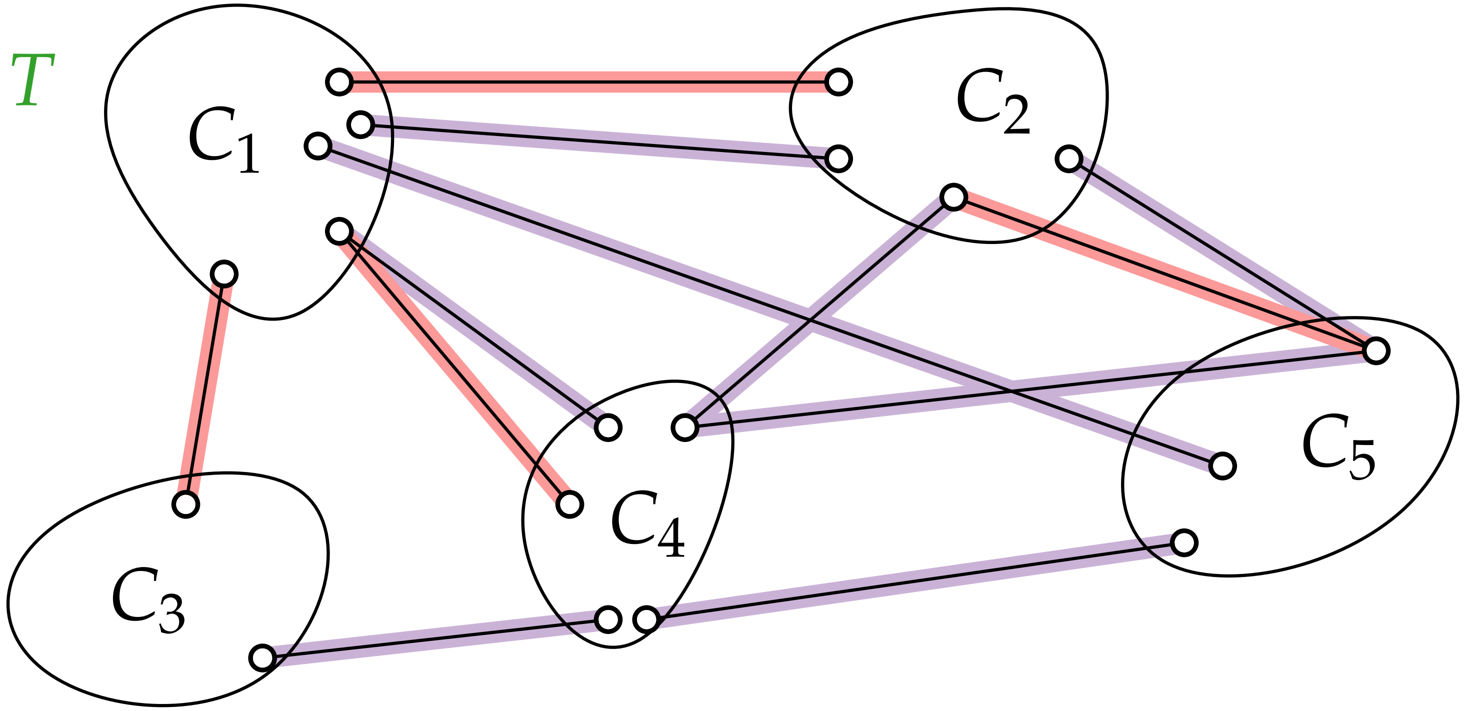


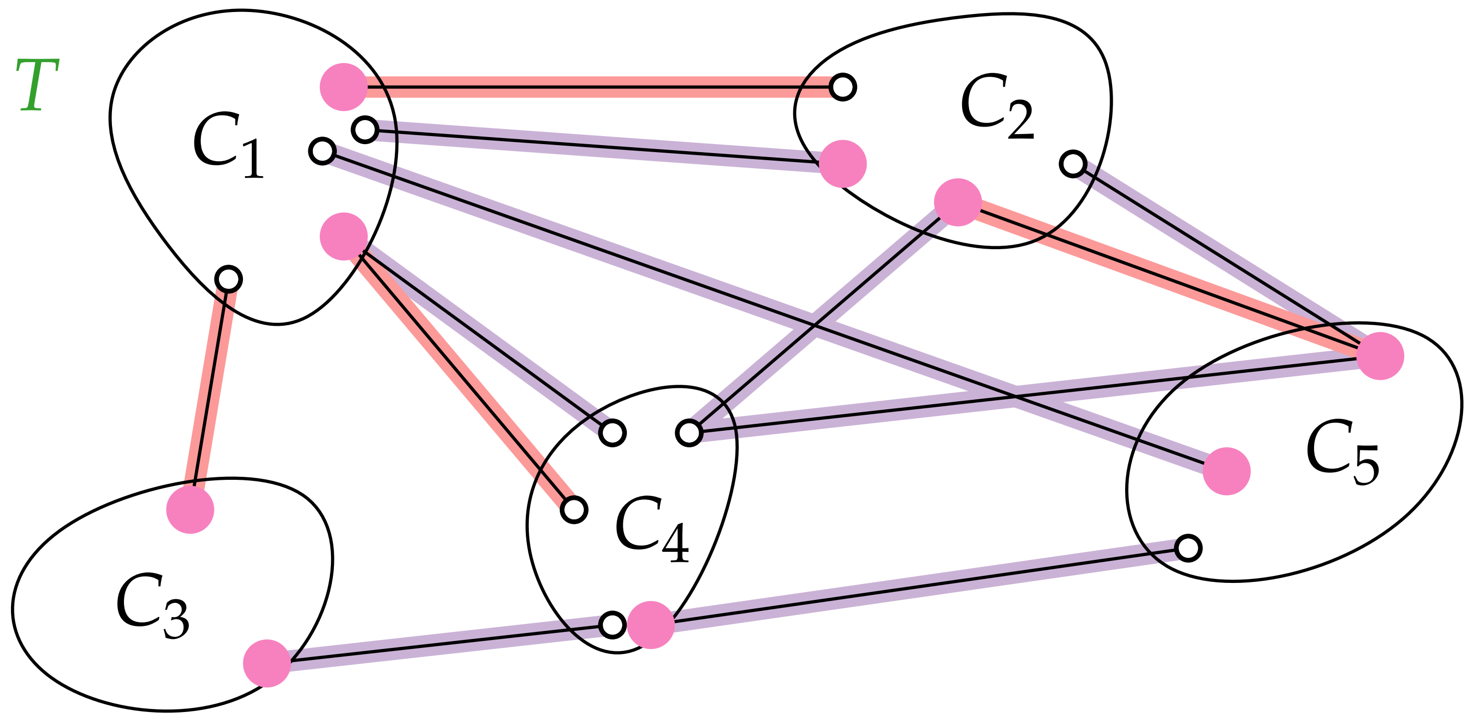
Decomposition

- Removing k edges decomposes T into $k+1$ components
- $E' = \{\text{edges in } G \text{ between different components } C_i \neq C_j\}$.



Decomposition

- Removing k edges decomposes T into $k+1$ components
- $E' = \{\text{edges in } G \text{ between different components } C_i \neq C_j\}$.
- $S :=$ vertex cover of E'.



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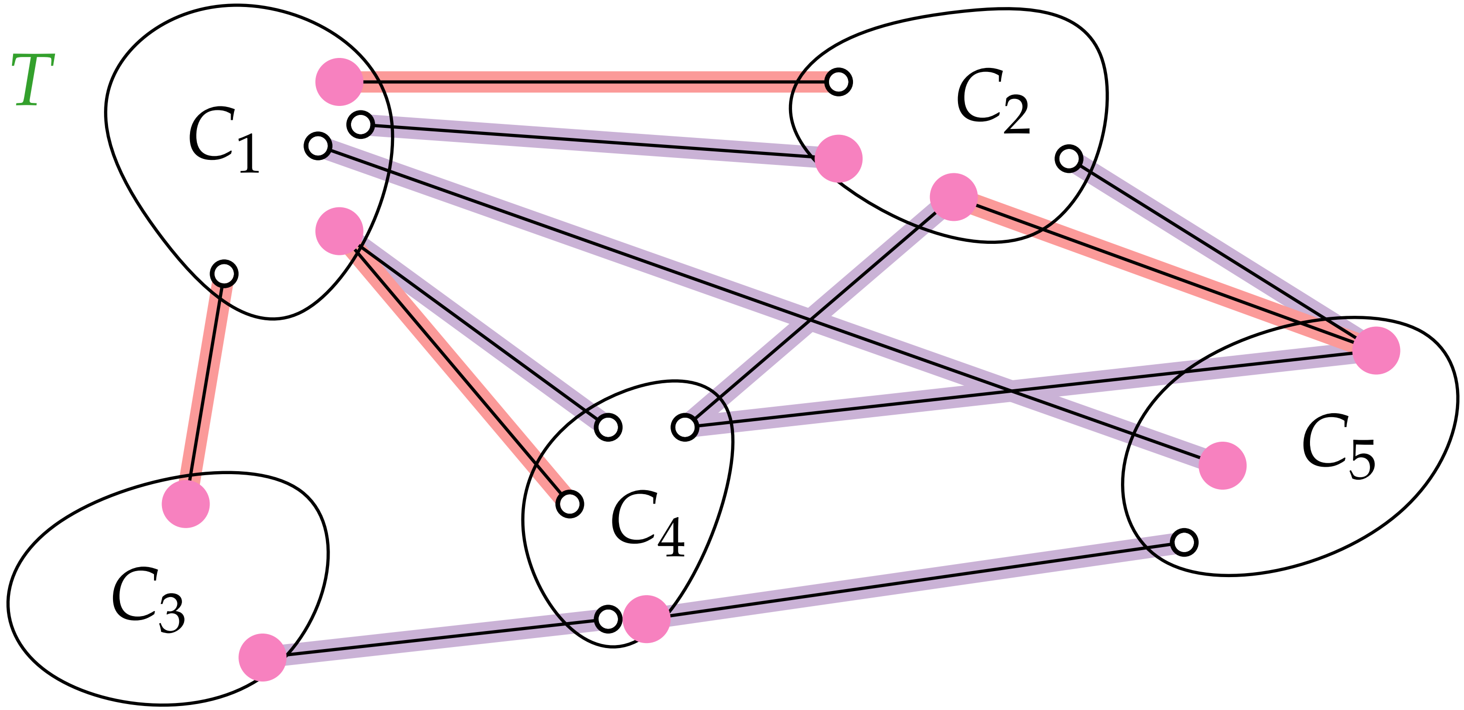

- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*

Decomposition

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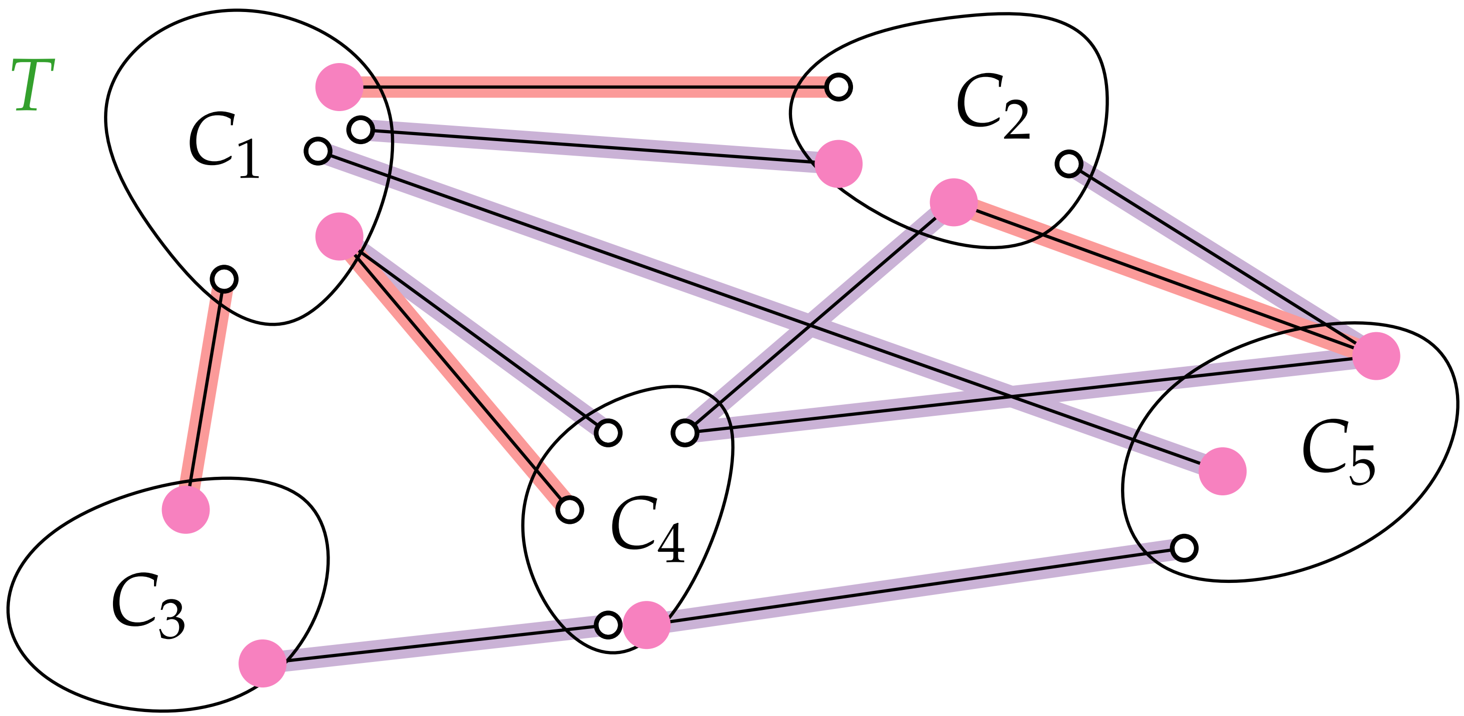

- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*
- $\sum_{v \in S} \deg_{T^*}(v) \geq k$

