Lehrstuhl für Informatik I
– Algorithmen und Komplexität –
Universität Würzburg

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Exercise Sheet #4

Advanced Algorithms (WS 2023/24)

Exercise 1 – Flow problem with vertex capacities

Let $G = (V, E)$ be a flow network where the capacity function c is not only defined on the edges but also on the vertices. More precisely, for an s–t flow f in G there is a capacity constraint $\sum_{u \in V} f(u, v) = \sum_{u \in V} f(v, u) \leq c(v)$ for each $v \in V \setminus \{s, t\}$.

Show that a maximum flow problem on a flow network with vertex and edge capacities can be reduced to a maximum flow problem in a normal flow network G', i.e., without vertex capacities. How many more vertices and edges does G' require precisely?

3 Points

Exercise 2 – Implementing the generic push–relabel algorithm

In the lecture we have only outlined how the methods PUSH and RELABEL work and not discussed how we actually find the next applicable operation at all.

a) Show how to implement PUSH in $\mathcal{O}(1)$ time, RELABEL in $\mathcal{O}(|V|)$, and how to select an applicable operation in $\mathcal{O}(1)$ time. In particular, what data structure would you use to maintain overflowing vertices? **6 Points**

b) Using these implementations, what is the resulting runtime of the push–relabel algorithm? **1 Point**

Exercise 3 – Push–relabel for integer flow network

Show that if each edge in a flow network has integral capacity, then the push–relabel algorithm finds an integral maximum flow. **2 Points**

Exercise 4 – Maximum bipartite matching

A *matching* in a graph $G = (V, E)$ is a subset of edges $M \subseteq E$ such that for all $v \in V$, at most one edge of M is incident to v. A *maximum matching* is a matching with maximum cardinality, that is, a matching M such that for any matching M', we have $|M| \geq |M'|$. A *bipartite* graph $G = (L, R, E)$ has two vertex sets L and R and each edge in E connects a vertex in L to a vertex in R.

a) Explain how we can transform the problem of finding a maximum matching in a bipartite graph $G = (L, R, E)$ into a maximum flow problem on a flow network N. **4 Points**

b) Prove that the cardinality of a maximum matching M in G equals the value of a maximum integer flow f in your constructed flow network N. **4 Points**

Please hand in your solutions on Wuecampus until the beginning of the next lecture, that is 14:15 on Wednesday, November 29.