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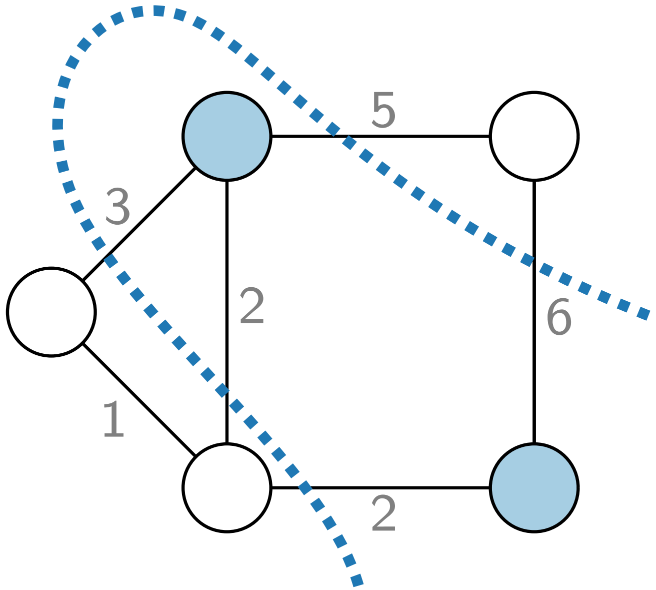
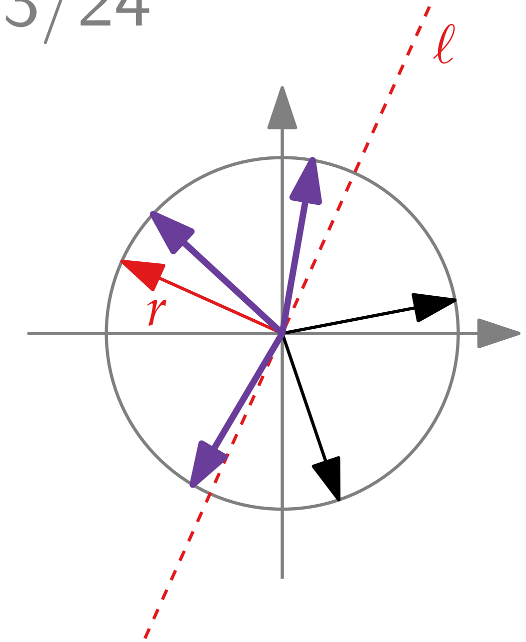
Advanced Algorithms

QP-Relaxation
for MaxCut

Johannes Zink · WS23/24

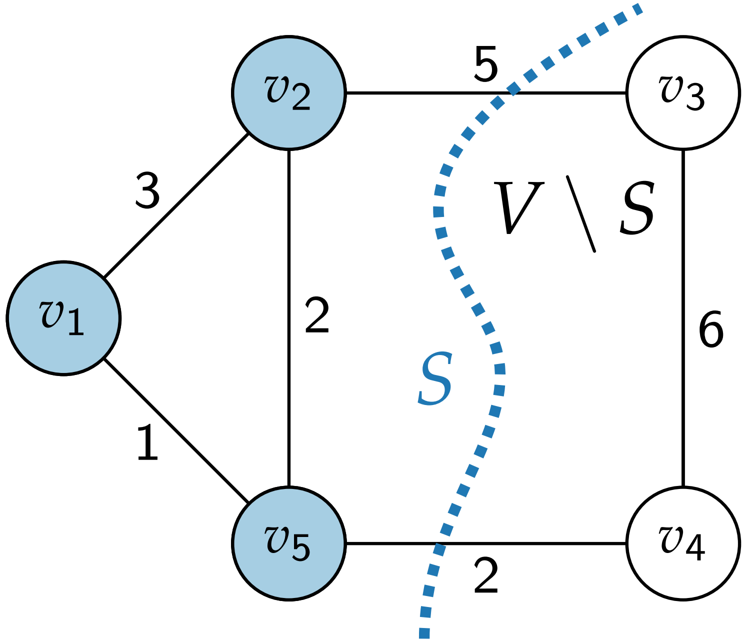




Cut

- Let $G = (V, E)$ be a graph with edge weights $w\colon E \to \mathbb{N}$.
- A **cut** of G is a partition $(S, V \setminus S)$ of V with $\emptyset \neq S \neq V$.
- The **weight** of a cut $(S, V \setminus S)$ is

$$w(S, V \setminus S) = \sum_{\substack{uv \in E,\\ u \in S, v \in V \setminus S}} w(uv)$$



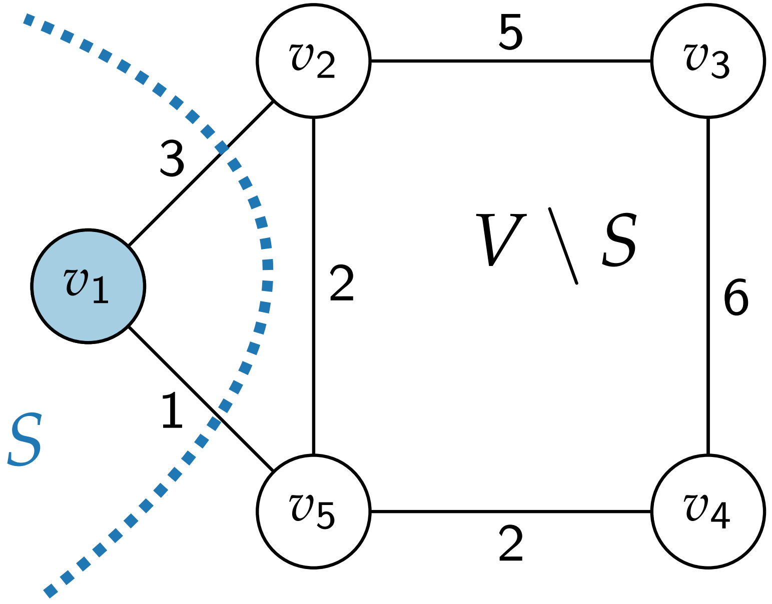
$$w(\{v_1, v_2, v_5\}, \{v_3, v_4\})$$
$$= w(v_2 v_3) + w(v_4 v_5) = 7$$

The **MinCut** Problem

Input. Graph $G = (V, E)$, edge weights $w \colon E \to \mathbb{N}$.