Decomposition $\Rightarrow$ Lower Bound for OPT

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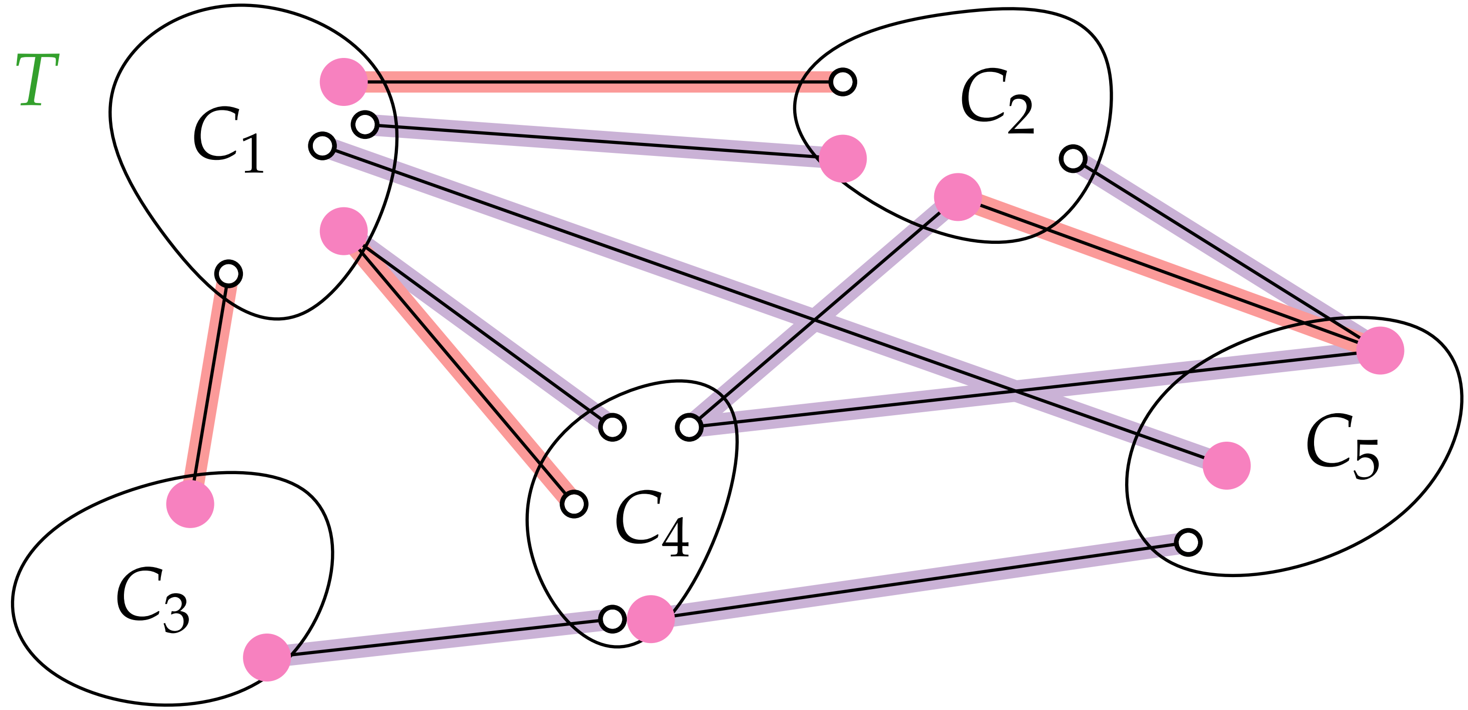

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- $\sum_{v \in S} \deg_{T^*}(v) \geq k$

Lemma 1.
$\underset{\text{Obs. 2}}{\Rightarrow}$ OPT $\geq$

Decomposition $\Rightarrow$ Lower Bound for OPT

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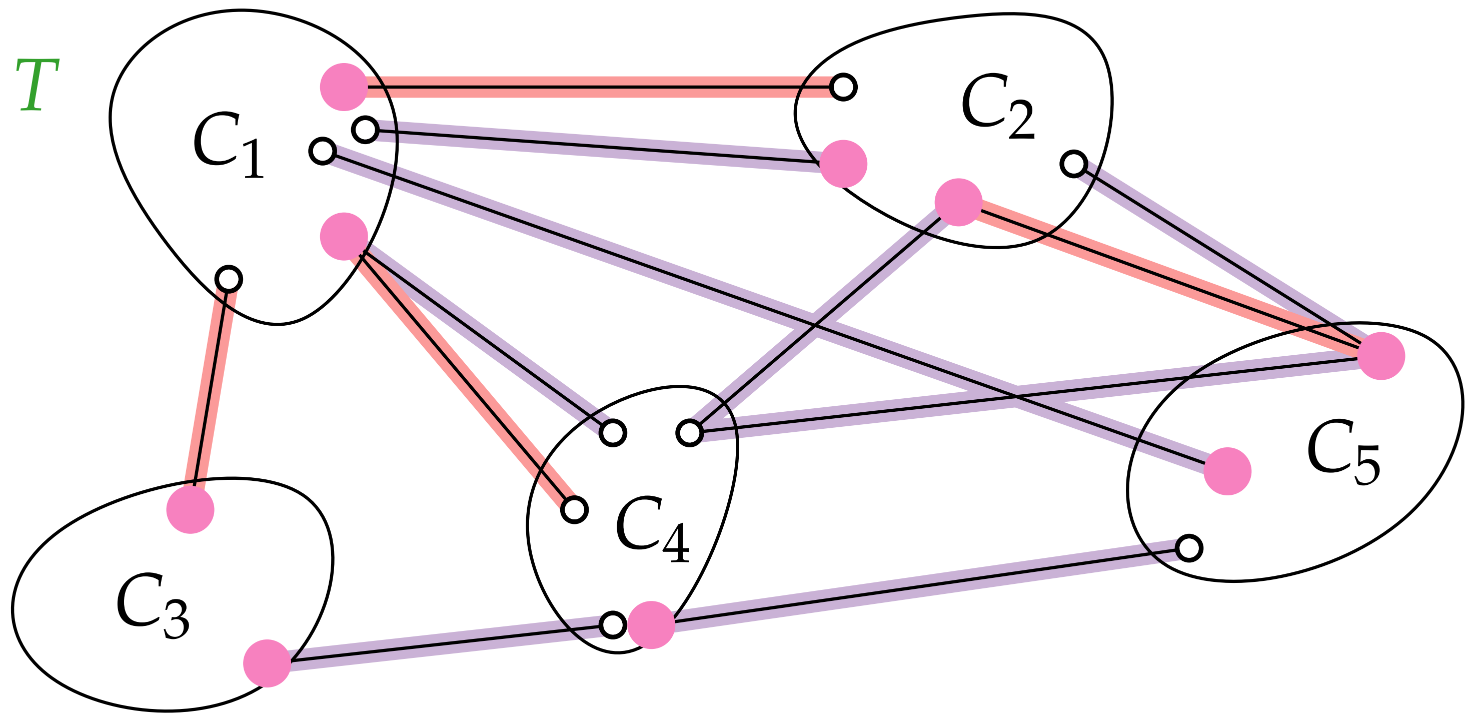

- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*
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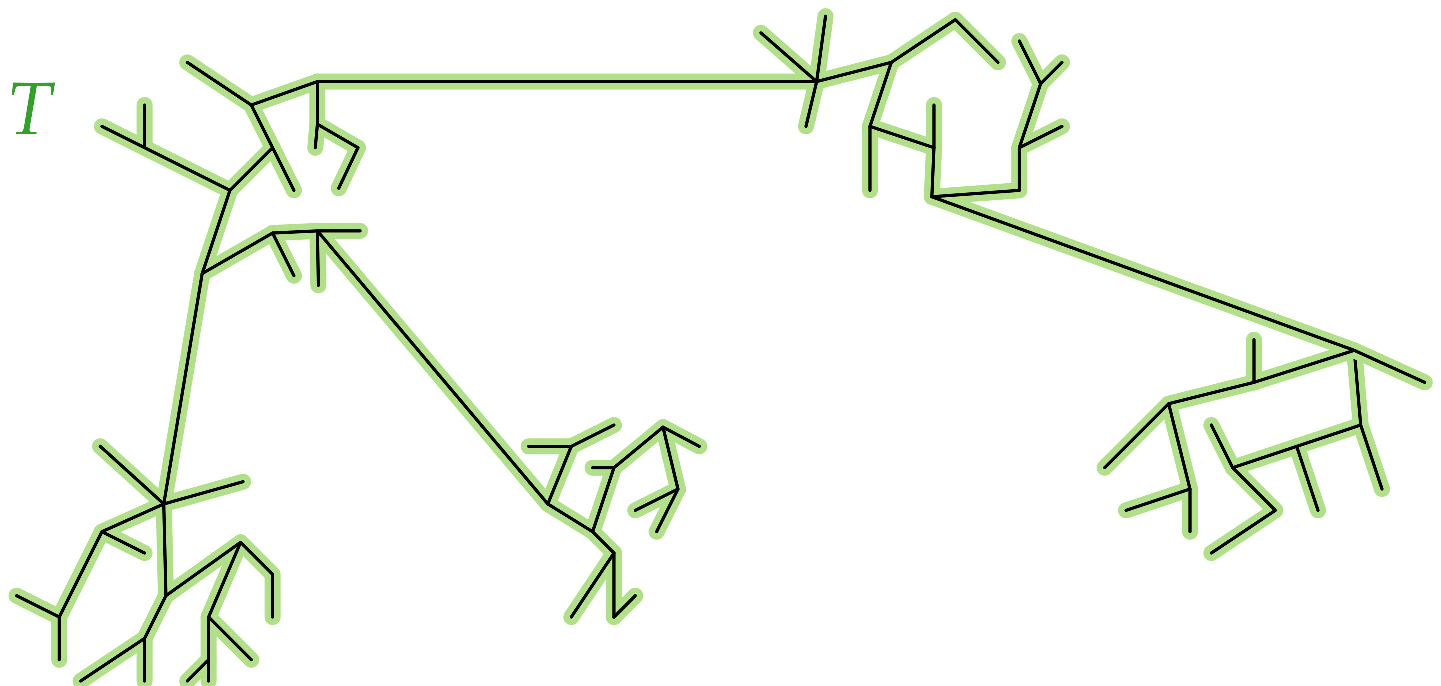
Lemma 1.
$\Rightarrow_{\text{Obs. 2}}$ OPT $\geq k/|S|$

Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

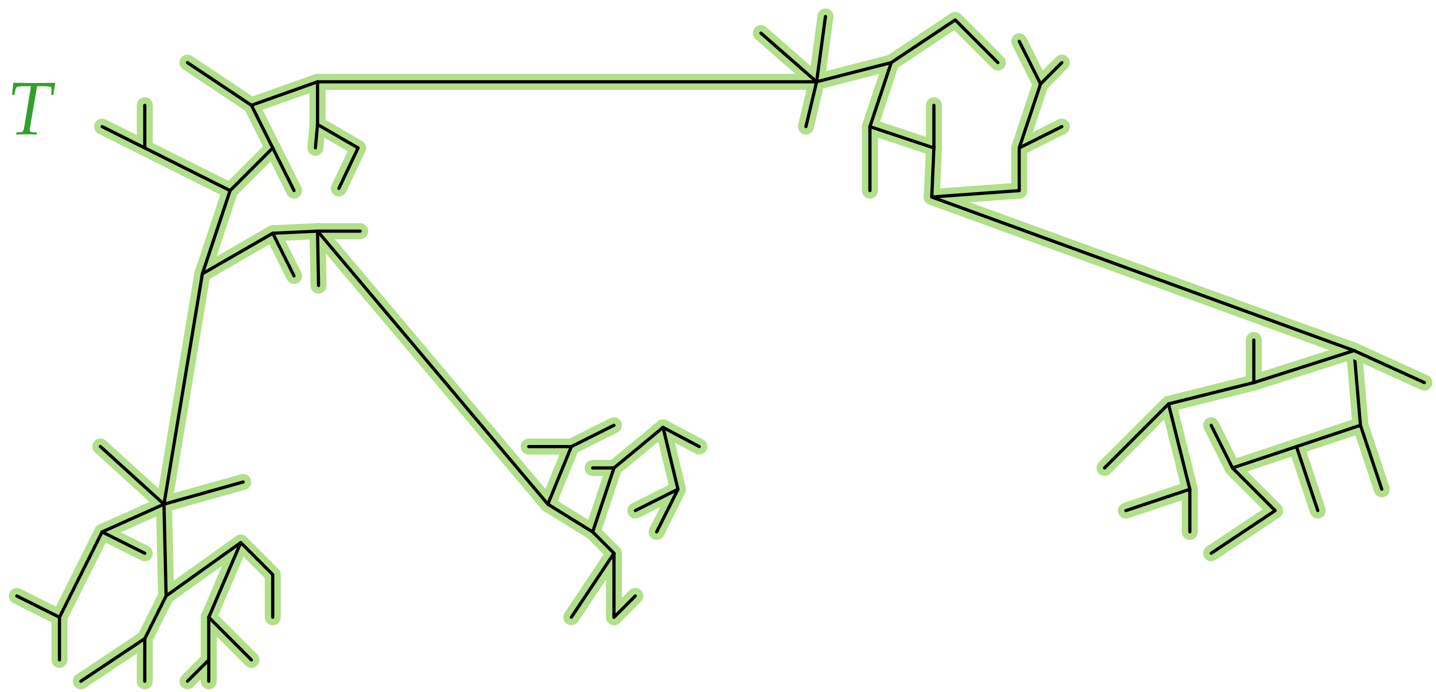
Part IV:
More Lemmas

More Lemmas



More Lemmas

Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.



