Chair of Computer Science I
Algorithms and Complexity
University of Würzburg

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# Homework Assignment #7
# Algorithms for Geographic Information Systems (SS 2023)

**Exercise 1 – Polynomial fitting**

Let $S = \{(0, -4), (1, 11), (-12, 1)\}$ be a set of points $(x, y) \in \mathbb{R}^2$. It seems as if the relation between the points is best described by a straight line, that is, a polynomial of the form $y = mx + b$ for some values $m$ and $b$.

Model the problem of identifying $m$ and $b$ as a linear least-square problem. What are $A$ and $L$? Then solve the problem. What are $m$ and $b$ in the solution, and what is $v$? **[4 points]**

**Exercise 2 – Matrices with small rank**

Let $A \in \mathbb{R}^{n\times u}$, and let $L \in \mathbb{R}^n$. In the lecture it was shown that if $\text{rank}(A) \geq n$, then there always exists a unique solution $X \in \mathbb{R}^u$ and a vector $v \in \mathbb{R}^n$ with $AX = L + v$ such that $v^T v$ is minimum.

Now assume that $\text{rank}(A) < u$. Prove or disprove:

a) There always exist a least-squares solution $X_0$ for $AX = L$. **[1 point]**

b) If there exists a least-squares solution $X_0 \in \mathbb{R}^u$ for $AX = L$, then this solution is unique. **[2 points]**

**Exercise 3 – Orthogonal relations**

Let $A \in \mathbb{R}^{n\times u}$, let $L \in \mathbb{R}^n$, let $X \in \mathbb{R}^u$, and let $v \in \mathbb{R}^n$ be a least-squares solution, that is, $AX = L + v$ and $v^T v$ is minimum. Show that $v$ is orthogonal to the linear subspace $\text{Im}(A) = \{Ax \mid x \in \mathbb{R}^u\}$ of $\mathbb{R}^u$. **[6 points]**

**Exercise 4 – Pairwise orthogonal vectors**

a) Let $A \in \mathbb{R}^{n\times u}$ be such that the colums $\{a_1, a_2, \ldots, a_u\}$ of $A$ are pairwise orthogonal, and let $L \in \mathbb{R}^n$. Let $X = \left(\frac{L\cdot a_1}{a_1\cdot a_1}, \frac{L\cdot a_2}{a_2\cdot a_2}, \ldots, \frac{L\cdot a_u}{a_u\cdot a_u}\right)^T$, where $\cdot$ denotes the scalar product.

Prove that $A^T AX = A^T L$ (meaning that $X$ is a least-squares solution). **[5 points]**

b) Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, and let $L = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$. Find the least-squares solution of $AX = L$. **[2 points]**

Please hand in your solutions on WueCampus until 12:00 on Thuesday, July 06.