Output. Cut $(S, V \setminus S)$ of G with **minimum** weight.

- Has applications in flow networks (*max-flow min-cut theorem*), finding a bottleneck in a network, graph partition problems, clustering, ...
- Can be solved optimally in polynomial time, e.g., by the Stoer–Wagner algorithm.

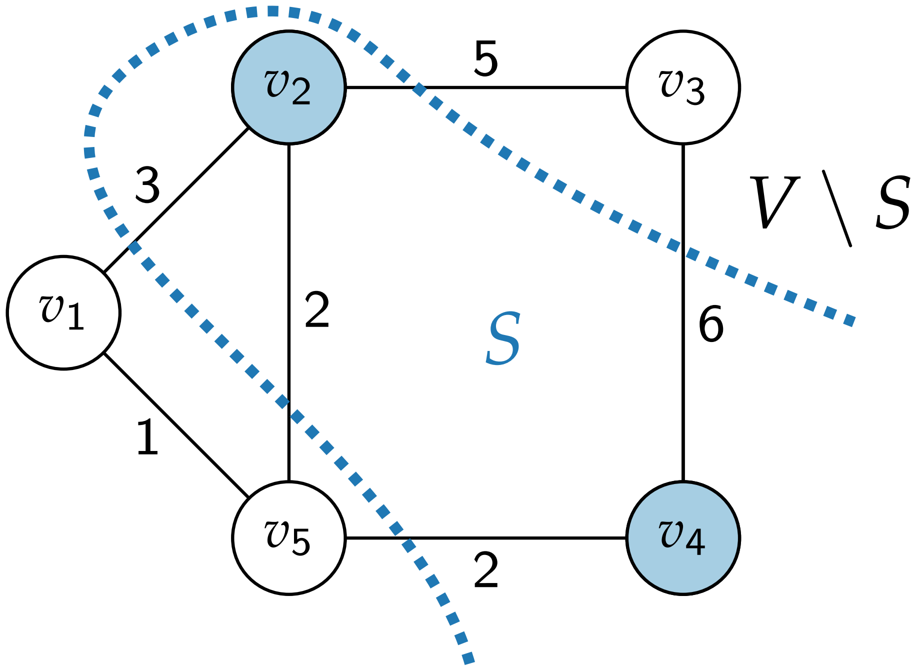


$w(S, V \setminus S) = 4$

The **MaxCut** Problem

Input. Graph $G = (V, E)$, edge weights $w\colon E \to \mathbb{N}$.
Output. Cut $(S, V \setminus S)$ of G with **maximum** weight.

- Has applications in binary classification (vertices are features and weighted edges are distances), statistical physics (equivalent to minimizing the "Hamiltonian" of a spin glass model), and integrated circuit design for computer chips (modeling a specific assignment problem as a graph problem).
- NP-complete to find a cut of maximum weight.



$w(S, V \setminus S) = 18$

Randomized 0.5-Approximation for (Unweighted) MaxCut

Theorem 1.
CoinFlipMaxCut is a randomized 0.5-approximation algorithm for MaxCut.

Proof.

- Runs in $O(n+m)$, where $n = |V|, m = |E|$.
- Compute expected weight of cut:

$$\mathsf{E}[w(\textsc{CoinFlipMaxCut}(G))] = \mathsf{E}\big[|E(S, V \setminus S)|\big] = \sum_{e \in E} \mathsf{P}[e \in E(S, V \setminus S)] = \sum_{e \in E} \frac{1}{2} = \frac{1}{2}|E| \geq \frac{1}{2}\mathsf{OPT}(G)$$

- Can be "de-randomized". Exercise.

```
CoinFlipMaxCut(G, w: E → 1)
  S ← ∅
  foreach v ∈ V do
    if coin flip shows Heads then
      S ← S ∪ {v}
  return w(S, V \ S), S
```

LP-Relaxation

Integer Linear Program

$$\begin{array}{llcl} \textbf{maximize} & c^\mathsf{T} x & & \\ \textbf{subject to} & Ax & \leq & b \\ & x & \geq & 0 \\ & x & \in & \mathbb{Z}^n \end{array}$$

LP-Relaxation

Linear Program

$$\begin{array}{llcl} \textbf{maximize} & c^\mathsf{T} x & & \\ \textbf{subject to} & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