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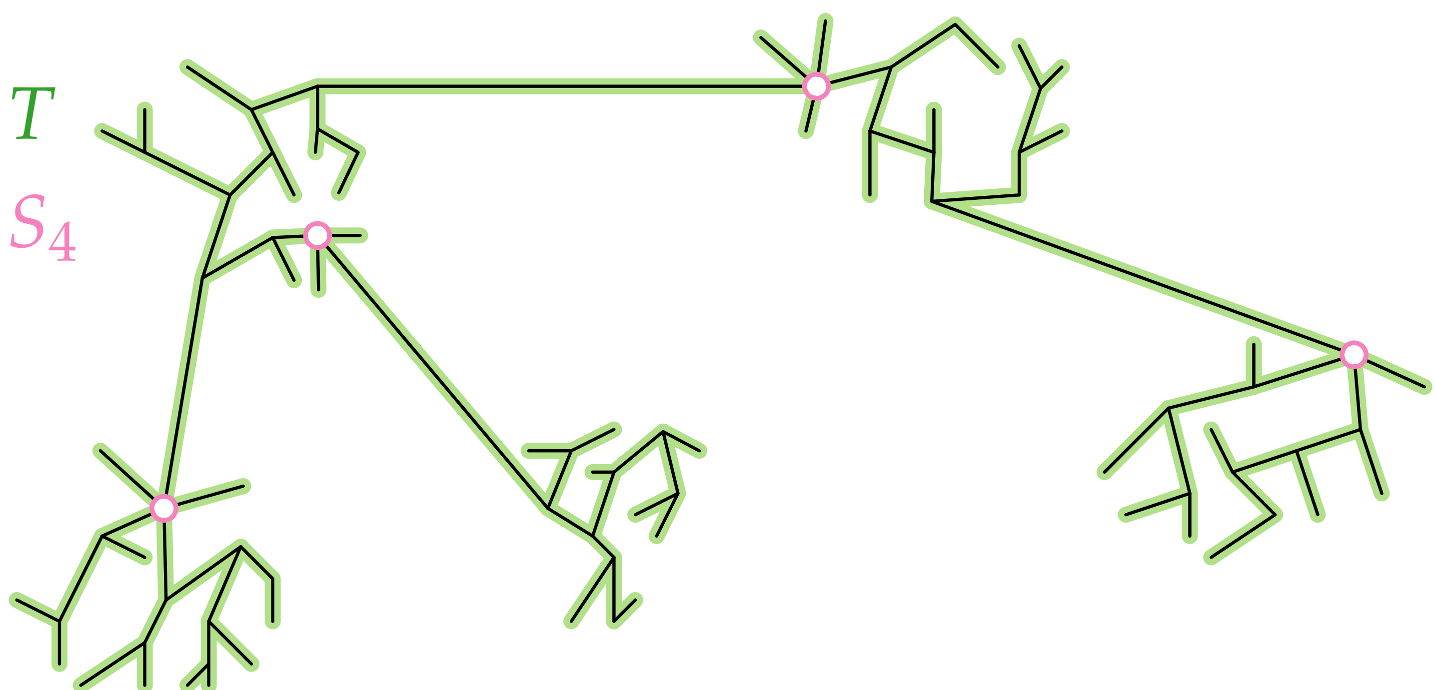

More Lemmas

Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
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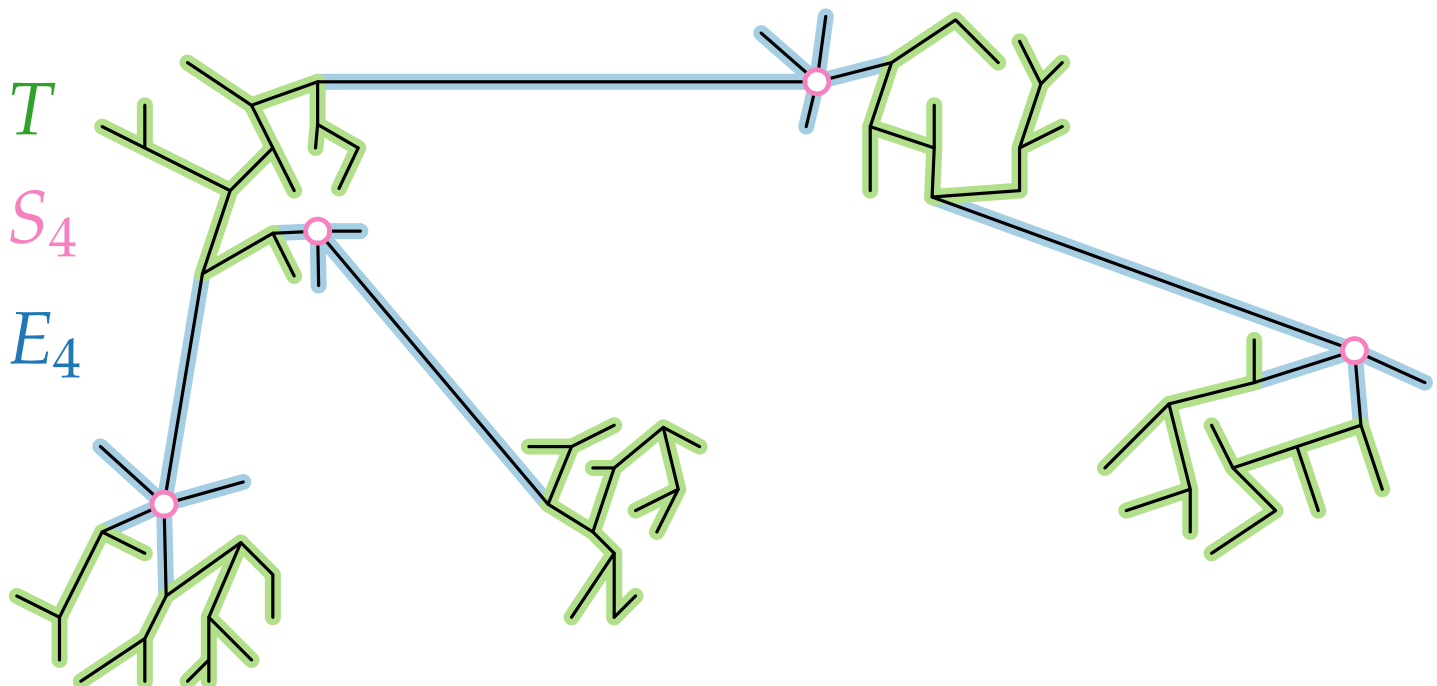

More Lemmas

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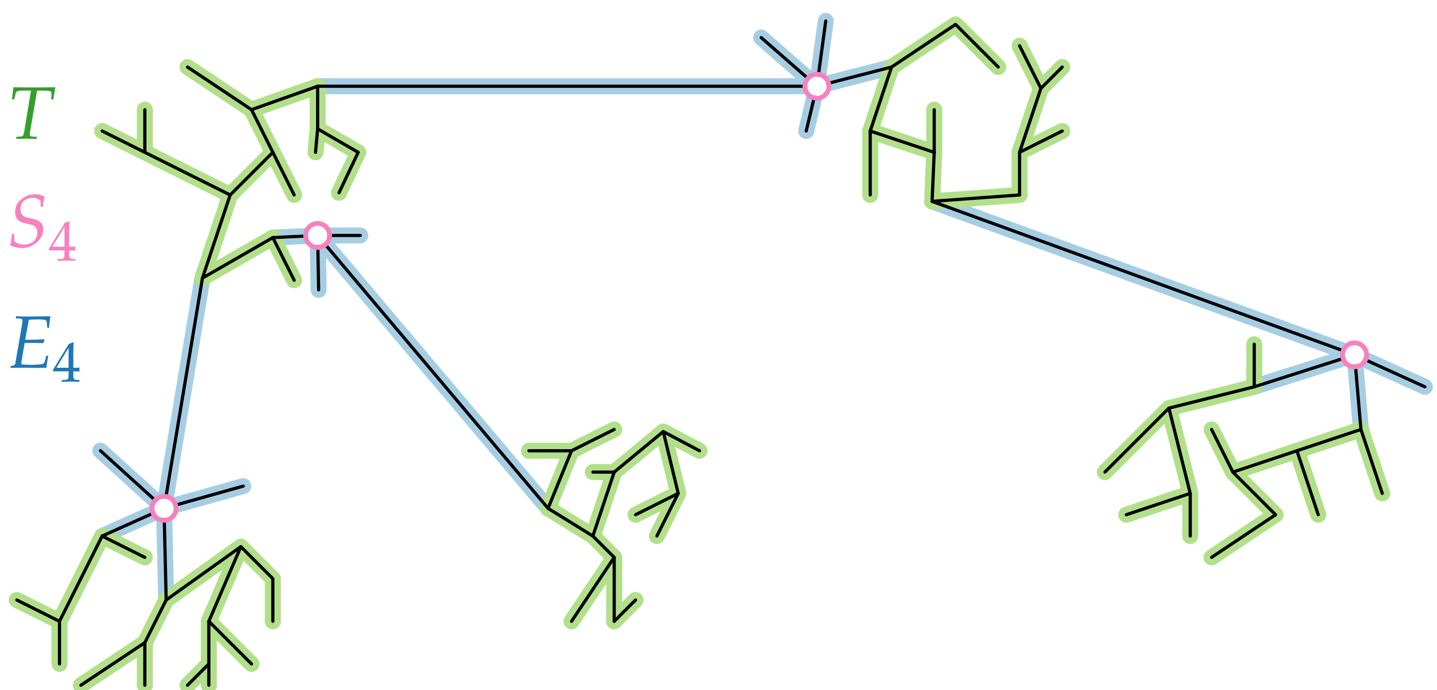

More Lemmas

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$

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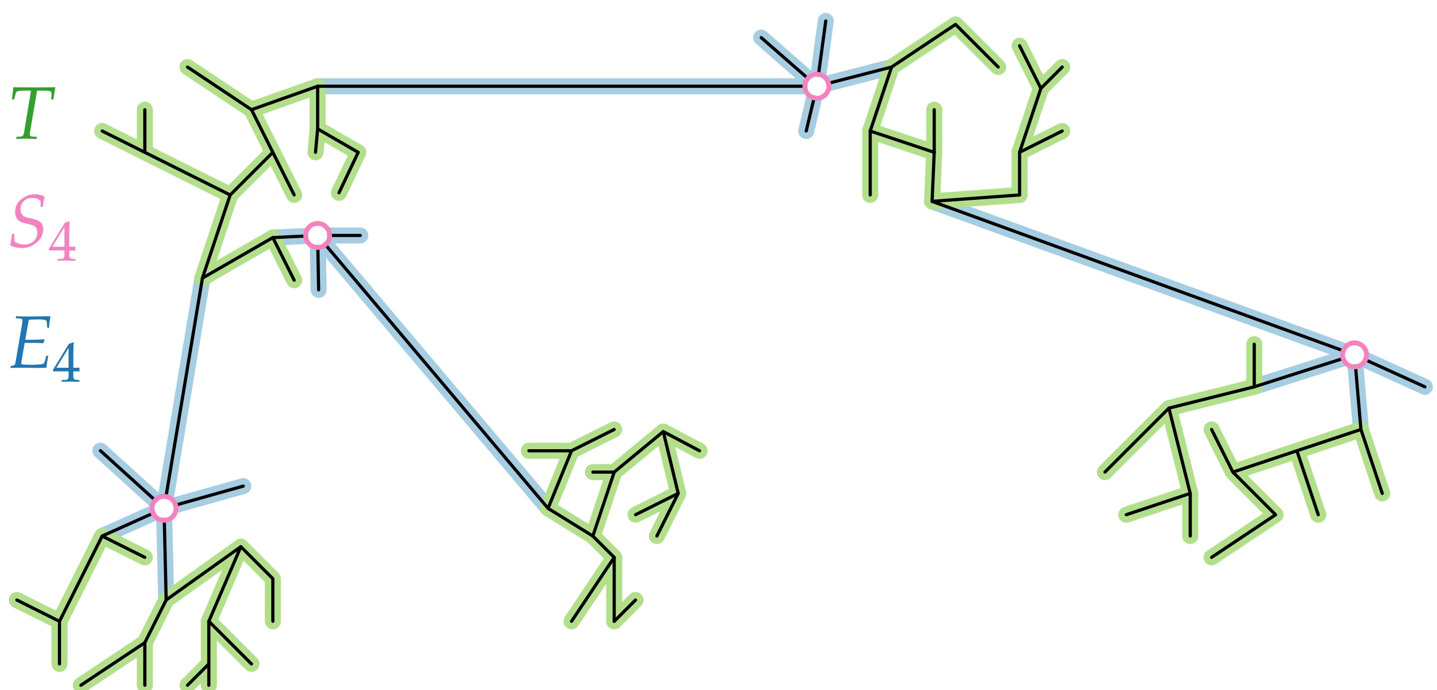

More Lemmas

$\Rightarrow S_1 \supseteq S_2 \supseteq \ldots$
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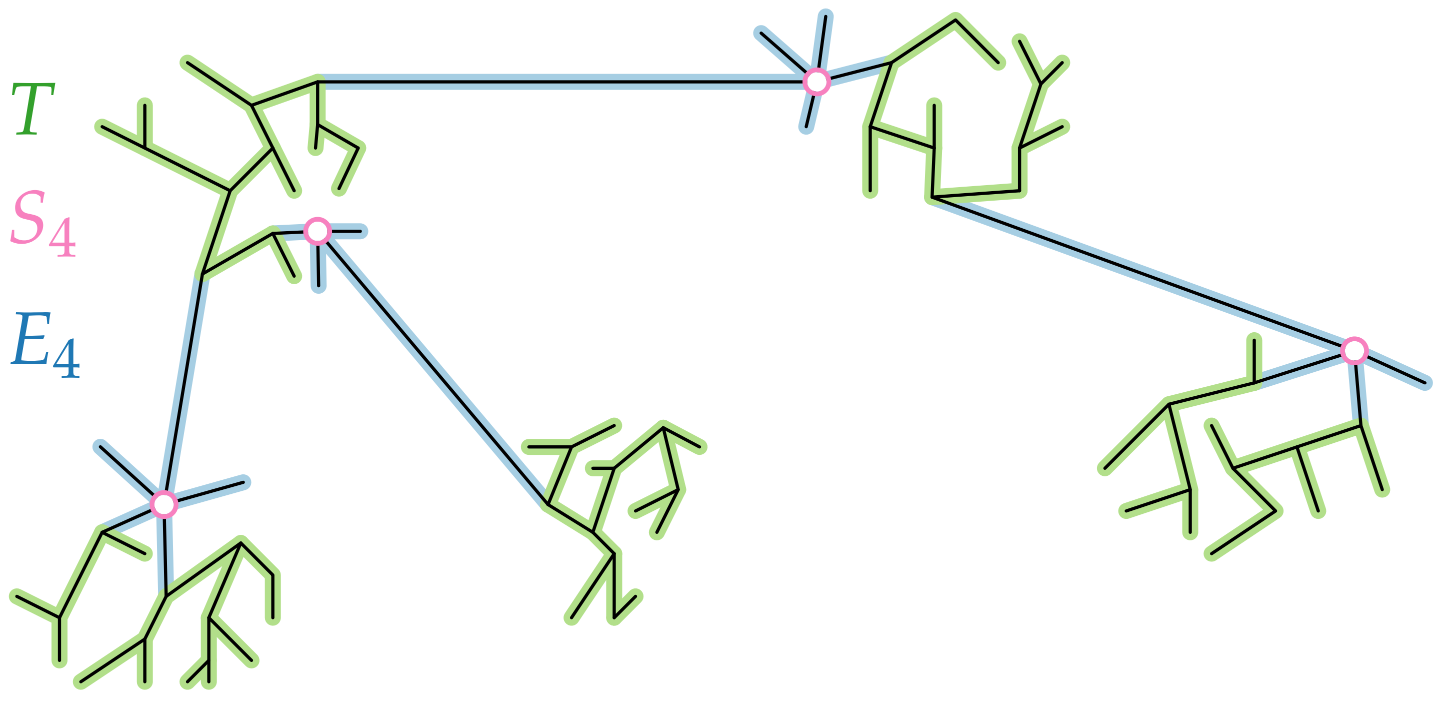

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Let S_i be the set of vertices v in T with $\deg_T(v) \geq i$.
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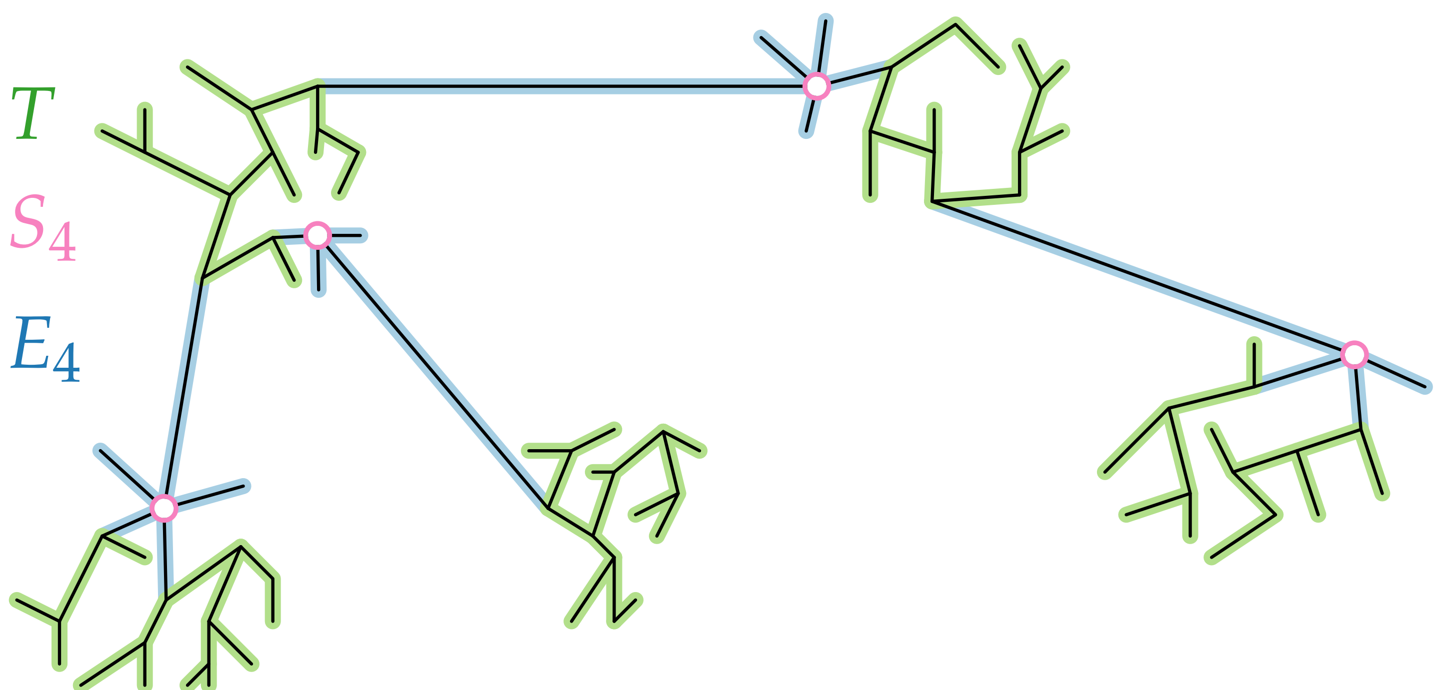
Lemma 2. $\exists i$ s.t. $\Delta(T) - \ell + 1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.



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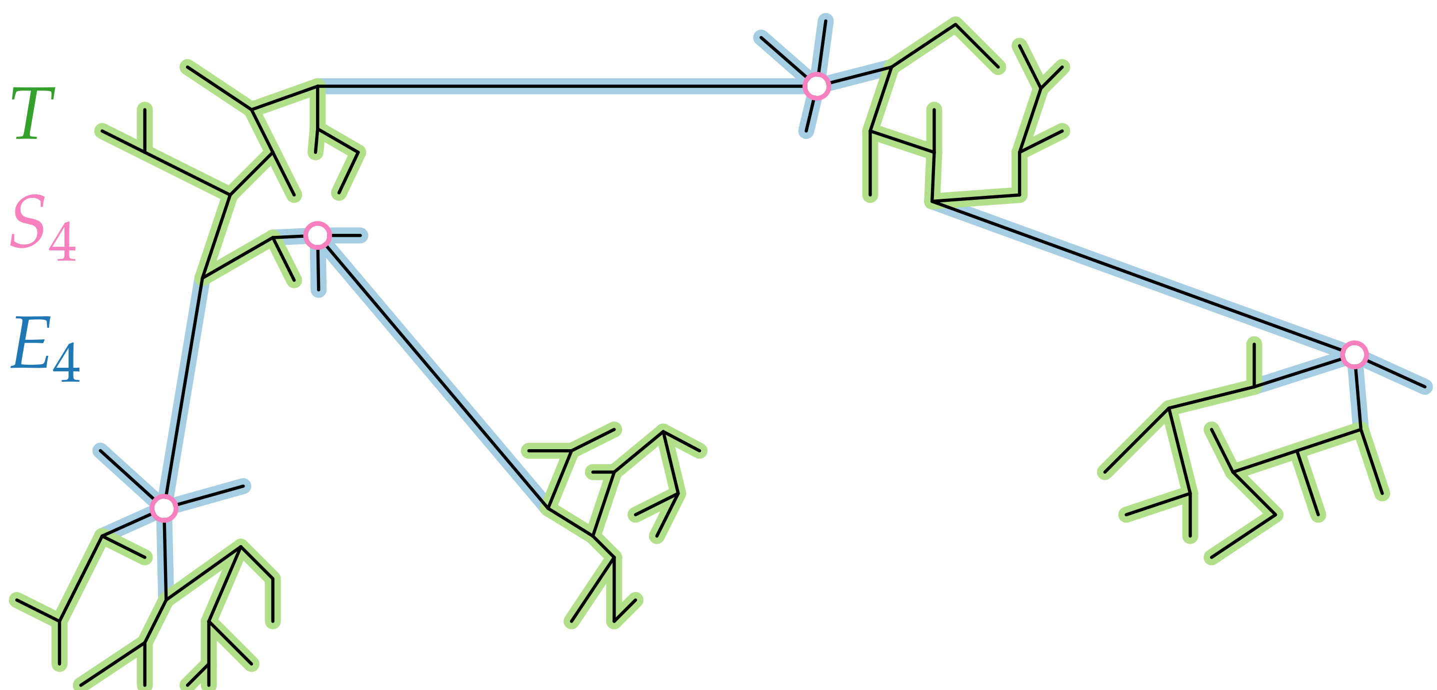
Lemma 2. $\exists i$ s.t. $\Delta(T)-\ell+1 \leq i \leq \Delta(T)$ with $|S_{i-1}| \leq 2|S_i|$.

Proof. $|S_{\Delta(T)-\ell}| > 2^\ell |S_{\Delta(T)}|$

Otherwise



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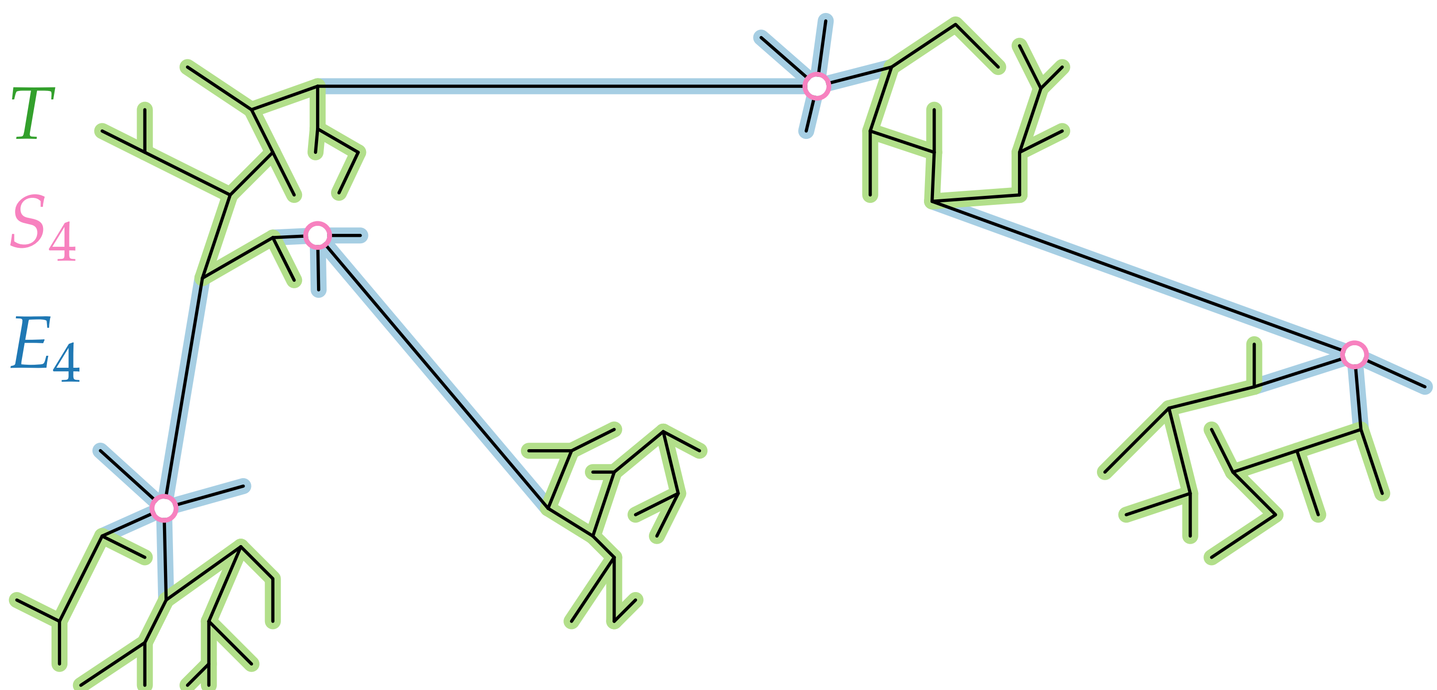
Proof. $|S_{\Delta(T)-\ell}| > 2^{\ell}|S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil}|S_{\Delta(T)}| \geq$

$\ell = \lceil \log_2 n \rceil$

Otherwise



More Lemmas

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Otherwise

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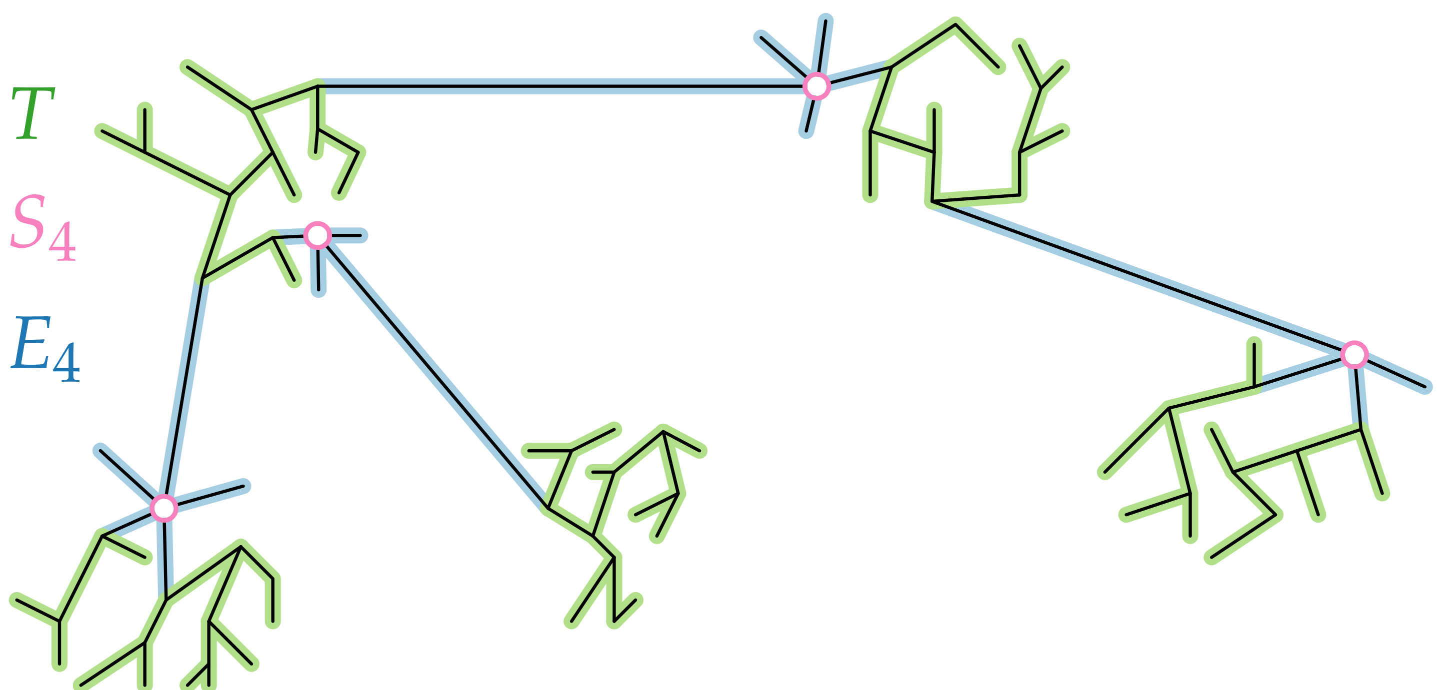