Solve in polynomial time

Solution for LP

$x^\star$

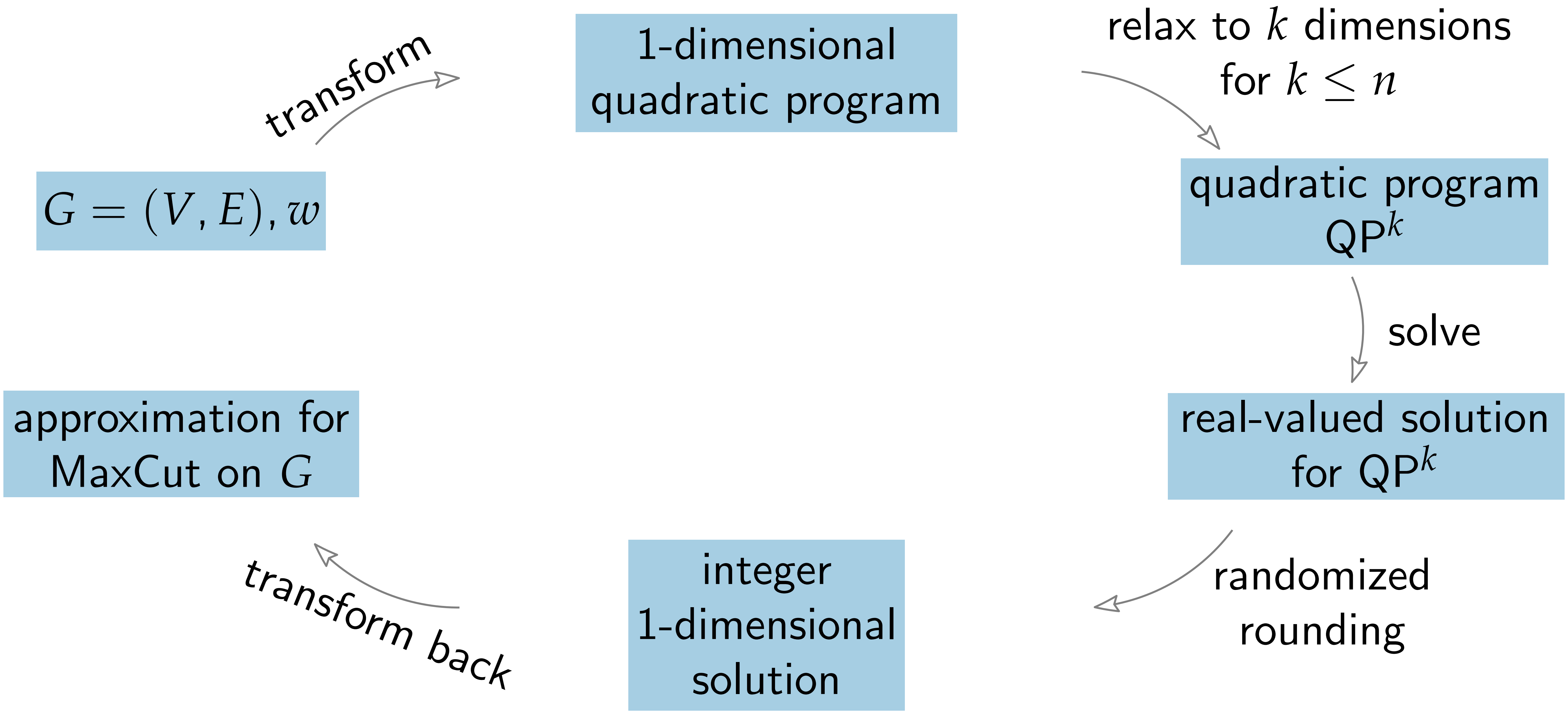
e.g. rounding

Assignment for ILP

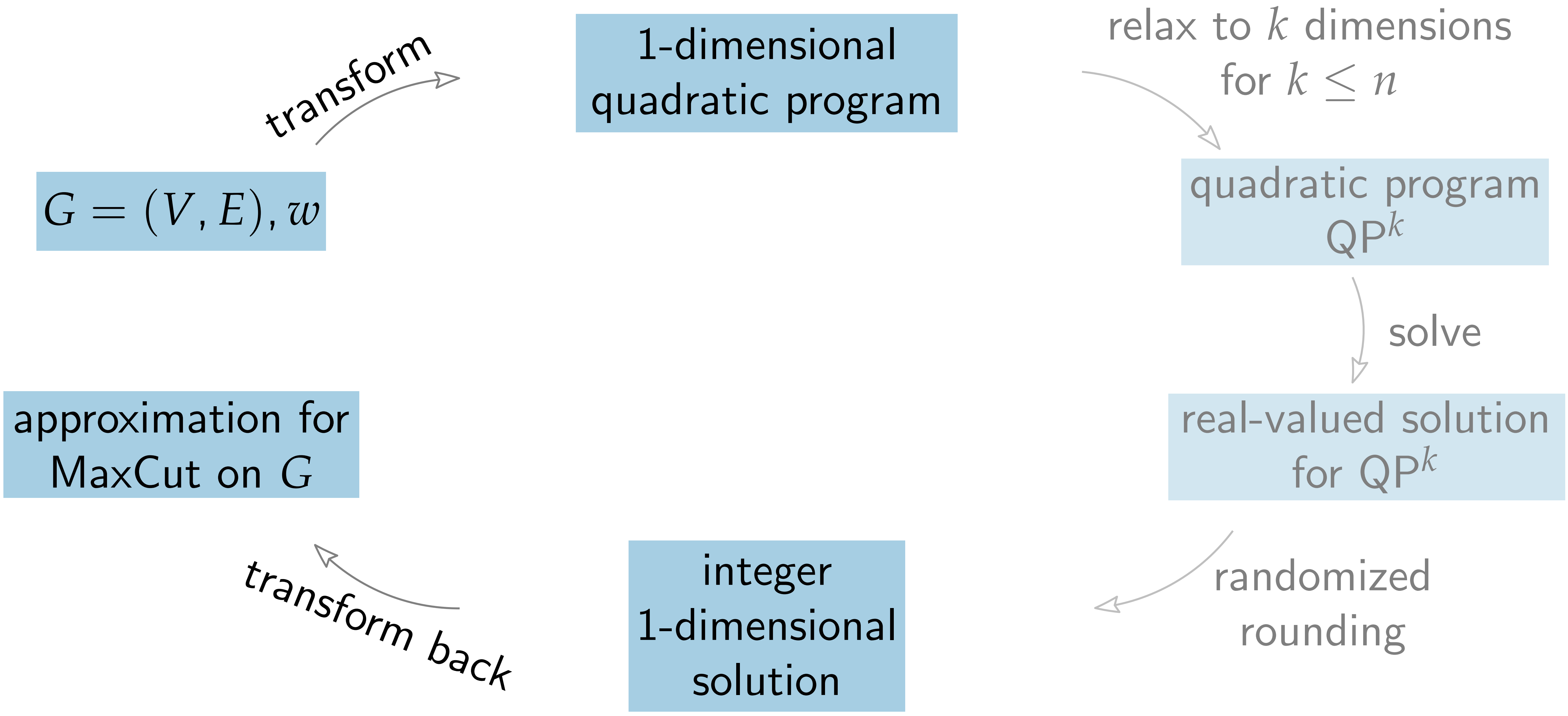
$x^\star$

Solution, approximation, or bound

Goemans-Williamson Algorithm for MaxCut



Goemans-Williamson Algorithm for MaxCut



QP(G, w)