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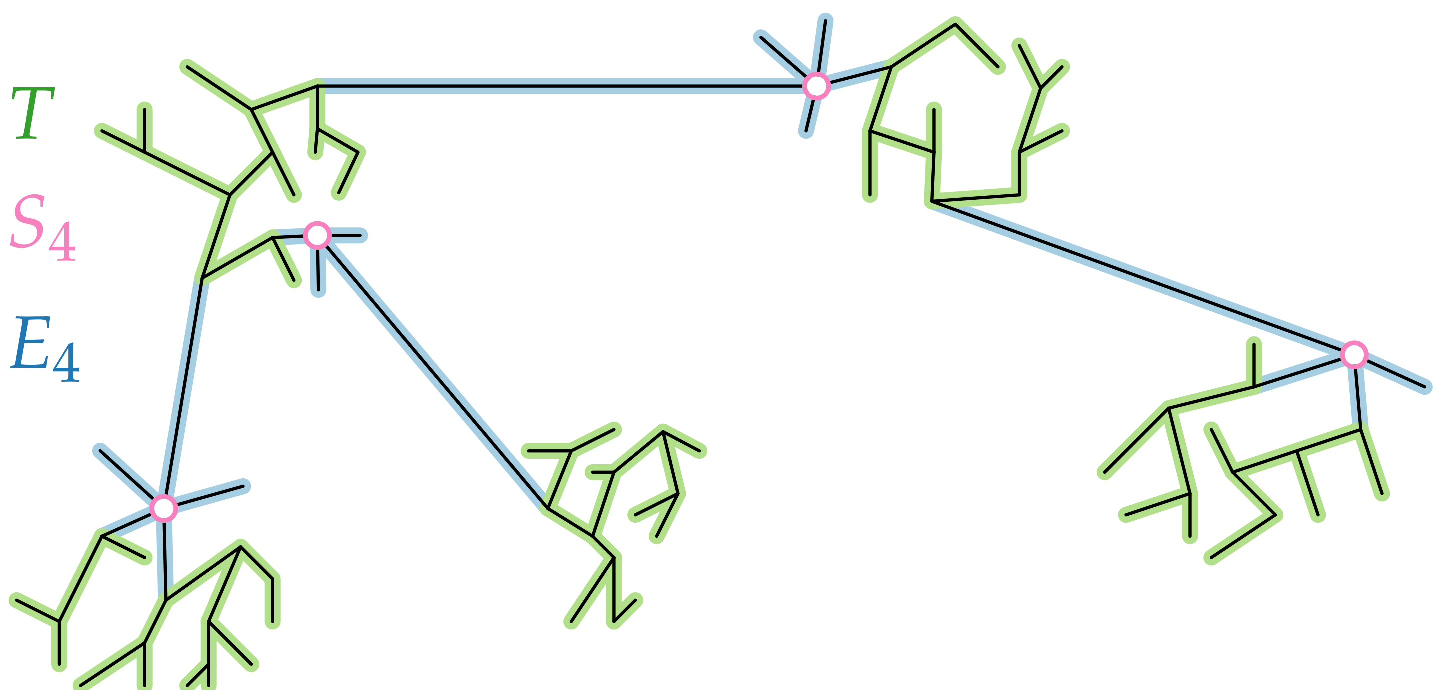
Otherwise



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Otherwise

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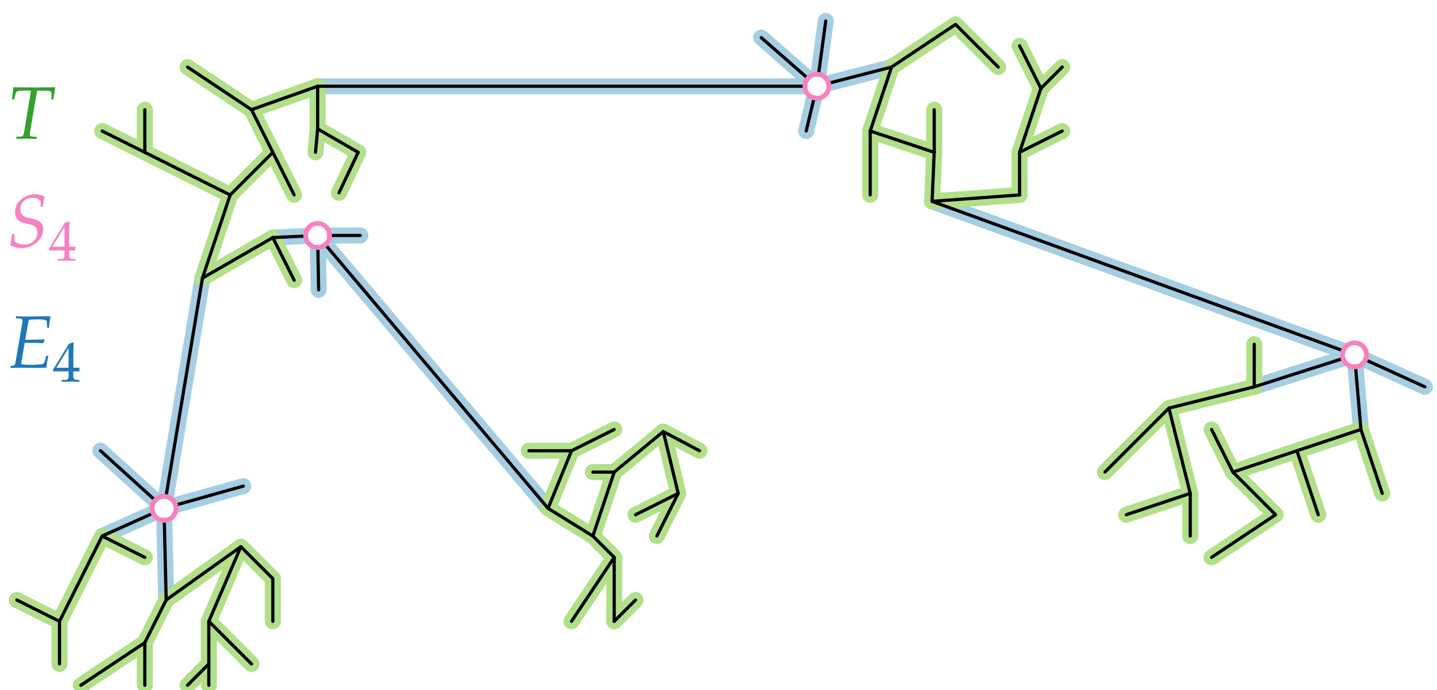
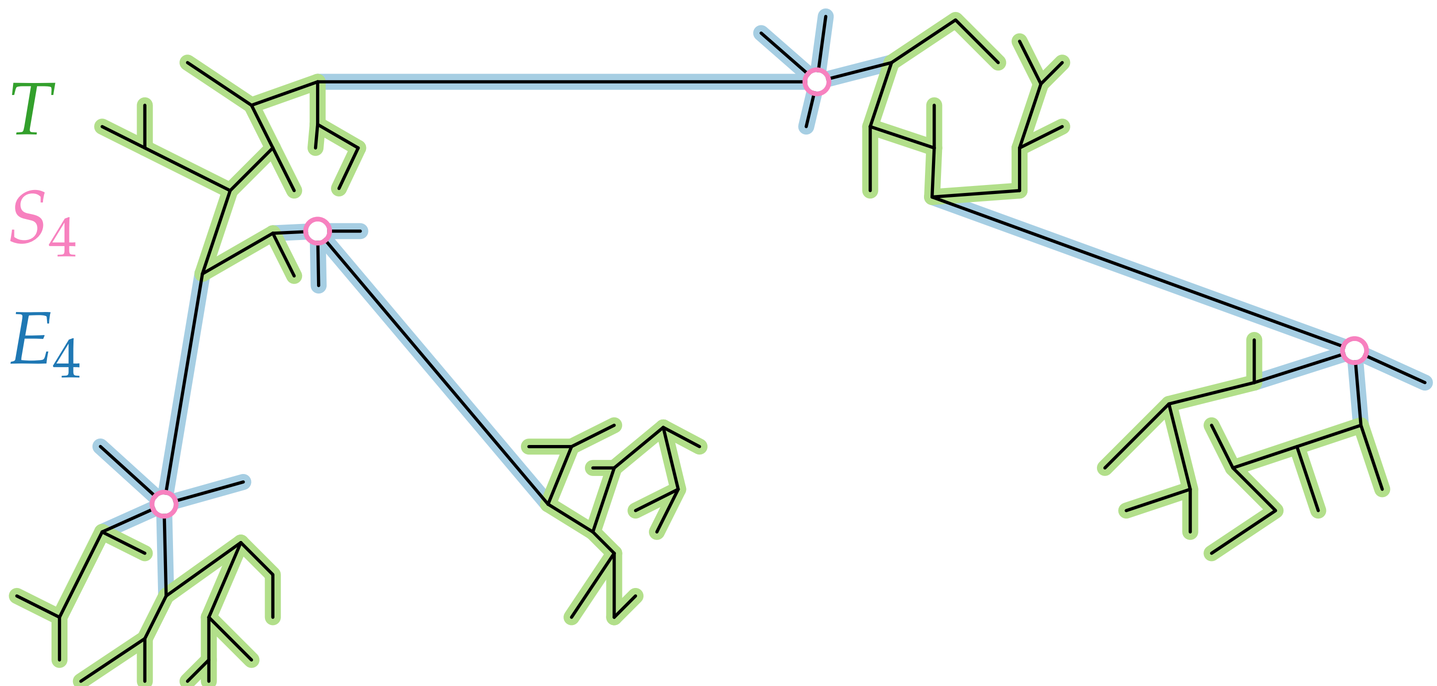
TODO: What if $\ell > \Delta(T)$?



More Lemmas

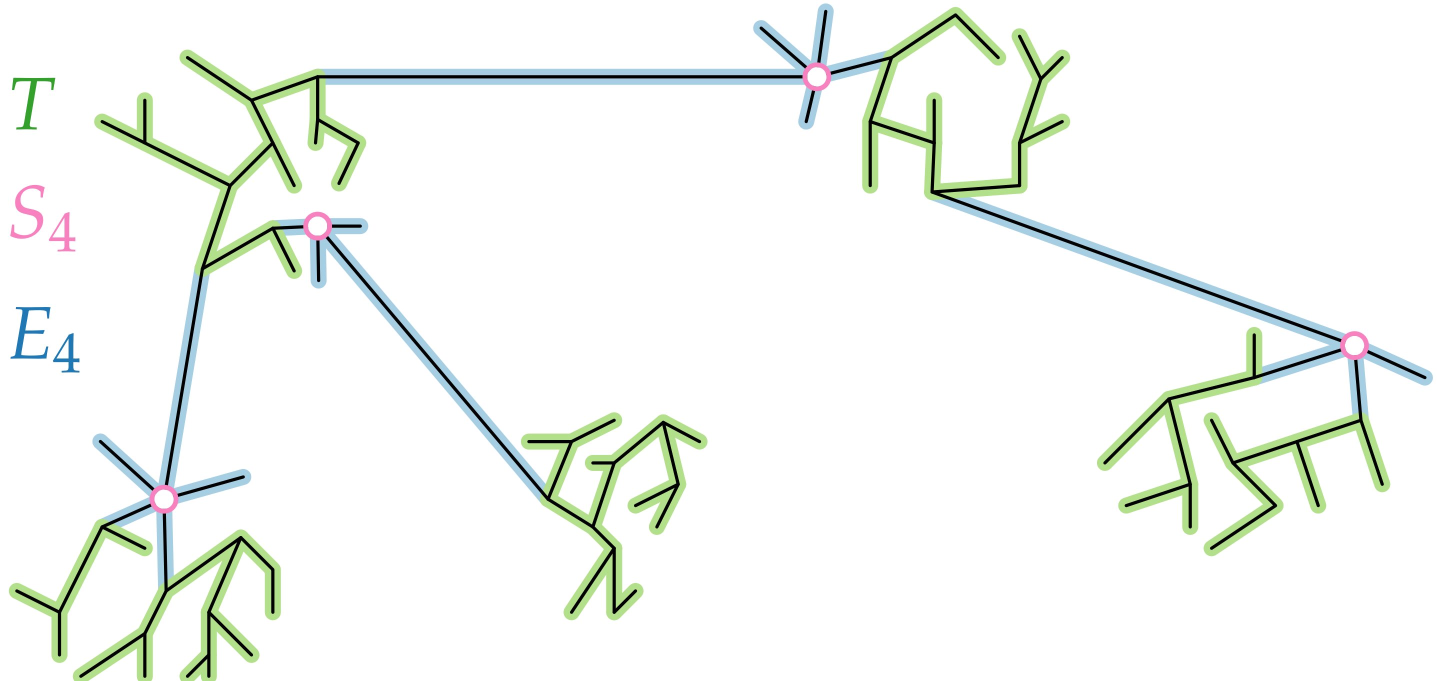
Lemma 3. For $i \geq \Delta(T) - \ell + 1$,



More Lemmas

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,

(i) $|E_i| \geq (i-1)|S_i| + 1$,



More Lemmas

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
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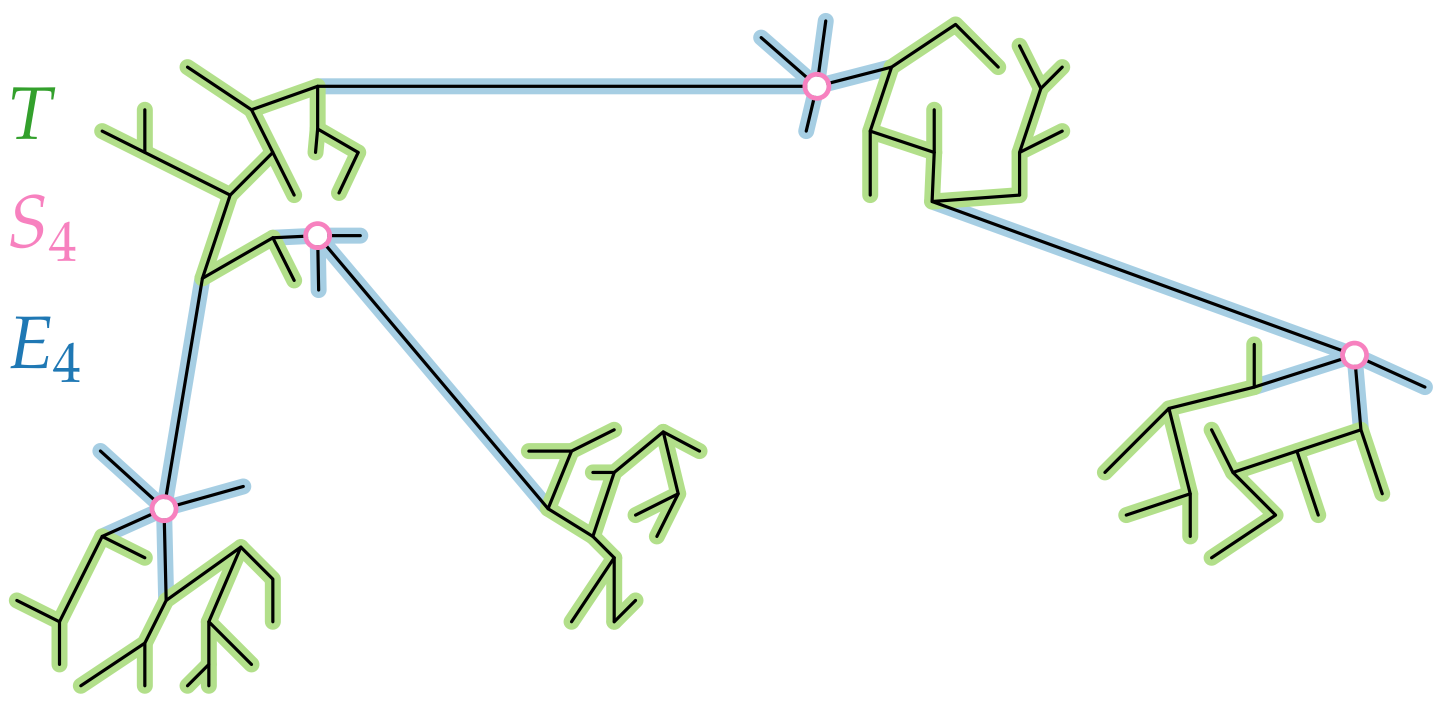

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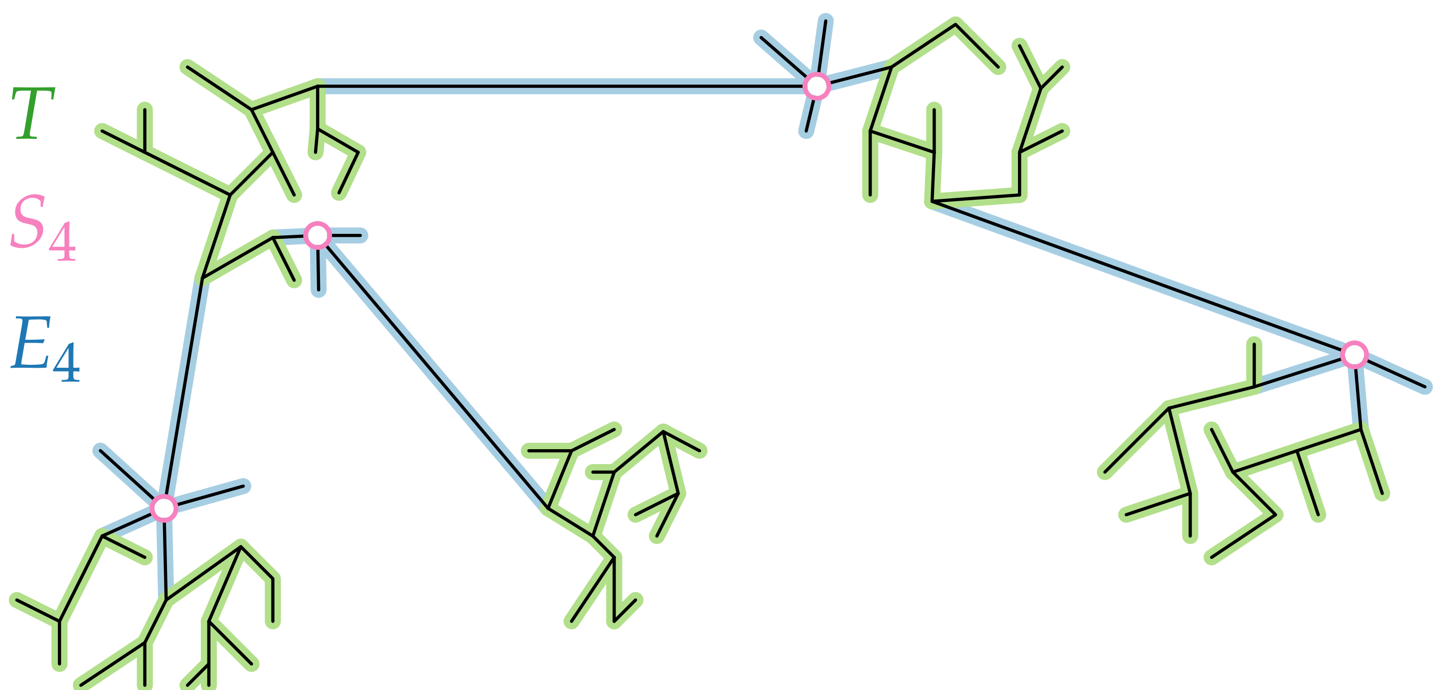
Proof. (i) $|E_i| \geq$



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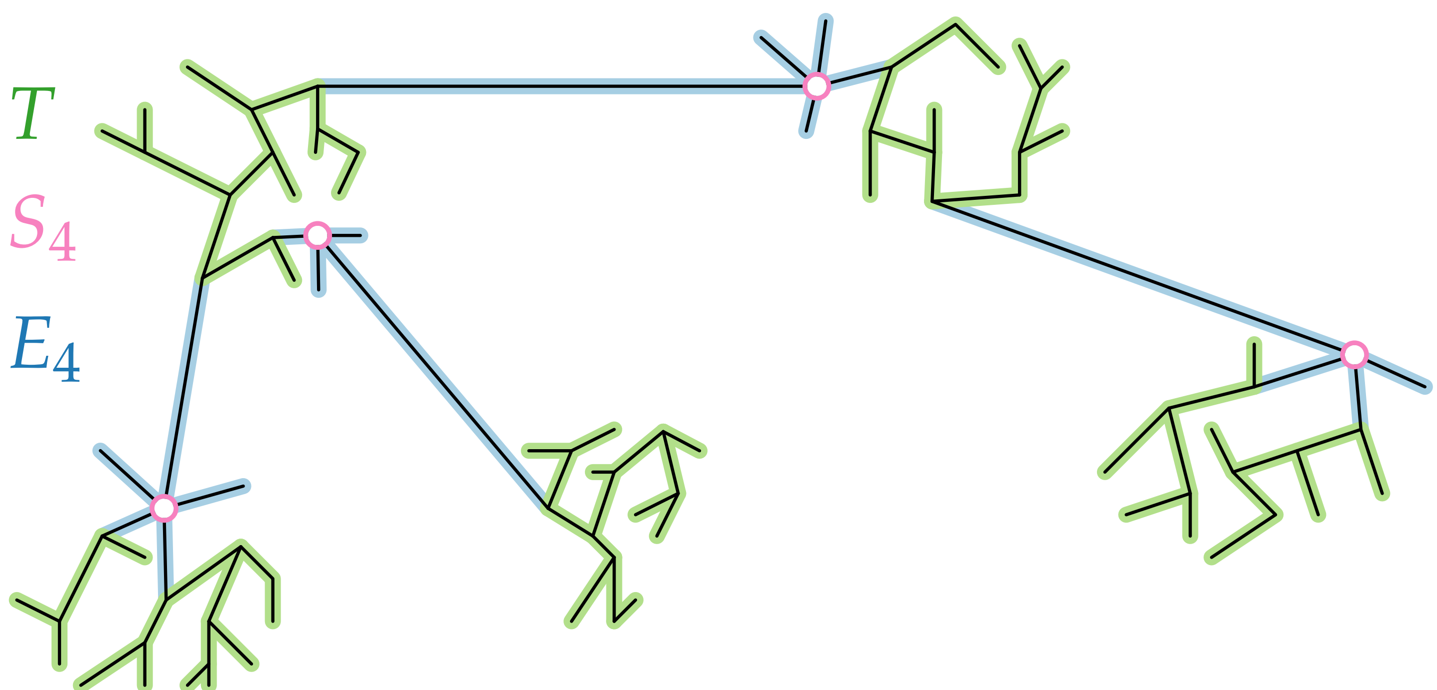
Proof. (i) $|E_i| \geq i|S_i|$
vertex-deg



More Lemmas

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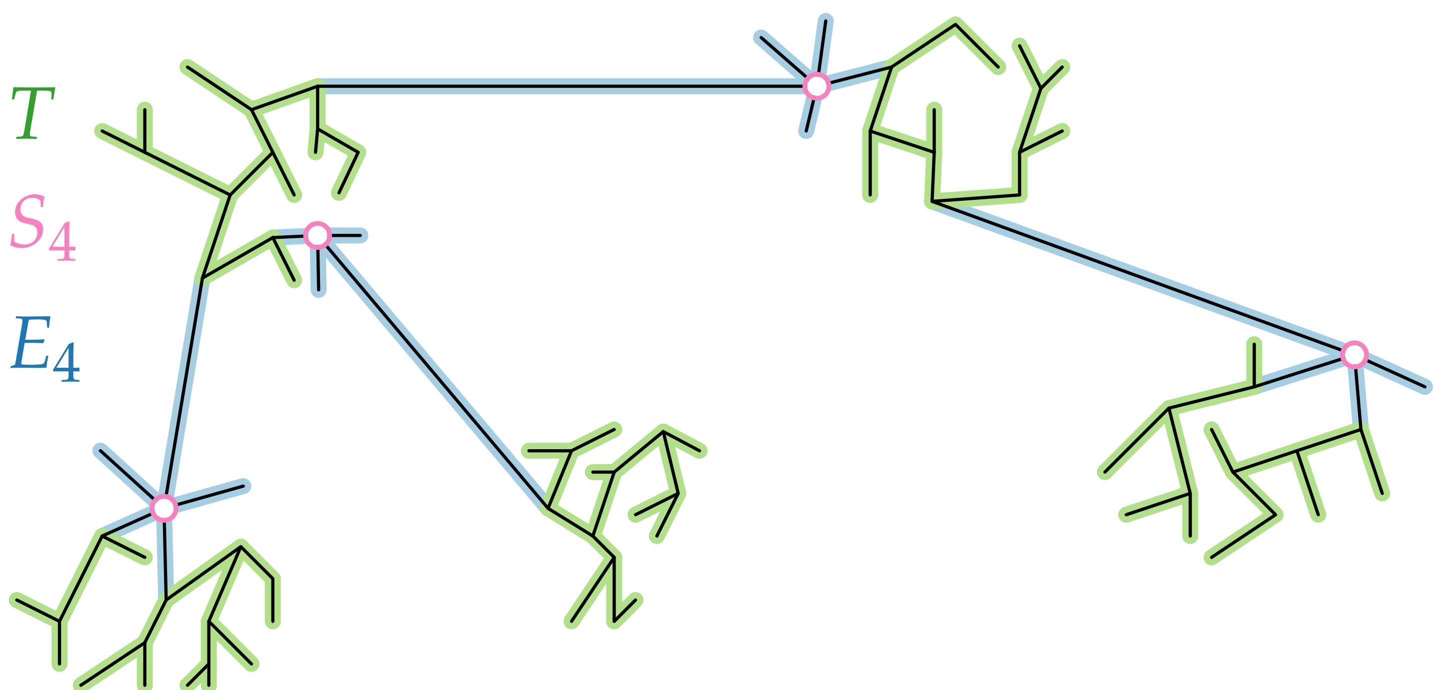
vertex-deg counted twice?