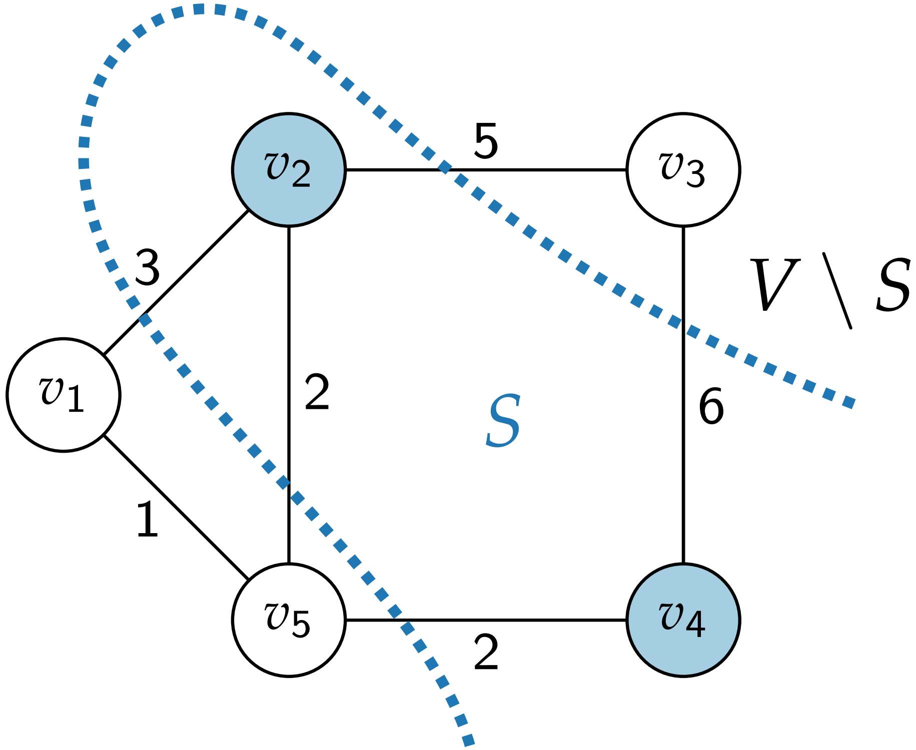
Idea.

- Indicator variable for each vertex v_i: $x_i \in \{1, -1\}$
- $x_i \cdot x_j = \begin{cases} 1 & \text{if } i, j \text{ in same partition} \\ -1 & \text{otherwise} \end{cases}$

QP(G, w)

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j)$

subject to $x_i^2 = 1$



- Weight matrix w_{ij}

	1	2	3	4	5
1		3			1
2	3		5		2
3		5		6	
4			6		2
5	1	2		2	

- Solution

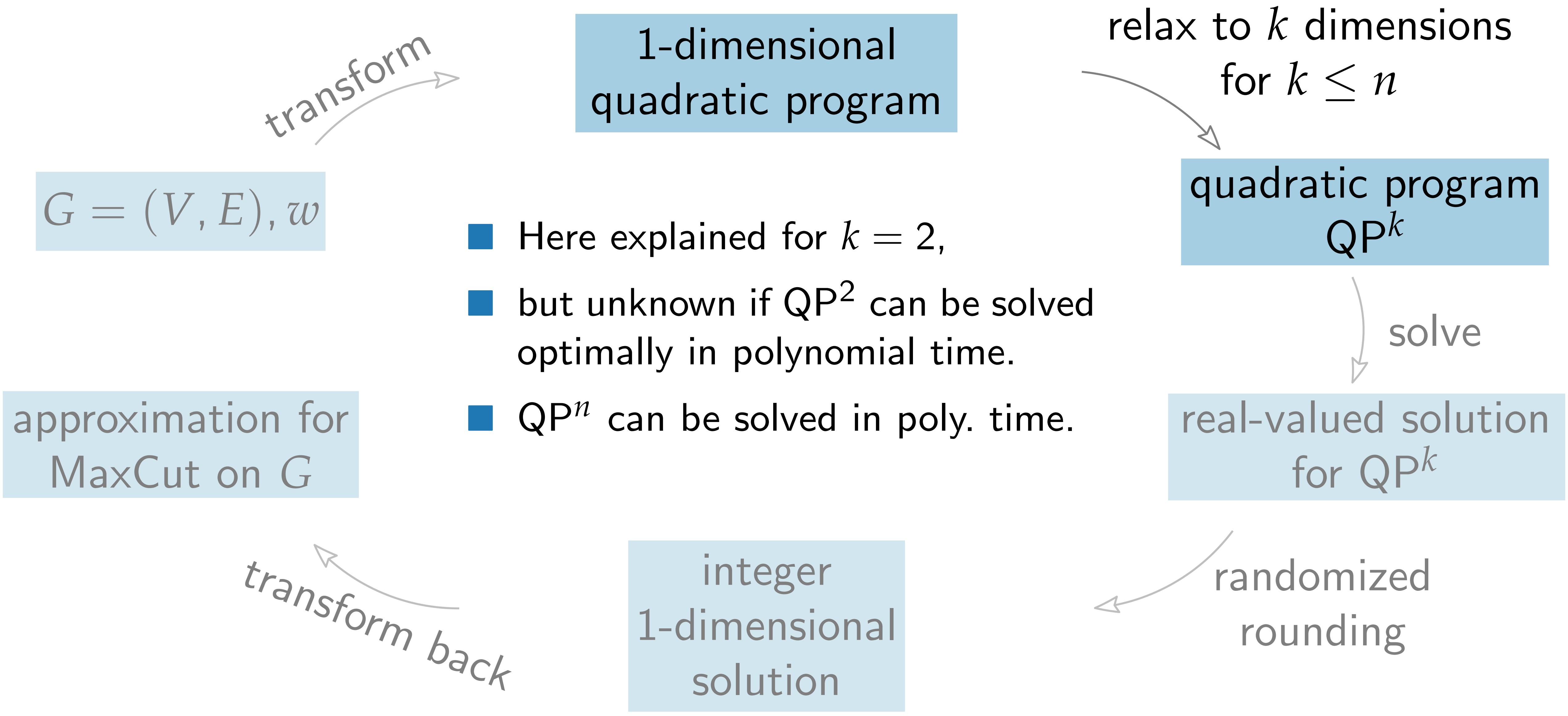
$x_2 = x_4 = 1$

$x_1 = x_3 = x_5 = -1$

Note.