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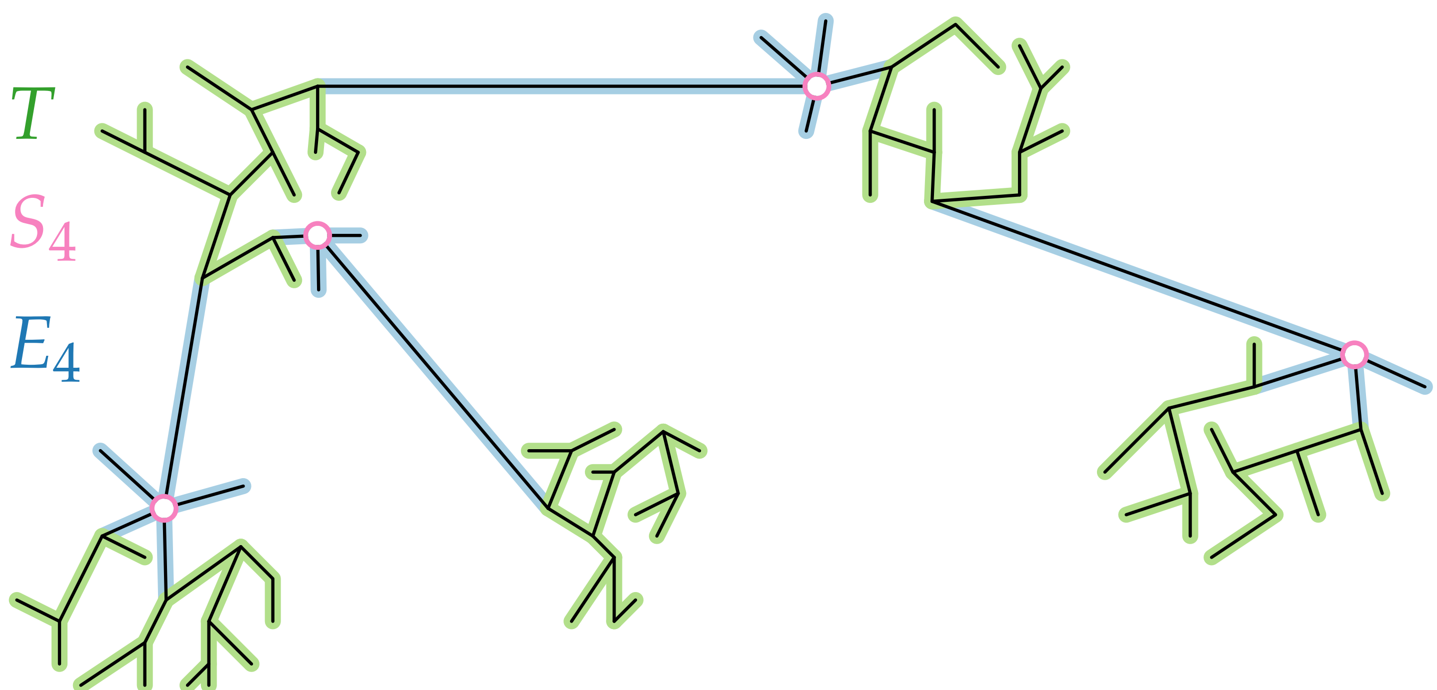

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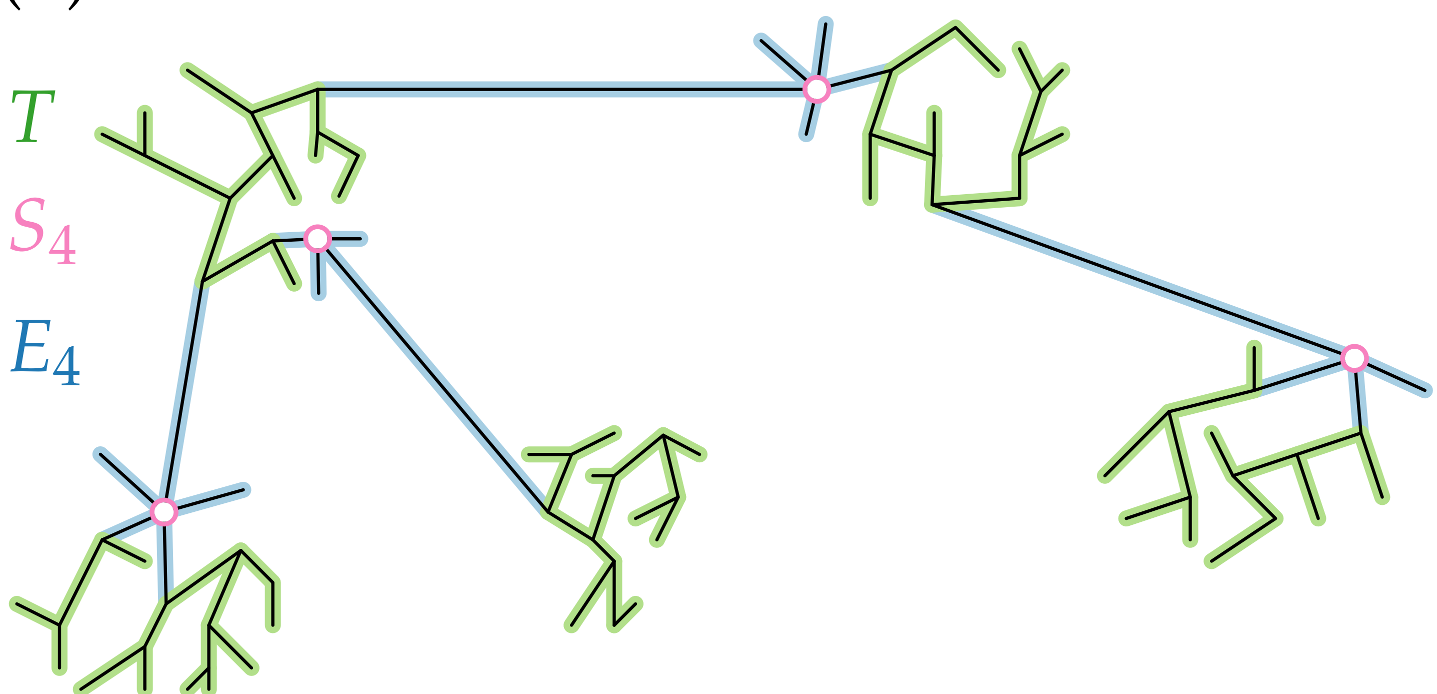

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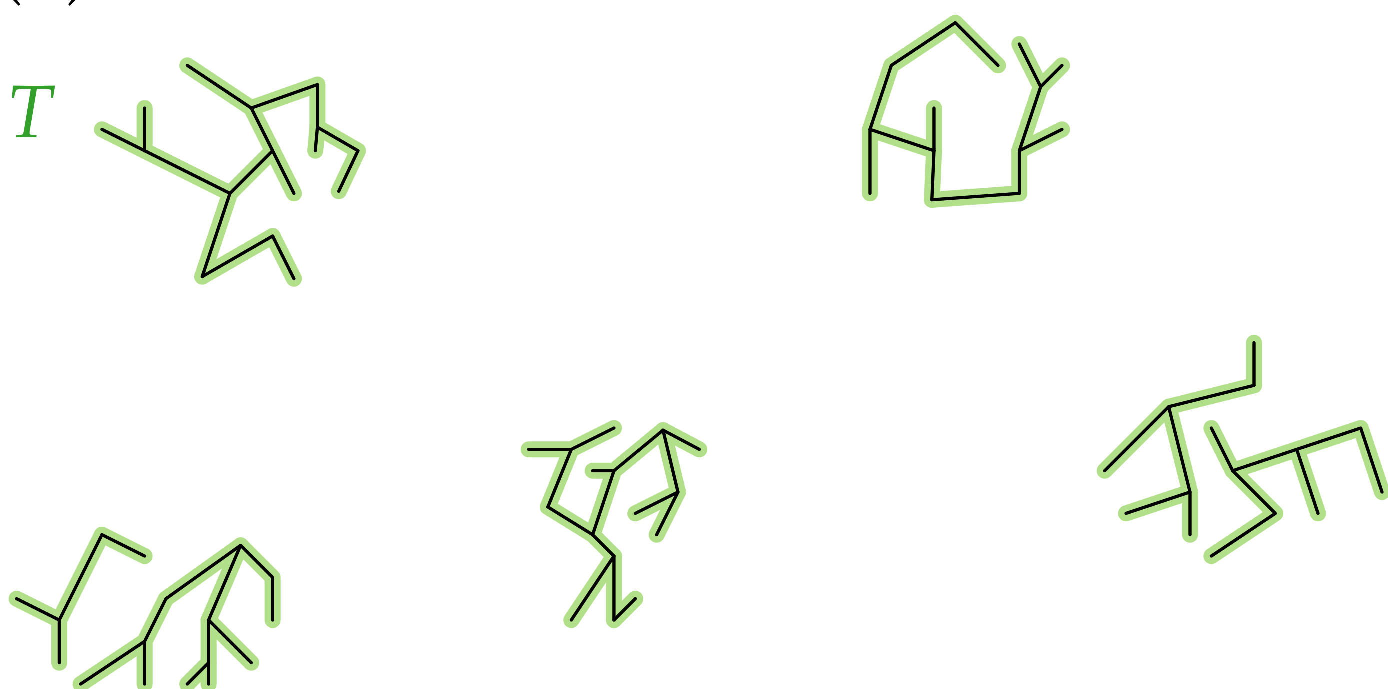

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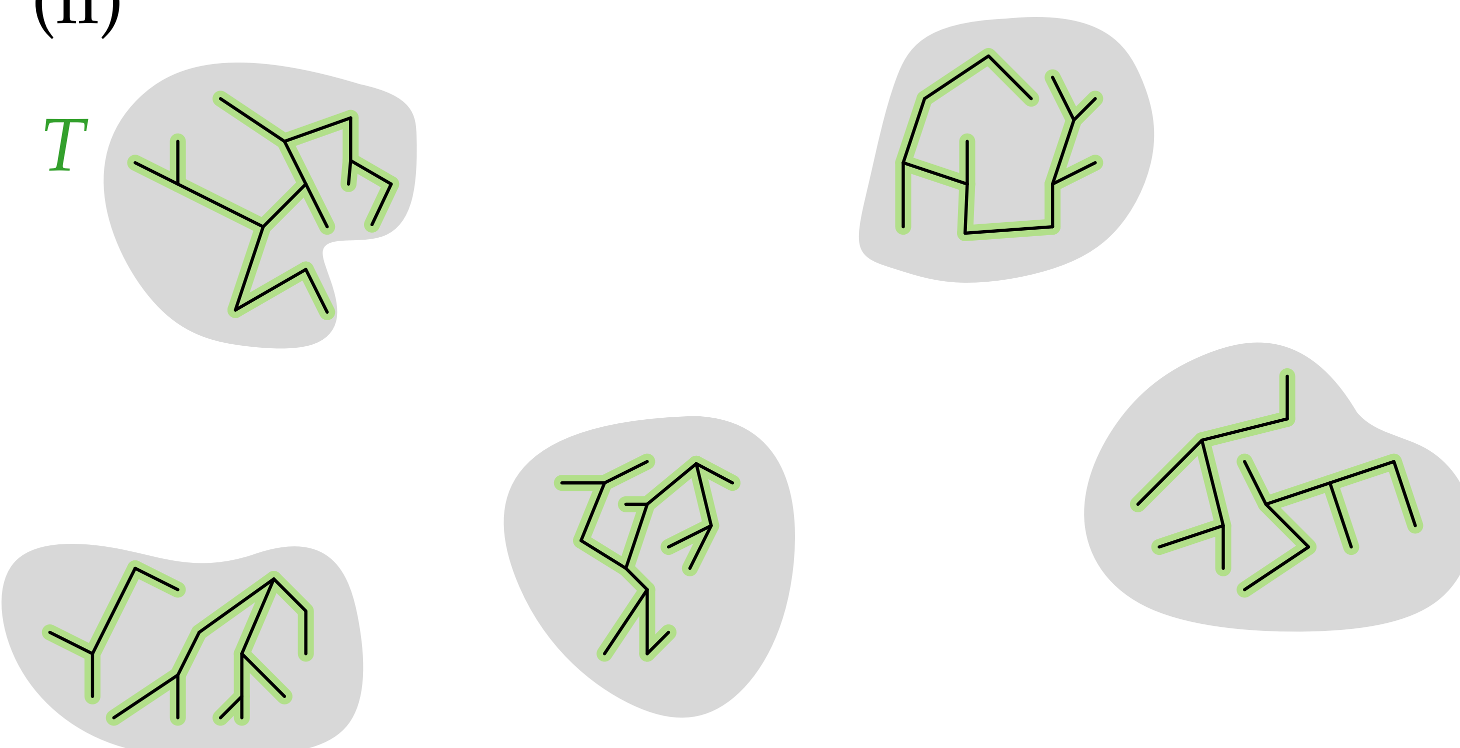