- Solving QP(G, w) is NP-hard.
- Otherwise MaxCut would not be NP-hard.

Goemans-Williamson Algorithm for MaxCut



- Here explained for $k = 2$,
- but unknown if QP^2 can be solved optimally in polynomial time.
- QP^n can be solved in poly. time.

Relaxation of QP(G,w)

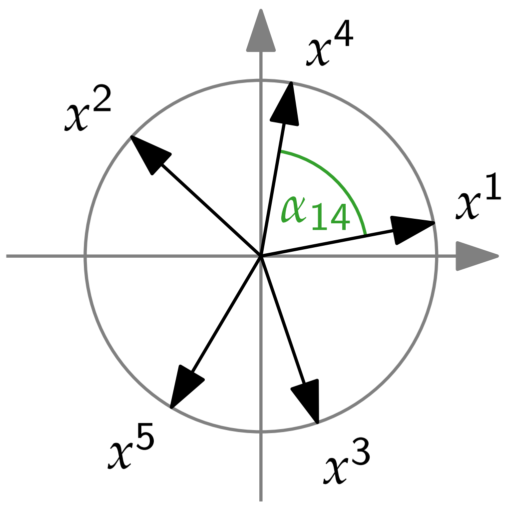
QP$^2(G,w)$

maximize $\quad \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to $\quad x^i \cdot x^i = 1$

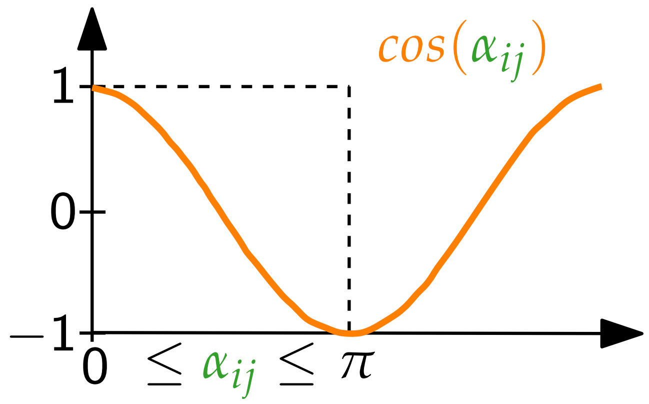
$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$

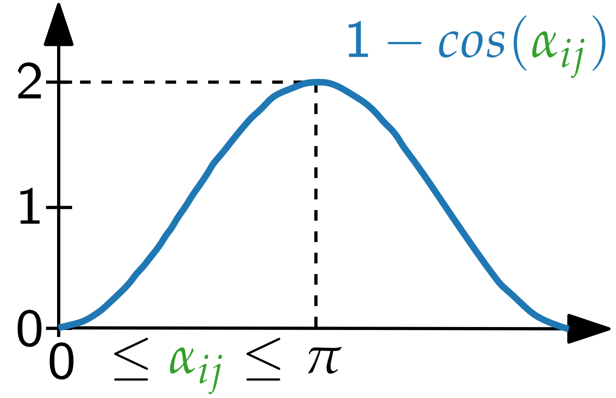
- " $\cdot$ " is scalar product.
- x^i lies on the unit circle.
- $x^i \cdot x^j = \|x^i\|\|x^j\| \cos(\alpha_{ij}) = \cos(\alpha_{ij})$ with $0 \leq \alpha_{ij} \leq \pi$.



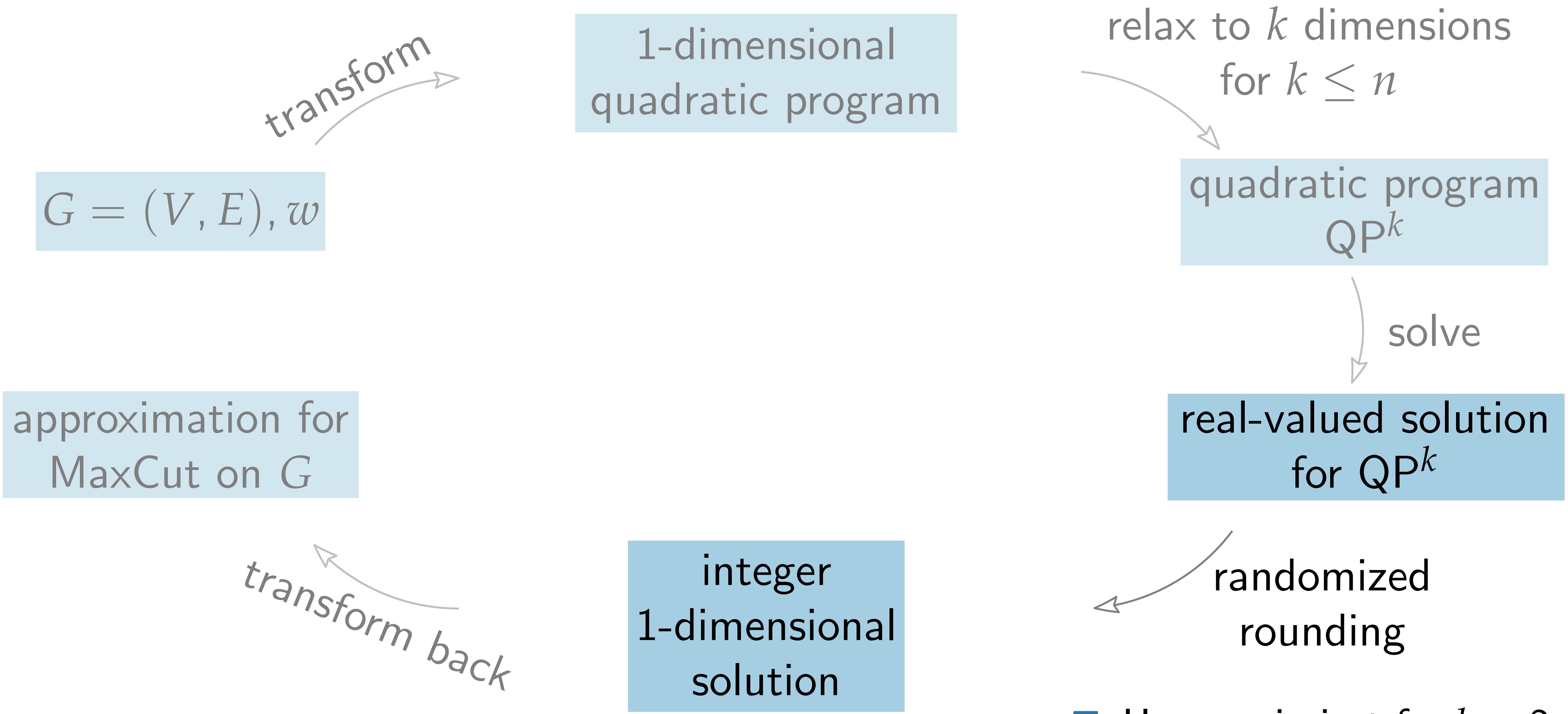
- The variables are 2-dimensional vectors.
- We maximize angles α_{ij} since larger α_{ij} increase the contribution of w_{ij}.
- Hence, our objective is:

$$\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{j-1} w_{ij}(1 - \cos(\alpha_{ij}))$$





Goemans-Williamson Algorithm for MaxCut



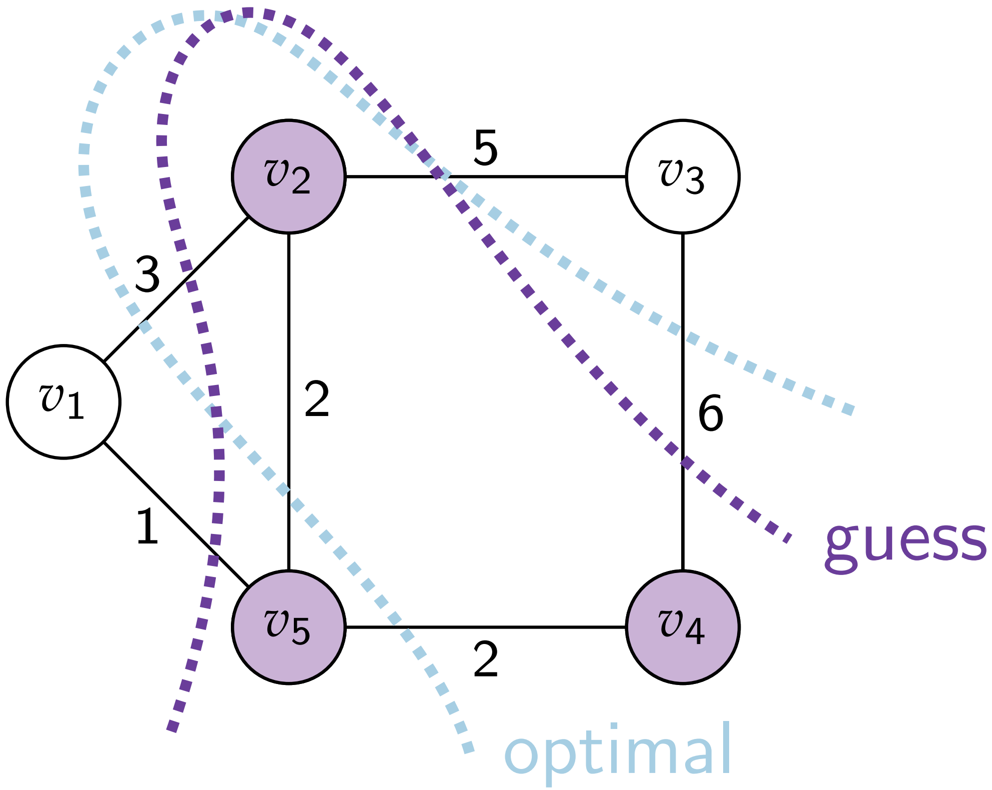
- Here again just for $k = 2$.

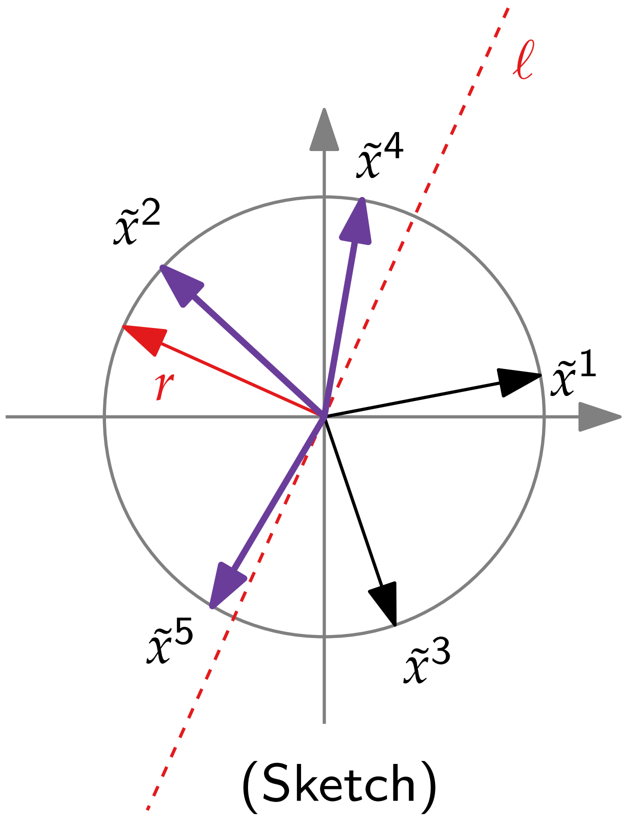
Algorithm RandomizedMaxCut

RandomizedMaxCut(G, w)

Compute optimal solution $(\tilde{x}^1, \ldots, \tilde{x}^n)$ for $\mathrm{QP}^2(G, w)$
Pick random vector $r \in \mathbb{R}^2$
$S \leftarrow \{v_i \in V : \tilde{x}^i \cdot r \geq 0\}$
return $c(S, V \setminus S)$

- $\tilde{x}^i$ lies above the line ℓ orthogonal to r



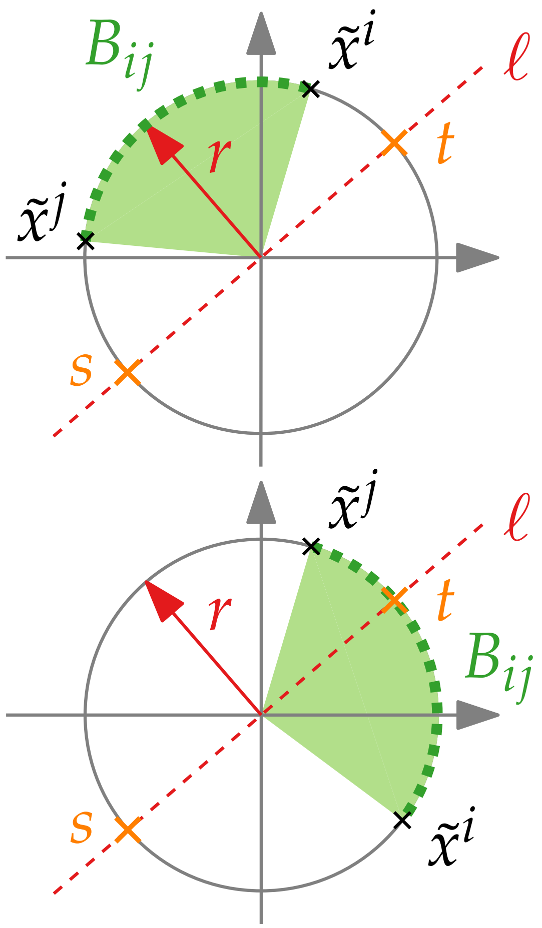


(Sketch)

RandomMaxCut – Expected Value

Lemma 2.
Let X be the solution of RandomizedMaxCut(G, w).
If r is picked uniformly at random, then

$$\mathrm{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi} .$$



Proof.

- $E[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}\, \mathrm{P}[\ell \text{ separates } \tilde{x}^i, \tilde{x}^j] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi}$
- $\mathrm{P}[\ell \text{ separates } \tilde{x}^i, \tilde{x}^j] = \mathrm{P}[s \text{ or } t \text{ lies on } B_{ij}] = \frac{\alpha_{ij}}{2\pi} + \frac{\alpha_{ij}}{2\pi} = \frac{\alpha_{ij}}{\pi}$
- B_{ij} has length α_{ij}.
- If $\tilde{x}^i$ (or $\tilde{x}^j$) lies $\leq \alpha_{ij}$ before s or t on the perimter of the unit disk, s or t lies on B_{ij}.

RandomMaxCut – Quality

Theorem 3.
Let X be the solution of RandomizedMaxCut(G, w). Then

$$\frac{\mathsf{E}[X]}{\mathsf{OPT}(G,w)} \geq 0.8785.$$

Proof.

- Lemma 2: $\mathsf{E}[X] = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi}$
- Optimal solution for QP^2:

$$\mathsf{QP}^2(G, w) = \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) = \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{1-\cos(\alpha_{ij})}{2}$$

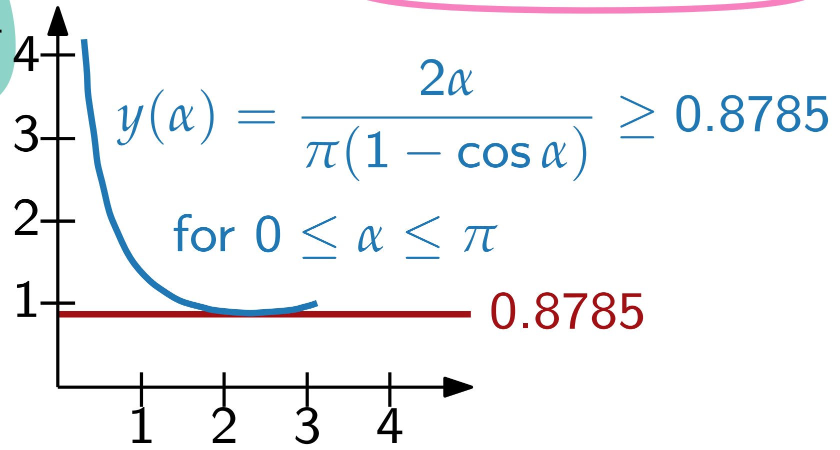
- $\mathsf{QP}^2(G, w)$ is relaxation of $\mathsf{QP}(G, w)$:

$$\mathsf{QP}^2(G, w) \geq \mathsf{QP}(G, w) = \mathsf{OPT}(G, w)$$

- $$\frac{\mathsf{E}[X]}{\mathsf{OPT}(G,w)} \geq \frac{\mathsf{E}[X]}{\mathsf{QP}^2(G,w)} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{\alpha_{ij}}{\pi}}{\sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij} \frac{1-\cos(\alpha_{ij})}{2}} \geq 0.8785$$

- $$\frac{\frac{\alpha_{ij}}{\pi}}{\frac{1-\cos(\alpha_{ij})}{2}} \geq 0.8785$$

$$\Leftrightarrow \frac{\alpha_{ij}}{\pi} \geq 0.8785 \frac{1-\cos(\alpha_{ij})}{2}$$