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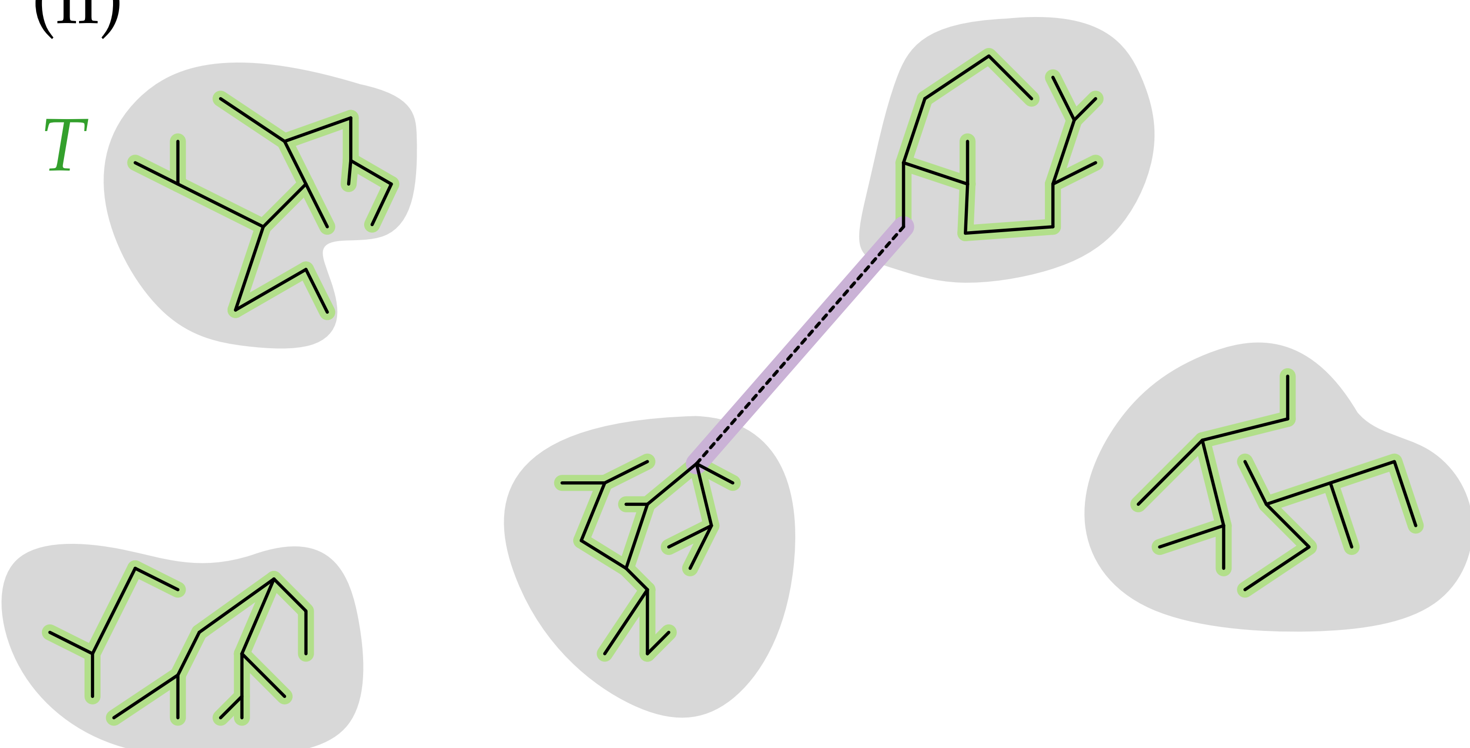

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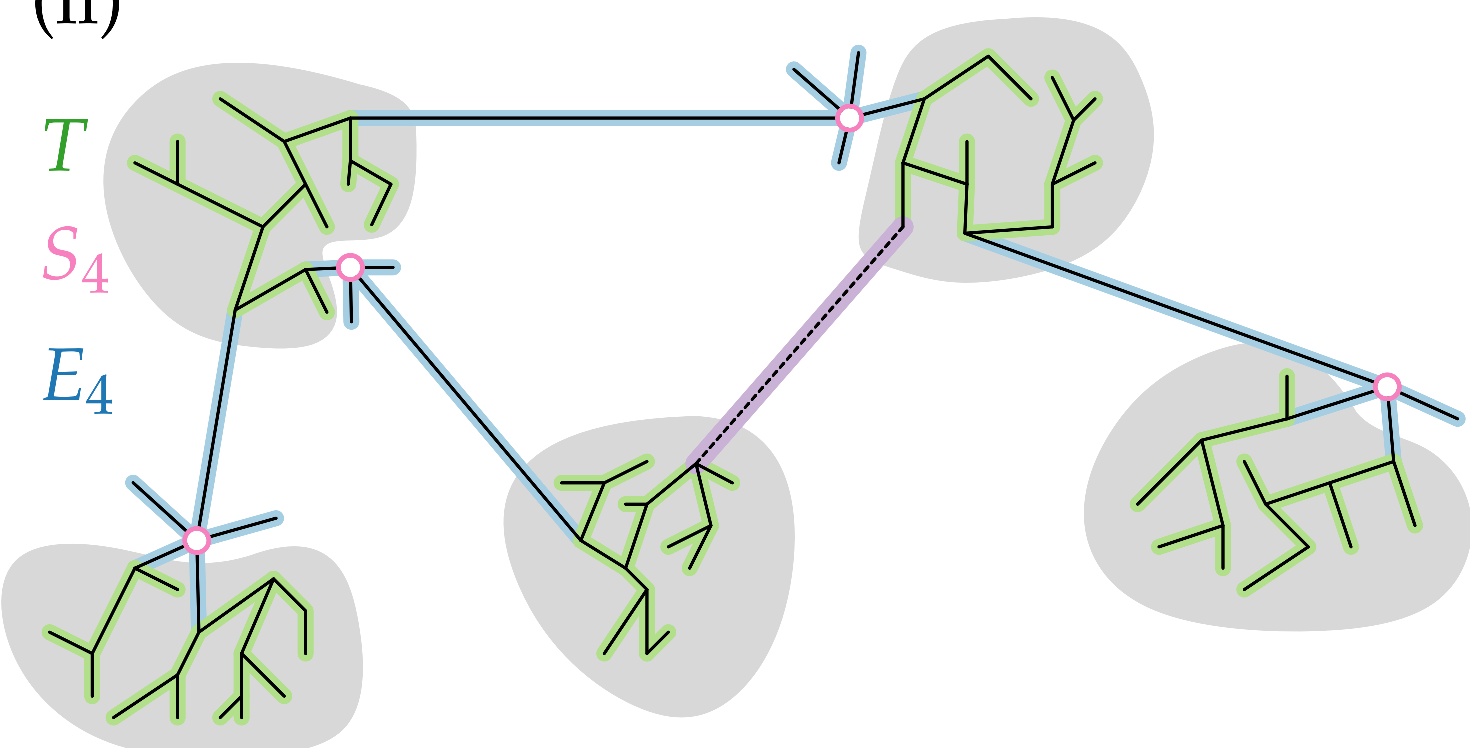

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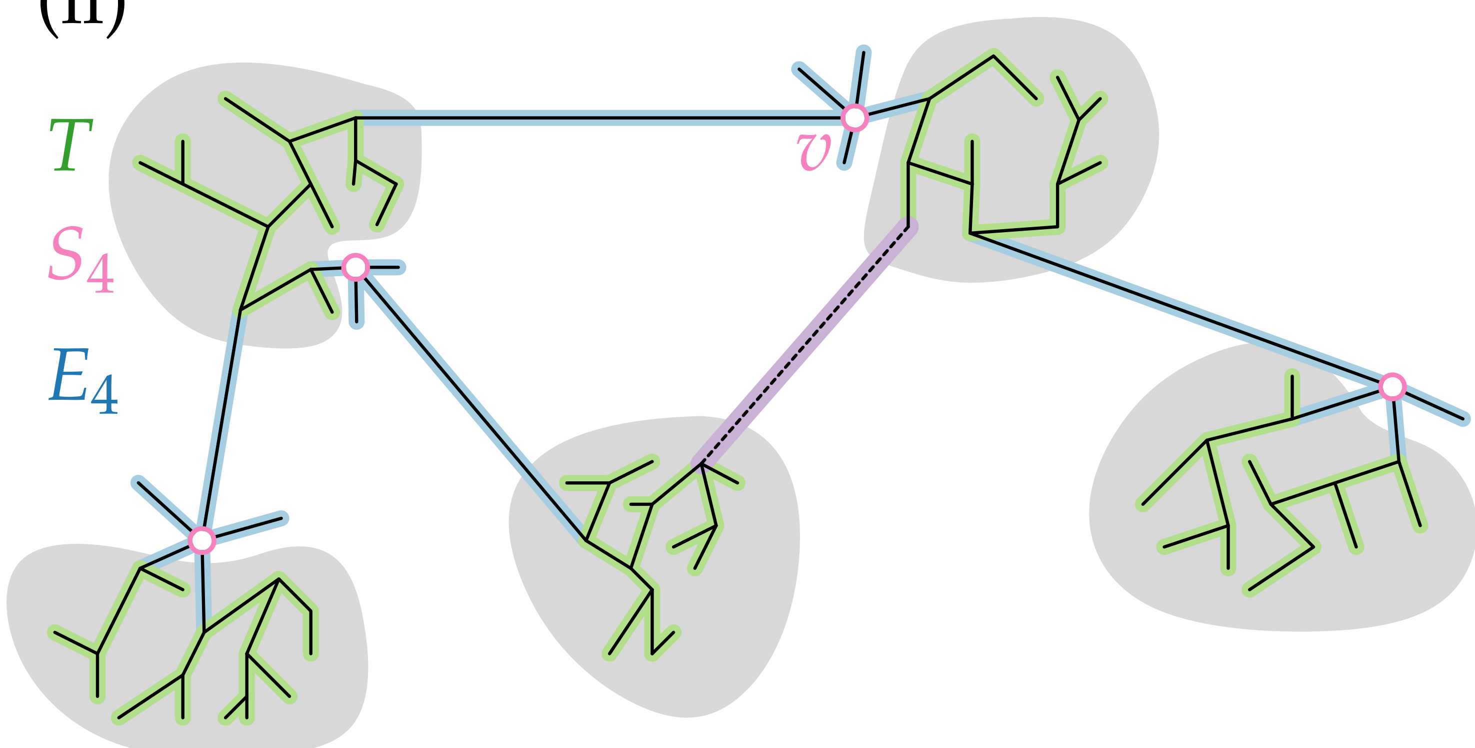

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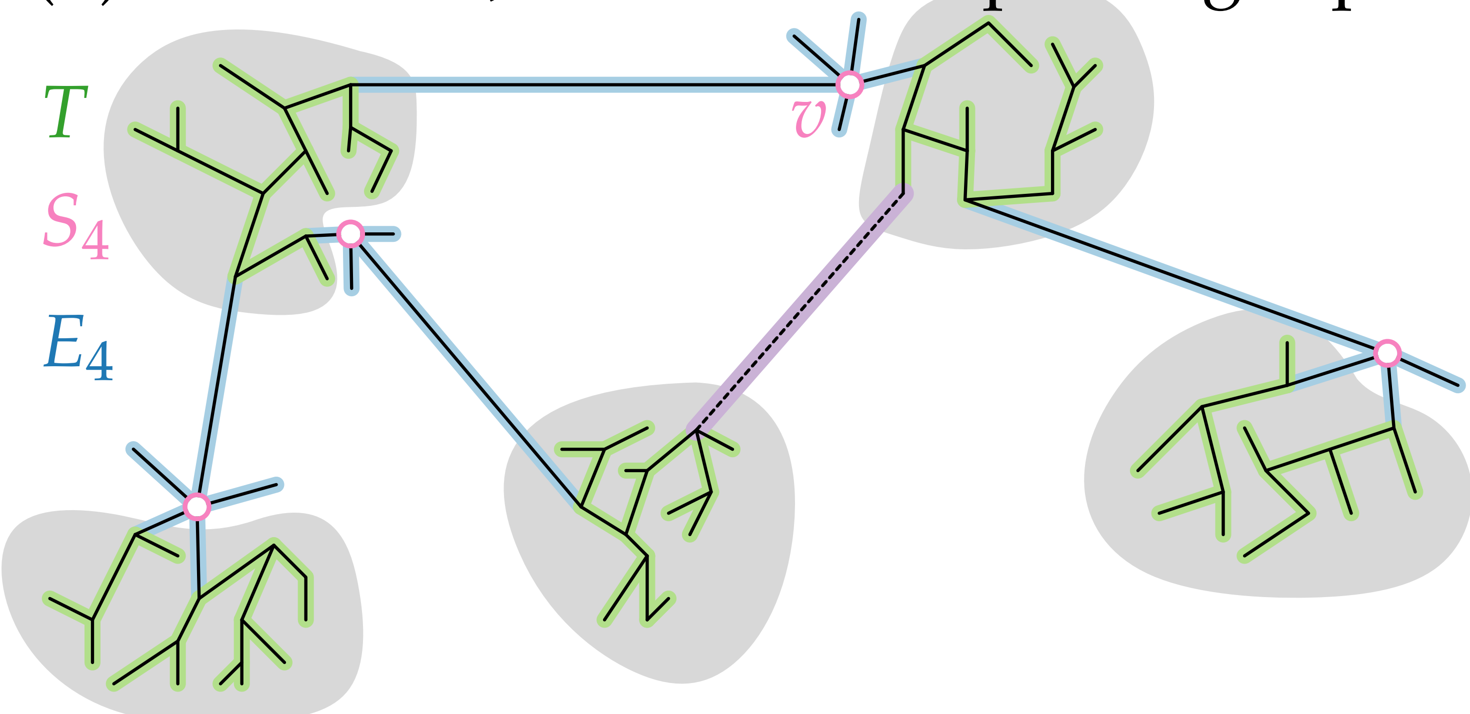
(ii) Otherwise, there is an improving flip for $v \in S_i$.



Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part V:
Approximation Factor

Approximation Factor

Approximation Factor

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. Let T be a locally optimal spanning tree.
Then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

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□

Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
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Part VI:
Termination, Running Time & Extensions

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree efficiently.

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Proof.

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- The function is bounded both from above and below.

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- Executing $f(n)$ iterations would exceed the lower bound.

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Theorem. The algorithm finds a locally optimal spanning tree after at most $f(n)$ iterations.

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Lemma. After each flip $T \to T'$, $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

Homework

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Lemma. For each spanning tree T, $\Phi(T) \in [3n, n3^n]$.

- Executing $f(n)$ iterations would exceed the lower bound.

How does $\Phi(T)$ change?

$\Phi(T)$ decreases by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)} =$

Goal: After $f(n)$ iterations: $\Phi(T) = n < 3n$

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after at most $f(n)$ iterations.

Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully): $\Phi(T) = \sum_{v \in V(G)} 3^{\deg_T(v)}$

- Each iteration decreases the potential of a solution.

Lemma. After each flip $T \to T'$, $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

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  Let $f(n) = \frac{27}{2}n^4 \cdot \ln 3$. How does $\Phi(T)$ change?
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Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after $O(n^4)$ iterations.

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Extensions

Corollary. For any constant $b > 1$ and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T with $\Delta(T) \leq b \cdot \text{OPT} + \lceil \log_b n \rceil$.

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Proof. Similar to previous pages. Homework $\square$

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Theorem. There is a local search algorithm that runs in $O(EV\alpha(E,V)\log V)$ time and produces a spanning tree T with $\Delta(T) \leq \mathrm{OPT} + 1$.

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- Further variants for directed graphs and Steiner tree.