Example

1. Step: Build QP

$$\begin{aligned} \textbf{maximize} \quad & \frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x_i x_j) \\ \textbf{subject to} \quad & x_i^2 = 1 \end{aligned}$$

Weight matrix w_{ij}

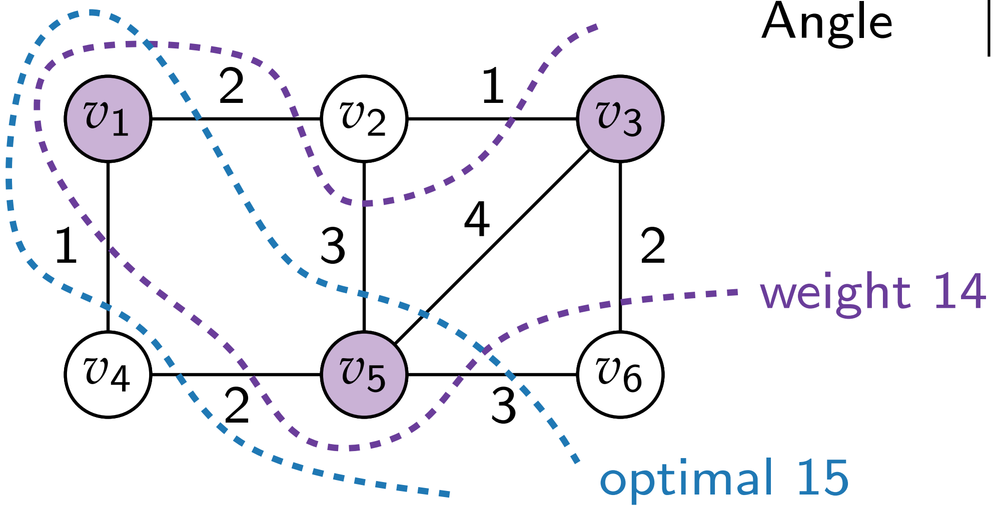
	1	2	3	4	5	6
1		2		1		
2	2		1		3	
3		1			4	2
4	1				2	
5		3	4	2		3
6			2		3	

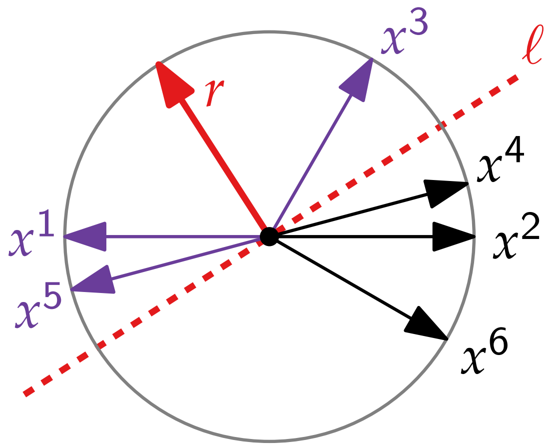
2. Step: Relax QP to QP^2

$$\begin{aligned} \textbf{maximize} \quad & \frac{1}{2}\sum_{j=1}^{6}\sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j) \\ \textbf{subject to} \quad & x^i \cdot x^i = 1 \\ & x^i = (x^i_1, x^i_2) \in \mathbb{R}^2 \end{aligned}$$

3. Step: Solve QP^2

Variable	x^1	x^2	x^3	x^4	x^5	x^6
Angle	0	180	120	165	345	210





4. Step: Guess r

5. Step: Derive S

Goemans-Williamson Algorithm for MaxCut

$G = (V, E), w$

transform

1-dimensional quadratic program

relax to k dimensions for $k \leq n$

quadratic program QP^k

solve

real-valued solution for QP^k

randomized rounding

integer 1-dimensional solution

transform back

approximation for MaxCut on G

- So far, $k = 2$.
- QP^n can be solved in polynomial time.

$\mathrm{QP}^n(G, w)$

$\mathrm{QP}^2(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i = (x^i_1, x^i_2) \in \mathbb{R}^2$

$\mathrm{QP}^n(G, w)$

maximize $\frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{j-1} w_{ij}(1 - x^i \cdot x^j)$

subject to $x^i \cdot x^i = 1$

$x^i \in \mathbb{R}^n$

- A matrix M is called **positive semidefinite** if for any vector $v \in \mathbb{R}^n$:
$$v^\intercal \cdot M \cdot v \geq 0$$
- $M = (m_{ij}) = (x^i \cdot x^j)$ is positive semidefinite.
- $\mathrm{QP}^n(G, w)$ becomes the problem $\textsc{SemiDefiniteCut}(G, w)$.
  - Can be approximated in time polynomial in (G, w) and $1/\varepsilon$ with additive guarantee ε.
  - Note that the approximation of $\mathrm{QP}(G, w)$ is an extra step we have seen before. (The approximation of $\mathrm{QP}(G, w)$ with factor 0.8785 works for $\mathrm{QP}^n(G, w)$, too)

Discussion

- If the *Unique Games Conjecture* is true, then the approximation ratio of ≈ 0.8785 achieved by SemiDefiniteCut (and RandomizedMaxCut) is best possible.
- Otherwise, no approximation ratio better than $\frac{16}{17} \approx 0.941$ is possible.
  In particular no polynomial-time approximation scheme (PTAS) exists.
- On planar graphs, the MaxCut problem can be solved optimally in polynomial time.

- Semidefinite programming is a powerful tool to develop approximation algorithms
- Whole book on this topic:
  - [Gärtner, Matoušek] "Approximation Algorithms and Semidefinite Progamming"

- Using randomness is another tool to design approximation algorithms.

$\rightarrow$ See future lectures, in particular the next lecture!

Literature

Original paper:

- [GW '95] "Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming"

Source:

- [Vazirani Ch26] "Approximation Algorithms"

Whole book on this topic:

- [Gärtner, Matoušek] "Approximation Algorithms and Semidefinite Progamming"